An Algebraic Approach to Internet Routing
Lectures 05 and 06

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Outline

1 Lecture 05: A closer look at the lexicographic product

2 Lecture 06: A gentle introduction to Metarouting

3 Bibliography
Revisit Lexicographic Semiring

[Lex Product Theorem] Assume \( \oplus_S \) is commutative and idempotent. Then

\[
\text{LD}(S \ctimes T) \iff \text{LD}(S) \land \text{LD}(T) \land (\text{LC}(S) \lor \text{LK}(T))
\]

But wait! How could any semiring satisfy either of these properties?

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC</td>
<td>( \forall a, b, c : c \otimes a = c \otimes b \Rightarrow a = b )</td>
</tr>
<tr>
<td>LK</td>
<td>( \forall a, b, c : c \otimes a = c \otimes b )</td>
</tr>
</tbody>
</table>

- For LC, note that we always have \( \overline{0} \otimes a = \overline{0} \otimes b \), so LC could only hold when \( S = \{ \overline{0} \} \).
- For LK, let \( a = \overline{1} \) and \( b = \overline{0} \) and LK leads to the conclusion that every \( c \) is equal to \( \overline{0} \) (again!). Thanks to Ramana Kumar for pointing this out!

My mistake! The theorem above was formulated in the context of a much more liberal algebraic setting [Sai70, GG07, Gur08] and I should not have introduced it in the context of semirings.

Bisemigroups – a more liberal setting

\((S, \oplus, \otimes)\) is a **bisemigroup** when

- \( \oplus \) is a associative
- \( \otimes \) is a associative

Each semiring properties may, or may not, hold

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>COMM(\oplus)</td>
<td>( \forall a, b : a \oplus b = b \oplus a )</td>
</tr>
<tr>
<td>(\exists \overline{0})</td>
<td>( \exists \overline{0} : \forall a : a \oplus \overline{0} = \overline{0} \oplus a = a )</td>
</tr>
<tr>
<td>(\exists \overline{1})</td>
<td>( \exists \overline{1} : \forall a : a \otimes \overline{1} = \overline{1} \otimes a = a )</td>
</tr>
<tr>
<td>(\text{ANN}\overline{0})</td>
<td>( \forall a : a \otimes \overline{0} = \overline{0} \otimes \overline{0} = \overline{0} )</td>
</tr>
<tr>
<td>LD</td>
<td>( \forall a, b, c : c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b) )</td>
</tr>
<tr>
<td>RD</td>
<td>( \forall a, b, c : (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) )</td>
</tr>
</tbody>
</table>
Some bisemigroups (that are not semirings)

<table>
<thead>
<tr>
<th>name</th>
<th>S</th>
<th>$\oplus$</th>
<th>$\otimes$</th>
<th>$\bar{0}$</th>
<th>$\bar{1}$</th>
<th>possible routing use</th>
</tr>
</thead>
<tbody>
<tr>
<td>min_plus</td>
<td>$\mathbb{N}$</td>
<td>$\min$</td>
<td>$+$</td>
<td>0</td>
<td></td>
<td>minimum-weight routing</td>
</tr>
<tr>
<td>$\text{left}(W)$</td>
<td>$2^W$</td>
<td>$\cup$</td>
<td>$\text{left}$</td>
<td>${\cdot}$</td>
<td></td>
<td>compute next-hop(s)</td>
</tr>
<tr>
<td>$\text{right}(W)$</td>
<td>$2^W$</td>
<td>$\cup$</td>
<td>$\text{right}$</td>
<td>${\cdot}$</td>
<td></td>
<td>compute origin(s)</td>
</tr>
</tbody>
</table>

Operation for inserting a zero

Suppose $\bar{0} \notin S$

\[
\text{add}_\text{zero}(\bar{0}, (S, \oplus, \otimes)) = (S \cup \{\bar{0}\}, \hat{\oplus}, \hat{\otimes})
\]

where

\[
\begin{align*}
\hat{\oplus}b &= \begin{cases} 
a & \text{(if } b = \bar{0} \text{)} 
b & \text{(if } a = \bar{0} \text{)} 
a \oplus b & \text{(otherwise)} \end{cases} \\
\hat{\otimes}b &= \begin{cases} 
\bar{0} & \text{(if } b = \bar{0} \text{)} 
\bar{0} & \text{(if } a = \bar{0} \text{)} 
a \otimes b & \text{(otherwise)} \end{cases}
\end{align*}
\]

\[
sp = \text{add}_\text{zero}(\infty, \text{min}_\text{plus}).
\]

In previous lecture, when I wrote $sp \not\geq bw$ it should have been

\[
\text{add}_\text{zero}(\infty, \text{min}_\text{plus} \not\geq bw)
\]
Operation for inserting a one

