Computable functions

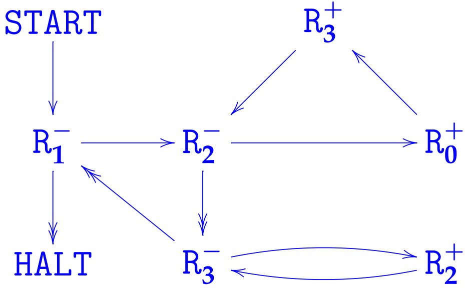
Recall:

Definition. $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ is (register machine) computable if there is a register machine M with at least $n+1$ registers R_0, R_1, . . . , R_n (and maybe more) such that for all $(x_1, \ldots, x_n) \in \mathbb{N}^n$ and all $y \in \mathbb{N}$,

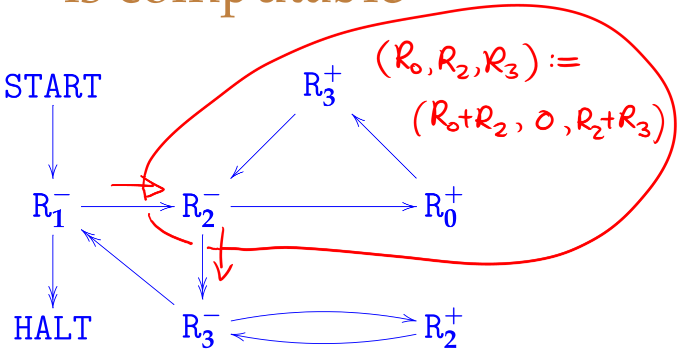
> the computation of M starting with $\mathrm{R}_0 = 0$, $\mathrm{R}_1 = x_1$, . . . , $\mathrm{R}_n = x_n$ and all other registers set to 0, halts with $\mathrm{R}_0 = y$

if and only if $f(x_1, \ldots, x_n) = y$.

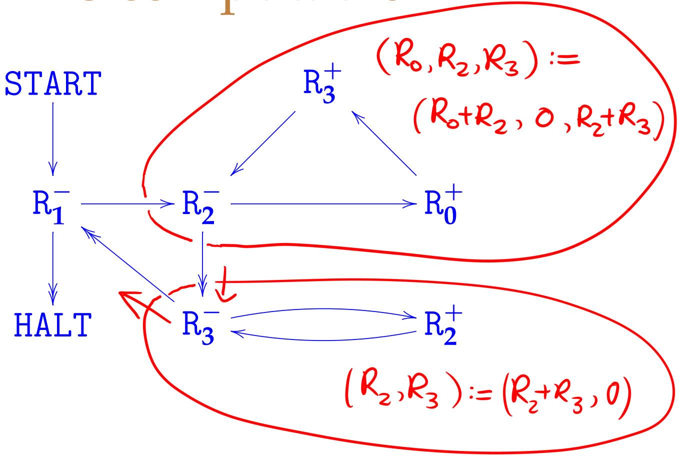
Multiplication $f(x, y) \triangleq xy$ is computable



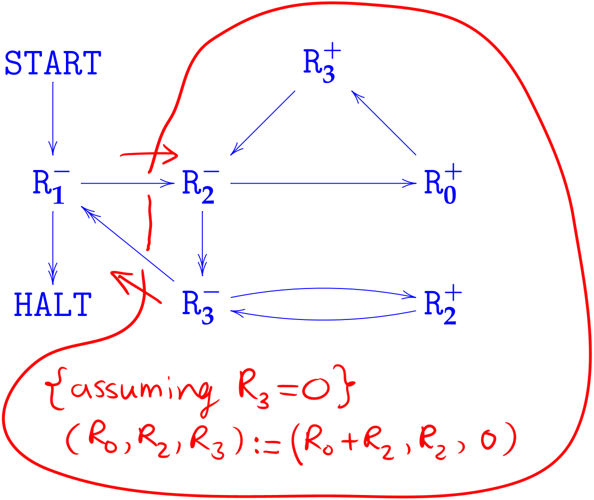
Multiplication $f(x,y) \triangleq xy$ is computable



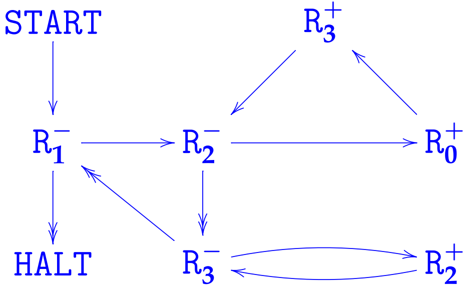
Multiplication $f(x, y) \triangleq xy$ is computable



Multiplication $f(x,y) \triangleq xy$ is computable



Multiplication $f(x, y) \triangleq xy$ is computable



If the machine is started with $(R_0, R_1, R_2, R_3) = (0, x, y, 0)$, it halts with $(R_0, R_1, R_2, R_3) = (xy, 0, y, 0)$.

