Cpos
= posets
+ lubs of ω-chains.

domains
= cpos with least element.

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lubs in $(X \rightharpoonup Y), \subseteq$

$$f_0 \subseteq f_1 \subseteq \cdots \subseteq f_n \subseteq \cdots$$

$$\underline{dom}(f_0) \subseteq \underline{dom}(f_1) \subseteq \cdots \underline{dom}(f_n) \subseteq \cdots$$

$\bigsqcup_n f_n$ has domain of definition

$$\bigcup_n \underline{dom}(f_n)$$

$\exists k.\ x \in \underline{dom}(f_k)$
iff
$x \in \bigcup_n \underline{dom}(f_n)$

$$\left(\bigsqcup_n f_n\right)(x) = \begin{cases} f_k(x) & x \in \bigcup_n \underline{dom}(f_n) \\ \uparrow & x \notin \bigcup_n \underline{dom}(f_n) \end{cases}$$

for $f_n \sqsubseteq \bigsqcup_n f_n$

it should happen that

$$\underline{\text{dom}}(f_n) \subseteq \underline{\text{dom}}(\bigsqcup_n f_n)$$

&

$$\forall x \in \underline{\text{dom}}(f_n).\ (\bigsqcup_n f_n)(x) = f_n(x)$$

D cpo

$d \in D$

$d \sqsubseteq d \sqsubseteq \cdots \sqsubseteq d \sqsubseteq \cdots$

$\bigsqcup_n d$ has the property

that $d \sqsubseteq \bigsqcup_n d$

&

$\forall x.\ d \sqsubseteq x \Rightarrow \bigsqcup_n d \sqsubseteq x$

so $\bigsqcup_n d = d$

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$$

$$d_N \sqsubseteq d_{N+1} \sqsubseteq d_{N+2} \sqsubseteq \cdots \sqsubseteq d_{N+k} \sqsubseteq \cdots$$

$$\Rightarrow \bigsqcup_{n \geq 0} d_n = \bigsqcup_{k \geq 0} d_{N+k}$$

We show:

(1) $\forall n \quad d_n \sqsubseteq \bigsqcup_{k \geq 0} d_{N+k}$

(2) $\forall x . (d_n \sqsubseteq x \quad \forall n)$

$$\Rightarrow \bigsqcup_{k \geq 0} d_{N+k} \sqsubseteq x$$

For (1):

$$\forall l . \quad d_{N+l} \sqsubseteq \bigsqcup_{k \geq 0} d_{N+k}$$

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_N \sqsubseteq \bigsqcup_{k \geq 0} d_{N+k}$$

For (2) assume $d_n \sqsubseteq x \ \forall n$

$$\Rightarrow \quad d_{N+k} \sqsubseteq x \quad \forall k$$

$$\Rightarrow \quad \bigsqcup_{k \geq 0} d_{N+k} \sqsubseteq x$$

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$$\begin{array}{ccccccccc}
e_0 \sqsubseteq & e_1 \sqsubseteq & \cdots & \sqsubseteq e_n \sqsubseteq & \cdots & \bigsqcup_n e_n \\
\sqcup\!\!\mid & \sqcup\!\!\mid & \cdots & \sqcup\!\!\mid & \cdots \Rightarrow & \sqcup\!\!\mid \\
d_0 \sqsubseteq & d_1 \sqsubseteq & \cdots & \sqsubseteq d_n \sqsubseteq & \cdots & \bigsqcup_n d_n
\end{array}$$

