[Chapter 3, p33]

Constructions on Domains

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \qquad \text{(lub1)}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

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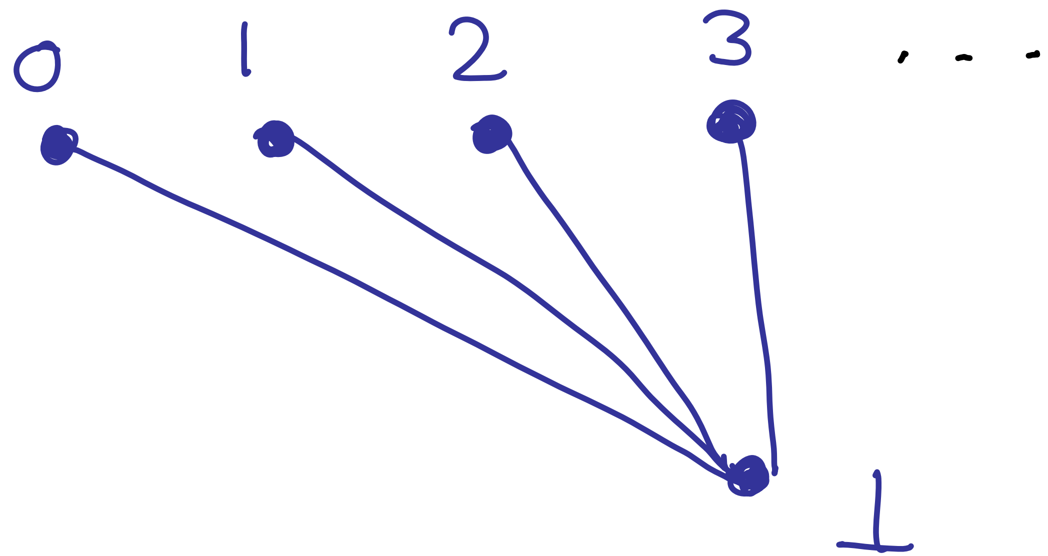
Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Eg $N_\perp$ looks like:



Note that every chain $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in $X_\perp$ is eventually constant (i.e. $\exists N.\ \forall n \geq N.\ d_n = d_N$) and so has a lub.

Note that every chain $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in $X_\perp$ is eventually constant (i.e. $\exists N. \forall n \geq N. \ d_n = d_N$) and so has a lub.

Hence

- $X_\perp$ does have lubs of chains

Note that every chain $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in $X_\perp$ is eventually constant (i.e. $\exists N.\, \forall n \geq N.\ d_n = d_N$) and so has a lub.

Hence

- $X_\perp$ does have lubs of chains
- a function $f : X_\perp \to D$ (with D a domain) is continuous if & only if it is monotone (iff $\forall x \in X.\ f(\perp) \sqsubseteq f(x)$)

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \ \stackrel{\text{def}}{\Leftrightarrow} \ d_1 \sqsubseteq_1 d_1' \ \& \ d_2 \sqsubseteq_2 d_2' \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}) \ .$$

Chain in $D_1 \times D_2$

$$(d_{1,1}, d_{2,1}) \sqsubseteq (d_{1,2}, d_{2,2}) \sqsubseteq (d_{1,3}, d_{2,3}) \sqsubseteq \cdots$$

$$\text{get} \begin{cases} d_{1,1} \sqsubseteq d_{1,2} \sqsubseteq d_{1,3} \sqsubseteq \cdots \text{ chain in } D_1 \\ d_{2,1} \sqsubseteq d_{2,2} \sqsubseteq d_{2,3} \sqsubseteq \cdots \text{ chain in } D_2 \end{cases}$$

Chain in $D_1 \times D_2$

$$(d_{1,1}, d_{2,1}) \sqsubseteq (d_{1,2}, d_{2,2}) \sqsubseteq (d_{1,3}, d_{2,3}) \sqsubseteq \cdots$$

get $\begin{cases} d_{1,1} \sqsubseteq d_{1,2} \sqsubseteq d_{1,3} \sqsubseteq \cdots \text{ chain in } D_1 \\ d_{2,1} \sqsubseteq d_{2,2} \sqsubseteq d_{2,3} \sqsubseteq \cdots \text{ chain in } D_2 \end{cases}$