Further examples

The following arithmetic functions are all computable. (Proof—left as an exercise!)

projection: $p(x, y) \triangleq x$

constant: $c(x) \triangleq n$

truncated subtraction: $x \mathbin{\dot{-}} y \triangleq \begin{cases} x - y & \text{if } y \leq x \\ 0 & \text{if } y > x \end{cases}$

Further examples

The following arithmetic functions are all computable. (Proof—left as an exercise!)

integer division:

$$x \mathit{\ div\ } y \triangleq \begin{cases} \textit{integer part of } x/y & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases}$$

integer remainder: $x \mathit{\ mod\ } y \triangleq x \dot{-} y(x \mathit{\ div\ } y)$

exponentiation base 2: $e(x) \triangleq 2^x$

logarithm base 2:

$$\log_2(x) \triangleq \begin{cases} \textit{greatest } y \textit{ such that } 2^y \leq x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Coding Programs as Numbers

Turing/Church solution of the Etscheidungsproblem uses the idea that (formal descriptions of) algorithms can be the data on which algorithms act.

To realize this idea with Register Machines we have to be able to code RM programs as numbers. (In general, such codings are often called Gödel numberings.)

To do that, first we have to code pairs of numbers and lists of numbers as numbers. There are many ways to do that. We fix upon one. . .

Numerical coding of pairs

For $x, y \in \mathbb{N}$, define $\begin{cases} \langle\!\langle x, y \rangle\!\rangle \triangleq 2^x(2y+1) \\ \langle x, y \rangle \triangleq 2^x(2y+1) - 1 \end{cases}$

So

$$\boxed{0\mathrm{b}\langle\!\langle x, y \rangle\!\rangle} = \boxed{0\mathrm{b}y \;|\; 1 \;|\; \overbrace{0 \cdots 0}^{x\ 0\text{'s}}}$$

$$\boxed{0\mathrm{b}\langle x, y \rangle} = \boxed{0\mathrm{b}y \;|\; 0 \;|\; \underbrace{1 \cdots 1}_{x\ 1\text{'s}}}$$

(Notation: $0\mathrm{b}x \triangleq$ *x in binary*.)

E.g. $27 = 0\mathrm{b}11011 = \langle\!\langle 0, 13 \rangle\!\rangle = \langle 2, 3 \rangle$

Numerical coding of pairs

For $x, y \in \mathbb{N}$, define $\begin{cases} \langle\!\langle x, y \rangle\!\rangle & \triangleq & 2^x(2y+1) \\ \langle x, y \rangle & \triangleq & 2^x(2y+1) - 1 \end{cases}$

So

$0\mathrm{b}\langle\!\langle x, y \rangle\!\rangle$	$=$	$0\mathrm{b}y$	1	$0 \cdots 0$
$0\mathrm{b}\langle x, y \rangle$	$=$	$0\mathrm{b}y$	0	$1 \cdots 1$

$\langle -, - \rangle$ gives a bijection (one-one correspondence) between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

$\langle\!\langle -, - \rangle\!\rangle$ gives a bijection between $\mathbb{N} \times \mathbb{N}$ and $\{n \in \mathbb{N} \mid n \neq 0\}$.

Numerical coding of lists

$\mathit{list}\,\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists:

- empty list: $[]$
- list-cons: $x :: \ell \in \mathit{list}\,\mathbb{N}$ (given $x \in \mathbb{N}$ and $\ell \in \mathit{list}\,\mathbb{N}$)
- $[x_1, x_2, \ldots, x_n] \triangleq x_1 :: (x_2 :: (\cdots x_n :: [] \cdots))$

Numerical coding of lists

$\mathit{list}\,\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists.

For $\ell \in \mathit{list}\,\mathbb{N}$, define $\ulcorner \ell \urcorner \in \mathbb{N}$ by induction on the length of the list ℓ:

$$\begin{cases} \ulcorner [] \urcorner & \triangleq \; 0 \\ \ulcorner x :: \ell \urcorner & \triangleq \; \langle\langle x, \ulcorner \ell \urcorner \rangle\rangle = 2^x (2 \cdot \ulcorner \ell \urcorner + 1) \end{cases}$$

Thus $\ulcorner [x_1, x_2, \ldots, x_n] \urcorner = \langle\langle x_1, \langle\langle x_2, \cdots \langle\langle x_n, 0 \rangle\rangle \cdots \rangle\rangle \rangle\rangle$

Numerical coding of lists

$list\,\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists.

