Cpo’s and domains

A chain complete poset, or cpo for short, is a poset \((D, \sqsubseteq)\) in which all countable increasing chains \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots\) have least upper bounds, \(\bigsqcup_{n \geq 0} d_n:\)

\[
\forall m \geq 0 . \, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{lub1}
\]

\[
\forall d \in D . \, (\forall m \geq 0 . \, d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{lub2}
\]

A domain is a cpo that possesses a least element, \(\bot:\)

\[
\forall d \in D . \, \bot \sqsubseteq d.
\]

"lub" = least upper bound
Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that

$$f\left(\bigsqcup_{n \geq 0} d_n\right) = \bigsqcup_{n \geq 0} f(d_n) \text{ in } E.$$
Continuity and strictness

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  $$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n)$$

  in $E$.

  \[\text{NB} \quad f(d_0) \leq f(d_1) \leq f(d_2) \leq \ldots \text{ if } f \text{ is monotone}\]

  \[\text{NB} \quad \forall i : d_i \sqsubseteq \bigcup_{n \geq 0} d_n \quad \text{monotonicity} \quad \forall i : f(d_i) \sqsubseteq f(\bigcup_{n \geq 0} d_n)\]

  \[\Rightarrow \bigcup_{i \geq 0} f(d_i) \sqsubseteq f(\bigcup_{n \geq 0} d_n)\]
Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
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\[\text{NB} \quad \forall i. \ d_i \subseteq \bigcup_{n \geq 0} d_n \quad \text{monotonicity} \quad \forall i. \ f(d_i) \subseteq f(\bigcup_{n \geq 0} d_n) \quad \Rightarrow \quad \bigcup_{i \geq 0} f(d_i) \subseteq f(\bigcup_{n \geq 0} d_n)\]

So given 1, for 2 just need $f(\bigcup_{n \geq 0} d_n) \subseteq \bigcup_{n \geq 0} f(d_n)$
### Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is **continuous** iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that

  $$f\left( \bigsqcup_{n \geq 0} d_n \right) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$ 

- If $D$ and $E$ have least elements, then the function $f$ is **strict** iff $f(\bot) = \bot$. 

Tarski’s Fixed Point Theorem

Let $f : D \rightarrow D$ be a continuous function on a domain $D$. Then

- $f$ possesses a least pre-fixed point, given by

$$\text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

- Moreover, $\text{fix}(f)$ is a fixed point of $f$, i.e. satisfies

$$f(\text{fix}(f)) = \text{fix}(f),$$

and hence is the least fixed point of $f$.

where

$$\begin{cases}
    f^0(\bot) \triangleq \bot \\
    f^{n+1}(\bot) \triangleq f(f^n(\bot))
\end{cases}$$
Pre-fixed points

Let $D$ be a poset and $f : D \rightarrow D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$.

The least pre-fixed point of $f$, if it exists, will be written $\text{fix}(f)$.

It is thus (uniquely) specified by the two properties:

\[ f(\text{fix}(f)) \sqsubseteq \text{fix}(f) \]  \hspace{1cm} (lfp1)
\[ \forall d \in D. \ f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d. \]  \hspace{1cm} (lfp2)
Proof of Tarski’s Theorem

\( \bot \leq f(\bot) \) because \( \bot \) is least elt of \( D \)
Proof of Tarski’s Theorem

\( \bot \subseteq f(\bot) \) because \( \bot \) is least elt of \( \mathbb{D} \)

so \( f(\bot) \subseteq f(f(\bot)) \equiv f^2(\bot) \) by monotonicity of \( f \)

so \( f^2(\bot) \subseteq f(f^2(\bot)) = f^3(\bot) \)

etc.
Proof of Tarski’s Theorem

\[ \bot \subseteq f(\bot) \text{ because } \bot \text{ is least elt of } D \]

so \[ f(\bot) \subseteq f(f(\bot)) \equiv f^2(\bot) \text{ by monotonicity of } f \]

so \[ f^2(\bot) \subseteq f(f^2(\bot)) = f^3(\bot) \]

etc.

