Introduction to Category Theory for Computer Scientists

Bartek Klin

bk291@cam.ac.uk
http://www.mimuw.edu.pl/~klin
office FS15

Literature

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- S. Awodey:
  Category Theory. Oxford University Press, 2006.

- J. Adamek, H. Herrlich, G. E. Strecker:
  Abstract and Concrete Categories: the Joy of Cats.
  `http://katmat.math.uni-bremen.de/acc/acc.pdf`
- M. Barr, C. Wells:
  Category Theory Lecture Notes.
  `http://folli.loria.fr/cds/1999/library/pdf/barrwells.pdf`

A glimpse of categorical thinking

The Cartesian product of sets A and B:

$$A \times B = \{\langle a, b\rangle \mid a \in A,\ b \in B\}$$

projections:

$$\pi_1 : A \times B \to A \qquad \pi_1(\langle a, b\rangle) = a$$

$$\pi_2 : A \times B \to B \qquad \pi_2(\langle a, b\rangle) = b$$

Set described in terms of its elements

Fact: for any set C with functions $f : C \to A$ and $g : C \to B$,

there exists a unique function $m : C \to A \times B$ such that:

$$\pi_1 \circ m = f \quad \text{and} \quad \pi_2 \circ m = g\ .$$

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dashed, "m"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

Set described in terms of functions to and from other sets

Reversing arrows

Coproduct of A and B:

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "f"] \arrow[r, "i_1"'] & D \arrow[u, dashed, "m"'] & B \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

a set D with functions $i_1 : A \to D$

$i_2 : B \to D$

such that for any set C with functions

$$f : A \to C \qquad g : B \to C$$

there exists a unique function $m : D \to C$ such that:

$$m \circ i_1 = f \qquad \text{and} \qquad m \circ i_2 = g .$$

Fact: The condition is satisfied by the disjoint sum of A and B:

$$A + B = \{\langle 1, a\rangle \mid a \in A\} \cup \{\langle 2, b\rangle \mid b \in B\}$$

Category Theory

An abstract theory of functions

- Mathematical objects studied in terms of their relations to other objects
- A unified view on different mathematical structures
- Developed in 1940s for use in algebraic topology
- In Computer Science:
  - semantics of computation
  - functional programming
  - logic
  - type theory

Plan

Universality

1. Category (today)

2. Limit (week 2-3)

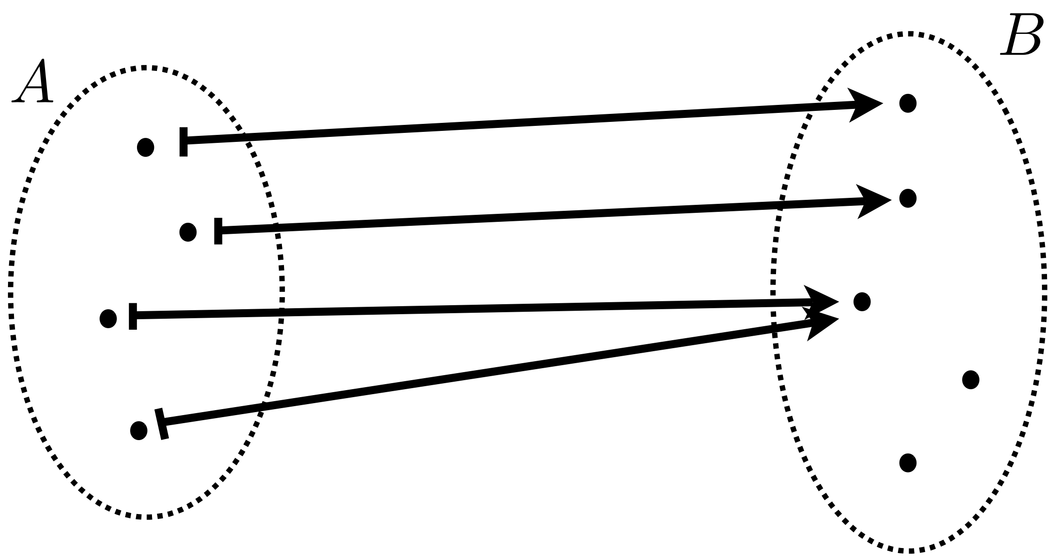
3. Functor (week 3-4)

Naturality

4. Natural transformation (week 5)

5. Adjunction (week 7)

Functions between sets



$$f : A \to B$$

$$A \xrightarrow{f} B$$

Composition:
$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, "g"'] & C \end{tikzcd} \qquad (g \circ f)(a) = g(f(a))$$

Identity:
$$1_A : A \to A \qquad 1_A(a) = a$$

Facts: for any $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

$(h \circ g) \circ f = h \circ (g \circ f)$ (composition is associative)