For $\ell \in list\,\mathbb{N}$, define $\ulcorner \ell \urcorner \in \mathbb{N}$ by induction on the length of the list ℓ:

$$\begin{cases} \ulcorner [\,] \urcorner & \triangleq \; \mathbf{0} \\ \ulcorner x :: \ell \urcorner & \triangleq \; \langle\!\langle x, \ulcorner \ell \urcorner \rangle\!\rangle = \mathbf{2}^x(\mathbf{2} \cdot \ulcorner \ell \urcorner + \mathbf{1}) \end{cases}$$

$$\boxed{0\mathrm{b}\ulcorner [x_1, x_2, \ldots, x_n] \urcorner} = \boxed{\mathbf{1}\;\big|\;\underbrace{\mathbf{0} \cdots \mathbf{0}}_{x_n\text{ 0's}}}\;\boxed{\mathbf{1}\;\big|\;\underbrace{\mathbf{0} \cdots \mathbf{0}}_{x_{n-1}\text{ 0's}}} \cdots \boxed{\mathbf{1}\;\big|\;\underbrace{\mathbf{0} \cdots \mathbf{0}}_{x_1\text{ 0's}}}$$

Numerical coding of lists

$\mathit{list}\,\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists.

For $\ell \in \mathit{list}\,\mathbb{N}$, define $\ulcorner \ell \urcorner \in \mathbb{N}$ by induction on the length of the list ℓ:

$$\begin{cases} \ulcorner [] \urcorner & \triangleq \; \mathbf{0} \\ \ulcorner x :: \ell \urcorner & \triangleq \; \langle\!\langle x, \ulcorner \ell \urcorner \rangle\!\rangle = \mathbf{2}^{x}(\mathbf{2} \cdot \ulcorner \ell \urcorner + \mathbf{1}) \end{cases}$$

$$\boxed{0\mathrm{b} \ulcorner [x_1, x_2, \ldots, x_n] \urcorner} = \boxed{\mathbf{1} \mid \mathbf{0} \cdots \mathbf{0}} \; \boxed{\mathbf{1} \mid \mathbf{0} \cdots \mathbf{0}} \cdots \boxed{\mathbf{1} \mid \mathbf{0} \cdots \mathbf{0}}$$

Hence $\ell \mapsto \ulcorner \ell \urcorner$ gives a bijection from $\mathit{list}\,\mathbb{N}$ to $\mathbb{N}$.

Numerical coding of lists

$\mathit{list}\,\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists.

For $\ell \in \mathit{list}\,\mathbb{N}$, define $\ulcorner \ell \urcorner \in \mathbb{N}$ by induction on the length of the list ℓ:

$$\begin{cases} \ulcorner [\,] \urcorner & \triangleq 0 \\ \ulcorner x :: \ell \urcorner & \triangleq \langle\!\langle x, \ulcorner \ell \urcorner \rangle\!\rangle = 2^x (2 \cdot \ulcorner \ell \urcorner + 1) \end{cases}$$

For example:

$\ulcorner [3] \urcorner = \ulcorner 3 :: [\,] \urcorner = \langle\!\langle 3, 0 \rangle\!\rangle = 2^3(2 \cdot 0 + 1) = 8 = 0\mathrm{b}1000$

$\ulcorner [1,3] \urcorner = \langle\!\langle 1, \ulcorner [3] \urcorner \rangle\!\rangle = \langle\!\langle 1, 8 \rangle\!\rangle = 34 = 0\mathrm{b}100010$

$\ulcorner [2,1,3] \urcorner = \langle\!\langle 2, \ulcorner [1,3] \urcorner \rangle\!\rangle = \langle\!\langle 2, 34 \rangle\!\rangle = 276 = 0\mathrm{b}100010100$

Numerical coding of lists

$\mathit{list}\,\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists.

For $\ell \in \mathit{list}\,\mathbb{N}$, define $\ulcorner \ell \urcorner \in \mathbb{N}$ by induction on the length of the list ℓ:

$$\begin{cases} \ulcorner [] \urcorner & \triangleq \; 0 \\ \ulcorner x :: \ell \urcorner & \triangleq \; \langle\!\langle x, \ulcorner \ell \urcorner \rangle\!\rangle = 2^x (2 \cdot \ulcorner \ell \urcorner + 1) \end{cases}$$

For example:

$\ulcorner [3] \urcorner = \ulcorner 3 :: [] \urcorner = \langle\!\langle 3, 0 \rangle\!\rangle = 2^3 (2 \cdot 0 + 1) = 8 = 0\mathrm{b}1000$

$\ulcorner [1,3] \urcorner = \langle\!\langle 1, \ulcorner [3] \urcorner \rangle\!\rangle = \langle\!\langle 1, 8 \rangle\!\rangle = 34 = 0\mathrm{b}100010$

