There are ten questions below. Each is worth 20 marks. Answer as many questions as you wish. Your best five answers will be added to give a total mark out of 100.

1. In the lecture course, we saw that the problem of deciding, given a graph $G$ and a sentence $\phi$ of first-order logic, whether $G \models \phi$ is PSPACE-complete. Here, you are to show that the same is true for Monadic Second-Order logic (MSO). That is, prove that the following decision problem is PSPACE-complete:

   **Input:** A graph $G$ and a sentence $\phi$ of MSO.
   **Decide:** whether or not $G \models \phi$.

   To do this,
   
   (a) explain why the problem is decidable in polynomial space; and
   (b) prove that it is PSPACE-hard.

2. Recall that a Boolean formula is in *conjunctive normal form* if it is the conjunction of a collection of *clauses*, each of which is the disjunction of a set of *literals*. Each literal is either a propositional variable or the negation of a propositional variable. We say that a formula is in *3-CNF* if it is in conjunctive normal form and each clause contains exactly 3 literals. It is in *2-CNF* if it is in conjunctive normal form and each clause contains exactly 2 literals.

   The problem of deciding whether a given formula in 3-CNF is satisfiable is known to be NP-complete. Here, the aim is to show that the problem of deciding whether a given formula in 2-CNF is satisfiable is in NL.

   (a) Show that every clause containing 2 literals can be written as an implication in exactly two ways.

   For any formula $\phi$ in 2-CNF, define the directed graph $G_\phi$ to be the graph whose set of vertices is the set of all literals that occur in $\phi$, and in which there is an edge from literal $x$ to literal $y$ if, and only if, the implication $(x \rightarrow y)$ is equivalent to one of the clauses in $\phi$.

   (b) Show that $\phi$ is *unsatisfiable* if, and only if, there is a literal $x$ such that there is a path in $G_\phi$ from $x$ to $\neg x$ and a path from $\neg x$ to $x$.

Recall that the problem of reachability in directed graphs is in NL and that NL = co-NL.
(c) Explain why from (a), (b) and the above statement, it follows that
the problem of determining whether a formula in 2-CNF is satisfiable is
in NL.

3. A clique in a graph \( G = (V, E) \) is a set \( X \subseteq V \) of vertices such that for
any \( u, v \in X \) if \( u \neq v \) then \( (u, v) \) is an edge in \( E \). The decision problem
CLIQUE is the problem of deciding, given a graph \( G \) and a positive integer
\( k \) whether or not \( G \) contains a clique with \( k \) or more elements. This
problem is known to be NP-complete.

We will represent this problem as a class of structures as follows. The
vocabulary consists of two binary relations \( E \) and \( < \) and one constant \( k \).
Consider structures \( G = (V, E, <, k) \) in this vocabulary where \( (V, E) \) is a
graph, \( < \) is a linear order on \( V \) and \( k \) is some element of \( V \). We say that
\( G \) is in CLIQUE if there is a set \( X \subseteq V \) of vertices which forms a clique in
the graph \( (V, E) \) and so that the number of elements in \( X \) is larger than
the number of elements in \( \{ v \in V \mid v < k \} \), i.e. the number of elements
before \( k \) in the linear order.

(a) Give a sentence of existential second-order logic that defines the class
of structures CLIQUE.

(b) Prove that there is no sentence of first-order logic that defines this
class of structures.

Hint: for the second part, you will want to construct, for each \( p \), a pair
of structures \( A_p \) and \( B_p \) one of which is in CLIQUE and the other is not,
but which are equivalent under \( \equiv_p \). For this, you might find it useful to
consider the proof, using Ehrenfeucht games, that evenness is not first-
order definable on linear orders.

(c) It is an open question whether or not CLIQUE as defined above is de-
finable in LFP. What would be the consequences if it was definable?
What would follow if you could prove it is not definable?

4. In this question, we will consider a different representation of the problem
CLIQUE from Question 3 above. Let the vocabulary consist of two unary
relations \( V \) and \( N \), two binary relations \( E \) and \( < \) and a constant \( k \). Say
that a structure \( A = (A, V, N, E, <, k) \) is in the class CLIQUE if: \( V \) and
\( N \) are subsets of \( A \); \( E \) is a binary relation on \( V \); \( < \) is a linear order on \( N \);
\( k \in N \); and the graph \( (V, E) \) contains a clique with more elements than
there are in the set \( \{ a \in N \mid a < k \} \).

(a) Give a sentence of existential second-order logic that defines the class
of structures CLIQUE in this sense.

(b) Prove that there is no sentence of LFP that defines this class of
structures.
Hint: for the second part, you will want to construct, for each $k$ a pair of structures $A_k$ and $B_k$, one of which is in $\text{CLIQUE}$ and the other is not, but which are equivalent under $\equiv^k$. For this, you might find it useful to consider the proof, using Ehrenfeucht games, that evenness is not definable in LFP on unordered structures.

