Topics in Logic and Complexity
Handout 11

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Capturing P

The *expressive power* of LFP is strictly weaker than P.

On the other hand, LFP can express all queries in P on *ordered structures*.

Thus, every query in P can be defined by a sentence of the form

$$\exists < \ (\mathrm{lo}(<) \wedge \phi)$$

where $\mathrm{lo}(<)$ is the first-order formula that says that $<$ is a linear order and ϕ is a sentence of LFP.

Capturing P

With a sentence of the form $\exists < \ (\mathrm{lo}(<) \wedge \phi)$, we can also define NP-complete problems.

$$\exists < (\mathrm{lo}(<) \wedge \forall xy[(y = x+1 \rightarrow E(x,y)) \wedge (x = \max \wedge y = \min \rightarrow E(x,y))]).$$

defines the graphs that contain a *Hamiltonian cycle*.

Partial Fixed Point Logic

For any formula $\phi(R)$ (not necessarily *positive* in R), we can define the operator F_ϕ in any structure of the appropriate vocabulary.

We can also define the iteration:

$$\begin{aligned} PF^0 &= \emptyset \\ PF^{i+1} &= F_\phi(PF^i). \end{aligned}$$

If $\mathbb{A}$ has n elements, then there are only 2^{n^k} distinct relations of arity k on $\mathbb{A}$, and therefore, for $j > 2^{n^k}$, there is an $i < j$ such that $PF^j = PF^i$, and we conclude that the sequence is eventually periodic.

We say that the sequence *converges* if the period is 1. That is, if for some i, $PF^{i+1} = PF^i$.

PFP

The logic PFP (or *partial fixed point logic*) is defined syntactically like IFP, except it uses the operator **pfp** in place of **ifp**

The semantics of the predicate expression $\mathbf{pfp}_{R,\mathbf{x}}\phi$ is given by the rule:

> If the sequence $PF^j (j \in \mathbb{N})$ converges, then the expression denotes the relation PF^i, where $PF^i = PF^{i+1}$, and it denotes the empty relation otherwise.

Example

Let $\phi(R, x, y)$ be $E(x, y) \vee \exists z(E(x, z) \wedge R(z, y))$

$F_\phi^m = IF_\phi^m = PF_\phi^m = \{(v, w) \mid \text{ there is a path } v - w \text{ of length } \leq m\}$

$F^\infty = IF^\infty = PF^\infty$ is the transitive closure of the graph

Let $\psi(R, x, y)$ be

$$(E(x, y) \wedge \forall x \forall y \neg R(x, y)) \vee \exists z(E(x, z) \wedge R(z, y)).$$

$IF_\phi^m = IF_\psi^m$

For the partial fixed point:

$$PF^m = \{(v, w) \mid \text{ there is a path } v - w \text{ of length } = m\}.$$

IFP vs. PFP

Every formula of IFP is equivalent to one of PFP.

The predicate expression

$$\mathbf{ifp}_{R,\mathbf{x}}\phi$$

is equivalent to

$$\mathbf{pfp}_{R,\mathbf{x}}(R(\mathbf{x}) \vee \phi).$$

Complexity of PFP

Every collection of finite structures definable in PFP is decidable by an algorithm in *polynomial space*.

To decide $\mathbb{A} \models \phi[\mathbf{a}]$, when $\phi \equiv \mathbf{pfp}_{R,\mathbf{x}}\psi(\mathbf{t})$

$R_{\text{old}} := \emptyset$; $R_{\text{new}} := \emptyset$; converge := false
for $i := 1$ **to** 2^{n^l} **do**
    $R_{\text{old}} := R_{\text{new}}$
    $R_{\text{new}} := F_\psi(R_{\text{new}})$
    if $R_{\text{new}} = R_{\text{old}}$ **then** converge := true **end**
if converge **and** $\mathbf{a} \in R$ **then** accept **else** reject

Capturing PSPACE

Every collection of finite *ordered* structures decidable in polynomial space is definable in PFP. **(Abiteboul, Vianu)**

Given a machine M and an integer k, we can define formulas ϕ_σ (for each *symbol* σ in the alphabet), ψ_q (for each *state* q of M) and η in k free variables so that:

if $T_{\sigma_1} \dots T_{\sigma_s}, S_{q_1} \dots S_{q_m}, H$ code the *current* configuration of M, then

$$\phi_{\sigma_1} \dots \phi_{\sigma_s}, \psi_{q_1} \dots \psi_{q_m}, \eta$$

code the *next* configuration.

