Examples ctd.

Consider the list functor: $L : \mathbf{Sets} \to \mathbf{Sets}$

$$L(X) = X^* \qquad L(f)(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$$

(BTW: $L = U \circ F$ for $\mathbf{Sets} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Mon}$)

Some natural transformations involving L:

- $\eta : \mathrm{Id} \to L$ $\quad \eta_X(x) = (x)$ (singleton)
- $\alpha : \mathrm{Id} \times L \to L$ $\quad \alpha_X(x, l) = x :: l$ (append)
- $\rho : L \to L$ $\quad \rho_X(l) = l^R$ (reverse)
- $\gamma : L \circ L \to L$ $\quad \gamma_X(l_1, \ldots, l_n) = l_1 \cdots l_n$ (concatenation)

Naturality means that these functions are *polymorphic*

Examples ctd.

Subsets vs. characteristic functions

- $\theta : \overleftarrow{\mathcal{P}} \to \mathrm{Hom}(-, 2) \qquad : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$

  $2 = \{\mathbf{tt}, \mathbf{ff}\}$

  $\theta_X : \mathcal{P}X \to 2^X \qquad \theta_X(Y)(x) = \mathbf{tt} \iff x \in Y$

  Fact. θ is a natural isomorphism.

- Recall $U : \mathbf{Pos} \to \mathbf{Sets}$. Define

$$\xi : U \to \mathrm{Hom}_{\mathbf{Pos}}((1, =), -)$$

$$\xi_{(A,\leq)} : A \to \mathrm{hom}_{\mathbf{Pos}}((1, =), (A, \leq))$$

$$\xi_{(A,\leq)}(a) : (1, =) \to (A, \leq)$$

$$\xi_{(A,\leq)}(a)(*) = a$$

Fact. ξ is a natural isomorphism.

Examples ctd.

- Remember $A \times B \cong B \times A$
  How to say that this isomorphism "does not depend on A, B"?

Remember $\times : \mathbf{Sets}^2 \to \mathbf{Sets}$ $\quad \times(A, B) = A \times B$

Define $\bar{\times} : \mathbf{Sets}^2 \to \mathbf{Sets}$ $\quad \bar{\times}(A, B) = B \times A$

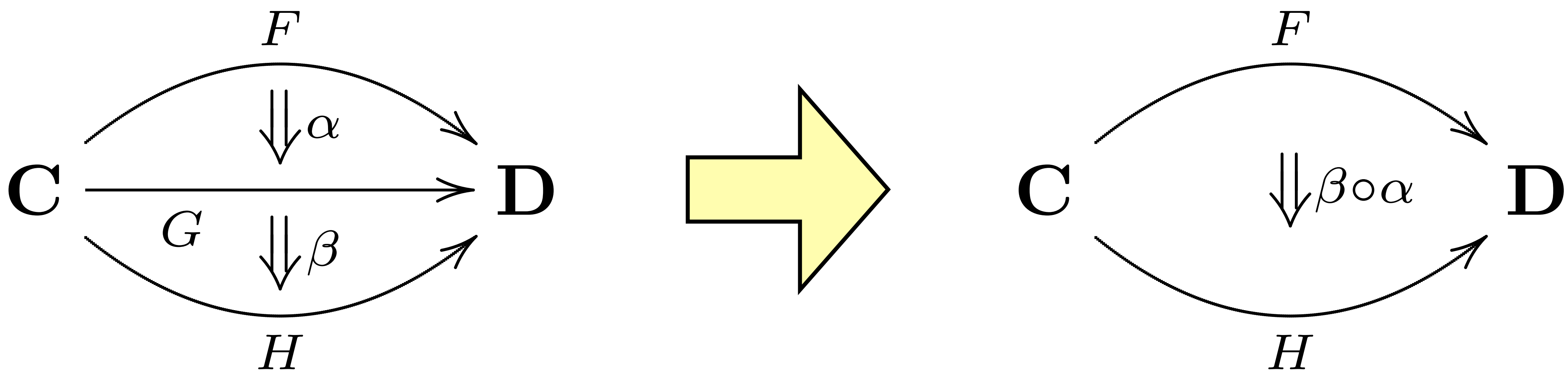
$\vartheta : \times \to \bar{\times}$ $\quad \vartheta_{X,Y}(x, y) = (y, x)$

Fact. ϑ is a natural isomorphism.

- Recall that the functor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$
  depends on a choice of products in $\mathbf{C}$.

Fact. All these product functors are naturally isomorphic.

