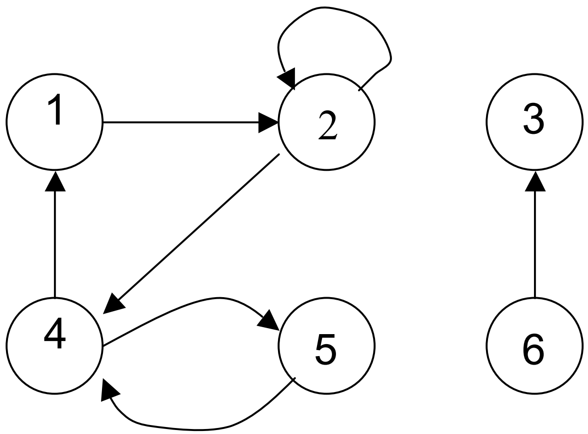
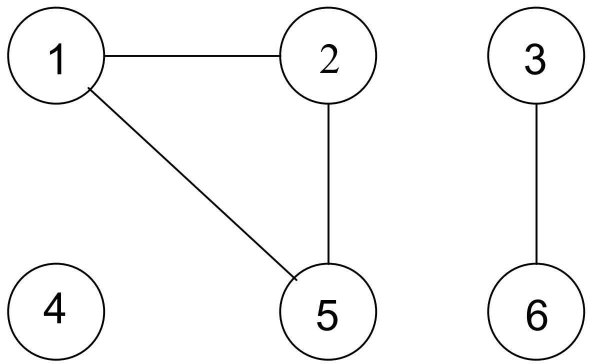
Introduction to Network Theory

What is a Network?

- Network = graph
- Informally a *graph* is a set of nodes joined by a set of lines or arrows.





Graph-based representations

- Representing a problem as a graph can provide a different point of view
- Representing a problem as a graph can make a problem much simpler
  - More accurately, it can provide the appropriate tools for solving the problem

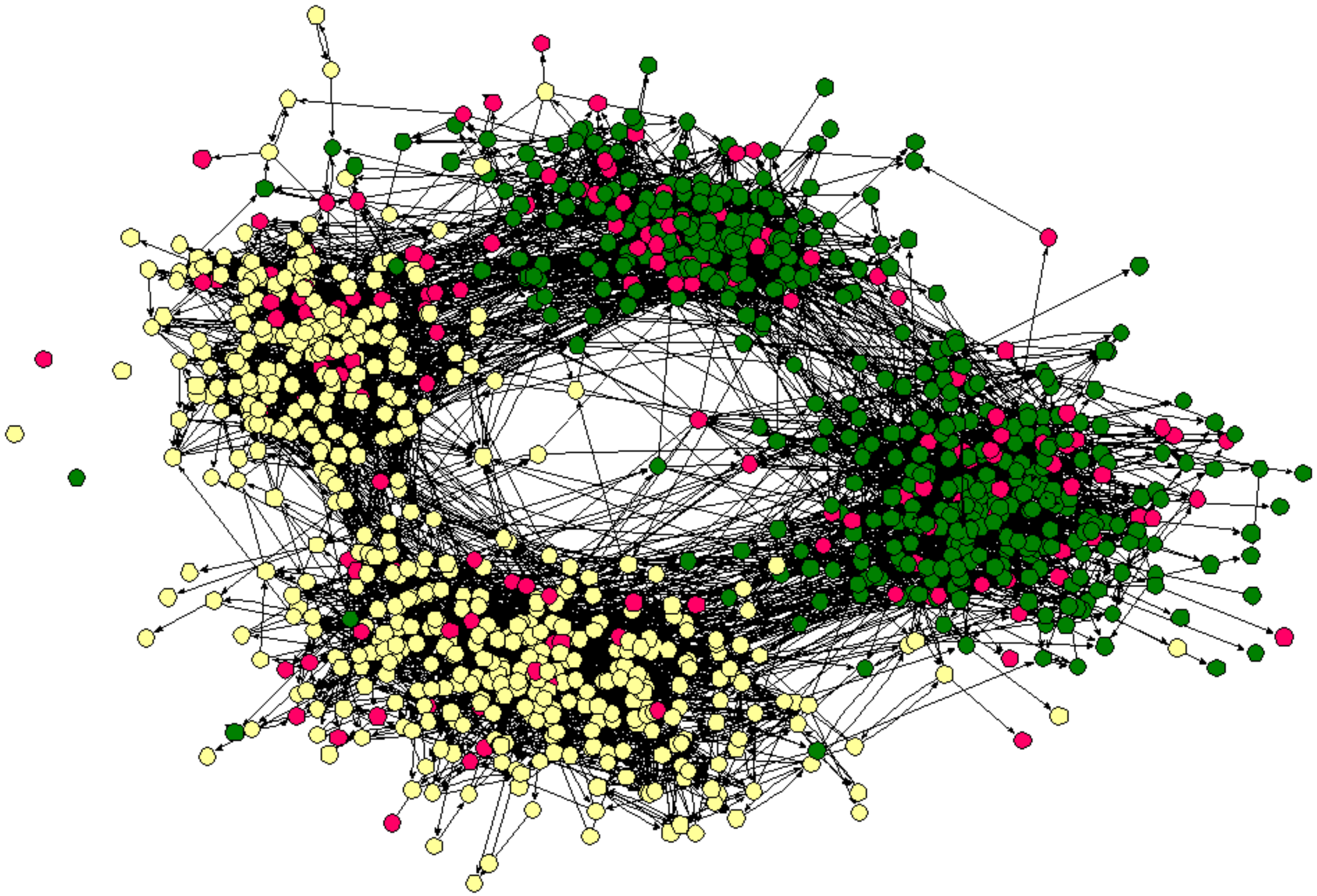
What is network theory?

- *Network theory* provides a set of techniques for analysing graphs
- *Complex systems network theory* provides techniques for analysing structure in a system of interacting agents, represented as a network
- Applying network theory to a system means using a graph-theoretic representation

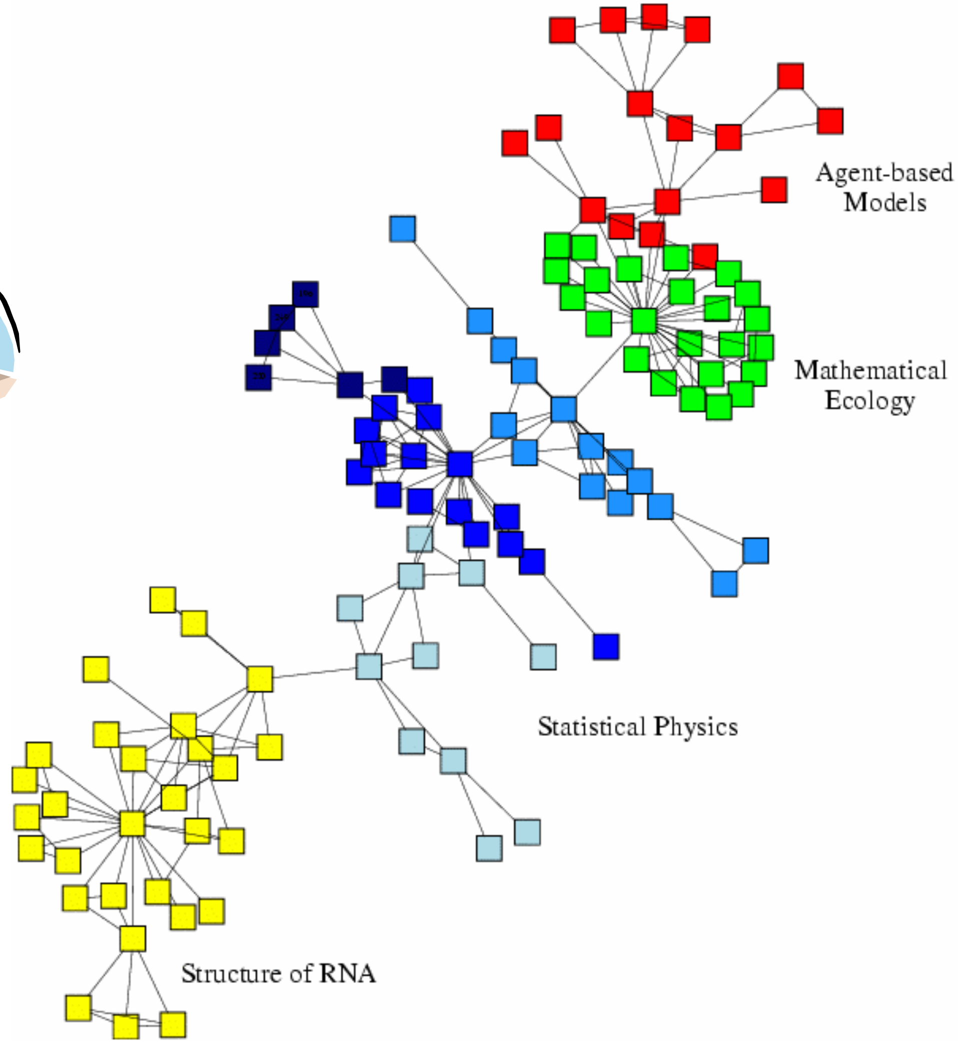
What makes a problem graph-like?

- There are two components to a graph
  - Nodes and edges
- In graph-like problems, these components have natural correspondences to problem elements
  - Entities are nodes and interactions between entities are edges
- Most complex systems are graph-like

