

**Theorem.** Given \( f \in \mathbb{N}^n \rightarrow \mathbb{N} \) and \( g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N} \), there is a unique \( h \in \mathbb{N}^{n+1} \rightarrow \mathbb{N} \) satisfying

\[
\begin{align*}
  h(\vec{x}, 0) &= f(\vec{x}) \\
  h(\vec{x}, x + 1) &= g(\vec{x}, x, h(\vec{x}, x))
\end{align*}
\]

for all \( \vec{x} \in \mathbb{N}^n \) and \( x \in \mathbb{N} \).

We write \( \rho^n(f, g) \) for \( h \) and call it the partial function defined by primitive recursion from \( f \) and \( g \).
Representing primitive recursion

If $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ is represented by a $\lambda$-term $F$ and $g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ is represented by a $\lambda$-term $G$, we want to show $\lambda$-definability of the unique $h \in \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ satisfying

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\begin{align*}
    h(\vec{a},0) &= f(\vec{a}) \\
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$$h(\bar{a}, a) = \text{if } a = 0 \text{ then } f(\bar{a})$$
$$\text{else } g(\bar{a}, a - 1, h(\bar{a}, a - 1))$$
Representing primitive recursion

If $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ is represented by a $\lambda$-term $F$ and $g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ is represented by a $\lambda$-term $G$, we want to show $\lambda$-definability of the unique $h \in \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ satisfying $h = \Phi_{f,g}(h)$

where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N}^{n+1} \rightarrow \mathbb{N})$ is given by

$$\Phi_{f,g}(h)(\vec{a}, a) \triangleq \begin{cases} f(\vec{a}) & \text{if } a = 0 \\ g(\vec{a}, a - 1, h(\vec{a}, a - 1)) & \text{else} \end{cases}$$
Representing primitive recursion

If $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ is represented by a $\lambda$-term $F$ and $g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ is represented by a $\lambda$-term $G$, we want to show $\lambda$-definability of the unique $h \in \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ satisfying $h = \Phi_{f,g}(h)$

where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N}^{n+1} \rightarrow \mathbb{N})$ is given by...

Strategy:

- show that $\Phi_{f,g}$ is $\lambda$-definable;
- show that we can solve fixed point equations $X = M X$ up to $\beta$-conversion in the $\lambda$-calculus.
Representing booleans

\[
\begin{align*}
\text{True} & \triangleq \lambda x y. x \\
\text{False} & \triangleq \lambda x y. y \\
\text{If} & \triangleq \lambda f x y. f x y
\end{align*}
\]

satisfy

\[
\begin{align*}
\text{If True } M N & \equiv_{\beta} \text{True } M N \equiv_{\beta} M \\
\text{If False } M N & \equiv_{\beta} \text{False } M N \equiv_{\beta} N
\end{align*}
\]
Representing test-for-zero

\[ \text{Eq}_0 \triangleq \lambda x. x(\lambda y. \text{False}) \text{ True} \]

satisfies

- \[ \text{Eq}_0 0 =_\beta 0 (\lambda y. \text{False}) \text{ True} \]
  \[ =_\beta \text{ True} \]

- \[ \text{Eq}_0 n + 1 =_\beta n + 1 (\lambda y. \text{False}) \text{ True} \]
  \[ =_\beta (\lambda y. \text{False})^{n+1} \text{ True} \]
  \[ =_\beta (\lambda y. \text{False})(((\lambda y. \text{False})^n \text{ True})) \]
  \[ =_\beta \text{ False} \]
Representing ordered pairs

\[ \text{Pair} \triangleq \lambda x y f. f x y \]
\[ \text{Fst} \triangleq \lambda f. f \text{True} \]
\[ \text{Snd} \triangleq \lambda f. f \text{False} \]

satisfy

\[ \text{Fst}(\text{Pair } M N) =_\beta \text{Fst}(\lambda f. f M N) \]
\[ =_\beta (\lambda f. f M N) \text{True} \]
\[ =_\beta \text{True } M N \]
\[ =_\beta M \]

\[ \text{Snd}(\text{Pair } M N) =_\beta \cdots =_\beta N \]
Representing predecessor

Want \( \lambda \)-term \( \text{Pred} \) satisfying

\[
\begin{align*}
\text{Pred} \ n + 1 & \ \beta \ \ n \\
\text{Pred} \ 0 & \ \beta \ \ 0
\end{align*}
\]

Have to show how to reduce the “\( n + 1 \)-iterator” \( n + 1 \) to the “\( n \)-iterator” \( n \).

**Idea:** given \( f \), iterating the function \( g_f : (x, y) \mapsto (f(x), x) \) \( n + 1 \) times starting from \( (x, x) \) gives the pair \( (f^{n+1}(x), f^n(x)) \). So we can get \( f^n(x) \) from \( f^{n+1}(x) \) *parametrically in \( f \) and \( x \),* by building \( g_f \) from \( f \), iterating \( n + 1 \) times from \( (x, x) \) and then taking the second component.

