Extensionality

[p 87 et seq.]

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   - Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

$$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

$$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.

Definition of $d \lhd_\tau M$ $\quad$ ($d \in [\![\tau]\!], M \in \mathrm{PCF}_\tau$)

$d \in \mathbb{N}_\perp$

$d = \perp \vee (d \in \mathbb{N} \ \& \ M \Downarrow \mathrm{succ}^d(0))$

$$d \lhd_{nat} M \overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow M \Downarrow_{nat} \mathbf{succ}^d(\mathbf{0}))$$

$d \in \mathbb{B}_\perp$

$d = \perp \vee (d = true \ \& \ M \Downarrow true) \vee (d = false \ \& \ M \Downarrow false)$

$$d \lhd_{bool} M \overset{\text{def}}{\Leftrightarrow} (d = true \Rightarrow M \Downarrow_{bool} \mathbf{true}) \ \& \ (d = false \Rightarrow M \Downarrow_{bool} \mathbf{false})$$

$$d \lhd_{\tau \to \tau'} M \overset{\text{def}}{\Leftrightarrow} \forall e, N \ (e \lhd_\tau N \Rightarrow d(e) \lhd_{\tau'} M\,N)$$

$d \in [\![\tau]\!] \to [\![\tau']\!]$

Contextual preorder between PCF terms

Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \leq_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V \in \mathrm{PCF}_\gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \implies \mathcal{C}[M_2] \Downarrow_\gamma V \ .$$

$$\boxed{\underline{\underline{\text{NB}}} \quad M_1 \cong_{ctx} M_2 \quad \text{iff} \quad M_1 \leq_{ctx} M_2 \ \& \ M_2 \leq_{ctx} M_1}$$

Contextual preorder from formal approximation

Proposition. *For all PCF types τ and all closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$,

$$[\![M_1]\!] \lhd_\tau M_2 \iff M_1 \leq_{\mathrm{ctx}} M_2 : \tau \ .$$

$$M_1 \leq_{ctx} M_2 : \tau \implies [\![M_1]\!] \lhd_\tau M_2$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M_1]\!] \lhd_\tau M_1}$$

$$M_1 \leq_{ctx} M_2 : \tau \Rightarrow [\![M_1]\!] \lhd_\tau M_2$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M_1]\!] \lhd_\tau M_1}$$

Can also prove (by induction on τ) that

$$\boxed{\text{if } d \lhd_\tau M_1 \ \& \ M_1 \leq_{ctx} M_2 : \tau, \text{ then } d \lhd_\tau M_2}$$

use
- $M_1 \leq_{ctx} M_2 : \gamma \ \& \ M_1 \Downarrow_\gamma V \Rightarrow M_2 \Downarrow_\gamma V$ $(\gamma = nat, bool)$
- $M_1 \leq_{ctx} M_2 : \tau \to \tau' \Rightarrow \forall M : \tau \, (M_1 M \leq_{ctx} M_2 M : \tau')$

to prove this

$$M_1 \leq_{ctx} M_2 : \tau \Rightarrow [\![M_1]\!] \lhd_\tau M_2$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M_1]\!] \lhd_\tau M_1}$$

Can also prove (by induction on τ) that

$$\boxed{\text{if } d \lhd_\tau M_1 \ \& \ M_1 \leq_{ctx} M_2 : \tau, \text{ then } d \lhd_\tau M_2}$$

take $d = [\![M_1]\!]$ to get

$$M_1 \leq_{ctx} M_2 : \tau \Rightarrow [\![M_1]\!] \lhd_\tau M_2$$

$$[\![M_1]\!] \lhd_\tau M_2 \Rightarrow M_1 \leq_{ctx} M_2 : \tau$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M]\!] \lhd_{\tau \to bool} M}$$

for any $M : \tau \to bool$

$$[\![M_1]\!] \lhd_\tau M_2 \Rightarrow M_1 \leq_{ctx} M_2 : \tau$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M]\!] \lhd_{\tau \to bool} M}$$ for any $M : \tau \to bool$