So we can form $\begin{cases} \bigsqcup_{i \geq 0} d_{1,i} \text{ lub in } D_1 \\ \bigsqcup_{j \geq 0} d_{2,j} \text{ lub in } D_2 \end{cases}$

if chain in $D_1 \times D_2$ has an upper bound

$$(d_{1,1}, d_{2,1}) \sqsubseteq (d_{1,2}, d_{2,2}) \sqsubseteq (d_{1,3}, d_{2,3}) \sqsubseteq \cdots \sqsubseteq (x_1, x_2)$$

then get

$$\begin{cases} d_{1,1} \sqsubseteq d_{1,2} \sqsubseteq d_{1,3} \sqsubseteq \cdots \sqsubseteq x_1 & D_1 \\ d_{2,1} \sqsubseteq d_{2,2} \sqsubseteq d_{2,3} \sqsubseteq \cdots \sqsubseteq x_2 & D_2 \end{cases}$$

hence

$$\begin{cases} \bigsqcup_{i \geq 0} d_{1,i} \sqsubseteq x_1 & D_1 \\ \bigsqcup_{j \geq 0} d_{2,j} \sqsubseteq x_2 & D_2 \end{cases}$$

and thus $\left(\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j} \right) \sqsubseteq (x_1, x_2)$ in $D_1 \times D_2$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1\times D_2} = (\perp_{D_1}, \perp_{D_2})$.

for all $(d_1, d_2) \in D_1 \times D_2$

$$\left.\begin{array}{ll} \perp_{D_1} \sqsubseteq d_1 & \text{in } D_1 \\ \perp_{D_2} \sqsubseteq d_2 & \text{in } D_2 \end{array}\right\} \text{so } (\perp, \perp) \sqsubseteq (d_1, d_2) \text{ in } D_1 \times D_2$$

Continuous functions of two arguments

Proposition. *Let* D, E, F *be cpo's. A function* $f : (D \times E) \to F$ *is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m \,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$
$$f(d \,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

If we just know $\begin{cases} \text{for all } d, d', e, e' : \\ d \sqsubseteq d' \Longrightarrow f(d, e) \sqsubseteq f(d', e) \\ e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e') \end{cases}$

then we get $f : D \times E \to F$ is monotone :

If we just know $\begin{cases} \text{for all } d, d', e, e' : \\ d \sqsubseteq d' \Rightarrow f(d,e) \sqsubseteq f(d',e) \\ e \sqsubseteq e' \Rightarrow f(d,e) \sqsubseteq f(d,e') \end{cases}$

then we get $f : D \times E \to F$ is monotone :

$$
\begin{aligned}
(d,e) \sqsubseteq (d',e') &\Rightarrow d \sqsubseteq d' \ \& \ e \sqsubseteq e' \\
&\Rightarrow f(d,e) \sqsubseteq f(d',e) \ \& \ e \sqsubseteq e' \\
&\Rightarrow \text{" "} \quad f(d',e) \sqsubseteq f(d',e') \\
&\Rightarrow f(d,e) \sqsubseteq f(d',e')
\end{aligned}
$$

If we just know $\left\{ \begin{array}{l} \text{for all } d, d', e, e' : \\ d \sqsubseteq d' \Rightarrow f(d,e) \sqsubseteq f(d',e) \\ e \sqsubseteq e' \Rightarrow f(d,e) \sqsubseteq f(d,e') \end{array} \right.$

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&\Rightarrow \text{" \quad "} \quad f(d',e) \sqsubseteq f(d',e') \\
&\Rightarrow f(d,e) \sqsubseteq f(d',e')
\end{aligned}$$

If we just know monotonicity + $\begin{cases} f\left(\bigsqcup_{m\geq 0} d_m\ ,\ e\right) \sqsubseteq \bigsqcup_{m\geq 0} f(d_m, e) \\ f\left(d, \bigsqcup_{n\geq 0} e_n\right) \sqsubseteq \bigsqcup_{n\geq 0} f(d, e_n) \end{cases}$

then we get ~~that~~ $f : D \times E \to F$ is continuous:

If we just know monotonicity +
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$$f\left(\bigsqcup_{n\geq 0} (d_n, e_n)\right) = f\left(\bigsqcup_{i\geq 0} d_i, \bigsqcup_{j\geq 0} e_j\right)$$

