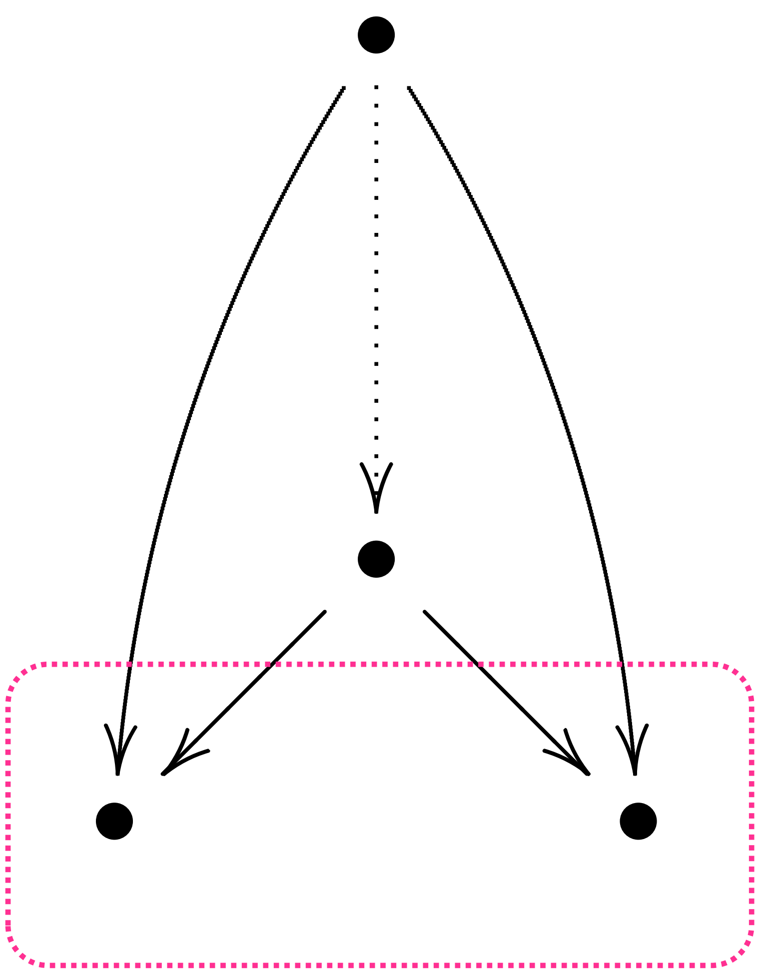
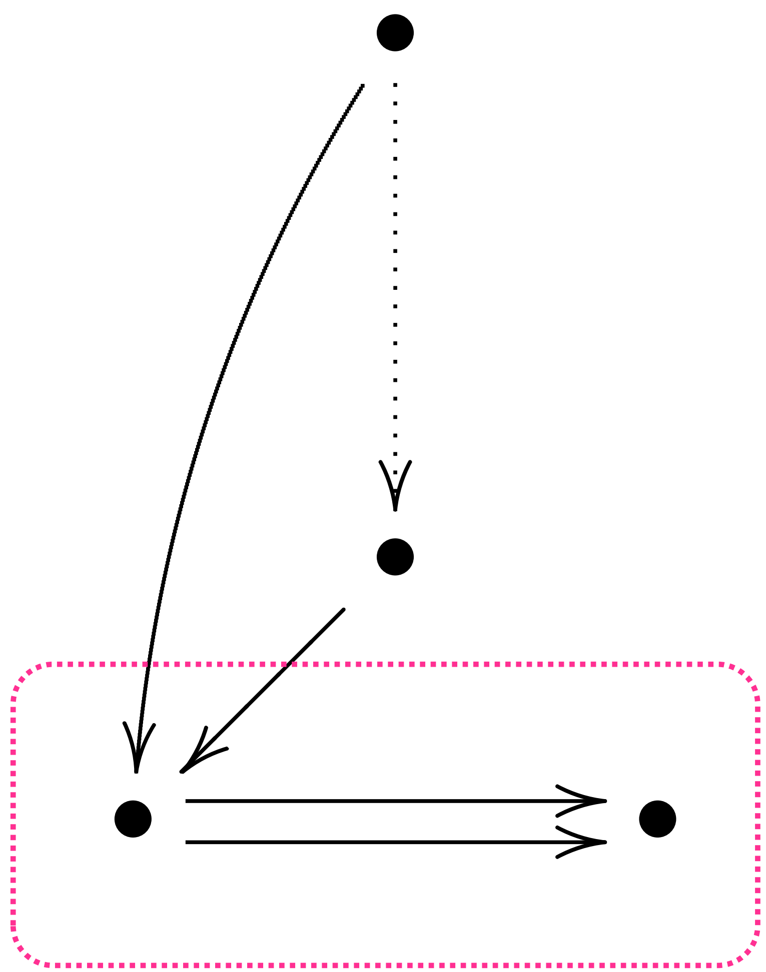
A common theme