So if $[\![M_1]\!] \lhd_\tau M_2$, by definition of $\lhd_{\tau \to bool}$ we get

$$[\![M]\!]([\![M_1]\!]) \lhd_{bool} M\, M_2$$

$$[\![M_1]\!] \lhd_\tau M_2 \Rightarrow M_1 \leq_{ctx} M_2 : \tau$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M]\!] \lhd_{\tau \to bool} M}$$

for any $M : \tau \to bool$

So if $[\![M_1]\!] \lhd_\tau M_2$, by definition of $\lhd_{\tau \to bool}$ we get

$$[\![M\,M_1]\!] \lhd_{bool} M\,M_2$$

$$[\![M_1]\!] \lhd_\tau M_2 \implies M_1 \leq_{ctx} M_2 : \tau$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M]\!] \lhd_{\tau \to bool} M}$$

for any $M : \tau \to bool$

So if $[\![M_1]\!] \lhd_\tau M_2$, by definition of $\lhd_{\tau \to bool}$ we get

$$[\![M\,M_1]\!] \lhd_{bool} M\,M_2$$

So by definition of $\lhd_{bool}$ we get

$$\forall v : bool\,(\,M\,M_1 \Downarrow_{bool} v \implies M\,M_2 \Downarrow_{bool} v\,)$$

$$[\![M_1]\!] \lhd_\tau M_2 \Rightarrow M_1 \leq_{ctx} M_2 : \tau$$

"Fundamental property" of $\lhd$ gives

$$\boxed{[\![M]\!] \lhd_{\tau \to bool} M}$$ for any $M : \tau \to bool$

So if $[\![M_1]\!] \lhd_\tau M_2$, by definition of $\lhd_{\tau \to bool}$ we get

$$[\![M\,M_1]\!] \lhd_{bool} M\,M_2$$

So by definition of $\lhd_{bool}$ we get

$$\boxed{\forall V : bool\,(\,M\,M_1 \Downarrow_{bool} V \Rightarrow M\,M_2 \Downarrow_{bool} V\,)}$$

not hard to see this is equivalent to $M_1 \leq_{ctx} M_2 : \tau$

Contextual preorder from formal approximation

Proposition. *For all PCF types* τ *and all closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$,

$$[\![M_1]\!] \lhd_\tau M_2 \iff M_1 \leq_{\mathrm{ctx}} M_2 : \tau \ .$$

Some useful corollaries of $M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2$

- $\Omega \stackrel{\text{def}}{=} \mathbf{fix}(\mathbf{fn}\, x : \tau \,.\, x)$ is least w.r.t. $\leq_{ctx}$

  $\forall M : \tau \; (\Omega \leq_{ctx} M : \tau)$

Some useful corollaries of $M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2$

- $\Omega \stackrel{\text{def}}{=} \mathbf{fix}(\mathbf{fn}\, x : \tau \,.\, x)$ is least w.r.t. $\leq_{ctx}$

  $\forall M : \tau \; (\, \Omega \leq_{ctx} M : \tau \,)$

  because $[\![\Omega]\!] = fix(\lambda d.d) = \bot \lhd_\tau M$

Some useful corollaries of $M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2$

- For each $M : \tau \to \tau$, $fix(M) : \tau$ is least pre-fixed point w.r.t. $\leq_{ctx}$

  $$\forall N : \tau \left(M N \leq_{ctx} N : \tau \implies fix(M) \leq_{ctx} N : \tau \right)$$

$\boxed{\text{Some useful corollaries of } M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2}$

- For each $M : \tau \to \tau$, $fix(M) : \tau$ is least pre-fixed point w.r.t. $\leq_{ctx}$

$$\boxed{\forall N : \tau \; (M\,N \leq_{ctx} N : \tau \Rightarrow fix(M) \leq_{ctx} N : \tau)}$$