If we just know monotonicity +
$$\begin{cases} f\left(\bigsqcup_{m\geq 0} d_m \,,\; e\right) = \bigsqcup_{m\geq 0} f(d_m, e) \\ f\left(d, \bigsqcup_{n\geq 0} e_n\right) = \bigsqcup_{n\geq 0} f(d, e_n) \end{cases}$$

then we get ~~that~~ $f : D \times E \to F$ is continuous:

$$f\left(\bigsqcup_{n\geq 0} (d_n, e_n)\right) = f\left(\bigsqcup_{i\geq 0} d_i \,,\quad e \quad\right)$$

where $e = \bigsqcup_{j\geq 0} e_j$

$$= \bigsqcup_{i\geq 0} f(d_i \,,\quad e \quad)$$

If we just know monotonicity + $\begin{cases} f\left(\bigsqcup_{m\geq 0} d_m , e\right) = \bigsqcup_{m\geq 0} f(d_m, e) \\ f\left(d, \bigsqcup_{n\geq 0} e_n\right) = \bigsqcup_{n\geq 0} f(d, e_n) \end{cases}$

then we get ~~that~~ $f : D \times E \to F$ is continuous:

$$f\left(\bigsqcup_{n\geq 0} (d_n, e_n)\right) = f\left(\bigsqcup_{i\geq 0} d_i , \bigsqcup_{j\geq 0} e_j\right)$$

$$= \bigsqcup_{i\geq 0} f\left(d_i , \bigsqcup_{j\geq 0} e_j\right)$$

$$= \bigsqcup_{i\geq 0} \left(\bigsqcup_{j\geq 0} f(d_i, e_j)\right)$$

If we just know monotonicity + $\begin{cases} f(\bigsqcup_{m\geq 0} d_m, e) = \bigsqcup_{m\geq 0} f(d_m, e) \\ f(d, \bigsqcup_{n\geq 0} e_n) = \bigsqcup_{n\geq 0} f(d, e_n) \end{cases}$

then we get ~~that~~ $f : D \times E \to F$ is continuous:

$$
\begin{aligned}
f\left(\bigsqcup_{n\geq 0} (d_n, e_n)\right) &= f\left(\bigsqcup_{i\geq 0} d_i , \bigsqcup_{j\geq 0} e_j\right) \\
&= \bigsqcup_{i\geq 0} f\left(d_i , \bigsqcup_{j\geq 0} e_j\right) \\
&= \bigsqcup_{i\geq 0}\left(\bigsqcup_{j\geq 0} f(d_i, e_j)\right) \\
&\overset{\text{See Slide 27}}{=} \bigsqcup_{k\geq 0} f(d_k, e_k)
\end{aligned}
$$

Diagonalising a double chain

Lemma. *Let* D *be a cpo. Suppose that the doubly-indexed family of elements* $d_{m,n} \in D$ *(*$m, n \geq 0$*) satisfies*

$$m \leq m' \;\&\; n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots$$

Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) .$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

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and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n \; = \; \lambda d \in D. \underbrace{\bigsqcup_{n\geq 0} f_n(d)}\;.$$

have to see ~~that~~ { this is a (well-defined) continuous function
it is a lub for $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$

If E is a domain, then so is $D \to E$ and $\bot_{D\to E}(d) = \bot_E$, all $d \in D$.

Given $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ in $D \to E$

for each $d \in D$ we get

$$f_0(d) \sqsubseteq f_1(d) \sqsubseteq f_2(d) \sqsubseteq \cdots \quad \text{chain in } E$$

and can form its lub $\bigsqcup_{n \geq 0} f_n(d)$.

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for each $d \in D$ we get

$$f_0(d) \sqsubseteq f_1(d) \sqsubseteq f_2(d) \sqsubseteq \cdots \quad \text{chain in } E$$

and can form its lub $\bigsqcup_{n \geq 0} f_n(d)$.

$\lambda d \in D . \bigsqcup_{n \geq 0} f_n(d)$ is monotone, because:

$$d \sqsubseteq d' \Rightarrow \forall n \geq 0 . f_n d \sqsubseteq f_n d'$$

$$\Rightarrow \bigsqcup_{n \geq 0} f_n d \sqsubseteq \bigsqcup_{n \geq 0} f_n d'$$

Given $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ in $D \to E$

for each $d \in D$ we get

$$f_0(d) \sqsubseteq f_1(d) \sqsubseteq f_2(d) \sqsubseteq \cdots \quad \text{chain in } E$$

and can form its lub $\bigsqcup_{n \geq 0} f_n(d)$.

$\lambda d \in D . \bigsqcup_{n \geq 0} f_n(d)$ is continuous, because:

$$\bigsqcup_{n\geq 0} f_n\left(\bigsqcup_{m \geq 0} d_m\right) = \bigsqcup_{n \geq 0}\left(\bigsqcup_{m \geq 0} f_n(d_m)\right) \quad \text{each } f_n \text{ is continuous}$$

$$= \bigsqcup_{k \geq 0} f_k(d_k) \quad \text{Slide 27}$$

$$= \bigsqcup_{m \geq 0}\left(\bigsqcup_{n \geq 0} f_n(d_m)\right) \quad \text{slide 27}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n = \lambda d \in D. \bigsqcup_{n\geq 0} f_n(d) \ .$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

