Probability

Computer Laboratory

Computer Science Tripos, Part IA

Lent Term 2010/11

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Problem sheet
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1. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ show the following results.

   (a) If $E_1, E_2, \ldots \in \mathcal{F}$ then $\cap_{i=1}^{\infty} E_i \in \mathcal{F}$.

   (b) If $E_1, E_2 \in \mathcal{F}$ then $E_1 \setminus E_2 \in \mathcal{F}$.

   (c) If $E \in \mathcal{F}$ then $\mathbb{P}(\Omega \setminus E) = 1 - \mathbb{P}(E)$.

   (d) If $E_1, E_2 \in \mathcal{F}$ and $E_1 \subset E_2$ then $\mathbb{P}(E_1) \leq \mathbb{P}(E_2)$.

2. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $E_1, E_2 \in \mathcal{F}$ show that

$$\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)\,.$$

3. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two *disjoint* events $E_1, E_2 \in \mathcal{F}$ show that if E_1 and E_2 are independent then at least one of the two events has zero probability.

4. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a fixed event $E' \in \mathcal{F}$ with $\mathbb{P}(E') > 0$ show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space where $\mathbb{Q} : \mathcal{F} \to \mathbb{R}$ is defined by

$$\mathbb{Q}(E) = \mathbb{P}(E|E') \qquad \forall\, E \in \mathcal{F}\,.$$

5. Consider a sample space $\Omega = \{1, 2, 3, 4\}$ with equally likely outcomes. That is, with the event space given by the powerset $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(E) = |E|/4$ for all events $E \in \mathcal{F}$. Show that the three events $E_1 = \{1, 2\}$, $E_2 = \{1, 3\}$ and $E_3 = \{2, 3\}$ are pairwise independent but not independent events.

6. The PWF contains two types of workstations labelled A and B, respectively. A workstation of type A has a probability of $1/10$ of being defective whereas a workstation of type B has a probability of being defective of $2/10$. The PWF has 140 workstations of type A and 60 of type B. You choose one of the workstations at random. What is the probability that the workstation is defective? Given that the workstation is defective what is the probability that it is of type A?

7. A campus router has been mis-configured in such a way that packets between two colleges C_1 and C_2 are routed off campus with probability $3/4$ and stay on campus with probability $1/4$. A packet routed off campus has a probability of being dropped of $1/3$ whereas a packet that doesn't leave the campus has a lower probability of being dropped of $1/4$. What is the probability that a packet travelling between C_1 and C_2 is dropped? Given that a packet is received at C_2 from C_1 without being dropped, what is the probability that the packet was routed off campus?

8. Suppose that one person in 1000 suffers a severe adverse reaction to some drug. A simple test is available that claims to be 95% reliable in the sense that if a person would suffer the reaction a positive test results with probability 95% and if they would not suffer the reaction a negative test results with probability 95%. Given that you have tested positive, what is the probability that you would suffer the adverse reaction to the drug? What do you make of the claim that the test is 95% reliable?

9. Suppose that X is a discrete random variable with the uniform distribution $X \sim U(1,n)$. Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

10. Suppose that X is a continuous random variable with the uniform distribution $X \sim U(a,b)$ with $a < b$. Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

11. Suppose that $X_1, X_2, \ldots, X_n$ is a sequence of independent and identically distributed random variables with $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Define the *sample mean*, $\overline{X}_n$, by

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \,.$$

Show that $\mathbb{E}(\overline{X}_n) = \mu$ and $\mathrm{Var}(X) = \sigma^2/n$. Define the *sample variance*, $\overline{S}_n$, by

$$\overline{S}_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \,.$$

Show that $\mathbb{E}(\overline{S}_n) = \sigma^2$.

12. Let X be a random variable with a geometric distribution with parameter p and let $q = 1-p$. Show that for $|z| < 1/q$, X has a probability generating function given by $G_X(z) = pz/(1-qz)$. Using the probability generating function $G_X(z)$ calculate the mean and variance of X.

13. Suppose that X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively.

    (a) Show that $X + Y \sim \mathrm{Pois}(\lambda_1 + \lambda_2)$.

    (b) Find the probability distribution of X conditional on the event that $X + Y = n$ where n is a fixed non-negative integer in the range $n = 0, 1, 2, \ldots$.

14. Consider a sequence of independent identically distributed random variables $Y_1, Y_2, \ldots$ with $\mathbb{P}(Y_i = 1) = p$ and $\mathbb{P}(Y_i = -1) = 1 - p$ with $p \in [0,1]$. Define the *simple random walk* X_n by

$$X_n = X_0 + Y_1 + Y_2 + \cdots + Y_n$$

where $X_0 \in \mathbb{R}$.

    (a) Find $\mathbb{E}(X_n)$ and $\mathrm{Var}(X_n)$ when $X_0 = 0$.

    (b) Find $\mathbb{P}(X_n = n + k)$ when $X_0 = k$.

15. (a) Consider the Gambler's ruin problem studied in lectures and construct both $\mathbb{P}(\text{A is ruined})$ and $\mathbb{P}(\text{B is ruined})$. What is $\mathbb{P}(\text{A is ruined}) + \mathbb{P}(\text{B is ruined})$?

    (b) Check that the solution given in lectures for the expected duration of the Gambler's ruin problem solves the stated difference equation.