

Isomorphisms

Fix an arbitrary category \mathbf{C} for a while.

Defn. An arrow $f : A \rightarrow B$ is an **isomorphism** (short: iso) if there exists $g : B \rightarrow A$ (the **inverse**) such that:

$$f \circ g = 1_B \quad \text{and} \quad g \circ f = 1_A.$$

Then A and B are **isomorphic**, written $A \cong B$.

Fact: The inverse, if it exists, is unique. (Denote it f^{-1} .)

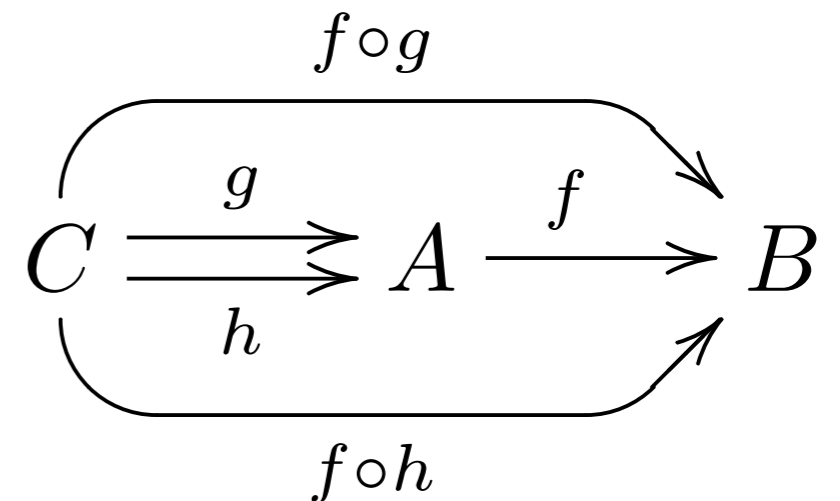
Fact: The composition of two isomorphisms is an isomorphism.

Example: Isomorphisms in Sets are exactly bijections.

Exercise: What are isomorphisms in a poset? In a monoid?

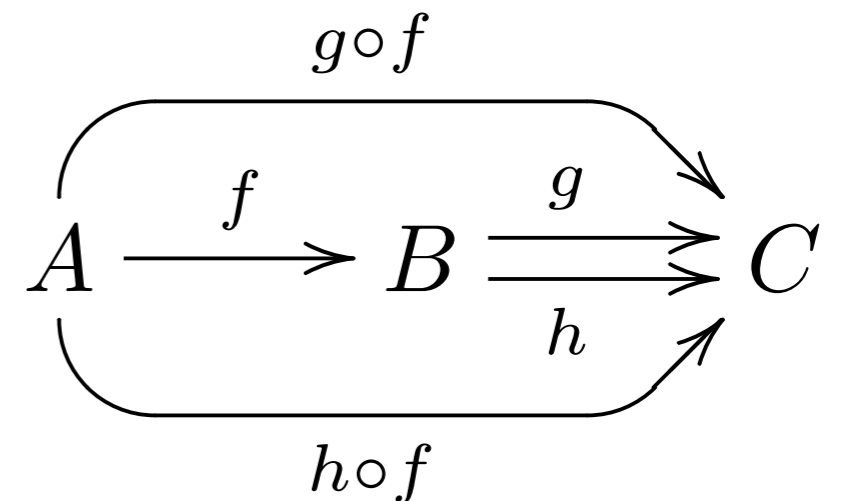
Monos and epis

Defn. An arrow $f : A \rightarrow B$ is a **monomorphism** (“a mono”) if for all $g, h : C \rightarrow A$, $f \circ g = f \circ h$ implies $g = h$.



Example: In Sets, monos are exactly the injective functions.

Defn. An arrow $f : A \rightarrow B$ is an **epimorphism** (“an epi”) if for all $g, h : B \rightarrow C$, $g \circ f = h \circ f$ implies $g = h$.



epi is co-mono

Example: In Sets, epis are exactly the surjective functions.

Monos and epis ctd.

Example: in **Pos**, **Mon**, **Grp**, monicity = injectivity.

But epis are not necessarily surjective e.g. in **Mon**, consider the inclusion $(\mathbb{N}, +, 0)$ in $(\mathbb{Z}, +, 0)$.

Exercise: what are monos/epis in a poset? In a monoid?

Fact: composition of two monos is a mono.

Fact: if $g \circ f$ is a mono then f is a mono.

Dualize!

Fact: all isomorphisms are both mono and epi.

Exercise: Is it the case that all arrows that are both mono and epi are isomorphisms?

Sections and retractions

Defn. An arrow $f : A \rightarrow B$ is a **section**
if there is $g : B \rightarrow A$ (a **left inverse**) s.t. $g \circ f = 1_A$

Defn. An arrow $f : A \rightarrow B$ is a **retraction**
if there is $g : B \rightarrow A$ (a **right inverse**) s.t. $f \circ g = 1_B$

Fact: every section is mono, every retraction is epi.
(sections are called **split monos**, retractions - **split epis**)

Example: in **Sets**, every epi is a retraction
and every mono with nonempty domain is a section.

Exercise: Find an epi in **Pos** that is not a retraction.

Fact: every isomorphism is both a section and a retraction.

Fact: every epi section is an isomorphism.

Dualize!

Initial and final objects

Defn. An object A is **initial** if for any object B , there is a *unique* arrow $f : A \rightarrow B$.

Dually: An object A is **final** if for any object B , there is a *unique* arrow $f : B \rightarrow A$.

Example: \emptyset is initial, and singletons are final in **Sets**

One-element monoids are *both* initial and final in **Mon.**

The least (greatest) element is initial (final) in a poset.

Fact: Initial objects, if they exist, are **unique up to isomorphism:**

- any two initial objects are isomorphic,
- any object isomorphic to an initial object is initial.

Dually, final objects are unique up to isomorphism.

Initial objects are denoted 0 , final objects are denoted 1 .