$(\lambda d. \perp) \sqsubseteq f$ because $\forall d.\ (\lambda d. \perp)(d) = \perp \sqsubseteq f(d)$

Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D .\, g\big(f(d)\big)$$

is continuous.

proof left as an exercise

Evaluation function $ev : (D \to E) \times D \to E$

$$ev(f, d) = f(d)$$

Monotone: if $f \sqsubseteq f'$ and $d \sqsubseteq d'$, then

$$ev(f,d) = f(d) \sqsubseteq f(d')$$

f is monotone

$$\sqsubseteq f'(d')$$

$$= ev(f', d')$$

definition of $f \sqsubseteq f'$

Evaluation function $\quad ev : (D \to E) \times D \to E$

$$ev(f, d) = f(d)$$

Continuous: if $\begin{cases} f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots & \text{in } D \to E \\ d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots & \text{in } D \end{cases}$ then

$$ev\left(\bigsqcup_{n \geq 0} (f_n, d_n)\right) = ev\left(\bigsqcup_{i \geq 0} f_i, \bigsqcup_{j \geq 0} d_j\right)$$

$\boxed{\text{lubs in } (D \to E) \times D}$

Evaluation function $ev : (D \to E) \times D \to E$

$$ev(f, d) = f(d)$$

Continuous : if $\begin{cases} f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots & \text{in } D \to E \\ d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots & \text{in } D \end{cases}$ then

$$ev\Big(\bigsqcup_{n \geq 0} (f_n, d_n)\Big) = ev\Big(\bigsqcup_{i \geq 0} f_i , \bigsqcup_{j \geq 0} d_j\Big)$$

$$= \Big(\bigsqcup_{i \geq 0} f_i\Big)\Big(\bigsqcup_{j \geq 0} d_j\Big)$$

$\boxed{\text{def}^{n}\text{: of } ev}$

Evaluation function $ev : (D \to E) \times D \to E$

$$ev(f, d) = f(d)$$

Continuous: if $\begin{cases} f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots & \text{in } D \to E \\ d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots & \text{in } D \end{cases}$ then

$$
\begin{aligned}
ev\left(\bigsqcup_{n \geq 0} (f_n, d_n)\right) &= ev\left(\bigsqcup_{i \geq 0} f_i, \bigsqcup_{j \geq 0} d_j\right) \\
&= \left(\bigsqcup_{i \geq 0} f_i\right)\left(\bigsqcup_{j \geq 0} d_j\right) \\
&= \bigsqcup_{i \geq 0} f_i\left(\bigsqcup_{j \geq 0} d_j\right)
\end{aligned}
$$

$\boxed{\text{lubs in } D \to E}$

Evaluation function $ev : (D \to E) \times D \to E$

$$ev(f, d) = f(d)$$

Continuous : if $\begin{cases} f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots & \text{in } D \to E \\ d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots & \text{in } D \end{cases}$ then

$$
\begin{aligned}
ev\left(\bigsqcup_{n \geq 0} (f_n, d_n)\right) &= ev\left(\bigsqcup_{i \geq 0} f_i, \bigsqcup_{j \geq 0} d_j\right) \\
&= \left(\bigsqcup_{i \geq 0} f_i\right)\left(\bigsqcup_{j \geq 0} d_j\right) \\
&= \bigsqcup_{i \geq 0} f_i\left(\bigsqcup_{j \geq 0} d_j\right) \\
&= \bigsqcup_{i \geq 0} \bigsqcup_{j \geq 0} f_i(d_j)
\end{aligned}
$$

each f_i is cts

Evaluation function $ev : (D \to E) \times D \to E$

$$ev(f, d) = f(d)$$

Continuous : if $\begin{cases} f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots & \text{in } D \to E \\ d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots & \text{in } D \end{cases}$ then

$$
\begin{aligned}
ev\left(\bigsqcup_{n \geq 0} (f_n, d_n)\right) &= ev\left(\bigsqcup_{i \geq 0} f_i, \bigsqcup_{j \geq 0} d_j\right) \\
&= \left(\bigsqcup_{i \geq 0} f_i\right)\left(\bigsqcup_{j \geq 0} d_j\right) \\
&= \bigsqcup_{i \geq 0} f_i\left(\bigsqcup_{j \geq 0} d_j\right) \\
&= \bigsqcup_{i \geq 0} \bigsqcup_{j \geq 0} f_i(d_j) \\
\boxed{\text{Slide 27}} \longrightarrow &= \bigsqcup_{k \geq 0} f_k(d_k) \\
\boxed{\text{def}^{\text{n}} \text{ of } ev} \longrightarrow &= \bigsqcup_{k \geq 0} ev(f_k, d_k)
\end{aligned}
$$

