Some Generalizations

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Part 1

About These Slides

Copyright

These slides constitute the lecture notes for

- MACS L111 Advanced Data Flow Analysis course at Cambridge University, and
- CS 618 Program Analysis course at IIT Bombay.

They have been made available under GNU FDL v1.2 or later (purely for academic or research use) as teaching material accompanying the book:

- Uday Khedker, Amitabha Sanyal, and Bageshri Karkare. *Data Flow Analysis: Theory and Practice*. CRC Press (Taylor and Francis Group). 2009.

Apart from the above book, some slides are based on the material from the following books

- M. S. Hecht. *Flow Analysis of Computer Programs*. Elsevier North-Holland Inc. 1977.
- F. Nielson, H. R. Nielson, and C. Hankin. *Principles of Program Analysis*. Springer-Verlag. 1998.

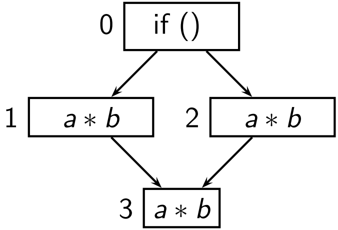
Outline

- Partial Redundancy Elimination
- Introduction to Constant Propagation
- Theoretical Abstractions in Data Flow Analysis
  - The world of data flow values
  - The world of functions and operations that compute data values (Not today)
  - Results of data flow analysis (Not today)
  - Algorithms for performing data flow analysis (Not today)

Part 2

Partial Redundancy Elimination

Precursor: Common Subexpression Elimination



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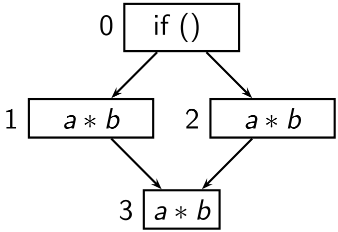

- a and b are not modified along paths $1 \rightarrow 3$ and $2 \rightarrow 3$

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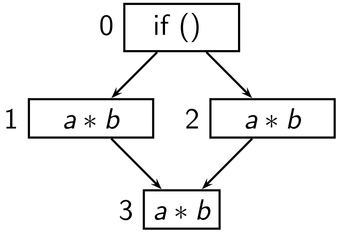

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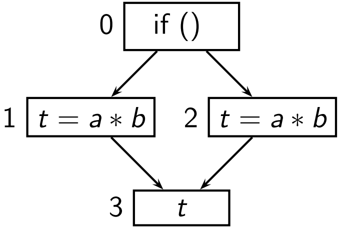
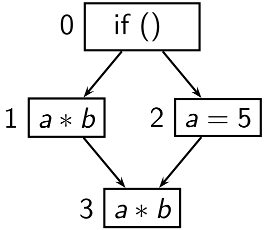


- a and b are not modified along paths $1 \rightarrow 3$ and $2 \rightarrow 3$
- Computation of $a * b$ in 3 is redundant
- Previous value can be used

Partial Redundancy Elimination



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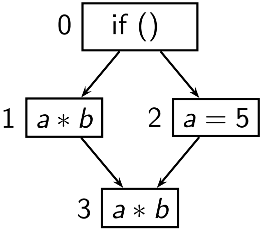

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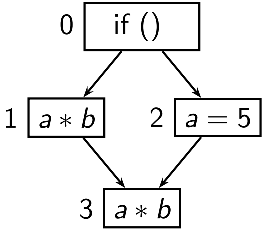

- Computation of $a * b$ in 3 is
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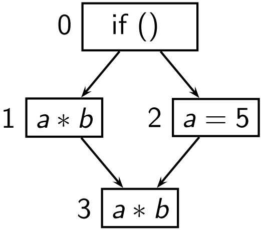
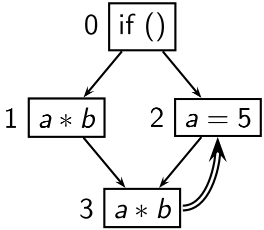


- Computation of $a * b$ in 3 is
  - redundant along path $1 \rightarrow 3$, but ...
  - not redundant along path $2 \rightarrow 3$

Code Hoisting for Partial Redundancy Elimination



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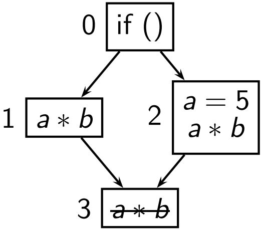
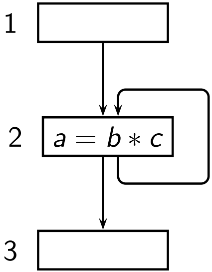


- Computation of $a * b$ in 3 becomes totally redundant
- Can be deleted

PRE Subsumes Loop Invariant Movement



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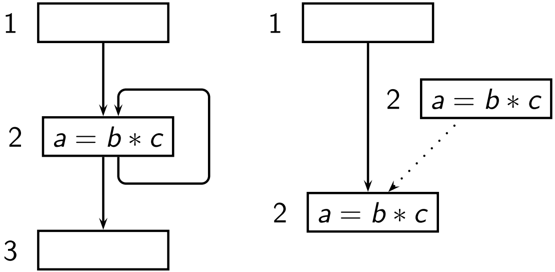

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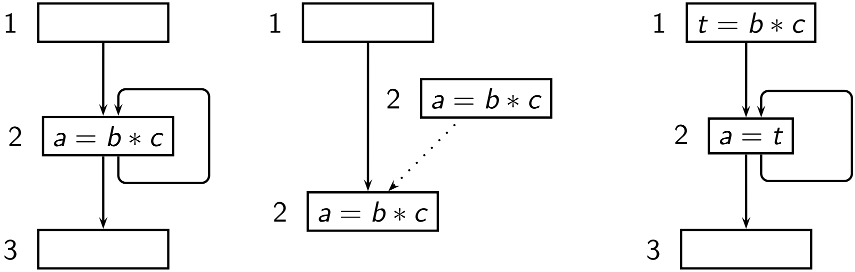
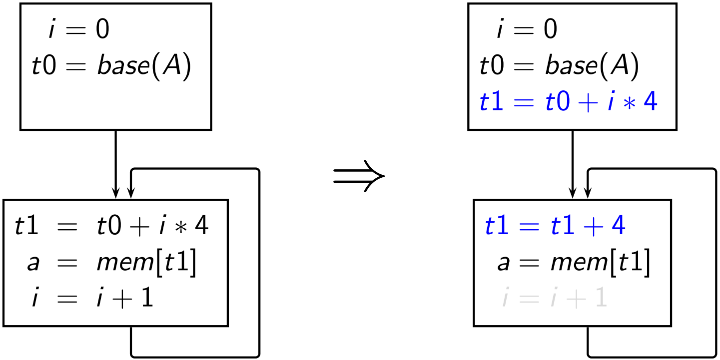


PRE Can be Used for Strength Reduction

$$
\begin{aligned}
i &= 0 \\
t0 &= base(A)
\end{aligned}
$$

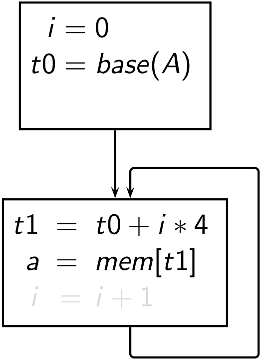
$$
\begin{aligned}
t1 &= t0 + i * 4 \\
a &= mem[t1] \\
i &= i + 1
\end{aligned}
$$

PRE Can be Used for Strength Reduction



- $*$ and $+$ in the loop have been replaced by $+$
- $i = i + 1$ in the loop has been eliminated

PRE Can be Used for Strength Reduction



- Delete $i = i + 1$

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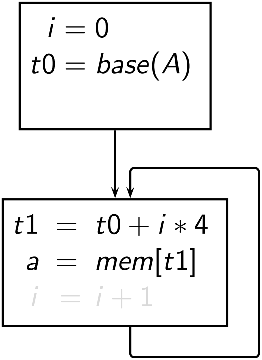

- Delete $i = i + 1$
- Expression $t0 + i * 4$ becomes loop invariant

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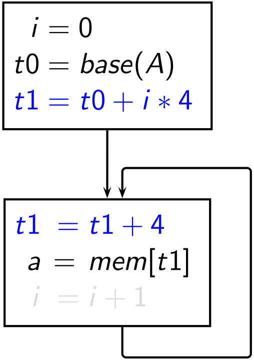
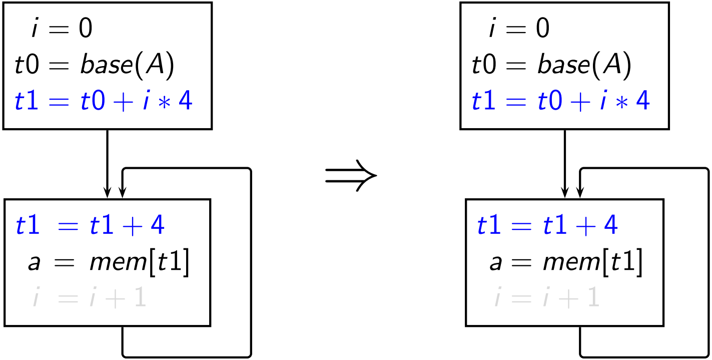


- Delete $i = i + 1$
- Expression $t0 + i * 4$ becomes loop invariant
- Hoist it and increment $t1$ in the loop

PRE Can be Used for Strength Reduction



- $*$ and $+$ in the loop have been replaced by $+$
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Performing Partial Redundancy Elimination

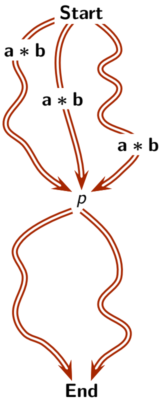
1. Identify partial redundancies
2. Identify program points where computations can be inserted
3. Insert expressions
4. Partial redundancies become total redundancies
   $\Longrightarrow$ Delete them.

Morel-Renvoise Algorithm (*CACM*, 1979.)

Defining Hoisting Criteria

- An expression can be safely inserted at a program point p if it is

Available at p



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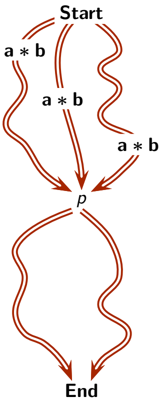
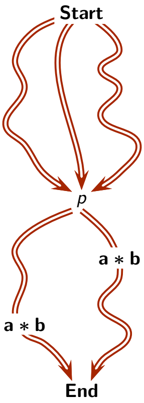


Anticipable at p



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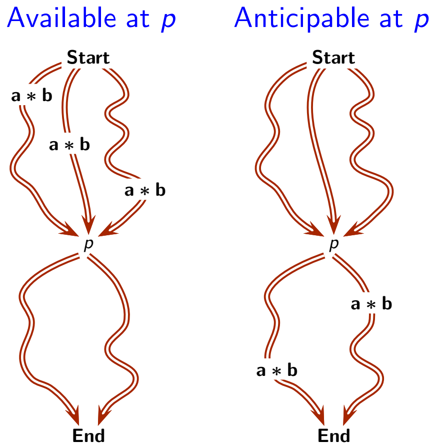

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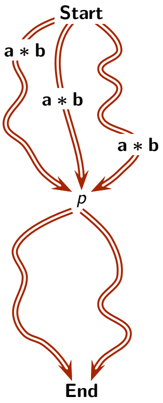

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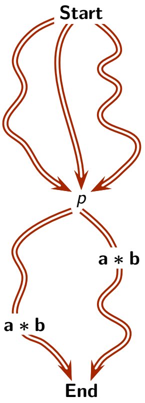

- If it is available at p, then there is no need to insert it at p.
- If it is anticipable at p then all such occurrence should be hoisted to p.

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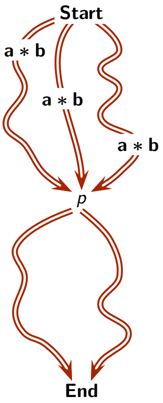
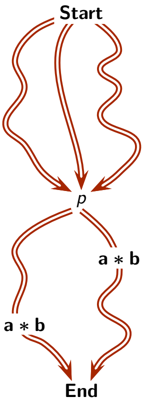


Anticipable at p



- If it is available at p, then there is no need to insert it at p.
- If it is anticipable at p then all such occurrence should be hoisted to p.
- *An expression should be hoisted to p provided it can be hoisted to p along all paths from p to exit.*

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- *Safety of hoisting to the exit of a block*.
  - S.1 Should be hoisted only if it can be hoisted to the entry of all succesors

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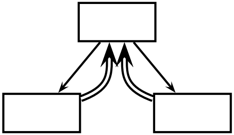

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$\boxed{a * c}$

$\boxed{\begin{array}{c} a * c \\ a = \end{array}}$

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$a =$

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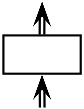

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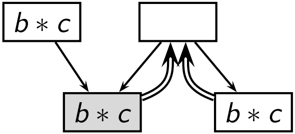

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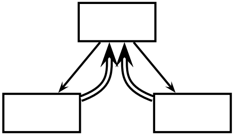

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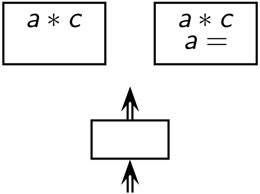

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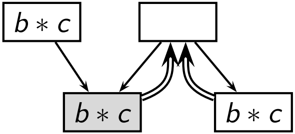

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What does this slide show?

