Denotational Semantics

8–12 lectures for Part II CST 2010/11

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Course web page:

http://www.cl.cam.ac.uk/teaching/1011/DenotSem/

Lecture 1

Introduction

What is this course about?

- General area.

  Formal methods: Mathematical techniques for the specification, development, and verification of software and hardware systems.

- Specific area.

  Formal semantics: Mathematical theories for ascribing meanings to computer languages.

Why do we care?

- Rigour.

  ... specification of programming languages

  ... justification of program transformations

- Insight.

  ... generalisations of notions computability

  ... higher-order functions

  ... data structures

- Feedback into language design.

  . . . continuations

  . . . monads

- Reasoning principles.

  . . . Scott induction

  . . . Logical relations

  . . . Co-induction

Styles of formal semantics

Operational.

Meanings for program phrases defined in terms of the *steps of computation* they can take during program execution.

Axiomatic.

Meanings for program phrases defined indirectly via the *axioms and rules* of some logic of program properties.

Denotational.

Concerned with giving *mathematical models* of programming languages. Meanings for program phrases defined abstractly as elements of some suitable mathematical structure.

Basic idea of denotational semantics

$$
\begin{array}{rcl}
\text{Syntax} & \xrightarrow{[\![-]\!]} & \text{Semantics} \\
\text{Recursive program} & \mapsto & \text{Partial recursive function} \\
\text{Boolean circuit} & \mapsto & \text{Boolean function} \\
P & \mapsto & [\![P]\!]
\end{array}
$$

Concerns:

- Abstract models (*i.e.* implementation/machine independent).

  $\rightsquigarrow$ Lectures 2, 3 and 4.

- Compositionality.

  $\rightsquigarrow$ Lectures 5 and 6.

- Relationship to computation (*e.g.* operational semantics).

  $\rightsquigarrow$ Lectures 7 and 8.

Characteristic features of a denotational semantics

- Each phrase (= part of a program), P, is given a denotation, $[\![P]\!]$ — a mathematical object representing the contribution of P to the meaning of *any* complete program in which it occurs.

- The denotation of a phrase is determined just by the denotations of its subphrases (one says that the semantics is compositional).

Basic example of denotational semantics (I)

IMP^{-} syntax

Arithmetic expressions

$$A \in \mathbf{Aexp} \quad ::= \quad \underline{n} \mid L \mid A + A \mid \ldots$$

where n ranges over *integers* and

L over a specified set of *locations* $\mathbb{L}$

Boolean expressions

$$\begin{aligned} B \in \mathbf{Bexp} \quad ::= &\quad \mathbf{true} \mid \mathbf{false} \mid A = A \mid \ldots \\ \mid &\quad \neg B \mid \ldots \end{aligned}$$

Commands

$$\begin{aligned} C \in \mathbf{Comm} \quad ::= &\quad \mathbf{skip} \mid L := A \mid C ; C \\ \mid &\quad \mathbf{if}\, B\, \mathbf{then}\, C\, \mathbf{else}\, C \end{aligned}$$

Basic example of denotational semantics (II)

Semantic functions

$$\begin{aligned}
\mathcal{A}: &\quad \mathbf{Aexp} \to (\mathit{State} \to \mathbb{Z}) \\
\mathcal{B}: &\quad \mathbf{Bexp} \to (\mathit{State} \to \mathbb{B}) \\
\mathcal{C}: &\quad \mathbf{Comm} \to (\mathit{State} \rightharpoonup \mathit{State})
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{Z} &= \{\ldots, -1, 0, 1, \ldots\} \\
\mathbb{B} &= \{\mathit{true}, \mathit{false}\} \\
\mathit{State} &= (\mathbb{L} \to \mathbb{Z})
\end{aligned}$$

Basic example of denotational semantics (III)

Semantic function $\mathcal{A}$

$$\mathcal{A}[\![\underline{n}]\!] = \lambda s \in State.\, n$$

$$\mathcal{A}[\![L]\!] = \lambda s \in State.\, s(L)$$

$$\mathcal{A}[\![A_1 + A_2]\!] = \lambda s \in State.\, \mathcal{A}[\![A_1]\!](s) + \mathcal{A}[\![A_2]\!](s)$$

