Lecture 3
Constructions on Domains

Discrete cpo’s and flat domains

For any set \( X \), the relation of equality

\[ x \sqsubseteq x' \overset{\text{def}}{=} x = x' \quad (x, x' \in X) \]

makes \((X, \sqsubseteq)\) into a cpo, called the discrete cpo with underlying set \( X \).

Let \( X \uplus \overset{\text{def}}{=} X \cup \{\bot\} \), where \( \bot \) is some element not in \( X \). Then

\[ d \sqsubseteq d' \overset{\text{def}}{=} (d = d') \lor (d = \bot) \quad (d, d' \in X \uplus) \]

makes \((X \uplus, \sqsubseteq)\) into a domain (with least element \( \bot \)), called the flat domain determined by \( X \).

Continuous functions of two arguments

**Proposition.** Let \( D, E, F \) be cpo’s. A function \( f : (D \times E) \to F \) is monotone if and only if it is monotone in each argument separately:

\[ \forall d, d' \in D, e \in E. d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e) \]
\[ \forall d \in D, e, e' \in E. e \sqsubseteq e' \Rightarrow f(d, e) \sqsubseteq f(d, e'). \]

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

\[ f(\bigsqcup_{m \geq 0} d_m, e) = \bigsqcup_{m \geq 0} f(d_m, e) \]
\[ f(d, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n). \]
Function cpo’s and domains

Given cpo’s \((D, \preceq_D)\) and \((E, \preceq_E)\), the function cpo \((D \to E, \preceq)\) has underlying set

\[
D \to E \overset{\text{def}}{=} \{ f \mid f : D \to E \text{ is a continuous function}\}
\]

and partial order: \(f \preceq f' \overset{\text{def}}{=} \forall d \in D : f(d) \preceq_E f'(d)\).

Lubs of chains are calculated ‘argumentwise’ (using lubs in \(E\)):

\[
\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d).\]

If \(E\) is a domain, then so is \(D \to E\) and \(\bot_{D \to E}(d) = \bot_E\), all \(d \in D\).

Continuity of composition

For cpo’s \(D, E, F\), the composition function

\[
\circ : \left( (E \to F) \times (D \to E) \right) \to (D \to F)
\]

defined by setting, for all \(f \in (D \to E)\) and \(g \in (E \to F)\),

\[
g \circ f = \lambda d \in D. g(f(d))
\]

is continuous.

Continuity of the fixpoint operator

Let \(D\) be a domain.

By Tarski’s Fixed Point Theorem we know that each continuous function \(f \in (D \to D)\) possesses a least fixed point, \(\text{fix}(f) \in D\).

**Proposition.** The function

\[
\text{fix} : (D \to D) \to D
\]

is continuous.