Computable functions

Recall:

**Definition.** \( f \in \mathbb{N}^n \rightarrow \mathbb{N} \) is (register machine) computable if there is a register machine \( M \) with at least \( n + 1 \) registers \( R_0, R_1, \ldots, R_n \) (and maybe more) such that for all \( (x_1, \ldots, x_n) \in \mathbb{N}^n \) and all \( y \in \mathbb{N} \), the computation of \( M \) starting with \( R_0 = 0 \), \( R_1 = x_1 \), \ldots, \( R_n = x_n \) and all other registers set to 0, halts with \( R_0 = y \) if and only if \( f(x_1, \ldots, x_n) = y \).
Multiplication $f(x, y) \triangleq xy$ is computable
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$\{\text{assuming } R_3 = 0\}$

$(R_0, R_2, R_3) := (R_0 + R_2, R_2, 0)$
Multiplication $f(x, y) \triangleq xy$ is computable

If the machine is started with $(R_0, R_1, R_2, R_3) = (0, x, y, 0)$, it halts with $(R_0, R_1, R_2, R_3) = (xy, 0, y, 0)$. 
Further examples

The following arithmetic functions are all computable. (Proof—left as an exercise!)

projection: \( p(x, y) \triangleq x \)
constant: \( c(x) \triangleq n \)

truncated subtraction: \( x \cdot y \triangleq \begin{cases} x - y & \text{if } y \leq x \\ 0 & \text{if } y > x \end{cases} \)
Further examples

The following arithmetic functions are all computable. (Proof—left as an exercise!)

integer division:

\[ x \text{ div } y \triangleq \begin{cases} \text{integer part of } x/y & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases} \]

integer remainder:

\[ x \text{ mod } y \triangleq x \div y(x \text{ div } y) \]

exponentiation base 2:

\[ e(x) \triangleq 2^x \]

logarithm base 2:

\[ \log_2(x) \triangleq \begin{cases} \text{greatest } y \text{ such that } 2^y \leq x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \]
Coding Programs as Numbers
Turing/Church solution of the Entscheidungsproblem uses the idea that (formal descriptions of) algorithms can be the data on which algorithms act.

To realize this idea with Register Machines we have to be able to code RM programs as numbers. (In general, such codings are often called Gödel numberings.)

To do that, first we have to code pairs of numbers and lists of numbers as numbers. There are many ways to do that. We fix upon one...
Numerical coding of pairs

For \(x, y \in \mathbb{N}\), define

\[
\langle x, y \rangle \triangleq 2^x(2y + 1)
\]

\[
\langle x, y \rangle \triangleq 2^x(2y + 1) - 1
\]

So

\[
0\text{b}\langle x, y \rangle = \begin{array}{c}
0by \\
0b\langle x, y \rangle
\end{array}
\]

\[
10\cdots0
\]

\[
0by \\
0by
\]

\[
1\cdots1
\]

(Notation: \(0\text{b}x \triangleq x \text{ in binary.}\))

E.g. \(27 = 0\text{b}11011 = \langle 0, 13 \rangle = \langle 2, 3 \rangle\)
Numerical coding of pairs

For \( x, y \in \mathbb{N} \), define

\[
\langle x, y \rangle \triangleq 2^x(2y + 1)
\]

\[
\langle x, y \rangle \triangleq 2^x(2y + 1) - 1
\]

So

\[
0b\langle x, y \rangle = \begin{array}{c} 0by \ 1 \ 0 \ \cdots \ 0 \\
0b\langle x, y \rangle = \begin{array}{c} 0by \ 0 \ 1 \ \cdots \ 1 \\
\end{array}
\end{array}
\]

\( \langle -, - \rangle \) gives a bijection (one-one correspondence) between \( \mathbb{N} \times \mathbb{N} \) and \( \mathbb{N} \).

\( \langle -, - \rangle \) gives a bijection between \( \mathbb{N} \times \mathbb{N} \) and \( \{ n \in \mathbb{N} \mid n \neq 0 \} \).
Numerical coding of lists

\(\text{list } \mathbb{N} \triangleq\) set of all finite lists of natural numbers, using ML notation for lists:

- empty list: \([\ ]\)
- list-cons: \(x :: \ell \in \text{list } \mathbb{N}\) (given \(x \in \mathbb{N}\) and \(\ell \in \text{list } \mathbb{N}\))
- \([x_1, x_2, \ldots, x_n] \triangleq x_1 :: (x_2 :: (\cdots x_n :: [\ ] \cdots ))\)
Numerical coding of lists

`list N` ≜ set of all finite lists of natural numbers, using ML notation for lists.

