Analysis

March 13, 2025

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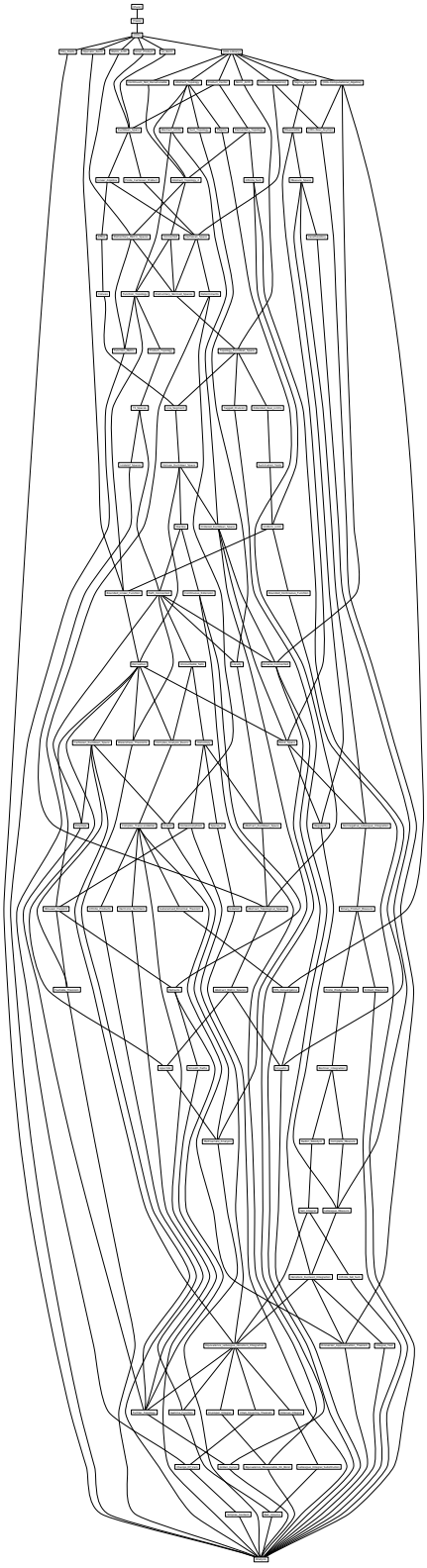

Chapter 1

Linear Algebra

theory *L2_Norm*
imports *Complex_Main*
begin

1.1 L2 Norm

definition *L2_set* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow real$ **where**
$L2\_set\ f\ A = sqrt\ (\sum i{\in}A.\ (f\ i)^2)$

lemma *L2_set_cong*:
$[\![A = B;\ \bigwedge x.\ x \in B \Longrightarrow f\ x = g\ x]\!] \Longrightarrow L2\_set\ f\ A = L2\_set\ g\ B$
unfolding *L2_set_def* **by** *simp*

lemma *L2_set_cong_simp*:
$[\![A = B;\ \bigwedge x.\ x \in B =simp=> f\ x = g\ x]\!] \Longrightarrow L2\_set\ f\ A = L2\_set\ g\ B$
unfolding *L2_set_def simp_implies_def* **by** *simp*

lemma *L2_set_infinite* [*simp*]: $\neg\ finite\ A \Longrightarrow L2\_set\ f\ A = 0$
unfolding *L2_set_def* **by** *simp*

lemma *L2_set_empty* [*simp*]: $L2\_set\ f\ \{\} = 0$
unfolding *L2_set_def* **by** *simp*

lemma *L2_set_insert* [*simp*]:
$[\![finite\ F;\ a \notin F]\!] \Longrightarrow$
$L2\_set\ f\ (insert\ a\ F) = sqrt\ ((f\ a)^2 + (L2\_set\ f\ F)^2)$
unfolding *L2_set_def* **by** (*simp add*: *sum_nonneg*)

lemma *L2_set_nonneg* [*simp*]: $0 \leq L2\_set\ f\ A$
unfolding *L2_set_def* **by** (*simp add*: *sum_nonneg*)

lemma *L2_set_0'*: $\forall\ a{\in}A.\ f\ a = 0 \Longrightarrow L2\_set\ f\ A = 0$
unfolding *L2_set_def* **by** *simp*

lemma *L2_set_constant*: *L2_set* (λx. *y*) *A* = *sqrt* (*of_nat* (*card A*)) * |*y*|
unfolding *L2_set_def* **by** (*simp add*: *real_sqrt_mult*)

lemma *L2_set_mono*:
assumes ⋀*i*. *i* ∈ *K* ⟹ *f i* ≤ *g i*
assumes ⋀*i*. *i* ∈ *K* ⟹ *0* ≤ *f i*
shows *L2_set f K* ≤ *L2_set g K*
unfolding *L2_set_def*
by (*simp add*: *sum_nonneg sum_mono power_mono assms*)

lemma *L2_set_strict_mono*:
assumes *finite K* **and** *K* ≠ {}
assumes ⋀*i*. *i* ∈ *K* ⟹ *f i* < *g i*
assumes ⋀*i*. *i* ∈ *K* ⟹ *0* ≤ *f i*
shows *L2_set f K* < *L2_set g K*
unfolding *L2_set_def*
by (*simp add*: *sum_strict_mono power_strict_mono assms*)

lemma *L2_set_right_distrib*:
0 ≤ *r* ⟹ *r* * *L2_set f A* = *L2_set* (λx. *r* * *f x*) *A*
unfolding *L2_set_def*
apply (*simp add*: *power_mult_distrib*)
apply (*simp add*: *sum_distrib_left* [*symmetric*])
apply (*simp add*: *real_sqrt_mult sum_nonneg*)
done

lemma *L2_set_left_distrib*:
0 ≤ *r* ⟹ *L2_set f A* * *r* = *L2_set* (λx. *f x* * *r*) *A*
unfolding *L2_set_def*
apply (*simp add*: *power_mult_distrib*)
apply (*simp add*: *sum_distrib_right* [*symmetric*])
apply (*simp add*: *real_sqrt_mult sum_nonneg*)
done

lemma *L2_set_eq_0_iff*: *finite A* ⟹ *L2_set f A* = *0* ⟷ (∀ x∈*A*. *f x* = *0*)
unfolding *L2_set_def*
by (*simp add*: *sum_nonneg sum_nonneg_eq_0_iff*)

proposition *L2_set_triangle_ineq*:
L2_set (λi. *f i* + *g i*) *A* ≤ *L2_set f A* + *L2_set g A*
proof (*cases finite A*)
case *False*
thus *?thesis* **by** *simp*
next
case *True*
thus *?thesis*
proof (*induct set*: *finite*)
case *empty*
show *?case* **by** *simp*

```
  next
    case (insert x F)
    hence sqrt ((f x + g x)^2 + (L2_set (λi. f i + g i) F)^2) ≤
          sqrt ((f x + g x)^2 + (L2_set f F + L2_set g F)^2)
      by (intro real_sqrt_le_mono add_left_mono power_mono insert
                L2_set_nonneg add_increasing zero_le_power2)
    also have
      ... ≤ sqrt ((f x)^2 + (L2_set f F)^2) + sqrt ((g x)^2 + (L2_set g F)^2)
      by (rule real_sqrt_sum_squares_triangle_ineq)
    finally show ?case
      using insert by simp
  qed
qed

lemma L2_set_le_sum [rule_format]:
  (∀ i∈A. 0 ≤ f i) ⟶ L2_set f A ≤ sum f A
  apply (cases finite A)
  apply (induct set: finite)
  apply simp
  apply clarsimp
  apply (erule order_trans [OF sqrt_sum_squares_le_sum])
  apply simp
  apply simp
  apply simp
  done

lemma L2_set_le_sum_abs: L2_set f A ≤ (∑ i∈A. |f i|)
  apply (cases finite A)
  apply (induct set: finite)
  apply simp
  apply simp
  apply (rule order_trans [OF sqrt_sum_squares_le_sum_abs])
  apply simp
  apply simp
  done

lemma L2_set_mult_ineq: (∑ i∈A. |f i| * |g i|) ≤ L2_set f A * L2_set g A
  apply (cases finite A)
  apply (induct set: finite)
  apply simp
  apply (rule power2_le_imp_le, simp)
  apply (rule order_trans)
  apply (rule power_mono)
  apply (erule add_left_mono)
  apply (simp add: sum_nonneg)
  apply (simp add: power2_sum)
  apply (simp add: power_mult_distrib)
  apply (simp add: distrib_left distrib_right)
  apply (rule ord_le_eq_trans)
```

apply (*rule L2_set_mult_ineq_lemma*)
apply *simp_all*
done

lemma *member_le_L2_set*: ⟦*finite A*; *i* ∈ *A*⟧ ⟹ *f i* ≤ *L2_set f A*
unfolding *L2_set_def*
by (*auto intro*!: *member_le_sum real_le_rsqrt*)

end

1.2 Inner Product Spaces and Gradient Derivative

theory *Inner_Product*
imports *Complex_Main*
begin

1.2.1 Real inner product spaces

Temporarily relax type constraints for *open*, *uniformity*, *dist*, and *norm*.

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*open*›, *SOME* ***typ*** ‹'*a*::*open set* ⇒ *bool*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*dist*›, *SOME* ***typ*** ‹'*a*::*dist* ⇒ '*a* ⇒ *real*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*uniformity*›, *SOME* ***typ*** ‹('*a*::*uniformity* × '*a*) *filter*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*norm*›, *SOME* ***typ*** ‹'*a*::*norm* ⇒ *real*›)›

class *real_inner* = *real_vector* + *sgn_div_norm* + *dist_norm* + *uniformity_dist*
+ *open_uniformity* +
fixes *inner* :: '*a* ⇒ '*a* ⇒ *real*
assumes *inner_commute*: *inner x y* = *inner y x*
and *inner_add_left*: *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
and *inner_scaleR_left* [*simp*]: *inner* (*scaleR r x*) *y* = *r* ∗ (*inner x y*)
and *inner_ge_zero* [*simp*]: *0* ≤ *inner x x*
and *inner_eq_zero_iff* [*simp*]: *inner x x* = *0* ⟷ *x* = *0*
and *norm_eq_sqrt_inner*: *norm x* = *sqrt* (*inner x x*)
begin

lemma *inner_zero_left* [*simp*]: *inner 0 x* = *0*
using *inner_add_left* [*of 0 0 x*] **by** *simp*

lemma *inner_minus_left* [*simp*]: *inner* (− *x*) *y* = − *inner x y*
using *inner_add_left* [*of x* − *x y*] **by** *simp*

lemma *inner_diff_left*: *inner* $(x - y)$ z = *inner* x z − *inner* y z
using *inner_add_left* [*of* $x - y$ z] **by** *simp*

lemma *inner_sum_left*: *inner* $(\sum x{\in}A.\ f\ x)$ y = $(\sum x{\in}A.$ *inner* $(f\ x)$ $y)$
by (*cases finite A*, *induct set*: *finite*, *simp_all add*: *inner_add_left*)

lemma *all_zero_iff* [*simp*]: $(\forall u.$ *inner* x $u = 0) \longleftrightarrow (x = 0)$
by *auto* (*use inner_eq_zero_iff* **in** *blast*)

Transfer distributivity rules to right argument.

lemma *inner_add_right*: *inner* x $(y + z)$ = *inner* x y + *inner* x z
using *inner_add_left* [*of* y z x] **by** (*simp only*: *inner_commute*)

lemma *inner_scaleR_right* [*simp*]: *inner* x $(scaleR\ r\ y) = r * (inner\ x\ y)$
using *inner_scaleR_left* [*of* r y x] **by** (*simp only*: *inner_commute*)

lemma *inner_zero_right* [*simp*]: *inner* x $0 = 0$
using *inner_zero_left* [*of* x] **by** (*simp only*: *inner_commute*)

lemma *inner_minus_right* [*simp*]: *inner* x $(-\ y) = -$ *inner* x y
using *inner_minus_left* [*of* y x] **by** (*simp only*: *inner_commute*)

lemma *inner_diff_right*: *inner* x $(y - z)$ = *inner* x y − *inner* x z
using *inner_diff_left* [*of* y z x] **by** (*simp only*: *inner_commute*)

lemma *inner_sum_right*: *inner* x $(\sum y{\in}A.\ f\ y) = (\sum y{\in}A.$ *inner* x $(f\ y))$
using *inner_sum_left* [*of f A x*] **by** (*simp only*: *inner_commute*)

lemmas *inner_add* [*algebra_simps*] = *inner_add_left inner_add_right*
lemmas *inner_diff* [*algebra_simps*] = *inner_diff_left inner_diff_right*
lemmas *inner_scaleR* = *inner_scaleR_left inner_scaleR_right*

Legacy theorem names

lemmas *inner_left_distrib* = *inner_add_left*
lemmas *inner_right_distrib* = *inner_add_right*
lemmas *inner_distrib* = *inner_left_distrib inner_right_distrib*

lemma *inner_gt_zero_iff* [*simp*]: $0 <$ *inner* x $x \longleftrightarrow x \neq 0$
by (*simp add*: *order_less_le*)

lemma *power2_norm_eq_inner*: $(norm\ x)^2$ = *inner* x x
by (*simp add*: *norm_eq_sqrt_inner*)

Identities involving real multiplication and division.

lemma *inner_mult_left*: *inner* $(of\_real\ m * a)$ $b = m * (inner\ a\ b)$
by (*metis real_inner_class.inner_scaleR_left scaleR_conv_of_real*)

lemma *inner_mult_right*: *inner* a $(of\_real\ m * b) = m * (inner\ a\ b)$
by (*metis real_inner_class.inner_scaleR_right scaleR_conv_of_real*)

lemma *inner_mult_left'*: *inner* (*a* ∗ *of_real m*) *b* = *m* ∗ (*inner a b*)
by (*simp add*: *of_real_def*)

lemma *inner_mult_right'*: *inner a* (*b* ∗ *of_real m*) = (*inner a b*) ∗ *m*
by (*simp add*: *of_real_def real_inner_class.inner_scaleR_right*)

lemma *Cauchy_Schwarz_ineq*:
(*inner x y*)^2 ≤ *inner x x* ∗ *inner y y*
proof (*cases*)
assume *y* = *0*
thus *?thesis* **by** *simp*
next
assume *y*: *y* ≠ *0*
let *?r* = *inner x y* / *inner y y*
have *0* ≤ *inner* (*x* − *scaleR ?r y*) (*x* − *scaleR ?r y*)
by (*rule inner_ge_zero*)
also have ... = *inner x x* − *inner y x* ∗ *?r*
by (*simp add*: *inner_diff*)
also have ... = *inner x x* − (*inner x y*)^2 / *inner y y*
by (*simp add*: *power2_eq_square inner_commute*)
finally have *0* ≤ *inner x x* − (*inner x y*)^2 / *inner y y* **.**
hence (*inner x y*)^2 / *inner y y* ≤ *inner x x*
by (*simp add*: *le_diff_eq*)
thus (*inner x y*)^2 ≤ *inner x x* ∗ *inner y y*
by (*simp add*: *pos_divide_le_eq y*)
qed

lemma *Cauchy_Schwarz_ineq2*:
|*inner x y*| ≤ *norm x* ∗ *norm y*
proof (*rule power2_le_imp_le*)
have (*inner x y*)^2 ≤ *inner x x* ∗ *inner y y*
using *Cauchy_Schwarz_ineq* **.**
thus |*inner x y*|^2 ≤ (*norm x* ∗ *norm y*)^2
by (*simp add*: *power_mult_distrib power2_norm_eq_inner*)
show *0* ≤ *norm x* ∗ *norm y*
unfolding *norm_eq_sqrt_inner*
by (*intro mult_nonneg_nonneg real_sqrt_ge_zero inner_ge_zero*)
qed

lemma *norm_cauchy_schwarz*: *inner x y* ≤ *norm x* ∗ *norm y*
using *Cauchy_Schwarz_ineq2* [*of x y*] **by** *auto*

subclass *real_normed_vector*
proof
fix *a* :: *real* **and** *x y* :: *'a*
show *norm x* = *0* ⟷ *x* = *0*
unfolding *norm_eq_sqrt_inner* **by** *simp*
show *norm* (*x* + *y*) ≤ *norm x* + *norm y*

```
    proof (rule power2_le_imp_le)
      have inner x y ≤ norm x * norm y
        by (rule norm_cauchy_schwarz)
      thus (norm (x + y))² ≤ (norm x + norm y)²
        unfolding power2_sum power2_norm_eq_inner
        by (simp add: inner_add inner_commute)
      show 0 ≤ norm x + norm y
        unfolding norm_eq_sqrt_inner by simp
    qed
  have sqrt (a² * inner x x) = |a| * sqrt (inner x x)
    by (simp add: real_sqrt_mult)
  then show norm (a *R x) = |a| * norm x
    unfolding norm_eq_sqrt_inner
    by (simp add: power2_eq_square mult.assoc)
qed

end

lemma square_bound_lemma:
  fixes x :: real
  shows x < (1 + x) * (1 + x)
proof −
  have (x + 1/2)² + 3/4 > 0
    using zero_le_power2[of x+1/2] by arith
  then show ?thesis
    by (simp add: field_simps power2_eq_square)
qed

lemma square_continuous:
  fixes e :: real
  shows e > 0 ⟹ ∃ d. 0 < d ∧ (∀ y. |y − x| < d ⟶ |y * y − x * x| < e)
  using isCont_power[OF continuous_ident, of x, unfolded isCont_def LIM_eq,
rule_format, of e 2]
  by (force simp add: power2_eq_square)

lemma norm_le: norm x ≤ norm y ⟷ inner x x ≤ inner y y
  by (simp add: norm_eq_sqrt_inner)

lemma norm_lt: norm x < norm y ⟷ inner x x < inner y y
  by (simp add: norm_eq_sqrt_inner)

lemma norm_eq: norm x = norm y ⟷ inner x x = inner y y
  apply (subst order_eq_iff)
  apply (auto simp: norm_le)
  done

lemma norm_eq_1: norm x = 1 ⟷ inner x x = 1
  by (simp add: norm_eq_sqrt_inner)
```

lemma *inner_divide_left*:
fixes *a* :: *'a* :: {*real_inner*,*real_div_algebra*}
shows *inner* (*a* / *of_real m*) *b* = (*inner a b*) / *m*
by (*metis* (*no_types*) *divide_inverse inner_commute inner_scaleR_right mult.left_neutral mult.right_neutral mult_scaleR_right of_real_inverse scaleR_conv_of_real times_divide_eq_left*)

lemma *inner_divide_right*:
fixes *a* :: *'a* :: {*real_inner*,*real_div_algebra*}
shows *inner a* (*b* / *of_real m*) = (*inner a b*) / *m*
by (*metis inner_commute inner_divide_left*)

Re-enable constraints for *open*, *uniformity*, *dist*, and *norm*.

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*open*›, *SOME* ***typ*** ‹*'a*::*topological_space set* ⇒ *bool*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*uniformity*›, *SOME* ***typ*** ‹(*'a*::*uniform_space* × *'a*) *filter*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*dist*›, *SOME* ***typ*** ‹*'a*::*metric_space* ⇒ *'a* ⇒ *real*›)›

setup ‹*Sign.add_const_constraint*
(***const_name*** ‹*norm*›, *SOME* ***typ*** ‹*'a*::*real_normed_vector* ⇒ *real*›)›

lemma *bounded_bilinear_inner*:
bounded_bilinear (*inner*::*'a*::*real_inner* ⇒ *'a* ⇒ *real*)
proof
fix *x y z* :: *'a* **and** *r* :: *real*
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
by (*rule inner_add_left*)
show *inner x* (*y* + *z*) = *inner x y* + *inner x z*
by (*rule inner_add_right*)
show *inner* (*scaleR r x*) *y* = *scaleR r* (*inner x y*)
unfolding *real_scaleR_def* **by** (*rule inner_scaleR_left*)
show *inner x* (*scaleR r y*) = *scaleR r* (*inner x y*)
unfolding *real_scaleR_def* **by** (*rule inner_scaleR_right*)
show ∃ *K*. ∀ *x y*::*'a*. *norm* (*inner x y*) ≤ *norm x* ∗ *norm y* ∗ *K*
proof
show ∀ *x y*::*'a*. *norm* (*inner x y*) ≤ *norm x* ∗ *norm y* ∗ *1*
by (*simp add*: *Cauchy_Schwarz_ineq2*)
qed
qed

lemmas *tendsto_inner* [*tendsto_intros*] =
bounded_bilinear.tendsto [*OF bounded_bilinear_inner*]

lemmas *isCont_inner* [*simp*] =
bounded_bilinear.isCont [*OF bounded_bilinear_inner*]

lemmas *has_derivative_inner* [*derivative_intros*] =
bounded_bilinear.FDERIV [*OF bounded_bilinear_inner*]

lemmas *bounded_linear_inner_left* =
bounded_bilinear.bounded_linear_left [*OF bounded_bilinear_inner*]

lemmas *bounded_linear_inner_right* =
bounded_bilinear.bounded_linear_right [*OF bounded_bilinear_inner*]

lemmas *bounded_linear_inner_left_comp* = *bounded_linear_inner_left*[*THEN bounded_linear_compose*]

lemmas *bounded_linear_inner_right_comp* = *bounded_linear_inner_right*[*THEN bounded_linear_compose*]

lemmas *has_derivative_inner_right* [*derivative_intros*] =
bounded_linear.has_derivative [*OF bounded_linear_inner_right*]

lemmas *has_derivative_inner_left* [*derivative_intros*] =
bounded_linear.has_derivative [*OF bounded_linear_inner_left*]

lemma *differentiable_inner* [*simp*]:
f differentiable (*at x within s*) $\Longrightarrow$ *g differentiable at x within s* $\Longrightarrow$ (λx. *inner* (*f x*) (*g x*)) *differentiable at x within s*
unfolding *differentiable_def* **by** (*blast intro*: *has_derivative_inner*)

1.2.2 Class instances

instantiation *real* :: *real_inner*
begin

definition *inner_real_def* [*simp*]: *inner* = (*)

instance
proof
fix *x y z r* :: *real*
show *inner x y* = *inner y x*
unfolding *inner_real_def* **by** (*rule mult.commute*)
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
unfolding *inner_real_def* **by** (*rule distrib_right*)
show *inner* (*scaleR r x*) *y* = *r* * *inner x y*
unfolding *inner_real_def real_scaleR_def* **by** (*rule mult.assoc*)
show *0* $\leq$ *inner x x*
unfolding *inner_real_def* **by** *simp*
show *inner x x* = *0* $\longleftrightarrow$ *x* = *0*
unfolding *inner_real_def* **by** *simp*
show *norm x* = *sqrt* (*inner x x*)
unfolding *inner_real_def* **by** *simp*
qed

end

lemma
shows *real_inner_1_left*[*simp*]: *inner 1 x = x*
and *real_inner_1_right*[*simp*]: *inner x 1 = x*
by *simp_all*

instantiation *complex* :: *real_inner*
begin

definition *inner_complex_def*:
*inner x y = Re x * Re y + Im x * Im y*

instance
proof
fix *x y z* :: *complex* **and** *r* :: *real*
show *inner x y = inner y x*
unfolding *inner_complex_def* **by** (*simp add*: *mult.commute*)
show *inner (x + y) z = inner x z + inner y z*
unfolding *inner_complex_def* **by** (*simp add*: *distrib_right*)
show *inner (scaleR r x) y = r * inner x y*
unfolding *inner_complex_def* **by** (*simp add*: *distrib_left*)
show *0 ≤ inner x x*
unfolding *inner_complex_def* **by** *simp*
show *inner x x = 0 ⟷ x = 0*
unfolding *inner_complex_def*
by (*simp add*: *add_nonneg_eq_0_iff complex_eq_iff*)
show *norm x = sqrt (inner x x)*
unfolding *inner_complex_def norm_complex_def*
by (*simp add*: *power2_eq_square*)
qed

end

lemma *complex_inner_1* [*simp*]: *inner 1 x = Re x*
unfolding *inner_complex_def* **by** *simp*

lemma *complex_inner_1_right* [*simp*]: *inner x 1 = Re x*
unfolding *inner_complex_def* **by** *simp*

lemma *complex_inner_i_left* [*simp*]: *inner* i *x = Im x*
unfolding *inner_complex_def* **by** *simp*

lemma *complex_inner_i_right* [*simp*]: *inner x* i *= Im x*
unfolding *inner_complex_def* **by** *simp*

lemma *dot_square_norm*: *inner x x = (norm x)*^{2}

by (*simp only*: *power2_norm_eq_inner*)

lemma *norm_eq_square*: *norm x* = *a* ⟷ *0* ≤ *a* ∧ *inner x x* = a^2
by (*auto simp add*: *norm_eq_sqrt_inner*)

lemma *norm_le_square*: *norm x* ≤ *a* ⟷ *0* ≤ *a* ∧ *inner x x* ≤ a^2
apply (*simp add*: *dot_square_norm abs_le_square_iff*[*symmetric*])
using *norm_ge_zero*[*of x*]
apply *arith*
done

lemma *norm_ge_square*: *norm x* ≥ *a* ⟷ *a* ≤ *0* ∨ *inner x x* ≥ a^2
apply (*simp add*: *dot_square_norm abs_le_square_iff*[*symmetric*])
using *norm_ge_zero*[*of x*]
apply *arith*
done

lemma *norm_lt_square*: *norm x* < *a* ⟷ *0* < *a* ∧ *inner x x* < a^2
by (*metis not_le norm_ge_square*)

lemma *norm_gt_square*: *norm x* > *a* ⟷ *a* < *0* ∨ *inner x x* > a^2
by (*metis norm_le_square not_less*)

Dot product in terms of the norm rather than conversely.

lemmas *inner_simps* = *inner_add_left inner_add_right inner_diff_right inner_diff_left inner_scaleR_left inner_scaleR_right*

lemma *dot_norm*: *inner x y* = ((*norm* (*x* + *y*))^2 − (*norm x*)^2 − (*norm y*)^2) / *2*
by (*simp only*: *power2_norm_eq_inner inner_simps inner_commute*) *auto*

lemma *dot_norm_neg*: *inner x y* = (((*norm x*)^2 + (*norm y*)^2) − (*norm* (*x* − *y*))^2) / *2*
by (*simp only*: *power2_norm_eq_inner inner_simps inner_commute*)
(*auto simp add*: *algebra_simps*)

lemma *of_real_inner_1* [*simp*]:
inner (*of_real x*) (*1* :: *'a* :: {*real_inner*, *real_normed_algebra_1*}) = *x*
by (*simp add*: *of_real_def dot_square_norm*)

lemma *summable_of_real_iff*:
summable (λ*x*. *of_real* (*f x*) :: *'a* :: {*real_normed_algebra_1*,*real_inner*}) ⟷ *summable f*
proof
assume ∗: *summable* (λ*x*. *of_real* (*f x*) :: *'a*)
interpret *bounded_linear* λ*x*::*'a*. *inner x 1*
by (*rule bounded_linear_inner_left*)
from *summable* [*OF* ∗] **show** *summable f* **by** *simp*
qed (*auto intro*: *summable_of_real*)

1.2.3 Gradient derivative

definition
gderiv :: ['*a*::*real_inner* ⇒ *real*, '*a*, '*a*] ⇒ *bool*
(‹(‹*notation*=‹*mixfix GDERIV*››*GDERIV* (_)/ (_)/ :> (_))› [*1000*, *1000*, *60*] *60*)
where
GDERIV f x :> D ⟷ *FDERIV f x :> (λh. inner h D)*

lemma *gderiv_deriv* [*simp*]: *GDERIV f x :> D* ⟷ *DERIV f x :> D*
by (*simp only*: *gderiv_def has_field_derivative_def inner_real_def mult_commute_abs*)

lemma *GDERIV_DERIV_compose*:
⟦*GDERIV f x :> df*; *DERIV g* (*f x*) :> *dg*⟧
⟹ *GDERIV* (*λx. g* (*f x*)) *x :> scaleR dg df*
unfolding *gderiv_def has_field_derivative_def*
apply (*drule* (*1*) *has_derivative_compose*)
apply (*simp add*: *ac_simps*)
done

lemma *has_derivative_subst*: ⟦*FDERIV f x :> df*; *df* = *d*⟧ ⟹ *FDERIV f x :> d*
by *simp*

lemma *GDERIV_subst*: ⟦*GDERIV f x :> df*; *df* = *d*⟧ ⟹ *GDERIV f x :> d*
by *simp*

lemma *GDERIV_const*: *GDERIV* (*λx. k*) *x :> 0*
unfolding *gderiv_def inner_zero_right* **by** (*rule has_derivative_const*)

lemma *GDERIV_add*:
⟦*GDERIV f x :> df*; *GDERIV g x :> dg*⟧
⟹ *GDERIV* (*λx. f x* + *g x*) *x :> df* + *dg*
unfolding *gderiv_def inner_add_right* **by** (*rule has_derivative_add*)

lemma *GDERIV_minus*:
GDERIV f x :> df ⟹ *GDERIV* (*λx.* − *f x*) *x :>* − *df*
unfolding *gderiv_def inner_minus_right* **by** (*rule has_derivative_minus*)

lemma *GDERIV_diff*:
⟦*GDERIV f x :> df*; *GDERIV g x :> dg*⟧
⟹ *GDERIV* (*λx. f x* − *g x*) *x :> df* − *dg*
unfolding *gderiv_def inner_diff_right* **by** (*rule has_derivative_diff*)

lemma *GDERIV_scaleR*:
⟦*DERIV f x :> df*; *GDERIV g x :> dg*⟧
⟹ *GDERIV* (*λx. scaleR* (*f x*) (*g x*)) *x*
:> (*scaleR* (*f x*) *dg* + *scaleR df* (*g x*))
unfolding *gderiv_def has_field_derivative_def inner_add_right inner_scaleR_right*
apply (*rule has_derivative_subst*)

apply (*erule* (*1*) *has_derivative_scaleR*)
apply (*simp add*: *ac_simps*)
done

lemma *GDERIV_mult*:
⟦*GDERIV f x :> df*; *GDERIV g x :> dg*⟧
⟹ *GDERIV* (λ*x*. *f x* ∗ *g x*) *x :> scaleR* (*f x*) *dg* + *scaleR* (*g x*) *df*
unfolding *gderiv_def*
apply (*rule has_derivative_subst*)
apply (*erule* (*1*) *has_derivative_mult*)
apply (*simp add*: *inner_add ac_simps*)
done

lemma *GDERIV_inverse*:
⟦*GDERIV f x :> df*; *f x* ≠ *0*⟧
⟹ *GDERIV* (λ*x*. *inverse* (*f x*)) *x :> −* (*inverse* (*f x*))^2 ∗$_R$ *df*
by (*metis DERIV_inverse GDERIV_DERIV_compose numerals*(*2*))

lemma *GDERIV_norm*:
assumes *x* ≠ *0* **shows** *GDERIV* (λ*x*. *norm x*) *x :> sgn x*
unfolding *gderiv_def norm_eq_sqrt_inner*
by (*rule derivative_eq_intros* | *force simp add*: *inner_commute sgn_div_norm norm_eq_sqrt_inner assms*)+

lemmas *has_derivative_norm* = *GDERIV_norm* [*unfolded gderiv_def*]

bundle *inner_syntax*
begin
notation *inner* (**infix** ‹•› *70*)
end

end

1.3 Cartesian Products as Vector Spaces

theory *Product_Vector*
imports
Complex_Main
HOL−Library.Product_Plus
begin

lemma *Times_eq_image_sum*:
fixes *S* :: *'a* :: *comm_monoid_add set* **and** *T* :: *'b* :: *comm_monoid_add set*
shows *S* × *T* = {*u* + *v* |*u v*. *u* ∈ (λ*x*. (*x*, *0*)) ‘ *S* ∧ *v* ∈ *Pair 0* ‘ *T*}
by *force*

1.3.1 Product is a Module

locale *module_prod* = *module_pair* **begin**

definition *scale* :: *'a* ⇒ *'b* × *'c* ⇒ *'b* × *'c*
where *scale a v* = (*s1 a* (*fst v*), *s2 a* (*snd v*))

lemma *scale_prod*: *scale x* (*a*, *b*) = (*s1 x a*, *s2 x b*)
by (*auto simp*: *scale_def*)

sublocale *p*: *module scale*
proof qed (*simp_all add*: *scale_def*
m1.scale_left_distrib m1.scale_right_distrib m2.scale_left_distrib m2.scale_right_distrib)

lemma *subspace_Times*: *m1.subspace A* ⟹ *m2.subspace B* ⟹ *p.subspace* (*A* ×
B)
unfolding *m1.subspace_def m2.subspace_def p.subspace_def*
by (*auto simp*: *zero_prod_def scale_def*)

lemma *module_hom_fst*: *module_hom scale s1 fst*
by *unfold_locales* (*auto simp*: *scale_def*)

lemma *module_hom_snd*: *module_hom scale s2 snd*
by *unfold_locales* (*auto simp*: *scale_def*)

end

locale *vector_space_prod* = *vector_space_pair* **begin**

sublocale *module_prod s1 s2*
rewrites *module_hom* = *Vector_Spaces.linear*
by *unfold_locales* (*fact module_hom_eq_linear*)

sublocale *p*: *vector_space scale* **by** *unfold_locales* (*auto simp*: *algebra_simps*)

lemmas *linear_fst* = *module_hom_fst*
and *linear_snd* = *module_hom_snd*

end

1.3.2 Product is a Real Vector Space

instantiation *prod* :: (*real_vector*, *real_vector*) *real_vector*
begin

definition *scaleR_prod_def*:
scaleR r A = (*scaleR r* (*fst A*), *scaleR r* (*snd A*))

lemma *fst_scaleR* [*simp*]: *fst* (*scaleR r A*) = *scaleR r* (*fst A*)
unfolding *scaleR_prod_def* **by** *simp*

lemma *snd_scaleR* [*simp*]: *snd* (*scaleR r A*) = *scaleR r* (*snd A*)

unfolding *scaleR_prod_def* **by** *simp*

proposition *scaleR_Pair* [*simp*]: *scaleR r* (*a*, *b*) = (*scaleR r a*, *scaleR r b*)
unfolding *scaleR_prod_def* **by** *simp*

instance
proof
fix *a b* :: *real* **and** *x y* :: *'a* × *'b*
show *scaleR a* (*x* + *y*) = *scaleR a x* + *scaleR a y*
by (*simp add*: *prod_eq_iff scaleR_right_distrib*)
show *scaleR* (*a* + *b*) *x* = *scaleR a x* + *scaleR b x*
by (*simp add*: *prod_eq_iff scaleR_left_distrib*)
show *scaleR a* (*scaleR b x*) = *scaleR* (*a* ∗ *b*) *x*
by (*simp add*: *prod_eq_iff*)
show *scaleR 1 x* = *x*
by (*simp add*: *prod_eq_iff*)
qed

end

lemma *module_prod_scale_eq_scaleR*: *module_prod.scale* (∗*R*) (∗*R*) = *scaleR*
apply (*rule ext*) **apply** (*rule ext*)
apply (*subst module_prod.scale_def*)
subgoal by *unfold_locales*
by (*simp add*: *scaleR_prod_def*)

interpretation *real_vector?*: *vector_space_prod scaleR*::_⇒_⇒*'a*::*real_vector scaleR*::_⇒_⇒*'b*::*real_vector*
rewrites *scale* = ((∗*R*)::_⇒_⇒(*'a* × *'b*))
and *module.dependent* (∗*R*) = *dependent*
and *module.representation* (∗*R*) = *representation*
and *module.subspace* (∗*R*) = *subspace*
and *module.span* (∗*R*) = *span*
and *vector_space.extend_basis* (∗*R*) = *extend_basis*
and *vector_space.dim* (∗*R*) = *dim*
and *Vector_Spaces.linear* (∗*R*) (∗*R*) = *linear*
subgoal by *unfold_locales*
subgoal by (*fact module_prod_scale_eq_scaleR*)
unfolding *dependent_raw_def representation_raw_def subspace_raw_def span_raw_def*
extend_basis_raw_def dim_raw_def linear_def
by (*rule refl*)+

1.3.3 Product is a Metric Space

instantiation *prod* :: (*metric_space*, *metric_space*) *dist*
begin

definition *dist_prod_def*[*code del*]:
dist x y = *sqrt* ((*dist* (*fst x*) (*fst y*))^2 + (*dist* (*snd x*) (*snd y*))^2)

instance ..
end

instantiation *prod* :: (*uniformity*, *uniformity*) *uniformity* **begin**

definition [*code del*]: ‹(*uniformity* :: (('*a* × '*b*) × ('*a* × '*b*)) *filter*) =
filtermap (λ((*x1*,*x2*),(*y1*,*y2*)). ((*x1*,*y1*),(*x2*,*y2*))) (*uniformity* ×$_F$ *uniformity*)›

instance..
end

Uniform spaces

instantiation *prod* :: (*uniform_space*, *uniform_space*) *uniform_space*
begin
instance
proof *standard*
fix *U* :: ‹('*a* × '*b*) *set*›
show ‹*open U* ⟷ (∀ *x*∈*U*. ∀$_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *U*)›
proof (*intro iffI ballI*)
fix *x* **assume** ‹*open U*› **and** ‹*x* ∈ *U*›
then obtain *A B* **where** ‹*open A*› ‹*open B*› ‹*x* ∈ *A*×*B*› ‹*A*×*B* ⊆ *U*›
by (*metis open_prod_elim*)
define *UA* **where** ‹*UA* = (λ(*x'*::'*a*,*y*). *x'* = *fst x* ⟶ *y* ∈ *A*)›
from ‹*open A*› ‹*x* ∈ *A*×*B*›
have ‹*eventually UA uniformity*›
unfolding *open_uniformity UA_def* **by** *auto*
define *UB* **where** ‹*UB* = (λ(*x'*::'*b*,*y*). *x'* = *snd x* ⟶ *y* ∈ *B*)›
from ‹*open A*› ‹*open B*› ‹*x* ∈ *A*×*B*›
have ‹*eventually UA uniformity*› ‹*eventually UB uniformity*›
unfolding *open_uniformity UA_def UB_def* **by** *auto*
then have ‹∀$_F$ ((*x'1*, *y1*), (*x'2*, *y2*)) *in uniformity* ×$_F$ *uniformity*. (*x'1*,*x'2*) = *x* ⟶ (*y1*,*y2*) ∈ *U*›
apply (*auto intro*!: *exI*[*of*_ *UA*] *exI*[*of*_ *UB*] *simp add*: *eventually_prod_filter*)
using ‹*A*×*B* ⊆ *U*› **by** (*auto simp*: *UA_def UB_def*)
then show ‹∀$_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *U*›
by (*simp add*: *uniformity_prod_def eventually_filtermap case_prod_unfold*)
next
assume *asm*: ‹∀ *x*∈*U*. ∀$_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *U*›
show ‹*open U*›
proof (*unfold open_prod_def*, *intro ballI*)
fix *x* **assume** ‹*x* ∈ *U*›
with *asm* **have** ‹∀$_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *U*›
by *auto*
then have ‹∀$_F$ ((*x'1*, *y1*), (*x'2*, *y2*)) *in uniformity* ×$_F$ *uniformity*. (*x'1*,*x'2*) = *x* ⟶ (*y1*,*y2*) ∈ *U*›
by (*simp add*: *uniformity_prod_def eventually_filtermap case_prod_unfold*)
then obtain *UA UB* **where** ‹*eventually UA uniformity*› **and** ‹*eventually UB*

uniformity›
and *UA_UB_U*: ‹*UA* (*a1*, *a2*) ⟹ *UB* (*b1*, *b2*) ⟹ (*a1*, *b1*) = *x* ⟹ (*a2*, *b2*) ∈ *U*› **for** *a1 a2 b1 b2*
apply *atomize_elim* **by** (*simp add*: *case_prod_beta eventually_prod_filter*)
have ‹*eventually* (λ*a*. *UA* (*fst x*, *a*)) (*nhds* (*fst x*))›
using ‹*eventually UA uniformity*› *eventually_mono eventually_nhds_uniformity* **by** *fastforce*
then obtain *A* **where** ‹*open A*› **and** *A_UA*: ‹*A* ⊆ {*a*. *UA* (*fst x*, *a*)}› **and** ‹*fst x* ∈ *A*›
by (*metis* (*mono_tags*, *lifting*) *eventually_nhds mem_Collect_eq subsetI*)
have ‹*eventually* (λ*b*. *UB* (*snd x*, *b*)) (*nhds* (*snd x*))›
using ‹*eventually UB uniformity*› *eventually_mono eventually_nhds_uniformity* **by** *fastforce*
then obtain *B* **where** ‹*open B*› **and** *B_UB*: ‹*B* ⊆ {*b*. *UB* (*snd x*, *b*)}› **and** ‹*snd x* ∈ *B*›
by (*metis* (*mono_tags*, *lifting*) *eventually_nhds mem_Collect_eq subsetI*)
have ‹*x* ∈ *A* × *B*›
by (*simp add*: ‹*fst x* ∈ *A*› ‹*snd x* ∈ *B*› *mem_Times_iff*)
have ‹*A* × *B* ⊆ *U*›
using *A_UA B_UB UA_UB_U* **by** *fastforce*
show ‹∃ *A B*. *open A* ∧ *open B* ∧ *x* ∈ *A* × *B* ∧ *A* × *B* ⊆ *U*›
using ‹*A* × *B* ⊆ *U*› ‹*open A*› ‹*open B*› ‹*x* ∈ *A* × *B*› **by** *auto*
qed
qed
next
show ‹*eventually E uniformity* ⟹ *E* (*x*, *x*)› **for** *E* **and** *x* :: ‹'*a* × '*b*›
apply (*simp add*: *uniformity_prod_def eventually_filtermap case_prod_unfold eventually_prod_filter*)
by (*metis surj_pair uniformity_refl*)
next
show ‹*eventually E uniformity* ⟹ ∀⇩*F* (*x*::'*a*×'*b*, *y*) *in uniformity*. *E* (*y*, *x*)› **for** *E*
apply (*simp only*: *uniformity_prod_def eventually_filtermap case_prod_unfold eventually_prod_filter*)
apply (*erule exE*, *erule exE*, *rename_tac Pf Pg*)
apply (*rule_tac x*=‹λ(*x*,*y*). *Pf* (*y*,*x*)› **in** *exI*)
apply (*rule_tac x*=‹λ(*x*,*y*). *Pg* (*y*,*x*)› **in** *exI*)
by (*auto simp add*: *uniformity_sym*)
next
show ‹∃ *D*. *eventually D uniformity* ∧ (∀ *x y z*. *D* (*x*::'*a*×'*b*, *y*) ⟶ *D* (*y*, *z*) ⟶ *E* (*x*, *z*))›
if ‹*eventually E uniformity*› **for** *E*
proof −
from *that*
obtain *EA EB* **where** ‹*eventually EA uniformity*› **and** ‹*eventually EB uniformity*›
and *EA_EB_E*: ‹*EA* (*a1*, *a2*) ⟹ *EB* (*b1*, *b2*) ⟹ *E* ((*a1*, *b1*), (*a2*, *b2*))› **for** *a1 a2 b1 b2*
by (*auto simp add*: *uniformity_prod_def eventually_filtermap case_prod_unfold*

eventually_prod_filter)
obtain *DA* **where** ‹*eventually DA uniformity*› **and** *DA_EA*: ‹*DA* (*x*,*y*) ⟹ *DA* (*y*,*z*) ⟹ *EA* (*x*,*z*)› **for** *x y z*
using ‹*eventually EA uniformity*› *uniformity_transE* **by** *blast*
obtain *DB* **where** ‹*eventually DB uniformity*› **and** *DB_EB*: ‹*DB* (*x*,*y*) ⟹ *DB* (*y*,*z*) ⟹ *EB* (*x*,*z*)› **for** *x y z*
using ‹*eventually EB uniformity*› *uniformity_transE* **by** *blast*
define *D* **where** ‹*D* = (λ((*a1*,*b1*),(*a2*,*b2*)). *DA* (*a1*,*a2*) ∧ *DB* (*b1*,*b2*))›
have ‹*eventually D uniformity*›
using ‹*eventually DA uniformity*› ‹*eventually DB uniformity*›
by (*auto simp add*: *uniformity_prod_def eventually_filtermap case_prod_unfold eventually_prod_filter D_def*)
moreover have ‹*D* ((*a1*, *b1*), (*a2*, *b2*)) ⟹ *D* ((*a2*, *b2*), (*a3*, *b3*)) ⟹ *E* ((*a1*, *b1*), (*a3*, *b3*))› **for** *a1 b1 a2 b2 a3 b3*
using *DA_EA DB_EB D_def EA_EB_E* **by** *blast*
ultimately show *?thesis*
by *auto*
qed
qed
end

lemma (**in** *uniform_space*) *nhds_eq_comap_uniformity*: *nhds x* = *filtercomap* (λ*y*. (*x*, *y*)) *uniformity*
proof −
have ∗: *eventually P* (*filtercomap* (λ*y*. (*x*, *y*)) *F*) ⟷
eventually (λ*z*. *fst z* = *x* ⟶ *P* (*snd z*)) *F* **for** *P* :: *'a* ⇒ *bool* **and** *F*
unfolding *eventually_filtercomap*
by (*smt* (*verit*) *eventually_elim2 fst_conv prod.collapse snd_conv*)
thus *?thesis*
unfolding *filter_eq_iff*
by (*subst* ∗) (*auto simp*: *eventually_nhds_uniformity case_prod_unfold*)
qed

lemma *uniformity_of_uniform_continuous_invariant*:
fixes *f* :: *'a* :: *uniform_space* ⇒ *'a* ⇒ *'a*
assumes *filterlim* (λ((*a*,*b*),(*c*,*d*)). (*f a c*, *f b d*)) *uniformity* (*uniformity* ×$_F$ *uniformity*)
assumes *eventually P uniformity*
obtains *Q* **where** *eventually Q uniformity* ⋀*a b c*. *Q* (*a*, *b*) ⟹ *P* (*f a c*, *f b c*)
using *eventually_compose_filterlim*[*OF assms*(*2*,*1*)] *uniformity_refl*
by (*fastforce simp*: *case_prod_unfold eventually_filtercomap eventually_prod_same*)

class *uniform_topological_monoid_add* = *topological_monoid_add* + *uniform_space* +
assumes *uniformly_continuous_add'*:
filterlim (λ((*a*,*b*), (*c*,*d*)). (*a* + *c*, *b* + *d*)) *uniformity* (*uniformity* ×$_F$ *uniformity*)

lemma *uniformly_continuous_add*:

uniformly_continuous_on UNIV (λ(*x* :: *'a* :: *uniform_topological_monoid_add*,*y*). *x* + *y*)
using *uniformly_continuous_add'*[**where** *?'a* = *'a*]
by (*simp add*: *uniformly_continuous_on_uniformity case_prod_unfold uniformity_prod_def filterlim_filtermap*)

lemma *filterlim_fst*: *filterlim fst F* (*F* ×$_F$ *G*)
by (*simp add*: *filterlim_def filtermap_fst_prod_filter*)

lemma *filterlim_snd*: *filterlim snd G* (*F* ×$_F$ *G*)
by (*simp add*: *filterlim_def filtermap_snd_prod_filter*)

class *uniform_topological_group_add* = *topological_group_add* + *uniform_topological_monoid_add* +
assumes *uniformly_continuous_uminus'*: *filterlim* (λ(*a*, *b*). (−*a*, −*b*)) *uniformity uniformity*
begin

lemma *uniformly_continuous_minus'*:
filterlim (λ((*a*,*b*), (*c*,*d*)). (*a* − *c*, *b* − *d*)) *uniformity* (*uniformity* ×$_F$ *uniformity*)
proof −
have *filterlim* ((λ((*a*,*b*), (*c*,*d*)). (*a* + *c*, *b* + *d*)) ∘ (λ((*a*,*b*), (*c*,*d*)). ((*a*, *b*), (−*c*, −*d*))))
uniformity (*uniformity* ×$_F$ *uniformity*)
unfolding *o_def* **using** *uniformly_continuous_uminus'*
by (*intro filterlim_compose*[*OF uniformly_continuous_add'*])
(*auto simp*: *case_prod_unfold intro*!: *filterlim_Pair*
filterlim_fst filterlim_compose[*OF _ filterlim_snd*])
thus *?thesis*
by (*simp add*: *o_def case_prod_unfold*)
qed

end

lemma *uniformly_continuous_uminus*:
uniformly_continuous_on UNIV (λ*x* :: *'a* :: *uniform_topological_group_add*. −*x*)
using *uniformly_continuous_uminus'*[**where** *?'a* = *'a*]
by (*simp add*: *uniformly_continuous_on_uniformity*)

lemma *uniformly_continuous_minus*:
uniformly_continuous_on UNIV (λ(*x* :: *'a* :: *uniform_topological_group_add*,*y*). *x* − *y*)
using *uniformly_continuous_minus'*[**where** *?'a* = *'a*]
by (*simp add*: *uniformly_continuous_on_uniformity case_prod_unfold uniformity_prod_def filterlim_filtermap*)

lemma *real_normed_vector_is_uniform_topological_group_add* [*Pure.intro*]:
OFCLASS('*a* :: *real_normed_vector*, *uniform_topological_group_add_class*)
proof
show *filterlim* (λ((*a*::'*a*,*b*), (*c*,*d*)). (*a* + *c*, *b* + *d*)) *uniformity* (*uniformity* ×$_F$ *uniformity*)
unfolding *filterlim_def le_filter_def eventually_filtermap case_prod_unfold*
proof *safe*
fix *P* :: '*a* × '*a* ⇒ *bool*
assume *eventually P uniformity*
then obtain ε **where** ε: ε > 0 ⋀*x y*. *dist x y* < ε ⟹ *P* (*x*, *y*)
by (*auto simp*: *eventually_uniformity_metric*)
define *Q* **where** *Q* = (λ(*x*::'*a*,*y*). *dist x y* < ε / 2)
have *Q*: *eventually Q uniformity*
unfolding *eventually_uniformity_metric Q_def* **using** ‹ε > 0›
by (*meson case_prodI divide_pos_pos zero_less_numeral*)
have *P* (*a* + *c*, *b* + *d*) **if** *Q* (*a*, *b*) *Q* (*c*, *d*) **for** *a b c d*
proof −
have *dist* (*a* + *c*) (*b* + *d*) ≤ *dist a b* + *dist c d*
by (*simp add*: *dist_norm norm_diff_triangle_ineq*)
also have ... < ε
using *that* **by** (*auto simp*: *Q_def*)
finally show *?thesis*
by (*intro* ε)
qed
thus ∀$_F$ *x in uniformity* ×$_F$ *uniformity*. *P* (*fst* (*fst x*) + *fst* (*snd x*), *snd* (*fst x*) + *snd* (*snd x*))
unfolding *eventually_prod_filter* **by** (*intro exI*[*of* _ *Q*] *conjI Q*) *auto*
qed
next
show *filterlim* (λ((*a*::'*a*), *b*). (−*a*, −*b*)) *uniformity uniformity*
unfolding *filterlim_def le_filter_def eventually_filtermap*
proof *safe*
fix *P* :: '*a* × '*a* ⇒ *bool*
assume *eventually P uniformity*
then obtain ε **where** ε: ε > 0 ⋀*x y*. *dist x y* < ε ⟹ *P* (*x*, *y*)
by (*auto simp*: *eventually_uniformity_metric*)
show ∀$_F$ *x in uniformity*. *P* (*case x of* (*a*, *b*) ⇒ (− *a*, − *b*))
unfolding *eventually_uniformity_metric*
by (*intro exI*[*of* _ ε]) (*auto intro*!: ε *simp*: *dist_norm norm_minus_commute*)
qed
qed

instance *real* :: *uniform_topological_group_add* ..
instance *complex* :: *uniform_topological_group_add* ..

lemma *cauchy_seq_finset_iff_vanishing*:
uniformity = *filtercomap* (λ(*x*,*y*). *y* − *x* :: '*a* :: *uniform_topological_group_add*) (*nhds* 0)
proof −

have *filtercomap* (λx. (0, *case* x *of* (x, y) ⇒ y − (x :: 'a))) *uniformity* ≤ *uniformity*
apply (*simp add*: *le_filter_def eventually_filtercomap*)
using *uniformity_of_uniform_continuous_invariant*[*OF uniformly_continuous_add'*]
by (*metis diff_self eq_diff_eq*)
moreover
have *uniformity* ≤ *filtercomap* (λx. (0, *case* x *of* (x, y) ⇒ y − (x :: 'a))) *uniformity*
apply (*simp add*: *le_filter_def eventually_filtercomap*)
using *uniformity_of_uniform_continuous_invariant*[*OF uniformly_continuous_minus'*]
by (*metis* (*mono_tags*) *diff_self eventually_mono surjective_pairing*)
ultimately show *?thesis*
by (*simp add*: *nhds_eq_comap_uniformity filtercomap_filtercomap*)
qed

Metric spaces

instantiation *prod* :: (*metric_space*, *metric_space*) *uniformity_dist* **begin**
instance
proof
show ‹*uniformity* = (*INF* e∈{0 <..}. *principal* {(x::'a×'b, y). *dist* x y < e})›
proof (*subst filter_eq_iff*, *intro allI iffI*)
fix *P* :: ‹('a × 'b) × ('a × 'b) ⇒ *bool*›

have *1*: ‹∃ e∈{0<..}.
{(x,y). *dist* x y < e} ⊆ {(x,y). *dist* x y < a} ∧
{(x,y). *dist* x y < e} ⊆ {(x,y). *dist* x y < b}› **if** ‹a>0› ‹b>0› **for** *a b*
apply (*rule bexI*[*of* _ ‹*min a b*›])
using *that* **by** *auto*
have *2*: ‹*mono* (λP. *eventually* (λx. *P* (*Q* x)) *F*)› **for** *F* :: ‹'z *filter*› **and** *Q* :: ‹'z ⇒ 'y›
unfolding *mono_def* **using** *eventually_mono le_funD* **by** *fastforce*
have ‹∀*F* ((*x1*::'a,*y1*),(*x2*::'b,*y2*)) *in uniformity* ×*F* *uniformity*. *dist x1 y1* < *e*/2 ∧ *dist x2 y2* < *e*/2› **if** ‹e>0› **for** *e*
by (*auto intro*!: *eventually_prodI exI*[*of* _ ‹*e*/2›] *simp*: *case_prod_unfold eventually_uniformity_metric that*)
then have *3*: ‹∀*F* ((*x1*::'a,*y1*),(*x2*::'b,*y2*)) *in uniformity* ×*F* *uniformity*. *dist* (*x1*,*x2*) (*y1*,*y2*) < *e*› **if** ‹e>0› **for** *e*
apply (*rule eventually_rev_mp*)
by (*auto intro*!: *that eventuallyI simp*: *case_prod_unfold dist_prod_def sqrt_sum_squares_half_less*)
show ‹*eventually P* (*INF* e∈{0<..}. *principal* {(x, y). *dist* x y < e}) ⟹ *eventually P uniformity*›
apply (*subst* (*asm*) *eventually_INF_base*)
using *1 3* **apply** (*auto simp*: *uniformity_prod_def case_prod_unfold eventually_filtermap 2 eventually_principal*)
by (*smt* (*verit*, *best*) *eventually_mono*)
next
fix *P* :: ‹('a × 'b) × ('a × 'b) ⇒ *bool*›

assume ‹*eventually P uniformity*›
then obtain *P1 P2* **where** ‹*eventually P1 uniformity*› ‹*eventually P2 uniformity*›
and *P1P2P*: ‹*P1* (*x1*, *y1*) ⟹ *P2* (*x2*, *y2*) ⟹ *P* ((*x1*, *x2*), (*y1*, *y2*))› **for** *x1 y1 x2 y2*
by (*auto simp*: *eventually_filtermap case_prod_beta eventually_prod_filter uniformity_prod_def*)
from ‹*eventually P1 uniformity*› **obtain** *e1* **where** ‹*e1*>*0*› **and** *e1P1*: ‹*dist x y* < *e1* ⟹ *P1* (*x*,*y*)› **for** *x y*
using *eventually_uniformity_metric* **by** *blast*
from ‹*eventually P2 uniformity*› **obtain** *e2* **where** ‹*e2*>*0*› **and** *e2P2*: ‹*dist x y* < *e2* ⟹ *P2* (*x*,*y*)› **for** *x y*
using *eventually_uniformity_metric* **by** *blast*
define *e* **where** ‹*e* = *min e1 e2*›
have ‹*e* > *0*›
using ‹*0* < *e1*› ‹*0* < *e2*› *e_def* **by** *auto*
have ‹*dist* (*x1*,*x2*) (*y1*,*y2*) < *e* ⟹ *dist x1 y1* < *e1*› **for** *x1 y1* :: *'a* **and** *x2 y2* :: *'b*
unfolding *dist_prod_def e_def* **apply** *auto*
by (*smt* (*verit*, *best*) *real_sqrt_sum_squares_ge1*)
moreover have ‹*dist* (*x1*,*x2*) (*y1*,*y2*) < *e* ⟹ *dist x2 y2* < *e2*› **for** *x1 y1* :: *'a* **and** *x2 y2* :: *'b*
unfolding *dist_prod_def e_def* **apply** *auto*
by (*smt* (*verit*, *best*) *real_sqrt_sum_squares_ge1*)
ultimately have ∗: ‹*dist* (*x1*,*x2*) (*y1*,*y2*) < *e* ⟹ *P* ((*x1*, *x2*), (*y1*, *y2*))› **for** *x1 y1 x2 y2*
using *e1P1 e2P2 P1P2P* **by** *auto*

show ‹*eventually P* (*INF e*∈{*0*<..}. *principal* {(*x*, *y*). *dist x y* < *e*})›
apply (*rule eventually_INF1*[**where** *i*=*e*])
using ‹*e* > *0*› ∗ **by** (*auto simp*: *eventually_principal*)
qed
qed
end

declare *uniformity_Abort*[**where** *'a*='*a* :: *metric_space* × *'b* :: *metric_space*, *code*]

instantiation *prod* :: (*metric_space*, *metric_space*) *metric_space*
begin

proposition *dist_Pair_Pair*: *dist* (*a*, *b*) (*c*, *d*) = *sqrt* ((*dist a c*)² + (*dist b d*)²)
unfolding *dist_prod_def* **by** *simp*

lemma *dist_fst_le*: *dist* (*fst x*) (*fst y*) ≤ *dist x y*
unfolding *dist_prod_def* **by** (*rule real_sqrt_sum_squares_ge1*)

lemma *dist_snd_le*: *dist* (*snd x*) (*snd y*) ≤ *dist x y*
unfolding *dist_prod_def* **by** (*rule real_sqrt_sum_squares_ge2*)

```
instance
proof
  fix x y :: 'a × 'b
  show dist x y = 0 ⟷ x = y
    unfolding dist_prod_def prod_eq_iff by simp
next
  fix x y z :: 'a × 'b
  show dist x y ≤ dist x z + dist y z
    unfolding dist_prod_def
    by (intro order_trans [OF _ real_sqrt_sum_squares_triangle_ineq]
        real_sqrt_le_mono add_mono power_mono dist_triangle2 zero_le_dist)
next
  fix S :: ('a × 'b) set
  have ∗: open S ⟷ (∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S)
  proof
    assume open S show ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S
    proof
      fix x assume x ∈ S
      obtain A B where open A open B x ∈ A × B A × B ⊆ S
        using ‹open S› and ‹x ∈ S› by (rule open_prod_elim)
      obtain r where r: 0 < r ∀ y. dist y (fst x) < r ⟶ y ∈ A
        using ‹open A› and ‹x ∈ A × B› unfolding open_dist by auto
      obtain s where s: 0 < s ∀ y. dist y (snd x) < s ⟶ y ∈ B
        using ‹open B› and ‹x ∈ A × B› unfolding open_dist by auto
      let ?e = min r s
      have 0 < ?e ∧ (∀ y. dist y x < ?e ⟶ y ∈ S)
      proof (intro allI impI conjI)
        show 0 < min r s by (simp add: r(1) s(1))
      next
        fix y assume dist y x < min r s
        hence dist y x < r and dist y x < s
          by simp_all
        hence dist (fst y) (fst x) < r and dist (snd y) (snd x) < s
          by (auto intro: le_less_trans dist_fst_le dist_snd_le)
        hence fst y ∈ A and snd y ∈ B
          by (simp_all add: r(2) s(2))
        hence y ∈ A × B by (induct y, simp)
        with ‹A × B ⊆ S› show y ∈ S ..
      qed
      thus ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S ..
    qed
  next
    assume ∗: ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S show open S
    proof (rule open_prod_intro)
      fix x assume x ∈ S
      then obtain e where 0 < e and S: ∀ y. dist y x < e ⟶ y ∈ S
        using ∗ by fast
      define r where r = e / sqrt 2
```

```
    define s where s = e / sqrt 2
    from ‹0 < e› have 0 < r and 0 < s
      unfolding r_def s_def by simp_all
    from ‹0 < e› have e = sqrt (r^2 + s^2)
      unfolding r_def s_def by (simp add: power_divide)
    define A where A = {y. dist (fst x) y < r}
    define B where B = {y. dist (snd x) y < s}
    have open A and open B
      unfolding A_def B_def by (simp_all add: open_ball)
    moreover have x ∈ A × B
      unfolding A_def B_def mem_Times_iff
      using ‹0 < r› and ‹0 < s› by simp
    moreover have A × B ⊆ S
    proof (clarify)
      fix a b assume a ∈ A and b ∈ B
      hence dist a (fst x) < r and dist b (snd x) < s
        unfolding A_def B_def by (simp_all add: dist_commute)
      hence dist (a, b) x < e
        unfolding dist_prod_def ‹e = sqrt (r^2 + s^2)›
        by (simp add: add_strict_mono power_strict_mono)
      thus (a, b) ∈ S
        by (simp add: S)
    qed
    ultimately show ∃ A B. open A ∧ open B ∧ x ∈ A × B ∧ A × B ⊆ S by
fast
  qed
 qed
qed

end

declare [[code abort: dist::('a::metric_space*'b::metric_space)⇒('a*'b) ⇒ real]]

lemma Cauchy_fst: Cauchy X ⟹ Cauchy (λn. fst (X n :: 'a::metric_space ×
'b::metric_space))
 unfolding Cauchy_def by (fast elim: le_less_trans [OF dist_fst_le])

lemma Cauchy_snd: Cauchy X ⟹ Cauchy (λn. snd (X n :: 'a::metric_space ×
'b::metric_space))
 unfolding Cauchy_def by (fast elim: le_less_trans [OF dist_snd_le])

lemma Cauchy_Pair:
 assumes Cauchy X and Cauchy Y
 shows Cauchy (λn. (X n :: 'a::metric_space, Y n :: 'a::metric_space))
proof (rule metric_CauchyI)
 fix r :: real assume 0 < r
 hence 0 < r / sqrt 2 (is 0 < ?s) by simp
 obtain M where M: ∀ m≥M. ∀ n≥M. dist (X m) (X n) < ?s
   using metric_CauchyD [OF ‹Cauchy X› ‹0 < ?s›] ..
```

obtain N **where** N: $\forall\, m{\geq}N.\ \forall\, n{\geq}N.$ *dist* $(Y\ m)\ (Y\ n) <$ *?s*
using *metric_CauchyD* [*OF* ‹*Cauchy* Y› ‹$0 <$ *?s*›] **..**
have $\forall\, m{\geq}$*max* $M\ N.\ \forall\, n{\geq}$*max* $M\ N.$ *dist* $(X\ m,\ Y\ m)\ (X\ n,\ Y\ n) < r$
using $M\ N$ **by** (*simp add*: *real_sqrt_sum_squares_less dist_Pair_Pair*)
then show $\exists\, n0.\ \forall\, m{\geq}n0.\ \forall\, n{\geq}n0.$ *dist* $(X\ m,\ Y\ m)\ (X\ n,\ Y\ n) < r$ **..**
qed

Analogue to *uniformly_continuous_on_def* for two-argument functions.

lemma *uniformly_continuous_on_prod_metric*:
fixes f :: ‹$('a$::*metric_space* $\times$ $'b$::*metric_space*) $\Rightarrow$ $'c$::*metric_space*›
shows ‹*uniformly_continuous_on* $(S{\times}T)\ f \longleftrightarrow (\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x{\in}S.\ \forall\, y{\in}S.\ \forall\, x'{\in}T.\ \forall\, y'{\in}T.$ *dist* $x\ y < d \longrightarrow$ *dist* $x'\ y' < d \longrightarrow$ *dist* $(f\ (x,\ x'))\ (f\ (y,\ y')) < e)$›
proof (*unfold uniformly_continuous_on_def*, *intro iffI impI allI*)
fix e :: *real*
assume ‹$e > 0$› **and** ‹$\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x{\in}S.\ \forall\, y{\in}S.\ \forall\, x'{\in}T.\ \forall\, y'{\in}T.$ *dist* $x\ y < d \longrightarrow$ *dist* $x'\ y' < d \longrightarrow$ *dist* $(f\ (x,\ x'))\ (f\ (y,\ y')) < e$›
then obtain d **where** ‹$d > 0$›
and d: ‹$\forall\, x{\in}S.\ \forall\, y{\in}S.\ \forall\, x'{\in}T.\ \forall\, y'{\in}T.$ *dist* $x\ y < d \longrightarrow$ *dist* $x'\ y' < d \longrightarrow$ *dist* $(f\ (x,\ x'))\ (f\ (y,\ y')) < e$›
by *auto*
show ‹$\exists\, d{>}0.\ \forall\, x{\in}S{\times}T.\ \forall\, y{\in}S{\times}T.$ *dist* $y\ x < d \longrightarrow$ *dist* $(f\ y)\ (f\ x) < e$›
apply (*rule exI*[*of* _ d])
using ‹$d{>}0$› d[*rule_format*] **apply** *auto*
by (*smt* (*verit*, *del_insts*) *dist_fst_le dist_snd_le fst_conv snd_conv*)
next
fix e :: *real*
assume ‹$e > 0$› **and** ‹$\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x{\in}S{\times}T.\ \forall\, x'{\in}S{\times}T.$ *dist* $x'\ x < d \longrightarrow$ *dist* $(f\ x')\ (f\ x) < e$›
then obtain d **where** ‹$d > 0$› **and** d: ‹$\forall\, x{\in}S{\times}T.\ \forall\, x'{\in}S{\times}T.$ *dist* $x'\ x < d \longrightarrow$ *dist* $(f\ x')\ (f\ x) < e$›
by *auto*
show ‹$\exists\, d{>}0.\ \forall\, x{\in}S.\ \forall\, y{\in}S.\ \forall\, x'{\in}T.\ \forall\, y'{\in}T.$ *dist* $x\ y < d \longrightarrow$ *dist* $x'\ y' < d \longrightarrow$ *dist* $(f\ (x,\ x'))\ (f\ (y,\ y')) < e$›
proof (*intro exI conjI impI ballI*)
from ‹$d > 0$› **show** ‹$d\ /\ 2 > 0$› **by** *auto*
fix $x\ y\ x'\ y'$
assume [*simp*]: ‹$x \in S$› ‹$y \in S$› ‹$x' \in T$› ‹$y' \in T$›
assume ‹*dist* $x\ y < d\ /\ 2$› **and** ‹*dist* $x'\ y' < d\ /\ 2$›
then have ‹*dist* $(x,\ x')\ (y,\ y') < d$›
by (*simp add*: *dist_Pair_Pair sqrt_sum_squares_half_less*)
with d **show** ‹*dist* $(f\ (x,\ x'))\ (f\ (y,\ y')) < e$›
by *auto*
qed
qed

Analogue to *isUCont_def* for two-argument functions.

lemma *isUCont_prod_metric*:
fixes f :: ‹$('a$::*metric_space* $\times$ $'b$::*metric_space*) $\Rightarrow$ $'c$::*metric_space*›

shows ‹*isUCont f* ⟷ (∀ *e*>*0*. ∃ *d*>*0*. ∀ *x*. ∀ *y*. ∀ *x'*. ∀ *y'*. *dist x y* < *d* ⟶ *dist x' y'* < *d* ⟶ *dist* (*f* (*x*, *x'*)) (*f* (*y*, *y'*)) < *e*)›
using *uniformly_continuous_on_prod_metric*[*of UNIV UNIV*]
by *auto*

This logically belong with the real vector spaces but we only have the necessary lemmas now.

lemma *isUCont_plus*[*simp*]:
shows ‹*isUCont* (λ(*x*::'*a*::*real_normed_vector*,*y*). *x*+*y*)›
proof (*rule isUCont_prod_metric*[*THEN iffD2*], *intro allI impI*, *simp*)
fix *e* :: *real* **assume** ‹*0* < *e*›
show ‹∃ *d*>*0*. ∀ *x y* :: '*a*. *dist x y* < *d* ⟶ (∀ *x' y'*. *dist x' y'* < *d* ⟶ *dist* (*x* + *x'*) (*y* + *y'*) < *e*)›
apply (*rule exI*[*of* _ ‹*e*/*2*›])
using ‹*0* < *e*› **apply** *auto*
by (*smt* (*verit*, *ccfv_SIG*) *dist_add_cancel dist_add_cancel2 dist_commute dist_triangle_lt*)
qed

1.3.4 Product is a Complete Metric Space

instance *prod* :: (*complete_space*, *complete_space*) *complete_space*
proof
fix *X* :: *nat* ⇒ '*a* × '*b* **assume** *Cauchy X*
have *1*: (λ*n*. *fst* (*X n*)) ⟶ *lim* (λ*n*. *fst* (*X n*))
using *Cauchy_fst* [*OF* ‹*Cauchy X*›]
by (*simp add*: *Cauchy_convergent_iff convergent_LIMSEQ_iff*)
have *2*: (λ*n*. *snd* (*X n*)) ⟶ *lim* (λ*n*. *snd* (*X n*))
using *Cauchy_snd* [*OF* ‹*Cauchy X*›]
by (*simp add*: *Cauchy_convergent_iff convergent_LIMSEQ_iff*)
have *X* ⟶ (*lim* (λ*n*. *fst* (*X n*)), *lim* (λ*n*. *snd* (*X n*)))
using *tendsto_Pair* [*OF 1 2*] **by** *simp*
then show *convergent X*
by (*rule convergentI*)
qed

1.3.5 Product is a Normed Vector Space

instantiation *prod* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
begin

definition *norm_prod_def*[*code del*]:
norm x = *sqrt* ((*norm* (*fst x*))^2 + (*norm* (*snd x*))^2)

definition *sgn_prod_def*:
sgn (*x*::'*a* × '*b*) = *scaleR* (*inverse* (*norm x*)) *x*

proposition *norm_Pair*: *norm* (*a*, *b*) = *sqrt* ((*norm a*)^2 + (*norm b*)^2)
unfolding *norm_prod_def* **by** *simp*

instance
proof
  fix *r* :: *real* **and** *x y* :: *'a* × *'b*
  show *norm x* = *0* ⟷ *x* = *0*
    unfolding *norm_prod_def*
    by (*simp add*: *prod_eq_iff*)
  show *norm* (*x* + *y*) ≤ *norm x* + *norm y*
    unfolding *norm_prod_def*
    apply (*rule order_trans* [*OF* _ *real_sqrt_sum_squares_triangle_ineq*])
    apply (*simp add*: *add_mono power_mono norm_triangle_ineq*)
    done
  show *norm* (*scaleR r x*) = |*r*| * *norm x*
    unfolding *norm_prod_def*
    apply (*simp add*: *power_mult_distrib*)
    apply (*simp add*: *distrib_left* [*symmetric*])
    apply (*simp add*: *real_sqrt_mult*)
    done
  show *sgn x* = *scaleR* (*inverse* (*norm x*)) *x*
    by (*rule sgn_prod_def*)
  show *dist x y* = *norm* (*x* − *y*)
    unfolding *dist_prod_def norm_prod_def*
    by (*simp add*: *dist_norm*)
qed

end

declare [[*code abort*: *norm*::(*'a*::*real_normed_vector*∗*'b*::*real_normed_vector*) ⇒ *real*]]

instance *prod* :: (*banach*, *banach*) *banach* **..**

Pair operations are linear

lemma *bounded_linear_fst*: *bounded_linear fst*
  using *fst_add fst_scaleR*
  by (*rule bounded_linear_intro* [**where** *K*=*1*], *simp add*: *norm_prod_def*)

lemma *bounded_linear_snd*: *bounded_linear snd*
  using *snd_add snd_scaleR*
  by (*rule bounded_linear_intro* [**where** *K*=*1*], *simp add*: *norm_prod_def*)

lemmas *bounded_linear_fst_comp* = *bounded_linear_fst*[*THEN bounded_linear_compose*]

lemmas *bounded_linear_snd_comp* = *bounded_linear_snd*[*THEN bounded_linear_compose*]

lemma *bounded_linear_Pair*:
  assumes *f*: *bounded_linear f*
  assumes *g*: *bounded_linear g*

shows *bounded_linear* (λx. (*f x*, *g x*))
proof
interpret *f*: *bounded_linear f* **by** *fact*
interpret *g*: *bounded_linear g* **by** *fact*
fix *x y* **and** *r* :: *real*
show (*f* (*x* + *y*), *g* (*x* + *y*)) = (*f x*, *g x*) + (*f y*, *g y*)
by (*simp add*: *f.add g.add*)
show (*f* (*r* ∗*R* *x*), *g* (*r* ∗*R* *x*)) = *r* ∗*R* (*f x*, *g x*)
by (*simp add*: *f.scale g.scale*)
obtain *Kf* **where** *0* < *Kf* **and** *norm_f*: ⋀*x*. *norm* (*f x*) ≤ *norm x* ∗ *Kf*
using *f.pos_bounded* **by** *fast*
obtain *Kg* **where** *0* < *Kg* **and** *norm_g*: ⋀*x*. *norm* (*g x*) ≤ *norm x* ∗ *Kg*
using *g.pos_bounded* **by** *fast*
have ∀ *x*. *norm* (*f x*, *g x*) ≤ *norm x* ∗ (*Kf* + *Kg*)
apply (*rule allI*)
apply (*simp add*: *norm_Pair*)
apply (*rule order_trans* [*OF sqrt_add_le_add_sqrt*], *simp*, *simp*)
apply (*simp add*: *distrib_left*)
apply (*rule add_mono* [*OF norm_f norm_g*])
done
then show ∃ *K*. ∀ *x*. *norm* (*f x*, *g x*) ≤ *norm x* ∗ *K* **..**
qed

Frechet derivatives involving pairs

proposition *has_derivative_Pair* [*derivative_intros*]:
assumes *f*: (*f has_derivative f′*) (*at x within s*)
and *g*: (*g has_derivative g′*) (*at x within s*)
shows ((λx. (*f x*, *g x*)) *has_derivative* (λh. (*f′ h*, *g′ h*))) (*at x within s*)
proof (*rule has_derivativeI_sandwich*[*of 1*])
show *bounded_linear* (λh. (*f′ h*, *g′ h*))
using *f g* **by** (*intro bounded_linear_Pair has_derivative_bounded_linear*)
let *?Rf* = λy. *f y* − *f x* − *f′* (*y* − *x*)
let *?Rg* = λy. *g y* − *g x* − *g′* (*y* − *x*)
let *?R* = λy. ((*f y*, *g y*) − (*f x*, *g x*) − (*f′* (*y* − *x*), *g′* (*y* − *x*)))

show ((λy. *norm* (*?Rf y*) / *norm* (*y* − *x*) + *norm* (*?Rg y*) / *norm* (*y* − *x*))
⟶ *0*) (*at x within s*)
using *f g* **by** (*intro tendsto_add_zero*) (*auto simp*: *has_derivative_iff_norm*)

fix *y* :: *′a* **assume** *y* ≠ *x*
show *norm* (*?R y*) / *norm* (*y* − *x*) ≤ *norm* (*?Rf y*) / *norm* (*y* − *x*) + *norm*
(*?Rg y*) / *norm* (*y* − *x*)
unfolding *add_divide_distrib* [*symmetric*]
by (*simp add*: *norm_Pair divide_right_mono order_trans* [*OF sqrt_add_le_add_sqrt*])
qed *simp*

lemma *differentiable_Pair* [*simp*, *derivative_intros*]:
f differentiable at x within s ⟹ *g differentiable at x within s* ⟹

$(\lambda x.\ (f\ x,\ g\ x))$ *differentiable at x within s*
unfolding *differentiable_def* **by** (*blast intro*: *has_derivative_Pair*)

lemmas *has_derivative_fst* [*derivative_intros*] = *bounded_linear.has_derivative* [*OF bounded_linear_fst*]
lemmas *has_derivative_snd* [*derivative_intros*] = *bounded_linear.has_derivative* [*OF bounded_linear_snd*]

lemma *has_derivative_split* [*derivative_intros*]:
$((\lambda p.\ f\ (fst\ p)\ (snd\ p))$ *has_derivative* $f')\ F \Longrightarrow ((\lambda(a,\ b).\ f\ a\ b)$ *has_derivative* $f')\ F$
unfolding *split_beta'* **.**

Vector derivatives involving pairs

lemma *has_vector_derivative_Pair*[*derivative_intros*]:
assumes (f *has_vector_derivative* f') (*at x within s*)
(g *has_vector_derivative* g') (*at x within s*)
shows $((\lambda x.\ (f\ x,\ g\ x))$ *has_vector_derivative* $(f',\ g'))$ (*at x within s*)
using *assms*
by (*auto simp*: *has_vector_derivative_def intro*!: *derivative_eq_intros*)

lemma
fixes $x :: 'a$::*real_normed_vector*
shows *norm_Pair1* [*simp*]: *norm* $(0,x) =$ *norm* x
and *norm_Pair2* [*simp*]: *norm* $(x,0) =$ *norm* x
by (*auto simp*: *norm_Pair*)

lemma *norm_commute*: *norm* $(x,y) =$ *norm* (y,x)
by (*simp add*: *norm_Pair*)

lemma *norm_fst_le*: *norm* $x \le$ *norm* (x,y)
by (*metis dist_fst_le fst_conv fst_zero norm_conv_dist*)

lemma *norm_snd_le*: *norm* $y \le$ *norm* (x,y)
by (*metis dist_snd_le snd_conv snd_zero norm_conv_dist*)

lemma *norm_Pair_le*:
shows *norm* $(x,\ y) \le$ *norm* $x +$ *norm* y
unfolding *norm_Pair*
by (*metis norm_ge_zero sqrt_sum_squares_le_sum*)

lemma (**in** *vector_space_prod*) *span_Times_sing1*: *p.span* $(\{0\} \times B) = \{0\} \times$ *vs2.span* B
apply (*rule p.span_unique*)
subgoal by (*auto intro*!: *vs1.span_base vs2.span_base*)
subgoal using *vs1.subspace_single_0 vs2.subspace_span* **by** (*rule subspace_Times*)
subgoal for T
proof *safe*

fix *b*
assume *subset_T*: $\{0\} \times B \subseteq T$ **and** *subspace*: *p.subspace T* **and** *b_span*: *b* $\in$ *vs2.span B*
then obtain *t r* **where** *b*: $b = (\sum a{\in}t.\ r\ a *b\ a)$ **and** *t*: *finite t* $t \subseteq B$
by (*auto simp*: *vs2.span_explicit*)
have $(0,\ b) = (\sum b{\in}t.\ scale\ (r\ b)\ (0,\ b))$
unfolding *b scale_prod sum_prod*
by *simp*
also have $\ldots \in T$
using ‹$t \subseteq B$› *subset_T*
by (*auto intro*!: *p.subspace_sum p.subspace_scale subspace*)
finally show $(0,\ b) \in T$ **.**
qed
done

lemma (**in** *vector_space_prod*) *span_Times_sing2*: *p.span* $(A \times \{0\}) =$ *vs1.span* $A \times \{0\}$
apply (*rule p.span_unique*)
subgoal by (*auto intro*!: *vs1.span_base vs2.span_base*)
subgoal using *vs1.subspace_span vs2.subspace_single_0* **by** (*rule subspace_Times*)
subgoal for *T*
proof *safe*
fix *a*
assume *subset_T*: $A \times \{0\} \subseteq T$ **and** *subspace*: *p.subspace T* **and** *a_span*: *a* $\in$ *vs1.span A*
then obtain *t r* **where** *a*: $a = (\sum a{\in}t.\ r\ a *a\ a)$ **and** *t*: *finite t* $t \subseteq A$
by (*auto simp*: *vs1.span_explicit*)
have $(a,\ 0) = (\sum a{\in}t.\ scale\ (r\ a)\ (a,\ 0))$
unfolding *a scale_prod sum_prod*
by *simp*
also have $\ldots \in T$
using ‹$t \subseteq A$› *subset_T*
by (*auto intro*!: *p.subspace_sum p.subspace_scale subspace*)
finally show $(a,\ 0) \in T$ **.**
qed
done

