

# Lecture 6

## Denotational Semantics of PCF

### Denotational semantics of PCF

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To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$\llbracket \Gamma \vdash M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$$

between domains.

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### Denotational semantics of PCF types

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$$\llbracket nat \rrbracket \stackrel{\text{def}}{=} \mathbb{N}_\perp \quad (\text{flat domain})$$

$$\llbracket bool \rrbracket \stackrel{\text{def}}{=} \mathbb{B}_\perp \quad (\text{flat domain})$$

where  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{B} = \{true, false\}$ .

### Denotational semantics of PCF types

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$$\llbracket nat \rrbracket \stackrel{\text{def}}{=} \mathbb{N}_\perp \quad (\text{flat domain})$$

$$\llbracket bool \rrbracket \stackrel{\text{def}}{=} \mathbb{B}_\perp \quad (\text{flat domain})$$

$$\llbracket \tau \rightarrow \tau' \rrbracket \stackrel{\text{def}}{=} \llbracket \tau \rrbracket \rightarrow \llbracket \tau' \rrbracket \quad (\text{function domain}).$$

where  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{B} = \{true, false\}$ .

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## Denotational semantics of PCF type environments

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$$\llbracket \Gamma \rrbracket \stackrel{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} \llbracket \Gamma(x) \rrbracket \quad (\Gamma\text{-environments})$$

## Denotational semantics of PCF type environments

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$$\begin{aligned} \llbracket \Gamma \rrbracket &\stackrel{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} \llbracket \Gamma(x) \rrbracket \quad (\Gamma\text{-environments}) \\ &= \text{the domain of partial functions } \rho \text{ from variables} \\ &\quad \text{to domains such that } \text{dom}(\rho) = \text{dom}(\Gamma) \text{ and} \\ &\quad \rho(x) \in \llbracket \Gamma(x) \rrbracket \text{ for all } x \in \text{dom}(\Gamma) \end{aligned}$$

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## Denotational semantics of PCF type environments

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$$\begin{aligned} \llbracket \Gamma \rrbracket &\stackrel{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} \llbracket \Gamma(x) \rrbracket \quad (\Gamma\text{-environments}) \\ &= \text{the domain of partial functions } \rho \text{ from variables} \\ &\quad \text{to domains such that } \text{dom}(\rho) = \text{dom}(\Gamma) \text{ and} \\ &\quad \rho(x) \in \llbracket \Gamma(x) \rrbracket \text{ for all } x \in \text{dom}(\Gamma) \end{aligned}$$

$$2. \llbracket \langle x \mapsto \tau \rangle \rrbracket = (\{x\} \rightarrow \llbracket \tau \rrbracket)$$

### Example:

1. For the empty type environment  $\emptyset$ ,

$$\llbracket \emptyset \rrbracket = \{ \perp \}$$

where  $\perp$  denotes the unique partial function with  $\text{dom}(\perp) = \emptyset$ .

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$$2. \llbracket \langle x \mapsto \tau \rangle \rrbracket = (\{x\} \rightarrow \llbracket \tau \rrbracket) \cong \llbracket \tau \rrbracket$$

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3.

$$\begin{aligned} & \llbracket \langle x_1 \mapsto \tau_1, \dots, x_n \mapsto \tau_n \rangle \rrbracket \\ & \cong (\{x_1\} \rightarrow \llbracket \tau_1 \rrbracket) \times \dots \times (\{x_n\} \rightarrow \llbracket \tau_n \rrbracket) \\ & \cong \llbracket \tau_1 \rrbracket \times \dots \times \llbracket \tau_n \rrbracket \end{aligned}$$

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### Denotational semantics of PCF terms, I

