Products

Defn. A product of two objects A, B is an object $A \times B$ with arrows $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, such that for any object C with arrows $f : C \to A$, $g : C \to B$ there is a *unique* arrow $m : C \to A \times B$ such that $\pi_1 \circ m = f$ and $\pi_2 \circ m = g$.

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dotted, "\exists!"', "m"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

(m is denoted $\langle f, g \rangle$)

Example. In $\mathbf{Sets}$, Cartesian product is a product.
In a poset, product = greatest lower bound.

Fact: Products, if they exist, are unique up to isomorphism.

Arrow pairing

Take $f : A \to B$, $g : C \to D$, and assume $A \times C$, $B \times D$ exist. The pairing of f and g:

$$f \times g : A \times C \to B \times D$$

is defined by $f \times g = \langle f \circ \pi_A, g \circ \pi_C \rangle$:

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times C \arrow[l, "\pi_A"'] \arrow[r, "\pi_C"] \arrow[d, dotted, "\exists! \; f \times g"] & C \arrow[d, "g"] \\
B & B \times D \arrow[l, "\pi_B"'] \arrow[r, "\pi_D"] & D
\end{tikzcd}$$

Fact: $(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g)$

Exercises

- Product is commutative up to iso: $A \times B \cong B \times A$

- Define a product of any family of objects.
  (What is a product of the empty family?)

- Product is associative up to iso: $A \times (B \times C) \cong (A \times B) \times C$

- What are products in $\mathbf{Mon}$? In $\mathbf{Pos}$?

- Define the category of sets and *partial* functions.
  Describe products in that category.

- Let 1 be a final object. Prove $1 \times A \cong A$.
- Let 0 be an initial object. Is $0 \times A$ always initial?

Coproducts

Defn. A coproduct of two objects A, B is an object $A + B$ with arrows $\iota_1 : A \to A + B$, $\iota_2 : B \to A + B$, such that for any object C with arrows $f : A \to C$, $g : B \to C$ there is a *unique* arrow $m : A + B \to C$ such that $m \circ \iota_1 = f$ and $m \circ \iota_2 = g$.

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "\iota_1"'] & A + B \arrow[u, dotted, "\exists!\ m"'] & B \arrow[ul, "g"'] \arrow[l, "\iota_2"] \end{tikzcd}$$

(m is denoted $[f, g]$)

Example. In $\mathbf{Sets}$, disjoint sum is a coproduct.
In a poset, coproduct = least upper bound.

Dualize facts and exercises about products

Equalizers

Defn. An equalizer of two arrows $f, g : A \to B$ is an arrow $e : E \to A$ such that:

- $f \circ e = g \circ e$, and
- for every $d : D \to A$ s.t. $f \circ d = g \circ d$ there is a *unique* $m : D \to E$ s.t. $e \circ m = d$

$$\begin{tikzcd} D \arrow[d, dotted, "\exists!"', "m"] \arrow[dr, "d"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Example. In $\mathbf{Sets}$, take $E = \{a \in A \mid f(a) = g(a)\} \subseteq A$.

Facts:
- Equalizers, if they exist, are unique up to isomorphism.
- Every equalizer is a monomorphism.
- Every epi equalizer is an isomorphism.

Coequalizers

Defn. A coequalizer of two arrows $f, g : A \to B$ is an arrow $c : B \to C$ such that:

- $c \circ f = c \circ g$, and
- for every $d : B \to D$ s.t. $d \circ f = d \circ g$ there is a *unique* $m : C \to D$ s.t. $m \circ c = d$

$$\begin{tikzcd} & & D \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[ur, "d"] \arrow[r, "c"'] & C \arrow[u, dotted, "\exists!", "m"'] \end{tikzcd}$$

Example. In $\mathbf{Sets}$, take $C = B/\!\equiv$, where $\equiv$ is the least equivalence such that $f(a) \equiv g(a)$ for all $a \in A$.

Facts:
- Coequalizers, if they exist, are unique up to isomorphism.
- Every coequalizer is an epimorphism.
- Every mono coequalizer is an isomorphism.