RECAP

$\Gamma \vdash M : \tau$

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!] \qquad (*)$$

$$[\![x_1 : \tau_1, \ldots, x_n : \tau_n]\!] = [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] .$$

defined by structural induction on terms.

The continuity of $(*)$ is proved by induction on terms.

$$[\![x_1:\tau_1, \ldots, x_n:\tau_n \vdash x_i : \tau_i]\!]$$

$$: [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] \longrightarrow [\![\tau_i]\!]$$

$$(d_1, \ldots\ldots, d_n) \longmapsto d_i$$

The projection functions.

$$\pi_i : D_1 \times \cdots \times D_n \to D_i \qquad (1 \le i \le n)$$

are continuous.

$$[\![\Gamma \vdash \underline{\text{succ}}(M)]\!] : [\![\Gamma]\!] \longrightarrow \mathbb{N}_\perp$$

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \longrightarrow \mathbb{N}_\perp$$

$$succ \circ [\![\Gamma \vdash M]\!]$$

where $succ : \mathbb{N}_\perp \to \mathbb{N}_\perp$

$$succ(x) = \begin{cases} \perp & \text{if } x = \perp \\ x+1 & \text{otw} \end{cases}$$

$$\circ : (E \to F) \times (D \to E) \longrightarrow (D \to F)$$

D, E, F domains.

is continuous.

$$\Downarrow$$

$$(f, g \text{ continuous} \Rightarrow g \circ f \text{ continuous}).$$

$$[\![\Gamma \vdash \mathrm{zero}(M)]\!] : [\![\Gamma]\!] \longrightarrow \mathbb{B}_\perp$$

$$=$$

$$\mathrm{zero} \circ [\![\Gamma \vdash M]\!]$$

<u>Monotonicity</u>

$$x \sqsubseteq y \overset{?}{\Rightarrow} (g \circ f)x \sqsubseteq (g \circ f)(y)$$

$$\Downarrow$$

$fx \sqsubseteq fy$ by mon. of f

$$\Downarrow$$

$gfx \sqsubseteq gfy$ by mon. of g

Continuity

$$\bigsqcup_n (g \circ f)(x_n) \stackrel{?}{=} (g \circ f)(\bigsqcup x_n)$$

$$\shortparallel$$

$$g(f(\bigsqcup x_n))$$

$$\shortparallel$$

$$g(\bigsqcup_n f x_n) \quad f \text{ cont.}$$

$$\shortparallel \quad g \text{ cont.}$$

$$\bigsqcup_n g f(x_n))$$

$$[\![\Gamma \vdash M_1 M_2]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau_2]\!]$$

$$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \longrightarrow ([\![\tau_1]\!] \to [\![\tau_2]\!])$$

$$[\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau_1]\!]$$

$$f : D \longrightarrow (E \to F)$$

$$g : D \longrightarrow E$$

Given $f : D \longrightarrow P$

and $g : D \longrightarrow Q$

define $\langle f, g \rangle : D \to P \times Q$

$$\langle f, g\rangle(d) \stackrel{\text{def}}{=} (f d, g d)$$

If f & g are cont. then $\langle f, g\rangle$ cont.

$$\langle f, g\rangle : D \to (E \to F) \times E$$

Define $\mathit{eval} : (E \to F) \times E \longrightarrow F$

$$\mathit{eval}\,(f, e) = f(e).$$

eval is continuous.

$$\begin{aligned}&[\![\Gamma \vdash M_1\, M_2]\!] \\ &\quad \| \\ &\quad \mathit{eval} \circ \langle [\![\Gamma \vdash M_1]\!], [\![\Gamma \vdash M_2]\!] \rangle\end{aligned}$$

$$[\![\Gamma \vdash \mathrm{fn}\, x \,.\, M]\!] : [\![\Gamma]\!] \to ([\![\tau_1]\!] \to [\![\tau_2]\!])$$

$$[\![\Gamma, x:\tau \vdash M]\!] : [\![\Gamma]\!] \times [\![\tau_1]\!] \longrightarrow [\![\tau_2]\!]$$

$$\hat{f} : D \to (E \to F)$$

$$f : D \times E \longrightarrow F$$

$$\hat{(-)} : (D \times E \to F) \longrightarrow (D \to (E \to F))$$

$$h \longmapsto \hat{h} = \lambda d . \lambda e . h(d,e)$$

is continuous.

$$= [\![\Gamma, x:\tau \vdash M]\!]^{\wedge}$$

$$[\![\Gamma \vdash \underline{fix}(M)]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$$

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \to [\![\tau]\!])$$

$$\underline{fix} : (D \to D) \longrightarrow D \quad \text{cont.}$$

$$= \underline{fix} \circ [\![\Gamma \vdash M]\!] .$$

$$[\![\vdash M]\!] : [\![\emptyset]\!] \longrightarrow [\![\tau]\!]$$

$$[\![\emptyset]\!] = \{\perp\}$$

$$[\![M]\!] \in [\![\tau]\!]$$

Soundness

- $$\frac{}{V \Downarrow V}$$

- $$\frac{M\,(\underline{fix}\; M) \Downarrow V}{\underline{fix}(M) \Downarrow V}$$

by ind. $[\![M\,(\underline{fix}\; M)]\!] = [\![V]\!]$

$$\begin{aligned}
&[\![M\,(\underline{fix}\; M)]\!] \\
&\quad\| \\
&[\![M]\!]\,([\![\underline{fix}\; M]\!]) \\
&\quad\| \\
&[\![M]\!]\,(\underline{fix}([\![M]\!])) \\
&\quad\| \\
&\underline{fix}\; [\![M]\!] \\
&\quad\| \\
&[\![\underline{fix}\; M]\!]
\end{aligned}$$

- $$\frac{M_1 \Downarrow \underline{fn}\, x.\, M \qquad M[{}^{M_2}/x] \Downarrow V}{M_1\, M_2 \Downarrow V}$$

$$[\![M_1\ M_2]\!] = [\![M_1]\!]\left([\![M_2]\!]\right)$$

$$[\![M_1]\!] \overset{ind}{=} [\![fn\, x.\, M]\!] = [\![x \vdash M]\!]$$

$$[\![M_1 M_2]\!]$$

$$= [\![x \vdash M]\!]\left([\![M_2]\!]\right)$$

$$\| \sim \text{lemma.}$$

$$[\![M[{}^{M_2}/x]]\!] = [\![V]\!]$$