1.3.6 Product is Finite Dimensional

lemma (**in** *finite_dimensional_vector_space*) *zero_not_in_Basis*[*simp*]: $0 \notin$ *Basis*
using *dependent_zero local.independent_Basis* **by** *blast*

locale *finite_dimensional_vector_space_prod* = *vector_space_prod* + *finite_dimensional_vector_space*
begin

definition *Basis_pair* = $B1 \times \{0\} \cup \{0\} \times B2$

sublocale *p*: *finite_dimensional_vector_space scale Basis_pair*

proof *unfold_locales*
show *finite Basis_pair*
by (*auto intro*!: *finite_cartesian_product vs1.finite_Basis vs2.finite_Basis simp*: *Basis_pair_def*)
show *p.independent Basis_pair*
unfolding *p.dependent_def Basis_pair_def*
proof *safe*
fix *a*
assume *a*: $a \in B1$
assume $(a, 0) \in p.span\ (B1 \times \{0\} \cup \{0\} \times B2 - \{(a, 0)\})$
also have $B1 \times \{0\} \cup \{0\} \times B2 - \{(a, 0)\} = (B1 - \{a\}) \times \{0\} \cup \{0\} \times B2$
by *auto*
finally show *False*
using *a vs1.dependent_def vs1.independent_Basis*
by (*auto simp*: *p.span_Un span_Times_sing1 span_Times_sing2*)
next
fix *b*
assume *b*: $b \in B2$
assume $(0, b) \in p.span\ (B1 \times \{0\} \cup \{0\} \times B2 - \{(0, b)\})$
also have $(B1 \times \{0\} \cup \{0\} \times B2 - \{(0, b)\}) = B1 \times \{0\} \cup \{0\} \times (B2 - \{b\})$
by *auto*
finally show *False*
using *b vs2.dependent_def vs2.independent_Basis*
by (*auto simp*: *p.span_Un span_Times_sing1 span_Times_sing2*)
qed
show $p.span\ Basis\_pair = UNIV$
by (*auto simp*: *p.span_Un span_Times_sing2 span_Times_sing1 vs1.span_Basis vs2.span_Basis*
Basis_pair_def)
qed

proposition *dim_Times*:
assumes *vs1.subspace S vs2.subspace T*
shows $p.dim(S \times T) = vs1.dim\ S + vs2.dim\ T$
proof −
interpret *p1*: *Vector_Spaces.linear s1 scale* $(\lambda x.\ (x, 0))$
by *unfold_locales* (*auto simp*: *scale_def*)
interpret *pair1*: *finite_dimensional_vector_space_pair* $(*a)$ *B1 scale Basis_pair*
by *unfold_locales*
interpret *p2*: *Vector_Spaces.linear s2 scale* $(\lambda x.\ (0, x))$
by *unfold_locales* (*auto simp*: *scale_def*)
interpret *pair2*: *finite_dimensional_vector_space_pair* $(*b)$ *B2 scale Basis_pair*
by *unfold_locales*
have *ss*: $p.subspace\ ((\lambda x.\ (x, 0))\ `\ S)\ p.subspace\ (Pair\ 0\ `\ T)$
by (*rule p1.subspace_image p2.subspace_image assms*)+
have $p.dim(S \times T) = p.dim(\{u + v \mid u\ v.\ u \in (\lambda x.\ (x, 0))\ `\ S \wedge v \in Pair\ 0\ `\ T\})$

by (*simp add*: *Times_eq_image_sum*)
moreover have *p.dim* ((λ*x*. (*x*, *0*::'*c*)) ‘ *S*) = *vs1.dim S p.dim* (*Pair* (*0*::'*b*) ‘ *T*) = *vs2.dim T*
by (*simp_all add*: *inj_on_def p1.linear_axioms pair1.dim_image_eq p2.linear_axioms pair2.dim_image_eq*)
moreover have *p.dim* ((λ*x*. (*x*, *0*)) ‘ *S* ∩ *Pair 0* ‘ *T*) = *0*
by (*subst p.dim_eq_0*) *auto*
ultimately show *?thesis*
using *p.dim_sums_Int* [*OF ss*] **by** *linarith*
qed

lemma *dimension_pair*: *p.dimension* = *vs1.dimension* + *vs2.dimension*
using *dim_Times*[*OF vs1.subspace_UNIV vs2.subspace_UNIV*]
by (*auto simp*: *p.dimension_def vs1.dimension_def vs2.dimension_def*)

end

end

1.4 Finite-Dimensional Inner Product Spaces

theory *Euclidean_Space*
imports
L2_Norm
Inner_Product
Product_Vector
begin

1.4.1 Interlude: Some properties of real sets

lemma *seq_mono_lemma*:
assumes ∀(*n*::*nat*) ≥ *m*. (*d n* :: *real*) < *e n*
and ∀ *n* ≥ *m*. *e n* ≤ *e m*
shows ∀ *n* ≥ *m*. *d n* < *e m*
using *assms* **by** *force*

1.4.2 Type class of Euclidean spaces

class *euclidean_space* = *real_inner* +
fixes *Basis* :: '*a set*
assumes *nonempty_Basis* [*simp*]: *Basis* ≠ {}
assumes *finite_Basis* [*simp*]: *finite Basis*
assumes *inner_Basis*:
⟦*u* ∈ *Basis*; *v* ∈ *Basis*⟧ ⟹ *inner u v* = (*if u* = *v then 1 else 0*)
assumes *euclidean_all_zero_iff*:
(∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ (*x* = *0*)

syntax *_type_dimension* :: *type* ⇒ *nat* (‹(‹*indent=1 notation=‹mixfix type dimension››DIM/(1'(_'))›)

syntax_consts *_type_dimension* ⇌ *card*
translations *DIM('a)* ↽ *CONST card (CONST Basis :: 'a set)*
typed_print_translation ‹
[(***const_syntax*** ‹*card*›,
*fn ctxt => fn _ => fn [Const (**const_syntax** ‹Basis›, Type (**type_name** ‹set›,*
[T]))] =>
*Syntax.const **syntax_const** ‹_type_dimension› $ Syntax_Phases.term_of_typ ctxt T)]*
›

lemma (**in** *euclidean_space*) *norm_Basis*[*simp*]: *u* ∈ *Basis* ⟹ *norm u* = *1*
unfolding *norm_eq_sqrt_inner* **by** (*simp add*: *inner_Basis*)

lemma (**in** *euclidean_space*) *inner_same_Basis*[*simp*]: *u* ∈ *Basis* ⟹ *inner u u* = *1*
by (*simp add*: *inner_Basis*)

lemma (**in** *euclidean_space*) *inner_not_same_Basis*: *u* ∈ *Basis* ⟹ *v* ∈ *Basis* ⟹ *u* ≠ *v* ⟹ *inner u v* = *0*
by (*simp add*: *inner_Basis*)

lemma (**in** *euclidean_space*) *sgn_Basis*: *u* ∈ *Basis* ⟹ *sgn u* = *u*
unfolding *sgn_div_norm* **by** (*simp add*: *scaleR_one*)

lemma *inner_sum_Basis*[*simp*]: *i* ∈ *Basis* ⟹ *inner* (∑ *Basis*) *i* = *1*
by (*simp add*: *inner_sum_left sum.If_cases inner_Basis*)

lemma (**in** *euclidean_space*) *Basis_zero* [*simp*]: *0* ∉ *Basis*
proof
assume *0* ∈ *Basis* **thus** *False*
using *inner_Basis* [*of 0 0*] **by** *simp*
qed

lemma (**in** *euclidean_space*) *nonzero_Basis*: *u* ∈ *Basis* ⟹ *u* ≠ *0*
by *clarsimp*

lemma (**in** *euclidean_space*) *SOME_Basis*: (*SOME i. i* ∈ *Basis*) ∈ *Basis*
by (*metis ex_in_conv nonempty_Basis someI_ex*)

lemma *norm_some_Basis* [*simp*]: *norm* (*SOME i. i* ∈ *Basis*) = *1*
by (*simp add*: *SOME_Basis*)

lemma (**in** *euclidean_space*) *inner_sum_left_Basis*[*simp*]:
b ∈ *Basis* ⟹ *inner* (∑ *i*∈*Basis*. *f i* ∗$_R$ *i*) *b* = *f b*
by (*simp add*: *inner_sum_left inner_Basis if_distrib comm_monoid_add_class.sum.If_cases*)

lemma (**in** *euclidean_space*) *euclidean_eqI*:
assumes *b*: ⋀*b*. *b* ∈ *Basis* ⟹ *inner x b* = *inner y b* **shows** *x* = *y*
proof −

from b **have** $\forall\, b{\in}Basis.\ inner\ (x - y)\ b = 0$
by (*simp add*: *inner_diff_left*)
then show $x = y$
by (*simp add*: *euclidean_all_zero_iff*)
qed

lemma (**in** *euclidean_space*) *euclidean_eq_iff*:
$x = y \longleftrightarrow (\forall\, b{\in}Basis.\ inner\ x\ b = inner\ y\ b)$
by (*auto intro*: *euclidean_eqI*)

lemma (**in** *euclidean_space*) *euclidean_representation_sum*:
$(\sum i{\in}Basis.\ f\ i *_R i) = b \longleftrightarrow (\forall\, i{\in}Basis.\ f\ i = inner\ b\ i)$
by (*subst euclidean_eq_iff*) *simp*

lemma (**in** *euclidean_space*) *euclidean_representation_sum'*:
$b = (\sum i{\in}Basis.\ f\ i *_R i) \longleftrightarrow (\forall\, i{\in}Basis.\ f\ i = inner\ b\ i)$
by (*auto simp add*: *euclidean_representation_sum*[*symmetric*])

lemma (**in** *euclidean_space*) *euclidean_representation*: $(\sum b{\in}Basis.\ inner\ x\ b *_R b) = x$
unfolding *euclidean_representation_sum* **by** *simp*

lemma (**in** *euclidean_space*) *euclidean_inner*: $inner\ x\ y = (\sum b{\in}Basis.\ (inner\ x\ b) * (inner\ y\ b))$
by (*subst* (*1 2*) *euclidean_representation* [*symmetric*])
(*simp add*: *inner_sum_right inner_Basis ac_simps*)

lemma (**in** *euclidean_space*) *choice_Basis_iff*:
fixes $P :: 'a \Rightarrow real \Rightarrow bool$
shows $(\forall\, i{\in}Basis.\ \exists\, x.\ P\ i\ x) \longleftrightarrow (\exists\, x.\ \forall\, i{\in}Basis.\ P\ i\ (inner\ x\ i))$
unfolding *bchoice_iff*
proof *safe*
fix f **assume** $\forall\, i{\in}Basis.\ P\ i\ (f\ i)$
then show $\exists\, x.\ \forall\, i{\in}Basis.\ P\ i\ (inner\ x\ i)$
by (*auto intro*!: *exI*[*of* _ $\sum i{\in}Basis.\ f\ i *_R i$])
qed *auto*

lemma (**in** *euclidean_space*) *bchoice_Basis_iff*:
fixes $P :: 'a \Rightarrow real \Rightarrow bool$
shows $(\forall\, i{\in}Basis.\ \exists\, x{\in}A.\ P\ i\ x) \longleftrightarrow (\exists\, x.\ \forall\, i{\in}Basis.\ inner\ x\ i \in A \wedge P\ i\ (inner\ x\ i))$
by (*simp add*: *choice_Basis_iff Bex_def*)

lemma (**in** *euclidean_space*) *euclidean_representation_sum_fun*:
$(\lambda x.\ \sum b{\in}Basis.\ inner\ (f\ x)\ b *_R b) = f$
by (*rule ext*) (*simp add*: *euclidean_representation_sum*)

lemma *euclidean_isCont*:
assumes $\bigwedge b.\ b \in Basis \Longrightarrow isCont\ (\lambda x.\ (inner\ (f\ x)\ b) *_R b)\ x$

shows *isCont f x*
apply (*subst euclidean_representation_sum_fun* [*symmetric*])
apply (*rule isCont_sum*)
apply (*blast intro*: *assms*)
done

lemma *DIM_positive* [*simp*]: *0 < DIM('a::euclidean_space)*
by (*simp add*: *card_gt_0_iff*)

lemma *DIM_ge_Suc0* [*simp*]: *Suc 0 ≤ card Basis*
by (*meson DIM_positive Suc_leI*)

lemma *sum_inner_Basis_scaleR* [*simp*]:
fixes *f* :: *'a::euclidean_space ⇒ 'b::real_vector*
assumes *b ∈ Basis* **shows** *(∑ i∈Basis. (inner i b) *R f i) = f b*
by (*simp add*: *comm_monoid_add_class.sum.remove* [*OF finite_Basis assms*]
assms inner_not_same_Basis comm_monoid_add_class.sum.neutral)

lemma *sum_inner_Basis_eq* [*simp*]:
assumes *b ∈ Basis* **shows** *(∑ i∈Basis. (inner i b) * f i) = f b*
by (*simp add*: *comm_monoid_add_class.sum.remove* [*OF finite_Basis assms*]
assms inner_not_same_Basis comm_monoid_add_class.sum.neutral)

lemma *sum_if_inner* [*simp*]:
assumes *i ∈ Basis j ∈ Basis*
shows *inner (∑ k∈Basis. if k = i then f i *R i else g k *R k) j = (if j=i then f j else g j)*
proof (*cases i=j*)
case *True*
with *assms* **show** *?thesis*
by (*auto simp*: *inner_sum_left if_distrib* [*of λx. inner x j*] *inner_Basis cong*: *if_cong*)
next
case *False*
have *(∑ k∈Basis. inner (if k = i then f i *R i else g k *R k) j) =*
(∑ k∈Basis. if k = j then g k else 0)
apply (*rule sum.cong*)
using *False assms* **by** (*auto simp*: *inner_Basis*)
also have *... = g j*
using *assms* **by** *auto*
finally show *?thesis*
using *False* **by** (*auto simp*: *inner_sum_left*)
qed

lemma *norm_le_componentwise*:
(⋀b. b ∈ Basis ⟹ abs(inner x b) ≤ abs(inner y b)) ⟹ norm x ≤ norm y
by (*auto simp*: *norm_le euclidean_inner* [*of x x*] *euclidean_inner* [*of y y*] *abs_le_square_iff power2_eq_square intro*!: *sum_mono*)

lemma *Basis_le_norm*: $b \in Basis \Longrightarrow |inner\ x\ b| \leq norm\ x$
by (*rule order_trans* [*OF Cauchy_Schwarz_ineq2*]) *simp*

lemma *norm_bound_Basis_le*: $b \in Basis \Longrightarrow norm\ x \leq e \Longrightarrow |inner\ x\ b| \leq e$
by (*metis Basis_le_norm order_trans*)

lemma *norm_bound_Basis_lt*: $b \in Basis \Longrightarrow norm\ x < e \Longrightarrow |inner\ x\ b| < e$
by (*metis Basis_le_norm le_less_trans*)

lemma *norm_le_l1*: $norm\ x \leq (\sum b{\in}Basis.\ |inner\ x\ b|)$
apply (*subst euclidean_representation*[*of x, symmetric*])
apply (*rule order_trans*[*OF norm_sum*])
apply (*auto intro*!: *sum_mono*)
done

lemma *sum_norm_allsubsets_bound*:
fixes $f :: 'a \Rightarrow 'n{::}euclidean\_space$
assumes *fP*: *finite P*
and *fPs*: $\bigwedge Q.\ Q \subseteq P \Longrightarrow norm\ (sum\ f\ Q) \leq e$
shows $(\sum x{\in}P.\ norm\ (f\ x)) \leq 2 * real\ DIM('n) * e$
proof −
have $(\sum x{\in}P.\ norm\ (f\ x)) \leq (\sum x{\in}P.\ \sum b{\in}Basis.\ |inner\ (f\ x)\ b|)$
by (*rule sum_mono*) (*rule norm_le_l1*)
also have $(\sum x{\in}P.\ \sum b{\in}Basis.\ |inner\ (f\ x)\ b|) = (\sum b{\in}Basis.\ \sum x{\in}P.\ |inner\ (f\ x)\ b|)$
by (*rule sum.swap*)
also have $\ldots \leq of\_nat\ (card\ (Basis :: 'n\ set)) * (2 * e)$
proof (*rule sum_bounded_above*)
fix $i :: 'n$
assume *i*: $i \in Basis$
have $norm\ (\sum x{\in}P.\ |inner\ (f\ x)\ i|) \leq$
$norm\ (inner\ (\sum x{\in}P \cap -\ \{x.\ inner\ (f\ x)\ i < 0\}.\ f\ x)\ i) + norm\ (inner\ (\sum x{\in}P \cap \{x.\ inner\ (f\ x)\ i < 0\}.\ f\ x)\ i)$
by (*simp add*: *abs_real_def sum.If_cases*[*OF fP*] *sum_negf norm_triangle_ineq4 inner_sum_left*
del: *real_norm_def*)
also have $\ldots \leq e + e$
unfolding *real_norm_def*
by (*intro add_mono norm_bound_Basis_le i fPs*) *auto*
finally show $(\sum x{\in}P.\ |inner\ (f\ x)\ i|) \leq 2{*}e$ **by** *simp*
qed
also have $\ldots = 2 * real\ DIM('n) * e$ **by** *simp*
finally show *?thesis* **.**
qed

1.4.3 Subclass relationships

instance *euclidean_space* ⊆ *perfect_space*

proof
fix $x :: 'a$ **show** $\neg$ *open* $\{x\}$
proof
assume *open* $\{x\}$
then obtain e **where** $0 < e$ **and** e: $\forall y.$ *dist* $y\ x < e \longrightarrow y = x$
unfolding *open_dist* **by** *fast*
define y **where** $y = x + scaleR\ (e/2)\ (SOME\ b.\ b \in Basis)$
have [*simp*]: $(SOME\ b.\ b \in Basis) \in Basis$
by (*rule someI_ex*) (*auto simp*: *ex_in_conv*)
from ‹$0 < e$› **have** $y \neq x$
unfolding *y_def* **by** (*auto intro*!: *nonzero_Basis*)
from ‹$0 < e$› **have** *dist* $y\ x < e$
unfolding *y_def* **by** (*simp add*: *dist_norm*)
from ‹$y \neq x$› **and** ‹*dist* $y\ x < e$› **show** *False*
using e **by** *simp*
qed
qed

1.4.4 Class instances

Type *real*

instantiation *real* :: *euclidean_space*
begin

definition
[*simp*]: *Basis* = $\{1::real\}$

instance
by *standard auto*

end

lemma *DIM_real*[*simp*]: *DIM*(*real*) = *1*
by *simp*

Type *complex*

instantiation *complex* :: *euclidean_space*
begin

definition *Basis_complex_def*: *Basis* = $\{1, \mathrm{i}\}$

instance
by *standard* (*auto simp add*: *Basis_complex_def intro*: *complex_eqI split*: *if_split_asm*)

end

lemma *DIM_complex*[*simp*]: *DIM*(*complex*) = *2*
unfolding *Basis_complex_def* **by** *simp*

lemma *complex_Basis_1* [*iff*]: (*1*::*complex*) ∈ *Basis*
by (*simp add*: *Basis_complex_def*)

lemma *complex_Basis_i* [*iff*]: i ∈ *Basis*
by (*simp add*: *Basis_complex_def*)

Type *'a* × *'b*

instantiation *prod* :: (*real_inner*, *real_inner*) *real_inner*
begin

definition *inner_prod_def*:
inner x y = *inner* (*fst x*) (*fst y*) + *inner* (*snd x*) (*snd y*)

lemma *inner_Pair* [*simp*]: *inner* (*a*, *b*) (*c*, *d*) = *inner a c* + *inner b d*
unfolding *inner_prod_def* **by** *simp*

instance
proof
fix *r* :: *real*
fix *x y z* :: *'a*::*real_inner* × *'b*::*real_inner*
show *inner x y* = *inner y x*
unfolding *inner_prod_def*
by (*simp add*: *inner_commute*)
show *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
unfolding *inner_prod_def*
by (*simp add*: *inner_add_left*)
show *inner* (*scaleR r x*) *y* = *r* ∗ *inner x y*
unfolding *inner_prod_def*
by (*simp add*: *distrib_left*)
show *0* ≤ *inner x x*
unfolding *inner_prod_def*
by (*intro add_nonneg_nonneg inner_ge_zero*)
show *inner x x* = *0* ⟷ *x* = *0*
unfolding *inner_prod_def prod_eq_iff*
by (*simp add*: *add_nonneg_eq_0_iff*)
show *norm x* = *sqrt* (*inner x x*)
unfolding *norm_prod_def inner_prod_def*
by (*simp add*: *power2_norm_eq_inner*)
qed

end

lemma *inner_Pair_0*: *inner x* (*0*, *b*) = *inner* (*snd x*) *b inner x* (*a*, *0*) = *inner* (*fst x*) *a*
by (*cases x*, *simp*)+

instantiation *prod* :: (*euclidean_space*, *euclidean_space*) *euclidean_space*

begin

definition
Basis = (λ*u*. (*u*, *0*)) ' *Basis* ∪ (λ*v*. (*0*, *v*)) ' *Basis*

lemma *sum_Basis_prod_eq*:
fixes *f*::('*a*∗'*b*)⇒('*a*∗'*b*)
shows *sum f Basis* = *sum* (λ*i*. *f* (*i*, *0*)) *Basis* + *sum* (λ*i*. *f* (*0*, *i*)) *Basis*
proof −
have *inj_on* (λ*u*. (*u*::'*a*, *0*::'*b*)) *Basis inj_on* (λ*u*. (*0*::'*a*, *u*::'*b*)) *Basis*
by (*auto intro*!: *inj_onI Pair_inject*)
thus *?thesis*
unfolding *Basis_prod_def*
by (*subst sum.union_disjoint*) (*auto simp*: *Basis_prod_def sum.reindex*)
qed

instance proof
show (*Basis* :: ('*a* × '*b*) *set*) ≠ {}
unfolding *Basis_prod_def* **by** *simp*
next
show *finite* (*Basis* :: ('*a* × '*b*) *set*)
unfolding *Basis_prod_def* **by** *simp*
next
fix *u v* :: '*a* × '*b*
assume *u* ∈ *Basis* **and** *v* ∈ *Basis*
thus *inner u v* = (*if u* = *v then 1 else 0*)
unfolding *Basis_prod_def inner_prod_def*
by (*auto simp add*: *inner_Basis split*: *if_split_asm*)
next
fix *x* :: '*a* × '*b*
show (∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ *x* = *0*
unfolding *Basis_prod_def ball_Un ball_simps*
by (*simp add*: *inner_prod_def prod_eq_iff euclidean_all_zero_iff*)
qed

lemma *DIM_prod*[*simp*]: *DIM*('*a* × '*b*) = *DIM*('*a*) + *DIM*('*b*)
unfolding *Basis_prod_def*
by (*subst card_Un_disjoint*) (*auto intro*!: *card_image arg_cong2*[**where** *f*=(+)] *inj_onI*)

end

1.4.5 Locale instances

lemma *finite_dimensional_vector_space_euclidean*:
finite_dimensional_vector_space (∗$_R$) *Basis*
proof *unfold_locales*
show *finite* (*Basis*::'*a set*) **by** (*metis finite_Basis*)
show *real_vector.independent* (*Basis*::'*a set*)

```
    unfolding dependent_def dependent_raw_def[symmetric]
    apply (subst span_finite)
    apply simp
    apply clarify
    apply (drule_tac f=inner a in arg_cong)
    apply (simp add: inner_Basis inner_sum_right eq_commute)
    done
  show module.span (*R) Basis = UNIV
    unfolding span_finite [OF finite_Basis] span_raw_def[symmetric]
    by (auto intro!: euclidean_representation[symmetric])
qed

interpretation eucl?: finite_dimensional_vector_space scaleR :: real => 'a =>
'a::euclidean_space Basis
  rewrites module.dependent (*R) = dependent
    and module.representation (*R) = representation
    and module.subspace (*R) = subspace
    and module.span (*R) = span
    and vector_space.extend_basis (*R) = extend_basis
    and vector_space.dim (*R) = dim
    and Vector_Spaces.linear (*R) (*R) = linear
    and Vector_Spaces.linear (*) (*R) = linear
    and finite_dimensional_vector_space.dimension Basis = DIM('a)
    and dimension = DIM('a)
  by (auto simp add: dependent_raw_def representation_raw_def
       subspace_raw_def span_raw_def extend_basis_raw_def dim_raw_def lin-
ear_def
      real_scaleR_def[abs_def]
      finite_dimensional_vector_space.dimension_def
      intro!: finite_dimensional_vector_space.dimension_def
      finite_dimensional_vector_space_euclidean)

interpretation eucl?: finite_dimensional_vector_space_pair_1
  scaleR::real⇒'a::euclidean_space⇒'a Basis
  scaleR::real⇒'b::real_vector ⇒ 'b
  by unfold_locales

interpretation eucl?: finite_dimensional_vector_space_prod scaleR scaleR Basis
Basis
  rewrites Basis_pair = Basis
    and module_prod.scale (*R) (*R) = (scaleR::_⇒_⇒('a × 'b))
proof −
  show finite_dimensional_vector_space_prod (*R) (*R) Basis Basis
    by unfold_locales
   interpret finite_dimensional_vector_space_prod (*R) (*R) Basis::'a set Ba-
sis::'b set
    by fact
  show Basis_pair = Basis
    unfolding Basis_pair_def Basis_prod_def by auto
```

show *module_prod.scale* $(*_R)$ $(*_R)$ = *scaleR*
by (*fact module_prod_scale_eq_scaleR*)
qed

end

1.5 Elementary Linear Algebra on Euclidean Spaces

theory *Linear_Algebra*
imports
Euclidean_Space
HOL−Library.Infinite_Set
begin

lemma *linear_simps*:
assumes *bounded_linear f*
shows
$f\ (a + b) = f\ a + f\ b$
$f\ (a - b) = f\ a - f\ b$
$f\ 0 = 0$
$f\ (-\ a) = -\ f\ a$
$f\ (s *_R v) = s *_R (f\ v)$
proof −
interpret *f*: *bounded_linear f* **by** *fact*
show $f\ (a + b) = f\ a + f\ b$ **by** (*rule f.add*)
show $f\ (a - b) = f\ a - f\ b$ **by** (*rule f.diff*)
show $f\ 0 = 0$ **by** (*rule f.zero*)
show $f\ (-\ a) = -\ f\ a$ **by** (*rule f.neg*)
show $f\ (s *_R v) = s *_R (f\ v)$ **by** (*rule f.scale*)
qed

lemma *finite_Atleast_Atmost_nat*[*simp*]: *finite* $\{f\ x\ |x.\ x \in (UNIV::'a::finite\ set)\}$
using *finite finite_image_set* **by** *blast*

lemma *substdbasis_expansion_unique*:
includes *inner_syntax*
assumes *d*: $d \subseteq Basis$
shows $(\sum i{\in}d.\ f\ i *_R i) = (x::'a::euclidean\_space) \longleftrightarrow$
$(\forall i{\in}Basis.\ (i \in d \longrightarrow f\ i = x \cdot i) \wedge (i \notin d \longrightarrow x \cdot i = 0))$
proof −
have ∗: $\bigwedge x\ a\ b\ P.\ x * (if\ P\ then\ a\ else\ b) = (if\ P\ then\ x * a\ else\ x * b)$
by *auto*
have ∗∗: *finite d*
by (*auto intro*: *finite_subset*[*OF assms*])
have ∗∗∗: $\bigwedge i.\ i \in Basis \Longrightarrow (\sum i{\in}d.\ f\ i *_R i) \cdot i = (\sum x{\in}d.\ if\ x = i\ then\ f\ x\ else\ 0)$
using *d*
by (*auto intro*!: *sum.cong simp*: *inner_Basis inner_sum_left*)
show *?thesis*

unfolding *euclidean_eq_iff*[**where** *'a='a*] **by** (*auto simp*: *sum.delta*[*OF* **] ***])
qed

lemma *independent_substdbasis*: $d \subseteq Basis \Longrightarrow independent\ d$
by (*rule independent_mono*[*OF independent_Basis*])

lemma *subset_translation_eq* [*simp*]:
fixes *a* :: *'a*::*real_vector* **shows** $(+)\ a\ `\ s \subseteq (+)\ a\ `\ t \longleftrightarrow s \subseteq t$
by *auto*

lemma *translate_inj_on*:
fixes *A* :: *'a*::*ab_group_add set*
shows *inj_on* $(\lambda x.\ a + x)\ A$
unfolding *inj_on_def* **by** *auto*

lemma *translation_assoc*:
fixes *a b* :: *'a*::*ab_group_add*
shows $(\lambda x.\ b + x)\ `\ ((\lambda x.\ a + x)\ `\ S) = (\lambda x.\ (a + b) + x)\ `\ S$
by *auto*

lemma *translation_invert*:
fixes *a* :: *'a*::*ab_group_add*
assumes $(\lambda x.\ a + x)\ `\ A = (\lambda x.\ a + x)\ `\ B$
shows $A = B$
using *assms translation_assoc* **by** *fastforce*

lemma *translation_galois*:
fixes *a* :: *'a*::*ab_group_add*
shows $T = ((\lambda x.\ a + x)\ `\ S) \longleftrightarrow S = ((\lambda x.\ (-\ a) + x)\ `\ T)$
by (*metis add.right_inverse group_cancel.rule0 translation_invert translation_assoc*)

lemma *translation_inverse_subset*:
assumes $((\lambda x.\ -\ a + x)\ `\ V) \leq (S :: 'n::ab\_group\_add\ set)$
shows $V \leq ((\lambda x.\ a + x)\ `\ S)$
by (*metis assms subset_image_iff translation_galois*)

1.5.1 More interesting properties of the norm

unbundle *inner_syntax*

Equality of vectors in terms of $(\cdot)$ products.

lemma *linear_componentwise*:
fixes *f*:: *'a*::*euclidean_space* $\Rightarrow$ *'b*::*real_inner*
assumes *lf*: *linear f*
shows $(f\ x) \cdot j = (\sum i{\in}Basis.\ (x{\cdot}i) * (f\ i{\cdot}j))$ (**is** *?lhs* = *?rhs*)
proof −
interpret *linear f* **by** *fact*
have *?rhs* = $(\sum i{\in}Basis.\ (x{\cdot}i) *_R (f\ i)){\cdot}j$

by (*simp add*: *inner_sum_left*)
then show *?thesis*
by (*simp add*: *euclidean_representation sum*[*symmetric*] *scale*[*symmetric*])
qed

lemma *vector_eq*: $x = y \longleftrightarrow x \cdot x = x \cdot y \wedge y \cdot y = x \cdot x$
by (*metis* (*no_types*, *opaque_lifting*) *inner_commute inner_diff_right inner_eq_zero_iff right_minus_eq*)

lemma *norm_triangle_half_r*:
$norm\ (y - x1) < e/2 \Longrightarrow norm\ (y - x2) < e/2 \Longrightarrow norm\ (x1 - x2) < e$
using *dist_triangle_half_r* **unfolding** *dist_norm*[*symmetric*] **by** *auto*

lemma *norm_triangle_half_l*:
assumes $norm\ (x - y) < e/2$ **and** $norm\ (x' - y) < e/2$
shows $norm\ (x - x') < e$
by (*metis assms dist_norm dist_triangle_half_l*)

lemma *abs_triangle_half_r*:
fixes $y :: 'a$::*linordered_field*
shows $abs\ (y - x1) < e/2 \Longrightarrow abs\ (y - x2) < e/2 \Longrightarrow abs\ (x1 - x2) < e$
by *linarith*

lemma *abs_triangle_half_l*:
fixes $y :: 'a$::*linordered_field*
assumes $abs\ (x - y) < e/2$ **and** $abs\ (x' - y) < e/2$
shows $abs\ (x - x') < e$
using *assms* **by** *linarith*

lemma *sum_clauses*:
shows $sum\ f\ \{\} = 0$
and *finite* $S \Longrightarrow sum\ f\ (insert\ x\ S) = (if\ x \in S\ then\ sum\ f\ S\ else\ f\ x + sum\ f\ S)$
by (*auto simp add*: *insert_absorb*)

lemma *vector_eq_ldot*: $(\forall x.\ x \cdot y = x \cdot z) \longleftrightarrow y = z$ **and** *vector_eq_rdot*: $(\forall z.\ x \cdot z = y \cdot z) \longleftrightarrow x = y$
by (*metis inner_commute vector_eq*)+

1.5.2 Substandard Basis

lemma *ex_card*:
assumes $n \leq card\ A$
shows $\exists S \subseteq A.\ card\ S = n$
by (*meson assms obtain_subset_with_card_n*)

lemma *subspace_substandard*: *subspace* $\{x::'a::euclidean\_space.\ (\forall i \in Basis.\ P\ i \longrightarrow x \cdot i = 0)\}$
by (*auto simp*: *subspace_def inner_add_left*)

lemma *dim_substandard*:
assumes *d*: $d \subseteq Basis$
shows *dim* $\{x::'a::euclidean\_space.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x{\cdot}i = 0\} = card\ d$ (**is** *dim ?A = _*)
proof (*rule dim_unique*)
from *d* **show** $d \subseteq ?A$
by (*auto simp*: *inner_Basis*)
from *d* **show** *independent d*
by (*rule independent_mono* [*OF independent_Basis*])
have $x \in span\ d$ **if** $\forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0$ **for** *x*
proof −
have *finite d*
by (*rule finite_subset* [*OF d finite_Basis*])
then have $(\sum i{\in}d.\ (x \cdot i) *_R i) \in span\ d$
by (*simp add*: *span_sum span_clauses*)
also have $(\sum i{\in}d.\ (x \cdot i) *_R i) = (\sum i{\in}Basis.\ (x \cdot i) *_R i)$
by (*rule sum.mono_neutral_cong_left* [*OF finite_Basis d*]) (*auto simp*: *that*)
finally show $x \in span\ d$
by (*simp only*: *euclidean_representation*)
qed
then show $?A \subseteq span\ d$ **by** *auto*
qed *simp*

1.5.3 Orthogonality

definition (**in** *real_inner*) *orthogonal x y* $\longleftrightarrow x \cdot y = 0$

context *real_inner*
begin

lemma *orthogonal_self*: *orthogonal x x* $\longleftrightarrow x = 0$
by (*simp add*: *orthogonal_def*)

lemma *orthogonal_clauses*:
orthogonal a 0
orthogonal a x $\Longrightarrow$ *orthogonal a* $(c *_R x)$
orthogonal a x $\Longrightarrow$ *orthogonal a* $(-x)$
orthogonal a x $\Longrightarrow$ *orthogonal a y* $\Longrightarrow$ *orthogonal a* $(x + y)$
orthogonal a x $\Longrightarrow$ *orthogonal a y* $\Longrightarrow$ *orthogonal a* $(x - y)$
orthogonal 0 a
orthogonal x a $\Longrightarrow$ *orthogonal* $(c *_R x)$ *a*
orthogonal x a $\Longrightarrow$ *orthogonal* $(-x)$ *a*
orthogonal x a $\Longrightarrow$ *orthogonal y a* $\Longrightarrow$ *orthogonal* $(x + y)$ *a*
orthogonal x a $\Longrightarrow$ *orthogonal y a* $\Longrightarrow$ *orthogonal* $(x - y)$ *a*
unfolding *orthogonal_def inner_add inner_diff* **by** *auto*

end

lemma *orthogonal_commute*: *orthogonal x y* $\longleftrightarrow$ *orthogonal y x*
by (*simp add*: *orthogonal_def inner_commute*)

lemma *orthogonal_scaleR* [*simp*]: $c \neq 0 \Longrightarrow$ *orthogonal* $(c *_R x)$ = *orthogonal x*
by (*rule ext*) (*simp add*: *orthogonal_def*)

lemma *pairwise_ortho_scaleR*:
pairwise $(\lambda i\ j.$ *orthogonal* $(f\ i)\ (g\ j))\ B$
$\Longrightarrow$ *pairwise* $(\lambda i\ j.$ *orthogonal* $(a\ i *_R f\ i)\ (a\ j *_R g\ j))\ B$
by (*auto simp*: *pairwise_def orthogonal_clauses*)

lemma *orthogonal_rvsum*:
$[\![$*finite s*; $\bigwedge y.\ y \in s \Longrightarrow$ *orthogonal x* $(f\ y)]\!] \Longrightarrow$ *orthogonal x* $(sum\ f\ s)$
by (*induction s rule*: *finite_induct*) (*auto simp*: *orthogonal_clauses*)

lemma *orthogonal_lvsum*:
$[\![$*finite s*; $\bigwedge x.\ x \in s \Longrightarrow$ *orthogonal* $(f\ x)\ y]\!] \Longrightarrow$ *orthogonal* $(sum\ f\ s)\ y$
by (*induction s rule*: *finite_induct*) (*auto simp*: *orthogonal_clauses*)

lemma *norm_add_Pythagorean*:
assumes *orthogonal a b*
shows $(norm\ (a + b))^2 = (norm\ a)^2 + (norm\ b)^2$
proof −
from *assms* **have** $(a - (0 - b)) \cdot (a - (0 - b)) = a \cdot a - (0 - b \cdot b)$
by (*simp add*: *algebra_simps orthogonal_def inner_commute*)
then show *?thesis*
by (*simp add*: *power2_norm_eq_inner*)
qed

lemma *norm_sum_Pythagorean*:
assumes *finite I pairwise* $(\lambda i\ j.$ *orthogonal* $(f\ i)\ (f\ j))\ I$
shows $(norm\ (sum\ f\ I))^2 = (\sum i \in I.\ (norm\ (f\ i))^2)$
using *assms*
proof (*induction I rule*: *finite_induct*)
case *empty* **then show** *?case* **by** *simp*
next
case (*insert x I*)
then have *orthogonal* $(f\ x)\ (sum\ f\ I)$
by (*metis pairwise_insert orthogonal_rvsum*)
with *insert* **show** *?case*
by (*simp add*: *pairwise_insert norm_add_Pythagorean*)
qed

1.5.4 Orthogonality of a transformation

definition *orthogonal_transformation f* $\longleftrightarrow$ *linear f* $\wedge$ $(\forall v\ w.\ f\ v \cdot f\ w = v \cdot w)$

lemma *orthogonal_transformation*:
orthogonal_transformation f $\longleftrightarrow$ *linear f* $\wedge$ $(\forall v.\ norm\ (f\ v) = norm\ v)$

by (*smt* (*verit*, *ccfv_threshold*) *dot_norm linear_add norm_eq_sqrt_inner orthogonal_transformation_def*)

lemma *orthogonal_transformation_id* [*simp*]: *orthogonal_transformation* (λ*x*. *x*)
by (*simp add*: *linear_iff orthogonal_transformation_def*)

lemma *orthogonal_orthogonal_transformation*:
orthogonal_transformation f ⟹ *orthogonal* (*f x*) (*f y*) ⟷ *orthogonal x y*
by (*simp add*: *orthogonal_def orthogonal_transformation_def*)

lemma *orthogonal_transformation_compose*:
⟦*orthogonal_transformation f*; *orthogonal_transformation g*⟧ ⟹ *orthogonal_transformation*(*f* ∘ *g*)
by (*auto simp*: *orthogonal_transformation_def linear_compose*)

lemma *orthogonal_transformation_neg*:
orthogonal_transformation(λ*x*. −(*f x*)) ⟷ *orthogonal_transformation f*
by (*auto simp*: *orthogonal_transformation_def dest*: *linear_compose_neg*)

lemma *orthogonal_transformation_scaleR*: *orthogonal_transformation f* ⟹ *f* (*c* *$_R$ *v*) = *c* *$_R$ *f v*
by (*simp add*: *linear_iff orthogonal_transformation_def*)

lemma *orthogonal_transformation_linear*:
orthogonal_transformation f ⟹ *linear f*
by (*simp add*: *orthogonal_transformation_def*)

lemma *orthogonal_transformation_inj*:
orthogonal_transformation f ⟹ *inj f*
unfolding *orthogonal_transformation_def inj_on_def*
by (*metis vector_eq*)

lemma *orthogonal_transformation_surj*:
orthogonal_transformation f ⟹ *surj f*
for *f* :: *'a*::*euclidean_space* ⇒ *'a*::*euclidean_space*
by (*simp add*: *linear_injective_imp_surjective orthogonal_transformation_inj orthogonal_transformation_linear*)

lemma *orthogonal_transformation_bij*:
orthogonal_transformation f ⟹ *bij f*
for *f* :: *'a*::*euclidean_space* ⇒ *'a*::*euclidean_space*
by (*simp add*: *bij_def orthogonal_transformation_inj orthogonal_transformation_surj*)

lemma *orthogonal_transformation_inv*:
orthogonal_transformation f ⟹ *orthogonal_transformation* (*inv f*)
for *f* :: *'a*::*euclidean_space* ⇒ *'a*::*euclidean_space*
by (*metis* (*no_types*, *opaque_lifting*) *bijection.inv_right bijection_def inj_linear_imp_inv_linear orthogonal_transformation orthogonal_transformation_bij orthogonal_transformation_inj*)

lemma *orthogonal_transformation_norm*:
orthogonal_transformation f $\Longrightarrow$ *norm* (*f x*) = *norm x*
by (*metis orthogonal_transformation*)

1.5.5 Bilinear functions

definition
bilinear :: ('*a*::*real_vector* $\Rightarrow$ '*b*::*real_vector* $\Rightarrow$ '*c*::*real_vector*) $\Rightarrow$ *bool* **where**
bilinear f $\longleftrightarrow$ ($\forall$ *x*. *linear* (λ*y*. *f x y*)) $\wedge$ ($\forall$ *y*. *linear* (λ*x*. *f x y*))

lemma *bilinear_ladd*: *bilinear h* $\Longrightarrow$ *h* (*x* + *y*) *z* = *h x z* + *h y z*
by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_radd*: *bilinear h* $\Longrightarrow$ *h x* (*y* + *z*) = *h x y* + *h x z*
by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_times*:
fixes *c*::'*a*::*real_algebra* **shows** *bilinear* (λ*x y*::'*a*. *x*∗*y*)
by (*auto simp*: *bilinear_def distrib_left distrib_right intro*!: *linearI*)

lemma *bilinear_lmul*: *bilinear h* $\Longrightarrow$ *h* (*c* $*_R$ *x*) *y* = *c* $*_R$ *h x y*
by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_rmul*: *bilinear h* $\Longrightarrow$ *h x* (*c* $*_R$ *y*) = *c* $*_R$ *h x y*
by (*simp add*: *bilinear_def linear_iff*)

lemma *bilinear_lneg*: *bilinear h* $\Longrightarrow$ *h* (− *x*) *y* = − *h x y*
by (*drule bilinear_lmul* [*of* _ − *1*]) *simp*

lemma *bilinear_rneg*: *bilinear h* $\Longrightarrow$ *h x* (− *y*) = − *h x y*
by (*drule bilinear_rmul* [*of* _ _ − *1*]) *simp*

lemma (**in** *ab_group_add*) *eq_add_iff*: *x* = *x* + *y* $\longleftrightarrow$ *y* = *0*
using *add_left_imp_eq*[*of x y 0*] **by** *auto*

lemma *bilinear_lzero*:
assumes *bilinear h*
shows *h 0 x* = *0*
using *bilinear_ladd* [*OF assms*, *of 0 0 x*] **by** (*simp add*: *eq_add_iff field_simps*)

lemma *bilinear_rzero*:
assumes *bilinear h*
shows *h x 0* = *0*
using *bilinear_radd* [*OF assms*, *of x 0 0*] **by** (*simp add*: *eq_add_iff field_simps*)

lemma *bilinear_lsub*: *bilinear h* $\Longrightarrow$ *h* (*x* − *y*) *z* = *h x z* − *h y z*
using *bilinear_ladd* [*of h x* − *y*] **by** (*simp add*: *bilinear_lneg*)

lemma *bilinear_rsub*: *bilinear h* $\Longrightarrow$ *h z* (*x* − *y*) = *h z x* − *h z y*

using *bilinear_radd* [*of* h _ $x - y$] **by** (*simp add*: *bilinear_rneg*)

lemma *bilinear_sum*:
assumes *bilinear* h
shows h (*sum* f S) (*sum* g T) = *sum* ($\lambda(i,j)$. h (f i) (g j)) ($S \times T$)
proof −
interpret l: *linear* λx. h x y **for** y **using** *assms* **by** (*simp add*: *bilinear_def*)
interpret r: *linear* λy. h x y **for** x **using** *assms* **by** (*simp add*: *bilinear_def*)
have h (*sum* f S) (*sum* g T) = *sum* (λx. h (f x) (*sum* g T)) S
by (*simp add*: *l.sum*)
also have ... = *sum* (λx. *sum* (λy. h (f x) (g y)) T) S
by (*rule sum.cong*) (*simp_all add*: *r.sum*)
finally show *?thesis*
unfolding *sum.cartesian_product* .
qed

1.5.6 Adjoints

definition *adjoint* :: (('*a*::*real_inner*) ⇒ ('*b*::*real_inner*)) ⇒ '*b* ⇒ '*a* **where**
adjoint f = (*SOME* f'. $\forall x\ y$. $f\ x \cdot y = x \cdot f'\ y$)

lemma *adjoint_unique*:
assumes $\forall x\ y$. *inner* (f x) y = *inner* x (g y)
shows *adjoint* f = g
unfolding *adjoint_def*
proof (*rule some_equality*)
show $\forall x\ y$. *inner* (f x) y = *inner* x (g y)
by (*rule assms*)
next
fix h
assume $\forall x\ y$. *inner* (f x) y = *inner* x (h y)
then show h = g
by (*metis assms ext vector_eq_ldot*)
qed

TODO: The following lemmas about adjoints should hold for any Hilbert space (i.e. complete inner product space). (see https://en.wikipedia.org/wiki/Hermitian_adjoint)

lemma *adjoint_works*:
fixes f :: '*n*::*euclidean_space* ⇒ '*m*::*euclidean_space*
assumes *lf*: *linear* f
shows $x \cdot$ *adjoint* f y = f $x \cdot y$
proof −
interpret *linear* f **by** *fact*
have $\forall y$. $\exists w$. $\forall x$. $f\ x \cdot y = x \cdot w$
proof (*intro allI exI*)
fix y :: '*m* **and** x
let *?w* = ($\sum$ *i*∈*Basis*. (f $i \cdot y$) $*_R$ i) :: '*n*
have $f\ x \cdot y = f$ ($\sum$ *i*∈*Basis*. ($x \cdot i$) $*_R$ i) $\cdot\ y$

by (*simp add*: *euclidean_representation*)
also have ... = ($\sum i \in Basis$. $(x \cdot i) *_R f\ i) \cdot y$
by (*simp add*: *sum scale*)
finally show $f\ x \cdot y = x \cdot ?w$
by (*simp add*: *inner_sum_left inner_sum_right mult.commute*)
qed
then show *?thesis*
unfolding *adjoint_def choice_iff*
by (*intro someI2_ex*[**where** $Q=\lambda f'.\ x \cdot f'\ y = f\ x \cdot y$]) *auto*
qed

lemma *adjoint_clauses*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
assumes *lf*: *linear f*
shows $x \cdot adjoint\ f\ y = f\ x \cdot y$
and $adjoint\ f\ y \cdot x = y \cdot f\ x$
by (*simp_all add*: *adjoint_works*[*OF lf*] *inner_commute*)

lemma *adjoint_linear*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
assumes *lf*: *linear f*
shows *linear* (*adjoint f*)
by (*simp add*: *lf linear_iff euclidean_eq_iff*[**where** $'a='n$] *euclidean_eq_iff*[**where** $'a='m$]
adjoint_clauses[*OF lf*] *inner_distrib*)

lemma *adjoint_adjoint*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
assumes *lf*: *linear f*
shows *adjoint* (*adjoint f*) = *f*
by (*rule adjoint_unique*, *simp add*: *adjoint_clauses* [*OF lf*])

1.5.7 Euclidean Spaces as Typeclass

lemma *independent_Basis*: *independent Basis*
by (*rule independent_Basis*)

lemma *span_Basis* [*simp*]: *span Basis* = *UNIV*
by (*rule span_Basis*)

lemma *in_span_Basis*: $x \in span\ Basis$
unfolding *span_Basis* **..**

1.5.8 Linearity and Bilinearity continued

lemma *linear_bounded*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::real\_normed\_vector$
assumes *lf*: *linear f*
shows $\exists B.\ \forall x.\ norm\ (f\ x) \leq B * norm\ x$
proof

interpret *linear f* **by** *fact*
let *?B* = $\sum$ *b*∈*Basis*. *norm* (*f b*)
show ∀ *x*. *norm* (*f x*) ≤ *?B* ∗ *norm x*
proof
fix *x* :: *'a*
let *?g* = λ*b*. (*x* · *b*) ∗$_R$ *f b*
have *norm* (*f x*) = *norm* (*f* ($\sum$ *b*∈*Basis*. (*x* · *b*) ∗$_R$ *b*))
unfolding *euclidean_representation* **..**
also have ... = *norm* (*sum ?g Basis*)
by (*simp add*: *sum scale*)
finally have *th0*: *norm* (*f x*) = *norm* (*sum ?g Basis*) **.**
have *th*: *norm* (*?g i*) ≤ *norm* (*f i*) ∗ *norm x* **if** *i* ∈ *Basis* **for** *i*
proof −
from *Basis_le_norm*[*OF that*, *of x*]
show *norm* (*?g i*) ≤ *norm* (*f i*) ∗ *norm x*
unfolding *norm_scaleR* **by** (*metis mult.commute mult_left_mono norm_ge_zero*)
qed
from *sum_norm_le*[*of* _ *?g*, *OF th*]
show *norm* (*f x*) ≤ *?B* ∗ *norm x*
by (*simp add*: *sum_distrib_right th0*)
qed
qed

lemma *linear_conv_bounded_linear*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
shows *linear f* ⟷ *bounded_linear f*
by (*metis mult.commute bounded_linear_axioms.intro bounded_linear_def linear_bounded*)

lemmas *linear_linear* = *linear_conv_bounded_linear*[*symmetric*]

lemma *inj_linear_imp_inv_bounded_linear*:
fixes *f*::*'a*::*euclidean_space* ⇒ *'a*
shows ⟦*bounded_linear f*; *inj f*⟧ ⟹ *bounded_linear* (*inv f*)
by (*simp add*: *inj_linear_imp_inv_linear linear_linear*)

lemma *linear_bounded_pos*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes *lf*: *linear f*
obtains *B* **where** *B* > *0* ⋀*x*. *norm* (*f x*) ≤ *B* ∗ *norm x*
by (*metis bounded_linear.pos_bounded lf linear_linear mult.commute*)

lemma *linear_invertible_bounded_below_pos*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *'b*::*euclidean_space*
assumes *linear f linear g* **and** *gf*: *g* ∘ *f* = *id*
obtains *B* **where** *B* > *0* ⋀*x*. *B* ∗ *norm x* ≤ *norm*(*f x*)
proof −
obtain *B* **where** *B* > *0* **and** *B*: ⋀*x*. *norm* (*g x*) ≤ *B* ∗ *norm x*
using *linear_bounded_pos* [*OF* ‹*linear g*›] **by** *blast*

```
  show thesis
  proof
    show 0 < 1/B
      by (simp add: ‹B > 0›)
    show 1/B * norm x ≤ norm (f x) for x
      by (smt (verit, ccfv_SIG) B ‹0 < B› gf comp_apply divide_inverse id_apply
inverse_eq_divide
            less_divide_eq mult.commute)
  qed
qed

lemma linear_inj_bounded_below_pos:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
  assumes linear f inj f
  obtains B where B > 0 ⋀x. B * norm x ≤ norm(f x)
  using linear_injective_left_inverse [OF assms]
    linear_invertible_bounded_below_pos assms by blast

lemma bounded_linearI':
  fixes f ::'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes ⋀x y. f (x + y) = f x + f y
    and ⋀c x. f (c *R x) = c *R f x
  shows bounded_linear f
  using assms linearI linear_conv_bounded_linear by blast

lemma bilinear_bounded:
  fixes h :: 'm::euclidean_space ⇒ 'n::euclidean_space ⇒ 'k::real_normed_vector
  assumes bh: bilinear h
  shows ∃ B. ∀ x y. norm (h x y) ≤ B * norm x * norm y
proof (clarify intro!: exI[of _ ∑ i∈Basis. ∑ j∈Basis. norm (h i j)])
  fix x :: 'm
  fix y :: 'n
  have norm (h x y) = norm (h (sum (λi. (x • i) *R i) Basis) (sum (λi. (y • i)
*R i) Basis))
    by (simp add: euclidean_representation)
  also have ... = norm (sum (λ (i,j). h ((x • i) *R i) ((y • j) *R j)) (Basis ×
Basis))
    unfolding bilinear_sum[OF bh] ..
  finally have th: norm (h x y) = ... .
  have ⋀i j. ⟦i ∈ Basis; j ∈ Basis⟧
        ⟹ |x • i| * (|y • j| * norm (h i j)) ≤ norm x * (norm y * norm (h i j))
    by (auto simp add: zero_le_mult_iff Basis_le_norm mult_mono)
  then show norm (h x y) ≤ (∑ i∈Basis. ∑ j∈Basis. norm (h i j)) * norm x *
norm y
    unfolding sum_distrib_right th sum.cartesian_product
    by (clarsimp simp add: bilinear_rmul[OF bh] bilinear_lmul[OF bh]
      field_simps simp del: scaleR_scaleR intro!: sum_norm_le)
qed
```

lemma *bilinear_conv_bounded_bilinear*:
fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*real_normed_vector*
shows *bilinear h* ⟷ *bounded_bilinear h*
proof
assume *bilinear h*
show *bounded_bilinear h*
proof
fix *x y z*
show *h* (*x* + *y*) *z* = *h x z* + *h y z*
using ‹*bilinear h*› **unfolding** *bilinear_def linear_iff* **by** *simp*
next
fix *x y z*
show *h x* (*y* + *z*) = *h x y* + *h x z*
using ‹*bilinear h*› **unfolding** *bilinear_def linear_iff* **by** *simp*
next
show *h* (*scaleR r x*) *y* = *scaleR r* (*h x y*) *h x* (*scaleR r y*) = *scaleR r* (*h x y*)
for *r x y*
using ‹*bilinear h*› **unfolding** *bilinear_def linear_iff*
by *simp_all*
next
have ∃ *B*. ∀ *x y*. *norm* (*h x y*) ≤ *B* ∗ *norm x* ∗ *norm y*
using ‹*bilinear h*› **by** (*rule bilinear_bounded*)
then show ∃ *K*. ∀ *x y*. *norm* (*h x y*) ≤ *norm x* ∗ *norm y* ∗ *K*
by (*simp add*: *ac_simps*)
qed
next
assume *bounded_bilinear h*
then interpret *h*: *bounded_bilinear h* **.**
show *bilinear h*
unfolding *bilinear_def linear_conv_bounded_linear*
using *h.bounded_linear_left h.bounded_linear_right* **by** *simp*
qed

lemma *bilinear_bounded_pos*:
fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*real_normed_vector*
assumes *bh*: *bilinear h*
shows ∃ *B* > *0*. ∀ *x y*. *norm* (*h x y*) ≤ *B* ∗ *norm x* ∗ *norm y*
by (*metis mult.assoc bh bilinear_conv_bounded_bilinear bounded_bilinear.pos_bounded mult.commute*)

lemma *bounded_linear_imp_has_derivative*:
bounded_linear f ⟹ (*f has_derivative f*) *net*
by (*auto simp add*: *has_derivative_def linear_diff linear_linear linear_def dest*: *bounded_linear.linear*)

lemma *linear_imp_has_derivative*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
shows *linear f* ⟹ (*f has_derivative f*) *net*
by (*simp add*: *bounded_linear_imp_has_derivative linear_conv_bounded_linear*)

lemma *bounded_linear_imp_differentiable*: *bounded_linear f* $\Longrightarrow$ *f differentiable net*
using *bounded_linear_imp_has_derivative differentiable_def* **by** *blast*

lemma *linear_imp_differentiable*:
fixes *f* :: *'a*::*euclidean_space* $\Rightarrow$ *'b*::*real_normed_vector*
shows *linear f* $\Longrightarrow$ *f differentiable net*
by (*metis linear_imp_has_derivative differentiable_def*)

lemma *of_real_differentiable* [*simp*,*derivative_intros*]: *of_real differentiable F*
by (*simp add*: *bounded_linear_imp_differentiable bounded_linear_of_real*)

1.5.9 We continue

lemma *independent_bound*:
fixes *S* :: *'a*::*euclidean_space set*
shows *independent S* $\Longrightarrow$ *finite S* $\wedge$ *card S* $\leq$ *DIM*(*'a*)
by (*metis dim_subset_UNIV finiteI_independent dim_span_eq_card_independent*)

lemmas *independent_imp_finite* = *finiteI_independent*

corollary *independent_card_le*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *independent S*
shows *card S* $\leq$ *DIM*(*'a*)
using *assms independent_bound* **by** *auto*

lemma *dependent_biggerset*:
fixes *S* :: *'a*::*euclidean_space set*
shows (*finite S* $\Longrightarrow$ *card S* $>$ *DIM*(*'a*)) $\Longrightarrow$ *dependent S*
by (*metis independent_bound not_less*)

Picking an orthogonal replacement for a spanning set.

lemma *vector_sub_project_orthogonal*:
fixes *b x* :: *'a*::*euclidean_space*
shows *b* $\cdot$ (*x* $-$ ((*b* $\cdot$ *x*) / (*b* $\cdot$ *b*)) $*_R$ *b*) = *0*
unfolding *inner_simps* **by** *auto*

lemma *pairwise_orthogonal_insert*:
assumes *pairwise orthogonal S*
and $\bigwedge y$. *y* $\in$ *S* $\Longrightarrow$ *orthogonal x y*
shows *pairwise orthogonal* (*insert x S*)
using *assms* **by** (*auto simp*: *pairwise_def orthogonal_commute*)

lemma *basis_orthogonal*:
fixes *B* :: *'a*::*real_inner set*
assumes *fB*: *finite B*
shows $\exists$ *C*. *finite C* $\wedge$ *card C* $\leq$ *card B* $\wedge$ *span C* = *span B* $\wedge$ *pairwise orthogonal*

```
C
  (is  ∃ C. ?P B C)
  using fB
proof (induct rule: finite_induct)
  case empty
  then show ?case
    using pairwise_empty by blast
next
  case (insert a B)
  note fB = ‹finite B› and aB = ‹a ∉ B›
  from ‹∃ C. finite C ∧ card C ≤ card B ∧ span C = span B ∧ pairwise orthogonal
C›
  obtain C where C: finite C card C ≤ card B
    span C = span B pairwise orthogonal C by blast
  let ?a = a − sum (λx. (x · a / (x · x)) *R x) C
  let ?C = insert ?a C
  from C(1) have fC: finite ?C
    by simp
  have cC: card ?C ≤ card (insert a B)
    using C aB card_insert_if local.insert(1) by fastforce
  {
    fix x k
    have th0: ⋀(a::'a) b c. a − (b − c) = c + (a − b)
      by (simp add: field_simps)
    have x − k *R (a − (∑ x∈C. (x · a / (x · x)) *R x)) ∈ span C ⟷ x − k *R
a ∈ span C
      unfolding scaleR_right_diff_distrib th0
      by (intro span_add_eq span_scale span_sum span_base)
  }
  then have SC: span ?C = span (insert a B)
    unfolding set_eq_iff span_breakdown_eq C(3)[symmetric] by auto
  {
    fix y
    assume yC: y ∈ C
    then have Cy: C = insert y (C − {y})
      by blast
    have fth: finite (C − {y})
      using C by simp
    have y ≠ 0 ⟹ ∀ x∈C − {y}. x · a ∗ (x · y) / (x · x) = 0
      using ‹pairwise orthogonal C›
        by (metis Cy DiffE div_0 insertCI mult_zero_right orthogonal_def pair-
wise_insert)
    then have orthogonal ?a y
      unfolding orthogonal_def
      unfolding inner_diff inner_sum_left right_minus_eq
      unfolding sum.remove [OF ‹finite C› ‹y ∈ C›]
      by (auto simp add: sum.neutral inner_commute[of y a])
  }
  with ‹pairwise orthogonal C› have CPO: pairwise orthogonal ?C
```

```
    by (rule pairwise_orthogonal_insert)
  from fC cC SC CPO have ?P (insert a B) ?C
    by blast
  then show ?case by blast
qed

lemma orthogonal_basis_exists:
  fixes V :: ('a::euclidean_space) set
  shows ∃ B. independent B ∧ B ⊆ span V ∧ V ⊆ span B ∧ (card B = dim V)
∧ pairwise orthogonal B
proof −
  from basis_exists[of V] obtain B where
    B: B ⊆ V independent B V ⊆ span B card B = dim V
    by force
  from B have fB: finite B card B = dim V
    using independent_bound by auto
  from basis_orthogonal[OF fB(1)] obtain C where
    C: finite C card C ≤ card B span C = span B pairwise orthogonal C
    by blast
  from C B have CSV: C ⊆ span V
    by (metis span_superset span_mono subset_trans)
  from span_mono[OF B(3)] C have SVC: span V ⊆ span C
    by (simp add: span_span)
  from C fB have card C ≤ dim V
    by simp
  moreover have dim V ≤ card C
    using span_card_ge_dim[OF CSV SVC C(1)]
    by simp
  ultimately have card C = dim V
    using C(1) by simp
  with C B CSV show ?thesis
    by (metis SVC card_eq_dim dim_span)
qed
```

Low-dimensional subset is in a hyperplane (weak orthogonal complement).

```
lemma span_not_UNIV_orthogonal:
  fixes S :: 'a::euclidean_space set
  assumes sU: span S ≠ UNIV
  shows ∃ a::'a. a ≠ 0 ∧ (∀ x ∈ span S. a · x = 0)
proof −
  from sU obtain a where a: a ∉ span S
    by blast
  from orthogonal_basis_exists obtain B where
    B: independent B B ⊆ span S S ⊆ span B card B = dim S pairwise orthogonal
B
    by blast
  from B have fB: finite B card B = dim S
    using independent_bound by auto
  have sSB: span S = span B
```

```
    by (simp add: B span_eq)
  let ?a = a − sum (λb. (a · b / (b · b)) *R b) B
  have sum (λb. (a · b / (b · b)) *R b) B ∈ span S
    by (simp add: sSB span_base span_mul span_sum)
  with a have a0:?a ≠ 0
    by auto
  have ?a · x = 0 if x∈span B for x
  proof (rule span_induct [OF that])
    show subspace {x. ?a · x = 0}
      by (auto simp add: subspace_def inner_add)
  next
    {
      fix x
      assume x: x ∈ B
      from x have B': B = insert x (B − {x})
        by blast
      have fth: finite (B − {x})
        using fB by simp
      have (∑ b∈B − {x}. a · b * (b · x) / (b · b)) = 0 if x ≠ 0
      by (smt (verit) B' B(5) DiffD2 divide_eq_0_iff inner_real_def inner_zero_right
insertCI orthogonal_def pairwise_insert sum.neutral)
      then have ?a · x = 0
        apply (subst B')
        using fB fth
        unfolding sum_clauses(2)[OF fth]
        by (auto simp add: inner_add_left inner_diff_left inner_sum_left)
    }
    then show ?a · x = 0 if x ∈ B for x
      using that by blast
    qed
  with a0 sSB show ?thesis
    by blast
qed

lemma span_not_univ_subset_hyperplane:
  fixes S :: 'a::euclidean_space set
  assumes SU: span S ≠ UNIV
  shows ∃ a. a ≠0 ∧ span S ⊆ {x. a · x = 0}
  using span_not_UNIV_orthogonal[OF SU] by auto

lemma lowdim_subset_hyperplane:
  fixes S :: 'a::euclidean_space set
  assumes d: dim S < DIM('a)
  shows ∃ a::'a. a ≠ 0 ∧ span S ⊆ {x. a · x = 0}
  using d dim_eq_full nless_le span_not_univ_subset_hyperplane by blast

lemma linear_eq_stdbasis:
  fixes f :: 'a::euclidean_space ⇒ _
  assumes lf: linear f
```

and *lg*: *linear g*
and *fg*: $\bigwedge b.\ b \in Basis \Longrightarrow f\ b = g\ b$
shows $f = g$
using *linear_eq_on_span*[*OF lf lg*, *of Basis*] *fg* **by** *auto*

Similar results for bilinear functions.

lemma *bilinear_eq*:
assumes *bf*: *bilinear f*
and *bg*: *bilinear g*
and *SB*: $S \subseteq span\ B$
and *TC*: $T \subseteq span\ C$
and $x{\in}S\ y{\in}T$
and *fg*: $\bigwedge x\ y.\ [\![x \in B; y{\in}\ C]\!] \Longrightarrow f\ x\ y = g\ x\ y$
shows $f\ x\ y = g\ x\ y$
proof −
let *?P* = $\{x.\ \forall y{\in}\ span\ C.\ f\ x\ y = g\ x\ y\}$
from *bf bg* **have** *sp*: *subspace ?P*
unfolding *bilinear_def linear_iff subspace_def bf bg*
by (*auto simp add*: *span_zero bilinear_lzero*[*OF bf*] *bilinear_lzero*[*OF bg*]
span_add Ball_def
intro: *bilinear_ladd*[*OF bf*])
have *sfg*: $\bigwedge x.\ x \in B \Longrightarrow subspace\ \{a.\ f\ x\ a = g\ x\ a\}$
by (*auto simp*: *subspace_def bf bg bilinear_rzero bilinear_radd bilinear_rmul*)
have $\forall y{\in}\ span\ C.\ f\ x\ y = g\ x\ y$ **if** $x \in span\ B$ **for** *x*
using *span_induct* [*OF that sp*] *fg sfg span_induct* **by** *blast*
then show *?thesis*
using *SB TC assms* **by** *auto*
qed

lemma *bilinear_eq_stdbasis*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ _
assumes *bf*: *bilinear f*
and *bg*: *bilinear g*
and *fg*: $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow f\ i\ j = g\ i\ j$
shows $f = g$
using *bilinear_eq*[*OF bf bg equalityD2*[*OF span_Basis*] *equalityD2*[*OF span_Basis*]]
fg **by** *blast*

1.5.10 Infinity norm

definition *infnorm* (*x*::*'a*::*euclidean_space*) = *Sup* $\{|x \cdot b|\ |b.\ b \in Basis\}$

lemma *infnorm_set_image*:
fixes *x* :: *'a*::*euclidean_space*
shows $\{|x \cdot i|\ |i.\ i \in Basis\} = (\lambda i.\ |x \cdot i|)\ `\ Basis$
by *blast*

lemma *infnorm_Max*:
fixes *x* :: *'a*::*euclidean_space*

shows *infnorm x = Max ((λi. |x · i|) ' Basis)*
by (*simp add*: *infnorm_def infnorm_set_image cSup_eq_Max*)

lemma *infnorm_set_lemma*:
fixes *x* :: *'a*::*euclidean_space*
shows *finite* {|*x* · *i*| |*i*. *i* ∈ *Basis*}
and {|*x* · *i*| |*i*. *i* ∈ *Basis*} ≠ {}
unfolding *infnorm_set_image* **by** *auto*

lemma *infnorm_pos_le*:
fixes *x* :: *'a*::*euclidean_space*
shows *0* ≤ *infnorm x*
by (*simp add*: *infnorm_Max Max_ge_iff ex_in_conv*)

lemma *infnorm_triangle*:
fixes *x* :: *'a*::*euclidean_space*
shows *infnorm* (*x* + *y*) ≤ *infnorm x* + *infnorm y*
proof −
have ∗: ⋀*a b c d* :: *real*. |*a*| ≤ *c* ⟹ |*b*| ≤ *d* ⟹ |*a* + *b*| ≤ *c* + *d*
by *simp*
show *?thesis*
by (*auto simp*: *infnorm_Max inner_add_left intro*!: ∗)
qed

lemma *infnorm_eq_0*:
fixes *x* :: *'a*::*euclidean_space*
shows *infnorm x = 0* ⟷ *x = 0*
proof −
have *infnorm x* ≤ *0* ⟷ *x = 0*
unfolding *infnorm_Max* **by** (*simp add*: *euclidean_all_zero_iff*)
then show *?thesis*
using *infnorm_pos_le*[*of x*] **by** *simp*
qed

lemma *infnorm_0*: *infnorm 0 = 0*
by (*simp add*: *infnorm_eq_0*)

lemma *infnorm_neg*: *infnorm* (− *x*) = *infnorm x*
unfolding *infnorm_def* **by** *simp*

lemma *infnorm_sub*: *infnorm* (*x* − *y*) = *infnorm* (*y* − *x*)
by (*metis infnorm_neg minus_diff_eq*)

lemma *absdiff_infnorm*: |*infnorm x* − *infnorm y*| ≤ *infnorm* (*x* − *y*)
by (*smt* (*verit*, *del_insts*) *diff_add_cancel infnorm_sub infnorm_triangle*)

lemma *real_abs_infnorm*: |*infnorm x*| = *infnorm x*
using *infnorm_pos_le*[*of x*] **by** *arith*

lemma *Basis_le_infnorm*:
fixes $x :: 'a::euclidean\_space$
shows $b \in Basis \Longrightarrow |x \cdot b| \leq infnorm\ x$
by (*simp add*: *infnorm_Max*)

lemma *infnorm_mul*: $infnorm\ (a *_R x) = |a| * infnorm\ x$
unfolding *infnorm_Max*
proof (*safe intro*!: *Max_eqI*)
let $?B = (\lambda i.\ |x \cdot i|)\ `\ Basis$
{ fix $b :: 'a$
assume $b \in Basis$
then show $|a *_R x \cdot b| \leq |a| * Max\ ?B$
by (*simp add*: *abs_mult mult_left_mono*)
next
from *Max_in*[*of* *?B*] **obtain** b **where** $b \in Basis\ Max\ ?B = |x \cdot b|$
by (*auto simp del*: *Max_in*)
then show $|a| * Max\ ((\lambda i.\ |x \cdot i|)\ `\ Basis) \in (\lambda i.\ |a *_R x \cdot i|)\ `\ Basis$
by (*intro image_eqI*[**where** $x{=}b$]) (*auto simp*: *abs_mult*)
}
qed *simp*

lemma *infnorm_mul_lemma*: $infnorm\ (a *_R x) \leq |a| * infnorm\ x$
unfolding *infnorm_mul* **..**

lemma *infnorm_pos_lt*: $infnorm\ x > 0 \longleftrightarrow x \neq 0$
using *infnorm_pos_le*[*of* *x*] *infnorm_eq_0*[*of* *x*] **by** *arith*

Prove that it differs only up to a bound from Euclidean norm.

lemma *infnorm_le_norm*: $infnorm\ x \leq norm\ x$
by (*simp add*: *Basis_le_norm infnorm_Max*)

lemma *norm_le_infnorm*:
fixes $x :: 'a::euclidean\_space$
shows $norm\ x \leq sqrt\ DIM('a) * infnorm\ x$
unfolding *norm_eq_sqrt_inner id_def*
proof (*rule real_le_lsqrt*)
show $sqrt\ DIM('a) * infnorm\ x \geq 0$
by (*simp add*: *zero_le_mult_iff infnorm_pos_le*)
have $x \cdot x \leq (\sum b{\in}Basis.\ x \cdot b * (x \cdot b))$
by (*metis euclidean_inner order_refl*)
also have $\ldots \leq DIM('a) * |infnorm\ x|^2$
by (*rule sum_bounded_above*) (*metis Basis_le_infnorm abs_le_square_iff power2_eq_square real_abs_infnorm*)
also have $\ldots \leq (sqrt\ DIM('a) * infnorm\ x)^2$
by (*simp add*: *power_mult_distrib*)
finally show $x \cdot x \leq (sqrt\ DIM('a) * infnorm\ x)^2$ **.**
qed

lemma *tendsto_infnorm* [*tendsto_intros*]:

assumes $(f \longrightarrow a)$ *F*
shows $((\lambda x.\ infnorm\ (f\ x)) \longrightarrow infnorm\ a)$ *F*
proof (*rule tendsto_compose* [*OF LIM_I assms*])
fix *r* :: *real*
assume $r > 0$
then show $\exists\, s{>}0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < s \longrightarrow norm\ (infnorm\ x - infnorm\ a) < r$
by (*metis real_norm_def le_less_trans absdiff_infnorm infnorm_le_norm*)
qed

Equality in Cauchy-Schwarz and triangle inequalities.

lemma *norm_cauchy_schwarz_eq*: $x \cdot y = norm\ x * norm\ y \longleftrightarrow norm\ x *_R y = norm\ y *_R x$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof (*cases x=0*)
case *True*
then show *?thesis*
by *auto*
next
case *False*
from *inner_eq_zero_iff*[*of norm y* $*_R$ *x* $-$ *norm x* $*_R$ *y*]
have *?rhs* $\longleftrightarrow$
(*norm y* $*$ (*norm y* $*$ *norm x* $*$ *norm x* $-$ *norm x* $*$ ($x \cdot y$)) $-$
norm x $*$ (*norm y* $*$ ($y \cdot x$) $-$ *norm x* $*$ *norm y* $*$ *norm y*) $= 0$)
using *False* **unfolding** *inner_simps*
by (*auto simp add*: *power2_norm_eq_inner*[*symmetric*] *power2_eq_square inner_commute field_simps*)
also have ... $\longleftrightarrow$ ($2 *$ *norm x* $*$ *norm y* $*$ (*norm x* $*$ *norm y* $- x \cdot y$) $= 0$)
using *False* **by** (*simp add*: *field_simps inner_commute*)
also have ... $\longleftrightarrow$ *?lhs*
using *False* **by** *auto*
finally show *?thesis* **by** *metis*
qed

lemma *norm_cauchy_schwarz_abs_eq*:
$|x \cdot y| = norm\ x * norm\ y \longleftrightarrow$
norm x $*_R$ *y* $=$ *norm y* $*_R$ *x* $\vee$ *norm x* $*_R$ *y* $= -$ *norm y* $*_R$ *x*
using *norm_cauchy_schwarz_eq* [*symmetric*, *of x y*]
using *norm_cauchy_schwarz_eq* [*symmetric*, *of* $-x$ *y*] *Cauchy_Schwarz_ineq2* [*of x y*]
by *auto*

lemma *norm_triangle_eq*:
fixes *x y* :: *'a*::*real_inner*
shows *norm* $(x + y) =$ *norm x* $+$ *norm y* $\longleftrightarrow$ *norm x* $*_R$ *y* $=$ *norm y* $*_R$ *x*
proof (*cases* $x = 0 \vee y = 0$)
case *True*
then show *?thesis*
by *force*

next
case *False*
then have *n*: *norm x > 0 norm y > 0*
by *auto*
have *norm (x + y) = norm x + norm y ⟷ (norm (x + y))² = (norm x + norm y)²*
by *simp*
also have *... ⟷ norm x *R y = norm y *R x*
by (*smt* (*verit*, *best*) *dot_norm inner_real_def inner_simps norm_cauchy_schwarz_eq power2_eq_square*)
finally show *?thesis* **.**
qed

lemma *dist_triangle_eq*:
fixes *x y z* :: *'a*::*real_inner*
shows *dist x z = dist x y + dist y z ⟷*
*norm (x − y) *R (y − z) = norm (y − z) *R (x − y)*
by (*metis* (*no_types*, *lifting*) *add_diff_eq diff_add_cancel dist_norm norm_triangle_eq*)