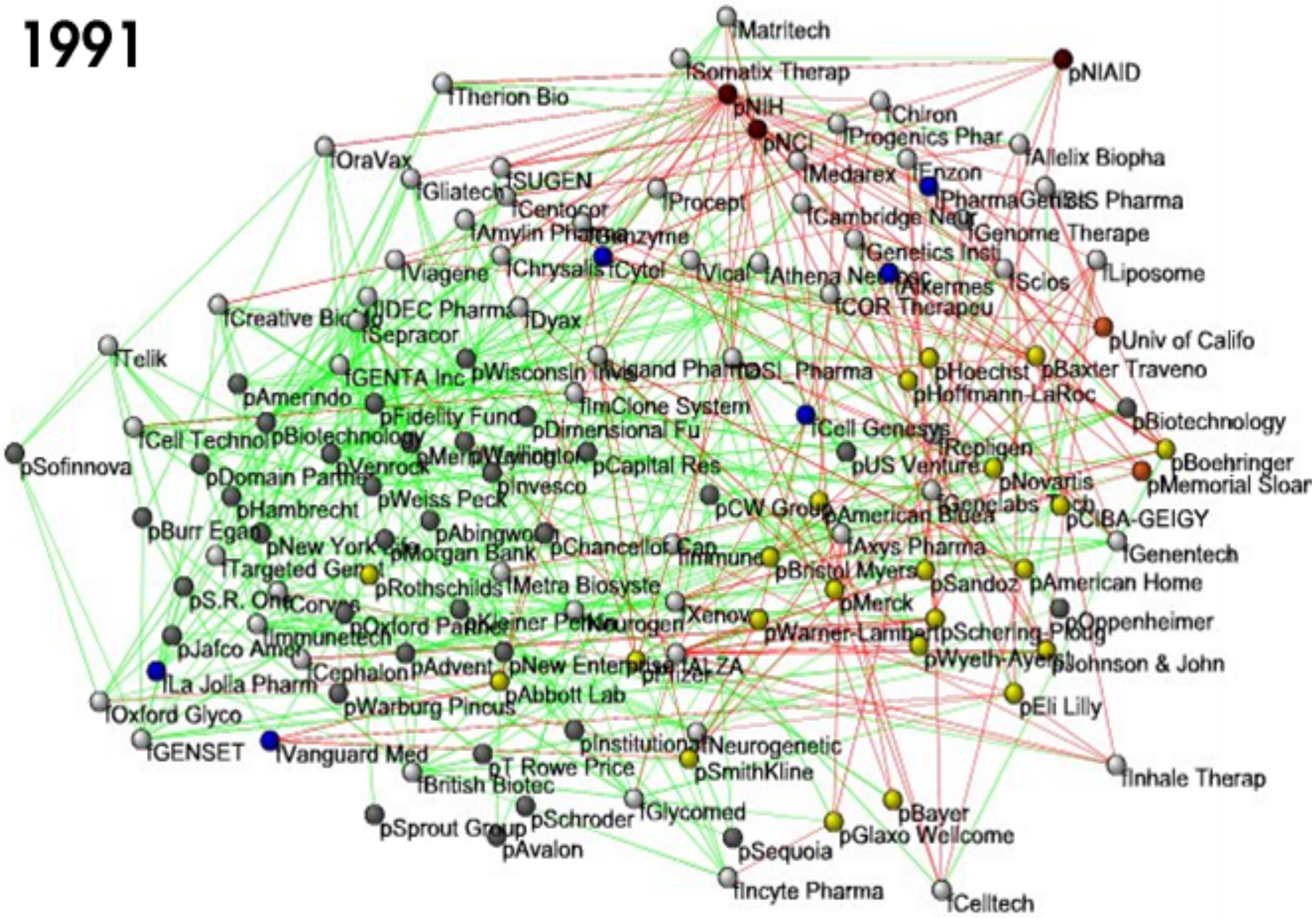
Friendship Network



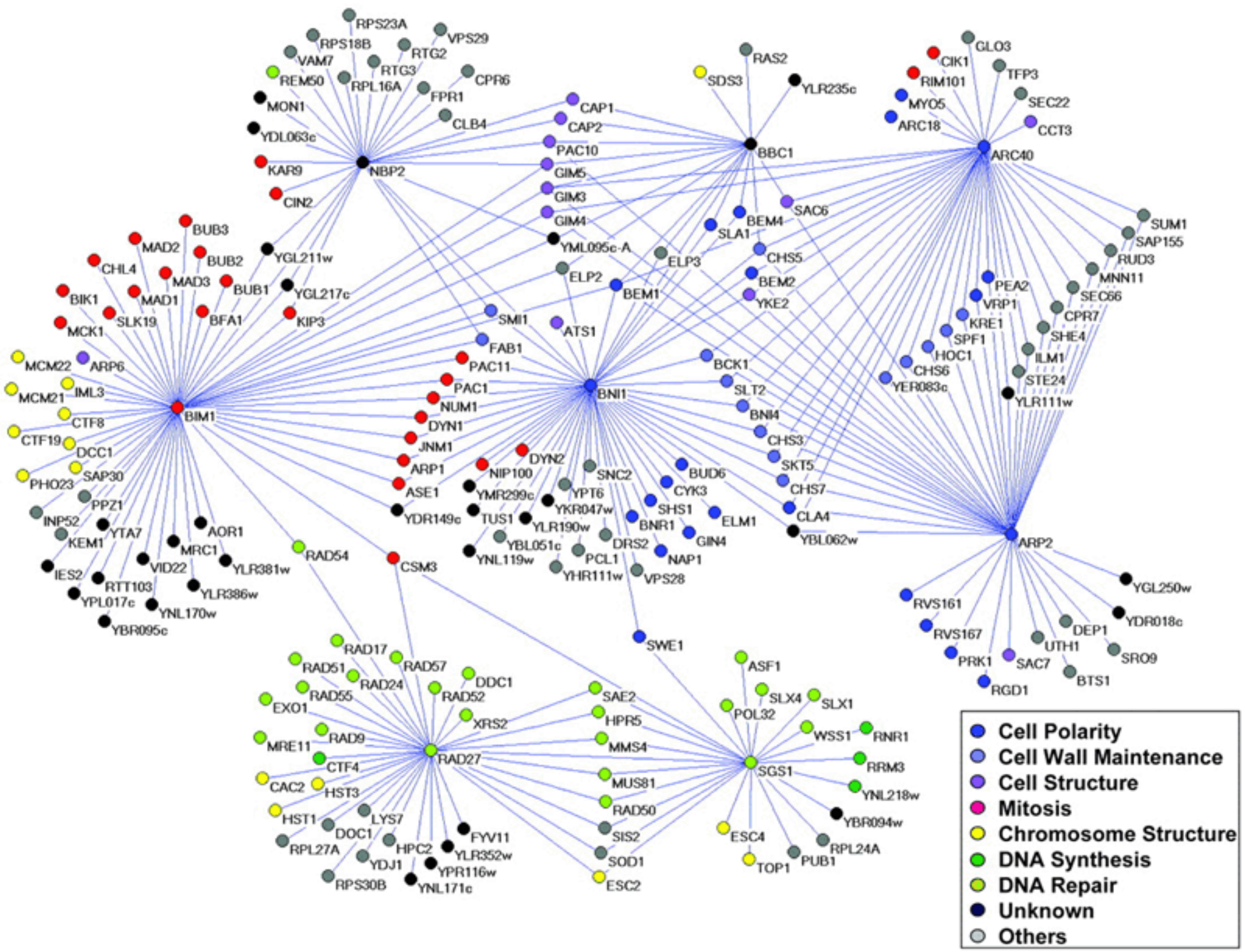
Scientific collaboration network



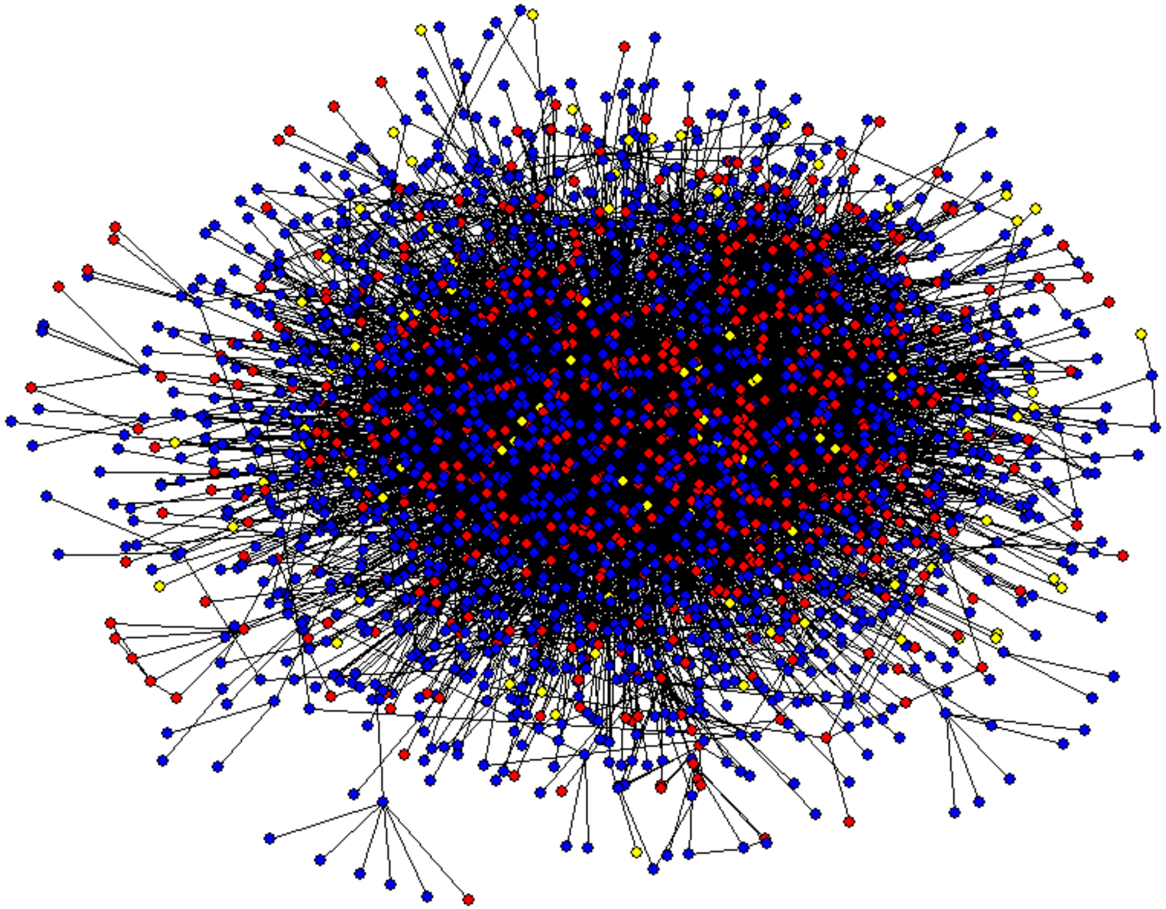
Business ties in US biotech-industry



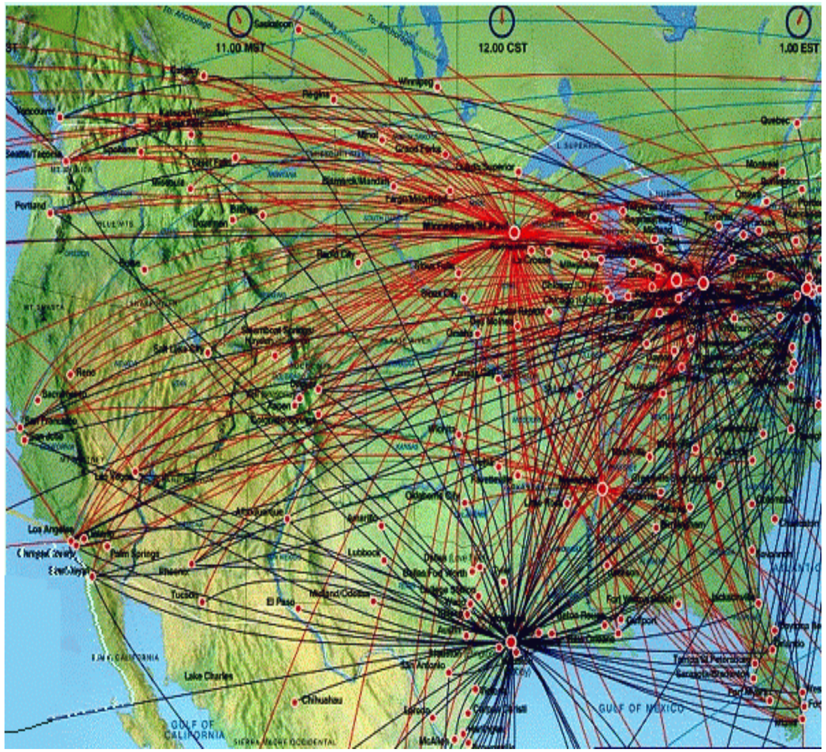
Genetic interaction network



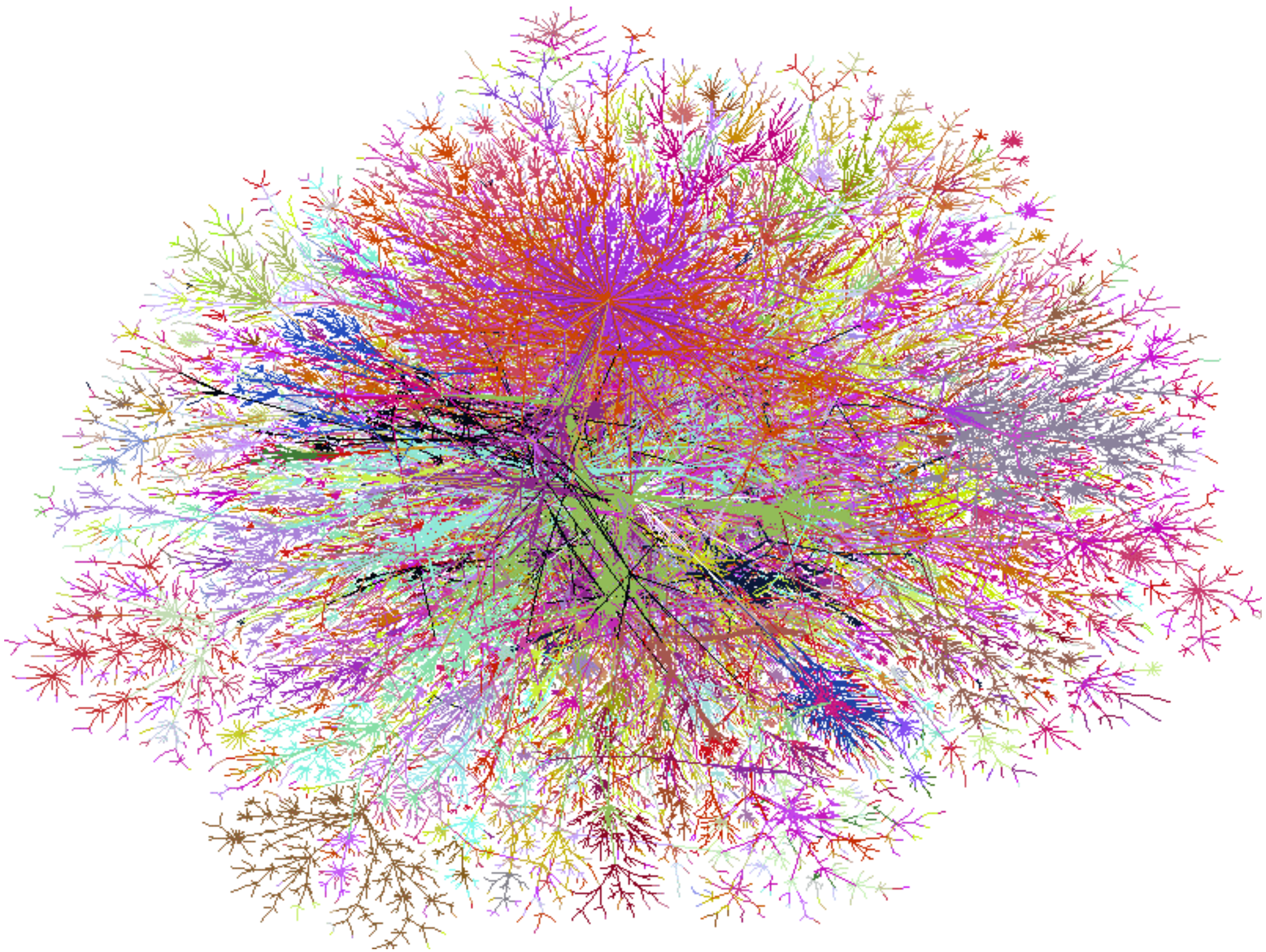
Protein-Protein Interaction Networks



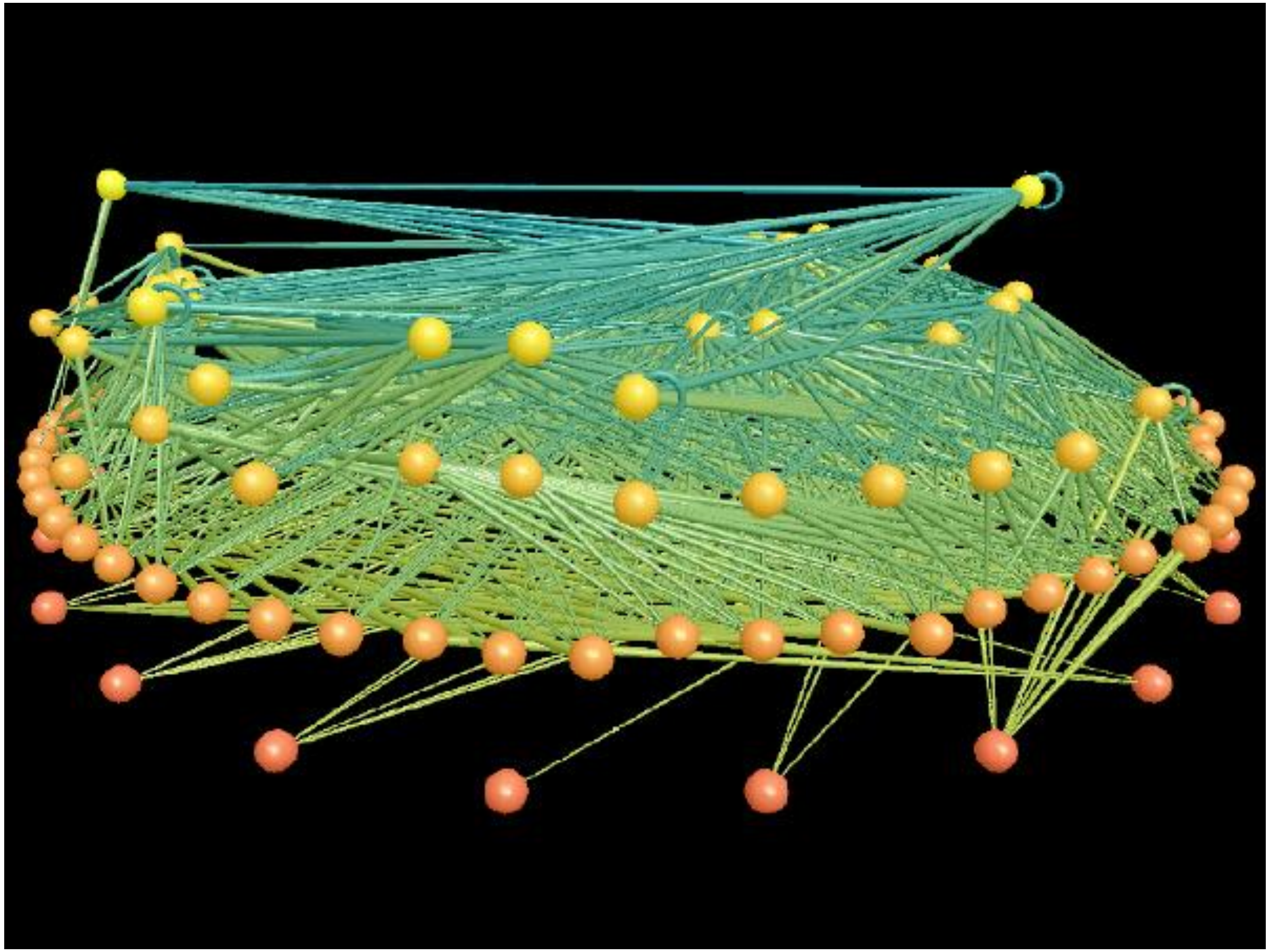
Transportation Networks



Internet



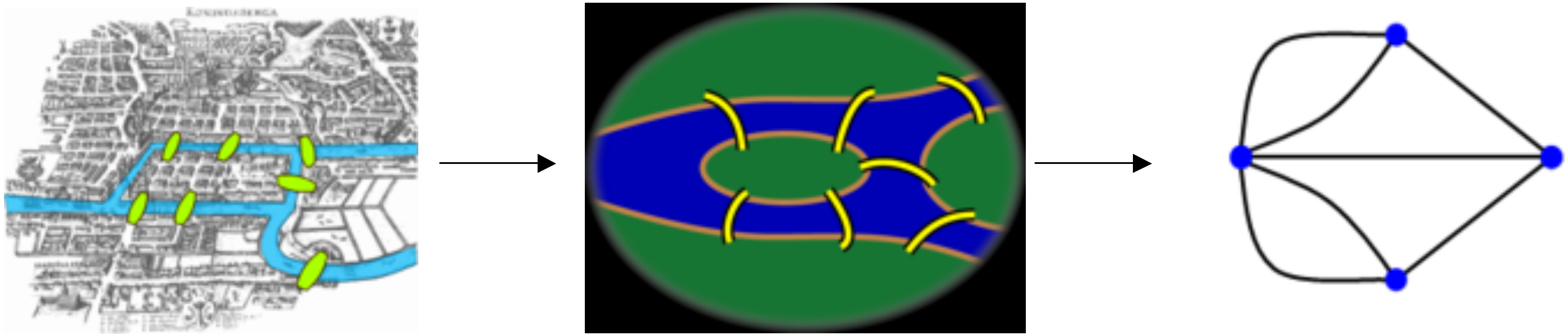
Ecological Networks



Graph Theory - History

Leonhard Euler's paper on “*Seven Bridges of Königsberg*” ,

published in 1736.