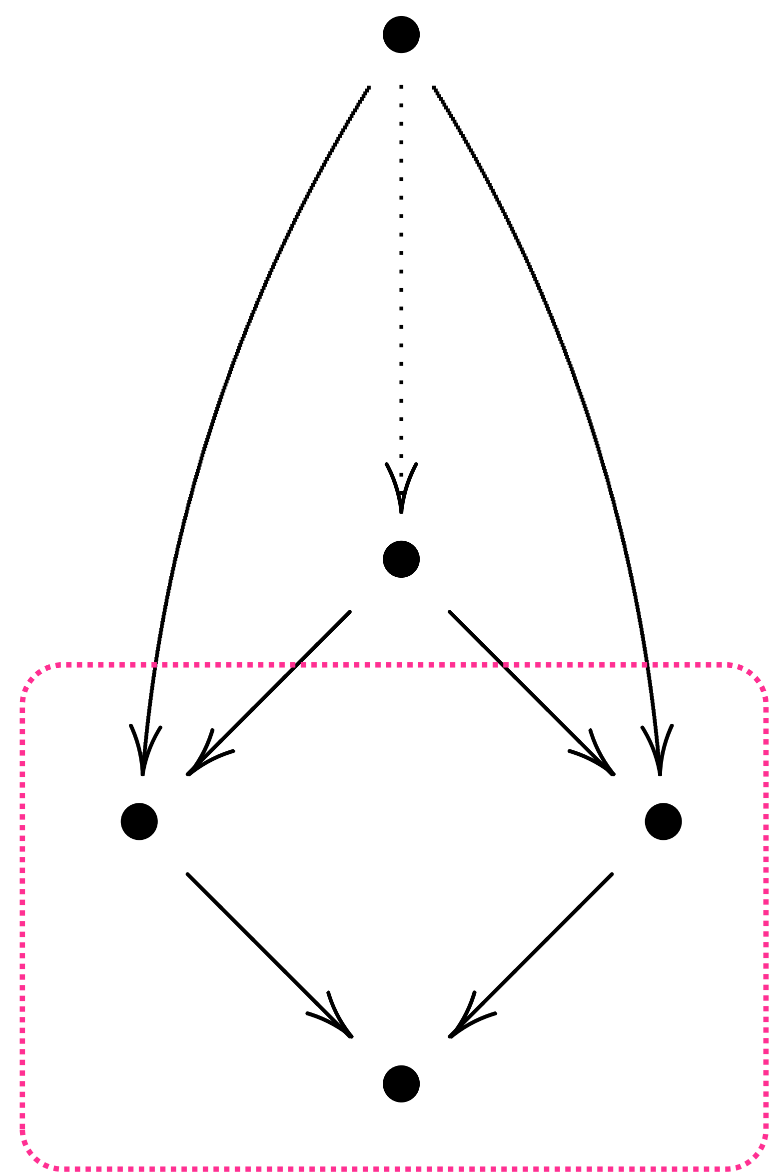
Product:



Equalizer:



Pullback:



Diagrams

Defn. A diagram D in a category $\mathbf{C}$ consists of:

- a graph $G(D)$,
- an object D_v for each vertex v in $G(D)$,
- an arrow D_e for each edge e in $G(D)$,

s.t. if $e : v \longrightarrow w$ is an edge then

$$\mathrm{dom}(D_e) = D_v \text{ and } \mathrm{cod}(D_e) = D_w.$$

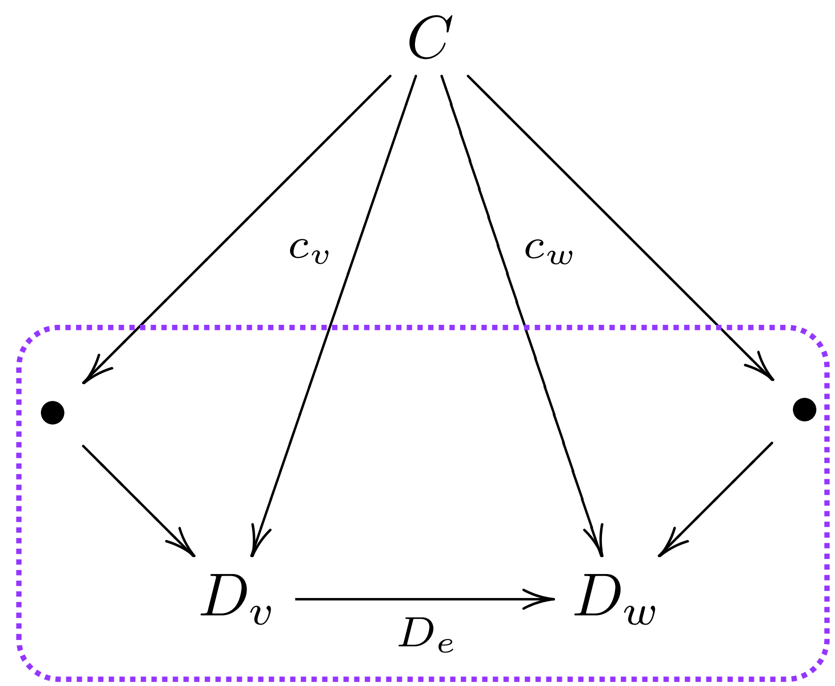
A diagram D is commutative (or commutes), if for any two paths in $G(D)$ with common source and target, the compositions of the corresponding sequences of arrows are equal.

Cones and cocones

Defn. A cone for a diagram D in $\mathbf{C}$ is:

- an object C in $\mathbf{C}$,
- an arrow $c_v : C \to D_v$ for each vertex v,

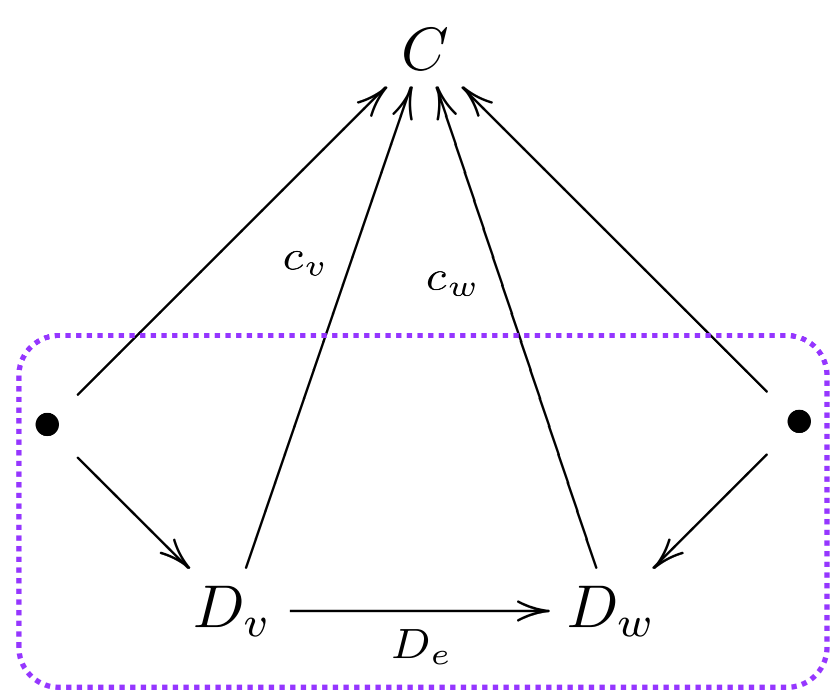
s.t. for each edge $e : v \to w$, $D_e \circ c_v = c_w$.