- Four examples
- For each example
  - statements in blue enable hoisting
  - statements in red prohibit hoisting

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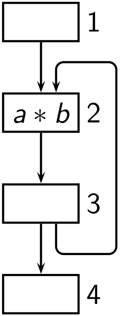

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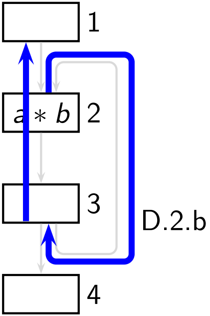

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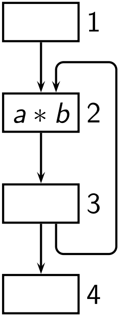
What if we insert

- $a = 2$ in 3?
- $a = 2$ in 3 and $a * b$ in 4?



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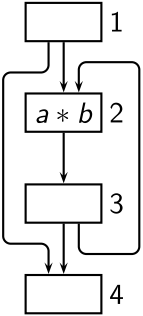

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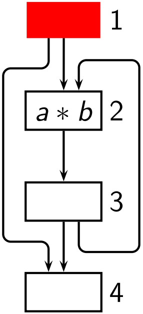

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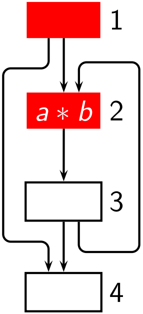

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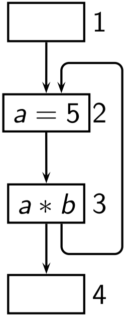
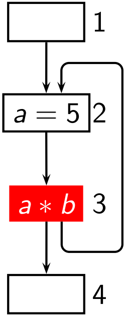


(Example 3)

Applying the Hoisting Criteria

- *Safety of hoisting to the exit of a block.*
  - S.1 Should be hoisted only if it can be hoisted to the entry of all succesors
- *Safety of hoisting to the entry of a block.* Should be hoisted only if
  - S.2 it is upwards exposed, or
  - S.3 it can be hoisted to its exit and is transparent in the block
- *Desirability of hoisting to the entry of a block.* Should be hoisted only if
  - D.1 it is partially available, and
  - D.2 For each predecessor
    - D.2.a it is hoisted to its exit, or
    - D.2.b is available at its exit.



(Example 3)

Applying the Hoisting Criteria

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  D.2 For each predecessor

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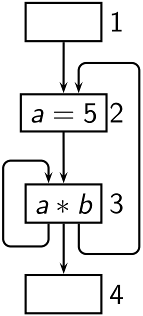

(Example 4)

Applying the Hoisting Criteria

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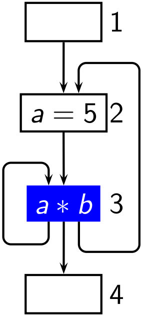

(Example 4)

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- *Safety of hoisting to the exit of a block.*

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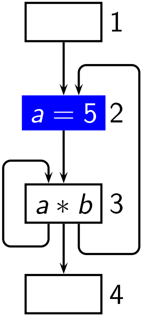

(Example 4)

Applying the Hoisting Criteria

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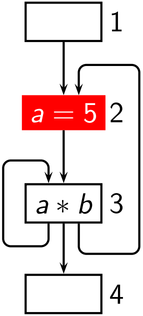


(Example 4)

First Level Global Data Flow Properties in PRE

- Partial Availability.

$$PavIn_n = \begin{cases} BI & n \text{ is } Start \text{ block} \\ \bigcup_{p \in pred(n)} PavOut_p & \text{otherwise} \end{cases}$$

$$PavOut_n = Gen_n \cup (PavIn_n - Kill_n)$$

- Total Availability.

$$AvIn_n = \begin{cases} BI & n \text{ is } Start \text{ block} \\ \bigcap_{p \in pred(n)} AvOut_p & \text{otherwise} \end{cases}$$

$$AvOut_n = Gen_n \cup (AvIn_n - Kill_n)$$

PRE Data Flow Equations

Desirability: D.1

$$In_n = \boxed{PavIn_n} \cap \left(AntGen_n \cup (Out_n - Kill_n) \right)$$
$$\bigcap_{p \in pred(n)} \left(Out_p \cup AvOut_p \right)$$

$$Out_n = \begin{cases} BI & n \text{ is } End \text{ block} \\ \bigcap_{s \in succ(n)} In_s & \text{otherwise} \end{cases}$$

Expressions should be partially available, and

PRE Data Flow Equations

Safety: S.2

$$In_n = PavIn_n \cap \Big(AntGen_n \cup (Out_n - Kill_n) \Big) \bigcap_{p \in pred(n)} \Big(Out_p \cup AvOut_p \Big)$$

$$Out_n = \begin{cases} BI & n \text{ is } End \text{ block} \\ \bigcap_{s \in succ(n)} In_s & \text{otherwise} \end{cases}$$

Expressions should be upwards exposed, or

PRE Data Flow Equations

Safety: S.3

$$In_n = PavIn_n \cap \Big(AntGen_n \cup \big(Out_n - Kill_n\big) \Big) \bigcap_{p \in pred(n)} \Big(Out_p \cup AvOut_p \Big)$$

$$Out_n = \begin{cases} BI & n \text{ is } End \text{ block} \\ \bigcap_{s \in succ(n)} In_s & \text{otherwise} \end{cases}$$

Expressions can be hoisted to the exit and are transparent in the block

PRE Data Flow Equations

Desirability: D.2.a

$$In_n = PavIn_n \cap \left(AntGen_n \cup (Out_n - Kill_n) \right) \bigcap_{p \in pred(n)} \left(Out_p \cup AvOut_p \right)$$

$$Out_n = \begin{cases} BI & n \text{ is } End \text{ block} \\ \bigcap_{s \in succ(n)} In_s & \text{otherwise} \end{cases}$$

For every predecessor, expressions can be hoisted to its exit, or

PRE Data Flow Equations

Desirability: D.2.b

$$In_n = PavIn_n \cap \left(AntGen_n \cup (Out_n - Kill_n) \right) \bigcap_{p \in pred(n)} \left(Out_p \cup AvOut_p \right)$$

$$Out_n = \begin{cases} BI & n \text{ is } End \text{ block} \\ \bigcap_{s \in succ(n)} In_s & \text{otherwise} \end{cases}$$

...expressions are available at the exit of the same predecessor

PRE Data Flow Equations

Safety: S.1

$$In_n = PavIn_n \cap \left(AntGen_n \cup (Out_n - Kill_n) \right) \bigcap_{p \in pred(n)} \left(Out_p \cup AvOut_p \right)$$

$$Out_n = \begin{cases} BI & n \text{ is } End \text{ block} \\ \bigcap_{s \in succ(n)} In_s & \text{otherwise} \end{cases}$$

Expressions should be hoisted to the exit of a block if they can be hoisted to the entry of all succesors

PRE Data Flow Equations

$$
\begin{aligned}
In_n &= PavIn_n \cap \Big(AntGen_n \cup (Out_n - Kill_n) \Big) \\
& \qquad \bigcap_{p \in pred(n)} \Big(Out_p \cup AvOut_p \Big) \\
Out_n &= \begin{cases} BI & n \text{ is } End \text{ block} \\ \displaystyle\bigcap_{s \in succ(n)} In_s & \text{otherwise} \end{cases}
\end{aligned}
$$

Deletion Criteria in PRE

- An expression is redundant in node n if
  - it can be placed at the entry (i.e. can be "hoisted" out) of n, AND
  - it is upwards exposed in node n.

$$Redundant_n = In_n \cap AntGen_n$$

- A hoisting path for an expression e begins at n if $e \in Redundant_n$
- This hoisting path extends against the control flow.

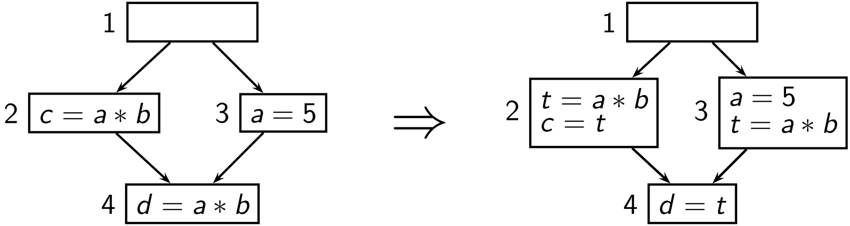
Insertion Criteria in PRE

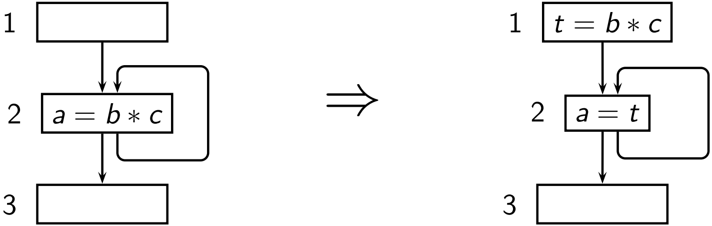
- An expression is inserted at the exit of node n is
  - it can be placed at the exit of n, AND
  - it is not available at the exit of n, AND
  - it cannot be hoisted out of n, OR it is modified in n.

$$Insert_n = Out_n \cap (\neg AvOut_n) \cap (\neg In_n \cup Kill_n)$$

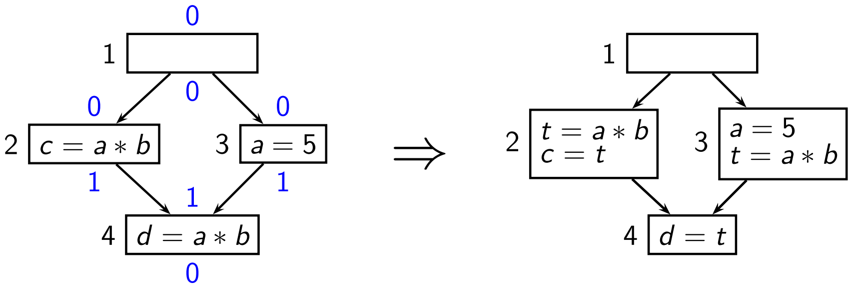
- A hoisting path for an expression e ends at n if $e \in Insert_n$

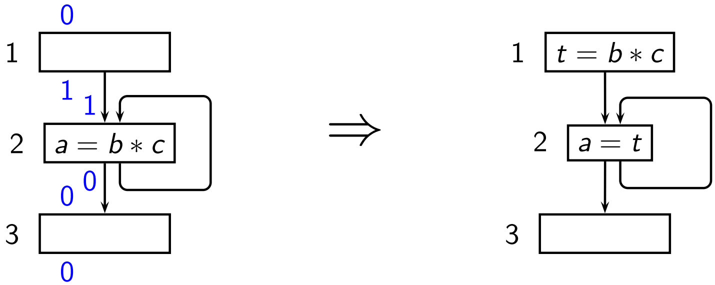
Performing PRE by Computing *In*/*Out*: Simple Cases



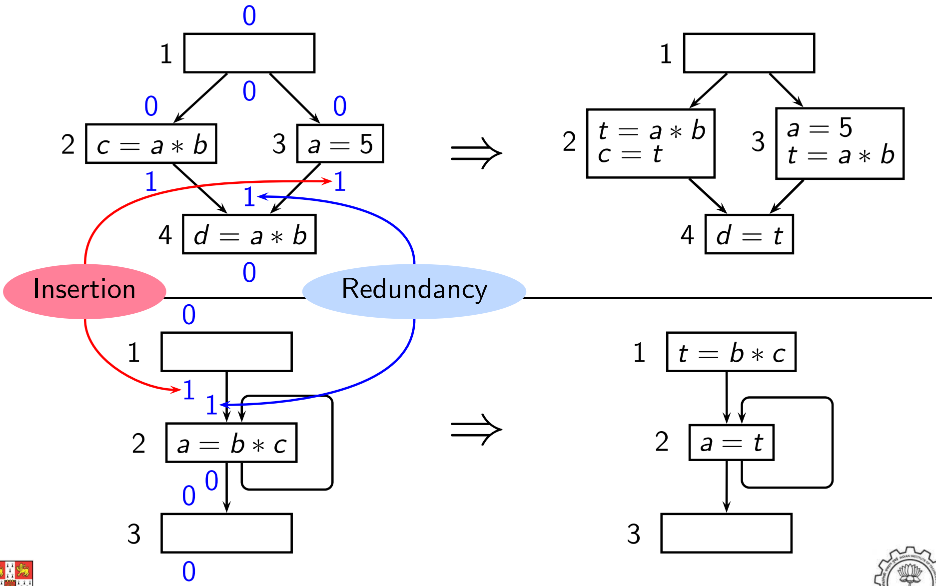


Performing PRE by Computing *In*/*Out*: Simple Cases

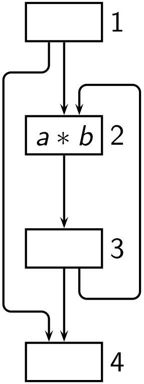




Performing PRE by Computing *In*/*Out*: Simple Cases

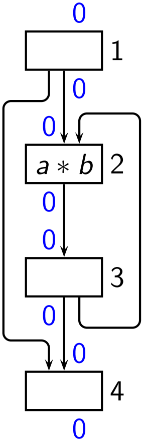


Tutorial Problems for PRE



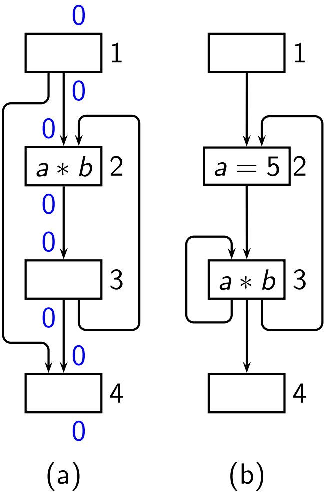
(a)

