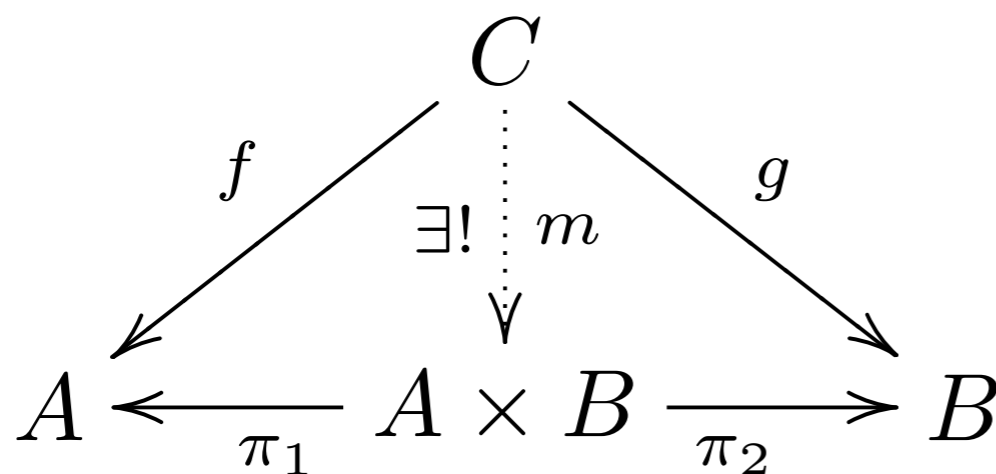


# Products

**Defn.** A **product** of two objects  $A, B$  is an object  $A \times B$  with arrows  $\pi_1 : A \times B \rightarrow A$ ,  $\pi_2 : A \times B \rightarrow B$ , such that for any object  $C$  with arrows  $f : C \rightarrow A$ ,  $g : C \rightarrow B$  there is a *unique* arrow  $m : C \rightarrow A \times B$  such that  $\pi_1 \circ m = f$  and  $\pi_2 \circ m = g$ .



(  $m$  is denoted  $\langle f, g \rangle$  )

**Example.** In Sets, Cartesian product is a product.

In a poset, product = greatest lower bound.

**Fact:** Products, if they exist, are unique up to isomorphism.

# Arrow pairing

Take  $f : A \rightarrow B$ ,  $g : C \rightarrow D$ , and assume  $A \times C$ ,  $B \times D$  exist. The **pairing** of  $f$  and  $g$ :

$$f \times g : A \times C \rightarrow B \times D$$

is defined by  $f \times g = \langle f \circ \pi_A, g \circ \pi_C \rangle$ :

$$\begin{array}{ccccc} A & \xleftarrow{\pi_A} & A \times C & \xrightarrow{\pi_C} & C \\ \downarrow f & & \downarrow \exists! f \times g & & \downarrow g \\ B & \xleftarrow{\pi_B} & B \times D & \xrightarrow{\pi_D} & D \end{array}$$

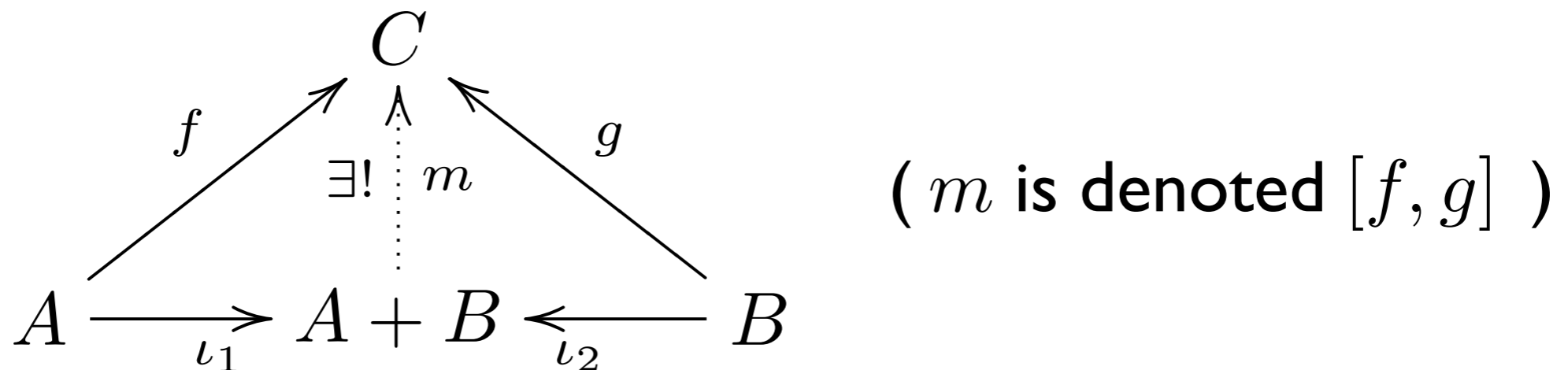
**Fact:**  $(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g)$

