An Albebraic Approach to Internet Routing
Lectures 04 — 08

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Outline

Lexicographic Semiring, example continued

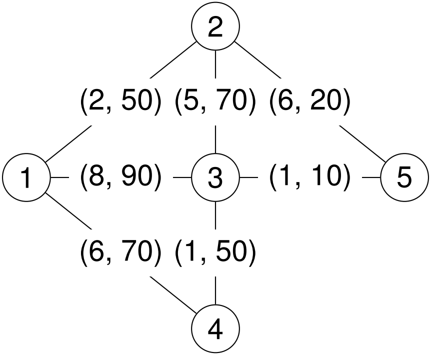
$\mathrm{sp} \vec{\times} \mathrm{bw}$

Let $(S,\ \oplus,\ \otimes,\ \overline{0},\ \overline{1}) = \mathrm{sp} \vec{\times} \mathrm{bw}$.

$$
\begin{array}{rcl}
\mathrm{sp} & = & (\mathbb{N}^{\infty},\ \mathrm{min},\ +,\ \infty,\ 0) \\
\mathrm{bw} & = & (\mathbb{N}^{\infty},\ \mathrm{max},\ \mathrm{min},\ 0,\ \infty) \\
\mathrm{sp} \vec{\times} \mathrm{bw} & = & (\mathbb{N}^{\infty} \times \mathbb{N}^{\infty},\ \mathrm{min} \vec{\times} \mathrm{max},\ + \times \mathrm{min},\ (\infty,\ 0),\ (0,\ \infty))
\end{array}
$$

$$
\begin{array}{rcl}
(17,\ 10) \oplus (21, 100) & = & (17,\ 10) \\
(17,\ 10) \oplus (17, 100) & = & (17,\ 100) \\
(17,\ 10) \otimes (21, 100) & = & (38,\ 10) \\
(17,\ 10) \otimes (17, 100) & = & (34,\ 10)
\end{array}
$$

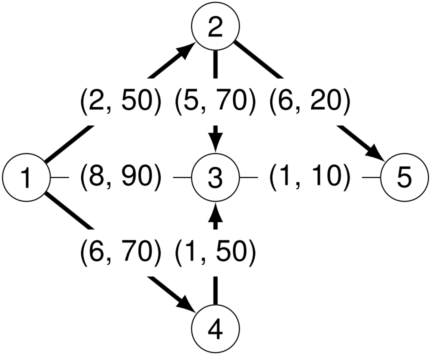
Sample instance for $\mathrm{sp} \vec{\times} \mathrm{bw}$



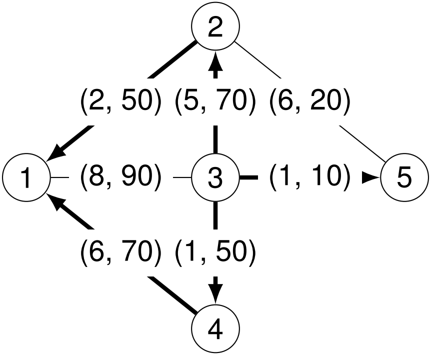
The adjacency matrix

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
(\infty,\ 0) & (2,\ 50) & (8,\ 90) & (6,\ 70) & (\infty,\ 0) \\
(2,\ 50) & (\infty,\ 0) & (5,\ 70) & (\infty,\ 0) & (6,\ 20) \\
(8,\ 90) & (5,\ 70) & (\infty,\ 0) & (1,\ 50) & (1,\ 10) \\
(6,\ 70) & (\infty,\ 0) & (1,\ 50) & (\infty,\ 0) & (\infty,\ 0) \\
(\infty,\ 0) & (6,\ 20) & (1,\ 10) & (\infty,\ 0) & (\infty,\ 0)
\end{array}\right]
\end{array}
$$

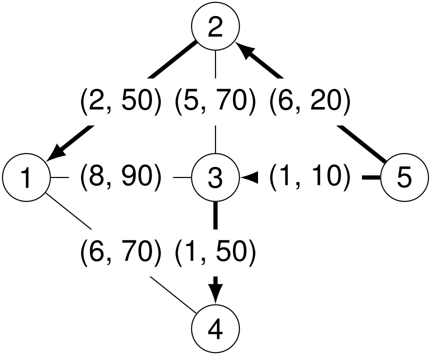
Shortest-path DAG rooted at 1



Shortest-path DAG rooted at 3



Shortest-path DAG rooted at 5



The routing matrix

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
(0,\ \infty) & (2,\ 50) & (7,\ 50) & (6,\ 70) & (8,\ 20) \\
(2,\ 50) & (0,\ \infty) & (5,\ 70) & (6,\ 50) & (6,\ 20) \\
(7,\ 50) & (5,\ 70) & (0,\ \infty) & (1,\ 50) & (1,\ 10) \\
(6,\ 70) & (6,\ 50) & (1,\ 50) & (0,\ \infty) & (2,\ 10) \\
(8,\ 20) & (6,\ 20) & (1,\ 10) & (2,\ 10) & (0,\ \infty)
\end{array}\right]
\end{array}
$$