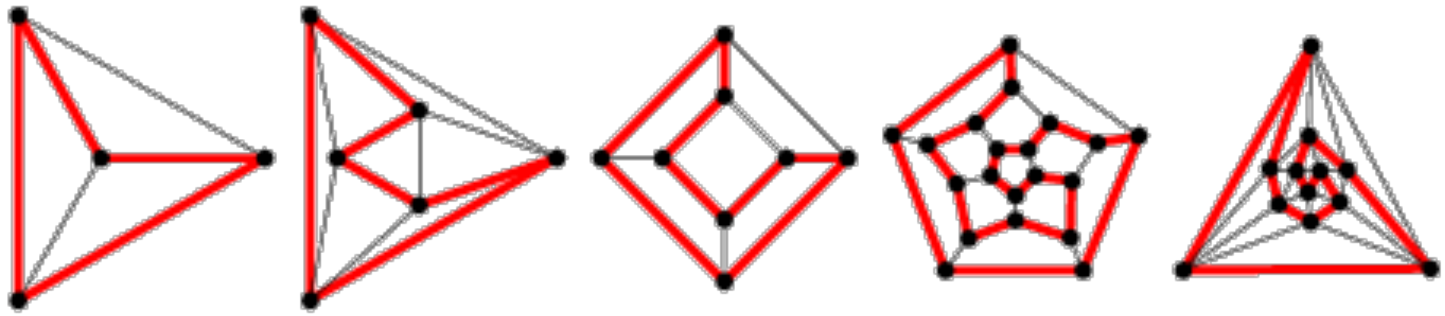


Graph Theory - History

Cycles in Polyhedra

Thomas P. Kirkman

William R. Hamilton

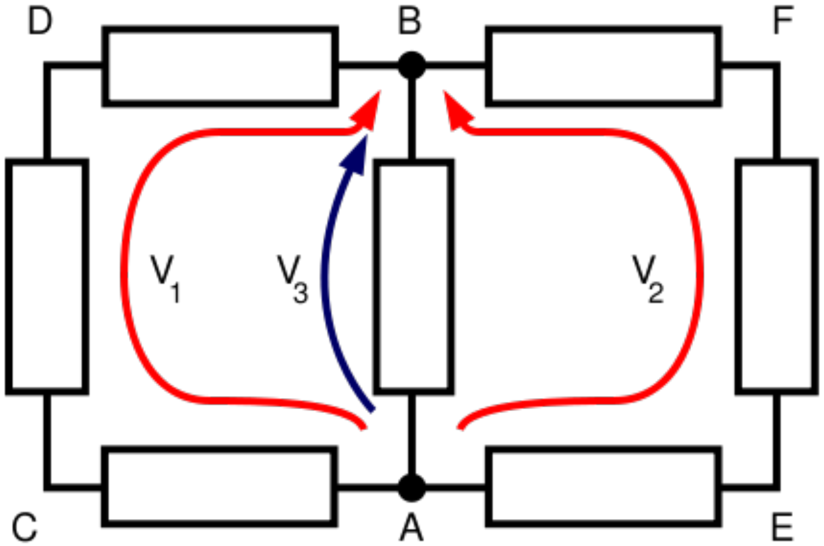


Hamiltonian cycles in Platonic graphs

Graph Theory - History

Trees in Electric Circuits

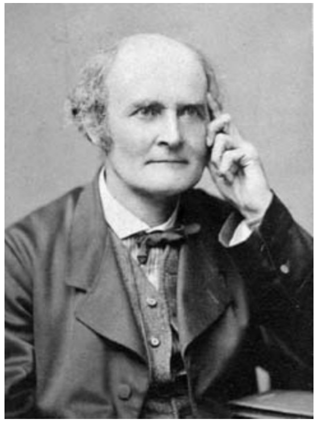
Gustav Kirchhoff



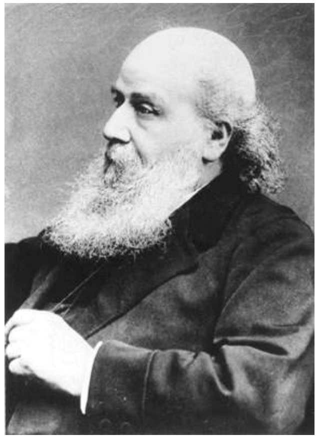
$$V_1 = V_2 = V_3$$

Graph Theory - History

Enumeration of Chemical Isomers

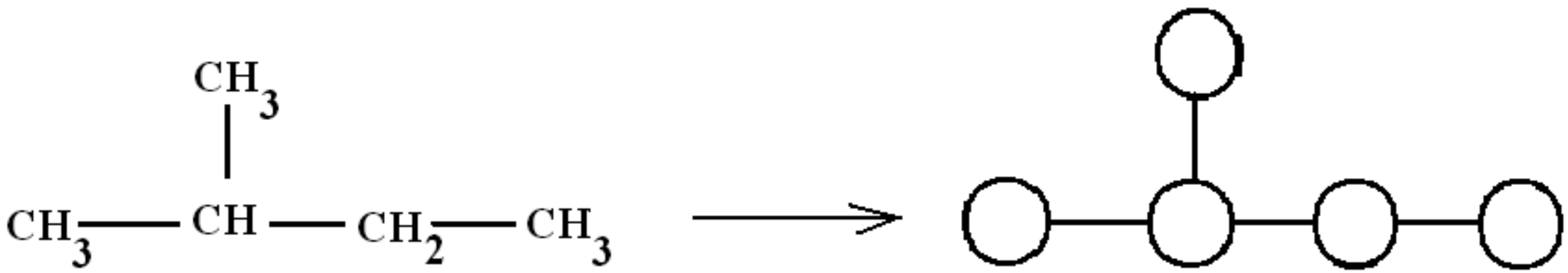


Arthur Cayley



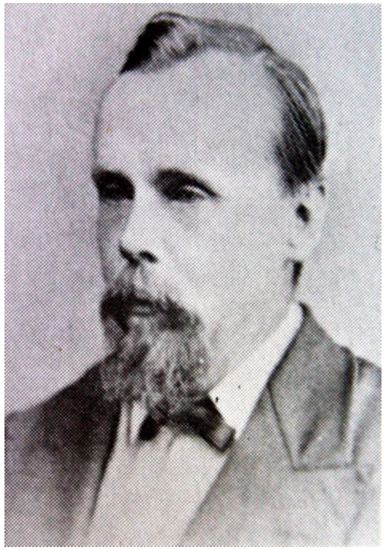
James J. Sylvester

George Polya



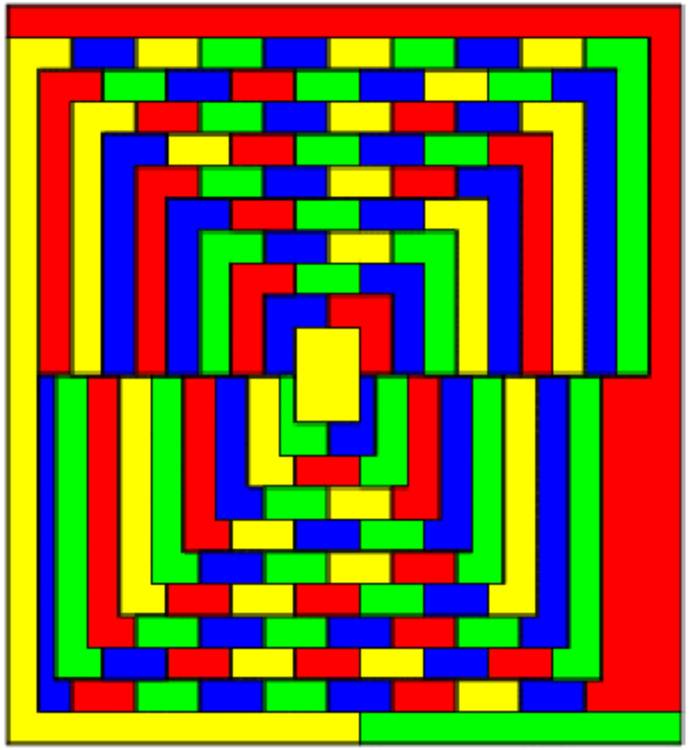
Graph Theory - History

Four Colors of Maps



Francis Guthrie

Auguste DeMorgan



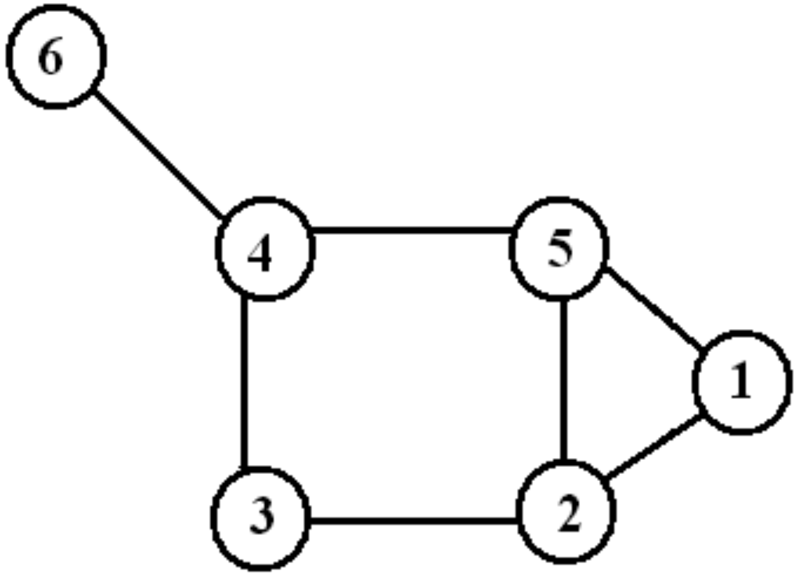
Definition: Graph

- G is an ordered triple G:=(V, E, f)
  - V is a set of nodes, points, or vertices.
  - E is a set, whose elements are known as edges or lines.
  - f is a function
    - maps each element of E
    - to an unordered pair of vertices in V.

Definitions

- Vertex
  - Basic Element
  - Drawn as a *node* or a *dot*.
  - **Vertex set** of G is usually denoted by $V(G)$, or V
- Edge
  - A set of two elements
  - Drawn as a line connecting two vertices, called end vertices, or endpoints.
  - The edge set of G is usually denoted by E(G), or E.

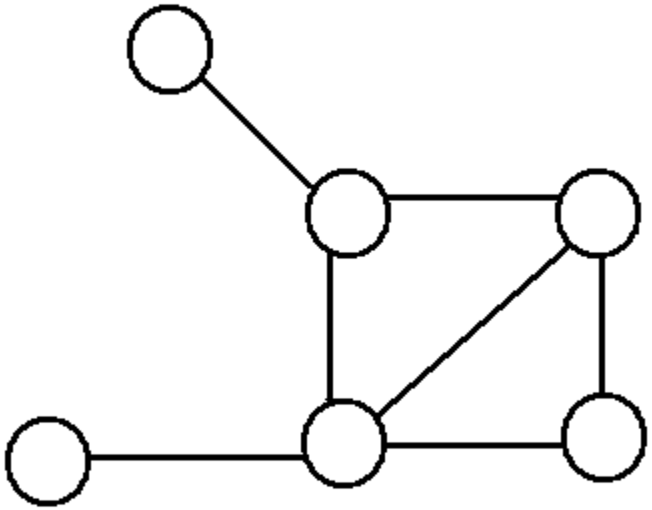
Example



- V:={1,2,3,4,5,6}
- E:={{1,2},{1,5},{2,3},{2,5},{3,4},{4,5},{4,6}}

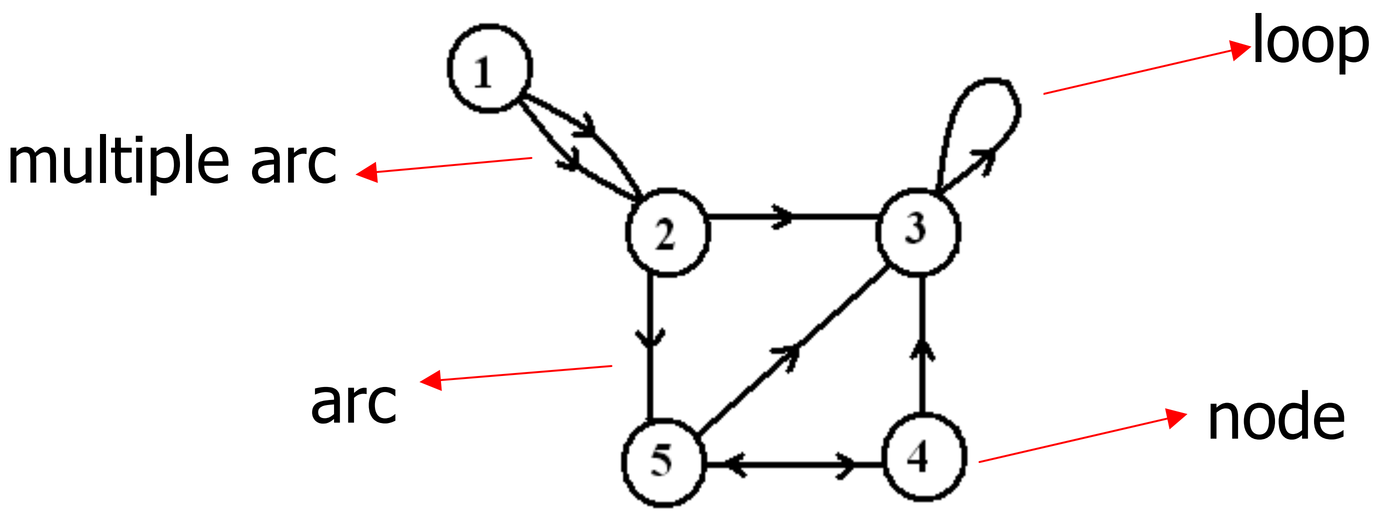
Simple Graphs

Simple graphs are graphs without multiple edges or self-loops.