Proof: use the fact ~~that~~ $\{d \in [\![\tau]\!] \mid d \lhd_\tau N\}$ is admissible.

$$\boxed{\text{Some useful corollaries of } \quad M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2}$$

- For each $M : \tau \to \tau$, $fix(M) : \tau$ is least pre-fixed point w.r.t. $\leq_{ctx}$

$$\boxed{\forall N : \tau \,(M N \leq_{ctx} N : \tau \Rightarrow fix(M) \leq_{ctx} N : \tau)}$$

Proof: use the fact ~~that~~ $\{ d \in [\![\tau]\!] \mid d \lhd_\tau N \}$ is admissible.

$d \lhd_\tau N \Rightarrow [\![M]\!](d) \lhd_\tau M N$ since $[\![M]\!] \lhd M$

$$\boxed{\text{Some useful corollaries of } \quad M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2}$$

- For each $M : \tau \to \tau$, $\mathrm{fix}(M) : \tau$ is least pre-fixed point w.r.t. $\leq_{ctx}$

$$\boxed{\forall N : \tau \left(MN \leq_{ctx} N : \tau \;\Rightarrow\; \mathrm{fix}(M) \leq_{ctx} N : \tau \right)}$$

Proof: use the fact ~~that~~ $\{ d \in [\![\tau]\!] \mid d \lhd_\tau N \}$ is admissible.

$d \lhd_\tau N \;\Rightarrow\; [\![M]\!](d) \lhd_\tau MN$ since $[\![M]\!] \lhd M$

$\qquad\qquad\;\Rightarrow\; [\![M]\!](d) \lhd_\tau N$ since $MN \leq_{ctx} N$

Some useful corollaries of $M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2$

- For each $M : \tau \to \tau$, $\mathit{fix}(M) : \tau$ is least pre-fixed point w.r.t. $\leq_{ctx}$

$$\boxed{\forall N : \tau \, (M N \leq_{ctx} N : \tau \Rightarrow \mathit{fix}(M) \leq_{ctx} N : \tau)}$$

Proof: use the fact that $\{ d \in [\![\tau]\!] \mid d \lhd_\tau N \}$ is admissible.

$$d \lhd_\tau N \Rightarrow [\![M]\!](d) \lhd_\tau M N \quad \text{since } [\![M]\!] \lhd M$$

$$\Rightarrow [\![M]\!](d) \lhd_\tau N \quad \text{since } M N \leq_{ctx} N$$

So $[\![\mathit{fix}(M)]\!] = \mathit{fix}([\![M]\!]) \lhd_\tau N$ by Scott Induction.

Some useful corollaries of $M_1 \leq_{ctx} M_2 \Leftrightarrow [\![M_1]\!] \lhd M_2$

- For each $M : \tau \to \tau$, $fix(M) : \tau$ is least pre-fixed point w.r.t. $\leq_{ctx}$

$$\boxed{\forall N : \tau \left(MN \leq_{ctx} N : \tau \Rightarrow fix(M) \leq_{ctx} N : \tau \right)}$$

Proof: use the fact ~~that~~ $\{d \in [\![\tau]\!] \mid d \lhd_\tau N\}$ is admissible.

$$d \lhd_\tau N \Rightarrow [\![M]\!](d) \lhd_\tau MN \quad \text{since } [\![M]\!] \lhd M$$

$$\Rightarrow [\![M]\!](d) \lhd_\tau N \quad \text{since } MN \leq_{ctx} N$$

So $[\![fix(M)]\!] = fix([\![M]\!]) \lhd_\tau N$ by Scott Induction.

Hence $fix(M) \leq_{ctx} N : \tau$ Q.E.D.