A Strange Lexicographic Semiring

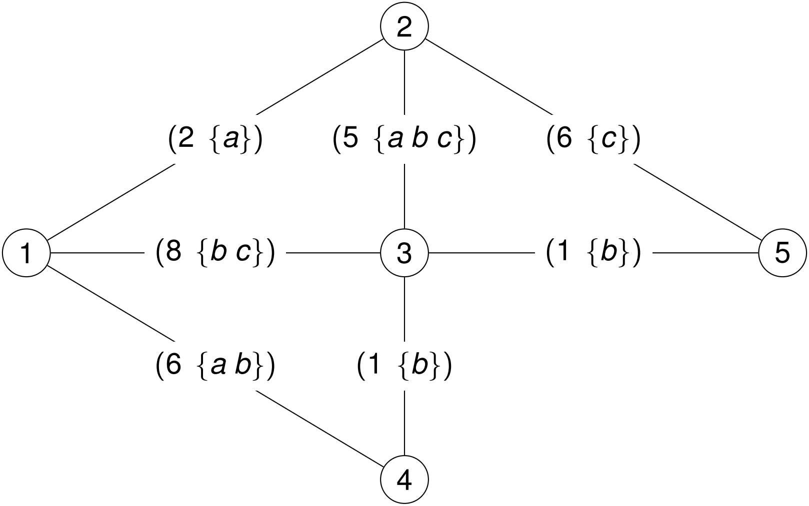
$\text{sp} \vec{\times} \text{oneforall}$

Let $(S, \oplus, \otimes, \overline{0}, \overline{1}) = \text{sp} \vec{\times} \text{oneforall}$.

$$
\begin{array}{rcl}
\text{sp} & = & (\mathbb{N}^{\infty}, \ \min, \ +, \ \infty, \ 0) \\
\text{oneforall} & = & (2^{\{a, \ b, \ c\}}, \ \cup, \ \cap, \ \{\}, \ \{a, \ b, \ c\}) \\
\text{sp} \vec{\times} \text{oneforall} & = & (\mathbb{N}^{\infty} \times 2^{\{a, \ b, \ c\}}, \ \min \vec{\times} \cup, \ + \times \cap, \ (\infty, \ \{\}), \ (0, \ \{a, \ b,
\end{array}
$$

$$
\begin{array}{rcl}
(17, \ \{a\}) \oplus (21, \{b\}) & = & (17, \ \{a\}) \\
(17, \ \{a\}) \oplus (17, \{b\}) & = & (17, \ \{a, \ b\}) \\
(17, \ \{a\}) \otimes (21, \{b\}) & = & (38, \ \{\}) \\
(17, \ \{a\}) \otimes (17, \{b\}) & = & (34, \ \{\})
\end{array}
$$

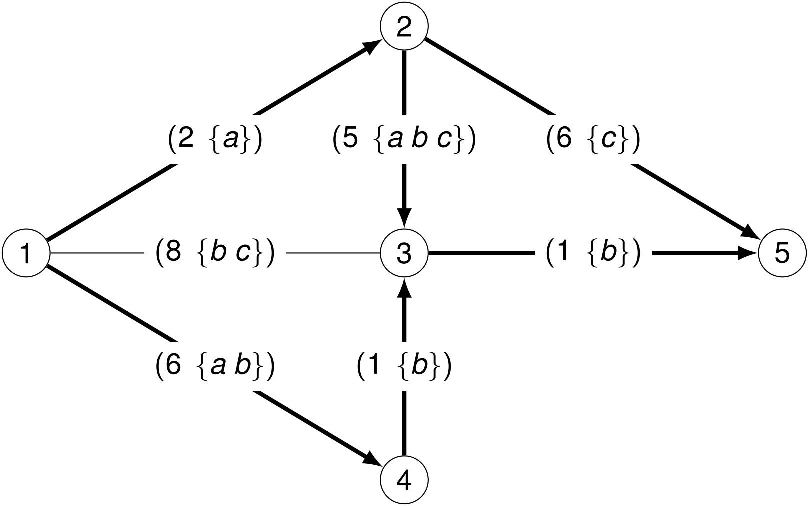
Sample instance for sp $\vec{\times}$ oneforall



The adjacency matrix

	1	2	3	4	5
1	$(\infty, \{\})$	$(2, \{a\})$	$(8, \{b, c\})$	$(6, \{a, b\})$	$(\infty, \{\})$
2	$(2, \{a\})$	$(\infty, \{\})$	$(5, \{a, b, c\})$	$(\infty, \{\})$	$(6, \{c\})$
3	$(8, \{b, c\})$	$(5, \{a, b, c\})$	$(\infty, \{\})$	$(1, \{b\})$	$(1, \{b\})$
4	$(6, \{a, b\})$	$(\infty, \{\})$	$(1, \{b\})$	$(\infty, \{\})$	$(\infty, \{\})$
5	$(\infty, \{\})$	$(6, \{c\})$	$(1, \{b\})$	$(\infty, \{\})$	$(\infty, \{\})$

Sample instance for $\mathrm{sp} \vec{\times} \mathrm{oneforall}$



Shotest paths — for the first component only — rooted at node 1

The routing matrix

If $\mathbf{R}(i, j) = (v, S)$ and $x \in S$, then there is at least one path of weight v from i to j with x in every arc weight along the path.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
(0, \{a\, b\, c\}) & (2, \{a\}) & (7, \{a, b\}) & (6, \{a, b\}) & (8, \{b\}) \\
(2, \{a\}) & (0, \{a\, b\, c\}) & (5, ?) & (6, ?) & (6, ?) \\
(7, \{a, b\}) & (5, ?) & (0, \{a\, b\, c\}) & (1, ?) & (1, ?) \\
(6, \{a, b\}) & (6, ?) & (1, ?) & (0, \{a\, b\, c\}) & (2, ?) \\
(8, \{b\}) & (6, ?) & (1, ?) & (2, ?) & (0, \{a\, b\, c\})
\end{array}\right]
\end{array}
$$

Please fill in the “?”...