Capturing PSPACE

Since PFP can express all queries in PSPACE on *ordered* structures, the collection of formulas of the form.

$$\exists < \ (\mathrm{lo}(<) \wedge \phi)$$

where $\mathrm{lo}(<)$ is the first-order formula that says that $<$ is a linear order and ϕ is a sentence of PFP, can express all queries in PSPACE.

Moreover, *every* such formula expresses a query in PSPACE.

So, this is a logic exactly capturing PSPACE.

Evenness

The collection of structures with an *even* number of elements is not definable in PFP.

Recall that $\mathcal{E}$ is the collection of all structures in the empty signature.

Lemma
For every PFP formula ϕ there is a first order formula ψ, such that for all structures $\mathbb{A}$ in $\mathcal{E}$, $\mathbb{A} \models (\phi \leftrightarrow \psi)$.

Defining the Stages

Given a formula $\psi(R, \mathbf{x})$ defining a (not necessarily monotone) operator.

ψ^1 is obtained from ψ by replacing all occurrences of subformulas of the form $R(\mathbf{t})$ by $t \neq t$.

ψ^{i+1} is obtained by replacing in ψ, all subformulas of the form $R(\mathbf{t})$ by $\psi^i(\mathbf{t}, \mathbf{y})$

ψ^i is a *first-order* formula defining PF^i.

On $\mathcal{E}$, there is a *fixed* bound p such that $\mathbf{pfp}_{R,\mathbf{x}}\psi$ is defined by:

$$\psi^p \wedge \forall \mathbf{x}(\psi^p \leftrightarrow \psi^{p+1})$$

Finite Variable Logic

We write L^k for the first order formulas using only the variables $x_1, \ldots, x_k$.

$$\mathbb{A} \equiv^k \mathbb{B}$$

denotes that $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k.

$$(\mathbb{A}, \mathbf{a}) \equiv^k (\mathbb{B}, \mathbf{b})$$

denotes that there is no formula ϕ of L^k such that $\mathbb{A} \models \phi[\mathbf{a}]$ and $\mathbb{B} \not\models \phi[\mathbf{b}]$

For a tuple $\mathbf{a}$ in $\mathbb{A}$, $\mathsf{Type}^k(\mathbb{A}, \mathbf{a})$ denotes the collection of all formulas $\phi \in L^k$ such that $\mathbb{A} \models \phi[\mathbf{a}]$.

Finite Variable Logic

For any k,

$$\mathbb{A} \equiv^k \mathbb{B} \quad \Rightarrow \quad \mathbb{A} \equiv_k \mathbb{B}$$

However, for any q, there are $\mathbb{A}$ and $\mathbb{B}$ such that

$$\mathbb{A} \equiv_q \mathbb{B} \quad \text{and} \quad \mathbb{A} \not\equiv^2 \mathbb{B}.$$

Take $\mathbb{A}$ and $\mathbb{B}$ to be linear orders longer than 2^q.

Stages

For every formula ϕ of LFP (or PFP), there is a k such that the query defined by ϕ is closed under $\equiv^k$.

Consider a formula $\psi(R, \mathbf{x})$ defining an operator.

Let the variables occurring in ψ be $x_1, \ldots, x_k$, with $\mathbf{x} = (x_1, \ldots, x_l)$, and $y_1, \ldots, y_l$ be new.

