MPhil Advanced Computer Science
Topics in Logic and Complexity

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Exercise Sheet 1

1. Show that, for every nondeterministic machine M which uses $O(\log n)$ work space, there is a machine R with three tapes (`input`, `work` and `output`) which works as follows. On input x, R produces on its output tape a description of the configuration graph for M, x, and R uses $O(\log |x|)$ space on its work tape.

   Explain why this means that if $\mathsf{Reachability}$ is in L, then $\mathsf{L} = \mathsf{NL}$.

2. Show that a language L is in $\mathsf{co\text{-}NP}$ if, and only if, there is a nondeterministic Turing machine M and a polynomial p such that M halts in time $p(n)$ for all inputs of length x, and L is exactly the set of strings x such that *all* computations of M on input x end in an accepting state.

3. Define a *strong* nondeterministic Turing machine as one where each computation has three possible outcomes: accept, reject or maybe. If M is such a machine, we say that it accepts L, if for every $x \in L$, every computation path of M on x ends in either accept or maybe, with at least one accept *and* for $x \notin L$, every computation path of M on x ends in reject or maybe, with at least one reject.

   Show that if L is decided by a strong nondeterministic Turing machine running in polynomial time, then $L \in \mathsf{NP} \cap \mathsf{co\text{-}NP}$.

4. *Geography* and *HEX* are examples of two-player games played on graphs for which the problem of deciding which of the two players has a winning strategy is PSpace-complete (see Handout 3, slide 6). The games are defined as follows.

   Geography We are given a directed graph $G = (V, E)$ with a distinguished start vertex $s \in V$. At the beginning of the game, s is *marked*. The players mover alternately. The player whose turn it is marks a previouasly unmarked vertex v such that there is an edge from u to v, where u is the vertex marked most recently by the other player. A player who gets stuck (i.e. the vertex most recently marked is u and all edges leaving u go to marked vertices) loses the game.

   HEX We are given a directed graph $G = (V, E)$ with two distinguished vertices $a, b \in V$. There are two players (*red* and *blue*) who take alternate turns. In each turn, the player chooses a vertex not previously coloured and colours it with its own colour (player *red* colours it red or player *blue* colours it blue). The game ends when all nodes have been coloured. If there is a path from a to b consisting entirely of red vertices, then player *red* has won, otherwise *blue* has won.

Explain why both these problems are in PSpace. Prove, by means of suitable reductions, that they are PSpace-complete.

5. A second-order Horn sentence ($\mathsf{SO\text{-}Horn}$ sentence, for short) is one of the form
$$Q_1 R_1 \dots Q_p R_p (\forall \mathbf{x} \bigwedge_i C_i)$$
where, each Q_i is either $\exists$ or $\forall$, each R_i is a relational variable and each C_i is a *Horn* clause, which is defined for our purposes as a disjunction of atomic and negated atomic formulas such that it contains at most one positive occurrence of a relational variable. A sentence is said to be $\mathsf{ESO\text{-}Horn}$ if it is as above, and all Q_i are $\exists$.

   (a) Show that any $\mathsf{ESO\text{-}Horn}$ sentence in a relational signature defines a class of structures decidable in polynomial time.

   (b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and
   $$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$
   is decidable in polynomial time, then there is an $\mathsf{ESO\text{-}Horn}$ sentence that defines K.

   (c) Show that any $\mathsf{SO\text{-}Horn}$ sentence is equivalent to an $\mathsf{ESO\text{-}Horn}$ sentence.

6. Show that *Cook's theorem*—that the problem SAT (see Handout 3, slide 4) is NP-complete—can be obtained as a corollary to Fagin's theorem.

7. A graph $G = (V, E)$ is said to be *Hamiltonian* if it contains a cycle which visits every vertex exactly once. The problem of determining whether a graph is Hamiltonian is known to be NP-complete. Write down a sentence of ESO that defines this property.

8. We have seen a sentence of ESO that defines the structures with an even number of elements (Handout 4, slide 13). Can you define the property in USO?

9. We have seen a sentence of ESO that defines the 3-colourable graphs (Handout 1, slide 7). We can, of course, write a similar sentence to define the 2-colourable graphs. However, the property of being 2-colourable is in P, since a graph is 2-colourable if, and only if, it has no cycles of odd length. Can you write a USO sentence that defines the 2-colourable graphs?

10. Recall that a graph is *planar* if it can be drawn in the plane without any crossing edges. It is decidable in polynomial time whether a given graph is planar. Can you write a USO sentence that defines the planar graphs? How about an ESO sentence?

11. Show that the levels of the polynomial hierarchy are closed under polynomial time reductions. That is to say, if L_1 is a decision problem in Σ_n (or Π_n) for some n and $L_2 \leq_P L_1$ then L_2 is also in Σ_n (or Π_n respectively).

12. Recall the definition of *quantified Boolean formulas* (Handout 4, slide 3). We now define the following restricted classes of formulas.

    - A quantified Boolean formula is said to be Σ_1 if it consists of a sequence of existential quantifiers followed by a Boolean formula without quantifiers.
    - A quantified Boolean formula is said to be Π_1 if it consists of a sequence of universal quantifiers followed by a Boolean formula without quantifiers.
    - A quantified Boolean formula is said to be Σ_{n+1} if it consists of a sequence of existential quantifiers followed by a Π_n formula.
    - A quantified Boolean formula is said to be Π_{n+1} if it consists of a sequence of universal quantifiers followed by a Σ_n formula.

    For each n define Σ_n-*QBF* to be the problem of determining, given a Σ_n formula without free variables, whether or not it evaluates to true. Π_n-*QBF* is defined similarly for Π_n formulas.

    Prove that Σ_n-*QBF* is complete for the complexity class Σ_n^1 (i.e. the nth existential level of the polynomial hierarchy), and that Π_n-*QBF* is complete for the complexity class Π_n^1.