Outline

Challenge

Construct a semiring path so that if $\mathbf{R}(i,\ j) = (v,\ W)$, then W is a set of all paths from i to j with weight v.

The Free Monoid over (the set) Σ

$$\mathrm{free}(\Sigma) = (\Sigma^*,\ \cdot,\ \epsilon)$$

where

- Σ^* is the set of all finite sequences over Σ,
- $\cdot$ is concatenation,
- ϵ is the empty sequence.

Given the graph $G = (V,\ E)$, we might consider using $\mathrm{free}(E)$ to represent paths.

A general construction

- $(S\otimes,\ \overline{1})$ a monoid.
- $\mathrm{uniontimes}(S,\ \otimes,\ \overline{1}) = (2^S,\ \cup,\ \otimes_\times, \{\},\ \{\overline{1}\})$, where

$$A \otimes_\times B = \{a \otimes b \mid a \in A,\ b \in B\}.$$

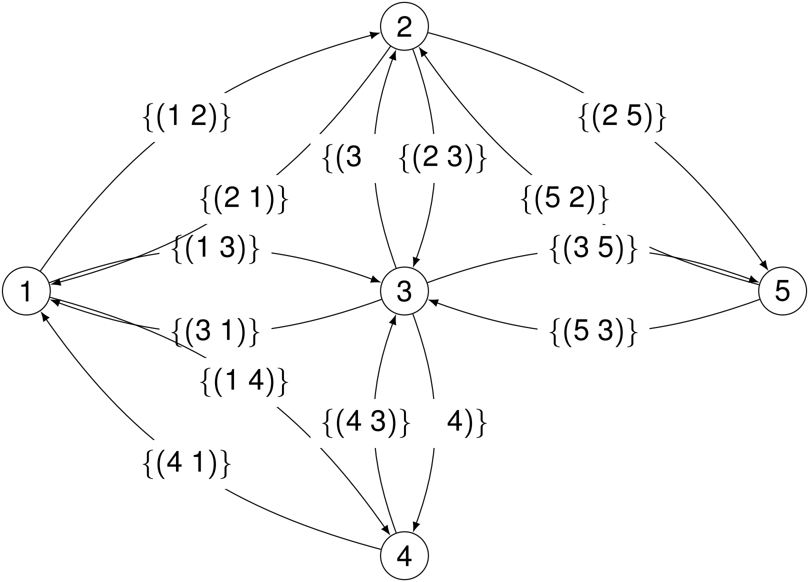
Claim

$\mathrm{uniontimes}(S,\ \otimes,\ \overline{1})$ is a semiring

Will this work?

$$\mathrm{paths} = \mathrm{uniontimes}(\mathrm{free}(E))$$

Sample instance for path



But is there a problem?

paths is not q-stable, for any q

$$
\begin{aligned}
\mathbf{R}(1,\ 5) \quad = \quad & \{(1,2)(2,5), \\
& (1,3)(3,5), \\
& (1,3)(3,1)(13)(3,5), \\
& (1,3)(3,1)(13)(3,5)(5,3)(3,2)(2,5), \\
& \ldots \\
& \}
\end{aligned}
$$

But what about $\text{sp} \vec{\times} \text{paths}$?

$$\begin{array}{rcl} \text{sp} & = & (\mathbb{N}^{\infty},\ \min,\ +,\ \infty,\ 0) \\ \text{paths} & = & (2^{E^*},\ \cup,\ \cdot_{\times},\ \{\},\ \{\epsilon\}) \\ \text{sp} \vec{\times} \text{paths} & = & (\mathbb{N}^{\infty} \times 2^{E^*},\ \min \vec{\times} \cup,\ + \times \cdot_{\times},\ (\infty,\ \{\}),\ (0,\ \{\epsilon\})) \end{array}$$

$$\begin{array}{rcl} (17,\ \{(1,2)(2,3)\}) \oplus (17, \{(1,3)\}) & = & (17,\ \{(1,2)(2,3)\}) \\ (17,\ \{(1,2)(2,3)\}) \oplus (17, \{(1,3)\}) & = & (17,\ \{(1,2)(2,3),\ (1,3)\}) \\ (17,\ \{(1,2)(2,3)\}) \otimes (21, \{(3,4),\ (3,5)\}) & = & (38,\ \{(1,2)(2,3)(3,4),\ (1,2)(2 \\ (17,\ 10) \otimes (17, 100) & = & (34,\ \{(1,2)(2,3)(3,4),\ (1,2)(2 \end{array}$$

Show that this “works”. What is going on? (on Exercises II list)

Outline

Path Weight with functions on arcs?

