**Isomorphisms**

Fix an arbitrary category $\mathcal{C}$ for a while.

**Defn.** An arrow $f : A \to B$ is an **isomorphism** (short: iso) if there exists $g : B \to A$ (the **inverse**) such that:

$$f \circ g = 1_B \quad \text{and} \quad g \circ f = 1_A.$$  

Then $A$ and $B$ are **isomorphic**, written $A \cong B$.

**Fact:** The inverse, if it exists, is unique.  (Denote it $f^{-1}$.)

**Fact:** The composition of two isomorphisms is an isomorphism.

**Example:** Isomorphisms in **Sets** are exactly bijections.

**Exercise:** What are isomorphisms in a poset? In a monoid?
Monos and epis

Defn. An arrow \( f : A \to B \)

is a **monomorphism** ("a mono")

if for all \( g, h : C \to A \),

\[ f \circ g = f \circ h \] implies \( g = h \).

**Example:** In \( \text{Sets} \), monos are exactly the injective functions.

Defn. An arrow \( f : A \to B \)

is an **epimorphism** ("an epi")

if for all \( g, h : B \to C \),

\[ g \circ f = h \circ f \] implies \( g = h \).

**Example:** In \( \text{Sets} \), epis are exactly the surjective functions.
Monos and epis ctd.

Example: in Pos, Mon, Grp, monicity = injectivity.

But epis are not necessarily surjective e.g. in Mon, consider the inclusion \((\mathbb{N}, +, 0)\) in \((\mathbb{Z}, +, 0)\).

Exercise: what are monos/epis in a poset? In a monoid?

Fact: composition of two monos is a mono.
Fact: if \(g \circ f\) is a mono then \(f\) is a mono.

Fact: all isomorphisms are both mono and epi.

Exercise: Is it the case that all arrows that are both mono and epi are isomorphisms?
Sections and retractions

Defn. An arrow $f : A \rightarrow B$ is a section if there is $g : B \rightarrow A$ (a left inverse) s.t. $g \circ f = 1_A$

Defn. An arrow $f : A \rightarrow B$ is a retraction if there is $g : B \rightarrow A$ (a right inverse) s.t. $f \circ g = 1_B$

Fact: every section is mono, every retraction is epi.
(sections are called split monos, retractions - split epis)

Example: in $\textbf{Sets}$, every epi is a retraction and every mono with nonempty domain is a section.

Exercise: Find an epi in $\textbf{Pos}$ that is not a retraction.

Fact: every isomorphism is both a section and a retraction.
Fact: every epi section is an isomorphism. Dualize!
**Initial and final objects**

**Defn.** An object $A$ is **initial** if for any object $B$, there is a *unique* arrow $f : A \to B$.

**Dually:** An object $A$ is **final** if for any object $B$, there is a *unique* arrow $f : B \to A$.

**Example:** $\emptyset$ is initial, and singletons are final in $\text{Sets}$.

One-element monoids are both initial and final in $\text{Mon}$.

The least (greatest) element is initial (final) in a poset.

**Fact:** Initial objects, if they exist, are **unique up to isomorphism**:
- any two initial objects are isomorphic,
- any object isomorphic to an initial objects is initial.

Dually, final objects are unique up to isomorphism.

Initial objects are denoted $0$, final objects are denoted $1$. 