Hence...
Representing predecessor

Want $\lambda$-term $\text{Pred}$ satisfying

\[
\begin{align*}
\text{Pred} n + 1 & \equiv_{\beta} n \\
\text{Pred} 0 & \equiv_{\beta} 0
\end{align*}
\]

$\text{Pred} \triangleq \lambda y f x. \text{Snd}(y (G f)(\text{Pair} x x))$

where

$G \triangleq \lambda f p. \text{Pair}(f(\text{Fst} p))(\text{Fst} p)$

has the required $\beta$-reduction properties. [Exercise]
Curry’s fixed point combinator Y

\[ Y \triangleq \lambda f. (\lambda x. f(x \, x)) (\lambda x. f(x \, x)) \]

satisfies \[ Y \, M \rightarrow (\lambda x. M(x \, x)) (\lambda x. M(x \, x)) \]
\[ \rightarrow M((\lambda x. M(x \, x)) (\lambda x. M(x \, x))) \]

hence \[ Y \, M \rightarrow M((\lambda x. M(x \, x)) (\lambda x. M(x \, x))) \leftrightarrow M(Y \, M). \]

So for all \( \lambda \)-terms \( M \) we have

\[ Y \, M =_{\beta} M(Y \, M) \]
Representing primitive recursion

If \( f \in \mathbb{N}^n \rightarrow \mathbb{N} \) is represented by a \( \lambda \)-term \( F \) and \( g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N} \) is represented by a \( \lambda \)-term \( G \), we want to show \( \lambda \)-definability of the unique \( h \in \mathbb{N}^{n+1} \rightarrow \mathbb{N} \) satisfying

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If \( f \in \mathbb{N}^n \rightarrow \mathbb{N} \) is represented by a \( \lambda \)-term \( F \) and \( g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N} \) is represented by a \( \lambda \)-term \( G \), we want to show \( \lambda \)-definability of the unique \( h \in \mathbb{N}^{n+1} \rightarrow \mathbb{N} \) satisfying \( h = \Phi_{f,g}(h) \)

where \( \Phi_{f,g} \in (\mathbb{N}^{n+1} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N}^{n+1} \rightarrow \mathbb{N}) \) is given by

\[
\Phi_{f,g}(h)(\vec{a}, a) \triangleq \text{if } a = 0 \text{ then } f(\vec{a}) \text{ else } g(\vec{a}, a - 1, h(\vec{a}, a - 1))
\]

We now know that \( h \) can be represented by

\[
Y(\lambda z \vec{x}. \text{If}(\text{Eq}_0 x)(F \vec{x})(G \vec{x}(\text{Pred} x)(z \vec{x}(\text{Pred} x)))) \]

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Representing primitive recursion

Recall that the class \( \text{PRIM} \) of primitive recursive functions is the smallest collection of (total) functions containing the basic functions and closed under the operations of composition and primitive recursion.

Combining the results about \( \lambda \)-definability so far, we have: every \( f \in \text{PRIM} \) is \( \lambda \)-definable.

So for \( \lambda \)-definability of all recursive functions, we just have to consider how to represent minimization. Recall...
Minimization

Given a partial function \( f \in \mathbb{N}^{n+1} \to \mathbb{N} \), define \( \mu^n f \in \mathbb{N}^n \to \mathbb{N} \) by

\[
\mu^n f(\vec{x}) \triangleq \text{least } x \text{ such that } f(\vec{x}, x) = 0 \text{ and for each } i = 0, \ldots, x - 1, f(\vec{x}, i) \text{ is defined and } > 0
\]

(undefined if there is no such \( x \))

Can express \( \mu^n f \) in terms of a fixed point equation:

\[
\mu^n f(\vec{x}) \equiv g(\vec{x}, 0) \text{ where } g \text{ satisfies } g = \Psi_f(g)
\]

with \( \Psi_f \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N}) \) defined by

\[
\Psi_f(g)(\vec{x}, x) \equiv \text{if } f(\vec{x}, x) = 0 \text{ then } x \text{ else } g(\vec{x}, x + 1)
\]
Suppose $f \in \mathbb{N}^{n+1} \to \mathbb{N}$ (totally defined function) satisfies $\forall \vec{a} \exists a \ (f(\vec{a}, a) = 0)$, so that $\mu^n f \in \mathbb{N}^n \to \mathbb{N}$ is totally defined.

Thus for all $\vec{a} \in \mathbb{N}^n$, $\mu^n f(\vec{a}) = g(\vec{a}, 0)$ with $g = \Psi_f(g)$ and $\Psi_f(g)(\vec{a}, a)$ given by

if $(f(\vec{a}, a) = 0)$ then $a$ else $g(\vec{a}, a + 1)$.

So if $f$ is represented by a $\lambda$-term $F$, then $\mu^n f$ is represented by

$$\lambda \vec{x}.Y(\lambda z \vec{x}x. \text{If}(\text{Eq}_0(F \vec{x}x)) x (z \vec{x}(\text{Succ} x))) \vec{x}0$$
Recursive implies $\lambda$-definable

**Fact:** every partial recursive $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ can be expressed in a standard form as $f = g \circ (\mu^n h)$ for some $g, h \in \text{PRIM}$. (Follows from the proof that computable $=$ partial-recursive.)

Hence every (total) recursive function is $\lambda$-definable.

More generally, every partial recursive function is $\lambda$-definable, but matching up $\uparrow$ with $\forall \beta - \text{nf}$ makes the representations more complicated than for total functions: see [Hindley, J.R. & Seldin, J.P. (CUP, 2008), chapter 4.]
Computable = $\lambda$-definable

**Theorem.** A partial function is computable if and only if it is $\lambda$-definable.

We already know that computable $= \text{partial recursive} \Rightarrow \lambda$-definable. So it just remains to see that $\lambda$-definable functions are RM computable. To show this one can

- code $\lambda$-terms as numbers (ensuring that operations for constructing and deconstructing terms are given by RM computable functions on codes)
- write a RM interpreter for (normal order) $\beta$-reduction.

The details are straightforward, if tedious.