University of Reading

CMOHB

Evolving Networks & Social Norm Dynamics

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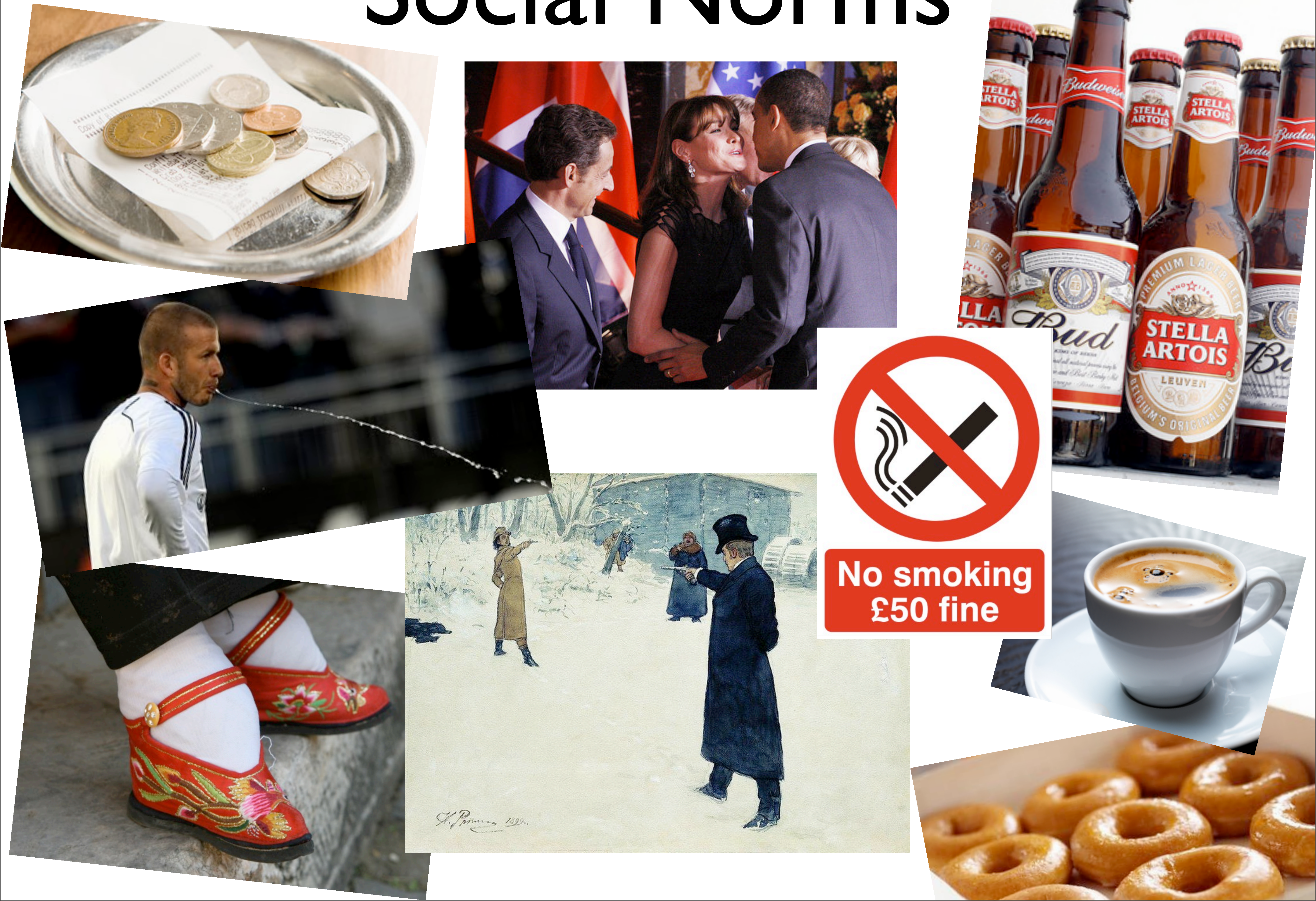
Social Norms



Modelling

"Social norms are shared understandings about actions that are obligatory, permitted or forbidden..."
E. Ostrom, 2000.