We get a chain \[ \bot \subseteq f(\bot) \subseteq f^2(\bot) \subseteq f^3(\bot) \subseteq \ldots \]

and can form its lub \[ \bigcup_{n \geq 0} f^n(\bot) \]
Proof of Tarski’s Theorem

Applying $f$ to

$\bot \subseteq f(\bot) \subseteq f^2(\bot) \subseteq \cdots \subseteq \bigcup_{n \geq 0} f^n(\bot)$

we get

$f(\bot) \subseteq f(f(\bot)) \subseteq f(f^2(\bot)) \subseteq \cdots \subseteq f\left(\bigcup_{n \geq 0} f^n(\bot)\right)$

by monotonicity of $f$
Proof of Tarski’s Theorem

Applying $f$ to $\bot \subseteq f(\bot) \subseteq f^2(\bot) \subseteq \ldots \subseteq \bigcup_{n \geq 0} f^n(\bot)$

we get $f(\bot) \subseteq f(f(\bot)) \subseteq f(f^2(\bot)) \subseteq \ldots \subseteq f\left(\bigcup_{n \geq 0} f^n(\bot)\right)$

by continuity of $f$ $\Rightarrow$ $\bigcup_{n \geq 0} f(f^n(\bot))$

$\Rightarrow$ $\bigcup_{n \geq 0} f^{n+1}(\bot)$

$\Rightarrow$ $\bigcup_{m \geq 1} f^m(\bot)$
Proof of Tarski’s Theorem

So \( \bigcup_{n \geq 0} f^n(\bot) \) is a (pre-)fixed point for \( f \)

\[
\bigcup_{n \geq 0} f^n(\bot) = \bigcup_{m \geq 1} f^m(\bot)
\]

\[
f(\bigcup_{n \geq 0} f^n(\bot)) = \bigcup_{n \geq 0} f(f^n(\bot)) = \bigcup_{n \geq 0} f^{n+1}(\bot)
\]
Proof of Tarski’s Theorem

For any pre-fixed point $f(d) \leq d$ we have $\bot \leq d$ because $\bot$ is least elt of $D$.
Proof of Tarski’s Theorem

For any pre-fixed point $f(d) \subseteq d$ we have $\bot \subseteq d$ because $\bot$ is least elt of $D$.

So $f(\bot) \subseteq f(d) \subseteq d$ monotonicity +
Proof of Tarski’s Theorem

For any pre-fixed point $f(d) \subseteq d$ we have $\bot \subseteq d$ because $\bot$ is least elt of $D$.

So $f(\bot) \subseteq f(d) \subseteq d$ by monotonicity.

So $f^2(\bot) = f(f(\bot)) \subseteq f(d) \subseteq d$.

Etc.
Proof of Tarski's Theorem

For any pre-fixed point \( f(d) \sqsubseteq d \) we have \( \bot \sqsubseteq d \) because \( \bot \) is least elt of \( D \)

So \( f(\bot) \sqsubseteq f(d) \sqsubseteq d \) \quad \text{monotonicity +}

So \( f^2(\bot) = f(f(\bot)) \sqsubseteq f(d) \sqsubseteq d \)

etc.

We get \( f^n(\bot) \sqsubseteq d \) for all \( n \geq 0 \)

So \( \bigsqcup_{n \geq 0} f^n(\bot) \sqsubseteq d \)
Proof of Tarski's Theorem

For any pre-fixed point \( f(d) \subseteq d \) we have

So \( \bigcup_{n \geq 0} f^n(\bot) \) is a least pre-fixed point

We get

\[ \bigcup_{n \geq 0} f^n(\bot) \subseteq d \]

QED
Example

Domain \( D = (\mathcal{P}(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \rightarrow \top \))

Function \( f : D \to D \)

\[
f(S) = \{0\} \cup \{x+2 \mid x \in S\}
\]
Example

Domain $D = (\mathcal{P}(\mathbb{N}), \subseteq)$ (same as $\mathbb{N} \rightarrow \mathbb{N}$)