Tutorial Problems for PRE



(a)

Tutorial Problems for PRE



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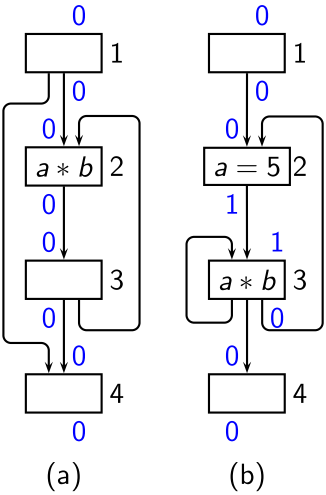
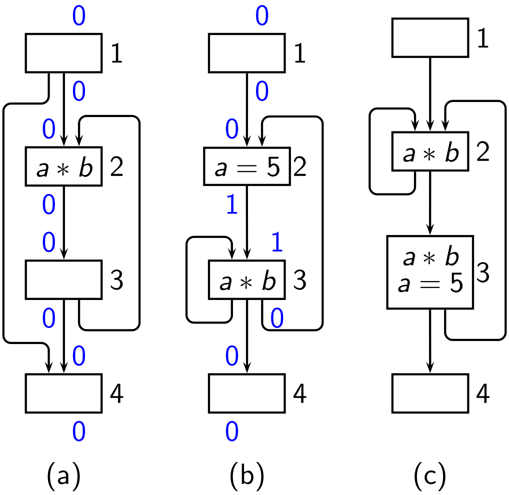


Tutorial Problems for PRE



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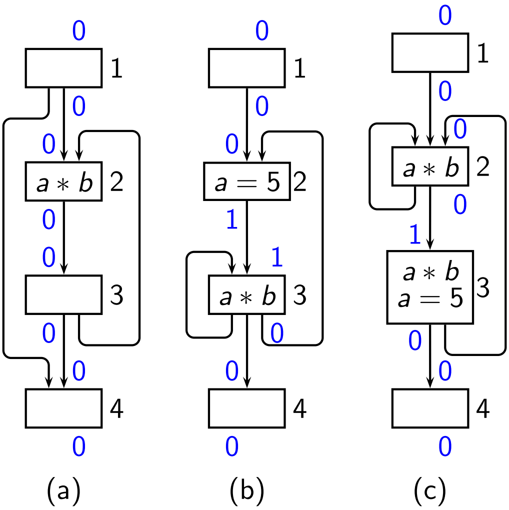

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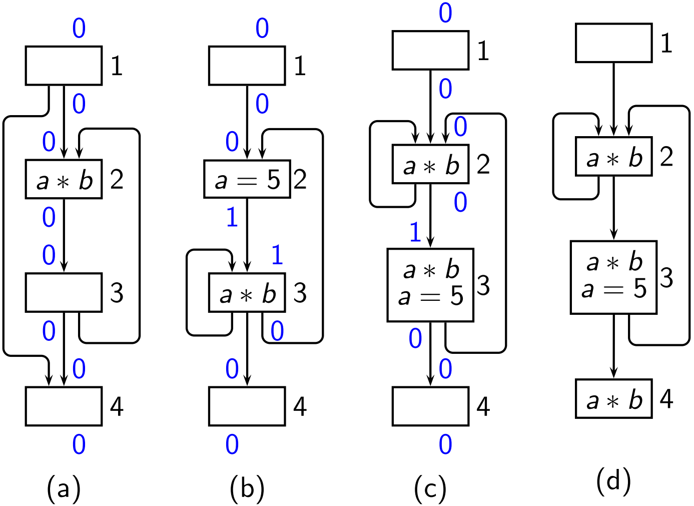

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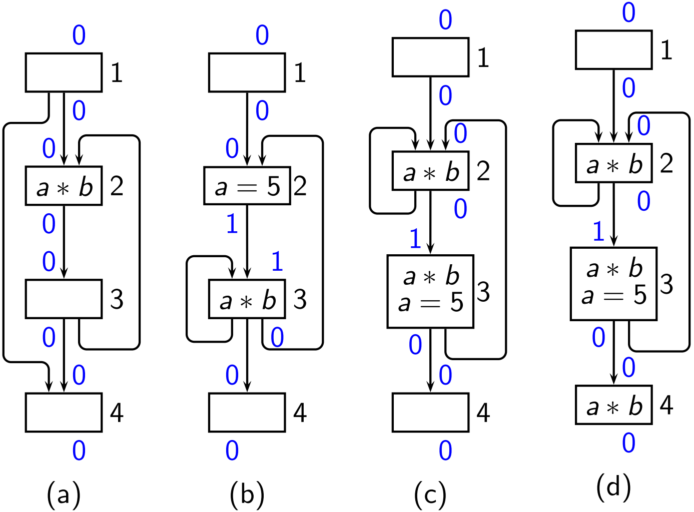

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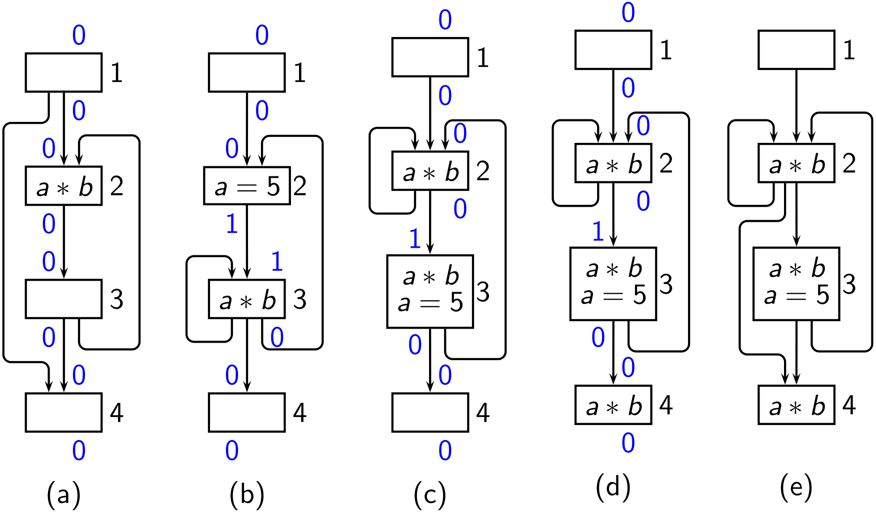

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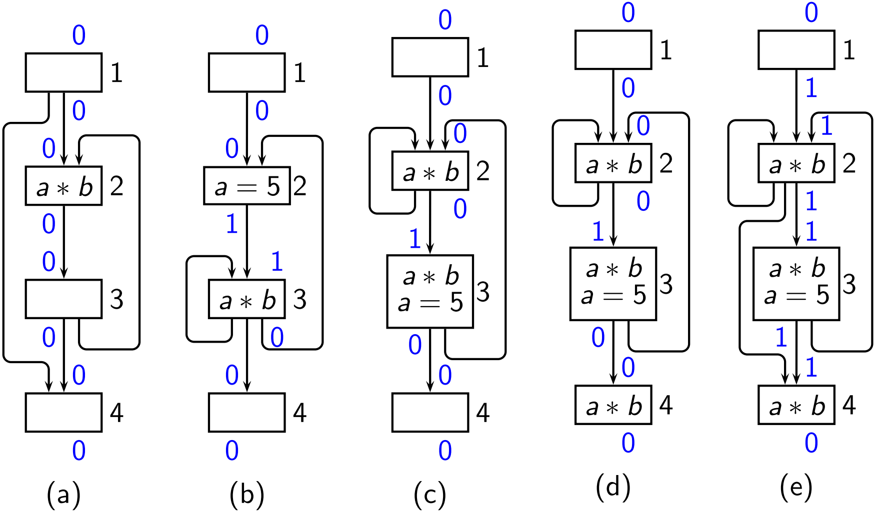

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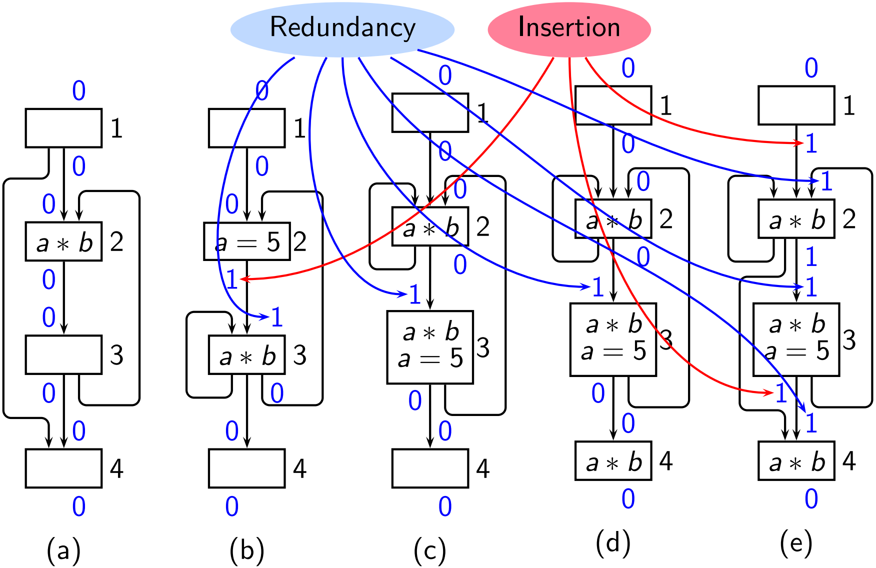
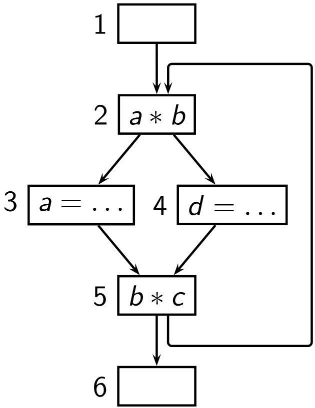


Further Tutorial Problem for PRE



Let $\{a * b, b * c\} \equiv$ bit string 11

Node n	$Kill_n$	$AntGen_n$	$PavIn_n$	$AvOut_n$
1	00	00	00	00
2	00	10	11	10
3	10	00	11	00
4	00	00	11	10
5	00	01	11	01
6	00	00	11	01

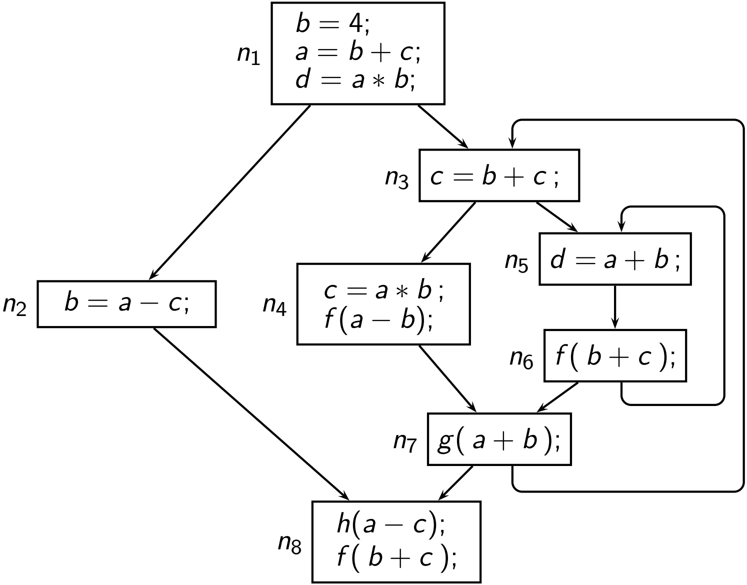
- Compute $In_n/Out_n/Redundant_n/Insert_n$
- Identify hoisting paths

Result of PRE Data Flow Analysis of the Running Example

Bit vector	$a * b$	$a + b$	$a - b$	$a - c$	$b + c$

Block	Global Information							
	Constant information		Iteration # 1		Changes in iteration # 2		Changes in iteration # 3	
	$PavIn_n$	$AvOut_n$	Out_n	In_n	Out_n	In_n	Out_n	In_n
n_8	11111	00011	00000	00011				00001
n_7	11101	11000	00011	01001	00001			
n_6	11101	11001	01001	01001			01000	
n_5	11101	11000	01001	01001		01000		
n_4	11100	10100	01001	11100		11000		
n_3	11101	10000	01000	01001		00001		
n_2	10001	00010	00011	00000			00001	
n_1	00000	10001	00000	00000				

Hoisting Paths for Some Expressions in the Running Example



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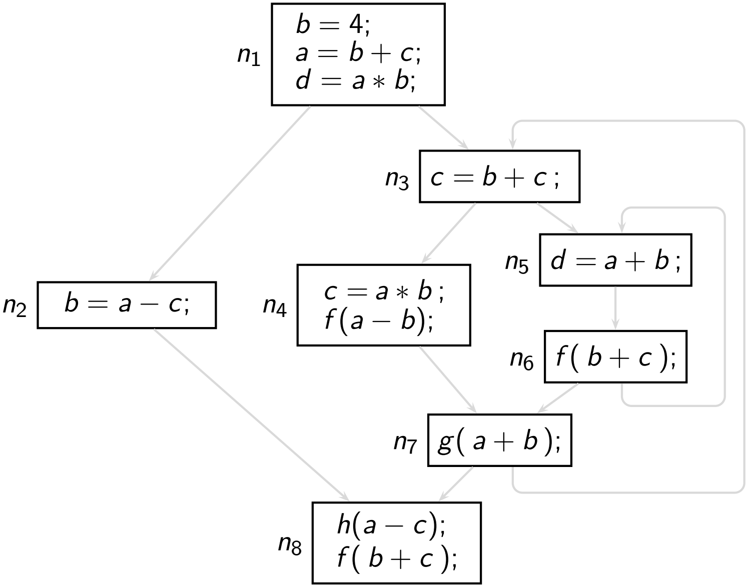

