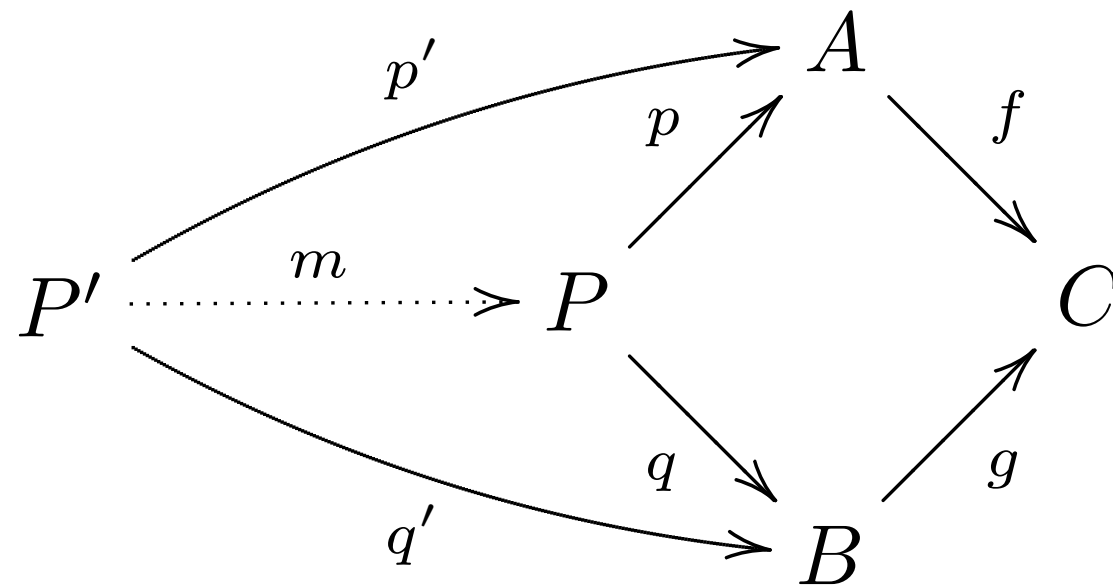


Pullbacks

Defn. A **pullback** of two arrows $A \xrightarrow{f} C \xleftarrow{g} B$ is an object P with arrows $A \xleftarrow{p} P \xrightarrow{q} B$ s.t.:

- $f \circ p = g \circ q$,
- for every $A \xleftarrow{p'} P' \xrightarrow{q'} B$ s.t. $f \circ p' = g \circ q'$ there is a *unique* $m : P' \rightarrow P$ s.t. $p \circ m = p'$, $q \circ m = q'$.



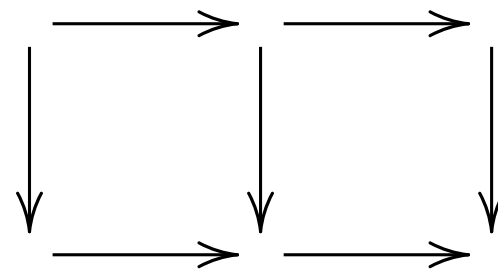
Example. In **Sets**, take $P = \{(a, b) \in A \times B \mid f(a) = g(b)\}$.
(Exercise: check how this works when f and/or g is a mono)

Facts about pullbacks

- Pullbacks, if they exist, are unique up to isomorphism.
- if \mathbf{C} has all products and equalizers, then it has all pullbacks.
- if \mathbf{C} has all pullbacks and a final object, then it has all (binary) products and equalizers.

- If $\begin{array}{ccc} & \longrightarrow & \\ g \downarrow & & \downarrow f \\ & \longrightarrow & \end{array}$ is a pullback and f is a mono then g is a mono.

- In a commuting diagram:



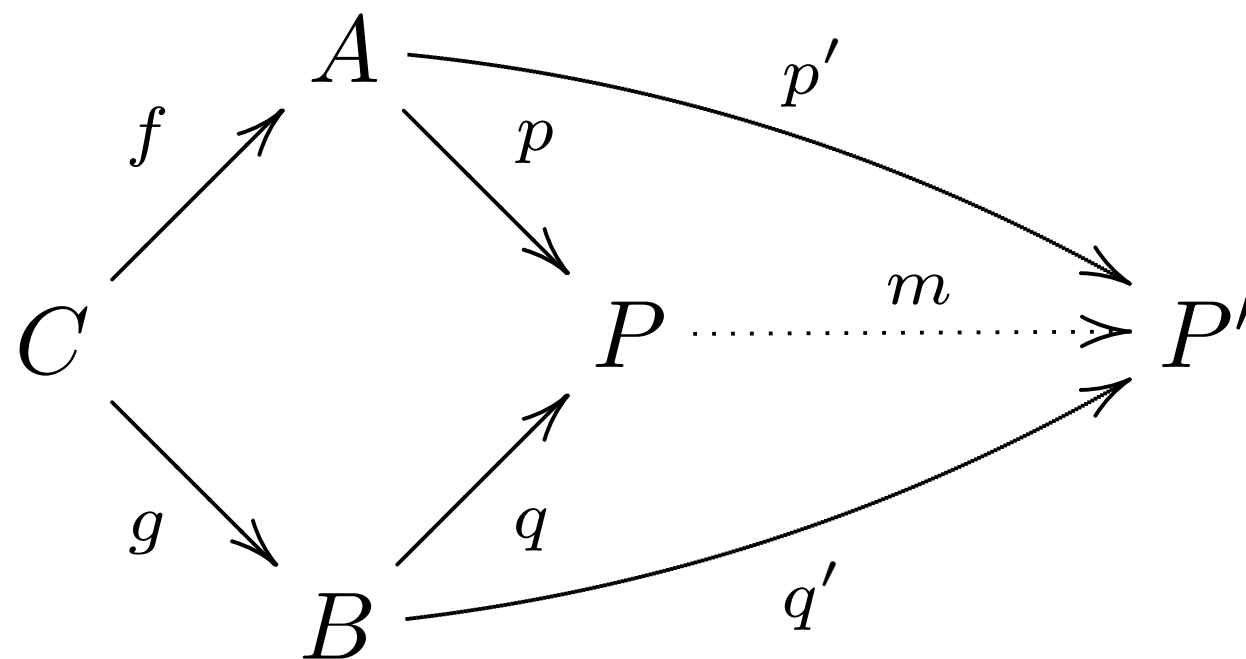
- if the two squares are pullbacks, so is the outer rectangle.
- if the right square and the outer rectangle are pullbacks, so is the left square.

