Constructions on Domains
Cpo’s and domains

A chain complete poset, or cpo for short, is a poset \((D, \sqsubseteq)\) in which all countable increasing chains \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots\) have least upper bounds, \(\bigsqcup_{n \geq 0} d_n:\)

\[
\forall m \geq 0 \cdot d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{lub1}
\]

\[
\forall d \in D . (\forall m \geq 0 . d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{lub2}
\]

A domain is a cpo that possesses a least element, \(\bot:\)

\[
\forall d \in D . \bot \sqsubseteq d.
\]
Discrete cpo’s and flat domains

For any set $X$, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\iff} x = x' \quad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set $X$. 
Discrete cpo’s and flat domains

For any set \( X \), the relation of equality

\[
x \sqsubseteq x' \overset{\text{def}}{\iff} x = x' \quad (x, x' \in X)
\]

makes \( (X, \sqsubseteq) \) into a cpo, called the discrete cpo with underlying set \( X \).

Let \( X_\bot \overset{\text{def}}{=} X \cup \{ \bot \} \), where \( \bot \) is some element not in \( X \). Then

\[
d \sqsubseteq d' \overset{\text{def}}{\iff} (d = d') \lor (d = \bot) \quad (d, d' \in X_\bot)
\]

makes \( (X_\bot, \sqsubseteq) \) into a domain (with least element \( \bot \)), called the flat domain determined by \( X \).
Eg \( N_1 \) looks like:

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & & & \\
\end{array}
\]
Note that every chain \( d_0 \subseteq d_1 \subseteq d_2 \subseteq \cdots \) in \( X \) is eventually constant (i.e. \( \exists N. \forall n \geq N. d_n = d_N \)) and so has a lub.
Note that every chain \( d_0 \subseteq d_1 \subseteq d_2 \subseteq \cdots \) in \( X \) is eventually constant (i.e., \( \exists N. \forall n \geq N. d_n = d_N \)) and so has a lub.

Hence

\( X \) does have lubs of chains
Note that every chain \( d_0 \subseteq d_1 \subseteq d_2 \subseteq \cdots \) in \( X_\bot \) is eventually constant (i.e. \( \exists N. \forall n \geq N. d_n = d_N \)) and so has a lub.

Hence

- \( X_\bot \) does have lubs of chains

- a function \( f : X_\bot \rightarrow D \) (with \( D \) a domain) is continuous if and only if it is monotone (iff \( \forall x \in X. f(\bot) \leq f(x) \))
Binary product of cpo’s and domains

The **product** of two cpo’s \((D_1, \sqsubseteq_1)\) and \((D_2, \sqsubseteq_2)\) has underlying set

\[
D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \& d_2 \in D_2\}
\]

and partial order \(\sqsubseteq\) defined by

\[
(d_1, d_2) \sqsubseteq (d'_1, d'_2) \iff d_1 \sqsubseteq_1 d'_1 \& d_2 \sqsubseteq_2 d'_2.
\]

\[
(x_1, x_2) \sqsubseteq (y_1, y_2) \implies x_1 \sqsubseteq_1 y_1 \& x_2 \sqsubseteq_2 y_2
\]
Lubs of chains are calculated componentwise:

\[
\biguplus_{n \geq 0} (d_{1,n}, d_{2,n}) = \left( \biguplus_{i \geq 0} d_{1,i}, \biguplus_{j \geq 0} d_{2,j} \right).
\]
Chain in $D_1 \times D_2$

$(d_{1,1}, d_{2,1}) \leq (d_{1,2}, d_{2,2}) \leq (d_{1,3}, d_{2,3}) \leq \cdots$

get

$\begin{cases} d_{1,1} \leq d_{1,2} \leq d_{1,3} \leq \cdots \text{ chain in } D_1 \\
 d_{2,1} \leq d_{2,2} \leq d_{2,3} \leq \cdots \text{ chain in } D_2 \end{cases}$
Chain in $D_1 \times D_2$

$\langle d_{1,1}, d_{2,1} \rangle \leq (d_{1,2}, d_{2,2}) \leq (d_{1,3}, d_{2,3}) \leq \ldots$

get

\[
\begin{align*}
&\langle d_{1,1}, \leq d_{1,2}, \leq d_{1,3} \leq \ldots \text{ chain in } D_1 \rangle \\
&\langle d_{2,1}, \leq d_{2,2}, \leq d_{2,3} \leq \ldots \text{ chain in } D_2 \rangle
\end{align*}
\]

