Facts about left adjoints

**Theorem:** Left adjoints to any fixed $G$, if they exist, are unique up to *natural* isomorphism.

**Theorem:** If $F$ is left adjoint to $G$, then:
- $F$ preserves colimits,
- $G$ preserves limits.

**Theorem:** Let $\mathcal{D}$ be locally small & complete (ie. have all limits).

A functor $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if:
- $G$ preserves limits,
- for every $C \in |\mathcal{C}|$ there exists a set 
  $$\{f_i : C \rightarrow GD_i \mid i \in \mathcal{I}\}$$
  of arrows such that
for each $D \in \mathcal{D}$ and $f : C \rightarrow GD$,
there exist $i \in \mathcal{I}$ and $g : D_i \rightarrow D$ such that
$f = Gg \circ f_i$. 

(i.e not a proper class)
Cofree objects

Consider a functor $F : C \to D$.

**Defn.** Given an object $D$ in $D$, a cofree object over $D$ w.r.t. $F$ is an object $C$ in $C$ with an arrow $\epsilon_D : FC \to D$ in $D$ (the counit arrow) such that

for every $B$ in $C$ with an arrow $f : FB \to D$

there exists a unique arrow $f^b : B \to C$ s.t. $\epsilon_D \circ Ff^b = f$.

\[ \begin{array}{cc}
C & \xrightarrow{F} & D \\
\downarrow & & \downarrow \\
FB & \xrightarrow{Ff^b} & FC \\
\downarrow & \nearrow f & \\ \\
Ff^b & \equiv & f \\
\end{array} \]

\[ \begin{array}{cc}
& \downarrow \epsilon_D & \\
FC & \xrightarrow{\epsilon_D} & D \\
\end{array} \]
Examples

- For a monotonic function \( f : C \to D \) between posets, the cofree element over \( d \in D \) is the greatest element \( c \in C \) such that \( f(c) \leq d \).

Exercise: What is the cofree object over \((A, B)\) wrt. the diagonal functor \( \Delta : C \to C \times C \)?

- Fix a set \( A \). For a functor \( A \times - : \text{Sets} \to \text{Sets} \), a cofree set over a set \( B \) is the set of functions \( B^A \).

- A cofree set over a monoid \((M, \star, 1)\) wrt. the free monoid functor \((-)^* : \text{Sets} \to \text{Mon}\) is \( M \).
Facts about cofree objects

**Fact**: For a functor $F : C \to D$, cofree objects over $D \in |D|$ are final objects in the comma category $F \downarrow K_D$ where $K_D : 1 \to D$ is the functor constant at $D$.

**Corollary**: Cofree objects, if they exist, are unique up to isomorphism.

**Fact**: If $C$ is cofree over $D$ wrt. $F : C \to D$ then for each $B \in |C|$ there is a bijection
\[
(-)^b : C(B, C) \cong D(FB, D)
\]
Cofree objects are functorial

Consider a functor $F : \mathcal{C} \to \mathcal{D}$.

If every $D \in |\mathcal{D}|$ has a cofree object $GD \in |\mathcal{C}|$ wrt. $F$ then the mapping

$$D \mapsto GD$$

$$f : D \to D' \mapsto (f \circ \epsilon_D)$$

defines a functor $G : \mathcal{D} \to \mathcal{C}$.

Further, $\epsilon : FG \to \text{Id}_D$ is a natural transformation.

\[\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
GD & \downarrow & GD' \\
Gf=(f\circ\epsilon_D) & \downarrow & Gf \quad \downarrow \\
GD' & & FGD' \xrightarrow{\epsilon_{D'}} D' \\
\end{array}\]
Right adjoints

**Defn.** A functor $G : D \to C$ is right adjoint to $F : C \to D$ with counit $\epsilon : FG \to \text{Id}_D$ if for every $D \in |D|$, $GD$ with $\epsilon_D$ is cofree over $D$ wrt. $F$.

**Examples:**

- “the” product functor $\times : C \times C \to C$ is right adjoint to the diagonal functor $\Delta : C \to C \times C$.

- For any set $A$, a right adjoint to $A \times - : \text{Sets} \to \text{Sets}$ is denoted $(-)^A$ and defined by:
  - $B^A$ - the set of functions from $A$ to $B$
  - for $f : B \to C$, $f^A = f \circ -$ : $B^A \to C^A$

**Defn:** A category is **cartesian closed** if it has final objects, products and if each functor $A \times -$ has a right adjoint.