Function $f : D \rightarrow D$

$$f(S) \triangleq \{0\} \cup \{x+2 \mid x \in S\}$$

$S \in D$ is a prefixed point of $f$ if

$$f(S) \subseteq S$$

i.e. $0 \in S$ & $x+2 \in S$ for all $x \in S$

i.e. $S$ is closed under the rules

$0 \in S$ & $\frac{x \in S}{x+2 \in S}$
Example

Domain $D = (\mathcal{P}(\mathbb{N}), \subseteq)$ (same as $\mathbb{N} \to \mathbb{N}$)

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i.e. $S$ is closed under the rules

$$0 \in S \quad \& \quad \frac{x \in S}{x+2 \in S}$$

So expect least pre-fixed point of $f$

to be $\text{Even} = \{2 \cdot x \mid x \in \mathbb{N}\}$
Example

Domain \( D = (\mathcal{P}(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \to 1 \))

Function \( f : D \to D \)

\[
f(S) = \{0\} \cup \{x+2 \mid x \in S\}
\]

\( f \) is monotone: \( S \subseteq S' \Rightarrow f(S) \subseteq f(S') \) \( \checkmark \)
Example

Domain \( D = (P(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \rightarrow \uparrow \))

Function \( f : D \rightarrow D \)

\[
f(S) = \{0\} \cup \{x+2 \mid x \in S\}
\]

\( f \) is monotone: \( S \subseteq S' \Rightarrow f(S) \subseteq f(S') \) √

\( f \) is continuous: \( f(\bigcup_{n \geq 0} S_n) = \{0\} \cup \{x+2 \mid x \in \bigcup_{n \geq 0} S_n\} \)

= \{0\} \cup \bigcup_{n \geq 0} \{x+2 \mid x \in S_n\} \)

= \bigcup_{n \geq 0} f(S_n) \) √
Example

Domain $D = (\mathcal{P}(\mathbb{N}), \subseteq)$ (same as $\mathbb{N} \rightarrow \top$)

Function $f : D \rightarrow D$

$$f(S) \triangleq \{0\} \cup \{x+2 \mid x \in S\}$$

Tarski's Theorem applies:

$$\text{fix}(f) = \bigcup_{n \geq 0} f^n(\emptyset)$$

$f(\emptyset) = \{0\}$

$f^2(\emptyset) = \{0\} \cup \{0+2\}$

$f^3(\emptyset) = \{0, 2, 4\}$

$f^n(\emptyset) = \{0, 2, 4, \ldots, 2(n-1)\}$
Example

Domain \( D = (P(\mathbb{N}), \subseteq) \) (same as \( \mathbb{N} \to \mathbb{N} \))

Function \( f : D \to D \)

\[
f(S) \triangleq \{0\} \cup \{x+2 \mid x \in S\}
\]

Tarski's Theorem applies:

\[
\text{fix}(f) = \bigcup_{n \geq 0} f^n(\emptyset) = \{0, 2, 4, 8, \ldots\} = \{2x \mid x \in \mathbb{N}\}
\]

\[
f(\emptyset) = \{0\}
\]

\[
f^2(\emptyset) = \{0\} \cup \{0+2\}
\]

\[
f^3(\emptyset) = \{0, 2, 4\}
\]

\[
f^n(\emptyset) = \{0, 2, 4, \ldots, 2(n-1)\}
\]

(as expected).
Fixed point property of 
\([\text{while } B \text{ do } C]\)

\[ [\text{while } B \text{ do } C] = f_{B,C}([\text{while } B \text{ do } C]) \]

where, for each \( b : \text{State} \rightarrow \{\text{true}, \text{false}\} \) and \( c : \text{State} \rightarrow \text{State} \), we define

\[ f_{b,c} : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State}) \]

as

\[ f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}). \lambda s \in \text{State}. \]

\[ \text{if } (b(s), w(c(s)), s). \]

- Why does \( w = f_{B,C}(w) \) have a solution?
- What if it has several solutions—which one do we take to be \([\text{while } B \text{ do } C]\)?
Fixed point property of
\[ \textbf{while } B \text{ do } C \]

\[ \texttt{[while } B \text{ do } C \texttt{]} = f_{[B],[C]}(\texttt{[while } B \text{ do } C \texttt{]}) \]

where, for each \( b : \text{State} \rightarrow \{\text{true, false}\} \) and \( c : \text{State} \rightarrow \text{State} \), we define

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as

\[ f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}). \lambda s \in \text{State}. \text{if} (b(s), w(c(s)), s). \]

\begin{itemize}
  
  \item Why does \( w = f_{[B],[C]}(w) \) have a solution?