# Exercises

- Product is commutative up to iso:  $A \times B \cong B \times A$
- Define a product of any family of objects.  
(What is a product of the empty family?)
- Product is associative up to iso:  $A \times (B \times C) \cong (A \times B) \times C$
- What are products in **Mon**? In **Pos**?
- Define the category of sets and *partial* functions.  
Describe products in that category.
- Let  $1$  be a final object. Prove  $1 \times A \cong A$ .
- Let  $0$  be an initial object. Is  $0 \times A$  always initial?

# Coproducts

**Defn.** A **coproduct** of two objects  $A, B$  is an object  $A + B$  with arrows  $\iota_1 : A \rightarrow A + B$ ,  $\iota_2 : B \rightarrow A + B$ , such that for any object  $C$  with arrows  $f : A \rightarrow C$ ,  $g : B \rightarrow C$  there is a *unique* arrow  $m : A + B \rightarrow C$  such that  $m \circ \iota_1 = f$  and  $m \circ \iota_2 = g$ .



**Example.** In **Sets**, disjoint sum is a coproduct.

In a poset, coproduct = least upper bound.

Dualize facts and exercises about products

# Equalizers

**Defn.** An **equalizer** of two arrows  $f, g : A \rightarrow B$  is an arrow  $e : E \rightarrow A$  such that:

- $f \circ e = g \circ e$ , and
- for every  $d : D \rightarrow A$  s.t.  $f \circ d = g \circ d$  there is a *unique*  $m : D \rightarrow E$  s.t.  $e \circ m = d$

$$\begin{array}{ccccc} D & & & & \\ \text{\scriptsize } \exists! \downarrow \text{\scriptsize } & \searrow d & & & \\ \vdots & & & & \\ E & \xrightarrow{e} & A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \end{array}$$

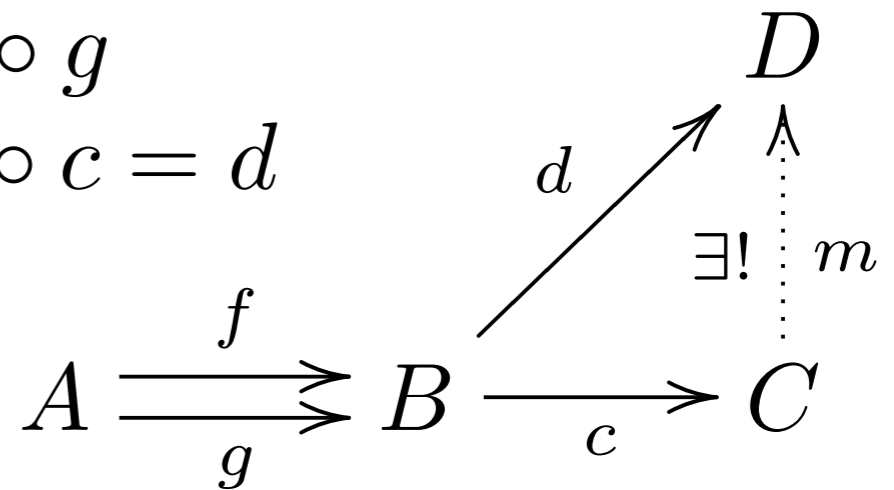
**Example.** In **Sets**, take  $E = \{a \in A \mid f(a) = g(a)\} \subseteq A$ .

- Facts:**
- Equalizers, if they exist, are unique up to isomorphism.
  - Every equalizer is a monomorphism.
  - Every epi equalizer is an isomorphism.

# Coequalizers

**Defn.** A **coequalizer** of two arrows  $f, g : A \rightarrow B$  is an arrow  $c : B \rightarrow C$  such that:

- $c \circ f = c \circ g$ , and
- for every  $d : B \rightarrow D$  s.t.  $d \circ f = d \circ g$  there is a unique  $m : C \rightarrow D$  s.t.  $m \circ c = d$



**Example.** In Sets, take  $C = B/\equiv$ , where  $\equiv$  is the least equivalence such that  $f(a) \equiv g(a)$  for all  $a \in A$ .

- Facts:**
- Coequalizers, if they exist, are unique up to isomorphism.
  - Every coequalizer is an epimorphism.
  - Every mono coequalizer is an isomorphism.