Suppose \( \overline{1} \notin S \)

\[
\text{add\_one}(\overline{1}, (S, \oplus, \otimes)) = (S \cup \{\overline{1}\}, \hat{\oplus}, \hat{\otimes})
\]

where

\[
a \hat{\oplus} b = \begin{cases} 
\overline{1} & \text{if } b = \overline{1} \\
\overline{1} & \text{if } a = \overline{1} \\
a \oplus b & \text{otherwise}
\end{cases}
\]

\[
a \hat{\otimes} b = \begin{cases} 
a & \text{if } b = \overline{1} \\
b & \text{if } a = \overline{1} \\
a \otimes b & \text{otherwise}
\end{cases}
\]

next hop semiring

For graph \( G = (V, E) \), let \( \text{nh} = \text{add\_one}(\text{self, left}(V)) \). To use, label each arc \((u, v) \in E\) as \( w(u, v) = \{v\} \).

Prove \( \text{LD}(S) \land \text{LD}(T) \land (\text{LC}(S) \lor \text{LK}(T)) \implies \text{LD}(S \times T) \)

Assume \( S \) and \( T \) are bisemigroups, \( \text{LD}(S) \land \text{LD}(T) \land (\text{LC}(S) \lor \text{LK}(T)) \), and

\[(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T.\]

Then (dropping operator subscripts for clarity) we have

\[
\text{lhs} = (s_1, t_1) \otimes ((s_2, t_2) \hat{\otimes} (s_3, t_3))
\]
\[
= (s_1, t_1) \otimes (s_2 \oplus s_3, t_{\text{lhs}})
\]
\[
= (s_1 \otimes (s_2 \oplus s_3), t_1 \otimes t_{\text{lhs}})
\]

\[
\text{rhs} = ((s_1, t_1) \otimes (s_2, t_2)) \hat{\oplus} ((s_1, t_1) \otimes (s_3, t_3))
\]
\[
= (s_1 \otimes s_2, t_1 \otimes t_2) \hat{\oplus} (s_1 \otimes s_3, t_1 \otimes t_3)
\]
\[
= ((s_1 \otimes s_2) \oplus_S (s_1 \otimes s_3), t_{\text{rhs}})
\]
\[
= (s_1 \otimes (s_2 \oplus s_3), t_{\text{rhs}})
\]

where \( t_{\text{lhs}} \) and \( t_{\text{rhs}} \) are determined by the definition of \( \hat{\oplus} \).

We need to show that \( \text{lhs} = \text{rhs} \), that is \( t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}} \).
Case 1: $\text{LC}(S)$

Note that from $\text{LCNZ}(S)$ we have

\[ (*) \quad \forall a, b, c : a \neq b \implies c \times a \neq c \times b \]

There are four sub-cases to consider.

Case 1.1: $s_2 = s_2 \oplus s_3 = s_3$. Then $t_{\text{lhs}} = t_2 \oplus t_3$ and
\[ t_1 \times t_{\text{lhs}} = t_1 \times (t_2 \oplus t_3) = (t_1 \times t_2) \oplus (t_1 \times t_3), \] by $\text{LD}(S)$. Also,
\[ s_1 \times s_2 = s_1 \times s_3 \text{ and } s_1 \times s_2 = s_1 \times (s_2 \oplus s_3) = (s_1 \times s_2) \oplus (s_1 \times s_3), \]
again by $\text{LD}(S)$. Therefore $t_{\text{rhs}} = (t_1 \times t_2) \oplus (t_1 \times t_3) = t_1 \times t_{\text{lhs}}$.

Case 1.2: $s_2 = s_2 \oplus s_3 \neq s_3$. Then $t_1 \times t_{\text{lhs}} = t_1 \times t_2$ Also
\[ s_2 = s_2 \oplus s_3 \implies s_1 \times s_2 = s_1 \times (s_2 \oplus s_3) \text{ and by } * \]
\[ s_2 \oplus s_3 \neq s_3 \implies s_1 \times (s_2 \oplus s_3) \neq s_1 \times s_3. \] Thus, by $\text{LD}(S)$,
\[ (s_1 \times s_2) \oplus (s_1 \times s_3) \neq s_1 \times s_3 \text{ and we get } t_{\text{rhs}} = t_1 \times t_2 = t_1 \times t_{\text{lhs}}. \]

Case 1: $\text{LC}(S)$ (continued)

Case 1.3: $s_2 \neq s_2 \oplus s_3 = s_3$. Similar to case 1.2.

Case 1.4: $s_2 \neq s_2 \oplus s_3 \neq s_3$. Then $t_{\text{lhs}} = \bar{0}$ and $t_1 \times t_{\text{lhs}} = \bar{0}$. Using $*$ (twice), we have $s_1 \times s_2 \neq (s_1 \times s_2) \oplus (s_1 \times s_3) \neq s_1 \times s_3$, so $t_{\text{rhs}} = \bar{0}$. 

\[ \begin{array}{c} \text{T. Griffin (dlcmacuka)} \quad \text{An Algebraic Approach to Internet Routing LC} \quad \text{T.G.Griffin} @ 2010 \quad 9 / 21 \end{array} \]
Case 2: $\text{LK}(T)$

Proving this case is problem 1 for problem set 2.