For graph $G = (V,\ E)$, and path $p = i_1, i_2, i_3, \cdots, i_k$.

Semiring Path Weight

Weight function $w : E \rightarrow S$

$$w(p) = w(i_1,\ i_2) \otimes w(i_2,\ i_3) \otimes \cdots \otimes w(i_{k-1},\ i_k).$$

How about functions on arcs?

Weight function $w : E \rightarrow (S \rightarrow S)$

$$w(p) = w(i_1,\ i_2)(w(i_2,\ i_3)(\cdots w(i_{k-1},\ i_k)(a)\cdots)),$$

where a is some value originated by node i_k

How can we make this work?

Algebra of Monoid Endomorphisms ([GM08])

A homomorphism is a function that preserves structure. An endomprhism is a homomorphism mapping a structure to itself.

Let $(S, \oplus, \overline{0})$ be a commutative monoid.

$(S, \oplus, F \subseteq S \rightarrow S, \overline{0}, i, \omega)$ is a algebra of monoid endomorphisms (AME) if

- $\forall f \in F \; \forall b, c \in S : f(b \oplus c) = f(b) \oplus f(c)$
- $\forall f \in F : f(\overline{0}) = \overline{0}$
- $\exists i \in F \; \forall a \in S : i(a) = a$
- $\exists \omega \in F \; \forall a \in S : \omega(a) = \overline{0}$

Solving (some) equations over a AMEs

We will be interested in solving for x equations of the form

$$x = f(x) \oplus b$$

Let

$$\begin{array}{rcl} f^0 & = & i \\ f^{k+1} & = & f \circ f^k \end{array}$$

and

$$\begin{array}{rcl} f^{(k)}(b) & = & f^0(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^k(b) \\ f^{(*)}(b) & = & f^0(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^k(b) \oplus \cdots \end{array}$$

Definition (q stability)

If there exists a q such that for all b $f^{(q)}(b) = f^{(q+1)}(b)$, then f is *q-stable*. Therefore, $f^{(*)}(b) = f^{(q)}(b)$.

Key result (again)

Lemma

If f is q-stable, then $x = f^{(*)}(b)$ *solves the AME equation*

$$x = f(x) \oplus b.$$

Proof: Substitute $f^{(*)}(b)$ for x to obtain

$$\begin{array}{ll} & f(f^{(*)}(b)) \oplus b \\ = & f(f^{(q)}(b)) \oplus b \\ = & f(f^0(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^q(b)) \oplus b \\ = & f^1(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^{q+1}(b) \oplus b \\ = & f^0(b) \oplus f^1(b) \oplus f^1(b) \oplus f^2(b) \oplus \cdots \oplus f^{q+1}(b) \\ = & f^{(q+1)}(b) \\ = & f^{(q)}(b) \\ = & f^{(*)}(b) \end{array}$$

AME of Matrices

Given an AME $S = (S,\ \oplus,\ F)$, define the semiring of $n \times n$-matrices over S,

$$\mathbb{M}_n(S) = (\mathbb{M}_n(S),\ \boxplus,\ G),$$

where for $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(S)$ we have

$$(\mathbf{A} \boxplus \mathbf{B})(i,\ j) = \mathbf{A}(i,\ j) \oplus \mathbf{B}(i,\ j).$$

Elements of the set G are represented by $n \times n$ matrices of functions in F. That is, each function in G is represented by a matrix $\mathbf{A}$ with $\mathbf{A}(i,\ j) \in F$. If $\mathbf{B} \in \mathbb{M}_n(S)$ then define $\mathbf{A}(\mathbf{B})$ so that

$$(\mathbf{A}(\mathbf{B}))(i,\ j) = \sum_{1 \le q \le n}^{\oplus} \mathbf{A}(i,\ q)(\mathbf{B}(q,\ j)).$$

Here we go again...

Path Weight

For graph $G = (V,\ E)$ with $w : E \to F$
The *weight* of a path $p = i_1, i_2, i_3, \cdots, i_k$ is then calculated as

$$w(p) = w(i_1,\ i_2)(w(i_2,\ i_3)(\cdots w(i_{k-1},\ i_k)(\omega_{\oplus})\cdots)).$$

adjacency matrix

$$\mathbf{A}(i,\ j) = \begin{cases} w(i,\ j) & \text{if } (i,\ j) \in E, \\ \omega & \text{otherwise} \end{cases}$$

We want to solve equations like these

$$\mathbf{X} = \mathbf{A}(\mathbf{X}) \boxplus \mathbf{B}$$

So why do we need Monoid Endomorphisms??

Monoid Endomorphisms can be viewed as semirings

Suppose $(S,\ \oplus,\ F)$ is a monoid of endomorphisms. We can turn it into a semiring

$$(F,\ \hat{\oplus},\ \circ)$$

where $(f \mathbin{\hat{\oplus}} g)(a) = f(a) \oplus g(a)$

Functions are hard to work with....

- All algorithms need to check equality over elements of semiring,
- $f = g$ means $\forall a \in S : f(a) = g(a)$,
- S can be very large, or infinite.