Basic example of denotational semantics (IV)

Semantic function $\mathcal{B}$

$$
\begin{array}{rcl}
\mathcal{B}[\![\mathbf{true}]\!] & = & \lambda s \in State.\, true \\
\mathcal{B}[\![\mathbf{false}]\!] & = & \lambda s \in State.\, false \\
\mathcal{B}[\![A_1 = A_2]\!] & = & \lambda s \in State.\, eq\big(\mathcal{A}[\![A_1]\!](s), \mathcal{A}[\![A_2]\!](s)\big)
\end{array}
$$

where $eq(a, a') = \begin{cases} true & \text{if } a = a' \\ false & \text{if } a \neq a' \end{cases}$

Basic example of denotational semantics (V)

Semantic function $\mathcal{C}$

$$[\![\mathbf{skip}]\!] \quad = \quad \lambda s \in State.\, s$$

NB: From now on the names of semantic functions are omitted!

A simple example of compositionality

Given partial functions $[\![C]\!], [\![C']\!] : \mathit{State} \rightharpoonup \mathit{State}$ and a function $[\![B]\!] : \mathit{State} \to \{\mathit{true}, \mathit{false}\}$, we can define

$$[\![\mathbf{if}\ B\ \mathbf{then}\ C\ \mathbf{else}\ C']\!] = \\ \lambda s \in \mathit{State}.\ \mathit{if}\big([\![B]\!](s), [\![C]\!](s), [\![C']\!](s)\big)$$

where

$$\mathit{if}(b, x, x') = \begin{cases} x & \text{if } b = \mathit{true} \\ x' & \text{if } b = \mathit{false} \end{cases}$$

Basic example of denotational semantics (VI)

Semantic function $\mathcal{C}$

$$[\![L := A]\!] \quad = \quad \lambda s \in State.\, \lambda \ell \in \mathbb{L}.\, if\big(\ell = L, [\![A]\!](s), s(\ell)\big)$$

Denotational semantics of sequential composition

Denotation of sequential composition $C; C'$ of two commands

$$[\![C; C']\!] = [\![C']\!] \circ [\![C]\!] = \lambda s \in State.\, [\![C']\!]\big([\![C]\!](s)\big)$$

given by composition of the partial functions from states to states $[\![C]\!], [\![C']\!] : State \rightharpoonup State$ which are the denotations of the commands.

Cf. operational semantics of sequential composition:

$$\frac{C, s \Downarrow s' \quad C', s' \Downarrow s''}{C; C', s \Downarrow s''} \; .$$

Fixed point property of $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\mathbf{while}\ B\ \mathbf{do}\ C]\!])$$

where, for each $b : \mathit{State} \to \{\mathit{true}, \mathit{false}\}$ and $c : \mathit{State} \rightharpoonup \mathit{State}$, we define

$$f_{b,c} : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$$

as

$$f_{b,c} = \lambda w \in (\mathit{State} \rightharpoonup \mathit{State}).\, \lambda s \in \mathit{State}.\, \mathit{if}\big(b(s), w(c(s)), s\big).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$?

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$
\begin{aligned}
& {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot) \\
& \quad = \quad \lambda s \in State. \\
& \qquad \begin{cases}
[\![C]\!]^{k}(s) & \text{if } \exists\, 0 \leq k < n.\, [\![B]\!]([\![C]\!]^{k}(s)) = \mathit{false} \\
& \text{and } \forall\, 0 \leq i < k.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true} \\
& \\
\uparrow & \text{if } \forall\, 0 \leq i < n.\, [\![B]\!]([\![C]\!]^{i}(s)) = \mathit{true}
\end{cases}
\end{aligned}
$$

$$D \stackrel{\text{def}}{=} (State \rightharpoonup State)$$

- **Partial order $\sqsubseteq$ on D:**

$w \sqsubseteq w'$ iff for all $s \in State$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

iff the graph of w is included in the graph of w'.

- **Least element $\perp \in D$ w.r.t. $\sqsubseteq$:**

$\perp$ = totally undefined partial function

= partial function with empty graph

(satisfies $\perp \sqsubseteq w$, for all $w \in D$).