Necessary condition for left distributivity?

How about this?

$$\text{LD}(S \boxtimes T) \implies \text{LD}(S) \land \text{LD}(T) \land (\text{LC}(S) \lor \text{LK}(T))$$

Problem: does not (directly) give a “bottom up” method of constructing counter examples.
Alternative

**Theorem**

\[ \text{NLD}(S) \lor \text{NLD}(T) \lor (\text{NLC}(S) \land \text{NLK}(T)) \implies \text{NLD}(S \boxtimes T) \]

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<td>NLD</td>
<td>( \exists a, b, c : c \otimes (a \oplus b) \neq (c \otimes a) \oplus (c \otimes b) )</td>
</tr>
<tr>
<td>NLC</td>
<td>( \exists a, b, c : c \otimes a = c \otimes b \land a \neq b )</td>
</tr>
<tr>
<td>NLK</td>
<td>( \exists a, b, c : c \otimes a \neq c \otimes b )</td>
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Proving this is problem 2 for problem set 2. For additional credit, show clearly how counter examples to \( \text{LD}(S \boxtimes T) \) can be constructed.

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1. Lecture 05: A closer look at the lexicographic product

2. Lecture 06: A gentle introduction to Metarouting

3. Bibliography
Define a little language (syntax!) $\mathcal{L}$ for bisemigroups.

\[ E ::= \cdots \]

with semantics

\[ \llbracket E \rrbracket = (S, \oplus, \otimes). \]

- Let $\mathcal{P}$ be the set of properties that we need or care about (yes, this is vague). We assume that for each property $Q \in \mathcal{P}$ there is a property $\neg Q \in \mathcal{P}$ where $\neg (Q \land \neg Q)$ holds.
- We may need a well-formedness predicate on language expressions, $\text{wf}(E)$.

Now for the hard part ...

**Closure**

The language $\mathcal{L}$ is closed w.r.t $\mathcal{P}$ if

\[ \forall Q \in \mathcal{P} : \forall E \in \mathcal{L} : \text{wf}(E) \implies (Q(\llbracket E \rrbracket) \lor \neg Q(\llbracket E \rrbracket)) \]

holds constructively.

**The Research Challenge**

Define $\mathcal{L}$, $\mathcal{P}$, and $\text{wf}(E)$ is such a way that

- $\mathcal{L}$ is expressive enough to model Internet protocols and more ...
- $\mathcal{L}$ is closed with respect to $\mathcal{P}$
The approach — bottom up construction of \( Q([A]) \lor \neg Q([A]) \)

For example, with \( S \times T \) we have

\[
LD(S) \lor LD(T) \lor (LC(S) \land LK(T)) \implies LD(S \times T)
\]

\[
NLD(S) \lor NLD(T) \lor (NL(C(S) \land NLK(T)) \implies NLD(S \times T)
\]

The ability to do this cleanly may hinge on the details!!

Example: suppose we make the mistake of defining Lexicographic Product of Semigroups this way....

**Definition \( (\times_0) \)**

Suppose \((S, \oplus_S, \bar{0}_S)\) is commutative idempotent monoid and \((T, \oplus_T, \bar{0}_T)\) is a monoid. The **lexicographic product with zero** is defined as the monoid

\[
(S, \oplus_S) \times_0 (T, \oplus_T) \equiv (((S - \{\bar{0}_S\}) \times T) \cup \{\bar{0}\}, \oplus_0, \bar{0})
\]

where \(\bar{0}\) is the identity for \(\oplus_0\) and

\[
(s_1, t_1) \oplus_0 (s_2, t_2) = \begin{cases} (s_1 \oplus_S s_2, t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2, t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\ (s_1 \oplus_S s_2, t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2, \bar{0}_T) & \text{otherwise.} \end{cases}
\]
The problem ...

If we restrict ourselves to Semirings, then our new lexicographic product requires rules such as:

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</tr>
<tr>
<td>LCNZ</td>
<td>( \forall a, b, c : (c \neq \emptyset \land c \otimes a = c \otimes b) \implies a = b )</td>
</tr>
<tr>
<td>LKNZ</td>
<td>( \forall a, b, c : (a \neq \emptyset \land b \neq \emptyset) \implies c \otimes a = c \otimes b )</td>
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</table>

These are very hard to work with!

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1. Lecture 05: A closer look at the lexicographic product
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3. Bibliography
Lexicographic products in metarouting.

Designing routing algebras with meta-languages.

[Sai70] Tôru Saitô.
Note on the lexicographic product of ordered semigroups.