Mathematical Methods for Computer Science

UNIVERSITY OF CAMBRIDGE

Computer Laboratory

Computer Science Tripos, Part IB

Lent Term 2008

R. J. Gibbens

Problem sheet

William Gates Building
15 JJ Thomson Avenue
Cambridge
CB3 0FD

http://www.cl.cam.ac.uk/

(Ia) Fourier and related methods

1. Given a complex linear space, V, define the notion of an *inner product* and in the case of $V = \mathbb{C}^n$ show that for any two vectors $x, y \in \mathbb{C}^n$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ defines an inner product.

2. Suppose that V is a complex inner product space. Show the *Cauchy-Schwarz inequality*, namely, that for all $u, v \in V$

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle \, .$$

Define the notion of a *norm* for V and show that

$$||v|| = \sqrt{\langle v, v \rangle}$$

is a norm.

3. Suppose that V is an inner product space and let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal system for V and let $W = \text{span}\{e_1, e_2, \ldots, e_n\}$. Using $\tilde{u} = \sum_{k=1}^{n} \langle u, e_k \rangle e_k$ for the *orthogonal projection* of $u \in V$ on W show that

$$||\tilde{u}||^2 = \sum_{k=1}^{n} |\langle u, e_k \rangle|^2 \leq ||u||^2 \, .$$

Now, consider the case of an infinite orthonormal system $\{e_1, e_2, \ldots\}$ and show that the infinite sum

$$\lim_{n \to \infty} \sum_{k=1}^{n} |\langle u, e_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2$$

exists and that the limit is bounded above with

$$\sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2 \leq ||u||^2 \, .$$

Hence deduce that

$$\lim_{k \to \infty} \langle u, e_k \rangle = 0 \, .$$

4. Calculate the Fourier series of the function $f(x)$ ($x \in [-\pi, \pi]$) defined by

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 0 & -\pi \leq x < 0 \, . \end{cases}$$

Find also the complex Fourier series for $f(x)$.

5. Suppose that $f(x)$ is a 2π-periodic function with complex Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}\,.$$

Now consider the shifted version of $f(x)$ given by

$$g(x) = f(x - x_0)$$

where x_0 is a constant. Find the relationship between the complex Fourier coefficients of $g(x)$ in terms of those of $f(x)$. How do the magnitudes of the corresponding coefficients compare?

6. Suppose that $f(x)$ and $g(x)$ are two functions defined for real x and that they have Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. Show that

$$\int_{-\infty}^{\infty} f(x)G(x)dx = \int_{-\infty}^{\infty} F(x)g(x)dx\,.$$

You may assume that the above integrals exist and that you may change the order of integration in your calculations.

7. Consider the functions $f(x)$ and $g(x)$ defined by

$$f(x) = \begin{cases} 0 & x > b \\ 1 & -b < x \le b \\ 0 & x \le -b \end{cases}$$

where $b > 0$ is a constant and

$$g(x) = \begin{cases} 0 & x > 4 \\ 1 & 3 < x \le 4 \\ 1.5 & 2 < x \le 3 \\ 1 & 1 < x \le 2 \\ 0 & x \le 1\,. \end{cases}$$

Use the Fourier transform of $f(x)$ (derived in lectures) together with properties of Fourier transforms (which you should state carefully) to construct the Fourier transform of $g(x)$.

8. Suppose that the N-point DFT of the sequence $f[n]$ is given by $F[k]$ where $f(n)$ is itself a N-periodic sequence, that is $f(n+N) = f(n)$ for $n = 0, 1, \ldots, N-1$. Show that the shifted sequence $f[n-m]$ has DFT

$$e^{-2\pi imk/N}F[k]$$

where m is a constant integer. Show also that $\overline{f[n]}$, the complex conjugate of $f[n]$, has DFT $\overline{F[-k]}$. Suppose that $f[-2] = -1, f[-1] = -2, f[0] = 0, f[1] = 2, f[2] = 1$. Find the 5-point DFT of $f[n]$. Can you explain why it is purely imaginary?

9. Suppose that the sequences $f[n]$ and $g[n]$ have N-point DFTs given by $F[k]$ and $G[k]$, respectively. By expanding $F[k]G[k]$ show that the *cyclical convolution*

$$\sum_{m=0}^{N-1} f[m]g[n-m]$$

has DFT $F[k]G[k]$.