1.5.11 Collinearity

definition *collinear* :: *'a*::*real_vector set ⇒ bool*
where *collinear S ⟷ (∃ u. ∀ x ∈ S. ∀ y ∈ S. ∃ c. x − y = c *R u)*

lemma *collinear_alt*:
*collinear S ⟷ (∃ u v. ∀ x ∈ S. ∃ c. x = u + c *R v)* (**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *collinear_def* **by** (*metis add.commute diff_add_cancel*)
next
assume *?rhs*
then obtain *u v* **where** *: ⋀x. x ∈ S ⟹ ∃ c. x = u + c *R v*
by *auto*
have *∃ c. x − y = c *R v* **if** *x ∈ S y ∈ S* **for** *x y*
by (*metis *[OF ‹x ∈ S›] *[OF ‹y ∈ S›] scaleR_left.diff add_diff_cancel_left*)
then show *?lhs*
using *collinear_def* **by** *blast*
qed

lemma *collinear*:
fixes *S* :: *'a*::{*perfect_space*,*real_vector*} *set*
shows *collinear S ⟷ (∃ u. u ≠ 0 ∧ (∀ x ∈ S. ∀ y ∈ S. ∃ c. x − y = c *R u))*
proof −
have *∃ v. v ≠ 0 ∧ (∀ x∈S. ∀ y∈S. ∃ c. x − y = c *R v)*
if *∀ x∈S. ∀ y∈S. ∃ c. x − y = c *R u u=0* **for** *u*
proof −
have *∀ x∈S. ∀ y∈S. x = y*
using *that* **by** *auto*

```
    moreover
    obtain v::'a where v ≠ 0
      using UNIV_not_singleton [of 0] by auto
    ultimately have ∀ x∈S. ∀ y∈S. ∃ c. x − y = c *R v
      by auto
    then show ?thesis
      using ‹v ≠ 0› by blast
  qed
  then show ?thesis
    by (metis collinear_def)
qed

lemma collinear_subset: ⟦collinear T; S ⊆ T⟧ ⟹ collinear S
  by (meson collinear_def subsetCE)

lemma collinear_empty [iff]: collinear {}
  by (simp add: collinear_def)

lemma collinear_sing [iff]: collinear {x}
  by (simp add: collinear_def)

lemma collinear_2 [iff]: collinear {x, y}
 by (simp add: collinear_def) (metis minus_diff_eq scaleR_left.minus scaleR_one)

lemma collinear_lemma: collinear {0, x, y} ⟷ x = 0 ∨ y = 0 ∨ (∃ c. y = c
*R x)
  (is ?lhs ⟷ ?rhs)
proof (cases x = 0 ∨ y = 0)
  case True
  then show ?thesis
    by (auto simp: insert_commute)
next
  case False
  show ?thesis
  proof
    assume h: ?lhs
    then obtain u where u: ∀ x∈ {0,x,y}. ∀ y∈ {0,x,y}. ∃ c. x − y = c *R u
      unfolding collinear_def by blast
    from u[rule_format, of x 0] u[rule_format, of y 0]
    obtain cx and cy where
      cx: x = cx *R u and cy: y = cy *R u
      by auto
    from cx cy False have cx0: cx ≠ 0 and cy0: cy ≠ 0 by auto
    let ?d = cy / cx
    from cx cy cx0 have y = ?d *R x
      by simp
    then show ?rhs using False by blast
  next
    assume h: ?rhs
```

```
    then obtain c where c: y = c *R x
      using False by blast
    show ?lhs
      apply (simp add: collinear_def c)
       by (metis (mono_tags, lifting) scaleR_left.minus scaleR_left_diff_distrib
scaleR_one)
  qed
qed

lemma collinear_iff_Reals: collinear {0::complex,w,z} ⟷ z/w ∈ ℝ
proof
  show z/w ∈ ℝ ⟹ collinear {0,w,z}
   by (metis Reals_cases collinear_lemma nonzero_divide_eq_eq scaleR_conv_of_real)
qed (auto simp: collinear_lemma scaleR_conv_of_real)

lemma collinear_scaleR_iff: collinear {0, α *R w, β *R z} ⟷ collinear {0,w,z}
∨ α=0 ∨ β=0
  (is ?lhs = ?rhs)
proof (cases α=0 ∨ β=0)
  case False
  then have (∃ c. β *R z = (c * α) *R w) = (∃ c. z = c *R w)
    by (metis mult.commute scaleR_scaleR vector_fraction_eq_iff)
  then show ?thesis
    by (auto simp add: collinear_lemma)
qed (auto simp: collinear_lemma)

lemma norm_cauchy_schwarz_equal: |x · y| = norm x * norm y ⟷ collinear
{0, x, y}
proof (cases x=0)
  case True
  then show ?thesis
    by (auto simp: insert_commute)
next
  case False
  then have nnz: norm x ≠ 0
    by auto
  show ?thesis
  proof
    assume |x · y| = norm x * norm y
    then show collinear {0, x, y}
      unfolding norm_cauchy_schwarz_abs_eq collinear_lemma
      by (meson eq_vector_fraction_iff nnz)
  next
    assume collinear {0, x, y}
    with False show |x · y| = norm x * norm y
      unfolding norm_cauchy_schwarz_abs_eq collinear_lemma  by (auto simp:
abs_if)
  qed
qed
```

lemma *norm_triangle_eq_imp_collinear*:
fixes *x y* :: *'a*::*real_inner*
assumes *norm* (*x* + *y*) = *norm x* + *norm y*
shows *collinear*{*0*,*x*,*y*}
using *assms norm_cauchy_schwarz_abs_eq norm_cauchy_schwarz_equal norm_triangle_eq*

by *blast*

1.5.12 Properties of special hyperplanes

lemma *subspace_hyperplane*: *subspace* {*x*. *a* · *x* = *0*}
by (*simp add*: *subspace_def inner_right_distrib*)

lemma *subspace_hyperplane2*: *subspace* {*x*. *x* · *a* = *0*}
by (*simp add*: *inner_commute inner_right_distrib subspace_def*)

lemma *special_hyperplane_span*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *k* ∈ *Basis*
shows {*x*. *k* · *x* = *0*} = *span* (*Basis* − {*k*})
proof −
have ∗: *x* ∈ *span* (*Basis* − {*k*}) **if** *k* · *x* = *0* **for** *x*
proof −
have *x* = (∑ *b*∈*Basis*. (*x* · *b*) ∗*R b*)
by (*simp add*: *euclidean_representation*)
also have ... = (∑ *b* ∈ *Basis* − {*k*}. (*x* · *b*) ∗*R b*)
by (*auto simp*: *sum.remove* [*of* _ *k*] *inner_commute assms that*)
finally have *x* = (∑ *b*∈*Basis* − {*k*}. (*x* · *b*) ∗*R b*) .
then show *?thesis*
by (*simp add*: *span_finite*)
qed
show *?thesis*
apply (*rule span_subspace* [*symmetric*])
using *assms*
apply (*auto simp*: *inner_not_same_Basis intro*: ∗ *subspace_hyperplane*)
done
qed

lemma *dim_special_hyperplane*:
fixes *k* :: *'n*::*euclidean_space*
shows *k* ∈ *Basis* ⟹ *dim* {*x*. *k* · *x* = *0*} = *DIM*(*'n*) − *1*
by (*metis Diff_subset card_Diff_singleton indep_card_eq_dim_span independent_substdbasis special_hyperplane_span*)

proposition *dim_hyperplane*:
fixes *a* :: *'a*::*euclidean_space*
assumes *a* ≠ *0*
shows *dim* {*x*. *a* · *x* = *0*} = *DIM*(*'a*) − *1*

proof −
have *span0*: *span* {*x*. *a* · *x* = *0*} = {*x*. *a* · *x* = *0*}
by (*rule span_unique*) (*auto simp*: *subspace_hyperplane*)
then obtain *B* **where** *independent B*
and *Bsub*: *B* ⊆ {*x*. *a* · *x* = *0*}
and *subspB*: {*x*. *a* · *x* = *0*} ⊆ *span B*
and *card0*: (*card B* = *dim* {*x*. *a* · *x* = *0*})
and *ortho*: *pairwise orthogonal B*
using *orthogonal_basis_exists* **by** *metis*
with *assms* **have** *a* ∉ *span B*
by (*metis* (*mono_tags*, *lifting*) *span_eq inner_eq_zero_iff mem_Collect_eq span0*)
then have *ind*: *independent* (*insert a B*)
by (*simp add*: ‹*independent B*› *independent_insert*)
have *finite B*
using ‹*independent B*› *independent_bound* **by** *blast*
have *UNIV* ⊆ *span* (*insert a B*)
proof fix *y*::*'a*
obtain *r z* **where** *y* = *r* ∗$_R$ *a* + *z a* · *z* = *0*
by (*metis add.commute diff_add_cancel vector_sub_project_orthogonal*)
then show *y* ∈ *span* (*insert a B*)
by (*metis* (*mono_tags*, *lifting*) *Bsub add_diff_cancel_left'*
mem_Collect_eq span0 span_breakdown_eq span_eq subspB)
qed
then have *DIM*(*'a*) = *dim*(*insert a B*)
by (*metis independent_Basis span_Basis dim_eq_card top.extremum_uniqueI*)
then show *?thesis*
by (*metis One_nat_def* ‹*a* ∉ *span B*› ‹*finite B*› *card0 card_insert_disjoint*
diff_Suc_Suc diff_zero dim_eq_card_independent ind span_base)
qed

lemma *lowdim_eq_hyperplane*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *dim S* = *DIM*(*'a*) − *1*
obtains *a* **where** *a* ≠ *0* **and** *span S* = {*x*. *a* · *x* = *0*}
proof −
obtain *b* **where** *b*: *b* ≠ *0 span S* ⊆ {*a*. *b* · *a* = *0*}
by (*metis DIM_positive assms diff_less zero_less_one lowdim_subset_hyperplane*)
then show *?thesis*
by (*metis assms dim_hyperplane dim_span dim_subset subspace_dim_equal subspace_hyperplane subspace_span that*)
qed

lemma *dim_eq_hyperplane*:
fixes *S* :: *'n*::*euclidean_space set*
shows *dim S* = *DIM*(*'n*) − *1* ⟷ (∃ *a*. *a* ≠ *0* ∧ *span S* = {*x*. *a* · *x* = *0*})
by (*metis One_nat_def dim_hyperplane dim_span lowdim_eq_hyperplane*)

1.5.13 Orthogonal bases and Gram-Schmidt process

lemma *pairwise_orthogonal_independent*:
assumes *pairwise orthogonal S* **and** *0* $\notin$ *S*
shows *independent S*
proof −
have *0*: $\bigwedge x\ y.\ [\![x \neq y;\ x \in S;\ y \in S]\!] \Longrightarrow x \cdot y = 0$
using *assms* **by** (*simp add*: *pairwise_def orthogonal_def*)
have *False* **if** $a \in S$ **and** *a*: $a \in$ *span* $(S - \{a\})$ **for** *a*
proof −
obtain *T U* **where** $T \subseteq S - \{a\}$ $a = (\sum v{\in}T.\ U\ v *_R v)$
using *a* **by** (*force simp*: *span_explicit*)
then have $a \cdot a = a \cdot (\sum v{\in}T.\ U\ v *_R v)$
by *simp*
also have $\ldots = 0$
apply (*simp add*: *inner_sum_right*)
by (*smt* (*verit*) *0 DiffE* ‹$T \subseteq S - \{a\}$› *in_mono insertCI mult_not_zero sum.neutral that*(*1*))
finally show *?thesis*
using ‹*0* $\notin$ *S*› ‹$a \in S$› **by** *auto*
qed
then show *?thesis*
by (*force simp*: *dependent_def*)
qed

lemma *pairwise_orthogonal_imp_finite*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *pairwise orthogonal S*
shows *finite S*
by (*metis Set.set_insert assms finite_insert independent_bound pairwise_insert*
pairwise_orthogonal_independent)

lemma *subspace_orthogonal_to_vector*: *subspace* $\{y.\ orthogonal\ x\ y\}$
by (*simp add*: *subspace_def orthogonal_clauses*)

lemma *subspace_orthogonal_to_vectors*: *subspace* $\{y.\ \forall x \in S.\ orthogonal\ x\ y\}$
by (*simp add*: *subspace_def orthogonal_clauses*)

lemma *orthogonal_to_span*:
assumes *a*: $a \in$ *span S* **and** *x*: $\bigwedge y.\ y \in S \Longrightarrow$ *orthogonal x y*
shows *orthogonal x a*
by (*metis a orthogonal_clauses*(*1*,*2*,*4*)
span_induct_alt x)

proposition *Gram_Schmidt_step*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S*: *pairwise orthogonal S* **and** *x*: $x \in$ *span S*
shows *orthogonal x* $(a - (\sum b{\in}S.\ (b \cdot a\ /\ (b \cdot b)) *_R b))$
proof −

```
  have finite S
    by (simp add: S pairwise_orthogonal_imp_finite)
  have orthogonal (a − (∑ b∈S. (b · a / (b · b)) *R b)) x
      if x ∈ S for x
  proof −
    have a · x = (∑ y∈S. if y = x then y · a else 0)
      by (simp add: ‹finite S› inner_commute that)
    also have ... = (∑ b∈S. b · a * (b · x) / (b · b))
      apply (rule sum.cong [OF refl], simp)
      by (meson S orthogonal_def pairwise_def that)
   finally show ?thesis
     by (simp add: orthogonal_def algebra_simps inner_sum_left)
  qed
  then show ?thesis
    using orthogonal_to_span orthogonal_commute x by blast
qed


lemma orthogonal_extension_aux:
  fixes S :: 'a::euclidean_space set
  assumes finite T finite S pairwise orthogonal S
    shows ∃ U. pairwise orthogonal (S ∪ U) ∧ span (S ∪ U) = span (S ∪ T)
using assms
proof (induction arbitrary: S)
  case empty then show ?case
    by simp (metis sup_bot_right)
next
  case (insert a T)
  have 0: ⋀x y. ⟦x ≠ y; x ∈ S; y ∈ S⟧ ⟹ x · y = 0
    using insert by (simp add: pairwise_def orthogonal_def)
  define a' where a' = a − (∑ b∈S. (b · a / (b · b)) *R b)
  obtain U where orthU: pairwise orthogonal (S ∪ insert a' U)
            and spanU: span (insert a' S ∪ U) = span (insert a' S ∪ T)
    by (rule exE [OF insert.IH [of insert a' S]])
      (auto simp: Gram_Schmidt_step a'_def insert.prems orthogonal_commute
        pairwise_orthogonal_insert span_clauses)
  have orthS: ⋀x. x ∈ S ⟹ a' · x = 0
    using Gram_Schmidt_step a'_def insert.prems orthogonal_commute orthogo-
nal_def span_base by blast
  have span (S ∪ insert a' U) = span (insert a' (S ∪ T))
    using spanU by simp
  also have ... = span (insert a (S ∪ T))
   by (simp add: a'_def span_neg span_sum span_base span_mul eq_span_insert_eq)
  also have ... = span (S ∪ insert a T)
    by simp
  finally show ?case
    using orthU by blast
qed
```

proposition *orthogonal_extension*:
fixes S :: *'a*::*euclidean_space set*
assumes S: *pairwise orthogonal S*
obtains U **where** *pairwise orthogonal* $(S \cup U)$ *span* $(S \cup U) = span\ (S \cup T)$
proof −
obtain B **where** *finite B span B* = *span T*
using *basis_subspace_exists* [*of span T*] *subspace_span* **by** *metis*
with *orthogonal_extension_aux* [*of B S*]
obtain U **where** *pairwise orthogonal* $(S \cup U)$ *span* $(S \cup U) = span\ (S \cup B)$
using *assms pairwise_orthogonal_imp_finite* **by** *auto*
with ‹*span B* = *span T*› **show** *?thesis*
by (*rule_tac U*=*U* **in** *that*) (*auto simp*: *span_Un*)
qed

corollary *orthogonal_extension_strong*:
fixes S :: *'a*::*euclidean_space set*
assumes S: *pairwise orthogonal S*
obtains U **where** $U \cap (insert\ 0\ S) = \{\}$ *pairwise orthogonal* $(S \cup U)$
span $(S \cup U) = span\ (S \cup T)$
proof −
obtain U **where** U: *pairwise orthogonal* $(S \cup U)$ *span* $(S \cup U) = span\ (S \cup T)$
using *orthogonal_extension assms* **by** *blast*
moreover have *pairwise orthogonal* $(S \cup (U - insert\ 0\ S))$
by (*smt* (*verit*, *best*) *Un_Diff_Int Un_iff U pairwise_def*)
ultimately show *?thesis*
by (*metis Diff_disjoint Un_Diff_cancel Un_insert_left inf_commute span_insert_0 that*)
qed

1.5.14 Decomposing a vector into parts in orthogonal subspaces

existence of orthonormal basis for a subspace.

lemma *orthogonal_spanningset_subspace*:
fixes S :: *'a* :: *euclidean_space set*
assumes *subspace S*
obtains B **where** $B \subseteq S$ *pairwise orthogonal B span B* = *S*
by (*metis assms basis_orthogonal basis_subspace_exists span_eq*)

lemma *orthogonal_basis_subspace*:
fixes S :: *'a* :: *euclidean_space set*
assumes *subspace S*
obtains B **where** $0 \notin B$ $B \subseteq S$ *pairwise orthogonal B independent B*
card B = *dim S span B* = *S*
by (*metis assms dependent_zero orthogonal_basis_exists span_eq span_eq_iff*)

proposition *orthonormal_basis_subspace*:

```
  fixes S :: 'a :: euclidean_space set
  assumes subspace S
  obtains B where B ⊆ S pairwise orthogonal B
             and ⋀x. x ∈ B ⟹ norm x = 1
             and independent B card B = dim S span B = S
proof −
  obtain B where 0 ∉ B B ⊆ S
            and orth: pairwise orthogonal B
            and independent B card B = dim S span B = S
    by (blast intro: orthogonal_basis_subspace [OF assms])
  have 1: (λx. x /R norm x) ‘ B ⊆ S
    using ‹span B = S› span_superset span_mul by fastforce
  have 2: pairwise orthogonal ((λx. x /R norm x) ‘ B)
    using orth by (force simp: pairwise_def orthogonal_clauses)
  have 3: ⋀x. x ∈ (λx. x /R norm x) ‘ B ⟹ norm x = 1
    by (metis (no_types, lifting) ‹0 ∉ B› image_iff norm_sgn sgn_div_norm)
  have 4: independent ((λx. x /R norm x) ‘ B)
    by (metis 2 3 norm_zero pairwise_orthogonal_independent zero_neq_one)
  have inj_on (λx. x /R norm x) B
  proof
    fix x y
    assume x ∈ B y ∈ B x /R norm x = y /R norm y
    moreover have ⋀i. i ∈ B ⟹ norm (i /R norm i) = 1
      using 3 by blast
    ultimately show x = y
      by (metis norm_eq_1 orth orthogonal_clauses(7) orthogonal_commute or-
thogonal_def pairwise_def zero_neq_one)
  qed
  then have 5: card ((λx. x /R norm x) ‘ B) = dim S
    by (metis ‹card B = dim S› card_image)
  have 6: span ((λx. x /R norm x) ‘ B) = S
    by (metis 1 4 5 assms card_eq_dim independent_imp_finite span_subspace)
  show ?thesis
    by (rule that [OF 1 2 3 4 5 6])
qed


proposition orthogonal_to_subspace_exists_gen:
  fixes S :: 'a :: euclidean_space set
  assumes span S ⊂ span T
  obtains x where x ≠ 0 x ∈ span T ⋀y. y ∈ span S ⟹ orthogonal x y
proof −
  obtain B where B ⊆ span S and orthB: pairwise orthogonal B
            and ⋀x. x ∈ B ⟹ norm x = 1
            and independent B card B = dim S span B = span S
    by (metis dim_span orthonormal_basis_subspace subspace_span)
  with assms obtain u where spanBT: span B ⊆ span T and u ∉ span B u ∈
span T
    by auto
```

obtain C **where** *orthBC*: *pairwise orthogonal* $(B \cup C)$ **and** *spanBC*: *span* $(B \cup C) = span\ (B \cup \{u\})$
by (*blast intro*: *orthogonal_extension* [*OF orthB*])
show *thesis*
proof (*cases* $C \subseteq insert\ 0\ B$)
case *True*
then have $C \subseteq span\ B$
using *span_eq*
by (*metis span_insert_0 subset_trans*)
moreover have $u \in span\ (B \cup C)$
using ‹*span* $(B \cup C) = span\ (B \cup \{u\})$› *span_superset* **by** *force*
ultimately show *?thesis*
using *True* ‹$u \notin span\ B$›
by (*metis Un_insert_left span_insert_0 sup.orderE*)
next
case *False*
then obtain x **where** $x \in C\ x \neq 0\ x \notin B$
by *blast*
then have $x \in span\ T$
by (*smt* (*verit*, *ccfv_SIG*) *Set.set_insert* ‹$u \in span\ T$› *empty_subsetI insert_subset*
le_sup_iff spanBC spanBT span_mono span_span span_superset subset_trans)
moreover have *orthogonal x y* **if** $y \in span\ B$ **for** y
using *that*
proof (*rule span_induct*)
show *subspace* $\{a.\ orthogonal\ x\ a\}$
by (*simp add*: *subspace_orthogonal_to_vector*)
show $\bigwedge b.\ b \in B \Longrightarrow orthogonal\ x\ b$
by (*metis Un_iff* ‹$x \in C$› ‹$x \notin B$› *orthBC pairwise_def*)
qed
ultimately show *?thesis*
using ‹$x \neq 0$› *that* ‹*span* $B = span\ S$› **by** *auto*
qed
qed

corollary *orthogonal_to_subspace_exists*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes $dim\ S < DIM('a)$
obtains x **where** $x \neq 0\ \bigwedge y.\ y \in span\ S \Longrightarrow orthogonal\ x\ y$
proof −
have $span\ S \subset UNIV$
by (*metis assms dim_eq_full order_less_imp_not_less top.not_eq_extremum*)
with *orthogonal_to_subspace_exists_gen* [*of S UNIV*] *that* **show** *?thesis*
by (*auto*)
qed

corollary *orthogonal_to_vector_exists*:
fixes $x :: 'a :: euclidean\_space$

assumes $2 \leq DIM('a)$
obtains y **where** $y \neq 0$ *orthogonal* x y
proof −
have *dim* $\{x\} < DIM('a)$
using *assms* **by** *auto*
then show *thesis*
by (*rule orthogonal_to_subspace_exists*) (*simp add*: *orthogonal_commute span_base that*)
qed

proposition *orthogonal_subspace_decomp_exists*:
fixes $S :: 'a :: euclidean\_space\ set$
obtains y z
where $y \in span\ S$
and $\bigwedge w.\ w \in span\ S \Longrightarrow orthogonal\ z\ w$
and $x = y + z$
proof −
obtain T **where** $0 \notin T$ $T \subseteq span\ S$ *pairwise orthogonal* T *independent* T
card $T = dim\ (span\ S)$ *span* $T = span\ S$
using *orthogonal_basis_subspace subspace_span* **by** *blast*
let $?a = \sum b{\in}T.\ (b \cdot x\ /\ (b \cdot b)) *_R b$
have *orth*: *orthogonal* $(x - ?a)\ w$ **if** $w \in span\ S$ **for** w
by (*simp add*: *Gram_Schmidt_step* ‹*pairwise orthogonal T*› ‹*span T = span S*›
orthogonal_commute that)
with *that*[*of* $?a\ x{-}?a$] ‹$T \subseteq span\ S$› **show** *?thesis*
by (*simp add*: *span_mul span_sum subsetD*)
qed

lemma *orthogonal_subspace_decomp_unique*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes $x + y = x' + y'$
and *ST*: $x \in span\ S\ x' \in span\ S\ y \in span\ T\ y' \in span\ T$
and *orth*: $\bigwedge a\ b.\ [\![a \in S;\ b \in T]\!] \Longrightarrow orthogonal\ a\ b$
shows $x = x' \wedge y = y'$
proof −
have $x + y - y' = x'$
by (*simp add*: *assms*)
moreover have $\bigwedge a\ b.\ [\![a \in span\ S;\ b \in span\ T]\!] \Longrightarrow orthogonal\ a\ b$
by (*meson orth orthogonal_commute orthogonal_to_span*)
ultimately have $0 = x' - x$
using *assms*
by (*metis add.commute add_diff_cancel_right' diff_right_commute orthogonal_self span_diff*)
with *assms* **show** *?thesis* **by** *auto*
qed

lemma *vector_in_orthogonal_spanningset*:
fixes $a :: 'a{::}euclidean\_space$

obtains S **where** $a \in S$ *pairwise orthogonal* S *span* $S = UNIV$
by (*metis UnI1 Un_UNIV_right insertI1 orthogonal_extension pairwise_singleton span_UNIV*)

lemma *vector_in_orthogonal_basis*:
fixes $a :: 'a$::*euclidean_space*
assumes $a \neq 0$
obtains S **where** $a \in S$ $0 \notin S$ *pairwise orthogonal* S *independent* S *finite* S
span $S = UNIV$ *card* $S = DIM('a)$
proof −
obtain S **where** S: $a \in S$ *pairwise orthogonal* S *span* $S = UNIV$
using *vector_in_orthogonal_spanningset* **.**
show *thesis*
proof
show *pairwise orthogonal* $(S - \{0\})$
using *pairwise_mono* $S(2)$ **by** *blast*
show *independent* $(S - \{0\})$
by (*simp add*: ‹*pairwise orthogonal* $(S - \{0\})$› *pairwise_orthogonal_independent*)
show *finite* $(S - \{0\})$
using ‹*independent* $(S - \{0\})$› *independent_imp_finite* **by** *blast*
show *card* $(S - \{0\}) = DIM('a)$
using *span_delete_0* [*of* S] S
by (*simp add*: ‹*independent* $(S - \{0\})$› *indep_card_eq_dim_span*)
qed (*use* S ‹$a \neq 0$› **in** *auto*)
qed

lemma *vector_in_orthonormal_basis*:
fixes $a :: 'a$::*euclidean_space*
assumes *norm* $a = 1$
obtains S **where** $a \in S$ *pairwise orthogonal* S $\bigwedge x.\ x \in S \Longrightarrow$ *norm* $x = 1$
independent S *card* $S = DIM('a)$ *span* $S = UNIV$
proof −
have $a \neq 0$
using *assms* **by** *auto*
then obtain S **where** $a \in S$ $0 \notin S$ *finite* S
and S: *pairwise orthogonal* S *independent* S *span* $S = UNIV$ *card* $S = DIM('a)$
by (*metis vector_in_orthogonal_basis*)
let *?S* $= (\lambda x.\ x\ /_R$ *norm* $x)$ ‘ S
show *thesis*
proof
show $a \in$ *?S*
using ‹$a \in S$› *assms image_iff* **by** *fastforce*
next
show *pairwise orthogonal ?S*
using ‹*pairwise orthogonal* S› **by** (*auto simp*: *pairwise_def orthogonal_def*)
show $\bigwedge x.\ x \in (\lambda x.\ x\ /_R$ *norm* $x)$ ‘ $S \Longrightarrow$ *norm* $x = 1$
using ‹$0 \notin S$› **by** (*auto simp*: *field_split_simps*)
then show *ind*: *independent ?S*

by (*metis* ‹*pairwise orthogonal* ((λ*x*. *x* /$_R$ *norm x*) ‘ *S*)› *norm_zero pairwise_orthogonal_independent zero_neq_one*)
have *inj_on* (λ*x*. *x* /$_R$ *norm x*) *S*
unfolding *inj_on_def*
by (*metis* (*full_types*) *S*(*1*) ‹*0* ∉ *S*› *inverse_nonzero_iff_nonzero norm_eq_zero orthogonal_scaleR orthogonal_self pairwise_def*)
then show *card ?S* = *DIM*('*a*)
by (*simp add*: *card_image S*)
then show *span ?S* = *UNIV*
by (*metis ind dim_eq_card dim_eq_full*)
qed
qed

proposition *dim_orthogonal_sum*:
fixes *A* :: '*a*::*euclidean_space set*
assumes ⋀*x y*. ⟦*x* ∈ *A*; *y* ∈ *B*⟧ ⟹ *x* ∙ *y* = *0*
shows *dim*(*A* ∪ *B*) = *dim A* + *dim B*
proof −
have *1*: ⋀*x y*. ⟦*x* ∈ *span A*; *y* ∈ *B*⟧ ⟹ *x* ∙ *y* = *0*
by (*erule span_induct* [*OF* _ *subspace_hyperplane2*]; *simp add*: *assms*)
have ⋀*x y*. ⟦*x* ∈ *span A*; *y* ∈ *span B*⟧ ⟹ *x* ∙ *y* = *0*
using *1* **by** (*simp add*: *span_induct* [*OF* _ *subspace_hyperplane*])
then have *0*: ⋀*x y*. ⟦*x* ∈ *span A*; *y* ∈ *span B*⟧ ⟹ *x* ∙ *y* = *0*
by *simp*
have *dim*(*A* ∪ *B*) = *dim* (*span* (*A* ∪ *B*))
by (*simp*)
also have *span* (*A* ∪ *B*) = ((λ(*a*, *b*). *a* + *b*) ‘ (*span A* × *span B*))
by (*auto simp add*: *span_Un image_def*)
also have *dim* ... = *dim* {*x* + *y* |*x y*. *x* ∈ *span A* ∧ *y* ∈ *span B*}
by (*auto intro*!: *arg_cong* [**where** *f*=*dim*])
also have ... = *dim* {*x* + *y* |*x y*. *x* ∈ *span A* ∧ *y* ∈ *span B*} + *dim*(*span A* ∩ *span B*)
by (*auto dest*: *0*)
also have ... = *dim A* + *dim B*
using *dim_sums_Int* **by** *fastforce*
finally show *?thesis* **.**
qed

lemma *dim_subspace_orthogonal_to_vectors*:
fixes *A* :: '*a*::*euclidean_space set*
assumes *subspace A subspace B A* ⊆ *B*
shows *dim* {*y* ∈ *B*. ∀ *x* ∈ *A*. *orthogonal x y*} + *dim A* = *dim B*
proof −
have *dim* (*span* ({*y* ∈ *B*. ∀ *x*∈*A*. *orthogonal x y*} ∪ *A*)) = *dim* (*span B*)
proof (*rule arg_cong* [**where** *f*=*dim*, *OF subset_antisym*])
show *span* ({*y* ∈ *B*. ∀ *x*∈*A*. *orthogonal x y*} ∪ *A*) ⊆ *span B*
by (*simp add*: ‹*A* ⊆ *B*› *Collect_restrict span_mono*)
next
have ∗: *x* ∈ *span* ({*y* ∈ *B*. ∀ *x*∈*A*. *orthogonal x y*} ∪ *A*)

if $x \in B$ **for** x
proof −
obtain y z **where** $x = y + z$ $y \in span\ A$ **and** $orth$: $\bigwedge w.\ w \in span\ A \Longrightarrow orthogonal\ z\ w$
using *orthogonal_subspace_decomp_exists* [*of A x*] *that* **by** *auto*
moreover
have $y \in span\ B$
using ‹$y \in span\ A$› *assms*(*3*) *span_mono* **by** *blast*
ultimately have $z \in B \wedge (\forall x.\ x \in A \longrightarrow orthogonal\ x\ z)$
using *assms* **by** (*metis orthogonal_commute span_add_eq span_eq_iff that*)
then have *z*: $z \in span\ \{y \in B.\ \forall x{\in}A.\ orthogonal\ x\ y\}$
by (*simp add*: *span_base*)
then show *?thesis*
by (*smt* (*verit*, *best*) ‹$x = y + z$› ‹$y \in span\ A$› *le_sup_iff span_add_eq span_subspace_induct*
span_superset subset_iff subspace_span)
qed
show $span\ B \subseteq span\ (\{y \in B.\ \forall x{\in}A.\ orthogonal\ x\ y\} \cup A)$
by (*rule span_minimal*) (*auto intro*: ∗ *span_minimal*)
qed
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *dim_orthogonal_sum dim_span mem_Collect_eq*
orthogonal_commute orthogonal_def)
qed

1.5.15 Linear functions are (uniformly) continuous on any set

1.5.16 Topological properties of linear functions

lemma *linear_lim_0*:
assumes *bounded_linear f*
shows $(f \longrightarrow 0)\ (at\ (0))$
proof −
interpret *f*: *bounded_linear f* **by** *fact*
have $(f \longrightarrow f\ 0)\ (at\ 0)$
using *tendsto_ident_at* **by** (*rule f.tendsto*)
then show *?thesis* **unfolding** *f.zero* **.**
qed

lemma *linear_continuous_at*:
bounded_linear f ⟹*continuous* (*at a*) *f*
by (*simp add*: *bounded_linear.isUCont isUCont_isCont*)

lemma *linear_continuous_within*:
bounded_linear f ⟹ *continuous* (*at x within s*) *f*
using *continuous_at_imp_continuous_at_within linear_continuous_at* **by** *blast*

lemma *linear_continuous_on*:
bounded_linear f ⟹ *continuous_on s f*

using *continuous_at_imp_continuous_on*[*of s f*] **using** *linear_continuous_at*[*of f*] **by** *auto*

lemma *Lim_linear*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* **and** *h* :: *'b* ⇒ *'c*::*real_normed_vector*
assumes (*f* ⟶ *l*) *F* *linear h*
shows ((λ*x*. *h*(*f x*)) ⟶ *h l*) *F*
proof −
obtain *B* **where** *B*: *B* > *0* ⋀*x*. *norm* (*h x*) ≤ *B* ∗ *norm x*
using *linear_bounded_pos* [*OF* ‹*linear h*›] **by** *blast*
show *?thesis*
unfolding *tendsto_iff*
by (*simp add*: *assms bounded_linear.tendsto linear_linear tendstoD*)
qed

lemma *linear_continuous_compose*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* **and** *g* :: *'b* ⇒ *'c*::*real_normed_vector*
assumes *continuous F f linear g*
shows *continuous F* (λ*x*. *g*(*f x*))
using *assms* **unfolding** *continuous_def* **by** (*rule Lim_linear*)

lemma *linear_continuous_on_compose*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* **and** *g* :: *'b* ⇒ *'c*::*real_normed_vector*
assumes *continuous_on S f linear g*
shows *continuous_on S* (λ*x*. *g*(*f x*))
using *assms* **by** (*simp add*: *continuous_on_eq_continuous_within linear_continuous_compose*)

Also bilinear functions, in composition form

lemma *bilinear_continuous_compose*:
fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*real_normed_vector*
assumes *continuous F f continuous F g bilinear h*
shows *continuous F* (λ*x*. *h* (*f x*) (*g x*))
using *assms bilinear_conv_bounded_bilinear bounded_bilinear.continuous* **by** *blast*

lemma *bilinear_continuous_on_compose*:
fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*real_normed_vector*
and *f* :: *'d*::*t2_space* ⇒ *'a*
assumes *continuous_on S f continuous_on S g bilinear h*
shows *continuous_on S* (λ*x*. *h* (*f x*) (*g x*))
using *assms* **by** (*simp add*: *continuous_on_eq_continuous_within bilinear_continuous_compose*)

end

1.6 Affine Sets

theory *Affine*
imports *Linear_Algebra*
begin

lemma *if_smult*: (*if P then x else* (*y*::*real*)) $*_R$ *v* = (*if P then x* $*_R$ *v else y* $*_R$ *v*)
by *simp*

lemma *sum_delta_notmem*:
assumes *x* ∉ *s*
shows *sum* (λ*y*. *if* (*y* = *x*) *then P x else Q y*) *s* = *sum Q s*
and *sum* (λ*y*. *if* (*x* = *y*) *then P x else Q y*) *s* = *sum Q s*
and *sum* (λ*y*. *if* (*y* = *x*) *then P y else Q y*) *s* = *sum Q s*
and *sum* (λ*y*. *if* (*x* = *y*) *then P y else Q y*) *s* = *sum Q s*
by (*smt* (*verit*, *best*) *assms sum.cong*)+

lemma *span_substd_basis*:
assumes *d*: *d* ⊆ *Basis*
shows *span d* = {*x*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*}
(**is** *_* = *?B*)
proof −
have *d* ⊆ *?B*
using *d* **by** (*auto simp*: *inner_Basis*)
moreover have *s*: *subspace ?B*
using *subspace_substandard*[*of* λ*i*. *i* ∉ *d*] **.**
ultimately have *span d* ⊆ *?B*
using *span_mono*[*of d ?B*] *span_eq_iff*[*of ?B*] **by** *blast*
moreover have ∗: *card d* ≤ *dim* (*span d*)
by (*simp add*: *d dim_eq_card_independent independent_substdbasis*)
moreover from ∗ **have** *dim ?B* ≤ *dim* (*span d*)
using *dim_substandard*[*OF assms*] **by** *auto*
ultimately show *?thesis*
by (*simp add*: *s subspace_dim_equal*)
qed

lemma *basis_to_substdbasis_subspace_isomorphism*:
fixes *B* :: *'a*::*euclidean_space set*
assumes *independent B*
shows ∃*f d*::*'a set*. *card d* = *card B* ∧ *linear f* ∧ *f* ‘ *B* = *d* ∧
f ‘ *span B* = {*x*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x* • *i* = *0*} ∧ *inj_on f* (*span B*) ∧ *d* ⊆
Basis
proof −
have *B*: *card B* = *dim B*
using *dim_unique*[*of B B card B*] *assms span_superset*[*of B*] **by** *auto*
have *dim B* ≤ *card* (*Basis* :: *'a set*)
using *dim_subset_UNIV*[*of B*] **by** *simp*
from *obtain_subset_with_card_n*[*OF this*]
obtain *d* :: *'a set* **where** *d*: *d* ⊆ *Basis* **and** *t*: *card d* = *dim B*
by *auto*
let *?t* = {*x*::*'a*::*euclidean_space*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*}
have ∃*f*. *linear f* ∧ *f* ‘ *B* = *d* ∧ *f* ‘ *span B* = *?t* ∧ *inj_on f* (*span B*)
proof (*intro basis_to_basis_subspace_isomorphism subspace_span subspace_substandard span_superset*)

show $d \subseteq \{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\}$
using *d inner_not_same_Basis* **by** *blast*
qed (*auto simp*: *span_substd_basis independent_substdbasis dim_substandard d t B assms*)
with *t* ‹*card B* = *dim B*› *d* **show** *?thesis* **by** *auto*
qed

1.6.1 Affine set and affine hull

definition *affine* :: *'a::real_vector set* ⇒ *bool*
where *affine S* ⟷ $(\forall x{\in}S.\ \forall y{\in}S.\ \forall u\ v.\ u + v = 1 \longrightarrow u *_R x + v *_R y \in S)$

lemma *affine_alt*: *affine S* ⟷ $(\forall x{\in}S.\ \forall y{\in}S.\ \forall u{::}real.\ (1 - u) *_R x + u *_R y \in S)$
unfolding *affine_def* **by** (*metis eq_diff_eq'*)

lemma *affine_empty* [*iff*]: *affine* {}
unfolding *affine_def* **by** *auto*

lemma *affine_sing* [*iff*]: *affine* {*x*}
unfolding *affine_alt* **by** (*auto simp*: *scaleR_left_distrib* [*symmetric*])

lemma *affine_UNIV* [*iff*]: *affine UNIV*
unfolding *affine_def* **by** *auto*

lemma *affine_Inter* [*intro*]: $(\bigwedge S.\ S{\in}\mathcal{F} \Longrightarrow affine\ S) \Longrightarrow affine\ (\bigcap \mathcal{F})$
unfolding *affine_def* **by** *auto*

lemma *affine_Int*[*intro*]: *affine S* ⟹ *affine T* ⟹ *affine* $(S \cap T)$
unfolding *affine_def* **by** *auto*

lemma *affine_scaling*: *affine S* ⟹ *affine* $((*_R)\ c\ `\ S)$
apply (*clarsimp simp*: *affine_def*)
apply (*rule_tac x*=*u* $*_R$ *x* + *v* $*_R$ *y* **in** *image_eqI*)
apply (*auto simp*: *algebra_simps*)
done

lemma *affine_affine_hull* [*simp*]: *affine*(*affine hull S*)
unfolding *hull_def*
using *affine_Inter*[*of* $\{T.\ affine\ T \wedge S \subseteq T\}$] **by** *auto*

lemma *affine_hull_eq*[*simp*]: (*affine hull s* = *s*) ⟷ *affine s*
by (*metis affine_affine_hull hull_same*)

lemma *affine_hyperplane*: *affine* $\{x.\ a \cdot x = b\}$
by (*simp add*: *affine_def algebra_simps*) (*metis distrib_right mult.left_neutral*)

Some explicit formulations

Formalized by Lars Schewe.

lemma *affine*:
fixes $V::'a::real\_vector\ set$
shows *affine* $V \longleftrightarrow$
$(\forall S\ u.\ finite\ S \wedge S \neq \{\} \wedge S \subseteq V \wedge sum\ u\ S = 1 \longrightarrow (\sum x{\in}S.\ u\ x *_R x) \in V)$
proof −
have $u *_R x + v *_R y \in V$ **if** $x \in V\ y \in V\ u + v = (1::real)$
and $*$: $\bigwedge S\ u.\ [\![finite\ S;\ S \neq \{\};\ S \subseteq V;\ sum\ u\ S = 1]\!] \Longrightarrow (\sum x{\in}S.\ u\ x *_R x) \in V$ **for** $x\ y\ u\ v$
proof (*cases* $x = y$)
case *True*
then show *?thesis*
using *that* **by** (*metis scaleR_add_left scaleR_one*)
next
case *False*
then show *?thesis*
using *that* $*[of\ \{x,y\}\ \lambda w.\ if\ w = x\ then\ u\ else\ v]$ **by** *auto*
qed
moreover have $(\sum x{\in}S.\ u\ x *_R x) \in V$
if $*$: $\bigwedge x\ y\ u\ v.\ [\![x{\in}V;\ y{\in}V;\ u + v = 1]\!] \Longrightarrow u *_R x + v *_R y \in V$
and *finite S* $S \neq \{\}$ $S \subseteq V$ *sum u S* $= 1$ **for** $S\ u$
proof −
define n **where** $n = card\ S$
consider $card\ S = 0 \mid card\ S = 1 \mid card\ S = 2 \mid card\ S > 2$ **by** *linarith*
then show $(\sum x{\in}S.\ u\ x *_R x) \in V$
proof *cases*
assume $card\ S = 1$
then obtain a **where** $S{=}\{a\}$
by (*auto simp*: *card_Suc_eq*)
then show *?thesis*
using *that* **by** *simp*
next
assume $card\ S = 2$
then obtain $a\ b$ **where** $S = \{a, b\}$
by (*metis Suc_1 card_1_singletonE card_Suc_eq*)
then show *?thesis*
using $*[of\ a\ b]$ *that*
by (*auto simp*: *sum_clauses*(*2*))
next
assume $card\ S > 2$
then show *?thesis* **using** *that n_def*
proof (*induct n arbitrary*: *u S*)
case *0*
then show *?case* **by** *auto*
next
case (*Suc n u S*)

```
      have sum u S = card S if ¬ (∃x∈S. u x ≠ 1)
        using that unfolding card_eq_sum by auto
      with Suc.prems obtain x where x ∈ S and x: u x ≠ 1 by force
      have c: card (S − {x}) = card S − 1
        by (simp add: Suc.prems(3) ‹x ∈ S›)
      have sum u (S − {x}) = 1 − u x
        by (simp add: Suc.prems sum_diff1 ‹x ∈ S›)
      with x have eq1: inverse (1 − u x) * sum u (S − {x}) = 1
        by auto
      have inV: (∑ y∈S − {x}. (inverse (1 − u x) * u y) *R y) ∈ V
      proof (cases card (S − {x}) > 2)
        case True
        then have S: S − {x} ≠ {} card (S − {x}) = n
          using Suc.prems c by force+
        show ?thesis
        proof (rule Suc.hyps)
          show (∑ a∈S − {x}. inverse (1 − u x) * u a) = 1
            by (auto simp: eq1 sum_distrib_left[symmetric])
        qed (use S Suc.prems True in auto)
      next
        case False
        then have card (S − {x}) = Suc (Suc 0)
          using Suc.prems c by auto
        then obtain a b where ab: (S − {x}) = {a, b} a≠b
          unfolding card_Suc_eq by auto
        then show ?thesis
          using eq1 ‹S ⊆ V›
         by (auto simp: sum_distrib_left distrib_left intro!: Suc.prems(2)[of a b])
      qed
      have u x + (1 − u x) = 1 ⟹
        u x *R x + (1 − u x) *R ((∑ y∈S − {x}. u y *R y) /R (1 − u x)) ∈ V
            by (rule Suc.prems) (use ‹x ∈ S› Suc.prems inV in ‹auto simp:
scaleR_right.sum›)
      moreover have (∑ a∈S. u a *R a) = u x *R x + (∑ a∈S − {x}. u a *R
a)
        by (meson Suc.prems(3) sum.remove ‹x ∈ S›)
      ultimately show (∑ x∈S. u x *R x) ∈ V
        by (simp add: x)
    qed
  qed (use ‹S≠{}› ‹finite S› in auto)
 qed
 ultimately show ?thesis
   unfolding affine_def by meson
qed


lemma affine_hull_explicit:
  affine hull p = {y. ∃S u. finite S ∧ S ≠ {} ∧ S ⊆ p ∧ sum u S = 1 ∧ sum (λv.
u v *R v) S = y}
```

```
  (is _ = ?rhs)
proof (rule hull_unique)
  have ⋀x. sum (λz. 1) {x} = 1
    by auto
  show p ⊆ ?rhs
  proof (intro subsetI CollectI exI conjI)
    show ⋀x. sum (λz. 1) {x} = 1
      by auto
  qed auto
  show ?rhs ⊆ T if p ⊆ T affine T for T
    using that unfolding affine by blast
  show affine ?rhs
    unfolding affine_def
  proof clarify
    fix u v :: real and sx ux sy uy
    assume uv: u + v = 1
      and x: finite sx sx ≠ {} sx ⊆ p sum ux sx = (1::real)
      and y: finite sy sy ≠ {} sy ⊆ p sum uy sy = (1::real)
    have **: (sx ∪ sy) ∩ sx = sx (sx ∪ sy) ∩ sy = sy
      by auto
    show ∃S w. finite S ∧ S ≠ {} ∧ S ⊆ p ∧
        sum w S = 1 ∧ (∑v∈S. w v *R v) = u *R (∑v∈sx. ux v *R v) + v *R
(∑v∈sy. uy v *R v)
    proof (intro exI conjI)
      show finite (sx ∪ sy)
        using x y by auto
      show sum (λi. (if i∈sx then u * ux i else 0) + (if i∈sy then v * uy i else 0))
(sx ∪ sy) = 1
        using x y uv
      by (simp add: sum_Un sum.distrib sum.inter_restrict[symmetric] sum_distrib_left
[symmetric] **)
      have (∑i∈sx ∪ sy. ((if i ∈ sx then u * ux i else 0) + (if i ∈ sy then v * uy
i else 0)) *R i)
          = (∑i∈sx. (u * ux i) *R i) + (∑i∈sy. (v * uy i) *R i)
        using x y
        unfolding scaleR_left_distrib scaleR_zero_left if_smult
        by (simp add: sum_Un sum.distrib sum.inter_restrict[symmetric]  **)
      also have ... = u *R (∑v∈sx. ux v *R v) + v *R (∑v∈sy. uy v *R v)
       unfolding scaleR_scaleR[symmetric] scaleR_right.sum [symmetric] by blast
      finally show (∑i∈sx ∪ sy. ((if i ∈ sx then u * ux i else 0) + (if i ∈ sy then
v * uy i else 0)) *R i)
                = u *R (∑v∈sx. ux v *R v) + v *R (∑v∈sy. uy v *R v) .
    qed (use x y in auto)
  qed
qed

lemma affine_hull_finite:
  assumes finite S
  shows affine hull S = {y. ∃u. sum u S = 1 ∧ sum (λv. u v *R v) S = y}
```

proof −
have ∗: ∃ *h*. *sum h S* = *1* ∧ (∑ *v*∈*S*. *h v* ∗$_R$ *v*) = *x*
if *F* ⊆ *S finite F F* ≠ {} **and** *sum*: *sum u F* = *1* **and** *x*: (∑ *v*∈*F*. *u v* ∗$_R$ *v*) = *x* **for** *x F u*
proof −
have *S* ∩ *F* = *F*
using *that* **by** *auto*
show *?thesis*
proof (*intro exI conjI*)
show (∑ *x*∈*S*. *if x* ∈ *F then u x else 0*) = *1*
by (*metis* (*mono_tags*, *lifting*) ‹*S* ∩ *F* = *F*› *assms sum.inter_restrict sum*)
show (∑ *v*∈*S*. (*if v* ∈ *F then u v else 0*) ∗$_R$ *v*) = *x*
by (*simp add*: *if_smult cong*: *if_cong*) (*metis* (*no_types*) ‹*S* ∩ *F* = *F*› *assms sum.inter_restrict x*)
qed
qed
show *?thesis*
unfolding *affine_hull_explicit* **using** *assms*
by (*fastforce dest*: ∗)
qed

Stepping theorems and hence small special cases

lemma *affine_hull_empty*[*simp*]: *affine hull* {} = {}
by *simp*

lemma *affine_hull_finite_step*:
fixes *y* :: *'a*::*real_vector*
shows *finite S* ⟹
(∃ *u*. *sum u* (*insert a S*) = *w* ∧ *sum* (λ*x*. *u x* ∗$_R$ *x*) (*insert a S*) = *y*) ⟷
(∃ *v u*. *sum u S* = *w* − *v* ∧ *sum* (λ*x*. *u x* ∗$_R$ *x*) *S* = *y* − *v* ∗$_R$ *a*) (**is** *_* ⟹ *?lhs* = *?rhs*)
proof −
assume *fin*: *finite S*
show *?lhs* = *?rhs*
proof
assume *?lhs*
then obtain *u* **where** *u*: *sum u* (*insert a S*) = *w* ∧ (∑ *x*∈*insert a S*. *u x* ∗$_R$ *x*) = *y*
by *auto*
show *?rhs*
proof (*cases a* ∈ *S*)
case *True*
then show *?thesis*
using *u* **by** (*simp add*: *insert_absorb*) (*metis diff_zero real_vector.scale_zero_left*)
next
case *False*
show *?thesis*
by (*rule exI* [**where** *x*=*u a*]) (*use u fin False* **in** *auto*)

```
    qed
  next
    assume ?rhs
    then obtain v u where vu: sum u S = w - v  (∑x∈S. u x *R x) = y - v
*R a
      by auto
    have *: ⋀x M. (if x = a then v else M) *R x = (if x = a then v *R x else M
*R x)
      by auto
    show ?lhs
    proof (cases a ∈ S)
      case True
      show ?thesis
        by (rule exI [where x=λx. (if x=a then v else 0) + u x])
            (simp add: True scaleR_left_distrib sum.distrib sum_clauses fin vu *
cong: if_cong)
    next
      case False
      then show ?thesis
        apply (rule_tac x=λx. if x=a then v else u x in exI)
        apply (simp add: vu sum_clauses(2)[OF fin] *)
        by (simp add: sum_delta_notmem(3) vu)
    qed
  qed
qed

lemma affine_hull_2:
  fixes a b :: 'a::real_vector
  shows affine hull {a,b} = {u *R a + v *R b| u v. (u + v = 1)}
  (is ?lhs = ?rhs)
proof -
  have *:
    ⋀x y z. z = x - y ⟷ y + z = (x::real)
    ⋀x y z. z = x - y ⟷ y + z = (x::'a) by auto
  have ?lhs = {y. ∃u. sum u {a, b} = 1 ∧ (∑ v∈{a, b}. u v *R v) = y}
    using affine_hull_finite[of {a,b}] by auto
  also have ... = {y. ∃v u. u b = 1 - v ∧ u b *R b = y - v *R a}
    by (simp add: affine_hull_finite_step[of {b} a])
  also have ... = ?rhs unfolding * by auto
  finally show ?thesis by auto
qed

lemma affine_hull_3:
  fixes a b c :: 'a::real_vector
  shows affine hull {a,b,c} = { u *R a + v *R b + w *R c| u v w. u + v + w =
1}
proof -
  have *:
    ⋀x y z. z = x - y ⟷ y + z = (x::real)
```

```
    ⋀x y z. z = x − y ⟷ y + z = (x::'a) by auto
  show ?thesis
    apply (simp add: affine_hull_finite affine_hull_finite_step)
    unfolding ∗
    apply safe
     apply (metis add.assoc)
    apply (rule_tac x=u in exI, force)
    done
qed

lemma mem_affine:
  assumes affine S x ∈ S y ∈ S u + v = 1
  shows u ∗R x + v ∗R y ∈ S
  using assms affine_def[of S] by auto

lemma mem_affine_3:
  assumes affine S x ∈ S y ∈ S z ∈ S u + v + w = 1
  shows u ∗R x + v ∗R y + w ∗R z ∈ S
proof −
  have u ∗R x + v ∗R y + w ∗R z ∈ affine hull {x, y, z}
    using affine_hull_3[of x y z] assms by auto
  moreover
  have affine hull {x, y, z} ⊆ affine hull S
    using hull_mono[of {x, y, z} S] assms by auto
  moreover
  have affine hull S = S
    using assms affine_hull_eq[of S] by auto
  ultimately show ?thesis by auto
qed

lemma mem_affine_3_minus:
  assumes affine S x ∈ S y ∈ S z ∈ S
  shows x + v ∗R (y−z) ∈ S
  using mem_affine_3[of S x y z 1 v −v] assms
  by (simp add: algebra_simps)

corollary mem_affine_3_minus2:
    ⟦affine S; x ∈ S; y ∈ S; z ∈ S⟧ ⟹ x − v ∗R (y−z) ∈ S
 by (metis add_uminus_conv_diff mem_affine_3_minus real_vector.scale_minus_left)
```

Some relations between affine hull and subspaces

```
lemma affine_hull_insert_subset_span:
  affine hull (insert a S) ⊆ {a + v| v . v ∈ span {x − a | x . x ∈ S}}
proof −
  have ∃v T u. x = a + v ∧ (finite T ∧ T ⊆ {x − a |x. x ∈ S} ∧ (∑v∈T. u v
∗R v) = v)
    if finite F F ≠ {} F ⊆ insert a S sum u F = 1 (∑v∈F. u v ∗R v) = x
    for x F u
```

```
  proof −
    have ∗: (λx. x − a) ‘ (F − {a}) ⊆ {x − a |x. x ∈ S}
      using that by auto
    show ?thesis
    proof (intro exI conjI)
      show finite ((λx. x − a) ‘ (F − {a}))
        by (simp add: that(1))
      show (∑ v∈(λx. x − a) ‘ (F − {a}). u(v+a) ∗R v) = x−a
        by (simp add: sum.reindex[unfolded inj_on_def] algebra_simps
            sum_subtractf scaleR_left.sum[symmetric] sum_diff1 that)
    qed (use ‹F ⊆ insert a S› in auto)
  qed
  then show ?thesis
    unfolding affine_hull_explicit span_explicit by fast
qed

lemma affine_hull_insert_span:
  assumes a ∉ S
  shows affine hull (insert a S) = {a + v | v . v ∈ span {x − a | x.  x ∈ S}}
proof −
  have ∗: ∃ G u. finite G ∧ G ≠ {} ∧ G ⊆ insert a S ∧ sum u G = 1 ∧ (∑ v∈G.
u v ∗R v) = y
    if v ∈ span {x − a |x. x ∈ S} y = a + v for y v
  proof −
    from that
    obtain T u where u: finite T T ⊆ {x − a |x. x ∈ S} a + (∑ v∈T. u v ∗R v)
= y
      unfolding span_explicit by auto
    define F where F = (λx. x + a) ‘ T
    have F: finite F F ⊆ S (∑ v∈F. u (v − a) ∗R (v − a)) = y − a
     unfolding F_def using u by (auto simp: sum.reindex[unfolded inj_on_def])
    have ∗: F ∩ {a} = {} F ∩ − {a} = F
      using F assms by auto
    show ∃ G u. finite G ∧ G ≠ {} ∧ G ⊆ insert a S ∧ sum u G = 1 ∧ (∑ v∈G.
u v ∗R v) = y
      apply (rule_tac x = insert a F in exI)
      apply (rule_tac x = λx. if x=a then 1 − sum (λx. u (x − a)) F else u (x −
a) in exI)
      using assms F
     apply (auto simp:  sum_clauses sum.If_cases if_smult sum_subtractf scaleR_left.sum
algebra_simps ∗)
      done
  qed
  show ?thesis
   by (intro subset_antisym affine_hull_insert_subset_span) (auto simp: affine_hull_explicit
dest!: ∗)
qed

lemma affine_hull_span:
```

assumes $a \in S$
shows *affine hull* $S = \{a + v \mid v.\ v \in$ *span* $\{x - a \mid x.\ x \in S - \{a\}\}\}$
using *affine_hull_insert_span*[*of* a $S - \{a\}$, *unfolded insert_Diff*[*OF assms*]]
by *auto*

Parallel affine sets

definition *affine_parallel* :: $'a$::*real_vector set* $\Rightarrow$ $'a$::*real_vector set* $\Rightarrow$ *bool*
where *affine_parallel* $S\ T \longleftrightarrow (\exists a.\ T = (\lambda x.\ a + x)\ `\ S)$

lemma *affine_parallel_expl_aux*:
fixes $S\ T$ :: $'a$::*real_vector set*
assumes $\bigwedge x.\ x \in S \longleftrightarrow a + x \in T$
shows $T = (\lambda x.\ a + x)\ `\ S$
proof −
have $x \in ((\lambda x.\ a + x)\ `\ S)$ **if** $x \in T$ **for** x
using *that*
by (*simp add*: *image_iff*) (*metis add.commute diff_add_cancel assms*)
moreover have $T \geq (\lambda x.\ a + x)\ `\ S$
using *assms* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *affine_parallel_expl*: *affine_parallel* $S\ T \longleftrightarrow (\exists a.\ \forall x.\ x \in S \longleftrightarrow a + x \in T)$
by (*auto simp add*: *affine_parallel_def*)
(*use affine_parallel_expl_aux* [*of* S _ T] **in** *blast*)

lemma *affine_parallel_reflex*: *affine_parallel* $S\ S$
unfolding *affine_parallel_def*
using *image_add_0* **by** *blast*

lemma *affine_parallel_commute*:
assumes *affine_parallel* $A\ B$
shows *affine_parallel* $B\ A$
by (*metis affine_parallel_def assms translation_galois*)

lemma *affine_parallel_assoc*:
assumes *affine_parallel* $A\ B$
and *affine_parallel* $B\ C$
shows *affine_parallel* $A\ C$
by (*metis affine_parallel_def assms translation_assoc*)

lemma *affine_translation_aux*:
fixes a :: $'a$::*real_vector*
assumes *affine* $((\lambda x.\ a + x)\ `\ S)$
shows *affine* S
proof −
{

fix *x y u v*
assume *xy*: $x \in S$ $y \in S$ $(u :: real) + v = 1$
then have $(a + x) \in ((\lambda x.\ a + x)\ `\ S)$ $(a + y) \in ((\lambda x.\ a + x)\ `\ S)$
by *auto*
then have *h1*: $u *_R\ (a + x) + v *_R (a + y) \in (\lambda x.\ a + x)\ `\ S$
using *xy assms* **unfolding** *affine_def* **by** *auto*
have $u *_R (a + x) + v *_R (a + y) = (u + v) *_R a + (u *_R x + v *_R y)$
by (*simp add*: *algebra_simps*)
also have $\ldots = a + (u *_R x + v *_R y)$
using ‹$u + v = 1$› **by** *auto*
ultimately have $a + (u *_R x + v *_R y) \in (\lambda x.\ a + x)\ `\ S$
using *h1* **by** *auto*
then have $u *_R x + v *_R y \in S$ **by** *auto*
}
then show *?thesis* **unfolding** *affine_def* **by** *auto*
qed

lemma *affine_translation*:
affine $S \longleftrightarrow$ *affine* $((+)\ a\ `\ S)$ **for** $a :: 'a$::*real_vector*
by (*metis affine_translation_aux translation_galois*)

lemma *parallel_is_affine*:
fixes $S\ T :: 'a$::*real_vector set*
assumes *affine S affine_parallel S T*
shows *affine T*
by (*metis affine_parallel_def affine_translation assms*)

lemma *subspace_imp_affine*: *subspace s* $\Longrightarrow$ *affine s*
unfolding *subspace_def affine_def* **by** *auto*

lemma *affine_hull_subset_span*: $(affine\ hull\ s) \subseteq (span\ s)$
by (*metis hull_minimal span_superset subspace_imp_affine subspace_span*)

Subspace parallel to an affine set

lemma *subspace_affine*: *subspace* $S \longleftrightarrow$ *affine* $S \wedge 0 \in S$
by (*metis add_cancel_right_left affine_alt diff_add_cancel mem_affine_3 scaleR_eq_0_iff subspace_def vector_space_assms*(4))

lemma *affine_diffs_subspace*:
assumes *affine* S $a \in S$
shows *subspace* $((\lambda x.\ (-a)+x)\ `\ S)$
by (*metis ab_left_minus affine_translation assms image_eqI subspace_affine*)

lemma *affine_diffs_subspace_subtract*:
subspace $((\lambda x.\ x - a)\ `\ S)$ **if** *affine* S $a \in S$
using *that affine_diffs_subspace* [*of* _ *a*] **by** *simp*

lemma *parallel_subspace_explicit*:

assumes *affine S*
and $a \in S$
assumes $L \equiv \{y.\ \exists x \in S.\ (-a) + x = y\}$
shows *subspace L* $\wedge$ *affine_parallel S L*
by (*smt* (*verit*) *Collect_cong ab_left_minus affine_parallel_def assms image_def mem_Collect_eq parallel_is_affine subspace_affine*)

lemma *parallel_subspace_aux*:
assumes *subspace A*
and *subspace B*
and *affine_parallel A B*
shows $A \supseteq B$
by (*metis add.right_neutral affine_parallel_expl assms subsetI subspace_def*)

lemma *parallel_subspace*:
assumes *subspace A*
and *subspace B*
and *affine_parallel A B*
shows $A = B$
by (*simp add*: *affine_parallel_commute assms parallel_subspace_aux subset_antisym*)

lemma *affine_parallel_subspace*:
assumes *affine S* $S \neq \{\}$
shows $\exists!L.$ *subspace L* $\wedge$ *affine_parallel S L*
by (*meson affine_parallel_assoc affine_parallel_commute assms equals0I parallel_subspace parallel_subspace_explicit*)

1.6.2 Affine Dependence

Formalized by Lars Schewe.

definition *affine_dependent* :: *'a*::*real_vector set* $\Rightarrow$ *bool*
where *affine_dependent S* $\longleftrightarrow$ $(\exists x{\in}S.\ x \in$ *affine hull* $(S - \{x\}))$

lemma *affine_dependent_imp_dependent*: *affine_dependent S* $\Longrightarrow$ *dependent S*
unfolding *affine_dependent_def dependent_def*
using *affine_hull_subset_span* **by** *auto*

lemma *affine_dependent_subset*:
$[\![$*affine_dependent S*; $S \subseteq T]\!] \Longrightarrow$ *affine_dependent T*
using *hull_mono* [*OF Diff_mono* [*OF _ subset_refl*]]
by (*smt* (*verit*) *affine_dependent_def subsetD*)

lemma *affine_independent_subset*:
shows $[\![\neg$ *affine_dependent T*; $S \subseteq T]\!] \Longrightarrow \neg$ *affine_dependent S*
by (*metis affine_dependent_subset*)

lemma *affine_independent_Diff*:
$\neg$ *affine_dependent S* $\Longrightarrow \neg$ *affine_dependent*$(S - T)$
by (*meson Diff_subset affine_dependent_subset*)

proposition *affine_dependent_explicit*:
affine_dependent p $\longleftrightarrow$
$(\exists\, S\ U.\ \textit{finite}\ S \wedge S \subseteq p \wedge \textit{sum}\ U\ S = 0 \wedge (\exists\, v{\in}S.\ U\ v \neq 0) \wedge \textit{sum}\ (\lambda v.\ U\ v *_R v)\ S = 0)$
proof −
have $\exists\, S\ U.\ \textit{finite}\ S \wedge S \subseteq p \wedge \textit{sum}\ U\ S = 0 \wedge (\exists\, v{\in}S.\ U\ v \neq 0) \wedge (\sum w{\in}S.\ U\ w *_R w) = 0$
if $(\sum w{\in}S.\ U\ w *_R w) = x\ x \in p\ \textit{finite}\ S\ S \neq \{\}\ S \subseteq p - \{x\}\ \textit{sum}\ U\ S = 1$ **for** *x S U*
proof (*intro exI conjI*)
have $x \notin S$
using *that* **by** *auto*
then show $(\sum v \in \textit{insert}\ x\ S.\ \textit{if}\ v = x\ \textit{then} - 1\ \textit{else}\ U\ v) = 0$
using *that* **by** (*simp add*: *sum_delta_notmem*)
show $(\sum w \in \textit{insert}\ x\ S.\ (\textit{if}\ w = x\ \textit{then} - 1\ \textit{else}\ U\ w) *_R w) = 0$
using *that* ‹$x \notin S$› **by** (*simp add*: *if_smult sum_delta_notmem cong*: *if_cong*)
qed (*use that* **in** *auto*)
moreover have $\exists\, x{\in}p.\ \exists\, S\ U.\ \textit{finite}\ S \wedge S \neq \{\} \wedge S \subseteq p - \{x\} \wedge \textit{sum}\ U\ S = 1 \wedge (\sum v{\in}S.\ U\ v *_R v) = x$
if $(\sum v{\in}S.\ U\ v *_R v) = 0\ \textit{finite}\ S\ S \subseteq p\ \textit{sum}\ U\ S = 0\ v \in S\ U\ v \neq 0$ **for** *S U v*
proof (*intro bexI exI conjI*)
have $S \neq \{v\}$
using *that* **by** *auto*
then show $S - \{v\} \neq \{\}$
using *that* **by** *auto*
show $(\sum x \in S - \{v\}.\ -\ (1\ /\ U\ v) * U\ x) = 1$
unfolding *sum_distrib_left*[*symmetric*] *sum_diff1*[*OF* ‹*finite S*›] **by** (*simp add*: *that*)
show $(\sum x{\in}S - \{v\}.\ (-\ (1\ /\ U\ v) * U\ x) *_R x) = v$
unfolding *sum_distrib_left* [*symmetric*] *scaleR_scaleR*[*symmetric*]
scaleR_right.sum [*symmetric*] *sum_diff1*[*OF* ‹*finite S*›]
using *that* **by** *auto*
show $S - \{v\} \subseteq p - \{v\}$
using *that* **by** *auto*
qed (*use that* **in** *auto*)
ultimately show *?thesis*
unfolding *affine_dependent_def affine_hull_explicit* **by** *auto*
qed

lemma *affine_dependent_explicit_finite*:
fixes *S* :: *'a*::*real_vector set*
assumes *finite S*
shows *affine_dependent S* $\longleftrightarrow$
$(\exists\, U.\ \textit{sum}\ U\ S = 0 \wedge (\exists\, v{\in}S.\ U\ v \neq 0) \wedge \textit{sum}\ (\lambda v.\ U\ v *_R v)\ S = 0)$
(**is** *?lhs* = *?rhs*)
proof
have $*$: $\bigwedge vt\ U\ v.\ (\textit{if}\ vt\ \textit{then}\ U\ v\ \textit{else}\ 0) *_R v = (\textit{if}\ vt\ \textit{then}\ (U\ v) *_R v\ \textit{else}\ 0{::}'a)$

```
    by auto
  assume ?lhs
  then obtain T U v where
    finite T T ⊆ S sum U T = 0 v∈T U v ≠ 0  (∑ v∈T. U v *R v) = 0
    unfolding affine_dependent_explicit by auto
  then show ?rhs
    apply (rule_tac x=λx. if x∈T then U x else 0 in exI)
   apply (auto simp: * sum.inter_restrict[OF assms, symmetric] Int_absorb1[OF
‹T⊆S›])
    done
next
  assume ?rhs
  then obtain U v where sum U S = 0  v∈S U v ≠ 0 (∑ v∈S. U v *R v) = 0
    by auto
  then show ?lhs unfolding affine_dependent_explicit
    using assms by auto
qed

lemma dependent_imp_affine_dependent:
  assumes dependent {x - a| x . x ∈ S}
    and a ∉ S
  shows affine_dependent (insert a S)
proof -
  from assms(1)[unfolded dependent_explicit] obtain S' U v
    where S: finite S' S' ⊆ {x - a |x. x ∈ S} v∈S' U v  ≠ 0 (∑ v∈S'. U v *R v)
= 0
    by auto
  define T where T = (λx. x + a) ` S'
  have inj: inj_on (λx. x + a) S'
    unfolding inj_on_def by auto
  have 0 ∉ S'
    using S(2) assms(2) unfolding subset_eq by auto
  have fin: finite T and T ⊆ S
    unfolding T_def using S(1,2) by auto
  then have finite (insert a T) and insert a T ⊆ insert a S
    by auto
  moreover have *: ⋀P Q. (∑ x∈T. (if x = a then P x else Q x)) = (∑ x∈T. Q
x)
    by (smt (verit, best) ‹T ⊆ S› assms(2) subsetD sum.cong)
  have (∑ x∈insert a T. if x = a then - (∑ x∈T. U (x - a)) else U (x - a)) =
0
    by (smt (verit) ‹T ⊆ S› assms(2) fin insert_absorb insert_subset sum.insert
sum_mono)
  moreover have ∃ v∈insert a T. (if v = a then - (∑ x∈T. U (x - a)) else U
(v - a)) ≠ 0
    using S(3,4) ‹0∉S'›
    by (rule_tac x=v + a in bexI) (auto simp: T_def)
 moreover have *: ⋀P Q. (∑ x∈T. (if x = a then P x else Q x) *R x) = (∑ x∈T.
Q x *R x)
```

```
    using ‹a∉S› ‹T⊆S› by (auto intro!: sum.cong)
  have (∑ x∈T. U (x − a)) *R a = (∑ v∈T. U (v − a) *R v)
    unfolding scaleR_left.sum
    unfolding T_def and sum.reindex[OF inj] and o_def
    using S(5)
    by (auto simp: sum.distrib scaleR_right_distrib)
  then have (∑ v∈insert a T. (if v = a then − (∑ x∈T. U (x − a)) else U (v −
a)) *R v) = 0
    unfolding sum_clauses(2)[OF fin] using ‹a∉S› ‹T⊆S› by (auto simp: *)
  ultimately show ?thesis
    unfolding affine_dependent_explicit
    by (force intro!: exI[where x=insert a T])
qed

lemma affine_dependent_biggerset:
  fixes S :: 'a::euclidean_space set
  assumes finite S card S ≥ DIM('a) + 2
  shows affine_dependent S
proof −
  have S ≠ {} using assms by auto
  then obtain a where a∈S by auto
  have *: {x − a |x. x ∈ S − {a}} = (λx. x − a) ‘ (S − {a})
    by auto
  have card {x − a |x. x ∈ S − {a}} = card (S − {a})
    unfolding * by (simp add: card_image inj_on_def)
  also have ... > DIM('a) using assms(2)
    unfolding card_Diff_singleton[OF ‹a∈S›] by auto
  finally  have affine_dependent (insert a (S − {a}))
    using dependent_biggerset dependent_imp_affine_dependent by blast
  then show ?thesis
    by (simp add: ‹a ∈ S› insert_absorb)
qed

lemma affine_dependent_biggerset_general:
  assumes finite (S :: 'a::euclidean_space set)
    and card S ≥ dim S + 2
  shows affine_dependent S
proof −
  from assms(2) have S ≠ {} by auto
  then obtain a where a∈S by auto
  have *: {x − a |x. x ∈ S − {a}} = (λx. x − a) ‘ (S − {a})
    by auto
  have **: card {x − a |x. x ∈ S − {a}} = card (S − {a})
    by (metis (no_types, lifting) * card_image diff_add_cancel inj_on_def)
  have dim {x − a |x. x ∈ S − {a}} ≤ dim S
    using ‹a∈S› by (auto simp: span_base span_diff intro: subset_le_dim)
  also have ... < dim S + 1 by auto
  also have ... ≤ card (S − {a})
    using assms card_Diff_singleton[OF ‹a∈S›] by auto
```

finally have *affine_dependent* (*insert a* (*S* − {*a*}))
by (*smt* (*verit*) *Collect_cong dependent_imp_affine_dependent dependent_biggerset_general* ** *Diff_iff insertCI*)
then show *?thesis*
by (*simp add*: ‹*a* ∈ *S*› *insert_absorb*)
qed

1.6.3 Some Properties of Affine Dependent Sets

lemma *affine_independent_0* [*simp*]: ¬ *affine_dependent* {}
by (*simp add*: *affine_dependent_def*)

lemma *affine_independent_1* [*simp*]: ¬ *affine_dependent* {*a*}
by (*simp add*: *affine_dependent_def*)

lemma *affine_independent_2* [*simp*]: ¬ *affine_dependent* {*a*,*b*}
by (*simp add*: *affine_dependent_def insert_Diff_if hull_same*)

lemma *affine_hull_translation*: *affine hull* ((λ*x*. *a* + *x*) ‘ *S*) = (λ*x*. *a* + *x*) ‘ (*affine hull S*)
proof −
have *affine* ((λ*x*. *a* + *x*) ‘ (*affine hull S*))
using *affine_translation affine_affine_hull* **by** *blast*
moreover have (λ*x*. *a* + *x*) ‘ *S* ⊆ (λ*x*. *a* + *x*) ‘ (*affine hull S*)
using *hull_subset*[*of S*] **by** *auto*
ultimately have *h1*: *affine hull* ((λ*x*. *a* + *x*) ‘ *S*) ⊆ (λ*x*. *a* + *x*) ‘ (*affine hull S*)
by (*metis hull_minimal*)
have *affine*((λ*x*. −*a* + *x*) ‘ (*affine hull* ((λ*x*. *a* + *x*) ‘ *S*)))
using *affine_translation affine_affine_hull* **by** *blast*
moreover have (λ*x*. −*a* + *x*) ‘ (λ*x*. *a* + *x*) ‘ *S* ⊆ (λ*x*. −*a* + *x*) ‘ (*affine hull* ((λ*x*. *a* + *x*) ‘ *S*))
using *hull_subset*[*of* (λ*x*. *a* + *x*) ‘ *S*] **by** *auto*
moreover have *S* = (λ*x*. −*a* + *x*) ‘ (λ*x*. *a* + *x*) ‘ *S*
using *translation_assoc*[*of* −*a a*] **by** *auto*
ultimately have (λ*x*. −*a* + *x*) ‘ (*affine hull* ((λ*x*. *a* + *x*) ‘ *S*)) >= (*affine hull S*)
by (*metis hull_minimal*)
then have *affine hull* ((λ*x*. *a* + *x*) ‘ *S*) >= (λ*x*. *a* + *x*) ‘ (*affine hull S*)
by *auto*
then show *?thesis* **using** *h1* **by** *auto*
qed

lemma *affine_dependent_translation*:
assumes *affine_dependent S*
shows *affine_dependent* ((λ*x*. *a* + *x*) ‘ *S*)
proof −
obtain *x* **where** *x*: *x* ∈ *S* ∧ *x* ∈ *affine hull* (*S* − {*x*})
using *assms affine_dependent_def* **by** *auto*

```
  have (+) a ‘ (S − {x}) = (+) a ‘ S − {a + x}
    by auto
  then have a + x ∈ affine hull ((λx. a + x) ‘ S − {a + x})
    using affine_hull_translation[of a S − {x}] x by auto
  moreover have a + x ∈ (λx. a + x) ‘ S
    using x by auto
  ultimately show ?thesis
    unfolding affine_dependent_def by auto
qed

lemma affine_dependent_translation_eq:
  affine_dependent S ⟷ affine_dependent ((λx. a + x) ‘ S)
  by (metis affine_dependent_translation translation_galois)

lemma affine_hull_0_dependent:
  assumes 0 ∈ affine hull S
  shows dependent S
proof −
  obtain s u where s_u: finite s ∧ s ≠ {} ∧ s ⊆ S ∧ sum u s = 1 ∧ (∑ v∈s. u
v *R v) = 0
    using assms affine_hull_explicit[of S] by auto
  then have ∃ v∈s. u v ≠ 0 by auto
  then have finite s ∧ s ⊆ S ∧ (∃ v∈s. u v ≠ 0 ∧ (∑ v∈s. u v *R v) = 0)
    using s_u by auto
  then show ?thesis
    unfolding dependent_explicit[of S] by auto
qed

lemma affine_dependent_imp_dependent2:
  assumes affine_dependent (insert 0 S)
  shows dependent S
proof −
  obtain x where x: x ∈ insert 0 S ∧ x ∈ affine hull (insert 0 S − {x})
    using affine_dependent_def[of (insert 0 S)] assms by blast
  then have x ∈ span (insert 0 S − {x})
    using affine_hull_subset_span by auto
  moreover have span (insert 0 S − {x}) = span (S − {x})
    using insert_Diff_if[of 0 S {x}] span_insert_0[of S−{x}] by auto
  ultimately have x ∈ span (S − {x}) by auto
  then have x ≠ 0 ⟹ dependent S
    using x dependent_def by auto
  moreover have dependent S if x = 0
   by (metis that affine_hull_0_dependent Diff_insert_absorb dependent_zero x)
  ultimately show ?thesis by auto
qed

lemma affine_dependent_iff_dependent:
  assumes a ∉ S
  shows affine_dependent (insert a S) ⟷ dependent ((λx. −a + x) ‘ S)
```

proof −
have $((+)\ (-\ a)\ `\ S) = \{x - a|\ x\ .\ x \in S\}$ **by** *auto*
then show *?thesis*
using *affine_dependent_translation_eq*[*of* (*insert a S*) −*a*]
affine_dependent_imp_dependent2 assms
dependent_imp_affine_dependent[*of a S*]
by (*auto simp del*: *uminus_add_conv_diff*)
qed

lemma *affine_dependent_iff_dependent2*:
assumes $a \in S$
shows *affine_dependent S* $\longleftrightarrow$ *dependent* $((\lambda x.\ -a + x)\ `\ (S-\{a\}))$
by (*metis Diff_iff affine_dependent_iff_dependent assms insert_Diff singletonI*)

lemma *affine_hull_insert_span_gen*:
affine hull (*insert a S*) = $(\lambda x.\ a + x)\ `$ *span* $((\lambda x.\ -\ a + x)\ `\ S)$
proof −
have *h1*: $\{x - a\ |x.\ x \in S\} = ((\lambda x.\ -a+x)\ `\ S)$
by *auto*
{
assume $a \notin S$
then have *?thesis*
using *affine_hull_insert_span*[*of a S*] *h1* **by** *auto*
}
moreover
{
assume *a1*: $a \in S$
then have *insert 0* $((\lambda x.\ -a+x)\ `\ (S - \{a\})) = (\lambda x.\ -a+x)\ `\ S$
by *auto*
then have *span* $((\lambda x.\ -a+x)\ `\ (S - \{a\})) =$ *span* $((\lambda x.\ -a+x)\ `\ S)$
using *span_insert_0*[*of* (+) (− *a*) ` (*S* − {*a*})]
by *presburger*
moreover have $\{x - a\ |x.\ x \in (S - \{a\})\} = ((\lambda x.\ -a+x)\ `\ (S - \{a\}))$
by *auto*
moreover have *insert a* $(S - \{a\}) =$ *insert a S*
by *auto*
ultimately have *?thesis*
using *affine_hull_insert_span*[*of a S*−{*a*}] **by** *auto*
}
ultimately show *?thesis* **by** *auto*
qed

lemma *affine_hull_span2*:
assumes $a \in S$
shows *affine hull S* = $(\lambda x.\ a+x)\ `$ *span* $((\lambda x.\ -a+x)\ `\ (S-\{a\}))$
by (*metis affine_hull_insert_span_gen assms insert_Diff*)

lemma *affine_hull_span_gen*:
assumes $a \in$ *affine hull S*

shows *affine hull S* = (λ*x*. *a*+*x*) ‘ *span* ((λ*x*. −*a*+*x*) ‘ *S*)
by (*metis affine_hull_insert_span_gen assms hull_redundant*)

lemma *affine_hull_span_0*:
assumes *0* ∈ *affine hull S*
shows *affine hull S* = *span S*
using *affine_hull_span_gen*[*of 0 S*] *assms* **by** *auto*

lemma *extend_to_affine_basis_nonempty*:
fixes *S V* :: *'n*::*real_vector set*
assumes ¬ *affine_dependent S S* ⊆ *V S* ≠ {}
shows ∃ *T*. ¬ *affine_dependent T* ∧ *S* ⊆ *T* ∧ *T* ⊆ *V* ∧ *affine hull T* = *affine hull V*
proof −
obtain *a* **where** *a*: *a* ∈ *S*
using *assms* **by** *auto*
then have *h0*: *independent* ((λ*x*. −*a* + *x*) ‘ (*S*−{*a*}))
using *affine_dependent_iff_dependent2 assms* **by** *auto*
obtain *B* **where** *B*:
(λ*x*. −*a*+*x*) ‘ (*S* − {*a*}) ⊆ *B* ∧ *B* ⊆ (λ*x*. −*a*+*x*) ‘ *V* ∧ *independent B* ∧ (λ*x*. −*a*+*x*) ‘ *V* ⊆ *span B*
using *assms*
by (*blast intro*: *maximal_independent_subset_extend*[*OF* _ *h0*, *of* (λ*x*. −*a* + *x*) ‘ *V*])
define *T* **where** *T* = (λ*x*. *a*+*x*) ‘ *insert 0 B*
then have *T* = *insert a* ((λ*x*. *a*+*x*) ‘ *B*)
by *auto*
then have *affine hull T* = (λ*x*. *a*+*x*) ‘ *span B*
using *affine_hull_insert_span_gen*[*of a* ((λ*x*. *a*+*x*) ‘ *B*)] *translation_assoc*[*of* −*a a B*]
by *auto*
then have *V* ⊆ *affine hull T*
using *B assms translation_inverse_subset*[*of a V span B*]
by *auto*
moreover have *T* ⊆ *V*
using *T_def B a assms* **by** *auto*
ultimately have *affine hull T* = *affine hull V*
by (*metis Int_absorb1 Int_absorb2 hull_hull hull_mono*)
moreover have *S* ⊆ *T*
using *T_def B translation_inverse_subset*[*of a S*−{*a*} *B*]
by *auto*
moreover have ¬ *affine_dependent T*
using *T_def affine_dependent_translation_eq*[*of insert 0 B*]
affine_dependent_imp_dependent2 B
by *auto*
ultimately show *?thesis* **using** ‹*T* ⊆ *V*› **by** *auto*
qed

lemma *affine_basis_exists*:

fixes $V :: 'n::real\_vector\ set$
shows $\exists B.\ B \subseteq V \wedge \neg\ affine\_dependent\ B \wedge affine\ hull\ V = affine\ hull\ B$
by (*metis affine_dependent_def affine_independent_1 empty_subsetI extend_to_affine_basis_nonempty insert_subset order_refl*)

proposition *extend_to_affine_basis*:
fixes $S\ V :: 'n::real\_vector\ set$
assumes $\neg\ affine\_dependent\ S\ S \subseteq V$
obtains T **where** $\neg\ affine\_dependent\ T\ S \subseteq T\ T \subseteq V\ affine\ hull\ T = affine\ hull\ V$
by (*metis affine_basis_exists assms(1) assms(2) bot.extremum extend_to_affine_basis_nonempty*)

