Non-uniform Refine and Smooth Subdivision for General Degree B-splines

Tom Cashman\textsuperscript{1}    Neil Dodgson\textsuperscript{1}    Malcolm Sabin\textsuperscript{2}

\textsuperscript{1}Computer Laboratory
University of Cambridge

\textsuperscript{2}Numerical Geometry Ltd.

8th November 2007
Outline

1 Introduction
   Motivation
   Background
   Blossoming

2 Approaches to non-uniform refine and smooth
   Degree independent
   Schaefer’s algorithm
   Symmetric algorithm
NURBS

- **Non-Uniform Rational B-Splines**
- NURBS curves are used in 2D (and 3D)
NURBS

- Non-Uniform Rational B-Splines
- NURBS curves are used in 2D (and 3D)
- NURBS surfaces use a rectangular control grid
- Industry standard for Computer-Aided Design (CAD)
Subdivision Surfaces

- Control meshes without a rigid rectangular grid
- Vertices with irregular valency are called *extraordinary points*
- Used heavily in animation since ‘Geri’s Game’ (Pixar, 1997)
A Brief History of Subdivision Surfaces

NURBS
Surfaces
A Brief History of Subdivision Surfaces

NURBS Surfaces

- Uniform, general degree
- Non-uniform, low degree

means "⊂"
A Brief History of Subdivision Surfaces

means “⊂”

NURBS Surfaces

Uniform, general degree

Non-uniform, low degree

Uniform, low degree
A Brief History of Subdivision Surfaces

- NURBS Surfaces
  - Uniform, general degree
  - Non-uniform, low degree
  - Uniform, low degree

- 1978
  - Catmull-Clark, Doo-Sabin
  - Surfaces with extraordinary points

- 1987
  - Loop

- 1990
  - Butterfly

- 1998
  - Sederberg et al.

- 2001
  - Zorin & Schröder, Stam

NURBS-compatible subdivision?
A Brief History of Subdivision Surfaces

Motivation

NURBS Surfaces
- Uniform, general degree
- Non-uniform, low degree
- Uniform, low degree

1978 Catmull-Clark, Doo-Sabin
1987 Loop, 1990 Butterfly
1998 Sederberg et al.
2001 Zorin & Schroeder, Stam

Surfaces with extraordinary points

Other regular surfaces

means “⊂”
A Brief History of Subdivision Surfaces

- 1978: Catmull-Clark, Doo-Sabin
- 1987: Loop, 1990: Butterfly
- 1998: Sederberg et al.
- 2001: Zorin & Schröder, Stam

Uniform, general degree
Non-uniform, low degree
Uniform, low degree
Non-uniform, low degree

NURBS Surfaces

Surfaces with extraordinary points

Other regular surfaces

Means "⊂"
A Brief History of Subdivision Surfaces

Motivation

- 1978: Catmull-Clark, Doo-Sabin
  - Non-uniform, low degree

- 1987: Loop
  - Uniform, general degree
  - Surfaces with extraordinary points

- 1990: Butterfly

- 1998: Sederberg et al.
  - NURBS-compatible subdivision?

- 2001: Zorin & Schröder, Stam

NURBS Surfaces

- Uniform, general degree
- Non-uniform, low degree
- Uniform, low degree

Other regular surfaces
Knot Insertion

- B-splines are pieces of polynomial meeting at knots
- Uniform B-splines have even spacing between the knots
- Subdivision inserts more knots
  - Bringing the control polygon closer to the curve
Knot Insertion

- B-splines are pieces of polynomial meeting at knots
- Uniform B-splines have even spacing between the knots
- Subdivision inserts more knots
  - Bringing the control polygon closer to the curve
Refine and smooth subdivision

- Lane-Riesenfeld – uniform B-spline subdivision
Refine and smooth subdivision

- Lane-Riesenfeld – uniform B-spline subdivision

- Refine
  - polygon lengthened by adding points
Refine and smooth subdivision

- Lane-Riesenfeld – uniform B-spline subdivision

- **Refine**
  - polygon lengthened by adding points

- and **Smooth**
  - each step creates another polygon
  - points moved using local filters

- More smoothing steps for higher degree
Why Refine and Smooth?

NURBS Surfaces

1978 Catmull-Clark, Doo-Sabin
1987 Loop, Butterfly
1998 Sederberg et al.
2001 Zorin & Schröder, Stam

Surfaces with extraordinary points

Uniform, low degree
Non-uniform, low degree
Uniform, general degree

means “⊂”

NURBS-compatible subdivision?