(IIa) Introductory probability

1. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ show the following results.
   (a) If $E_1, E_2 \in \mathcal{F}$ then $E_1 \setminus E_2 \in \mathcal{F}$.
   (b) If $E_1, E_2, \ldots \in \mathcal{F}$ then $\cap_{i=1}^{\infty} E_i \in \mathcal{F}$.
   (c) If $E \in \mathcal{F}$ then $\mathbb{P}(\Omega \setminus E) = 1 - \mathbb{P}(E)$.
   (d) If $E_1, E_2 \in \mathcal{F}$ and $E_1 \subset E_2$ then $\mathbb{P}(E_1) \leq \mathbb{P}(E_2)$.

2. The PWF contains two types of workstations labelled A and B, respectively. A workstation of type A has a probability of $1/10$ of being defective whereas a workstation of type B has a probability of being defective of $2/10$. The PWF has 140 workstations of type A and 60 of type B. You choose one of the workstations at random. What is the probability that the workstation is defective? Given that the workstation is defective what is the probability that it is of type A?

3. A campus router has been mis-configured in such a way that packets between two colleges C_1 and C_2 are routed off campus with probability $3/4$ and stay on campus with probability $1/4$. A packet routed off campus has a probability of being dropped of $1/3$ whereas a packet that doesn't leave the campus has a lower probability of being dropped of $1/4$. What is the probability that a packet travelling between C_1 and C_2 is dropped? Given that a packet is received at C_2 from C_1 without being dropped, what is the probability that the packet was routed off campus?

4. Suppose that one person in 1000 suffers a severe adverse reaction to some drug. A simple test is available that claims to be 95% reliable in the sense that if a person would suffer the reaction a positive test results with probability 95% and if they would not suffer the reaction a negative test results with probability 95%. Given that you have tested positive, what is the probability that you would suffer the adverse reaction to the drug? What do you make of the claim that the test is 95% reliable?

5. Let X be a random variable with a geometric distribution with parameter p and let $q = 1-p$. Show that for $|z| < 1/q$, X has a probability generating function given by $G_X(z) = pz/(1-qz)$. Using the probability generating function $G_X(z)$, calculate the mean and variance of X.

6. A binary symmetric channel takes an input $X \in \{0,1\}$ and produces an output value $Y \in \{0,1\}$. Suppose that an input of X is correctly received as an output Y with probability 0.9 but with probability 0.1 an input of 1 is received as an output of 0 and an input of 0 is received as an output of 1. Suppose that the inputs $X = 0$ and $X = 1$ occur with probabilities 0.4 and 0.6, respectively. What are the probabilities of receiving the outputs $Y = 0$ and $Y = 1$? What is the probability that a one bit error occurs in the channel?

7. Suppose that X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively. Find the probability distribution of X, conditional on the event that $X + Y = n$ where n is fixed in the range $n = 0, 1, 2, \ldots$.

8. (a) Consider the Gambler's ruin problem studied in lectures and construct both $\mathbb{P}(\text{A is ruined})$ and $\mathbb{P}(\text{B is ruined})$. What is $\mathbb{P}(\text{A is ruined}) + \mathbb{P}(\text{B is ruined})$?

   (b) Check that the solution given in lectures for the expected duration of the Gambler's ruin problem solves the stated difference equation.

(IIb) Limits and inequalities

1. Suppose that X is a random variable with the $U(-1,1)$ distribution. Find the exact value of $\mathbb{P}(|X| > a)$ for each $a > 0$ and compare it to the upper bounds obtained from the Markov and Chebychev inequalities.

2. Let X be the random variable giving the number of heads obtained in a sequence of n fair coin flips. Compare the upper bounds on $\mathbb{P}(X > 3n/4)$ obtained from the Markov and Chebychev inequalities.

3. Let A_i $(i = 1, 2, \ldots, n)$ be a collection of random events and set $N = \sum_{i=1}^{n} \mathbb{I}(A_i)$. By considering Markov's inequality applied to $\mathbb{P}(N \geq 1)$ show Boole's inequality, namely,

$$\mathbb{P}\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)\,.$$

4. Let $h : \mathbb{R} \to [0, \infty)$ be a non-negative function. Show that

$$\mathbb{P}(h(X) \geq a) \leq \frac{\mathbb{E}(h(X))}{a} \qquad \text{for all} \qquad a > 0\,.$$

   By making suitable choices of $h(x)$, show that we may obtain the Markov and Chebychev inequalities as special cases.

