Complexity Theory
Lecture 6

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http://www.cl.cam.ac.uk/teaching/1112/Complexity/

3SAT

A Boolean expression is in 3CNF if it is in conjunctive normal form and each clause contains at most 3 literals.

3SAT is defined as the language consisting of those expressions in 3CNF that are satisfiable.

3SAT is NP-complete, as there is a polynomial time reduction from CNF-SAT to 3SAT.

Composing Reductions

Polynomial time reductions are clearly closed under composition.

So, if $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then we also have $L_1 \leq_P L_3$.

Note, this is also true of $\leq_L$, though less obvious.

If we show, for some problem A in NP that

$$\mathsf{SAT} \leq_P A$$

or

$$\mathsf{3SAT} \leq_P A$$

it follows that A is also NP-complete.

Independent Set

Given a graph $G = (V, E)$, a subset $X \subseteq V$ of the vertices is said to be an *independent set*, if there are no edges (u, v) for $u, v \in X$.

The natural algorithmic problem is, given a graph, find the largest independent set.

To turn this *optimisation problem* into a *decision problem*, we define IND as:

> The set of pairs (G, K), where G is a graph, and K is an integer, such that G contains an independent set with K or more vertices.

IND is clearly in NP. We now show it is NP-complete.

Reduction

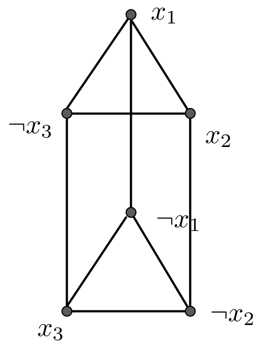
We can construct a reduction from 3SAT to IND.

A Boolean expression ϕ in 3CNF with m clauses is mapped by the reduction to the pair (G, m), where G is the graph obtained from ϕ as follows:

> G contains m triangles, one for each clause of ϕ, with each node representing one of the literals in the clause.
>
> Additionally, there is an edge between two nodes in different triangles if they represent literals where one is the negation of the other.

Example

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$$



Clique

Given a graph $G = (V, E)$, a subset $X \subseteq V$ of the vertices is called a *clique*, if for every $u, v \in X$, (u, v) is an edge.

As with IND, we can define a decision problem version:

CLIQUE is defined as:

> The set of pairs (G, K), where G is a graph, and K is an integer, such that G contains a clique with K or more vertices.

Clique 2

CLIQUE is in NP by the algorithm which *guesses* a clique and then verifies it.

CLIQUE is NP-complete, since

IND $\leq_P$ CLIQUE

by the reduction that maps the pair (G, K) to $(\bar{G}, K)$, where $\bar{G}$ is the complement graph of G.

k-Colourability

A graph $G = (V, E)$ is k-colourable, if there is a function

$$\chi : V \to \{1, \ldots, k\}$$

such that, for each $u, v \in V$, if $(u, v) \in E$,

$$\chi(u) \neq \chi(v)$$

This gives rise to a decision problem for each k.

2-colourability is in P.

For all $k > 2$, k-colourability is NP-complete.

3-Colourability

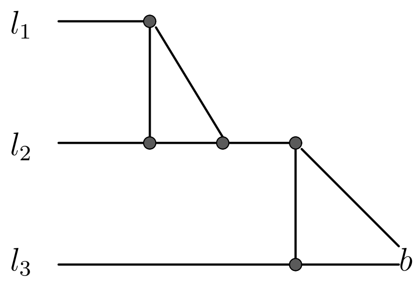
3-Colourability is in NP, as we can *guess* a colouring and verify it.

To show NP-completeness, we can construct a reduction from 3SAT to 3-Colourability.

For each variable x, have two vertices x, $\bar{x}$ which are connected in a triangle with the vertex a (common to all variables).

In addition, for each clause containing the literals l_1, l_2 and l_3 we have a gadget.

Gadget



With a further edge from a to b.

Hamiltonian Graphs

Recall the definition of HAM—the language of Hamiltonian graphs.

Given a graph $G = (V, E)$, a *Hamiltonian cycle* in G is a path in the graph, starting and ending at the same node, such that every node in V appears on the cycle *exactly once*.

A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The language HAM is the set of encodings of Hamiltonian graphs.

Hamiltonian Cycle

We can construct a reduction from 3SAT to HAM

Essentially, this involves coding up a Boolean expression as a graph, so that every satisfying truth assignment to the expression corresponds to a Hamiltonian circuit of the graph.

This reduction is much more intricate than the one for IND.

Travelling Salesman

Recall the travelling salesman problem

Given

- V — a set of nodes.
- $c : V \times V \to \mathbb{N}$ — a cost matrix.

Find an ordering $v_1, \ldots, v_n$ of V for which the total cost:

$$c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})$$

is the smallest possible.

Travelling Salesman

As with other optimisation problems, we can make a decision problem version of the Travelling Salesman problem.

The problem TSP consists of the set of triples

$$(V, c : V \times V \to \mathbb{N}, t)$$

such that there is a tour of the set of vertices V, which under the cost matrix c, has cost t or less.

Reduction

There is a simple reduction from HAM to TSP, mapping a graph (V, E) to the triple $(V, c : V \times V \to \mathbb{N}, n)$, where

$$c(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \textit{otherwise} \end{cases}$$

and n is the size of V.