Basic Rewriting Theory

Exercises from 'Term *Re*writing and *All That*' by F. Baader and T. Nipkow

1. Let $T_\Sigma(\{x\})$ be the set of terms from a signature Σ over one variable x. The reduction relation $\rightarrow_I$ on $T_\Sigma(\{x\})$ is defined by $s \rightarrow_I t$ iff s is an instance of t and $s \neq t$. Show that $\rightarrow_I$ is confluent and terminating. Can this result be generalised to the case of more than one variable?

2. Let e be a constant, i a unary function symbol, and m a binary function symbol, and consider the equational presentation G given by

$$\mathsf{m}(x, \mathsf{m}(y, z)) \approx \mathsf{m}(\mathsf{m}(x, y), z)\ ,\ \ \mathsf{m}(\mathsf{e}, x) \approx x\ ,\ \ \mathsf{m}(\mathsf{i}(x), x) \approx \mathsf{e}$$

Show that $\mathsf{m}(x, \mathsf{e}) \overset{*}{\leftrightarrow}_G x$.

3. Let f be a binary function symbol, and consider the equational presentation E given by

$$\mathsf{f}(x, \mathsf{f}(y, z)) \approx \mathsf{f}(\mathsf{f}(x, y), z)\ ,\ \ \mathsf{f}(\mathsf{f}(x, y), x) \approx x$$

Show that $\mathsf{f}(x, x) \overset{*}{\leftrightarrow}_E x$ and $\mathsf{f}(\mathsf{f}(x, y), z) \overset{*}{\leftrightarrow}_E \mathsf{f}(x, z)$.

4. Show that a reduction relation is confluent and normalising iff every element has a unique normal form.

5. For D the equational presentation

$$x * (y + z) \approx (x * y) + (x * z)\ ,\ \ (x + y) * z \approx x * z + y * z$$

show that $\approx_D$, equality modulo distributivity, is decidable.

6. For substitutions σ, σ', θ, show that if $\sigma \precsim \sigma'$ then $\sigma\theta \precsim \sigma'\theta$. Does $\sigma \precsim \sigma'$ imply $\theta\sigma \precsim \theta\sigma'$?

6. For substitutions σ, σ' and a term s, show that if $\sigma s = \sigma' s$ then σ and σ' coincide on the variables of s.

7. Let S and T be unification problems. Show that if σ is a m.g.u. of S and θ a m.g.u. of $\sigma(T)$ then $\theta\sigma$ is a m.g.u. of $S \cup T$.

8. Check if the following unification problems are solvable.

(*i*) $\mathsf{f}(x, y) \overset{?}{=} \mathsf{f}(\mathsf{h}(\mathsf{a}), x)$

(*ii*) $\mathsf{f}(x, y) \overset{?}{=} \mathsf{f}(\mathsf{h}(x), x)$

(*iii*) $\mathsf{f}(x, \mathsf{b}) \overset{?}{=} \mathsf{f}(\mathsf{h}(y), z)$

(*iv*) $\mathsf{f}(x, x) \overset{?}{=} \mathsf{f}(\mathsf{h}(y), y)$

9. Check if the following matching problems are solvable.

(i) $\mathsf{f}(x, y) \overset{?}{\precsim} \mathsf{f}(\mathsf{h}(\mathsf{a}), x)$

(ii) $\mathsf{f}(x, y) \overset{?}{\precsim} \mathsf{f}(\mathsf{h}(x), x)$

(iii) $\mathsf{f}(x, \mathsf{b}) \overset{?}{\precsim} \mathsf{f}(\mathsf{h}(y), z)$

(iv) $\mathsf{f}(x, x) \overset{?}{\precsim} \mathsf{f}(\mathsf{h}(y), y)$

10. Modify the transformation rules for unification such that they directly solve the matching problem (rather than first replacing all variables on the right-hand sides by constants). Allow for variables on both sides and detect unsovability as early as possible.

11. Show that $\xrightarrow{+}$ is terminating iff so is $\rightarrow$.

12. A *strict order* is a transitive and irreflective relation. Show that $\xrightarrow{+}$ is a strict partial order iff $\rightarrow$ is acyclic.

13. A relation is called *bounded* if for each element the length of all paths starting from it is bounded; that is, $\forall\, x.\, \exists\, n.\, x \overset{n}{\not\rightarrow}$.

(i) Is every terminating relation bounded?

(ii) Show that a finitely branching relation terminates iff it is bounded.