$1_B \circ f = f = f \circ 1_A$ (identity is unit for composition)

Categories

Definition. A category consists of:

- objects $A, B, C, \ldots$
- arrows $f, g, h, \ldots$ (also called morphisms)
- for each arrow f, there are objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$

  (we write $f : A \to B$ to say that $\mathrm{dom}(f) = A, \mathrm{cod}(f) = B$)
- for arrows $f : A \to B$, $g : B \to C$, there is an arrow

  $$g \circ f : A \to C$$

  composition
- for each A, there is an arrow $1_A : A \to A$, identity

subject to the following laws:

- $h \circ (g \circ f) = (h \circ g) \circ f$ for $f : A \to B,\ g : B \to C,\ h : C \to D$
- $1_B \circ f = f = f \circ 1_A$ for $f : A \to B.$

Bits of notation

Given a category $\mathbf{C}$,

- the collection of its objects is denoted $|\mathbf{C}|$
- the collection of its arrows is denoted $\mathrm{Ar}(\mathbf{C})$
- for any objects A, B the collection of arrows $f : A \to B$ is denoted $\hom(A, B)$ or $\mathbf{C}(A, B)$.

An equivalent definition of category:

- a collection $A, B, C, \ldots$ of objects,
- for any objects A, B, a collection $\hom(A, B)$ of arrows,
- for any A, B, C, a function

$$\circ : \hom(B, C) \times \hom(A, B) \to \hom(A, C)$$

- for any A, a distinguished arrow $1_A \in \hom(A, A)$,

such that etc.

Examples

Some finite categories:

- $\mathbf{0}$: (the empty category)

- $\mathbf{1}$: $*$

- $\mathbf{2}$: $* \longrightarrow \star$

- $\mathbf{3}$:
$$\begin{tikzcd} * \arrow[r] \arrow[rr, bend left] & \star \arrow[r] & \bullet \end{tikzcd}$$

...

(identities omitted)

- $$\begin{tikzcd} * \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \star \end{tikzcd}$$

- $$\begin{tikzcd} * \arrow[r, bend left, "f"] & \star \arrow[l, bend left, "g"] \end{tikzcd}$$ where $g \circ f = 1_*$, $f \circ g = 1_\star$

- a discrete category: category with no arrows except identities.

Categories vs. graphs

Definition. A (directed multi-)graph consists of:

- a set V of vertices
- a set E of edges
- source and target functions $s, t : E \to V$

Definition. Path in $G = (V, E, s, t)$ is a finite sequence of edges:

$$e_1 e_2 e_3 \ldots e_n \text{ s.t. } t(e_i) = s(e_{i+1}) \text{ for } i = 1..n-1$$

The category of paths on G:

- objects = vertices, arrows = paths
- $\mathrm{dom}(e_1 \ldots e_n) = s(e_1)$ and $\mathrm{cod}(e_1 \ldots e_n) = t(e_n)$
- composition = path concatenation
- identity = empty path

(a separate empty path for every vertex is needed)

Partial orders as categories

Defn. A binary relation $\leq$ on a set A is a preorder if:

- $a \leq a$ for $a \in A$ (reflexivity),
- if $a \leq b$, $b \leq c$ then $a \leq c$, for $a, b, c \in A$ (transitivity).

If, additionally,

- $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry),

then $\leq$ is a partial order (and $(A, \leq)$ is a poset).

Fact: a preorder (hence every poset) can be seen as a category.

Fact: a category s.t. for any objects A, B there is *at most one* arrow $f : A \to B$, is a preorder.

(up to size issues)

Idea: $A \to B \iff A \leq B$

Monoids as categories

Defn. A monoid consists of:

- a set M (the carrier)
- an operation $\cdot : M \times M \to M$ (the multiplication)
- an element $1 \in M$ (the unit)

such that for $x, y, z \in M$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad 1 \cdot x = x = x \cdot 1$$

Fact. Categories with one object are exactly monoids.

(up to size issues)

Idea:

- exactly one object $\bullet$
- arrows = elements of the carrier
- composition = multiplication
- identity = unit

More examples

- $\mathbf{Sets}$: sets and functions

functions tagged with codomains

- $\mathbf{Sets}_{\mathrm{fin}}$: finite sets and functions

- $\mathbf{Sets}_{1\text{-}1}$:sets and *injective* functions
  ($f : A \to B$ injective if $a \neq a'$ implies $f(a) \neq f(a')$)

- sets and *surjective* functions
  ($f : A \to B$ surjective if $\forall b \in B. \exists a \in A.\ f(a) = b$)

How about:

- sets and functions $f : A \to B$ such that
  $|f^{-1}(b)| \leq 2$ for every $b \in B$?

These functions do not compose

- sets and *non-surjective* functions?

Identity is not such