Exam briefing

This module will be assessed via a two-hour written test. There will be a choice of questions on the test paper, consisting of a mixture of “essay-style” and “problem-style” questions, with more weight given to the former than the latter.

To prepare for the problem-style questions, try some of the Exercises below. The essay-style questions will ask you to write about material that was covered in the lectures. To prepare, review the lecture slides and read the recommended texts, Operational Semantics and Program Equivalence and Relational Properties of Domains. Examples of essay-style questions:

I. Describe the operational semantics of a fragment of OCaml with (at least) integer storage locations and recursive functions, both in terms of an evaluation (“big-step”) and a transition (“small-step”) relation. What is the relationship between the two styles of operational semantics? Give a type system for the language and state preservation and progress properties for the operational semantics with respect to the type system. (Proofs of these properties are not required.)

II. Let $\mathcal{D}$ be the category whose objects are $\omega$-chain complete partial orders with least element and whose morphisms are strict continuous functions. Explain what is meant by a locally continuous functor $F : \mathcal{D}^{op} \times \mathcal{D} \rightarrow \mathcal{D}$ and a minimal invariant for such an $F$. Show that a minimal invariant is a regular free di-algebra in the sense of Freyd. (Standard properties of least fixed points of continuous functions may be used without proof if clearly stated.)

Exercises


Denotational Semantics (Lectures 9-16) [Throughout, “domain” means $\omega$-chain complete partial order with least element.]
1. Suppose that $D$, $E$, and $F$ are domains. Show that a function $f : D \times E \rightarrow E$ is continuous if and only if $f(d,-) : E \rightarrow F$ and $f(-,e) : D \rightarrow F$ are continuous for all $d \in D$ and $e \in E$.

2. Let $D$ and $E$ be domains and $f : D \times E \rightarrow E$ be a continuous function. Show that there is a continuous function $Y f : D \rightarrow E$ such that for all $d \in D$, $(Y f)(d) = f(d, (Y f)(d))$.

3. Let $D$ and $E$ be domains and let $f : D \rightarrow E$ and $g : E \rightarrow D$ be continuous functions. Prove that $\text{fix}(g \circ f) = g(\text{fix}(f \circ g))$.

4. Suppose that $D$ and $E$ are domains and that $f : D \rightarrow D$ and and $g : D \times E \rightarrow E$ are continuous functions. Let $(d,e)$ be the least element of $D \times E$ satisfying

$$\begin{cases} d = f(d) \\ e = g(d, e) \end{cases}$$

Prove that $d = \text{fix}(f)$.

5. Given domains $D$ and $E$ show how to make the set $D \rightarrow E$ of continuous functions from $D$ to $E$ into a domain with the following property: for all continuous functions $f : F \times D \rightarrow E$, there is a unique continuous function $\hat{f} : F \rightarrow (D \rightarrow E)$ such that for all $z \in F$ and $x \in D$, $(\hat{f} z) x = f(z, x)$.

6. Consider the denotational semantics of the untyped $\lambda$-calculus in a domain satisfying $D \cong (D \rightarrow D)_\perp$ (lecture 10). Show that the usual law of $\eta$-expansion, $e = \lambda x. e x$ (where $x$ not free in $e$) is not satisfied up to denotational equality. Adapt the denotational semantics to use a domain $D'$ satisfying $D' \cong D' \rightarrow D'$ and show that this does satisfy $\eta$-expansion (as well as $\beta$-reduction, $(\lambda x. e) e' = e[e/x]$). How many elements does any minimal invariant solution of the domain equation $X \cong X \times X$ have?

7. Let $\Lambda_0$ be the set of closed $\lambda$-terms and let $\Rightarrow \subseteq \Lambda_0 \times \Lambda_0$ be the usual call-by-name evaluation relation inductively defined by the rules

$$\frac{}{\lambda x. e \Rightarrow \lambda x. e} \quad \frac{e_1 \Rightarrow \lambda x. e \quad e_2[x/x] \Rightarrow c}{e_1 e_2 \Rightarrow c}$$

A subset $S \subseteq \Lambda_0 \times \Lambda_0$ is called an applicative simulation if it satisfies: $(e, e') \in S$ and $e \Rightarrow \lambda x. e_1$ implies $e' \Rightarrow \lambda x. e'_1$ for some $e'_1$ such that $\forall e_2 \in \Lambda_0. (e_1[e_2/x], e'_1[e_2/x]) \in S$.

Given $e, e' \in \Lambda_0$, we write $e \preceq e'$ if there is some applicative simulation $S$ with $(e, e') \in S$. (The relation $\preceq$ is called applicative similarity.)

(a) Show that $\preceq$ is a reflexive and transitive relation.

(b) Let $\triangleleft \subseteq D \times \Lambda_0$ be the relation constructed in lecture 13 between the minimal invariant domain $D \cong (D \rightarrow D)_\perp$ and closed $\lambda$-terms. Show that $\{(e, e') \mid \llbracket e \rrbracket \triangleleft e'\}$ is an applicative simulation.

(c) Show that $d \sqsubseteq d' \triangleleft e' \preceq e$ implies $d \triangleleft e$. [Hint: use the fact that $(\triangleleft, \triangleleft)$ is the least pre-fixed point of the monotone function $\Phi^\triangleleft$ defined in lecture 13; show that $\Phi^\triangleleft(R, \triangleleft) \subseteq (R, \triangleleft)$ where $R = \{(d, e) \mid \exists d', e'. d \sqsubseteq d' \triangleleft e' \preceq e\}$

(d) Combine (b) and (c) to show that $e \preceq e'$ if and only if $\llbracket e \rrbracket \triangleleft e'$. Deduce that $\preceq$ is a congruence, in the sense that $e \preceq e'$ implies $e''[e/x] \preceq e''[e'/x]$ (for any $\lambda$-term $e''$ with at most one free variable $x$).