(c) What can you conclude about whether $\text{CLIQUE}$ is definable in PFP?

5. We say that a graph $G = (V, E)$ contains a perfect matching if there is a set $M \subseteq E$ of edges such that for every vertex $v \in V$, there is exactly one edge $e \in M$ such that $e$ is incident on $v$.

(a) Give a sentence of existential second-order logic that defines the class of graphs that contain a perfect matching.

(b) Prove that there is no sentence of LFP that defines this class of structures.

Hint: for the second part, recall the proof from the lecture that shows that Hamiltonicity is not definable in LFP.

6. Consider a vocabulary consisting of two unary relations $P$ and $O$, one binary relation $E$ and two constants $s$ and $t$. We say that a structure $\mathcal{A} = (A, P, O, E, s, t)$ in this vocabulary is an arena if $P \cup O = A$ and $P \cap O = \emptyset$. That is, $P$ and $O$ partition the universe into two disjoint sets.

An arena defines the following game played between a player and an opponent. The game involves a token that is initially placed on the element $s$. At each move, if the token is currently on an element of $P$ it is player who plays and if it is on an element of $O$, it is opponent who plays. At each move, if the token is on an element $a$, the one who plays choses an element $b$ such that $(a, b) \in E$ and moves the token from $a$ to $b$. If the token reaches $t$ at any point then player has won the game.

We define $\text{GAME}$ to be the class of arenas for which player has a strategy for winning the game. Note that in an arena $\mathcal{A} = (A, P, O, E, s, t)$, player has a strategy to win from an element $a$ if either $a \in P$ and there is some move from $a$ so that player still has a strategy to win after that move or $a \in O$ and for every move from $a$, player can win after that move.

(a) Give a sentence of LFP that defines the class of structures $\text{GAME}$.

We say that a collection $\mathcal{C}$ of decision problems is closed under logarithmic space reductions if whenever $A \in \mathcal{C}$ and $B \leq_L A$ (i.e. $B$ is reducible to $A$ by a logarithmic-space reduction) then $B \in \mathcal{C}$.

The class of structures $\text{GAME}$ defined above is known to be P-complete under logarithmic-space reductions.

(b) Explain why this, together with (a) implies that the class of problems definable in LFP is not closed under logarithmic-space reductions.
7. If $\sigma$ is a relational signature (i.e. it contains no function or constant symbols), and $A$ and $B$ are $\sigma$-structures, write $A + B$ for the structure whose universe is the disjoint union of the universes of $A$ and $B$ and where each relation symbol $R$ of $\sigma$ is interpreted by the corresponding union of its interpretations in $A$ and $B$. Similarly, write $nA$ for the disjoint union of $n$ copies of $A$.

(a) Show that, if $A \equiv qA'$ and $B \equiv qB'$, then $A + B \equiv qA' + B'$.

(b) Show that, for $n, m \geq q$, $nA \equiv qmA$.

8. Suppose $\phi$ is formula of PFP, $R$ is a relational variable, and $O$ is the class of structures (in a vocabulary $\sigma$ containing a binary relation symbol $<$) that interpret the symbol $<$ as a linear order.

(a) Show there is a formula of PFP that is equivalent to $\exists R\phi$ on all structures in $O$.

(b) Use this fact to conclude that a class $K$ of structures (not necessarily ordered) is definable by a sentence of the form $\exists R\phi$ (where $\phi$ is in PFP) if, and only if, $\{|A|_< \mid A \in K \text{ and } < \text{ is any order on } A\}$ is in PSPACE.

9. Consider a structure $E = (A, E)$ where $E$ is an equivalence relation on the set $A$, and let $e_i$ denote the number of equivalence classes of $E$ with exactly $i$ elements. Define the $k$-index of $E$ to be the $k$-tuple $(n_1, \ldots, n_k)$ where, for $i < k$, $n_i = \min(k, e_i)$ and $n_k = \min(k, \sum_{i \geq k} e_i)$.

(a) Show that if $E_1$ and $E_2$ are two such structures with the same $k$-index, then $E_1 \equiv^k E_2$.

It is known that if $C$ is a class of structures where for each $k$ there are only finitely many equivalence classes of structures with respect to $\equiv^k$, then every formula of LFP is equivalent, on $C$, to a formula of first-order logic. Use this and (a) to

(b) prove that LFP is no more expressive than first-order logic on the class of finite equivalence relations.

10. We define the 3-chordal graph on the set of vertices $\{0, \ldots, n - 1\}$ to be the graph in which there is an edge from $i$ to $j$ if, and only if, $j = i + 1 \pmod n$ or $j = i + 2 \pmod n$.

(a) Prove that the 3-chordal graph on $\{0, \ldots, n - 1\}$ is 3-colourable if, and only if, $n$ is a multiple of 3.

(b) Use this to show that 3-colourability is not definable in first-order logic.

Hint: for the second part, you may wish to consider Hanf’s theorem and the proof that connectivity is not first-order definable.