So we can form

\[
\begin{align*}
&\bigcup_{i \geq 0} d_{1,i} \text{ lub in } D_1 \\
&\bigcup_{j \geq 0} d_{2,j} \text{ lub in } D_2
\end{align*}
\]
if chain in $D_1 \times D_2$ has an upper bound

$$(d_{1,1}, d_{2,1}) \leq (d_{1,2}, d_{2,2}) \leq (d_{1,3}, d_{2,3}) \leq \cdots \leq (x_1, x_2)$$

then get

$$\begin{cases} d_{1,1} \leq d_{1,2} \leq d_{1,3} \leq \cdots \leq x_1 & D_1 \\ d_{2,1} \leq d_{2,2} \leq d_{2,3} \leq \cdots \leq x_2 & D_2 \end{cases}$$

hence

$$\begin{cases} \bigcup_{i \geq 0} d_{1,i} \leq x_1 & D_1 \\ \bigcup_{j \geq 0} d_{2,j} \leq x_2 & D_2 \end{cases}$$

and thus

$$\left( \bigcup_{i \geq 0} d_{1,i} , \bigcup_{j \geq 0} d_{2,j} \right) \leq (x_1, x_2) \text{ in } D_1 \times D_2$$
Lubs of chains are calculated componentwise:

\[
\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = \left( \bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j} \right).
\]

If \((D_1, \sqsubseteq_1)\) and \((D_2, \sqsubseteq_2)\) are domains so is \((D_1 \times D_2, \sqsubseteq)\) and \(\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})\).

\[\downarrow\] for all \((d_1, d_2) \in D_1 \times D_2\)

\[\perp_{D_1} \leq d_1 \quad \text{in} \quad D_1, \quad \perp_{D_2} \leq d_2 \quad \text{in} \quad D_2\]

\(\Rightarrow\) so \((\perp, \perp) \leq (d_1, d_2)\) in \(D_1 \times D_2\).
Continuous functions of two arguments

Proposition. Let $D$, $E$, $F$ be cpo’s. A function $f : (D \times E) \rightarrow F$ is monotone if and only if it is monotone in each argument separately:

$\forall d, d' \in D, e \in E. d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)$

$\forall d \in D, e, e' \in E. e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e')$.

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$f(\bigsqcup_{m \geq 0} d_m, e) = \bigsqcup_{m \geq 0} f(d_m, e)$

$f(d, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n)$.
If we just know \[
\begin{align*}
\forall d, d', e, e' : \\
& d \subseteq d' \Rightarrow f(d, e) \subseteq f(d', e) \\
& e \subseteq e' \Rightarrow f(d, e) \subseteq f(d, e')
\end{align*}
\]
then we get \[f : D \times E \rightarrow F\] is monotone.
If we just know \[
\begin{align*}
\text{for all } d, d', e, e' : \\
d \subseteq d' \implies f(d, e) \subseteq f(d', e) \\
e \subseteq e' \implies f(d, e) \subseteq f(d, e')
\end{align*}
\]
then we get \( f : D \times E \to F \) is monotone:

\[
(d, e) \subseteq (d', e') \implies d \subseteq d' \land e \subseteq e'
\]

\[
\implies f(d, e) \subseteq f(d', e) \land e \subseteq e'
\]

\[
\implies \quad \quad f(d', e) \subseteq f(d', e')
\]

\[
\implies f(d, e) \subseteq f(d', e')
\]
If we just know \[
\begin{align*}
&\text{for all } d, d', e, e' : \\
&d \subseteq d' \implies f(d, e) \subseteq f(d', e) \\
&e \subseteq e' \implies f(d, e) \subseteq f(d, e')
\end{align*}
\]
then we get \( f : D \times E \to F \) is monotone:
\[
\begin{align*}
(d, e) &\subseteq (d', e') \implies d \subseteq d' \text{ & } e \subseteq e' \\
&\implies f(d, e) \subseteq f(d', e) \text{ & } e \subseteq e' \\
&\implies \quad \quad f(d', e) \subseteq f(d', e') \\
&\implies f(d, e) \subseteq f(d', e')
\end{align*}
\]
If we just know \[
\begin{align*}
\text{for all } d,d', e,e' : \\
& d \subseteq d' \implies f(d, e) \subseteq f(d', e) \\
& e \subseteq e' \implies f(d, e) \subseteq f(d, e')
\end{align*}
\]
then we get \( f : D \times E \to F \) is monotone:

\[
(d, e) \subseteq (d', e') \implies d \subseteq d' \land e \subseteq e' \\
\implies f(d, e) \subseteq f(d', e) \land e \subseteq e' \\
\implies \ldots \implies f(d', e) \subseteq f(d', e') \\
\implies f(d, e) \subseteq f(d', e')
\]
If we just know monotonicity +

\[
\begin{align*}
    f(\bigcup_{m \geq 0} d_m, e) &\subseteq \bigcup_{m \geq 0} f(d_m, e) \\
    f(d, \bigcup_{n \geq 0} e_n) &\subseteq \bigcup_{n \geq 0} f(d, e_n)
\end{align*}
\]

then we get that \( f : D \times E \rightarrow F \) is continuous.
If we just know monotonicity, then we get that $f : D \times E \to F$ is continuous:

$$f(\bigcup_{n \geq 0} (d_n, e_n)) = f \left( \bigcup_{i \geq 0} d_i, \bigcup_{j \geq 0} e_j \right)$$
If we just know monotonicity +

\[
\begin{align*}
  f(\bigcup_{m \geq o} d_m, e) &= \bigcup_{m \geq o} f(d_m, e) \\
  f(d, \bigcup_{n \geq o} e_n) &= \bigcup_{n \geq o} f(d, e_n)
\end{align*}
\]

then we get that \( f : D \times E \to F \) is continuous:

\[
\begin{align*}
  f(\bigcup_{n \geq o} (d_n, e_n)) &= f(\bigcup_{i \geq o} d_i, e) \\
  &= \bigcup_{i \geq o} f(d_i, e)
\end{align*}
\]

where \( e = \bigcup_{j \geq o} e_j \)
If we just know monotonicity + 

\[ \begin{align*} 
    f(\bigcup_{m>0} d_m, e) &= \bigcup_{m>0} f(d_m, e) \\
    f(d, \bigcup_{n>0} e_n) &= \bigcup_{n>0} f(d, e_n) 
\end{align*} \]

then we get that \( f : D \times E \to F \) is continuous:

\[ \begin{align*} 
    f(\bigcup_{n>0} (d_n, e_n)) &= f\left(\bigcup_{i>0} d_i, \bigcup_{j>0} e_j\right) \\
    &= \bigcup_{i>0} f(d_i, \bigcup_{j>0} e_j) \\
    &= \bigcup_{i>0} \left(\bigcup_{j>0} f(d_i, e_j)\right) 
\end{align*} \]
If we just know monotonicity + 
\[
\begin{align*}
  f(\bigcup_{m \geq 0} d_m, e) &= \bigcup_{m \geq 0} f(d_m, e) \\
  f(d, \bigcup_{n \geq 0} e_n) &= \bigcup_{n \geq 0} f(d, e_n)
\end{align*}
\]

then we get that \( f : D \times E \to F \) is continuous:

\[
\begin{align*}
  f(\bigcup_{n \geq 0} (d_n, e_n)) &= f(\bigcup_{i \geq 0} d_i, \bigcup_{j \geq 0} e_j) \\
  &= \bigcup_{i \geq 0} f(d_i, \bigcup_{j \geq 0} e_j) \\
  &= \bigcup_{i \geq 0} \big( \bigcup_{j \geq 0} f(d_i, e_j) \big) \\
  &= \bigcup_{k \geq 0} f(d_k, e_k)
\end{align*}
\]

See Slide 27
Lemma. Let $D$ be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies

$$m \leq m' \& n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}.$$  (†)

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots$$

Moreover

$$\bigsqcup_{m \geq 0} \left( \bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left( \bigsqcup_{m \geq 0} d_{m,n} \right).$$
Function cpo’s and domains

Given cpo’s \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\), the function cpo \((D \to E, \sqsubseteq)\) has underlying set

\[
(D \to E) \overset{\text{def}}{=} \{ f \mid f : D \to E \text{ is a continuous function} \}
\]

and partial order: \(f \sqsubseteq f' \overset{\text{def}}{\iff} \forall d \in D . f(d) \sqsubseteq_E f'(d)\).
Function cpo’s and domains

Given cpo’s \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\), the function cpo \((D \rightarrow E, \sqsubseteq)\) has underlying set

\[
(D \rightarrow E) \overset{\text{def}}{=} \{ f \mid f : D \rightarrow E \text{ is a continuous function} \}
\]

and partial order: \(f \sqsubseteq f' \Leftrightarrow \forall d \in D. f(d) \sqsubseteq_E f'(d)\).

