Lecture 2

Least Fixed Points

Partially ordered sets

A binary relation $\sqsubseteq$ on a set $D$ is a partial order iff it is

- **reflexive**: $\forall d \in D. d \sqsubseteq d$
- **transitive**: $\forall d, d', d'' \in D. d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$
- **anti-symmetric**: $\forall d, d' \in D. d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

Monotonicity

- A function $f : D \to E$ between posets is monotone iff
  $$\forall d, d' \in D. d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$
Pre-fixed points

Let $D$ be a poset and $f : D \rightarrow D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$. 

The least pre-fixed point of $f$, if it exists, will be written $\mathit{fix}(f)$.

It is thus (uniquely) specified by the two properties:

\begin{align*}
  f(\mathit{fix}(f)) & \sqsubseteq \mathit{fix}(f) \quad \text{(lfp1)} \\
  \forall d \in D. \, f(d) \sqsubseteq d & \Rightarrow \mathit{fix}(f) \sqsubseteq d. \quad \text{(lfp2)}
\end{align*}

Proof principle

Let $D$ be a poset and let $f : D \rightarrow D$ be a monotone function with a least pre-fixed point $\mathit{fix}(f) \in D$.

For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

Thesis*

All domains of computation are complete partial orders with a least element.

All computable functions are continuous.
Cpo’s and domains

A chain complete poset, or cpo for short, is a poset \((D, \sqsubseteq)\) in which all countable increasing chains \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots\) have least upper bounds, \(\bigsqcup_{n \geq 0} d_n:\)

\[
\forall m \geq 0 . \ d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \quad \text{(lub1)}
\]

\[
\forall d \in D . (\forall m \geq 0 . d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \quad \text{(lub2)}
\]

A domain is a cpo that possesses a least element, \(\bot:\)

\[
\forall d \in D . \ \bot \sqsubseteq d.
\]

Some properties of lubs of chains

Let \(D\) be a cpo.

1. For \(d \in D\), \(\bigsqcup_n d = d\).

2. For every chain \(d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots\) in \(D\),

\[
\bigsqcup_n d_n = \bigsqcup_n d_{N+n}
\]

for all \(N \in \mathbb{N}\).

3. For every pair of chains \(d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots\) and \(e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots\) in \(D\),

if \(d_n \sqsubseteq e_n\) for all \(n \in \mathbb{N}\) then \(\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n\).

Domain of partial functions, \(X \to Y\)

Underlying set: all partial functions, \(f\), with domain of definition \(\text{dom}(f) \subseteq X\) and taking values in \(Y\).

Partial order:

\[
f \sqsubseteq g \quad \text{iff} \quad \text{dom}(f) \subseteq \text{dom}(g) \quad \text{and} \quad \forall x \in \text{dom}(f) . f(x) = g(x)
\]

iff graph\((f) \subseteq \text{graph}(g)\)

Lub of chain \(f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots\) is the partial function \(f\) with \(\text{dom}(f) = \bigcup_{n \geq 0} \text{dom}(f_n)\) and

\[
f(x) = \begin{cases} f_n(x) & \text{if } x \in \text{dom}(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}
\]

Least element \(\bot\) is the totally undefined partial function.

Diagonalising a double chain

Lemma. Let \(D\) be a cpo. Suppose that the doubly-indexed family of elements \(d_{m,n} \in D\) \((m, n \geq 0)\) satisfies

\[
m \leq m' \ & \& \ n \leq n' \Rightarrow \ d_{m,n} \sqsubseteq d_{m',n'}. \quad (\dagger)
\]

Then

\[
\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots
\]

and

\[
\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots
\]

Moreover

\[
\bigsqcup_{m \geq 0} \left( \bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left( \bigsqcup_{m \geq 0} d_{m,n} \right).
\]
**Continuity and strictness**

- If $D$ and $E$ are cpo’s, the function $f$ is **continuous** iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \subseteq d_1 \subseteq \ldots$ in $D$, it is the case that $f\left( \bigsqcup_{n \geq 0} d_n \right) = \bigsqcup_{n \geq 0} f(d_n)$ in $E$.

- If $D$ and $E$ have least elements, then the function $f$ is **strict** iff $f(\bot) = \bot$.

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**Tarski’s Fixed Point Theorem**

Let $f : D \rightarrow D$ be a continuous function on a domain $D$. Then

- $f$ possesses a least pre-fixed point, given by $\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot)$.

- Moreover, $\mathit{fix}(f)$ is a fixed point of $f$, *i.e.* satisfies $f(\mathit{fix}(f)) = \mathit{fix}(f)$, and hence is the **least fixed point** of $f$.

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**[while $B$ do $C$]**

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[while $B$ do $C$]

= \mathit{fix}(f_{[B], [C]})

= \bigsqcup_{n \geq 0} f_{[B], [C]}^n(\bot)

= \lambda s \in \text{State}. \begin{cases} 
\llbracket C \rrbracket^k(s) & \text{if } \forall 0 \leq i < k \in \mathbb{N}. [B](\llbracket C \rrbracket^i(s)) = \text{true} \\
\text{and } [B](\llbracket C \rrbracket^k(s)) = \text{false} \\
\uparrow & \text{if } \forall i \in \mathbb{N}. [B](\llbracket C \rrbracket^i(s)) = \text{true} 
\end{cases}
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