UoCCL MPhil in Advanced Computer Science

L16: Semantics of HOT Languages

Lent Term 2010

Exam Briefing and Exercises

Exam briefing

This module will be assessed via a two-hour written test. There will be a choice of questions on the test paper, consisting of a mixture of "essay-style" and "problem-style" questions, with more weight given to the former than the latter.

To prepare for the problem-style questions, try some of the Exercises below. The essay-style questions will ask you to write about material that was covered in the lectures. To prepare, review the lecture slides and read the recommended texts, *Operational Semantics and Program Equivalence* and *Relational Properties of Domains*. Examples of essay-style questions:

I. Describe the operational semantics of a fragment of OCaml with (at least) integer storage locations and recursive functions, both in terms of an evaluation ("big-step") and a transition ("small-step") relation. What is the relationship between the two styles of operational semantics? Give a type system for the language and state *preservation* and *progress* properties for the operational semantics with respect to the type system. (Proofs of these properties are not required.)

II. Let $\mathcal{D}$ be the category whose objects are ω-chain complete partial orders with least element and whose morphisms are strict continuous functions. Explain what is meant by a *locally continuous functor* $F : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{D}$ and a *minimal invariant* for such an F. Show that a minimal invariant is a *regular free di-algebra* in the sense of Freyd. (Standard properties of least fixed points of continuous functions may be used without proof if clearly stated.)

Exercises

Operational Semantics (Lectures 1-8) See exercises B.1–B.7 on pp 409 and 410 of *Operational Semantics and Program Equivalence*.

Denotational Semantics (Lectures 9-16) [Throughout, "domain" means ω-chain complete partial order with least element.]

1. Suppose that D, E, and F are domains. Show that a function $f : D \times E \to E$ is continuous if and only if $f(d, -) : E \to F$ and $f(-, e) : D \to F$ are continuous for all $d \in D$ and $e \in E$.

2. Let D and E be domains and $f : D \times E \to E$ be a continuous function. Show that there is a continuous function $Y f : D \to E$ such that for all $d \in D$, $(Y f)(d) = f(d, (Y f)(d))$.

3. Let D and E be domains and let $f : D \to E$ and $g : E \to D$ be continuous functions. Prove that $\mathit{fix}(g \circ f) = g(\mathit{fix}(f \circ g))$.

4. Suppose that D and E are domains and that $f : D \to D$ and and $g : D \times E \to E$ are continuous functions. Let (d, e) be the least element of $D \times E$ satisfying

$$\begin{cases} d &= f(d) \\ e &= g(d, e). \end{cases}$$

   Prove that $d = \mathit{fix}(f)$.

5. Given domains D and E show how to make the set $D \to E$ of continuous functions from D to E into a domain with the following property: for all continuous functions $f : F \times D \to E$, there is a unique continuous function $\hat{f} : F \to (D \to E)$ such that for all $z \in F$ and $x \in D$, $(\hat{f}\, z)\, x = f(z, x)$.

6. Consider the denotational semantics of the untyped λ-calculus in a domain satisfying $D \cong (D \to D)_\perp$ (lecture 10). Show that the usual law of *η-expansion*, $e = \lambda x.\, e\, x$ (where x not free in e) is *not* satisfied up to denotational equality. Adapt the denotational semantics to use a domain D' satisfying $D' \cong D' \to D'$ and show that this does satisfy η-expansion (as well as β-reduction, $(\lambda x.\, e)e' = e[e'/x]$). How many elements does any minimal invariant solution of the domain equation $X \cong X \to X$ have?

7. Let Λ_0 be the set of closed λ-terms and let $\Rightarrow\ \subseteq \Lambda_0 \times \Lambda_0$ be the usual call-by-name evaluation relation inductively defined by the rules

$$\frac{}{\lambda x.\, e \Rightarrow \lambda x.\, e} \qquad \frac{e_1 \Rightarrow \lambda x.\, e \qquad e[e_2/x] \Rightarrow c}{e_1\, e_2 \Rightarrow c}$$

   A subset $S \subseteq \Lambda_0 \times \Lambda_0$ is called an *applicative simulation* if it satisfies: $(e, e') \in S$ and $e \Rightarrow \lambda x.\, e_1$ implies $e' \Rightarrow \lambda x.\, e'_1$ for some e'_1 such that $\forall e_2 \in \Lambda_0.\ (e_1[e_2/x], e'_1[e_2/x]) \in S$. Given $e, e' \in \Lambda_0$, we write $e \preceq e'$ if there is some applicative simulation S with $(e, e') \in S$. (The relation $\preceq$ is called *applicative similarity*.)

   (a) Show that $\preceq$ is a reflexive and transitive relation.

   (b) Let $\lhd \subseteq D \times \Lambda_0$ be the relation constructed in lecture 13 between the minimal invariant domain $D \cong (D \to D)_\perp$ and closed λ-terms. Show that $\{(e, e') \mid [\![e]\!] \lhd e'\}$ is an applicative simulation.

   (c) Show that $d \sqsubseteq d' \lhd e' \preceq e$ implies $d \lhd e$. [Hint: use the fact that $(\lhd, \lhd)$ is the least pre-fixed point of the monotone function $\Phi^{\S}$ defined in lecture 13; show that $\Phi^{\S}(R, \lhd) \leq (R, \lhd)$ where $R = \{(d, e) \mid \exists d', e'.\, d \sqsubseteq d' \lhd e' \preceq e\}$.]

   (d) Combine (b) and (c) to show that $e \preceq e'$ if and only if $[\![e]\!] \lhd e'$. Deduce that $\preceq$ is a congruence, in the sense that $e \preceq e'$ implies $e''[e/x] \preceq e''[e'/x]$ (for any λ-term e'' with at most one free variable x).