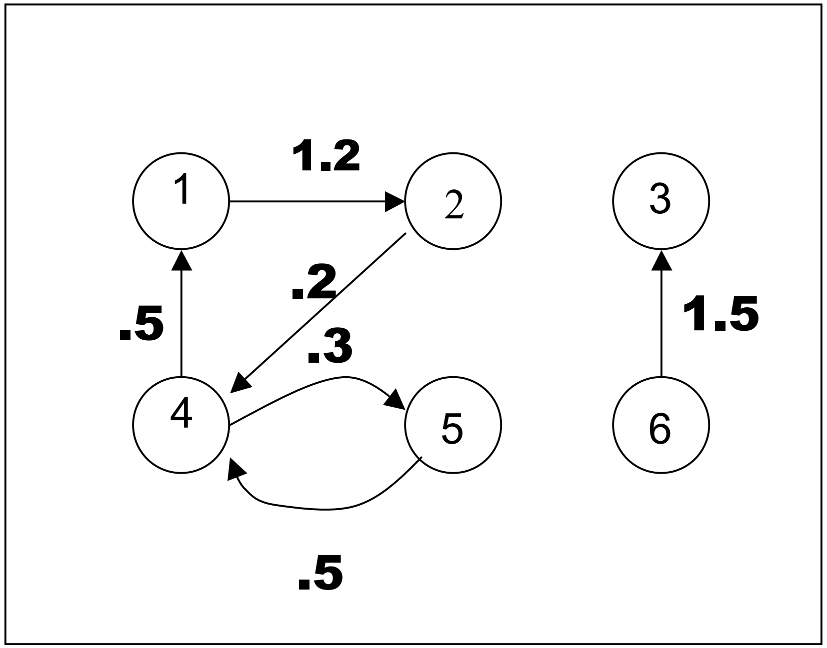
Directed Graph (digraph)

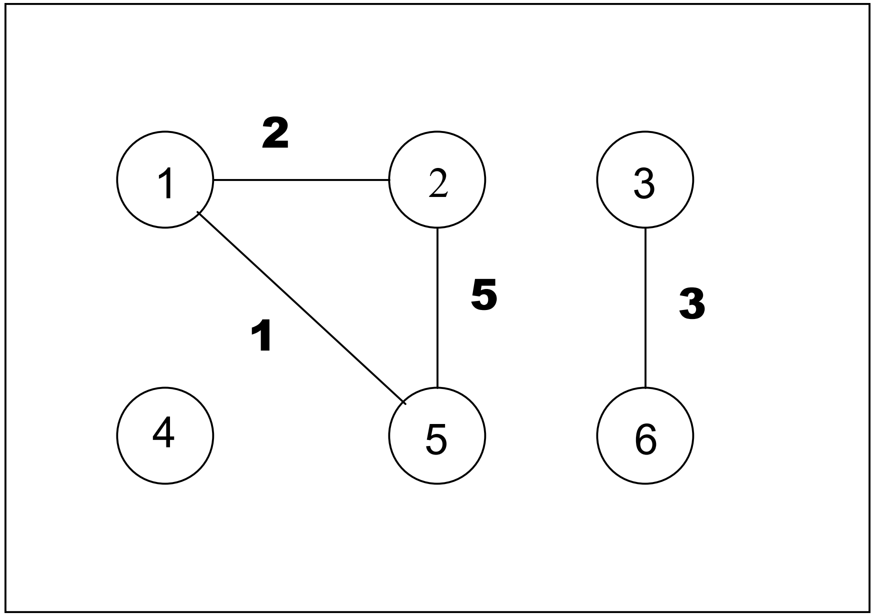
- Edges have directions
  - An edge is an *ordered* pair of nodes



Weighted graphs

- is a graph for which each edge has an associated ***weight***, usually given by a ***weight function*** $w: E \rightarrow \mathbf{R}$.





Structures and structural metrics

- Graph structures are used to isolate interesting or important sections of a graph
- Structural metrics provide a measurement of a structural property of a graph
  - Global metrics refer to a whole graph
  - Local metrics refer to a single node in a graph

Graph structures

- Identify interesting sections of a graph
  - Interesting because they form a significant domain-specific structure, or because they significantly contribute to graph properties
- A subset of the nodes and edges in a graph that possess certain characteristics, or relate to each other in particular ways

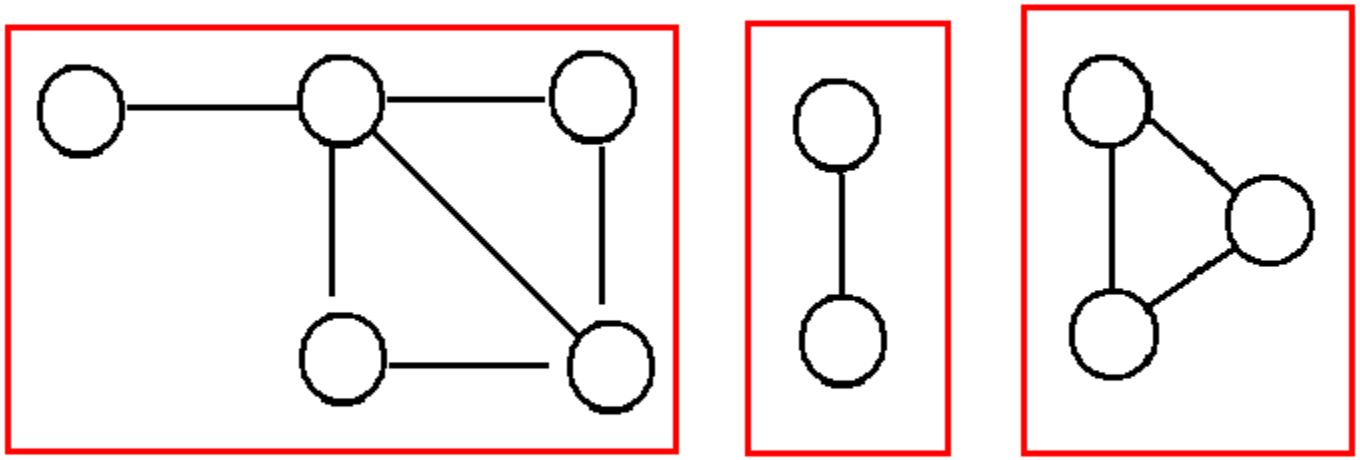
Connectivity

- a graph is ***connected*** if
  - you can get from any node to any other by following a sequence of edges OR
  - any two nodes are connected by a path.

- A directed graph is ***strongly connected*** if there is a directed path from any node to any other node.

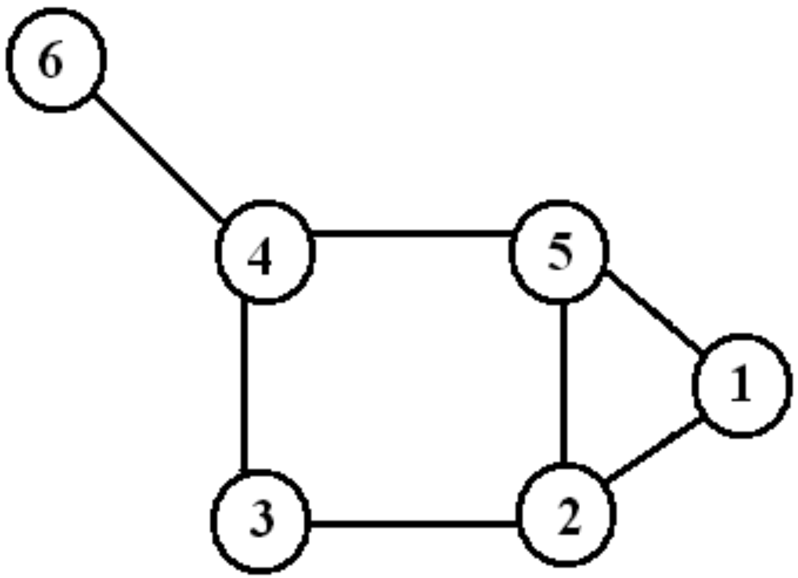
Component

- Every disconnected graph can be split up into a number of connected ***components***.



Degree

- Number of edges incident on a node

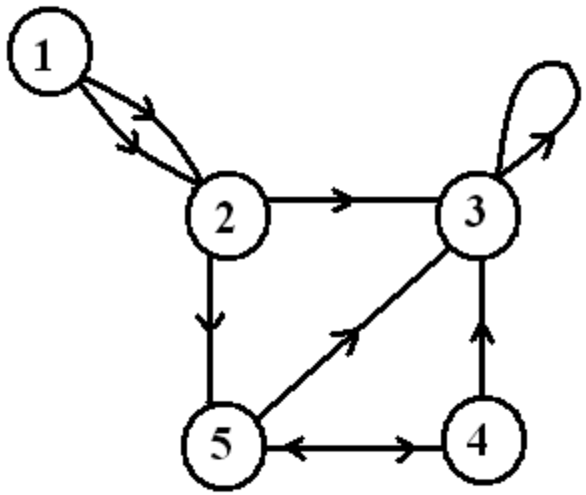


The degree of 5 is 3

Degree (Directed Graphs)

- In-degree: Number of edges entering
- Out-degree: Number of edges leaving

- Degree = indeg + outdeg



outdeg(1)=2
indeg(1)=0

outdeg(2)=2
indeg(2)=2

outdeg(3)=1
indeg(3)=4

Degree: Simple Facts

- If G is a graph with m edges, then

$$\sum \deg(v) = 2m = 2\,|E|$$

- If G is a digraph then

$$\sum \mathrm{indeg}(v) = \sum \mathrm{outdeg}(v) = |E|$$

- Number of Odd degree Nodes is even

Walks

A ***walk of length k*** in a graph is a succession of k (not necessarily different) edges of the form

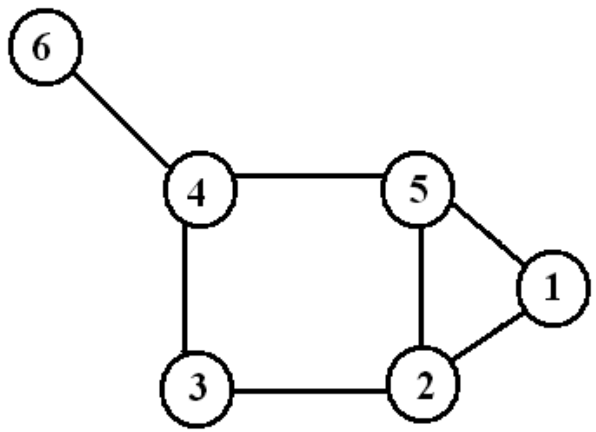
uv,vw,wx,…,yz.

This walk is denote by uvwx…xz, and is referred to as a ***walk between u and z***.

A walk is ***closed*** is u=z.

Path

- A *path* is a walk in which all the edges and all the nodes are different.

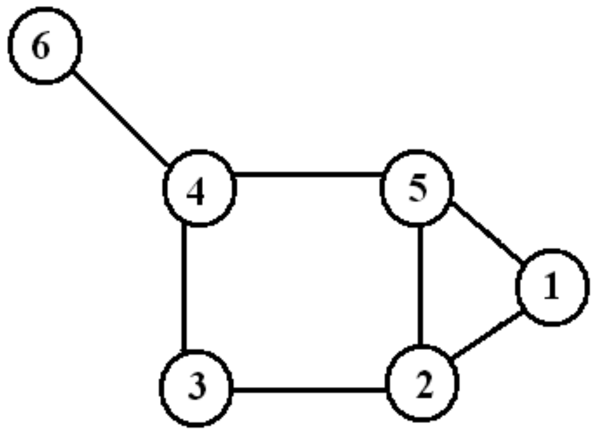


Walks and Paths

1,2,5,2,3,4	1,2,5,2,3,2,1	1,2,3,4,6
walk of length 5	CW of length 6	path of length 4

Cycle

- A ***cycle*** is a closed path in which all the edges are different.

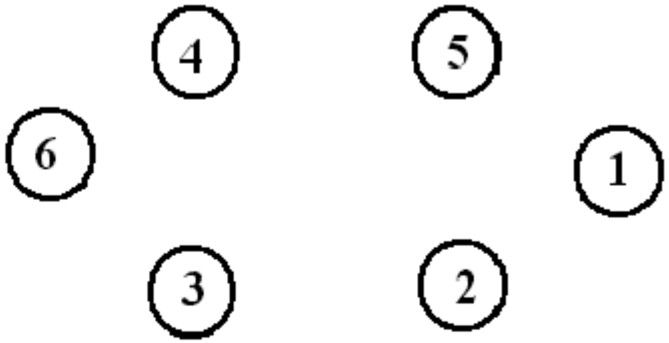


1,2,5,1
3-cycle

2,3,4,5,2
4-cycle

Special Types of Graphs

- Empty Graph / Edgeless graph
  - No edge

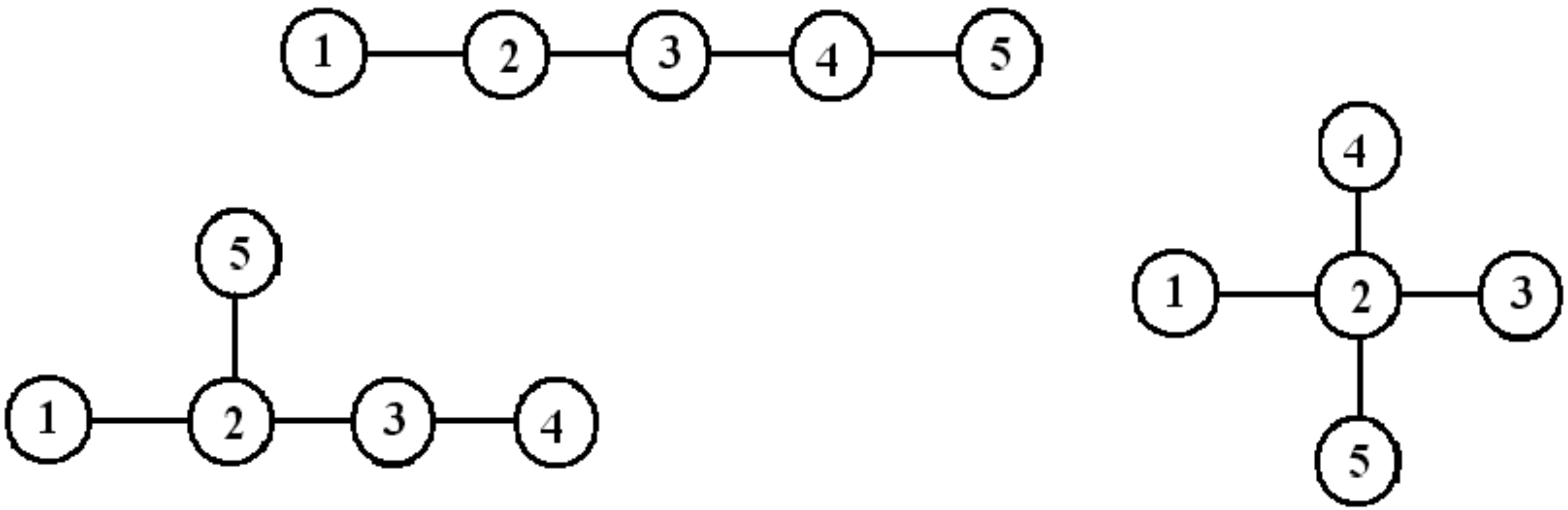


- Null graph
  - No nodes
  - Obviously no edge

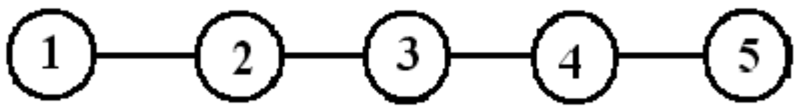
Trees

- Connected Acyclic Graph

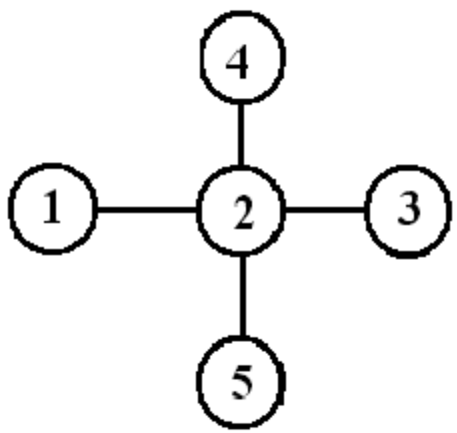
- Two nodes have *exactly* one path between them



