Primitive recursion

Theorem. Given $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ and $g \in \mathbb{N}^{n+2} \rightharpoonup \mathbb{N}$, there is a unique $h \in \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$ satisfying

$$\begin{cases} h(\vec{x}, 0) & \equiv f(\vec{x}) \\ h(\vec{x}, x+1) & \equiv g(\vec{x}, x, h(\vec{x}, x)) \end{cases}$$

for all $\vec{x} \in \mathbb{N}^n$ and $x \in \mathbb{N}$.

We write $\rho^n(f, g)$ for h and call it the partial function defined by primitive recursion from f and g.

Representing primitive recursion

If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ-term F and $g \in \mathbb{N}^{n+2} \to \mathbb{N}$ is represented by a λ-term G,

we want to show λ-definability of the unique $h \in \mathbb{N}^{n+1} \to \mathbb{N}$ satisfying

$$\begin{cases} h(\vec{a}, 0) & = f(\vec{a}) \\ h(\vec{a}, a+1) & = g(\vec{a}, a, h(\vec{a}, a)) \end{cases}$$

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$$\begin{aligned} h(\vec{a}, a) = {} & \mathit{if}\ a = 0\ \mathit{then}\ f(\vec{a}) \\ & \mathit{else}\ g(\vec{a}, a-1, h(\vec{a}, a-1)) \end{aligned}$$

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$h \in \mathbb{N}^{n+1} \to \mathbb{N}$ satisfying $\boxed{h = \Phi_{f,g}(h)}$

where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by

$$\begin{aligned}\Phi_{f,g}(h)(\vec{a}, a) \triangleq\ & \textit{if}\ a = 0\ \textit{then}\ f(\vec{a}) \\ & \textit{else}\ g(\vec{a}, a-1, h(\vec{a}, a-1))\end{aligned}$$

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Strategy:

- show that $\Phi_{f,g}$ is λ-definable;
- show that we can solve fixed point equations $\boxed{X = M\,X}$ up to β-conversion in the λ-calculus.

Representing booleans

$$\begin{aligned}\textsf{True} &\triangleq \lambda x\, y.\, x\\ \textsf{False} &\triangleq \lambda x\, y.\, y\\ \textsf{If} &\triangleq \lambda f\, x\, y.\, f\, x\, y\end{aligned}$$

satisfy

- **If True** $M\ N =_\beta$ **True** $M\ N =_\beta M$
- **If False** $M\ N =_\beta$ **False** $M\ N =_\beta N$

Representing test-for-zero

$$\mathsf{Eq}_0 \triangleq \lambda x.\, x(\lambda y.\,\mathsf{False})\,\mathsf{True}$$

satisfies

- $\mathsf{Eq}_0\, \underline{0} =_\beta \underline{0}\,(\lambda y.\,\mathsf{False})\,\mathsf{True}$
  $=_\beta \mathsf{True}$
- $\mathsf{Eq}_0\, \underline{n+1} =_\beta \underline{n+1}\,(\lambda y.\,\mathsf{False})\,\mathsf{True}$
  $=_\beta (\lambda y.\,\mathsf{False})^{n+1}\,\mathsf{True}$
  $=_\beta (\lambda y.\,\mathsf{False})((\lambda y.\,\mathsf{False})^{n}\,\mathsf{True})$
  $=_\beta \mathsf{False}$

Representing ordered pairs

$$\begin{aligned} \mathsf{Pair} &\triangleq \lambda x\, y\, f .\, f\, x\, y \\ \mathsf{Fst} &\triangleq \lambda f .\, f\, \mathsf{True} \\ \mathsf{Snd} &\triangleq \lambda f .\, f\, \mathsf{False} \end{aligned}$$

satisfy

- $$\begin{aligned} \mathsf{Fst}(\mathsf{Pair}\, M\, N) &=_\beta \mathsf{Fst}(\lambda f .\, f\, M\, N) \\ &=_\beta (\lambda f .\, f\, M\, N)\, \mathsf{True} \\ &=_\beta \mathsf{True}\, M\, N \\ &=_\beta M \end{aligned}$$
- $\mathsf{Snd}(\mathsf{Pair}\, M\, N) =_\beta \cdots =_\beta N$

Representing predecessor

Want λ-term **Pred** satisfying

$$\begin{aligned} \mathbf{Pred}\, \underline{n+1} &=_\beta \underline{n} \\ \mathbf{Pred}\, \underline{0} &=_\beta \underline{0} \end{aligned}$$

Have to show how to reduce the "$n+1$-iterator" $\underline{n+1}$ to the "n-iterator" $\underline{n}$.

Idea: given f, iterating the function $g_f : (x, y) \mapsto (f(x), x)$ $n+1$ times starting from (x, x) gives the pair $(f^{n+1}(x), f^n(x))$. So we can get $f^n(x)$ from $f^{n+1}(x)$ *parametrically in* f *and* x, by building g_f from f, iterating $n+1$ times from (x, x) and then taking the second component.

Hence...

