An Algebraic Approach to Internet Routing
Lecture 12
Metric-neutral partitions

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Hybrid Link-State, Path-Vector Routing
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Motivation

- Corporate networks routing objectives include:
  - Fast convergence
  - Minimize resource usage (computation, memory, bandwidth)
  - Least-cost paths
  - Load balancing (by traffic engineering)
  - Fault isolation
  - Scalability

- Link-state routing offers fast convergence, but has very high resource requirements.

- OSPF and IS–IS use link-state with partitioning.

- They use complicated routing metrics (e.g., intra-area and inter-area and L1, L2, L1→L2, and L2→L1).

- Breaks global optimality and prevents traffic engineering for load balancing.
An example

- The following graph is labelled with shortest-path \( (\mathbb{N}^+, \min, +, 0, \infty) \) algebra

![Graph Image]

- The adjacency matrix and the all-pair shortest-paths matrix (solution to Equation (??)) are

\[
A = \begin{bmatrix}
\infty & 7 & 3 & 1 & \infty \\
7 & \infty & 3 & 3 & \infty \\
3 & 3 & \infty & \infty & \infty \\
1 & 3 & \infty & \infty & 7 \\
\infty & 2 & \infty & 7 & \infty \\
\end{bmatrix}
\]

\[
A^* = \begin{bmatrix}
0 & 4 & 3 & 1 & 6 \\
4 & 0 & 3 & 3 & 2 \\
3 & 3 & 0 & 4 & 5 \\
1 & 3 & 4 & 0 & 5 \\
6 & 2 & 5 & 5 & 0 \\
\end{bmatrix}
\]
Routing vs. Forwarding

Routing solves \( X = AX \oplus I \) with router adjacencies \( A \).

Forwarding solves \( F = AF \oplus M \), where \( M \) is the mapping matrix.

Link-state: \( F = A^*M \).

Path-vector: \( F^{[0]} = M, F^{[k+1]} = AF^{[k]} \oplus M \).
Partitioning

- \( \pi(G) = \{ V_1, V_2, \cdots, V_m \} \).
- \( \pi(G) \Rightarrow E_{s,t} = \{ (u, v) \in E \mid u \in V_s \land v \in V_t \} \).
- Region subgraph \( G_r = (V_r, E_{r,r}) \).
- Border node \( u \in V_r \) if \( \exists v \in V_{s \neq r} \land (u, v) \in E_{r,s} \) and \( E_{r,s} \) is inter-region edge.
- Boundary subgraph \( G_b = (V_b, E_{r,s}) \), may not be connected.
- Adjacency matrix of a partitioned graph

\[
A = \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m,1} & A_{m,2} & \cdots & A_{m,m}
\end{bmatrix}.
\]
Algebraic Model of AS Partitioning

- The all-region adjacency matrix

\[
R = \begin{bmatrix}
A_{1,1} & O_{1,2} & \cdots & O_{1,m} \\
O_{2,1} & A_{2,2} & \cdots & O_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
O_{m,1} & O_{m,2} & \cdots & A_{m,m}
\end{bmatrix},
\]

where \(O_{s,t}\) for \(s \neq t\) is a sub-matrix of all \(0\)s.

- The routing solution of all the regions

\[
R^* = \begin{bmatrix}
A^*_{1,1} & O_{1,2} & \cdots & O_{1,m} \\
O_{2,1} & A^*_{2,2} & \cdots & O_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
O_{m,1} & O_{m,2} & \cdots & A^*_{m,m}
\end{bmatrix}.
\]
Algebraic Model of AS Partitioning

- The adjacency matrix of of the boundary subgraph

\[ B = \begin{bmatrix}
O_{1,1} & A_{1,2} & \cdots & A_{1,m} \\
A_{2,1} & O_{2,2} & \cdots & A_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m,1} & A_{m,2} & \cdots & O_{m,m}
\end{bmatrix} \]

- We can then define the *transit matrix* \( T \) to capture the weights virtual links between border nodes \( u \) and \( v \).

\[
T(u, v) = \begin{cases} 
A^*,r(u, v) & \text{if } (u, v) \text{ is a transit arc in region } r \\
0 & \text{otherwise}
\end{cases}
\]

- Define (the adjacency matrix of) the core graph

\[ C = B \oplus T. \]

where \( T \) represents all the virtual links in the graph.
Hybrid Routing?