1.6.4 Affine Dimension of a Set

definition $aff\_dim :: ('a::euclidean\_space)\ set \Rightarrow int$
where $aff\_dim\ V =$
$(SOME\ d :: int.$
$\exists B.\ affine\ hull\ B = affine\ hull\ V \wedge \neg\ affine\_dependent\ B \wedge of\_nat\ (card\ B) = d + 1)$

lemma *aff_dim_basis_exists*:
fixes $V :: ('n::euclidean\_space)\ set$
shows $\exists B.\ affine\ hull\ B = affine\ hull\ V \wedge \neg\ affine\_dependent\ B \wedge of\_nat\ (card\ B) = aff\_dim\ V + 1$
proof −
obtain B **where** $\neg\ affine\_dependent\ B \wedge affine\ hull\ B = affine\ hull\ V$
using *affine_basis_exists*[*of V*] **by** *auto*
then show *?thesis*
unfolding *aff_dim_def*
some_eq_ex[*of* $\lambda d.\ \exists B.\ affine\ hull\ B = affine\ hull\ V \wedge \neg\ affine\_dependent\ B \wedge of\_nat\ (card\ B) = d + 1$]
apply *auto*
apply (*rule exI*[*of* _ *int* (*card B*) − (*1* :: *int*)])
apply (*rule exI*[*of* _ *B*], *auto*)
done
qed

lemma *affine_hull_eq_empty* [*simp*]: $affine\ hull\ S = \{\} \longleftrightarrow S = \{\}$
by (*metis affine_empty subset_empty subset_hull*)

lemma *empty_eq_affine_hull*[*simp*]: $\{\} = affine\ hull\ S \longleftrightarrow S = \{\}$
by (*metis affine_hull_eq_empty*)

lemma *aff_dim_parallel_subspace_aux*:
fixes $B :: 'n::euclidean\_space\ set$
assumes $\neg\ affine\_dependent\ B\ a \in B$
shows $finite\ B \wedge ((card\ B) - 1 = dim\ (span\ ((\lambda x.\ -a+x)\ `\ (B-\{a\}))))$
proof −
have $independent\ ((\lambda x.\ -a + x)\ `\ (B-\{a\}))$

```
    using affine_dependent_iff_dependent2 assms by auto
  then have fin: dim (span ((λx. −a+x) ‘ (B−{a}))) = card ((λx. −a + x) ‘
(B−{a}))
    finite ((λx. −a + x) ‘ (B − {a}))
    using indep_card_eq_dim_span[of (λx. −a+x) ‘ (B−{a})] by auto
  show ?thesis
  proof (cases (λx. −a + x) ‘ (B − {a}) = {})
    case True
    have B = insert a ((λx. a + x) ‘ (λx. −a + x) ‘ (B − {a}))
      using translation_assoc[of a −a (B − {a})] assms by auto
    then have B = {a} using True by auto
    then show ?thesis using assms fin by auto
  next
    case False
    then have card ((λx. −a + x) ‘ (B − {a})) > 0
      using fin by auto
    moreover have h1: card ((λx. −a + x) ‘ (B−{a})) = card (B−{a})
      by (rule card_image) (use translate_inj_on in blast)
    ultimately have card (B−{a}) > 0 by auto
    then have ∗: finite (B − {a})
      using card_gt_0_iff[of (B − {a})] by auto
    then have card (B − {a}) = card B − 1
      using card_Diff_singleton assms by auto
    with ∗ show ?thesis using fin h1 by auto
  qed
qed

lemma aff_dim_parallel_subspace:
  fixes V L :: 'n::euclidean_space set
  assumes V ≠ {}
    and subspace L
    and affine_parallel (affine hull V) L
  shows aff_dim V = int (dim L)
proof −
  obtain B where
    B: affine hull B = affine hull V ∧ ¬ affine_dependent B ∧ int (card B) =
aff_dim V + 1
    using aff_dim_basis_exists by auto
  then have B ≠ {}
    using assms B
    by auto
  then obtain a where a: a ∈ B by auto
  define Lb where Lb = span ((λx. −a+x) ‘ (B−{a}))
  moreover have affine_parallel (affine hull B) Lb
    using Lb_def B assms affine_hull_span2[of a B] a
      affine_parallel_commute[of Lb (affine hull B)]
    unfolding affine_parallel_def
    by auto
  moreover have subspace Lb
```

```
    using Lb_def subspace_span by auto
  moreover have affine hull B ≠ {}
    using assms B by auto
  ultimately have L = Lb
    using assms affine_parallel_subspace[of affine hull B] affine_affine_hull[of B]
B
    by auto
  then have dim L = dim Lb
    by auto
  moreover have card B − 1 = dim Lb and finite B
    using Lb_def aff_dim_parallel_subspace_aux a B by auto
  ultimately show ?thesis
    using B ‹B ≠ {}› card_gt_0_iff[of B] by auto
qed

lemma aff_independent_finite:
  fixes B :: 'n::euclidean_space set
  assumes ¬ affine_dependent B
  shows finite B
  using aff_dim_parallel_subspace_aux assms finite.simps by fastforce


lemma aff_dim_empty:
  fixes S :: 'n::euclidean_space set
  shows S = {} ⟷ aff_dim S = −1
proof −
  obtain B where ∗: affine hull B = affine hull S
    and ¬ affine_dependent B
    and int (card B) = aff_dim S + 1
    using aff_dim_basis_exists by auto
  moreover
  from ∗ have S = {} ⟷ B = {}
    by auto
  ultimately show ?thesis
    using aff_independent_finite[of B] card_gt_0_iff[of B] by auto
qed

lemma aff_dim_empty_eq [simp]: aff_dim ({}::'a::euclidean_space set) = −1
  using aff_dim_empty by blast

lemma aff_dim_affine_hull [simp]: aff_dim (affine hull S) = aff_dim S
  unfolding aff_dim_def using hull_hull[of _ S] by auto

lemma aff_dim_affine_hull2:
  assumes affine hull S = affine hull T
  shows aff_dim S = aff_dim T
  unfolding aff_dim_def using assms by auto

lemma aff_dim_unique:
```

fixes B V :: $'n$::*euclidean_space set*
assumes *affine hull* B = *affine hull* $V \wedge \neg$ *affine_dependent* B
shows *of_nat* (*card* B) = *aff_dim* V + *1*
proof (*cases* $B = \{\}$)
case *True*
then have $V = \{\}$
using *assms*
by *auto*
then have *aff_dim* V = (-1::*int*)
using *aff_dim_empty* **by** *auto*
then show *?thesis*
using ‹$B = \{\}$› **by** *auto*
next
case *False*
then obtain *a* **where** *a*: $a \in B$ **by** *auto*
define *Lb* **where** *Lb* = *span* ((λ*x*. $-a$+*x*) ‘ (B−{*a*}))
have *affine_parallel* (*affine hull B*) *Lb*
using *Lb_def affine_hull_span2*[*of a B*] *a*
affine_parallel_commute[*of Lb* (*affine hull B*)]
unfolding *affine_parallel_def* **by** *auto*
moreover have *subspace Lb*
using *Lb_def subspace_span* **by** *auto*
ultimately have *aff_dim B* = *int*(*dim Lb*)
using *aff_dim_parallel_subspace*[*of B Lb*] ‹$B \neq \{\}$› **by** *auto*
moreover have (*card B*) − *1* = *dim Lb finite B*
using *Lb_def aff_dim_parallel_subspace_aux a assms* **by** *auto*
ultimately have *of_nat* (*card B*) = *aff_dim B* + *1*
using ‹$B \neq \{\}$› *card_gt_0_iff*[*of B*] **by** *auto*
then show *?thesis*
using *aff_dim_affine_hull2 assms* **by** *auto*
qed

lemma *aff_dim_affine_independent*:
fixes B :: $'n$::*euclidean_space set*
assumes ¬ *affine_dependent B*
shows *of_nat* (*card B*) = *aff_dim B* + *1*
using *aff_dim_unique*[*of B B*] *assms* **by** *auto*

lemma *affine_independent_iff_card*:
fixes S :: $'a$::*euclidean_space set*
shows ¬ *affine_dependent S* ⟷ *finite S* ∧ *aff_dim S* = *int*(*card S*) − *1* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*simp add*: *aff_dim_affine_independent aff_independent_finite*)
show *?rhs* ⟹ *?lhs*
by (*metis of_nat_eq_iff affine_basis_exists aff_dim_unique card_subset_eq diff_add_cancel*)
qed

lemma *aff_dim_sing* [*simp*]:
fixes *a* :: *'n*::*euclidean_space*
shows *aff_dim* {*a*} = *0*
using *aff_dim_affine_independent*[*of* {*a*}] *affine_independent_1* **by** *auto*

lemma *aff_dim_2* [*simp*]:
fixes *a* :: *'n*::*euclidean_space*
shows *aff_dim* {*a*,*b*} = (*if a* = *b then 0 else 1*)
proof (*clarsimp*)
assume *a* ≠ *b*
then have *aff_dim*{*a*,*b*} = *card*{*a*,*b*} − *1*
using *affine_independent_2* [*of a b*] *aff_dim_affine_independent* **by** *fastforce*
also have ... = *1*
using ‹*a* ≠ *b*› **by** *simp*
finally show *aff_dim* {*a*, *b*} = *1* **.**
qed

lemma *aff_dim_inner_basis_exists*:
fixes *V* :: (*'n*::*euclidean_space*) *set*
shows ∃ *B*. *B* ⊆ *V* ∧ *affine hull B* = *affine hull V* ∧
¬ *affine_dependent B* ∧ *of_nat* (*card B*) = *aff_dim V* + *1*
by (*metis aff_dim_unique affine_basis_exists*)

lemma *aff_dim_le_card*:
fixes *V* :: *'n*::*euclidean_space set*
assumes *finite V*
shows *aff_dim V* ≤ *of_nat* (*card V*) − *1*
by (*metis aff_dim_inner_basis_exists assms card_mono le_diff_eq of_nat_le_iff*)

lemma *aff_dim_parallel_le*:
fixes *S T* :: *'n*::*euclidean_space set*
assumes *affine_parallel* (*affine hull S*) (*affine hull T*)
shows *aff_dim S* ≤ *aff_dim T*
proof (*cases S*={} ∨ *T*={})
case *True*
then show *?thesis*
by (*smt* (*verit*, *best*) *aff_dim_affine_hull2 affine_hull_empty affine_parallel_def assms empty_is_image*)
next
case *False*
then obtain *L* **where** *L*: *subspace L affine_parallel* (*affine hull T*) *L*
by (*metis affine_affine_hull affine_hull_eq_empty affine_parallel_subspace*)
with *False* **show** *?thesis*
by (*metis aff_dim_parallel_subspace affine_parallel_assoc assms dual_order.refl*)
qed

lemma *aff_dim_parallel_eq*:
fixes *S T* :: *'n*::*euclidean_space set*

assumes *affine_parallel* (*affine hull S*) (*affine hull T*)
shows *aff_dim S* = *aff_dim T*
by (*smt* (*verit*, *del_insts*) *aff_dim_parallel_le affine_parallel_commute assms*)

lemma *aff_dim_translation_eq*:
aff_dim ((+) *a* ‘ *S*) = *aff_dim S* **for** *a* :: *'n*::*euclidean_space*
by (*metis aff_dim_parallel_eq affine_hull_translation affine_parallel_def*)

lemma *aff_dim_translation_eq_subtract*:
aff_dim ((λ*x*. *x* − *a*) ‘ *S*) = *aff_dim S* **for** *a* :: *'n*::*euclidean_space*
using *aff_dim_translation_eq* [*of* − *a*] **by** (*simp cong*: *image_cong_simp*)

lemma *aff_dim_affine*:
fixes *S L* :: *'n*::*euclidean_space set*
assumes *affine S subspace L affine_parallel S L S* ≠ {}
shows *aff_dim S* = *int* (*dim L*)
by (*simp add*: *aff_dim_parallel_subspace assms hull_same*)

lemma *dim_affine_hull* [*simp*]:
fixes *S* :: *'n*::*euclidean_space set*
shows *dim* (*affine hull S*) = *dim S*
by (*metis affine_hull_subset_span dim_eq_span dim_mono hull_subset span_eq_dim*)

lemma *aff_dim_subspace*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *subspace S*
shows *aff_dim S* = *int* (*dim S*)
by (*metis aff_dim_affine affine_parallel_subspace assms empty_iff parallel_subspace subspace_affine*)

lemma *aff_dim_zero*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *0* ∈ *affine hull S*
shows *aff_dim S* = *int* (*dim S*)
by (*metis aff_dim_affine_hull aff_dim_subspace affine_hull_span_0 assms dim_span subspace_span*)

lemma *aff_dim_eq_dim*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *a* ∈ *affine hull S*
shows *aff_dim S* = *int* (*dim* ((+) (− *a*) ‘ *S*))
by (*metis ab_left_minus aff_dim_translation_eq aff_dim_zero affine_hull_translation image_eqI assms*)

lemma *aff_dim_eq_dim_subtract*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *a* ∈ *affine hull S*
shows *aff_dim S* = *int* (*dim* ((λ*x*. *x* − *a*) ‘ *S*))
using *aff_dim_eq_dim assms* **by** *auto*

lemma *aff_dim_UNIV* [*simp*]: *aff_dim* (*UNIV* :: *'n*::*euclidean_space set*) = *int*(*DIM*(*'n*))
by (*simp add*: *aff_dim_subspace*)

lemma *aff_dim_geq*:
fixes *V* :: *'n*::*euclidean_space set*
shows *aff_dim V* ≥ −*1*
by (*metis add_le_cancel_right aff_dim_basis_exists diff_self of_nat_0_le_iff uminus_add_conv_diff*)

lemma *aff_dim_negative_iff* [*simp*]:
fixes *S* :: *'n*::*euclidean_space set*
shows *aff_dim S* < *0* ⟷ *S* = {}
by (*metis aff_dim_empty aff_dim_geq diff_0 eq_iff zle_diff1_eq*)

lemma *aff_lowdim_subset_hyperplane*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *aff_dim S* < *DIM*(*'a*)
obtains *a b* **where** *a* ≠ *0 S* ⊆ {*x*. *a* • *x* = *b*}
proof (*cases S*={})
case *True*
moreover
have (*SOME b*. *b* ∈ *Basis*) ≠ *0*
by (*metis norm_some_Basis norm_zero zero_neq_one*)
ultimately show *?thesis*
using *that* **by** *blast*
next
case *False*
then obtain *c S'* **where** *c* ∉ *S' S* = *insert c S'*
by (*meson equals0I mk_disjoint_insert*)
have *dim* ((+) (−*c*) ‘ *S*) < *DIM*(*'a*)
by (*metis* ‹*S* = *insert c S'*› *aff_dim_eq_dim assms hull_inc insertI1 of_nat_less_imp_less*)
then obtain *a* **where** *a* ≠ *0 span* ((+) (−*c*) ‘ *S*) ⊆ {*x*. *a* • *x* = *0*}
using *lowdim_subset_hyperplane* **by** *blast*
moreover
have *a* • *w* = *a* • *c* **if** *span* ((+) (− *c*) ‘ *S*) ⊆ {*x*. *a* • *x* = *0*} *w* ∈ *S* **for** *w*
proof −
have *w*−*c* ∈ *span* ((+) (− *c*) ‘ *S*)
by (*simp add*: *span_base* ‹*w* ∈ *S*›)
with *that* **have** *w*−*c* ∈ {*x*. *a* • *x* = *0*}
by *blast*
then show *?thesis*
by (*auto simp*: *algebra_simps*)
qed
ultimately have *S* ⊆ {*x*. *a* • *x* = *a* • *c*}
by *blast*
then show *?thesis*
by (*rule that*[*OF* ‹*a* ≠ *0*›])
qed

lemma *affine_independent_card_dim_diffs*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes $\neg\ affine\_dependent\ S\ a \in S$
shows $card\ S = dim\ ((\lambda x.\ x - a)\ `\ S) + 1$
using *aff_dim_affine_independent aff_dim_eq_dim_subtract assms hull_subset*
by *fastforce*

lemma *independent_card_le_aff_dim*:
fixes $B :: 'n::euclidean\_space\ set$
assumes $B \subseteq V$
assumes $\neg\ affine\_dependent\ B$
shows $int\ (card\ B) \leq aff\_dim\ V + 1$
by (*metis aff_dim_unique aff_independent_finite assms card_mono extend_to_affine_basis of_nat_mono*)

lemma *aff_dim_subset*:
fixes $S\ T :: 'n::euclidean\_space\ set$
assumes $S \subseteq T$
shows $aff\_dim\ S \leq aff\_dim\ T$
by (*metis add_le_cancel_right aff_dim_inner_basis_exists assms dual_order.trans independent_card_le_aff_dim*)

lemma *aff_dim_le_DIM*:
fixes $S :: 'n::euclidean\_space\ set$
shows $aff\_dim\ S \leq int\ (DIM('n))$
by (*metis aff_dim_UNIV aff_dim_subset top_greatest*)

lemma *affine_dim_equal*:
fixes $S :: 'n::euclidean\_space\ set$
assumes $affine\ S\ affine\ T\ S \neq \{\}\ S \subseteq T\ aff\_dim\ S = aff\_dim\ T$
shows $S = T$
proof −
obtain a **where** $a \in S\ a \in T\ T \neq \{\}$ **using** *assms* **by** *auto*
define LS **where** $LS = \{y.\ \exists x \in S.\ (-a) + x = y\}$
then have *ls*: $subspace\ LS\ affine\_parallel\ S\ LS$
using *assms parallel_subspace_explicit*[*of S a LS*] ‹$a \in S$› **by** *auto*
then have *h1*: $int(dim\ LS) = aff\_dim\ S$
using *assms aff_dim_affine*[*of S LS*] **by** *auto*
define LT **where** $LT = \{y.\ \exists x \in T.\ (-a) + x = y\}$
then have *lt*: $subspace\ LT \wedge affine\_parallel\ T\ LT$
using *assms parallel_subspace_explicit*[*of T a LT*] ‹$a \in T$› **by** *auto*
then have $dim\ LS = dim\ LT$
using *assms aff_dim_affine*[*of T LT*] ‹$T \neq \{\}$› *h1* **by** *auto*
moreover have $LS \leq LT$
using *LS_def LT_def assms* **by** *auto*
ultimately have $LS = LT$
using *subspace_dim_equal*[*of LS LT*] *ls lt* **by** *auto*
moreover have $S = \{x.\ \exists y \in LS.\ a{+}y{=}x\}\ T = \{x.\ \exists y \in LT.\ a{+}y{=}x\}$

```
    using LS_def LT_def by auto
  ultimately show ?thesis by auto
qed

lemma aff_dim_eq_0:
  fixes S :: 'a::euclidean_space set
  shows aff_dim S = 0 ⟷ (∃ a. S = {a})
proof (cases S = {})
  case False
  then obtain a where a ∈ S by auto
  show ?thesis
  proof safe
    assume 0: aff_dim S = 0
    have ¬ {a,b} ⊆ S if b ≠ a for b
      by (metis 0 aff_dim_2 aff_dim_subset not_one_le_zero that)
    then show ∃ a. S = {a}
      using ‹a ∈ S› by blast
  qed auto
qed auto

lemma affine_hull_UNIV:
  fixes S :: 'n::euclidean_space set
  assumes aff_dim S = int(DIM('n))
  shows affine hull S = (UNIV :: ('n::euclidean_space) set)
  by (simp add: aff_dim_empty affine_dim_equal assms)

lemma disjoint_affine_hull:
  fixes S :: 'n::euclidean_space set
  assumes ¬ affine_dependent S T ⊆ S U ⊆ S T ∩ U = {}
    shows (affine hull T) ∩ (affine hull U) = {}
proof −
  obtain finite S finite T finite U
    using assms by (simp add: aff_independent_finite finite_subset)
  have False if y ∈ affine hull T and y ∈ affine hull U for y
  proof −
    from that obtain a b
      where a1 [simp]: sum a T = 1
        and [simp]: sum (λv. a v *R v) T = y
        and [simp]: sum b U = 1 sum (λv. b v *R v) U = y
      by (auto simp: affine_hull_finite ‹finite T› ‹finite U›)
    define c where c x = (if x ∈ T then a x else if x ∈ U then −(b x) else 0) for
x
    have [simp]: S ∩ T = T S ∩ − T ∩ U = U
      using assms by auto
    have sum c S = 0
    by (simp add: c_def comm_monoid_add_class.sum.If_cases ‹finite S› sum_negf)
    moreover have ¬ (∀ v∈S. c v = 0)
    by (metis (no_types) IntD1 ‹S ∩ T = T› a1 c_def sum.neutral zero_neq_one)
    moreover have (∑ v∈S. c v *R v) = 0
```

by (*simp add*: *c_def if_smult sum_negf comm_monoid_add_class.sum.If_cases* ‹*finite S*›)
ultimately show *?thesis*
using *assms*(*1*) ‹*finite S*› **by** (*auto simp*: *affine_dependent_explicit*)
qed
then show *?thesis* **by** *blast*
qed

end

1.7 Convex Sets and Functions

theory *Convex*
imports
Affine *HOL−Library.Set_Algebras* *HOL−Library.FuncSet*
begin

1.7.1 Convex Sets

definition *convex* :: *'a*::*real_vector set* ⇒ *bool*
where *convex s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*≥*0*. ∀ *v*≥*0*. *u* + *v* = *1* ⟶ *u* ∗*R* *x* + *v* ∗*R* *y* ∈ *s*)

lemma *convexI*:
assumes ⋀*x y u v*. *x* ∈ *s* ⟹ *y* ∈ *s* ⟹ *0* ≤ *u* ⟹ *0* ≤ *v* ⟹ *u* + *v* = *1* ⟹ *u* ∗*R* *x* + *v* ∗*R* *y* ∈ *s*
shows *convex s*
by (*simp add*: *assms convex_def*)

lemma *convexD*:
assumes *convex s* **and** *x* ∈ *s* **and** *y* ∈ *s* **and** *0* ≤ *u* **and** *0* ≤ *v* **and** *u* + *v* = *1*
shows *u* ∗*R* *x* + *v* ∗*R* *y* ∈ *s*
using *assms* **unfolding** *convex_def* **by** *fast*

lemma *convex_alt*: *convex s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*. *0* ≤ *u* ∧ *u* ≤ *1* ⟶ ((*1* − *u*) ∗*R* *x* + *u* ∗*R* *y*) ∈ *s*)
(**is** *_* ⟷ *?alt*)
by (*smt* (*verit*) *convexD convexI*)

lemma *convexD_alt*:
assumes *convex s a* ∈ *s b* ∈ *s 0* ≤ *u u* ≤ *1*
shows ((*1* − *u*) ∗*R* *a* + *u* ∗*R* *b*) ∈ *s*
using *assms* **unfolding** *convex_alt* **by** *auto*

lemma *mem_convex_alt*:
assumes *convex S x* ∈ *S y* ∈ *S u* ≥ *0 v* ≥ *0 u* + *v* > *0*
shows ((*u*/(*u*+*v*)) ∗*R* *x* + (*v*/(*u*+*v*)) ∗*R* *y*) ∈ *S*
using *assms*
by (*simp add*: *convex_def zero_le_divide_iff add_divide_distrib* [*symmetric*])

lemma *convex_empty*[*intro*,*simp*]: *convex* {}
unfolding *convex_def* **by** *simp*

lemma *convex_singleton*[*intro*,*simp*]: *convex* {*a*}
unfolding *convex_def* **by** (*auto simp*: *scaleR_left_distrib*[*symmetric*])

lemma *convex_UNIV*[*intro*,*simp*]: *convex UNIV*
unfolding *convex_def* **by** *auto*

lemma *convex_Inter*: $(\bigwedge s.\ s{\in}f \Longrightarrow \mathit{convex}\ s) \Longrightarrow \mathit{convex}(\bigcap f)$
unfolding *convex_def* **by** *auto*

lemma *convex_Int*: *convex s* $\Longrightarrow$ *convex t* $\Longrightarrow$ *convex* $(s \cap t)$
unfolding *convex_def* **by** *auto*

lemma *convex_INT*: $(\bigwedge i.\ i \in A \Longrightarrow \mathit{convex}\ (B\ i)) \Longrightarrow \mathit{convex}\ (\bigcap i{\in}A.\ B\ i)$
unfolding *convex_def* **by** *auto*

lemma *convex_Times*: *convex s* $\Longrightarrow$ *convex t* $\Longrightarrow$ *convex* $(s \times t)$
unfolding *convex_def* **by** *auto*

lemma *convex_halfspace_le*: *convex* $\{x.\ \mathit{inner}\ a\ x \leq b\}$
unfolding *convex_def*
by (*auto simp*: *inner_add intro*!: *convex_bound_le*)

lemma *convex_halfspace_ge*: *convex* $\{x.\ \mathit{inner}\ a\ x \geq b\}$
proof −
have ∗: $\{x.\ \mathit{inner}\ a\ x \geq b\} = \{x.\ \mathit{inner}\ (-a)\ x \leq -b\}$
by *auto*
show *?thesis*
unfolding ∗ **using** *convex_halfspace_le*[*of* $-a$ $-b$] **by** *auto*
qed

lemma *convex_halfspace_abs_le*: *convex* $\{x.\ |\mathit{inner}\ a\ x| \leq b\}$
proof −
have ∗: $\{x.\ |\mathit{inner}\ a\ x| \leq b\} = \{x.\ \mathit{inner}\ a\ x \leq b\} \cap \{x.\ -b \leq \mathit{inner}\ a\ x\}$
by *auto*
show *?thesis*
unfolding ∗ **by** (*simp add*: *convex_Int convex_halfspace_ge convex_halfspace_le*)
qed

lemma *convex_hyperplane*: *convex* $\{x.\ \mathit{inner}\ a\ x = b\}$
proof −
have ∗: $\{x.\ \mathit{inner}\ a\ x = b\} = \{x.\ \mathit{inner}\ a\ x \leq b\} \cap \{x.\ \mathit{inner}\ a\ x \geq b\}$
by *auto*
show *?thesis* **using** *convex_halfspace_le convex_halfspace_ge*
by (*auto intro*!: *convex_Int simp*: ∗)
qed

lemma *convex_halfspace_lt*: *convex* {*x*. *inner a x* < *b*}
unfolding *convex_def*
by (*auto simp*: *convex_bound_lt inner_add*)

lemma *convex_halfspace_gt*: *convex* {*x*. *inner a x* > *b*}
using *convex_halfspace_lt*[*of* −*a* −*b*] **by** *auto*

lemma *convex_halfspace_Re_ge*: *convex* {*x*. *Re x* ≥ *b*}
using *convex_halfspace_ge*[*of b 1*::*complex*] **by** *simp*

lemma *convex_halfspace_Re_le*: *convex* {*x*. *Re x* ≤ *b*}
using *convex_halfspace_le*[*of 1*::*complex b*] **by** *simp*

lemma *convex_halfspace_Im_ge*: *convex* {*x*. *Im x* ≥ *b*}
using *convex_halfspace_ge*[*of b* i] **by** *simp*

lemma *convex_halfspace_Im_le*: *convex* {*x*. *Im x* ≤ *b*}
using *convex_halfspace_le*[*of* i *b*] **by** *simp*

lemma *convex_halfspace_Re_gt*: *convex* {*x*. *Re x* > *b*}
using *convex_halfspace_gt*[*of b 1*::*complex*] **by** *simp*

lemma *convex_halfspace_Re_lt*: *convex* {*x*. *Re x* < *b*}
using *convex_halfspace_lt*[*of 1*::*complex b*] **by** *simp*

lemma *convex_halfspace_Im_gt*: *convex* {*x*. *Im x* > *b*}
using *convex_halfspace_gt*[*of b* i] **by** *simp*

lemma *convex_halfspace_Im_lt*: *convex* {*x*. *Im x* < *b*}
using *convex_halfspace_lt*[*of* i *b*] **by** *simp*

lemma *convex_real_interval* [*iff*]:
fixes *a b* :: *real*
shows *convex* {*a*..} **and** *convex* {..*b*}
and *convex* {*a*<..} **and** *convex* {..<*b*}
and *convex* {*a*..*b*} **and** *convex* {*a*<..*b*}
and *convex* {*a*..<*b*} **and** *convex* {*a*<..<*b*}
proof −
have {*a*..} = {*x*. *a* ≤ *inner 1 x*}
by *auto*
then show *1*: *convex* {*a*..}
by (*simp only*: *convex_halfspace_ge*)
have {..*b*} = {*x*. *inner 1 x* ≤ *b*}
by *auto*
then show *2*: *convex* {..*b*}
by (*simp only*: *convex_halfspace_le*)
have {*a*<..} = {*x*. *a* < *inner 1 x*}
by *auto*

```
  then show 3: convex {a<..}
    by (simp only: convex_halfspace_gt)
  have {..<b} = {x. inner 1 x < b}
    by auto
  then show 4: convex {..<b}
    by (simp only: convex_halfspace_lt)
  have {a..b} = {a..} ∩ {..b}
    by auto
  then show convex {a..b}
    by (simp only: convex_Int 1 2)
  have {a<..b} = {a<..} ∩ {..b}
    by auto
  then show convex {a<..b}
    by (simp only: convex_Int 3 2)
  have {a..<b} = {a..} ∩ {..<b}
    by auto
  then show convex {a..<b}
    by (simp only: convex_Int 1 4)
  have {a<..<b} = {a<..} ∩ {..<b}
    by auto
  then show convex {a<..<b}
    by (simp only: convex_Int 3 4)
qed

lemma convex_Reals: convex ℝ
  by (simp add: convex_def scaleR_conv_of_real)
```

1.7.2 Explicit expressions for convexity in terms of arbitrary sums

```
lemma convex_sum:
  fixes C :: 'a::real_vector set
  assumes finite S
    and convex C
    and a: (∑ i ∈ S. a i) = 1 ⋀i. i ∈ S ⟹ a i ≥ 0
    and C: ⋀i. i ∈ S ⟹ y i ∈ C
  shows (∑ j ∈ S. a j *R y j) ∈ C
  using ‹finite S› a C
proof (induction arbitrary: a set: finite)
  case empty
  then show ?case by simp
next
  case (insert i S)
  then have 0 ≤ sum a S
    by (simp add: sum_nonneg)
  have a i *R y i + (∑j∈S. a j *R y j) ∈ C
  proof (cases sum a S = 0)
    case True with insert show ?thesis
      by (simp add: sum_nonneg_eq_0_iff)
```

```
  next
    case False
    with ‹0 ≤ sum a S› have 0 < sum a S
      by simp
    then have (∑j∈S. (a j / sum a S) *R y j) ∈ C
      using insert
      by (simp add: insert.IH flip: sum_divide_distrib)
    with ‹convex C› insert ‹0 ≤ sum a S›
    have a i *R y i + sum a S *R (∑j∈S. (a j / sum a S) *R y j) ∈ C
      by (simp add: convex_def)
    then show ?thesis
      by (simp add: scaleR_sum_right False)
  qed
  then show ?case using ‹finite S› and ‹i ∉ S›
    by simp
qed

lemma convex:
  convex S ⟷
    (∀(k::nat) u x. (∀i. 1≤i ∧ i≤k ⟶ 0 ≤ u i ∧ x i ∈S) ∧ (sum u {1..k} = 1)
      ⟶ sum (λi. u i *R x i) {1..k} ∈ S)
  (is ?lhs = ?rhs)
proof
  show ?lhs ⟹ ?rhs
    by (metis (full_types) atLeastAtMost_iff convex_sum finite_atLeastAtMost)
  assume *: ∀k u x. (∀ i :: nat. 1 ≤ i ∧ i ≤ k ⟶ 0 ≤ u i ∧ x i ∈ S) ∧ sum u
{1..k} = 1
    ⟶ (∑ i = 1..k. u i *R (x i :: 'a)) ∈ S
  {
    fix μ :: real
    fix x y :: 'a
    assume xy: x ∈ S y ∈ S
    assume mu: μ ≥ 0 μ ≤ 1
    let ?u = λi. if (i :: nat) = 1 then μ else 1 − μ
    let ?x = λi. if (i :: nat) = 1 then x else y
    have {1 :: nat .. 2} ∩ − {x. x = 1} = {2}
      by auto
    then have S: (∑j ∈ {1..2}. ?u j *R ?x j) ∈ S
      using sum.If_cases[of {(1 :: nat) .. 2} λx. x = 1 λx. μ λx. 1 − μ]
      using mu xy * by auto
    have grarr: (∑j ∈ {Suc (Suc 0)..2}. ?u j *R ?x j) = (1 − μ) *R y
     using sum.atLeast_Suc_atMost[of Suc (Suc 0) 2 λ j. (1 − μ) *R y] by auto
    with sum.atLeast_Suc_atMost
    have (∑j ∈ {1..2}. ?u j *R ?x j) = μ *R x + (1 − μ) *R y
      by (smt (verit, best) Suc_1 Suc_eq_plus1 add_0 le_add1)
    then have (1 − μ) *R y + μ *R x ∈ S
      using S by (auto simp: add.commute)
  }
  then show convex S
```

unfolding *convex_alt* **by** *auto*
qed

lemma *convex_explicit*:
fixes S :: *'a*::*real_vector set*
shows *convex* $S \longleftrightarrow$
$(\forall\, t\ u.\ \mathit{finite}\ t \wedge t \subseteq S \wedge (\forall\, x{\in}t.\ 0 \leq u\ x) \wedge \mathit{sum}\ u\ t = 1 \longrightarrow \mathit{sum}\ (\lambda x.\ u\ x *_R x)\ t \in S)$
proof *safe*
fix t
fix u :: *'a* $\Rightarrow$ *real*
assume *convex* S
and *finite* t
and $t \subseteq S\ \forall\, x{\in}t.\ 0 \leq u\ x\ \mathit{sum}\ u\ t = 1$
then show $(\sum x{\in}t.\ u\ x *_R x) \in S$
by (*simp add*: *convex_sum subsetD*)
next
assume $*$: $\forall\, t.\ \forall\ u.\ \mathit{finite}\ t \wedge t \subseteq S \wedge (\forall\, x{\in}t.\ 0 \leq u\ x) \wedge$
$\mathit{sum}\ u\ t = 1 \longrightarrow (\sum x{\in}t.\ u\ x *_R x) \in S$
show *convex* S
unfolding *convex_alt*
proof *safe*
fix $x\ y$
fix μ :: *real*
assume $**$: $x \in S\ y \in S\ 0 \leq \mu\ \mu \leq 1$
show $(1 - \mu) *_R x + \mu *_R y \in S$
proof (*cases* $x = y$)
case *False*
then show *?thesis*
using $*$[*rule_format*, *of* $\{x, y\}\ \lambda\ z.$ *if* $z = x$ *then* $1 - \mu$ *else* μ] $**$
by *auto*
next
case *True*
then show *?thesis*
using $*$[*rule_format*, *of* $\{x, y\}\ \lambda\ z.\ 1$] $**$
by (*auto simp*: *field_simps real_vector.scale_left_diff_distrib*)
qed
qed
qed

lemma *convex_finite*:
assumes *finite* S
shows *convex* $S \longleftrightarrow (\forall\, u.\ (\forall\, x{\in}S.\ 0 \leq u\ x) \wedge \mathit{sum}\ u\ S = 1 \longrightarrow \mathit{sum}\ (\lambda x.\ u\ x *_R x)\ S \in S)$
(**is** *?lhs* = *?rhs*)
proof
{ **have** *if_distrib_arg*: $\bigwedge P\ f\ g\ x.\ (\mathit{if}\ P\ \mathit{then}\ f\ \mathit{else}\ g)\ x = (\mathit{if}\ P\ \mathit{then}\ f\ x\ \mathit{else}\ g\ x)$
by *simp*

fix $T :: 'a\ set$ **and** $u :: 'a \Rightarrow real$
assume *sum*: $\forall u.\ (\forall x{\in}S.\ 0 \leq u\ x) \wedge sum\ u\ S = 1 \longrightarrow (\sum x{\in}S.\ u\ x *_R x) \in S$
assume $*$: $\forall x{\in}T.\ 0 \leq u\ x\ sum\ u\ T = 1$
assume $T \subseteq S$
then have $S \cap T = T$ **by** *auto*
with *sum*[*THEN spec*[**where** $x{=}\lambda x.\ if\ x{\in}T\ then\ u\ x\ else\ 0$]] $*$
have $(\sum x{\in}T.\ u\ x *_R x) \in S$
by (*auto simp*: *assms sum.If_cases if_distrib if_distrib_arg*) **}**
moreover assume *?rhs*
ultimately show *?lhs*
unfolding *convex_explicit* **by** *auto*
qed (*auto simp*: *convex_explicit assms*)

1.7.3 Convex Functions on a Set

definition *convex_on* :: $'a$::*real_vector set* $\Rightarrow$ ($'a \Rightarrow real$) $\Rightarrow$ *bool*
where *convex_on S f* $\longleftrightarrow$ *convex S* $\wedge$
$(\forall x{\in}S.\ \forall y{\in}S.\ \forall u{\geq}0.\ \forall v{\geq}0.\ u + v = 1 \longrightarrow f\ (u *_R x + v *_R y) \leq u * f\ x + v * f\ y)$

definition *concave_on* :: $'a$::*real_vector set* $\Rightarrow$ ($'a \Rightarrow real$) $\Rightarrow$ *bool*
where *concave_on S f* $\equiv$ *convex_on S* $(\lambda x.\ -f\ x)$

lemma *convex_on_iff_concave*: *convex_on S f* $=$ *concave_on S* $(\lambda x.\ -f\ x)$
by (*simp add*: *concave_on_def*)

lemma *concave_on_iff*:
concave_on S f $\longleftrightarrow$ *convex S* $\wedge$
$(\forall x{\in}S.\ \forall y{\in}S.\ \forall u{\geq}0.\ \forall v{\geq}0.\ u + v = 1 \longrightarrow f\ (u *_R x + v *_R y) \geq u * f\ x + v * f\ y)$
by (*auto simp*: *concave_on_def convex_on_def algebra_simps*)

lemma *concave_onD*:
assumes *concave_on A f*
shows $\bigwedge t\ x\ y.\ t \geq 0 \Longrightarrow t \leq 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \geq (1 - t) * f\ x + t * f\ y$
using *assms* **by** (*auto simp*: *concave_on_iff*)

lemma *convex_onI* [*intro?*]:
assumes $\bigwedge t\ x\ y.\ t > 0 \Longrightarrow t < 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\ x + t * f\ y$
and *convex A*
shows *convex_on A f*
unfolding *convex_on_def*
by (*smt* (*verit*, *del_insts*) *assms mult_cancel_right1 mult_eq_0_iff scaleR_collapse scaleR_eq_0_iff*)

lemma *convex_onD*:

assumes *convex_on A f*
shows $\bigwedge t\ x\ y.\ t \geq 0 \Longrightarrow t \leq 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\ x + t * f\ y$
using *assms* **by** (*auto simp*: *convex_on_def*)

lemma *convex_on_linorderI* [*intro?*]:
fixes $A :: ('a::\{linorder,real\_vector\})\ set$
assumes $\bigwedge t\ x\ y.\ t > 0 \Longrightarrow t < 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x < y \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\ x + t * f\ y$
and *convex A*
shows *convex_on A f*
by (*smt* (*verit*, *best*) *add.commute assms convex_onI distrib_left linorder_cases mult.commute mult_cancel_left2 scaleR_collapse*)

lemma *concave_on_linorderI* [*intro?*]:
fixes $A :: ('a::\{linorder,real\_vector\})\ set$
assumes $\bigwedge t\ x\ y.\ t > 0 \Longrightarrow t < 1 \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x < y \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \geq (1 - t) * f\ x + t * f\ y$
and *convex A*
shows *concave_on A f*
by (*smt* (*verit*) *assms concave_on_def convex_on_linorderI mult_minus_right*)

lemma *convex_on_imp_convex*: *convex_on A f* $\Longrightarrow$ *convex A*
by (*auto simp*: *convex_on_def*)

lemma *concave_on_imp_convex*: *concave_on A f* $\Longrightarrow$ *convex A*
by (*simp add*: *concave_on_def convex_on_imp_convex*)

lemma *convex_onD_Icc*:
assumes *convex_on* $\{x..y\}\ f\ x \leq (y :: \_ :: \{real\_vector,preorder\})$
shows $\bigwedge t.\ t \geq 0 \Longrightarrow t \leq 1 \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \leq (1 - t) * f\ x + t * f\ y$
using *assms*(*2*) **by** (*intro convex_onD* [*OF assms*(*1*)]) *simp_all*

lemma *convex_on_subset*: $[\![$*convex_on T f*; $S \subseteq T$; *convex S*$]\!] \Longrightarrow$ *convex_on S f*
by (*simp add*: *convex_on_def subset_iff*)

lemma *convex_on_add* [*intro*]:
assumes *convex_on S f*
and *convex_on S g*
shows *convex_on S* $(\lambda x.\ f\ x + g\ x)$
proof −
{
fix *x y*
assume $x \in S\ y \in S$
moreover
fix *u v* :: *real*
assume $0 \leq u\ 0 \leq v\ u + v = 1$

```
    ultimately
    have f (u *R x + v *R y) + g (u *R x + v *R y) ≤ (u * f x + v * f y) + (u
* g x + v * g y)
      using assms unfolding convex_on_def by (auto simp: add_mono)
    then have f (u *R x + v *R y) + g (u *R x + v *R y) ≤ u * (f x + g x) + v
* (f y + g y)
      by (simp add: field_simps)
  }
  with assms show ?thesis
    unfolding convex_on_def by auto
qed

lemma convex_on_ident: convex_on S (λx. x) ⟷ convex S
  by (simp add: convex_on_def)

lemma concave_on_ident: concave_on S (λx. x) ⟷ convex S
  by (simp add: concave_on_iff)

lemma convex_on_const: convex_on S (λx. a) ⟷ convex S
  by (simp add: convex_on_def flip: distrib_right)

lemma concave_on_const: concave_on S (λx. a) ⟷ convex S
  by (simp add: concave_on_iff flip: distrib_right)

lemma convex_on_diff:
  assumes convex_on S f and concave_on S g
  shows convex_on S (λx. f x − g x)
  using assms concave_on_def convex_on_add by fastforce

lemma concave_on_diff:
  assumes concave_on S f
    and convex_on S g
  shows concave_on S (λx. f x − g x)
  using convex_on_diff assms concave_on_def by fastforce

lemma concave_on_add:
  assumes concave_on S f
    and concave_on S g
  shows concave_on S (λx. f x + g x)
  using assms convex_on_iff_concave concave_on_diff concave_on_def by fast-
force

lemma convex_on_mul:
  fixes S::real set
  assumes convex_on S f convex_on S g
  assumes mono_on S f mono_on S g
  assumes fty: f ∈ S → {0..} and gty: g ∈ S → {0..}
  shows convex_on S (λx. f x*g x)
proof (intro convex_on_linorderI)
```

show *convex S*
using *assms convex_on_imp_convex* **by** *auto*
fix *t*::*real* **and** *x y*
assume *t*: *0 < t t < 1* **and** *xy*: *x ∈ S y ∈ S x<y*
have ∗: *t∗(1−t) ∗ f x ∗ g y + t∗(1−t) ∗ f y ∗ g x ≤ t∗(1−t) ∗ f x ∗ g x + t∗(1−t) ∗ f y ∗ g y*
using *t ‹mono_on S f› ‹mono_on S g› xy*
by (*smt* (*verit*, *ccfv_SIG*) *left_diff_distrib mono_onD mult_left_less_imp_less zero_le_mult_iff*)
have *inS*: *(1−t)∗x + t∗y ∈ S*
using *t xy ‹convex S›* **by** (*simp add*: *convex_alt*)
then have *f ((1−t)∗x + t∗y) ∗ g ((1−t)∗x + t∗y) ≤ ((1−t) ∗ f x + t ∗ f y)∗g ((1−t)∗x + t∗y)*
using *convex_onD* [*OF ‹convex_on S f›*, *of t x y*] *t xy fty gty*
by (*intro mult_mono add_nonneg_nonneg*) (*auto simp*: *Pi_iff zero_le_mult_iff*)
also have *... ≤ ((1−t) ∗ f x + t ∗ f y) ∗ ((1−t)∗g x + t∗g y)*
using *convex_onD* [*OF ‹convex_on S g›*, *of t x y*] *t xy fty gty inS*
by (*intro mult_mono add_nonneg_nonneg*) (*auto simp*: *Pi_iff zero_le_mult_iff*)
also have *... ≤ (1−t) ∗ (f x∗g x) + t ∗ (f y∗g y)*
using ∗ **by** (*simp add*: *algebra_simps*)
finally show *f ((1−t) ∗R x + t ∗R y) ∗ g ((1−t) ∗R x + t ∗R y) ≤ (1−t)∗(f x∗g x) + t∗(f y∗g y)*
by *simp*
qed

lemma *convex_on_cmul* [*intro*]:
fixes *c* :: *real*
assumes *0 ≤ c*
and *convex_on S f*
shows *convex_on S* (λ*x*. *c ∗ f x*)
proof −
have ∗: *u ∗ (c ∗ fx) + v ∗ (c ∗ fy) = c ∗ (u ∗ fx + v ∗ fy)*
for *u c fx v fy* :: *real*
by (*simp add*: *field_simps*)
show *?thesis* **using** *assms(2)* **and** *mult_left_mono* [*OF _ assms(1)*]
unfolding *convex_on_def* **and** ∗ **by** *auto*
qed

lemma *convex_on_cdiv* [*intro*]:
fixes *c* :: *real*
assumes *0 ≤ c* **and** *convex_on S f*
shows *convex_on S* (λ*x*. *f x / c*)
unfolding *divide_inverse*
using *convex_on_cmul* [*of inverse c S f*] **by** (*simp add*: *mult.commute assms*)

lemma *convex_lower*:
assumes *convex_on S f*
and *x ∈ S*
and *y ∈ S*

```
    and 0 ≤ u
    and 0 ≤ v
    and u + v = 1
  shows f (u *R x + v *R y) ≤ max (f x) (f y)
proof −
  let ?m = max (f x) (f y)
  have u * f x + v * f y ≤ u * max (f x) (f y) + v * max (f x) (f y)
    using assms(4,5) by (auto simp: mult_left_mono add_mono)
  also have ... = max (f x) (f y)
    using assms(6) by (simp add: distrib_right [symmetric])
  finally show ?thesis
    using assms unfolding convex_on_def by fastforce
qed

lemma convex_on_dist [intro]:
  fixes S :: 'a::real_normed_vector set
  assumes convex S
  shows convex_on S (λx. dist a x)
unfolding convex_on_def dist_norm
proof (intro conjI strip)
  fix x y
  assume x ∈ S y ∈ S
  fix u v :: real
  assume 0 ≤ u
  assume 0 ≤ v
  assume u + v = 1
  have a = u *R a + v *R a
    by (metis ‹u + v = 1› scaleR_left.add scaleR_one)
  then have a − (u *R x + v *R y) = (u *R (a − x)) + (v *R (a − y))
    by (auto simp: algebra_simps)
  then show norm (a − (u *R x + v *R y)) ≤ u * norm (a − x) + v * norm (a
− y)
    by (smt (verit, best) ‹0 ≤ u› ‹0 ≤ v› norm_scaleR norm_triangle_ineq)
qed (use assms in auto)

lemma concave_on_mul:
  fixes S::real set
  assumes f: concave_on S f and g: concave_on S g
  assumes mono_on S f antimono_on S g
  assumes fty: f ∈ S → {0..} and gty: g ∈ S → {0..}
  shows concave_on S (λx. f x * g x)
proof (intro concave_on_linorderI)
  show convex S
    using concave_on_imp_convex f by blast
  fix t::real and x y
  assume t: 0 < t t < 1 and xy: x ∈ S y ∈ S x<y
  have inS: (1−t)*x + t*y ∈ S
    using t xy ‹convex S› by (simp add: convex_alt)
  have f x * g y + f y * g x ≥ f x * g x + f y * g y
```

```
    using ‹mono_on S f› ‹antimono_on S g›
   unfolding monotone_on_def by (smt (verit, best) left_diff_distrib mult_left_mono
xy)
  with t have *: t*(1−t) * f x * g y + t*(1−t) * f y * g x ≥ t*(1−t) * f x * g x
+ t*(1−t) * f y * g y
   by (smt (verit, ccfv_SIG) distrib_left mult_left_mono diff_ge_0_iff_ge mult.assoc)
  have (1 − t) * (f x * g x) + t * (f y * g y) ≤ ((1−t) * f x + t * f y) * ((1−t)
* g x + t * g y)
    using * by (simp add: algebra_simps)
  also have ... ≤ ((1−t) * f x + t * f y)*g ((1−t)*x + t*y)
    using concave_onD [OF ‹concave_on S g›, of t x y] t xy fty gty inS
   by (intro mult_mono add_nonneg_nonneg) (auto simp: Pi_iff zero_le_mult_iff)
  also have ... ≤ f ((1−t)*x + t*y) * g ((1−t)*x + t*y)
    using concave_onD [OF ‹concave_on S f›, of t x y] t xy fty gty inS
   by (intro mult_mono add_nonneg_nonneg) (auto simp: Pi_iff zero_le_mult_iff)
  finally show (1 − t) * (f x * g x) + t * (f y * g y)
        ≤ f ((1 − t) *R x + t *R y) * g ((1 − t) *R x + t *R y)
    by simp
qed

lemma concave_on_cmul [intro]:
  fixes c :: real
  assumes 0 ≤ c and concave_on S f
  shows concave_on S (λx. c * f x)
  using assms convex_on_cmul [of c S λx. − f x]
  by (auto simp: concave_on_def)

lemma concave_on_cdiv [intro]:
  fixes c :: real
  assumes 0 ≤ c and concave_on S f
  shows concave_on S (λx. f x / c)
  unfolding divide_inverse
  using concave_on_cmul [of inverse c S f] by (simp add: mult.commute assms)
```

1.7.4 Arithmetic operations on sets preserve convexity

```
lemma convex_linear_image:
  assumes linear f and convex S
  shows convex (f ‘ S)
proof −
  interpret f: linear f by fact
  from ‹convex S› show convex (f ‘ S)
    by (simp add: convex_def f.scaleR [symmetric] f.add [symmetric])
qed

lemma convex_linear_vimage:
  assumes linear f and convex S
  shows convex (f −‘ S)
proof −
```

interpret f: *linear* f **by** *fact*
from ‹*convex* S› **show** *convex* $(f\ -`\ S)$
by (*simp add*: *convex_def f.add f.scaleR*)
qed

lemma *convex_scaling*:
assumes *convex* S
shows *convex* $((\lambda x.\ c *_R x)\ `\ S)$
by (*simp add*: *assms convex_linear_image*)

lemma *convex_scaled*:
assumes *convex* S
shows *convex* $((\lambda x.\ x *_R c)\ `\ S)$
by (*simp add*: *assms convex_linear_image*)

lemma *convex_negations*:
assumes *convex* S
shows *convex* $((\lambda x.\ -\ x)\ `\ S)$
by (*simp add*: *assms convex_linear_image module_hom_uminus*)

lemma *convex_sums*:
assumes *convex* S
and *convex* T
shows *convex* $(\bigcup x{\in}\ S.\ \bigcup y \in T.\ \{x + y\})$
proof −
have *linear* $(\lambda(x,\ y).\ x + y)$
by (*auto intro*: *linearI simp*: *scaleR_add_right*)
with *assms* **have** *convex* $((\lambda(x,\ y).\ x + y)\ `\ (S \times T))$
by (*intro convex_linear_image convex_Times*)
also have $((\lambda(x,\ y).\ x + y)\ `\ (S \times T)) = (\bigcup x{\in}\ S.\ \bigcup y \in T.\ \{x + y\})$
by *auto*
finally show *?thesis* **.**
qed

lemma *convex_differences*:
assumes *convex* S *convex* T
shows *convex* $(\bigcup x{\in}\ S.\ \bigcup y \in T.\ \{x - y\})$
proof −
have $\{x - y|\ x\ y.\ x \in S \wedge y \in T\} = \{x + y\ |x\ y.\ x \in S \wedge y \in uminus\ `\ T\}$
by (*auto simp*: *diff_conv_add_uminus simp del*: *add_uminus_conv_diff*)
then show *?thesis*
using *convex_sums*[*OF assms*(*1*) *convex_negations*[*OF assms*(*2*)]] **by** *auto*
qed

lemma *convex_translation*:
convex $((+)\ a\ `\ S)$ **if** *convex* S
proof −
have $(\bigcup\ x{\in}\ \{a\}.\ \bigcup y \in S.\ \{x + y\}) = (+)\ a\ `\ S$
by *auto*

```
  then show ?thesis
    using convex_sums [OF convex_singleton [of a] that] by auto
qed

lemma convex_translation_subtract:
  convex ((λb. b − a) ‘ S) if convex S
  using convex_translation [of S − a] that by (simp cong: image_cong_simp)

lemma convex_affinity:
  assumes convex S
  shows convex ((λx. a + c *R x) ‘ S)
proof −
  have (λx. a + c *R x) ‘ S = (+) a ‘ (*R) c ‘ S
    by auto
  then show ?thesis
    using convex_translation[OF convex_scaling[OF assms], of a c] by auto
qed

lemma convex_on_sum:
  fixes a :: 'a ⇒ real
    and y :: 'a ⇒ 'b::real_vector
    and f :: 'b ⇒ real
  assumes finite S S ≠ {}
    and convex_on C f
    and (∑ i ∈ S. a i) = 1
    and ⋀i. i ∈ S ⟹ a i ≥ 0
    and ⋀i. i ∈ S ⟹ y i ∈ C
  shows f (∑ i ∈ S. a i *R y i) ≤ (∑ i ∈ S. a i * f (y i))
  using assms
proof (induct S arbitrary: a rule: finite_ne_induct)
  case (singleton i)
  then show ?case
    by auto
next
  case (insert i S)
  then have yai: y i ∈ C a i ≥ 0
    by auto
  with insert have conv: ⋀x y μ. x ∈ C ⟹ y ∈ C ⟹ 0 ≤ μ ⟹ μ ≤ 1 ⟹
      f (μ *R x + (1 − μ) *R y) ≤ μ * f x + (1 − μ) * f y
    by (simp add: convex_on_def)
  show ?case
  proof (cases a i = 1)
    case True
    with insert have (∑ j ∈ S. a j) = 0
      by auto
    with insert show ?thesis
      by (simp add: sum_nonneg_eq_0_iff)
  next
    case False
```

```
    then have ai1: a i < 1
      using sum_nonneg_leq_bound[of insert i S a] insert by force
    then have i0: 1 − a i > 0
      by auto
    let ?a = λj. a j / (1 − a i)
    have a_nonneg: ?a j ≥ 0 if j ∈ S for j
      using i0 insert that by fastforce
    have (∑ j ∈ insert i S. a j) = 1
      using insert by auto
    then have (∑ j ∈ S. a j) = 1 − a i
      using sum.insert insert by fastforce
    then have (∑ j ∈ S. a j) / (1 − a i) = 1
      using i0 by auto
    then have a1: (∑ j ∈ S. ?a j) = 1
      unfolding sum_divide_distrib by simp
    have convex C
      using ‹convex_on C f› by (simp add: convex_on_def)
    have asum: (∑ j ∈ S. ?a j *R y j) ∈ C
      using insert convex_sum [OF ‹finite S› ‹convex C› a1 a_nonneg] by auto
    have asum_le: f (∑ j ∈ S. ?a j *R y j) ≤ (∑ j ∈ S. ?a j * f (y j))
      using a_nonneg a1 insert by blast
    have f (∑ j ∈ insert i S. a j *R y j) = f ((∑ j ∈ S. a j *R y j) + a i *R y i)
      by (simp add: add.commute insert.hyps)
    also have ... = f (((1 − a i) * inverse (1 − a i)) *R (∑ j ∈ S. a j *R y j)
+ a i *R y i)
      using i0 by auto
    also have ... = f ((1 − a i) *R (∑ j ∈ S. (a j * inverse (1 − a i)) *R y j)
+ a i *R y i)
      using scaleR_right.sum[of inverse (1 − a i) λ j. a j *R y j S, symmetric]
      by (auto simp: algebra_simps)
    also have ... = f ((1 − a i) *R (∑ j ∈ S. ?a j *R y j) + a i *R y i)
      by (auto simp: divide_inverse)
    also have ... ≤ (1 − a i) *R f ((∑ j ∈ S. ?a j *R y j)) + a i * f (y i)
      using ai1 by (smt (verit) asum conv real_scaleR_def yai)
    also have ... ≤ (1 − a i) * (∑ j ∈ S. ?a j * f (y j)) + a i * f (y i)
      using asum_le i0 by fastforce
    also have ... = (∑ j ∈ S. a j * f (y j)) + a i * f (y i)
      using i0 by (auto simp: sum_distrib_left)
    finally show ?thesis
      using insert by auto
  qed
qed

lemma concave_on_sum:
  fixes a :: 'a ⇒ real
    and y :: 'a ⇒ 'b::real_vector
    and f :: 'b ⇒ real
  assumes finite S S ≠ {}
    and concave_on C f
```

```
    and (∑ i ∈ S. a i) = 1
    and ⋀i. i ∈ S ⟹ a i ≥ 0
    and ⋀i. i ∈ S ⟹ y i ∈ C
  shows f (∑ i ∈ S. a i *R y i) ≥ (∑ i ∈ S. a i * f (y i))
proof −
  have (uminus ∘ f) (∑ i∈S. a i *R y i) ≤ (∑ i∈S. a i * (uminus ∘ f) (y i))
  proof (intro convex_on_sum)
    show convex_on C (uminus ∘ f)
      by (simp add: assms convex_on_iff_concave)
  qed (use assms in auto)
  then show ?thesis
    by (simp add: sum_negf o_def)
qed

lemma convex_on_alt:
  fixes C :: 'a::real_vector set
  shows convex_on C f ⟷ convex C ∧
        (∀ x ∈ C. ∀ y ∈ C. ∀ μ :: real. μ ≥ 0 ∧ μ ≤ 1 ⟶
         f (μ *R x + (1 − μ) *R y) ≤ μ * f x + (1 − μ) * f y)
  by (smt (verit) convex_on_def)

lemma convex_on_slope_le:
  fixes f :: real ⇒ real
  assumes f: convex_on I f
    and I: x ∈ I y ∈ I
    and t: x < t t < y
  shows (f x − f t) / (x − t) ≤ (f x − f y) / (x − y)
    and (f x − f y) / (x − y) ≤ (f t − f y) / (t − y)
proof −
  define a where a ≡ (t − y) / (x − y)
  with t have 0 ≤ a 0 ≤ 1 − a
    by (auto simp: field_simps)
  with f ‹x ∈ I› ‹y ∈ I› have cvx: f (a * x + (1 − a) * y) ≤ a * f x + (1 − a)
* f y
    by (auto simp: convex_on_def)
  have a * x + (1 − a) * y = a * (x − y) + y
    by (simp add: field_simps)
  also have ... = t
    unfolding a_def using ‹x < t› ‹t < y› by simp
  finally have f t ≤ a * f x + (1 − a) * f y
    using cvx by simp
  also have ... = a * (f x − f y) + f y
    by (simp add: field_simps)
  finally have f t − f y ≤ a * (f x − f y)
    by simp
  with t show (f x − f t) / (x − t) ≤ (f x − f y) / (x − y)
    by (simp add: le_divide_eq divide_le_eq field_simps a_def)
  with t show (f x − f y) / (x − y) ≤ (f t − f y) / (t − y)
    by (simp add: le_divide_eq divide_le_eq field_simps)
```

qed

lemma *pos_convex_function*:
fixes $f :: real \Rightarrow real$
assumes *convex C*
and *leq*: $\bigwedge x\ y.\ x \in C \Longrightarrow y \in C \Longrightarrow f'\ x * (y - x) \leq f\ y - f\ x$
shows *convex_on C f*
unfolding *convex_on_alt*
using *assms*
proof *safe*
fix $x\ y\ \mu :: real$
let $?x = \mu *_R x + (1 - \mu) *_R y$
assume $*$: *convex C* $x \in C$ $y \in C$ $\mu \geq 0$ $\mu \leq 1$
then have $1 - \mu \geq 0$ **by** *auto*
then have *xpos*: $?x \in C$
using $*$ **unfolding** *convex_alt* **by** *fastforce*
have *geq*: $\mu * (f\ x - f\ ?x) + (1 - \mu) * (f\ y - f\ ?x) \geq$
$\mu * f'\ ?x * (x - ?x) + (1 - \mu) * f'\ ?x * (y - ?x)$
using *add_mono* [*OF mult_left_mono* [*OF leq* [*OF xpos* $*(2)$] ‹$\mu \geq 0$›]
mult_left_mono [*OF leq* [*OF xpos* $*(3)$] ‹$1 - \mu \geq 0$›]]
by *auto*
then have $\mu * f\ x + (1 - \mu) * f\ y - f\ ?x \geq 0$
by (*auto simp*: *field_simps*)
then show $f\ (\mu *_R x + (1 - \mu) *_R y) \leq \mu * f\ x + (1 - \mu) * f\ y$
by *auto*
qed

lemma *atMostAtLeast_subset_convex*:
fixes $C :: real\ set$
assumes *convex C*
and $x \in C\ y \in C\ x < y$
shows $\{x\ ..\ y\} \subseteq C$
proof *safe*
fix z **assume** z: $z \in \{x\ ..\ y\}$
have *less*: $z \in C$ **if** $*$: $x < z\ z < y$
proof −
let $?\mu = (y - z)\ /\ (y - x)$
have $0 \leq ?\mu\ ?\mu \leq 1$
using *assms* $*$ **by** (*auto simp*: *field_simps*)
then have *comb*: $?\mu * x + (1 - ?\mu) * y \in C$
using *assms iffD1*[*OF convex_alt*, *rule_format*, *of C y x* $?\mu$]
by (*simp add*: *algebra_simps*)
have $?\mu * x + (1 - ?\mu) * y = (y - z) * x\ /\ (y - x) + (1 - (y - z)\ /\ (y - x)) * y$
by (*auto simp*: *field_simps*)
also have $\ldots = ((y - z) * x + (y - x - (y - z)) * y)\ /\ (y - x)$
using *assms* **by** (*simp only*: *add_divide_distrib*) (*auto simp*: *field_simps*)
also have $\ldots = z$
using *assms* **by** (*auto simp*: *field_simps*)

```
    finally show ?thesis
      using comb by auto
  qed
  show z ∈ C
    using z less assms by (auto simp: le_less)
qed

lemma f''_imp_f':
  fixes f :: real ⇒ real
  assumes convex C
    and f': ⋀x. x ∈ C ⟹ DERIV f x :> (f' x)
    and f'': ⋀x. x ∈ C ⟹ DERIV f' x :> (f'' x)
    and pos: ⋀x. x ∈ C ⟹ f'' x ≥ 0
    and x: x ∈ C
    and y: y ∈ C
  shows f' x * (y - x) ≤ f y - f x
  using assms
proof -
  have f y - f x ≥ f' x * (y - x) f' y * (x - y) ≤ f x - f y
    if *: x ∈ C y ∈ C y > x for x y :: real
  proof -
    from * have ge: y - x > 0 y - x ≥ 0 and le: x - y < 0 x - y ≤ 0
      by auto
    then obtain z1 where z1: z1 > x z1 < y f y - f x = (y - x) * f' z1
      using subsetD[OF atMostAtLeast_subset_convex[OF ‹convex C› ‹x ∈ C› ‹y
∈ C› ‹x < y›],
          THEN f', THEN MVT2[OF ‹x < y›, rule_format, unfolded atLeastAt-
Most_iff[symmetric]]]
      by auto
    then have z1 ∈ C
      using atMostAtLeast_subset_convex ‹convex C› ‹x ∈ C› ‹y ∈ C› ‹x < y›
      by fastforce
    obtain z2 where z2: z2 > x z2 < z1 f' z1 - f' x = (z1 - x) * f'' z2
      using subsetD[OF atMostAtLeast_subset_convex[OF ‹convex C› ‹x ∈ C› ‹z1
∈ C› ‹x < z1›],
          THEN f'', THEN MVT2[OF ‹x < z1›, rule_format, unfolded atLeastAt-
Most_iff[symmetric]]] z1
      by auto
    obtain z3 where z3: z3 > z1 z3 < y f' y - f' z1 = (y - z1) * f'' z3
      using subsetD[OF atMostAtLeast_subset_convex[OF ‹convex C› ‹z1 ∈ C›
‹y ∈ C› ‹z1 < y›],
          THEN f'', THEN MVT2[OF ‹z1 < y›, rule_format, unfolded atLeastAt-
Most_iff[symmetric]]] z1
      by auto
    from z1 have f x - f y = (x - y) * f' z1
      by (simp add: field_simps)
    then have cool': f' y - (f x - f y) / (x - y) = (y - z1) * f'' z3
      using le(1) z3(3) by auto
    have z3 ∈ C
```

```
    using z3 * atMostAtLeast_subset_convex ‹convex C› ‹x ∈ C› ‹z1 ∈ C› ‹x
< z1›
    by fastforce
  then have B': f'' z3 ≥ 0
    using assms by auto
  with cool' have f' y - (f x - f y) / (x - y) ≥ 0
    using z1 by auto
  then have res: f' y * (x - y) ≤ f x - f y
    by (meson diff_ge_0_iff_ge le(1) neg_divide_le_eq)
  have cool: (f y - f x) / (y - x) - f' x = (z1 - x) * f'' z2
    using le(1) z1(3) z2(3) by auto
  have z2 ∈ C
    using z2 z1 * atMostAtLeast_subset_convex ‹convex C› ‹z1 ∈ C› ‹y ∈ C›
‹z1 < y›
    by fastforce
  with z1 assms have (z1 - x) * f'' z2 ≥ 0
    by auto
  then show f y - f x ≥ f' x * (y - x) f' y * (x - y) ≤ f x - f y
    using that(3) z1(3) res cool by auto
 qed
 then show ?thesis
  using x y by fastforce
qed

lemma f''_ge0_imp_convex:
 fixes f :: real ⇒ real
 assumes convex C
  and ⋀x. x ∈ C ⟹ DERIV f x :> (f' x)
  and ⋀x. x ∈ C ⟹ DERIV f' x :> (f'' x)
  and ⋀x. x ∈ C ⟹ f'' x ≥ 0
 shows convex_on C f
 by (metis assms f''_imp_f' pos_convex_function)

lemma f''_le0_imp_concave:
 fixes f :: real ⇒ real
 assumes convex C
  and ⋀x. x ∈ C ⟹ DERIV f x :> (f' x)
  and ⋀x. x ∈ C ⟹ DERIV f' x :> (f'' x)
  and ⋀x. x ∈ C ⟹ f'' x ≤ 0
 shows concave_on C f
 unfolding concave_on_def
 by (rule assms f''_ge0_imp_convex derivative_eq_intros | simp)+

lemma convex_power_even:
 assumes even n
 shows convex_on (UNIV::real set) (λx. x^n)
proof (intro f''_ge0_imp_convex)
 show ((λx. x ^ n) has_real_derivative of_nat n * x^(n-1)) (at x) for x
  by (rule derivative_eq_intros | simp)+
```

show $((\lambda x.\ of\_nat\ n * x\hat{}(n-1))\ has\_real\_derivative\ of\_nat\ n * of\_nat\ (n-1) * x\hat{}(n-2))\ (at\ x)$ **for** x
by (*rule derivative_eq_intros* | *simp add*: *eval_nat_numeral*)+
show $\bigwedge x.\ 0 \leq real\ n * real\ (n - 1) * x\ \hat{}\ (n - 2)$
using *assms* **by** (*auto simp*: *zero_le_mult_iff zero_le_even_power*)
qed *auto*

lemma *convex_power_odd*:
assumes *odd n*
shows *convex_on* $\{0::real..\}$ $(\lambda x.\ x\hat{}n)$
proof (*intro f''_ge0_imp_convex*)
show $((\lambda x.\ x\ \hat{}\ n)\ has\_real\_derivative\ of\_nat\ n * x\hat{}(n-1))\ (at\ x)$ **for** x
by (*rule derivative_eq_intros* | *simp*)+
show $((\lambda x.\ of\_nat\ n * x\hat{}(n-1))\ has\_real\_derivative\ of\_nat\ n * of\_nat\ (n-1) * x\hat{}(n-2))\ (at\ x)$ **for** x
by (*rule derivative_eq_intros* | *simp add*: *eval_nat_numeral*)+
show $\bigwedge x.\ x \in \{0::real..\} \Longrightarrow 0 \leq real\ n * real\ (n - 1) * x\ \hat{}\ (n - 2)$
using *assms* **by** (*auto simp*: *zero_le_mult_iff zero_le_even_power*)
qed *auto*

lemma *convex_power2*: *convex_on* (*UNIV*::*real set*) *power2*
by (*simp add*: *convex_power_even*)

lemma *log_concave*:
fixes *b* :: *real*
assumes *b* > *1*
shows *concave_on* $\{0<..\}$ $(\lambda\ x.\ log\ b\ x)$
using *assms*
by (*intro f''_le0_imp_concave derivative_eq_intros* | *simp*)+

lemma *ln_concave*: *concave_on* $\{0<..\}$ *ln*
unfolding *log_ln* **by** (*simp add*: *log_concave*)

lemma *minus_log_convex*:
fixes *b* :: *real*
assumes *b* > *1*
shows *convex_on* $\{0 <..\}$ $(\lambda\ x.\ -\ log\ b\ x)$
using *assms concave_on_def log_concave* **by** *blast*

lemma *powr_convex*:
assumes $p \geq 1$ **shows** *convex_on* $\{0<..\}$ $(\lambda x.\ x\ powr\ p)$
using *assms*
by (*intro f''_ge0_imp_convex derivative_eq_intros* | *simp*)+

lemma *exp_convex*: *convex_on UNIV exp*
by (*intro f''_ge0_imp_convex derivative_eq_intros* | *simp*)+

The AM-GM inequality: the arithmetic mean exceeds the geometric mean.

lemma *arith_geom_mean*:

fixes $x :: 'a \Rightarrow real$
assumes *finite S* $S \neq \{\}$
and x: $\bigwedge i.\ i \in S \Longrightarrow x\ i \geq 0$
shows $(\sum i \in S.\ x\ i\ /\ card\ S) \geq (\prod i \in S.\ x\ i)\ powr\ (1\ /\ card\ S)$
proof (*cases* $\exists i{\in}S.\ x\ i = 0$)
case *True*
then have $(\prod i \in S.\ x\ i) = 0$
by (*simp add*: ‹*finite S*›)
moreover have $(\sum i \in S.\ x\ i\ /\ card\ S) \geq 0$
by (*simp add*: *sum_nonneg x*)
ultimately show *?thesis*
by *simp*
next
case *False*
have *ln* $(\sum i \in S.\ (1\ /\ card\ S) *_R x\ i) \geq (\sum i \in S.\ (1\ /\ card\ S) * ln\ (x\ i))$
proof (*intro concave_on_sum*)
show *concave_on* {*0*<..} *ln*
by (*simp add*: *ln_concave*)
show $\bigwedge i.\ i{\in}S \Longrightarrow x\ i \in \{0<..\}$
using *False x* **by** *fastforce*
qed (*use assms False* **in** *auto*)
moreover have $(\sum i \in S.\ (1\ /\ card\ S) *_R x\ i) > 0$
using *False assms* **by** (*simp add*: *card_gt_0_iff less_eq_real_def sum_pos*)
ultimately have $(\sum i \in S.\ (1\ /\ card\ S) *_R x\ i) \geq exp\ (\sum i \in S.\ (1\ /\ card\ S)$
$*\ ln\ (x\ i))$
using *ln_ge_iff* **by** *blast*
then have $(\sum i \in S.\ x\ i\ /\ card\ S) \geq exp\ (\sum i \in S.\ ln\ (x\ i)\ /\ card\ S)$
by (*simp add*: *divide_simps*)
then show *?thesis*
using *assms False*
by (*smt* (*verit*, *ccfv_SIG*) *divide_inverse exp_ln exp_powr_real exp_sum in-*
verse_eq_divide prod.cong prod_powr_distrib)
qed

1.7.5 Convexity of real functions

lemma *convex_on_realI*:
assumes *connected A*
and $\bigwedge x.\ x \in A \Longrightarrow (f\ has\_real\_derivative\ f'\ x)\ (at\ x)$
and $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \leq y \Longrightarrow f'\ x \leq f'\ y$
shows *convex_on A f*
proof (*rule convex_on_linorderI*)
show *convex A*
using ‹*connected A*› *convex_real_interval interval_cases*
by (*smt* (*verit*, *ccfv_SIG*) *connectedD_interval convex_UNIV convex_empty*)
— the equivalence of "connected" and "convex" for real intervals is proved later
next
fix *t x y* :: *real*
assume *t*: $t > 0\ t < 1$

```
  assume xy: x ∈ A y ∈ A x < y
  define z where z = (1 − t) * x + t * y
  with ‹connected A› and xy have ivl: {x..y} ⊆ A
    using connected_contains_Icc by blast

  from xy t have xz: z > x
    by (simp add: z_def algebra_simps)
  have y − z = (1 − t) * (y − x)
    by (simp add: z_def algebra_simps)
  also from xy t have ... > 0
    by (intro mult_pos_pos) simp_all
  finally have yz: z < y
    by simp

  from assms xz yz ivl t have ∃ξ. ξ > x ∧ ξ < z ∧ f z − f x = (z − x) * f' ξ
    by (intro MVT2) (auto intro!: assms(2))
  then obtain ξ where ξ: ξ > x ξ < z f' ξ = (f z − f x) / (z − x)
    by auto
  from assms xz yz ivl t have ∃η. η > z ∧ η < y ∧ f y − f z = (y − z) * f' η
    by (intro MVT2) (auto intro!: assms(2))
  then obtain η where η: η > z η < y f' η = (f y − f z) / (y − z)
    by auto

  from η(3) have (f y − f z) / (y − z) = f' η ..
  also from ξ η ivl have ξ ∈ A η ∈ A
    by auto
  with ξ η have f' η ≥ f' ξ
    by (intro assms(3)) auto
  also from ξ(3) have f' ξ = (f z − f x) / (z − x) .
  finally have (f y − f z) * (z − x) ≥ (f z − f x) * (y − z)
    using xz yz by (simp add: field_simps)
  also have z − x = t * (y − x)
    by (simp add: z_def algebra_simps)
  also have y − z = (1 − t) * (y − x)
    by (simp add: z_def algebra_simps)
  finally have (f y − f z) * t ≥ (f z − f x) * (1 − t)
    using xy by simp
  then show (1 − t) * f x + t * f y ≥ f ((1 − t) *R x + t *R y)
    by (simp add: z_def algebra_simps)
qed

lemma convex_on_inverse:
  fixes A :: real set
  assumes A ⊆ {0<..} convex A
  shows convex_on A inverse
proof −
  have convex_on {0::real<..} inverse
  proof (intro convex_on_realI)
    fix u v :: real
```

```
    assume u ∈ {0<..} v ∈ {0<..} u ≤ v
    with assms show −inverse (u^2) ≤ −inverse (v^2)
      by simp
  next
    show ⋀x. x ∈ {0<..} ⟹ (inverse has_real_derivative − inverse (x^2)) (at x)
      by (rule derivative_eq_intros | simp add: power2_eq_square)+
  qed auto
  then show ?thesis
    using assms convex_on_subset by blast
qed

lemma convex_onD_Icc':
  assumes convex_on {x..y} f c ∈ {x..y}
  defines d ≡ y − x
  shows f c ≤ (f y − f x) / d ∗ (c − x) + f x
proof (cases x y rule: linorder_cases)
  case less
  then have d: d > 0
    by (simp add: d_def)
  from assms(2) less have A: 0 ≤ (c − x) / d (c − x) / d ≤ 1
    by (simp_all add: d_def field_split_simps)
  have f c = f (x + (c − x) ∗ 1)
    by simp
  also from less have 1 = ((y − x) / d)
    by (simp add: d_def)
  also from d have x + (c − x) ∗ ... = (1 − (c − x) / d) ∗R x + ((c − x) / d)
∗R y
    by (simp add: field_simps)
  also have f ... ≤ (1 − (c − x) / d) ∗ f x + (c − x) / d ∗ f y
    using assms less by (intro convex_onD_Icc) simp_all
  also from d have ... = (f y − f x) / d ∗ (c − x) + f x
    by (simp add: field_simps)
  finally show ?thesis .
qed (use assms in auto)

lemma convex_onD_Icc'':
  assumes convex_on {x..y} f c ∈ {x..y}
  defines d ≡ y − x
  shows f c ≤ (f x − f y) / d ∗ (y − c) + f y
proof (cases x y rule: linorder_cases)
  case less
  then have d: d > 0
    by (simp add: d_def)
  from assms(2) less have A: 0 ≤ (y − c) / d (y − c) / d ≤ 1
    by (simp_all add: d_def field_split_simps)
  have f c = f (y − (y − c) ∗ 1)
    by simp
  also from less have 1 = ((y − x) / d)
    by (simp add: d_def)
```

also from d **have** $y - (y - c) * \ldots = (1 - (1 - (y - c) / d)) *_R x + (1 - (y - c) / d) *_R y$
by (*simp add*: *field_simps*)
also have $f \ldots \leq (1 - (1 - (y - c) / d)) * f\ x + (1 - (y - c) / d) * f\ y$
using *assms less* **by** (*intro convex_onD_Icc*) (*simp_all add*: *field_simps*)
also from d **have** $\ldots = (f\ x - f\ y) / d * (y - c) + f\ y$
by (*simp add*: *field_simps*)
finally show *?thesis* **.**
qed (*use assms* **in** *auto*)

lemma *concave_onD_Icc*:
assumes *concave_on* $\{x..y\}$ f $x \leq (y :: \_ :: \{real\_vector, preorder\})$
shows $\bigwedge t.\ t \geq 0 \Longrightarrow t \leq 1 \Longrightarrow$
$f\ ((1 - t) *_R x + t *_R y) \geq (1 - t) * f\ x + t * f\ y$
using *assms*(*2*) **by** (*intro concave_onD* [*OF assms*(*1*)]) *simp_all*

lemma *concave_onD_Icc'*:
assumes *concave_on* $\{x..y\}$ f $c \in \{x..y\}$
defines $d \equiv y - x$
shows $f\ c \geq (f\ y - f\ x) / d * (c - x) + f\ x$
proof −
have $- f\ c \leq (f\ x - f\ y) / d * (c - x) - f\ x$
using *assms convex_onD_Icc'* [*of x y* $\lambda x. - f\ x$ *c*]
by (*simp add*: *concave_on_def*)
then show *?thesis*
by (*smt* (*verit*, *best*) *divide_minus_left mult_minus_left*)
qed

lemma *concave_onD_Icc''*:
assumes *concave_on* $\{x..y\}$ f $c \in \{x..y\}$
defines $d \equiv y - x$
shows $f\ c \geq (f\ x - f\ y) / d * (y - c) + f\ y$
proof −
have $- f\ c \leq (f\ y - f\ x) / d * (y - c) - f\ y$
using *assms convex_onD_Icc''* [*of x y* $\lambda x. - f\ x$ *c*]
by (*simp add*: *concave_on_def*)
then show *?thesis*
by (*smt* (*verit*, *best*) *divide_minus_left mult_minus_left*)
qed

lemma *convex_on_le_max*:
fixes *a*::*real*
assumes *convex_on* $\{x..y\}$ f **and** *a*: $a \in \{x..y\}$
shows $f\ a \leq max\ (f\ x)\ (f\ y)$
proof −
have $*$: $(f\ y - f\ x) * (a - x) \leq (f\ y - f\ x) * (y - x)$ **if** $f\ x \leq f\ y$
using *a that* **by** (*intro mult_left_mono*) *auto*
have $f\ a \leq (f\ y - f\ x) / (y - x) * (a - x) + f\ x$
using *assms convex_onD_Icc'* **by** *blast*

also have ... $\leq$ *max* (*f x*) (*f y*)
using *a* ∗
by (*simp add*: *divide_le_0_iff mult_le_0_iff zero_le_mult_iff max_def add.commute mult.commute scaling_mono*)
finally show *?thesis* **.**
qed

lemma *concave_on_ge_min*:
fixes *a*::*real*
assumes *concave_on* {*x*..*y*} *f* **and** *a*: *a* ∈ {*x*..*y*}
shows *f a* ≥ *min* (*f x*) (*f y*)
proof −
have ∗: (*f y* − *f x*) ∗ (*a* − *x*) ≥ (*f y* − *f x*) ∗ (*y* − *x*) **if** *f x* ≥ *f y*
using *a that* **by** (*intro mult_left_mono_neg*) *auto*
have *min* (*f x*) (*f y*) ≤ (*f y* − *f x*) / (*y* − *x*) ∗ (*a* − *x*) + *f x*
using *a* ∗ **apply** (*simp add*: *zero_le_divide_iff mult_le_0_iff zero_le_mult_iff min_def*)
by (*smt* (*verit*, *best*) *nonzero_eq_divide_eq pos_divide_le_eq*)
also have ... ≤ *f a*
using *assms concave_onD_Icc′* **by** *blast*
finally show *?thesis* **.**
qed