Lexicographic product of AMEs

$$(S,\ \oplus_S,\ F)\ \vec{\times}\ (T,\ \oplus_T,\ G) = (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ F \times G)$$

Theorem ([Sai70, GG07, Gur08])

$$\mathrm{M}(S\ \vec{\times}\ T) \iff \mathrm{M}(S) \wedge \mathrm{M}(T) \wedge (\mathrm{C}(S) \vee \mathrm{K}(T))$$

Where

Property	Definition
M	$\forall a, b, f : f(a \oplus b) = f(a) \oplus f(b)$
C	$\forall a, b, f : f(a) = f(b) \implies a = b$
K	$\forall a, b, f : f(a) = f(b)$

Functional Union of AMEs

$$(S,\ \oplus,\ F) +_{\mathrm{m}} (S,\ \oplus,\ G) = (S,\ \oplus,\ F + G)$$

Fact

$$\mathrm{M}(S +_{\mathrm{m}} T) \iff \mathrm{M}(S) \wedge \mathrm{M}(T)$$

Where

Property	Definition
M	$\forall a, b, f : f(a \oplus b) = f(a) \oplus f(b)$

Left and Right

right

$$\textbf{right}(S, \oplus, F) = (S, \oplus, \{i\})$$

left

$$\textbf{left}(S, \oplus, F) = (S, \oplus, K(S))$$

where $K(S)$ represents all constant functions over S. For $a \in S$, define the function $\kappa_a(b) = a$. Then $K(S) = \{\kappa_a \mid a \in S\}$.

Facts

The following are always true.

$$
\begin{array}{ll}
\text{M}(\textbf{right}(S)) & \\
\text{M}(\textbf{left}(S)) & \quad (\text{assuming } \oplus \text{ is idempotent}) \\
\text{C}(\textbf{right}(S)) & \\
\text{K}(\textbf{left}(S)) &
\end{array}
$$

Scoped Product

$$S \Theta T = (S \vec{\times} \mathbf{left}(T)) +_{\mathrm{m}} (\mathbf{right}(S) \vec{\times} T)$$

Theorem

$$\mathrm{M}(S \Theta T) \iff \mathrm{M}(S) \wedge \mathrm{M}(T).$$

Proof.

$$\begin{aligned}
& \mathrm{M}(S \Theta T) \\
& \mathrm{M}((S \vec{\times} \mathbf{left}(T)) +_{\mathrm{m}} (\mathbf{right}(S) \vec{\times} T)) \\
\iff & \mathrm{M}(S \vec{\times} \mathbf{left}(T)) \wedge \mathrm{M}(\mathbf{right}(S) \vec{\times} T) \\
\iff & \mathrm{M}(S) \wedge \mathrm{M}(\mathbf{left}(T)) \wedge (\mathrm{C}(S) \vee \mathrm{K}(\mathbf{left}(T))) \\
& \quad \wedge \mathrm{M}(\mathbf{right}(S)) \wedge \mathrm{M}(T) \wedge (\mathrm{C}(\mathbf{right}(S)) \vee \mathrm{K}(T)) \\
\iff & \mathrm{M}(S) \wedge \mathrm{M}(T)
\end{aligned}$$

Delta Product (OSPF-like?)

$$S \Delta T = (S \vec{\times} T) +_{\mathrm{m}} (\mathbf{right}(S) \vec{\times} T)$$

Theorem

$$\textsc{m}(S \Delta T) \iff \textsc{m}(S) \wedge \textsc{m}(T) \wedge (\textsc{c}(S) \vee \textsc{k}(T)).$$

Proof.

$$\begin{aligned}
& \textsc{m}(S \Theta T) \\
& \textsc{m}((S \vec{\times} T) +_{\mathrm{m}} (\mathbf{right}(S) \vec{\times} T)) \\
\iff & \textsc{m}(S \vec{\times} T) \wedge \textsc{m}(\mathbf{right}(S) \vec{\times} T) \\
\iff & \textsc{m}(S) \wedge \textsc{m}(\mathbf{left}(T)) \wedge (\textsc{c}(S) \vee \textsc{k}(T)) \\
& \quad \wedge \textsc{m}(\mathbf{right}(S)) \wedge \textsc{m}(T) \wedge (\textsc{c}(\mathbf{right}(S)) \vee \textsc{k}(T)) \\
\iff & \textsc{m}(S) \wedge \textsc{m}(T) \wedge (\textsc{c}(S) \vee \textsc{k}(T))
\end{aligned}$$

How do we represent functions?

Definition (transforms (indexed functions))

A set of transforms $(S,\ L,\ \rhd)$ is made up of non-empty sets S and L, and a function

$$\rhd \in L \rightarrow (S \rightarrow S).$$

We normally write $l \rhd s$ rather than $\rhd(l)(s)$. We can think of $l \in L$ as the index for a function $f_l(s) = l \rhd s$, so $(S,\ L,\ \rhd)$ represents the set of function $F = \{f_l \mid l \in L\}$.