Special Trees



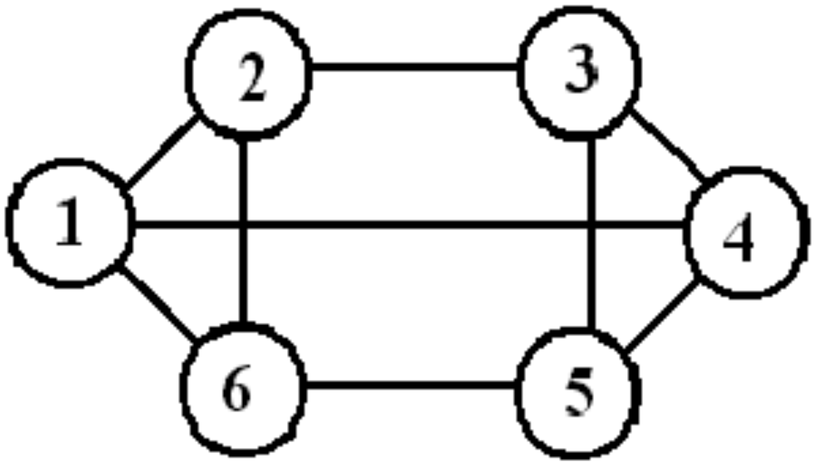
Paths



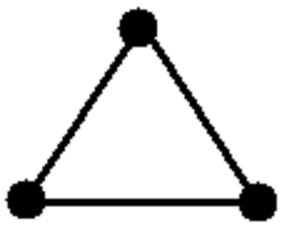
Stars

Regular

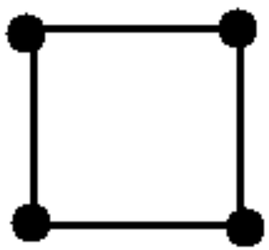
- Connected Graph
- All nodes have the same degree



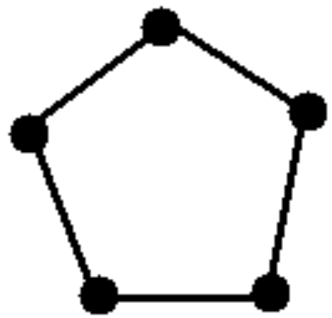
Special Regular Graphs: Cycles



C_3



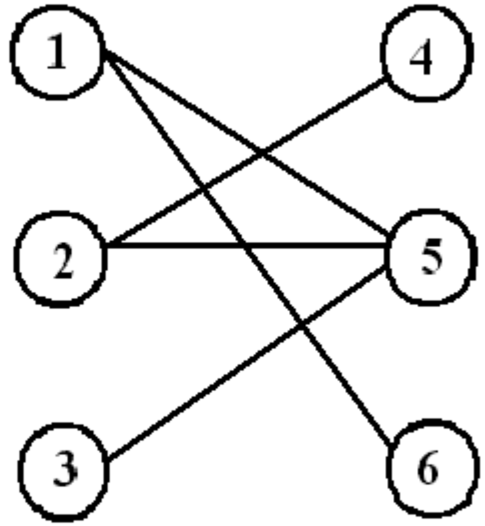
C_4



C_5

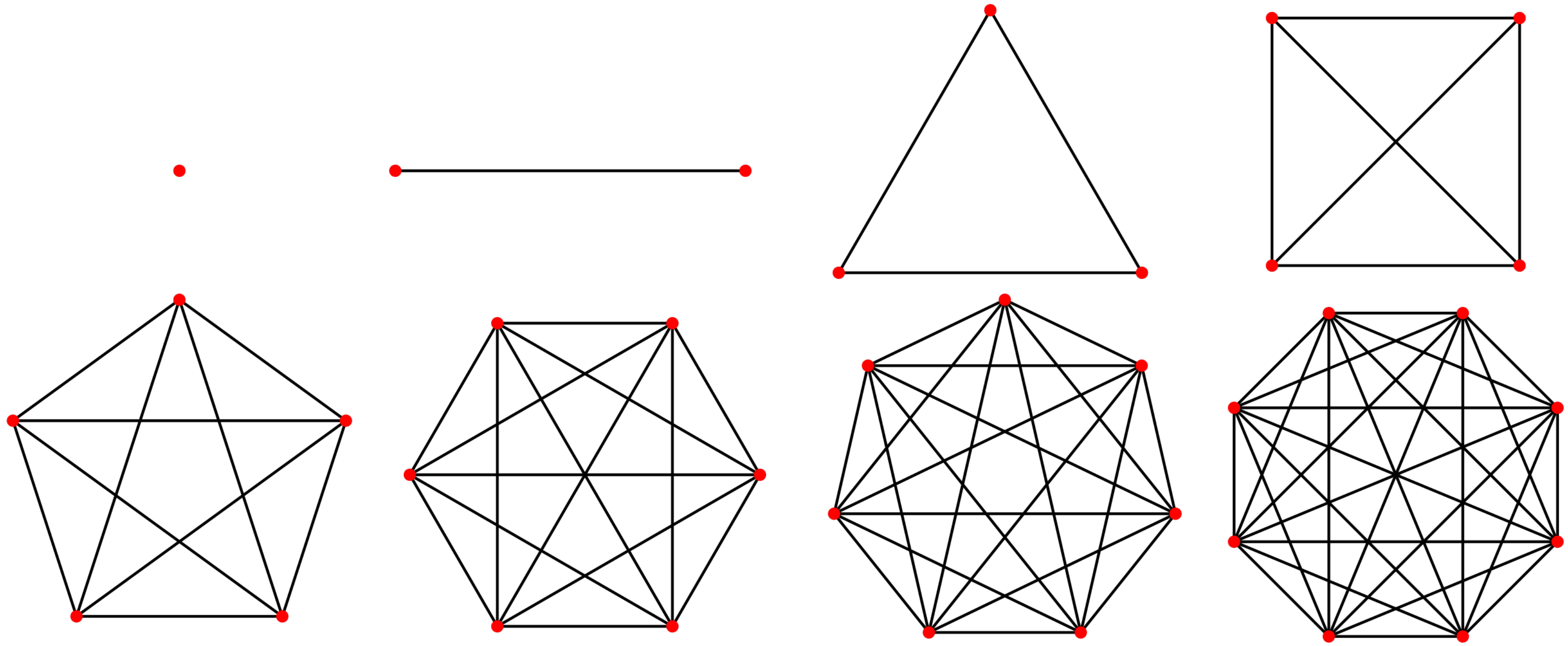
Bipartite graph

- V can be partitioned into 2 sets V_1 and V_2
  such that $(u,v) \in E$ implies
  - either $u \in V_1$ and $v \in V_2$
  - OR $v \in V_1$ and $u \in V_2$.



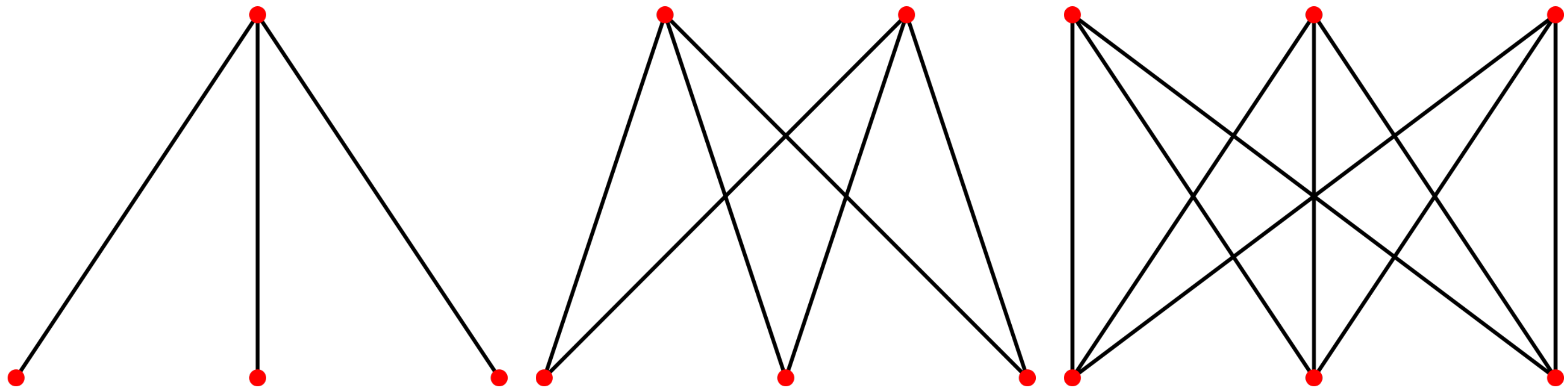
Complete Graph

- Every pair of vertices are adjacent
- Has n(n-1)/2 edges



Complete Bipartite Graph

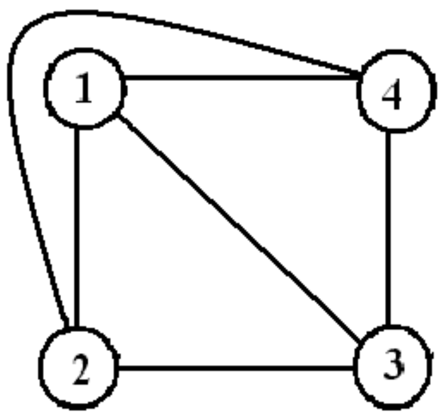
- Bipartite Variation of Complete Graph
- Every node of one set is connected to every other node on the other set

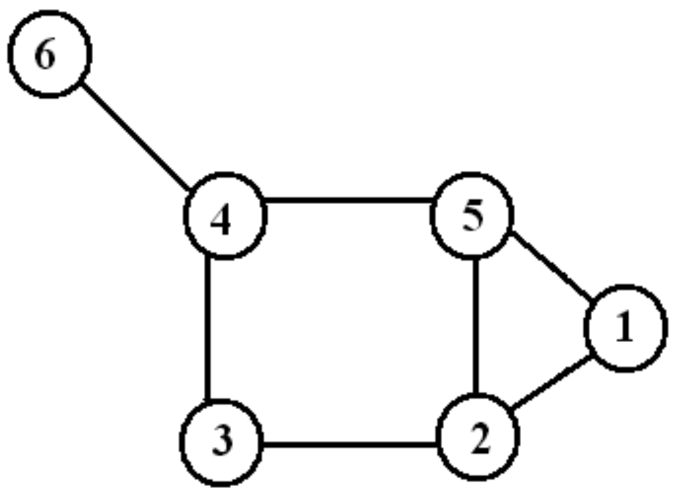


Stars

Planar Graphs

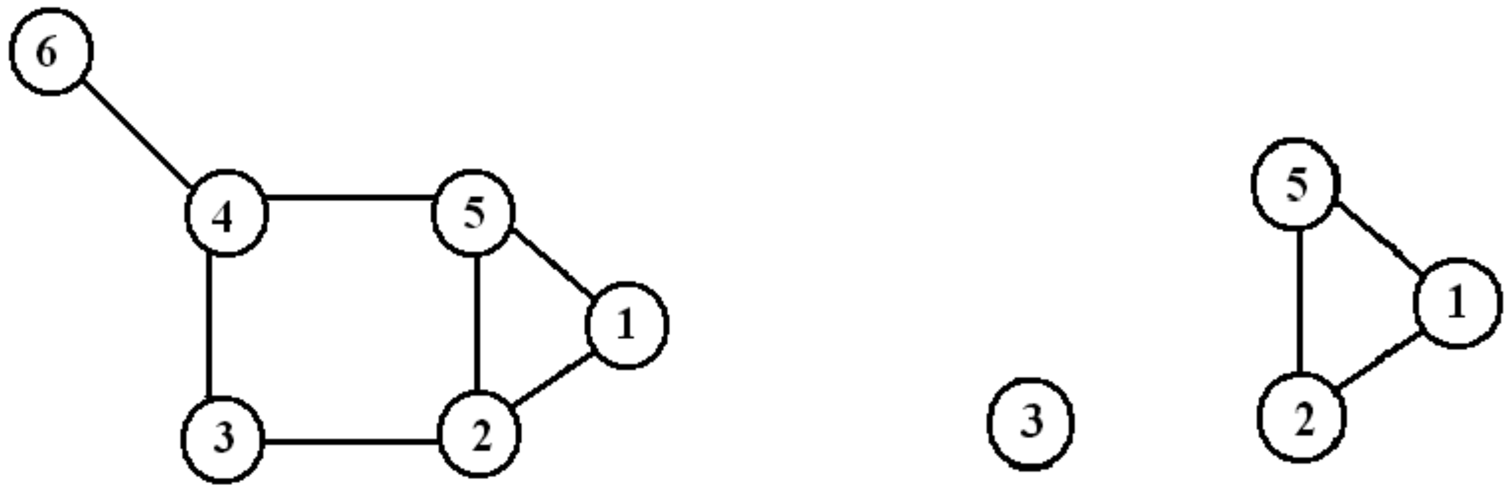
- Can be drawn on a plane such that no two edges intersect
- K_4 is the largest complete graph that is planar





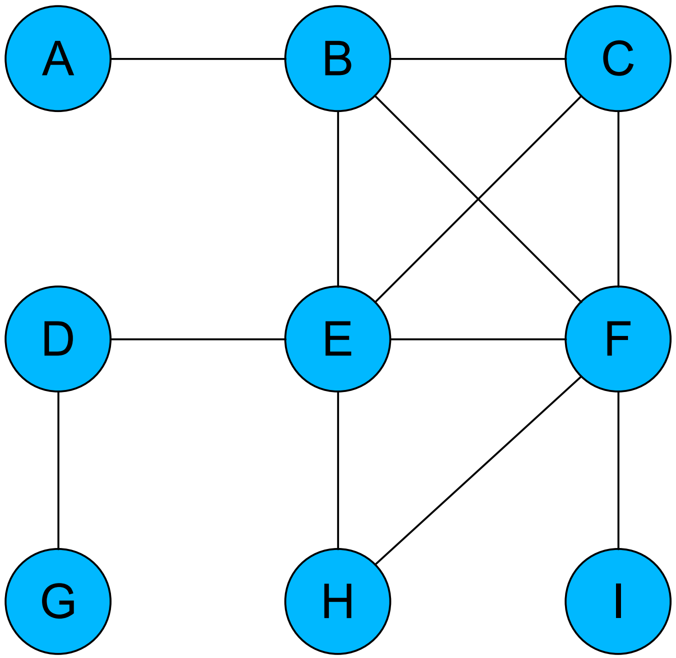
Subgraph

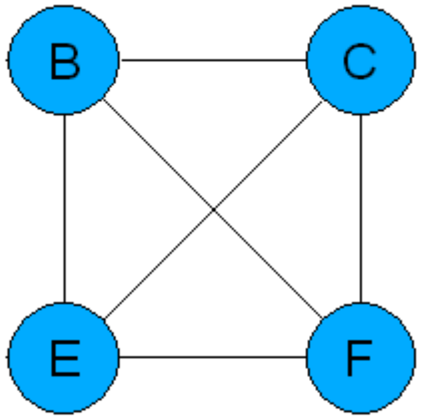
- Vertex and edge sets are subsets of those of G
  - a *supergraph* of a graph G is a graph that contains G as a subgraph.