Defn. A cocone for a diagram D in $\mathbf{C}$ is:

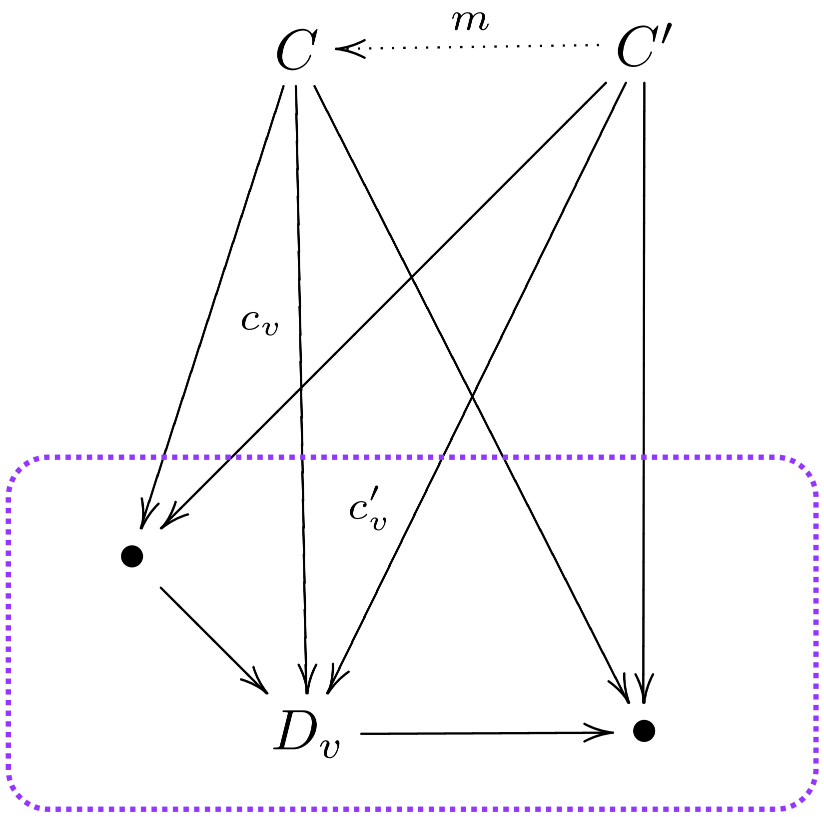
- an object C in $\mathbf{C}$,
- an arrow $c_v : D_v \to C$ for each vertex v,

s.t. for each edge $e : v \to w$, $c_w \circ D_e = c_v$.