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$,

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

$$\forall\, V \in \mathrm{PCF}_\gamma\ (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V)\ .$$

At a function type $\tau \to \tau'$,

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall\, M \in \mathrm{PCF}_\tau\ (M_1\, M \leq_{\mathrm{ctx}} M_2\, M : \tau')\ .$$

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

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At a function type $\tau \to \tau'$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall\, M \in \mathrm{PCF}_\tau\ (M_1\, M \leq_{\mathrm{ctx}} M_2\, M : \tau')\ .$$

If $M_1 \leq_{ctx} M_2 : nat$, then taking

$\mathcal{C} = [-]$ we get

$$M_1 = \mathcal{C}[M_1] \Downarrow_{nat} V \implies \mathcal{C}[M_2] \Downarrow_{nat} V$$

$$\implies M_2 \Downarrow_{nat} V$$

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

$$\forall V \in \mathrm{PCF}_\gamma \; (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V) \; .$$

At a function type $\tau \to \tau'$**,**

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall M \in \mathrm{PCF}_\tau \; (M_1 \, M \leq_{\mathrm{ctx}} M_2 \, M : \tau') \; .$$

If $M_1 \leq_{ctx} M_2 : \tau \to \tau'$, then

$$\mathcal{C}[M_1\, M] \Downarrow_\gamma V \implies \mathcal{C}'[M_1] \Downarrow_\gamma V$$

where $\mathcal{C}' = \mathcal{C}[-\, M]$

$$\implies \mathcal{C}'[M_2] \Downarrow_\gamma V$$

$$\implies \mathcal{C}[M_2\, M] \Downarrow_\gamma V$$

If $M_1 \leq_{ctx} M_2 : \tau \to \tau'$, then

$$\mathcal{C}[M_1 M] \Downarrow_\gamma V \Rightarrow \mathcal{C}'[M_1] \Downarrow_\gamma V$$

where $\mathcal{C}' = \mathcal{C}[- M]$

$$\Rightarrow \mathcal{C}'[M_2] \Downarrow_\gamma V$$

$$\Rightarrow \mathcal{C}[M_2 M] \Downarrow_\gamma V$$

So $M_1 M \leq_{ctx} M_2 M : \tau'$

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$,

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

$$\forall V \in \mathrm{PCF}_\gamma \; (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V) \; .$$

At a function type $\tau \to \tau'$,

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall M \in \mathrm{PCF}_\tau \; (M_1 \, M \leq_{\mathrm{ctx}} M_2 \, M : \tau') \; .$$

If $\forall V : nat \, (M_1 \Downarrow_{nat} V \Rightarrow M_2 \Downarrow_{nat} V)$

putting $d = [\![M_1]\!] \in \mathbb{N}_\perp$

if $d \neq \perp$ then $[\![M_1]\!] = d = [\![succ^d(0)]\!]$

so $M_1 \Downarrow_{nat} succ^d(0)$ (adequacy)

If $\forall V : nat \, (M_1 \Downarrow_{nat} V \Rightarrow M_2 \Downarrow_{nat} V)$ (*)

putting $d = [\![M_1]\!] \in \mathbb{N}_\perp$

if $d \neq \perp$ then $[\![M_1]\!] = d = [\![succ^d(0)]\!]$

so $M_1 \Downarrow_{nat} succ^d(0)$

so $M_2 \Downarrow_{nat} succ^d(0)$ (by (*))

If $\forall V : nat \, (M_1 \Downarrow_{nat} V \Rightarrow M_2 \Downarrow_{nat} V)$

putting $d = [\![M_1]\!] \in \mathbb{N}_\perp$

if $d \neq \perp$ then $[\![M_1]\!] = d = [\![succ^d(0)]\!]$

so $M_1 \Downarrow_{nat} succ^d(0)$

so $M_2 \Downarrow_{nat} succ^d(0)$

By definition of $\triangleleft_{nat}$, this means

$$d \triangleleft_{nat} M_2$$

So $[\![M_1]\!] \triangleleft_{nat} M_2$

So $M_1 \leq_{ctx} M_2 : nat$ (slide 68)