14. Does confluence imply the following property?

$$y_1 \leftarrow x \rightarrow y_2 \implies y1 \overset{=}{\rightarrow} z \overset{=}{\leftarrow} y_2 \text{ for some } z$$

where $u \overset{=}{\rightarrow} v \iff u \rightarrow v$ or $u = v$. Give a proof or a counterexample.

15. Show that if $\rightarrow$ has the diammond property, every element is in normal form or has no normal form.

16. Let $\rightarrow$ satisfy the following weak form of the diammond property: if $y \leftarrow x \rightarrow z$ and $y \neq z$ then there is a u such that $y \rightarrow u \leftarrow z$. Show that if an element a has a normal form, then (i) there is no infinite reduction sequence starting from a, and (ii) all reductions from a to its normal form have the same length.

17. Show that if $(l \approx r) \in E$ and $\mathrm{Var}(l) \supseteq \mathrm{Var}(r)$ does not hold, then $\rightarrow_E$ is non-terminating.

18. Find r_1 and r_2 such that the rewriting system given by

$$\mathsf{f}(\mathsf{g}(x)) \rightarrow r_1\ ,\ \ \mathsf{g}(\mathsf{h}(x)) \rightarrow r_2$$

is confluent.

19. Is the rewriting system R given by

$$\mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \to \mathsf{g}(x)$$

confluent? Find a convergent rewriting system R' such that $\approx_R$ and $\approx_{R'}$ are equal.

20. Compute all critical pairs for each of the following rewriting systems.

(*i*) $\mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \to x$, $\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{f}(x))$

(*ii*) $\mathsf{0} + y \to y$, $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$, $x + \mathsf{0} \to x$, $x + \mathsf{s}(y) \to \mathsf{s}(x + y)$

(*iii*) $\mathsf{f}(x, x) \to \mathsf{a}$, $\mathsf{f}(x, \mathsf{g}(x)) \to \mathsf{b}$

(*iv*) $\mathsf{m}(\mathsf{m}(x, y), z) \to \mathsf{m}(x, \mathsf{m}(y, z))$, $\mathsf{m}(x, \mathsf{1}) \to x$

(*v*) $\mathsf{m}(\mathsf{m}(x, y), z) \to \mathsf{m}(x, \mathsf{m}(y, z))$, $\mathsf{m}(\mathsf{1}, x) \to x$

(*vi*) $\mathsf{f}(x, \mathsf{f}(y, z)) \to \mathsf{f}(\mathsf{f}(x, y), \mathsf{f}(x, z))$, $\mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(\mathsf{f}(x, z), \mathsf{f}(y, z))$,
$\mathsf{f}(\mathsf{f}(x, y), \mathsf{f}(y, z)) \to y$

Which systems are locally confluent? Which ones are convergent?

21. Show that the following system is convergent

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x) \leftarrow \mathsf{g}(\mathsf{g}(x)) \ , \ \mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(x) \leftarrow \mathsf{g}(\mathsf{f}(x))$$

Can you determine the normal form of a term as a function of the number of fs and gs in it?

22. Find a measure function into $\mathbb{N}$ which proves termination of the rewriting relation

$$ubav \to uabv \qquad (u, v \in \{a, b\}^*) \ .$$

23. Find a measure function into $\mathbb{N}$ which proves termination of $\to_1 \cup \to_2$, for

$$\begin{array}{c} uabbv \to_1 uaav \\ uabv \to_2 vbu \end{array} \qquad (a, b \in A, u, v \in A^*) \ .$$

24. Prove that the lexicographic product of two terminating relations is terminating by well-founded induction.

25. Show that the following process always terminates. There is a box full of black and white balls. Each step consists of removing an arbitrary ball from the box. If it happens to be a black ball, one also adds an arbitrary finite number of white balls to the box.

26. Given a relation $>$ on a set A, define the relation $>_{Lex}$ on A^* as follows: $w_1 >_{Lex} w_2$ if w_2 is a proper prefix of w_1 or if $w_1 = uav$, $w_2 = ubw$ $(a, b \in A, u, v \in A^*)$, and $a > b$. Does termination of $>$ imply termination of $>_{Lex}$?

27. Consider the identity $E = \{ \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \approx x \}$. Choose an appropriate reduction order $>$ and apply the basic completion procedure to the input $(E, >)$.

28. Consider the identity $E = \{ \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \approx \mathsf{f}(\mathsf{g}(x)) \}$. Choose an appropriate reduction order $>$ and apply the basic completion procedure to the input $(E, >)$.

29. Show that the TRS

$$(x * y) * (y * z) \to y \ , \ \ x * ((x * y) * z) \to x * y \ , \ \ (x * (y * z)) * z \to y * z$$

is confluent.