1.7.6 Convexity of the generalised binomial

lemma *mono_on_mul*:
fixes *f*::*′a*::*ord* ⇒ *′b*::*ordered_semiring*
assumes *mono_on S f mono_on S g*
assumes *fty*: *f* ∈ *S* → {*0*..} **and** *gty*: *g* ∈ *S* → {*0*..}
shows *mono_on S* (λ*x*. *f x* ∗ *g x*)
using *assms* **by** (*auto simp*: *Pi_iff monotone_on_def intro*!: *mult_mono*)

lemma *mono_on_prod*:
fixes *f*::*′i* ⇒ *′a*::*ord* ⇒ *′b*::*linordered_idom*
assumes ⋀*i*. *i* ∈ *I* ⟹ *mono_on S* (*f i*)
assumes ⋀*i*. *i* ∈ *I* ⟹ *f i* ∈ *S* → {*0*..}
shows *mono_on S* (λ*x*. *prod* (λ*i*. *f i x*) *I*)
using *assms*
by (*induction I rule*: *infinite_finite_induct*)
(*auto simp*: *mono_on_const Pi_iff prod_nonneg mono_on_mul mono_onI*)

lemma *convex_gchoose_aux*: *convex_on* {*k*−*1*..} (λ*a*. *prod* (λ*i*. *a* − *of_nat i*) {*0*..<*k*})
proof (*induction k*)
case *0*
then show *?case*
by (*simp add*: *convex_on_def*)
next
case (*Suc k*)

```
  have convex_on {real k..} (λa. (∏ i = 0..<k. a − real i) * (a − real k))
  proof (intro convex_on_mul convex_on_diff)
    show convex_on {real k..} (λx. ∏ i = 0..<k. x − real i)
      using Suc convex_on_subset by fastforce
    show mono_on {real k..} (λx. ∏ i = 0..<k. x − real i)
      by (force simp: monotone_on_def intro!: prod_mono)
  next
    show (λx. ∏ i = 0..<k. x − real i) ∈ {real k..} → {0..}
      by (auto intro!: prod_nonneg)
  qed (auto simp: convex_on_ident concave_on_const mono_onI)
  then show ?case
    by simp
qed

lemma convex_gchoose: convex_on {k−1..} (λx. x gchoose k)
  by (simp add: gbinomial_prod_rev convex_on_cdiv convex_gchoose_aux)
```

1.7.7 Some inequalities: Applications of convexity

```
lemma Youngs_inequality_0:
  fixes a::real
  assumes 0 ≤ α 0 ≤ β α+β = 1 a>0 b>0
  shows a powr α * b powr β ≤ α*a + β*b
proof −
  have α * ln a + β * ln b ≤ ln (α * a + β * b)
    using assms ln_concave by (simp add: concave_on_iff)
  moreover have 0 < α * a + β * b
    using assms by (smt (verit) mult_pos_pos split_mult_pos_le)
  ultimately show ?thesis
    using assms by (simp add: powr_def mult_exp_exp flip: ln_ge_iff)
qed

lemma Youngs_inequality:
  fixes p::real
  assumes p>1 q>1 1/p + 1/q = 1 a≥0 b≥0
  shows a * b ≤ a powr p / p + b powr q / q
proof (cases a=0 ∨ b=0)
  case False
  then show ?thesis
  using Youngs_inequality_0 [of 1/p 1/q a powr p b powr q] assms
  by (simp add: powr_powr)
qed (use assms in auto)

lemma Cauchy_Schwarz_ineq_sum:
  fixes a :: 'a ⇒ 'b::linordered_field
  shows (∑ i∈I. a i * b i)^2 ≤ (∑ i∈I. (a i)^2) * (∑ i∈I. (b i)^2)
proof (cases (∑ i∈I. (b i)^2) > 0)
  case False
  then consider ⋀i. i∈I ⟹ b i = 0 | infinite I
```

by (*metis* (*mono_tags*, *lifting*) *sum_pos2 zero_le_power2 zero_less_power2*)
thus *?thesis*
by *fastforce*
next
case *True*
define r **where** $r \equiv (\sum i{\in}I.\ a\ i * b\ i) \ /\ (\sum i{\in}I.\ (b\ i)^2)$
have $0 \le (\sum i{\in}I.\ (a\ i - r * b\ i)^2)$
by (*simp add*: *sum_nonneg*)
also have ... $= (\sum i{\in}I.\ (a\ i)^2) - 2 * r * (\sum i{\in}I.\ a\ i * b\ i) + r^2 * (\sum i{\in}I.\ (b\ i)^2)$
by (*simp add*: *algebra_simps power2_eq_square sum_distrib_left flip*: *sum.distrib*)
also have ... $= (\sum i{\in}I.\ (a\ i)^2) - ((\sum i{\in}I.\ a\ i * b\ i))^2\ /\ (\sum i{\in}I.\ (b\ i)^2)$
by (*simp add*: *r_def power2_eq_square*)
finally have $0 \le (\sum i{\in}I.\ (a\ i)^2) - ((\sum i{\in}I.\ a\ i * b\ i))^2\ /\ (\sum i{\in}I.\ (b\ i)^2)$ **.**
hence $((\sum i{\in}I.\ a\ i * b\ i))^2\ /\ (\sum i{\in}I.\ (b\ i)^2) \le (\sum i{\in}I.\ (a\ i)^2)$
by (*simp add*: *le_diff_eq*)
thus $((\sum i{\in}I.\ a\ i * b\ i))^2 \le (\sum i{\in}I.\ (a\ i)^2) * (\sum i{\in}I.\ (b\ i)^2)$
by (*simp add*: *pos_divide_le_eq True*)
qed

lemma *sum_squared_le_sum_of_squares*:
fixes $f :: 'a \Rightarrow real$
assumes $\bigwedge i.\ i{\in}I \Longrightarrow f\ i \ge 0$ *finite* I $I \neq \{\}$
shows $(\sum i{\in}I.\ f\ i)^2 \le (\sum y{\in}I.\ (f\ y)^2) * card\ I$
proof (*cases finite* $I \wedge I \neq \{\}$)
case *True*
have $(\sum i{\in}I.\ f\ i\ /\ real\ (card\ I))^2 \le (\sum i{\in}I.\ (f\ i)^2\ /\ real\ (card\ I))$
using *assms convex_on_sum* [*OF* _ _ *convex_power2*, **where** $a = \lambda x.\ 1\ /\ real(card\ I)$ **and** $S{=}I$]
by *simp*
then show *?thesis*
using *assms*
by (*simp add*: *divide_simps power2_eq_square split*: *if_split_asm flip*: *sum_divide_distrib*)
qed *auto*

1.7.8 Misc related lemmas

lemma *convex_translation_eq* [*simp*]:
$convex\ ((+)\ a\ `\ s) \longleftrightarrow convex\ s$
by (*metis convex_translation translation_galois*)

lemma *convex_translation_subtract_eq* [*simp*]:
$convex\ ((\lambda b.\ b - a)\ `\ s) \longleftrightarrow convex\ s$
using *convex_translation_eq* [*of* $-a$] **by** (*simp cong*: *image_cong_simp*)

lemma *convex_linear_image_eq* [*simp*]:
fixes $f :: 'a{::}real\_vector \Rightarrow 'b{::}real\_vector$
shows $[\![linear\ f;\ inj\ f]\!] \Longrightarrow convex\ (f\ `\ s) \longleftrightarrow convex\ s$
by (*metis* (*no_types*) *convex_linear_image convex_linear_vimage inj_vimage_image_eq*)

lemma *vector_choose_size*:
assumes $0 \leq c$
obtains $x :: 'a::\{real\_normed\_vector, perfect\_space\}$ **where** $norm\ x = c$
proof −
obtain $a::'a$ **where** $a \neq 0$
using *UNIV_not_singleton UNIV_eq_I set_zero singletonI* **by** *fastforce*
then show *?thesis*
by (*rule_tac x=scaleR* (*c / norm a*) *a* **in** *that*) (*simp add*: *assms*)
qed

lemma *vector_choose_dist*:
assumes $0 \leq c$
obtains $y :: 'a::\{real\_normed\_vector, perfect\_space\}$ **where** $dist\ x\ y = c$
by (*metis add_diff_cancel_left' assms dist_commute dist_norm vector_choose_size*)

lemma *sum_delta''*:
fixes $s::'a::real\_vector\ set$
assumes *finite s*
shows $(\sum x{\in}s.\ (if\ y = x\ then\ f\ x\ else\ 0) *_R x) = (if\ y{\in}s\ then\ (f\ y) *_R y\ else\ 0)$
proof −
have $*: \bigwedge x\ y.\ (if\ y = x\ then\ f\ x\ else\ (0::real)) *_R x = (if\ x{=}y\ then\ (f\ x) *_R x\ else\ 0)$
by *auto*
show *?thesis*
unfolding $*$ **using** *sum.delta*[*OF assms*, *of y* $\lambda x.\ f\ x *_R x$] **by** *auto*
qed

1.7.9 Cones

definition $cone :: 'a::real\_vector\ set \Rightarrow bool$
where $cone\ s \longleftrightarrow (\forall x{\in}s.\ \forall c{\geq}0.\ c *_R x \in s)$

lemma *cone_empty*[*intro*, *simp*]: *cone* {}
unfolding *cone_def* **by** *auto*

lemma *cone_univ*[*intro*, *simp*]: *cone UNIV*
unfolding *cone_def* **by** *auto*

lemma *cone_Inter*[*intro*]: $\forall s{\in}f.\ cone\ s \Longrightarrow cone\ (\bigcap f)$
unfolding *cone_def* **by** *auto*

lemma *subspace_imp_cone*: $subspace\ S \Longrightarrow cone\ S$
by (*simp add*: *cone_def subspace_scale*)

Conic hull

lemma *cone_cone_hull*: *cone* (*cone hull S*)
unfolding *hull_def* **by** *auto*

lemma *cone_hull_eq*: *cone hull S = S ⟷ cone S*
by (*metis cone_cone_hull hull_same*)

lemma *mem_cone*:
assumes *cone S x ∈ S c ≥ 0*
shows *c *$_R$ *x ∈ S*
using *assms cone_def*[*of S*] **by** *auto*

lemma *cone_contains_0*:
assumes *cone S*
shows *S ≠ {} ⟷ 0 ∈ S*
using *assms mem_cone* **by** *fastforce*

lemma *cone_0*: *cone {0}*
unfolding *cone_def* **by** *auto*

lemma *cone_Union*[*intro*]: *(∀ s∈f. cone s) ⟶ cone (⋃f)*
unfolding *cone_def* **by** *blast*

lemma *cone_iff*:
assumes *S ≠ {}*
shows *cone S ⟷ 0 ∈ S ∧ (∀ c. c > 0 ⟶ ((*$_R$*) c) ‘ S = S)* (**is** *_ = ?rhs*)
proof
assume *cone S*
{
fix *c :: real*
assume *c > 0*
have *x ∈ ((*$_R$*) c) ‘ S* **if** *x ∈ S* **for** *x*
using ‹*cone S*› ‹*c>0*› *mem_cone*[*of S x 1/c*] *that*
exI[*of* (λ*t. t ∈ S ∧ x = c *$_R$ *t*) (1 / c) *$_R$ *x*]
by *auto*
then have *((*$_R$*) c) ‘ S = S*
using ‹*0 < c*› ‹*cone S*› *mem_cone* **by** *fastforce*
}
then show *0 ∈ S ∧ (∀ c. c > 0 ⟶ ((*$_R$*) c) ‘ S = S)*
using ‹*cone S*› *cone_contains_0*[*of S*] *assms* **by** *auto*
next
show *?rhs ⟹ cone S*
by (*metis Convex.cone_def imageI order_neq_le_trans scaleR_zero_left*)
qed

lemma *cone_hull_empty*: *cone hull {} = {}*
by (*metis cone_empty cone_hull_eq*)

lemma *cone_hull_empty_iff*: *S = {} ⟷ cone hull S = {}*
by (*metis cone_hull_empty hull_subset subset_empty*)

lemma *cone_hull_contains_0*: *S ≠ {} ⟷ 0 ∈ cone hull S*
by (*metis cone_cone_hull cone_contains_0 cone_hull_empty_iff*)

lemma *mem_cone_hull*:
assumes $x \in S$ $c \geq 0$
shows $c *_R x \in$ *cone hull* S
by (*metis assms cone_cone_hull hull_inc mem_cone*)

proposition *cone_hull_expl*: *cone hull* $S = \{c *_R x \mid c\ x.\ c \geq 0 \wedge x \in S\}$
(**is** *?lhs* = *?rhs*)
proof
have *?rhs* $\in$ *Collect cone*
using *Convex.cone_def* **by** *fastforce*
moreover have $S \subseteq$ *?rhs*
by (*smt* (*verit*) *mem_Collect_eq scaleR_one subsetI*)
ultimately show *?lhs* $\subseteq$ *?rhs*
using *hull_minimal* **by** *blast*
qed (*use mem_cone_hull* **in** *auto*)

lemma *convex_cone*:
convex $S \wedge$ *cone* $S \longleftrightarrow (\forall x{\in}S.\ \forall y{\in}S.\ (x + y) \in S) \wedge (\forall x{\in}S.\ \forall c{\geq}0.\ (c *_R x) \in S)$
(**is** *?lhs* = *?rhs*)
proof −
{
fix $x\ y$
assume $x{\in}S$ $y{\in}S$ **and** *?lhs*
then have $2 *_R x \in S$ $2 *_R y \in S$ *convex* S
unfolding *cone_def* **by** *auto*
then have $x + y \in S$
using *convexD* [*OF* ‹*convex S*›, *of* $2{*_R}\ x$ $2{*_R}\ y$]
by (*smt* (*verit*, *ccfv_threshold*) *field_sum_of_halves scaleR_2 scaleR_half_double*)
}
then show *?thesis*
unfolding *convex_def cone_def* **by** *blast*
qed

1.7.10 Connectedness of convex sets

lemma *convex_connected*:
fixes S :: *'a*::*real_normed_vector set*
assumes *convex* S
shows *connected* S
proof (*rule connectedI*)
fix $A\ B$
assume *open* A *open* B $A \cap B \cap S = \{\}$ $S \subseteq A \cup B$
moreover
assume $A \cap S \neq \{\}$ $B \cap S \neq \{\}$
then obtain $a\ b$ **where** a: $a \in A$ $a \in S$ **and** b: $b \in B$ $b \in S$ **by** *auto*
define f **where** [*abs_def*]: $f\ u = u *_R a + (1 - u) *_R b$ **for** u
then have *continuous_on* $\{0\ ..\ 1\}$ f

by (*auto intro*!: *continuous_intros*)
then have *connected* (*f* ‘ {*0* .. *1*})
by (*auto intro*!: *connected_continuous_image*)
note *connectedD*[*OF this*, *of A B*]
moreover have *a* ∈ *A* ∩ *f* ‘ {*0* .. *1*}
using *a* **by** (*auto intro*!: *image_eqI*[*of _ _ 1*] *simp*: *f_def*)
moreover have *b* ∈ *B* ∩ *f* ‘ {*0* .. *1*}
using *b* **by** (*auto intro*!: *image_eqI*[*of _ _ 0*] *simp*: *f_def*)
moreover have *f* ‘ {*0* .. *1*} ⊆ *S*
using ‹*convex S*› *a b* **unfolding** *convex_def f_def* **by** *auto*
ultimately show *False* **by** *auto*
qed

corollary *connected_UNIV*[*intro*]: *connected* (*UNIV* :: *'a*::*real_normed_vector set*)
by (*simp add*: *convex_connected*)

lemma *convex_prod*:
assumes ⋀*i*. *i* ∈ *Basis* ⟹ *convex* {*x*. *P i x*}
shows *convex* {*x*. ∀ *i*∈*Basis*. *P i* (*x*•*i*)}
using *assms* **by** (*auto simp*: *inner_add_left convex_def*)

lemma *convex_positive_orthant*: *convex* {*x*::*'a*::*euclidean_space*. (∀ *i*∈*Basis*. *0* ≤ *x*•*i*)}
by (*rule convex_prod*) (*simp flip*: *atLeast_def*)

1.7.11 Convex hull

lemma *convex_convex_hull* [*iff*]: *convex* (*convex hull s*)
by (*metis* (*mono_tags*) *convex_Inter hull_def mem_Collect_eq*)

lemma *convex_hull_subset*:
s ⊆ *convex hull t* ⟹ *convex hull s* ⊆ *convex hull t*
by (*simp add*: *subset_hull*)

lemma *convex_hull_eq*: *convex hull s* = *s* ⟷ *convex s*
by (*metis convex_convex_hull hull_same*)

Convex hull is "preserved" by a linear function

lemma *convex_hull_linear_image*:
assumes *f*: *linear f*
shows *f* ‘ (*convex hull S*) = *convex hull* (*f* ‘ *S*)
proof
show *convex hull* (*f* ‘ *S*) ⊆ *f* ‘ (*convex hull S*)
by (*intro hull_minimal image_mono hull_subset convex_linear_image assms convex_convex_hull*)
show *f* ‘ (*convex hull S*) ⊆ *convex hull* (*f* ‘ *S*)
by (*meson convex_convex_hull convex_linear_vimage f hull_minimal hull_subset image_subset_iff_subset_vimage*)
qed

lemma *in_convex_hull_linear_image*:
assumes *linear f x* ∈ *convex hull S*
shows *f x* ∈ *convex hull* (*f* ‘ *S*)
using *assms convex_hull_linear_image image_eqI* **by** *blast*

lemma *convex_hull_Times*:
convex hull (*S* × *T*) = (*convex hull S*) × (*convex hull T*)
proof
show *convex hull* (*S* × *T*) ⊆ (*convex hull S*) × (*convex hull T*)
by (*intro hull_minimal Sigma_mono hull_subset convex_Times convex_convex_hull*)
have (*x*, *y*) ∈ *convex hull* (*S* × *T*) **if** *x*: *x* ∈ *convex hull S* **and** *y*: *y* ∈ *convex hull T* **for** *x y*
proof (*rule hull_induct* [*OF x*], *rule hull_induct* [*OF y*])
fix *x y* **assume** *x* ∈ *S* **and** *y* ∈ *T*
then show (*x*, *y*) ∈ *convex hull* (*S* × *T*)
by (*simp add*: *hull_inc*)
next
fix *x* **let** *?S* = ((λ*y*. (*0*, *y*)) −‘ (λ*p*. (− *x*, *0*) + *p*) ‘ (*convex hull S* × *T*))
have *convex ?S*
by (*intro convex_linear_vimage convex_translation convex_convex_hull*,
simp add: *linear_iff*)
also have *?S* = {*y*. (*x*, *y*) ∈ *convex hull* (*S* × *T*)}
by (*auto simp*: *image_def Bex_def*)
finally show *convex* {*y*. (*x*, *y*) ∈ *convex hull* (*S* × *T*)} **.**
next
show *convex* {*x*. (*x*, *y*) ∈ *convex hull S* × *T*}
proof −
fix *y* **let** *?S* = ((λ*x*. (*x*, *0*)) −‘ (λ*p*. (*0*, − *y*) + *p*) ‘ (*convex hull S* × *T*))
have *convex ?S*
by (*intro convex_linear_vimage convex_translation convex_convex_hull*,
simp add: *linear_iff*)
also have *?S* = {*x*. (*x*, *y*) ∈ *convex hull* (*S* × *T*)}
by (*auto simp*: *image_def Bex_def*)
finally show *convex* {*x*. (*x*, *y*) ∈ *convex hull* (*S* × *T*)} **.**
qed
qed
then show (*convex hull S*) × (*convex hull T*) ⊆ *convex hull* (*S* × *T*)
unfolding *subset_eq split_paired_Ball_Sigma* **by** *blast*
qed

Stepping theorems for convex hulls of finite sets

lemma *convex_hull_empty*[*simp*]: *convex hull* {} = {}
by (*simp add*: *hull_same*)

lemma *convex_hull_singleton*[*simp*]: *convex hull* {*a*} = {*a*}
by (*simp add*: *hull_same*)

lemma *convex_hull_insert*:
fixes S :: *'a*::*real_vector set*
assumes $S \neq \{\}$
shows *convex hull* (*insert a S*) =
$\{x.\ \exists u{\geq}0.\ \exists v{\geq}0.\ \exists b.\ (u + v = 1) \wedge b \in (\textit{convex hull } S) \wedge (x = u *_R a + v *_R b)\}$
(**is** _ = *?hull*)
proof (*intro equalityI hull_minimal subsetI*)
fix *x*
assume $x \in$ *insert a S*
then show $x \in$ *?hull*
unfolding *insert_iff*
proof
assume $x = a$
then show *?thesis*
by (*smt* (*verit*, *del_insts*) *add.right_neutral assms ex_in_conv hull_inc mem_Collect_eq scaleR_one scaleR_zero_left*)
next
assume $x \in S$
with *hull_subset* **show** *?thesis*
by *force*
qed
next
fix *x*
assume $x \in$ *?hull*
then obtain *u v b* **where** *obt*: $u{\geq}0\ v{\geq}0\ u + v = 1\ b \in \textit{convex hull } S\ x = u *_R a + v *_R b$
by *auto*
have $a \in$ *convex hull insert a S b* $\in$ *convex hull insert a S*
using *hull_mono*[*of S insert a S convex*] *hull_mono*[*of* {*a*} *insert a S convex*] **and** *obt*(*4*)
by *auto*
then show $x \in$ *convex hull insert a S*
unfolding *obt*(*5*) **using** *obt*(*1*−*3*)
by (*rule convexD* [*OF convex_convex_hull*])
next
show *convex ?hull*
proof (*rule convexI*)
fix *x y u v*
assume *as*: $(0{::}\textit{real}) \leq u\ 0 \leq v\ u + v = 1$ **and** *x*: $x \in$ *?hull* **and** *y*: $y \in$ *?hull*
from *x* **obtain** *u1 v1 b1* **where**
obt1: $u1{\geq}0\ v1{\geq}0\ u1 + v1 = 1\ b1 \in \textit{convex hull } S$ **and** *xeq*: $x = u1 *_R a + v1 *_R b1$
by *auto*
from *y* **obtain** *u2 v2 b2* **where**
obt2: $u2{\geq}0\ v2{\geq}0\ u2 + v2 = 1\ b2 \in \textit{convex hull } S$ **and** *yeq*: $y = u2 *_R a + v2 *_R b2$
by *auto*
have $*$: $\bigwedge(x{::}'a)\ s1\ s2.\ x - s1 *_R x - s2 *_R x = ((1{::}\textit{real}) - (s1 + s2)) *_R x$

```
    by (auto simp: algebra_simps)
  have ∃b∈convex hull S. u *R x + v *R y =
    (u * u1) *R a + (v * u2) *R a + (b - (u * u1) *R b - (v * u2) *R b)
  proof (cases u * v1 + v * v2 = 0)
    case True
    have *: ⋀(x::'a) s1 s2. x - s1 *R x - s2 *R x = ((1::real) - (s1 + s2)) *R
x
      by (auto simp: algebra_simps)
    have eq0: u * v1 = 0 v * v2 = 0
    using True mult_nonneg_nonneg[OF ‹u≥0› ‹v1≥0›] mult_nonneg_nonneg[OF
‹v≥0› ‹v2≥0›]
      by arith+
    then have u * u1 + v * u2 = 1
      using as(3) obt1(3) obt2(3) by auto
    then show ?thesis
      using * eq0 as obt1(4) xeq yeq by auto
  next
    case False
    have 1 - (u * u1 + v * u2) = (u + v) - (u * u1 + v * u2)
      by (simp add: as(3))
    also have ... = u * v1 + v * v2
      by (smt (verit, ccfv_SIG) distrib_left mult_cancel_left1 obt1(3) obt2(3))
    finally have **:1 - (u * u1 + v * u2) = u * v1 + v * v2 .
    let ?b = ((u * v1) / (u * v1 + v * v2)) *R b1 + ((v * v2) / (u * v1 + v *
v2)) *R b2
    have zeroes: 0 ≤ u * v1 + v * v2 0 ≤ u * v1 0 ≤ u * v1 + v * v2 0 ≤ v *
v2
      using as obt1 obt2 by auto
    show ?thesis
    proof
      show u *R x + v *R y = (u * u1) *R a + (v * u2) *R a + (?b - (u * u1)
*R ?b - (v * u2) *R ?b)
        unfolding xeq yeq * **
        using False by (auto simp: scaleR_left_distrib scaleR_right_distrib)
      show ?b ∈ convex hull S
          using False mem_convex_alt obt1(4) obt2(4) zeroes(2) zeroes(4) by
fastforce
    qed
  qed
  then obtain b where b: b ∈ convex hull S
    u *R x + v *R y = (u * u1) *R a + (v * u2) *R a + (b - (u * u1) *R b -
(v * u2) *R b) ..
  obtain u1: u1 ≤ 1 and u2: u2 ≤ 1
    using obt1 obt2 by auto
  have u1 * u + u2 * v ≤ max u1 u2 * u + max u1 u2 * v
    by (smt (verit, ccfv_SIG) as mult_right_mono)
  also have ... ≤ 1
    unfolding distrib_left[symmetric] and as(3) using u1 u2 by auto
  finally have le1: u1 * u + u2 * v ≤ 1 .
```

```
    show u *R x + v *R y ∈ ?hull
    proof (intro CollectI exI conjI)
      show 0 ≤ u * u1 + v * u2
        by (simp add: as obt1(1) obt2(1))
      show 0 ≤ 1 − u * u1 − v * u2
        by (simp add: le1 diff_diff_add mult.commute)
    qed (use b in ‹auto simp: algebra_simps›)
  qed
qed

lemma convex_hull_insert_alt:
   convex hull (insert a S) =
     (if S = {} then {a}
      else {(1 − u) *R a + u *R x |x u. 0 ≤ u ∧ u ≤ 1 ∧ x ∈ convex hull S})
  apply (simp add: convex_hull_insert)
  using diff_add_cancel diff_ge_0_iff_ge
  by (smt (verit, del_insts) Collect_cong)
```

Explicit expression for convex hull

```
proposition convex_hull_indexed:
  fixes S :: 'a::real_vector set
  shows convex hull S =
    {y. ∃k u x. (∀ i∈{1::nat .. k}. 0 ≤ u i ∧ x i ∈ S) ∧
              (sum u {1..k} = 1) ∧ (∑ i = 1..k. u i *R x i) = y}
    (is ?xyz = ?hull)
proof (rule hull_unique [OF _ convexI])
  show S ⊆ ?hull
    by (clarsimp, rule_tac x=1 in exI, rule_tac x=λx. 1 in exI, auto)
next
  fix T
  assume S ⊆ T convex T
  then show ?hull ⊆ T
    by (blast intro: convex_sum)
next
  fix x y u v
  assume uv: 0 ≤ u 0 ≤ v u + v = (1::real)
  assume xy: x ∈ ?hull y ∈ ?hull
  from xy obtain k1 u1 x1 where
    x [rule_format]: ∀ i∈{1::nat..k1}. 0≤u1 i ∧ x1 i ∈ S
                  sum u1 {Suc 0..k1} = 1 (∑ i = Suc 0..k1. u1 i *R x1 i) = x
    by auto
  from xy obtain k2 u2 x2 where
    y [rule_format]: ∀ i∈{1::nat..k2}. 0≤u2 i ∧ x2 i ∈ S
                  sum u2 {Suc 0..k2} = 1 (∑ i = Suc 0..k2. u2 i *R x2 i) = y
    by auto
  have *: ⋀P (x::'a) y s t i. (if P i then s else t) *R (if P i then x else y) = (if P
i then s *R x else t *R y)
        {1..k1 + k2} ∩ {1..k1} = {1..k1} {1..k1 + k2} ∩ − {1..k1} = (λi. i +
```

```
k1) ‘ {1..k2}
    by auto
  have inj: inj_on (λi. i + k1) {1..k2}
    unfolding inj_on_def by auto
  let ?uu = λi. if i ∈ {1..k1} then u * u1 i else v * u2 (i − k1)
  let ?xx = λi. if i ∈ {1..k1} then x1 i else x2 (i − k1)
  show u *R x + v *R y ∈ ?hull
  proof (intro CollectI exI conjI ballI)
    show 0 ≤ ?uu i ?xx i ∈ S if i ∈ {1..k1+k2} for i
      using that by (auto simp add: le_diff_conv uv(1) x(1) uv(2) y(1))
    show (∑ i = 1..k1 + k2. ?uu i) = 1  (∑ i = 1..k1 + k2. ?uu i *R ?xx i) =
u *R x + v *R y
      unfolding * sum.If_cases[OF finite_atLeastAtMost[of 1 k1 + k2]]
        sum.reindex[OF inj] Collect_mem_eq o_def
      unfolding scaleR_scaleR[symmetric] scaleR_right.sum [symmetric] sum_distrib_left[symmetric]
      by (simp_all add: sum_distrib_left[symmetric]  x(2,3) y(2,3) uv(3))
  qed
qed

lemma convex_hull_finite:
  fixes S :: 'a::real_vector set
  assumes finite S
  shows convex hull S = {y. ∃ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = 1 ∧ sum (λx. u
x *R x) S = y}
  (is ?HULL = _)
proof (rule hull_unique [OF _ convexI]; clarify)
  fix x
  assume x ∈ S
  then show ∃ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = 1 ∧ (∑ x∈S. u x *R x) = x
    by (rule_tac x=λy. if x=y then 1 else 0 in exI) (auto simp: sum.delta'[OF
assms] sum_delta''[OF assms])
next
  fix u v :: real
  assume uv: 0 ≤ u 0 ≤ v u + v = 1
  fix ux assume ux [rule_format]: ∀ x∈S. 0 ≤ ux x sum ux S = (1::real)
  fix uy assume uy [rule_format]: ∀ x∈S. 0 ≤ uy x sum uy S = (1::real)
  have 0 ≤ u * ux x + v * uy x if x∈S for x
    by (simp add: that uv ux(1) uy(1))
  moreover
  have (∑ x∈S. u * ux x + v * uy x) = 1
    unfolding sum.distrib and sum_distrib_left[symmetric] ux(2) uy(2)
    using uv(3) by auto
  moreover
  have (∑ x∈S. (u * ux x + v * uy x) *R x) = u *R (∑ x∈S. ux x *R x) + v *R
(∑ x∈S. uy x *R x)
    unfolding scaleR_left_distrib sum.distrib scaleR_scaleR[symmetric] scaleR_right.sum
[symmetric]
    by auto
  ultimately
```

show $\exists uc.\ (\forall x{\in}S.\ 0 \le uc\ x) \wedge sum\ uc\ S = 1 \wedge$
$(\sum x{\in}S.\ uc\ x *_R x) = u *_R (\sum x{\in}S.\ ux\ x *_R x) + v *_R (\sum x{\in}S.\ uy\ x *_R x)$
by (*rule_tac* $x{=}\lambda x.\ u * ux\ x + v * uy\ x$ **in** *exI*, *auto*)
qed (*use assms* **in** ‹*auto simp*: *convex_explicit*›)

Another formulation

Formalized by Lars Schewe.

lemma *convex_hull_explicit*:
fixes p :: *'a*::*real_vector set*
shows *convex hull p* =
$\{y.\ \exists S\ u.\ finite\ S \wedge S \subseteq p \wedge (\forall x{\in}S.\ 0 \le u\ x) \wedge sum\ u\ S = 1 \wedge sum\ (\lambda v.\ u\ v *_R v)\ S = y\}$
(**is** *?lhs* = *?rhs*)
proof (*intro subset_antisym subsetI*)
fix x
assume $x \in$ *convex hull p*
then obtain $k\ u\ y$ **where**
obt: $\forall i{\in}\{1{::}nat..k\}.\ 0 \le u\ i \wedge y\ i \in p\ sum\ u\ \{1..k\} = 1\ (\sum i = 1..k.\ u\ i *_R y\ i) = x$
unfolding *convex_hull_indexed* **by** *auto*
have *fin*: *finite* $\{1..k\}$ **by** *auto*
{
fix j
assume $j{\in}\{1..k\}$
then have $y\ j \in p \wedge 0 \le sum\ u\ \{i.\ Suc\ 0 \le i \wedge i \le k \wedge y\ i = y\ j\}$
by (*metis* (*mono_tags*, *lifting*) *One_nat_def atLeastAtMost_iff mem_Collect_eq obt*(*1*) *sum_nonneg*)
}
moreover have $(\sum v{\in}y\ `\ \{1..k\}.\ sum\ u\ \{i \in \{1..k\}.\ y\ i = v\}) = 1$
unfolding *sum.image_gen*[*OF fin*, *symmetric*] **using** *obt*(*2*) **by** *auto*
moreover have $(\sum v{\in}y\ `\ \{1..k\}.\ sum\ u\ \{i \in \{1..k\}.\ y\ i = v\} *_R v) = x$
using *sum.image_gen*[*OF fin*, *of* $\lambda i.\ u\ i *_R y\ i$ *y*, *symmetric*]
unfolding *scaleR_left.sum* **using** *obt*(*3*) **by** *auto*
ultimately
have $\exists S\ u.\ finite\ S \wedge S \subseteq p \wedge (\forall x{\in}S.\ 0 \le u\ x) \wedge sum\ u\ S = 1 \wedge (\sum v{\in}S.\ u\ v *_R v) = x$
by (*smt* (*verit*, *ccfv_SIG*) *imageE mem_Collect_eq obt*(*1*) *subsetI sum.cong sum.infinite sum_nonneg*)
then show $x \in$ *?rhs* **by** *auto*
next
fix y
assume $y \in$ *?rhs*
then obtain $S\ u$ **where**
S: *finite S* $S \subseteq p\ \forall x{\in}S.\ 0 \le u\ x\ sum\ u\ S = 1\ (\sum v{\in}S.\ u\ v *_R v) = y$
by *auto*
obtain f **where** f: *inj_on f* $\{1..card\ S\}\ f\ `\ \{1..card\ S\} = S$
using *ex_bij_betw_nat_finite_1*[*OF S*(*1*)] **unfolding** *bij_betw_def* **by** *auto*

```
  then have 0 ≤ u (f i) f i ∈ p if i ∈ {1..card S} for i
    using S ‹i ∈ {1..card S}› by blast+
  moreover
  {
    fix y
    assume y∈S
    then obtain i where i∈{1..card S} f i = y
      by (metis f(2) image_iff)
    then have {x. Suc 0 ≤ x ∧ x ≤ card S ∧ f x = y} = {i}
      using f(1) inj_onD by fastforce
    then have (∑ x∈{x ∈ {1..card S}. f x = y}. u (f x)) = u y
      (∑ x∈{x ∈ {1..card S}. f x = y}. u (f x) *R f x) = u y *R y
      by (simp_all add: sum_constant_scaleR ‹f i = y›)
  }
  then have (∑ x = 1..card S. u (f x)) = 1 (∑ i = 1..card S. u (f i) *R f i) = y
    by (metis (mono_tags, lifting) S(4,5) f sum.reindex_cong)+
  ultimately
  show y ∈ convex hull p
    unfolding convex_hull_indexed
    by (smt (verit, del_insts) mem_Collect_eq sum.cong)
qed
```

A stepping theorem for that expansion

```
lemma convex_hull_finite_step:
  fixes S :: 'a::real_vector set
  assumes finite S
  shows
    (∃ u. (∀ x∈insert a S. 0 ≤ u x) ∧ sum u (insert a S) = w ∧ sum (λx. u x *R
x) (insert a S) = y)
      ⟷ (∃ v≥0. ∃ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = w − v ∧ sum (λx. u x *R
x) S = y − v *R a)
  (is ?lhs = ?rhs)
proof (cases a ∈ S)
  case True
  then have *: insert a S = S by auto
  show ?thesis
  proof
    assume ?lhs
    then show ?rhs
      unfolding * by force
  next
    have fin: finite (insert a S) using assms by auto
    assume ?rhs
    then obtain v u where uv: v≥0 ∀ x∈S. 0 ≤ u x sum u S = w − v (∑ x∈S.
u x *R x) = y − v *R a
      by auto
    then show ?lhs
      using uv True assms
```

apply (*rule_tac* $x = \lambda x.\ (\mathit{if}\ a = x\ \mathit{then}\ v\ \mathit{else}\ 0) + u\ x$ **in** *exI*)
apply (*auto simp*: *sum_clauses scaleR_left_distrib sum.distrib sum_delta''*[*OF fin*])
done
qed
next
case *False*
show *?thesis*
proof
assume *?lhs*
then obtain u **where** u: $\forall x{\in}\mathit{insert}\ a\ S.\ 0 \le u\ x\ \mathit{sum}\ u\ (\mathit{insert}\ a\ S) = w\ (\sum x{\in}\mathit{insert}\ a\ S.\ u\ x *_R x) = y$
by *auto*
then show *?rhs*
using u ‹$a{\notin}S$› **by** (*rule_tac* $x{=}u\ a$ **in** *exI*) (*auto simp*: *sum_clauses assms*)
next
assume *?rhs*
then obtain $v\ u$ **where** uv: $v{\ge}0\ \forall x{\in}S.\ 0 \le u\ x\ \mathit{sum}\ u\ S = w - v\ (\sum x{\in}S.\ u\ x *_R x) = y - v *_R a$
by *auto*
moreover
have $(\sum x{\in}S.\ \mathit{if}\ a = x\ \mathit{then}\ v\ \mathit{else}\ u\ x) = \mathit{sum}\ u\ S\ \ (\sum x{\in}S.\ (\mathit{if}\ a = x\ \mathit{then}\ v\ \mathit{else}\ u\ x) *_R x) = (\sum x{\in}S.\ u\ x *_R x)$
using *False* **by** (*auto intro*!: *sum.cong*)
ultimately show *?lhs*
using *False* **by** (*rule_tac* $x{=}\lambda x.\ \mathit{if}\ a = x\ \mathit{then}\ v\ \mathit{else}\ u\ x$ **in** *exI*) (*auto simp*: *sum_clauses*(*2*)[*OF assms*])
qed
qed

Hence some special cases

lemma *convex_hull_2*: *convex hull* $\{a,b\} = \{u *_R a + v *_R b \mid u\ v.\ 0 \le u \wedge 0 \le v \wedge u + v = 1\}$
(**is** *?lhs* = *?rhs*)
proof −
have $**$: *finite* $\{b\}$ **by** *auto*
have $\bigwedge x\ v\ u.\ [\![0 \le v;\ v \le 1;\ (1 - v) *_R b = x - v *_R a]\!] \Longrightarrow \exists u\ v.\ x = u *_R a + v *_R b \wedge 0 \le u \wedge 0 \le v \wedge u + v = 1$
by (*metis add.commute diff_add_cancel diff_ge_0_iff_ge*)
moreover
have $\bigwedge u\ v.\ [\![0 \le u;\ 0 \le v;\ u + v = 1]\!] \Longrightarrow \exists p{\ge}0.\ \exists q.\ 0 \le q\ b \wedge q\ b = 1 - p \wedge q\ b *_R b = u *_R a + v *_R b - p *_R a$
apply (*rule_tac* $x{=}u$ **in** *exI*, *simp*)
apply (*rule_tac* $x{=}\lambda x.\ v$ **in** *exI*, *simp*)
done
ultimately show *?thesis*
using *convex_hull_finite_step*[*OF* $**$, *of a 1*]

by (*auto simp add*: *convex_hull_finite*)
qed

lemma *convex_hull_2_alt*: *convex hull* {*a*,*b*} = {*a* + *u* ∗*R* (*b* − *a*) | *u*. 0 ≤ *u* ∧ *u* ≤ *1*}
unfolding *convex_hull_2*
proof (*rule Collect_cong*)
have ∗: ⋀*x y* ::*real*. *x* + *y* = *1* ⟷ *x* = *1* − *y*
by *auto*
fix *x*
show (∃ *v u*. *x* = *v* ∗*R* *a* + *u* ∗*R* *b* ∧ 0 ≤ *v* ∧ 0 ≤ *u* ∧ *v* + *u* = *1*) ⟷
(∃ *u*. *x* = *a* + *u* ∗*R* (*b* − *a*) ∧ 0 ≤ *u* ∧ *u* ≤ *1*)
apply (*simp add*: ∗)
by (*rule ex_cong1*) (*auto simp*: *algebra_simps*)
qed

lemma *convex_hull_3*:
convex hull {*a*,*b*,*c*} = { *u* ∗*R* *a* + *v* ∗*R* *b* + *w* ∗*R* *c* | *u v w*. 0 ≤ *u* ∧ 0 ≤ *v* ∧ 0 ≤ *w* ∧ *u* + *v* + *w* = *1*}
proof −
have *fin*: *finite* {*a*,*b*,*c*} *finite* {*b*,*c*} *finite* {*c*}
by *auto*
have ∗: ⋀*x y z* ::*real*. *x* + *y* + *z* = *1* ⟷ *x* = *1* − *y* − *z*
by (*auto simp*: *field_simps*)
show *?thesis*
unfolding *convex_hull_finite*[*OF fin*(*1*)] **and** *convex_hull_finite_step*[*OF fin*(*2*)] **and** ∗
unfolding *convex_hull_finite_step*[*OF fin*(*3*)]
apply (*rule Collect_cong*, *simp*)
apply *auto*
apply (*rule_tac x*=*va* **in** *exI*)
apply (*rule_tac x*=*u c* **in** *exI*, *simp*)
apply (*rule_tac x*=*1* − *v* − *w* **in** *exI*, *simp*)
apply (*rule_tac x*=*v* **in** *exI*, *simp*)
apply (*rule_tac x*=λ*x*. *w* **in** *exI*, *simp*)
done
qed

lemma *convex_hull_3_alt*:
convex hull {*a*,*b*,*c*} = {*a* + *u* ∗*R* (*b* − *a*) + *v* ∗*R* (*c* − *a*) | *u v*. 0 ≤ *u* ∧ 0 ≤ *v* ∧ *u* + *v* ≤ *1*}
proof −
have ∗: ⋀*x y z* ::*real*. *x* + *y* + *z* = *1* ⟷ *x* = *1* − *y* − *z*
by *auto*
show *?thesis*
unfolding *convex_hull_3*
apply (*auto simp*: ∗)
apply (*rule_tac x*=*v* **in** *exI*)
apply (*rule_tac x*=*w* **in** *exI*)

apply (*simp add*: *algebra_simps*)
apply (*rule_tac x*=*u* **in** *exI*)
apply (*rule_tac x*=*v* **in** *exI*)
apply (*simp add*: *algebra_simps*)
done
qed

1.7.12 Relations among closure notions and corresponding hulls

lemma *affine_imp_convex*: *affine s* $\Longrightarrow$ *convex s*
unfolding *affine_def convex_def* **by** *auto*

lemma *convex_affine_hull* [*simp*]: *convex* (*affine hull S*)
by (*simp add*: *affine_imp_convex*)

lemma *subspace_imp_convex*: *subspace s* $\Longrightarrow$ *convex s*
using *subspace_imp_affine affine_imp_convex* **by** *auto*

lemma *convex_hull_subset_span*: (*convex hull s*) $\subseteq$ (*span s*)
by (*metis hull_minimal span_superset subspace_imp_convex subspace_span*)

lemma *convex_hull_subset_affine_hull*: (*convex hull s*) $\subseteq$ (*affine hull s*)
by (*metis affine_affine_hull affine_imp_convex hull_minimal hull_subset*)

lemma *aff_dim_convex_hull*:
fixes *S* :: *'n*::*euclidean_space set*
shows *aff_dim* (*convex hull S*) = *aff_dim S*
by (*smt* (*verit*) *aff_dim_affine_hull aff_dim_subset convex_hull_subset_affine_hull hull_subset*)

1.7.13 Caratheodory's theorem

lemma *convex_hull_caratheodory_aff_dim*:
fixes *p* :: (*'a*::*euclidean_space*) *set*
shows *convex hull p* =
{*y*. $\exists$ *S u*. *finite S* $\wedge$ *S* $\subseteq$ *p* $\wedge$ *card S* $\leq$ *aff_dim p* + *1* $\wedge$
($\forall$ *x*$\in$*S*. *0* $\leq$ *u x*) $\wedge$ *sum u S* = *1* $\wedge$ *sum* (λ*v*. *u v* $*_R$ *v*) *S* = *y*}
unfolding *convex_hull_explicit set_eq_iff mem_Collect_eq*
proof (*intro allI iffI*)
fix *y*
let *?P* = λ*n*. $\exists$ *S u*. *finite S* $\wedge$ *card S* = *n* $\wedge$ *S* $\subseteq$ *p* $\wedge$ ($\forall$ *x*$\in$*S*. *0* $\leq$ *u x*) $\wedge$
sum u S = *1* $\wedge$ ($\sum$ *v*$\in$*S*. *u v* $*_R$ *v*) = *y*
assume $\exists$ *S u*. *finite S* $\wedge$ *S* $\subseteq$ *p* $\wedge$ ($\forall$ *x*$\in$*S*. *0* $\leq$ *u x*) $\wedge$ *sum u S* = *1* $\wedge$ ($\sum$ *v*$\in$*S*. *u v* $*_R$ *v*) = *y*
then obtain *N* **where** *?P N* **by** *auto*
then have $\exists$ *n*$\leq$*N*. ($\forall$ *k*<*n*. $\neg$ *?P k*) $\wedge$ *?P n*
by (*rule_tac ex_least_nat_le*, *auto*)
then obtain *n* **where** *?P n* **and** *smallest*: $\forall$ *k*<*n*. $\neg$ *?P k*

```
    by blast
  then obtain S u where S: finite S card S = n S⊆p
    and u: ∀x∈S. 0 ≤ u x sum u S = 1  (∑v∈S. u v *R v) = y by auto

  have card S ≤ aff_dim p + 1
  proof (rule ccontr, simp only: not_le)
    assume aff_dim p + 1 < card S
    then have affine_dependent S
      by (smt (verit) independent_card_le_aff_dim S(3))
    then obtain w v where wv: sum w S = 0 v∈S w v ≠ 0 (∑v∈S. w v *R v)
= 0
      using affine_dependent_explicit_finite[OF S(1)] by auto
    define i where i = (λv. (u v) / (− w v)) ` {v∈S. w v < 0}
    define t where t = Min i
    have ∃x∈S. w x < 0
      by (smt (verit, best) S(1) sum_pos2 wv)
    then have i ≠ {} unfolding i_def by auto
    then have t ≥ 0
      using Min_ge_iff[of i 0] and S(1) u[unfolded le_less]
      unfolding t_def i_def
      by (auto simp: divide_le_0_iff)
    have t: ∀v∈S. u v + t * w v ≥ 0
    proof
      fix v
      assume v ∈ S
      then have v: 0 ≤ u v
        using u(1) by blast
      show 0 ≤ u v + t * w v
      proof (cases w v < 0)
        case False
        thus ?thesis using v ‹t≥0› by auto
      next
        case True
        then have t ≤ u v / (− w v)
          using ‹v∈S› S unfolding t_def i_def by (auto intro: Min_le)
        then show ?thesis
          unfolding real_0_le_add_iff
          using True neg_le_minus_divide_eq by auto
      qed
    qed
    obtain a where a ∈ S and t = (λv. (u v) / (− w v)) a and w a < 0
      using Min_in[OF _ ‹i≠{}›] and S(1) unfolding i_def t_def by auto
    then have a: a ∈ S u a + t * w a = 0 by auto
    have *: ⋀f. sum f (S − {a}) = sum f S − ((f a)::'b::ab_group_add)
      unfolding sum.remove[OF S(1) ‹a∈S›] by auto
    have (∑v∈S. u v + t * w v) = 1
      by (metis add.right_neutral mult_zero_right sum.distrib sum_distrib_left
u(2) wv(1))
    moreover have (∑v∈S. u v *R v + (t * w v) *R v) − (u a *R a + (t * w a)
```

$*_R$ $a) = y$
unfolding *sum.distrib* *u(3)* *scaleR_scaleR*[*symmetric*] *scaleR_right.sum* [*symmetric*] *wv(4)*
using *a(2)* [*THEN eq_neg_iff_add_eq_0* [*THEN iffD2*]] **by** *simp*
ultimately have *?P* $(n - 1)$
apply (*rule_tac* x=$(S - \{a\})$ **in** *exI*)
apply (*rule_tac* x=$\lambda v.\ u\ v + t * w\ v$ **in** *exI*)
using $S\ t\ a$
apply (*auto simp*: $*$ *scaleR_left_distrib*)
done
then show *False*
using *smallest*[*THEN spec*[**where** x=$n - 1$]] **by** *auto*
qed
then show $\exists S\ u.$ *finite* $S \wedge S \subseteq p \wedge$ *card* $S \leq$ *aff_dim* $p + 1 \wedge$
$(\forall x{\in}S.\ 0 \leq u\ x) \wedge$ *sum* $u\ S = 1 \wedge (\sum v{\in}S.\ u\ v *_R v) = y$
using $S\ u$ **by** *auto*
qed *auto*

lemma *caratheodory_aff_dim*:
fixes p :: ('*a*::*euclidean_space*) *set*
shows *convex hull* $p = \{x.\ \exists S.$ *finite* $S \wedge S \subseteq p \wedge$ *card* $S \leq$ *aff_dim* $p + 1 \wedge x \in$ *convex hull* $S\}$
(**is** *?lhs* = *?rhs*)
proof
have $\bigwedge x\ S\ u.$ $[\![$*finite* S; $S \subseteq p$; *int* (*card* S) $\leq$ *aff_dim* $p + 1$; $\forall x{\in}S.\ 0 \leq u\ x$; *sum* $u\ S = 1]\!]$
$\Longrightarrow (\sum v{\in}S.\ u\ v *_R v) \in$ *convex hull* S
by (*metis* (*mono_tags*, *lifting*) *convex_convex_hull convex_explicit hull_subset*)
then show *?lhs* $\subseteq$ *?rhs*
by (*subst convex_hull_caratheodory_aff_dim*, *auto*)
qed (*use hull_mono* **in** *auto*)

lemma *convex_hull_caratheodory*:
fixes p :: ('*a*::*euclidean_space*) *set*
shows *convex hull* $p =$
$\{y.\ \exists S\ u.$ *finite* $S \wedge S \subseteq p \wedge$ *card* $S \leq$ *DIM*('*a*) $+ 1 \wedge$
$(\forall x{\in}S.\ 0 \leq u\ x) \wedge$ *sum* $u\ S = 1 \wedge$ *sum* $(\lambda v.\ u\ v *_R v)\ S = y\}$
(**is** *?lhs* = *?rhs*)
proof (*intro set_eqI iffI*)
fix x
assume $x \in$ *?lhs* **then show** $x \in$ *?rhs*
unfolding *convex_hull_caratheodory_aff_dim*
using *aff_dim_le_DIM* [*of p*] **by** *fastforce*
qed (*auto simp*: *convex_hull_explicit*)

theorem *caratheodory*:
convex hull $p =$
$\{x$::'*a*::*euclidean_space*. $\exists S.$ *finite* $S \wedge S \subseteq p \wedge$ *card* $S \leq$ *DIM*('*a*) $+ 1 \wedge x \in$ *convex hull* $S\}$

proof *safe*
fix *x*
assume *x* ∈ *convex hull p*
then obtain *S u* **where** *finite S S* ⊆ *p card S* ≤ *DIM*('*a*) + *1*
∀ *x*∈*S*. *0* ≤ *u x sum u S* = *1* (∑ *v*∈*S*. *u v* ∗$_R$ *v*) = *x*
unfolding *convex_hull_caratheodory* **by** *auto*
then show ∃ *S*. *finite S* ∧ *S* ⊆ *p* ∧ *card S* ≤ *DIM*('*a*) + *1* ∧ *x* ∈ *convex hull S*
using *convex_hull_finite* **by** *fastforce*
qed (*use hull_mono* **in** *force*)

1.7.14 Some Properties of subset of standard basis

lemma *affine_hull_substd_basis*:
assumes *d* ⊆ *Basis*
shows *affine hull* (*insert 0 d*) = {*x*::'*a*::*euclidean_space*. ∀ *i*∈*Basis*. *i* ∉ *d* ⟶ *x*•*i* = *0*}
(**is** *affine hull* (*insert 0 ?A*) = *?B*)
proof −
have ∗: ⋀*A*. (+) (*0*::'*a*) ' *A* = *A* ⋀*A*. (+) (− (*0*::'*a*)) ' *A* = *A*
by *auto*
show *?thesis*
unfolding *affine_hull_insert_span_gen span_substd_basis*[*OF assms*,*symmetric*] ∗ **..**
qed

lemma *affine_hull_convex_hull* [*simp*]: *affine hull* (*convex hull S*) = *affine hull S*
by (*metis Int_absorb1 Int_absorb2 convex_hull_subset_affine_hull hull_hull hull_mono hull_subset*)

1.7.15 Moving and scaling convex hulls

lemma *convex_hull_set_plus*:
convex hull (*S* + *T*) = *convex hull S* + *convex hull T*
by (*simp add*: *set_plus_image linear_iff scaleR_right_distrib convex_hull_Times*
flip: *convex_hull_linear_image*)

lemma *translation_eq_singleton_plus*: (λ*x*. *a* + *x*) ' *T* = {*a*} + *T*
unfolding *set_plus_def* **by** *auto*

lemma *convex_hull_translation*:
convex hull ((λ*x*. *a* + *x*) ' *S*) = (λ*x*. *a* + *x*) ' (*convex hull S*)
by (*simp add*: *convex_hull_set_plus translation_eq_singleton_plus*)

lemma *convex_hull_scaling*:
convex hull ((λ*x*. *c* ∗$_R$ *x*) ' *S*) = (λ*x*. *c* ∗$_R$ *x*) ' (*convex hull S*)
by (*simp add*: *convex_hull_linear_image*)

lemma *convex_hull_affinity*:

convex hull ((λ*x*. *a* + *c* ∗$_R$ *x*) ‘ *S*) = (λ*x*. *a* + *c* ∗$_R$ *x*) ‘ (*convex hull S*)
by (*metis convex_hull_scaling convex_hull_translation image_image*)

1.7.16 Convexity of cone hulls

lemma *convex_cone_hull*:
assumes *convex S*
shows *convex* (*cone hull S*)
proof (*rule convexI*)
fix *x y*
assume *xy*: *x* ∈ *cone hull S y* ∈ *cone hull S*
then have *S* ≠ {}
using *cone_hull_empty_iff*[*of S*] **by** *auto*
fix *u v* :: *real*
assume *uv*: *u* ≥ *0 v* ≥ *0 u* + *v* = *1*
then have ∗: *u* ∗$_R$ *x* ∈ *cone hull S v* ∗$_R$ *y* ∈ *cone hull S*
by (*simp_all add*: *cone_cone_hull mem_cone uv xy*)
then obtain *cx* :: *real* **and** *xx*
and *cy* :: *real* **and** *yy* **where** *x*: *u* ∗$_R$ *x* = *cx* ∗$_R$ *xx cx* ≥ *0 xx* ∈ *S*
and *y*: *v* ∗$_R$ *y* = *cy* ∗$_R$ *yy cy* ≥ *0 yy* ∈ *S*
using *cone_hull_expl*[*of S*] **by** *auto*

have *u* ∗$_R$ *x* + *v* ∗$_R$ *y* ∈ *cone hull S* **if** *cx* + *cy* ≤ *0*
using ∗(*1*) *nless_le that x*(*2*) *y* **by** *fastforce*
moreover
have *u* ∗$_R$ *x* + *v* ∗$_R$ *y* ∈ *cone hull S* **if** *cx* + *cy* > *0*
proof −
have (*cx* / (*cx* + *cy*)) ∗$_R$ *xx* + (*cy* / (*cx* + *cy*)) ∗$_R$ *yy* ∈ *S*
using *assms mem_convex_alt*[*of S xx yy cx cy*] *x y that* **by** *auto*
then have *cx* ∗$_R$ *xx* + *cy* ∗$_R$ *yy* ∈ *cone hull S*
using *mem_cone_hull*[*of* (*cx*/(*cx*+*cy*)) ∗$_R$ *xx* + (*cy*/(*cx*+*cy*)) ∗$_R$ *yy S cx*+*cy*]
‹*cx*+*cy*>*0*›
by (*auto simp*: *scaleR_right_distrib*)
then show *?thesis*
using *x y* **by** *auto*
qed
moreover have *cx* + *cy* ≤ *0* ∨ *cx* + *cy* > *0* **by** *auto*
ultimately show *u* ∗$_R$ *x* + *v* ∗$_R$ *y* ∈ *cone hull S* **by** *blast*
qed

lemma *cone_convex_hull*:
assumes *cone S*
shows *cone* (*convex hull S*)
by (*metis* (*no_types*, *lifting*) *affine_hull_convex_hull affine_hull_eq_empty assms cone_iff convex_hull_scaling hull_inc*)

1.8 Conic sets and conic hull

definition *conic* :: ′*a*::*real_vector set* ⇒ *bool*

where *conic S* ≡ ∀ *x c*. *x* ∈ *S* ⟶ *0* ≤ *c* ⟶ (*c* *$_R$ *x*) ∈ *S*

lemma *conicD*: ⟦*conic S*; *x* ∈ *S*; *0* ≤ *c*⟧ ⟹ (*c* *$_R$ *x*) ∈ *S*
by (*meson conic_def*)

lemma *subspace_imp_conic*: *subspace S* ⟹ *conic S*
by (*simp add*: *conic_def subspace_def*)

lemma *conic_empty* [*simp*]: *conic* {}
using *conic_def* **by** *blast*

lemma *conic_UNIV*: *conic UNIV*
by (*simp add*: *conic_def*)

lemma *conic_Inter*: (⋀*S*. *S* ∈ ℱ ⟹ *conic S*) ⟹ *conic*(⋂ℱ)
by (*simp add*: *conic_def*)

lemma *conic_linear_image*:
⟦*conic S*; *linear f*⟧ ⟹ *conic*(*f* ‘ *S*)
by (*smt* (*verit*) *conic_def image_iff linear.scaleR*)

lemma *conic_linear_image_eq*:
⟦*linear f*; *inj f*⟧ ⟹ *conic* (*f* ‘ *S*) ⟷ *conic S*
by (*smt* (*verit*) *conic_def conic_linear_image inj_image_mem_iff linear_cmul*)

lemma *conic_mul*: ⟦*conic S*; *x* ∈ *S*; *0* ≤ *c*⟧ ⟹ (*c* *$_R$ *x*) ∈ *S*
using *conic_def* **by** *blast*

lemma *conic_conic_hull*: *conic*(*conic hull S*)
by (*metis* (*no_types*, *lifting*) *conic_Inter hull_def mem_Collect_eq*)

lemma *conic_hull_eq*: (*conic hull S* = *S*) ⟷ *conic S*
by (*metis conic_conic_hull hull_same*)

lemma *conic_hull_UNIV* [*simp*]: *conic hull UNIV* = *UNIV*
by *simp*

lemma *conic_negations*: *conic S* ⟹ *conic* (*image uminus S*)
by (*auto simp*: *conic_def image_iff*)

lemma *conic_span* [*iff*]: *conic*(*span S*)
by (*simp add*: *subspace_imp_conic*)

lemma *conic_hull_explicit*:
conic hull S = {*c* *$_R$ *x*| *c x*. *0* ≤ *c* ∧ *x* ∈ *S*}
proof (*rule hull_unique*)
show *S* ⊆ {*c* *$_R$ *x* |*c x*. *0* ≤ *c* ∧ *x* ∈ *S*}
by (*metis* (*no_types*) *cone_hull_expl hull_subset*)
show *conic* {*c* *$_R$ *x* |*c x*. *0* ≤ *c* ∧ *x* ∈ *S*}

```
    using mult_nonneg_nonneg by (force simp: conic_def)
qed (auto simp: conic_def)

lemma conic_hull_as_image:
   conic hull S = (λz. fst z *R snd z) ‘ ({0..} × S)
  by (force simp: conic_hull_explicit)

lemma conic_hull_linear_image:
   linear f ⟹ conic hull f ‘ S = f ‘ (conic hull S)
  by (force simp: conic_hull_explicit image_iff set_eq_iff linear_scale)

lemma conic_hull_image_scale:
  assumes ⋀x. x ∈ S ⟹ 0 < c x
  shows   conic hull (λx. c x *R x) ‘ S = conic hull S
proof
  show conic hull (λx. c x *R x) ‘ S ⊆ conic hull S
  proof (rule hull_minimal)
    show (λx. c x *R x) ‘ S ⊆ conic hull S
      using assms conic_hull_explicit by fastforce
  qed (simp add: conic_conic_hull)
  show conic hull S ⊆ conic hull (λx. c x *R x) ‘ S
  proof (rule hull_minimal)
    show S ⊆ conic hull (λx. c x *R x) ‘ S
    proof clarsimp
      fix x
      assume x ∈ S
      then have x = inverse(c x) *R c x *R x
        using assms by fastforce
      then show x ∈ conic hull (λx. c x *R x) ‘ S
        by (smt (verit, best) ‹x ∈ S› assms conic_conic_hull conic_mul hull_inc
image_eqI inverse_nonpositive_iff_nonpositive)
    qed
  qed (simp add: conic_conic_hull)
qed

lemma convex_conic_hull:
  assumes convex S
  shows convex (conic hull S)
proof (clarsimp simp add: conic_hull_explicit convex_alt)
  fix c x d y and u :: real
  assume §: (0::real) ≤ c x ∈ S (0::real) ≤ d y ∈ S 0 ≤ u u ≤ 1
  show ∃ c'' x''. ((1 − u) * c) *R x + (u * d) *R y = c'' *R x'' ∧ 0 ≤ c'' ∧ x''
∈ S
  proof (cases (1 − u) * c = 0)
    case True
    with ‹0 ≤ d› ‹y ∈ S›‹0 ≤ u›
    show ?thesis by force
  next
    case False
```

```
    define ξ where ξ ≡ (1 − u) ∗ c + u ∗ d
    have ∗: c ∗ u ≤ c
      by (simp add: § mult_left_le)
    have ξ > 0
      using False § by (smt (verit, best) ξ_def split_mult_pos_le)
    then have ∗∗: c + d ∗ u = ξ + c ∗ u
      by (simp add: ξ_def mult.commute right_diff_distrib′)
    show ?thesis
    proof (intro exI conjI)
      show 0 ≤ ξ
        using ‹0 < ξ› by auto
      show ((1 − u) ∗ c) ∗R x + (u ∗ d) ∗R y = ξ ∗R (((1 − u) ∗ c / ξ) ∗R x +
(u ∗ d / ξ) ∗R y)
        using ‹ξ > 0› by (simp add: algebra_simps diff_divide_distrib)
      show ((1 − u) ∗ c / ξ) ∗R x + (u ∗ d / ξ) ∗R y ∈ S
        using ‹0 < ξ›
        by (intro convexD [OF assms]) (auto simp: § field_split_simps ∗ ∗∗)
    qed
  qed
qed

lemma conic_halfspace_le: conic {x. a • x ≤ 0}
  by (auto simp: conic_def mult_le_0_iff)

lemma conic_halfspace_ge: conic {x. a • x ≥ 0}
  by (auto simp: conic_def mult_le_0_iff)

lemma conic_hull_empty [simp]: conic hull {} = {}
  by (simp add: conic_hull_eq)

lemma conic_contains_0: conic S ⟹ (0 ∈ S ⟷ S ≠ {})
  by (simp add: Convex.cone_def cone_contains_0 conic_def)

lemma conic_hull_eq_empty: conic hull S = {} ⟷ (S = {})
  using conic_hull_explicit by fastforce

lemma conic_sums: ⟦conic S; conic T⟧ ⟹ conic (⋃x∈ S. ⋃y ∈ T. {x + y})
  by (simp add: conic_def) (metis scaleR_right_distrib)

lemma conic_Times: ⟦conic S; conic T⟧ ⟹ conic(S × T)
  by (auto simp: conic_def)

lemma conic_Times_eq:
   conic(S × T) ⟷ S = {} ∨ T = {} ∨ conic S ∧ conic T (is ?lhs = ?rhs)
proof
  show ?lhs ⟹ ?rhs
    by (force simp: conic_def)
  show ?rhs ⟹ ?lhs
    by (force simp: conic_Times)
```

qed

lemma *conic_hull_0* [*simp*]: *conic hull* $\{0\} = \{0\}$
by (*simp add*: *conic_hull_eq subspace_imp_conic*)

lemma *conic_hull_contains_0* [*simp*]: $0 \in$ *conic hull* $S \longleftrightarrow (S \neq \{\})$
by (*simp add*: *conic_conic_hull conic_contains_0 conic_hull_eq_empty*)

lemma *conic_hull_eq_sing*:
conic hull $S = \{x\} \longleftrightarrow S = \{0\} \wedge x = 0$
proof
show *conic hull* $S = \{x\} \Longrightarrow S = \{0\} \wedge x = 0$
by (*metis conic_conic_hull conic_contains_0 conic_def conic_hull_eq hull_inc insert_not_empty singleton_iff*)
qed *simp*

lemma *conic_hull_Int_affine_hull*:
assumes $T \subseteq S$ $0 \notin$ *affine hull* S
shows (*conic hull* T) $\cap$ (*affine hull* S) $= T$
proof −
have *TaffS*: $T \subseteq$ *affine hull* S
using ‹$T \subseteq S$› *hull_subset* **by** *fastforce*
moreover
have *conic hull* $T \cap$ *affine hull* $S \subseteq T$
proof (*clarsimp simp*: *conic_hull_explicit*)
fix c x
assume $c *_R x \in$ *affine hull* S
and $0 \leq c$
and $x \in T$
show $c *_R x \in T$
proof (*cases c=1*)
case *True*
then show *?thesis*
by (*simp add*: ‹$x \in T$›)
next
case *False*
then have $x /_R (1 - c) = x + (c * inverse\ (1 - c)) *_R x$
by (*smt* (*verit*, *ccfv_SIG*) *diff_add_cancel mult.commute real_vector_affinity_eq scaleR_collapse scaleR_scaleR*)
then have $0 = inverse(1 - c) *_R c *_R x + (1 - inverse(1 - c)) *_R x$
by (*simp add*: *algebra_simps*)
then have $0 \in$ *affine hull* S
by (*smt* (*verit*) ‹$c *_R x \in$ *affine hull* S› ‹$x \in T$› *affine_affine_hull TaffS in_mono mem_affine*)
then show *?thesis*
using *assms* **by** *auto*
qed
qed
ultimately show *?thesis*

by (*auto simp*: *hull_inc*)
qed

1.9 Convex cones and corresponding hulls

definition *convex_cone* :: *'a::real_vector set* ⇒ *bool*
where *convex_cone* ≡ λ*S*. *S* ≠ {} ∧ *convex S* ∧ *conic S*

lemma *convex_cone_iff*:
convex_cone S ⟷
0 ∈ *S* ∧ (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *x* + *y* ∈ *S*) ∧ (∀ *x* ∈ *S*. ∀ *c*≥*0*. *c* ∗*R* *x* ∈ *S*)
by (*metis cone_def conic_contains_0 conic_def convex_cone convex_cone_def*)

lemma *convex_cone_add*: ⟦*convex_cone S*; *x* ∈ *S*; *y* ∈ *S*⟧ ⟹ *x*+*y* ∈ *S*
by (*simp add*: *convex_cone_iff*)

lemma *convex_cone_scaleR*: ⟦*convex_cone S*; *0* ≤ *c*; *x* ∈ *S*⟧ ⟹ *c* ∗*R* *x* ∈ *S*
by (*simp add*: *convex_cone_iff*)

lemma *convex_cone_nonempty*: *convex_cone S* ⟹ *S* ≠ {}
by (*simp add*: *convex_cone_def*)

lemma *convex_cone_linear_image*:
convex_cone S ∧ *linear f* ⟹ *convex_cone*(*f* ‘ *S*)
by (*simp add*: *conic_linear_image convex_cone_def convex_linear_image*)

lemma *convex_cone_linear_image_eq*:
⟦*linear f*; *inj f*⟧ ⟹ (*convex_cone*(*f* ‘ *S*) ⟷ *convex_cone S*)
by (*simp add*: *conic_linear_image_eq convex_cone_def*)

lemma *convex_cone_halfspace_ge*: *convex_cone* {*x*. *a* ∙ *x* ≥ *0*}
by (*simp add*: *convex_cone_iff inner_simps*(*2*))

lemma *convex_cone_halfspace_le*: *convex_cone* {*x*. *a* ∙ *x* ≤ *0*}
by (*simp add*: *convex_cone_iff inner_right_distrib mult_nonneg_nonpos*)

lemma *convex_cone_contains_0*: *convex_cone S* ⟹ *0* ∈ *S*
using *convex_cone_iff* **by** *blast*

lemma *convex_cone_Inter*:
(⋀*S*. *S* ∈ *f* ⟹ *convex_cone S*) ⟹ *convex_cone*(⋂ *f*)
by (*simp add*: *convex_cone_iff*)

lemma *convex_cone_convex_cone_hull*: *convex_cone*(*convex_cone hull S*)
by (*metis* (*no_types*, *lifting*) *convex_cone_Inter hull_def mem_Collect_eq*)

lemma *convex_convex_cone_hull*: *convex*(*convex_cone hull S*)
by (*meson convex_cone_convex_cone_hull convex_cone_def*)

lemma *conic_convex_cone_hull*: *conic*(*convex_cone hull S*)
by (*metis convex_cone_convex_cone_hull convex_cone_def*)

lemma *convex_cone_hull_nonempty*: *convex_cone hull S* ≠ {}
by (*simp add*: *convex_cone_convex_cone_hull convex_cone_nonempty*)

lemma *convex_cone_hull_contains_0*: *0* ∈ *convex_cone hull S*
by (*simp add*: *convex_cone_contains_0 convex_cone_convex_cone_hull*)

lemma *convex_cone_hull_add*:
⟦*x* ∈ *convex_cone hull S*; *y* ∈ *convex_cone hull S*⟧ ⟹ *x* + *y* ∈ *convex_cone hull S*
by (*simp add*: *convex_cone_add convex_cone_convex_cone_hull*)

lemma *convex_cone_hull_mul*:
⟦*x* ∈ *convex_cone hull S*; *0* ≤ *c*⟧ ⟹ (*c* *$_R$ *x*) ∈ *convex_cone hull S*
by (*simp add*: *conic_convex_cone_hull conic_mul*)

thm *convex_sums*
lemma *convex_cone_sums*:
⟦*convex_cone S*; *convex_cone T*⟧ ⟹ *convex_cone* (⋃*x*∈ *S*. ⋃*y* ∈ *T*. {*x* + *y*})
by (*simp add*: *convex_cone_def conic_sums convex_sums*)

lemma *convex_cone_Times*:
⟦*convex_cone S*; *convex_cone T*⟧ ⟹ *convex_cone*(*S* × *T*)
by (*simp add*: *conic_Times convex_Times convex_cone_def*)

lemma *convex_cone_Times_D1*: *convex_cone* (*S* × *T*) ⟹ *convex_cone S*
by (*metis Times_empty conic_Times_eq convex_cone_def convex_convex_hull convex_hull_Times hull_same times_eq_iff*)

lemma *convex_cone_Times_eq*:
convex_cone(*S* × *T*) ⟷ *convex_cone S* ∧ *convex_cone T*
proof (*cases S*={} ∨ *T*={})
case *True*
then show *?thesis*
by (*auto dest*: *convex_cone_nonempty*)
next
case *False*
then have *convex_cone* (*S* × *T*) ⟹ *convex_cone T*
by (*metis conic_Times_eq convex_cone_def convex_convex_hull convex_hull_Times hull_same times_eq_iff*)
then show *?thesis*
using *convex_cone_Times convex_cone_Times_D1* **by** *blast*
qed

lemma *convex_cone_hull_Un*:
convex_cone hull(*S* ∪ *T*) = (⋃*x* ∈ *convex_cone hull S*. ⋃*y* ∈ *convex_cone hull*

$T.\ \{x + y\})$
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
proof (*rule hull_minimal*)
show $S \cup T \subseteq (\bigcup x{\in}$*convex_cone hull S*$.\ \bigcup y{\in}$*convex_cone hull T*$.\ \{x + y\})$
apply (*clarsimp simp*: *subset_iff*)
by (*metis add_0 convex_cone_hull_contains_0 group_cancel.rule0 hull_inc*)
show *convex_cone* $(\bigcup x{\in}$*convex_cone hull S*$.\ \bigcup y{\in}$*convex_cone hull T*$.\ \{x + y\})$
by (*simp add*: *convex_cone_convex_cone_hull convex_cone_sums*)
qed
next
show *?rhs* ⊆ *?lhs*
by *clarify* (*metis convex_cone_hull_add hull_mono le_sup_iff subsetD subsetI*)
qed

lemma *convex_cone_singleton* [*iff*]: *convex_cone* $\{0\}$
by (*simp add*: *convex_cone_iff*)

lemma *convex_hull_subset_convex_cone_hull*:
convex hull S ⊆ *convex_cone hull S*
by (*simp add*: *convex_convex_cone_hull hull_minimal hull_subset*)

lemma *conic_hull_subset_convex_cone_hull*:
conic hull S ⊆ *convex_cone hull S*
by (*simp add*: *conic_convex_cone_hull hull_minimal hull_subset*)

lemma *subspace_imp_convex_cone*: *subspace S* ⟹ *convex_cone S*
by (*simp add*: *convex_cone_iff subspace_def*)

lemma *convex_cone_span*: *convex_cone*(*span S*)
by (*simp add*: *subspace_imp_convex_cone*)

lemma *convex_cone_negations*:
convex_cone S ⟹ *convex_cone* (*image uminus S*)
by (*simp add*: *convex_cone_linear_image module_hom_uminus*)

lemma *subspace_convex_cone_symmetric*:
subspace S ⟷ *convex_cone S* ∧ $(\forall x \in S.\ -x \in S)$
by (*smt* (*verit*) *convex_cone_iff scaleR_left.minus subspace_def subspace_neg*)

lemma *convex_cone_hull_separate_nonempty*:
assumes $S \neq \{\}$
shows *convex_cone hull S* = *conic hull* (*convex hull S*) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*metis assms conic_conic_hull convex_cone_def convex_conic_hull convex_convex_hull hull_subset subset_empty subset_hull*)

show *?rhs* ⊆ *?lhs*
by (*simp add*: *conic_convex_cone_hull convex_hull_subset_convex_cone_hull subset_hull*)
qed

lemma *convex_cone_hull_empty* [*simp*]: *convex_cone hull* {} = {*0*}
by (*metis convex_cone_hull_contains_0 convex_cone_singleton hull_redundant hull_same*)

lemma *convex_cone_hull_separate*:
convex_cone hull S = *insert 0* (*conic hull* (*convex hull S*))
proof(*cases S*={})
case *False*
then show *?thesis*
using *convex_cone_hull_contains_0 convex_cone_hull_separate_nonempty*
by *blast*
qed *auto*

lemma *convex_cone_hull_convex_hull_nonempty*:
S ≠ {} ⟹ *convex_cone hull S* = (⋃ *x* ∈ *convex hull S*. ⋃ *c*∈{*0*..}. {*c* ∗*R* *x*})
by (*force simp*: *convex_cone_hull_separate_nonempty conic_hull_as_image*)

lemma *convex_cone_hull_convex_hull*:
convex_cone hull S = *insert 0* (⋃ *x* ∈ *convex hull S*. ⋃ *c*∈{*0*..}. {*c* ∗*R* *x*})
by (*force simp*: *convex_cone_hull_separate conic_hull_as_image*)

lemma *convex_cone_hull_linear_image*:
linear f ⟹ *convex_cone hull* (*f* ‘ *S*) = *image f* (*convex_cone hull S*)
by (*metis* (*no_types*, *lifting*) *conic_hull_linear_image convex_cone_hull_separate convex_hull_linear_image image_insert linear_0*)

1.9.1 Radon’s theorem

Formalized by Lars Schewe.

lemma *Radon_ex_lemma*:
assumes *finite c affine_dependent c*
shows ∃ *u*. *sum u c* = *0* ∧ (∃ *v*∈*c*. *u v* ≠ *0*) ∧ *sum* (λ*v*. *u v* ∗*R* *v*) *c* = *0*
using *affine_dependent_explicit_finite assms* **by** *blast*

lemma *Radon_s_lemma*:
assumes *finite S*
and *sum f S* = (*0*::*real*)
shows *sum f* {*x*∈*S*. *0* < *f x*} = − *sum f* {*x*∈*S*. *f x* < *0*}
proof −
have ⋀*x*. (*if f x* < *0 then f x else 0*) + (*if 0* < *f x then f x else 0*) = *f x*
by *auto*
then show *?thesis*
using *assms* **by** (*simp add*: *sum.inter_filter flip*: *sum.distrib add_eq_0_iff*)
qed

lemma *Radon_v_lemma*:
assumes *finite S*
and *sum f S = 0*
and $\forall x.\ g\ x = (0::real) \longrightarrow f\ x = (0::'a::euclidean\_space)$
shows $(sum\ f\ \{x{\in}S.\ 0 < g\ x\}) = -\ sum\ f\ \{x{\in}S.\ g\ x < 0\}$
proof −
have $\bigwedge x.\ (if\ 0 < g\ x\ then\ f\ x\ else\ 0) + (if\ g\ x < 0\ then\ f\ x\ else\ 0) = f\ x$
using *assms* **by** *auto*
then show *?thesis*
using *assms* **by** (*simp add*: *sum.inter_filter eq_neg_iff_add_eq_0 flip*: *sum.distrib add_eq_0_iff*)
qed

lemma *Radon_partition*:
assumes *finite C affine_dependent C*
shows $\exists M\ P.\ M \cap P = \{\} \wedge M \cup P = C \wedge (convex\ hull\ M) \cap (convex\ hull\ P) \neq \{\}$
proof −
obtain *u v* **where** *uv*: *sum u C = 0* $v{\in}C\ u\ v \neq 0$ $(\sum v{\in}C.\ u\ v *_R v) = 0$
using *Radon_ex_lemma*[*OF assms*] **by** *auto*
have *fin*: *finite* $\{x \in C.\ 0 < u\ x\}$ *finite* $\{x \in C.\ 0 > u\ x\}$
using *assms*(*1*) **by** *auto*
define *z* **where** $z = inverse\ (sum\ u\ \{x{\in}C.\ u\ x > 0\}) *_R sum\ (\lambda x.\ u\ x *_R x)\ \{x{\in}C.\ u\ x > 0\}$
have $sum\ u\ \{x \in C.\ 0 < u\ x\} \neq 0$
proof (*cases u v* $\geq$ *0*)
case *False*
then have *u v < 0* **by** *auto*
then show *?thesis*
by (*smt* (*verit*) *assms*(*1*) *fin*(*1*) *mem_Collect_eq sum.neutral_const sum_mono_inv uv*)
next
case *True*
with *fin uv* **show** $sum\ u\ \{x \in C.\ 0 < u\ x\} \neq 0$
by (*smt* (*verit*) *fin*(*1*) *mem_Collect_eq sum_nonneg_eq_0_iff uv*)
qed
then have $*$: $sum\ u\ \{x{\in}C.\ u\ x > 0\} > 0$
unfolding *less_le* **by** (*metis* (*no_types*, *lifting*) *mem_Collect_eq sum_nonneg*)
moreover have $sum\ u\ (\{x \in C.\ 0 < u\ x\} \cup \{x \in C.\ u\ x < 0\}) = sum\ u\ C$
$(\sum x{\in}\{x \in C.\ 0 < u\ x\} \cup \{x \in C.\ u\ x < 0\}.\ u\ x *_R x) = (\sum x{\in}C.\ u\ x *_R x)$
using *assms*(*1*)
by (*rule_tac*[!] *sum.mono_neutral_left*, *auto*)
then have $sum\ u\ \{x \in C.\ 0 < u\ x\} = -\ sum\ u\ \{x \in C.\ 0 > u\ x\}$
$(\sum x{\in}\{x \in C.\ 0 < u\ x\}.\ u\ x *_R x) = -\ (\sum x{\in}\{x \in C.\ 0 > u\ x\}.\ u\ x *_R x)$
unfolding *eq_neg_iff_add_eq_0*
using *uv*(*1*,*4*)
by (*auto simp*: *sum.union_inter_neutral*[*OF fin*, *symmetric*])
moreover have $\forall x{\in}\{v \in C.\ u\ v < 0\}.\ 0 \leq inverse\ (sum\ u\ \{x \in C.\ 0 < u\ x\})$

```
* - u x
    using * by (fastforce intro: mult_nonneg_nonneg)
  ultimately have z ∈ convex hull {v ∈ C. u v ≤ 0}
    unfolding convex_hull_explicit mem_Collect_eq
    apply (rule_tac x={v ∈ C. u v < 0} in exI)
    apply (rule_tac x=λy. inverse (sum u {x∈C. u x > 0}) * - u y in exI)
   using assms(1) unfolding scaleR_scaleR[symmetric] scaleR_right.sum [symmetric]

    by (auto simp: z_def sum_negf sum_distrib_left[symmetric])
  moreover have ∀x∈{v ∈ C. 0 < u v}. 0 ≤ inverse (sum u {x ∈ C. 0 < u x})
* u x
    using * by (fastforce intro: mult_nonneg_nonneg)
  then have z ∈ convex hull {v ∈ C. u v > 0}
    unfolding convex_hull_explicit mem_Collect_eq
    apply (rule_tac x={v ∈ C. 0 < u v} in exI)
    apply (rule_tac x=λy. inverse (sum u {x∈C. u x > 0}) * u y in exI)
    using assms(1)
    unfolding scaleR_scaleR[symmetric] scaleR_right.sum [symmetric]
    using * by (auto simp: z_def sum_negf sum_distrib_left[symmetric])
  ultimately show ?thesis
    apply (rule_tac x={v∈C. u v ≤ 0} in exI)
    apply (rule_tac x={v∈C. u v > 0} in exI, auto)
    done
qed

theorem Radon:
  assumes affine_dependent c
  obtains M P where M ⊆ c P ⊆ c M ∩ P = {} (convex hull M) ∩ (convex hull
P) ≠ {}
  by (smt (verit) Radon_partition affine_dependent_explicit affine_dependent_explicit_finite
assms le_sup_iff)
```