Vertical composition



Defn. $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$

Defn. For any $\mathbf{C}, \mathbf{D}$, the functor category $\mathbf{D}^{\mathbf{C}}$ has:

- functors $F : \mathbf{C} \to \mathbf{D}$ as objects,
- natural transformations as arrows.

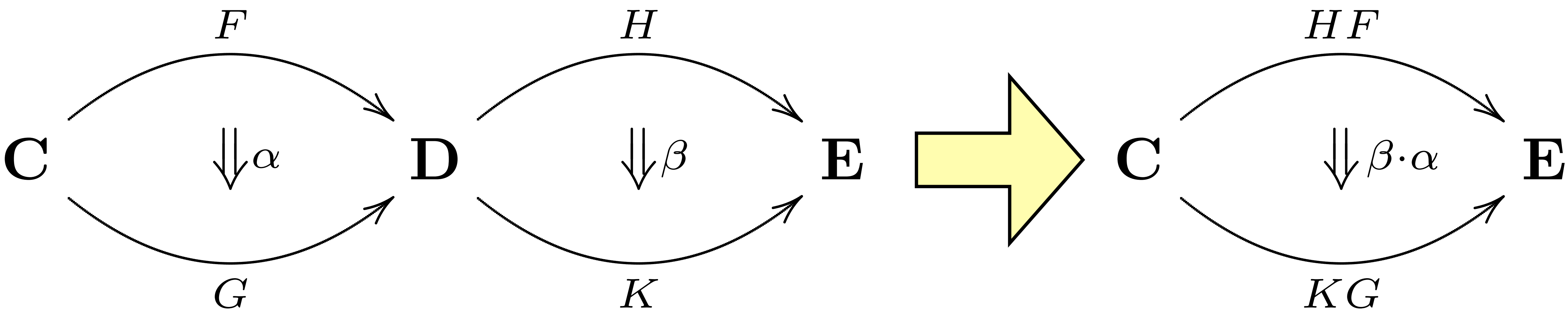
Composition is vertical, identity transformations are identities.

Example: $\mathbf{C} \times \mathbf{C} \cong \mathbf{C}^{\mathbf{2}}$ ($\mathbf{2}$ the 2-object discrete category)

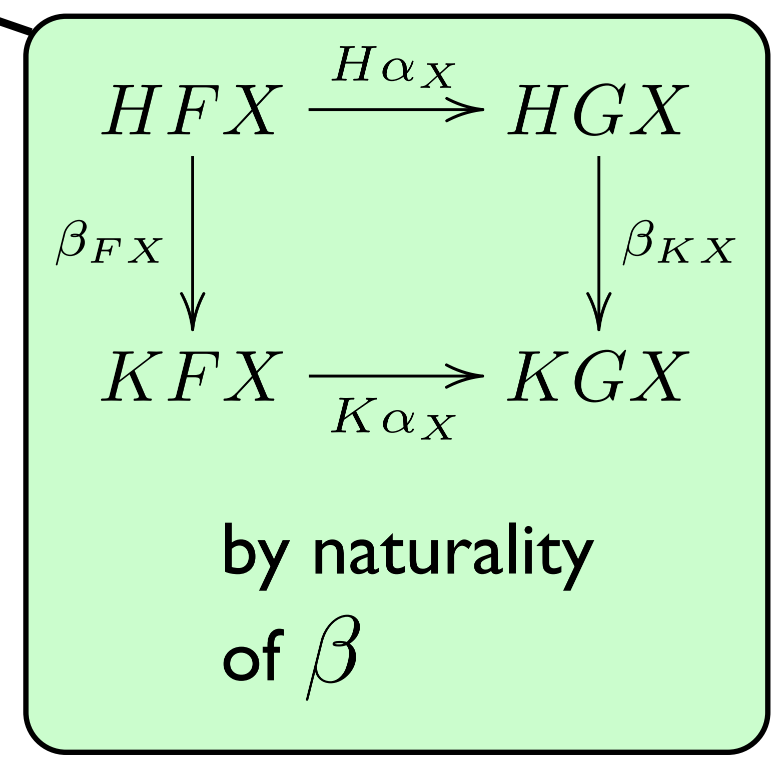
Fact: If $\mathbf{D}$ has (co)products then $\mathbf{D}^{\mathbf{C}}$ has them too.

(calculated pointwise)

Horizontal composition



Defn. $(\beta\cdot\alpha)_X = \beta_{GX}\circ H(\alpha_X) = K(\alpha_X)\circ\beta_{FX}$



Multiplication by functor: we write

$$H\alpha = \mathrm{id}_H\cdot\alpha \qquad : HF\to HG$$
$$\beta F = \beta\cdot\mathrm{id}_F \qquad : HF\to KF$$