Fundamental Properties of Lambda-calculus

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Contents

1 Basic definitions of Lambda-calculus **4**
- 1.1 Lambda-terms in de Bruijn notation and substitution 4
- 1.2 Beta-reduction 5
- 1.3 Congruence rules 5
- 1.4 Substitution-lemmas 5
- 1.5 Equivalence proof for optimized substitution 6
- 1.6 Preservation theorems 6

2 Abstract commutation and confluence notions **7**
- 2.1 Basic definitions 7
- 2.2 Basic lemmas 8
- 2.3 Church-Rosser 9
- 2.4 Newman's lemma 9

3 Parallel reduction and a complete developments **10**
- 3.1 Parallel reduction 10
- 3.2 Inclusions 10
- 3.3 Misc properties of *par-beta* 10
- 3.4 Confluence (directly) 10
- 3.5 Complete developments 11
- 3.6 Confluence (via complete developments) 11

4 Eta-reduction **11**
- 4.1 Definition of eta-reduction and relatives 11
- 4.2 Properties of *eta*, *subst* and *free* 12
- 4.3 Confluence of *eta* 12
- 4.4 Congruence rules for eta^* 12
- 4.5 Commutation of *beta* and *eta* 13
- 4.6 Implicit definition of *eta* 13
- 4.7 Eta-postponement theorem 14

5 Application of a term to a list of terms **14**

6 Simply-typed lambda terms **16**
6.1 Environments 16
6.2 Types and typing rules 16
6.3 Some examples 17
6.4 Lists of types 17
6.5 n-ary function types 18
6.6 Lifting preserves well-typedness 18
6.7 Substitution lemmas 19
6.8 Subject reduction 19
6.9 Alternative induction rule for types 19

7 Lifting an order to lists of elements **19**

8 Lifting beta-reduction to lists **21**

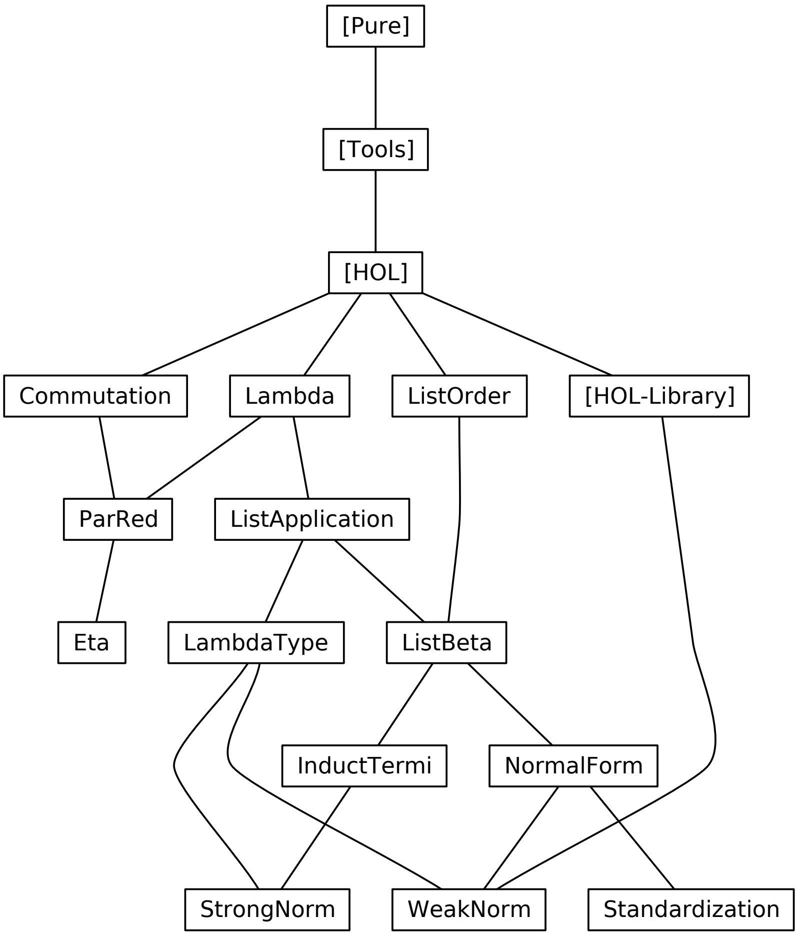
9 Inductive characterization of terminating lambda terms **21**
9.1 Terminating lambda terms 21
9.2 Every term in *IT* terminates 22
9.3 Every terminating term is in *IT* 22

10 Strong normalization for simply-typed lambda calculus **22**
10.1 Properties of *IT* 22
10.2 Well-typed substitution preserves termination 23
10.3 Well-typed terms are strongly normalizing 23

11 Inductive characterization of lambda terms in normal form **23**
11.1 Terms in normal form 23
11.2 Properties of *NF* 25

12 Standardization **25**
12.1 Standard reduction relation 25
12.2 Leftmost reduction and weakly normalizing terms 27

13 Weak normalization for simply-typed lambda calculus **28**
13.1 Main theorems 28
13.2 Extracting the program 29
13.3 Generating executable code 32


[Pure]
[Tools]
[HOL]
Commutation
Lambda
ListOrder
[HOL-Library]
ParRed
ListApplication
Eta
LambdaType
ListBeta
InductTermi
NormalForm
StrongNorm
WeakNorm
Standardization

1 Basic definitions of Lambda-calculus

theory *Lambda*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

1.1 Lambda-terms in de Bruijn notation and substitution

```
datatype dB =
    Var nat
  | App dB dB (infixl ‹°› 200)
  | Abs dB

primrec
  lift :: [dB, nat] => dB
where
    lift (Var i) k = (if i < k then Var i else Var (i + 1))
  | lift (s ° t) k = lift s k ° lift t k
  | lift (Abs s) k = Abs (lift s (k + 1))

primrec
  subst :: [dB, dB, nat] => dB  (‹-[-'/-]› [300, 0, 0] 300)
where
    subst-Var: (Var i)[s/k] =
      (if k < i then Var (i − 1) else if i = k then s else Var i)
  | subst-App: (t ° u)[s/k] = t[s/k] ° u[s/k]
  | subst-Abs: (Abs t)[s/k] = Abs (t[lift s 0 / k+1])

declare subst-Var [simp del]
```

Optimized versions of *subst* and *lift*.

```
primrec
  liftn :: [nat, dB, nat] => dB
where
    liftn n (Var i) k = (if i < k then Var i else Var (i + n))
  | liftn n (s ° t) k = liftn n s k ° liftn n t k
  | liftn n (Abs s) k = Abs (liftn n s (k + 1))

primrec
  substn :: [dB, dB, nat] => dB
where
    substn (Var i) s k =
      (if k < i then Var (i − 1) else if i = k then liftn k s 0 else Var i)
  | substn (t ° u) s k = substn t s k ° substn u s k
  | substn (Abs t) s k = Abs (substn t s (k + 1))
```

1.2 Beta-reduction

inductive *beta* :: [*dB*, *dB*] => *bool* (**infixl** ‹$\rightarrow_\beta$› *50*)
where
beta [*simp*, *intro*!]: $Abs\ s \circ t \rightarrow_\beta s[t/0]$
| *appL* [*simp*, *intro*!]: $s \rightarrow_\beta t \Longrightarrow s \circ u \rightarrow_\beta t \circ u$
| *appR* [*simp*, *intro*!]: $s \rightarrow_\beta t \Longrightarrow u \circ s \rightarrow_\beta u \circ t$
| *abs* [*simp*, *intro*!]: $s \rightarrow_\beta t \Longrightarrow Abs\ s \rightarrow_\beta Abs\ t$

abbreviation
beta-reds :: [*dB*, *dB*] => *bool* (**infixl** ‹$\rightarrow_\beta{}^*$› *50*) **where**
$s \rightarrow_\beta{}^* t == beta^{**}\ s\ t$

inductive-cases *beta-cases* [*elim*!]:
$Var\ i \rightarrow_\beta t$
$Abs\ r \rightarrow_\beta s$
$s \circ t \rightarrow_\beta u$

declare *if-not-P* [*simp*] *not-less-eq* [*simp*]
— don't add *r-into-rtrancl*[*intro*!]

1.3 Congruence rules

lemma *rtrancl-beta-Abs* [*intro*!]:
$s \rightarrow_\beta{}^* s' \Longrightarrow Abs\ s \rightarrow_\beta{}^* Abs\ s'$
⟨*proof*⟩

lemma *rtrancl-beta-AppL*:
$s \rightarrow_\beta{}^* s' \Longrightarrow s \circ t \rightarrow_\beta{}^* s' \circ t$
⟨*proof*⟩

lemma *rtrancl-beta-AppR*:
$t \rightarrow_\beta{}^* t' \Longrightarrow s \circ t \rightarrow_\beta{}^* s \circ t'$
⟨*proof*⟩

lemma *rtrancl-beta-App* [*intro*]:
$[\![s \rightarrow_\beta{}^* s';\ t \rightarrow_\beta{}^* t']\!] \Longrightarrow s \circ t \rightarrow_\beta{}^* s' \circ t'$
⟨*proof*⟩

1.4 Substitution-lemmas

lemma *subst-eq* [*simp*]: $(Var\ k)[u/k] = u$
⟨*proof*⟩

lemma *subst-gt* [*simp*]: $i < j \Longrightarrow (Var\ j)[u/i] = Var\ (j - 1)$
⟨*proof*⟩

lemma *subst-lt* [*simp*]: $j < i \Longrightarrow (Var\ j)[u/i] = Var\ j$
⟨*proof*⟩

lemma *lift-lift*:
$i < k + 1 \Longrightarrow \mathit{lift}\ (\mathit{lift}\ t\ i)\ (\mathit{Suc}\ k) = \mathit{lift}\ (\mathit{lift}\ t\ k)\ i$
⟨*proof*⟩

lemma *lift-subst* [*simp*]:
$j < i + 1 \Longrightarrow \mathit{lift}\ (t[s/j])\ i = (\mathit{lift}\ t\ (i + 1))\ [\mathit{lift}\ s\ i\ /\ j]$
⟨*proof*⟩

lemma *lift-subst-lt*:
$i < j + 1 \Longrightarrow \mathit{lift}\ (t[s/j])\ i = (\mathit{lift}\ t\ i)\ [\mathit{lift}\ s\ i\ /\ j + 1]$
⟨*proof*⟩

lemma *subst-lift* [*simp*]:
$(\mathit{lift}\ t\ k)[s/k] = t$
⟨*proof*⟩

lemma *subst-subst*:
$i < j + 1 \Longrightarrow t[\mathit{lift}\ v\ i\ /\ \mathit{Suc}\ j][u[v/j]/i] = t[u/i][v/j]$
⟨*proof*⟩

1.5 Equivalence proof for optimized substitution

lemma *liftn-0* [*simp*]: $\mathit{liftn}\ 0\ t\ k = t$
⟨*proof*⟩

lemma *liftn-lift* [*simp*]: $\mathit{liftn}\ (\mathit{Suc}\ n)\ t\ k = \mathit{lift}\ (\mathit{liftn}\ n\ t\ k)\ k$
⟨*proof*⟩

lemma *substn-subst-n* [*simp*]: $\mathit{substn}\ t\ s\ n = t[\mathit{liftn}\ n\ s\ 0\ /\ n]$
⟨*proof*⟩

theorem *substn-subst-0*: $\mathit{substn}\ t\ s\ 0 = t[s/0]$
⟨*proof*⟩

1.6 Preservation theorems

Not used in Church-Rosser proof, but in Strong Normalization.

theorem *subst-preserves-beta* [*simp*]:
$r \to_\beta s \Longrightarrow r[t/i] \to_\beta s[t/i]$
⟨*proof*⟩

theorem *subst-preserves-beta'*: $r \to_\beta{}^* s \Longrightarrow r[t/i] \to_\beta{}^* s[t/i]$
⟨*proof*⟩

theorem *lift-preserves-beta* [*simp*]:
$r \to_\beta s \Longrightarrow \mathit{lift}\ r\ i \to_\beta \mathit{lift}\ s\ i$
⟨*proof*⟩

theorem *lift-preserves-beta'*: $r \to_\beta{}^* \ s \Longrightarrow \mathit{lift}\ r\ i \to_\beta{}^* \ \mathit{lift}\ s\ i$
⟨*proof*⟩

theorem *subst-preserves-beta2* [*simp*]: $r \to_\beta s \Longrightarrow t[r/i] \to_\beta{}^* \ t[s/i]$
⟨*proof*⟩

theorem *subst-preserves-beta2'*: $r \to_\beta{}^* \ s \Longrightarrow t[r/i] \to_\beta{}^* \ t[s/i]$
⟨*proof*⟩

end

2 Abstract commutation and confluence notions

theory *Commutation*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

2.1 Basic definitions

definition
square :: [*'a* => *'a* => *bool*, *'a* => *'a* => *bool*, *'a* => *'a* => *bool*, *'a* => *'a* => *bool*] => *bool* **where**
square R S T U =
$(\forall x\ y.\ R\ x\ y \ {-}{-}{>}\ (\forall z.\ S\ x\ z \ {-}{-}{>}\ (\exists u.\ T\ y\ u \land U\ z\ u)))$

definition
commute :: [*'a* => *'a* => *bool*, *'a* => *'a* => *bool*] => *bool* **where**
commute R S = *square R S S R*

definition
diamond :: (*'a* => *'a* => *bool*) => *bool* **where**
diamond R = *commute R R*

definition
Church-Rosser :: (*'a* => *'a* => *bool*) => *bool* **where**
Church-Rosser R =
$(\forall x\ y.\ (\mathit{sup}\ R\ (R^{-1-1}))^{**}\ x\ y \longrightarrow (\exists z.\ R^{**}\ x\ z \land R^{**}\ y\ z))$

abbreviation
confluent :: (*'a* => *'a* => *bool*) => *bool* **where**
confluent R == *diamond* (R^{**})

2.2 Basic lemmas

square

lemma *square-sym*: *square R S T U ==> square S R U T*
⟨*proof*⟩

lemma *square-subset*:
$[\!|$ *square R S T U*; $T \leq T'$ $|\!]$ ==> *square R S* T' *U*
⟨*proof*⟩

lemma *square-reflcl*:
$[\!|$ *square R S T* $(R^{==})$; $S \leq T$ $|\!]$ ==> *square* $(R^{==})$ *S T* $(R^{==})$
⟨*proof*⟩

lemma *square-rtrancl*:
square R S S T ==> *square* (R^{**}) *S S* (T^{**})
⟨*proof*⟩

lemma *square-rtrancl-reflcl-commute*:
square R S (S^{**}) $(R^{==})$ ==> *commute* (R^{**}) (S^{**})
⟨*proof*⟩

commute

lemma *commute-sym*: *commute R S ==> commute S R*
⟨*proof*⟩

lemma *commute-rtrancl*: *commute R S* ==> *commute* (R^{**}) (S^{**})
⟨*proof*⟩

lemma *commute-Un*:
$[\!|$ *commute R T*; *commute S T* $|\!]$ ==> *commute* (*sup R S*) *T*
⟨*proof*⟩

diamond, *confluence*, **and** *union*

lemma *diamond-Un*:
$[\!|$ *diamond R*; *diamond S*; *commute R S* $|\!]$ ==> *diamond* (*sup R S*)
⟨*proof*⟩

lemma *diamond-confluent*: *diamond R ==> confluent R*
⟨*proof*⟩

lemma *square-reflcl-confluent*:
square R R $(R^{==})$ $(R^{==})$ ==> *confluent R*
⟨*proof*⟩

lemma *confluent-Un*:
$[\!|$ *confluent R*; *confluent S*; *commute* (R^{**}) (S^{**}) $|\!]$ ==> *confluent* (*sup R S*)

⟨*proof*⟩

lemma *diamond-to-confluence*:
[| *diamond* R; $T \leq R$; $R \leq T^{**}$ |] ==> *confluent* T
⟨*proof*⟩

2.3 Church-Rosser

lemma *Church-Rosser-confluent*: *Church-Rosser* R = *confluent* R
⟨*proof*⟩

2.4 Newman's lemma

Proof by Stefan Berghofer

theorem *newman*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

Alternative version. Partly automated by Tobias Nipkow. Takes 2 minutes (2002).

This is the maximal amount of automation possible using *blast*.

theorem *newman'*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

Using the coherent logic prover, the proof of the induction step is completely automatic.

lemma *eq-imp-rtranclp*: $x = y \Longrightarrow r^{**}\ x\ y$
⟨*proof*⟩

theorem *newman''*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

end

3 Parallel reduction and a complete developments

theory *ParRed* **imports** *Lambda Commutation* **begin**

3.1 Parallel reduction

inductive *par-beta* :: $[dB, dB] => bool$ (**infixl** ‹=>› *50*)
  where
    var [*simp*, *intro*!]: $Var\ n => Var\ n$
  | *abs* [*simp*, *intro*!]: $s => t ==> Abs\ s => Abs\ t$
  | *app* [*simp*, *intro*!]: $[|\ s => s';\ t => t'\ |] ==> s \circ t => s' \circ t'$
  | *beta* [*simp*, *intro*!]: $[|\ s => s';\ t => t'\ |] ==> (Abs\ s) \circ t => s'[t'/0]$

inductive-cases *par-beta-cases* [*elim*!]:
  $Var\ n => t$
  $Abs\ s => Abs\ t$
  $(Abs\ s) \circ t => u$
  $s \circ t => u$
  $Abs\ s => t$

3.2 Inclusions

$beta \subseteq par\text{-}beta \subseteq beta^*$

lemma *par-beta-varL* [*simp*]:
  $(Var\ n => t) = (t = Var\ n)$
  ⟨*proof*⟩

lemma *par-beta-refl* [*simp*]: $t => t$
  ⟨*proof*⟩

lemma *beta-subset-par-beta*: *beta* <= *par-beta*
  ⟨*proof*⟩

lemma *par-beta-subset-beta*: $par\text{-}beta \leq beta^{**}$
  ⟨*proof*⟩

3.3 Misc properties of *par-beta*

lemma *par-beta-lift* [*simp*]:
  $t => t' \Longrightarrow lift\ t\ n => lift\ t'\ n$
  ⟨*proof*⟩

lemma *par-beta-subst*:
  $s => s' \Longrightarrow t => t' \Longrightarrow t[s/n] => t'[s'/n]$
  ⟨*proof*⟩

3.4 Confluence (directly)

lemma *diamond-par-beta*: *diamond par-beta*

⟨*proof*⟩

3.5 Complete developments

fun
cd :: *dB* => *dB*
where
cd (*Var n*) = *Var n*
| *cd* (*Var n* ° *t*) = *Var n* ° *cd t*
| *cd* ((*s1* ° *s2*) ° *t*) = *cd* (*s1* ° *s2*) ° *cd t*
| *cd* (*Abs u* ° *t*) = (*cd u*)[*cd t*/*0*]
| *cd* (*Abs s*) = *Abs* (*cd s*)

lemma *par-beta-cd*: *s* => *t* ⟹ *t* => *cd s*
⟨*proof*⟩

3.6 Confluence (via complete developments)

lemma *diamond-par-beta2*: *diamond par-beta*
⟨*proof*⟩

theorem *beta-confluent*: *confluent beta*
⟨*proof*⟩

end

4 Eta-reduction

theory *Eta* **imports** *ParRed* **begin**

4.1 Definition of eta-reduction and relatives

primrec
free :: *dB* => *nat* => *bool*
where
free (*Var j*) *i* = (*j* = *i*)
| *free* (*s* ° *t*) *i* = (*free s i* ∨ *free t i*)
| *free* (*Abs s*) *i* = *free s* (*i* + *1*)

inductive
eta :: [*dB*, *dB*] => *bool* (**infixl** ‹$\to_\eta$› *50*)
where
eta [*simp*, *intro*]: ¬ *free s 0* ==> *Abs* (*s* ° *Var 0*) $\to_\eta$ *s*[*dummy*/*0*]
| *appL* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *s* ° *u* $\to_\eta$ *t* ° *u*
| *appR* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *u* ° *s* $\to_\eta$ *u* ° *t*
| *abs* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *Abs s* $\to_\eta$ *Abs t*

abbreviation
eta-reds :: [*dB*, *dB*] => *bool* (**infixl** ‹$\to_\eta{}^*$› *50*) **where**

$s \rightarrow_\eta{}^* \ t == eta^{**} \ s \ t$

abbreviation
$eta\text{-}red0 :: [dB,\ dB] => bool$ (**infixl** $\langle \rightarrow_\eta{}^= \rangle$ *50*) **where**
$s \rightarrow_\eta{}^= \ t == eta^{==} \ s \ t$

inductive-cases *eta-cases* [*elim*!]:
$Abs \ s \rightarrow_\eta z$
$s \circ t \rightarrow_\eta u$
$Var \ i \rightarrow_\eta t$

4.2 Properties of *eta*, *subst* and *free*

lemma *subst-not-free* [*simp*]: $\neg$ *free s i* $\Longrightarrow s[t/i] = s[u/i]$
⟨*proof*⟩

lemma *free-lift* [*simp*]:
$free \ (lift \ t \ k) \ i = (i < k \wedge free \ t \ i \vee k < i \wedge free \ t \ (i - 1))$
⟨*proof*⟩

lemma *free-subst* [*simp*]:
$free \ (s[t/k]) \ i =$
$(free \ s \ k \wedge free \ t \ i \vee free \ s \ (if \ i < k \ then \ i \ else \ i + 1))$
⟨*proof*⟩

lemma *free-eta*: $s \rightarrow_\eta t ==> free \ t \ i = free \ s \ i$
⟨*proof*⟩

lemma *not-free-eta*:
$[| \ s \rightarrow_\eta t; \neg \ free \ s \ i \ |] ==> \neg \ free \ t \ i$
⟨*proof*⟩

lemma *eta-subst* [*simp*]:
$s \rightarrow_\eta t ==> s[u/i] \rightarrow_\eta t[u/i]$
⟨*proof*⟩

theorem *lift-subst-dummy*: $\neg$ *free s i* $\Longrightarrow lift \ (s[dummy/i]) \ i = s$
⟨*proof*⟩

4.3 Confluence of *eta*

lemma *square-eta*: *square eta eta* $(eta^{==})$ $(eta^{==})$
⟨*proof*⟩

theorem *eta-confluent*: *confluent eta*
⟨*proof*⟩

4.4 Congruence rules for eta^*

lemma *rtrancl-eta-Abs*: $s \rightarrow_\eta{}^* \ s' ==> Abs \ s \rightarrow_\eta{}^* \ Abs \ s'$

$\langle proof \rangle$

lemma *rtrancl-eta-AppL*: $s \to_\eta{}^* s' ==> s \circ t \to_\eta{}^* s' \circ t$
$\langle proof \rangle$

lemma *rtrancl-eta-AppR*: $t \to_\eta{}^* t' ==> s \circ t \to_\eta{}^* s \circ t'$
$\langle proof \rangle$

lemma *rtrancl-eta-App*:
$[| s \to_\eta{}^* s'; t \to_\eta{}^* t' |] ==> s \circ t \to_\eta{}^* s' \circ t'$
$\langle proof \rangle$

4.5 Commutation of *beta* and *eta*

lemma *free-beta*:
$s \to_\beta t ==> free\ t\ i \Longrightarrow free\ s\ i$
$\langle proof \rangle$

lemma *beta-subst* [*intro*]: $s \to_\beta t ==> s[u/i] \to_\beta t[u/i]$
$\langle proof \rangle$

lemma *subst-Var-Suc* [*simp*]: $t[Var\ i/i] = t[Var(i)/i + 1]$
$\langle proof \rangle$

lemma *eta-lift* [*simp*]: $s \to_\eta t ==> lift\ s\ i \to_\eta lift\ t\ i$
$\langle proof \rangle$

lemma *rtrancl-eta-subst*: $s \to_\eta t \Longrightarrow u[s/i] \to_\eta{}^* u[t/i]$
$\langle proof \rangle$

lemma *rtrancl-eta-subst'*:
fixes $s\ t :: dB$
assumes *eta*: $s \to_\eta{}^* t$
shows $s[u/i] \to_\eta{}^* t[u/i]$ $\langle proof \rangle$

lemma *rtrancl-eta-subst''*:
fixes $s\ t :: dB$
assumes *eta*: $s \to_\eta{}^* t$
shows $u[s/i] \to_\eta{}^* u[t/i]$ $\langle proof \rangle$

lemma *square-beta-eta*: *square beta eta* (eta^{**}) $(beta^{==})$
$\langle proof \rangle$

lemma *confluent-beta-eta*: *confluent* (*sup beta eta*)
$\langle proof \rangle$

4.6 Implicit definition of *eta*

$Abs\ (lift\ s\ 0 \circ Var\ 0) \to_\eta s$

lemma *not-free-iff-lifted*:
  $(\neg\ free\ s\ i) = (\exists\, t.\ s = lift\ t\ i)$
$\langle proof \rangle$

theorem *explicit-is-implicit*:
$(\forall\, s\ u.\ (\neg\ free\ s\ 0) \ {-}{-}{>}\ R\ (Abs\ (s\ {\circ}\ Var\ 0))\ (s[u/0])) =$
  $(\forall\, s.\ R\ (Abs\ (lift\ s\ 0\ {\circ}\ Var\ 0))\ s)$
$\langle proof \rangle$

4.7 Eta-postponement theorem

Based on a paper proof due to Andreas Abel. Unlike the proof by Masako Takahashi [4], it does not use parallel eta reduction, which only seems to complicate matters unnecessarily.

theorem *eta-case*:
fixes $s :: dB$
assumes *free*: $\neg\ free\ s\ 0$
and *s*: $s[dummy/0] \Rightarrow u$
shows $\exists\, t'.\ Abs\ (s\ {\circ}\ Var\ 0) \Rightarrow t' \wedge t' \rightarrow_\eta{}^*\ u$
$\langle proof \rangle$

theorem *eta-par-beta*:
assumes *st*: $s \rightarrow_\eta t$
and *tu*: $t \Rightarrow u$
shows $\exists\, t'.\ s \Rightarrow t' \wedge t' \rightarrow_\eta{}^*\ u$ $\langle proof \rangle$

theorem *eta-postponement'*:
assumes *eta*: $s \rightarrow_\eta{}^*\ t$ **and** *beta*: $t \Rightarrow u$
shows $\exists\, t'.\ s \Rightarrow t' \wedge t' \rightarrow_\eta{}^*\ u$ $\langle proof \rangle$

theorem *eta-postponement*:
assumes $(sup\ beta\ eta)^{**}\ s\ t$
shows $(beta^{**}\ OO\ eta^{**})\ s\ t$ $\langle proof \rangle$

end

5 Application of a term to a list of terms

theory *ListApplication* **imports** *Lambda* **begin**

abbreviation
list-application :: $dB \Rightarrow dB\ list \Rightarrow dB$ (**infixl** ‹°°› *150*) **where**
$t\ {}^{\circ\circ}\ ts == foldl\ ({}^\circ)\ t\ ts$

lemma *apps-eq-tail-conv* [*iff*]: $(r\ {}^{\circ\circ}\ ts = s\ {}^{\circ\circ}\ ts) = (r = s)$
$\langle proof \rangle$

lemma *Var-eq-apps-conv* [*iff*]: $(Var\ m = s\ {}^{\circ\circ}\ ss) = (Var\ m = s \wedge ss = [])$

$\langle proof \rangle$

lemma *Var-apps-eq-Var-apps-conv* [*iff*]:
$(Var\ m\ {}^{\circ\circ}\ rs = Var\ n\ {}^{\circ\circ}\ ss) = (m = n \wedge rs = ss)$
$\langle proof \rangle$

lemma *App-eq-foldl-conv*:
$(r\ {}^{\circ}\ s = t\ {}^{\circ\circ}\ ts) =$
$(if\ ts = []\ then\ r\ {}^{\circ}\ s = t$
$else\ (\exists\, ss.\ ts = ss\ @\ [s] \wedge r = t\ {}^{\circ\circ}\ ss))$
$\langle proof \rangle$

lemma *Abs-eq-apps-conv* [*iff*]:
$(Abs\ r = s\ {}^{\circ\circ}\ ss) = (Abs\ r = s \wedge ss = [])$
$\langle proof \rangle$

lemma *apps-eq-Abs-conv* [*iff*]: $(s\ {}^{\circ\circ}\ ss = Abs\ r) = (s = Abs\ r \wedge ss = [])$
$\langle proof \rangle$

lemma *Abs-apps-eq-Abs-apps-conv* [*iff*]:
$(Abs\ r\ {}^{\circ\circ}\ rs = Abs\ s\ {}^{\circ\circ}\ ss) = (r = s \wedge rs = ss)$
$\langle proof \rangle$

lemma *Abs-App-neq-Var-apps* [*iff*]:
$Abs\ s\ {}^{\circ}\ t \neq Var\ n\ {}^{\circ\circ}\ ss$
$\langle proof \rangle$

lemma *Var-apps-neq-Abs-apps* [*iff*]:
$Var\ n\ {}^{\circ\circ}\ ts \neq Abs\ r\ {}^{\circ\circ}\ ss$
$\langle proof \rangle$

lemma *ex-head-tail*:
$\exists\, ts\ h.\ t = h\ {}^{\circ\circ}\ ts \wedge ((\exists\, n.\ h = Var\ n) \vee (\exists\, u.\ h = Abs\ u))$
$\langle proof \rangle$

lemma *size-apps* [*simp*]:
$size\ (r\ {}^{\circ\circ}\ rs) = size\ r + foldl\ (+)\ 0\ (map\ size\ rs) + length\ rs$
$\langle proof \rangle$

lemma *lem0*: $[|\ (0::nat) < k;\ m <= n\ |] ==> m < n + k$
$\langle proof \rangle$

lemma *lift-map* [*simp*]:
$lift\ (t\ {}^{\circ\circ}\ ts)\ i = lift\ t\ i\ {}^{\circ\circ}\ map\ (\lambda t.\ lift\ t\ i)\ ts$
$\langle proof \rangle$

lemma *subst-map* [*simp*]:
$subst\ (t\ {}^{\circ\circ}\ ts)\ u\ i = subst\ t\ u\ i\ {}^{\circ\circ}\ map\ (\lambda t.\ subst\ t\ u\ i)\ ts$
$\langle proof \rangle$

lemma *app-last*: $(t \circ\circ ts) \circ u = t \circ\circ (ts\ @\ [u])$
⟨*proof*⟩

A customized induction schema for $\circ\circ$.

lemma *lem*:
assumes $!!n\ ts.\ \forall t \in set\ ts.\ P\ t ==> P\ (Var\ n \circ\circ ts)$
and $!!u\ ts.\ [|\ P\ u; \forall t \in set\ ts.\ P\ t\ |] ==> P\ (Abs\ u \circ\circ ts)$
shows $size\ t = n \Longrightarrow P\ t$
⟨*proof*⟩

theorem *Apps-dB-induct*:
assumes $!!n\ ts.\ \forall t \in set\ ts.\ P\ t ==> P\ (Var\ n \circ\circ ts)$
and $!!u\ ts.\ [|\ P\ u; \forall t \in set\ ts.\ P\ t\ |] ==> P\ (Abs\ u \circ\circ ts)$
shows $P\ t$
⟨*proof*⟩

end

6 Simply-typed lambda terms

theory *LambdaType* **imports** *ListApplication* **begin**

6.1 Environments

definition
$shift :: (nat \Rightarrow 'a) \Rightarrow nat \Rightarrow 'a \Rightarrow nat \Rightarrow 'a$ (‹-⟨-:-⟩› $[90, 0, 0]\ 91$) **where**
$e\langle i{:}a\rangle = (\lambda j.\ if\ j < i\ then\ e\ j\ else\ if\ j = i\ then\ a\ else\ e\ (j - 1))$

lemma *shift-eq* [*simp*]: $i = j \Longrightarrow (e\langle i{:}T\rangle)\ j = T$
⟨*proof*⟩

lemma *shift-gt* [*simp*]: $j < i \Longrightarrow (e\langle i{:}T\rangle)\ j = e\ j$
⟨*proof*⟩

lemma *shift-lt* [*simp*]: $i < j \Longrightarrow (e\langle i{:}T\rangle)\ j = e\ (j - 1)$
⟨*proof*⟩

lemma *shift-commute* [*simp*]: $e\langle i{:}U\rangle\langle 0{:}T\rangle = e\langle 0{:}T\rangle\langle Suc\ i{:}U\rangle$
⟨*proof*⟩

6.2 Types and typing rules

datatype *type* =
Atom nat
| *Fun type type* (**infixr** ‹⇒› *200*)

inductive *typing* :: $(nat \Rightarrow type) \Rightarrow dB \Rightarrow type \Rightarrow bool$ (‹- ⊢ - : -› $[50, 50, 50]$ 50)

where
  $Var\ [intro!]\colon env\ x = T \Longrightarrow env \vdash Var\ x : T$
| $Abs\ [intro!]\colon env\langle 0{:}T\rangle \vdash t : U \Longrightarrow env \vdash Abs\ t : (T \Rightarrow U)$
| $App\ [intro!]\colon env \vdash s : T \Rightarrow U \Longrightarrow env \vdash t : T \Longrightarrow env \vdash (s \circ t) : U$

inductive-cases *typing-elims* [*elim*!]:
  $e \vdash Var\ i : T$
  $e \vdash t \circ u : T$
  $e \vdash Abs\ t : T$

primrec
  $typings :: (nat \Rightarrow type) \Rightarrow dB\ list \Rightarrow type\ list \Rightarrow bool$
where
  $typings\ e\ []\ Ts = (Ts = [])$
| $typings\ e\ (t\ \#\ ts)\ Ts =$
    $(case\ Ts\ of$
      $[] \Rightarrow False$
    $|\ T\ \#\ Ts \Rightarrow e \vdash t : T \wedge typings\ e\ ts\ Ts)$

abbreviation
  $typings\text{-}rel :: (nat \Rightarrow type) \Rightarrow dB\ list \Rightarrow type\ list \Rightarrow bool$
  (‹- ⊩ - : -› [50, 50, 50] 50) **where**
  $env \Vdash ts : Ts == typings\ env\ ts\ Ts$

abbreviation
  $funs :: type\ list \Rightarrow type \Rightarrow type$ (**infixr** ‹⇛› 200) **where**
  $Ts \Rrightarrow T == foldr\ Fun\ Ts\ T$

6.3 Some examples

schematic-goal $e \vdash Abs\ (Abs\ (Abs\ (Var\ 1 \circ (Var\ 2 \circ Var\ 1 \circ Var\ 0)))) : ?T$
  ⟨*proof*⟩

schematic-goal $e \vdash Abs\ (Abs\ (Abs\ (Var\ 2 \circ Var\ 0 \circ (Var\ 1 \circ Var\ 0)))) : ?T$
  ⟨*proof*⟩

6.4 Lists of types

lemma *lists-typings*:
  $e \Vdash ts : Ts \Longrightarrow listsp\ (\lambda t.\ \exists T.\ e \vdash t : T)\ ts$
⟨*proof*⟩

lemma *types-snoc*: $e \Vdash ts : Ts \Longrightarrow e \vdash t : T \Longrightarrow e \Vdash ts\ @\ [t] : Ts\ @\ [T]$
  ⟨*proof*⟩

lemma *types-snoc-eq*: $e \Vdash ts\ @\ [t] : Ts\ @\ [T] =$
  $(e \Vdash ts : Ts \wedge e \vdash t : T)$
⟨*proof*⟩

Cannot use *rev-exhaust* from the *List* theory, since it is not constructive

lemma *rev-exhaust2* [*extraction-expand*]:
obtains (*Nil*) $xs = []$ | (*snoc*) *ys y* **where** $xs = ys\ @\ [y]$
⟨*proof*⟩

lemma *types-snocE*:
assumes ‹$e \Vdash ts\ @\ [t] : Ts$›
obtains *Us* **and** *U* **where** ‹$Ts = Us\ @\ [U]$› ‹$e \Vdash ts : Us$› ‹$e \vdash t : U$›
⟨*proof*⟩

6.5 n-ary function types

lemma *list-app-typeD*:
$e \vdash t \degree\degree ts : T \Longrightarrow \exists Ts.\ e \vdash t : Ts \Rrightarrow T \wedge e \Vdash ts : Ts$
⟨*proof*⟩

lemma *list-app-typeE*:
$e \vdash t \degree\degree ts : T \Longrightarrow (\bigwedge Ts.\ e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

lemma *list-app-typeI*:
$e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow e \vdash t \degree\degree ts : T$
⟨*proof*⟩

For the specific case where the head of the term is a variable, the following theorems allow to infer the types of the arguments without analyzing the typing derivation. This is crucial for program extraction.

theorem *var-app-type-eq*:
$e \vdash Var\ i \degree\degree ts : T \Longrightarrow e \vdash Var\ i \degree\degree ts : U \Longrightarrow T = U$
⟨*proof*⟩

lemma *var-app-types*: $e \vdash Var\ i \degree\degree ts \degree\degree us : T \Longrightarrow e \Vdash ts : Ts \Longrightarrow$
$e \vdash Var\ i \degree\degree ts : U \Longrightarrow \exists Us.\ U = Us \Rrightarrow T \wedge e \Vdash us : Us$
⟨*proof*⟩

lemma *var-app-typesE*: $e \vdash Var\ i \degree\degree ts : T \Longrightarrow$
$(\bigwedge Ts.\ e \vdash Var\ i : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *abs-typeE*:
assumes $e \vdash Abs\ t : T\ \bigwedge U\ V.\ e\langle 0{:}U\rangle \vdash t : V \Longrightarrow P$
shows P
⟨*proof*⟩

6.6 Lifting preserves well-typedness

lemma *lift-type* [*intro!*]: $e \vdash t : T \Longrightarrow e\langle i{:}U\rangle \vdash lift\ t\ i : T$
⟨*proof*⟩

lemma *lift-types*:

$e \Vdash ts : Ts \Longrightarrow e\langle i{:}U\rangle \Vdash (map\ (\lambda t.\ lift\ t\ i)\ ts) : Ts$
$\langle proof\rangle$

6.7 Substitution lemmas

lemma *subst-lemma*:
$e \vdash t : T \Longrightarrow e' \vdash u : U \Longrightarrow e = e'\langle i{:}U\rangle \Longrightarrow e' \vdash t[u/i] : T$
$\langle proof\rangle$

lemma *substs-lemma*:
$e \vdash u : T \Longrightarrow e\langle i{:}T\rangle \Vdash ts : Ts \Longrightarrow$
$e \Vdash (map\ (\lambda t.\ t[u/i])\ ts) : Ts$
$\langle proof\rangle$

6.8 Subject reduction

lemma *subject-reduction*: $e \vdash t : T \Longrightarrow t \rightarrow_\beta t' \Longrightarrow e \vdash t' : T$
$\langle proof\rangle$

theorem *subject-reduction'*: $t \rightarrow_\beta{}^* t' \Longrightarrow e \vdash t : T \Longrightarrow e \vdash t' : T$
$\langle proof\rangle$

6.9 Alternative induction rule for types

lemma *type-induct* [*induct type*]:
assumes
$(\bigwedge T.\ (\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T1) \Longrightarrow$
$(\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T2) \Longrightarrow P\ T)$
shows $P\ T$
$\langle proof\rangle$

end

7 Lifting an order to lists of elements

theory *ListOrder*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

Lifting an order to lists of elements, relating exactly one element.

definition
step1 :: ('*a* => '*a* => *bool*) => '*a list* => '*a list* => *bool* **where**
step1 *r* =
$(\lambda ys\ xs.\ \exists us\ z\ z'\ vs.\ xs = us\ @\ z\ \#\ vs \wedge r\ z'\ z \wedge ys =$
$us\ @\ z'\ \#\ vs)$

lemma *step1-converse* [*simp*]: *step1* $(r^{-1-1}) = (\mathit{step1}\ r)^{-1-1}$
⟨*proof*⟩

lemma *in-step1-converse* [*iff*]: $(\mathit{step1}\ (r^{-1-1})\ x\ y) = ((\mathit{step1}\ r)^{-1-1}\ x\ y)$
⟨*proof*⟩

lemma *not-Nil-step1* [*iff*]: ¬ *step1 r* [] *xs*
⟨*proof*⟩

lemma *not-step1-Nil* [*iff*]: ¬ *step1 r xs* []
⟨*proof*⟩

lemma *Cons-step1-Cons* [*iff*]:
(*step1 r* (*y* # *ys*) (*x* # *xs*)) =
(*r y x* ∧ *xs* = *ys* ∨ *x* = *y* ∧ *step1 r ys xs*)
⟨*proof*⟩

lemma *append-step1I*:
step1 r ys xs ∧ *vs* = *us* ∨ *ys* = *xs* ∧ *step1 r vs us*
==> *step1 r* (*ys* @ *vs*) (*xs* @ *us*)
⟨*proof*⟩

lemma *Cons-step1E* [*elim*!]:
assumes *step1 r ys* (*x* # *xs*)
and !!*y*. *ys* = *y* # *xs* ⟹ *r y x* ⟹ *R*
and !!*zs*. *ys* = *x* # *zs* ⟹ *step1 r zs xs* ⟹ *R*
shows *R*
⟨*proof*⟩

lemma *Snoc-step1-SnocD*:
step1 r (*ys* @ [*y*]) (*xs* @ [*x*])
==> (*step1 r ys xs* ∧ *y* = *x* ∨ *ys* = *xs* ∧ *r y x*)
⟨*proof*⟩

lemma *Cons-acc-step1I* [*intro*!]:
Wellfounded.accp r x ==> *Wellfounded.accp* (*step1 r*) *xs* ⟹ *Wellfounded.accp* (*step1 r*) (*x* # *xs*)
⟨*proof*⟩

lemma *lists-accD*: *listsp* (*Wellfounded.accp r*) *xs* ==> *Wellfounded.accp* (*step1 r*) *xs*
⟨*proof*⟩

lemma *ex-step1I*:
[| *x* ∈ *set xs*; *r y x* |]
==> ∃ *ys*. *step1 r ys xs* ∧ *y* ∈ *set ys*
⟨*proof*⟩

lemma *lists-accI*: *Wellfounded.accp* (*step1 r*) *xs* ==> *listsp* (*Wellfounded.accp r*)

xs
⟨*proof*⟩

end

8 Lifting beta-reduction to lists

theory *ListBeta* **imports** *ListApplication ListOrder* **begin**

Lifting beta-reduction to lists of terms, reducing exactly one element.

abbreviation
list-beta :: *dB list => dB list => bool* (**infixl** ‹=>› *50*) **where**
rs => ss == step1 beta rs ss

lemma *head-Var-reduction*:
$Var\ n\ {}^{\circ\circ}\ rs \to_\beta v \Longrightarrow \exists ss.\ rs => ss \land v = Var\ n\ {}^{\circ\circ}\ ss$
⟨*proof*⟩

lemma *apps-betasE* [*elim*!]:
assumes *major*: $r\ {}^{\circ\circ}\ rs \to_\beta s$
and *cases*: $!!r'.\ [|\ r \to_\beta r';\ s = r'\ {}^{\circ\circ}\ rs\ |] ==> R$
$!!rs'.\ [|\ rs => rs';\ s = r\ {}^{\circ\circ}\ rs'\ |] ==> R$
$!!t\ u\ us.\ [|\ r = Abs\ t;\ rs = u\ \#\ us;\ s = t[u/0]\ {}^{\circ\circ}\ us\ |] ==> R$
shows *R*
⟨*proof*⟩

lemma *apps-preserves-beta* [*simp*]:
$r \to_\beta s ==> r\ {}^{\circ\circ}\ ss \to_\beta s\ {}^{\circ\circ}\ ss$
⟨*proof*⟩

lemma *apps-preserves-beta2* [*simp*]:
$r \to_\beta{}^* s ==> r\ {}^{\circ\circ}\ ss \to_\beta{}^* s\ {}^{\circ\circ}\ ss$
⟨*proof*⟩

lemma *apps-preserves-betas* [*simp*]:
$rs => ss \Longrightarrow r\ {}^{\circ\circ}\ rs \to_\beta r\ {}^{\circ\circ}\ ss$
⟨*proof*⟩

end

9 Inductive characterization of terminating lambda terms

theory *InductTermi* **imports** *ListBeta* **begin**

9.1 Terminating lambda terms

inductive *IT* :: *dB => bool*

where
  Var [*intro*]: *listsp IT rs ==> IT* (*Var n* °° *rs*)
| *Lambda* [*intro*]: *IT r ==> IT* (*Abs r*)
| *Beta* [*intro*]: *IT* ((*r*[*s/0*]) °° *ss*) *==> IT s ==> IT* ((*Abs r* ° *s*) °° *ss*)

9.2 Every term in *IT* terminates

lemma *double-induction-lemma* [*rule-format*]:
  termip beta s ==> ∀ *t. termip beta t -->*
    (∀ *r ss. t* = *r*[*s/0*] °° *ss --> termip beta* (*Abs r* ° *s* °° *ss*))
  ⟨*proof*⟩

lemma *IT-implies-termi*: *IT t ==> termip beta t*
  ⟨*proof*⟩

9.3 Every terminating term is in *IT*

declare *Var-apps-neq-Abs-apps* [*symmetric*, *simp*]

lemma [*simp*, *THEN not-sym*, *simp*]: *Var n* °° *ss* ≠ *Abs r* ° *s* °° *ts*
  ⟨*proof*⟩

lemma [*simp*]:
  (*Abs r* ° *s* °° *ss* = *Abs r′* ° *s′* °° *ss′*) = (*r* = *r′* ∧ *s* = *s′* ∧ *ss* = *ss′*)
  ⟨*proof*⟩

inductive-cases [*elim*!]:
  IT (*Var n* °° *ss*)
  IT (*Abs t*)
  IT (*Abs r* ° *s* °° *ts*)

theorem *termi-implies-IT*: *termip beta r ==> IT r*
  ⟨*proof*⟩

end

10 Strong normalization for simply-typed lambda calculus

theory *StrongNorm* **imports** *LambdaType InductTermi* **begin**

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

10.1 Properties of *IT*

lemma *lift-IT* [*intro*!]: *IT t* ⟹ *IT* (*lift t i*)
  ⟨*proof*⟩

lemma *lifts-IT*: $listsp\ IT\ ts \Longrightarrow listsp\ IT\ (map\ (\lambda t.\ lift\ t\ 0)\ ts)$
⟨*proof*⟩

lemma *subst-Var-IT*: $IT\ r \Longrightarrow IT\ (r[Var\ i/j])$
⟨*proof*⟩

lemma *Var-IT*: $IT\ (Var\ n)$
⟨*proof*⟩

lemma *app-Var-IT*: $IT\ t \Longrightarrow IT\ (t\ {}^\circ\ Var\ i)$
⟨*proof*⟩

10.2 Well-typed substitution preserves termination

lemma *subst-type-IT*:
$\bigwedge t\ e\ T\ u\ i.\ IT\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow$
$IT\ u \Longrightarrow e \vdash u : U \Longrightarrow IT\ (t[u/i])$
(**is** $PROP\ ?P\ U$ **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
⟨*proof*⟩

10.3 Well-typed terms are strongly normalizing

lemma *type-implies-IT*:
assumes $e \vdash t : T$
shows $IT\ t$
⟨*proof*⟩

theorem *type-implies-termi*: $e \vdash t : T \Longrightarrow termip\ beta\ t$
⟨*proof*⟩

end

11 Inductive characterization of lambda terms in normal form

theory *NormalForm*
imports *ListBeta*
begin

11.1 Terms in normal form

definition
$listall :: ('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
$listall\ P\ xs \equiv (\forall i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$

declare *listall-def* [*extraction-expand-def*]

theorem *listall-nil*: $listall\ P\ []$

⟨*proof*⟩

theorem *listall-nil-eq* [*simp*]: *listall P* [] = *True*
⟨*proof*⟩

theorem *listall-cons*: $P\ x \Longrightarrow listall\ P\ xs \Longrightarrow listall\ P\ (x\ \#\ xs)$
⟨*proof*⟩

theorem *listall-cons-eq* [*simp*]: $listall\ P\ (x\ \#\ xs) = (P\ x \wedge listall\ P\ xs)$
⟨*proof*⟩

lemma *listall-conj1*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ P\ xs$
⟨*proof*⟩

lemma *listall-conj2*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ Q\ xs$
⟨*proof*⟩

lemma *listall-app*: $listall\ P\ (xs\ @\ ys) = (listall\ P\ xs \wedge listall\ P\ ys)$
⟨*proof*⟩

lemma *listall-snoc* [*simp*]: $listall\ P\ (xs\ @\ [x]) = (listall\ P\ xs \wedge P\ x)$
⟨*proof*⟩

lemma *listall-cong* [*cong*, *extraction-expand*]:
$xs = ys \Longrightarrow listall\ P\ xs = listall\ P\ ys$
— Currently needed for strange technical reasons
⟨*proof*⟩

listsp is equivalent to *listall*, but cannot be used for program extraction.

lemma *listall-listsp-eq*: $listall\ P\ xs = listsp\ P\ xs$
⟨*proof*⟩

inductive $NF :: dB \Rightarrow bool$
where
App: $listall\ NF\ ts \Longrightarrow NF\ (Var\ x\ {}^{\circ\circ}\ ts)$
| *Abs*: $NF\ t \Longrightarrow NF\ (Abs\ t)$
monos *listall-def*

lemma *nat-eq-dec*: $\bigwedge n::nat.\ m = n \vee m \neq n$
⟨*proof*⟩

lemma *nat-le-dec*: $\bigwedge n::nat.\ m < n \vee \neg\ (m < n)$
⟨*proof*⟩

lemma *App-NF-D*: **assumes** *NF*: $NF\ (Var\ n\ {}^{\circ\circ}\ ts)$
shows *listall NF ts* ⟨*proof*⟩

11.2 Properties of *NF*

lemma *Var-NF*: $NF\ (Var\ n)$
$\langle proof\rangle$

lemma *Abs-NF*:
  assumes *NF*: $NF\ (Abs\ t\ {}^{\circ\circ}\ ts)$
  shows $ts = []$ $\langle proof\rangle$

lemma *subst-terms-NF*: $listall\ NF\ ts \Longrightarrow$
    $listall\ (\lambda t.\ \forall\, i\ j.\ NF\ (t[Var\ i/j]))\ ts \Longrightarrow$
    $listall\ NF\ (map\ (\lambda t.\ t[Var\ i/j])\ ts)$
  $\langle proof\rangle$

lemma *subst-Var-NF*: $NF\ t \Longrightarrow NF\ (t[Var\ i/j])$
  $\langle proof\rangle$

lemma *app-Var-NF*: $NF\ t \Longrightarrow \exists\, t'.\ t\ {}^{\circ}\ Var\ i \rightarrow_{\beta}{}^{*}\ t' \wedge NF\ t'$
  $\langle proof\rangle$

lemma *lift-terms-NF*: $listall\ NF\ ts \Longrightarrow$
    $listall\ (\lambda t.\ \forall\, i.\ NF\ (lift\ t\ i))\ ts \Longrightarrow$
    $listall\ NF\ (map\ (\lambda t.\ lift\ t\ i)\ ts)$
  $\langle proof\rangle$

lemma *lift-NF*: $NF\ t \Longrightarrow NF\ (lift\ t\ i)$
  $\langle proof\rangle$

NF characterizes exactly the terms that are in normal form.

lemma *NF-eq*: $NF\ t = (\forall\, t'.\ \neg\ t \rightarrow_{\beta}\ t')$
$\langle proof\rangle$

end

12 Standardization

theory *Standardization*
imports *NormalForm*
begin

Based on lecture notes by Ralph Matthes [3], original proof idea due to Ralph Loader [2].

12.1 Standard reduction relation

declare *listrel-mono* [*mono-set*]

inductive
  $sred :: dB \Rightarrow dB \Rightarrow bool$ (**infixl** $\langle \rightarrow_s \rangle$ *50*)

and *sredlist* :: *dB list* ⇒ *dB list* ⇒ *bool* (**infixl** ‹$[\to_s]$› *50*)
where
$s\ [\to_s]\ t \equiv listrelp\ (\to_s)\ s\ t$
| *Var*: $rs\ [\to_s]\ rs' \Longrightarrow Var\ x\ {}^{\circ\circ}\ rs \to_s Var\ x\ {}^{\circ\circ}\ rs'$
| *Abs*: $r \to_s r' \Longrightarrow ss\ [\to_s]\ ss' \Longrightarrow Abs\ r\ {}^{\circ\circ}\ ss \to_s Abs\ r'\ {}^{\circ\circ}\ ss'$
| *Beta*: $r[s/0]\ {}^{\circ\circ}\ ss \to_s t \Longrightarrow Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ss \to_s t$

lemma *refl-listrelp*: $\forall x \in set\ xs.\ R\ x\ x \Longrightarrow listrelp\ R\ xs\ xs$
⟨*proof*⟩

lemma *refl-sred*: $t \to_s t$
⟨*proof*⟩

lemma *refl-sreds*: $ts\ [\to_s]\ ts$
⟨*proof*⟩

lemma *listrelp-conj1*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ R\ x\ y$
⟨*proof*⟩

lemma *listrelp-conj2*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ S\ x\ y$
⟨*proof*⟩

lemma *listrelp-app*:
assumes *xsys*: $listrelp\ R\ xs\ ys$
shows $listrelp\ R\ xs'\ ys' \Longrightarrow listrelp\ R\ (xs\ @\ xs')\ (ys\ @\ ys')$ ⟨*proof*⟩

lemma *lemma1*:
assumes *r*: $r \to_s r'$ **and** *s*: $s \to_s s'$
shows $r\ {}^{\circ}\ s \to_s r'\ {}^{\circ}\ s'$ ⟨*proof*⟩

lemma *lemma1'*:
assumes *ts*: $ts\ [\to_s]\ ts'$
shows $r \to_s r' \Longrightarrow r\ {}^{\circ\circ}\ ts \to_s r'\ {}^{\circ\circ}\ ts'$ ⟨*proof*⟩

lemma *lemma2-1*:
assumes *beta*: $t \to_\beta u$
shows $t \to_s u$ ⟨*proof*⟩

lemma *listrelp-betas*:
assumes *ts*: $listrelp\ (\to_\beta{}^*)\ ts\ ts'$
shows $\bigwedge t\ t'.\ t \to_\beta{}^*\ t' \Longrightarrow t\ {}^{\circ\circ}\ ts \to_\beta{}^*\ t'\ {}^{\circ\circ}\ ts'$ ⟨*proof*⟩

lemma *lemma2-2*:
assumes *t*: $t \to_s u$
shows $t \to_\beta{}^*\ u$ ⟨*proof*⟩

lemma *sred-lift*:
assumes *s*: $s \to_s t$
shows $lift\ s\ i \to_s lift\ t\ i$ ⟨*proof*⟩

lemma *lemma3*:
  assumes r: $r \to_s r'$
  shows $s \to_s s' \Longrightarrow r[s/x] \to_s r'[s'/x]$ ⟨*proof*⟩

lemma *lemma4-aux*:
  assumes rs: $listrelp\ (\lambda t\ u.\ t \to_s u \wedge (\forall r.\ u \to_\beta r \longrightarrow t \to_s r))\ rs\ rs'$
  shows $rs' => ss \Longrightarrow rs\ [\to_s]\ ss$ ⟨*proof*⟩

lemma *lemma4*:
  assumes r: $r \to_s r'$
  shows $r' \to_\beta r'' \Longrightarrow r \to_s r''$ ⟨*proof*⟩

lemma *rtrancl-beta-sred*:
  assumes r: $r \to_\beta{}^* r'$
  shows $r \to_s r'$ ⟨*proof*⟩

12.2 Leftmost reduction and weakly normalizing terms

inductive
  $lred :: dB \Rightarrow dB \Rightarrow bool$ (**infixl** $\langle\to_l\rangle$ *50*)
  and $lredlist :: dB\ list \Rightarrow dB\ list \Rightarrow bool$ (**infixl** $\langle[\to_l]\rangle$ *50*)
where
  $s\ [\to_l]\ t \equiv listrelp\ (\to_l)\ s\ t$
| *Var*: $rs\ [\to_l]\ rs' \Longrightarrow Var\ x \circ\circ rs \to_l Var\ x \circ\circ rs'$
| *Abs*: $r \to_l r' \Longrightarrow Abs\ r \to_l Abs\ r'$
| *Beta*: $r[s/0] \circ\circ ss \to_l t \Longrightarrow Abs\ r \circ s \circ\circ ss \to_l t$

lemma *lred-imp-sred*:
  assumes *lred*: $s \to_l t$
  shows $s \to_s t$ ⟨*proof*⟩

inductive $WN :: dB => bool$
  where
    Var: $listsp\ WN\ rs \Longrightarrow WN\ (Var\ n \circ\circ rs)$
  | *Lambda*: $WN\ r \Longrightarrow WN\ (Abs\ r)$
  | *Beta*: $WN\ ((r[s/0]) \circ\circ ss) \Longrightarrow WN\ ((Abs\ r \circ s) \circ\circ ss)$

lemma *listrelp-imp-listsp1*:
  assumes H: $listrelp\ (\lambda x\ y.\ P\ x)\ xs\ ys$
  shows $listsp\ P\ xs$ ⟨*proof*⟩

lemma *listrelp-imp-listsp2*:
  assumes H: $listrelp\ (\lambda x\ y.\ P\ y)\ xs\ ys$
  shows $listsp\ P\ ys$ ⟨*proof*⟩

lemma *lemma5*:
  assumes *lred*: $r \to_l r'$
  shows $WN\ r$ **and** $NF\ r'$ ⟨*proof*⟩

lemma *lemma6*:
  assumes *wn*: $WN\ r$
  shows $\exists\, r'.\ r \rightarrow_l r'$ ⟨*proof*⟩

lemma *lemma7*:
  assumes *r*: $r \rightarrow_s r'$
  shows $NF\ r' \Longrightarrow r \rightarrow_l r'$ ⟨*proof*⟩

lemma *WN-eq*: $WN\ t = (\exists\, t'.\ t \rightarrow_\beta{}^* t' \wedge NF\ t')$
⟨*proof*⟩

end

13 Weak normalization for simply-typed lambda calculus

theory *WeakNorm*
imports *LambdaType NormalForm HOL−Library.Realizers HOL−Library.Code-Target-Int*
begin

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

13.1 Main theorems

lemma *norm-list*:
  assumes *f-compat*: $\bigwedge t\ t'.\ t \rightarrow_\beta{}^* t' \Longrightarrow f\ t \rightarrow_\beta{}^* f\ t'$
  and *f-NF*: $\bigwedge t.\ NF\ t \Longrightarrow NF\ (f\ t)$
  and *uNF*: $NF\ u$ **and** *uT*: $e \vdash u : T$
  shows $\bigwedge Us.\ e\langle i{:}T\rangle \Vdash as : Us \Longrightarrow$
    $listall\ (\lambda t.\ \forall\, e\ T'\ u\ i.\ e\langle i{:}T\rangle \vdash t : T' \longrightarrow$
      $NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists\, t'.\ t[u/i] \rightarrow_\beta{}^* t' \wedge NF\ t'))\ as \Longrightarrow$
    $\exists\, as'.\ \forall\, j.\ Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ as \rightarrow_\beta{}^*$
      $Var\ j\ {}^{\circ\circ}\ map\ f\ as' \wedge NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ as')$
  (**is** $\bigwedge Us.$ - $\Longrightarrow listall\ ?R\ as \Longrightarrow \exists\, as'.\ ?ex\ Us\ as\ as'$)
⟨*proof*⟩

lemma *subst-type-NF*:
  $\bigwedge t\ e\ T\ u\ i.\ NF\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow NF\ u \Longrightarrow e \vdash u : U \Longrightarrow \exists\, t'.\ t[u/i]$
$\rightarrow_\beta{}^* t' \wedge NF\ t'$
  (**is** $PROP\ ?P\ U$ **is** $\bigwedge t\ e\ T\ u\ i.$ - $\Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
⟨*proof*⟩
inductive *rtyping* :: $(nat \Rightarrow type) \Rightarrow dB \Rightarrow type \Rightarrow bool$ (‹- $\vdash_R$ - : -› [50, 50, 50]
50)
  where
    Var: $e\ x = T \Longrightarrow e \vdash_R Var\ x : T$
  | *Abs*: $e\langle 0{:}T\rangle \vdash_R t : U \Longrightarrow e \vdash_R Abs\ t : (T \Rightarrow U)$

| *App*: $e \vdash_R s : T \Rightarrow U \Longrightarrow e \vdash_R t : T \Longrightarrow e \vdash_R (s \circ t) : U$

lemma *rtyping-imp-typing*: $e \vdash_R t : T \Longrightarrow e \vdash t : T$
⟨*proof*⟩

theorem *type-NF*:
assumes $e \vdash_R t : T$
shows $\exists\, t'.\ t \rightarrow_\beta{}^* t' \wedge NF\ t'$ ⟨*proof*⟩

13.2 Extracting the program

declare *NF.induct* [*ind-realizer*]
declare *rtranclp.induct* [*ind-realizer irrelevant*]
declare *rtyping.induct* [*ind-realizer*]
lemmas [*extraction-expand*] = *conj-assoc listall-cons-eq subst-all equal-allI*

extract *type-NF*

lemma *rtranclR-rtrancl-eq*: $rtranclpR\ r\ a\ b = r^{**}\ a\ b$
⟨*proof*⟩

lemma *NFR-imp-NF*: $NFR\ nf\ t \Longrightarrow NF\ t$
⟨*proof*⟩

The program corresponding to the proof of the central lemma, which performs substitution and normalization, is shown in Figure 1. The correctness theorem corresponding to the program *subst-type-NF* is

$$\bigwedge x.\ NFR\ x\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow (\bigwedge xa.\ NFR\ xa\ u \Longrightarrow e \vdash u : U \Longrightarrow t[u/i] \rightarrow_\beta{}^* fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa) \wedge NFR\ (snd\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa))\ (fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa)))$$

where *NFR* is the realizability predicate corresponding to the datatype *NFT*, which is inductively defined by the rules

```
subst-type-NF ≡
λx xa xb xc xd xe H Ha.
  type-induct-P xc
   (λx H2 H2a xa xaa xb xc xd H.
       compat-NFT.rec-split-NFT default
        (λts xa xaa r xb xc xd xe H.
            var-app-typesE-P (xb⟨xe:x⟩) xa ts
             (λUs--. case nat-eq-dec xa xe of
                    Left ⇒ case ts of [] ⇒ (xd, H)
                           | a # list ⇒
                              case Us-- of [] ⇒ default
                              | T''-- # Ts-- ⇒
                                 let (x, y) =
                                       norm-list (λt. lift t 0) xd xb xe list Ts--
                                        (λt. lift-NF 0) H
                                        (listall-conj2-P-Q list (λi. (xaa (Suc i), r (Suc i))));
                                     (xa, ya) = snd (xaa 0, r 0) xb T''-- xd xe H;
                                     (xd, yb) = app-Var-NF 0 (lift-NF 0 H);
                                     (xa, ya) =
                                       H2 T''-- (Ts-- ⇛ xc) xd xb (Ts-- ⇛ xc) xa 0 yb ya;
                                     (x, y) =
                                       H2a T''-- (Ts-- ⇛ xc) (dB.Var 0 °° map (λt. lift t 0) x)
                                        xb xc xa 0 (y 0) ya
                                 in (x, y)
                    | Right ⇒
                       let (x, y) =
                             let (x, y) =
                                   norm-list (λt. t) xd xb xe ts Us-- (λx H. H) H
                                    (listall-conj2-P-Q ts (λz. (xaa z, r z)))
                             in (x, λx. y x)
                       in case nat-le-dec xe xa of
                          Left ⇒ (dB.Var (xa − Suc 0) °° x, y (xa − Suc 0))
                          | Right ⇒ (dB.Var xa °° x, y xa)))
        (λt x r xa xaa xb xc H.
            abs-typeE-P xaa
             (λU V. let (x, y) =
                          let (x, y) = r (λa. (xa⟨0:U⟩) a) V (lift xb 0) (Suc xc) (lift-NF 0 H)
                          in (dB.Abs x, NFT.Abs x y)
                    in (x, y)))
        H (λa. xaa a) xb xc xd)
   x xa xd xe xb H Ha
```

Figure 1: Program extracted from *subst-type-NF*

```
subst-Var-NF ≡
λx xa H.
  compat-NFT.rec-split-NFT default
   (λts x xa r xb xc.
      case nat-eq-dec x xc of
      Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) xb
             (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
               (listall-conj2-P-Q ts (λz. (xa z, r z))))
      | Right ⇒
         case nat-le-dec xc x of
         Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) (x − Suc 0)
                (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                  (listall-conj2-P-Q ts (λz. (xa z, r z))))
         | Right ⇒
            NFT.App (map (λt. t[dB.Var xb/xc]) ts) x
             (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
               (listall-conj2-P-Q ts (λz. (xa z, r z)))))
   (λt x r xa xaa. NFT.Abs (t[dB.Var (Suc xa)/Suc xaa]) (r (Suc xa) (Suc xaa))) H x xa

app-Var-NF ≡
λx. compat-NFT.rec-split-NFT default
     (λts xa xaa r.
        (dB.Var xa °° (ts @ [dB.Var x]),
         NFT.App (ts @ [dB.Var x]) xa
          (snd (listall-app-P ts)
            (listall-conj1-P-Q ts (λz. (xaa z, r z)),
             listall-cons-P (Var-NF x) listall-nil-eq-P))))
     (λt xa r. let (xb, y) = r in (t[dB.Var x/0], subst-Var-NF x 0 xa))

lift-NF ≡
λx H. compat-NFT.rec-split-NFT default
        (λts x xa r xb.
           case nat-le-dec x xb of
           Left ⇒ NFT.App (map (λt. lift t xb) ts) x
                  (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z))))
           | Right ⇒
              NFT.App (map (λt. lift t xb) ts) (Suc x)
               (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                 (listall-conj2-P-Q ts (λz. (xa z, r z)))))
        (λt x r xa. NFT.Abs (lift t (Suc xa)) (r (Suc xa))) H x

type-NF ≡
λH. rec-rtypingT (λe x T. (dB.Var x, Var-NF x))
      (λe T t U x r. let (x, y) = r in (dB.Abs x, NFT.Abs x y))
      (λe s T U t x xa r ra.
          let (x, y) = r; (xa, ya) = ra;
              (x, y) =
                let (x, y) =
                      subst-type-NF (dB.Var 0 ° lift xa 0) e 0 (T ⇒ U) U x
                       (NFT.App [lift xa 0] 0 (listall-cons-P (lift-NF 0 ya) listall-nil-P)) y
                in (x, y)
          in (x, y))
      H
```

Figure 2: Program extracted from lemmas and main theorem

$\forall\, i<length\ ts.\ NFR\ (nfs\ i)\ (ts\ !\ i) \Longrightarrow NFR\ (NFT.App\ ts\ x\ nfs)\ (dB.Var\ x\ {}^{\circ\circ}\ ts)$
$NFR\ nf\ t \Longrightarrow NFR\ (NFT.Abs\ t\ nf)\ (dB.Abs\ t)$

The programs corresponding to the main theorem *type-NF*, as well as to some lemmas, are shown in Figure 2. The correctness statement for the main function *type-NF* is

$\bigwedge x.\ rtypingR\ x\ e\ t\ T \Longrightarrow t \to_\beta{}^*\ fst\ (type\text{-}NF\ x) \wedge NFR\ (snd\ (type\text{-}NF\ x))\ (fst\ (type\text{-}NF\ x))$

where the realizability predicate *rtypingR* corresponding to the computationally relevant version of the typing judgement is inductively defined by the rules

$e\ x = T \Longrightarrow rtypingR\ (rtypingT.Var\ e\ x\ T)\ e\ (dB.Var\ x)\ T$
$rtypingR\ ty\ (e\langle 0{:}T\rangle)\ t\ U \Longrightarrow rtypingR\ (rtypingT.Abs\ e\ T\ t\ U\ ty)\ e\ (dB.Abs\ t)\ (T \Rightarrow U)$
$rtypingR\ ty\ e\ s\ (T \Rightarrow U) \Longrightarrow$
$rtypingR\ ty'\ e\ t\ T \Longrightarrow rtypingR\ (rtypingT.App\ e\ s\ T\ U\ t\ ty\ ty')\ e\ (s\ {}^\circ\ t)\ U$

13.3 Generating executable code

instantiation *NFT* :: *default*
begin

definition *default* = *Dummy* ()

instance ⟨*proof*⟩

end

instantiation *dB* :: *default*
begin

definition *default* = *dB.Var 0*

instance ⟨*proof*⟩

end

instantiation *prod* :: (*default*, *default*) *default*
begin

definition *default* = (*default*, *default*)

instance ⟨*proof*⟩

end

instantiation *list* :: (*type*) *default*
begin

definition *default* = []

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *default*) *default*
begin

definition *default* = (λx. *default*)

instance ⟨*proof*⟩

end

definition *int-of-nat* :: *nat* ⇒ *int* **where**
int-of-nat = *of-nat*

The following functions convert between Isabelle's built-in `term` datatype and the generated `dB` datatype. This allows to generate example terms using Isabelle's parser and inspect normalized terms using Isabelle's pretty printer.

⟨*ML*⟩

end

References

[1] F. Joachimski and R. Matthes. Short proofs of normalization for the simply-typed λ-calculus, permutative conversions and Gödel's T. *Archive for Mathematical Logic*, 42(1):59–87, 2003.

[2] R. Loader. Notes on Simply Typed Lambda Calculus. Technical Report ECS-LFCS-98-381, Laboratory for Foundations of Computer Science, School of Informatics, University of Edinburgh, 1998.

[3] R. Matthes. Lambda Calculus: A Case for Inductive Definitions. In *Lecture notes of the 12th European Summer School in Logic, Language and Information (ESSLLI 2000)*. School of Computer Science, University of Birmingham, August 2000.

[4] M. Takahashi. Parallel reductions in λ-calculus. *Information and Computation*, 118(1):120–127, April 1995.