Expressive Power of Logics

We have seen that the expressive power of first-order logic, in terms of computational complexity is weak.

Second-order logic allows us to express all properties in the polynomial hierarchy.

Are there interesting logics intermediate between these two?

We have seen one—monadic second-order logic.

We now examine another—LFP—the logic of least fixed points.

Inductive Definitions

LFP is a logic that formalises inductive definitions.

Unlike in second-order logic, we cannot quantify over arbitrary relations, but we can build new relations inductively.

Inductive definitions are pervasive in mathematics and computer science.

The syntax and semantics of various formal languages are typically defined inductively.

viz. the definitions of the syntax and semantics of first-order logic seen earlier.

Transitive Closure

The transitive closure of a binary relation $E$ is the smallest relation $T$ satisfying:

- $E \subseteq T$; and
- if $(x, y) \in T$ and $(y, z) \in E$ then $(x, z) \in T$.

This constitutes an inductive definition of $T$ and, as we have already seen, there is no first-order formula that can define $T$ in terms of $E$. 
Monotone Operators

In order to introduce LFP, we briefly look at the theory of monotone operators, in our restricted context.

We write \( \text{Pow}(A) \) for the powerset of \( A \).
An operator in \( A \) is a function
\[
F : \text{Pow}(A) \rightarrow \text{Pow}(A).
\]

\( F \) is monotone if
\[
\text{if } S \subseteq T, \text{ then } F(S) \subseteq F(T).
\]

Least and Greatest Fixed Points

A fixed point of \( F \) is any set \( S \subseteq A \) such that \( F(S) = S \).

\( S \) is the least fixed point of \( F \), if for all fixed points \( T \) of \( F \), \( S \subseteq T \).

\( S \) is the greatest fixed point of \( F \), if for all fixed points \( T \) of \( F \), \( T \subseteq S \).

Least and Greatest Fixed Points

For any monotone operator \( F \), define the collection of its pre-fixed points as:
\[
\text{Pre} = \{ S \subseteq A \mid F(S) \subseteq S \}.
\]

Note: \( A \in \text{Pre} \).

Taking
\[
L = \bigcap \text{Pre},
\]
we can show that \( L \) is a fixed point of \( F \).

Fixed Points

For any set \( S \in \text{Pre} \),
\[
L \subseteq S \quad \text{by definition of } L.
\]
\[
F(L) \subseteq F(S) \quad \text{by monotonicity of } F.
\]
\[
F(L) \subseteq S \quad \text{by definition of } F(S).
\]
\[
F(L) \subseteq L \quad \text{by definition of } L.
\]
\[
F(F(L)) \subseteq F(L) \quad \text{by monotonicity of } F.
\]
\[
F(L) \in \text{Pre} \quad \text{by definition of } F(L).
\]
\[
L \subseteq F(L) \quad \text{by definition of } L.
\]
Least and Greatest Fixed Points

\( L \) is a fixed point of \( F \).

Every fixed point \( P \) of \( F \) is in \( \text{Pre} \), and therefore \( L \subseteq P \).

Thus, \( L \) is the least fixed point of \( F \).

Similarly, the greatest fixed point is given by:

\[ G = \bigcup \{ S \subseteq A \mid S \subseteq F(S) \} \]

Iteration

Let \( A \) be a finite set and \( F \) be a monotone operator on \( A \).

Define for \( i \in \mathbb{N} \):

\[
\begin{align*}
F^0 &= \emptyset \\
F^{i+1} &= F(F^i).
\end{align*}
\]

For each \( i \), \( F^i \subseteq F^{i+1} \) (proved by induction).

Fixed-Point by Iteration

If \( A \) has \( n \) elements, then

\[ F^n = F^{n+1} = F^m \quad \text{for all} \quad m > n \]

Thus, \( F^n \) is a fixed point of \( F \).

Let \( P \) be any fixed point of \( F \). We can show induction on \( i \), that \( F^i \subseteq P \).

If \( F^i \subseteq P \), then

\[ F^{i+1} = F(F^i) \subseteq F(P) = P \]

Thus \( F^n \) is the least fixed point of \( F \).
Defined Operators

Suppose \( \phi \) contains a relation symbol \( R \) (of arity \( k \)) not interpreted in the structure \( A \) and let \( x \) be a tuple of \( k \) free variables of \( \phi \).

For any relation \( P \subseteq A^k \), \( \phi \) defines a new relation:

\[
F_P = \{ a \mid (A, P) \models \phi[a] \}.
\]

The operator \( F_{\phi} : \text{Pow}(A^k) \to \text{Pow}(A^k) \) defined by \( \phi \) is given by the map

\[
P \mapsto F_P.
\]

Or, \( F_{\phi,b} \) if we fix parameters \( b \).

Positive Formulas

Definition

A formula \( \phi \) is \textit{positive} in the relation symbol \( R \), if every occurrence of \( R \) in \( \phi \) is within the scope of an even number of negation signs.

Lemma

For any structure \( A \) not interpreting the symbol \( R \), any formula \( \phi \) which is positive in \( R \), and any tuple \( b \) of elements of \( A \), the operator \( F_{\phi,b} : \text{Pow}(A^k) \to \text{Pow}(A^k) \) is monotone.

Reading List for this Handout

1. Ebbinghaus and Flum. Section 8.1.
2. Libkin. Sections 10.1 and 10.2.
3. Grädel et al. Section 3.3.