Other regular surfaces
Problem statement

We want a knot insertion algorithm that is

- non-uniform,
- general degree, and uses
- refine and smooth
The polar form of a polynomial

Polynomials of degree $d$ \iff\ Symmetric $d$-affine maps

(polar form or \textit{blossom})
The polar form of a polynomial

Polynomials of degree $d$ $\Leftrightarrow$ Symmetric $d$-affine maps (polar form or blossom)

$P(t) = p(t, t, \ldots, t, t)$
The polar form of a polynomial

Polynomials of degree $d$ $\Rightarrow$ Symmetric $d$-affine maps

$P(t) = p(t, t, \ldots, t, t)$

$bx^2 + cx + d = b(x_1 x_2) + c\left(\frac{x_1 + x_2}{2}\right) + d$
The polar form of a polynomial

Polynomials of degree \( d \) \( \Rightarrow \) Symmetric \( d \)-affine maps

\[[\text{polar form or } \text{blossom}]\]

\[P(t) = p(t, t, \ldots, t, t)\]

\[bx^2 + cx + d = b(x_1x_2) + c\left(\frac{x_1+x_2}{2}\right) + d\]

\[ax^3 + bx^2 + cx + d = a(x_1x_2x_3) + b\left(\frac{x_1x_2+x_2x_3+x_1x_3}{3}\right) + c\left(\frac{x_1+x_2+x_3}{3}\right) + d\]

Properties: symmetric, multiaffine, diagonal
Blossoming and B-spline control points

Control points are the blossom evaluated at consecutive knots

\[ p(1, 3, 4) \]
\[ p(3, 4, 7) \]
\[ p(7, 8, 9) \]
\[ p(4, 7, 8) \]
\[ p(8, 9, 13) \]
\[ p(9, 13, 15) \]
\[ p(15, 16, 19) \]
Blossoming and knot insertion

\[ \tau_0 \quad \tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4 \]
Blossoming and knot insertion

Insert knots $t_i$: $\tau_i \leq t_i \leq \tau_{i+1}$
Blossoming and knot insertion

Insert knots $t_i$: $\tau_i \leq t_i \leq \tau_{i+1}$

\[ C = \frac{\tau_k - t_j}{\tau_k - \tau_i} A + \frac{t_j - \tau_i}{\tau_k - \tau_i} B \]
Blossoming and knot insertion

Insert knots $t_i$: $\tau_i \leq t_i \leq \tau_{i+1}$

\[ C = \frac{\tau_k - t_j}{\tau_k - \tau_i} A + \frac{t_j - \tau_i}{\tau_k - \tau_i} B \]

\[ J = \frac{\tau_m - t_k}{\tau_m - \tau_i} D + \frac{t_k - t_j}{\tau_m - \tau_i} E + \frac{t_j - \tau_i}{\tau_m - \tau_i} F \]
Blossoming and knot insertion

Insert knots $t_i$: $\tau_i \leq t_i \leq \tau_{i+1}$

\[
C = \frac{\tau_k - t_j}{\tau_k - \tau_i} A + \frac{t_j - \tau_i}{\tau_k - \tau_i} B
\]

\[
G = \frac{\tau_m - t_k}{\tau_m - x} D + \frac{t_k - x}{\tau_m - x} E
\]

\[
H = \frac{x - t_j}{x - \tau_i} E + \frac{t_j - \tau_i}{x - \tau_i} F
\]

\[
J = \frac{\tau_m - x}{\tau_m - \tau_i} G + \frac{x - \tau_i}{\tau_m - \tau_i} H
\]
Outline

1 Introduction
   - Motivation
   - Background
   - Blossoming

2 Approaches to non-uniform refine and smooth
   - Degree independent
   - Schaefer’s algorithm
   - Symmetric algorithm
Adapting Lane-Riesenfeld

For Lane-Riesenfeld...

- Subdivision uses one \textit{refine} and multiple \textit{smooth} steps
- Smoothing filters compute a new point from \textit{two old points}
- Smoothing filters compute the \textit{midpoint} of two old points
Adapting Lane-Riesenfeld

For Lane-Riesenfeld...

- Subdivision uses one refine and multiple smooth steps
- Smoothing filters compute a new point from two old points
- Smoothing filters compute the midpoint of two old points
- Intermediate smoothing steps compute the subdivision result for lower degree
We want a non-uniform algorithm, where...

- Subdivision uses one \textit{refine} and multiple \textit{smooth} steps
- Smoothing filters compute a new point from \textit{two old points}
- Smoothing filters compute the \textit{midpoint} of two old points
- Intermediate smoothing steps compute the subdivision result \textit{for lower degree}
Maintaining degree independence

- But intermediate polygons that compute lower degree subdivision introduce constraints on $\tau$ and $t$
But intermediate polygons that compute lower degree subdivision introduce constraints on $\tau$ and $t$. 
Maintaining degree independence

- But intermediate polygons that compute lower degree subdivision introduce constraints on $\tau$ and $t$
Maintaining degree independence

- But intermediate polygons that compute lower degree subdivision introduce constraints on $\tau$ and $t$
Revisiting our requirements

We want a non-uniform algorithm, where...

- Subdivision uses one *refine* and multiple *smooth* steps
- Smoothing filters compute a new point from *two old points*

- Intermediate smoothing steps compute the subdivision result for lower degree
Revisiting our requirements

We want a non-uniform algorithm, where...

- Subdivision uses one `refine` and multiple `smooth` steps
- Smoothing filters compute a new point from two old points
- Intermediate smoothing steps compute the subdivision result for lower degree
\[ \mathcal{T}_0 \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \quad \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \mathcal{T}_5 \quad \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \mathcal{T}_5 \quad \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \mathcal{T}_5 \mathcal{T}_6 \]
Approaches to non-uniform refine and smooth

Schaefer’s algorithm
Approaches to non-uniform refine and smooth

Schaefer’s algorithm
Approaches to non-uniform refine and smooth

Schaefer’s algorithm

Cashman, Dodgson, Sabin (Cambridge)

Non-uniform Refine and Smooth

8th November 2007
Approaches to non-uniform refine and smooth

Schaefer’s algorithm

\[ \tau_0 \tau_1 \tau_2 \tau_3 \tau_4 \]  
\[ \tau_1 \tau_2 \tau_3 \tau_4 \]  
\[ t_0 \tau_1 \tau_2 \tau_3 \tau_4 \]  
\[ t_1 \tau_2 \tau_3 \tau_4 \]  
\[ t_2 \tau_3 \tau_4 \]  
\[ t_1 \tau_2 \tau_3 \tau_4 \]  
\[ t_1 \tau_2 \tau_3 \tau_4 \]  
\[ t_1 \tau_2 \tau_3 \tau_4 \]  
\[ t_2 \tau_3 \tau_4 \]  
\[ t_2 \tau_3 \tau_4 \]  
\[ t_2 \tau_3 \tau_4 \]  
\[ t_2 \tau_3 \tau_4 \]  
\[ t_2 \tau_3 \tau_4 \]
Approaches to non-uniform refine and smooth

Schaefer’s algorithm
Schaefer’s algorithm is asymmetric

The asymmetry in Schaefer’s algorithm makes it hard to use on surfaces
Revisiting our requirements

We want a non-uniform algorithm, where...

- Subdivision uses one refine and multiple smooth steps
- Smoothing filters compute a new point from two old points
Revisiting our requirements

We want a non-uniform algorithm, where...

- Subdivision uses one **refine** and multiple **smooth** steps
- Smoothing filters compute a new point from **two old points**
- Intermediate smoothing steps are **symmetric**
Approaches to non-uniform refine and smooth

Symmetric algorithm

Cashman, Dodgson, Sabin (Cambridge)

8th November 2007
Approaches to non-uniform refine and smooth

Symmetric algorithm

Cashman, Dodgson, Sabin (Cambridge)
Approaches to non-uniform refine and smooth

Symmetric algorithm

Cashman, Dodgson, Sabin (Cambridge)
Approaches to non-uniform refine and smooth

Symmetric algorithm

Cashman, Dodgson, Sabin (Cambridge)

8th November 2007 21 / 23
Even degree... and multiple knots

\[ \tau_1 \tau_2 \tau_3 \tau_4 \quad \tau_1 \tau_2 \tau_3 \tau_4 \quad \tau_2 \tau_3 \tau_4 \tau_5 \quad \tau_2 \tau_3 \tau_4 \tau_5 \]
Even degree... and multiple knots
Even degree... and multiple knots
Even degree... and multiple knots
Even degree... and multiple knots
Summary

- There are non-uniform analogues of the Lane-Riesenfeld refine and smooth algorithm
- In fact there are several, each for different requirements
- A symmetric algorithm may lead to subdivision schemes generalising NURBS
- Multiple knots fit into a common framework

- Next step: extraordinary points
Extraordinary points
Extraordinary points
Extraordinary points
Extraordinary points