"Currying"

From continuous $f : D' \times D \to E$

we get

$$\boxed{cur(f) = \lambda d' \in D.\, \lambda d \in D.\, f(d', d)}$$

- for each $d' \in D$, $cur(f)(d') \in D \to E$ (i.e. is continuous)

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From continuous $f : D' \times D \to E$

we get

$$\boxed{cur(f) = \lambda d' \in D'.\, \lambda d \in D.\, f(d', d)}$$

- for each $d' \in D'$, $cur(f)(d') \in D \to E$ (i.e. is continuous)

$$
\begin{aligned}
cur(f)(d')\left(\bigsqcup_{n \geq 0} d_n\right) &= f\left(d', \bigsqcup_{n \geq 0} d_n\right) \\
&= f\left(\bigsqcup_{n \geq 0} (d', d_n)\right) \\
&= \bigsqcup_{n \geq 0} f(d', d_n) \\
&= \bigsqcup_{n \geq 0} cur(f)(d')(d_n)
\end{aligned}
$$

"Currying"

From continuous $f : D' \times D \to E$

we get

$$\boxed{cur(f) = \lambda d' \in D.\, \lambda d \in D.\, f(d', d)}$$

- for each $d' \in D$, $cur(f)(d') \in D \to E$ (i.e. is continuous)
- $cur(f) \in D' \to (D \to E)$ (i.e. is continuous)

"Currying"

From continuous $f : D' \times D \to E$

we get

$$\boxed{cur(f) = \lambda d' \in D. \lambda d \in D. f(d', d)}$$

- for each $d' \in D$, $cur(f)(d') \in D \to E$ (i.e. is continuous)
- $cur(f) \in D' \to (D \to E)$ (i.e. is continuous)

$$\begin{aligned} cur(f)\left(\bigsqcup_{m \geq 0} d'_m\right)(d) &= f\left(\bigsqcup_{m \geq 0} d'_m, d\right) \\ &= \bigsqcup_{m \geq 0} f(d'_m, d) \\ &= \left(\bigsqcup_{m \geq 0} cur(f)(d'_m)\right)(d) \end{aligned}$$

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $fix(f) \in D$.

Proposition. *The function*

$$fix : (D \to D) \to D$$

is continuous.

Proof just uses defining properties of fix — (lfp1) & (lfp2) rather than the explicit construction $fix(f) = \bigsqcup_{n \geq 0} f^n(\bot)$.

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ \mathit{fix}(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

$$\boxed{fix : (D \to D) \to D}$$

is monotone : if $f \sqsubseteq f'$ in $D \to D$, then

$$f(fix\,f') \sqsubseteq f'(fix\,f') \sqsubseteq fix\,f'$$

$$\mathrm{fix} : (D \to D) \to D$$

is <u>monotone</u>: if $f \sqsubseteq f'$ in $D \to D$, then

$$f(\mathrm{fix}\, f') \sqsubseteq f'(\mathrm{fix}\, f') \sqsubseteq \mathrm{fix}\, f'$$

(lfp1)

so $\mathrm{fix}\, f'$ is a pre-fixed point of f

So by (lfp2) $\quad \mathrm{fix}\, f \sqsubseteq \mathrm{fix}\, f'$

$$\mathrm{fix} : (D \to D) \to D$$

is <u>continuous</u>: given $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ in $D \to D$

want to show $\mathrm{fix}(\bigsqcup_{n\geq 0} f_n) \sqsubseteq \bigsqcup_{n\geq 0} \mathrm{fix}(f_n)$

By (lfp2), enough to show

$$(\bigsqcup_{n\geq 0} f_n)(d) \sqsubseteq d \quad \text{for} \quad d = \bigsqcup_{n\geq 0} \mathrm{fix}(f_n)$$

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But

$$\begin{aligned}
(\textstyle\bigsqcup_{n\geq 0} f_n)(d) &= (\textstyle\bigsqcup_{n\geq 0} f_n)(\textstyle\bigsqcup_{m\geq 0} \mathrm{fix}(f_m)) \\
&= \textstyle\bigsqcup_{n\geq 0} \bigsqcup_{m\geq 0} f_n(\mathrm{fix}(f_m)) \\
&= \textstyle\bigsqcup_{k\geq 0} f_k(\mathrm{fix}(f_k)) \\
&\sqsubseteq \textstyle\bigsqcup_{k\geq 0} \mathrm{fix}(f_k) \\
&= d
\end{aligned}$$

(lfp1) for each f_k