An Algebraic Approach to Internet Routing
Lecture 09
A mini-metalanguage

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Outline

Redefine operation for inserting a zero

$$\mathrm{add\_zero}(\overline{0},\ (S,\ \oplus,\ \otimes)) = (S \uplus \{\overline{0}\}, \hat{\oplus},\ \hat{\otimes})$$

where

$$a \hat{\oplus} b \quad = \quad \begin{cases} a & (\text{if } b = \mathrm{inr}(\overline{0})) \\ b & (\text{if } a = \mathrm{inr}(\overline{0})) \\ \mathrm{inl}(a' \oplus b') & (\text{if } a = \mathrm{inl}(a'),\ b = \mathrm{inl}(b')) \end{cases}$$

$$a \hat{\otimes} b \quad = \quad \begin{cases} \mathrm{inr}(\overline{0}) & (\text{if } b = \mathrm{inr}(\overline{0})) \\ \mathrm{inr}(\overline{0}) & (\text{if } a = \mathrm{inr}(\overline{0})) \\ \mathrm{inl}(a' \otimes b') & (\text{if } a = \mathrm{inl}(a'),\ b = \mathrm{inl}(b')) \end{cases}$$

disjoint union

$$A \uplus B \equiv \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$$

Redefine operation for inserting a one

$$\mathrm{add\_one}(\overline{1},\ (S,\ \oplus,\ \otimes)) = (S \uplus \{\overline{1}\}, \hat{\oplus},\ \hat{\otimes})$$

where

$$a \hat{\oplus} b \quad = \quad \begin{cases} \mathrm{inr}(\overline{1}) & \text{(if } b = \mathrm{inr}(\overline{1})\text{)} \\ \mathrm{inr}(\overline{1}) & \text{(if } a = \mathrm{inr}(\overline{1})\text{)} \\ \mathrm{inl}(a' \oplus b') & \text{(if } a = \mathrm{inl}(a'),\ b = \mathrm{inl}(b')\text{)} \end{cases}$$

$$a \hat{\otimes} b \quad = \quad \begin{cases} a & \text{(if } b = \mathrm{inr}(\overline{1})\text{)} \\ b & \text{(if } a = \mathrm{inr}(\overline{1})\text{)} \\ \mathrm{inl}(a' \otimes b') & \text{(if } a = \mathrm{inl}(a'),\ b = \mathrm{inl}(b')\text{)} \end{cases}$$

Our (fuzzy wuzzy) goals

Define an expressive language for expressions E, a well-formedness condition $\textsc{wf}E$, so that for a very interesting set of properties $\mathcal{P}$ we have

$$\forall \mathrm{Q} \in P : \forall E \in \mathcal{L} : \mathrm{WF}(E) \implies (\mathrm{Q}([\![E]\!]) \vee \neg \mathrm{Q}([\![E]\!]))$$

But first, we will just define things so that

Soundness

$$\forall E \in \mathcal{L} : \mathrm{WF}(E) \implies (\mathrm{NTRIV}([\![E]\!]) \wedge \mathrm{AA}([\![E]\!]) \wedge \mathrm{AM}([\![E]\!]) \wedge \mathrm{CA}([\![E]\!]))$$

Property	Definition
NTRIV	$\exists a, b : a \neq b$
AA	$\forall a, b, c : a \oplus (b \oplus c) = (a \oplus b) \oplus c$
AM	$\forall a, b, c : a \otimes (b \otimes c) = (a \otimes b) \otimes c$
CA	$\forall a, b : a \oplus b = b \oplus a$

Base bisemigroups

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	comments
min_plus	$\mathbb{N}$	min	$+$		0	minimum-weight routing
max_min	$\mathbb{N}$	max	min	0		greatest-capacity routing
rel	[0, 1]	max	$\times$	0	1	most-reliable routing
shla(W)	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes
shpa(W)	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes
seq(W)	2^{W^*}	$\cup$	$\circ$	W	$\{\epsilon\}$	sets of sequences, where $A \circ B = \{a \circ b \mid a \in A, b \in B\}$

$$B ::= \texttt{min\_plus} \mid \texttt{max\_min} \mid \texttt{rel} \mid \texttt{shla}(W) \mid \texttt{shpa}(W) \mid \texttt{seq}(W)$$

A simple grammar for a mini-metalanguage [NG10]

$$
\begin{array}{rcl}
E & ::= & B \\
 & | & \texttt{add\_zero}(c,\ E\,) \\
 & | & \texttt{add\_one}(c,\ E\,) \\
 & | & \texttt{right}(\,E\,) \\
 & | & \texttt{left}(\,E\,) \\
 & | & (\,E\ \texttt{times}\ E\,) \\
 & | & (\,E\ \texttt{lex}\ E\,)
\end{array}
$$