For `ℓ ∈ list N`, define `⌜ ℓ ⌝ ∈ N` by induction on the length of the list `ℓ`:

\[
\begin{align*}
\text{⌜}[]\text{⌝} & \triangleq 0 \\
\text{⌜ x :: ℓ ⌝} & \triangleq \langle\langle x, \text{⌜ ℓ ⌝} \rangle\rangle = 2^x (2 \cdot \text{⌜ ℓ ⌝} + 1)
\end{align*}
\]

Thus `⌜ [x_1, x_2, \ldots, x_n] ⌝ = \langle\langle x_1, \langle\langle x_2, \ldots, \langle\langle x_n, 0 \rangle \rangle \ldots \rangle \rangle \rangle`
Numerical coding of lists

\textit{list} \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using ML notation for lists.}

For \( \ell \in \text{list} \mathbb{N} \), define \( \ulcorner \ell \urcorner \in \mathbb{N} \) by induction on the length of the list \( \ell \):

\[
\begin{cases}
\ulcorner [] \urcorner \triangleq 0 \\
\ulcorner x :: \ell \urcorner \triangleq \langle x, \ulcorner \ell \urcorner \rangle = 2^x (2 \cdot \ulcorner \ell \urcorner + 1)
\end{cases}
\]

\[
\text{Ob} \ulcorner [x_1, x_2, \ldots, x_n] \urcorner = \begin{array}{cccc}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 \\
\end{array}
\]

\( x_n \) 0's

\( x_{n-1} \) 0's

\( x_1 \) 0's
Numerical coding of lists

\[ \text{list} \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using ML notation for lists.} \]

For \( \ell \in \text{list} \mathbb{N} \), define \( \|^\ell \| \in \mathbb{N} \) by induction on the length of the list \( \ell \):

\[
\begin{align*}
\| [] \| & \triangleq 0 \\
\| x :: \ell \| & \triangleq 2^x (2 \cdot \| \ell \| + 1)
\end{align*}
\]

Hence \( \ell \mapsto \|^\ell \| \) gives a bijection from \( \text{list} \mathbb{N} \) to \( \mathbb{N} \).
**Numerical coding of lists**

**list** $\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists.

For $\ell \in \text{list} \mathbb{N}$, define $\llbracket \ell \rrbracket \in \mathbb{N}$ by induction on the length of the list $\ell$:

\[
\begin{align*}
\llbracket [] \rrbracket & \triangleq 0 \\
\llbracket x :: \ell \rrbracket & \triangleq \langle x, \llbracket \ell \rrbracket \rangle = 2^x (2 \cdot \llbracket \ell \rrbracket + 1)
\end{align*}
\]

For example:

- $\llbracket [3] \rrbracket = \llbracket 3 :: [] \rrbracket = \langle 3, 0 \rangle = 2^3 (2 \cdot 0 + 1) = 8 = 0b1000$
- $\llbracket [1, 3] \rrbracket = \langle 1, \llbracket [3] \rrbracket \rangle = \langle 1, 8 \rangle = 34 = 0b100010$
- $\llbracket [2, 1, 3] \rrbracket = \langle 2, \llbracket [1, 3] \rrbracket \rangle = \langle 2, 34 \rangle = 276 = 0b100010100$
Numerical coding of lists

$\text{list } \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using ML notation for lists.}$

For $\ell \in \text{list } \mathbb{N}$, define $\llbracket \ell \rrbracket \in \mathbb{N}$ by induction on the length of the list $\ell$:

$$
\begin{align*}
\llbracket [\ ] \rrbracket & \triangleq 0 \\
\llbracket x :: \ell \rrbracket & \triangleq \langle x, \llbracket \ell \rrbracket \rangle = 2^x (2 \cdot \llbracket \ell \rrbracket + 1)
\end{align*}
$$