- A derived rule:

\[
\frac{f \sqsubseteq_{(D \rightarrow E)} g \quad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}
\]
Lubs of chains are calculated ‘argumentwise’ (using lubs in $E$):

$$\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d).$$

If $E$ is a domain, then so is $D \to E$ and $\bot_{D \to E}(d) = \bot_E$, all $d \in D$. 

Note: This is a well-defined continuous function.
Given $f_0 \leq f_1 \leq f_2 \leq \ldots$ in $D \rightarrow E$, for each $d \in D$ we get

$$f_0(d) \leq f_1(d) \leq f_2(d) \leq \ldots$$

chain in $E$ and can form its lub $\bigcup_{n \geq 0} f_n(d)$. 
Given \( f_0 \preceq f_1 \preceq f_2 \preceq \ldots \) in \( D \rightarrow E \)
for each \( d \in D \) we get
\[
f_0(d) \preceq f_1(d) \preceq f_2(d) \preceq \ldots \quad \text{chain in } E
\]
and can form its \( \text{lub } \bigcup_{n \geq 0} f_n(d) \).

\( \lambda d \in D \cdot \bigcup_{n \geq 0} f_n(d) \) is monotone, because:

\[
d \preceq d' \Rightarrow \forall n \geq 0. f_n d \preceq f_n d'
\]

\[
\Rightarrow \bigcup_{n \geq 0} f_n d \preceq \bigcup_{n \geq 0} f_n d'
\]
Given \( f_0 \leq f_1 \leq f_2 \leq \ldots \) in \( D \to E \)
for each \( d \in D \) we get

\[
f_0(d) \subseteq f_1(d) \subseteq f_2(d) \subseteq \ldots \text{ chain in } E
\]

and can form its \( \text{lub} \) \( \bigcup_{n \geq 0} f_n(d) \).

\( \lambda d \in D \cdot \bigcup_{n \geq 0} f_n(d) \) is continuous, because:

\[
\bigcup_{n \geq 0} f_n \left( \bigcup_{m \geq 0} d_m \right) = \bigcup_{n \geq 0} \left( \bigcup_{m \geq 0} f_n(d_m) \right)
\]

\[
= \bigcup_{k \geq 0} f_k(d_k)
\]

\[
= \bigcup_{m \geq 0} \left( \bigcup_{n \geq 0} f_n(d_m) \right)
\]

each \( f_n \) is continuous

Slide 27
Lubs of chains are calculated ‘argumentwise’ (using lubs in $E$):

$$\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d).$$

● A derived rule:

$$\left( \bigsqcup_{n} f_n \right) \left( \bigsqcup_{m} x_m \right) = \bigsqcup_{k} f_k(x_k)$$

If $E$ is a domain, then so is $D \rightarrow E$ and $\bot_{D \rightarrow E}(d) = \bot_E$, all $d \in D$.

$$\left( \lambda d. \bot \right) \in f \text{ because } \forall d. \ (\lambda d. \bot)(d) = \bot \subseteq f(d)$$
Continuity of composition

For cpo’s $D$, $E$, $F$, the composition function

$$\circ : ( (E \to F) \times (D \to E) ) \to (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f = \lambda d \in D. g(f(d))$$

is continuous.