Main Claim

\[ A^* = R^* \oplus R^*C^*R^* \] (1)

Interpretation

1. Compute \( R^* \) – solve the region routing problem.
2. Construct the core graph, matrix \( C = B \oplus T \).
3. Compute \( C^* \) – solve the core routing problem.
4. Construct \( A^* \) by computing \( R^*(I \oplus C^*R^*) \).

Note that different algorithms – link-state or path-vector – can be used in the 1st and the 3rd steps.
A Partitioning Example

A = \begin{bmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2}
\end{bmatrix}

= \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
\infty & 7 & 3 & 1 & \infty \\
2 & \infty & 3 & 3 & 2 \\
3 & 3 & \infty & \infty & \infty \\
4 & 1 & 3 & \infty & \infty \\
5 & \infty & 2 & \infty & 7 \\
\infty & \infty & \infty & \infty & \infty
\end{bmatrix}
A Partitioning Example

- The all-region adjacency matrix and corresponding routing solution

\[
R = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & \infty & 7 & 3 & \infty \\
2 & \infty & 7 & \infty & \infty \\
3 & \infty & \infty & 7 & \infty \\
4 & \infty & \infty & \infty & \infty \\
5 & \infty & \infty & \infty & \infty \\
\end{bmatrix} \Rightarrow R^* = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 6 & 3 & \infty \\
2 & 6 & 0 & 3 & \infty \\
3 & \infty & \infty & 0 & 7 \\
4 & \infty & \infty & \infty & \infty \\
5 & \infty & \infty & \infty & 7 \\
\end{bmatrix}
\]

- The boundary matrix and the transit matrix

\[
B = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & \infty & \infty & \infty & 1 \\
2 & \infty & \infty & \infty & 3 \\
3 & \infty & \infty & \infty & \infty \\
4 & 1 & 3 & \infty & \infty \\
5 & \infty & 2 & \infty & \infty \\
\end{bmatrix}, \quad T = \begin{bmatrix}
1 & 2 & 4 & 5 \\
1 & \infty & 6 & \infty & \infty \\
2 & \infty & \infty & \infty & \infty \\
3 & \infty & \infty & \infty & \infty \\
4 & \infty & \infty & 7 & \infty \\
5 & \infty & \infty & \infty & \infty \\
\end{bmatrix}
\]
A Partitioning Example

- The core graph with virtual links

- Core graph adjacency matrix and the routing solution

\[
C = \begin{bmatrix}
\infty & 6 & 1 & \infty \\
6 & \infty & 3 & 2 \\
1 & 3 & \infty & 7 \\
\infty & 2 & 7 & \infty \\
\end{bmatrix}
\Rightarrow
C^* = \begin{bmatrix}
1 & 2 & 4 & 5 \\
0 & 4 & 1 & 6 \\
4 & 0 & 3 & 2 \\
1 & 3 & 0 & 5 \\
6 & 2 & 5 & 0 \\
\end{bmatrix}
\]
Equations of Hybrid Model

Using the distinctions of routing and forwarding:

1. Solve $F_1 = RF_1 \oplus I$ for region routing $R^*$. 
2. Build the core graph $C = B \oplus T$. 
3. Solve $F_2 = CF_2 \oplus F_1$ for core routing $C^*$ and exporting region internal routes to the core $C^*R^*$. 
4. Solve $F = F_1 \oplus F_1 F_2$ for importing inter-region routes $R^*(I \oplus C^*R^*)$.

Combinations of link-state and path-vector mechanisms

<table>
<thead>
<tr>
<th>Hybrid</th>
<th>Region</th>
<th>Core</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-over-D</td>
<td>link-state</td>
<td>link-state</td>
</tr>
<tr>
<td>B-over-D</td>
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<td>path-vector</td>
</tr>
<tr>
<td>D-over-B</td>
<td>path-vector</td>
<td>link-state</td>
</tr>
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</tbody>
</table>
Problem Set III (Due 1 December)

1. Construct an interesting weighted graph using the *scoped product* \((S \Theta T)\). Show adjacency and routing matrix. A picture might help.

2. Construct an interesting weighted graph using the *metric-neutral partitions* (this lecture). Show adjacency and routing matrix.

3. Prove this: \(((\cdot) X \oplus Y) = X^*(YX^*)^*\).

4. Prove the **Main Claim** above.