1. In the lecture, a proof was sketched showing a $\Omega(n \log n)$ lower bound on the complexity of the sorting problem. It was also stated that a similar analysis could be used to establish the same bound for the Travelling Salesman Problem. Give a detailed sketch of such an argument. Can you think of a way to improve the lower bound?

2. Consider the language $\text{Unary-Prime}$ in the one letter alphabet $\{a\}$ defined by

$$\text{Unary-Prime} = \{a^n \mid n \text{ is prime}\}.$$ 

Show that this language is in $P$.

3. We say that a propositional formula $\phi$ is in $2\text{CNF}$ if it is a conjunction of clauses, each of which contains exactly 2 literals. The point of this problem is to show that the satisfiability problem for formulas in $2\text{CNF}$ can be solved by a polynomial time algorithm.

First note that any clause with 2 literals can be written as an implication in exactly two ways. For instance $(p \lor \neg q)$ is equivalent to $(q \rightarrow p)$ and $(\neg p \rightarrow \neg q)$, and $(p \lor q)$ is equivalent to $(\neg p \rightarrow q)$ and $(\neg q \rightarrow p)$.

For any formula $\phi$, define the directed graph $G_\phi$ to be the graph whose set of vertices is the set of all literals that occur in $\phi$, and in which there is an edge from literal $x$ to literal $y$ if, and only if, the implication $(x \rightarrow y)$ is equivalent to one of the clauses in $\phi$.

(a) If $\phi$ has $n$ variables and $m$ clauses, give an upper bound on the number of vertices and edges in $G_\phi$.

(b) Show that $\phi$ is unsatisfiable if, and only if, there is a literal $x$ such that there is a path in $G_\phi$ from $x$ to $\neg x$ and a path from $\neg x$ to $x$.

(c) Give an algorithm for verifying that a graph $G_\phi$ satisfies the property stated in (b) above. What is the complexity of your algorithm?

(d) From (c) deduce that there is a polynomial time algorithm for testing whether or not a $2\text{CNF}$ propositional formula is satisfiable.

(e) Why does this idea not work if we have 3 literals per clause?

4. A clause (i.e. a disjunction of literals) is called a Horn clause, if it contains at most one positive literal. Such a clause can be written as an implication: $(x \lor (\neg y) \lor (\neg w) \lor (\neg z))$ is equivalent to $((y \land w \land z) \rightarrow x))$. HORNSAT is the problem of deciding whether a given Boolean expression that is a conjunction of Horn clauses is satisfiable.
(a) Show that there is a polynomial time algorithm for solving HORNSAT. (Hint: if a variable is the only literal in a clause, it must be set to true; if all the negative variables in a clause have been set to true, then the positive one must also be set to true. Continue this procedure until a contradiction is reached or a satisfying truth assignment is found).

(b) In the proof of the NP-completeness of SAT it was shown how to construct, for every nondeterministic machine $M$, integer $k$ and string $x$ a Boolean expression $\phi$ which is satisfiable if, and only if, $M$ accepts $x$ within $n^k$ steps. Show that, if $M$ is deterministic, than $\phi$ can be chosen to be a conjunction of Horn clauses.

(c) Conclude from (b) that the problem HORNSAT is P-complete under L-reductions.

5. In general $k$-colourability is the problem of deciding, given a graph $G = (V,E)$, whether there is a colouring $\chi : V \rightarrow \{1, \ldots, k\}$ of the vertices such that if $(u,v) \in E$, then $\chi(u) \neq \chi(v)$. That is, adjacent vertices do not have the same colour.

(a) Show that there is a polynomial time algorithm for solving 2-colourability.

(b) Show that, for each $k$, $k$-colourability is reducible to $k + 1$-colourability. What can you conclude from this about the complexity of 4-colourability?