- Attitudes/beliefs
- Agent based

Ingredients:

1. Social influence
2. Homophily

- Similar attitudes = more frequent interactions
- Contagion/Diffusion

Homophily:

- McPherson, Smith-Loving, Cook, Annu. Rev. Sociol. 2001

Models:

- Carley, A. Soc. Rev. 1991
- Axelrod, J. Conf. Resolut. 1997
- Macy, Kitts, Flache, Dynamic Social Networks, 2003
- Centola et al, J. Conf. Resolut. 2007

Model Framework - 1

Dynamic of individual's attitudes

$$\frac{du_i}{dt} = \boxed{f(u_i, v_i)}$$

$$\frac{dv_i}{dt} = \boxed{g(u_i, v_i)}$$

Model Framework - 1

Dynamic of individual's attitudes

STABLE

$$\frac{du_i}{dt} = f(u_i, v_i)$$

$$\frac{dv_i}{dt} = g(u_i, v_i)$$

Model Framework - 1

Dynamic of individual's attitudes

STABLE

$$\frac{du_i}{dt} = f(u_i, v_i) - D\sum_{j=1}^{N} A_{ij}(u_i - u_j)$$

$$\frac{dv_i}{dt} = g(u_i, v_i) - \sigma D\sum_{j=1}^{N} A_{ij}(v_i - v_j)$$

Diffusive coupling

Weighted Adjacency Matrix

$$A_{ij}(t) \in (0, 1)$$

$$k_i = \sum_{j=1}^{N} A_{ij}(t)$$

Model Framework - 1

Dynamic of individual's attitudes

STABLE

$$\frac{du_i}{dt} = \boxed{f(u_i, v_i)} - \boxed{D \sum_{j=1}^{N} L_{ij} u_j}$$

$$\frac{dv_i}{dt} = \boxed{g(u_i, v_i)} - \boxed{\sigma D \sum_{j=1}^{N} L_{ij} v_j}$$

Diffusive coupling

Degree

Weighted Graph Laplacian $\rightarrow L_{ij} = k_i \delta_{ij} - A_{ij}$

Weighted Adjacency Matrix

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Model Framework - 1

Dynamic of individual's attitudes

STABLE

Weighted Graph Laplacian → $L_{ij} = k_i \delta_{ij} - A_{ij}$

Degree (k_i)

Weighted Adjacency Matrix (A_{ij})

Jacobian:

$$\begin{pmatrix} + & - \\ + & - \end{pmatrix} \quad \begin{pmatrix} - & + \\ - & + \end{pmatrix}$$

$$\begin{pmatrix} + & + \\ - & - \end{pmatrix} \quad \begin{pmatrix} - & - \\ + & + \end{pmatrix}$$

$$\frac{du_i}{dt} = \boxed{f(u_i, v_i)} - \boxed{D \sum_{j=1}^{N} L_{ij} u_j}$$

$$\frac{dv_i}{dt} = \boxed{g(u_i, v_i)} - \boxed{\sigma D \sum_{j=1}^{N} L_{ij} v_j}$$

TURING INSTABILITY

Diffusive coupling

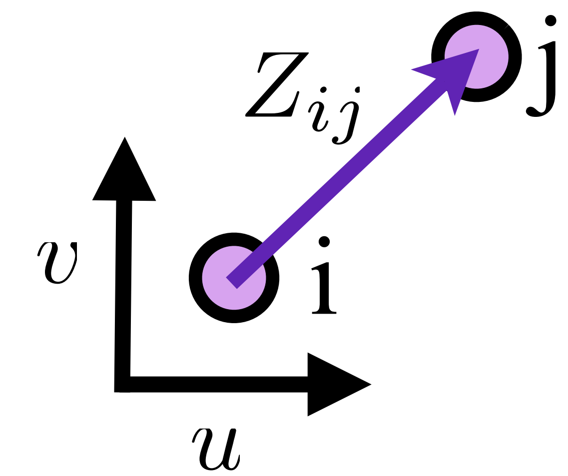
Weighted Adjacency Matrix

$$A_{ij}(t) \in (0, 1)$$

$$k_i = \sum_{j=1}^{N} A_{ij}(t)$$

ogy - norms depend
equency of
nters

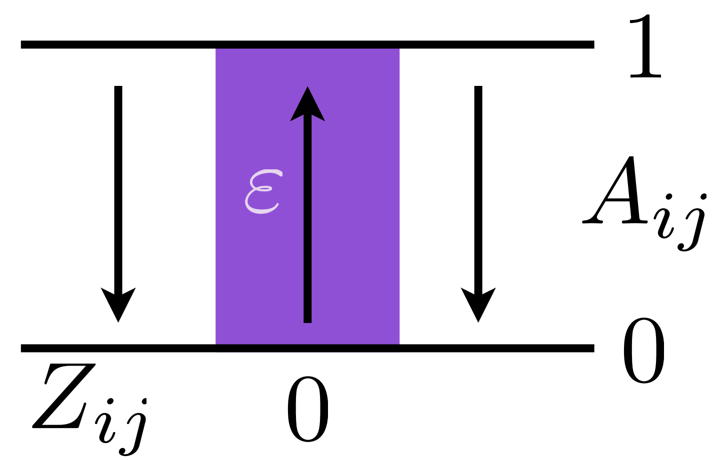
Model Framework - 2



- Homophily: ‘Birds of a feather flock together’
- Evolving/dynamic network model

$$\dot{A} = \alpha A \circ (1 - A) \circ (\varepsilon - \Phi(Z))$$

- e.g. $\Phi_{ij} = [u_i - u_j]^2$



Stochastic Edge Birth and Death

$$E(A(t+\delta t)|A(t)) = A(t) + \mu\delta t\Big[\underbrace{-\omega A(t) \circ (1 - \Phi(\mathbf{Z}))}_{\text{death}} + \overbrace{(1 - A(t)) \circ \Phi(\mathbf{Z})}^{\text{birth}}\Big]$$

Grindrod, Higham, Parsons; *Bistable Evolving Networks*, 2011.