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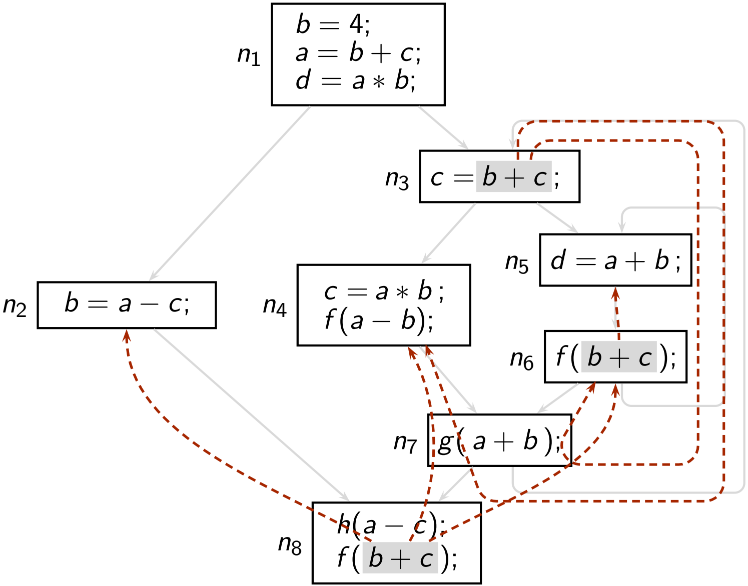

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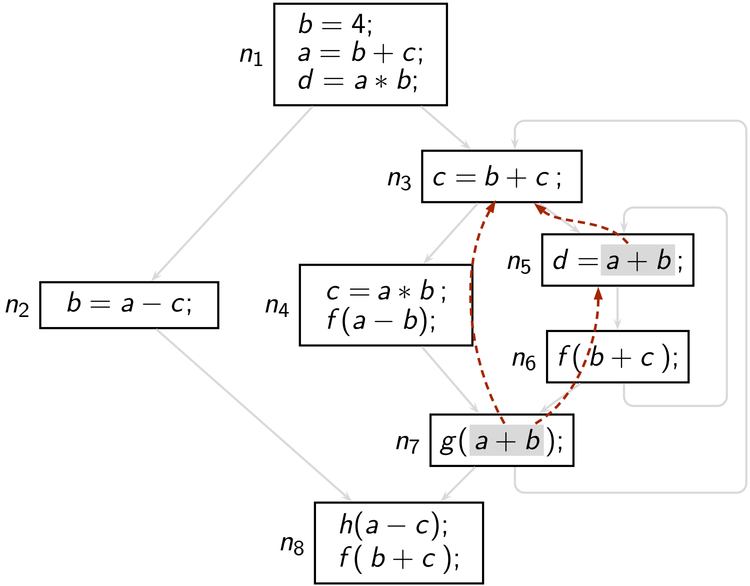

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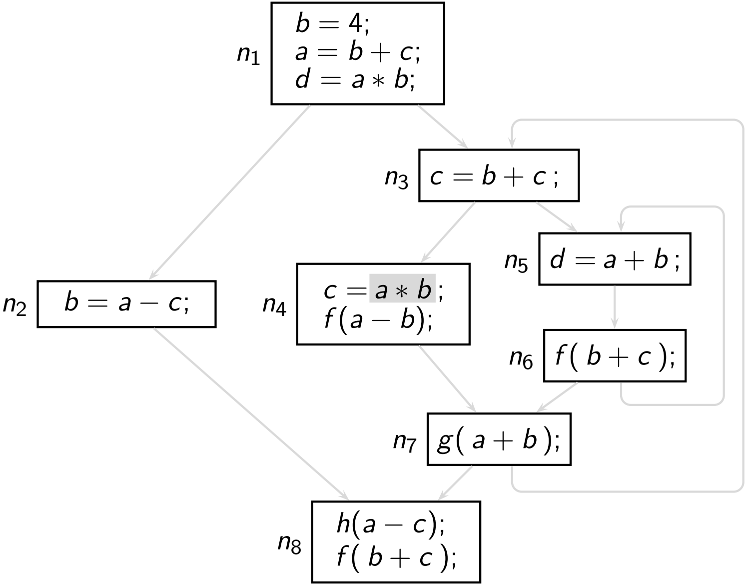

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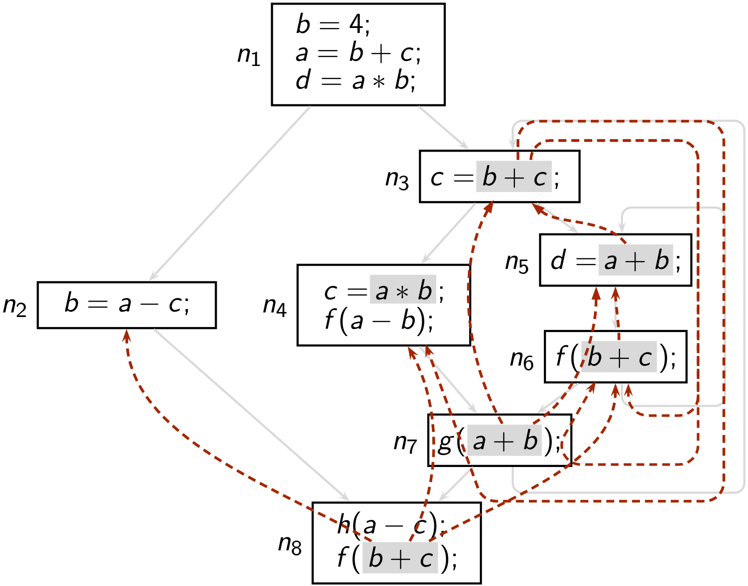
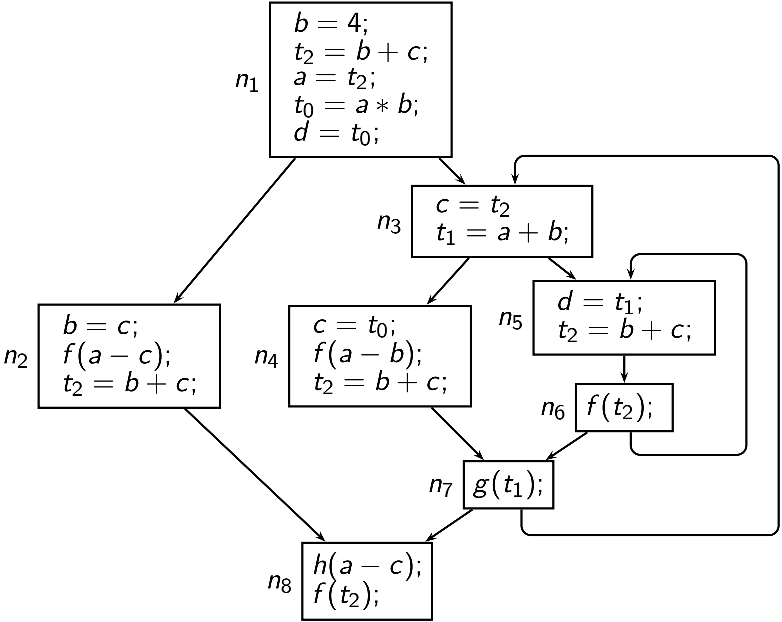


Optimized Version of the Running Example



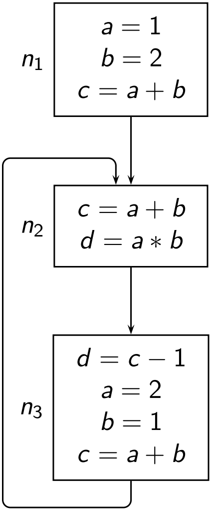
Part 3

The Need for a More General Setting

What We Have Seen So Far . . .

Analysis	Entity	Attribute at p	Paths	
Live variables	Variables	Use	Starting at p	Some
Available expressions	Expressions	Availability	Reaching p	All
Partially available expressions	Expressions	Availability	Reaching p	Some
Anticipable expressions	Expressions	Use	Starting at p	All
Reaching definitions	Definitions	Availability	Reaching p	Some
Partial redundancy elimination	Expressions	Profitable hoistability	Involving p	All

An Introduction to Constant Propagation



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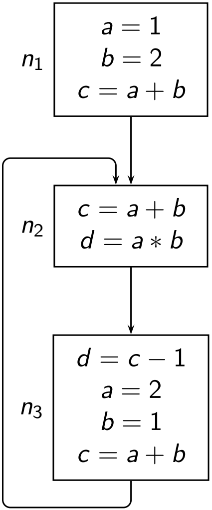
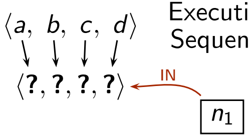


Execution Sequence



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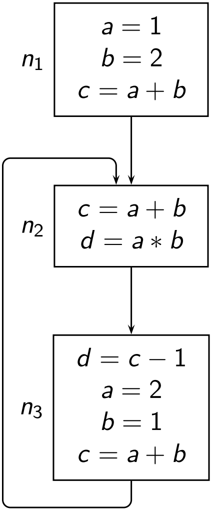

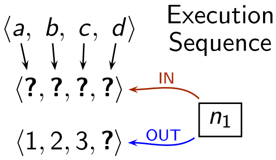

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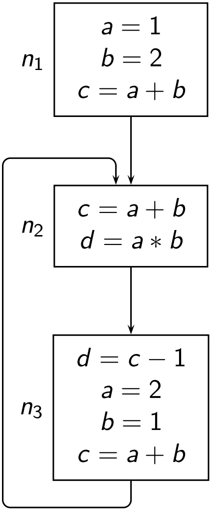

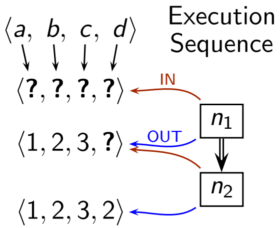

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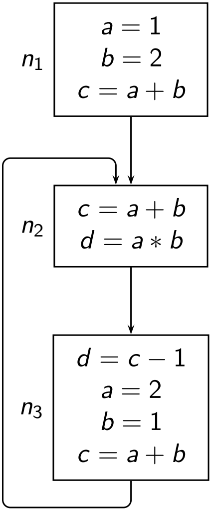

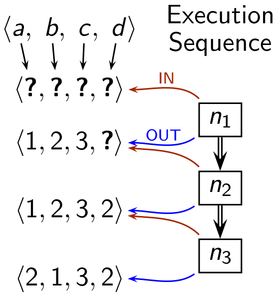

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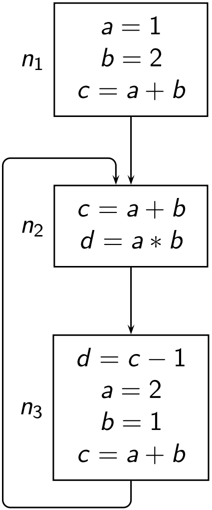

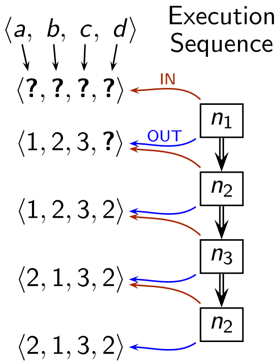

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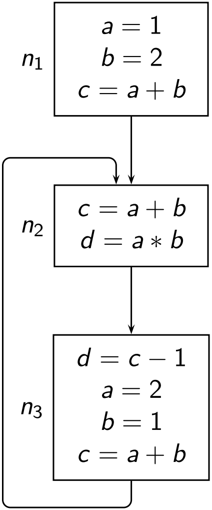

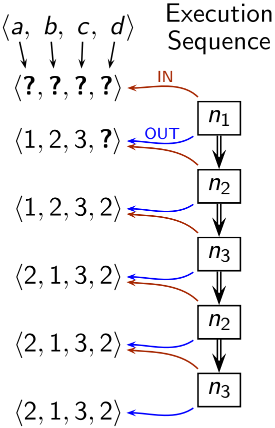

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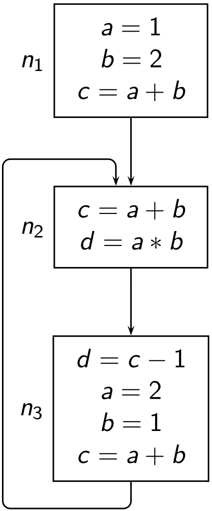