5. Show the following properties of the moment generating function.
   (a) If X has mgf $M_X(t)$ then $Y = aX + b$ has mgf $M_Y(t) = e^{bt} M_X(at)$.
   (b) If X and Y are independent then $X + Y$ has mgf $M_{X+Y}(t) = M_X(t) M_Y(t)$.
   (c) $\mathbb{E}(X^n) = M_X^{(n)}(0)$ where $M_X^{(n)}$ is the n^{th} derivative of M_X.
   (d) If X is a discrete random variable taking values $0, 1, 2, \ldots$ with probability generating function $G_X(z) = \mathbb{E}(z^X)$ then $M_X(t) = G_X(e^t)$.

6. Let X be a random variable with moment generating function $M_X(t)$ which you may assume exists for any value of t. Show that for any $a > 0$

$$\mathbb{P}(X \leq a) \leq e^{-ta} M_X(t) \qquad \text{for all} \qquad t < 0\,.$$

7. Show that, if $X_n \xrightarrow{D} X$, where X is a degenerate random variable (that is, $\mathbb{P}(X = \mu) = 1$ for some constant μ) then $X_n \xrightarrow{P} X$.

8. Suppose that you estimate your monthly phone bill by rounding all amounts to the nearest pound. If all rounding errors are independent and distributed as $U(-0.5, 0.5)$, estimate the probability that the total error exceeds one pound when your bill has 12 items. How does this procedure suggest an approximate method for constructing Normal random variables?

(IIc) Markov chains

1. Suppose that (X_n) is a Markov chain with n-step transition matrix, $P^{(n)}$, and let $\lambda_i^{(n)} = \mathbb{P}(X_n = i)$ be the elements of a row vector $\lambda^{(n)}$ $(n = 0, 1, 2, \ldots)$. Show that
   (a) $P^{(m+n)} = P^{(m)} P^{(n)}$ for $m, n = 0, 1, 2, \ldots$
   (b) $\lambda^{(n)} = \lambda^{(0)} P^{(n)}$ for $n = 0, 1, 2, \ldots$.

2. Suppose that (X_n) is a Markov chain with transition matrix P. Define the relations *"state j is accessible from state i"* and *"states i and j communicate"*. Show that the second relation is an equivalence relation and define the communicating classes as the equivalence classes under this relation. What is meant by the terms *closed class*, *absorbing class* and *irreducible*?

3. Show that
$$P_{ij}(z) = \delta_{ij} + F_{ij}(z) P_{jj}(z)$$
where
$$P_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n, \qquad F_{ij}(z) = \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n$$
and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. [You should assume that $p_{ij}^{(n)}$ and $f_{ij}^{(n)}$ are as defined in lectures with $p_{ij}^{(0)} = \delta_{ij}$ and $f_{ij}^{(0)} = 0$ for all states i, j.]

4. Suppose that (X_n) is a finite state Markov chain and that for some state i and for all states j
$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$$
for some collection of numbers (π_j). Show that $\pi = (\pi_j)$ is a stationary distribution.

5. Consider the Markov chain with transition matrix
$$P = \begin{pmatrix} 0.128 & 0.872 \\ 0.663 & 0.337 \end{pmatrix}$$
for Markov's example of a chain on the two states $\{\text{vowel}, \text{consonant}\}$ for consecutive letters in a passage of text. Find the stationary distribution for this Markov chain. What are the mean recurrence times for the two states?

6. Define what is meant by saying that (X_n) is a reversible Markov chain and write down the local balance conditions. Show that if a vector π is a distribution over the states of the Markov chain that satisfies the local balance conditions then it is a stationary distribution.

7. Consider the Erhenfest model for m balls moving between two containers with transition matrix
$$p_{i,i+1} = 1 - \frac{i}{m}, \qquad p_{i,i-1} = \frac{i}{m}$$
where i $(0 \le i \le m)$ is the number of balls in a given container. Show that the Markov chain is irreducible and periodic with period 2. Derive the stationary distribution.

8. Consider a random walk, (X_n), on the states $i = 0, 1, 2, \ldots$ with transition matrix

$$\begin{aligned} p_{i,i-1} &= p \qquad i = 1, 2, \ldots \\ p_{i,i+1} &= 1 - p \qquad i = 0, 1, \ldots \\ p_{0,0} &= p \end{aligned}$$

where $0 < p < 1$. Show that the Markov chain is irreducible and aperiodic. Find a condition on p to make the Markov chain positive recurrent and find the stationary distribution in this case.

9. Describe PageRank as a Markov chain model for the motion between nodes in a graph. Explain the main mathematical results that underpin PageRank's connection to a notion of web page *"importance"*.