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$$\llbracket \Gamma \vdash \mathbf{0} \rrbracket(\rho) \stackrel{\text{def}}{=} 0 \in \llbracket \text{nat} \rrbracket$$

$$\llbracket \Gamma \vdash \mathbf{0} \rrbracket(\rho) \stackrel{\text{def}}{=} 0 \in \llbracket \text{nat} \rrbracket$$

$$\llbracket \Gamma \vdash \mathbf{true} \rrbracket(\rho) \stackrel{\text{def}}{=} \text{true} \in \llbracket \text{bool} \rrbracket$$

$$\llbracket \Gamma \vdash \mathbf{true} \rrbracket(\rho) \stackrel{\text{def}}{=} \text{true} \in \llbracket \text{bool} \rrbracket$$

$$\llbracket \Gamma \vdash \mathbf{false} \rrbracket(\rho) \stackrel{\text{def}}{=} \text{false} \in \llbracket \text{bool} \rrbracket$$

$$\llbracket \Gamma \vdash \mathbf{false} \rrbracket(\rho) \stackrel{\text{def}}{=} \text{false} \in \llbracket \text{bool} \rrbracket$$

$$\llbracket \Gamma \vdash x \rrbracket(\rho) \stackrel{\text{def}}{=} \rho(x) \in \llbracket \Gamma(x) \rrbracket \quad (x \in \text{dom}(\Gamma))$$

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$\llbracket \Gamma \vdash \mathbf{succ}(M) \rrbracket(\rho)$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \perp \\ \perp & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \perp \end{cases}$$

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$\llbracket \Gamma \vdash \mathbf{succ}(M) \rrbracket(\rho)$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \perp \\ \perp & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \perp \end{cases}$$

$\llbracket \Gamma \vdash \mathbf{pred}(M) \rrbracket(\rho)$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) - 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) > 0 \\ \perp & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = 0, \perp \end{cases}$$

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$\llbracket \Gamma \vdash \mathbf{succ}(M) \rrbracket(\rho)$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \perp \\ \perp & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \perp \end{cases}$$

$\llbracket \Gamma \vdash \mathbf{pred}(M) \rrbracket(\rho)$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) - 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) > 0 \\ \perp & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = 0, \perp \end{cases}$$

$$\llbracket \Gamma \vdash \mathbf{zero}(M) \rrbracket(\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = 0 \\ \mathit{false} & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) > 0 \\ \perp & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \perp \end{cases}$$

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$\llbracket \Gamma \vdash \mathbf{if } M_1 \mathbf{ then } M_2 \mathbf{ else } M_3 \rrbracket(\rho)$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M_2 \rrbracket(\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket(\rho) = \mathit{true} \\ \llbracket \Gamma \vdash M_3 \rrbracket(\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket(\rho) = \mathit{false} \\ \perp & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket(\rho) = \perp \end{cases}$$

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### Denotational semantics of PCF terms, III

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$$\begin{aligned} & \llbracket \Gamma \vdash \text{if } M_1 \text{ then } M_2 \text{ else } M_3 \rrbracket(\rho) \\ & \stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M_2 \rrbracket(\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket(\rho) = \text{true} \\ \llbracket \Gamma \vdash M_3 \rrbracket(\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket(\rho) = \text{false} \\ \perp & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket(\rho) = \perp \end{cases} \end{aligned}$$

$$\llbracket \Gamma \vdash M_1 M_2 \rrbracket(\rho) \stackrel{\text{def}}{=} (\llbracket \Gamma \vdash M_1 \rrbracket(\rho)) (\llbracket \Gamma \vdash M_2 \rrbracket(\rho))$$

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### Denotational semantics of PCF terms, V

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$$\llbracket \Gamma \vdash \text{fix}(M) \rrbracket(\rho) \stackrel{\text{def}}{=} \text{fix}(\llbracket \Gamma \vdash M \rrbracket(\rho))$$

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Recall that *fix* is the function assigning least fixed points to continuous functions.