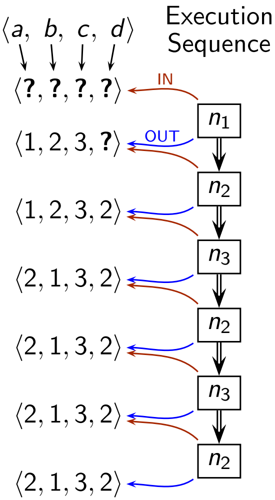

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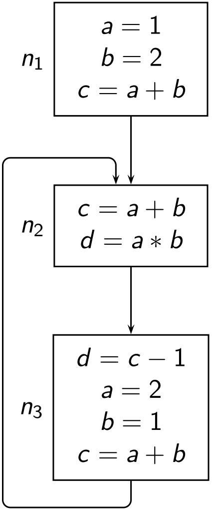
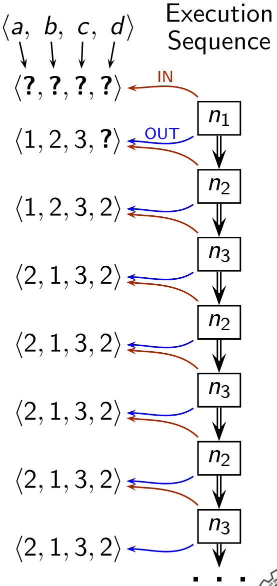
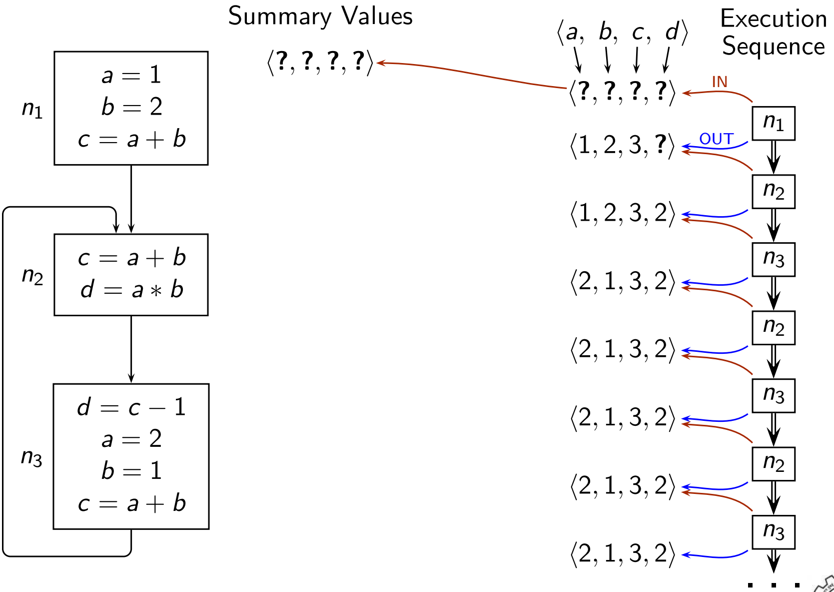




An Introduction to Constant Propagation



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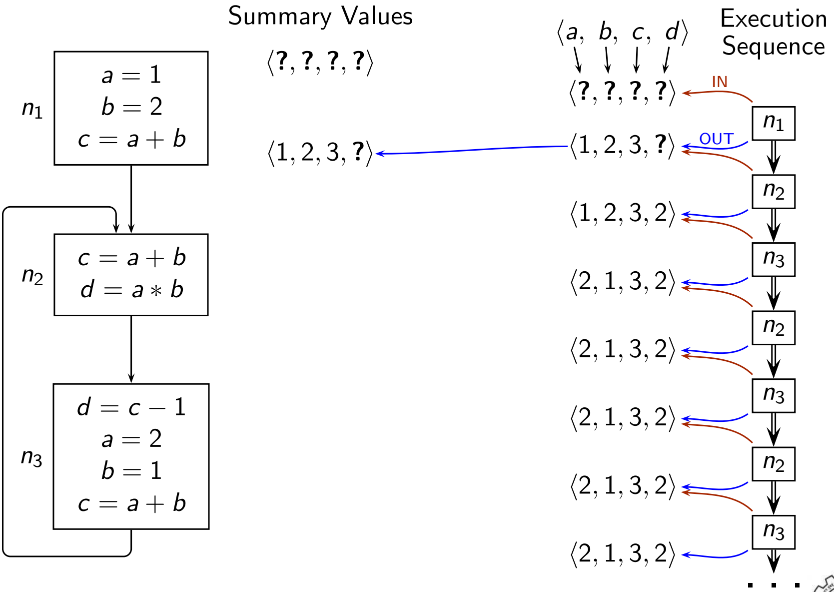

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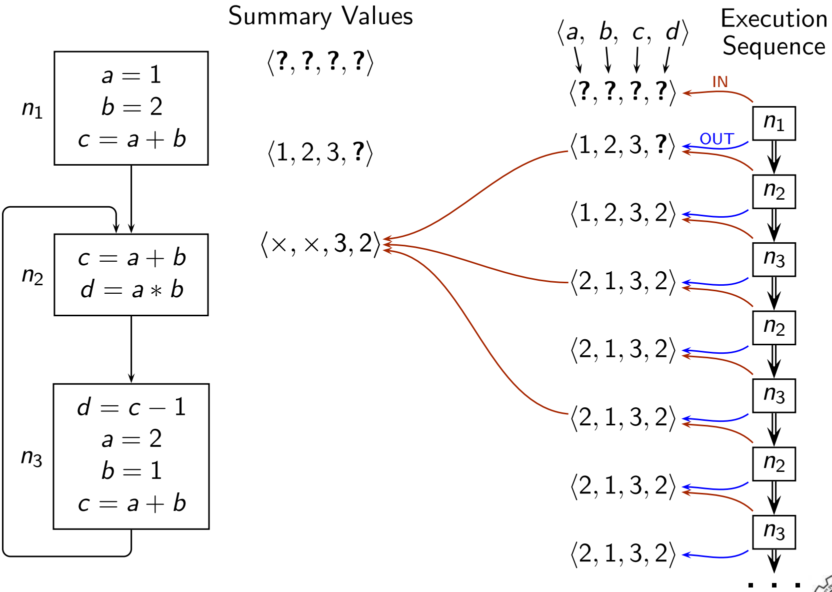

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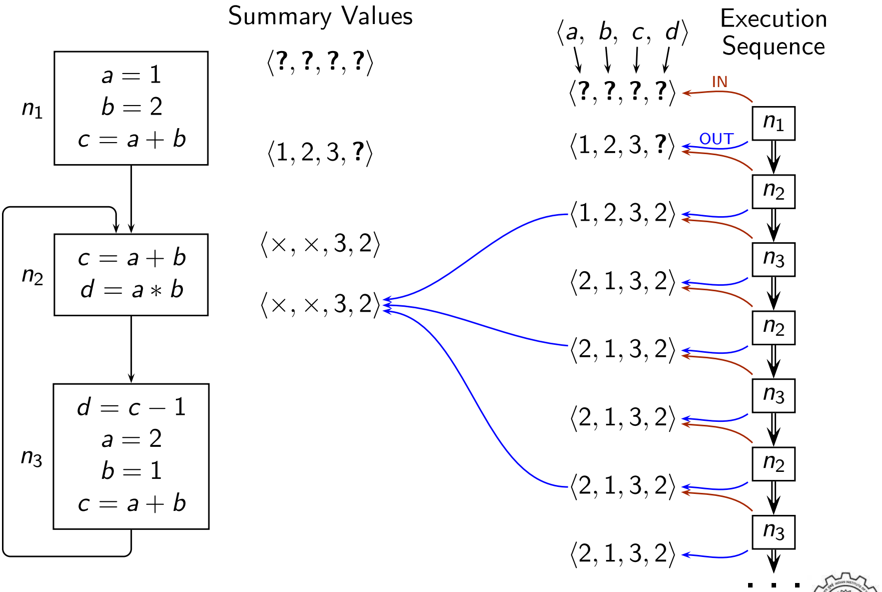

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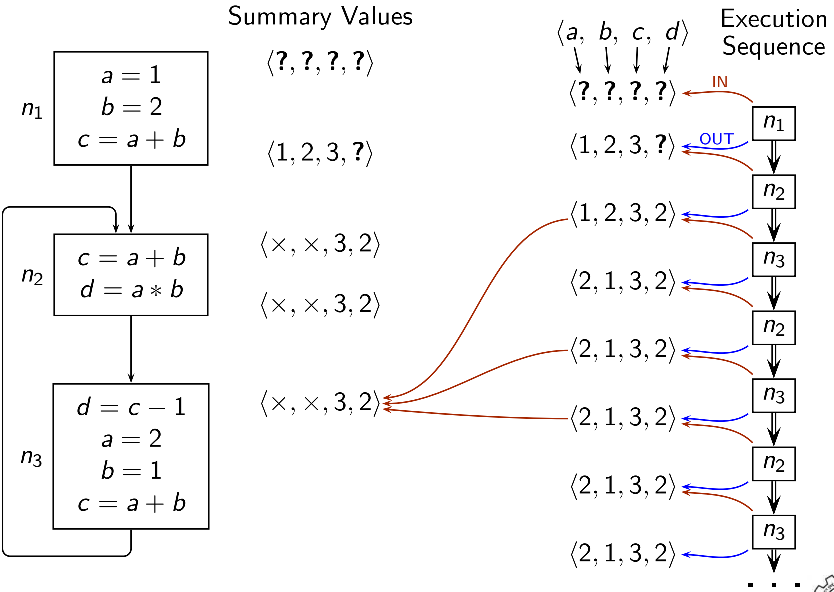

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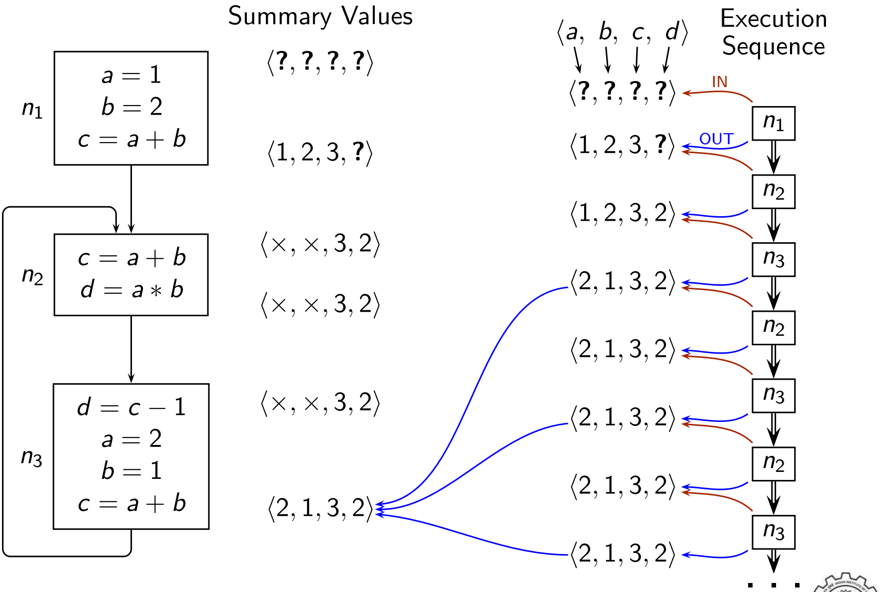

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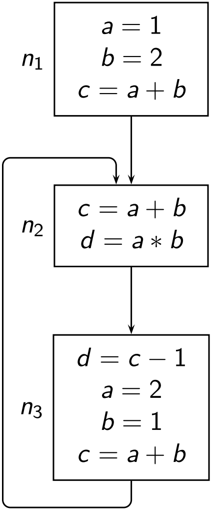


Summary Values

$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$

$\langle 1, 2, 3, \mathbf{?} \rangle$

$\langle \times, \times, 3, 2 \rangle$

$\langle \times, \times, 3, 2 \rangle$

$\langle \times, \times, 3, 2 \rangle$

$\langle 2, 1, 3, 2 \rangle$

Desired Solution

Data Flow Values for Constant Propagation

- Tuples of the form $\langle \xi_1, \xi_2, \ldots, \xi_k \rangle$ where ξ_i is the data flow value for i^{th} variable.

  Unlike bit vector frameworks, value ξ_i is not 0 or 1 (i.e. true or false). Instead, it is one of the following:
  - **?** indicating that not much is known about the constantness of variable v_i
  - $\times$ indicating that variable v_i does not have a constant value
  - An integer constant c_1 if the value of v_i is known to be c_1 at compile time

- Alternatively, sets of pairs $\langle v_i, \xi_i \rangle$ for each variable v_i.

Confluence Operation for Constant Propagation

- Confluence operation $\langle a, c_1 \rangle \sqcap \langle a, c_2 \rangle$

$\sqcap$	$\langle a, \mathbf{?} \rangle$	$\langle a, \times \rangle$	$\langle a, c_1 \rangle$
$\langle a, \mathbf{?} \rangle$	$\langle a, \mathbf{?} \rangle$	$\langle a, \times \rangle$	$\langle a, c_1 \rangle$
$\langle a, \times \rangle$	$\langle a, \times \rangle$	$\langle a, \times \rangle$	$\langle a, \times \rangle$
$\langle a, c_2 \rangle$	$\langle a, c_2 \rangle$	$\langle a, \times \rangle$	If $c_1 = c_2$ $\langle a, c_1 \rangle$ Otherwise $\langle a, \times \rangle$

- This is neither $\cap$ nor $\cup$.

  What are its properties?