Linear Analysis

FACTS

Graph Laplacian spectrum

- All eigenvalues are real and non-negative
- Zero is always an eigenvalue
- Multiplicity of zero eigenvalues = Number of connected components
- Eigenvalues of N-node connected graph: 0 with multiplicity 1, N with multiplicity N-1

a+d trace
ad-bc determinant

REFERENCE: Nakao

Linear Analysis

- Ansatz: Eigenvectors of Laplacian

$$\lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a - Dk_i & b \\ c & d - \sigma D k_i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

FACTS

Graph Laplacian spectrum

- All eigenvalues are real and non-negative
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Node Dynamics

Linearisation:

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Stability:

$$a + d < 0$$

$$ad - bc > 0$$

a+d trace
ad-bc determinant

REFERENCE: Nakao

Nakao & Mikhailov, Nature Physics, 2010

Linear Analysis

- Ansatz: Eigenvectors of Laplacian

$$\lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a - Dk_i & b \\ c & d - \sigma D k_i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Node Dynamics

Linearisation:

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Stability:

$$a + d < 0$$

$$ad - bc > 0$$

- Characteristic equation:

$$\lambda^2 - \underbrace{[a + d - D(1+\sigma)k_i]}_{<0} \lambda + \sigma D^2 h(k_i) = 0$$

