Review of constraint satisfaction problems (CSPs)

We have:

- A set of $n$ variables $V_1, V_2, \ldots, V_n$.
- For each $V_i$ a domain $D_i$ specifying the values that $V_i$ can take.
- A set of $m$ constraints $C_1, C_2, \ldots, C_m$.

Each constraint $C_i$ involves a set of variables and specifies an allowable collection of values.

- A state is an assignment of specific values to some or all of the variables.
- An assignment is consistent if it violates no constraints.
- An assignment is complete if it gives a value to every variable.

A solution is a consistent and complete assignment.
**Example**

We will use the problem of *colouring the nodes of a graph* as a running example.

Each node corresponds to a *variable*. We have three colours and directly connected nodes should have different colours.

*Caution required:* later on, edges will have a different meaning.
Example

This translates easily to a CSP formulation:

- The variables are the nodes
  \[ V_i = \text{node } i \]

- The domain for each variable contains the values black, red and cyan
  \[ D_i = \{B, R, C\} \]

- The constraints enforce the idea that directly connected nodes must have different colours. For example, for variables \( V_1 \) and \( V_2 \) the constraints specify
  \[ (B, R), (B, C), (R, B), (R, C), (C, B), (C, R) \]

- Variable \( V_8 \) is unconstrained.
Different kinds of CSP

This is an example of the simplest kind of CSP: it is discrete with finite domains. We will concentrate on these.

We will also concentrate on binary constraints; that is, constraints between pairs of variables.

- Constraints on single variables—unary constraints—can be handled by adjusting the variable’s domain. For example, if we don’t want $V_i$ to be red, then we just remove that possibility from $D_i$.

- Higher-order constraints applying to three or more variables can certainly be considered, but...

- ...when dealing with finite domains they can always be converted to sets of binary constraints by introducing extra auxiliary variables.

How does that work?
The state-variable representation

Another planning language: the *state-variable representation*. Things of interest such as people, places, objects *etc* are divided into *domains*:

\[
D_1 = \{\text{climber1, climber2}\} \\
D_2 = \{\text{home, jokeShop, hardwareStore, pavement, spire, hospital}\} \\
D_3 = \{\text{rope, inflatableGorilla}\}
\]

Part of the specification of a planning problem involves stating which domain a particular item is in. For example

\[
D_1(\text{climber1})
\]

and so on.

Relations and functions have arguments chosen from unions of these domains.

\[
\text{above}(x, y) \subseteq D_1^{\text{above}} \times D_2^{\text{above}}
\]

is a relation. The \(D_i^{\text{above}}\) are unions of one or more \(D_i\).
The state-variable representation

The relation above is in fact a rigid relation (RR), as it is unchanging: it does not depend upon state. (Remember fluents in situation calculus?)

Similarly, we have functions

\[ \text{at}(x_1, s) : D_1^{at} \times S \rightarrow D^{at}. \]

Here, \( \text{at}(x, s) \) is a state-variable. The domain \( D_1^{at} \) and range \( D^{at} \) are unions of one or more \( D_i \). In general these can have multiple parameters

\[ \text{sv}(x_1, \ldots, x_n, s) : D_1^{sv} \times \cdots \times D_n^{sv} \times S \rightarrow D^{sv}. \]

A state-variable denotes assertions such as

\[ \text{at}(\text{gorilla}, s) = \text{jokeShop} \]

where \( s \) denotes a state and the set \( S \) of all states will be defined later.

The state variable allows things such as locations to change—again, much like fluents in the situation calculus.

Variables appearing in relations and functions are considered to be typed.
The state-variable representation

Note:

- For properties such as a location a function might be considerably more suitable than a relation.
- For locations, everything has to be somewhere and it can only be in one place at a time.

So a function is perfect and immediately solves some of the problems seen earlier.
The state-variable representation

Actions as usual, have a name, a set of preconditions and a set of effects.

- *Names* are unique, and followed by a list of variables involved in the action.
- *Preconditions* are expressions involving state variables and relations.
- *Effects* are assignments to state variables.

For example:

<table>
<thead>
<tr>
<th>buy(x, y, l)</th>
<th>Preconditions</th>
<th>Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>at(x, s) = l</td>
<td>has(y, s) = x</td>
</tr>
<tr>
<td></td>
<td>sells(l, y)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>has(y, s) = l</td>
<td></td>
</tr>
</tbody>
</table>

\[ at(x, s) = l \]
\[ sells(l, y) \]
\[ has(y, s) = l \]
\[ has(y, s) = x \]
The state-variable representation

*Goals* are sets of *expressions* involving *state variables*.

For example:

<table>
<thead>
<tr>
<th>Goal:</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{at}(\text{climber}, s) = \text{home}</td>
</tr>
<tr>
<td>\text{has}(\text{rope}, s) = \text{climber}</td>
</tr>
<tr>
<td>\text{at}(\text{gorilla}, s) = \text{spire}</td>
</tr>
</tbody>
</table>

From now on we will generally suppress the state $s$ when writing state variables.
The state-variable representation

We can essentially regard a state as just a statement of what values the state variables take at a given time.

Formally:

- For each state variable sv we can consider all ground instances such as—sv(climber, rope)—with arguments that are consistent with the rigid relations.
  Define $X$ to be the set of all such ground instances.

- A state $s$ is then just a set

$$s = \{ (v = c) | v \in X \}$$

where $c$ is in the range of $v$.

This allows us to define the effect of an action.

A planning problem also needs a start state $s_0$, which can be defined in this way.
The state-variable representation

Considering all the ground actions consistent with the rigid relations:

• An action is applicable in s if all expressions \( v = c \) appearing in the set of preconditions also appear in s.

Finally, there is a function \( \gamma \) that maps a state and an action to a new state

\[
\gamma(s, a) = s'
\]

Specifically, we have

\[
\gamma(s, a) = \{(v = c) | v \in X\}
\]

where either \( c \) is specified in an effect of \( a \), or otherwise \( v = c \) is a member of \( s \).

Note: the definition of \( \gamma \) implicitly solves the frame problem.
The state-variable representation

A solution to a planning problem is a sequence \((a_0, a_1, \ldots, a_n)\) of actions such that...

- \(a_0\) is applicable in \(s_0\) and for each \(i\), \(a_i\) is applicable in \(s_i = \gamma(s_{i-1}, a_{i-1})\).
- For each goal \(g\) we have \(g \in \gamma(s_n, a_n)\).

What we need now is a method for transforming a problem described in this language into a CSP.

We’ll once again do this for a fixed upper limit \(T\) on the number of steps in the plan.
Converting to a CSP

Step 1: encode actions as CSP variables.

For each time step $t$ where $0 \leq t \leq T - 1$, the CSP has a variable $\text{action}^t$

with domain

$$D_{\text{action}^t} = \{a | a \text{ is the ground instance of an action}\} \cup \{\text{none}\}$$

Example: at some point in searching for a plan we might attempt to find the solution to the corresponding CSP involving

$$\text{action}^5 = \text{attach(inflatableGorilla, spire)}$$

Warning: be careful in what follows to distinguish between state variables, actions etc in the planning problem and variables in the CSP.
Converting to a CSP

**Step 2:** encode *ground state variables* as *CSP variables*, with a complete copy of all the state variables *for each time step*.

So, for each $t$ where $0 \leq t \leq T$ we have a CSP variable

$$sv_i^t(c_1, \ldots, c_n)$$

with domain $D_{sv_i}$. (That is, the *domain* of the CSP variable is the *range* of the state variable.)

**Example:** at some point in searching for a plan we might attempt to find the solution to the corresponding CSP involving

$$\text{location}^9(\text{climber1}) = \text{hospital}.$$
Converting to a CSP

Step 3: encode the preconditions for actions in the planning problem as constraints in the CSP problem.

For each time step t and for each ground action \( a(c_1, \ldots, c_n) \) with arguments consistent with the rigid relations in its preconditions:

For a precondition of the form \( sv_i = v \) include constraint pairs

\[
\text{(action}^t = a(c_1, \ldots, c_n), \quad sv_t^i = v)
\]

Example: consider the action \( \text{buy}(x, y, l) \) introduced above, and having the preconditions \( \text{at}(x) = l, \text{sells}(l, y) \) and \( \text{has}(y) = l \).

Assume \( \text{sells}(y, l) \) is only true for

\[ l = \text{jokeShop} \]

and

\[ y = \text{inflatableGorilla} \]

(it’s a very strange town) so we only consider these values for \( l \) and \( y \). Then for each time step \( t \) we have the constraints...
Converting to a CSP

\[
\begin{align*}
\text{action}^t &= \text{buy}(\text{climber1, inflatableGorilla, jokeShop}) \\
           &\quad \text{paired with} \\
           &\quad \text{at}^t(\text{climber1}) = \text{jokeShop} \\
\text{action}^t &= \text{buy}(\text{climber1, inflatableGorilla, jokeShop}) \\
           &\quad \text{paired with} \\
           &\quad \text{has}^t(\text{inflatableGorilla}) = \text{jokeShop} \\
\end{align*}
\]

\[
\begin{align*}
\text{action}^t &= \text{buy}(\text{climber2, inflatableGorilla, jokeShop}) \\
           &\quad \text{paired with} \\
           &\quad \text{at}^t(\text{climber2}) = \text{jokeShop} \\
\text{action}^t &= \text{buy}(\text{climber2, inflatableGorilla, jokeShop}) \\
           &\quad \text{paired with} \\
           &\quad \text{has}^t(\text{inflatableGorilla}) = \text{jokeShop} \\
           &\quad \text{and so on...}
\end{align*}
\]
Converting to a CSP

**Step 4:** encode the effects of actions in the planning problem as constraints in the CSP problem.

For each time step $t$ and for each ground action $a(c_1, \ldots, c_n)$ with arguments consistent with the rigid relations in its preconditions:

For an effect of the form $sv_i = v$ include constraint pairs

$$(\text{action}^t = a(c_1, \ldots, c_n),$$

$$sv_i^{t+1} = v)$$

**Example:** continuing with the previous example, we will include constraints

<table>
<thead>
<tr>
<th>$\text{action}^t = \text{buy}(\text{climber1, inflatableGorilla, jokeShop})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>paired with $\text{has}^{t+1}(\text{inflatableGorilla}) = \text{climber1}$</td>
</tr>
<tr>
<td>$\text{action}^t = \text{buy}(\text{climber2, inflatableGorilla, jokeShop})$</td>
</tr>
<tr>
<td>paired with $\text{has}^{t+1}(\text{inflatableGorilla}) = \text{climber2}$</td>
</tr>
<tr>
<td>and so on...</td>
</tr>
</tbody>
</table>
Converting to a CSP

Step 5: encode the frame axioms as constraints in the CSP problem.
An action must not change things not appearing in its effects. So:

For:

1. Each time step $t$.
2. Each ground action $a(c_1, \ldots, c_n)$ with arguments consistent with the rigid relations in its preconditions.
3. Each $sv_i$ that does not appear in the effects of $a$, and each $v \in D^{sv_i}$

include in the CSP the ternary constraint

$$(\text{action}^t = a(c_1, \ldots, c_n), \quad sv_i^t = v, \quad sv_i^{t+1} = v)$$
Finding a plan

Finally, having encoded a planning problem into a CSP, we solve the CSP. The scheme has the following property:

*A solution to the planning problem with at most T steps exists if and only if there is a solution to the corresponding CSP.*

Assume the CSP has a solution.

Then we can extract a plan simply by looking at the values assigned to the action\(^t\) variables in the solution of the CSP.

It is also the case that:

*There is a solution to the planning problem with at most T steps if and only if there is a solution to the corresponding CSP from which the solution can be extracted in this way.*

For a proof see:

*Automated Planning: Theory and Practice*