Flow Functions for Constant Propagation

- Flow function for $r = a_1 * a_2$

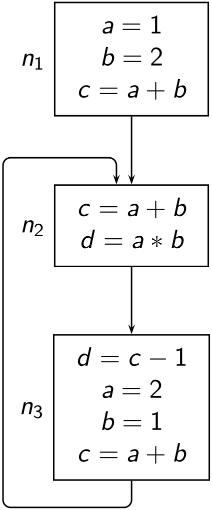
$mult$	$\langle a_1, \mathbf{?} \rangle$	$\langle a_1, \times \rangle$	$\langle a_1, c_1 \rangle$
$\langle a_2, \mathbf{?} \rangle$	$\langle r, \mathbf{?} \rangle$	$\langle r, \times \rangle$	$\langle r, \mathbf{?} \rangle$
$\langle a_2, \times \rangle$	$\langle r, \times \rangle$	$\langle r, \times \rangle$	$\langle r, \times \rangle$
$\langle a_2, c_2 \rangle$	$\langle r, \mathbf{?} \rangle$	$\langle r, \times \rangle$	$\langle r, (c_1 * c_2) \rangle$

- This cannot be expressed in the form

$$f_n(X) = Gen_n \cup (X - Kill_n)$$

where Gen_n and $Kill_n$ are constant effects of block n.

Round Robin Iterative Analysis for Constant Propagation



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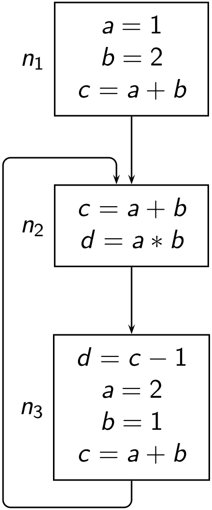
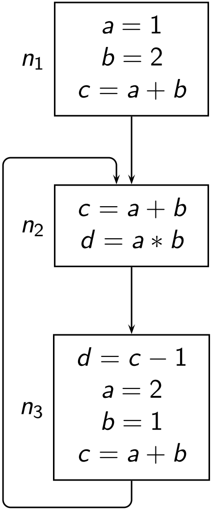


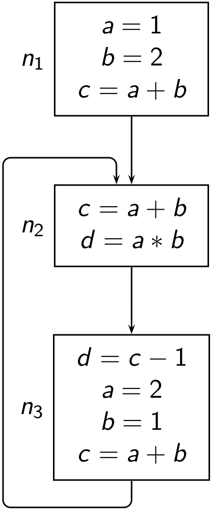
Iteration #1
$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$
$\langle 1, 2, 3, \mathbf{?} \rangle$
$\langle 1, 2, 3, \mathbf{?} \rangle$
$\langle 1, 2, 3, 2 \rangle$
$\langle 1, 2, 3, 2 \rangle$
$\langle 2, 1, 3, 2 \rangle$

Round Robin Iterative Analysis for Constant Propagation



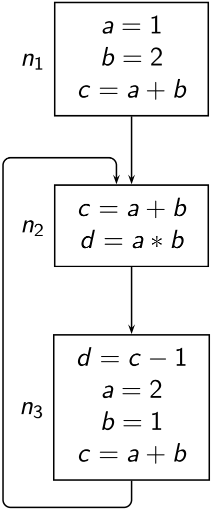
Program point	Iteration #1	Iteration #2
Entry of n_1	$\langle \textbf{?}, \textbf{?}, \textbf{?}, \textbf{?} \rangle$	$\langle \textbf{?}, \textbf{?}, \textbf{?}, \textbf{?} \rangle$
Exit of n_1	$\langle 1, 2, 3, \textbf{?} \rangle$	$\langle 1, 2, 3, \textbf{?} \rangle$
Entry of n_2	$\langle 1, 2, 3, \textbf{?} \rangle$	$\langle \times, \times, 3, 2 \rangle$
Exit of n_2	$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$
Entry of n_3	$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$
Exit of n_3	$\langle 2, 1, 3, 2 \rangle$	$\langle 2, 1, 3, \times \rangle$

Round Robin Iterative Analysis for Constant Propagation



Iteration #1	Iteration #2	Iteration #3
$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$	$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$	$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$
$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle 1, 2, 3, \mathbf{?} \rangle$
$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle \times, \times, 3, 2 \rangle$	$\langle \times, \times, 3, \times \rangle$
$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, \times, \times \rangle$
$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, \times, \times \rangle$
$\langle 2, 1, 3, 2 \rangle$	$\langle 2, 1, 3, \times \rangle$	$\langle 2, 1, 3, \times \rangle$

Round Robin Iterative Analysis for Constant Propagation

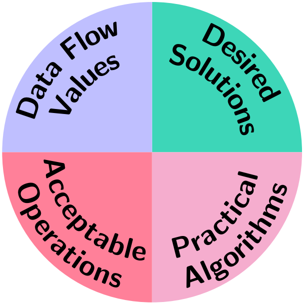


Iteration #1	Iteration #2	Iteration #3	Desired solution
$\langle ?, ?, ?, ? \rangle$	$\langle ?, ?, ?, ? \rangle$	$\langle ?, ?, ?, ? \rangle$	$\langle ?, ?, ?, ? \rangle$
$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$
$\langle 1, 2, 3, ? \rangle$	$\langle \times, \times, 3, 2 \rangle$	$\langle \times, \times, 3, \times \rangle$	$\langle \times, \times, 3, 2 \rangle$
$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, 3, 2 \rangle$
$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, 3, 2 \rangle$
$\langle 2, 1, 3, 2 \rangle$	$\langle 2, 1, 3, \times \rangle$	$\langle 2, 1, 3, \times \rangle$	$\langle 2, 1, 3, 2 \rangle$

Round Robin Iterative Analysis for Constant Propagation

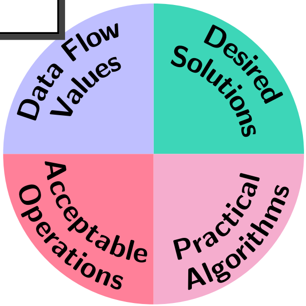
Node	Statements		Iteration #1	Iteration #2	Iteration #3	Desired solution
n_1	$a = 1$ $b = 2$ $c = a + b$	In	$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$	$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$	$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$	$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$
		Out	$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle 1, 2, 3, \mathbf{?} \rangle$
n_2	$c = a + b$ $d = a * b$	In	$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle \times, \times, 3, 2 \rangle$	$\langle \times, \times, 3, \times \rangle$	$\langle \times, \times, 3, 2 \rangle$
		Out	$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, 3, 2 \rangle$
n_3	$d = c - 1$ $a = 2$ $b = 1$ $c = a + b$	In	$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, 3, 2 \rangle$
		Out	$\langle 2, 1, 3, 2 \rangle$	$\langle 2, 1, 3, \times \rangle$	$\langle 2, 1, 3, \times \rangle$	$\langle 2, 1, 3, 2 \rangle$

Issues in Data Flow Analysis



Issues in Data Flow Analysis

- Representation
- Approximation: Partial Order, Lattices



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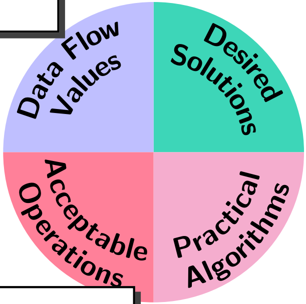
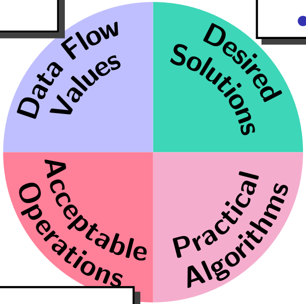


- Merge: Commutativity, Associativity, Idempotence
- Flow Functions: Monotonicity, Distributivity, Boundedness, Separability

Issues in Data Flow Analysis

- Representation
- Approximation: Partial Order, Lattices

- Existence
- Safety (soundness)
- Precision



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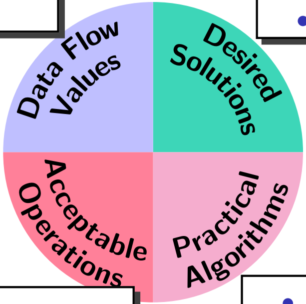


- Merge: Commutativity, Associativity, Idempotence
- Flow Functions: Monotonicity, Distributivity, Boundedness, Separability

- Complexity, efficiency
- Convergence
- Initialization

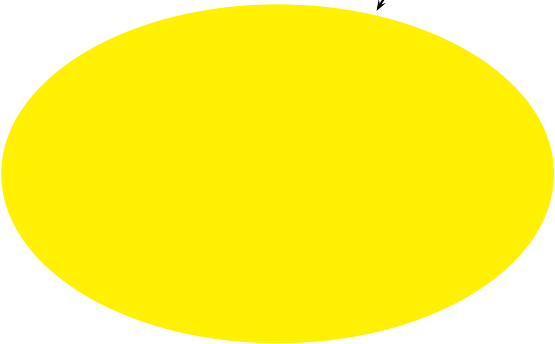
Part 4

Data Flow Values: An Overview

Data Flow Values: An Outline of Our Discussion

- The need to define the notion of abstraction
- Lattices, variants of lattices
- Relevance of lattices for data flow analysis
  - Partial order relation as approximation of data flow values
  - Meet operations as confluence of data flow values
- Cartesian product of lattices
- Example of lattices

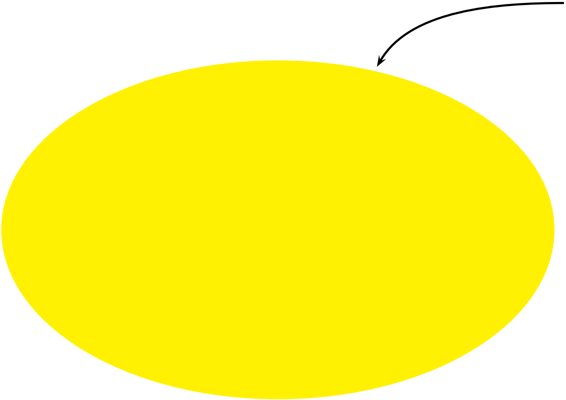
Partially Ordered Sets and Lattices



Partially ordered sets

Partial order $\sqsubseteq$ is reflexive, transitive, and antisymmetric

Partially Ordered Sets and Lattices



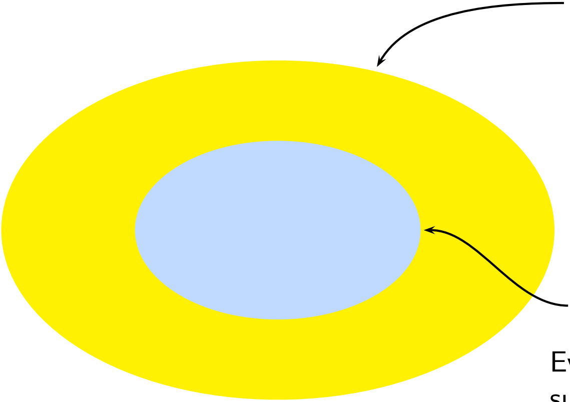
Partially ordered sets

Partial order $\sqsubseteq$ is reflexive, transitive, and antisymmetric

A lower bound of x, y is u s.t. $u \sqsubseteq x$ and $u \sqsubseteq y$

An upper bound of x, y is u s.t. $x \sqsubseteq u$ and $y \sqsubseteq u$

Partially Ordered Sets and Lattices



Partially ordered sets

Partial order $\sqsubseteq$ is reflexive, transitive, and antisymmetric

Lattices

Every non-empty finite subset has a greatest lower bound (glb) and a least upper bound (lub)

Partially Ordered Sets

Set $\{1, 2, 3, 4, 9\}$ with $\sqsubseteq$ relation as "divides" (i.e. $a \sqsubseteq b$ iff a divides b)

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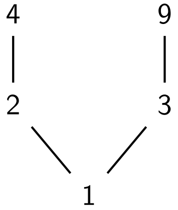

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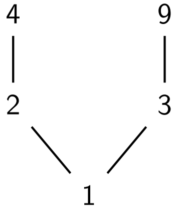
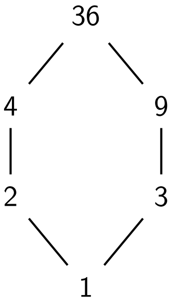


Subsets $\{4, 9\}$ and $\{2, 3\}$ do not have an upper bound in the set

Lattice

Set $\{1, 2, 3, 4, 9, 36\}$ with $\sqsubseteq$ relation as "divides" (i.e. $a \sqsubseteq b$ iff a divides b)



Complete Lattice

- Lattice: A partially ordered set such that every non-empty finite subset has a glb and a lub.

  Example:
  Lattice $\mathbb{Z}$ of integers under $\leq$ relation. All finite subsets have a glb and a lub. Infinite subsets do not have a glb or a lub.

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- Complete Lattice: A lattice in which even $\emptyset$ and infinite subsets have a glb and a lub.

  Example:
  Lattice $\mathbb{Z}$ of integers under $\leq$ relation with ∞ and $-\infty$.
  - ∞ is the top element denoted $\top$: $\forall i \in \mathbb{Z},\ i \leq \top$.
  - $-\infty$ is the bottom element denoted $\bot$: $\forall i \in \mathbb{Z},\ \bot \leq i$.

$\mathbb{Z} \cup \{\infty, -\infty\}$ is a Complete Lattice

- Infinite subsets of $\mathbb{Z} \cup \{\infty, -\infty\}$ have a glb and lub.

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    Every element of $\mathbb{Z} \cup \{\infty, -\infty\}$ is vacuously a lower bound of an element in $\emptyset$ (because there is no element in $\emptyset$).