1.9.2 Helly's theorem

```
lemma Helly_induct:
  fixes F :: 'a::euclidean_space set set
  assumes card F = n
    and n ≥ DIM('a) + 1
    and ∀S∈F. convex S ∀T⊆F. card T = DIM('a) + 1 ⟶ ⋂ T ≠ {}
  shows ⋂ F ≠ {}
  using assms
proof (induction n arbitrary: F)
  case 0
  then show ?case by auto
next
  case (Suc n)
  have finite F
    using ‹card F = Suc n› by (auto intro: card_ge_0_finite)
  show ⋂ F ≠ {}
```

proof (*cases* $n = DIM('a)$)
case *True*
then show *?thesis*
by (*simp add*: *Suc.prems*)
next
case *False*
have $\bigcap(\mathcal{F} - \{S\}) \neq \{\}$ **if** $S \in \mathcal{F}$ **for** S
proof (*rule Suc.IH*[*rule_format*])
show *card* $(\mathcal{F} - \{S\}) = n$
by (*simp add*: *Suc.prems*(*1*) ‹*finite* $\mathcal{F}$› *that*)
show $DIM('a) + 1 \leq n$
using *False Suc.prems*(*2*) **by** *linarith*
show $\bigwedge t.\ [\![t \subseteq \mathcal{F} - \{S\};\ card\ t = DIM('a) + 1]\!] \Longrightarrow \bigcap t \neq \{\}$
by (*simp add*: *Suc.prems*(*4*) *subset_Diff_insert*)
qed (*use Suc* **in** *auto*)
then have $\forall S{\in}\mathcal{F}.\ \exists x.\ x \in \bigcap(\mathcal{F} - \{S\})$
by *blast*
then obtain X **where** X: $\bigwedge S.\ S{\in}\mathcal{F} \Longrightarrow X\ S \in \bigcap(\mathcal{F} - \{S\})$
by *metis*
show *?thesis*
proof (*cases inj_on X* $\mathcal{F}$)
case *False*
then obtain $S\ T$ **where** $S{\neq}T$ **and** *st*: $S{\in}\mathcal{F}\ T{\in}\mathcal{F}\ X\ S = X\ T$
unfolding *inj_on_def* **by** *auto*
then have $*$: $\bigcap\mathcal{F} = \bigcap(\mathcal{F} - \{S\}) \cap \bigcap(\mathcal{F} - \{T\})$ **by** *auto*
show *?thesis*
by (*metis* $*$ *X disjoint_iff_not_equal st*)
next
case *True*
then obtain $M\ P$ **where** *mp*: $M \cap P = \{\}\ M \cup P = X\ `\ \mathcal{F}$ *convex hull* $M \cap$ *convex hull* $P \neq \{\}$
using *Radon_partition*[*of* $X\ `\ \mathcal{F}$] **and** *affine_dependent_biggerset*[*of* $X\ `\ \mathcal{F}$]
unfolding *card_image*[*OF True*] **and** ‹*card* $\mathcal{F}$ = *Suc n*›
using *Suc*(*3*) ‹*finite* $\mathcal{F}$› **and** *False*
by *auto*
have $M \subseteq X\ `\ \mathcal{F}\ P \subseteq X\ `\ \mathcal{F}$
using *mp*(*2*) **by** *auto*
then obtain $\mathcal{G}\ \mathcal{H}$ **where** *gh*:$M = X\ `\ \mathcal{G}\ P = X\ `\ \mathcal{H}\ \mathcal{G} \subseteq \mathcal{F}\ \mathcal{H} \subseteq \mathcal{F}$
unfolding *subset_image_iff* **by** *auto*
then have $\mathcal{F} \cup (\mathcal{G} \cup \mathcal{H}) = \mathcal{F}$ **by** *auto*
then have $\mathcal{F}$: $\mathcal{F} = \mathcal{G} \cup \mathcal{H}$
using *inj_on_Un_image_eq_iff*[*of X* $\mathcal{F}$ $\mathcal{G} \cup \mathcal{H}$] **and** *True*
unfolding *mp*(*2*)[*unfolded image_Un*[*symmetric*] *gh*]
by *auto*
have $*$: $\mathcal{G} \cap \mathcal{H} = \{\}$
using *gh local.mp*(*1*) **by** *blast*
have *convex hull* $(X\ `\ \mathcal{H}) \subseteq \bigcap\mathcal{G}$ *convex hull* $(X\ `\ \mathcal{G}) \subseteq \bigcap\mathcal{H}$
by (*rule hull_minimal*; *use* X $*$ $\mathcal{F}$ **in** ‹*auto simp*: *Suc.prems*(*3*) *convex_Inter*›)+

then show *?thesis*
unfolding $\mathcal{F}$ **using** *mp(3)[unfolded gh]* **by** *blast*
qed
qed
qed

theorem *Helly*:
fixes $\mathcal{F}$ :: *'a::euclidean_space set set*
assumes *card* $\mathcal{F} \geq$ *DIM('a) + 1* $\forall s{\in}\mathcal{F}$. *convex s*
and $\bigwedge t$. $[\![t{\subseteq}\mathcal{F}$; *card t = DIM('a) + 1*$]\!] \Longrightarrow \bigcap t \neq \{\}$
shows $\bigcap \mathcal{F} \neq \{\}$
using *Helly_induct assms* **by** *blast*

1.9.3 Epigraphs of convex functions

definition *epigraph S* (*f* :: _ ⇒ *real*) = {*xy*. *fst xy* ∈ *S* ∧ *f* (*fst xy*) ≤ *snd xy*}

lemma *mem_epigraph*: (*x*, *y*) ∈ *epigraph S f* ⟷ *x* ∈ *S* ∧ *f x* ≤ *y*
unfolding *epigraph_def* **by** *auto*

lemma *convex_epigraph*: *convex* (*epigraph S f*) ⟷ *convex_on S f*
proof *safe*
assume *L*: *convex* (*epigraph S f*)
then show *convex_on S f*
by (*fastforce simp*: *convex_def convex_on_def epigraph_def*)
next
assume *convex_on S f*
then show *convex* (*epigraph S f*)
unfolding *convex_def convex_on_def epigraph_def*
apply *safe*
apply (*rule_tac* [*2*] *y*=*u* ∗ *f a* + *v* ∗ *f aa* **in** *order_trans*)
apply (*auto intro*!:*mult_left_mono add_mono*)
done
qed

lemma *convex_epigraphI*: *convex_on S f* ⟹ *convex* (*epigraph S f*)
unfolding *convex_epigraph* **by** *auto*

lemma *convex_epigraph_convex*: *convex_on S f* ⟷ *convex*(*epigraph S f*)
by (*simp add*: *convex_epigraph*)

Use this to derive general bound property of convex function

lemma *convex_on*:
assumes *convex S*
shows *convex_on S f* ⟷
(∀ *k u x*. (∀ *i*∈{*1..k*::*nat*}. *0* ≤ *u i* ∧ *x i* ∈ *S*) ∧ *sum u* {*1..k*} = *1* ⟶
f (*sum* (λ*i*. *u i* ∗$_R$ *x i*) {*1..k*}) ≤ *sum* (λ*i*. *u i* ∗ *f*(*x i*)) {*1..k*})
(**is** *?lhs* = (∀ *k u x*. *?rhs k u x*))
proof

assume *?lhs*
then have §: *convex* {*xy*. *fst xy* ∈ *S* ∧ *f* (*fst xy*) ≤ *snd xy*}
by (*metis assms convex_epigraph epigraph_def*)
show ∀ *k u x*. *?rhs k u x*
proof (*intro allI*)
fix *k u x*
show *?rhs k u x*
using §
unfolding *convex mem_Collect_eq fst_sum snd_sum*
apply *safe*
apply (*drule_tac x*=*k* **in** *spec*)
apply (*drule_tac x*=*u* **in** *spec*)
apply (*drule_tac x*=λ*i*. (*x i*, *f* (*x i*)) **in** *spec*)
apply *simp*
done
qed
next
assume ∀ *k u x*. *?rhs k u x*
then show *?lhs*
unfolding *convex_epigraph_convex convex epigraph_def Ball_def mem_Collect_eq fst_sum snd_sum*
using *assms*[*unfolded convex*] **apply** *clarsimp*
apply (*rule_tac y*=∑ *i* = *1*..*k*. *u i* ∗ *f* (*fst* (*x i*)) **in** *order_trans*)
by (*auto simp add*: *mult_left_mono intro*: *sum_mono*)
qed

1.9.4 A bound within a convex hull

lemma *convex_on_convex_hull_bound*:
assumes *convex_on* (*convex hull S*) *f*
and ∀ *x*∈*S*. *f x* ≤ *b*
shows ∀ *x*∈ *convex hull S*. *f x* ≤ *b*
proof
fix *x*
assume *x* ∈ *convex hull S*
then obtain *k u v* **where**
u: ∀ *i*∈{*1*..*k*::*nat*}. *0* ≤ *u i* ∧ *v i* ∈ *S sum u* {*1*..*k*} = *1* (∑ *i* = *1*..*k*. *u i* ∗$_R$ *v i*) = *x*
unfolding *convex_hull_indexed mem_Collect_eq* **by** *auto*
have (∑ *i* = *1*..*k*. *u i* ∗ *f* (*v i*)) ≤ *b*
using *sum_mono*[*of* {*1*..*k*} λ*i*. *u i* ∗ *f* (*v i*) λ*i*. *u i* ∗ *b*]
unfolding *sum_distrib_right*[*symmetric*] *u*(*2*) *mult_1*
using *assms*(*2*) *mult_left_mono u*(*1*) **by** *blast*
then show *f x* ≤ *b*
using *assms*(*1*)[*unfolded convex_on*[*OF convex_convex_hull*], *rule_format*, *of k u v*]
using *hull_inc u* **by** *fastforce*
qed

lemma *convex_set_plus*:
assumes *convex S* **and** *convex T* **shows** *convex* (*S* + *T*)
by (*metis assms convex_hull_eq convex_hull_set_plus*)

lemma *convex_set_sum*:
assumes $\bigwedge i.\ i \in A \Longrightarrow$ *convex* (*B i*)
shows *convex* ($\sum i{\in}A.\ B\ i$)
using *assms*
by (*induction A rule*: *infinite_finite_induct*) (*auto simp*: *convex_set_plus*)

lemma *finite_set_sum*:
assumes $\forall\, i{\in}A.$ *finite* (*B i*) **shows** *finite* ($\sum i{\in}A.\ B\ i$)
using *assms*
by (*induction A rule*: *infinite_finite_induct*) (*auto simp*: *finite_set_plus*)

lemma *box_eq_set_sum_Basis*:
$\{x.\ \forall\, i{\in}Basis.\ x{\cdot}i \in B\ i\} = (\sum i{\in}Basis.\ (\lambda x.\ x *_R i)\ `\ (B\ i))$ (**is** *?lhs* = *?rhs*)
proof −
have $\bigwedge x.\ \forall\, i{\in}Basis.\ x \cdot i \in B\ i \Longrightarrow$
$\exists\, s.\ x = sum\ s\ Basis \wedge (\forall\, i{\in}Basis.\ s\ i \in (\lambda x.\ x *_R i)\ `\ B\ i)$
by (*metis* (*mono_tags*, *lifting*) *euclidean_representation image_iff*)
moreover
have *sum f Basis* · *i* ∈ *B i* **if** *i* ∈ *Basis* **and** *f*: $\forall\, i{\in}Basis.\ f\ i \in (\lambda x.\ x *_R i)\ `\ B\ i$ **for** *i f*
proof −
have ($\sum x{\in}Basis - \{i\}.\ f\ x \cdot i$) = *0*
proof (*intro strip sum.neutral*)
show *f x* · *i* = *0* **if** *x* ∈ *Basis* − {*i*} **for** *x*
using *that f* ‹*i* ∈ *Basis*› *inner_Basis that* **by** *fastforce*
qed
then have ($\sum x{\in}Basis.\ f\ x \cdot i$) = *f i* · *i*
by (*metis* (*no_types*) ‹*i* ∈ *Basis*› *add.right_neutral sum.remove* [*OF finite_Basis*])
then have ($\sum x{\in}Basis.\ f\ x \cdot i$) ∈ *B i*
using *f that*(*1*) **by** *auto*
then show *?thesis*
by (*simp add*: *inner_sum_left*)
qed
ultimately show *?thesis*
by (*subst set_sum_alt* [*OF finite_Basis*]) *auto*
qed

lemma *convex_hull_set_sum*:
convex hull ($\sum i{\in}A.\ B\ i$) = ($\sum i{\in}A.$ *convex hull* (*B i*))
by (*induction A rule*: *infinite_finite_induct*) (*auto simp*: *convex_hull_set_plus*)

end

1.10 Definition of Finite Cartesian Product Type

theory *Finite_Cartesian_Product*
imports
Euclidean_Space
L2_Norm
HOL−Library.Numeral_Type
HOL−Library.Countable_Set
HOL−Library.FuncSet
begin

1.10.1 Finite Cartesian products, with indexing and lambdas

typedef $('a, 'b)$ *vec* = *UNIV* :: $('b::finite \Rightarrow 'a)$ *set*
morphisms *vec_nth vec_lambda* **..**

declare *vec_lambda_inject* [*simplified*, *simp*]

open_bundle *vec_syntax*
begin
notation *vec_nth* (**infixl** ‹$› *90*) **and** *vec_lambda* (**binder** ‹χ› *10*)
end

Concrete syntax for $('a, 'b)$ *vec*:

- $'a\hat{}'b$ becomes $('a, 'b::finite)$ *vec*
- $'a\hat{}'b::\_$ becomes $('a, 'b)$ *vec* without extra sort-constraint

syntax *_vec_type* :: *type* ⇒ *type* ⇒ *type* (**infixl** ‹^› *15*)
syntax_types *_vec_type* ⇌ *vec*
parse_translation ‹
let
fun vec t u = Syntax.const ***type_syntax*** ‹*vec*› $ *t* $ *u*;
fun finite_vec_tr [*t*, *u*] =
(*case Term_Position.strip_positions u of*
v as Free (*x*, *_*) =>
if Lexicon.is_tid x then
vec t (*Syntax.const* ***syntax_const*** ‹*_ofsort*› $ *v* $
Syntax.const ***class_syntax*** ‹*finite*›)
else vec t u
| *_* => *vec t u*)
in
[(***syntax_const*** ‹*_vec_type*›, *K finite_vec_tr*)]
end
›

lemma *vec_eq_iff*: $(x = y) \longleftrightarrow (\forall i.\ x\$i = y\$i)$
by (*simp add*: *vec_nth_inject* [*symmetric*] *fun_eq_iff*)

lemma *vec_lambda_beta* [*simp*]: *vec_lambda g $ i = g i*
by (*simp add*: *vec_lambda_inverse*)

lemma *vec_lambda_unique*: (∀ *i*. *f$i = g i*) ⟷ *vec_lambda g = f*
by (*auto simp add*: *vec_eq_iff*)

lemma *vec_lambda_eta* [*simp*]: (χ *i*. (*g$i*)) = *g*
by (*simp add*: *vec_eq_iff*)

1.10.2 Cardinality of vectors

instance *vec* :: (*finite*, *finite*) *finite*
proof
show *finite* (*UNIV* :: ('*a*, '*b*) *vec set*)
proof (*subst bij_betw_finite*)
show *bij_betw vec_nth UNIV* (*Pi* (*UNIV* :: '*b set*) (λ_. *UNIV* :: '*a set*))
by (*intro bij_betwI*[*of* _ _ _ *vec_lambda*]) (*auto simp*: *vec_eq_iff*)
have *finite* (*PiE* (*UNIV* :: '*b set*) (λ_. *UNIV* :: '*a set*))
by (*intro finite_PiE*) *auto*
also have (*PiE* (*UNIV* :: '*b set*) (λ_. *UNIV* :: '*a set*)) = *Pi UNIV* (λ_. *UNIV*)
by *auto*
finally show *finite* ... **.**
qed
qed

lemma *countable_PiE*:
finite I ⟹ (⋀*i*. *i* ∈ *I* ⟹ *countable* (*F i*)) ⟹ *countable* (*Pi*$_E$ *I F*)
by (*induct I arbitrary*: *F rule*: *finite_induct*) (*auto simp*: *PiE_insert_eq*)

instance *vec* :: (*countable*, *finite*) *countable*
proof
have *countable* (*UNIV* :: ('*a*, '*b*) *vec set*)
proof (*rule countableI_bij2*)
show *bij_betw vec_nth UNIV* (*Pi* (*UNIV* :: '*b set*) (λ_. *UNIV* :: '*a set*))
by (*intro bij_betwI*[*of* _ _ _ *vec_lambda*]) (*auto simp*: *vec_eq_iff*)
have *countable* (*PiE* (*UNIV* :: '*b set*) (λ_. *UNIV* :: '*a set*))
by (*intro countable_PiE*) *auto*
also have (*PiE* (*UNIV* :: '*b set*) (λ_. *UNIV* :: '*a set*)) = *Pi UNIV* (λ_. *UNIV*)
by *auto*
finally show *countable* ... **.**
qed
thus ∃ *t*::('*a*, '*b*) *vec* ⇒ *nat*. *inj t*
by (*auto elim*!: *countableE*)
qed

lemma *infinite_UNIV_vec*:
assumes *infinite* (*UNIV* :: '*a set*)
shows *infinite* (*UNIV* :: ('*a*^'*b*) *set*)

proof (*subst bij_betw_finite*)
show *bij_betw vec_nth UNIV* (*Pi* (*UNIV* :: *'b set*) (λ_. *UNIV* :: *'a set*))
by (*intro bij_betwI*[*of* _ _ _ *vec_lambda*]) (*auto simp*: *vec_eq_iff*)
have *infinite* (*PiE* (*UNIV* :: *'b set*) (λ_. *UNIV* :: *'a set*)) (**is** *infinite ?A*)
proof
assume *finite ?A*
hence *finite* ((λ*f*. *f undefined*) ‘ *?A*)
by (*rule finite_imageI*)
also have (λ*f*. *f undefined*) ‘ *?A* = *UNIV*
by *auto*
finally show *False*
using ‹*infinite* (*UNIV* :: *'a set*)› **by** *contradiction*
qed
also have *?A* = *Pi UNIV* (λ_. *UNIV*)
by *auto*
finally show *infinite* (*Pi* (*UNIV* :: *'b set*) (λ_. *UNIV* :: *'a set*)) **.**
qed

proposition *CARD_vec* [*simp*]:
CARD(*'a*^*'b*) = *CARD*(*'a*) ^ *CARD*(*'b*)
proof (*cases finite* (*UNIV* :: *'a set*))
case *True*
show *?thesis*
proof (*subst bij_betw_same_card*)
show *bij_betw vec_nth UNIV* (*Pi* (*UNIV* :: *'b set*) (λ_. *UNIV* :: *'a set*))
by (*intro bij_betwI*[*of* _ _ _ *vec_lambda*]) (*auto simp*: *vec_eq_iff*)
have *CARD*(*'a*) ^ *CARD*(*'b*) = *card* (*PiE* (*UNIV* :: *'b set*) (λ_. *UNIV* :: *'a set*))
(**is** _ = *card ?A*)
by (*subst card_PiE*) (*auto*)
also have *?A* = *Pi UNIV* (λ_. *UNIV*)
by *auto*
finally show *card* ... = *CARD*(*'a*) ^ *CARD*(*'b*) **..**
qed
qed (*simp_all add*: *infinite_UNIV_vec*)

lemma *countable_vector*:
fixes *B*:: *'n*::*finite* ⇒ *'a set*
assumes ⋀*i*. *countable* (*B i*)
shows *countable* {*V*. ∀ *i*::*'n*::*finite*. *V* $ *i* ∈ *B i*}
proof −
have *f* ∈ ($) ‘ {*V*. ∀ *i*. *V* $ *i* ∈ *B i*} **if** *f* ∈ Pi_E *UNIV B* **for** *f*
proof −
have ∃ *W*. (∀ *i*. *W* $ *i* ∈ *B i*) ∧ ($) *W* = *f*
by (*metis that PiE_iff UNIV_I vec_lambda_inverse*)
then show *f* ∈ ($) ‘ {*v*. ∀ *i*. *v* $ *i* ∈ *B i*}
by *blast*
qed
then have Pi_E *UNIV B* = *vec_nth* ‘ {*V*. ∀ *i*::*'n*. *V* $ *i* ∈ *B i*}

by *blast*
then have *countable* (*vec_nth* ‘ {*V*. ∀ *i*. *V* $ *i* ∈ *B i*})
by (*metis finite_class.finite_UNIV countable_PiE assms*)
then have *countable* (*vec_lambda* ‘ *vec_nth* ‘ {*V*. ∀ *i*. *V* $ *i* ∈ *B i*})
by *auto*
then show *?thesis*
by (*simp add*: *image_comp o_def vec_nth_inverse*)
qed

1.10.3 Group operations and class instances

instantiation *vec* :: (*zero*, *finite*) *zero*
begin
definition *0* ≡ (χ *i*. *0*)
instance ..
end

instantiation *vec* :: (*plus*, *finite*) *plus*
begin
definition (+) ≡ (λ *x y*. (χ *i*. *x*$*i* + *y*$*i*))
instance ..
end

instantiation *vec* :: (*minus*, *finite*) *minus*
begin
definition (−) ≡ (λ *x y*. (χ *i*. *x*$*i* − *y*$*i*))
instance ..
end

instantiation *vec* :: (*uminus*, *finite*) *uminus*
begin
definition *uminus* ≡ (λ *x*. (χ *i*. − (*x*$*i*)))
instance ..
end

lemma *zero_index* [*simp*]: *0* $ *i* = *0*
unfolding *zero_vec_def* **by** *simp*

lemma *vector_add_component* [*simp*]: (*x* + *y*)$*i* = *x*$*i* + *y*$*i*
unfolding *plus_vec_def* **by** *simp*

lemma *vector_minus_component* [*simp*]: (*x* − *y*)$*i* = *x*$*i* − *y*$*i*
unfolding *minus_vec_def* **by** *simp*

lemma *vector_uminus_component* [*simp*]: (− *x*)$*i* = − (*x*$*i*)
unfolding *uminus_vec_def* **by** *simp*

instance *vec* :: (*semigroup_add*, *finite*) *semigroup_add*
by *standard* (*simp add*: *vec_eq_iff add.assoc*)

instance *vec* :: (*ab_semigroup_add*, *finite*) *ab_semigroup_add*
by *standard* (*simp add*: *vec_eq_iff add.commute*)

instance *vec* :: (*monoid_add*, *finite*) *monoid_add*
by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*comm_monoid_add*, *finite*) *comm_monoid_add*
by *standard* (*simp add*: *vec_eq_iff*)

instance *vec* :: (*cancel_semigroup_add*, *finite*) *cancel_semigroup_add*
by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*cancel_ab_semigroup_add*, *finite*) *cancel_ab_semigroup_add*
by *standard* (*simp_all add*: *vec_eq_iff diff_diff_eq*)

instance *vec* :: (*cancel_comm_monoid_add*, *finite*) *cancel_comm_monoid_add*
..

instance *vec* :: (*group_add*, *finite*) *group_add*
by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*ab_group_add*, *finite*) *ab_group_add*
by *standard* (*simp_all add*: *vec_eq_iff*)

1.10.4 Basic componentwise operations on vectors

instantiation *vec* :: (*times*, *finite*) *times*
begin

definition $(*) \equiv (\lambda\ x\ y.\ \ (\chi\ i.\ (x\$i) * (y\$i)))$
instance ..

end

instantiation *vec* :: (*one*, *finite*) *one*
begin

definition $1 \equiv (\chi\ i.\ 1)$
instance ..

end

instantiation *vec* :: (*ord*, *finite*) *ord*
begin

definition $x \leq y \longleftrightarrow (\forall\, i.\ x\$i \leq y\$i)$
definition $x < (y::'a\hat{}'b) \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
instance ..

end

The ordering on one-dimensional vectors is linear.

instance *vec*:: (*order*, *finite*) *order*
by *standard* (*auto simp*: *less_eq_vec_def less_vec_def vec_eq_iff*
intro: *order.trans order.antisym order.strict_implies_order*)

instance *vec* :: (*linorder*, *CARD_1*) *linorder*
proof
obtain *a* :: *'b* **where** *all*: $\bigwedge P.\ (\forall\, i.\ P\ i) \longleftrightarrow P\ a$
proof −
have *CARD* (*'b*) = *1* **by** (*rule CARD_1*)
then obtain *b* :: *'b* **where** *UNIV* = {*b*} **by** (*auto iff*: *card_Suc_eq*)
then have $\bigwedge P.\ (\forall\, i{\in}UNIV.\ P\ i) \longleftrightarrow P\ b$ **by** *auto*
then show *thesis* **by** (*auto intro*: *that*)
qed
fix *x y* :: *'a^'b*::*CARD_1*
note [*simp*] = *less_eq_vec_def less_vec_def all vec_eq_iff field_simps*
show $x \leq y \vee y \leq x$ **by** *auto*
qed

Constant Vectors

definition *vec x* = (χ *i*. *x*)

Also the scalar-vector multiplication.

definition *vector_scalar_mult*:: *'a*::*times* ⇒ *'a* ^ *'n* ⇒ *'a* ^ *'n* (**infixl** ‹∗*s*› *70*)
where *c* ∗*s x* = (χ *i*. *c* ∗ (*x*$*i*))

scalar product

definition *scalar_product* :: *'a* :: *semiring_1* ^ *'n* ⇒ *'a* ^ *'n* ⇒ *'a* **where**
scalar_product v w = ($\sum$ *i* ∈ *UNIV*. *v* $ *i* ∗ *w* $ *i*)

1.10.5 Real vector space

instantiation *vec* :: (*real_vector*, *finite*) *real_vector*
begin

definition *scaleR* ≡ (λ *r x*. (χ *i*. *scaleR r* (*x*$*i*)))

lemma *vector_scaleR_component* [*simp*]: (*scaleR r x*)$*i* = *scaleR r* (*x*$*i*)
unfolding *scaleR_vec_def* **by** *simp*

instance
by *standard* (*simp_all add*: *vec_eq_iff scaleR_left_distrib scaleR_right_distrib*)

end

1.10.6 Topological space

instantiation *vec* :: (*topological_space*, *finite*) *topological_space*
begin

definition [*code del*]:
  open (*S* :: ('*a* ^ '*b*) *set*) ⟷
    (∀ *x*∈*S*. ∃ *A*. (∀ *i*. *open* (*A i*) ∧ *x*$*i* ∈ *A i*) ∧
      (∀ *y*. (∀ *i*. *y*$*i* ∈ *A i*) ⟶ *y* ∈ *S*))

instance proof
  show *open* (*UNIV* :: ('*a* ^ '*b*) *set*)
    unfolding *open_vec_def* **by** *auto*
next
  fix *S T* :: ('*a* ^ '*b*) *set*
  assume *open S open T* **thus** *open* (*S* ∩ *T*)
    unfolding *open_vec_def*
    apply *clarify*
    apply (*drule* (*1*) *bspec*)+
    apply (*clarify*, *rename_tac Sa Ta*)
    apply (*rule_tac x*=λ*i*. *Sa i* ∩ *Ta i* **in** *exI*)
    apply (*simp add*: *open_Int*)
    done
next
  fix *K* :: ('*a* ^ '*b*) *set set*
  assume ∀ *S*∈*K*. *open S* **thus** *open* (⋃ *K*)
    unfolding *open_vec_def*
    by (*metis Union_iff*)
qed

end

lemma *open_vector_box*: ∀ *i*. *open* (*S i*) ⟹ *open* {*x*. ∀ *i*. *x* $ *i* ∈ *S i*}
  unfolding *open_vec_def* **by** *auto*

lemma *open_vimage_vec_nth*: *open S* ⟹ *open* ((λ*x*. *x* $ *i*) −‘ *S*)
  unfolding *open_vec_def*
  apply *clarify*
  apply (*rule_tac x*=λ*k*. *if k* = *i then S else UNIV* **in** *exI*, *simp*)
  done

lemma *closed_vimage_vec_nth*: *closed S* ⟹ *closed* ((λ*x*. *x* $ *i*) −‘ *S*)
  unfolding *closed_open vimage_Compl* [*symmetric*]
  by (*rule open_vimage_vec_nth*)

lemma *closed_vector_box*: ∀ *i*. *closed* (*S i*) ⟹ *closed* {*x*. ∀ *i*. *x* $ *i* ∈ *S i*}
proof −
  have {*x*. ∀ *i*. *x* $ *i* ∈ *S i*} = (⋂ *i*. (λ*x*. *x* $ *i*) −‘ *S i*) **by** *auto*
  thus ∀ *i*. *closed* (*S i*) ⟹ *closed* {*x*. ∀ *i*. *x* $ *i* ∈ *S i*}
    by (*simp add*: *closed_INT closed_vimage_vec_nth*)

qed

lemma *tendsto_vec_nth* [*tendsto_intros*]:
assumes $((\lambda x.\ f\ x) \longrightarrow a)$ *net*
shows $((\lambda x.\ f\ x\ \$\ i) \longrightarrow a\ \$\ i)$ *net*
proof (*rule topological_tendstoI*)
fix *S* **assume** *open S a* $\$$ *i* $\in$ *S*
then have *open* $((\lambda y.\ y\ \$\ i) -`\ S)$ *a* $\in$ $((\lambda y.\ y\ \$\ i) -`\ S)$
by (*simp_all add*: *open_vimage_vec_nth*)
with *assms* **have** *eventually* $(\lambda x.\ f\ x \in (\lambda y.\ y\ \$\ i) -`\ S)$ *net*
by (*rule topological_tendstoD*)
then show *eventually* $(\lambda x.\ f\ x\ \$\ i \in S)$ *net*
by *simp*
qed

lemma *isCont_vec_nth* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* $(\lambda x.\ f\ x\ \$\ i)$ *a*
unfolding *isCont_def* **by** (*rule tendsto_vec_nth*)

lemma *vec_tendstoI*:
assumes $\bigwedge i.\ ((\lambda x.\ f\ x\ \$\ i) \longrightarrow a\ \$\ i)$ *net*
shows $((\lambda x.\ f\ x) \longrightarrow a)$ *net*
proof (*rule topological_tendstoI*)
fix *S* **assume** *open S* **and** *a* $\in$ *S*
then obtain *A* **where** *A*: $\bigwedge i.$ *open* $(A\ i)$ $\bigwedge i.\ a\ \$\ i \in A\ i$
and *S*: $\bigwedge y.\ \forall i.\ y\ \$\ i \in A\ i \Longrightarrow y \in S$
unfolding *open_vec_def* **by** *metis*
have $\bigwedge i.$ *eventually* $(\lambda x.\ f\ x\ \$\ i \in A\ i)$ *net*
using *assms A* **by** (*rule topological_tendstoD*)
hence *eventually* $(\lambda x.\ \forall i.\ f\ x\ \$\ i \in A\ i)$ *net*
by (*rule eventually_all_finite*)
thus *eventually* $(\lambda x.\ f\ x \in S)$ *net*
by (*rule eventually_mono*, *simp add*: *S*)
qed

lemma *tendsto_vec_lambda* [*tendsto_intros*]:
assumes $\bigwedge i.\ ((\lambda x.\ f\ x\ i) \longrightarrow a\ i)$ *net*
shows $((\lambda x.\ \chi\ i.\ f\ x\ i) \longrightarrow (\chi\ i.\ a\ i))$ *net*
using *assms* **by** (*simp add*: *vec_tendstoI*)

lemma *open_image_vec_nth*: **assumes** *open S* **shows** *open* $((\lambda x.\ x\ \$\ i)\ `\ S)$
proof (*rule openI*)
fix *a* **assume** *a* $\in$ $(\lambda x.\ x\ \$\ i)\ `\ S$
then obtain *z* **where** *a* = *z* $\$$ *i* **and** *z* $\in$ *S* **..**
then obtain *A* **where** *A*: $\forall i.$ *open* $(A\ i) \wedge z\ \$\ i \in A\ i$
and *S*: $\forall y.\ (\forall i.\ y\ \$\ i \in A\ i) \longrightarrow y \in S$
using ‹*open S*› **unfolding** *open_vec_def* **by** *auto*
hence *A i* $\subseteq$ $(\lambda x.\ x\ \$\ i)\ `\ S$
by (*clarsimp*, *rule_tac x*=χ *j*. *if j* = *i then x else z* $\$$ *j* **in** *image_eqI*,
simp_all)

```
  hence open (A i) ∧ a ∈ A i ∧ A i ⊆ (λx. x $ i) ‘ S
    using A ‹a = z $ i› by simp
  then show ∃ T. open T ∧ a ∈ T ∧ T ⊆ (λx. x $ i) ‘ S by − (rule exI)
qed

instance vec :: (perfect_space, finite) perfect_space
proof
  fix x :: 'a ^ 'b show ¬ open {x}
  proof
    assume open {x}
    hence ∀ i. open ((λx. x $ i) ‘ {x}) by (fast intro: open_image_vec_nth)
    hence ∀ i. open {x $ i} by simp
    thus False by (simp add: not_open_singleton)
  qed
qed
```

1.10.7 Metric space

```
instantiation vec :: (metric_space, finite) dist
begin

definition
  dist x y = L2_set (λi. dist (x$i) (y$i)) UNIV

instance ..
end

instantiation vec :: (metric_space, finite) uniformity_dist
begin

definition [code del]:
  (uniformity :: (('a^'b::_) × ('a^'b::_)) filter) =
    (INF e∈{0 <..}. principal {(x, y). dist x y < e})

instance
  by standard (rule uniformity_vec_def)
end

declare uniformity_Abort[where 'a='a :: metric_space ^ 'b :: finite, code]

instantiation vec :: (metric_space, finite) metric_space
begin

proposition dist_vec_nth_le: dist (x $ i) (y $ i) ≤ dist x y
  unfolding dist_vec_def by (rule member_le_L2_set) simp_all

instance proof
  fix x y :: 'a ^ 'b
  show dist x y = 0 ⟷ x = y
```

```
    unfolding dist_vec_def
    by (simp add: L2_set_eq_0_iff vec_eq_iff)
next
  fix x y z :: 'a ^ 'b
  show dist x y ≤ dist x z + dist y z
    unfolding dist_vec_def
    apply (rule order_trans [OF _ L2_set_triangle_ineq])
    apply (simp add: L2_set_mono dist_triangle2)
    done
next
  fix S :: ('a ^ 'b) set
  have ∗: open S ⟷ (∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S)
  proof
    assume open S show ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S
    proof
      fix x assume x ∈ S
      obtain A where A: ∀ i. open (A i) ∀ i. x $ i ∈ A i
        and S: ∀ y. (∀ i. y $ i ∈ A i) ⟶ y ∈ S
        using ‹open S› and ‹x ∈ S› unfolding open_vec_def by metis
      have ∀ i∈UNIV. ∃ r>0. ∀ y. dist y (x $ i) < r ⟶ y ∈ A i
        using A unfolding open_dist by simp
      hence ∃ r. ∀ i∈UNIV. 0 < r i ∧ (∀ y. dist y (x $ i) < r i ⟶ y ∈ A i)
        by (rule finite_set_choice [OF finite])
      then obtain r where r1: ∀ i. 0 < r i
        and r2: ∀ i y. dist y (x $ i) < r i ⟶ y ∈ A i by fast
      have 0 < Min (range r) ∧ (∀ y. dist y x < Min (range r) ⟶ y ∈ S)
        by (simp add: r1 r2 S le_less_trans [OF dist_vec_nth_le])
      thus ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S ..
    qed
  next
    assume ∗: ∀ x∈S. ∃ e>0. ∀ y. dist y x < e ⟶ y ∈ S show open S
    proof (unfold open_vec_def, rule)
      fix x assume x ∈ S
      then obtain e where 0 < e and S: ∀ y. dist y x < e ⟶ y ∈ S
        using ∗ by fast
      define r where [abs_def]: r i = e / sqrt (of_nat CARD('b)) for i :: 'b
      from ‹0 < e› have r: ∀ i. 0 < r i
        unfolding r_def by simp_all
      from ‹0 < e› have e: e = L2_set r UNIV
        unfolding r_def by (simp add: L2_set_constant)
      define A where A i = {y. dist (x $ i) y < r i} for i
      have ∀ i. open (A i) ∧ x $ i ∈ A i
        unfolding A_def by (simp add: open_ball r)
      moreover have ∀ y. (∀ i. y $ i ∈ A i) ⟶ y ∈ S
       by (simp add: A_def S dist_vec_def e L2_set_strict_mono dist_commute)
      ultimately show ∃ A. (∀ i. open (A i) ∧ x $ i ∈ A i) ∧
        (∀ y. (∀ i. y $ i ∈ A i) ⟶ y ∈ S) by metis
    qed
  qed
```

```
  show open S = (∀x∈S. ∀F (x', y) in uniformity. x' = x ⟶ y ∈ S)
    unfolding * eventually_uniformity_metric
    by (simp del: split_paired_All add: dist_vec_def dist_commute)
qed

end

lemma Cauchy_vec_nth:
  Cauchy (λn. X n) ⟹ Cauchy (λn. X n $ i)
  unfolding Cauchy_def by (fast intro: le_less_trans [OF dist_vec_nth_le])

lemma vec_CauchyI:
  fixes X :: nat ⇒ 'a::metric_space ^ 'n
  assumes X: ⋀i. Cauchy (λn. X n $ i)
  shows Cauchy (λn. X n)
proof (rule metric_CauchyI)
  fix r :: real assume 0 < r
  hence 0 < r / of_nat CARD('n) (is 0 < ?s) by simp
  define N where N i = (LEAST N. ∀m≥N. ∀n≥N. dist (X m $ i) (X n $ i)
< ?s) for i
  define M where M = Max (range N)
  have ⋀i. ∃N. ∀m≥N. ∀n≥N. dist (X m $ i) (X n $ i) < ?s
    using X ‹0 < ?s› by (rule metric_CauchyD)
  hence ⋀i. ∀m≥N i. ∀n≥N i. dist (X m $ i) (X n $ i) < ?s
    unfolding N_def by (rule LeastI_ex)
  hence M: ⋀i. ∀m≥M. ∀n≥M. dist (X m $ i) (X n $ i) < ?s
    unfolding M_def by simp
  {
    fix m n :: nat
    assume M ≤ m M ≤ n
    have dist (X m) (X n) = L2_set (λi. dist (X m $ i) (X n $ i)) UNIV
      unfolding dist_vec_def ..
    also have ... ≤ sum (λi. dist (X m $ i) (X n $ i)) UNIV
      by (rule L2_set_le_sum [OF zero_le_dist])
    also have ... < sum (λi::'n. ?s) UNIV
      by (rule sum_strict_mono, simp_all add: M ‹M ≤ m› ‹M ≤ n›)
    also have ... = r
      by simp
    finally have dist (X m) (X n) < r .
  }
  hence ∀m≥M. ∀n≥M. dist (X m) (X n) < r
    by simp
  then show ∃M. ∀m≥M. ∀n≥M. dist (X m) (X n) < r ..
qed

instance vec :: (complete_space, finite) complete_space
proof
  fix X :: nat ⇒ 'a ^ 'b assume Cauchy X
  have ⋀i. (λn. X n $ i) ⟶⟶ lim (λn. X n $ i)
```

using *Cauchy_vec_nth* [*OF* ‹*Cauchy X*›]
by (*simp add*: *Cauchy_convergent_iff convergent_LIMSEQ_iff*)
hence *X* ⟶ *vec_lambda* (λ*i*. *lim* (λ*n*. *X n* $ *i*))
by (*simp add*: *vec_tendstoI*)
then show *convergent X*
by (*rule convergentI*)
qed

1.10.8 Normed vector space

instantiation *vec* :: (*real_normed_vector*, *finite*) *real_normed_vector*
begin

definition *norm x* = *L2_set* (λ*i*. *norm* (*x*$*i*)) *UNIV*

definition *sgn* (*x*::'*a*^'*b*) = *scaleR* (*inverse* (*norm x*)) *x*

instance proof
fix *a* :: *real* **and** *x y* :: '*a* ^ '*b*
show *norm x* = *0* ⟷ *x* = *0*
unfolding *norm_vec_def*
by (*simp add*: *L2_set_eq_0_iff vec_eq_iff*)
show *norm* (*x* + *y*) ≤ *norm x* + *norm y*
unfolding *norm_vec_def*
apply (*rule order_trans* [*OF* _ *L2_set_triangle_ineq*])
apply (*simp add*: *L2_set_mono norm_triangle_ineq*)
done
show *norm* (*scaleR a x*) = |*a*| ∗ *norm x*
unfolding *norm_vec_def*
by (*simp add*: *L2_set_right_distrib*)
show *sgn x* = *scaleR* (*inverse* (*norm x*)) *x*
by (*rule sgn_vec_def*)
show *dist x y* = *norm* (*x* − *y*)
unfolding *dist_vec_def norm_vec_def*
by (*simp add*: *dist_norm*)
qed

end

lemma *norm_nth_le*: *norm* (*x* $ *i*) ≤ *norm x*
unfolding *norm_vec_def*
by (*rule member_le_L2_set*) *simp_all*

lemma *norm_le_componentwise_cart*:
fixes *x* :: '*a*::*real_normed_vector*^'*n*
assumes ⋀*i*. *norm*(*x*$*i*) ≤ *norm*(*y*$*i*)
shows *norm x* ≤ *norm y*
unfolding *norm_vec_def*
by (*rule L2_set_mono*) (*auto simp*: *assms*)

lemma *component_le_norm_cart*: $|x\$i| \leq norm\ x$
by (*metis norm_nth_le real_norm_def*)

lemma *norm_bound_component_le_cart*: $norm\ x \leq e \Longrightarrow |x\$i| \leq e$
by (*metis component_le_norm_cart order_trans*)

lemma *norm_bound_component_lt_cart*: $norm\ x < e \Longrightarrow |x\$i| < e$
by (*metis component_le_norm_cart le_less_trans*)

lemma *norm_le_l1_cart*: $norm\ x \leq sum(\lambda i.\ |x\$i|)\ UNIV$
by (*simp add*: *norm_vec_def L2_set_le_sum*)

lemma *bounded_linear_vec_nth*[*intro*]: *bounded_linear* $(\lambda x.\ x\ \$\ i)$
proof
show $\exists K.\ \forall x.\ norm\ (x\ \$\ i) \leq norm\ x * K$
by (*metis mult.commute mult.left_neutral norm_nth_le*)
qed *auto*

instance *vec* :: (*banach*, *finite*) *banach* **..**

1.10.9 Inner product space

instantiation *vec* :: (*real_inner*, *finite*) *real_inner*
begin

definition $inner\ x\ y = sum\ (\lambda i.\ inner\ (x\$i)\ (y\$i))\ UNIV$

instance proof
fix r :: *real* **and** $x\ y\ z$ :: $'a\ \hat{}\ 'b$
show $inner\ x\ y = inner\ y\ x$
unfolding *inner_vec_def*
by (*simp add*: *inner_commute*)
show $inner\ (x + y)\ z = inner\ x\ z + inner\ y\ z$
unfolding *inner_vec_def*
by (*simp add*: *inner_add_left sum.distrib*)
show $inner\ (scaleR\ r\ x)\ y = r * inner\ x\ y$
unfolding *inner_vec_def*
by (*simp add*: *sum_distrib_left*)
show $0 \leq inner\ x\ x$
unfolding *inner_vec_def*
by (*simp add*: *sum_nonneg*)
show $inner\ x\ x = 0 \longleftrightarrow x = 0$
unfolding *inner_vec_def*
by (*simp add*: *vec_eq_iff sum_nonneg_eq_0_iff*)
show $norm\ x = sqrt\ (inner\ x\ x)$
unfolding *inner_vec_def norm_vec_def L2_set_def*
by (*simp add*: *power2_norm_eq_inner*)
qed

end

1.10.10 Euclidean space

Vectors pointing along a single axis.

definition *axis k x* = (*χ i. if i* = *k then x else 0*)

lemma *axis_nth* [*simp*]: *axis i x* $ *i* = *x*
unfolding *axis_def* **by** *simp*

lemma *axis_eq_axis*: *axis i x* = *axis j y* ⟷ *x* = *y* ∧ *i* = *j* ∨ *x* = *0* ∧ *y* = *0*
unfolding *axis_def vec_eq_iff* **by** *auto*

lemma *inner_axis_axis*:
inner (*axis i x*) (*axis j y*) = (*if i* = *j then inner x y else 0*)
by (*simp add*: *inner_vec_def axis_def sum.neutral sum.remove* [*of* _ *j*])

lemma *inner_axis*: *inner x* (*axis i y*) = *inner* (*x* $ *i*) *y*
by (*simp add*: *inner_vec_def axis_def sum.remove* [**where** *x*=*i*])

lemma *inner_axis′*: *inner*(*axis i y*) *x* = *inner y* (*x* $ *i*)
by (*simp add*: *inner_axis inner_commute*)

instantiation *vec* :: (*euclidean_space*, *finite*) *euclidean_space*
begin

definition *Basis* = (⋃ *i*. ⋃ *u*∈*Basis*. {*axis i u*})

instance proof
show (*Basis* :: (′*a* ^ ′*b*) *set*) ≠ {}
unfolding *Basis_vec_def* **by** *simp*
next
show *finite* (*Basis* :: (′*a* ^ ′*b*) *set*)
unfolding *Basis_vec_def* **by** *simp*
next
fix *u v* :: ′*a* ^ ′*b*
assume *u* ∈ *Basis* **and** *v* ∈ *Basis*
thus *inner u v* = (*if u* = *v then 1 else 0*)
unfolding *Basis_vec_def*
by (*auto simp add*: *inner_axis_axis axis_eq_axis inner_Basis*)
next
fix *x* :: ′*a* ^ ′*b*
show (∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ *x* = *0*
unfolding *Basis_vec_def*
by (*simp add*: *inner_axis euclidean_all_zero_iff vec_eq_iff*)
qed

proposition *DIM_cart* [*simp*]: *DIM*(′*a*^′*b*) = *CARD*(′*b*) ∗ *DIM*(′*a*)

proof −
have *card* ($\bigcup i{::}'b.\ \bigcup u{::}'a{\in}Basis.\ \{axis\ i\ u\}$) = ($\sum i{::}'b{\in}UNIV.\ card\ (\bigcup u{::}'a{\in}Basis.\ \{axis\ i\ u\})$)
by (*rule card_UN_disjoint*) (*auto simp*: *axis_eq_axis*)
also have ... = *CARD*('*b*) ∗ *DIM*('*a*)
by (*subst card_UN_disjoint*) (*auto simp*: *axis_eq_axis*)
finally show *?thesis*
by (*simp add*: *Basis_vec_def*)
qed

end

lemma *norm_axis_1* [*simp*]: *norm* (*axis m* (*1*::*real*)) = *1*
by (*simp add*: *inner_axis*′ *norm_eq_1*)

lemma *sum_norm_allsubsets_bound_cart*:
fixes *f*:: '*a* ⇒ *real* ^′*n*
assumes *fP*: *finite P* **and** *fPs*: ⋀*Q*. *Q* ⊆ *P* ⟹ *norm* (*sum f Q*) ≤ *e*
shows *sum* (λ*x*. *norm* (*f x*)) *P* ≤ *2* ∗ *real CARD*('*n*) ∗ *e*
using *sum_norm_allsubsets_bound*[*OF assms*]
by *simp*

lemma *cart_eq_inner_axis*: *a* $ *i* = *inner a* (*axis i 1*)
by (*simp add*: *inner_axis*)

lemma *axis_eq_0_iff* [*simp*]:
shows *axis m x* = *0* ⟷ *x* = *0*
by (*simp add*: *axis_def vec_eq_iff*)

lemma *axis_in_Basis_iff* [*simp*]: *axis i a* ∈ *Basis* ⟷ *a* ∈ *Basis*
by (*auto simp*: *Basis_vec_def axis_eq_axis*)

Mapping each basis element to the corresponding finite index

definition *axis_index* :: ('*a*::*comm_ring_1*) ^′*n* ⇒ '*n* **where** *axis_index v* ≡ *SOME i*. *v* = *axis i 1*

lemma *axis_inverse*:
fixes *v* :: *real* ^′*n*
assumes *v* ∈ *Basis*
shows ∃ *i*. *v* = *axis i 1*
proof −
have *v* ∈ ($\bigcup n.\ \bigcup r{\in}Basis.\ \{axis\ n\ r\}$)
using *assms Basis_vec_def* **by** *blast*
then show *?thesis*
by (*force simp add*: *vec_eq_iff*)
qed

lemma *axis_index*:
fixes *v* :: *real* ^′*n*

assumes $v \in Basis$
shows $v = axis\ (axis\_index\ v)\ 1$
by (*metis* (*mono_tags*) *assms axis_inverse axis_index_def someI_ex*)

lemma *axis_index_axis* [*simp*]:
fixes $UU :: real^{\sim 'n}$
shows $(axis\_index\ (axis\ u\ 1 :: real^{\sim 'n})) = (u::'n)$
by (*simp add*: *axis_eq_axis axis_index_def*)

1.10.11 A naive proof procedure to lift really trivial arithmetic stuff from the basis of the vector space

lemma *sum_cong_aux*:
$(\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x) \Longrightarrow sum\ f\ A = sum\ g\ A$
by (*auto intro*: *sum.cong*)

hide_fact (**open**) *sum_cong_aux*

method_setup *vector* = ‹
let
val ss1 =
simpset_of (*put_simpset HOL_basic_ss* ***context***
addsimps [@{*thm sum.distrib*} *RS sym*,
@{*thm sum_subtractf*} *RS sym*, @{*thm sum_distrib_left*},
@{*thm sum_distrib_right*}, @{*thm sum_negf*} *RS sym*])
val ss2 =
simpset_of (***context*** *addsimps*
[@{*thm plus_vec_def*}, @{*thm times_vec_def*},
@{*thm minus_vec_def*}, @{*thm uminus_vec_def*},
@{*thm one_vec_def*}, @{*thm zero_vec_def*}, @{*thm vec_def*},
@{*thm scaleR_vec_def*}, @{*thm vector_scalar_mult_def*}])
fun vector_arith_tac ctxt ths =
simp_tac (*put_simpset ss1 ctxt*)
THEN' (*fn i => resolve_tac ctxt* @{*thms Finite_Cartesian_Product.sum_cong_aux*}
i
ORELSE resolve_tac ctxt @{*thms sum.neutral*} *i*
ORELSE simp_tac (*put_simpset HOL_basic_ss ctxt addsimps* [@{*thm*
vec_eq_iff}]) *i*)
(∗ *THEN' TRY o clarify_tac HOL_cs THEN'* (*TRY o rtac* @{*thm iffI*}) ∗)
THEN' asm_full_simp_tac (*put_simpset ss2 ctxt addsimps ths*)
in
Attrib.thms >> (*fn ths => fn ctxt => SIMPLE_METHOD'* (*vector_arith_tac*
ctxt ths))
end
› *lift trivial vector statements to real arith statements*

lemma *vec_0*[*simp*]: *vec 0 = 0* **by** *vector*
lemma *vec_1*[*simp*]: *vec 1 = 1* **by** *vector*

lemma *vec_inj*[*simp*]: *vec x* = *vec y* ⟷ *x* = *y* **by** *vector*

lemma *vec_in_image_vec*: *vec x* ∈ (*vec* ‘ *S*) ⟷ *x* ∈ *S* **by** *auto*

lemma *vec_add*: *vec*(*x* + *y*) = *vec x* + *vec y* **by** *vector*
lemma *vec_sub*: *vec*(*x* − *y*) = *vec x* − *vec y* **by** *vector*
lemma *vec_cmul*: *vec*(*c* ∗ *x*) = *c* ∗*s vec x* **by** *vector*
lemma *vec_neg*: *vec*(− *x*) = − *vec x* **by** *vector*

lemma *vec_scaleR*: *vec*(*c* ∗ *x*) = *c* ∗$_R$ *vec x*
by *vector*

lemma *vec_sum*:
assumes *finite S*
shows *vec*(*sum f S*) = *sum* (*vec* ∘ *f*) *S*
using *assms*
proof *induct*
case *empty*
then show *?case* **by** *simp*
next
case *insert*
then show *?case* **by** (*auto simp add*: *vec_add*)
qed

Obvious "component-pushing".

lemma *vec_component* [*simp*]: *vec x* $ *i* = *x*
by *vector*

lemma *vector_mult_component* [*simp*]: (*x* ∗ *y*)$*i* = *x*$*i* ∗ *y*$*i*
by *vector*

lemma *vector_smult_component* [*simp*]: (*c* ∗*s y*)$*i* = *c* ∗ (*y*$*i*)
by *vector*

lemma *cond_component*: (*if b then x else y*)$*i* = (*if b then x*$*i else y*$*i*) **by** *vector*

lemmas *vector_component* =
vec_component vector_add_component vector_mult_component
vector_smult_component vector_minus_component vector_uminus_component
vector_scaleR_component cond_component

1.10.12 Some frequently useful arithmetic lemmas over vectors

instance *vec* :: (*semigroup_mult*, *finite*) *semigroup_mult*
by *standard* (*vector mult.assoc*)

instance *vec* :: (*monoid_mult*, *finite*) *monoid_mult*
by *standard vector*+

instance *vec* :: (*ab_semigroup_mult*, *finite*) *ab_semigroup_mult*
by *standard* (*vector mult.commute*)

instance *vec* :: (*comm_monoid_mult*, *finite*) *comm_monoid_mult*
by *standard vector*

instance *vec* :: (*semiring*, *finite*) *semiring*
by *standard* (*vector field_simps*)+

instance *vec* :: (*semiring_0*, *finite*) *semiring_0*
by *standard* (*vector field_simps*)+
instance *vec* :: (*semiring_1*, *finite*) *semiring_1*
by *standard vector*
instance *vec* :: (*comm_semiring*, *finite*) *comm_semiring*
by *standard* (*vector field_simps*)+

instance *vec* :: (*comm_semiring_0*, *finite*) *comm_semiring_0* **..**
instance *vec* :: (*semiring_0_cancel*, *finite*) *semiring_0_cancel* **..**
instance *vec* :: (*comm_semiring_0_cancel*, *finite*) *comm_semiring_0_cancel* **..**
instance *vec* :: (*ring*, *finite*) *ring* **..**
instance *vec* :: (*semiring_1_cancel*, *finite*) *semiring_1_cancel* **..**
instance *vec* :: (*comm_semiring_1*, *finite*) *comm_semiring_1* **..**

instance *vec* :: (*ring_1*, *finite*) *ring_1* **..**

instance *vec* :: (*real_algebra*, *finite*) *real_algebra*
by *standard* (*simp_all add*: *vec_eq_iff*)

instance *vec* :: (*real_algebra_1*, *finite*) *real_algebra_1* **..**

lemma *of_nat_index*: (*of_nat n* :: *'a*::*semiring_1* ^ *'n*)$*i* = *of_nat n*
proof (*induct n*)
case *0*
then show *?case* **by** *vector*
next
case *Suc*
then show *?case* **by** *vector*
qed

lemma *one_index* [*simp*]: (*1* :: *'a* :: *one* ^ *'n*) $ *i* = *1*
by *vector*

lemma *neg_one_index* [*simp*]: (− *1* :: *'a* :: {*one*, *uminus*} ^ *'n*) $ *i* = − *1*
by *vector*

instance *vec* :: (*semiring_char_0*, *finite*) *semiring_char_0*
proof
fix *m n* :: *nat*

show *inj* (*of_nat* :: *nat* ⇒ *'a ^ 'b*)
by (*auto intro*!: *injI simp add*: *vec_eq_iff of_nat_index*)
qed

instance *vec* :: (*numeral*, *finite*) *numeral* **..**
instance *vec* :: (*semiring_numeral*, *finite*) *semiring_numeral* **..**

lemma *numeral_index* [*simp*]: *numeral w $ i* = *numeral w*
by (*induct w*) (*simp_all only*: *numeral.simps vector_add_component one_index*)

lemma *neg_numeral_index* [*simp*]: − *numeral w $ i* = − *numeral w*
by (*simp only*: *vector_uminus_component numeral_index*)

instance *vec* :: (*comm_ring_1*, *finite*) *comm_ring_1* **..**
instance *vec* :: (*ring_char_0*, *finite*) *ring_char_0* **..**

lemma *vector_smult_assoc*: *a ∗s* (*b ∗s x*) = ((*a*::*'a*::*semigroup_mult*) ∗ *b*) *∗s x*
by (*vector mult.assoc*)
lemma *vector_sadd_rdistrib*: ((*a*::*'a*::*semiring*) + *b*) *∗s x* = *a ∗s x* + *b ∗s x*
by (*vector field_simps*)
lemma *vector_add_ldistrib*: (*c*::*'a*::*semiring*) *∗s* (*x* + *y*) = *c ∗s x* + *c ∗s y*
by (*vector field_simps*)
lemma *vector_smult_lzero*[*simp*]: (*0*::*'a*::*mult_zero*) *∗s x* = *0* **by** *vector*
lemma *vector_smult_lid*[*simp*]: (*1*::*'a*::*monoid_mult*) *∗s x* = *x* **by** *vector*
lemma *vector_ssub_ldistrib*: (*c*::*'a*::*ring*) *∗s* (*x* − *y*) = *c ∗s x* − *c ∗s y*
by (*vector field_simps*)
lemma *vector_smult_rneg*: (*c*::*'a*::*ring*) *∗s* −*x* = −(*c ∗s x*) **by** *vector*
lemma *vector_smult_lneg*: − (*c*::*'a*::*ring*) *∗s x* = −(*c ∗s x*) **by** *vector*
lemma *vector_sneg_minus1*: −*x* = (−*1*::*'a*::*ring_1*) *∗s x* **by** *vector*
lemma *vector_smult_rzero*[*simp*]: *c ∗s 0* = (*0*::*'a*::*mult_zero ^ 'n*) **by** *vector*
lemma *vector_sub_rdistrib*: ((*a*::*'a*::*ring*) − *b*) *∗s x* = *a ∗s x* − *b ∗s x*
by (*vector field_simps*)

lemma *vec_eq*[*simp*]: (*vec m* = *vec n*) ⟷ (*m* = *n*)
by (*simp add*: *vec_eq_iff*)

lemma *Vector_Spaces_linear_vec* [*simp*]: *Vector_Spaces.linear* (∗) *vector_scalar_mult vec*
by *unfold_locales* (*vector algebra_simps*)+

lemma *vector_mul_eq_0*[*simp*]: (*a ∗s x* = *0*) ⟷ *a* = (*0*::*'a*::*idom*) ∨ *x* = *0*
by *vector*

lemma *vector_mul_lcancel*[*simp*]: *a ∗s x* = *a ∗s y* ⟷ *a* = (*0*::*'a*::*field*) ∨ *x* = *y*
by (*metis eq_iff_diff_eq_0 vector_mul_eq_0 vector_ssub_ldistrib*)

lemma *vector_mul_rcancel*[*simp*]: *a ∗s x* = *b ∗s x* ⟷ (*a*::*'a*::*field*) = *b* ∨ *x* = *0*
by (*subst eq_iff_diff_eq_0*, *subst vector_sub_rdistrib* [*symmetric*]) *simp*

lemma *scalar_mult_eq_scaleR* [*abs_def*]: $c *s x = c *_R x$
unfolding *scaleR_vec_def vector_scalar_mult_def* **by** *simp*

lemma *dist_mul*[*simp*]: *dist* $(c *s x)$ $(c *s y) = |c| * dist\ x\ y$
unfolding *dist_norm scalar_mult_eq_scaleR*
unfolding *scaleR_right_diff_distrib*[*symmetric*] **by** *simp*

lemma *sum_component* [*simp*]:
fixes $f:: 'a \Rightarrow ('b::comm\_monoid\_add)\ ^\backprime n$
shows $(sum\ f\ S)\$i = sum\ (\lambda x.\ (f\ x)\$i)\ S$
proof (*cases finite S*)
case *True*
then show *?thesis* **by** *induct simp_all*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *sum_eq*: $sum\ f\ S = (\chi\ i.\ sum\ (\lambda x.\ (f\ x)\$i\ )\ S)$
by (*simp add*: *vec_eq_iff*)

lemma *sum_cmul*:
fixes $f:: 'c \Rightarrow ('a::semiring\_1)\ ^\backprime n$
shows $sum\ (\lambda x.\ c *s f\ x)\ S = c *s sum\ f\ S$
by (*simp add*: *vec_eq_iff sum_distrib_left*)

lemma *linear_vec* [*simp*]: *linear vec*
using *Vector_Spaces_linear_vec*
by *unfold_locales* (*vector algebra_simps*)+

1.10.13 Matrix operations

Matrix notation. NB: an MxN matrix is of type $(('a, 'n)\ vec, 'm)\ vec$, not $(('a, 'm)\ vec, 'n)\ vec$

definition *map_matrix*::$('a \Rightarrow 'b) \Rightarrow (('a, 'i::finite)vec, 'j::finite)\ vec \Rightarrow (('b, 'i)vec, 'j)\ vec$ **where**
$map\_matrix\ f\ x = (\chi\ i\ j.\ f\ (x\ \$\ i\ \$\ j))$

lemma *nth_map_matrix*[*simp*]: $map\_matrix\ f\ x\ \$\ i\ \$\ j = f\ (x\ \$\ i\ \$\ j)$
by (*simp add*: *map_matrix_def*)

definition *matrix_matrix_mult* :: $('a::semiring\_1)\ ^\backprime n^\backprime m \Rightarrow 'a\ ^\backprime p^\backprime n \Rightarrow 'a\ ^\backprime p\ ^\backprime m$
(**infixl** ‹**›) *70*)
where $m ** m' == (\chi\ i\ j.\ sum\ (\lambda k.\ ((m\$i)\$k) * ((m'\$k)\$j))\ (UNIV :: 'n\ set))$ $::'a\ ^\backprime p\ ^\backprime m$

definition *matrix_vector_mult* :: $('a::semiring\_1)\ ^\backprime n^\backprime m \Rightarrow 'a\ ^\backprime n \Rightarrow 'a\ ^\backprime m$
(**infixl** ‹*v›) *70*)

where $m *v x \equiv (\chi\ i.\ sum\ (\lambda j.\ ((m\$i)\$j) * (x\$j))\ (UNIV :: 'n\ set)) :: 'a\hat{}'m$

definition *vector_matrix_mult* :: $'a\ \hat{}\ 'm \Rightarrow ('a::semiring\_1)\ \hat{}'n\hat{}'m \Rightarrow 'a\ \hat{}'n$
(**infixl** ‹*v**› 70)
where $v\ v*\ m == (\chi\ j.\ sum\ (\lambda i.\ ((v\$i) * (m\$i)\$j))\ (UNIV :: 'm\ set)) :: 'a\hat{}'n$

definition $(mat::'a::zero => 'a\ \hat{}'n\hat{}'n)\ k = (\chi\ i\ j.\ if\ i = j\ then\ k\ else\ 0)$
definition *transpose* **where**
$(transpose::'a\hat{}'n\hat{}'m \Rightarrow 'a\hat{}'m\hat{}'n)\ A = (\chi\ i\ j.\ ((A\$j)\$i))$
definition $(row::'m => 'a\ \hat{}'n\hat{}'m \Rightarrow 'a\ \hat{}'n)\ i\ A = (\chi\ j.\ ((A\$i)\$j))$
definition $(column::'n =>'a\hat{}'n\hat{}'m =>'a\hat{}'m)\ j\ A = (\chi\ i.\ ((A\$i)\$j))$
definition $rows(A::'a\hat{}'n\hat{}'m) = \{\ row\ i\ A \mid i.\ i \in (UNIV :: 'm\ set)\}$
definition $columns(A::'a\hat{}'n\hat{}'m) = \{\ column\ i\ A \mid i.\ i \in (UNIV :: 'n\ set)\}$

lemma *times0_left* [*simp*]: $(0::'a::semiring\_1\hat{}'n\hat{}'m) ** (A::'a\ \hat{}'p\hat{}'n) = 0$
by (*simp add*: *matrix_matrix_mult_def zero_vec_def*)

lemma *times0_right* [*simp*]: $(A::'a::semiring\_1\hat{}'n\hat{}'m) ** (0::'a\ \hat{}'p\hat{}'n) = 0$
by (*simp add*: *matrix_matrix_mult_def zero_vec_def*)

lemma *mat_0*[*simp*]: *mat 0 = 0* **by** (*vector mat_def*)
lemma *matrix_add_ldistrib*: $(A ** (B + C)) = (A ** B) + (A ** C)$
by (*vector matrix_matrix_mult_def sum.distrib*[*symmetric*] *field_simps*)

lemma *matrix_mul_lid* [*simp*]:
fixes $A :: 'a::semiring\_1\ \hat{}\ 'm\ \hat{}\ 'n$
shows *mat 1 ** A = A*
unfolding *matrix_matrix_mult_def mat_def*
by (*auto simp*: *if_distrib if_distribR sum.delta'*[*OF finite*] *cong*: *if_cong*)

lemma *matrix_mul_rid* [*simp*]:
fixes $A :: 'a::semiring\_1\ \hat{}\ 'm\ \hat{}\ 'n$
shows *A ** mat 1 = A*
unfolding *matrix_matrix_mult_def mat_def*
by (*auto simp*: *if_distrib if_distribR sum.delta'*[*OF finite*] *cong*: *if_cong*)

lemma *matrix_mul_assoc*: $A ** (B ** C) = (A ** B) ** C$
apply (*vector matrix_matrix_mult_def sum_distrib_left sum_distrib_right mult.assoc*)
using *sum.swap* **by** *fastforce*

lemma *matrix_vector_mul_assoc*: $A *v (B *v x) = (A ** B) *v x$
apply (*vector matrix_matrix_mult_def matrix_vector_mult_def*
sum_distrib_left sum_distrib_right mult.assoc)
using *sum.swap* **by** *fastforce*

lemma *vector_matrix_mul_assoc*: $(x\ v*\ A)\ v*\ B = x\ v*\ (A**B)$
apply (*vector matrix_matrix_mult_def vector_matrix_mult_def*
sum_distrib_left sum_distrib_right mult.assoc)
using *sum.swap* **by** *fastforce*

lemma *scalar_matrix_assoc*:
fixes $A :: ('a::real\_algebra\_1)\hat{}'m\hat{}'n$
shows $k *_R (A ** B) = (k *_R A) ** B$
by (*simp add*: *matrix_matrix_mult_def sum_distrib_left mult_ac vec_eq_iff scaleR_sum_right*)

lemma *matrix_scalar_ac*:
fixes $A :: ('a::real\_algebra\_1)\hat{}'m\hat{}'n$
shows $A ** (k *_R B) = k *_R A ** B$
by (*simp add*: *matrix_matrix_mult_def sum_distrib_left mult_ac vec_eq_iff*)

lemma *matrix_vector_mul_lid* [*simp*]: $mat\ 1 *v x = (x::'a::semiring\_1\ \hat{}\ 'n)$
apply (*vector matrix_vector_mult_def mat_def*)
apply (*simp add*: *if_distrib if_distribR cong del*: *if_weak_cong*)
done

lemma *vector_matrix_mul_rid* [*simp*]:
fixes $v :: ('a::semiring\_1)\hat{}'n$
shows $v\ v* mat\ 1 = v$
apply (*vector vector_matrix_mult_def mat_def*)
apply (*simp add*: *if_distrib if_distribR cong del*: *if_weak_cong*)
done

lemma *matrix_transpose_mul*:
$transpose(A ** B) = transpose\ B ** transpose\ (A::'a::comm\_semiring\_1\hat{}\_\hat{}\_)$
by (*simp add*: *matrix_matrix_mult_def transpose_def vec_eq_iff mult.commute*)

lemma *matrix_mult_transpose_dot_column*:
shows $transpose\ A ** A = (\chi\ i\ j.\ inner\ (column\ i\ A)\ (column\ j\ A))$
by (*simp add*: *matrix_matrix_mult_def vec_eq_iff transpose_def column_def inner_vec_def*)

lemma *matrix_mult_transpose_dot_row*:
shows $A ** transpose\ A = (\chi\ i\ j.\ inner\ (row\ i\ A)\ (row\ j\ A))$
by (*simp add*: *matrix_matrix_mult_def vec_eq_iff transpose_def row_def inner_vec_def*)

lemma *matrix_eq*:
fixes $A\ B :: 'a::semiring\_1\ \hat{}\ 'n\ \hat{}\ 'm$
shows $A = B \longleftrightarrow (\forall x.\ A *v x = B *v x)$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?rhs*
then show *?lhs*
apply (*subst vec_eq_iff*)
apply (*clarsimp simp add*: *matrix_vector_mult_def if_distrib if_distribR vec_eq_iff cong*: *if_cong*)
apply (*erule_tac x=axis ia 1* **in** *allE*)
apply (*erule_tac x=i* **in** *allE*)

apply (*auto simp add*: *if_distrib if_distribR axis_def*
sum.delta[*OF finite*] *cong del*: *if_weak_cong*)
done
qed *auto*

lemma *matrix_vector_mul_component*: (*A* ∗*v* *x*)$*k* = *inner* (*A*$*k*) *x*
by (*simp add*: *matrix_vector_mult_def inner_vec_def*)

lemma *dot_lmul_matrix*: *inner* ((*x*::*real* ^_) *v*∗ *A*) *y* = *inner x* (*A* ∗*v* *y*)
apply (*simp add*: *inner_vec_def matrix_vector_mult_def vector_matrix_mult_def*
sum_distrib_right sum_distrib_left ac_simps)
apply (*subst sum.swap*)
apply *simp*
done

lemma *transpose_mat* [*simp*]: *transpose* (*mat n*) = *mat n*
by (*vector transpose_def mat_def*)

lemma *transpose_transpose* [*simp*]: *transpose*(*transpose A*) = *A*
by (*vector transpose_def*)

lemma *row_transpose* [*simp*]: *row i* (*transpose A*) = *column i A*
by (*simp add*: *row_def column_def transpose_def vec_eq_iff*)

lemma *column_transpose* [*simp*]: *column i* (*transpose A*) = *row i A*
by (*simp add*: *row_def column_def transpose_def vec_eq_iff*)

lemma *rows_transpose* [*simp*]: *rows*(*transpose A*) = *columns A*
by (*auto simp add*: *rows_def columns_def intro*: *set_eqI*)

lemma *columns_transpose* [*simp*]: *columns*(*transpose A*) = *rows A*
by (*metis transpose_transpose rows_transpose*)

lemma *transpose_scalar*: *transpose* (*k* ∗*R* *A*) = *k* ∗*R* *transpose A*
unfolding *transpose_def*
by (*simp add*: *vec_eq_iff*)

lemma *transpose_iff* [*iff*]: *transpose A* = *transpose B* ⟷ *A* = *B*
by (*metis transpose_transpose*)

lemma *matrix_mult_sum*:
(*A*::′*a*::*comm_semiring_1*^′*n*^′*m*) ∗*v* *x* = *sum* (λ*i*. (*x*$*i*) ∗*s* *column i A*) (*UNIV*::
′*n set*)
by (*simp add*: *matrix_vector_mult_def vec_eq_iff column_def mult.commute*)

lemma *vector_componentwise*:
(*x*::′*a*::*ring_1*^′*n*) = (χ *j*. ∑ *i*∈*UNIV*. (*x*$*i*) ∗ (*axis i 1* :: ′*a*^′*n*) $ *j*)
by (*simp add*: *axis_def if_distrib sum.If_cases vec_eq_iff*)

lemma *basis_expansion*: *sum* (λi. ($x\$i$) *∗s axis i 1*) *UNIV* = (x::('*a*::*ring_1*) ^'*n*)
by (*auto simp add*: *axis_def vec_eq_iff if_distrib sum.If_cases cong del*: *if_weak_cong*)

Correspondence between matrices and linear operators.

definition *matrix* :: ('*a*::{*plus*,*times*, *one*, *zero*}^'*m* ⇒ '*a* ^ '*n*) ⇒ '*a*^'*m*^'*n*
where *matrix f* = (χ *i j*. (*f*(*axis j 1*))$\$i$)

lemma *matrix_id_mat_1*: *matrix id* = *mat 1*
by (*simp add*: *mat_def matrix_def axis_def*)

lemma *matrix_scaleR*: (*matrix* ((∗$_R$) *r*)) = *mat r*
by (*simp add*: *mat_def matrix_def axis_def if_distrib cong*: *if_cong*)

lemma *matrix_vector_mul_linear*[*intro*, *simp*]: *linear* (λx. *A ∗v* (x::'*a*::*real_algebra_1* ^ _))
by (*simp add*: *linear_iff matrix_vector_mult_def vec_eq_iff field_simps sum_distrib_left sum.distrib scaleR_right.sum*)

lemma *vector_matrix_left_distrib* [*algebra_simps*]:
shows (*x* + *y*) *v∗ A* = *x v∗ A* + *y v∗ A*
unfolding *vector_matrix_mult_def*
by (*simp add*: *algebra_simps sum.distrib vec_eq_iff*)

lemma *vector_matrix_mult_diff_distrib* [*algebra_simps*]:
fixes *A* :: '*a*::*ring_1*^'*n*^'*m*
shows (*x* − *y*) *v∗ A* = *x v∗ A* − *y v∗ A*
by (*vector vector_matrix_mult_def sum_subtractf left_diff_distrib*)

lemma *matrix_vector_right_distrib* [*algebra_simps*]:
A ∗v (*x* + *y*) = *A ∗v x* + *A ∗v y*
by (*vector matrix_vector_mult_def sum.distrib distrib_left*)

lemma *matrix_vector_mult_diff_distrib* [*algebra_simps*]:
fixes *A* :: '*a*::*ring_1*^'*n*^'*m*
shows *A ∗v* (*x* − *y*) = *A ∗v x* − *A ∗v y*
by (*vector matrix_vector_mult_def sum_subtractf right_diff_distrib*)

lemma *matrix_vector_mult_scaleR*[*algebra_simps*]:
fixes *A* :: *real*^'*n*^'*m*
shows *A ∗v* (*c* ∗$_R$ *x*) = *c* ∗$_R$ (*A ∗v x*)
using *linear_iff matrix_vector_mul_linear* **by** *blast*

lemma *matrix_vector_mult_0_right* [*simp*]: *A ∗v 0* = *0*
by (*simp add*: *matrix_vector_mult_def vec_eq_iff*)

lemma *matrix_vector_mult_0* [*simp*]: *0 ∗v w* = *0*
by (*simp add*: *matrix_vector_mult_def vec_eq_iff*)

lemma *matrix_vector_mult_add_rdistrib* [*algebra_simps*]:

$(A + B) *v x = (A *v x) + (B *v x)$
by (*vector matrix_vector_mult_def sum.distrib distrib_right*)

lemma *matrix_vector_mult_diff_rdistrib* [*algebra_simps*]:
fixes $A :: 'a :: ring\_1^{'}n^{'}m$
shows $(A - B) *v x = (A *v x) - (B *v x)$
by (*vector matrix_vector_mult_def sum_subtractf left_diff_distrib*)

lemma *vector_matrix_mult_add_rdistrib* [*algebra_simps*]:
$x\ v* (A + B) = (x\ v* A) + (x\ v* B)$
by (*vector vector_matrix_mult_def sum.distrib distrib_left*)

lemma *vector_matrix_mult_diff_rdistrib* [*algebra_simps*]:
fixes $A :: 'a :: ring\_1^{'}n^{'}m$
shows $x\ v* (A - B) = (x\ v* A) - (x\ v* B)$
by (*vector vector_matrix_mult_def sum_subtractf right_diff_distrib*)

lemma *matrix_vector_column*:
$(A::'a::comm\_semiring\_1^{'}n^{'}\_) *v x = sum\ (\lambda i.\ (x\$i) *s ((transpose\ A)\$i))\ (UNIV::'n\ set)$
by (*simp add*: *matrix_vector_mult_def transpose_def vec_eq_iff mult.commute*)

1.10.14 Inverse matrices (not necessarily square)

definition
$invertible(A::'a::semiring\_1^{'}n^{'}m) \longleftrightarrow (\exists A'::'a^{'}m^{'}n.\ A ** A' = mat\ 1 \wedge A' ** A = mat\ 1)$

definition
$matrix\_inv(A::'a::semiring\_1^{'}n^{'}m) =$
$(SOME\ A'::'a^{'}m^{'}n.\ A ** A' = mat\ 1 \wedge A' ** A = mat\ 1)$

lemma *inj_matrix_vector_mult*:
fixes $A::'a::field^{'}n^{'}m$
assumes *invertible A*
shows *inj* $((*v)\ A)$
by (*metis assms inj_on_inverseI invertible_def matrix_vector_mul_assoc matrix_vector_mul_lid*)

lemma *scalar_invertible*:
fixes $A :: ('a::real\_algebra\_1)^{'}m^{'}n$
assumes $k \neq 0$ **and** *invertible A*
shows *invertible* $(k *_R A)$
proof −
obtain A' **where** $A ** A' = mat\ 1$ **and** $A' ** A = mat\ 1$
using *assms* **unfolding** *invertible_def* **by** *auto*
with ‹$k \neq 0$›
have $(k *_R A) ** ((1/k) *_R A') = mat\ 1\ ((1/k) *_R A') ** (k *_R A) = mat\ 1$
by (*simp_all add*: *assms matrix_scalar_ac*)

```
  thus invertible (k *R A)
    unfolding invertible_def by auto
qed

lemma scalar_invertible_iff:
  fixes A :: ('a::real_algebra_1)^'m^'n
  assumes k ≠ 0 and invertible A
  shows invertible (k *R A) ⟷ k ≠ 0 ∧ invertible A
  by (simp add: assms scalar_invertible)

lemma vector_transpose_matrix [simp]: x v* transpose A = A *v (x:: 'a::{comm_semiring_1}^'n)
  unfolding transpose_def vector_matrix_mult_def matrix_vector_mult_def
  by (simp add: mult.commute)

lemma transpose_matrix_vector [simp]: transpose A *v x = x v* (A:: 'a::{comm_semiring_1}^'m^'n)
  unfolding transpose_def vector_matrix_mult_def matrix_vector_mult_def
  by (simp add: mult.commute)

lemma vector_scalar_commute:
  fixes A :: 'a::{field}^'m^'n
  shows A *v (c *s x) = c *s (A *v x)
 by (simp add: vector_scalar_mult_def matrix_vector_mult_def mult_ac sum_distrib_left)

lemma scalar_vector_matrix_assoc:
  fixes k :: 'a::{field} and x :: 'a::{field}^'n and A :: 'a^'m^'n
  shows (k *s x) v* A = k *s (x v* A)
  by (metis transpose_matrix_vector vector_scalar_commute)

lemma vector_matrix_mult_0 [simp]: 0 v* A = 0
  unfolding vector_matrix_mult_def by (simp add: zero_vec_def)

lemma vector_matrix_mult_0_right [simp]: x v* 0 = 0
  unfolding vector_matrix_mult_def by (simp add: zero_vec_def)

lemma scaleR_vector_matrix_assoc:
  fixes k :: real and x :: real^'n and A :: real^'m^'n
  shows (k *R x) v* A = k *R (x v* A)
  by (metis matrix_vector_mult_scaleR transpose_matrix_vector)

lemma vector_scaleR_matrix_ac:
  fixes k :: real and x :: real^'n and A :: real^'m^'n
  shows x v* (k *R A) = k *R (x v* A)
proof -
  have x v* (k *R A) = (k *R x) v* A
    by (simp add: vector_matrix_mult_def algebra_simps)
  with scaleR_vector_matrix_assoc
  show x v* (k *R A) = k *R (x v* A)
    by auto
qed
```

end

1.11 Linear Algebra on Finite Cartesian Products

theory *Cartesian_Space*
imports
HOL−Combinatorics.Transposition
Finite_Cartesian_Product
Linear_Algebra
begin