Examples

Example 1: Trivial

Let $(S, \otimes)$ be a semigroup.

$$\mathrm{transform}(S, \oplus) = (S, S, \rhd_{\otimes}),$$

where $a \rhd_{\otimes} b = a \otimes b$

Example 2: Restriction

For $T \subset S$,

$$\mathrm{Restrict}(T, (S, \oplus)) = (S, T, \rhd_{\otimes}),$$

where $a \rhd_{\otimes} b = a \otimes b$

Example 3 : mildly abstract description of BGP's ASPATHs

Let $\mathrm{apaths}(X) = (\mathcal{E}(\Sigma^*) \cup \{\infty\},\ \Sigma \times \Sigma,\ \rhd)$ where

$$\begin{array}{rcl} \mathcal{E}(\Sigma^*) & = & \text{finite, elementary sequences over } \Sigma \text{ (no repeats)} \\ (m,\ n) \rhd \infty & = & \infty \\ (m,\ n) \rhd l & = & \begin{cases} n \cdot l & (\text{if } m \notin n \cdot l) \\ \infty & (\text{otherwise}) \end{cases} \end{array}$$

Exercises II

1. Complete the routing matrix for the instance of $\text{sp} \vec{\times} \text{oneforall}$ in Lecture 04.
2. Try to explain why our instance of $\text{sp} \vec{\times} \text{paths}$ (Lecture 05) has a finite routing matrix. Is the semiring 0-stable?
3. Prove that $\text{uniontimes}(S, \otimes, \overline{1})$ is a semiring.
4. Show that $(F, \hat{\oplus}, \circ)$ — from Lecture 06 — is a semiring.
5. Construct two interesting instances of the *scoped product* (Lecture 07!), each with adjacency and routing matrix.

Outline

Minimal Sets (finite anti-chains)

$\min_{\leq}(A)$

Suppose that $(S, \leq)$ is a pre-ordered set. Let $A \subseteq S$ be finite. Define

$$\min_{\leq}(A) \equiv \{a \in A \mid \forall b \in A : \neg(b < a)\}$$

Example 1

$$\begin{aligned} (S, \leq) &= (2^{\{a,\ b,\ c\}}, \subseteq) \\ \min_{\subseteq}(\{\{a,\ b,\ c\},\ \{a\}\}) &= \{\{a\}\} \\ \min_{\subseteq}(\{\{a,\ b,\ c\},\ \{a\}, \{a,\ b\},\ \{b, c\}\}) &= \{\{a\}, \{b, c\}\} \end{aligned}$$

Example 2

$$
\begin{array}{rcl}
(S, \leq) & = & (V^*, \leq) \\
V^* & = & \text{finite sequences of nodes fro} \\
p \leq q & \Longleftrightarrow & \mid p \mid \leq \mid q \mid \\
\min_{\leq}(\{(1, 3, 17), (4, 5)\}) & = & \{(4, 5)\} \\
\min_{\leq}(\{(1, 3, 17), (4, 5), (7, 8)\}) & = & \{(4, 5), (7, 8)\}
\end{array}
$$

Minimal Sets (continued)

Suppose that $(S, \leq)$ is a pre-ordered set.

$$\mathcal{P}_{\mathrm{min}}(S, \leq) \;\equiv\; \{A \subseteq S \mid A \text{ is finite and } \mathrm{min}_{\leq}(A) = A\}$$

The minset semigroup construction

$$\mathrm{minset}(S, \leq) = (\mathcal{P}^{\leq}_{\mathrm{min}}(S), \oplus^{\leq}_{\mathrm{min}})$$

is the semigroup where

$$A \oplus^{\leq}_{\mathrm{min}} B \;\equiv\; \mathrm{min}_{\leq}(A \cup B).$$

Martelli's semiring ([Mar74, Mar76])

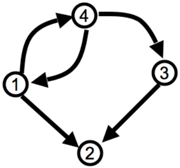
- A cut set $C \subseteq E$ for nodes i and j is a set of edges such there is no path from i to j in the graph $(V, E - C)$.
- C is minimal if no proper subset of C is a cut set.
- Martelli's semiring is such that $\mathbf{A}^{(*)}(i, j)$ is the set of all minimal cut sets for i and j.
- The arc (i, j) is has weight $w(i, j) = \{\{(i, j)\}\}$.
- S is the set of all subsets of the power set of E.
- $X \oplus Y$ is $\{x \cup y \mid x \in X, y \in Y\}$ with any non-minimal sets removed.
- $X \otimes Y$ is $X \cup Y$ with any non-minimal sets removed.

Example

$$\begin{aligned} X &= \{\{(2, 3\}, \{(1, 3), (2, 4)\}\} \\ Y &= \{\{(1, 3), (2, 3\}, \{(1, 3), (2, 4)\}\} \\ X \oplus Y &= \{\{(1, 3), (2, 3\}, \{(1, 3), (2, 4)\}\} \\ X \otimes Y &= \{\{(2, 3\}, \{(1, 3), (2, 4)\}\} \end{aligned}$$

Martelli

$$(i, j) \in E \rightarrow w(i, j) = \{\{(i, j)\}\}$$

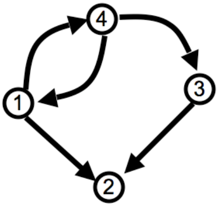


$$A = \begin{bmatrix} \{\phi\} & \{\{(1,2)\}\} & \{\phi\} & \{\{(1,4)\}\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\{(3,2)\}\} & \{\phi\} & \{\phi\} \\ \{\{(4,1)\}\} & \{\phi\} & \{\{(4,3)\}\} & \{\phi\} \end{bmatrix}$$