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      Every element of $\mathbb{Z} \cup \{\infty, -\infty\}$ is vacuously a lower bound of an element in $\emptyset$ (because there is no element in $\emptyset$).
      The greatest among these lower bounds is $\top$.
    - lub($\emptyset$) is $\perp$

Finite Lattices are Complete

- Any given set of elements has a glb and a lub

Available Expressions Analysis

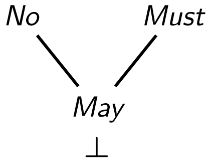
$$\begin{tikzcd}[column sep=small]
 & (\top) & \\
 & \{e_1, e_2, e_3\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{e_1, e_2\} \arrow[d] \arrow[dr] & \{e_1, e_3\} \arrow[dl] \arrow[dr] & \{e_2, e_3\} \arrow[dl] \arrow[d] \\
\{e_1\} \arrow[dr] & \{e_2\} \arrow[d] & \{e_3\} \arrow[dl] \\
 & \emptyset & \\
 & (\bot) &
\end{tikzcd}$$

Partially Available Expressions Analysis

$$\begin{tikzcd}[column sep=small]
 & (\top) & \\
 & \emptyset \arrow[dl] \arrow[d] \arrow[dr] & \\
\{e_1\} \arrow[d] \arrow[dr] & \{e_2\} \arrow[dl] \arrow[dr] & \{e_3\} \arrow[dl] \arrow[d] \\
\{e_1, e_2\} \arrow[dr] & \{e_1, e_3\} \arrow[d] & \{e_2, e_3\} \arrow[dl] \\
 & \{e_1, e_2, e_3\} & \\
 & (\bot) &
\end{tikzcd}$$

Lattice for May-Must Analysis

- There is no $\top$ among the natural values



Interpreting data flow values

- *No.* Information does not hold along any path
- *Must.* Information must hold along all paths
- *May.* Information may hold along some path

- An artificial $\top$ can be added
  However, a lub may not exist for arbitrary sets

Some Variants of Lattices

A poset L is

- A lattice iff each non-empty finite subset of L has a glb and lub.
- A complete lattice iff each subset of L has a glb and lub.
- A meet semilattice iff each non-empty finite subset of L has a glb.
- A join semilattice iff each non-empty finite subset of L has a lub.

Ascending and Descending Chains

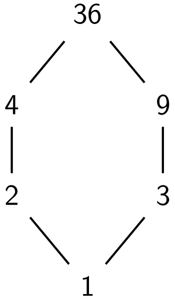
- Strictly ascending chain. $x \sqsubset y \sqsubset \cdots \sqsubset z$
- Strictly descending chain. $x \sqsupset y \sqsupset \cdots \sqsupset z$
- DCC: Descending Chain Condition
  All strictly descending chains are finite.
- ACC: Ascending Chain Condition
  All strictly ascending chains are finite.

Complete Lattice and Ascending and Descending Chains

- If L satisfies acc and dcc, then
  - L has finite height, and
  - L is complete.
- A complete lattice need not have finite height (i.e. strict chains may not be finite).
  Example:
  Lattice of integers under $\leq$ relation with ∞ as $\top$ and $-\infty$ as $\perp$.

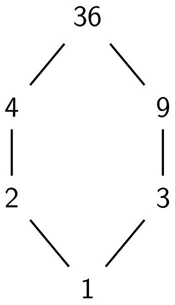
Operations on Lattices

- Meet ($\sqcap$) and Join ($\sqcup$)



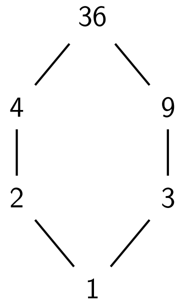
Operations on Lattices

- Meet ($\sqcap$) and Join ($\sqcup$)
  - $x \sqcap y$ computes the glb of x and y.
    $z = x \sqcap y \Rightarrow z \sqsubseteq x \land z \sqsubseteq y$



Operations on Lattices

- Meet ($\sqcap$) and Join ($\sqcup$)
  - $x \sqcap y$ computes the glb of x and y.
    $z = x \sqcap y \Rightarrow z \sqsubseteq x \wedge z \sqsubseteq y$
  - $x \sqcup y$ computes the lub of x and y.
    $z = x \sqcup y \Rightarrow z \sqsupseteq x \wedge z \sqsupseteq y$



Operations on Lattices

- Meet ($\sqcap$) and Join ($\sqcup$)
  - $x \sqcap y$ computes the glb of x and y.
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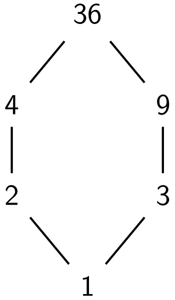

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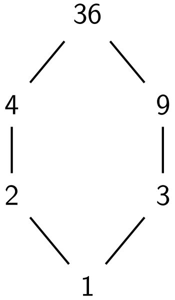
  - $\sqcap$ and $\sqcup$ are commutative, associative, and idempotent.
- Top ($\top$) and Bottom ($\bot$) elements

$$
\begin{aligned}
\forall x \in L,\ x \sqcap \top &= x \\
\forall x \in L,\ x \sqcup \top &= \top \\
\forall x \in L,\ x \sqcap \bot &= \bot \\
\forall x \in L,\ x \sqcup \bot &= x
\end{aligned}
$$



Operations on Lattices

Greatest common divisor (or highest common factor) **in the lattice**

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\begin{aligned}
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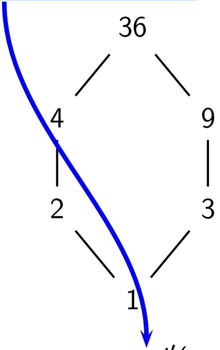
\end{aligned}
$$



$x \sqcap y = gcd'(x, y)$

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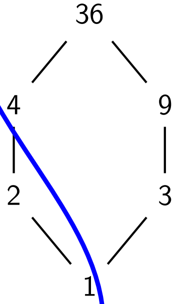
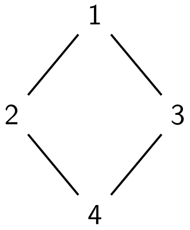
Greatest common divisor (or highest common factor) **in the lattice**



$$x \sqcap y = gcd'(x, y)$$
$$x \sqcup y = lcm'(x, y)$$

Lowest common multiple **in the lattice**

Cartesian Product of Lattice



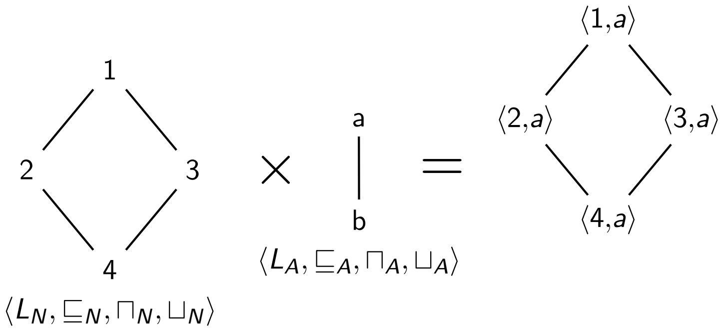
$\langle L_N, \sqsubseteq_N, \sqcap_N, \sqcup_N \rangle$

$\times$

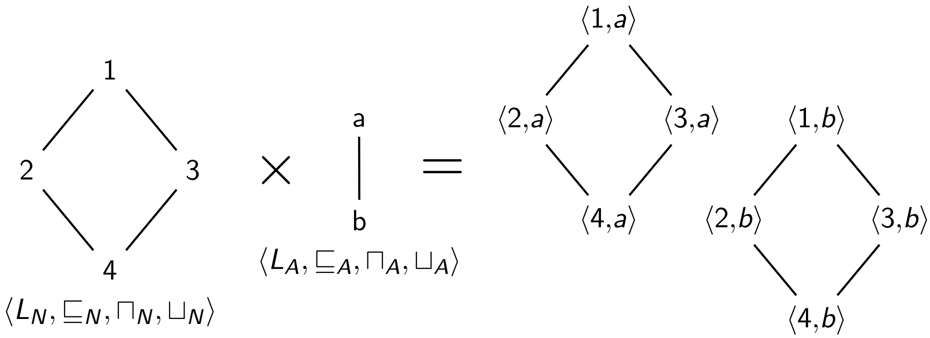
$\langle L_A, \sqsubseteq_A, \sqcap_A, \sqcup_A \rangle$

$=$

Cartesian Product of Lattice



Cartesian Product of Lattice



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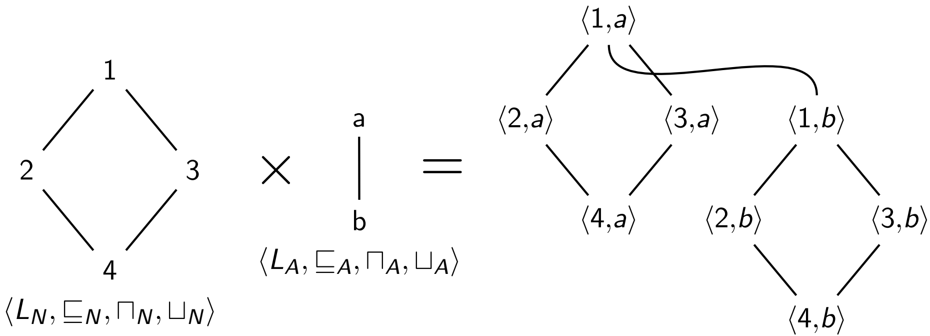

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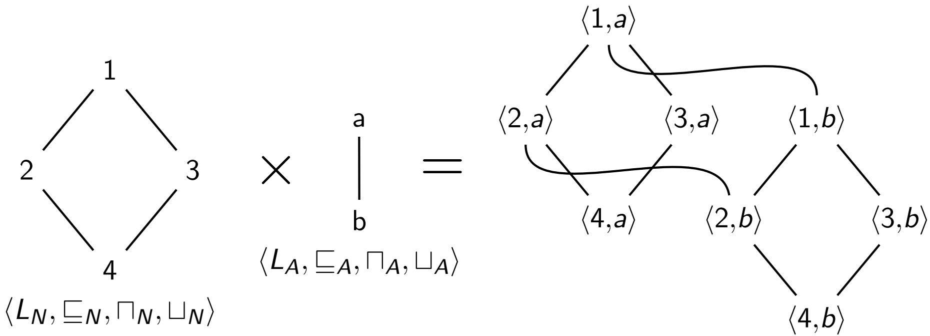

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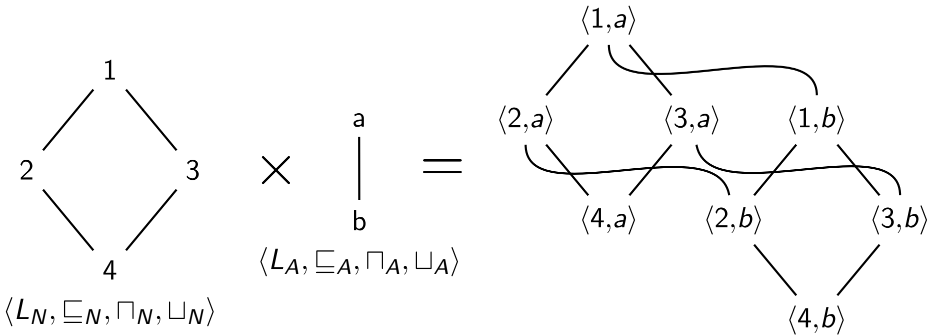

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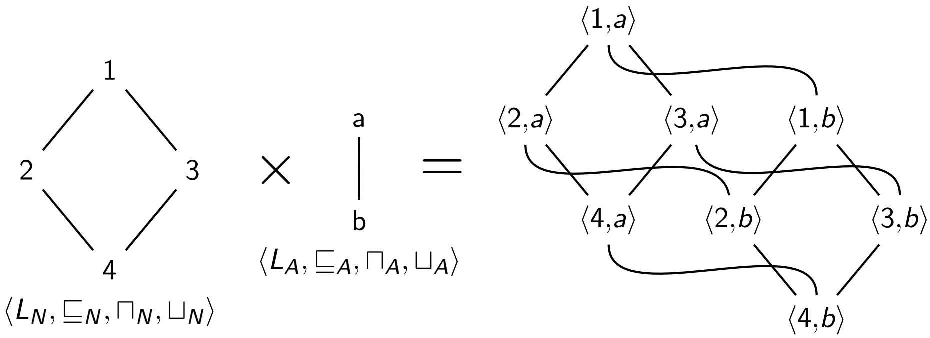

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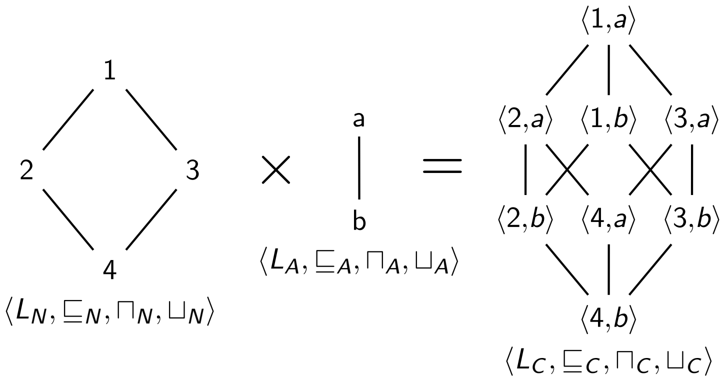

