Isomorphisms

Fix an arbitrary category $\mathbf{C}$ for a while.

Defn. An arrow $f : A \to B$ is an isomorphism (short: iso) if there exists $g : B \to A$ (the inverse) such that:

$$f \circ g = 1_B \qquad \text{and} \qquad g \circ f = 1_A .$$

Then A and B are isomorphic, written $A \cong B$.

Fact: The inverse, if it exists, is unique. (Denote it f^{-1}.)

Fact: The composition of two isomorphisms is an isomorphism.

Example: Isomorphisms in $\mathbf{Sets}$ are exactly bijections.

Exercise: What are isomorphisms in a poset? In a monoid?

Monos and epis

Defn. An arrow $f : A \to B$
is a monomorphism ("a mono")
if for all $g, h : C \to A$,
$f \circ g = f \circ h$ implies $g = h$.

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[rr, bend left, "f \circ g"] \arrow[rr, bend right, "f \circ h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Example: In $\mathbf{Sets}$, monos are exactly the injective functions.

Defn. An arrow $f : A \to B$
is an epimorphism ("an epi")
if for all $g, h : B \to C$,
$g \circ f = h \circ f$ implies $g = h$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] \arrow[rr, bend right, "h \circ f"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

epi is co-*mono*

Example: In $\mathbf{Sets}$, epis are exactly the surjective functions.

Monos and epis ctd.

Example: in $\mathbf{Pos}$, $\mathbf{Mon}$, $\mathbf{Grp}$, monicity = injectivity.

But epis are not necessarily surjective e.g. in $\mathbf{Mon}$,
consider the inclusion $(\mathbb{N}, +, 0)$ in $(\mathbb{Z}, +, 0)$.

Exercise: what are monos/epis in a poset? In a monoid?

Fact: composition of two monos is a mono.
Fact: if $g \circ f$ is a mono then f is a mono.

Dualize!

Fact: all isomorphisms are both mono and epi.

Exercise: Is it the case that all arrows that are both mono and epi are isomorphisms?

Sections and retractions

Defn. An arrow $f : A \to B$ is a section
if there is $g : B \to A$ (a left inverse) s.t. $g \circ f = 1_A$

Defn. An arrow $f : A \to B$ is a retraction
if there is $g : B \to A$ (a right inverse) s.t. $f \circ g = 1_B$

Fact: every section is mono, every retraction is epi.
(sections are called split monos, retractions - split epis)

Example: in $\mathbf{Sets}$, every epi is a retraction
and every mono with nonempty domain is a section.

Exercise: Find an epi in $\mathbf{Pos}$ that is not a retraction.

Fact: every isomorphism is both a section and a retraction.

Fact: every epi section is an isomorphism.

Dualize!

Initial and final objects

Defn. An object A is initial if for any object B,
there is a *unique* arrow $f : A \to B$.

Dually: An object A is final if for any object B,
there is a *unique* arrow $f : B \to A$.

Example: $\emptyset$ is initial, and singletons are final in $\mathbf{Sets}$

One-element monoids are *both* initial and final in $\mathbf{Mon}$.

The least (greatest) element is initial (final) in a poset.

Fact: Initial objects, if they exist, are unique up to isomorphism:

- any two initial objects are isomporphic,
- any object isomorphic to an initial objects is initial.

Dually, final objects are unique up to isomorphism.

Initial objects are denoted 0, final objects are denoted 1.