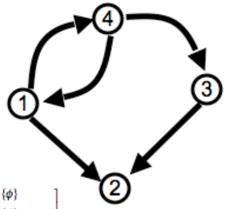
Martelli

$$A^2 = A \otimes A = \begin{bmatrix} \{\phi\} & \{\{(1,2)\}\} & \{\phi\} & \{\{(1,4)\}\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\{(3,2)\}\} & \{\phi\} & \{\phi\} \\ \{\{(4,1)\}\} & \{\phi\} & \{\{(4,3)\}\} & \{\phi\} \end{bmatrix} \otimes \begin{bmatrix} \{\phi\} & \{\{(1,2)\}\} & \{\phi\} & \{\{(1,4)\}\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\{(3,2)\}\} & \{\phi\} & \{\phi\} \\ \{\{(4,1)\}\} & \{\phi\} & \{\{(4,3)\}\} & \{\phi\} \end{bmatrix}$$

$$= \begin{bmatrix} \{\{(1,4)\},\{(4,1)\}\} & \{\phi\} & \{\{(1,4)\},\{(4,3)\}\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\{(1,2),(3,2)\},\{(1,2),(4,3)\},\{(4,1),(3,2)\},\{(4,1),(4,3)\}\} & \{\phi\} & \{\{(1,4)\},\{(4,1)\}\} \end{bmatrix}$$



Martelli



$$A = \begin{bmatrix} \{\phi\} & \{\{(1,2)\}\} & \{\phi\} & \{\{(1,4)\}\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\{(3,2)\}\} & \{\phi\} & \{\phi\} \\ \{\{(4,1)\}\} & \{\phi\} & \{\{(4,3)\}\} & \{\phi\} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \{\{(1,4)\},\{(4,1)\}\} & \{\phi\} & \{\{(1,4)\},\{(4,3)\}\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\{(1,2),(3,2)\},\{(1,2),(4,3)\},\{(4,1),(3,2)\},\{(4,1),(4,3)\}\} & \{\phi\} & \{\{(1,4)\},\{(4,1)\}\} \end{bmatrix}$$

$$A^3 = A^5 = \begin{bmatrix} \{\phi\} & \{\{(1,4)\},\{(1,2),(3,2)\},\{(1,2),(4,3)\},\{(4,1),(3,2)\},\{(4,1),(4,3)\}\} & \{\phi\} & \{\{(1,4)\},\{(4,1)\}\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\{(1,4)\},\{(4,1)\}\} & \{\phi\} & \{\{(4,1)\},\{(1,4)\},\{(4,3)\}\} & \{\phi\} \end{bmatrix}$$

$$A^4 = \begin{bmatrix} \{\{(1,4)\},\{(4,1)\}\} & \{\phi\} & \{\{(1,4)\},\{(1,4),(4,3)\}\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\phi\} & \{\phi\} & \{\phi\} \\ \{\phi\} & \{\{(4,1)\},\{(1,4)\},\{(1,2),(3,2)\},\{(1,2),(4,3)\}\} & \{\phi\} & \{\{(1,4)\},\{(1,4)\}\} \end{bmatrix}$$

$$A^{(4)} = \begin{bmatrix} \phi & \{\{(1,2),(1,4)\},\{\{(1,2),(3,2)\},\{(1,2),(4,3)\}\} & \{\{(1,4)\},\{(4,3)\}\} & \{\{(1,4)\}\} \\ \{\phi\} & \phi & \{\phi\} & \{\phi\} \\ \{\phi\} & \{(3,2)\} & \phi & \{\phi\} \\ \{\{(4,1)\}\} & \{\{(1,2),(3,2)\},\{(1,2),(4,3)\},\{(4,1),(3,2)\},\{(4,1),(4,3)\}\} & \{\{(4,3)\}\} & \phi \end{bmatrix}$$

More minset constructions (many details omitted ...)

For semirings

Suppose that $T = (S,\ \oplus,\ \otimes)$ is a semiring.

$$\begin{aligned} \mathrm{minsetL}(T) &\equiv (\mathcal{P}^{\leq^{\mathrm{L}}}_{\mathrm{min}}(S),\ \oplus^{\leq^{\mathrm{L}}}_{\mathrm{min}},\ \otimes^{\leq^{\mathrm{L}}}_{\mathrm{min}}) \\ \mathrm{minsetR}(T) &\equiv (\mathcal{P}^{\leq^{\mathrm{R}}}_{\mathrm{min}}(S),\ \oplus^{\leq^{\mathrm{R}}}_{\mathrm{min}},\ \otimes^{\leq^{\mathrm{R}}}_{\mathrm{min}}) \end{aligned}$$

where $a \leq^{\mathrm{L}} b \iff a = a \oplus b$, $a \leq^{\mathrm{R}} b \iff b = a \oplus b$, and $A \otimes^{\leq}_{\mathrm{min}} B = \mathrm{min}_{\leq}\{a \otimes b \mid a \in A,\ b \in B\}$.

