We know \((a, b) = (b, a) \Rightarrow a = b\) for pairs.

so why not lift the result to set product?

Theorem \((A \times B = B \times A) \Rightarrow A = B\)

Proof?

The first components of the pairs in \(A \times B\) are from \(A\).

The first components of the pairs in \(B \times A\) are from \(B\).

If \(A \times B = B \times A\) then these must be the same, so \(A = B\).

Theorem \((A \times B = B \times A) \Rightarrow A = B\)

Proof?

1. Assume \(A \times B = B \times A\)

We prove \(A = B\), i.e. \(\forall x. x \in A \Leftrightarrow x \in B\)

2. Consider an arbitrary \(x\).

3. Assume \(x \in A\).

4. Consider an arbitrary \(y \in B\).

5. \((x,y) \in A \times B\) by defn \(\times\)

6. \((x,y) \in B \times A\) by 1

7. \(x \in B\) by defn \(\times\)

8. \(x \in A \Rightarrow x \in B\) from 3–7 by \(\Rightarrow\)-introduction

9. The proof of the \(\Leftarrow\) implication is symmetric

10. \(\forall x. x \in A \Leftrightarrow x \in B\) from 2–9 by \(\forall\)-introduction

Theorem \((A \times B = B \times A) \land (\exists x. x \in A) \land (\exists y. y \in B) \Rightarrow A = B\)

Proof

1. Assume \(A \times B = B \times A\) \land (\exists x. x \in A) \land (\exists y. y \in B)

1a. \(A \times B = B \times A\) from 1 by \&-elimination

1b. (\exists x. x \in A) from 1 by \&-elimination

1c. (\exists y. y \in B) from 1 by \&-elimination

We prove \(A = B\), i.e. \(\forall x. x \in A \Leftrightarrow x \in B\)

2. Consider an arbitrary \(x\).

3. Assume \(x \in A\).

4. We have actual \(y \in B\) from 1c by \exists-elimination

5. \((x,y) \in A \times B\) by defn \(\times\)

6. \((x,y) \in B \times A\) by 1a

7. \(x \in B\) by defn \(\times\)

8. \(x \in A \Rightarrow x \in B\) from 3–7 by \(\Rightarrow\)-introduction

9. The proof of the \(\Leftarrow\) implication is symmetric

10. \(\forall x. x \in A \Leftrightarrow x \in B\) from 2–9 by \(\forall\)-introduction

Theorem \((A \times B = B \times A) \land (\exists x. x \in A) \land (\exists y. y \in B) \Rightarrow A = B\)

or equivalently

Theorem \((A \times B = B \times A) \Rightarrow A = B \lor A = \emptyset \lor B = \emptyset\)

using \(((P \land R) \Rightarrow Q) \iff (P \Rightarrow Q \lor R)\) and De Morgan