A few examples

`add_zero(`INF, `( min_plus lex min_plus ) )`

`add_zero(`INF, `( min_plus lex add_zero(`SELF, `max_min ) ) )`

Starting point : Semiring properties, and a few more

These properties are called $\mathcal{P}_0$

Property	Definition
EZ	$\exists \overline{0} : \forall a : a \oplus \overline{0} = \overline{0} \oplus a = a$
EO	$\exists \overline{1} : \forall a : a \otimes \overline{1} = \overline{1} \otimes a = a$
ZA	$\forall a : a \otimes \overline{0} = \overline{0} \otimes a = \overline{0}$
LD	$\forall a, b, c : c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$
RD	$\forall a, b, c : (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$
IA	$\forall a : a \oplus a = a$
SA	$\forall a, b : a \oplus b = a \vee a \oplus b = b$
OA	$\forall a : a \oplus \overline{1} = \overline{1} \oplus a = \overline{1}$
LINF	$\forall a, b : a = a \oplus (b \otimes a)$ (that is, $a \leq_{\oplus}^{L} b \otimes a$)
SLINF	$\forall a, b : a = a \oplus (b \otimes a) \neq b \otimes a$ (that is, $a <_{\oplus}^{L} b \otimes a$)

Definition of $[\![E]\!]$

$$
\begin{array}{rcl}
[\![B]\!] & = & \text{see } \textit{Base bisemigroup} \text{ slide} \\
[\![\texttt{add\_zero}(c,\ E\ )]\!] & = & \mathrm{add\_zero}(c,\ [\![E]\!]) \\
[\![\texttt{add\_one}(c,\ E\ )]\!] & = & \mathrm{add\_one}(c,\ [\![E]\!]) \\
[\![\texttt{right}(\ E\ )]\!] & = & \mathrm{right}([\![E]\!]) \\
[\![\texttt{left}(\ E\ )]\!] & = & \mathrm{left}([\![E]\!]) \\
[\![(\ E_1\ \texttt{times}\ E_2\ )]\!] & = & ([\![E_1]\!] \times [\![E_2]\!]) \\
[\![(\ E_1\ \texttt{lex}\ E_2\ )]\!] & = & ([\![E_1]\!] \vec{\times} [\![E_2]\!])
\end{array}
$$

Definitions of $\mathrm{WF}(E)$

$$
\begin{array}{rcl}
\mathrm{WF}(B) & = & \text{true} \\
\mathrm{WF}(\texttt{add\_zero}(c,\ E\ )) & = & \mathrm{WF}(E) \\
\mathrm{WF}(\texttt{add\_one}(c,\ E\ )) & = & \mathrm{WF}(E) \\
\mathrm{WF}(\texttt{right}(\ E\ )) & = & \mathrm{WF}(E) \\
\mathrm{WF}(\texttt{left}(\ E\ )) & = & \mathrm{WF}(E) \\
\mathrm{WF}((\ E_1\ \texttt{times}\ E_2\ )) & = & \mathrm{WF}(E_1) \wedge \mathrm{WF}(E_2) \\
\mathrm{WF}((\ E_1\ \texttt{lex}\ E_2\ )) & = & \mathrm{WF}(E_1) \wedge \mathrm{WF}(E_2) \wedge \\
 & & \mathrm{CA}([\![E_1]\!]) \wedge \mathrm{IA}([\![E_1]\!]) \wedge \\
 & & (\mathrm{SA}([\![E_1]\!]) \vee \mathrm{EZ}([\![E_2]\!]))
\end{array}
$$

Proving Soundness?

By induction on the structure of E.

This might be on the next problem set ....

Marching towards closure ...

Once we have fixed $\mathcal{P}_0$, grammar for expressions E, definitions of $[\![E]\!]$ and $\textsc{wf}(E)$.

- $\mathcal{P} := \mathcal{P}_0$
- For each $\mathtt{Q} \in \mathcal{P}$ and each construction $\mathrm{op}\,(E_1, \cdots E_k)$ in the language, attempt to construct a boolean formula F such that

$$\textsc{wf}(\mathrm{op}\,(E_1, \cdots E_k)) \implies (\mathtt{Q}([\![\mathrm{op}\,(E_1, \cdots E_k)]\!]) \iff F([\![E_1]\!], \cdots [\![E_k]\!]$$

- if no new properties are required, then stop.
- otherwise, add the new properties to $\mathcal{P}$ and continue.

Good Luck!

Closing $\mathcal{P}_0$ introduces auxiliary properties $\mathcal{P}_1$

Properties $\mathcal{P}_1$

Property	Definition
LC	$\forall a, b, c : c \otimes a = c \otimes b \implies a = b$
RC	$\forall a, b, c : a \otimes c = b \otimes c \implies a = b$
LK	$\forall a, b, c : c \otimes a = c \otimes b$
RK	$\forall a, b, c : a \otimes c = b \otimes c$

And closing $\mathcal{P}_1$ introduces auxiliary properties $\mathcal{P}_2$.

$\mathcal{P}_2$

Property	Definition
NOTLEFT	$\forall a, b, : a \otimes b \neq a$
NOTRIGHT	$\forall a, b, : a \otimes b \neq b$

Outline

Bibliography I

[NG10] Vilius Naudžiūnas and Timothy G. Griffin.
A small language for generating semirings.
Work in Progress, 2010.