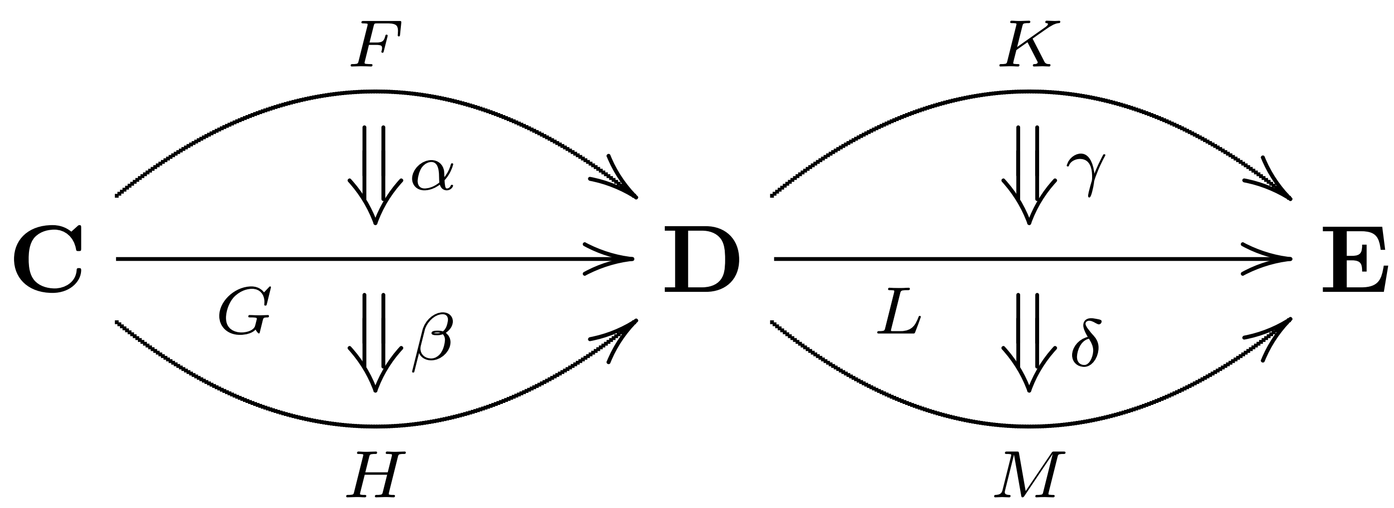
Double law

Fact: Given



there is:

$$(\delta \circ \gamma) \cdot (\beta \circ \alpha) = (\delta \cdot \beta) \circ (\gamma \cdot \alpha)$$

Coroll.: Functor composition is a functor $\cdot : \mathbf{D}^{\mathbf{C}} \times \mathbf{E}^{\mathbf{D}} \to \mathbf{E}^{\mathbf{C}}$

Defn. A 2-category $\mathbb{C}$ consists of:

- a collection $|\mathbb{C}|$ of objects
- for each A, B, a category $\mathbf{Hom}(A, B)$ with identity objects,
- composition functors:

$$\mathbf{Hom}(B, C) \times \mathbf{Hom}(A, B) \to \mathbf{Hom}(A, C)$$

s.t. ...

"The same" but non-isomorphic

- $\mathbf{Par}$: category of sets and partial functions
  -- arrow $f : A \rightharpoonup B$ is a function $f : C \to B$ for some $C \subseteq A$

- $\mathbf{Sets}_*$: category of pointed sets
  -- objects are pairs (A, a) s.t. $a \in A$
  -- arrow $f : (A, a) \to (B, b)$ is a function $f : A \to B$ s.t. $f(a) = b$

There are functors: $\mathbf{Par} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Set}_*$

$F(A) = (A + \{*\}, *)$ $\quad F(f)(a) = \begin{cases} f(a) & \text{if } a \in \mathrm{dom}(f) \\ * & \text{otherwise} \end{cases}$

$G(A, a) = A \setminus \{a\}$ $\quad G(f)(c) = \begin{cases} f(c) & \text{if } f(c) \neq b \\ \text{undefined otherwise} & \end{cases}$

But they are not mutually inverse.

Equivalence of categories

Defn. Categories $\mathbf{C}, \mathbf{D}$ are equivalent if

there exist functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ such that:

$G \circ F \cong \mathrm{Id}_{\mathbf{C}}$ $\quad F \circ G \cong \mathrm{Id}_{\mathbf{D}}$ (natural isomorphisms)

Example. $\mathbf{Par}$ and $\mathbf{Sets}_*$ are equivalent.

Theorem. $F : \mathbf{C} \to \mathbf{D}$ is (a part of) an equivalence iff it is:

- full and faithful,
- essentially surjective on objects:

$$\forall D \in |\mathbf{D}|.\ \exists C \in |\mathbf{C}|.\ F(C) \cong D$$

Equivalent categories have the same categorical properties

Exercise. If $\mathbf{C}, \mathbf{D}$ are equivalent and $\mathbf{C}$ has products then $\mathbf{D}$ has products.

Yoneda Lemma

An arrow $f : A \to B$ induces a natural transformation:

$$\mathrm{Hom}(-, f) : \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B)$$

defined by: $\mathrm{Hom}(-, f)_X(g : X \to A) = f \circ g$

Question: Are there any other nat. transfs. of this type? No!

$$Nat(\mathrm{Hom}(-, A), \mathrm{Hom}(-, B)) \cong \mathrm{hom}(A, B)$$

In fact, we can replace $\mathrm{Hom}(-, B)$ by any functor:

Yoneda Lemma: For any functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$, there is a bijection

$$Nat(\mathrm{Hom}(-, A), F) \cong FA$$

Moreover, the bijection is natural in F and X.