  \item What if it has several solutions—which one do we take to be \( \texttt{[while } B \text{ do } C \texttt{]}? \)

  \end{itemize}
Continuity of $f_{b,c}$

Suppose $c_0 \subseteq c \subseteq c_2 \subseteq \ldots$ in State $\rightarrow$ State

$$f_{b,c}(U_{n \geq 0} c_n) = \lambda s \in \text{State}. \text{ if } (b(s), (U_{n \geq 0} c_n)(c(s)), s)$$

that is

$$f_{b,c}(U_{n \geq 0} c_n) = \left\{ (s, s') \middle| \begin{array}{l}
  b(s) = \text{true} \land \exists s'' : c(s) = s'' \land \(U_{n \geq 0} c_n)(s'') = s', \\
  b(s) = \text{false} \land s = s'
\end{array} \right\}$$
Continuity of $f_{b,c}$

Suppose $c_0 \subseteq c \subseteq c_2 \subseteq \ldots$ in State $\rightarrow$ State

$$f_{b,c}(U_{n \geq 0} C_n) = \lambda s \in \text{State}. \text{if}(b(s), (U_{n \geq 0} C_n)(c(s)), s)$$

That is

$$f_{b,c}(U_{n \geq 0} C_n) = \left\{ (s, s') \mid \begin{align*}
    b(s) &= \text{true} \land \exists s''. c(s) = s'' \land \\
    \exists n \geq 0. \ c_n(s'') &= s' \\
    \lor \quad b(s) &= \text{false} \land s = s'
\end{align*} \right\}$$
Suppose \( c_0 \leq c \leq c_2 \leq \ldots \) in State \( \rightarrow \) State.

\[
\lim_{n \to \infty} c_n = \lambda s \in \text{State. if } (b(s), (U_{n>0} c_n)(c(s)), s) = (s, s')
\]

That is

\[
\lim_{n \to \infty} c_n = \left\{ (s, s') \mid \exists n \geq 0. b(s) = \text{true} \land \exists s''. (c(s) = s'' \land C_n(s'') = s') \land b(s) = \text{false} \land s = s' \right\}
\]
Continuity of $f_{b,c}$

Suppose $c_0 \subseteq c_1 \subseteq c_2 \subseteq \ldots$ in $\text{State} \rightarrow \text{State}$

$$f_{b,c}(U_{n \geq 0} c_n) = \lambda s \in \text{State}. \text{if } (b(s), (U_{n \geq 0} c_n)(c(s)), s)$$

That is

$$f_{b,c}(U_{n \geq 0} c_n) = \left\{ (s, s') \middle| \exists n \geq 0. b(s) = \text{true} \land \exists s''. c(s) = s'' \land c_n(s'') = s' \right\} \lor$$

$$b(s) = \text{false} \land s = s'$$

$$= U_{n \geq 0} \left\{ (s, s') \middle| \text{if } (b(s), c_n(c(s)), s) = s' \right\}$$

$$= U_{n \geq 0} f_{b,c}(c_n)$$

QED
[[while $B$ do $C$]]

$\left[ \right.$

[[while $B$ do $C$]]

$= fix(f_{[B],[C]})$

$= \bigsqcup_{n \geq 0} f_{[B],[C]}^n(\bot)$

$= \lambda s \in \text{State.}$

\[
\begin{cases} 
[C]^k(s) & \text{if } k \geq 0 \text{ is such that } [B](C)^k(s) = false \text{ and } [B](C)^i(s) = true \text{ for all } 0 \leq i < k \\
\text{undefined} & \text{if } [B](C)^i(s) = true \text{ for all } i \geq 0
\end{cases}
\]

Tarski Theorem

requires proof...