**Defn.** A **product** of two objects $A, B$ is an object $A \times B$ with arrows $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, such that for any object $C$ with arrows $f : C \to A$, $g : C \to B$ there is a **unique** arrow $m : C \to A \times B$ such that $\pi_1 \circ m = f$ and $\pi_2 \circ m = g$.

\[ C \xleftarrow{f} \exists! \xrightarrow{m} \xrightarrow{g} C \] (m is denoted $\langle f, g \rangle$)

\[ A \leftarrow \pi_1 \quad A \times B \quad \pi_2 \rightarrow B \]

**Example.** In **Sets**, Cartesian product is a product.

In a poset, product = greatest lower bound.

**Fact:** Products, if they exist, are unique up to isomorphism.
Arrow pairing

Take \( f : A \rightarrow B \), \( g : C \rightarrow D \), and assume \( A \times C \), \( B \times D \) exist. The pairing of \( f \) and \( g \):

\[
f \times g : A \times C \rightarrow B \times D
\]

is defined by \( f \times g = \langle f \circ \pi_A, g \circ \pi_C \rangle \):

\[
\begin{aligned}
A & \xleftarrow{\pi_A} A \times C \xrightarrow{\pi_C} C \\
B & \xleftarrow{\pi_B} B \times D \xrightarrow{\pi_D} D
\end{aligned}
\]

Fact: \((f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g)\)
Exercises

- Product is commutative up to iso: $A \times B \cong B \times A$

- Define a product of any family of objects. (What is a product of the empty family?)

- Product is associative up to iso: $A \times (B \times C) \cong (A \times B) \times C$

- What are products in $\text{Mon}$? In $\text{Pos}$?

- Define the category of sets and partial functions. Describe products in that category.

- Let 1 be a final object. Prove $1 \times A \cong A$.

- Let 0 be an initial object. Is $0 \times A$ always initial?
Coproducts

**Defn.** A coproduct of two objects $A, B$ is an object $A + B$ with arrows $\iota_1 : A \to A + B$, $\iota_2 : B \to A + B$, such that for any object $C$ with arrows $f : A \to C$, $g : B \to C$ there is a unique arrow $m : A + B \to C$ such that $m \circ \iota_1 = f$ and $m \circ \iota_2 = g$.

Example. In $\textbf{Sets}$, disjoint sum is a coproduct.
In a poset, coproduct = least upper bound.

Dualize facts and exercises about products.
Equalizers

**Defn.** An equalizer of two arrows $f, g : A \to B$ is an arrow $e : E \to A$ such that:
- $f \circ e = g \circ e$, and
- for every $d : D \to A$ s.t. $f \circ d = g \circ d$, there is a unique $m : D \to E$ s.t. $e \circ m = d$.

![Diagram of equalizers]

**Example.** In **Sets**, take $E = \{a \in A \mid f(a) = g(a)\} \subseteq A$.

**Facts:**
- Equalizers, if they exist, are unique up to isomorphism.
- Every equalizer is a monomorphism.
- Every epi equalizer is an isomorphism.
Coequalizers

**Defn.** A coequalizer of two arrows $f, g : A \rightarrow B$ is an arrow $c : B \rightarrow C$ such that:
- $c \circ f = c \circ g$, and
- for every $d : B \rightarrow D$ s.t. $d \circ f = d \circ g$
  there is a unique $m : C \rightarrow D$ s.t. $m \circ c = d$

**Example.** In $\textbf{Sets}$, take $C = B/\equiv$, where $\equiv$ is the least equivalence such that $f(a) \equiv g(a)$ for all $a \in A$.

**Facts:**
- Coequalizers, if they exist, are unique up to isomorphism.
- Every coequalizer is an epimorphism.
- Every mono coequalizer is an isomorphism.