Stages

Define, by induction, the formulas ψ^m.

$$\psi^0 = \exists x \, x \neq x$$

ψ^{m+1} is obtained from $\psi(R, \mathbf{x})$ by replacing all sub-formulas $R(t_1, \ldots, t_l)$ with

$$\exists y_1 \ldots \exists y_l \, (\bigwedge_{1 \le i \le l} y_i = t_i) \wedge \phi^m(\mathbf{y})$$

Note that each ψ^m has at most $k+1$ variables.

LFP and PFP

If $(\mathbb{A}, \mathbf{a}) \equiv^{k+l} (\mathbb{B}, \mathbf{b})$, then *for all* m:

$$\mathbb{A} \models \psi^m[\mathbf{a}] \quad \text{if, and only if,} \quad \mathbb{B} \models \psi^m[\mathbf{b}].$$

So, $(\mathbb{A}, \mathbf{a})$ and $(\mathbb{B}, \mathbf{b})$ are not distinguished by $\mathbf{lfp}_{R,\mathbf{x}}\psi$.

Also

$$\mathbb{A} \models \forall \mathbf{x}(\psi^m \leftrightarrow \psi^{m+1}) \quad \text{if, and only if,} \quad \mathbb{B} \models \forall \mathbf{x}(\psi^m \leftrightarrow \psi^{m+1}).$$

So, $(\mathbb{A}, \mathbf{a})$ and $(\mathbb{B}, \mathbf{b})$ are not distinguished by $\mathbf{pfp}_{R,\mathbf{x}}\psi$.

Pebble Games

The k-pebble game is played on two structures $\mathbb{A}$ and $\mathbb{B}$, by two players—*Spoiler* and *Duplicator*—using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

Spoiler moves by picking a pebble and placing it on an element (a_i on anelement of $\mathbb{A}$ or b_i on an element of $\mathbb{B}$).

Duplicator responds by picking the matching pebble and placing it on an element of the other structure

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ definedby the pebble pairs is not a partial isomorphism

If *Duplicator* has a winning strategy for q moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k of quantifier rank at most q. **(Barwise)**

Using Pebble Games

To show that a class of structures S is not definable in first-order logic:

$$\forall k\ \forall q\ \exists \mathbb{A}, \mathbb{B}\ (\mathbb{A} \in S \wedge \mathbb{B} \notin S \wedge \mathbb{A} \equiv_q^k \mathbb{B})$$

Since $\mathbb{A} \equiv_q^q \mathbb{B} \Rightarrow \mathbb{A} \equiv_q \mathbb{B}$, we can ignore the parameter k

To show that S is not closed under any $\equiv^k$ (and hence not definable in LFP or PFP):

$$\forall k\ \exists \mathbb{A}, \mathbb{B}\ \forall q\ (\mathbb{A} \in S \wedge \mathbb{B} \notin S \wedge \mathbb{A} \equiv_q^k \mathbb{B})$$

If $\mathbb{A} \equiv_q^k \mathbb{B}$ holds for all q, then *Duplicator* actually wins an *infinite* game. That is, she has a strategy to play forever.

Evenness

To show that *Evenness* is not definable in PFP, it suffices to show that:

for every k, there are structures $\mathbb{A}_k$ and $\mathbb{B}_k$ such that $\mathbb{A}_k$ has an even number of elements, $\mathbb{B}_k$ has an odd number of elements and

$$\mathbb{A} \equiv^k \mathbb{B}.$$

It is easily seen that *Duplicator* has a strategy to play forever when one structure is a set containing k elements (and no other relations) and the other structure has $k+1$ elements.

Hamiltonicity

Take $K_{k,k}$—the complete bipartite graph on two sets of k vertices.

and $K_{k,k+1}$—the complete bipartite graph on two sets, one of k vertices, the other of $k+1$.

These two graphs are $\equiv^k$ equivalent, yet one has a Hamiltonian cycle, and the other does not.

Reading List for this Handout

1. Libkin. Sections 11.1 and 11.2
2. Grädel et al. Section 2.7