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### Denotational semantics of PCF terms, IV

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$$\begin{aligned} & \llbracket \Gamma \vdash \text{fn } x : \tau . M \rrbracket(\rho) \\ & \stackrel{\text{def}}{=} \lambda d \in \llbracket \tau \rrbracket . \llbracket \Gamma[x \mapsto \tau] \vdash M \rrbracket(\rho[x \mapsto d]) \quad (x \notin \text{dom}(\Gamma)) \end{aligned}$$

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**NB:**  $\rho[x \mapsto d] \in \llbracket \Gamma[x \mapsto \tau] \rrbracket$  is the function mapping  $x$  to  $d \in \llbracket \tau \rrbracket$  and otherwise acting like  $\rho$ .

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### Denotational semantics of PCF

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**Proposition.** For all typing judgements  $\Gamma \vdash M : \tau$ , the denotation

$$\llbracket \Gamma \vdash M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$$

is a well-defined continuous function.

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## Denotations of closed terms

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For a closed term  $M \in \text{PCF}_\tau$ , we get

$$\llbracket \emptyset \vdash M \rrbracket : \llbracket \emptyset \rrbracket \rightarrow \llbracket \tau \rrbracket$$

and, since  $\llbracket \emptyset \rrbracket = \{\perp\}$ , we have

$$\llbracket M \rrbracket \stackrel{\text{def}}{=} \llbracket \emptyset \vdash M \rrbracket (\perp) \in \llbracket \tau \rrbracket \quad (M \in \text{PCF}_\tau)$$

## Soundness

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**Proposition.** For all closed terms  $M, V \in \text{PCF}_\tau$ ,

if  $M \Downarrow_\tau V$  then  $\llbracket M \rrbracket = \llbracket V \rrbracket \in \llbracket \tau \rrbracket$ .

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## Compositionality

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**Proposition.** For all typing judgements  $\Gamma \vdash M : \tau$  and  $\Gamma \vdash M' : \tau$ , and all contexts  $\mathcal{C}[-]$  such that  $\Gamma' \vdash \mathcal{C}[M] : \tau'$  and  $\Gamma' \vdash \mathcal{C}[M'] : \tau'$ ,

if  $\llbracket \Gamma \vdash M \rrbracket = \llbracket \Gamma \vdash M' \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$

then  $\llbracket \Gamma' \vdash \mathcal{C}[M] \rrbracket = \llbracket \Gamma' \vdash \mathcal{C}[M'] \rrbracket : \llbracket \Gamma' \rrbracket \rightarrow \llbracket \tau' \rrbracket$

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## Substitution property

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**Proposition.** Suppose that  $\Gamma \vdash M : \tau$  and that  $\Gamma[x \mapsto \tau] \vdash M' : \tau'$ , so that we also have  $\Gamma \vdash M'[M/x] : \tau'$ .

Then,

$$\begin{aligned} & \llbracket \Gamma \vdash M'[M/x] \rrbracket (\rho) \\ &= \llbracket \Gamma[x \mapsto \tau] \vdash M' \rrbracket (\rho[x \mapsto \llbracket \Gamma \vdash M \rrbracket]) \end{aligned}$$

for all  $\rho \in \llbracket \Gamma \rrbracket$ .

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## Substitution property

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**Proposition.** Suppose that  $\Gamma \vdash M : \tau$  and that  $\Gamma[x \mapsto \tau] \vdash M' : \tau'$ , so that we also have  $\Gamma \vdash M'[M/x] : \tau'$ .

Then,

$$\begin{aligned} & \llbracket \Gamma \vdash M'[M/x] \rrbracket (\rho) \\ &= \llbracket \Gamma[x \mapsto \tau] \vdash M' \rrbracket (\rho[x \mapsto \llbracket \Gamma \vdash M \rrbracket]) \end{aligned}$$

for all  $\rho \in \llbracket \Gamma \rrbracket$ .

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In particular when  $\Gamma = \emptyset$ ,  $\llbracket \langle x \mapsto \tau \rangle \vdash M' \rrbracket : \llbracket \tau \rrbracket \rightarrow \llbracket \tau' \rrbracket$  and

$$\llbracket M'[M/x] \rrbracket = \llbracket \langle x \mapsto \tau \rangle \vdash M' \rrbracket (\llbracket M \rrbracket)$$