$\ulcorner [2,1,3] \urcorner = \langle\!\langle 2, \ulcorner [1,3] \urcorner \rangle\!\rangle = \langle\!\langle 2, 34 \rangle\!\rangle = 276 = 0\mathrm{b}100010100$

Numerical coding of programs

If P is the RM program $\boxed{\begin{array}{c} L_0 : body_0 \\ L_1 : body_1 \\ \vdots \\ L_n : body_n \end{array}}$

then its numerical code is

$$\ulcorner P \urcorner \triangleq \ulcorner [\ulcorner body_0 \urcorner, \ldots, \ulcorner body_n \urcorner] \urcorner$$

where the numerical code $\ulcorner body \urcorner$ of an instruction body is defined by:

$$\begin{cases} \ulcorner R_i^+ \to L_j \urcorner & \triangleq & \langle\!\langle 2i, j \rangle\!\rangle \\ \ulcorner R_i^- \to L_j, L_k \urcorner & \triangleq & \langle\!\langle 2i+1, \langle j, k \rangle \rangle\!\rangle \\ \ulcorner \mathtt{HALT} \urcorner & \triangleq & 0 \end{cases}$$

Any $x \in \mathbb{N}$ decodes to a unique instruction $body(x)$:

if $x = 0$ then $body(x)$ is HALT,
else ($x > 0$ and) let $x = \langle\langle y, z \rangle\rangle$ in
if $y = 2i$ is even, then
$body(x)$ is $R_i^+ \to L_z$,
else $y = 2i + 1$ is odd, let $z = \langle j, k \rangle$ in
$body(x)$ is $R_i^- \to L_j, L_k$

So any $e \in \mathbb{N}$ decodes to a unique program $prog(e)$, called the register machine program with index e:

$$prog(e) \triangleq \boxed{\begin{array}{c} L_0 : body(x_0) \\ \vdots \\ L_n : body(x_n) \end{array}} \quad \text{where } e = \ulcorner [x_0, \ldots, x_n] \urcorner$$

Example of $prog(e)$

- $786432 = 2^{19} + 2^{18} = 0b11\underbrace{0\ldots0}_{18\ "0"s} = \ulcorner[18,0]\urcorner$
- $18 = 0b10010 = \langle\langle 1,4\rangle\rangle = \langle\langle 1,\langle 0,2\rangle\rangle\rangle = \ulcorner \mathtt{R}_0^- \to \mathtt{L}_0, \mathtt{L}_2 \urcorner$
- $0 = \ulcorner\mathtt{HALT}\urcorner$

So $prog(786432) = \boxed{\begin{array}{l} \mathtt{L}_0 : \mathtt{R}_0^- \to \mathtt{L}_0, \mathtt{L}_2 \\ \mathtt{L}_1 : \mathtt{HALT} \end{array}}$

Example of $prog(e)$

- $786432 = 2^{19} + 2^{18} = 0b11\underbrace{0\ldots0}_{18\ ''0''s} = \ulcorner[18,0]\urcorner$
- $18 = 0b10010 = \langle\langle 1,4 \rangle\rangle = \langle\langle 1, \langle 0,2 \rangle \rangle\rangle = \ulcorner R_0^- \to L_0, L_2 \urcorner$
- $0 = \ulcorner \mathtt{HALT} \urcorner$

So $prog(786432) = $

$$\boxed{\begin{array}{l} L_0 : R_0^- \to L_0, L_2 \\ L_1 : \mathtt{HALT} \end{array}}$$

N.B. jump to label with no body (erroneous halt)

Example of $prog(e)$

- $786432 = 2^{19} + 2^{18} = 0\text{b}11\underbrace{0\ldots0}_{18\ \text{"0"s}} = \ulcorner[18,0]\urcorner$
- $18 = 0\text{b}10010 = \langle\!\langle 1,4 \rangle\!\rangle = \langle\!\langle 1, \langle 0,2 \rangle \rangle\!\rangle = \ulcorner \mathtt{R}_0^- \to \mathtt{L}_0, \mathtt{L}_2 \urcorner$
- $0 = \ulcorner \mathtt{HALT} \urcorner$

So $prog(786432) = $ $\boxed{\begin{array}{l} \mathtt{L}_0 : \mathtt{R}_0^- \to \mathtt{L}_0, \mathtt{L}_2 \\ \mathtt{L}_1 : \mathtt{HALT} \end{array}}$

N.B. In case $e = 0$ we have $0 = \ulcorner[\,]\urcorner$, so $prog(0)$ is the program with an empty list of instructions, which by convention we regard as a RM that does nothing (i.e. that halts immediately).