1.11.1 Type ('a, 'n) vec and fields as vector spaces

definition *cart_basis* = {*axis i 1* | *i*. *i*∈*UNIV*}

lemma *finite_cart_basis*: *finite* (*cart_basis*) **unfolding** *cart_basis_def*
using *finite_Atleast_Atmost_nat* **by** *fastforce*

lemma *card_cart_basis*: *card* (*cart_basis*::('*a*::*zero_neq_one*^'*i*) *set*) = *CARD*('*i*)
unfolding *cart_basis_def Setcompr_eq_image*
by (*rule card_image*) (*auto simp*: *inj_on_def axis_eq_axis*)

interpretation *vec*: *vector_space* (∗*s*)
by *unfold_locales* (*vector algebra_simps*)+

lemma *independent_cart_basis*: *vec.independent* (*cart_basis*)
proof (*rule vec.independent_if_scalars_zero*)
show *finite* (*cart_basis*) **using** *finite_cart_basis* **.**
fix *f*::('*a*, '*b*) *vec* ⇒ '*a* **and** *x*::('*a*, '*b*) *vec*
assume *eq_0*: (∑ *x*∈*cart_basis*. *f x* ∗*s x*) = *0* **and** *x_in*: *x* ∈ *cart_basis*
obtain *i* **where** *x*: *x* = *axis i 1* **using** *x_in* **unfolding** *cart_basis_def* **by** *auto*
have *sum_eq_0*: (∑ *x*∈(*cart_basis*) − {*x*}. *f x* ∗ (*x* $ *i*)) = *0*
proof (*intro sum.neutral ballI*)
fix *y* **assume** *y*: *y* ∈ *cart_basis* − {*x*}
obtain *a* **where** *a*: *y* = *axis a 1* **and** *a_not_i*: *a* ≠ *i*
using *y x* **unfolding** *cart_basis_def* **by** *auto*
have *y* $ *i* = *0* **unfolding** *a axis_def* **using** *a_not_i* **by** *auto*
thus *f y* ∗ *y* $ *i* = *0* **by** *simp*
qed
have *0* = (∑ *x*∈*cart_basis*. *f x* ∗*s x*) $ *i* **using** *eq_0* **by** *simp*
also have ... = (∑ *x*∈*cart_basis*. (*f x* ∗*s x*) $ *i*) **unfolding** *sum_component* **..**
also have ... = (∑ *x*∈*cart_basis*. *f x* ∗ (*x* $ *i*)) **unfolding** *vector_smult_component*
..
also have ... = *f x* ∗ (*x* $ *i*) + (∑ *x*∈(*cart_basis*) − {*x*}. *f x* ∗ (*x* $ *i*))
by (*rule sum.remove*[*OF finite_cart_basis x_in*])
also have ... = *f x* ∗ (*x* $ *i*) **unfolding** *sum_eq_0* **by** *simp*
also have ... = *f x* **unfolding** *x axis_def* **by** *auto*
finally show *f x* = *0* **..**

qed

lemma *span_cart_basis* [*simp*]: *vec.span* (*cart_basis*) = *UNIV*
proof −
have *x* ∈ *vec.span cart_basis* **for** *x* :: ('*a*, '*b*) *vec*
proof −
let *?f*=λ*v*. *x* $ (*THE i*. *v* = *axis i 1*)
have *x* $ *i* = (∑ *v*∈*cart_basis*. *x* $ (*THE i*. *v* = *axis i 1*) ∗*s v*) $ *i* **for** *i*::'*b*
proof −
let *?w* = *axis i* (*1*::'*a*)
have *the_eq_i*: (*THE a*. *?w* = *axis a 1*) = *i*
by (*rule the_equality*, *auto simp*: *axis_eq_axis*)
have *sum_eq_0*: (∑ *v*∈(*cart_basis*) − {*?w*}. *x* $ (*THE i*. *v* = *axis i 1*) ∗ *v* $ *i*) = *0*
proof (*intro sum.neutral ballI*)
fix *y*:: ('*a*, '*b*) *vec*
assume *y*: *y* ∈ *cart_basis* − {*?w*}
obtain *j* **where** *j*: *y* = *axis j 1* **and** *i_not_j*: *i* ≠ *j*
using *y* **unfolding** *cart_basis_def* **by** *auto*
have *the_eq_j*: (*THE i*. *y* = *axis i 1*) = *j*
by (*simp add*: *axis_eq_axis j*)
show *x* $ (*THE i*. *y* = *axis i 1*) ∗ *y* $ *i* = *0*
by (*simp add*: *axis_def i_not_j j*)
qed
have (∑ *v*∈*cart_basis*. *x* $ (*THE i*. *v* = *axis i 1*) ∗*s v*) $ *i*
= (∑ *v*∈*cart_basis*. *x* $ (*THE i*. *v* = *axis i 1*) ∗ *v* $ *i*)
by *force*
also have ... = *x* $ (*THE a*. *?w* = *axis a 1*) ∗ *?w* $ *i* + (∑ *v*∈(*cart_basis*) − {*?w*}. *x* $ (*THE i*. *v* = *axis i 1*) ∗ *v* $ *i*)
by (*rule sum.remove*[*OF finite_cart_basis*], *auto simp add*: *cart_basis_def*)
also have ... = *x* $ (*THE a*. *?w* = *axis a 1*) ∗ *?w* $ *i*
unfolding *sum_eq_0* **by** *simp*
also have ... = *x* $ *i*
unfolding *the_eq_i* **unfolding** *axis_def* **by** *auto*
finally show *?thesis* **by** *simp*
qed
then show *x* ∈ *vec.span* (*cart_basis*)
by (*metis* (*no_types*, *lifting*) *vec.span_base vec.span_scale vec.span_sum vec_eq_iff*)
qed
then show *?thesis* **by** *auto*
qed

interpretation *vec*: *finite_dimensional_vector_space* (∗*s*) *cart_basis*
by (*unfold_locales*, *auto simp add*: *finite_cart_basis independent_cart_basis span_cart_basis*)

lemma *matrix_vector_mul_linear_gen*[*intro*, *simp*]:

Vector_Spaces.linear (∗s) (∗s) ((∗v) A)
by *unfold_locales*
(*vector matrix_vector_mult_def sum.distrib algebra_simps*)+

lemma *span_vec_eq*: *vec.span X = span X*
and *dim_vec_eq*: *vec.dim X = dim X*
and *dependent_vec_eq*: *vec.dependent X = dependent X*
and *subspace_vec_eq*: *vec.subspace X = subspace X*
for *X*::(*real^'n*) *set*
unfolding *span_raw_def dim_raw_def dependent_raw_def subspace_raw_def*
by (*auto simp*: *scalar_mult_eq_scaleR*)

lemma *linear_componentwise*:
fixes *f*:: *'a*::*field ^'m* ⇒ *'a ^'n*
assumes *lf*: *Vector_Spaces.linear* (∗s) (∗s) *f*
shows (*f x*)$*j* = *sum* (λ*i*. (*x*$*i*) ∗ (*f* (*axis i 1*)$*j*)) (*UNIV* :: *'m set*) (**is** *?lhs* = *?rhs*)
proof −
interpret *lf*: *Vector_Spaces.linear* (∗s) (∗s) *f*
using *lf* **.**
let *?M* = (*UNIV* :: *'m set*)
let *?N* = (*UNIV* :: *'n set*)
have *fM*: *finite ?M* **by** *simp*
have *?rhs* = (*sum* (λ*i*. (*x*$*i*) ∗s (*f* (*axis i 1*))) *?M*)$*j*
unfolding *sum_component* **by** *simp*
then show *?thesis*
unfolding *lf.sum*[*symmetric*] *lf.scale*[*symmetric*]
unfolding *basis_expansion* **by** *auto*
qed

interpretation *vec*: *Vector_Spaces.linear* (∗s) (∗s) (∗v) *A*
using *matrix_vector_mul_linear_gen***.**

interpretation *vec*: *finite_dimensional_vector_space_pair* (∗s) *cart_basis* (∗s) *cart_basis* **..**

lemma *matrix_works*:
assumes *lf*: *Vector_Spaces.linear* (∗s) (∗s) *f*
shows *matrix f* ∗v *x* = *f* (*x*::*'a*::*field ^'n*)
proof −
have ∀ *i*. (∑ *j*∈*UNIV*. *x* $ *j* ∗ *f* (*axis j 1*) $ *i*) = *f x* $ *i*
by (*simp add*: *Cartesian_Space.linear_componentwise lf*)
then show *?thesis*
by (*simp add*: *matrix_def matrix_vector_mult_def vec_eq_iff mult.commute*)
qed

lemma *matrix_of_matrix_vector_mul*[*simp*]: *matrix*(λ*x*. *A* ∗v (*x* :: *'a*::*field ^'n*)) = *A*
by (*simp add*: *matrix_eq matrix_works*)

lemma *matrix_compose_gen*:
assumes *lf*: *Vector_Spaces.linear* (∗*s*) (∗*s*) (*f*::*'a*::{*field*}^*'n* ⇒ *'a*^*'m*)
and *lg*: *Vector_Spaces.linear* (∗*s*) (∗*s*) (*g*::*'a*^*'m* ⇒ *'a*^_)
shows *matrix* (*g o f*) = *matrix g* ∗∗ *matrix f*
using *lf lg Vector_Spaces.linear_compose*[*OF lf lg*] *matrix_works*[*OF Vector_Spaces.linear_compose*[*O*
lf lg]]
by (*simp add*: *matrix_eq matrix_works matrix_vector_mul_assoc*[*symmetric*]
o_def)

lemma *matrix_compose*:
assumes *linear* (*f*::*real*^*'n* ⇒ *real*^*'m*) *linear* (*g*::*real*^*'m* ⇒ *real*^_)
shows *matrix* (*g o f*) = *matrix g* ∗∗ *matrix f*
using *matrix_compose_gen*[*of f g*] *assms*
by (*simp add*: *linear_def scalar_mult_eq_scaleR*)

lemma *left_invertible_transpose*:
(∃(*B*). *B* ∗∗ *transpose* (*A*) = *mat* (*1*::*'a*::*comm_semiring_1*)) ⟷ (∃(*B*). *A* ∗∗
B = *mat 1*)
by (*metis matrix_transpose_mul transpose_mat transpose_transpose*)

lemma *right_invertible_transpose*:
(∃(*B*). *transpose* (*A*) ∗∗ *B* = *mat* (*1*::*'a*::*comm_semiring_1*)) ⟷ (∃(*B*). *B* ∗∗
A = *mat 1*)
by (*metis matrix_transpose_mul transpose_mat transpose_transpose*)

lemma *linear_matrix_vector_mul_eq*:
Vector_Spaces.linear (∗*s*) (∗*s*) *f* ⟷ *linear* (*f* :: *real*^*'n* ⇒ *real* ^*'m*)
by (*simp add*: *scalar_mult_eq_scaleR linear_def*)

lemma *matrix_vector_mul*[*simp*]:
Vector_Spaces.linear (∗*s*) (∗*s*) *g* ⟹ (λ*y*. *matrix g* ∗*v y*) = *g*
linear f ⟹ (λ*x*. *matrix f* ∗*v x*) = *f*
bounded_linear f ⟹ (λ*x*. *matrix f* ∗*v x*) = *f*
for *f* :: *real*^*'n* ⇒ *real* ^*'m*
by (*simp_all add*: *ext matrix_works linear_matrix_vector_mul_eq linear_linear*)

lemma *matrix_left_invertible_injective*:
fixes *A* :: *'a*::*field*^*'n*^*'m*
shows (∃ *B*. *B* ∗∗ *A* = *mat 1*) ⟷ *inj* ((∗*v*) *A*)
proof *safe*
fix *B*
assume *B*: *B* ∗∗ *A* = *mat 1*
show *inj* ((∗*v*) *A*)
unfolding *inj_on_def*
by (*metis B matrix_vector_mul_assoc matrix_vector_mul_lid*)
next
assume *inj* ((∗*v*) *A*)
from *vec.linear_injective_left_inverse*[*OF matrix_vector_mul_linear_gen this*]

obtain *g* **where** *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* **and** *g* ∘ (∗*v*) *A* = *id*
by *blast*
then have *matrix g* ∗∗ *A* = *mat 1*
by (*metis matrix_compose_gen matrix_id_mat_1 matrix_of_matrix_vector_mul vec.linear_axioms*)
then show ∃ *B*. *B* ∗∗ *A* = *mat 1*
by *metis*
qed

lemma *matrix_left_invertible_ker*:
(∃ *B*. (*B*::′*a*::{*field*} ^′*m*^′*n*) ∗∗ (*A*::′*a*::{*field*}^′*n*^′*m*) = *mat 1*) ⟷ (∀ *x*. *A* ∗*v* *x* = *0* ⟶ *x* = *0*)
by (*simp add*: *matrix_left_invertible_injective vec.inj_iff_eq_0*)

lemma *matrix_right_invertible_surjective*:
(∃ *B*. (*A*::′*a*::*field*^′*n*^′*m*) ∗∗ (*B*::′*a*::*field*^′*m*^′*n*) = *mat 1*) ⟷ *surj* (λ*x*. *A* ∗*v* *x*)
proof −
have ⋀*B x*. *A* ∗∗ *B* = *mat 1* ⟹ ∃ *y*. *x* = *A* ∗*v* *y*
by (*metis matrix_vector_mul_assoc matrix_vector_mul_lid*)
moreover have ∀ *x*. ∃ *xa*. *x* = *A* ∗*v* *xa* ⟹ ∃ *B*. *A* ∗∗ *B* = *mat 1*
by (*metis* (*mono_tags*, *lifting*) *matrix_compose_gen matrix_id_mat_1 matrix_of_matrix_vector_mul surj_def vec.linear_axioms vec.linear_surjective_right_inverse*)
ultimately show *?thesis*
by (*auto simp*: *image_def set_eq_iff*)
qed

lemma *matrix_left_invertible_independent_columns*:
fixes *A* :: ′*a*::{*field*}^′*n*^′*m*
shows (∃ (*B*::′*a* ^′*m*^′*n*). *B* ∗∗ *A* = *mat 1*) ⟷
(∀ *c*. *sum* (λ*i*. *c i* ∗*s* *column i A*) (*UNIV* :: ′*n set*) = *0* ⟶ (∀ *i*. *c i* = *0*))
(**is** *?lhs* ⟷ *?rhs*)
proof −
let *?U* = *UNIV* :: ′*n set*
have *c i* = *0*
if ∀ *x*. *A* ∗*v* *x* = *0* ⟶ *x* = *0 sum* (λ*i*. *c i* ∗*s* *column i A*) *?U* = *0* **for** *c i*
by (*metis* (*no_types*) *UNIV_I matrix_mult_sum vec_lambda_eta vec_nth_cases zero_vec_def that*)
moreover have *x* = *0* **if** *A* ∗*v* *x* = *0 ?rhs* **for** *x*
by (*metis* (*full_types*) *matrix_mult_sum that vec_eq_iff zero_index*)
ultimately show *?thesis*
unfolding *matrix_left_invertible_ker* **by** *auto*
qed

lemma *matrix_right_invertible_independent_rows*:
fixes *A* :: ′*a*::{*field*}^′*n*^′*m*
shows (∃ (*B*::′*a*^′*m*^′*n*). *A* ∗∗ *B* = *mat 1*) ⟷
(∀ *c*. *sum* (λ*i*::′*m*. *c i* ∗*s* *row i A*) *UNIV* = *0* ⟶ (∀ *i*. *c i* = *0*))
by (*simp add*: *matrix_left_invertible_independent_columns flip*: *left_invertible_transpose*)

```
lemma matrix_right_invertible_span_columns:
  (∃(B::'a::field ^'n^'m). (A::'a ^'m^'n) ** B = mat 1) ⟷
    vec.span (columns A) = UNIV (is ?lhs = ?rhs)
proof −
  let ?U = UNIV :: 'm set
  have fU: finite ?U by simp
  have lhseq: ?lhs ⟷ (∀ y. ∃(x::'a^'m). sum (λi. (x$i) *s column i A) ?U = y)
    unfolding matrix_right_invertible_surjective matrix_mult_sum surj_def
    by (simp add: eq_commute)
  have rhseq: ?rhs ⟷ (∀ x. x ∈ vec.span (columns A)) by blast
  { assume h: ?lhs
    { fix x:: 'a ^'n
      obtain y :: 'a ^'m where y: sum (λi. (y$i) *s column i A) ?U = x
        using h lhseq by blast
      then have x ∈ vec.span (columns A)
       by (metis (mono_tags, lifting) columns_def mem_Collect_eq vec.span_base
vec.span_scale vec.span_sum)
    }
    then have ?rhs unfolding rhseq by blast }
  moreover
  { assume h:?rhs
    let ?P = λ(y::'a ^'n). ∃(x::'a^'m). sum (λi. (x$i) *s column i A) ?U = y
    { fix y
      have y ∈ vec.span (columns A)
        unfolding h by blast
      then have ?P y
      proof (induction rule: vec.span_induct_alt)
        case base
        then show ?case
          by (metis (full_types) matrix_mult_sum matrix_vector_mult_0_right)
      next
        case (step c y1 y2)
        from step obtain i where i: i ∈ ?U y1 = column i A
          unfolding columns_def by blast
        obtain x:: 'a ^'m where x: sum (λi. (x$i) *s column i A) ?U = y2
          using step by blast
        let ?x = (χ j. if j = i then c + (x$i) else (x$j))::'a^'m
        show ?case
          proof (rule exI[where x= ?x], vector, auto simp add: i x[symmetric]
if_distrib distrib_left if_distribR cong del: if_weak_cong)
          fix j
          have th: ∀ xa ∈ ?U. (if xa = i then (c + (x$i)) * ((column xa A)$j)
              else (x$xa) * ((column xa A$j))) = (if xa = i then c * ((column i A)$j)
else 0) + ((x$xa) * ((column xa A)$j))
            using i(1) by (simp add: field_simps)
          have sum (λxa. if xa = i then (c + (x$i)) * ((column xa A)$j)
              else (x$xa) * ((column xa A$j))) ?U = sum (λxa. (if xa = i then c *
((column i A)$j) else 0) + ((x$xa) * ((column xa A)$j))) ?U
            using th by force
```

```
      also have ... = sum (λxa. if xa = i then c * ((column i A)$j) else 0) ?U
+ sum (λxa. ((x$xa) * ((column xa A)$j))) ?U
        by (simp add: sum.distrib)
      also have ... = c * ((column i A)$j) + sum (λxa. ((x$xa) * ((column xa
A)$j))) ?U
        unfolding sum.delta[OF fU] using i(1) by simp
      finally show sum (λxa. if xa = i then (c + (x$i)) * ((column xa A)$j)
                                       else (x$xa) * ((column xa A$j))) ?U
                  = c * ((column i A)$j) + sum (λxa. ((x$xa) * ((column xa
A)$j))) ?U .
    qed
   qed
  }
  then have ?lhs unfolding lhseq ..
 }
 ultimately show ?thesis by blast
qed

lemma matrix_left_invertible_span_rows_gen:
 (∃(B::'a^'m^'n). B ** (A::'a::field^'n^'m) = mat 1) ⟷ vec.span (rows A) =
UNIV
 by (metis columns_transpose matrix_right_invertible_span_columns right_invertible_transpose)

lemma matrix_left_invertible_span_rows:
 (∃(B::real^'m^'n). B ** (A::real^'n^'m) = mat 1) ⟷ span (rows A) = UNIV
 using matrix_left_invertible_span_rows_gen[of A] by (simp add: span_vec_eq)

lemma matrix_left_right_inverse:
 fixes A A' :: 'a::{field}^'n^'n
 shows A ** A' = mat 1 ⟷ A' ** A = mat 1
proof −
 { fix A A' :: 'a ^'n^'n
  assume AA': A ** A' = mat 1
  have sA: surj ((*v) A)
   using AA' matrix_right_invertible_surjective by auto
  obtain f' :: 'a ^'n ⇒ 'a ^'n
   where f': Vector_Spaces.linear (*s) (*s) f' ∀ x. f' (A *v x) = x ∀ x. A *v f'
x = x
   using sA vec.linear_surjective_isomorphism by blast
  have matrix f' ** A = mat 1
   by (metis f' matrix_eq matrix_vector_mul_assoc matrix_vector_mul_lid
matrix_works)
  hence A' ** A = mat 1
   by (metis AA' matrix_mul_assoc matrix_mul_lid)
 }
 then show ?thesis by blast
qed

lemma invertible_left_inverse:
```

fixes $A :: 'a::\{field\}\hat{}'n\hat{}'n$
shows *invertible* $A \longleftrightarrow (\exists (B::'a\hat{}'n\hat{}'n).\ B ** A = mat\ 1)$
by (*metis invertible_def matrix_left_right_inverse*)

lemma *invertible_right_inverse*:
fixes $A :: 'a::\{field\}\hat{}'n\hat{}'n$
shows *invertible* $A \longleftrightarrow (\exists (B::'a\hat{}'n\hat{}'n).\ A** B = mat\ 1)$
by (*metis invertible_def matrix_left_right_inverse*)

lemma *invertible_mult*:
assumes *inv_A*: *invertible A*
and *inv_B*: *invertible B*
shows *invertible* $(A**B)$
proof −
obtain A' **where** AA': $A ** A' = mat\ 1$ **and** $A'A$: $A' ** A = mat\ 1$
using *inv_A* **unfolding** *invertible_def* **by** *blast*
obtain B' **where** BB': $B ** B' = mat\ 1$ **and** $B'B$: $B' ** B = mat\ 1$
using *inv_B* **unfolding** *invertible_def* **by** *blast*
have $A ** B ** (B' ** A') = mat\ 1$
by (*metis AA' BB' matrix_mul_assoc matrix_mul_rid*)
moreover have $B' ** A' ** (A ** B) = mat\ 1$
by (*metis A'A B'B matrix_mul_assoc matrix_mul_rid*)
ultimately show *?thesis*
using *invertible_def* **by** *blast*
qed

lemma *transpose_invertible*:
fixes $A :: real\hat{}'n\hat{}'n$
assumes *invertible A*
shows *invertible* (*transpose A*)
by (*meson assms invertible_def matrix_left_right_inverse right_invertible_transpose*)

lemma *matrix_scaleR_vector_ac*:
fixes $A :: real\hat{}('m::finite)\hat{}'n$
shows $A *v (k *_R v) = k *_R A *v v$
by (*metis matrix_vector_mult_scaleR transpose_scalar vector_scaleR_matrix_ac vector_transpose_matrix*)

lemma *scaleR_matrix_vector_assoc*:
fixes $A :: real\hat{}('m::finite)\hat{}'n$
shows $k *_R (A *v v) = k *_R A *v v$
by (*metis matrix_scaleR_vector_ac matrix_vector_mult_scaleR*)

1.11.2 Some interesting theorems and interpretations

locale *linear_first_finite_dimensional_vector_space* =
l?: *Vector_Spaces.linear scaleB scaleC f* +
B?: *finite_dimensional_vector_space scaleB BasisB*
for *scaleB* :: ('*a*::*field* => '*b*::*ab_group_add* => '*b*) (**infixr** ‹∗*b*› *75*)

and *scaleC* :: (*'a* => *'c*::*ab_group_add* => *'c*) (**infixr** ‹∗*c*› *75*)
and *BasisB* :: (*'b set*)
and *f* :: (*'b*=>*'c*)

lemma *vec_dim_card*: *vec.dim* (*UNIV*::(*'a*::{*field*}^*'n*) *set*) = *CARD* (*'n*)
by (*simp add*: *card_cart_basis*)

interpretation *vector_space_over_itself*: *vector_space* (∗) :: *'a*::*field* ⇒ *'a* ⇒ *'a*
by *unfold_locales* (*simp_all add*: *algebra_simps*)

lemmas [*simp del*] = *vector_space_over_itself.scale_scale*

interpretation *vector_space_over_itself*: *finite_dimensional_vector_space*
(∗) :: *'a*::*field* => *'a* => *'a* {*1*}
by *unfold_locales* (*auto simp*: *vector_space_over_itself.span_singleton*)

lemma *dimension_eq_1*[*code_unfold*]: *vector_space_over_itself.dimension TYPE*(*'a*::*field*)=
1
unfolding *vector_space_over_itself.dimension_def* **by** *simp*

lemma *dim_subset_UNIV_cart_gen*:
fixes *S* :: (*'a*::*field*^*'n*) *set*
shows *vec.dim S* ≤ *CARD*(*'n*)
by (*metis vec.dim_eq_full vec.dim_subset_UNIV vec.span_UNIV vec_dim_card*)

lemma *dim_subset_UNIV_cart*:
fixes *S* :: (*real*^*'n*) *set*
shows *dim S* ≤ *CARD*(*'n*)
using *dim_subset_UNIV_cart_gen*[*of S*] **by** (*simp add*: *dim_vec_eq*)

Two sometimes fruitful ways of looking at matrix-vector multiplication.

lemma *matrix_mult_dot*: *A* ∗*v x* = (χ *i*. *inner* (*A*$*i*) *x*)
by (*simp add*: *matrix_vector_mult_def inner_vec_def*)

lemma *adjoint_matrix*: *adjoint*(λ*x*. (*A*::*real*^*'n*^*'m*) ∗*v x*) = (λ*x*. *transpose A* ∗*v*
x)
by (*metis adjoint_unique dot_lmul_matrix vector_transpose_matrix*)

lemma *matrix_adjoint*:
assumes *lf*: *linear* (*f* :: *real*^*'n* ⇒ *real* ^*'m*)
shows *matrix*(*adjoint f*) = *transpose*(*matrix f*)
by (*metis adjoint_matrix assms matrix_of_matrix_vector_mul matrix_vector_mul*(*2*))

1.11.3 Rank of a matrix

Equivalence of row and column rank is taken from George Mackiw's paper, Mathematics Magazine 1995, p. 285.

lemma *matrix_vector_mult_in_columnspace_gen*:

fixes $A :: 'a::field^{\sim}'n^{\sim}'m$
shows $(A *v x) \in vec.span(columns\ A)$
unfolding *columns_def*
by (*metis* (*mono_tags*, *lifting*) *matrix_mult_sum mem_Collect_eq vec.span_base vec.span_scale vec.span_sum*)

lemma *matrix_vector_mult_in_columnspace*:
fixes $A :: real^{\sim}'n^{\sim}'m$
shows $(A *v x) \in span(columns\ A)$
using *matrix_vector_mult_in_columnspace_gen*[*of A x*] **by** (*simp add*: *span_vec_eq*)

lemma *subspace_orthogonal_to_vector*: *subspace* $\{y.\ orthogonal\ x\ y\}$
by (*simp add*: *subspace_def orthogonal_clauses*)

lemma *orthogonal_nullspace_rowspace*:
fixes $A :: real^{\sim}'n^{\sim}'m$
assumes *0*: $A *v x = 0$ **and** *y*: $y \in span(rows\ A)$
shows *orthogonal x y*
using *y*
proof (*induction rule*: *span_induct*)
case *base*
then show *?case*
by (*simp add*: *subspace_orthogonal_to_vector*)
next
case (*step v*)
then obtain *i* **where** $v = row\ i\ A$
by (*auto simp*: *rows_def*)
with *0* **show** *?case*
unfolding *orthogonal_def inner_vec_def matrix_vector_mult_def row_def*
by (*simp add*: *mult.commute*) (*metis* (*no_types*) *vec_lambda_beta zero_index*)
qed

lemma *nullspace_inter_rowspace*:
fixes $A :: real^{\sim}'n^{\sim}'m$
shows $A *v x = 0 \wedge x \in span(rows\ A) \longleftrightarrow x = 0$
using *orthogonal_nullspace_rowspace orthogonal_self span_zero matrix_vector_mult_0_right*
by *blast*

lemma *matrix_vector_mul_injective_on_rowspace*:
fixes $A :: real^{\sim}'n^{\sim}'m$
shows $[\![A *v x = A *v y;\ x \in span(rows\ A);\ y \in span(rows\ A)]\!] \Longrightarrow x = y$
using *nullspace_inter_rowspace* [*of A x−y*]
by (*metis diff_eq_diff_eq diff_self matrix_vector_mult_diff_distrib span_diff*)

definition $rank :: 'a::field^{\sim}'n^{\sim}'m => nat$
where *row_rank_def_gen*: $rank\ A \equiv vec.dim(rows\ A)$

lemma *row_rank_def*: $rank\ A = dim\ (rows\ A)$ **for** $A::real^{\sim}'n^{\sim}'m$
by (*auto simp*: *row_rank_def_gen dim_vec_eq*)

lemma *dim_rows_le_dim_columns*:
fixes $A :: real^{\wedge}'n^{\wedge}'m$
shows $dim(rows\ A) \leq dim(columns\ A)$
proof −
have $dim\ (span\ (rows\ A)) \leq dim\ (span\ (columns\ A))$
proof −
obtain B **where** *independent* B $span(rows\ A) \subseteq span\ B$
and B: $B \subseteq span(rows\ A) card\ B = dim\ (span(rows\ A))$
using *basis_exists* [*of* $span(rows\ A)$] **by** *metis*
with *span_subspace* **have** *eq*: $span\ B = span(rows\ A)$
by *auto*
then have *inj*: *inj_on* $((*v)\ A)\ (span\ B)$
by (*simp add*: *inj_on_def matrix_vector_mul_injective_on_rowspace*)
then have *ind*: *independent* $((*v)\ A\ `\ B)$
by (*rule linear_independent_injective_image* [*OF Finite_Cartesian_Product.matrix_vector_mul_linear* ‹*independent B*›])
have $dim\ (span\ (rows\ A)) \leq card\ ((*v)\ A\ `\ B)$
by (*metis* $B(2)$ *card_image inj inj_on_subset order.refl span_superset*)
also have $\ldots \leq dim\ (span\ (columns\ A))$
using _ *ind*
by (*rule independent_card_le_dim*) (*auto intro*!: *matrix_vector_mult_in_columnspace*)
finally show *?thesis* **.**
qed
then show *?thesis*
by (*simp*)
qed

lemma *column_rank_def*:
fixes $A :: real^{\wedge}'n^{\wedge}'m$
shows $rank\ A = dim(columns\ A)$
unfolding *row_rank_def*
by (*metis columns_transpose dim_rows_le_dim_columns le_antisym rows_transpose*)

lemma *rank_transpose*:
fixes $A :: real^{\wedge}'n^{\wedge}'m$
shows $rank(transpose\ A) = rank\ A$
by (*metis column_rank_def row_rank_def rows_transpose*)

lemma *matrix_vector_mult_basis*:
fixes $A :: real^{\wedge}'n^{\wedge}'m$
shows $A *v\ (axis\ k\ 1) = column\ k\ A$
by (*simp add*: *cart_eq_inner_axis column_def matrix_mult_dot*)

lemma *columns_image_basis*:
fixes $A :: real^{\wedge}'n^{\wedge}'m$
shows $columns\ A = (*v)\ A\ `\ (range\ (\lambda i.\ axis\ i\ 1))$
by (*force simp*: *columns_def matrix_vector_mult_basis* [*symmetric*])

lemma *rank_dim_range*:
fixes $A :: real^{\sim 'n^{\sim 'm}}$
shows $rank\ A = dim(range\ (\lambda x.\ A\ *v\ x))$
unfolding *column_rank_def*
by (*smt* (*verit*, *best*) *columns_image_basis dim_span image_subset_iff iso_tuple_UNIV_I matrix_vector_mult_in_columnspace span_eq*)

lemma *rank_bound*:
fixes $A :: real^{\sim 'n^{\sim 'm}}$
shows $rank\ A \leq min\ CARD('m)\ (CARD('n))$
by (*metis* (*mono_tags*, *lifting*) *dim_subset_UNIV_cart min.bounded_iff column_rank_def row_rank_def*)

lemma *full_rank_injective*:
fixes $A :: real^{\sim 'n^{\sim 'm}}$
shows $rank\ A = CARD('n) \longleftrightarrow inj\ ((*v)\ A)$
by (*simp add*: *matrix_left_invertible_injective* [*symmetric*] *matrix_left_invertible_span_rows row_rank_def*
dim_eq_full [*symmetric*] *card_cart_basis vec.dimension_def*)

lemma *full_rank_surjective*:
fixes $A :: real^{\sim 'n^{\sim 'm}}$
shows $rank\ A = CARD('m) \longleftrightarrow surj\ ((*v)\ A)$
by (*metis* (*no_types*, *opaque_lifting*) *dim_eq_full dim_vec_eq rank_dim_range span_vec_eq vec.span_UNIV vec.span_image vec_dim_card*)

lemma *rank_I*: $rank(mat\ 1::real^{\sim 'n^{\sim 'n}}) = CARD('n)$
by (*simp add*: *full_rank_injective inj_on_def*)

lemma *less_rank_noninjective*:
fixes $A :: real^{\sim 'n^{\sim 'm}}$
shows $rank\ A < CARD('n) \longleftrightarrow \neg\ inj\ ((*v)\ A)$
using *less_le rank_bound* **by** (*auto simp*: *full_rank_injective* [*symmetric*])

lemma *matrix_nonfull_linear_equations_eq*:
fixes $A :: real^{\sim 'n^{\sim 'm}}$
shows $(\exists x.\ (x \neq 0) \wedge A\ *v\ x = 0) \longleftrightarrow rank\ A \neq CARD('n)$
by (*meson matrix_left_invertible_injective full_rank_injective matrix_left_invertible_ker*)

lemma *rank_eq_0*: $rank\ A = 0 \longleftrightarrow A = 0$ **and** *rank_0* [*simp*]: $rank\ (0::real^{\sim 'n^{\sim 'm}}) = 0$
for $A :: real^{\sim 'n^{\sim 'm}}$
by (*auto simp*: *rank_dim_range matrix_eq*)

lemma *rank_mul_le_right*:
fixes $A :: real^{\sim 'n^{\sim 'm}}$ **and** $B :: real^{\sim 'p^{\sim 'n}}$
shows $rank(A\ **\ B) \leq rank\ B$
proof −
have $rank(A\ **\ B) \leq dim\ ((*v)\ A\ `\ range\ ((*v)\ B))$

```
  by (auto simp: rank_dim_range image_comp o_def matrix_vector_mul_assoc)
 also have ... ≤ rank B
  by (simp add: rank_dim_range dim_image_le)
 finally show ?thesis .
qed

lemma rank_mul_le_left:
 fixes A :: real^'n^'m and B :: real^'p^'n
 shows rank(A ** B) ≤ rank A
 by (metis matrix_transpose_mul rank_mul_le_right rank_transpose)
```

1.11.4 Lemmas for working on *real^1/2/3/4*

```
lemma exhaust_2:
 fixes x :: 2
 shows x = 1 ∨ x = 2
proof (induct x)
 case (of_int z)
 then have z = 0 | z = 1
  by fastforce
 then show ?case
  by auto
qed

lemma forall_2: (∀ i::2. P i) ⟷ P 1 ∧ P 2
 by (metis exhaust_2)

lemma exhaust_3:
 fixes x :: 3
 shows x = 1 ∨ x = 2 ∨ x = 3
proof (induct x)
 case (of_int z)
 then have z = 0 ∨ z = 1 ∨ z = 2 by fastforce
 then show ?case by auto
qed

lemma forall_3: (∀ i::3. P i) ⟷ P 1 ∧ P 2 ∧ P 3
 by (metis exhaust_3)

lemma exhaust_4:
 fixes x :: 4
 shows x = 1 ∨ x = 2 ∨ x = 3 ∨ x = 4
proof (induct x)
 case (of_int z)
 then have z = 0 ∨ z = 1 ∨ z = 2 ∨ z = 3 by fastforce
 then show ?case by auto
qed

lemma forall_4: (∀ i::4. P i) ⟷ P 1 ∧ P 2 ∧ P 3 ∧ P 4
```

by (*metis exhaust_4*)

lemma *UNIV_1* [*simp*]: *UNIV* = {*1*::*1*}
by (*auto simp add*: *num1_eq_iff*)

lemma *UNIV_2*: *UNIV* = {*1*::*2*, *2*::*2*}
using *exhaust_2* **by** *auto*

lemma *UNIV_3*: *UNIV* = {*1*::*3*, *2*::*3*, *3*::*3*}
using *exhaust_3* **by** *auto*

lemma *UNIV_4*: *UNIV* = {*1*::*4*, *2*::*4*, *3*::*4*, *4*::*4*}
using *exhaust_4* **by** *auto*

lemma *sum_1*: *sum f* (*UNIV*::*1 set*) = *f 1*
unfolding *UNIV_1* **by** *simp*

lemma *sum_2*: *sum f* (*UNIV*::*2 set*) = *f 1* + *f 2*
unfolding *UNIV_2* **by** *simp*

lemma *sum_3*: *sum f* (*UNIV*::*3 set*) = *f 1* + *f 2* + *f 3*
unfolding *UNIV_3* **by** (*simp add*: *ac_simps*)

lemma *sum_4*: *sum f* (*UNIV*::*4 set*) = *f 1* + *f 2* + *f 3* + *f 4*
unfolding *UNIV_4* **by** (*simp add*: *ac_simps*)

1.11.5 The collapse of the general concepts to dimension one

lemma *vector_one*: (*x*::*'a* ^*1*) = (χ *i*. (*x*$*1*))
by (*simp add*: *vec_eq_iff*)

lemma *forall_one*: (∀(*x*::*'a* ^*1*). *P x*) ⟷ (∀*x*. *P*(χ *i*. *x*))
by (*metis vector_one*)

lemma *norm_vector_1*: *norm* (*x* :: _^*1*) = *norm* (*x*$*1*)
by (*simp add*: *norm_vec_def*)

lemma *dist_vector_1*:
fixes *x* :: *'a*::*real_normed_vector*^*1*
shows *dist x y* = *dist* (*x*$*1*) (*y*$*1*)
by (*simp add*: *dist_norm norm_vector_1*)

lemma *norm_real*: *norm*(*x*::*real* ^ *1*) = |*x*$*1*|
by (*simp add*: *norm_vector_1*)

lemma *dist_real*: *dist*(*x*::*real* ^ *1*) *y* = |(*x*$*1*) − (*y*$*1*)|
by (*auto simp add*: *norm_real dist_norm*)

1.11.6 Routine results connecting the types (*real*, *1*) *vec* and *real*

lemma *vector_one_nth* [*simp*]:
fixes $x :: 'a\hat{}1$ **shows** *vec* $(x \$ 1) = x$
by (*metis vec_def vector_one*)

lemma *tendsto_at_within_vector_1*:
fixes $S :: 'a :: metric\_space\ set$
assumes $(f \longrightarrow fx)$ $(at\ x\ within\ S)$
shows $((\lambda y::'a\hat{}1.\ \chi\ i.\ f\ (y \$ 1)) \longrightarrow (vec\ fx::'a\hat{}1))$ $(at\ (vec\ x)\ within\ vec\ `\ S)$
proof (*rule topological_tendstoI*)
fix $T :: ('a\hat{}1)\ set$
assume *open T vec fx* $\in T$
have $\forall_F\ x\ in\ at\ x\ within\ S.\ f\ x \in (\lambda x.\ x \$ 1)\ `\ T$
using ‹*open T*› ‹*vec fx* $\in T$› *assms open_image_vec_nth tendsto_def* **by** *fastforce*
then show $\forall_F\ x::'a\hat{}1\ in\ at\ (vec\ x)\ within\ vec\ `\ S.\ (\chi\ i.\ f\ (x \$ 1)) \in T$
unfolding *eventually_at dist_norm* [*symmetric*]
by (*rule ex_forward*)
(*use* ‹*open T*› **in**
‹*fastforce simp*: *dist_norm dist_vec_def L2_set_def image_iff vector_one open_vec_def*›)
qed

lemma *has_derivative_vector_1*:
assumes *der_g*: $(g\ has\_derivative\ (\lambda x.\ x * g'\ a))$ $(at\ a\ within\ S)$
shows $((\lambda x.\ vec\ (g\ (x \$ 1)))\ has\_derivative\ (*_R)\ (g'\ a))$
$(at\ ((vec\ a)::real\hat{}1)\ within\ vec\ `\ S)$
using *der_g*
apply (*clarsimp simp*: *Deriv.has_derivative_within bounded_linear_scaleR_right norm_vector_1*)
apply (*drule tendsto_at_within_vector_1*, *vector*)
apply (*auto simp*: *algebra_simps eventually_at tendsto_def*)
done

1.11.7 Explicit vector construction from lists

definition *vector* $l = (\chi\ i.\ foldr\ (\lambda x\ f\ n.\ fun\_upd\ (f\ (n+1))\ n\ x)\ l\ (\lambda n\ x.\ 0)\ 1\ i)$

lemma *vector_1* [*simp*]: $(vector[x]) \$1 = x$
unfolding *vector_def* **by** *simp*

lemma *vector_2* [*simp*]: $(vector[x,y]) \$1 = x$ $(vector[x,y] :: 'a\hat{}2)\$2 = (y::'a::zero)$
unfolding *vector_def* **by** *simp_all*

lemma *vector_3* [*simp*]:
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$1 = x$
$(vector\ [x,y,z] ::('a::zero)\hat{}3)\$2 = y$

(*vector* [*x*,*y*,*z*] ::('*a*::*zero*)^*3*)$*3* = *z*
unfolding *vector_def* **by** *simp_all*

lemma *forall_vector_1*: (∀ *v*::'*a*::*zero*^*1*. *P v*) ⟷ (∀ *x*. *P*(*vector*[*x*])) **by** (*metis vector_1 vector_one*)

lemma *forall_vector_2*: (∀ *v*::'*a*::*zero*^*2*. *P v*) ⟷ (∀ *x y*. *P*(*vector*[*x*, *y*]))
proof −
have *P v* **if** ⋀*x y*. *P* (*vector* [*x*, *y*]) **for** *v*
proof −
have *vector* [*v*$*1*, *v*$*2*] = *v*
by (*smt* (*verit*, *best*) *exhaust_2 vec_eq_iff vector_2*)
then show *?thesis*
by (*metis that*)
qed
then show *?thesis* **by** *auto*
qed

lemma *forall_vector_3*: (∀ *v*::'*a*::*zero*^*3*. *P v*) ⟷ (∀ *x y z*. *P*(*vector*[*x*, *y*, *z*]))
proof −
have *P v* **if** ⋀*x y z*. *P* (*vector* [*x*, *y*, *z*]) **for** *v*
proof −
have *vector* [*v*$*1*, *v*$*2*, *v*$*3*] = *v*
by (*smt* (*verit*, *best*) *exhaust_3 vec_eq_iff vector_3*)
then show *?thesis*
by (*metis that*)
qed
then show *?thesis* **by** *auto*
qed

1.11.8 lambda skolemization on cartesian products

lemma *lambda_skolem*: (∀ *i*. ∃ *x*. *P i x*) ⟷ (∃ *x*::'*a* ^ '*n*. ∀ *i*. *P i* (*x* $ *i*)) **by** (*metis vec_lambda_beta*)

The same result in terms of square matrices.

Considering an n-element vector as an n-by-1 or 1-by-n matrix.

definition *rowvector v* = (χ *i j*. (*v*$*j*))

definition *columnvector v* = (χ *i j*. (*v*$*i*))

lemma *transpose_columnvector*: *transpose*(*columnvector v*) = *rowvector v* **by** (*simp add*: *transpose_def rowvector_def columnvector_def vec_eq_iff*)

lemma *transpose_rowvector*: *transpose*(*rowvector v*) = *columnvector v* **by** (*simp add*: *transpose_def columnvector_def rowvector_def vec_eq_iff*)

lemma *dot_rowvector_columnvector*: *columnvector* (*A* ∗*v v*) = *A* ∗∗ *columnvector v*

by (*vector columnvector_def matrix_matrix_mult_def matrix_vector_mult_def*)

lemma *dot_matrix_product*:
(*x*::*real*^'*n*) ⋅ *y* = (((*rowvector x* ::*real*^'*n*^*1*) ** (*columnvector y* :: *real*^*1*^'*n*))$*1*)$*1*
by (*vector matrix_matrix_mult_def rowvector_def columnvector_def inner_vec_def*)

lemma *dot_matrix_vector_mul*:
fixes *A B* :: *real* ^'*n* ^'*n* **and** *x y* :: *real* ^'*n*
shows (*A* *v *x*) ⋅ (*B* *v *y*) =
(((*rowvector x* :: *real*^'*n*^*1*) ** ((*transpose A* ** *B*) ** (*columnvector y* :: *real* ^*1*^'*n*)))$*1*)$*1*
by (*metis dot_lmul_matrix dot_matrix_product dot_rowvector_columnvector matrix_mul_assoc vector_transpose_matrix*)

lemma *dim_substandard_cart*: *vec.dim* {*x*::'*a*::*field*^'*n*. ∀ *i*. *i* ∉ *d* ⟶ *x*$*i* = *0*} = *card d*
(**is** *vec.dim ?A* = _)
proof (*rule vec.dim_unique*)
let *?B* = ((λ*x*. *axis x 1*) ` *d*)
have *subset_basis*: *?B* ⊆ *cart_basis*
by (*auto simp*: *cart_basis_def*)
show *?B* ⊆ *?A*
by (*auto simp*: *axis_def*)
show *vec.independent* ((λ*x*. *axis x 1*) ` *d*)
using *subset_basis*
by (*rule vec.independent_mono*[*OF vec.independent_Basis*])
have *x* ∈ *vec.span ?B* **if** ∀ *i*. *i* ∉ *d* ⟶ *x* $ *i* = *0* **for** *x*::'*a*^'*n*
proof −
have *finite ?B*
using *subset_basis finite_cart_basis*
by (*rule finite_subset*)
have *x* = (∑ *i*∈*UNIV*. *x* $ *i* *s *axis i 1*)
by (*rule basis_expansion*[*symmetric*])
also have ... = (∑ *i*∈*d*. (*x* $ *i*) *s *axis i 1*)
by (*rule sum.mono_neutral_cong_right*) (*auto simp*: *that*)
also have ... ∈ *vec.span ?B*
by (*simp add*: *vec.span_sum vec.span_clauses*)
finally show *x* ∈ *vec.span ?B* **.**
qed
then show *?A* ⊆ *vec.span ?B* **by** *auto*
qed (*simp add*: *card_image inj_on_def axis_eq_axis*)

lemma *affinity_inverses*:
assumes *m0*: *m* ≠ (*0*::'*a*::*field*)
shows (λ*x*. *m* *s *x* + *c*) ∘ (λ*x*. *inverse*(*m*) *s *x* + (−(*inverse*(*m*) *s *c*))) = *id*
(λ*x*. *inverse*(*m*) *s *x* + (−(*inverse*(*m*) *s *c*))) ∘ (λ*x*. *m* *s *x* + *c*) = *id*
using *m0*
by (*auto simp add*: *fun_eq_iff vector_add_ldistrib diff_conv_add_uminus simp*

del: *add_uminus_conv_diff*)

lemma *vector_affinity_eq*:
assumes *m0*: (*m*::'*a*::*field*) ≠ *0*
shows *m* ∗*s* *x* + *c* = *y* ⟷ *x* = *inverse m* ∗*s* *y* + −(*inverse m* ∗*s* *c*)
proof
assume *h*: *m* ∗*s* *x* + *c* = *y*
hence *m* ∗*s* *x* = *y* − *c* **by** (*simp add*: *field_simps*)
hence *inverse m* ∗*s* (*m* ∗*s* *x*) = *inverse m* ∗*s* (*y* − *c*) **by** *simp*
then show *x* = *inverse m* ∗*s* *y* + − (*inverse m* ∗*s* *c*)
by (*simp add*: *m0 vec.scale_right_diff_distrib*)
next
assume *h*: *x* = *inverse m* ∗*s* *y* + − (*inverse m* ∗*s* *c*)
show *m* ∗*s* *x* + *c* = *y* **unfolding** *h*
using *m0* **by** (*simp add*: *vector_smult_assoc vector_ssub_ldistrib*)
qed

lemma *vector_eq_affinity*:
(*m*::'*a*::*field*) ≠ *0* ==> (*y* = *m* ∗*s* *x* + *c* ⟷ *inverse*(*m*) ∗*s* *y* + −(*inverse*(*m*) ∗*s* *c*) = *x*)
by (*metis vector_affinity_eq*)

lemma *vector_cart*:
fixes *f* :: *real*^'*n* ⇒ *real*
shows (*χ* *i*. *f* (*axis i 1*)) = (∑ *i*∈*Basis*. *f i* ∗*R* *i*)
unfolding *euclidean_eq_iff*[**where** '*a*=*real*^'*n*]
by *simp* (*simp add*: *Basis_vec_def inner_axis*)

lemma *const_vector_cart*:((*χ* *i*. *d*)::*real*^'*n*) = (∑ *i*∈*Basis*. *d* ∗*R* *i*)
by (*rule vector_cart*)

1.11.9 Explicit formulas for low dimensions

lemma *prod_neutral_const*: *prod f* {(*1*::*nat*)..*1*} = *f 1*
by *simp*

lemma *prod_2*: *prod f* {(*1*::*nat*)..*2*} = *f 1* ∗ *f 2*
by (*simp add*: *eval_nat_numeral atLeastAtMostSuc_conv mult.commute*)

lemma *prod_3*: *prod f* {(*1*::*nat*)..*3*} = *f 1* ∗ *f 2* ∗ *f 3*
by (*simp add*: *eval_nat_numeral atLeastAtMostSuc_conv mult.commute*)

1.11.10 Orthogonality of a matrix

definition *orthogonal_matrix* (*Q*::'*a*::*semiring_1*^'*n*^'*n*) ⟷
transpose Q ∗∗ *Q* = *mat 1* ∧ *Q* ∗∗ *transpose Q* = *mat 1*

lemma *orthogonal_matrix*: *orthogonal_matrix* (*Q*:: *real* ^'*n*^'*n*) ⟷ *transpose Q* ∗∗ *Q* = *mat 1*
by (*metis matrix_left_right_inverse orthogonal_matrix_def*)

lemma *orthogonal_matrix_id*: *orthogonal_matrix* (*mat 1* :: _^'n^'n)
by (*simp add*: *orthogonal_matrix_def*)

proposition *orthogonal_matrix_mul*:
fixes *A* :: *real* ^'n^'n
assumes *orthogonal_matrix A orthogonal_matrix B*
shows *orthogonal_matrix*(*A* ** *B*)
using *assms*
by (*simp add*: *orthogonal_matrix matrix_transpose_mul matrix_left_right_inverse matrix_mul_assoc*)

proposition *orthogonal_transformation_matrix*:
fixes *f*:: *real*^'n ⇒ *real*^'n
shows *orthogonal_transformation f* ⟷ *linear f* ∧ *orthogonal_matrix*(*matrix f*)
(**is** *?lhs* ⟷ *?rhs*)
proof −
let *?mf* = *matrix f*
let *?ot* = *orthogonal_transformation f*
let *?U* = *UNIV* :: *'n set*
have *fU*: *finite ?U* **by** *simp*
let *?m1* = *mat 1* :: *real* ^'n^'n
{
assume *ot*: *?ot*
from *ot* **have** *lf*: *Vector_Spaces.linear* (∗*s*) (∗*s*) *f* **and** *fd*: ⋀*v w*. *f v* ∙ *f w* = *v* ∙ *w*
unfolding *orthogonal_transformation_def orthogonal_matrix linear_def scalar_mult_eq_scaleR*
by *blast*+
{
fix *i j*
let *?A* = *transpose ?mf* ** *?mf*
have *th0*: ⋀*b* (*x*::*'a*::*comm_ring_1*). (*if b then 1 else 0*)∗*x* = (*if b then x else 0*)
⋀*b* (*x*::*'a*::*comm_ring_1*). *x*∗(*if b then 1 else 0*) = (*if b then x else 0*)
by *simp_all*
from *fd*[*of axis i 1 axis j 1*,
simplified matrix_works[*OF lf*, *symmetric*] *dot_matrix_vector_mul*]
have *?A*$*i*$*j* = *?m1* $ *i* $ *j*
by (*simp add*: *inner_vec_def matrix_matrix_mult_def columnvector_def rowvector_def*
th0 sum.delta[*OF fU*] *mat_def axis_def*)
}
then have *orthogonal_matrix ?mf*
unfolding *orthogonal_matrix*
by *vector*
with *lf* **have** *?rhs*
unfolding *linear_def scalar_mult_eq_scaleR*
by *blast*

}
moreover
have *?lhs* **if** *Vector_Spaces.linear* (∗*s*) (∗*s*) *f* **and** *orthogonal_matrix ?mf*
using *that* **unfolding** *orthogonal_matrix_def norm_eq orthogonal_transformation*
by (*metis dot_matrix_product dot_matrix_vector_mul linear_matrix_vector_mul_eq matrix_mul_lid matrix_vector_mul*(*2*))
ultimately show *?thesis*
by (*auto simp*: *linear_def scalar_mult_eq_scaleR*)
qed

1.11.11 Finding an Orthogonal Matrix

We can find an orthogonal matrix taking any unit vector to any other.

lemma *orthogonal_matrix_transpose* [*simp*]:
orthogonal_matrix(*transpose A*) ⟷ *orthogonal_matrix A*
by (*auto simp*: *orthogonal_matrix_def*)

lemma *orthogonal_matrix_orthonormal_columns*:
fixes *A* :: *real*^'*n*^'*n*
shows *orthogonal_matrix A* ⟷
(∀ *i*. *norm*(*column i A*) = *1*) ∧
(∀ *i j*. *i* ≠ *j* ⟶ *orthogonal* (*column i A*) (*column j A*))
by (*auto simp*: *orthogonal_matrix matrix_mult_transpose_dot_column vec_eq_iff mat_def norm_eq_1 orthogonal_def*)

lemma *orthogonal_matrix_orthonormal_rows*:
fixes *A* :: *real*^'*n*^'*n*
shows *orthogonal_matrix A* ⟷
(∀ *i*. *norm*(*row i A*) = *1*) ∧
(∀ *i j*. *i* ≠ *j* ⟶ *orthogonal* (*row i A*) (*row j A*))
using *orthogonal_matrix_orthonormal_columns* [*of transpose A*] **by** *simp*

proposition *orthogonal_matrix_exists_basis*:
fixes *a* :: *real*^'*n*
assumes *norm a* = *1*
obtains *A* **where** *orthogonal_matrix A A* ∗*v* (*axis k 1*) = *a*
proof −
obtain *S* **where** *a* ∈ *S pairwise orthogonal S* **and** *noS*: ⋀*x*. *x* ∈ *S* ⟹ *norm x* = *1*
and *independent S card S* = *CARD*('*n*) *span S* = *UNIV*
using *vector_in_orthonormal_basis assms* **by** *force*
then obtain *f0* **where** *bij_betw f0* (*UNIV*::'*n set*) *S*
by (*metis finite_class.finite_UNIV finite_same_card_bij finiteI_independent*)
then obtain *f* **where** *f*: *bij_betw f* (*UNIV*::'*n set*) *S* **and** *a*: *a* = *f k*
using *bij_swap_iff* [*of f0 k inv f0 a*]
by (*metis UNIV_I* ‹*a* ∈ *S*› *bij_betw_inv_into_right bij_betw_swap_iff swap_apply*(*1*))
show *thesis*
proof
have [*simp*]: ⋀*i*. *norm* (*f i*) = *1*

```
    using bij_betwE [OF ‹bij_betw f UNIV S›] by (blast intro: noS)
  have [simp]: ⋀i j. i ≠ j ⟹ orthogonal (f i) (f j)
    using ‹pairwise orthogonal S› ‹bij_betw f UNIV S›
    by (auto simp: pairwise_def bij_betw_def inj_on_def)
  show orthogonal_matrix (χ i j. f j $ i)
    by (simp add: orthogonal_matrix_orthonormal_columns column_def)
  show (χ i j. f j $ i) *v axis k 1 = a
    by (simp add: matrix_vector_mult_def axis_def a if_distrib cong: if_cong)
 qed
qed

lemma orthogonal_transformation_exists_1:
 fixes a b :: real^'n
 assumes norm a = 1 norm b = 1
 obtains f where orthogonal_transformation f f a = b
proof −
 obtain k A B where AB: orthogonal_matrix A orthogonal_matrix B and eq:
A *v (axis k 1) = a B *v (axis k 1) = b
  using orthogonal_matrix_exists_basis assms by metis
 let ?f = λx. (B ** transpose A) *v x
 show thesis
 proof
  show orthogonal_transformation ?f
   by (simp add: AB orthogonal_matrix_mul orthogonal_transformation_matrix)
 next
  show ?f a = b
   using ‹orthogonal_matrix A› unfolding orthogonal_matrix_def
   by (metis eq matrix_mul_rid matrix_vector_mul_assoc)
 qed
qed

proposition orthogonal_transformation_exists:
 fixes a b :: real^'n
 assumes norm a = norm b
 obtains f where orthogonal_transformation f f a = b
proof (cases a = 0 ∨ b = 0)
 case True
 with assms show ?thesis
  using that by force
next
 case False
 then obtain f where f: orthogonal_transformation f and eq: f (a /R norm a)
= (b /R norm b)
  by (auto intro: orthogonal_transformation_exists_1 [of a /R norm a b /R norm
b])
 show ?thesis
  using False assms eq f orthogonal_transformation_scaleR that by fastforce
qed
```

1.11.12 Scaling and isometry

proposition *scaling_linear*:
fixes $f :: 'a::real\_inner \Rightarrow 'a::real\_inner$
assumes *f0*: $f\ 0 = 0$
and *fd*: $\forall x\ y.\ dist\ (f\ x)\ (f\ y) = c * dist\ x\ y$
shows *linear f*
proof −
{
fix *v w*
have *norm* $(f\ x) = c * norm\ x$ **for** *x*
by (*metis dist_0_norm f0 fd*)
then have $f\ v \cdot f\ w = c^2 * (v \cdot w)$
unfolding *dot_norm_neg dist_norm*[*symmetric*]
by (*simp add*: *fd power2_eq_square field_simps*)
}
then show *?thesis*
unfolding *linear_iff vector_eq*[**where** *'a='a*] *scalar_mult_eq_scaleR*
by (*simp add*: *inner_add field_simps*)
qed

lemma *isometry_linear*:
$f\ (0::'a::real\_inner) = (0::'a) \Longrightarrow \forall x\ y.\ dist(f\ x)\ (f\ y) = dist\ x\ y \Longrightarrow linear\ f$
by (*rule scaling_linear*[**where** *c=1*]) *simp_all*

Hence another formulation of orthogonal transformation

proposition *orthogonal_transformation_isometry*:
orthogonal_transformation f $\longleftrightarrow f(0::'a::real\_inner) = (0::'a) \wedge (\forall x\ y.\ dist(f\ x)\ (f\ y) = dist\ x\ y)$
unfolding *orthogonal_transformation*
by (*metis dist_0_norm dist_norm isometry_linear linear_0 linear_diff*)

Can extend an isometry from unit sphere:

lemma *isometry_sphere_extend*:
fixes $f::\ 'a::real\_inner \Rightarrow 'a$
assumes *f1*: $\bigwedge x.\ norm\ x = 1 \Longrightarrow norm\ (f\ x) = 1$
and *fd1*: $\bigwedge x\ y.\ [\![norm\ x = 1;\ norm\ y = 1]\!] \Longrightarrow dist\ (f\ x)\ (f\ y) = dist\ x\ y$
shows $\exists g.\ orthogonal\_transformation\ g \wedge (\forall x.\ norm\ x = 1 \longrightarrow g\ x = f\ x)$
proof −
{
fix $x\ y\ x'\ y'\ u\ v\ u'\ v' :: 'a$
assume *H*: $x = norm\ x *_R u\ y = norm\ y *_R v$
$x' = norm\ x *_R u'\ y' = norm\ y *_R v'$
and *J*: *norm u = 1 norm u' = 1 norm v = 1 norm v' = 1* $norm(u' - v') = norm(u - v)$
then have $*$: $u \cdot v = u' \cdot v' + v' \cdot u' - v \cdot u$
by (*simp add*: *norm_eq norm_eq_1 inner_add inner_diff*)
have $norm\ (norm\ x *_R u' - norm\ y *_R v') = norm\ (norm\ x *_R u - norm\ y *_R v)$
using *J* **by** (*simp add*: *norm_eq norm_eq_1 inner_diff* $*$ *field_simps*)

```
    then have norm(x' − y') = norm(x − y)
      using H by metis
  }
  note norm_eq = this
  let ?g = λx. if x = 0 then 0 else norm x *R f (x /R norm x)
  have thfg: ?g x = f x if norm x = 1 for x
    using that by auto
  have thd: dist (?g x) (?g y) = dist x y for x y
  proof (cases x=0 ∨ y=0)
    case False
    show dist (?g x) (?g y) = dist x y
      unfolding dist_norm
    proof (rule norm_eq)
      show x = norm x *R (x /R norm x) y = norm y *R (y /R norm y)
          norm (f (x /R norm x)) = 1 norm (f (y /R norm y)) = 1
        using False f1 by auto
    qed (use False in ‹auto simp: field_simps intro: f1 fd1[unfolded dist_norm]›)
  qed (auto simp: f1)
  show ?thesis
    unfolding orthogonal_transformation_isometry
    by (rule exI[where x= ?g]) (metis thfg thd)
qed
```

1.11.13 Induction on matrix row operations

```
lemma induct_matrix_row_operations:
  fixes P :: real^'n^'n ⇒ bool
  assumes zero_row: ⋀A i. row i A = 0 ⟹ P A
    and diagonal: ⋀A. (⋀i j. i ≠ j ⟹ A$i$j = 0) ⟹ P A
    and swap_cols: ⋀A m n. ⟦P A; m ≠ n⟧ ⟹ P(χ i j. A $ i $ Transposition.transpose m n j)
    and row_op: ⋀A m n c. ⟦P A; m ≠ n⟧
                ⟹ P(χ i. if i = m then row m A + c *R row n A else row i A)
  shows P A
proof −
  have P A if (⋀i j. ⟦j ∈ −K;  i ≠ j⟧ ⟹ A$i$j = 0) for A K
  proof −
    have finite K
      by simp
    then show ?thesis using that
    proof (induction arbitrary: A rule: finite_induct)
      case empty
      with diagonal show ?case
        by simp
    next
      case (insert k K)
      note insertK = insert
      have P A if kk: A$k$k ≠ 0
        and 0: ⋀i j. ⟦j ∈ − insert k K; i ≠ j⟧ ⟹ A$i$j = 0
```

```
          ⋀i. ⟦i ∈ −L; i ≠ k⟧ ⟹ A$i$k = 0 for A L
    proof −
      have finite L
        by simp
      then show ?thesis using 0 kk
      proof (induction arbitrary: A rule: finite_induct)
        case (empty B)
        show ?case
        proof (rule insertK)
          fix i j
          assume i ∈ − K j ≠ i
          show B $ j $ i = 0
            using ‹j ≠ i› ‹i ∈ − K› empty
           by (metis ComplD ComplI Compl_eq_Diff_UNIV Diff_empty UNIV_I
insert_iff)
        qed
      next
        case (insert l L B)
        show ?case
        proof (cases k = l)
          case True
          with insert show ?thesis
            by auto
        next
          case False
          let ?C = χ i. if i = l then row l B − (B $ l $ k / B $ k $ k) *R row k
B else row i B
          have 1: ⟦j ∈ − insert k K; i ≠ j⟧ ⟹ ?C $ i $ j = 0 for j i
            by (auto simp: insert.prems(1) row_def)
          have 2: ?C $ i $ k = 0
            if i ∈ − L i ≠ k for i
          proof (cases i=l)
            case True
            with that insert.prems show ?thesis
              by (simp add: row_def)
          next
            case False
            with that show ?thesis
              by (simp add: insert.prems(2) row_def)
          qed
          have 3: ?C $ k $ k ≠ 0
            by (auto simp: insert.prems row_def ‹k ≠ l›)
          have PC: P ?C
            using insert.IH [OF 1 2 3] by auto
          have eqB: (χ i. if i = l then row l ?C + (B $ l $ k / B $ k $ k) *R row
k ?C else row i ?C) = B
            using ‹k ≠ l› by (simp add: vec_eq_iff row_def)
          show ?thesis
           using row_op [OF PC, of l k, where c = B$l$k / B$k$k] eqB ‹k ≠ l›
```

```
            by (simp add: cong: if_cong)
          qed
        qed
      qed
      then have nonzero_hyp: P A
        if kk: A$k$k ≠ 0 and zeroes: ⋀i j. j ∈ − insert k K ∧ i≠j ⟹ A$i$j = 0
for A
        by (auto simp: intro!: kk zeroes)
      show ?case
      proof (cases row k A = 0)
        case True
        with zero_row show ?thesis by auto
      next
        case False
        then obtain l where l: A$k$l ≠ 0
          by (auto simp: row_def zero_vec_def vec_eq_iff)
        show ?thesis
        proof (cases k = l)
          case True
          with l nonzero_hyp insert.prems show ?thesis
            by blast
        next
          case False
          have ∗: A $ i $ Transposition.transpose k l j = 0 if j ≠ k j ∉ K i ≠ j for
i j
            using False l insert.prems that
            by (auto simp add: Transposition.transpose_def)
              have P (χ i j. (χ i j. A $ i $ Transposition.transpose k l j) $ i $
Transposition.transpose k l j)
            by (rule swap_cols [OF nonzero_hyp False]) (auto simp: l ∗)
          moreover
          have (χ i j. (χ i j. A $ i $ Transposition.transpose k l j) $ i $ Transposi-
tion.transpose k l j) = A
            by simp
          ultimately show ?thesis
            by simp
        qed
      qed
    qed
  qed
  then show ?thesis
    by blast
qed

lemma induct_matrix_elementary:
  fixes P :: real^'n^'n ⇒ bool
  assumes mult: ⋀A B. [[P A; P B]] ⟹ P(A ∗∗ B)
    and zero_row: ⋀A i. row i A = 0 ⟹ P A
    and diagonal: ⋀A. (⋀i j. i ≠ j ⟹ A$i$j = 0) ⟹ P A
```

```
    and swap1: ⋀m n. m ≠ n ⟹ P(χ i j. mat 1 $ i $ Transposition.transpose m
n j)
    and idplus: ⋀m n c. m ≠ n ⟹ P(χ i j. if i = m ∧ j = n then c else of_bool
(i = j))
  shows P A
proof −
  have swap: P (χ i j. A $ i $ Transposition.transpose m n j)  (is P ?C)
    if P A m ≠ n for A m n
  proof −
    have A ** (χ i j. mat 1 $ i $ Transposition.transpose m n j) = ?C
    by (simp add: matrix_matrix_mult_def mat_def vec_eq_iff if_distrib sum.delta_remove)
    then show ?thesis
      using mult swap1 that by metis
  qed
  have row: P (χ i. if i = m then row m A + c *R row n A else row i A)  (is P
?C)
    if P A m ≠ n for A m n c
  proof −
    let ?B = χ i j. if i = m ∧ j = n then c else of_bool (i = j)
    have ?B ** A = ?C
      using ‹m ≠ n› unfolding matrix_matrix_mult_def row_def of_bool_def
      by (auto simp: vec_eq_iff if_distrib [of λx. x * y for y] sum.remove cong:
if_cong)
    then show ?thesis
      by (rule subst) (auto simp: that mult idplus)
  qed
  show ?thesis
    by (rule induct_matrix_row_operations [OF zero_row diagonal swap row])
qed

lemma induct_matrix_elementary_alt:
  fixes P :: real^'n^'n ⇒ bool
  assumes mult: ⋀A B. ⟦P A; P B⟧ ⟹ P(A ** B)
    and zero_row: ⋀A i. row i A = 0 ⟹ P A
    and diagonal: ⋀A. (⋀i j. i ≠ j ⟹ A$i$j = 0) ⟹ P A
    and swap1: ⋀m n. m ≠ n ⟹ P(χ i j. mat 1 $ i $ Transposition.transpose m
n j)
    and idplus: ⋀m n. m ≠ n ⟹ P(χ i j. of_bool (i = m ∧ j = n ∨ i = j))
  shows P A
proof −
  have ∗: P (χ i j. if i = m ∧ j = n then c else of_bool (i = j))
    if m ≠ n for m n c
  proof (cases c = 0)
    case True
    with diagonal show ?thesis by auto
  next
    case False
    then have eq: (χ i j. if i = m ∧ j = n then c else of_bool (i = j)) =
                  (χ i j. if i = j then (if j = n then inverse c else 1) else 0) **
```

```
                (χ i j. of_bool (i = m ∧ j = n ∨ i = j)) **
                (χ i j. if i = j then if j = n then c else 1 else 0)
    using ‹m ≠ n›
    apply (simp add: matrix_matrix_mult_def vec_eq_iff of_bool_def if_distrib
[of λx. y * x for y] cong: if_cong)
    apply (simp add: if_if_eq_conj sum.neutral conj_commute cong: conj_cong)
    done
  show ?thesis
    unfolding eq by (intro mult idplus that) (auto intro: diagonal)
 qed
 show ?thesis
  by (rule induct_matrix_elementary) (auto intro: assms *)
qed

lemma matrix_vector_mult_matrix_matrix_mult_compose:
 (*v) (A ** B) = (*v) A ∘ (*v) B
 by (auto simp: matrix_vector_mul_assoc)

lemma induct_linear_elementary:
 fixes f :: real^'n ⇒ real^'n
 assumes linear f
  and comp: ⋀f g. ⟦linear f; linear g; P f; P g⟧ ⟹ P(f ∘ g)
  and zeroes: ⋀f i. ⟦linear f; ⋀x. (f x) $ i = 0⟧ ⟹ P f
  and const: ⋀c. P(λx. χ i. c i * x$i)
  and swap: ⋀m n::'n. m ≠ n ⟹ P(λx. χ i. x $ Transposition.transpose m n i)
  and idplus: ⋀m n::'n. m ≠ n ⟹ P(λx. χ i. if i = m then x$m + x$n else
x$i)
 shows P f
proof −
 have P ((*v) A) for A
 proof (rule induct_matrix_elementary_alt)
  fix A B
  assume P ((*v) A) and P ((*v) B)
  then show P ((*v) (A ** B))
  by (auto simp add: matrix_vector_mult_matrix_matrix_mult_compose intro!:
comp)
 next
  fix A :: real^'n^'n and i
  assume row i A = 0
  show P ((*v) A)
   using matrix_vector_mul_linear
   by (rule zeroes[where i=i])
      (metis ‹row i A = 0› inner_zero_left matrix_vector_mul_component
row_def vec_lambda_eta)
 next
  fix A :: real^'n^'n
  assume 0: ⋀i j. i ≠ j ⟹ A $ i $ j = 0
  have A $ i $ i * x $ i = (∑j∈UNIV. A $ i $ j * x $ j) for x and i :: 'n
   by (simp add: 0 comm_monoid_add_class.sum.remove [where x=i])
```

```
    then have (λx. χ i. A $ i $ i * x $ i) = ((*v) A)
      by (auto simp: 0 matrix_vector_mult_def)
    then show P ((*v) A)
      using const [of λi. A $ i $ i] by simp
  next
    fix m n :: 'n
    assume m ≠ n
    have eq: (∑ j∈UNIV. if i = Transposition.transpose m n j then x $ j else 0) =
              (∑ j∈UNIV. if j = Transposition.transpose m n i then x $ j else 0)
      for i and x :: real^'n
      by (rule sum.cong) (auto simp add: swap_id_eq)
    have (λx::real^'n. χ i. x $ Transposition.transpose m n i) = ((*v) (χ i j. if i
= Transposition.transpose m n j then 1 else 0))
      by (auto simp: mat_def matrix_vector_mult_def eq if_distrib [of λx. x * y
for y] cong: if_cong)
      with swap [OF ‹m ≠ n›] show P ((*v) (χ i j. mat 1 $ i $ Transposi-
tion.transpose m n j))
      by (simp add: mat_def matrix_vector_mult_def)
  next
    fix m n :: 'n
    assume m ≠ n
    then have x $ m + x $ n = (∑ j∈UNIV. of_bool (j = n ∨ m = j) * x $ j)
for x :: real^'n
      by (auto simp: of_bool_def if_distrib [of λx. x * y for y] sum.remove cong:
if_cong)
    then have (λx::real^'n. χ i. if i = m then x $ m + x $ n else x $ i) =
              ((*v) (χ i j. of_bool (i = m ∧ j = n ∨ i = j)))
      unfolding matrix_vector_mult_def of_bool_def
      by (auto simp: vec_eq_iff if_distrib [of λx. x * y for y] cong: if_cong)
    then show P ((*v) (χ i j. of_bool (i = m ∧ j = n ∨ i = j)))
      using idplus [OF ‹m ≠ n›] by simp
  qed
  then show ?thesis
    by (metis ‹linear f› matrix_vector_mul(2))
qed

end
```

1.12 Traces and Determinants of Square Matrices

```
theory Determinants
imports
  HOL−Combinatorics.Permutations
  Cartesian_Space
begin
```

1.12.1 Trace

```
definition trace :: 'a::semiring_1^'n^'n ⇒ 'a
```

where *trace A = sum* (λ*i*. ((*A*$*i*)$*i*)) (*UNIV*::′*n set*)

lemma *trace_0*: *trace* (*mat 0*) = *0*
by (*simp add*: *trace_def mat_def*)

lemma *trace_I*: *trace* (*mat 1* :: ′*a*::*semiring_1*^′*n*^′*n*) = *of_nat*(*CARD*(′*n*))
by (*simp add*: *trace_def mat_def*)

lemma *trace_add*: *trace* ((*A*::′*a*::*comm_semiring_1*^′*n*^′*n*) + *B*) = *trace A* + *trace B*
by (*simp add*: *trace_def sum.distrib*)

lemma *trace_sub*: *trace* ((*A*::′*a*::*comm_ring_1*^′*n*^′*n*) − *B*) = *trace A* − *trace B*
by (*simp add*: *trace_def sum_subtractf*)

lemma *trace_mul_sym*: *trace* ((*A*::′*a*::*comm_semiring_1*^′*n*^′*m*) ** *B*) = *trace* (*B***A*)
apply (*simp add*: *trace_def matrix_matrix_mult_def*)
apply (*subst sum.swap*)
apply (*simp add*: *mult.commute*)
done

Definition of determinant

definition *det*:: ′*a*::*comm_ring_1*^′*n*^′*n* ⇒ ′*a* **where**
det A =
sum (λ*p*. *of_int* (*sign p*) * *prod* (λ*i*. *A*$*i*$*p i*) (*UNIV* :: ′*n set*))
{*p*. *p permutes* (*UNIV* :: ′*n set*)}