Proof left as an exercise.
Evaluation function \( \text{ev} : (D \rightarrow E) \times D \rightarrow E \)

\[
\text{ev}(f, d) = f(d)
\]

Monotone: if \( f \leq f' \) and \( d \leq d' \), then

\[
\text{ev}(f, d) = f(d) \leq f(d') \\
\leq f'(d') \\
= \text{ev}(f', d')
\]

\( f \) is monotone

Definition of \( f \leq f' \)
Evaluation function $\text{ev}: (\mathbb{D} \to \mathbb{E}) \times \mathbb{D} \to \mathbb{E}$

$\text{ev}(f, d) = f(d)$

Continuous: if $\{f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots \}$ in $\mathbb{D} \to \mathbb{E}$ and $\{d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots \}$ in $\mathbb{D}$

$\text{ev}(\bigcup_{n>0}(f_n, d_n)) = \text{ev}(\bigcup_{i>0} f_i, \bigcup_{j>0} d_j)$
Evaluation function $\text{ev}: (D \to E) \times D \to E$

$\text{ev}(f, d) = f(d)$

Continuous: if $\{f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots \}$ in $D \to E$ and $d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots$ in $D$

$\text{ev}(\bigcup_{n \geq 0} (f_n, d_n)) = \text{ev}(\bigcup_{i \geq 0} f_i, \bigcup_{j \geq 0} d_j)$

Definition of $\text{ev}$
Evaluation function \( \text{ev} : (D \rightarrow E) \times D \rightarrow E \)

\[
\text{ev}(f, d) = f(d)
\]

Continuous: if \( \{f_0 \leq f_1 \leq f_2 \leq \ldots \text{ in } D \rightarrow E \} \) and \( \{d_0 \leq d_1 \leq d_2 \leq \ldots \text{ in } D \} \), then

\[
\text{ev}(\bigcup_{n \geq 0}(f_n, d_n)) = \text{ev}(\bigcup_{i \geq 0} f_i, \bigcup_{j \geq 0} d_j)
\]

\[
= (\bigcup_{i \geq 0} f_i)(\bigcup_{j \geq 0} d_j)
\]

\[
= \bigcup_{i \geq 0} f_i(\bigcup_{j \geq 0} d_j)
\]

\( \text{lbs in } D \rightarrow E \)
Evaluation function $\text{ev}: (D \to E) \times D \to E$

$$\text{ev}(f, d) = f(d)$$

Continuous: if $\{f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots \text{ in } D \to E\}$ and $d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots \text{ in } D$

$$\text{ev}(\bigcup_{n \geq 0} (f_n, d_n)) = \text{ev}(\bigcup_{i \geq 0} f_i, \bigcup_{j \geq 0} d_j)$$

$$= (\bigcup_{i \geq 0} f_i)(\bigcup_{j \geq 0} d_j)$$

$$= \bigcup_{i \geq 0} f_i(\bigcup_{j \geq 0} d_j)$$

$$= \bigcup_{i \geq 0} \bigcup_{j \geq 0} f_i(d_j)$$

Each $f_i$ is $d$-s
Evaluation function \( \text{ev} : (D \to E) \times D \to E \)

\[
\text{ev}(f, d) = f(d)
\]

Continuous: if \( \{ f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots \} \text{ in } D \Rightarrow E \) \( d_0 \leq d_1 \leq d_2 \leq \ldots \) in D

\[
\text{ev}(\bigcup_{n>0}(f_n, d_n)) = \text{ev}(\bigcup_{i>0} f_i, \bigcup_{j>0} d_j)
\]

\[
= (\bigcup_{i>0} f_i)(\bigcup_{j>0} d_j)
\]

\[
= \bigcup_{i>0} f_i(\bigcup_{j>0} d_j)
\]

\[
= \bigcup_{i>0} \bigcup_{j>0} f_i(d_j)
\]

\[
= \bigcup_{k>0} f_k(d_k)
\]

\[
= \bigcup_{k>0} \text{ev}(f_k, d_k)
\]
"Currying"

From continuous $f : D' \times D \rightarrow E$ we get

$$\text{cur}(f) = \lambda d' \in D. \lambda d \in D. f(d', d)$$

- for each $d' \in D$, $\text{cur}(f)(d') \in D \rightarrow E$ (i.e. is continuous)
"Currying" 

From continuous \( f : D' \times D \rightarrow E \) we get 

\[
\text{cur}(f) = \lambda d' \in D. \lambda d \in D. f(d', d)
\]

- for each \( d' \in D \), \( \text{cur}(f)(d') \in D \rightarrow E \) (i.e. is continuous)

\[
\text{cur}(f)(d')(\bigcup_{n \geq 0} d_n) = f(d', \bigcup_{n \geq 0} d_n)
\]

\[
= f\left(\bigcup_{n \geq 0} (d', d_n)\right)
\]

\[
= \bigcup_{n \geq 0} f(d', d_n)
\]

\[
= \bigcup_{n \geq 0} \text{cur}(f)(d')(d_n)
\]
"Currying"

