Lecture 7

Relating Denotational and Operational Semantics

Adequacy

For any closed PCF terms $M$ and $V$ of ground type $\gamma \in \{nat, bool\}$ with $V$ a value

$\llbracket M \rrbracket = \llbracket V \rrbracket \in \llbracket \gamma \rrbracket \ implies \ M \Downarrow_{\gamma} V.$

NB. Adequacy does not hold at function types:

$\llbracket \text{fn} \ x : \tau. \ (\text{fn} \ y : \tau. y) \ x \rrbracket = \llbracket \text{fn} \ x : \tau. x \rrbracket : [\tau] \rightarrow [\tau]$
For any closed PCF terms $M$ and $V$ of ground type $\gamma \in \{\text{nat, bool}\}$ with $V$ a value
\[ [M] = [V] \in [\gamma] \implies M \downarrow_\gamma V. \]

**NB.** Adequacy does not hold at function types:
\[
[\text{fn } x : \tau. (\text{fn } y : \tau. y) x] = [\text{fn } x : \tau. x] : [\tau] \rightarrow [\tau]
\]
but
\[
\text{fn } x : \tau. (\text{fn } y : \tau. y) x \not\downarrow_{\tau \rightarrow \tau} \text{fn } x : \tau. x
\]

**Adequacy proof idea**

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.
   - Consider $M$ to be $M_1 M_2$, $\text{fix}(M')$, $\text{fn } x : \tau. M'$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.
Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.
   - Consider $M$ to be $M_1 M_2$, $\text{fix}(M')$, $\text{fn } x : \tau . M'$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.
   This statement roughly takes the form:
   \[
   \llbracket M \rrbracket \llp \tau \rrp M \text{ for all types } \tau \text{ and all } M \in \text{PCF}_{\tau}
   \]
   where the formal approximation relations
   \[
   \llp \tau \rrp \subseteq \llbracket \tau \rrbracket \times \text{PCF}_{\tau}
   \]
   are logically chosen to allow a proof by induction.

Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{\text{nat}, \text{bool}\}$,

\[
\llbracket M \rrbracket \llp \gamma \rrp M \text{ implies } \forall V (\llbracket M \rrbracket = \llbracket V \rrbracket \implies M \Downarrow V)
\]

Proof of: $\llbracket M \rrbracket \llp \gamma \rrp M$ implies adequacy

Case $\gamma = \text{nat}$.

\[
\llbracket M \rrbracket = \llbracket V \rrbracket
\implies \llbracket M \rrbracket = \llbracket \text{succ}^n(0) \rrbracket \text{ for some } n \in \mathbb{N}
\implies n = \llbracket M \rrbracket \llp \gamma \rrp M
\implies M \Downarrow \text{succ}^n(0)
\]

by definition of $\llp \text{nat} \rrp(0)$.

Case $\gamma = \text{bool}$ is similar.
We want to be able to proceed by induction.

Consider the case $M = M_1 M_2$.

\[ f \prec_{\tau \rightarrow \tau'} M \quad (f \in ([\tau] \rightarrow [\tau']), \ M \in \text{PCF}_{\tau \rightarrow \tau'}) \]

We want to be able to proceed by induction.

Consider the case $M = \text{fix}(M')$. 

\[ f \prec_{\tau \rightarrow \tau'} M \quad (f \in ([\tau] \rightarrow [\tau']), \ M \in \text{PCF}_{\tau \rightarrow \tau'}) \]
Admissibility property

**Lemma.** For all types $\tau$ and $M \in \text{PCF}_\tau$, the set
\[
\{ d \in \llbracket \tau \rrbracket \mid d \triangleleft_\tau M \}
\]
is an admissible subset of $\llbracket \tau \rrbracket$.

Further properties

**Lemma.** For all types $\tau$, elements $d, d' \in \llbracket \tau \rrbracket$, and terms $M, N, V \in \text{PCF}_\tau$,
1. If $d \subseteq d'$ and $d' \triangleleft_\tau M$ then $d \triangleleft_\tau M$.
2. If $d \triangleleft_\tau M$ and $\forall V (M \downarrow_\tau V \implies N \downarrow_\tau V)$ then $d \triangleleft_\tau N$.

Fundamental property

**Theorem.** For all $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ and all $\Gamma \vdash M : \tau$, if $d_1 \triangleleft_{\tau_1} M_1, \ldots, d_n \triangleleft_{\tau_n} M_n$ then $\llbracket \Gamma \vdash M \rrbracket[x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \triangleleft_\tau M[M_1/x_1, \ldots, M_n/x_n]$.

Requirements on the formal approximation relations, IV

We want to be able to proceed by induction.

- Consider the case $M = \text{fn } x : \tau. M'$.
  \[ \leadsto \text{substitutivity property for open terms} \]
Fundamental property

Theorem. For all $\Gamma = \langle x_1 \leftrightarrow \tau_1, \ldots, x_n \leftrightarrow \tau_n \rangle$ and all
$\Gamma \vdash M : \tau$, if $d_1 \triangleleft_{\tau_1} M_1, \ldots, d_n \triangleleft_{\tau_n} M_n$ then
$[\Gamma \vdash M][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \triangleleft_{\tau} M[M_1/x_1, \ldots, M_n/x_n]$.

NB. The case $\Gamma = \emptyset$ reduces to

$[M] \triangleleft_{\tau} M$

for all $M \in \text{PCF}_\tau$.

Contextual preorder from formal approximation

Proposition. For all PCF types $\tau$ and all closed terms $M_1, M_2 \in \text{PCF}_\tau$,

$[M_1] \triangleleft_{\tau} M_2 \iff M_1 \leq_{\text{ctx}} M_2 : \tau$.

Contextual preorder between PCF terms

Given PCF terms $M_1, M_2$, PCF type $\tau$, and a type environment $\Gamma$, the relation $\Gamma \vdash M_1 \leq_{\text{ctx}} M_2 : \tau$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $C$ for which $C[M_1]$ and $C[M_2]$ are
  closed terms of type $\gamma$, where $\gamma = \text{nat}$ or $\gamma = \text{bool}$, and
  for all values $V \in \text{PCF}_\gamma$,

  $C[M_1] \triangleright_{\gamma} V \implies C[M_2] \triangleright_{\gamma} V$.

Extensionality properties of $\leq_{\text{ctx}}$

At a ground type $\gamma \in \{\text{bool}, \text{nat}\}$,

$M_1 \leq_{\text{ctx}} M_2 : \gamma$ holds if and only if

$\forall V \in \text{PCF}_\gamma \ (M_1 \triangleright_{\gamma} V \implies M_2 \triangleright_{\gamma} V)$.

At a function type $\tau \rightarrow \tau'$,

$M_1 \leq_{\text{ctx}} M_2 : \tau \rightarrow \tau'$ holds if and only if

$\forall M \in \text{PCF}_\tau \ (M_1 M \leq_{\text{ctx}} M_2 M : \tau')$. 