Special Subgraphs: Cliques

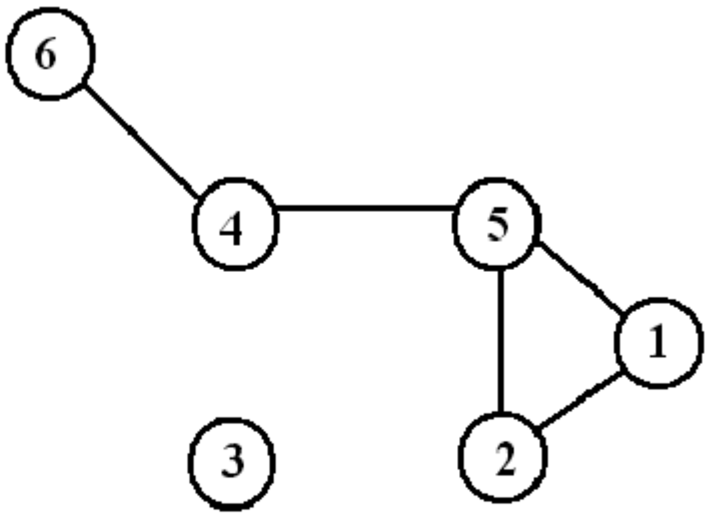
A **clique** is a maximum complete connected subgraph.





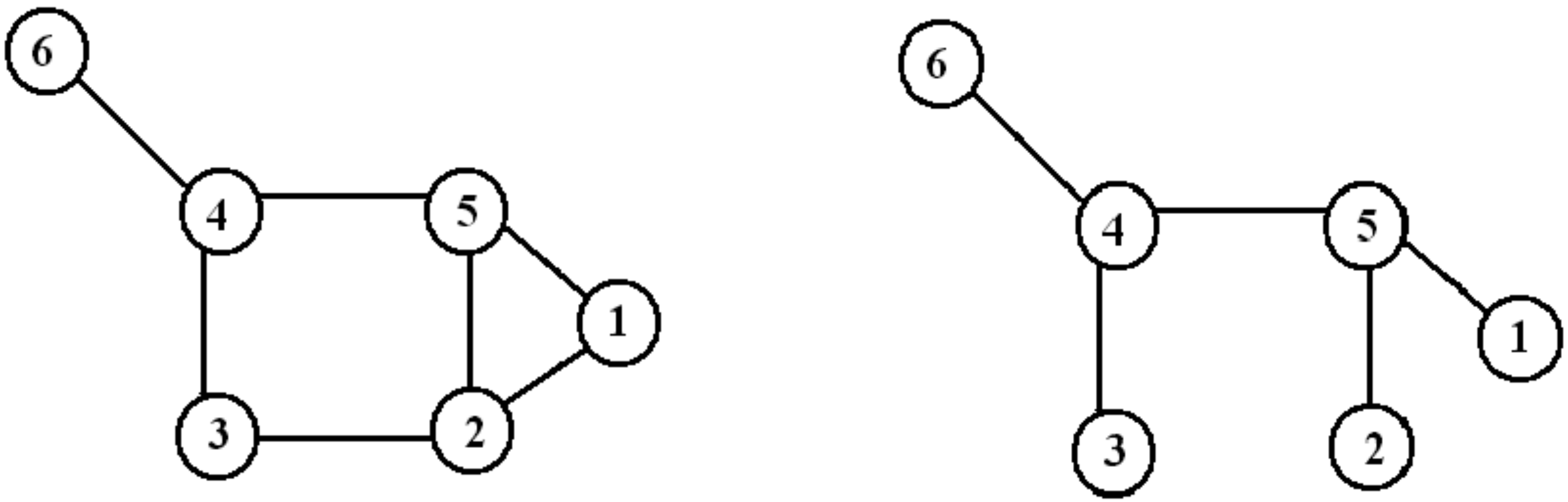
Spanning subgraph

- Subgraph H has the same vertex set as G.
  - Possibly not all the edges
  - “H spans G”.



Spanning tree

- Let G be a connected graph. Then a ***spanning tree*** in G is a subgraph of G that includes every node and is also a tree.



Isomorphism

- Bijection, i.e., a one-to-one mapping:

  f : V(G) -> V(H)

  u and v from G are adjacent if and only if f(u) and f(v) are adjacent in H.

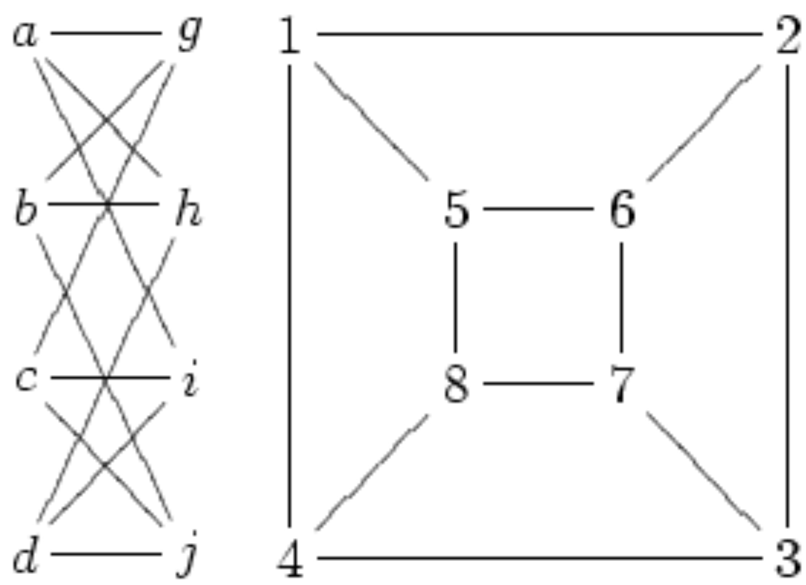
- If an isomorphism can be constructed between two graphs, then we say those graphs are ***isomorphic***.

Isomorphism Problem

- Determining whether two graphs are isomorphic
- Although these graphs look very different, they are isomorphic; one isomorphism between them is

  f(a)=1 f(b)=6 f(c)=8 f(d)=3

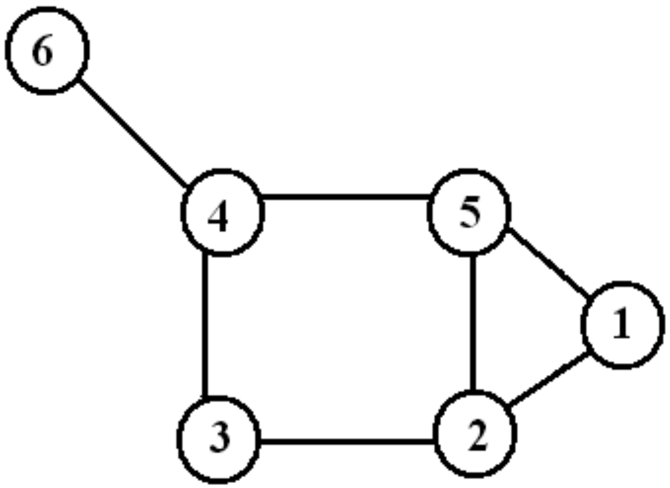
  f(g)=5 f(h)=2 f(i)=4 f(j)=7



Representation (Matrix)

- Incidence Matrix
  - V x E
  - [vertex, edges] contains the edge's data
- Adjacency Matrix
  - V x V
  - Boolean values (adjacent or not)
  - Or Edge Weights

Matrices



$$
\begin{array}{c|ccccccc}
 & 1,2 & 1,5 & 2,3 & 2,5 & 3,4 & 4,5 & 4,6 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
5 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
2 & 1 & 0 & 1 & 0 & 1 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 1 & 0 & 1 & 1 \\
5 & 1 & 1 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

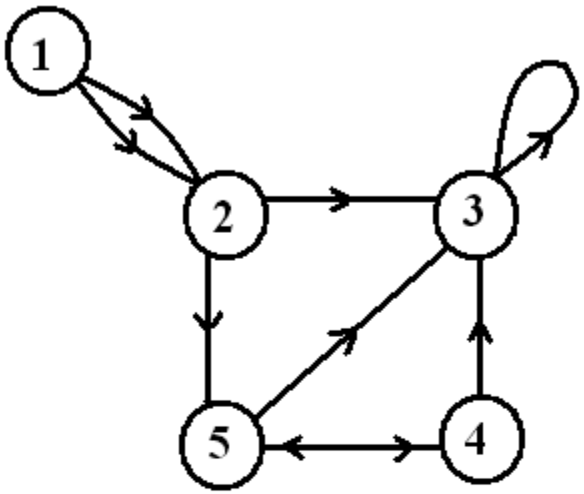
Representation (List)

- Edge List
  - pairs (ordered if directed) of vertices
  - Optionally weight and other data
- Adjacency List (node list)

Implementation of a Graph.

- ***Adjacency-list representation***
  - an array of $|V|$ lists, one for each vertex in V.
  - For each $u \in V$, $ADJ[u]$ points to all its adjacent vertices.

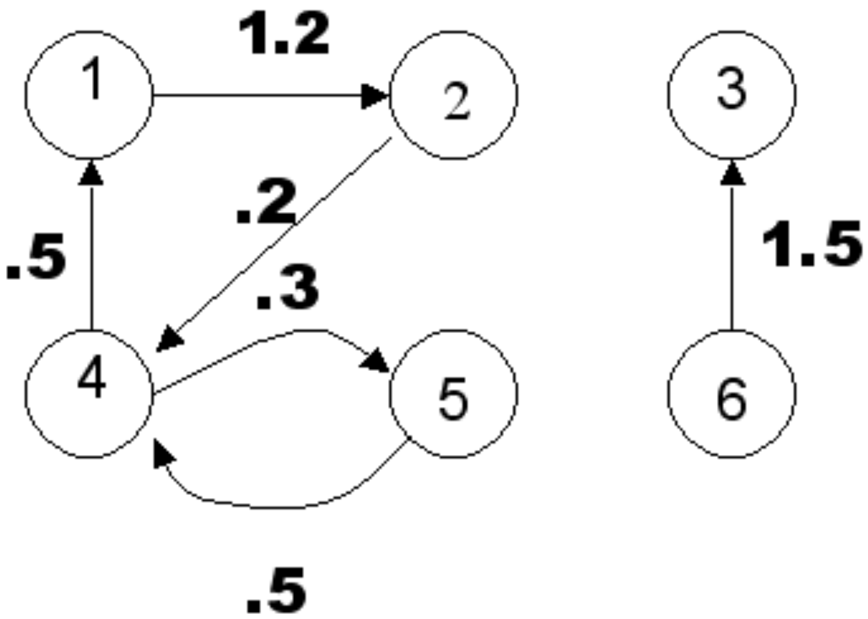
Edge and Node Lists



Edge List
1 2
1 2
2 3
2 5
3 3
4 3
4 5
5 3
5 4

Node List
1 2 2
2 3 5
3 3
4 3 5
5 3 4

Edge Lists for Weighted Graphs



<u>Edge List</u>

1 2 1.2
2 4 0.2
4 5 0.3
4 1 0.5
5 4 0.5
6 3 1.5

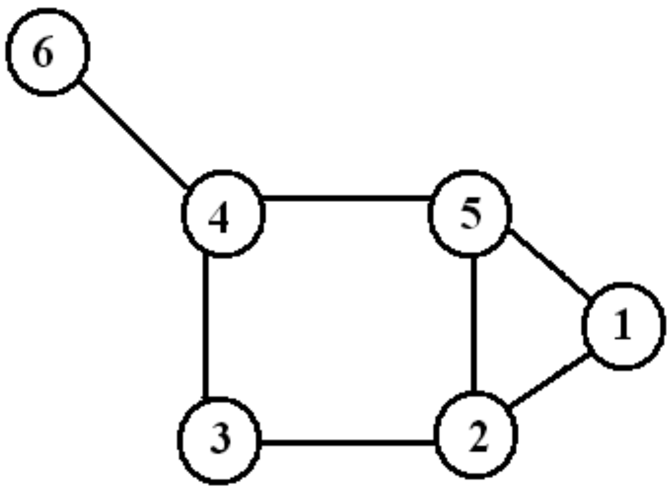
Topological Distance

- A shortest path is the minimum path connecting two nodes.

- The number of edges in the shortest path connecting *p* and *q* is the ***topological distance*** between these two nodes, $d_{p,q}$

Distance Matrix

- $|V| \times |V|$ matrix $D = (d_{ij})$ such that d_{ij} is the topological distance between i and j.



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 2 \\ 3 & 3 & 2 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

Random Graphs

Erdős and Renyi (1959)

N nodes

A pair of nodes has probability *p* of being connected.

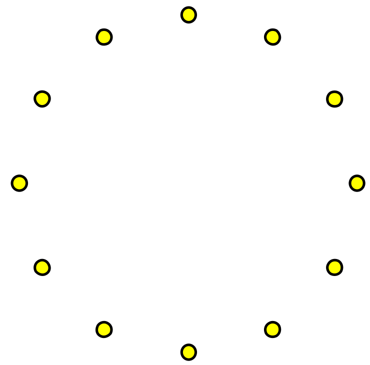
Average degree, $k \approx pN$

What interesting things can be said for different values of p or k ?