Pushouts

pushout = co-pullback

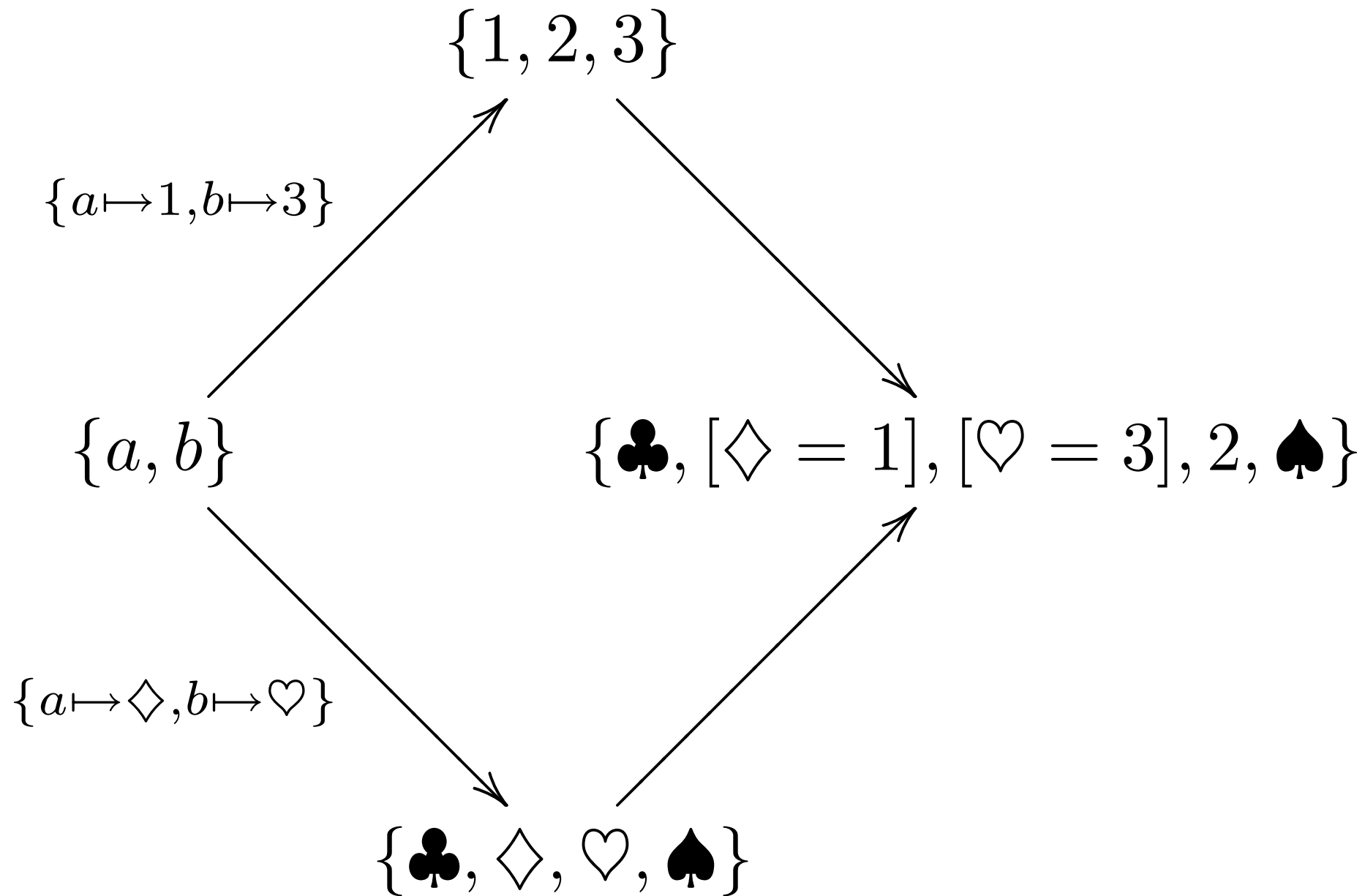
Defn. A **pushout** of two arrows $A \xleftarrow{f} C \xrightarrow{g} B$ is an object P with arrows $A \xrightarrow{p} P \xleftarrow{q} B$ s.t.:

- $p \circ f = q \circ g$,
- for every $A \xrightarrow{p'} P' \xleftarrow{q'} B$ s.t. $p' \circ f = q' \circ g$ there is a *unique* $m : P \rightarrow P'$ s.t. $m \circ p = p'$, $m \circ q = q'$.



Example. In **Sets**, take $P = (A + B) / \equiv$, where \equiv is the least equivalence on $A + B$ s.t. $f(c) \equiv g(c)$ for all $c \in C$.

Example



Pushouts glue together “shared” elements