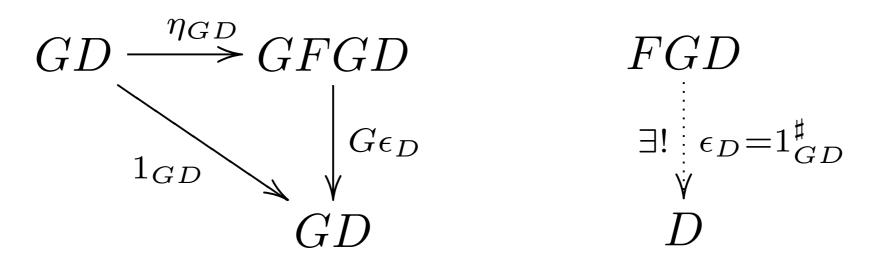
## From left to right adjoints

Let  $F: \mathbf{C} \to \mathbf{D}$  be left adjoint to  $G: \mathbf{D} \to \mathbf{C}$ 

with unit  $\eta: \mathrm{Id}_{\mathbf{C}} \to GF$ .

For any  $D \in |\mathbf{D}|$ , define  $\epsilon_D : FGD \to D$  by  $\epsilon_D = (1_{GD})^\sharp$ 

$$\mathbf{C} \leftarrow \mathcal{D}$$



Fact:  $\epsilon: FG \to \mathrm{Id}_D$  is a natural transformation.

Fact: G is right adjoint to F with counit  $\epsilon$ .

Moreover,  $G\epsilon \circ \eta G = \mathrm{id}_G$  and  $\epsilon F \circ F \eta = \mathrm{id}_F$ .

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## Adjunctions

Defn: An adjunction between categories  ${f C}$  and  ${f D}$  consists of:

- functors  $F: \mathbf{C} \to \mathbf{D}$  and  $G: \mathbf{D} \to \mathbf{C}$ ,
- natural transformations  $\eta: \mathrm{Id}_{\mathbf{C}} o GF$  ,  $\epsilon: FG o \mathrm{Id}_D$  , (unit) (counit)

such that  $G\epsilon \circ \eta G = \mathrm{id}_G$  and  $\epsilon F \circ F \eta = \mathrm{id}_F$ .

Such an adjunction is denoted  $F \dashv G$ .

Fact: In an adjunction  $F \dashv G$ , F is left adjoint to G and G is right adjoint to F.

## **Examples:**

- An adjunction where  $\eta$ ,  $\epsilon$  are natural isomorphisms is an equivalence of categories.
- Adjunctions between posets are Galois connections.

## Transposing along adjunctions

Fact: Any adjunction  $F \dashv G$  yields a bijection:

$$\hom_{\mathbf{C}}(C,GD)\cong \hom_{\mathbf{D}}(FC,D)$$
 for any  $C\in |\mathbf{C}|$ ,  $D\in |\mathbf{D}|$ 

Moreover, this bijection is natural in C and D:

$$\operatorname{Hom}_{\mathbf{C}}(-,G-) \cong \operatorname{Hom}_{\mathbf{D}}(F-,-)$$
 ( $\mathbf{C}^{\operatorname{op}} \times \mathbf{D} \to \mathbf{Sets}$ )

Fact: For any functors  $\mathbf{C} \xrightarrow{F} \mathbf{D}$ , a natural bijection as above induces an adjunction  $F \dashv G$ .

Equivalent defn: An adjunction is a pair of functors  $\mathbf{C} \xrightarrow{F} \mathbf{D}$ 

with a bijection 
$$\hom_{\mathbf{C}}(C,GD)\cong \hom_{\mathbf{D}}(FC,D)$$
 natural in  $C$  and  $D$ . right adjoint left adjoint