Extensionality properties of $\leq_{\mathrm{ctx}}$

At a ground type $\gamma \in \{bool, nat\}$,

$M_1 \leq_{\mathrm{ctx}} M_2 : \gamma$ holds if and only if

$$\forall V \in \mathrm{PCF}_\gamma \, (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V) \; .$$

At a function type $\tau \to \tau'$,

$M_1 \leq_{\mathrm{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall M \in \mathrm{PCF}_\tau \, (M_1 \, M \leq_{\mathrm{ctx}} M_2 \, M : \tau') \; .$$

If $\forall M : \tau \, (M_1 M \leq_{ctx} M_2 M : \tau')$

then

$d \lhd_\tau M \Rightarrow [\![M_1]\!](d) \lhd_{\tau'} M_1 M$ (since $[\![M_1]\!] \lhd_{\tau \to \tau'} M_1$)

If $\forall M:\tau\,(M_1 M \leq_{ctx} M_2 M : \tau')$ $(\ast)$

then

$$d \lhd_\tau M \Rightarrow [\![M_1]\!](d) \lhd_{\tau'} M_1 M$$

$$\Rightarrow [\![M_1]\!](d) \lhd_{\tau'} M_2 M \quad (\text{by } (\ast))$$

If $\forall M:\tau\,(M_1 M \leq_{ctx} M_2 M : \tau')$

then

$$d \lhd_\tau M \Rightarrow [\![M_1]\!](d) \lhd_{\tau'} M_1 M$$

$$\Rightarrow [\![M_1]\!](d) \lhd_{\tau'} M_2 M$$

So by definition of $\lhd_{\tau \to \tau'}$ this means

$$[\![M_1]\!] \lhd_{\tau \to \tau'} M_2$$

so $\quad M_1 \leq_{ctx} M_2 : \tau \to \tau'$ (slide 68)

We've seen

Compositionality + soundness + adequacy $\Rightarrow$

$$[\![M_1]\!] = [\![M_2]\!] \in [\![\tau]\!] \implies M_1 \cong_{ctx} M_2 : \tau$$

Similarly

$$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \text{ in } [\![\tau]\!] \implies M_1 \leq_{ctx} M_2 : \tau$$

We've seen

Compositionality + soundness + adequacy $\Rightarrow$

$$[\![M_1]\!] = [\![M_2]\!] \in [\![\tau]\!] \Rightarrow M_1 \cong_{ctx} M_2 : \tau$$

Similarly

$$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \text{ in } [\![\tau]\!] \Rightarrow M_1 \leq_{ctx} M_2 : \tau$$

What about the converse implications?

Full abstraction

A denotational model is said to be *fully abstract* whenever denotational equality characterises contextual equivalence.

At type nat

slide 69

$$M_1 \leq_{ctx} M_2 : nat \iff \forall n \in \mathbb{N} \left(M_1 \Downarrow_{nat} succ^n(0) \Rightarrow M_2 \Downarrow_{nat} succ^n(0) \right)$$

At type nat

$$M_1 \leq_{ctx} M_2 : nat \iff \forall n \in \mathbb{N} \, (M_1 \Downarrow_{nat} succ^n(0) \Rightarrow M_2 \Downarrow_{nat} succ^n(0))$$

soundness + adequacy

$$\iff \forall n \in \mathbb{N} \, ([\![M_1]\!] = n \Rightarrow [\![M_2]\!] = n)$$

At type nat

$$M_1 \leq_{ctx} M_2 : nat \iff \forall n \in \mathbb{N} \, (M_1 \Downarrow_{nat} succ^n(0) \Rightarrow M_2 \Downarrow_{nat} succ^n(0))$$

$$\iff \forall n \in \mathbb{N} \, (\llbracket M_1 \rrbracket = n \Rightarrow \llbracket M_2 \rrbracket = n)$$

def^n of $\sqsubseteq$ for $\mathbb{N}_\perp$

$$\iff \llbracket M_1 \rrbracket \sqsubseteq \llbracket M_2 \rrbracket \text{ in } \mathbb{N}_\perp$$