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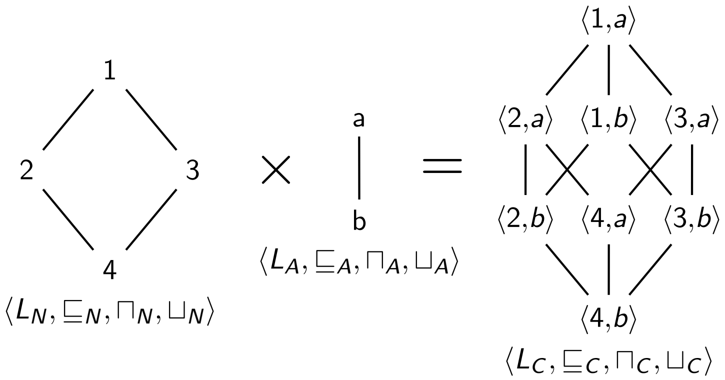

$$
\begin{aligned}
\langle x_1, y_1 \rangle \sqsubseteq_C \langle x_2, y_2 \rangle &\Leftrightarrow x_1 \sqsubseteq_N x_2 \wedge y_1 \sqsubseteq_A y_2 \\
\langle x_1, y_1 \rangle \sqcap_C \langle x_2, y_2 \rangle &= \langle x_1 \sqcap_N x_2, y_1 \sqcap_A y_2 \rangle \\
\langle x_1, y_1 \rangle \sqcup_C \langle x_2, y_2 \rangle &= \langle x_1 \sqcup_N x_2, y_1 \sqcup_A y_2 \rangle
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Meet semilattices satisfying the descending chain condition

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  - lub of arbitrary elements may not exist

The Set of Data Flow Values For Available Expressions Analysis

- The powerset of the universal set of expressions
- Partial order is the subset relation

$$
\begin{tikzcd}
 & & \{e_1, e_2, e_3\} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\{e_1, e_2\} \arrow[d] \arrow[drr] & & \{e_1, e_3\} \arrow[dll] \arrow[drr] & & \{e_2, e_3\} \arrow[dll] \arrow[d] \\
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 & & \emptyset & &
\end{tikzcd}
$$

Set View of the Lattice

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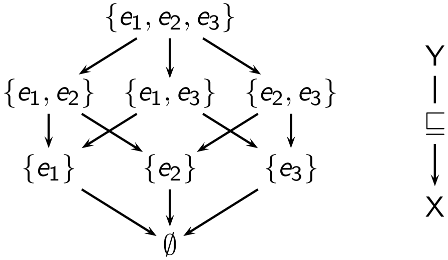

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 & \emptyset &
\end{tikzcd}$$

Set View of the Lattice

$$\begin{tikzcd}
Y \arrow[d, "\sqsubseteq" description] \\
X
\end{tikzcd}$$

$$\begin{tikzcd}
 & 111 \arrow[dl] \arrow[d] \arrow[dr] & \\
110 \arrow[d] \arrow[dr] & 101 \arrow[dl] \arrow[dr] & 011 \arrow[dl] \arrow[d] \\
100 \arrow[dr] & 010 \arrow[d] & 001 \arrow[dl] \\
 & 000 &
\end{tikzcd}$$

Bit Vector View

The Concept of Approximation

- x approximates y *iff*

  x can be used in place of y without causing any problems.

- Validity of approximation is context specific

  x may be approximated by y in one context and by z in another

  - Earnings : Rs. 1050 can be safely approximated by Rs. 1000.
  - Expenses : Rs. 1050 can be safely approximated by Rs. 1100.

Two Important Objectives in Data Flow Analysis

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- The intended use of data flow information ($\equiv$ context) determines validity of approximations

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Spurious Inclusion

Spurious Exclusion

Partial Order Captures Approximation

- $\sqsubseteq$ captures valid approximations for safety

  $x \sqsubseteq y \;\Rightarrow\; x$ is *weaker than* y

  - The data flow information represented by x can be safely used in place of the data flow information represented by y
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We want most exhaustive information which is also safe.

Most Approximate Values in a Complete Lattice

- *Top*. $\forall x \in L,\ x \sqsubseteq \top$. The most exhaustive value.

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Appropriate orientation chosen by design.

Setting Up Lattices

Available Expressions Analysis	Live Variables Analysis
$\begin{tikzcd}[column sep=small] & & \{e_1,e_2,e_3\} \arrow[dll] \arrow[d] \arrow[drr] & & \\ \{e_1,e_2\} \arrow[d] \arrow[drr] & & \{e_1,e_3\} \arrow[dll] \arrow[drr] & & \{e_2,e_3\} \arrow[dll] \arrow[d] \\ \{e_1\} \arrow[drr] & & \{e_2\} \arrow[d] & & \{e_3\} \arrow[dll] \\ & & \emptyset & & \end{tikzcd}$	$\begin{tikzcd}[column sep=small] & & \emptyset \arrow[dll] \arrow[d] \arrow[drr] & & \\ \{v_1\} \arrow[d] \arrow[drr] & & \{v_2\} \arrow[dll] \arrow[drr] & & \{v_3\} \arrow[dll] \arrow[d] \\ \{v_1,v_2\} \arrow[drr] & & \{v_1,v_3\} \arrow[d] & & \{v_2,v_3\} \arrow[dll] \\ & & \{v_1,v_2,v_3\} & & \end{tikzcd}$
$\sqsubseteq$ is $\subseteq$	$\sqsubseteq$ is $\supseteq$
$\sqcap$ is $\cap$	$\sqcap$ is $\cup$

Partial Order Relation

Reflexive	$x \sqsubseteq x$
Transitive	$x \sqsubseteq y, y \sqsubseteq z$ $\Rightarrow x \sqsubseteq z$
Antisymmetric	$x \sqsubseteq y, y \sqsubseteq x$ $\Leftrightarrow x = y$

Partial Order Relation

Reflexive	$x \sqsubseteq x$	x can be safely used in place of x
Transitive	$x \sqsubseteq y, y \sqsubseteq z \Rightarrow x \sqsubseteq z$	If x can be safely used in place of y and y can be safely used in place of z, then x can be safely used in place of z
Antisymmetric	$x \sqsubseteq y, y \sqsubseteq x \Leftrightarrow x = y$	If x can be safely used in place of y and y can be safely used in place of x, then x must be same as y

Merging Information

- $x \sqcap y$ computes the *greatest lower bound* of x and y i.e. largest z such that $z \sqsubseteq x$ and $z \sqsubseteq y$

  The largest safe approximation of combining data flow information x and y

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- Commutative $x \sqcap y = y \sqcap x$

  Associative $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$

  Idempotent $x \sqcap x = x$

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Commutative	$x \sqcap y = y \sqcap x$	The order in which the data flow information is merged, does not matter
Associative	$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	Allow n-ary merging without any restriction on the order
Idempotent	$x \sqcap x = x$	No loss of information if x is merged with itself

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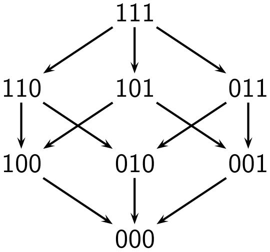
- $\top$ is the identity of $\sqcap$
  - Presence of loops $\Rightarrow$ self dependence of data flow information
  - Using $\top$ as the initial value ensure exhaustiveness

More on Lattices in Data Flow Analysis

L = Lattice for all expressions	$\widehat{L}$ = Lattice for a single expression
	(Expression e is available) 1 or $\{e\}$ $\downarrow$ 0 or $\emptyset$ (Expressions e is not available)

More on Lattices in Data Flow Analysis

L = Lattice for all expressions

$$\begin{tikzcd}
& 111 \arrow[dl] \arrow[d] \arrow[dr] & \\
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& 000 &
\end{tikzcd}$$

$\widehat{L}$ = Lattice for a single expression

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1 \text{ or } \{e\} \arrow[d] \\
0 \text{ or } \emptyset \\
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\end{tikzcd}$$

Cartesian products if sets are used, vectors (or tuples) if bit are used.

- $L = \widehat{L} \times \widehat{L} \times \widehat{L}$ and $x = \langle \widehat{x}_1, \widehat{x}_2, \widehat{x}_3 \rangle \in L$ where $\widehat{x}_i \in \widehat{L}$
- $\sqsubseteq = \widehat{\sqsubseteq} \times \widehat{\sqsubseteq} \times \widehat{\sqsubseteq}$ and $\sqcap = \widehat{\sqcap} \times \widehat{\sqcap} \times \widehat{\sqcap}$
- $\top = \widehat{\top} \times \widehat{\top} \times \widehat{\top}$ and $\bot = \widehat{\bot} \times \widehat{\bot} \times \widehat{\bot}$

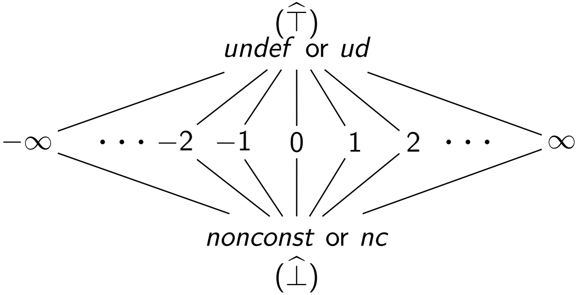
Component Lattice for Data Flow Information Represented By Bit Vectors

$$\begin{array}{c} (\widehat{\top}) \\ 1 \\ | \\ 0 \\ (\widehat{\bot}) \end{array} \qquad\qquad \begin{array}{c} (\widehat{\top}) \\ 0 \\ | \\ 1 \\ (\widehat{\bot}) \end{array}$$

$\sqcap$ is $\cap$ or Boolean AND

$\sqcap$ is $\cup$ or Boolean OR

Component Lattice for Integer Constant Propagation



- Overall lattice L is the product of $\widehat{L}$ for all variables.
- $\sqcap$ and $\widehat{\sqcap}$ get defined by $\sqsubseteq$ and $\widehat{\sqsubseteq}$.

$\widehat{\sqcap}$	$\langle a, ud\rangle$	$\langle a, nc\rangle$	$\langle a, c_1\rangle$
$\langle a, ud\rangle$	$\langle a, ud\rangle$	$\langle a, nc\rangle$	$\langle a, c_1\rangle$
$\langle a, nc\rangle$	$\langle a, nc\rangle$	$\langle a, nc\rangle$	$\langle a, nc\rangle$
$\langle a, c_2\rangle$	$\langle a, c_2\rangle$	$\langle a, nc\rangle$	If $c_1 = c_2$ then $\langle a, c_1\rangle$ else $\langle a, nc\rangle$

Component Lattice for May Points-To Analysis

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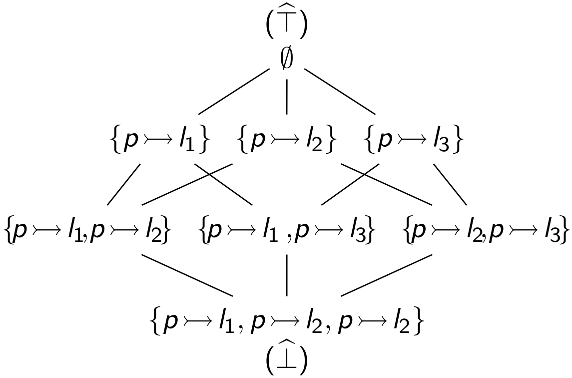

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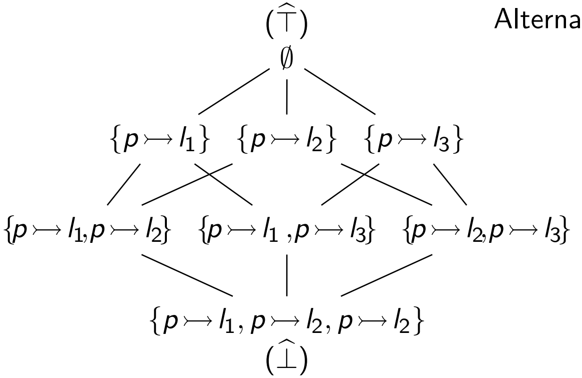
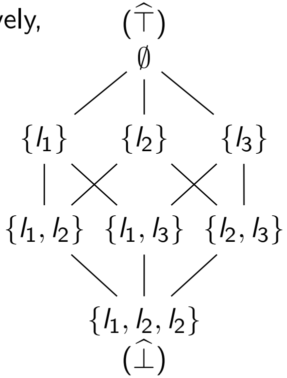
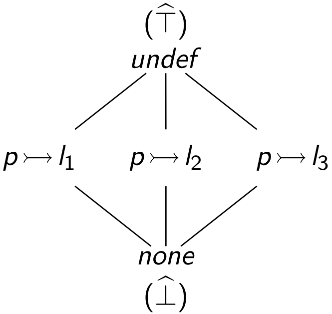


Alternatively,

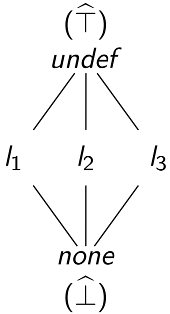


Component Lattice for Must Points-To Analysis

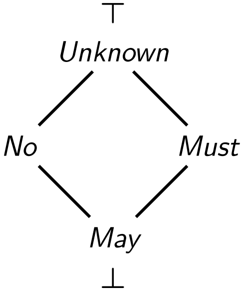
- A pointer can point to at most one location.



Alternatively,



General Lattice for May-Must Analysis



Interpreting data flow values

- *Unknown.* Nothing is known as yet
- *No.* Information does not hold along any path
- *Must.* Information must hold along all paths
- *May.* Information may hold along some path

Possible Applications

- Pointer Analysis : No need of separate of *May* and *Must* analyses eg. $(p \rightarrowtail l, \mathit{May})$, $(p \rightarrowtail l, \mathit{Must})$, $(p \rightarrowtail l, \mathit{No})$, or $(p \rightarrowtail l, \mathit{Unknown})$.
- Type Inferencing for Dynamically Checked Languages