Problem 1.
(a) Consider the commutative diagram

\[
\begin{array}{ccc}
B & \rightarrow & E \\
\downarrow & & \downarrow \\
A & \rightarrow & C & \rightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
D & \rightarrow & C & \rightarrow & E
\end{array}
\]

where \( m \) is a mono. Prove that if (*) is a pullback then so is the outer square.

(b) Consider the commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & D \\
\downarrow & & \downarrow \\
B & \rightarrow & C & \rightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
C & \rightarrow & C & \rightarrow & E
\end{array}
\]

where \( m \) is a mono.

Prove that if \( \xymatrix{A \ar[r]^q \ar[d]_p & D \ar[d]^g \\ C \ar[r]^f & E \ar[r]^{\circ m} \ar[d]^e } \) is a pullback, then so is \( \xymatrix{A \ar[r]^q \ar[d]_p & D \ar[d]^g \\ B \ar[r]^e \ar[d]^u & D \ar[d]^v } \).

Problem 2.

- A mono \( m : X \rightarrow Y \) is regular if there exists a diagram \( Y \xrightarrow{f} Z \) for which \( m \) is an equalizer.

- A mono \( m : X \rightarrow Y \) is strong, if for every commutative diagram as on the left below where \( e : U \rightarrow V \) is epi

\[
\begin{array}{ccc}
U & \rightarrow & V \\
\downarrow & & \downarrow \\
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
\]

there exists exactly one arrow $d : V \to X$ as on the right above such that both triangles commute.

- A mono $m : X \to Y$ is *extremal* if for every commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{e} & X \\
\downarrow v & & \downarrow m \\
X & \xrightarrow{\cdot} & Y
\end{array}
\]

where $e : X \to V$ is epi, $e$ is an isomorphism.

Prove that the following implications between properties of monomorphisms hold in any category:

section $\Rightarrow$ regular $\Rightarrow$ strong $\Rightarrow$ extremal.

(None of these is in general an equivalence, but that is another story.)

**Problem 3.** Given an object $A$ in a small category $C$, consider the following diagram in the category $\textbf{Cat}$ of small categories and functors:

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{\text{Cod}} & \text{1} \\
\downarrow K_A & & \downarrow K_A \\
\text{C} & & \text{C}
\end{array}
\]

where:

- $\text{C}^-$ is the arrow category of $C$ and Cod is the *codomain functor*, defined by:
  - $\text{Cod}(f) = \text{cod}(f)$ on objects in $\text{C}^-$,
  - $\text{Cod}(h, k) = k$ on arrows in $\text{C}^-$.

- $\textbf{1}$ is the category with one object $*$ and one arrow $1_*$, and $K_A$ is the constant functor defined by:
  - $K_A(*) = A$,
  - $K_A(1_*) = 1_A$.

Prove that the slice category $\text{C}/A$, together with suitable functors to $\text{C}^-$ and $\text{1}$, is a pullback of the above diagram.
Problem 4. Fix a set $X$ with $\ast \not\in X$.

For $n \in \mathbb{N}$, define

$$X_n \overset{\text{def}}{=} \{(x_1, \ldots, x_n) \in (X \cup \{\ast\})^n \mid x_i = \ast \implies x_{i+1} = \ast \text{ for all } i = 1, \ldots, n-1\}.$$

(a) Consider the following infinite diagram in the category \textbf{Sets} of sets and functions:

$$X_1 \xrightarrow{\iota_1} X_2 \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_n} X_n \xrightarrow{\iota_{n+1}} \cdots \quad (n \in \mathbb{N}) \quad (1)$$

where $\iota_n : X_n \rightarrow X_{n+1}$ is the function given by

$$\iota_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \ast).$$

Give a simple description of a colimit of (1), and prove its universal property.

(b) Consider the following infinite diagram in the category \textbf{Sets} of sets and functions:

$$X_1 \leftarrow p_1 X_2 \leftarrow p_2 \cdots \leftarrow X_n \leftarrow p_n X_{n+1} \leftarrow \cdots \quad (n \in \mathbb{N}) \quad (2)$$

where $p_n : X_{n+1} \rightarrow X_n$ is the function given by

$$p_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n).$$

Give a simple description of a limit of (2), and prove its universal property.