$$\cfrac{\cfrac{}{d_n \sqsubseteq e_n} \qquad \cfrac{}{e_n \sqsubseteq \bigsqcup_n e_n}}{\cfrac{\forall n \ d_n \sqsubseteq \bigsqcup_n e_n}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}}$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & \bigsqcup_n\left(\bigsqcup_m d_{mn}\right) \\
\bigsqcup_m d_{m0} & \sqsubseteq & \bigsqcup_m d_{m1} & \sqsubseteq & \cdots & \sqsubseteq & \bigsqcup_m d_{mn} & \sqsubseteq & \cdots & & & & \parallel \\
 & & & & & & & & & \bigsqcup_k d_{kk} & = & & \bigsqcup_m\left(\bigsqcup_n d_{mn}\right) \\
 & & & & & & & & & & & & \sqcup\!\!\mid \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid \\
d_{m,0} & \sqsubseteq & d_{m,1} & \sqsubseteq & d_{m,2} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,n} & \sqsubseteq & \cdots & & \bigsqcup_n d_{m,n} \\
\sqcup\!\!\mid & & & & & & & & & & & & \sqcup\!\!\mid \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid \\
d_{1,0} & \sqsubseteq & d_{1,1} & \sqsubseteq & d_{12} & \sqsubseteq & \cdots & \sqsubseteq & d_{1n} & \sqsubseteq & \cdots & & \bigsqcup_n d_{1,n} \\
\sqcup\!\!\mid & & \sqcup\!\!\mid & & \sqcup\!\!\mid & & \cdots & & \sqcup\!\!\mid & & & & \sqcup\!\!\mid \\
d_{0,0} & \sqsubseteq & d_{0,1} & \sqsubseteq & d_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & d_{0,n} & \sqsubseteq & \cdots & & \bigsqcup_n d_{0n}
\end{array}
$$

D, E cpos

$f : D \to E$ cont.

if ① it is monotone

$$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y),$$

② preserves lubs of chains

$$f(\sqcup d_n) \sqsubseteq \sqcup f(d_n)$$

$$f(\sqcup_n d_n) \sqsupseteq \sqcup_n f(d_n)$$

$$d_n \sqsubseteq \sqcup_n d_n \Rightarrow f(d_n) \sqsubseteq f(\sqcup d_n)$$

$$\begin{array}{ccc}
\sqcup_n d_n & & f(\sqcup_n d_n) \\
\vdots & & \vdots \\
\sqcup\!\!| & & \sqcup\!\!| \\
d_n & & f(d_n) \\
\sqcup\!\!| & & \sqcup\!\!| \\
\vdots & & \vdots \\
\sqcup\!\!| & & \sqcup\!\!| \\
d_1 & & f(d_1) \\
\sqcup\!\!| & & \sqcup\!\!| \\
d_0 & & f(d_0) \\
D & \xrightarrow{f} & E
\end{array}$$

Tarski's FPT

D domain

$f : D \to D$ continuous

Then f has a least pre fixed point (which is in fact a fixed point)

Define $fix(f) = \bigsqcup_n f^n(\bot)$

$\bot \sqsubseteq f\bot$

$f\bot \sqsubseteq ff\bot$

$ff\bot \sqsubseteq f^3\bot$

...

PREFIXED POINT PROPERTY

$$f(\underline{fix}(f)) \overset{?}{\sqsubseteq} \underline{fix}(f)$$

$$\underline{fix}(f) = \bigsqcup_{k\geq 0} f^k(\bot)$$

$$f(\underline{fix}(f)) = f\left(\bigsqcup_n f^n(\bot)\right)$$

$$= \bigsqcup_{n\geq 0} f(f^n(\bot))$$

$$= \bigsqcup_{n\geq 1} f^n(\bot)$$

$$\bigsqcup_{k\geq 0} f^k(\bot) = \bigsqcup_{n\geq 1} f^n(\bot)$$

$$\bot \sqsubseteq f\bot \sqsubseteq f^2\bot \sqsubseteq \cdots \sqsubseteq f^k\bot \sqsubseteq \cdots \bigsqcup_{k\geq 0} f^k\bot$$

$$\bigsqcup_{k\geq 0} f^k\bot = \bigsqcup_{n\geq 1} f^n\bot$$

$$f\bot \sqsubseteq f^2(\bot) \sqsubseteq \cdots \sqsubseteq f^{k+1}(\bot) \sqsubseteq \cdots \bigsqcup_{n\geq 1} f^n\bot$$