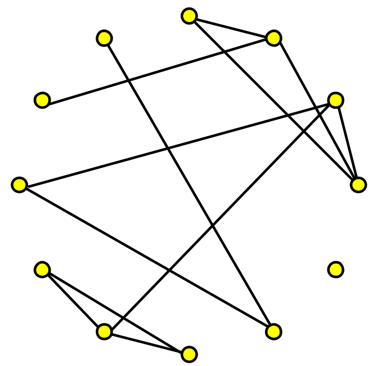
(that are true as N → ∞)

$N = 12$

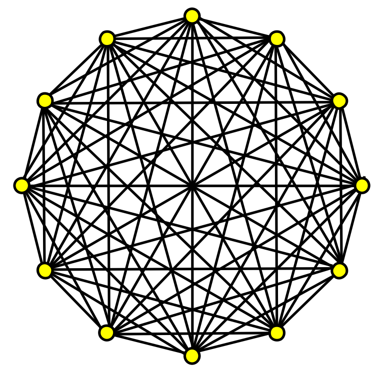
$p = 0.0$; $k = 0$



$p = 0.09$; $k = 1$



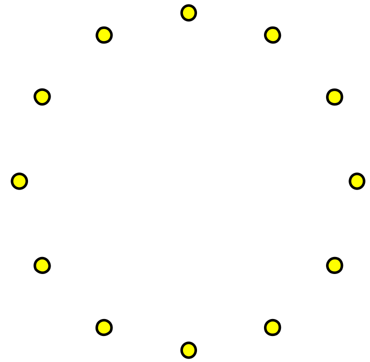
$p = 1.0$; $k \approx \frac{1}{2}N^2$

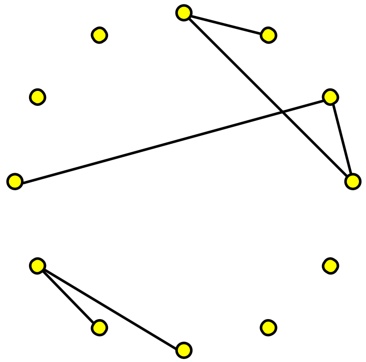


Random Graphs

Erdős and Renyi (1959)

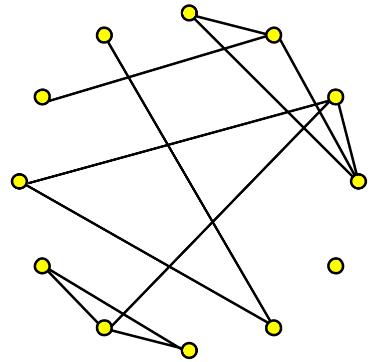
p = 0.0 ; k = 0



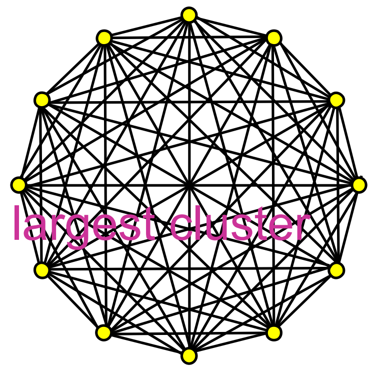


p = 0.045 ; k = 0.5

p = 0.09 ; k = 1



p = 1.0 ; $k \approx \frac{1}{2}N^2$

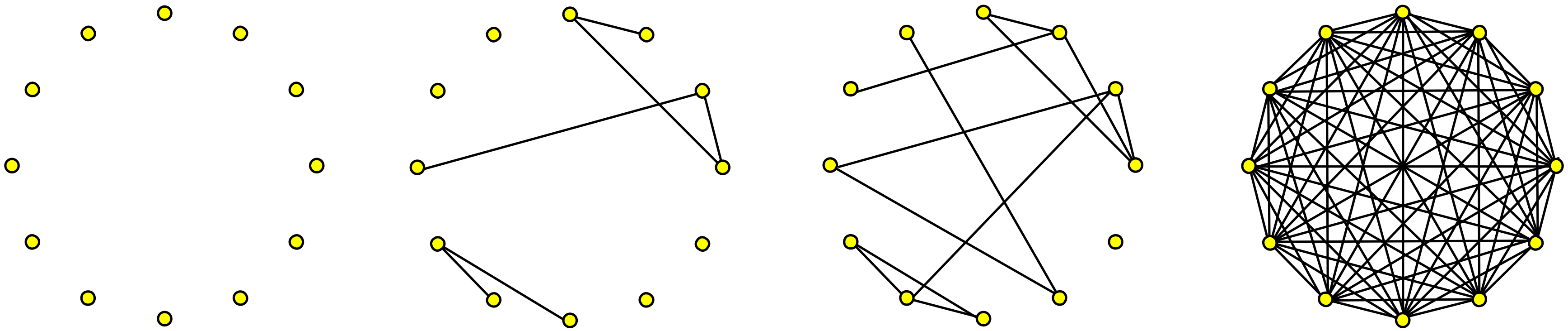


Let's look at…

- Size of the largest connected cluster
- Diameter (maximum path length between nodes) of the largest cluster
- Average path length between nodes (if a path exists)

Random Graphs

Erdős and Renyi (1959)



$p = 0.0$; $k = 0$ | $p = 0.045$; $k = 0.5$ | $p = 0.09$; $k = 1$ | $p = 1.0$; $k \approx ½N^2$

Size of largest component

1 | 5 | 11 | 12

Diameter of largest component

0 | 4 | 7 | 1

Average path length between nodes

0.0 | 2.0 | 4.2 | 1.0

Random Graphs

Erdős and Renyi (1959)

If $k < 1$:

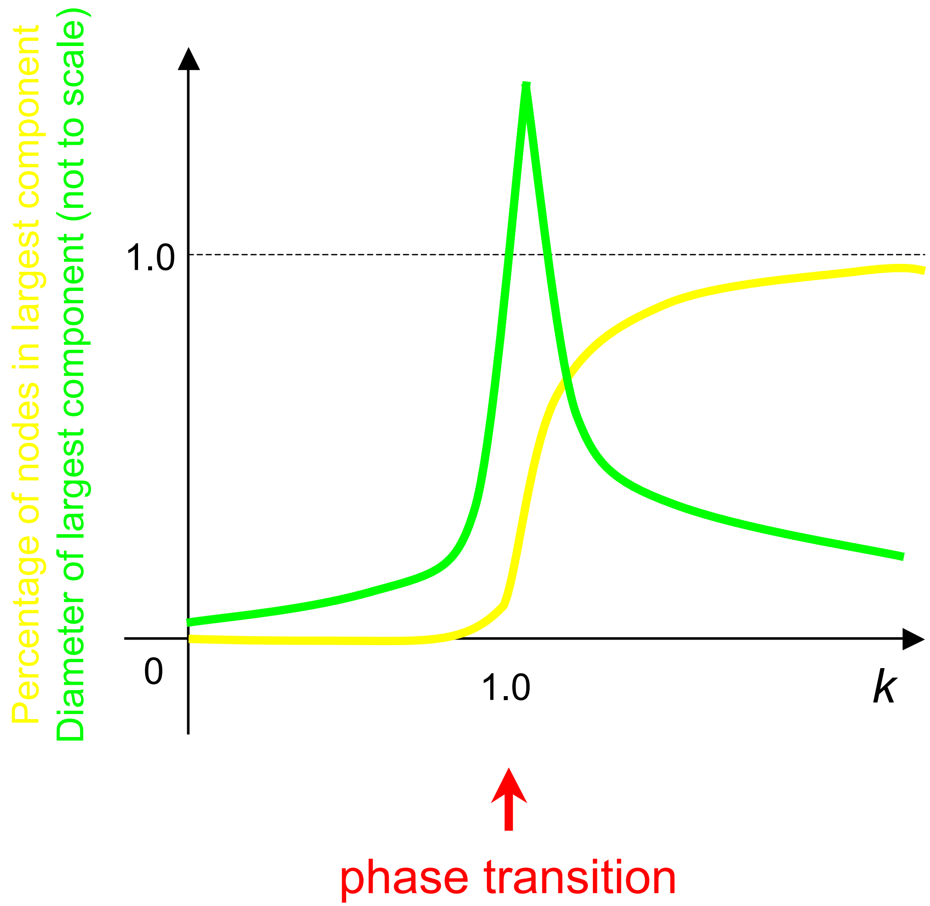
- small, isolated clusters
- small diameters
- short path lengths

At k = 1:

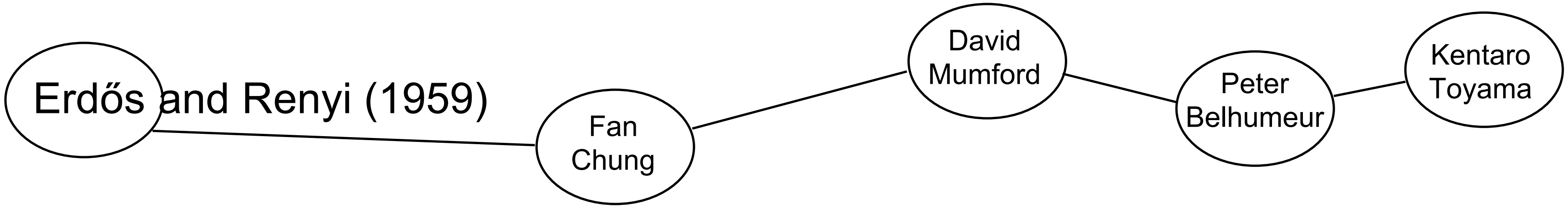
- a *giant component* appears
- diameter peaks
- path lengths are high

For k > 1:

- almost all nodes connected
- diameter shrinks
- path lengths shorten



Random Graphs



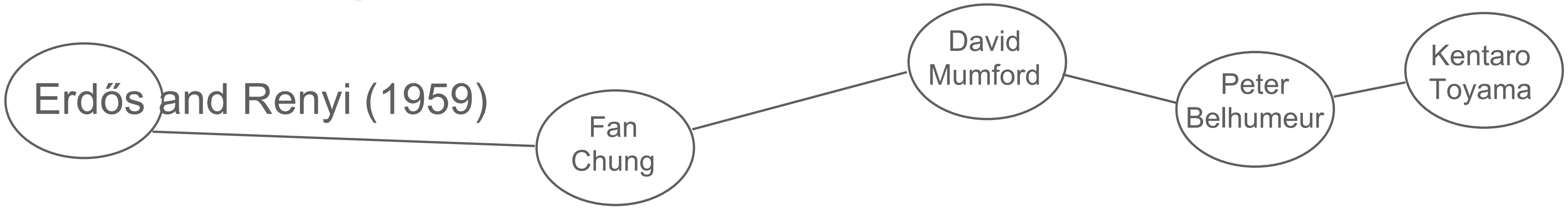
What does this mean?

- If connections between people can be modeled as a random graph, then…
  - Because the average person easily knows more than one person ($k >> 1$),
  - We live in a "small world" where within a few links, we are connected to anyone in the world.
  - Erdős and Renyi showed that average path length between connected nodes is

$$\frac{\ln N}{\ln k}$$

Random Graphs

Erdős and Renyi (1959)



What does this mean?

BIG "IF"!!!

- If connections between people can be modeled as a random graph, then…
  - Because the average person easily knows more than one person ($k \gg 1$),
  - We live in a "small world" where within a few links, we are connected to anyone in the world.
  - Erdős and Renyi computed average path length between connected nodes to be:

$$\frac{\ln N}{\ln k}$$

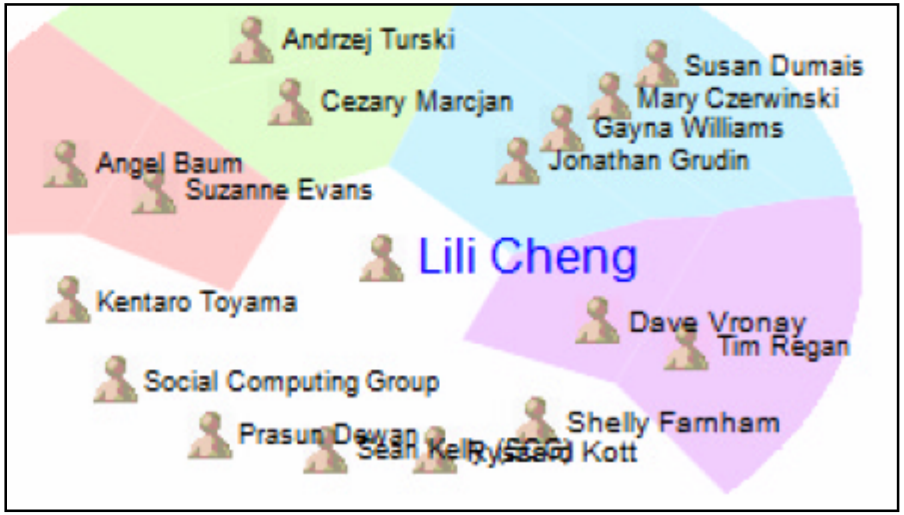
The Alpha Model

Watts (1999)

The people you know aren't randomly chosen.

People tend to get to know those who are two links away (Rapoport *, 1957).

The real world exhibits a lot of *clustering.*

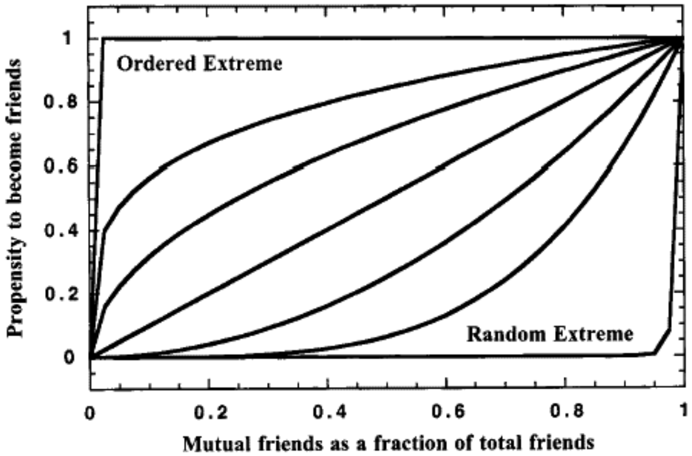


The Personal Map
by MSR Redmond's Social Computing Group

* Same Anatol Rapoport, known for TIT FOR TAT!

The Alpha Model

Watts (1999)



Probability of linkage as a function of number of mutual friends (α is 0 in upper left, 1 in diagonal, and ∞ in bottom right curves.)

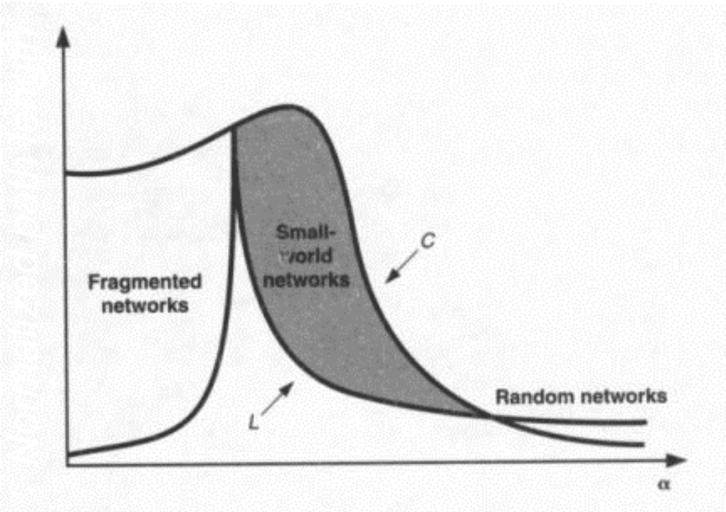
α model: Add edges to nodes, as in random graphs, but makes links more likely when two nodes have a common friend.