Controls **system** stability

a+d trace
ad-bc determinant

REFERENCE: Nakao

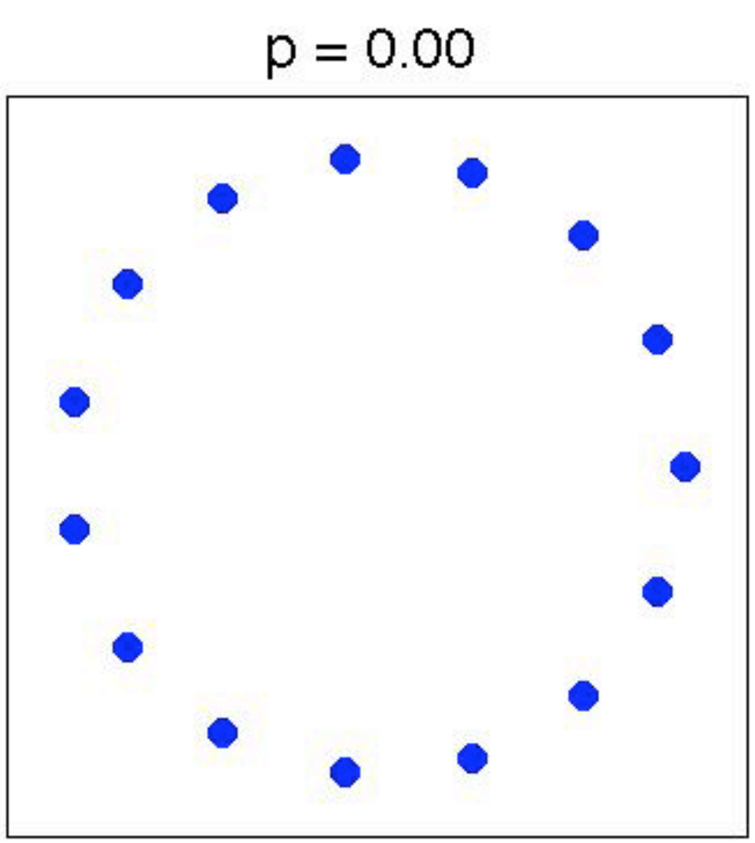
Nakao & Mikhailov, Nature Physics, 2010

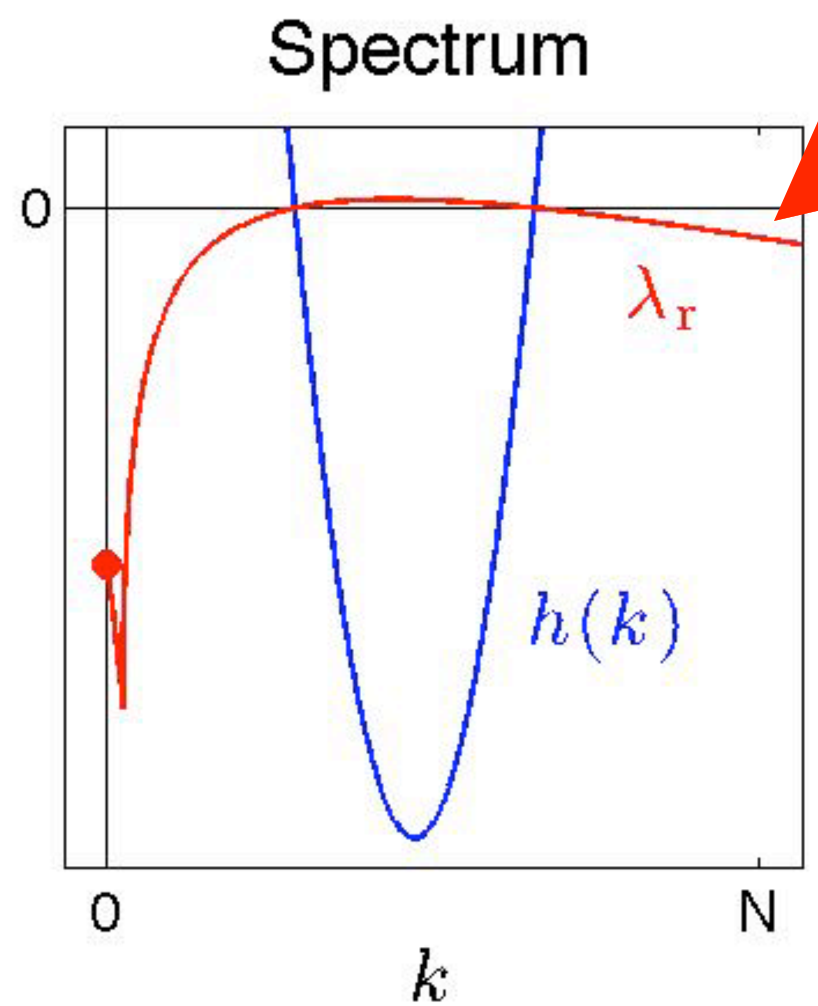
Linear Analysis

System stability controlled by determinant of linearisation:

$$h(k) = k^2 - \frac{(\sigma a + d)}{\sigma D} k + \frac{ad - bc}{\sigma D^2}$$

Fully connected graph





Example: Schnakenberg

Reaction Equations

$P \to U$

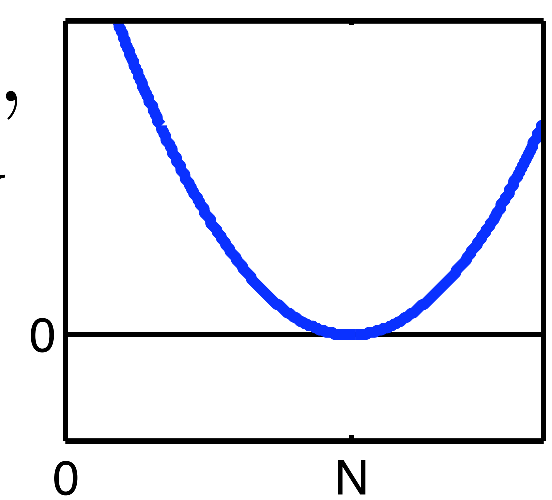
$Q \leftrightharpoons V$

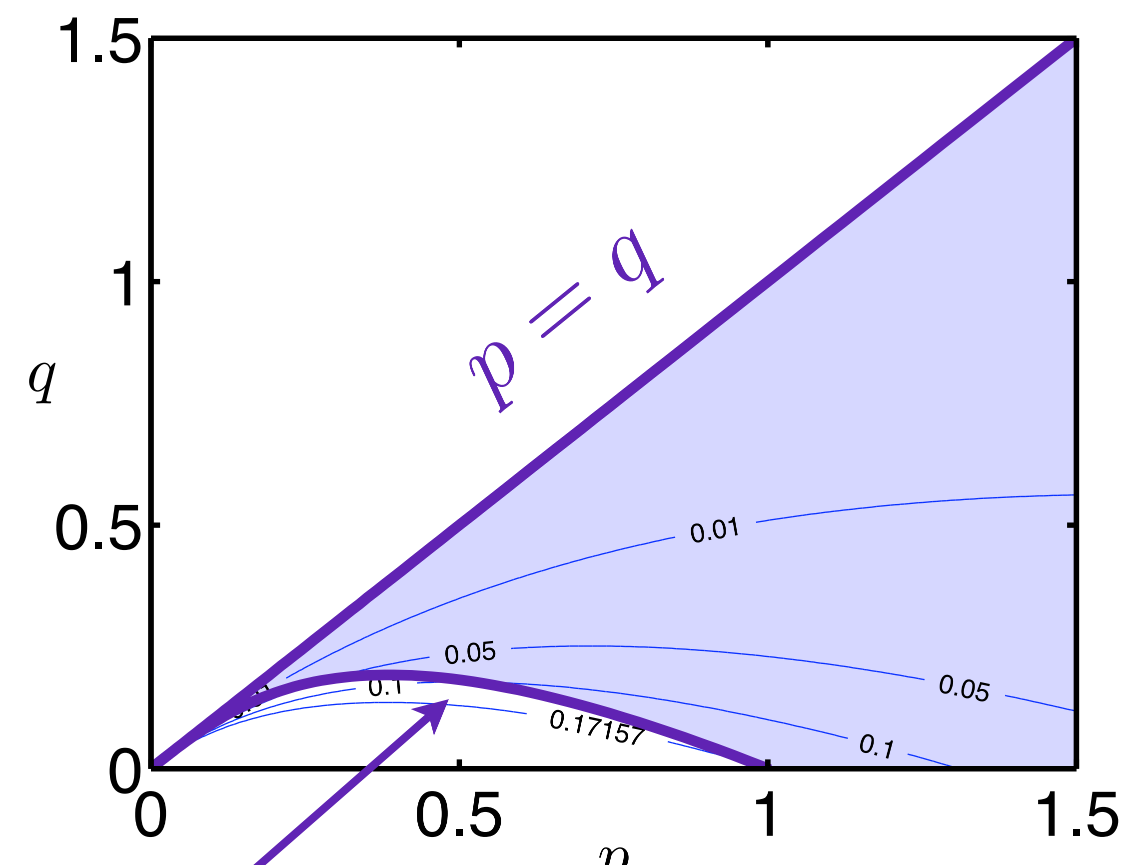
$U + 2V \to 3V$

$$f(u,v) = p - uv^2$$
$$g(u,v) = q - v + uv^2$$

Parameter Conditions:

For given p and q, control instability window with σ and D



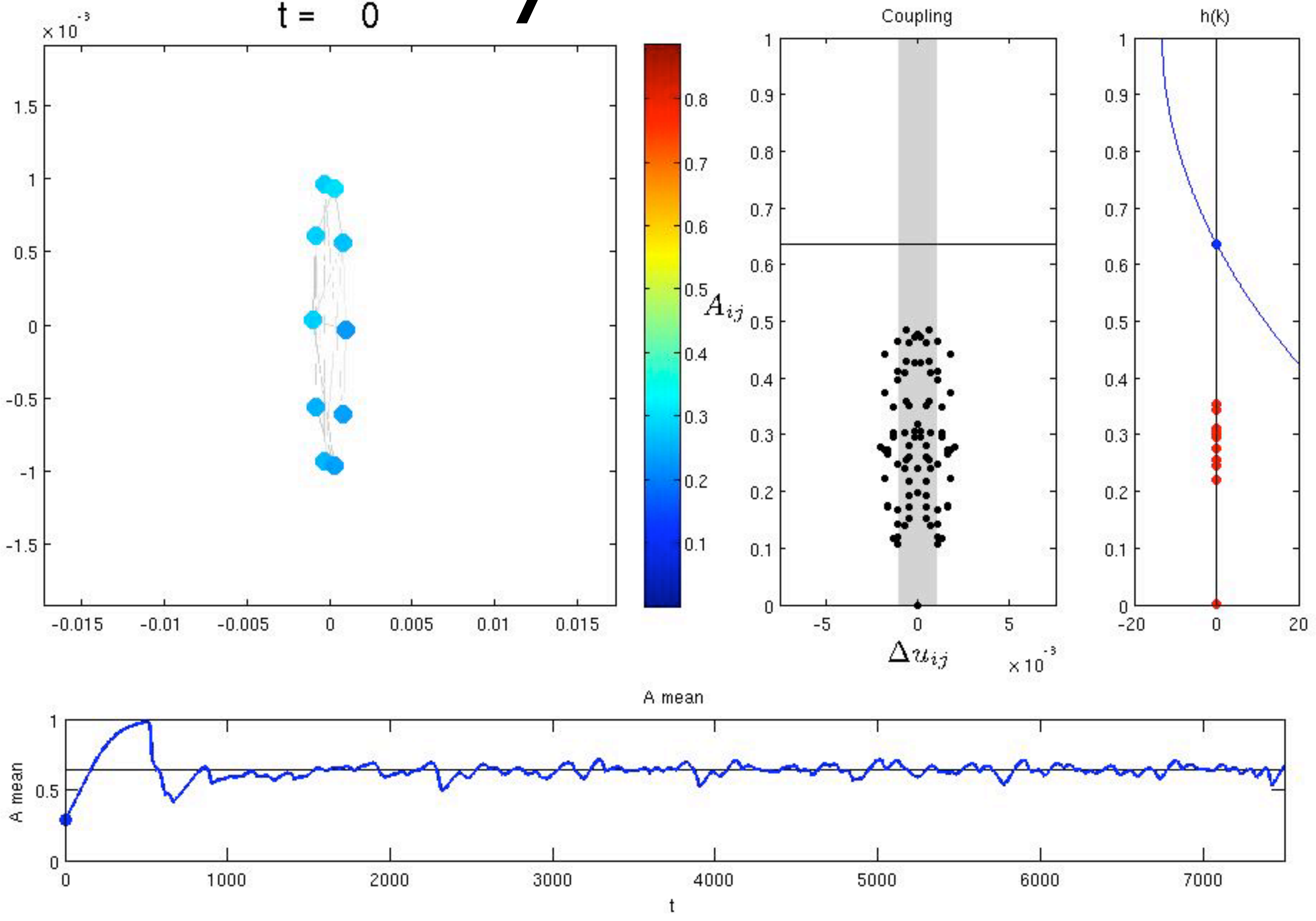


$p - q = (p+q)^3$

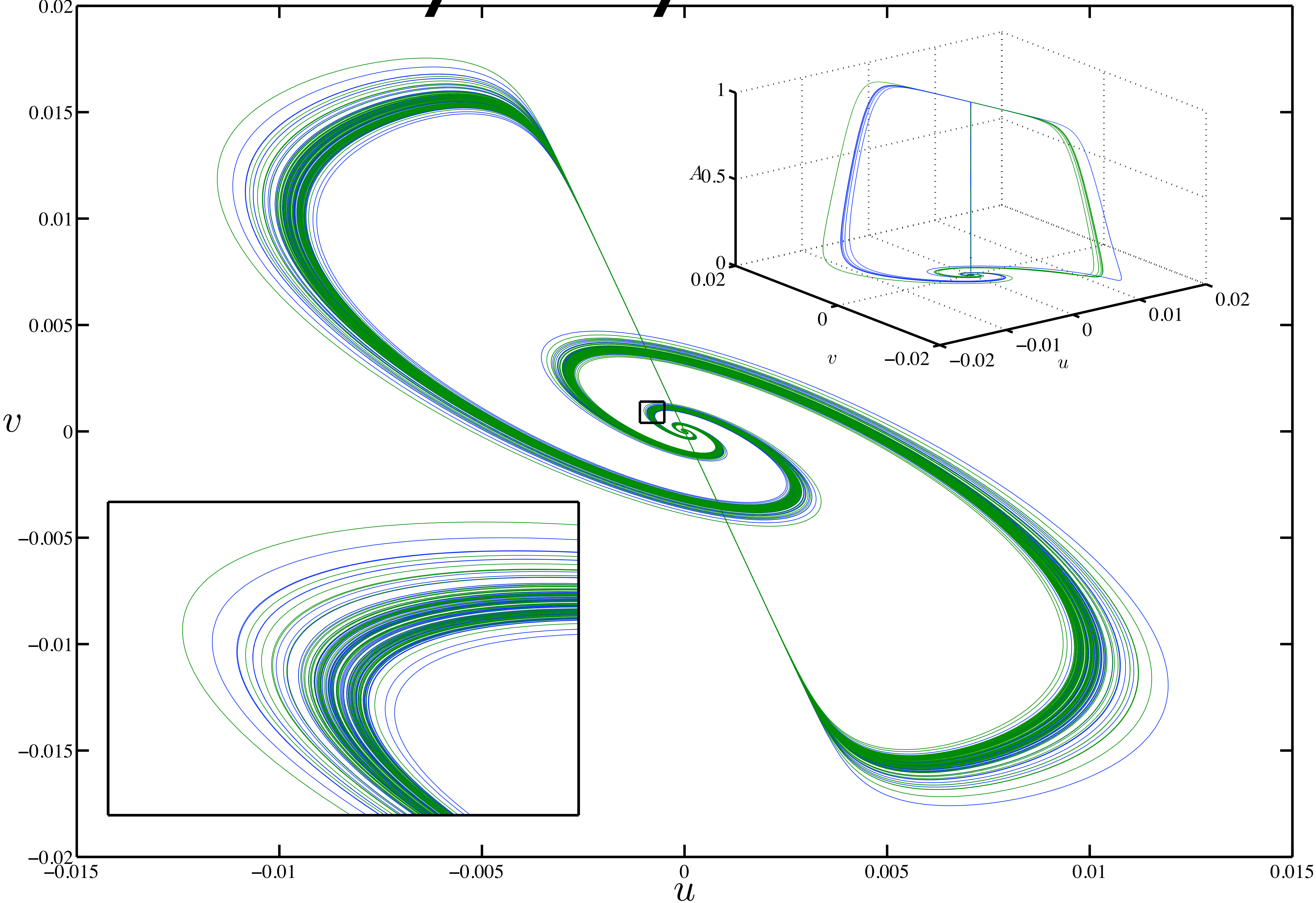
Dynamics

Only one dynamic - when clique is unstable

Dynamics



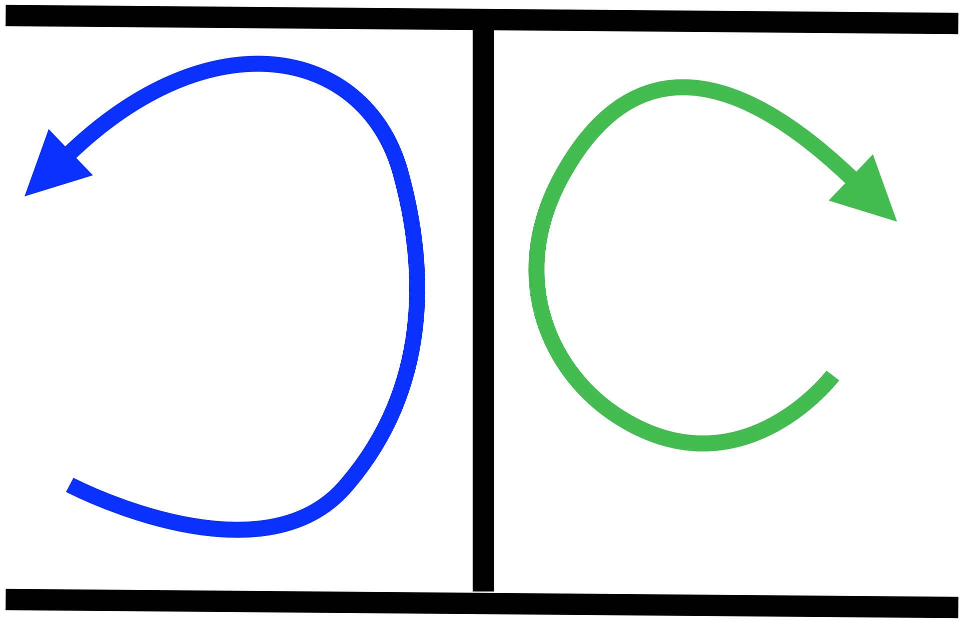
Dyad Dynamics



Symmetry

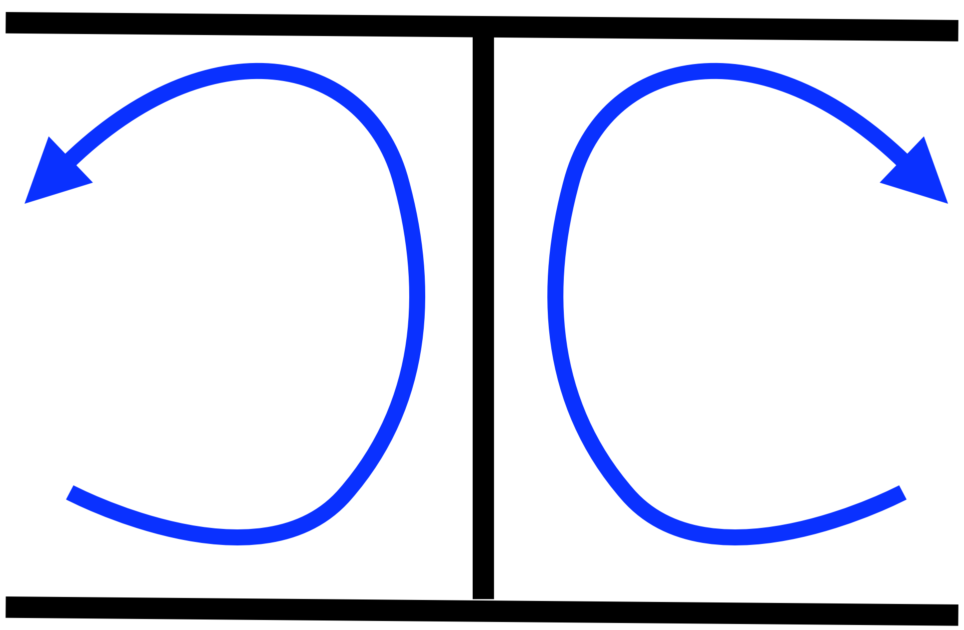
Two agents:

$$\dot{u}_1 = f(u_1, v_1) - 2DA(u_1 - u_2)$$
$$\dot{u}_2 = f(u_2, v_2) + 2DA(u_1 - u_2)$$
$$\dot{v}_1 = g(u_1, v_1) - 2\sigma DA(v_1 - v_2)$$
$$\dot{v}_2 = g(u_2, v_2) + 2\sigma DA(v_1 - v_2)$$
$$\dot{A} = \alpha A(1 - A)\left(\varepsilon - [u_1 - u_2]^2\right)$$



$$(u_1, u_2, v_1, v_2, A) \mapsto (u_2, u_1, v_2, v_1, A)$$

Odd symmetry reduces system:

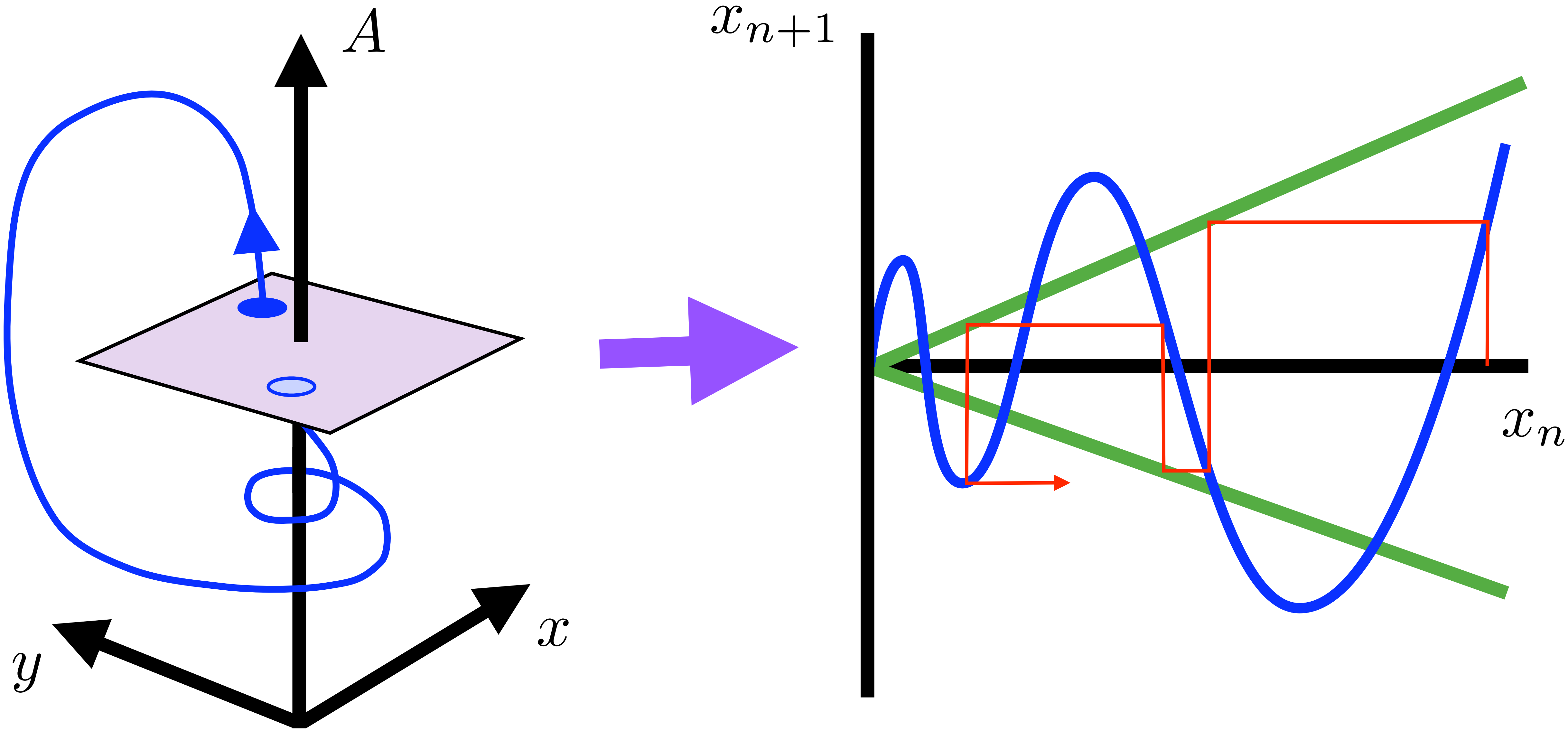


$$(u_1, u_2, v_1, v_2, A) \mapsto (-u_1, -u_2, -v_1, -v_2, A)$$

$$\dot{x} = ax + by - xy^2 - 2DAx$$
$$\dot{y} = cx + dy + xy^2 - 2DAy$$
$$\dot{A} = \alpha A(1 - A)(\varepsilon - x^2)$$

Maps

- First return map



- Reduce to symbolic dynamics: ...,L,R,L,R,R,L,...
- Find sensitive dependence to initial conditions

Summary/Future

- Deterministic model has complex dynamics
- Large scale: Predictable
- Small scale: Unpredictable

Future Work:

- Polarisation phenomena?
- Embedded network
- Triangulation
- Homoclinic/Heteroclinic

Thanks for listening!

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