Representing predecessor

Want λ-term **Pred** satisfying

$$\begin{aligned} \mathbf{Pred}\,\underline{n+1} &=_\beta \underline{n} \\ \mathbf{Pred}\,\underline{0} &=_\beta \underline{0} \end{aligned}$$

$$\mathbf{Pred} \triangleq \lambda y\, f\, x.\, \mathbf{Snd}(y\,(G\,f)\,(\mathbf{Pair}\, x\, x))$$

where

$$G \triangleq \lambda f\, p.\, \mathbf{Pair}(f\,(\mathbf{Fst}\, p))\,(\mathbf{Fst}\, p)$$

has the required β-reduction properties. [Exercise]

Curry's fixed point combinator Y

$$\mathbf{Y} \triangleq \lambda f.\,(\lambda x.\,f(x\,x))(\lambda x.\,f(x\,x))$$

satisfies $\mathbf{Y}\,M \to (\lambda x.\,M(x\,x))(\lambda x.\,M(x\,x))$
$\to M((\lambda x.\,M(x\,x))(\lambda x.\,M(x\,x)))$

hence $\mathbf{Y}\,M \twoheadrightarrow M((\lambda x.\,M(x\,x))(\lambda x.\,M(x\,x))) \twoheadleftarrow M(\mathbf{Y}\,M)$.

So for all λ-terms M we have

$$\boxed{\mathbf{Y}\,M =_\beta M(\mathbf{Y}\,M)}$$

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where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by

$$\begin{aligned}\Phi_{f,g}(h)(\vec{a}, a) \triangleq\ & \mathit{if}\ a = 0\ \mathit{then}\ f(\vec{a}) \\ & \mathit{else}\ g(\vec{a}, a-1, h(\vec{a}, a-1))\end{aligned}$$

We now know that h can be represented by

$Y(\lambda z\vec{x}x.\, \mathbf{If}(\mathbf{Eq}_0\, x)(F\,\vec{x})(G\,\vec{x}\,(\mathbf{Pred}\, x)(z\,\vec{x}\,(\mathbf{Pred}\, x))))$.

Representing primitive recursion

Recall that the class **PRIM** of primitive recursive functions is the smallest collection of (total) functions containing the basic functions and closed under the operations of composition and primitive recursion.

Combining the results about λ-definability so far, we have: **every $f \in \mathbf{PRIM}$ is λ-definable**.

So for λ-definability of all recursive functions, we just have to consider how to represent minimization. Recall…

Minimization

Given a partial function $f \in \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$, define $\mu^n f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ by

$$\mu^n f(\vec{x}) \triangleq \text{least } x \text{ such that } f(\vec{x}, x) = 0 \text{ and for each } i = 0, \ldots, x-1, \; f(\vec{x}, i) \text{ is defined and } > 0 \text{ (undefined if there is no such } x\text{)}$$

Can express $\mu^n f$ in terms of a fixed point equation:

$\mu^n f(\vec{x}) \equiv g(\vec{x}, 0)$ where g satisfies $\boxed{g = \Psi_f(g)}$

with $\Psi_f \in (\mathbb{N}^{n+1} \rightharpoonup \mathbb{N}) \to (\mathbb{N}^{n+1} \rightharpoonup \mathbb{N})$ defined by

$$\Psi_f(g)(\vec{x}, x) \equiv \mathit{if}\ f(\vec{x}, x) = 0\ \mathit{then}\ x\ \mathit{else}\ g(\vec{x}, x+1)$$

Representing minimization

Suppose $f \in \mathbb{N}^{n+1} \to \mathbb{N}$ (totally defined function) satisfies $\forall \vec{a}\, \exists a\, (f(\vec{a}, a) = 0)$, so that $\mu^n f \in \mathbb{N}^n \to \mathbb{N}$ is totally defined.

Thus for all $\vec{a} \in \mathbb{N}^n$, $\mu^n f(\vec{a}) = g(\vec{a}, 0)$ with $g = \Psi_f(g)$ and $\Psi_f(g)(\vec{a}, a)$ given by $\textit{if}\ (f(\vec{a}, a) = 0)\ \textit{then}\ a\ \textit{else}\ g(\vec{a}, a+1)$.

So if f is represented by a λ-term F, then $\mu^n f$ is represented by

$$\lambda \vec{x}.\mathbf{Y}(\lambda z\, \vec{x}\, x.\, \mathbf{If}(\mathbf{Eq}_0(F\, \vec{x}\, x))\, x\, (z\, \vec{x}\, (\mathbf{Succ}\, x)))\, \vec{x}\, \underline{0}$$

Recursive implies λ-definable

Fact: every partial recursive $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ can be expressed in a standard form as $f = g \circ (\mu^n h)$ for some $g, h \in \mathbf{PRIM}$. (Follows from the proof that computable = partial-recursive.)

Hence every (total) recursive function is λ-definable.

More generally, every partial recursive function is λ-definable, but matching up $\uparrow$ with $\nexists \beta\text{−}\mathbf{nf}$ makes the representations more complicated than for total functions: see [Hindley, J.R. & Seldin, J.P. (CUP, 2008), chapter 4.]

Computable = λ-definable

Theorem. A partial function is computable if and only if it is λ-definable.

We already know that computable = partial recursive $\Rightarrow$ λ-definable. So it just remains to see that **λ-definable functions are RM computable**. To show this one can

- code λ-terms as numbers (ensuring that operations for constructing and deconstructing terms are given by RM computable functions on codes)
- write a RM interpreter for (normal order) β-reduction.

The details are straightforward, if tedious.