From continuous \( f: D' \times D \to E \) we get

\[ \text{cur}(f) = \lambda d' \in D. \lambda d \in D. f(d', d) \]

- For each \( d' \in D \), \( \text{cur}(f)(d') \in D \to E \) (i.e. is continuous)
- \( \text{cur}(f) \in D' \to (D \to E) \) (i.e. is continuous)
"Currying"

From continuous \( f : D' \times D \to E \) we get

\[
\text{\textbf{cur}}(f) = \lambda d' \in D. \lambda d \in D. f(d', d)
\]

- for each \( d' \in D \), \( \text{\textbf{cur}}(f)(d') \in D \to E \) (i.e. is continuous)
- \( \text{\textbf{cur}}(f) \in D' \to (D \to E) \) (i.e. is continuous)

\[
\text{cur}(f)(\bigcup m \overrightarrow{d_m})(d) = f(\bigcup m \overrightarrow{d_m}, d) = \bigcup m \overrightarrow{f(d_m', d)} = (\bigcup m \overrightarrow{\text{cur}(f)(d_m')})(d)
\]
Continuity of the fixpoint operator

Let $D$ be a domain.

By Tarski’s Fixed Point Theorem we know that each continuous function $f \in (D \rightarrow D)$ possesses a least fixed point, $\text{fix}(f) \in D$.

**Proposition.** The function

$$\text{fix} : (D \rightarrow D) \rightarrow D$$

is continuous.

Proof just uses defining properties of fix — $\text{fix}$ is continuous rather than the explicit construction $\text{fix}(f) = \bigcup_{n \geq 0} f^n(\bot)$.
Pre-fixed points

Let $D$ be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$.

The least pre-fixed point of $f$, if it exists, will be written $\text{fix}(f)$.

It is thus (uniquely) specified by the two properties:

- $f(\text{fix}(f)) \sqsubseteq \text{fix}(f)$ (lfp1)
- $\forall d \in D. \ f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$. (lfp2)
\( \text{fix} : (D \to D) \to D \)

is monotone: if \( f \preceq f' \) in \( D \to D \), then

\[ f(\text{fix } f') \preceq f'(\text{fix } f') \preceq \text{fix } f' \]
\( \text{fix} : (D \to D) \to D \)

is monotone: if \( f \preceq f' \) in \( D \to D \), then

\[
f(\text{fix} f') \preceq f'(\text{fix} f') \preceq \text{fix} f'
\]

so \( \text{fix} f' \) is a pre-fixed point of \( f \)

so by (lfp2) \( \text{fix} f \preceq \text{fix} f' \)
\[ \text{fix} : (D \to D) \to D \]

is continuous: given \( f_0 \leq f_1 \leq f_2 \leq \cdots \) in \( D \to D \)

want to show \( \text{fix} (\bigcup_{n \geq 0} f_n) \subseteq \bigcup_{n \geq 0} \text{fix}(f_n) \)

By (lfp2), enough to show

\( (\bigcup_{n \geq 0} f_n)(d) \leq d \) for \( d = \bigcup_{n \geq 0} \text{fix}(f_n) \)
\[ \text{fix} : (D \to D) \to D \]

is continuous: given \( f_0 \leq f_1 \leq f_2 \leq \ldots \) in \( D \to D \)

want to show \( \text{fix} \left( \bigcup_{n \geq 0} f_n \right) \subseteq \bigcup_{n \geq 0} \text{fix}(f_n) \)

By (lfp2), enough to show

\[ (\bigcup_{n \geq 0} f_n)(d) \leq d \quad \text{for} \quad d = \bigcup_{n \geq 0} \text{fix}(f_n) \]

But

\[ (\bigcup_{n \geq 0} f_n)(d) = (\bigcup_{n \geq 0} f_n)(\bigcup_{m \geq 0} \text{fix}(f_m)) \]

\[ = \bigcup_{n \geq 0} \bigcup_{m \geq 0} f_n \left( \text{fix}(f_m) \right) \]

\[ = \bigcup_{k \geq 0} f_k \left( \text{fix}(f_k) \right) \subseteq \bigcup_{k \geq 0} \text{fix}(f_k) \]

(1fp1) for each \( f_k \)