So $\sqsubseteq$ & $\leq_{ctx}$ coincide at type nat

$\boxed{\text{At type } nat \to nat}$

$$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \text{ in } [\![nat \to nat]\!] = \mathbb{N}_\perp \to \mathbb{N}_\perp$$

$$\text{iff} \quad \forall d \in \mathbb{N}_\perp \left([\![M_1]\!](d) \sqsubseteq [\![M_2]\!](d) \right)$$

$\boxed{\text{At type } nat \to nat}$

$[\![M_1]\!] \sqsubseteq [\![M_2]\!]$ in $[\![nat \to nat]\!] = \mathbb{N}_\perp \to \mathbb{N}_\perp$

iff $\forall d \in \mathbb{N}_\perp \, ([\![M_1]\!](d) \sqsubseteq [\![M_2]\!](d))$

But every $d \in \mathbb{N}_\perp$ is of the form $d = [\![M]\!]$ for some $M : nat$

$\boxed{\text{At type } nat \to nat}$

$$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \text{ in } [\![nat \to nat]\!] = \mathbb{N}_\perp \to \mathbb{N}_\perp$$

$$\text{iff } \forall M : nat \left([\![M_1]\!]([\![M]\!]) \sqsubseteq [\![M_2]\!]([\![M]\!]) \right)$$

$\boxed{\text{At type } nat \to nat}$

$$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \text{ in } [\![nat \to nat]\!] = \mathbb{N}_\perp \to \mathbb{N}_\perp$$
$$\text{iff } \forall M : nat \big([\![M_1\ M]\!] \sqsubseteq [\![M_2\ M]\!] \big)$$

$\boxed{\text{At type } nat \to nat}$

$$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \text{ in } [\![nat \to nat]\!] = \mathbb{N}_\perp \to \mathbb{N}_\perp$$

iff $\forall M : nat\,(\,[\![M_1\ M]\!]) \sqsubseteq [\![M_2\ M]\!])\,)$

iff $\forall M : nat\,(\,M_1\,M \leq_{ctx} M_2\,M : nat\,)$

Since we now know $\sqsubseteq$ & $\leq_{ctx}$ coincide at type nat

$\boxed{\text{At type } nat \to nat}$

$$[\![M_1]\!] \sqsubseteq [\![M_2]\!] \text{ in } [\![nat \to nat]\!] = \mathbb{N}_\perp \to \mathbb{N}_\perp$$

$$\text{iff } \forall M : nat \, ([\![M_1 \, M]\!] \sqsubseteq [\![M_2 \; M]\!])$$

$$\text{iff } \forall M : nat \, (M_1 \, M \leq_{ctx} M_2 \, M : nat)$$

$$\text{iff } M_1 \leq_{ctx} M_2 : nat \to nat \quad \text{(by slide 69)}$$

$\boxed{\text{so } \sqsubseteq \text{ \& } \leq_{ctx} \text{ coincide at type } nat \to nat}$

$\boxed{\text{At type } nat \to nat}$ $\sqsubseteq$ & $\leq_{ctx}$ coincide,

essentially because of

$[\![M_1]\!] \sqsubseteq [\![M_2]\!]$ in $[\![nat \to nat]\!] = \mathbb{N}_\perp \to \mathbb{N}_\perp$

iff $\forall d \in \mathbb{N}_\perp \, ([\![M_1]\!](d) \sqsubseteq [\![M_2]\!](d))$

But every $d \in \mathbb{N}_\perp$ is of the form $d = [\![M]\!]$ for some $M : nat$

i.e. every element of $[\![nat]\!]$ is definable by a PCF term. (Ditto for $bool$.)