For a range of α values:

The Alpha Model

Watts (1999)

Clustering coefficient /



Clustering coefficient (*C*) and average path length (*L*) plotted against α

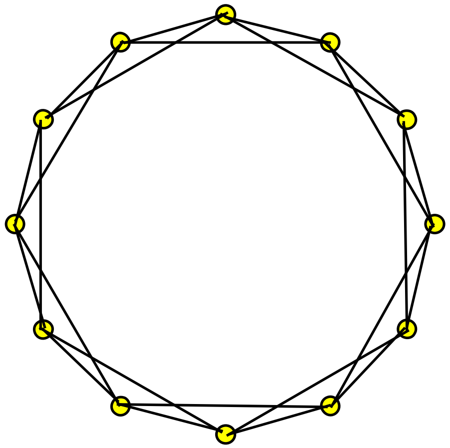
α model: Add edges to nodes, as in random graphs, but makes links more likely when two nodes have a common friend.

For a range of α values:

- The world is small (average path length is short), and
- Groups tend to form (high clustering coefficient).

The Beta Model

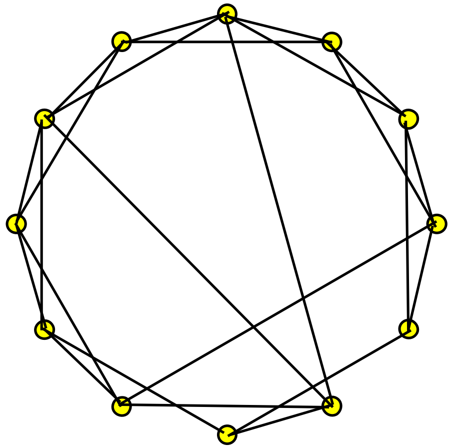
Watts and Strogatz (1998)



$\beta = 0$

People know
their neighbors.

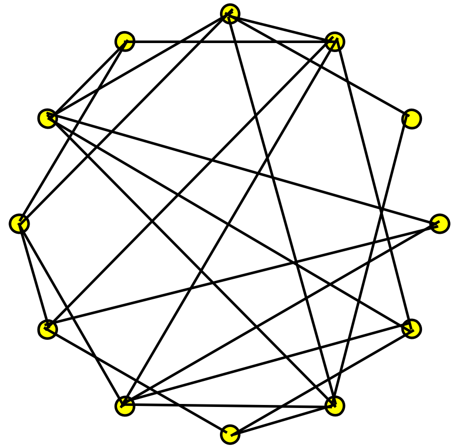
Clustered, but
not a "small world"



$\beta = 0.125$

People know
their neighbors,
and a few distant people.

Clustered and
"small world"

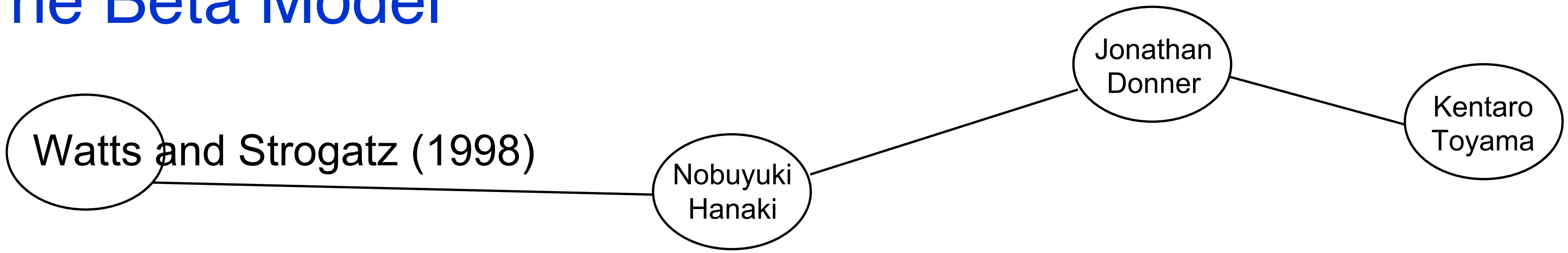


$\beta = 1$

People know
others at
random.

Not clustered,
but "small world"

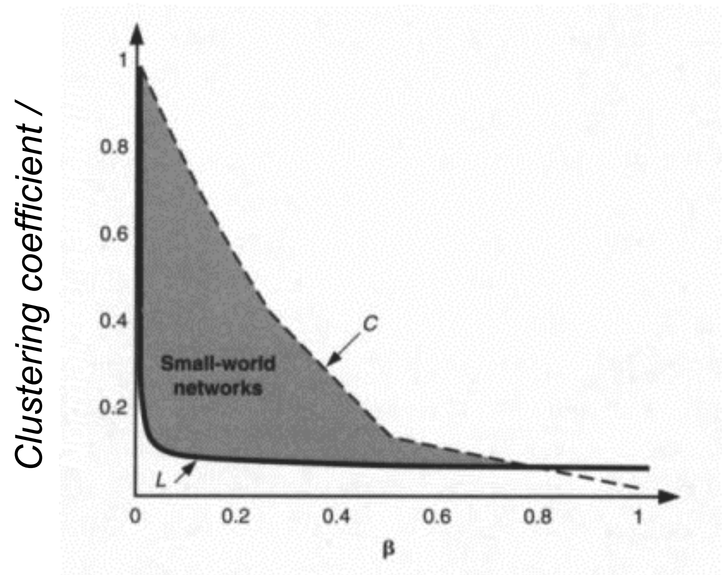
The Beta Model



First five random links reduce the average path length of the network by half, regardless of *N*!

Both α and β models reproduce short-path results of random graphs, but also allow for clustering.

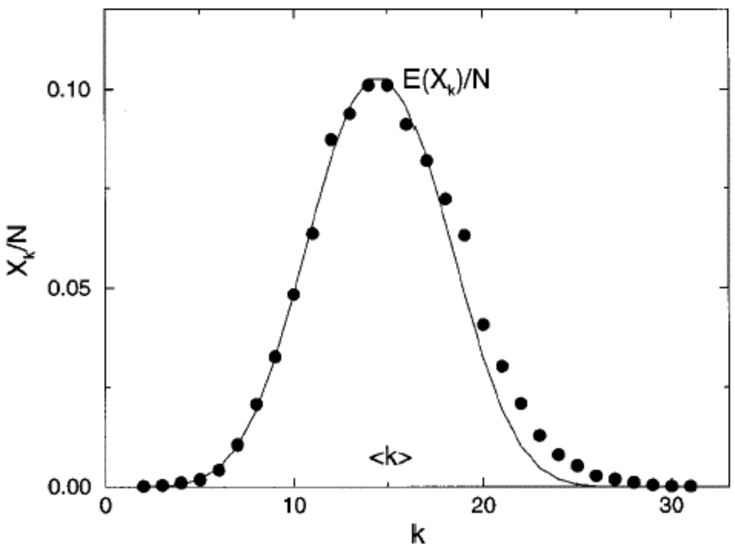
Small-world phenomena occur at threshold between order and chaos.



Clustering coefficient (*C*) and average path length (*L*) plotted against β

Power Laws

Albert and Barabasi (1999)



Degree distribution of a random graph,
N = 10,000 p = 0.0015 k = 15.
(Curve is a Poisson curve, for comparison.)

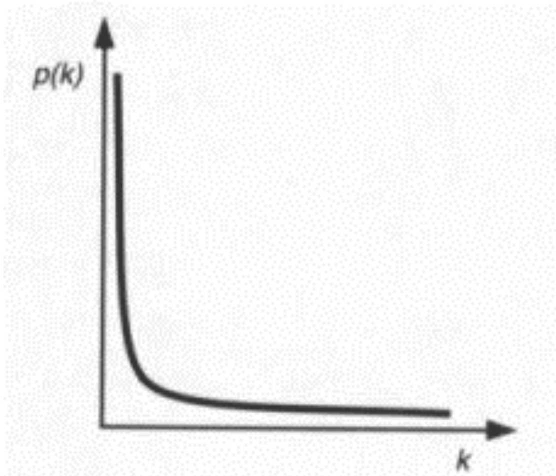
What's the degree (number of edges) distribution over a graph, for real-world graphs?

Random-graph model results in Poisson distribution.

But, many real-world networks exhibit a *power-law* distribution.

Power Laws

Albert and Barabasi (1999)



Typical shape of a power-law distribution.

What's the degree (number of edges) distribution over a graph, for real-world graphs?

Random-graph model results in Poisson distribution.

But, many real-world networks exhibit a *power-law* distribution.

Power Laws

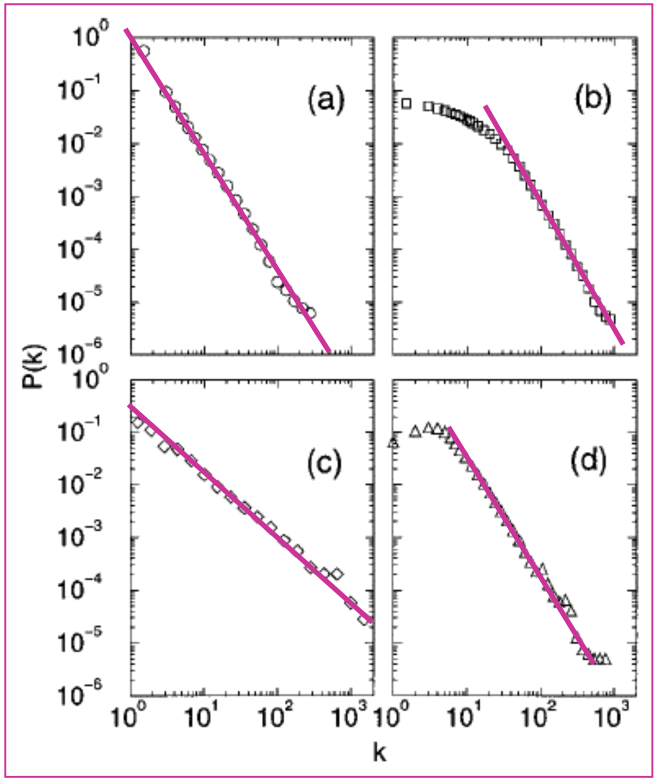
Albert and Barabasi (1999)

Power-law distributions are straight lines in log-log space.

How should random graphs be generated to create a power-law distribution of node degrees?

Hint:

Pareto's* Law: Wealth distribution follows a power law.



Power laws in real networks:
(a) WWW hyperlinks
(b) co-starring in movies
(c) co-authorship of physicists
(d) co-authorship of neuroscientists

** Same Velfredo Pareto, who defined Pareto optimality in game theory.*

Power Laws

Albert and Barabasi (1999)

Jennifer Chayes

Anandan

Kentaro Toyama

“The rich get richer!”

“Map of the Internet” poster

Power-law distribution of node distribution arises if

- Number of nodes grow;
- Edges are added in proportion to the number of edges a node already has.

Additional variable fitness coefficient allows for some nodes to grow faster than others.

Searchable Networks

Kleinberg (2000)

Just because a short path exists, doesn't mean you can easily find it.

You don't know all of the people whom your friends know.

Under what conditions is a network *searchable*?

Searchable Networks

Kleinberg (2000)

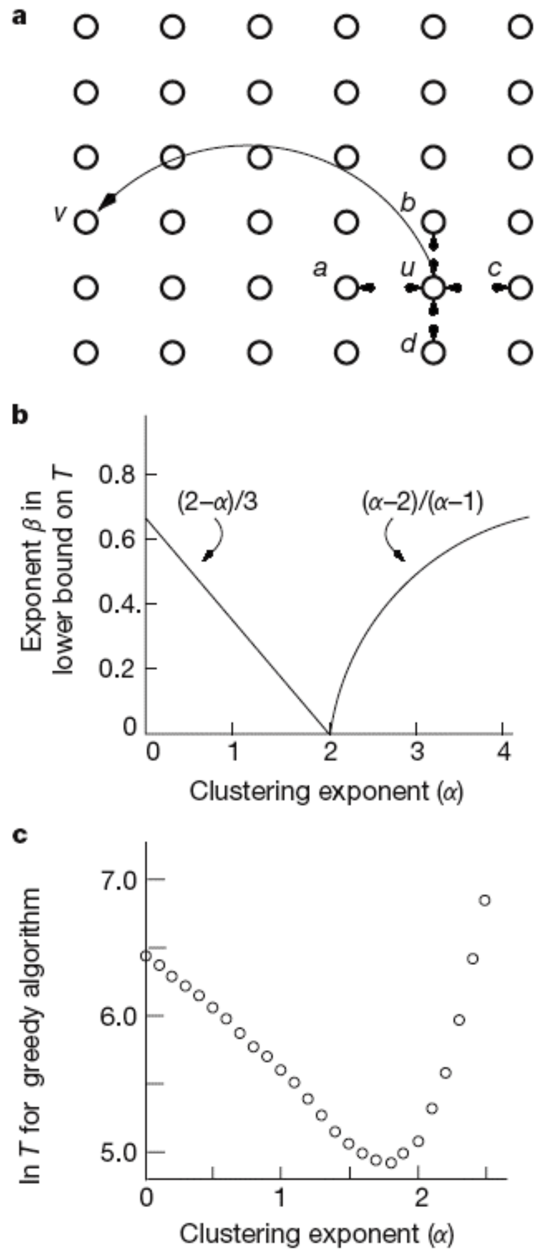
a) Variation of Watts's β model:

- Lattice is d-dimensional (d=2).
- One random link per node.
- Parameter α controls probability of random link – greater for closer nodes.

b) For d=2, dip in time-to-search at α=2

- For low α, random graph; no "geographic" correlation in links
- For high α, not a small world; no short paths to be found.

c) Searchability dips at α=2, in simulation



Searchable Networks

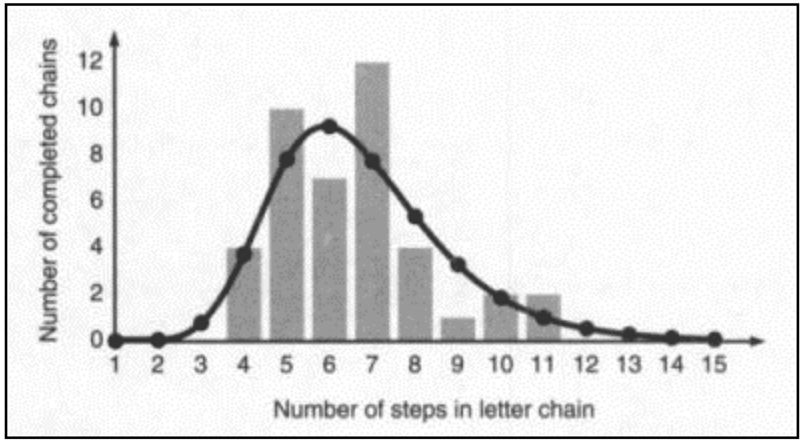
Kleinberg (2000)

Ramin
Zabih

Kentaro
Toyama

Watts, Dodds, Newman (2002) show that for *d* = 2 or 3, real networks are quite searchable.

Killworth and Bernard (1978) found that people tended to search their networks by *d* = 2: geography and profession.



The Watts-Dodds-Newman model closely fitting a real-world experiment

References

ldous & Wilson, *Graphs and Applications. An Introductory Approach*, Springer, 2000.

Wasserman & Faust, *Social Network Analysis*, Cambridge University Press, 2008.