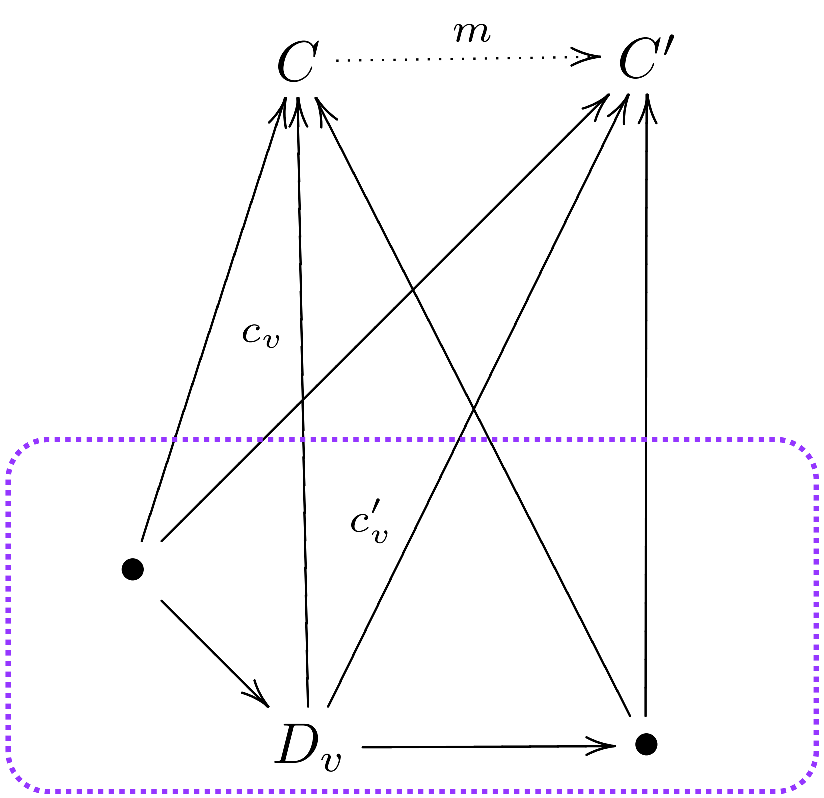


Limits and colimits

Defn. A cone C is a limit for D
if for any cone C' for D
there is a *unique* arrow $m : C' \to C$
s.t. $c_v \circ m = c'_v$ for each vertex v.



Defn. A cocone C is a colimit for D
if for any cocone C' for D
there is a *unique* arrow $m : C \to C'$
s.t. $m \circ c_v = c'_v$ for each vertex v.



Examples

Diagram shape	Limit	Colimit
$\emptyset$	Final object	Initial object
$\bullet \quad \bullet$	Product	Coproduct
$\bullet \rightrightarrows \bullet$	Equalizer	Coequalizer
$\bullet \longrightarrow \bullet \longleftarrow \bullet$	Pullback	
$\bullet \longleftarrow \bullet \longrightarrow \bullet$		Pushout

Some facts about (co)limits

Fact. (Co)Limits, if they exist, are unique up to isomorphism.

Defn. A category is (finitely) (co)complete
if it has all (finite) (co)limits.

Fact. If a category has products of arbitrary families of objects,
and equalizers, then it is complete.

Fact. If a category has a final object, all binary products
and equalizers, then it is finitely complete.

Note: We only consider (co)limits of *small* diagrams.

Fact. A category with limits of all (large) diagrams
is necessarily a preorder.

Also: A small complete category is necessarily a preorder.

Limits and colimits in $\mathbf{Sets}$.

Fact. $\mathbf{Sets}$ is complete and cocomplete.

Limits: For a diagram D, take the set

$$\{\langle x_v, x_w, \ldots \rangle \in D_v \times D_w \times \cdots \mid \forall e : v \to w.\ D_e(x_v) = x_w\}$$

with obvious projections as the limiting cone.

Colimits: For a diagram D, take the set

$$(D_v + D_w + \cdots)/{\equiv}$$

where $\equiv$ is the least equivalence s.t.

$$D_e(x_v) \equiv x_v \text{ for all } e : v \to w,\ x_v \in D_v\,,$$

with obvious injections as the colimiting cocone.

limit = subset of product, colimit = quotient of sum