

**MPhil Advanced Computer Science**  
**Topics in Logic and Complexity**

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Exercise Sheet 2

1. Page 118 of Handout 1 contains an illustration of a construction to show that *acyclicity* of graphs is not definable in first-order logic. Write out a proof of this result.

Prove that *acyclicity* is not definable in  $\text{Mon.}\Sigma_1^1$ . Is it definable in  $\text{Mon.}\Pi_1^1$ ?

2. Prove (using Hanf's theorem or otherwise) that 3-colourability of graphs is not definable in first-order logic.

Graph 3-colourability (and, indeed, 2-colourability) are definable in  $\text{Mon.}\Sigma_1^1$ . Can you show they are not definable in  $\text{Mon.}\Pi_1^1$ ? Are they definable in *universal second-order logic*?

3. Prove the lemma on page 143 of Handout 1. That is, show that any formula that is positive in the relation symbol  $R$  defines a monotone operator.
4. Prove that the formula of LFP given on page 152 of Handout 1 does, indeed, define the greatest fixed point of the operator defined by  $\phi$ .
5. On pages 153–156 of Handout 1, we saw how definitions by simultaneous induction can be replaced by a single application of the **lfp** operator. In this exercise, you are asked to show the same for *nested* applications of the **lfp** operator.

Suppose  $\phi(\mathbf{x}, \mathbf{y}, S, T)$  is a formula in which the relational variables  $S$  (of arity  $s$ ) and  $T$  (of arity  $t$ ) only appear positively, and  $\mathbf{x}$  and  $\mathbf{y}$  are tuples of variables of length  $s$  and  $t$  respectively. Show that (for any  $t$ -tuple of terms  $\mathbf{t}$ ) the predicate expression

$$[\mathbf{lfp}_{S,\mathbf{x}}([\mathbf{lfp}_{T,\mathbf{y}}\phi](\mathbf{t}))]$$

is equivalent to an expression with just one application of **lfp**.

6. Consider a vocabulary consisting of two unary relations  $P$  and  $O$ , one binary relation  $E$  and two constants  $s$  and  $t$ . We say that a structure  $\mathbb{A} = (A, P, O, E, s, t)$  in this vocabulary is an *arena* if  $P \cup O = A$  and  $P \cap O = \emptyset$ . That is,  $P$  and  $O$  partition the universe into two disjoint sets.

An arena defines the following game played between a *player* and an *opponent*. The game involves a *token* that is initially placed on the element  $s$ . At each move, if the token is currently on an element of  $P$  it is *player* who plays and if it is on an element of  $O$ , it is *opponent* who plays. At each move, if the token is on an element  $a$ , the one who plays chooses an element  $b$  such that  $(a, b) \in E$  and moves the token from  $a$  to  $b$ . If the token reaches  $t$  at any point then *player* has won the game.

We define **GAME** to be the class of arenas for which *player* has a strategy for winning the game. Note that in an arena  $\mathbb{A} = (A, P, O, E, s, t)$ , *player* has a strategy to win from an element  $a$  if *either*  $a \in P$  and there is some move from  $a$  so that *player* still has a strategy to win after that move *or*  $a \in O$  and for every move from  $a$ , *player* can win after that move.

- (a) Give a sentence of **LFP** that defines the class of structures **GAME**.

We say that a collection  $\mathcal{C}$  of decision problems is *closed under logarithmic space reductions* if whenever  $A \in \mathcal{C}$  and  $B \leq_L A$  (i.e.  $B$  is reducible to  $A$  by a logarithmic-space reduction) then  $B \in \mathcal{C}$ .

The class of structures **GAME** defined above is known to be **P**-complete under logarithmic-space reductions.

- (b) Explain why this, together with (a) implies that the class of problems definable in **LFP** is *not* closed under logarithmic-space reductions.

7. Give a sentence of **LFP** that defines the class of linear orders with an even number of elements.
8. The *directed graph reachability problem* is the problem of deciding, given a structure  $(V, E, s, t)$  where  $E$  is an arbitrary binary relation on  $V$ , and  $s, t \in V$ , whether  $(s, t)$  is in the reflexive-transitive closure of  $E$ . This problem is known to be decidable in **NL**.

Transitive closure logic is the extension of first-order logic with an operator **tc** which allows us to form formulae

$$\phi \equiv [\mathbf{tc}_{\mathbf{x}, \mathbf{y}} \psi](\mathbf{t}_1, \mathbf{t}_2)$$

where  $\mathbf{x}$  and  $\mathbf{y}$  are  $k$ -tuples of variables and  $\mathbf{t}_1$  and  $\mathbf{t}_2$  are  $k$ -tuples of terms, for some  $k$ ; and all occurrences of variables  $\mathbf{x}$  and  $\mathbf{y}$  in  $\psi$  are bound in  $\phi$ . The semantics is given by saying, if  $\mathbf{a}$  is an interpretation for the free variables of  $\phi$ , then  $\mathcal{A} \models \phi[\mathbf{a}]$  just in case  $(\mathbf{t}_1^{\mathbf{a}}, \mathbf{t}_2^{\mathbf{a}})$  is in the reflexive-transitive closure of the binary relation defined by  $\psi(\mathbf{x}, \mathbf{y})$  on  $A^k$ .

- (a) Show that any class of structures definable by a sentence  $\phi$ , as above, where  $\psi$  is first-order, is decidable in **NL**.
- (b) Show that, if  $K$  is an isomorphism-closed class of structures in a relational signature including  $<$ , such that each structure in  $K$  interprets  $<$  as a linear order and

$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

is decidable in **NL**, then there is a sentence of transitive-closure logic that defines  $K$ .

9. For a binary relation  $E$  on a set  $A$ , define its *deterministic transitive closure* to be the set of pairs  $(a, b)$  for which there are  $c_1, \dots, c_n \in A$  such

that  $a = c_1$ ,  $b = c_n$  and for each  $i < n$ ,  $c_{i+1}$  is the *unique* element of  $A$  with  $(c_i, c_{i+1}) \in E$ .

Let **DTC** denote the logic formed by extending first-order logic with an operator **dtc** with syntax analogous to **tc** above, where  $[\mathbf{dtc}_{\mathbf{x},\mathbf{y}}\psi]$  defines the deterministic transitive closure of  $\psi(\mathbf{x}, \mathbf{y})$ .

- (a) Show that every sentence of DTC defines a class of structures decidable in  $L$ .
- (b) Show that, if  $K$  is an isomorphism-closed class of structures in a relational signature including  $<$ , such that each structure in  $K$  interprets  $<$  as a linear order and

$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

is decidable in  $L$ , then there is a sentence of DTC that defines  $K$ .

- 10. Prove the strengthened version of Hanf's Theorem stated on page 14 of Handout 3.
- 11. We proved in the lectures that 3-colourability is NP-complete, even when restricted to graphs of degree at most 5. In this exercise, you are asked to do the same, for graphs of degree at most 4. That is, prove that on  $\mathcal{D}_4$ , 3-colourability is NP-complete.

Also show that every graph in  $\mathcal{D}_3$  is 4-colourable.

- 12. Let  $\mathcal{E}$  be the class of graphs which are disjoint unions of cliques. That is, the vertices of each graph  $G \in \mathcal{E}$  can be partitioned into cliques in such a way that there are no edges between the parts.

Show that satisfaction of FO is FPT on  $\mathcal{E}$ .