Scott induction

$$\frac{x \in S \Rightarrow f(x) \in S}{\underline{\text{fix}}(f) \in S} \qquad \begin{array}{l} f : D \to D \\ S \subseteq D \\ \text{admissible} \end{array}$$

Scott induction $\Rightarrow$ lfp property 2

Admissible

= chain closed

\+ contains $\perp$

if $x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$ in S

then $\bigsqcup_n x_n \in S$

Find $S \subseteq D$ admissible

$\underline{fix}(-) \in S \iff \underline{fix} f \sqsubseteq fix(g)$

?

Idea 1: f $\downarrow(\underline{fix}\, g)$

$$\frac{x \sqsubseteq \underline{fix}(g) \overset{?}{\Rightarrow} f(x) \sqsubseteq \underline{fix}(g)}{\underline{fix}(f) \sqsubseteq \underline{fix}(g)} \qquad S = \downarrow(\underline{fix}\, g)$$

$$x \sqsubseteq \underline{fix}(g) \Longrightarrow f(x) \sqsubseteq f(\underline{fix}\, g)$$

$$= f(g\ \underline{fix}\ g)$$

$$\sqsubseteq g\,(f(\underline{fix}\, g))$$

Idea 2:

$$D \xrightarrow{\langle f, g\rangle} D \times D$$

defined as $\langle f, g\rangle(x) = (fx, gx)$

Since f and g are continuous then so is $\langle f, g\rangle$.

$\sqsubseteq$ is an admissible subset of $D \times D$

So $\langle f, g\rangle^{-1}(\sqsubseteq)$ is chain closed

$$= \{x \in D \mid \langle f, g\rangle x \in \sqsubseteq\}$$

$$\parallel \{x \in D \mid fx \sqsubseteq gx\}$$

Assuming $f\bot \sqsubseteq g\bot$

$$S = \{x \in D \mid fx \sqsubseteq gx\}$$

is a admissible subset of D.

$$\begin{array}{ccc} fx \sqsubseteq gx & & f(gx) \sqsubseteq g(gx) \\ \parallel\!\mid & & \parallel\!\mid \\ x \in S & \overset{?}{\Rightarrow} & g(x) \in S \end{array}$$

$$\overline{\qquad \underline{\mathrm{fix}}(g) \in S \qquad}$$

This implication holds

so $\underline{\mathrm{fix}}(g) \in S$

$$\equiv \; f(\underline{\mathrm{fix}}\, g) \sqsubseteq g(\underline{\mathrm{fix}}\, g) = \underline{\mathrm{fix}}(g)$$

So $\underline{\mathrm{fix}}(g)$ is a prefixed point of f

and since $\underline{\mathrm{fix}}(f)$ is the least such

then $\underline{\mathrm{fix}}(f) \sqsubseteq \underline{\mathrm{fix}}(g)$.

Idea 3: Consider

$$f \times g : D \times D \to D \times D$$

defined as

$$(f \times g)(x, y) = (fx, gy)$$

$\sqsubseteq$ is a admissible subset of $D \times D$

and $f \times g$ is continuous.

So $(f \times g)^{-1}(\sqsubseteq)$ is chain closed

$$\begin{array}{c} \parallel \\ \{(x,y) \in D \times D \mid fx \sqsubseteq gy\} \end{array} \qquad (*)$$

Exercise Prove the result by Scott induction using (*) as admissible subset.

Considering $f \times g : D \times D \to D \times D$.

[You'll need to show

$$\mathit{fix}(f \times g) = (\mathit{fix}\, f, \mathit{fix}\, g) \,.]$$

PCF

Contexts

$$\Gamma \equiv (x_1 : \tau_1, \ldots, x_n : \tau_n)$$

$$\Gamma[x \mapsto \tau] \equiv (x_1 : \tau_1, \ldots, x_n : \tau_n, x : \tau)$$

$$\frac{x : \tau \vdash M : \tau'}{\emptyset \vdash \underline{\mathrm{fn}}\ x : \tau . M : \tau \to \tau'}$$

$$[\![\Gamma \vdash M : \tau]\!]$$

$h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ $\qquad$ $f : \mathbb{N} \to \mathbb{N}$

$\hat{h} : \underline{nat} \to \underline{nat} \to nat$

$\hat{f} : \underline{nat} \to \underline{nat}$

$\hat{g} : \underline{nat} \to \underline{nat} \to \underline{nat} \to \underline{nat}$

$\hat{h} = \underline{fix}\Big(\qquad$ PCF code $\qquad \Big)$

$\underline{fix}\Big(\lambda h.\, \lambda x.y.\ \underline{if}\ y = 0\ \underline{then}\ f(x)$
$\qquad \underline{else}\ g(x, y-1, h(x, y-1))\Big)$

Minimisation.

$$m(x) = m'(x,0)$$

$$m'(x,y) = \underline{\text{if}}\ k(x,y)=0\ \underline{\text{then}}\ y\ \underline{\text{else}}\ m'(x,y+1)$$

$$m(x) = \underline{\text{fix}}\big(\lambda m'.\,\lambda x.\,\lambda y.\ \underline{\text{if}}\ k(x,y)=0\ \underline{\text{then}}\ y\ \underline{\text{else}}\ m'(x,y+1)\big)\,x\,(0)$$