Basic determinant properties

lemma *det_transpose* [*simp*]: *det* (*transpose A*) = *det* (*A*::′*a*::*comm_ring_1* ^′*n*^′*n*)
proof −
let *?di* = λ*A i j*. *A*$*i*$*j*
let *?U* = (*UNIV* :: ′*n set*)
have *fU*: *finite ?U* **by** *simp*
{
fix *p*
assume *p*: *p* ∈ {*p*. *p permutes ?U*}
from *p* **have** *pU*: *p permutes ?U*
by *blast*
have *sth*: *sign* (*inv p*) = *sign p*
by (*metis sign_inverse fU p mem_Collect_eq permutation_permutes*)
from *permutes_inj*[*OF pU*]
have *pi*: *inj_on p ?U*
by (*blast intro*: *subset_inj_on*)
from *permutes_image*[*OF pU*]
have *prod* (λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) *?U* =
prod (λ*i*. *?di* (*transpose A*) *i* (*inv p i*)) (*p* ‘ *?U*)
by *simp*

```
    also have ... = prod ((λi. ?di (transpose A) i (inv p i)) ∘ p) ?U
      unfolding prod.reindex[OF pi] ..
    also have ... = prod (λi. ?di A i (p i)) ?U
    proof −
     have ((λi. ?di (transpose A) i (inv p i)) ∘ p) i = ?di A i (p i) if i ∈ ?U for i
       using that permutes_inv_o[OF pU] permutes_in_image[OF pU]
       unfolding transpose_def by (simp add: fun_eq_iff)
     then show prod ((λi. ?di (transpose A) i (inv p i)) ∘ p) ?U = prod (λi. ?di
A i (p i)) ?U
       by (auto intro: prod.cong)
    qed
    finally have of_int (sign (inv p)) * (prod (λi. ?di (transpose A) i (inv p i))
?U) =
      of_int (sign p) * (prod (λi. ?di A i (p i)) ?U)
      using sth by simp
  }
  then show ?thesis
    unfolding det_def
    by (subst sum_permutations_inverse) (blast intro: sum.cong)
qed

lemma det_lowerdiagonal:
  fixes A :: 'a::comm_ring_1^('n::{finite,wellorder})^('n::{finite,wellorder})
  assumes ld: ⋀i j. i < j ⟹ A$i$j = 0
  shows det A = prod (λi. A$i$i) (UNIV:: 'n set)
proof −
  let ?U = UNIV:: 'n set
  let ?PU = {p. p permutes ?U}
  let ?pp = λp. of_int (sign p) * prod (λi. A$i$p i) (UNIV :: 'n set)
  have fU: finite ?U
    by simp
  have id0: {id} ⊆ ?PU
    by (auto simp: permutes_id)
  have p0: ∀ p ∈ ?PU − {id}. ?pp p = 0
  proof
    fix p
    assume p ∈ ?PU − {id}
    then obtain i where i: p i > i
      by clarify (meson leI permutes_natset_le)
    from ld[OF i] have ∃ i ∈ ?U. A$i$p i = 0
      by blast
    with prod_zero[OF fU] show ?pp p = 0
      by force
  qed
 from sum.mono_neutral_cong_left[OF finite_permutations[OF fU] id0 p0] show
?thesis
    unfolding det_def by (simp add: sign_id)
qed
```

lemma *det_upperdiagonal*:
fixes $A :: 'a::comm\_ring\_1^{\wedge}'n::\{finite,wellorder\}^{\wedge}'n::\{finite,wellorder\}$
assumes *ld*: $\bigwedge i\ j.\ i > j \Longrightarrow A\$i\$j = 0$
shows $det\ A = prod\ (\lambda i.\ A\$i\$i)\ (UNIV :: 'n\ set)$
proof −
let $?U = UNIV :: 'n\ set$
let $?PU = \{p.\ p\ permutes\ ?U\}$
let $?pp = (\lambda p.\ of\_int\ (sign\ p) * prod\ (\lambda i.\ A\$i\$p\ i)\ (UNIV :: 'n\ set))$
have *fU*: *finite ?U*
by *simp*
have *id0*: $\{id\} \subseteq ?PU$
by (*auto simp*: *permutes_id*)
have *p0*: $\forall p \in ?PU - \{id\}.\ ?pp\ p = 0$
proof
fix *p*
assume *p*: $p \in ?PU - \{id\}$
then obtain *i* **where** *i*: $p\ i < i$
by *clarify* (*meson leI permutes_natset_ge*)
from *ld*[*OF i*] **have** $\exists i \in ?U.\ A\$i\$p\ i = 0$
by *blast*
with *prod_zero*[*OF fU*] **show** $?pp\ p = 0$
by *force*
qed
from *sum.mono_neutral_cong_left*[*OF finite_permutations*[*OF fU*] *id0 p0*] **show** *?thesis*
unfolding *det_def* **by** (*simp add*: *sign_id*)
qed

proposition *det_diagonal*:
fixes $A :: 'a::comm\_ring\_1^{\wedge}'n^{\wedge}'n$
assumes *ld*: $\bigwedge i\ j.\ i \neq j \Longrightarrow A\$i\$j = 0$
shows $det\ A = prod\ (\lambda i.\ A\$i\$i)\ (UNIV::'n\ set)$
proof −
let $?U = UNIV :: 'n\ set$
let $?PU = \{p.\ p\ permutes\ ?U\}$
let $?pp = \lambda p.\ of\_int\ (sign\ p) * prod\ (\lambda i.\ A\$i\$p\ i)\ (UNIV :: 'n\ set)$
have *fU*: *finite ?U* **by** *simp*
from *finite_permutations*[*OF fU*] **have** *fPU*: *finite ?PU* .
have *id0*: $\{id\} \subseteq ?PU$
by (*auto simp*: *permutes_id*)
have *p0*: $\forall p \in ?PU - \{id\}.\ ?pp\ p = 0$
proof
fix *p*
assume *p*: $p \in ?PU - \{id\}$
then obtain *i* **where** *i*: $p\ i \neq i$
by *fastforce*
with *ld* **have** $\exists i \in ?U.\ A\$i\$p\ i = 0$
by (*metis UNIV_I*)
with *prod_zero* [*OF fU*] **show** $?pp\ p = 0$

```
    by force
  qed
  from sum.mono_neutral_cong_left[OF fPU id0 p0] show ?thesis
    unfolding det_def by (simp add: sign_id)
qed

lemma det_I [simp]: det (mat 1 :: 'a::comm_ring_1^'n^'n) = 1
  by (simp add: det_diagonal mat_def)

lemma det_0 [simp]: det (mat 0 :: 'a::comm_ring_1^'n^'n) = 0
  by (simp add: det_def prod_zero power_0_left)

lemma det_permute_rows:
  fixes A :: 'a::comm_ring_1^'n^'n
  assumes p: p permutes (UNIV :: 'n::finite set)
  shows det (χ i. A$p i :: 'a^'n^'n) = of_int (sign p) * det A
proof −
  let ?U = UNIV :: 'n set
  let ?PU = {p. p permutes ?U}
  have *: (∑ q∈?PU. of_int (sign (q ∘ p)) * (∏ i∈?U. A $ p i $ (q ∘ p) i)) =
          (∑ n∈?PU. of_int (sign p) * of_int (sign n) * (∏ i∈?U. A $ i $ n i))
  proof (rule sum.cong)
    fix q
    assume qPU: q ∈ ?PU
    have fU: finite ?U
      by simp
    from qPU have q: q permutes ?U
      by blast
    have prod (λi. A$p i$ (q ∘ p) i) ?U = prod ((λi. A$p i$(q ∘ p) i) ∘ inv p) ?U
      by (simp only: prod.permute[OF permutes_inv[OF p], symmetric])
    also have ... = prod (λi. A $ (p ∘ inv p) i $ (q ∘ (p ∘ inv p)) i) ?U
      by (simp only: o_def)
    also have ... = prod (λi. A$i$q i) ?U
      by (simp only: o_def permutes_inverses[OF p])
    finally have thp: prod (λi. A$p i$ (q ∘ p) i) ?U = prod (λi. A$i$q i) ?U
      by blast
    from p q have pp: permutation p and qp: permutation q
      by (metis fU permutation_permutes)+
    show of_int (sign (q ∘ p)) * prod (λi. A$ p i$ (q ∘ p) i) ?U =
         of_int (sign p) * of_int (sign q) * prod (λi. A$i$q i) ?U
      by (simp only: thp sign_compose[OF qp pp] mult.commute of_int_mult)
  qed auto
  show ?thesis
    apply (simp add: det_def sum_distrib_left mult.assoc[symmetric])
    apply (subst sum_permutations_compose_right[OF p])
    apply (rule *)
    done
qed
```

lemma *det_permute_columns*:
fixes $A :: 'a::comm\_ring\_1^{\wedge}'n^{\wedge}'n$
assumes p: p *permutes* $(UNIV :: 'n\ set)$
shows $det(\chi\ i\ j.\ A\$i\$\ p\ j :: 'a^{\wedge}'n^{\wedge}'n) = of\_int\ (sign\ p) * det\ A$
proof −
let $?Ap = \chi\ i\ j.\ A\$i\$\ p\ j :: 'a^{\wedge}'n^{\wedge}'n$
let $?At = transpose\ A$
have $of\_int\ (sign\ p) * det\ A = det\ (transpose\ (\chi\ i.\ transpose\ A\ \$\ p\ i))$
unfolding *det_permute_rows*[*OF p*, *of ?At*] *det_transpose* **..**
moreover
have $?Ap = transpose\ (\chi\ i.\ transpose\ A\ \$\ p\ i)$
by (*simp add*: *transpose_def vec_eq_iff*)
ultimately show *?thesis*
by *simp*
qed

lemma *det_identical_columns*:
fixes $A :: 'a::comm\_ring\_1^{\wedge}'n^{\wedge}'n$
assumes jk: $j \neq k$
and r: *column* $j\ A = column\ k\ A$
shows $det\ A = 0$
proof −
let $?U = UNIV :: 'n\ set$
let $?t\_jk = Transposition.transpose\ j\ k$
let $?PU = \{p.\ p\ permutes\ ?U\}$
let $?S1 = \{p.\ p \in ?PU \wedge evenperm\ p\}$
let $?S2 = \{(?t\_jk \circ p)\ |p.\ p \in ?S1\}$
let $?f = \lambda p.\ of\_int\ (sign\ p) * (\prod i \in UNIV.\ A\ \$\ i\ \$\ p\ i)$
let $?g = \lambda p.\ ?t\_jk \circ p$
have *g_S1*: *?S2* = *?g*‘ *?S1* **by** *auto*
have *inj_g*: *inj_on ?g ?S1*
proof (*unfold inj_on_def*, *auto*)
fix $x\ y$ **assume** x: x *permutes* *?U* **and** *even_x*: *evenperm x*
and y: y *permutes* *?U* **and** *even_y*: *evenperm y* **and** *eq*: $?t\_jk \circ x = ?t\_jk \circ y$
show $x = y$ **by** (*metis* (*opaque_lifting*, *no_types*) *comp_assoc eq id_comp swap_id_idempotent*)
qed
have *tjk_permutes*: *?t_jk permutes ?U*
by (*auto simp add*: *permutes_def dest*: *transpose_eq_imp_eq*) (*meson transpose_involutory*)
have *tjk_eq*: $\forall i\ l.\ A\ \$\ i\ \$\ ?t\_jk\ l = A\ \$\ i\ \$\ l$
using *r jk*
unfolding *column_def vec_eq_iff* **by** (*simp add*: *Transposition.transpose_def*)

have *sign_tjk*: *sign ?t_jk* = −*1* **using** *sign_swap_id*[*of j k*] *jk* **by** *auto*
{fix x
assume x: $x \in ?S1$
have *sign* (*?t_jk* ∘ *x*) = *sign* (*?t_jk*) ∗ *sign x*

```
    by (metis (lifting) finite_class.finite_UNIV mem_Collect_eq
      permutation_permutes permutation_swap_id sign_compose x)
   also have ... = - sign x using sign_tjk by simp
   also have ... ≠ sign x unfolding sign_def by simp
   finally have sign (?t_jk ∘ x) ≠ sign x and (?t_jk ∘ x) ∈ ?S2
    using x by force+
 }
 hence disjoint: ?S1 ∩ ?S2 = {}
  by (force simp: sign_def)
 have PU_decomposition: ?PU = ?S1 ∪ ?S2
 proof (auto)
  fix x
   assume x: x permutes ?U and ∀ p. p permutes ?U ⟶ x = Transposi-
tion.transpose j k ∘ p ⟶ ¬ evenperm p
  then obtain p where p: p permutes UNIV and x_eq: x = Transposition.transpose
j k ∘ p
   and odd_p: ¬ evenperm p
   by (metis (mono_tags) id_o o_assoc permutes_compose swap_id_idempotent
tjk_permutes)
  thus evenperm x
   by (meson evenperm_comp evenperm_swap finite_class.finite_UNIV
     jk permutation_permutes permutation_swap_id)
 next
  fix p assume p: p permutes ?U
 show Transposition.transpose j k ∘ p permutes UNIV by (metis p permutes_compose
tjk_permutes)
 qed
 have sum ?f ?S2 = sum ((λp. of_int (sign p) * (∏ i∈UNIV. A $ i $ p i))
 ∘ (∘) (Transposition.transpose j k)) {p ∈ {p. p permutes UNIV}. evenperm p}
  unfolding g_S1 by (rule sum.reindex[OF inj_g])
 also have ... = sum (λp. of_int (sign (?t_jk ∘ p)) * (∏ i∈UNIV. A $ i $ p i))
?S1
  unfolding o_def by (rule sum.cong, auto simp: tjk_eq)
 also have ... = sum (λp. - ?f p) ?S1
 proof (rule sum.cong, auto)
  fix x assume x: x permutes ?U
   and even_x: evenperm x
  hence perm_x: permutation x and perm_tjk: permutation ?t_jk
   using permutation_permutes[of x] permutation_permutes[of ?t_jk] permuta-
tion_swap_id
   by (metis finite_code)+
  have (sign (?t_jk ∘ x)) = - (sign x)
   unfolding sign_compose[OF perm_tjk perm_x] sign_tjk by auto
  thus of_int (sign (?t_jk ∘ x)) * (∏ i∈UNIV. A $ i $ x i)
   = - (of_int (sign x) * (∏ i∈UNIV. A $ i $ x i))
   by auto
 qed
 also have ...= - sum ?f ?S1 unfolding sum_negf ..
 finally have *: sum ?f ?S2 = - sum ?f ?S1 .
```

```
  have det A = (∑ p | p permutes UNIV. of_int (sign p) * (∏ i∈UNIV. A $ i $
p i))
    unfolding det_def ..
  also have ... = sum ?f ?S1 + sum ?f ?S2
    by (subst PU_decomposition, rule sum.union_disjoint[OF _ _ disjoint], auto)
  also have ... = sum ?f ?S1 - sum ?f ?S1  unfolding * by auto
  also have ... = 0 by simp
  finally show det A = 0 by simp
qed

lemma det_identical_rows:
  fixes A :: 'a::comm_ring_1^'n^'n
  assumes ij: i ≠ j and r: row i A = row j A
  shows det A = 0
  by (metis column_transpose det_identical_columns det_transpose ij r)

lemma det_zero_row:
  fixes A :: 'a::{idom, ring_char_0}^'n^'n and F :: 'b::{field}^'m^'m
  shows row i A = 0 ⟹ det A = 0 and row j F = 0 ⟹ det F = 0
  by (force simp: row_def det_def vec_eq_iff sign_nz intro!: sum.neutral)+

lemma det_zero_column:
  fixes A :: 'a::{idom, ring_char_0}^'n^'n and F :: 'b::{field}^'m^'m
  shows column i A = 0 ⟹ det A = 0 and column j F = 0 ⟹ det F = 0
  unfolding atomize_conj atomize_imp
  by (metis det_transpose det_zero_row row_transpose)

lemma det_row_add:
  fixes a b c :: 'n::finite ⇒ _ ^ 'n
  shows det((χ i. if i = k then a i + b i else c i)::'a::comm_ring_1^'n^'n) =
    det((χ i. if i = k then a i else c i)::'a::comm_ring_1^'n^'n) +
    det((χ i. if i = k then b i else c i)::'a::comm_ring_1^'n^'n)
  unfolding det_def vec_lambda_beta sum.distrib[symmetric]
proof (rule sum.cong)
  let ?U = UNIV :: 'n set
  let ?pU = {p. p permutes ?U}
  let ?f = (λi. if i = k then a i + b i else c i)::'n ⇒ 'a::comm_ring_1^'n
  let ?g = (λ i. if i = k then a i else c i)::'n ⇒ 'a::comm_ring_1^'n
  let ?h = (λ i. if i = k then b i else c i)::'n ⇒ 'a::comm_ring_1^'n
  fix p
  assume p: p ∈ ?pU
  let ?Uk = ?U - {k}
  from p have pU: p permutes ?U
    by blast
  have kU: ?U = insert k ?Uk
    by blast
  have eq: prod (λi. ?f i $ p i) ?Uk = prod (λi. ?g i $ p i) ?Uk
        prod (λi. ?f i $ p i) ?Uk = prod (λi. ?h i $ p i) ?Uk
    by auto
```

```
  have Uk: finite ?Uk k ∉ ?Uk
    by auto
  have prod (λi. ?f i $ p i) ?U = prod (λi. ?f i $ p i) (insert k ?Uk)
    unfolding kU[symmetric] ..
  also have ... = ?f k $ p k * prod (λi. ?f i $ p i) ?Uk
    by (rule prod.insert) auto
  also have ... = (a k $ p k * prod (λi. ?f i $ p i) ?Uk) + (b k$ p k * prod (λi.
?f i $ p i) ?Uk)
    by (simp add: field_simps)
  also have ... = (a k $ p k * prod (λi. ?g i $ p i) ?Uk) + (b k$ p k * prod (λi.
?h i $ p i) ?Uk)
    by (metis eq)
   also have ... = prod (λi. ?g i $ p i) (insert k ?Uk) + prod (λi. ?h i $ p i)
(insert k ?Uk)
    unfolding  prod.insert[OF Uk] by simp
  finally have prod (λi. ?f i $ p i) ?U = prod (λi. ?g i $ p i) ?U + prod (λi. ?h
i $ p i) ?U
    unfolding kU[symmetric] .
  then show of_int (sign p) * prod (λi. ?f i $ p i) ?U =
    of_int (sign p) * prod (λi. ?g i $ p i) ?U + of_int (sign p) * prod (λi. ?h i $
p i) ?U
    by (simp add: field_simps)
qed auto

lemma  det_row_mul:
  fixes a b :: 'n::finite ⇒ _ ^ 'n
  shows det((χ i. if i = k then c *s a i else b i)::'a::comm_ring_1^'n^'n) =
    c * det((χ i. if i = k then a i else b i)::'a::comm_ring_1^'n^'n)
  unfolding det_def vec_lambda_beta sum_distrib_left
proof (rule sum.cong)
  let ?U = UNIV :: 'n set
  let ?pU = {p. p permutes ?U}
  let ?f = (λi. if i = k then c*s a i else b i)::'n ⇒ 'a::comm_ring_1^'n
  let ?g = (λ i. if i = k then a i else b i)::'n ⇒ 'a::comm_ring_1^'n
  fix p
  assume p: p ∈ ?pU
  let ?Uk = ?U - {k}
  from p have pU: p permutes ?U
    by blast
  have kU: ?U = insert k ?Uk
    by blast
  have eq: prod (λi. ?f i $ p i) ?Uk = prod (λi. ?g i $ p i) ?Uk
    by auto
  have Uk: finite ?Uk k ∉ ?Uk
    by auto
  have prod (λi. ?f i $ p i) ?U = prod (λi. ?f i $ p i) (insert k ?Uk)
    unfolding kU[symmetric] ..
  also have ... = ?f k $ p k  * prod (λi. ?f i $ p i) ?Uk
    by (rule prod.insert) auto
```

```
  also have ... = (c*s a k) $ p k * prod (λi. ?f i $ p i) ?Uk
    by (simp add: field_simps)
  also have ... = c* (a k $ p k * prod (λi. ?g i $ p i) ?Uk)
    unfolding eq by (simp add: ac_simps)
  also have ... = c* (prod (λi. ?g i $ p i) (insert k ?Uk))
    unfolding prod.insert[OF Uk] by simp
  finally have prod (λi. ?f i $ p i) ?U = c* (prod (λi. ?g i $ p i) ?U)
    unfolding kU[symmetric] .
  then show of_int (sign p) * prod (λi. ?f i $ p i) ?U = c * (of_int (sign p) *
prod (λi. ?g i $ p i) ?U)
    by (simp add: field_simps)
qed auto

lemma det_row_0:
  fixes b :: 'n::finite ⇒ _ ^ 'n
  shows det((χ i. if i = k then 0 else b i)::'a::comm_ring_1^'n^'n) = 0
  using det_row_mul[of k 0 λi. 1 b]
  apply simp
  apply (simp only: vector_smult_lzero)
  done

lemma det_row_operation:
  fixes A :: 'a::{comm_ring_1}^'n^'n
  assumes ij: i ≠ j
  shows det (χ k. if k = i then row i A + c *s row j A else row k A) = det A
proof −
  let ?Z = (χ k. if k = i then row j A else row k A) :: 'a ^'n^'n
  have th: row i ?Z = row j ?Z by (vector row_def)
  have th2: ((χ k. if k = i then row i A else row k A) :: 'a^'n^'n) = A
    by (vector row_def)
  show ?thesis
    unfolding det_row_add [of i] det_row_mul[of i] det_identical_rows[OF ij th]
th2
    by simp
qed

lemma det_row_span:
  fixes A :: 'a::{field}^'n^'n
  assumes x: x ∈ vec.span {row j A |j. j ≠ i}
  shows det (χ k. if k = i then row i A + x else row k A) = det A
  using x
proof (induction rule: vec.span_induct_alt)
  case base
  have (if k = i then row i A + 0 else row k A) = row k A for k
    by simp
  then show ?case
    by (simp add: row_def)
next
  case (step c z y)
```

```
  then obtain j where j: z = row j A i ≠ j
    by blast
  let ?w = row i A + y
  have th0: row i A + (c*s z + y) = ?w + c*s z
    by vector
  let ?d = λx. det (χ k. if k = i then x else row k A)
  have thz: ?d z = 0
    apply (rule det_identical_rows[OF j(2)])
    using j
    apply (vector row_def)
    done
  have ?d (row i A + (c*s z + y)) = ?d (?w + c*s z)
    unfolding th0 ..
  then have ?d (row i A + (c*s z + y)) = det A
    unfolding thz step.IH det_row_mul[of i] det_row_add[of i] by simp
  then show ?case
    unfolding scalar_mult_eq_scaleR .
qed

lemma matrix_id [simp]: det (matrix id) = 1
  by (simp add: matrix_id_mat_1)

proposition det_matrix_scaleR [simp]: det (matrix (((*R) r)) :: real^'n^'n) = r
^ CARD('n::finite)
  apply (subst det_diagonal)
   apply (auto simp: matrix_def mat_def)
  apply (simp add: cart_eq_inner_axis inner_axis_axis)
  done
```

May as well do this, though it's a bit unsatisfactory since it ignores exact duplicates by considering the rows/columns as a set.

```
lemma det_dependent_rows:
  fixes A:: 'a::{field}^'n^'n
  assumes d: vec.dependent (rows A)
  shows det A = 0
proof -
  let ?U = UNIV :: 'n set
  from d obtain i where i: row i A ∈ vec.span (rows A - {row i A})
    unfolding vec.dependent_def rows_def by blast
  show ?thesis
  proof (cases ∀ i j. i ≠ j ⟶ row i A ≠ row j A)
    case True
    with i have vec.span (rows A - {row i A}) ⊆ vec.span {row j A |j. j ≠ i}
      by (auto simp: rows_def intro!: vec.span_mono)
    then have - row i A ∈ vec.span {row j A|j. j ≠ i}
      by (meson i subsetCE vec.span_neg)
    from det_row_span[OF this]
    have det A = det (χ k. if k = i then 0 *s 1 else row k A)
      unfolding right_minus vector_smult_lzero ..
```

```
    with det_row_mul[of i 0 λi. 1]
    show ?thesis by simp
  next
    case False
    then obtain j k where jk: j ≠ k row j A = row k A
      by auto
    from det_identical_rows[OF jk] show ?thesis .
  qed
qed

lemma det_dependent_columns:
  assumes d: vec.dependent (columns (A::real^'n^'n))
  shows det A = 0
  by (metis d det_dependent_rows rows_transpose det_transpose)
```

Multilinearity and the multiplication formula

```
lemma Cart_lambda_cong: (⋀x. f x = g x) ⟹ (vec_lambda f::'a^'n) = (vec_lambda
g :: 'a^'n)
  by auto

lemma det_linear_row_sum:
  assumes fS: finite S
  shows det ((χ i. if i = k then sum (a i) S else c i)::'a::comm_ring_1^'n^'n) =
    sum (λj. det ((χ i. if i = k then a  i j else c i)::'a^'n^'n)) S
   using fS  by (induct rule: finite_induct; simp add: det_row_0 det_row_add
cong: if_cong)

lemma finite_bounded_functions:
  assumes fS: finite S
  shows finite {f. (∀ i ∈ {1.. (k::nat)}. f i ∈ S) ∧ (∀ i. i ∉ {1 .. k} ⟶ f i = i)}
proof (induct k)
  case 0
  have *: {f. ∀ i. f i = i} = {id}
    by auto
  show ?case
    by (auto simp: *)
next
  case (Suc k)
  let ?f = λ(y::nat,g) i. if i = Suc k then y else g i
  let ?S = ?f ' (S × {f. (∀ i∈{1..k}. f i ∈ S) ∧ (∀ i. i ∉ {1..k} ⟶ f i = i)})
  have ?S = {f. (∀ i∈{1.. Suc k}. f i ∈ S) ∧ (∀ i. i ∉ {1.. Suc k} ⟶ f i = i)}
    apply (auto simp: image_iff)
    apply (rename_tac f)
    apply (rule_tac x=f (Suc k) in bexI)
    apply (rule_tac x = λi. if i = Suc k then i else f i in exI, auto)
    done
  with finite_imageI[OF finite_cartesian_product[OF fS Suc.hyps(1)], of ?f]
  show ?case
    by metis
```

qed

lemma *det_linear_rows_sum_lemma*:
assumes *fS*: *finite S*
and *fT*: *finite T*
shows *det* ((χ *i*. *if i* ∈ *T then sum* (*a i*) *S else c i*):: *'a*::*comm_ring_1*^*'n*^*'n*) =
sum (λ*f*. *det*((χ *i*. *if i* ∈ *T then a i* (*f i*) *else c i*)::*'a*^*'n*^*'n*))
{*f*. (∀ *i* ∈ *T*. *f i* ∈ *S*) ∧ (∀ *i*. *i* ∉ *T* ⟶ *f i* = *i*)}
using *fT*
proof (*induct T arbitrary*: *a c set*: *finite*)
case *empty*
have *th0*: ⋀*x y*. (χ *i*. *if i* ∈ {} *then x i else y i*) = (χ *i*. *y i*)
by *vector*
from *empty.prems* **show** *?case*
unfolding *th0* **by** (*simp add*: *eq_id_iff*)
next
case (*insert z T a c*)
let *?F* = λ*T*. {*f*. (∀ *i* ∈ *T*. *f i* ∈ *S*) ∧ (∀ *i*. *i* ∉ *T* ⟶ *f i* = *i*)}
let *?h* = λ(*y*,*g*) *i*. *if i* = *z then y else g i*
let *?k* = λ*h*. (*h*(*z*),(λ*i*. *if i* = *z then i else h i*))
let *?s* = λ *k a c f*. *det*((χ *i*. *if i* ∈ *T then a i* (*f i*) *else c i*)::*'a*^*'n*^*'n*)
let *?c* = λ*j i*. *if i* = *z then a i j else c i*
have *thif*: ⋀*a b c d*. (*if a* ∨ *b then c else d*) = (*if a then c else if b then c else d*)
by *simp*
have *thif2*: ⋀*a b c d e*. (*if a then b else if c then d else e*) =
(*if c then* (*if a then b else d*) *else* (*if a then b else e*))
by *simp*
from ‹*z* ∉ *T*› **have** *nz*: ⋀*i*. *i* ∈ *T* ⟹ *i* ≠ *z*
by *auto*
have *det* (χ *i*. *if i* ∈ *insert z T then sum* (*a i*) *S else c i*) =
det (χ *i*. *if i* = *z then sum* (*a i*) *S else if i* ∈ *T then sum* (*a i*) *S else c i*)
unfolding *insert_iff thif* **..**
also have ... = (∑*j*∈*S*. *det* (χ *i*. *if i* ∈ *T then sum* (*a i*) *S else if i* = *z then a*
i j else c i))
unfolding *det_linear_row_sum*[*OF fS*]
by (*subst thif2*) (*simp add*: *nz cong*: *if_cong*)
finally have *tha*:
det (χ *i*. *if i* ∈ *insert z T then sum* (*a i*) *S else c i*) =
(∑ (*j*, *f*)∈*S* × *?F T*. *det* (χ *i*. *if i* ∈ *T then a i* (*f i*)
else if i = *z then a i j*
else c i))
unfolding *insert.hyps* **unfolding** *sum.cartesian_product* **by** *blast*
show *?case* **unfolding** *tha*
using ‹*z* ∉ *T*›
by (*intro sum.reindex_bij_witness*[**where** *i*=*?k* **and** *j*=*?h*])
(*auto intro*!: *cong*[*OF refl*[*of det*]] *simp*: *vec_eq_iff*)
qed

lemma *det_linear_rows_sum*:
fixes $S :: 'n::finite\ set$
assumes *fS*: *finite S*
shows $det\ (\chi\ i.\ sum\ (a\ i)\ S) =$
$sum\ (\lambda f.\ det\ (\chi\ i.\ a\ i\ (f\ i)) :: 'a::comm\_ring\_1\ \hat{}\ 'n\hat{}'n))\ \{f.\ \forall i.\ f\ i \in S\}$
proof −
have *th0*: $\bigwedge x\ y.\ ((\chi\ i.\ if\ i \in (UNIV :: 'n\ set)\ then\ x\ i\ else\ y\ i) :: 'a\hat{}'n\hat{}'n) = (\chi\ i.\ x\ i)$
by *vector*
from *det_linear_rows_sum_lemma*[*OF fS*, *of UNIV* :: *'n set a*, *unfolded th0*, *OF finite*]
show *?thesis* **by** *simp*
qed

lemma *matrix_mul_sum_alt*:
fixes $A\ B :: 'a::comm\_ring\_1\hat{}'n\hat{}'n$
shows $A\ **\ B = (\chi\ i.\ sum\ (\lambda k.\ A\$i\$k * s\ B\ \$\ k)\ (UNIV :: 'n\ set))$
by (*vector matrix_matrix_mult_def sum_component*)

lemma *det_rows_mul*:
$det((\chi\ i.\ c\ i\ *s\ a\ i)::'a::comm\_ring\_1\hat{}'n\hat{}'n) =$
$prod\ (\lambda i.\ c\ i)\ (UNIV :: 'n\ set) * det((\chi\ i.\ a\ i)::'a\hat{}'n\hat{}'n)$
proof (*simp add*: *det_def sum_distrib_left cong add*: *prod.cong*, *rule sum.cong*)
let $?U = UNIV :: 'n\ set$
let $?PU = \{p.\ p\ permutes\ ?U\}$
fix *p*
assume *pU*: $p \in ?PU$
let $?s = of\_int\ (sign\ p)$
from *pU* **have** *p*: *p permutes ?U*
by *blast*
have $prod\ (\lambda i.\ c\ i * a\ i\ \$\ p\ i)\ ?U = prod\ c\ ?U * prod\ (\lambda i.\ a\ i\ \$\ p\ i)\ ?U$
unfolding *prod.distrib* **..**
then show $?s * (\prod xa \in ?U.\ c\ xa * a\ xa\ \$\ p\ xa) =$
$prod\ c\ ?U * (?s * (\prod xa \in ?U.\ a\ xa\ \$\ p\ xa))$
by (*simp add*: *field_simps*)
qed *rule*

proposition *det_mul*:
fixes $A\ B :: 'a::comm\_ring\_1\hat{}'n\hat{}'n$
shows $det\ (A\ **\ B) = det\ A * det\ B$
proof −
let $?U = UNIV :: 'n\ set$
let $?F = \{f.\ (\forall i \in ?U.\ f\ i \in ?U) \wedge (\forall i.\ i \notin ?U \longrightarrow f\ i = i)\}$
let $?PU = \{p.\ p\ permutes\ ?U\}$
have $p \in ?F$ **if** *p permutes ?U* **for** *p*
by *simp*
then have *PUF*: $?PU \subseteq ?F$ **by** *blast*
{
fix *f*

```
    assume fPU: f ∈ ?F − ?PU
    have fUU: f ‘ ?U ⊆ ?U
      using fPU by auto
    from fPU have f: ∀ i ∈ ?U. f i ∈ ?U ∀ i. i ∉ ?U ⟶ f i = i ¬(∀ y. ∃!x. f x
= y)
      unfolding permutes_def by auto

    let ?A = (χ i. A$i$f i *s B$f i) :: 'a^'n^'n
    let ?B = (χ i. B$f i) :: 'a^'n^'n
    {
      assume fni: ¬ inj_on f ?U
      then obtain i j where ij: f i = f j i ≠ j
        unfolding inj_on_def by blast
      then have row i ?B = row j ?B
        by (vector row_def)
      with det_identical_rows[OF ij(2)]
      have det (χ i. A$i$f i *s B$f i) = 0
        unfolding det_rows_mul by force
    }
    moreover
    {
      assume fi: inj_on f ?U
      from f fi have fith: ⋀i j. f i = f j ⟹ i = j
        unfolding inj_on_def by metis
       note fs = fi[unfolded surjective_iff_injective_gen[OF finite finite refl fUU,
symmetric]]
      have ∃!x. f x = y for y
        using fith fs by blast
      with f(3) have det (χ i. A$i$f i *s B$f i) = 0
        by blast
    }
    ultimately have det (χ i. A$i$f i *s B$f i) = 0
      by blast
  }
  then have zth: ∀ f∈ ?F − ?PU. det (χ i. A$i$f i *s B$f i) = 0
    by simp
  {
    fix p
    assume pU: p ∈ ?PU
    from pU have p: p permutes ?U
      by blast
    let ?s = λp. of_int (sign p)
    let ?f = λq. ?s p * (∏ i∈ ?U. A $ i $ p i) * (?s q * (∏ i∈ ?U. B $ i $ q i))
    have (sum (λq. ?s q *
        (∏ i∈ ?U. (χ i. A $ i $ p i *s B $ p i :: 'a^'n^'n) $ i $ q i)) ?PU) =
      (sum (λq. ?s p * (∏ i∈ ?U. A $ i $ p i) * (?s q * (∏ i∈ ?U. B $ i $ q i)))
?PU)
      unfolding sum_permutations_compose_right[OF permutes_inv[OF p], of ?f]
    proof (rule sum.cong)
```

```
    fix q
    assume qU: q ∈ ?PU
    then have q: q permutes ?U
      by blast
    from p q have pp: permutation p and pq: permutation q
      unfolding permutation_permutes by auto
    have th00: of_int (sign p) * of_int (sign p) = (1::'a)
      ⋀a. of_int (sign p) * (of_int (sign p) * a) = a
      unfolding mult.assoc[symmetric]
      unfolding of_int_mult[symmetric]
      by (simp_all add: sign_idempotent)
    have ths: ?s q = ?s p * ?s (q ∘ inv p)
      using pp pq permutation_inverse[OF pp] sign_inverse[OF pp]
      by (simp add: th00 ac_simps sign_idempotent sign_compose)
    have th001: prod (λi. B$i$ q (inv p i)) ?U = prod ((λi. B$i$ q (inv p i)) ∘
p) ?U
      by (rule prod.permute[OF p])
    have thp: prod (λi. (χ i. A$i$p i *s B$p i :: 'a^'n^'n) $i $ q i) ?U =
      prod (λi. A$i$p i) ?U * prod (λi. B$i$ q (inv p i)) ?U
      unfolding th001 prod.distrib[symmetric] o_def permutes_inverses[OF p]
      apply (rule prod.cong[OF refl])
      using permutes_in_image[OF q]
      apply vector
      done
    show ?s q * prod (λi. (((χ i. A$i$p i *s B$p i) :: 'a^'n^'n)$i$q i)) ?U =
      ?s p * (prod (λi. A$i$p i) ?U) * (?s (q ∘ inv p) * prod (λi. B$i$(q ∘ inv p)
i) ?U)
      using ths thp pp pq permutation_inverse[OF pp] sign_inverse[OF pp]
      by (simp add: sign_nz th00 field_simps sign_idempotent sign_compose)
  qed rule
 }
 then have th2: sum (λf. det (χ i. A$i$f i *s B$f i)) ?PU = det A * det B
  unfolding det_def sum_product
  by (rule sum.cong [OF refl])
 have det (A**B) = sum (λf.  det (χ i. A $ i $ f i *s B $ f i)) ?F
  unfolding matrix_mul_sum_alt det_linear_rows_sum[OF finite]
  by simp
 also have ... = sum (λf. det (χ i. A$i$f i *s B$f i)) ?PU
  using sum.mono_neutral_cong_left[OF finite PUF zth, symmetric]
  unfolding det_rows_mul by auto
 finally show ?thesis unfolding th2 .
qed
```

1.12.2 Relation to invertibility

```
proposition invertible_det_nz:
 fixes A::'a::{field}^'n^'n
 shows invertible A ⟷ det A ≠ 0
proof (cases invertible A)
```

```
  case True
  then obtain B :: 'a^'n^'n where B: A ** B = mat 1
    unfolding invertible_right_inverse by blast
  then have det (A ** B) = det (mat 1 :: 'a^'n^'n)
    by simp
  then show ?thesis
    by (metis True det_I det_mul mult_zero_left one_neq_zero)
next
  case False
  let ?U = UNIV :: 'n set
  have fU: finite ?U
    by simp
  from False obtain c i where c: sum (λi. c i *s row i A) ?U = 0 and iU: i ∈
?U and ci: c i ≠ 0
   unfolding invertible_right_inverse matrix_right_invertible_independent_rows
    by blast
  have thr0: − row i A = sum (λj. (1/ c i) *s (c j *s row j A)) (?U − {i})
    unfolding sum_cmul  using c ci
    by (auto simp: sum.remove[OF fU iU] eq_vector_fraction_iff add_eq_0_iff)
  have thr: − row i A ∈ vec.span {row j A| j. j ≠ i}
    unfolding thr0 by (auto intro: vec.span_base vec.span_scale vec.span_sum)
  let ?B = (χ k. if k = i then 0 else row k A) :: 'a^'n^'n
  have thrb: row i ?B = 0 using iU by (vector row_def)
  have det A = 0
    unfolding det_row_span[OF thr, symmetric] right_minus
    unfolding det_zero_row(2)[OF thrb] ..
  then show ?thesis
    by (simp add: False)
qed


lemma  det_nz_iff_inj_gen:
  fixes f :: 'a::field^'n ⇒ 'a::field^'n
  assumes Vector_Spaces.linear (*s) (*s) f
  shows det (matrix f) ≠ 0 ⟷ inj f
proof
  assume det (matrix f) ≠ 0
  then show inj f
    using assms invertible_det_nz inj_matrix_vector_mult by force
next
  assume inj f
  show det (matrix f) ≠ 0
    using vec.linear_injective_left_inverse [OF assms ‹inj f›]
   by (metis assms invertible_det_nz invertible_left_inverse matrix_compose_gen
matrix_id_mat_1)
qed

lemma  det_nz_iff_inj:
  fixes f :: real^'n ⇒ real^'n
```

assumes *linear f*
shows *det (matrix f) ≠ 0 ⟷ inj f*
using *det_nz_iff_inj_gen[of f] assms*
unfolding *linear_matrix_vector_mul_eq* **.**

lemma *det_eq_0_rank*:
fixes *A* :: *real^'n^'n*
shows *det A = 0 ⟷ rank A < CARD('n)*
using *invertible_det_nz [of A]*
by (*auto simp*: *matrix_left_invertible_injective invertible_left_inverse less_rank_noninjective*)

Invertibility of matrices and corresponding linear functions

lemma *matrix_left_invertible_gen*:
fixes *f* :: *'a::field^'m ⇒ 'a::field^'n*
assumes *Vector_Spaces.linear (∗s) (∗s) f*
shows ((∃ *B*. *B ∗∗ matrix f = mat 1*) ⟷ (∃ *g*. *Vector_Spaces.linear (∗s) (∗s) g ∧ g ∘ f = id*))
proof *safe*
fix *B*
assume *1*: *B ∗∗ matrix f = mat 1*
show ∃ *g*. *Vector_Spaces.linear (∗s) (∗s) g ∧ g ∘ f = id*
proof (*intro exI conjI*)
show *Vector_Spaces.linear (∗s) (∗s) (λy. B ∗v y)*
by *simp*
show *((∗v) B) ∘ f = id*
unfolding *o_def*
by (*metis assms 1 eq_id_iff matrix_vector_mul(1) matrix_vector_mul_assoc matrix_vector_mul_lid*)
qed
next
fix *g*
assume *Vector_Spaces.linear (∗s) (∗s) g g ∘ f = id*
then have *matrix g ∗∗ matrix f = mat 1*
by (*metis assms matrix_compose_gen matrix_id_mat_1*)
then show ∃ *B*. *B ∗∗ matrix f = mat 1* **..**
qed

lemma *matrix_left_invertible*:
linear f ⟹ ((∃ *B*. *B ∗∗ matrix f = mat 1*) ⟷ (∃ *g*. *linear g ∧ g ∘ f = id*)) **for** *f*::*real^'m ⇒ real^'n*
using *matrix_left_invertible_gen[of f]*
by (*auto simp*: *linear_matrix_vector_mul_eq*)

lemma *matrix_right_invertible_gen*:
fixes *f* :: *'a::field^'m ⇒ 'a^'n*
assumes *Vector_Spaces.linear (∗s) (∗s) f*
shows ((∃ *B*. *matrix f ∗∗ B = mat 1*) ⟷ (∃ *g*. *Vector_Spaces.linear (∗s) (∗s) g ∧ f ∘ g = id*))

proof *safe*
fix *B*
assume *1*: *matrix f ** B = mat 1*
show ∃ *g*. *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* ∧ *f* ∘ *g* = *id*
proof (*intro exI conjI*)
show *Vector_Spaces.linear* (∗*s*) (∗*s*) ((∗*v*) *B*)
by *simp*
show *f* ∘ (∗*v*) *B* = *id*
using *1 assms comp_apply eq_id_iff vec.linear_id matrix_id_mat_1 matrix_vector_mul_assoc matrix_works*
by (*metis* (*no_types*, *opaque_lifting*))
qed
next
fix *g*
assume *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* **and** *f* ∘ *g* = *id*
then have *matrix f ** *matrix g* = *mat 1*
by (*metis assms matrix_compose_gen matrix_id_mat_1*)
then show ∃ *B*. *matrix f ** *B* = *mat 1* **..**
qed

lemma *matrix_right_invertible*:
linear f ⟹ ((∃ *B*. *matrix f ** *B* = *mat 1*) ⟷ (∃ *g*. *linear g* ∧ *f* ∘ *g* = *id*)) **for** *f*::*real*^'*m* ⇒ *real*^'*n*
using *matrix_right_invertible_gen*[*of f*]
by (*auto simp*: *linear_matrix_vector_mul_eq*)

lemma *matrix_invertible_gen*:
fixes *f* :: '*a*::*field*^'*m* ⇒ '*a*::*field*^'*n*
assumes *Vector_Spaces.linear* (∗*s*) (∗*s*) *f*
shows *invertible* (*matrix f*) ⟷ (∃ *g*. *Vector_Spaces.linear* (∗*s*) (∗*s*) *g* ∧ *f* ∘ *g* = *id* ∧ *g* ∘ *f* = *id*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis assms invertible_def left_right_inverse_eq matrix_left_invertible_gen matrix_right_invertible_gen*)
next
assume *?rhs* **then show** *?lhs*
by (*metis assms invertible_def matrix_compose_gen matrix_id_mat_1*)
qed

lemma *matrix_invertible*:
linear f ⟹ *invertible* (*matrix f*) ⟷ (∃ *g*. *linear g* ∧ *f* ∘ *g* = *id* ∧ *g* ∘ *f* = *id*)
for *f*::*real*^'*m* ⇒ *real*^'*n*
using *matrix_invertible_gen*[*of f*]
by (*auto simp*: *linear_matrix_vector_mul_eq*)

lemma *invertible_eq_bij*:
fixes *m* :: '*a*::*field*^'*m*^'*n*

shows *invertible m* $\longleftrightarrow$ *bij* ((∗*v*) *m*)
using *matrix_invertible_gen*[*OF matrix_vector_mul_linear_gen*, *of m*, *simplified matrix_of_matrix_vector_mul*]
by (*metis bij_betw_def left_right_inverse_eq matrix_vector_mul_linear_gen o_bij*
vec.linear_injective_left_inverse vec.linear_surjective_right_inverse)

1.12.3 Cramer's rule

lemma *cramer_lemma_transpose*:
fixes *A*:: *'a*::{*field*}^*'n*^*'n*
and *x* :: *'a*::{*field*}^*'n*
shows *det* ((χ *i*. *if i* = *k then sum* (λ*i*. *x*$*i* ∗*s row i A*) (*UNIV*::*'n set*)
else row i A)::*'a*::{*field*}^*'n*^*'n*) = *x*$*k* ∗ *det A*
(**is** *?lhs* = *?rhs*)
proof −
let *?U* = *UNIV* :: *'n set*
let *?Uk* = *?U* − {*k*}
have *U*: *?U* = *insert k ?Uk*
by *blast*
have *kUk*: *k* ∉ *?Uk*
by *simp*
have *th00*: ⋀*k s*. *x*$*k* ∗*s row k A* + *s* = (*x*$*k* − *1*) ∗*s row k A* + *row k A* + *s*
by (*vector field_simps*)
have *th001*: ⋀*f k* . (λ*x*. *if x* = *k then f k else f x*) = *f*
by *auto*
have (χ *i*. *row i A*) = *A* **by** (*vector row_def*)
then have *thd1*: *det* (χ *i*. *row i A*) = *det A*
by *simp*
have *thd0*: *det* (χ *i*. *if i* = *k then row k A* + (∑ *i* ∈ *?Uk*. *x* $ *i* ∗*s row i A*) *else row i A*) = *det A*
by (*force intro*: *det_row_span vec.span_sum vec.span_scale vec.span_base*)
show *?lhs* = *x*$*k* ∗ *det A*
apply (*subst U*)
unfolding *sum.insert*[*OF finite kUk*]
apply (*subst th00*)
unfolding *add.assoc*
apply (*subst det_row_add*)
unfolding *thd0*
unfolding *det_row_mul*
unfolding *th001*[*of k* λ*i*. *row i A*]
unfolding *thd1*
apply (*simp add*: *field_simps*)
done
qed

proposition *cramer_lemma*:
fixes *A* :: *'a*::{*field*}^*'n*^*'n*
shows *det*((χ *i j*. *if j* = *k then* (*A* ∗*v x*)$*i else A*$*i*$*j*):: *'a*::{*field*}^*'n*^*'n*) = *x*$*k*

```
* det A
proof −
  let ?U = UNIV :: 'n set
  have *: ⋀c. sum (λi. c i *s row i (transpose A)) ?U = sum (λi. c i *s column i
A) ?U
    by (auto intro: sum.cong)
  show ?thesis
    unfolding matrix_mult_sum
   unfolding cramer_lemma_transpose[of k x transpose A, unfolded det_transpose,
symmetric]
    unfolding *[of λi. x$i]
    apply (subst det_transpose[symmetric])
    apply (rule cong[OF refl[of det]])
    apply (vector transpose_def column_def row_def)
    done
qed

proposition cramer:
  fixes A ::'a::{field}^'n^'n
  assumes d0: det A ≠ 0
  shows A *v x = b ⟷ x = (χ k. det(χ i j. if j=k then b$i else A$i$j) / det A)
proof −
  from d0 obtain B where B: A ** B = mat 1 B ** A = mat 1
    unfolding invertible_det_nz[symmetric] invertible_def
    by blast
  have (A ** B) *v b = b
    by (simp add: B)
  then have A *v (B *v b) = b
    by (simp add: matrix_vector_mul_assoc)
  then have xe: ∃x. A *v x = b
    by blast
  {
    fix x
    assume x: A *v x = b
    have x = (χ k. det(χ i j. if j=k then b$i else A$i$j) / det A)
      unfolding x[symmetric]
      using d0 by (simp add: vec_eq_iff cramer_lemma field_simps)
  }
  with xe show ?thesis
    by auto
qed

lemma det_1: det (A::'a::comm_ring_1^1^1) = A$1$1
  by (simp add: det_def sign_id)

lemma det_2: det (A::'a::comm_ring_1^2^2) = A$1$1 * A$2$2 − A$1$2 *
A$2$1
proof −
  have f12: finite {2::2} 1 ∉ {2::2} by auto
```

```
  show ?thesis
    unfolding det_def UNIV_2
    unfolding sum_over_permutations_insert[OF f12]
    unfolding permutes_sing
    by (simp add: sign_swap_id sign_id swap_id_eq)
qed

lemma det_3:
  det (A::'a::comm_ring_1^3^3) =
    A$1$1 * A$2$2 * A$3$3 +
    A$1$2 * A$2$3 * A$3$1 +
    A$1$3 * A$2$1 * A$3$2 -
    A$1$1 * A$2$3 * A$3$2 -
    A$1$2 * A$2$1 * A$3$3 -
    A$1$3 * A$2$2 * A$3$1
proof -
  have f123: finite {2::3, 3} 1 ∉ {2::3, 3}
    by auto
  have f23: finite {3::3} 2 ∉ {3::3}
    by auto

  show ?thesis
    unfolding det_def UNIV_3
    unfolding sum_over_permutations_insert[OF f123]
    unfolding sum_over_permutations_insert[OF f23]
    unfolding permutes_sing
      by (simp add: sign_swap_id permutation_swap_id sign_compose sign_id
swap_id_eq)
qed

proposition det_orthogonal_matrix:
  fixes Q:: 'a::linordered_idom^'n^'n
  assumes oQ: orthogonal_matrix Q
  shows det Q = 1 ∨ det Q = - 1
proof -
  have Q ** transpose Q = mat 1
    by (metis oQ orthogonal_matrix_def)
  then have det (Q ** transpose Q) = det (mat 1:: 'a^'n^'n)
    by simp
  then have det Q * det Q = 1
    by (simp add: det_mul)
  then show ?thesis
    by (simp add: square_eq_1_iff)
qed

proposition orthogonal_transformation_det [simp]:
  fixes f :: real^'n ⇒ real^'n
  shows orthogonal_transformation f ⟹ |det (matrix f)| = 1
  using det_orthogonal_matrix orthogonal_transformation_matrix by fastforce
```

1.12.4 Rotation, reflection, rotoinversion

definition *rotation_matrix Q* $\longleftrightarrow$ *orthogonal_matrix Q* $\wedge$ *det Q = 1*
definition *rotoinversion_matrix Q* $\longleftrightarrow$ *orthogonal_matrix Q* $\wedge$ *det Q = − 1*

lemma *orthogonal_rotation_or_rotoinversion*:
fixes *Q* :: *'a::linordered_idom^'n^'n*
shows *orthogonal_matrix Q* $\longleftrightarrow$ *rotation_matrix Q* $\vee$ *rotoinversion_matrix Q*
by (*metis rotoinversion_matrix_def rotation_matrix_def det_orthogonal_matrix*)

Slightly stronger results giving rotation, but only in two or more dimensions

lemma *rotation_matrix_exists_basis*:
fixes *a* :: *real^'n*
assumes *2*: *2* $\leq$ *CARD('n)* **and** *norm a = 1*
obtains *A* **where** *rotation_matrix A A ∗v (axis k 1) = a*
proof −
obtain *A* **where** *orthogonal_matrix A* **and** *A*: *A ∗v (axis k 1) = a*
using *orthogonal_matrix_exists_basis assms* **by** *metis*
with *orthogonal_rotation_or_rotoinversion*
consider *rotation_matrix A* | *rotoinversion_matrix A*
by *metis*
then show *thesis*
proof *cases*
assume *rotation_matrix A*
then show *?thesis*
using ‹*A ∗v axis k 1 = a*› *that* **by** *auto*
next
from *obtain_subset_with_card_n*[*OF 2*] **obtain** *h i*::*'n* **where** *h ≠ i*
by (*fastforce simp add*: *eval_nat_numeral card_Suc_eq*)
then obtain *j* **where** *j ≠ k*
by (*metis* (*full_types*))
let *?TA = transpose A*
let *?A = χ i. if i = j then − 1 ∗R* (*?TA $ i*) *else ?TA $i*
assume *rotoinversion_matrix A*
then have [*simp*]: *det A = −1*
by (*simp add*: *rotoinversion_matrix_def*)
show *?thesis*
proof
have [*simp*]: *row i* (*χ i. if i = j then − 1 ∗R ?TA $ i else ?TA $ i*) = (*if i = j then − row i ?TA else row i ?TA*) **for** *i*
by (*auto simp*: *row_def*)
have *orthogonal_matrix ?A*
unfolding *orthogonal_matrix_orthonormal_rows*
using ‹*orthogonal_matrix A*› **by** (*auto simp*: *orthogonal_matrix_orthonormal_columns orthogonal_clauses*)
then show *rotation_matrix* (*transpose ?A*)
unfolding *rotation_matrix_def*
by (*simp add*: *det_row_mul*[*of j _ λi. ?TA $ i, unfolded scalar_mult_eq_scaleR*])
show *transpose ?A ∗v axis k 1 = a*
using ‹*j ≠ k*› *A* **by** (*simp add*: *matrix_vector_column axis_def scalar_mult_eq_scaleR*

```
if_distrib [of λz. z *R c for c] cong: if_cong)
    qed
  qed
qed

lemma rotation_exists_1:
  fixes a :: real^'n
  assumes 2 ≤ CARD('n) norm a = 1 norm b = 1
  obtains f where orthogonal_transformation f det(matrix f) = 1 f a = b
proof −
  obtain k::'n where True
    by simp
  obtain A B where AB: rotation_matrix A rotation_matrix B
              and eq: A *v (axis k 1) = a B *v (axis k 1) = b
    using rotation_matrix_exists_basis assms by metis
  let ?f = λx. (B ** transpose A) *v x
  show thesis
  proof
    show orthogonal_transformation ?f
      using AB orthogonal_matrix_mul orthogonal_transformation_matrix rota-
tion_matrix_def matrix_vector_mul_linear by force
    show det (matrix ?f) = 1
      using AB by (auto simp: det_mul rotation_matrix_def)
    show ?f a = b
      using AB unfolding orthogonal_matrix_def rotation_matrix_def
      by (metis eq matrix_mul_rid matrix_vector_mul_assoc)
  qed
qed

lemma rotation_exists:
  fixes a :: real^'n
  assumes 2: 2 ≤ CARD('n) and eq: norm a = norm b
  obtains f where orthogonal_transformation f det(matrix f) = 1 f a = b
proof (cases a = 0 ∨ b = 0)
  case True
  with assms have a = 0 b = 0
    by auto
  then show ?thesis
    by (metis eq_id_iff matrix_id orthogonal_transformation_id that)
next
  case False
  then obtain f where f: orthogonal_transformation f det (matrix f) = 1
    and f': f (a /R norm a) = b /R norm b
    using rotation_exists_1 [of a /R norm a b /R norm b, OF 2] by auto
  then interpret linear f by (simp add: orthogonal_transformation)
  have f a = b
    using f' False
    by (simp add: eq scale)
  with f show thesis ..
```

qed

lemma *rotation_rightward_line*:
fixes *a* :: *real^'n*
obtains *f* **where** *orthogonal_transformation f 2* ≤ *CARD('n)* ⟹ *det(matrix f) = 1*
*f(norm a *R axis k 1) = a*
proof (*cases CARD('n) = 1*)
case *True*
obtain *f* **where** *orthogonal_transformation f f (norm a *R axis k (1::real)) = a*
proof (*rule orthogonal_transformation_exists*)
show *norm (norm a *R axis k (1::real)) = norm a*
by *simp*
qed *auto*
then show *thesis*
using *True that* **by** *auto*
next
case *False*
obtain *f* **where** *orthogonal_transformation f det(matrix f) = 1 f (norm a *R axis k 1) = a*
proof (*rule rotation_exists*)
show *2* ≤ *CARD('n)*
using *False one_le_card_finite* [**where** *'a='n*] **by** *linarith*
show *norm (norm a *R axis k (1::real)) = norm a*
by *simp*
qed *auto*
then show *thesis*
using *that* **by** *blast*
qed

end

1.13 Operators involving abstract topology

theory *Abstract_Topology*
imports
Complex_Main
HOL−Library.Set_Idioms
HOL−Library.FuncSet
begin

1.13.1 General notion of a topology as a value

definition *istopology* :: (*'a set* ⇒ *bool*) ⇒ *bool* **where**
istopology L ≡ (∀ *S T*. *L S* ⟶ *L T* ⟶ *L (S* ∩ *T)*) ∧ (∀ *K*. (∀ *K*∈*K*. *L K*) ⟶ *L* (⋃ *K*))

typedef *'a topology* = {*L*::(*'a set*) ⇒ *bool*. *istopology L*}
morphisms *openin topology*

unfolding *istopology_def* **by** *blast*

lemma *istopology_openin*[*iff*]: *istopology*(*openin U*)
using *openin*[*of U*] **by** *blast*

lemma *istopology_open*[*iff*]: *istopology open*
by (*auto simp*: *istopology_def*)

lemma *topology_inverse'* [*simp*]: *istopology U* $\Longrightarrow$ *openin* (*topology U*) = *U*
using *topology_inverse*[*unfolded mem_Collect_eq*] **.**

lemma *topology_inverse_iff*: *istopology U* $\longleftrightarrow$ *openin* (*topology U*) = *U*
by (*metis istopology_openin topology_inverse'*)

lemma *topology_eq*: *T1* = *T2* $\longleftrightarrow$ ($\forall$ *S*. *openin T1 S* $\longleftrightarrow$ *openin T2 S*)
proof
assume *T1* = *T2*
then show $\forall$ *S*. *openin T1 S* $\longleftrightarrow$ *openin T2 S* **by** *simp*
next
assume *H*: $\forall$ *S*. *openin T1 S* $\longleftrightarrow$ *openin T2 S*
then have *openin T1* = *openin T2* **by** (*simp add*: *fun_eq_iff*)
then have *topology* (*openin T1*) = *topology* (*openin T2*) **by** *simp*
then show *T1* = *T2* **unfolding** *openin_inverse* **.**
qed

The "universe": the union of all sets in the topology.

definition *topspace T* = $\bigcup$ {*S*. *openin T S*}

Main properties of open sets

proposition *openin_clauses*:
fixes *U* :: *'a topology*
shows
openin U {}
$\bigwedge$*S T*. *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* (*S*$\cap$*T*)
$\bigwedge$*K*. ($\forall$ *S* $\in$ *K*. *openin U S*) $\Longrightarrow$ *openin U* ($\bigcup$ *K*)
using *openin*[*of U*] **unfolding** *istopology_def* **by** *auto*

lemma *openin_subset*: *openin U S* $\Longrightarrow$ *S* $\subseteq$ *topspace U*
unfolding *topspace_def* **by** *blast*

lemma *openin_empty*[*simp*]: *openin U* {}
by (*rule openin_clauses*)

lemma *openin_Int*[*intro*]: *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* (*S* $\cap$ *T*)
by (*rule openin_clauses*)

lemma *openin_Union*[*intro*]: ($\bigwedge$*S*. *S* $\in$ *K* $\Longrightarrow$ *openin U S*) $\Longrightarrow$ *openin U* ($\bigcup$ *K*)
using *openin_clauses* **by** *blast*

lemma *openin_Un*[*intro*]: *openin U S* $\Longrightarrow$ *openin U T* $\Longrightarrow$ *openin U* (*S* $\cup$ *T*)
using *openin_Union*[*of* {*S*,*T*} *U*] **by** *auto*

lemma *openin_topspace*[*intro*, *simp*]: *openin U* (*topspace U*)
by (*force simp*: *openin_Union topspace_def*)

lemma *openin_subopen*: *openin U S* $\longleftrightarrow$ ($\forall$ *x* $\in$ *S*. $\exists$ *T*. *openin U T* $\wedge$ *x* $\in$ *T* $\wedge$ *T* $\subseteq$ *S*)
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then show *?rhs* **by** *auto*
next
assume *H*: *?rhs*
let *?t* = $\bigcup$ {*T*. *openin U T* $\wedge$ *T* $\subseteq$ *S*}
have *openin U ?t* **by** (*force simp*: *openin_Union*)
also have *?t* = *S* **using** *H* **by** *auto*
finally show *openin U S* **.**
qed

lemma *openin_INT* [*intro*]:
assumes *finite I*
$\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *openin T* (*U i*)
shows *openin T* (($\bigcap$ *i* $\in$ *I*. *U i*) $\cap$ *topspace T*)
using *assms* **by** (*induct*, *auto simp*: *inf_sup_aci*(*2*) *openin_Int*)

lemma *openin_INT2* [*intro*]:
assumes *finite I I* $\neq$ {}
$\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *openin T* (*U i*)
shows *openin T* ($\bigcap$ *i* $\in$ *I*. *U i*)
proof −
have ($\bigcap$ *i* $\in$ *I*. *U i*) $\subseteq$ *topspace T*
using ‹*I* $\neq$ {}› *openin_subset*[*OF assms*(*3*)] **by** *auto*
then show *?thesis*
using *openin_INT*[*of* _ _ *U*, *OF assms*(*1*) *assms*(*3*)] **by** (*simp add*: *inf.absorb2 inf_commute*)
qed

lemma *openin_Inter* [*intro*]:
assumes *finite* $\mathcal{F}$ $\mathcal{F}$ $\neq$ {} $\bigwedge$*X*. *X* $\in$ $\mathcal{F}$ $\Longrightarrow$ *openin T X* **shows** *openin T* ($\bigcap$ $\mathcal{F}$)
by (*metis* (*full_types*) *assms openin_INT2 image_ident*)

lemma *openin_Int_Inter*:
assumes *finite* $\mathcal{F}$ *openin T U* $\bigwedge$*X*. *X* $\in$ $\mathcal{F}$ $\Longrightarrow$ *openin T X* **shows** *openin T* (*U* $\cap$ $\bigcap$ $\mathcal{F}$)
using *openin_Inter* [*of insert U* $\mathcal{F}$] *assms* **by** *auto*

Closed sets

definition *closedin* :: *'a topology* ⇒ *'a set* ⇒ *bool* **where**
closedin U S ⟷ *S* ⊆ *topspace U* ∧ *openin U* (*topspace U* − *S*)

lemma *closedin_subset*: *closedin U S* ⟹ *S* ⊆ *topspace U*
by (*metis closedin_def*)

lemma *closedin_empty*[*simp*]: *closedin U* {}
by (*simp add*: *closedin_def*)

lemma *closedin_topspace*[*intro*, *simp*]: *closedin U* (*topspace U*)
by (*simp add*: *closedin_def*)

lemma *closedin_Un*[*intro*]: *closedin U S* ⟹ *closedin U T* ⟹ *closedin U* (*S* ∪ *T*)
by (*auto simp*: *Diff_Un closedin_def*)

lemma *Diff_Inter*[*intro*]: *A* − ⋂ *S* = ⋃ {*A* − *s*|*s*. *s*∈*S*}
by *auto*

lemma *closedin_Union*:
assumes *finite S* ⋀*T*. *T* ∈ *S* ⟹ *closedin U T*
shows *closedin U* (⋃ *S*)
using *assms* **by** *induction auto*

lemma *closedin_Inter*[*intro*]:
assumes *Ke*: *K* ≠ {}
and *Kc*: ⋀*S*. *S* ∈*K* ⟹ *closedin U S*
shows *closedin U* (⋂ *K*)
using *Ke Kc* **unfolding** *closedin_def Diff_Inter* **by** *auto*

lemma *closedin_INT*[*intro*]:
assumes *A* ≠ {} ⋀*x*. *x* ∈ *A* ⟹ *closedin U* (*B x*)
shows *closedin U* (⋂ *x*∈*A*. *B x*)
using *assms* **by** *blast*

lemma *closedin_Int*[*intro*]: *closedin U S* ⟹ *closedin U T* ⟹ *closedin U* (*S* ∩ *T*)
using *closedin_Inter*[*of* {*S*,*T*} *U*] **by** *auto*

lemma *openin_closedin_eq*: *openin U S* ⟷ *S* ⊆ *topspace U* ∧ *closedin U* (*topspace U* − *S*)
by (*metis Diff_subset closedin_def double_diff equalityD1 openin_subset*)

lemma *topology_finer_closedin*:
topspace X = *topspace Y* ⟹ (∀ *S*. *openin Y S* ⟶ *openin X S*) ⟷ (∀ *S*. *closedin Y S* ⟶ *closedin X S*)
by (*metis closedin_def openin_closedin_eq*)

lemma *openin_closedin*: $S \subseteq$ *topspace* $U \Longrightarrow$ (*openin* U $S \longleftrightarrow$ *closedin* U (*topspace* $U - S$))
by (*simp add*: *openin_closedin_eq*)

lemma *openin_diff*[*intro*]:
assumes *oS*: *openin* U S
and *cT*: *closedin* U T
shows *openin* U ($S - T$)
by (*metis Int_Diff cT closedin_def inf.orderE oS openin_Int openin_subset*)

lemma *closedin_diff*[*intro*]:
assumes *oS*: *closedin* U S
and *cT*: *openin* U T
shows *closedin* U ($S - T$)
by (*metis Int_Diff cT closedin_Int closedin_subset inf.orderE oS openin_closedin_eq*)

lemma *all_openin*: ($\forall U.$ *openin* X $U \longrightarrow P$ U) $\longleftrightarrow$ ($\forall U.$ *closedin* X $U \longrightarrow P$ (*topspace* $X - U$))
by (*metis Diff_Diff_Int closedin_def inf.absorb_iff2 openin_closedin_eq*)

lemma *all_closedin*: ($\forall U.$ *closedin* X $U \longrightarrow P$ U) $\longleftrightarrow$ ($\forall U.$ *openin* X $U \longrightarrow P$ (*topspace* $X - U$))
by (*metis Diff_Diff_Int closedin_subset inf.absorb_iff2 openin_closedin_eq*)

lemma *ex_openin*: ($\exists U.$ *openin* X $U \wedge P$ U) $\longleftrightarrow$ ($\exists U.$ *closedin* X $U \wedge P$ (*topspace* $X - U$))
by (*metis Diff_Diff_Int closedin_def inf.absorb_iff2 openin_closedin_eq*)

lemma *ex_closedin*: ($\exists U.$ *closedin* X $U \wedge P$ U) $\longleftrightarrow$ ($\exists U.$ *openin* X $U \wedge P$ (*topspace* $X - U$))
by (*metis Diff_Diff_Int closedin_subset inf.absorb_iff2 openin_closedin_eq*)

1.13.2 The discrete topology

definition *discrete_topology* **where** *discrete_topology* $U \equiv$ *topology* ($\lambda S.$ $S \subseteq U$)

lemma *openin_discrete_topology* [*simp*]: *openin* (*discrete_topology* U) $S \longleftrightarrow S \subseteq U$
proof −
have *istopology* ($\lambda S.$ $S \subseteq U$)
by (*auto simp*: *istopology_def*)
then show *?thesis*
by (*simp add*: *discrete_topology_def topology_inverse'*)
qed

lemma *topspace_discrete_topology* [*simp*]: *topspace*(*discrete_topology* U) $= U$
by (*meson openin_discrete_topology openin_subset openin_topspace order_refl subset_antisym*)

lemma *closedin_discrete_topology* [*simp*]: *closedin* (*discrete_topology U*) *S* ⟷ *S* ⊆ *U*
by (*simp add*: *closedin_def*)

lemma *discrete_topology_unique*:
discrete_topology U = *X* ⟷ *topspace X* = *U* ∧ (∀ *x* ∈ *U*. *openin X* {*x*}) (**is** *?lhs* = *?rhs*)
proof
assume *R*: *?rhs*
then have *openin X S* **if** *S* ⊆ *U* **for** *S*
using *openin_subopen subsetD that* **by** *fastforce*
then show *?lhs*
by (*metis R openin_discrete_topology openin_subset topology_eq*)
qed *auto*

lemma *discrete_topology_unique_alt*:
discrete_topology U = *X* ⟷ *topspace X* ⊆ *U* ∧ (∀ *x* ∈ *U*. *openin X* {*x*})
using *openin_subset*
by (*auto simp*: *discrete_topology_unique*)

lemma *subtopology_eq_discrete_topology_empty*:
X = *discrete_topology* {} ⟷ *topspace X* = {}
using *discrete_topology_unique* [*of* {} *X*] **by** *auto*

lemma *subtopology_eq_discrete_topology_sing*:
X = *discrete_topology* {*a*} ⟷ *topspace X* = {*a*}
by (*metis discrete_topology_unique openin_topspace singletonD*)

abbreviation *trivial_topology* **where** *trivial_topology* ≡ *discrete_topology* {}

lemma *null_topspace_iff_trivial* [*simp*]: *topspace X* = {} ⟷ *X* = *trivial_topology*
by (*simp add*: *subtopology_eq_discrete_topology_empty*)

1.13.3 Subspace topology

definition *subtopology* :: *'a topology* ⇒ *'a set* ⇒ *'a topology*
where *subtopology U V* = *topology* (λ*T*. ∃ *S*. *T* = *S* ∩ *V* ∧ *openin U S*)

lemma *istopology_subtopology*: *istopology* (λ*T*. ∃ *S*. *T* = *S* ∩ *V* ∧ *openin U S*)
(**is** *istopology ?L*)
proof −
have *?L* {} **by** *blast*
{
fix *A B*
assume *A*: *?L A* **and** *B*: *?L B*
from *A B* **obtain** *Sa* **and** *Sb* **where** *Sa*: *openin U Sa A* = *Sa* ∩ *V* **and** *Sb*: *openin U Sb B* = *Sb* ∩ *V*
by *blast*
have *A* ∩ *B* = (*Sa* ∩ *Sb*) ∩ *V openin U* (*Sa* ∩ *Sb*)

```
    using Sa Sb by blast+
   then have ?L (A ∩ B) by blast
  }
  moreover
  {
   fix K
   assume K: K ⊆ Collect ?L
   have th0: Collect ?L = (λS. S ∩ V) ' Collect (openin U)
    by blast
   from K[unfolded th0 subset_image_iff]
   obtain Sk where Sk: Sk ⊆ Collect (openin U) K = (λS. S ∩ V) ' Sk
    by blast
   have ⋃K = (⋃Sk) ∩ V
    using Sk by auto
   moreover have openin U (⋃Sk)
    using Sk by (auto simp: subset_eq)
   ultimately have ?L (⋃K) by blast
  }
  ultimately show ?thesis
   unfolding subset_eq mem_Collect_eq istopology_def by auto
qed

lemma openin_subtopology: openin (subtopology U V) S ⟷ (∃ T. openin U T ∧
S = T ∩ V)
  unfolding subtopology_def topology_inverse'[OF istopology_subtopology]
  by auto

lemma subset_openin_subtopology:
   ⟦openin X S; S ⊆ T⟧ ⟹ openin (subtopology X T) S
  by (metis inf.orderE openin_subtopology)

lemma openin_subtopology_Int:
   openin X S ⟹ openin (subtopology X T) (S ∩ T)
  using openin_subtopology by auto

lemma openin_subtopology_Int2:
   openin X T ⟹ openin (subtopology X S) (S ∩ T)
  using openin_subtopology by auto

lemma openin_subtopology_diff_closed:
   ⟦S ⊆ topspace X; closedin X T⟧ ⟹ openin (subtopology X S) (S − T)
  unfolding closedin_def openin_subtopology
  by (rule_tac x=topspace X − T in exI) auto

lemma openin_relative_to: (openin X relative_to S) = openin (subtopology X S)
  by (force simp: relative_to_def openin_subtopology)

lemma topspace_subtopology [simp]: topspace (subtopology U V) = topspace U ∩
V
```

by (*auto simp*: *topspace_def openin_subtopology*)

lemma *topspace_subtopology_subset*:
$S \subseteq$ *topspace* $X \Longrightarrow$ *topspace*(*subtopology* X S) $= S$
by (*simp add*: *inf.absorb_iff2*)

lemma *closedin_subtopology*: *closedin* (*subtopology* U V) $S \longleftrightarrow (\exists T.$ *closedin* U $T \wedge S = T \cap V)$
unfolding *closedin_def topspace_subtopology*
by (*auto simp*: *openin_subtopology*)

lemma *closedin_subtopology_Int_closed*:
closedin X $T \Longrightarrow$ *closedin* (*subtopology* X S) $(S \cap T)$
using *closedin_subtopology inf_commute* **by** *blast*

lemma *closedin_subset_topspace*:
$[\![$*closedin* X S; $S \subseteq T]\!] \Longrightarrow$ *closedin* (*subtopology* X T) S
using *closedin_subtopology* **by** *fastforce*

lemma *closedin_relative_to*:
(*closedin* X *relative_to* S) $=$ *closedin* (*subtopology* X S)
by (*force simp*: *relative_to_def closedin_subtopology*)

lemma *openin_subtopology_refl*: *openin* (*subtopology* U V) $V \longleftrightarrow V \subseteq$ *topspace* U
unfolding *openin_subtopology*
by *auto* (*metis IntD1 in_mono openin_subset*)

lemma *subtopology_trivial_iff*: *subtopology* X S $=$ *trivial_topology* $\longleftrightarrow X =$ *trivial_topology* $\vee$ *topspace* $X \cap S = \{\}$
by (*auto simp flip*: *null_topspace_iff_trivial*)

lemma *subtopology_subtopology*:
subtopology (*subtopology* X S) $T =$ *subtopology* X $(S \cap T)$
proof −
have *eq*: $\bigwedge T'.$ $(\exists S'.\ T' = S' \cap T \wedge (\exists T.$ *openin* X $T \wedge S' = T \cap S)) = (\exists Sa.$ $T' = Sa \cap (S \cap T) \wedge$ *openin* X $Sa)$
by (*metis inf_assoc*)
have *subtopology* (*subtopology* X S) $T =$ *topology* $(\lambda Ta.\ \exists Sa.\ Ta = Sa \cap T \wedge$ *openin* (*subtopology* X S) $Sa)$
by (*simp add*: *subtopology_def*)
also have $\ldots =$ *subtopology* X $(S \cap T)$
by (*simp add*: *openin_subtopology eq*) (*simp add*: *subtopology_def*)
finally show *?thesis* **.**
qed

lemma *openin_subtopology_alt*:
openin (*subtopology* X U) $S \longleftrightarrow S \in (\lambda T.\ U \cap T)$ ‘ *Collect* (*openin* X)
by (*simp add*: *image_iff inf_commute openin_subtopology*)

lemma *closedin_subtopology_alt*:
  closedin (*subtopology X U*) *S* ⟷ *S* ∈ (λ*T*. *U* ∩ *T*) ‘ *Collect* (*closedin X*)
 by (*simp add*: *image_iff inf_commute closedin_subtopology*)

lemma *subtopology_superset*:
 assumes *UV*: *topspace U* ⊆ *V*
 shows *subtopology U V* = *U*
proof −
 { fix *S*
  have *openin U S* **if** *openin U T S* = *T* ∩ *V* **for** *T*
   by (*metis Int_subset_iff assms inf.orderE openin_subset that*)
  then have (∃ *T*. *openin U T* ∧ *S* = *T* ∩ *V*) ⟷ *openin U S*
   by (*metis assms inf.orderE inf_assoc openin_subset*)
 }
 then show *?thesis*
  unfolding *topology_eq openin_subtopology* **by** *blast*
qed

lemma *subtopology_topspace*[*simp*]: *subtopology U* (*topspace U*) = *U*
 by (*simp add*: *subtopology_superset*)

lemma *subtopology_UNIV*[*simp*]: *subtopology U UNIV* = *U*
 by (*simp add*: *subtopology_superset*)

lemma *subtopology_empty_iff_trivial* [*simp*]: *subtopology X* {} = *trivial_topology*
 by (*simp add*: *subtopology_eq_discrete_topology_empty*)

lemma *subtopology_restrict*:
  subtopology X (*topspace X* ∩ *S*) = *subtopology X S*
 by (*metis subtopology_subtopology subtopology_topspace*)

lemma *openin_subtopology_empty*:
 openin (*subtopology U* {}) *S* ⟷ *S* = {}
 by (*metis Int_empty_right openin_empty openin_subtopology*)

lemma *closedin_subtopology_empty*:
 closedin (*subtopology U* {}) *S* ⟷ *S* = {}
 by (*metis Int_empty_right closedin_empty closedin_subtopology*)

lemma *closedin_subtopology_refl* [*simp*]:
 closedin (*subtopology U X*) *X* ⟷ *X* ⊆ *topspace U*
by (*metis closedin_def closedin_topspace inf.absorb_iff2 le_inf_iff topspace_subtopology*)

lemma *closedin_topspace_empty* [*simp*]: *closedin trivial_topology S* ⟷ *S* = {}
 by (*simp add*: *closedin_def*)

lemma *openin_topspace_empty* [*simp*]:
  openin trivial_topology S ⟷ *S* = {}

by (*simp add*: *openin_closedin_eq*)

lemma *openin_imp_subset*:
openin (*subtopology U S*) *T* ⟹ *T* ⊆ *S*
by (*metis Int_iff openin_subtopology subsetI*)

lemma *closedin_imp_subset*:
closedin (*subtopology U S*) *T* ⟹ *T* ⊆ *S*
by (*simp add*: *closedin_def*)

lemma *openin_open_subtopology*:
openin X S ⟹ *openin* (*subtopology X S*) *T* ⟷ *openin X T* ∧ *T* ⊆ *S*
by (*metis inf.orderE openin_Int openin_imp_subset openin_subtopology*)

lemma *closedin_closed_subtopology*:
closedin X S ⟹ (*closedin* (*subtopology X S*) *T* ⟷ *closedin X T* ∧ *T* ⊆ *S*)
by (*metis closedin_Int closedin_imp_subset closedin_subtopology inf.orderE*)

lemma *closedin_trans_full*:
⟦*closedin* (*subtopology X U*) *S*; *closedin X U*⟧ ⟹ *closedin X S*
using *closedin_closed_subtopology* **by** *blast*

lemma *openin_subtopology_Un*:
⟦*openin* (*subtopology X T*) *S*; *openin* (*subtopology X U*) *S*⟧
⟹ *openin* (*subtopology X* (*T* ∪ *U*)) *S*
by (*simp add*: *openin_subtopology*) *blast*

lemma *closedin_subtopology_Un*:
⟦*closedin* (*subtopology X T*) *S*; *closedin* (*subtopology X U*) *S*⟧
⟹ *closedin* (*subtopology X* (*T* ∪ *U*)) *S*
by (*simp add*: *closedin_subtopology*) *blast*

lemma *openin_trans_full*:
⟦*openin* (*subtopology X U*) *S*; *openin X U*⟧ ⟹ *openin X S*
by (*simp add*: *openin_open_subtopology*)

1.13.4 The canonical topology from the underlying type class

abbreviation *euclidean* :: *'a*::*topological_space topology*
where *euclidean* ≡ *topology open*

abbreviation *top_of_set* :: *'a*::*topological_space set* ⇒ *'a topology*
where *top_of_set* ≡ *subtopology* (*topology open*)

lemma *open_openin*: *open S* ⟷ *openin euclidean S*
by *simp*

declare *open_openin* [*symmetric*, *simp*]

lemma *topspace_euclidean* [*simp*]: *topspace euclidean = UNIV*
by (*force simp*: *topspace_def*)

lemma *topspace_euclidean_subtopology*[*simp*]: *topspace* (*top_of_set S*) = *S*
by (*simp*)

lemma *closed_closedin*: *closed S* ⟷ *closedin euclidean S*
by (*simp add*: *closed_def closedin_def Compl_eq_Diff_UNIV*)

declare *closed_closedin* [*symmetric*, *simp*]

lemma *openin_subtopology_self* [*simp*]: *openin* (*top_of_set S*) *S*
by (*metis openin_topspace topspace_euclidean_subtopology*)

lemma *euclidean_nontrivial* [*simp*]: *euclidean* ≠ *trivial_topology*
by (*simp add*: *subtopology_eq_discrete_topology_empty*)

The most basic facts about the usual topology and metric on R

abbreviation *euclideanreal* :: *real topology*
where *euclideanreal* ≡ *topology open*

1.13.5 Basic "localization" results are handy for connectedness.

lemma *openin_open*: *openin* (*top_of_set U*) *S* ⟷ (∃ *T*. *open T* ∧ (*S* = *U* ∩ *T*))
by (*auto simp*: *openin_subtopology*)

lemma *openin_Int_open*:
⟦*openin* (*top_of_set U*) *S*; *open T*⟧
⟹ *openin* (*top_of_set U*) (*S* ∩ *T*)
by (*metis open_Int Int_assoc openin_open*)

lemma *openin_open_Int*[*intro*]: *open S* ⟹ *openin* (*top_of_set U*) (*U* ∩ *S*)
by (*auto simp*: *openin_open*)

lemma *open_openin_trans*[*trans*]:
open S ⟹ *open T* ⟹ *T* ⊆ *S* ⟹ *openin* (*top_of_set S*) *T*
by (*metis Int_absorb1 openin_open_Int*)

lemma *open_subset*: *S* ⊆ *T* ⟹ *open S* ⟹ *openin* (*top_of_set T*) *S*
by (*auto simp*: *openin_open*)

lemma *closedin_closed*: *closedin* (*top_of_set U*) *S* ⟷ (∃ *T*. *closed T* ∧ *S* = *U* ∩ *T*)
by (*simp add*: *closedin_subtopology Int_ac*)

lemma *closedin_closed_Int*: *closed S* ⟹ *closedin* (*top_of_set U*) (*U* ∩ *S*)
by (*metis closedin_closed*)

lemma *closed_subset*: $S \subseteq T \Longrightarrow$ *closed* $S \Longrightarrow$ *closedin* (*top_of_set* T) S
by (*auto simp*: *closedin_closed*)

lemma *closedin_closed_subset*:
$[\![$*closedin* (*top_of_set* U) V; $T \subseteq U$; $S = V \cap T]\!]$
$\Longrightarrow$ *closedin* (*top_of_set* T) S
by (*metis* (*no_types*, *lifting*) *Int_assoc Int_commute closedin_closed inf.orderE*)

lemma *finite_imp_closedin*:
fixes S :: *'a*::*t1_space set*
shows $[\![$*finite* S; $S \subseteq T]\!] \Longrightarrow$ *closedin* (*top_of_set* T) S
by (*simp add*: *finite_imp_closed closed_subset*)

lemma *closedin_singleton* [*simp*]:
fixes a :: *'a*::*t1_space*
shows *closedin* (*top_of_set* U) $\{a\} \longleftrightarrow a \in U$
using *closedin_subset* **by** (*force intro*: *closed_subset*)

lemma *openin_euclidean_subtopology_iff*:
fixes S U :: *'a*::*metric_space set*
shows *openin* (*top_of_set* U) $S \longleftrightarrow$
$S \subseteq U \wedge (\forall x{\in}S.\ \exists e{>}0.\ \forall x'{\in}U.\ dist\ x'\ x < e \longrightarrow x'{\in} S)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *openin_open open_dist* **by** *blast*
next
define T **where** $T = \{x.\ \exists a{\in}S.\ \exists d{>}0.\ (\forall y{\in}U.\ dist\ y\ a < d \longrightarrow y \in S) \wedge dist\ x\ a < d\}$
have *1*: $\forall x{\in}T.\ \exists e{>}0.\ \forall y.\ dist\ y\ x < e \longrightarrow y \in T$
unfolding *T_def*
apply *clarsimp*
apply (*rule_tac* $x{=}d - dist\ x\ a$ **in** *exI*)
by (*metis add_0_left dist_commute dist_triangle_lt less_diff_eq*)
assume *?rhs* **then have** *2*: $S = U \cap T$
unfolding *T_def*
by *auto* (*metis dist_self*)
from *1 2* **show** *?lhs*
unfolding *openin_open open_dist* **by** *fast*
qed

lemma *connected_openin*:
connected $S \longleftrightarrow$
$\neg(\exists E1\ E2.\ openin\ (top\_of\_set\ S)\ E1 \wedge$
openin (*top_of_set* S) $E2 \wedge$
$S \subseteq E1 \cup E2 \wedge E1 \cap E2 = \{\} \wedge E1 \neq \{\} \wedge E2 \neq \{\})$
unfolding *connected_def openin_open disjoint_iff_not_equal* **by** *blast*

lemma *connected_openin_eq*:
  connected S ