Pullbacks

Defn. A pullback of two arrows $A \xrightarrow{f} C \xleftarrow{g} B$ is an object P with arrows $A \xleftarrow{p} P \xrightarrow{q} B$ s.t.:

- $f \circ p = g \circ q$,
- for every $A \xleftarrow{p'} P' \xrightarrow{q'} B$ s.t. $f \circ p' = g \circ q'$ there is a *unique* $m : P' \to P$ s.t. $p \circ m = p'$, $q \circ m = q'$.

$$
\begin{tikzcd}
& & A \arrow[dr, "f"] & \\
P' \arrow[r, dotted, "m"] \arrow[urr, bend left, "p'"] \arrow[drr, bend right, "q'"'] & P \arrow[ur, "p"] \arrow[dr, "q"'] & & C \\
& & B \arrow[ur, "g"'] &
\end{tikzcd}
$$

Example. In $\mathbf{Sets}$, take $P = \{(a, b) \in A \times B \mid f(a) = g(b)\}$.

(**Exercise**: check how this works when f and/or g is a mono)

Facts about pullbacks

- Pullbacks, if they exist, are unique up to isomorphism.
- if $\mathbf{C}$ has all products and equalizers, then it has all pullbacks.
- if $\mathbf{C}$ has all pullbacks and a final object, then it has all (binary) products and equalizers.
- If $\begin{tikzcd} {} \arrow[r] \arrow[d, "g"'] & {} \arrow[d, "f"] \\ {} \arrow[r] & {} \end{tikzcd}$ is a pullback and f is a mono then g is a mono.
- In a commuting diagram: $\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$
  - -- if the two squares are pullbacks, so is the outer rectangle.
  - -- if the right square and the outer rectangle are pullbacks, so is the left square.

Pushouts

pushout = *co*-pullback

Defn. A pushout of two arrows $A \xleftarrow{f} C \xrightarrow{g} B$ is an object P with arrows $A \xrightarrow{p} P \xleftarrow{q} B$ s.t.:

- $p \circ f = q \circ g$,
- for every $A \xrightarrow{p'} P' \xleftarrow{q'} B$ s.t. $p' \circ f = q \circ g'$ there is a *unique* $m : P \to P'$ s.t. $m \circ p = p', m \circ q = q'$.

$$\begin{tikzcd}
& A \arrow[dr, "p"] \arrow[drr, bend left, "p'"] & & \\
C \arrow[ur, "f"] \arrow[dr, "g"'] & & P \arrow[r, dotted, "m"] & P' \\
& B \arrow[ur, "q"'] \arrow[urr, bend right, "q'"'] & &
\end{tikzcd}$$

Example. In $\mathbf{Sets}$, take $P = (A + B)/{\equiv}$, where $\equiv$ is the least equivalence on $A + B$ s.t. $f(c) \equiv g(c)$ for all $c \in C$.

Example

$$
\begin{tikzcd}
 & \{1,2,3\} \arrow[dr] & \\
\{a,b\} \arrow[ur, "\{a\mapsto 1, b\mapsto 3\}"] \arrow[dr, "\{a\mapsto \diamondsuit, b\mapsto \heartsuit\}"'] & & \{\clubsuit, [\diamondsuit = 1], [\heartsuit = 3], 2, \spadesuit\} \\
 & \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\} \arrow[ur] &
\end{tikzcd}
$$

Pushouts glue together “shared” elements