Introduction:
Contextual Equivalence

Styles of PL semantics

- Program logics
  - basis for verification
  - validated by operational/denotational semantics

- Denotational semantics
  - foundations and structure
  - involves sophisticated mathematics

- Operational semantics
  - basis for implementation
  - involves (deceptively) simple mathematics

The three approaches are inter-linked.
Styles of semantics

▶ Program logics
▶ Denotational semantics
▶ Operational semantics

Let’s compare their answers to a fundamental question:

When are two program phrases equal?
When are two program phrases semantically equal?

**Program Logic:**
when they satisfy the same logical assertions.

E.g. $C \cong C'$ iff for all pre-, post-conditions $P$, $Q$

$$\{P\} C \{Q\} \iff \{P\} C' \{Q\}$$

**Denotational semantics:**
when they have equal denotations.
When are two program phrases semantically equal?

**Program Logic:**
when they satisfy the same logical assertions.

**Denotational semantics:**
when they have equal denotations.

**Operational semantics:**
when they are contextually equivalent.

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**Contextual equivalence**

Two phrases of a programming language are ("Morris style") contextually equivalent ($\simeq_{ctx}$) if occurrences of the first phrase in any program can be replaced by the second phrase without affecting the observable results of executing the program.

We assume the programming language comes with an operational semantics as part of its definition.
Two phrases of a programming language are ("Morris style") contextually equivalent ($\equiv_{ctx}$) if occurrences of the first phrase in any program can be replaced by the second phrase without affecting the observable results of executing the program.

Different choices lead to possibly different notions of contextual equivalence.

Gottfried Wilhelm Leibniz (1646–1716): two mathematical objects are equal if there is no test to distinguish them.
Contextual equivalence

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Contextual Equivalence for HOT Programming Languages

First-class functions.
Types: higher-order, polymorphic, recursive.
+ local mutable state, modules, objects,
  concurrency, proof search, ...
Are these OCaml expressions contextually equivalent?

\[
\begin{align*}
\text{let } a &= \text{ref } n \text{ in} \\
\text{fun } x \to a &:= !a + x; \\
& \quad !a
\end{align*}
\]

\[
\begin{align*}
\text{let } b &= \text{ref}(\neg n) \text{ in} \\
\text{fun } y \to b &:= !b - y; \\
& \quad -(!b)
\end{align*}
\]

Etymology of “OCaml” (caml.inria.fr/ocaml):

ML = “Meta-language” (for LCF prover)
Are these OCaml expressions contextually equivalent?

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Etymology of “OCaml” (caml.inria.fr/ocaml):

\[\text{CAM = "Categorical Abstract Machine"} \]

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Etymology of “OCaml” (caml.inria.fr/ocaml):

\[O = "Objective" \]
Are these OCaml expressions contextually equivalent?

\[
H \triangleq \begin{array}{l}
\text{let } a = \text{ref } n \text{ in } \\
\text{fun } x \rightarrow a := !a + x; \\
\quad !a
\end{array}
\]

\[
K \triangleq \begin{array}{l}
\text{let } b = \text{ref } (-n) \text{ in } \\
\text{fun } y \rightarrow b := !b - y; \\
\quad -(!b)
\end{array}
\]

Yes, \( H \simeq_{\text{ctx}} K \), in the sense that

for all states \( s \) and all well-typed, closing contexts \( C[-] \),

\[
\exists s'. \langle s, C[H] \rangle \rightarrow^* \langle s', \text{true} \rangle \\
\iff \exists s'' . \langle s, C[K] \rangle \rightarrow^* \langle s'', \text{true} \rangle
\]
Are these OCaml expressions contextually equivalent?

\[ H \triangleq \]
let \( a = \text{ref } n \) in
fun \( x \rightarrow a := !a + x; \)
\hrule

\[ K \triangleq \]
let \( b = \text{ref } (\neg n) \) in
fun \( y \rightarrow b := !b - y; \)
\hrule

Yes, \( H \preceq_{\text{ctx}} K \), in the sense that

for all states \( s \) and all well-typed, closing contexts \( C[\_\_] \),

\[ \exists s'. \langle s, C[H] \rangle \rightarrow^* \langle s', \text{true} \rangle \]
\[ \iff \exists s''. \langle s, C[K] \rangle \rightarrow^* \langle s'', \text{true} \rangle \]

ACS L16, lecture 2 6/10

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Philosophically important:
operational behaviour is a characteristic feature of programming language semantics that distinguishes it from related areas of logic.
(Proof Theory, Model Theory, Recursion Theory)

Pragmatically important:
Contextual equivalence is used in verification of many programming language correctness properties.
(E.g. compiler optimisations, correctness of ADTs, information hiding and security properties, . . . )

What is special about HOT languages?

“When one attempts to combine language concepts, unexpected and counterintuitive interactions arise. At this point, even the most experienced designer’s intuition must be butressed by a rigorous definition of what the language means.”

John Reynolds, 1990
Why contextual equivalence matters

What is special about HOT languages?

- type-directed “laws” for contextual equivalence :-(
- higher-order types ⇒ programs can make use of constituent phrases in dynamically complicated ways :-(

\[
e_1 \cong_{\text{ctx}} e_2 : \tau \rightarrow \tau' \iff (\forall e : \tau) \, e_1 \, e \cong_{\text{ctx}} e_2 \, e : \tau'
\]
Are these OCaml expressions contextually equivalent?

\[
F \triangleq \\
\text{let } a = \text{ref()in} \\
\text{let } b = \text{ref()in} \\
\text{fun } x \rightarrow \\
\text{if } x == a \text{ then } b \\
\text{else } a
\]

\[
G \triangleq \\
\text{let } c = \text{ref()in} \\
\text{let } d = \text{ref()in} \\
\text{fun } y \rightarrow \\
\text{if } y == d \text{ then } d \\
\text{else } c
\]

No!

For \( T \triangleq \text{fun } f \rightarrow \text{let } x = \text{ref()in} f(f\ x) == f\ x, \)
\( TF \) has value \( \text{false} \), whereas \( TG \) has value \( \text{true} \), so \( F \not\sim_{ctx} G \).
Are these OCaml expressions contextually equivalent?

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\[ K \triangleq \]
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\begin{align*}
\text{let } & b = \text{ref } (-n) \text{ in} \\
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& -(!b)
\end{align*}
\]

Yes, \( H \cong_{ctx} K \), in the sense that

for all states \( s \) and all well-typed, closing contexts \( C[-] \),

\[
\exists s'. \langle s, C[H] \rangle \to^* \langle s', \text{true} \rangle \\
\Leftrightarrow \exists s''. \langle s, C[K] \rangle \to^* \langle s'', \text{true} \rangle
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How does one prove such statements?

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