For example:

$$
\begin{align*}
\llbracket [3] \rrbracket & = \llbracket 3 :: [\ ] \rrbracket = \langle 3, 0 \rangle = 2^3 (2 \cdot 0 + 1) = 8 = 0b1000 \\
\llbracket [1, 3] \rrbracket & = \langle 1, \llbracket [3] \rrbracket \rangle = \langle 1, 8 \rangle = 34 = 0b100010 \\
\llbracket [2, 1, 3] \rrbracket & = \langle 2, \llbracket [1, 3] \rrbracket \rangle = \langle 2, 34 \rangle = 276 = 0b100010100
\end{align*}
$$
Numerical coding of programs

If $P$ is the RM program

\[
L_0 : body_0 \\
L_1 : body_1 \\
\vdots \\
L_n : body_n
\]

then its numerical code is

\[
\llbracket P \rrbracket \triangleq \llbracket body_0 \rrbracket, \ldots, \llbracket body_n \rrbracket
\]

where the numerical code $\llbracket body \rrbracket$ of an instruction body

\[
\begin{cases}
\llbracket R_i^+ \rightarrow L_j \rrbracket \triangleq \langle 2i, j \rangle \\
\llbracket R_i^- \rightarrow L_j, L_k \rrbracket \triangleq \langle 2i + 1, \langle j, k \rangle \rangle \\
\llbracket HALT \rrbracket \triangleq 0
\end{cases}
\]
Any $x \in \mathbb{N}$ decodes to a unique instruction $\text{body}(x)$:

if $x = 0$ then $\text{body}(x)$ is HALT, 
else ($x > 0$ and) let $x = \langle y, z \rangle$ in 
  if $y = 2i$ is even, then 
    $\text{body}(x)$ is $R_i^+ \rightarrow L_z,$
  else $y = 2i + 1$ is odd, let $z = \langle j, k \rangle$ in 
    $\text{body}(x)$ is $R_i^- \rightarrow L_j, L_k$

So any $e \in \mathbb{N}$ decodes to a unique program $\text{prog}(e)$, called the register machine program with index $e$:

$$\text{prog}(e) \triangleq \begin{align*}
L_0 & : \text{body}(x_0) \\
\vdots \\
L_n & : \text{body}(x_n)
\end{align*}$$

where $e = \upharpoonright [x_0, \ldots, x_n]$
Example of \textit{prog}(e)

- \(786432 = 2^{19} + 2^{18} = 0b110\ldots0 = \llbracket18, 0\rrbracket\)
  - 18 "0"s

- \(18 = 0b10010 = \langle1, 4\rangle = \langle1, \langle0, 2\rangle\rangle = \llbracket R_0^- \rightarrow L_0, L_2\rrbracket\)

- \(0 = \llbracket \text{HALT} \rrbracket\)

So \(\text{prog}(786432) = \begin{array}{c}
L_0 : R_0^- \rightarrow L_0, L_2 \\
L_1 : \text{HALT}
\end{array}\)
Example of $\text{prog}(e)$

- $786432 = 2^{19} + 2^{18} = 0b110\ldots0 = \lceil [18, 0] \rceil$
  - 18 "0"s
- $18 = 0b10010 = \langle 1, 4 \rangle = \langle 1, \langle 0, 2 \rangle \rangle = \lceil R_0^- \rightarrow L_0, L_2 \rceil$
- $0 = \lceil \text{HALT} \rceil$

So $\text{prog}(786432) =$

- $L_0 : R_0^- \rightarrow L_0, L_2$
- $L_1 : \text{HALT}$

N.B. jump to label with no body (erroneous halt)
Example of $\text{prog}(e)$

- $786432 = 2^{19} + 2^{18} = 0b110\ldots0 = \lceil[18, 0]\rceil$
  
  - $18$ "0"s

- $18 = 0b10010 = \langle 1, 4 \rangle = \langle 1, \langle 0, 2 \rangle \rangle = \lceil R_{0}^{-} \rightarrow L_{0}, L_{2} \rceil$

- $0 = \lceil \text{HALT} \rceil$

So $\text{prog}(786432) = \begin{cases} L_{0} : R_{0}^{-} \rightarrow L_{0}, L_{2} \\ L_{1} : \text{HALT} \end{cases}$

N.B. In case $e = 0$ we have $0 = \lceil [] \rceil$, so $\text{prog}(0)$ is the program with an empty list of instructions, which by convention we regard as a RM that does nothing (i.e. that halts immediately).