For ordered semigroups

Suppose that $T = (S,\ \leq,\ \otimes)$ is a semiring.

$$\mathrm{minset}(T) \equiv (\mathcal{P}^{\leq}_{\mathrm{min}}(S),\ \oplus^{\leq}_{\mathrm{min}},\ \otimes^{\leq}_{\mathrm{min}})$$

Yet another minset constructions (many details omitted ...)

For "routing algebras"

Suppose that $T = (S,\ L,\ \leq,\ \rhd \in (L \times S) \rightarrow S)$ a routing algebra in the style of Sobrinho [Sob03, Sob05]. Then

$$\mathrm{minset}(T) \equiv (\mathcal{P}^{\leq}_{\mathrm{min}}(S),\ L,\ \oplus^{\leq}_{\mathrm{min}},\ \rhd^{\leq}_{\mathrm{min}})$$

where $\lambda \rhd^{\leq}_{\mathrm{min}} A = \mathrm{min}_{\leq}\{\lambda \rhd \mid a \in A\}$

Martelli's semiring expressed in a small language?

[NGG09]

$$\text{martelli} \quad = \quad \text{swap}(\text{minset}(\text{sg2osr}(2^E, \cup)))$$

where

$$\begin{array}{rcl}
\text{swap}(S, \oplus, \otimes) & = & (S, \otimes, \oplus) \\
\text{sg2osr}(S, \oplus) & = & (S, \leq^{\text{r}}_{\oplus}, \oplus) \\
\text{minset}(S, \leq, \otimes) & = & (\mathcal{P}^{\leq}_{\text{min}}(S), \oplus^{\leq}_{\text{min}}, \otimes^{\leq}_{\text{min}}) \\
\text{minset}(S, \leq) & = & (\mathcal{P}^{\leq}_{\text{min}}(S), \oplus^{\leq}_{\text{min}})
\end{array}$$

Outline

Local Optimality

Say that $\mathbf{R}$ is a locally optimal solution when

$$\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{R}(i,\ j) = \bigoplus_{q \in V} \mathbf{A}(i,\ q) \otimes \mathbf{R}(q,\ j) = \bigoplus_{q \in N(i)} w(i,\ q) \otimes \mathbf{R}(q,\ j),$$

where $N(i) = \{q \mid (i,\ q) \in E\}$ is the set of neighbors of i.

In other words, $\mathbf{R}(i,\ j)$ is the best possible value given the values $\mathbf{R}(q,\ j)$, for all neighbors q of i.

With Distributivity

A is an adjacency matrix over semiring S.

For Semirings, the following two problems are essentially the same — locally optimal solutions are globally optimal solutions.

Global Optimality	Local Optimality
Find **R** such that	Find **R** such that
$\mathbf{R}(i, j) = \sum^{\oplus}_{p \in P(i, j)} w(p)$	$\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}$

Prove this!

Without Distributivity

When $\otimes$ does not distribute over $\oplus$, the following two problems are distinct.

Global Optimality	Local Optimality
Find $\mathbf{R}$ such that $\mathbf{R}(i,\ j) = \sum^{\oplus}_{p \in P(i,\ j)} w(p)$	Find $\mathbf{R}$ such that $\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}$

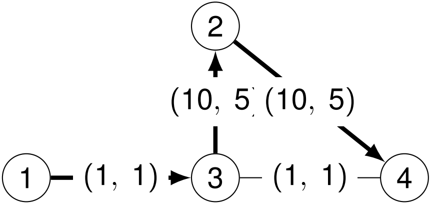
Global Optimality

This has been studied, for example [LT91b, LT91a] in the context of circuit layout. I do not know of any application of this problem to network routing. (Yet!)

Local Optimality

At a very high level, this is the type of problem that BGP attempts to solve!!

Example of local optima for $\text{bw} \vec{\times} \text{sp}$



- Node 1 would prefer the path $1 \to 3 \to 4$ with weight $(1,\ 2)$.
- But it is stuck with the best it can get: the path $1 \to 3 \to 2 \to 4$, with weight $(1,\ 11)$.

What are the conditions needed to guarantee existence of local optima?

For a non-distributed structure $S = (S, \oplus, \otimes)$, can be used to find local optima when the following property holds.

Increasing

$$\text{I} : \forall a, b \in S : a \neq \overline{0} \implies a < b \otimes a$$

where $a \leq b$ means $a = a \oplus b$.

Non-decreasing

In order to derive I we often need the non-decreasing property:

$$\text{ND} : \forall a, b \in S : a \leq b \otimes a$$

Finding local optima with the iterative method

$$
\begin{aligned}
\mathbf{A}^{[0]}(\mathbf{B}) &= \mathbf{B} \\
\mathbf{A}^{[k+1]}(\mathbf{B}) &= (\mathbf{A} \otimes \mathbf{A}^{[k]}(\mathbf{B})) \oplus \mathbf{B}
\end{aligned}
$$

Think of the iterative version as a very abstract implementation of "vectoring"....

When distributivity holds we have $A^{(k)} \otimes \mathbf{B} = A^{[k]}(\mathbf{B})$.

Claim

When S is increasing and $\oplus$ is selective and idempotent, then $\mathbf{A}^{[k]}(\mathbf{B})$ converges to a locally optimal solution.

For various flavors of proof see [GG08, kCGG06, Sob03, Sob05].
OPEN PROBLEM : no bounds are yet known!

Outline

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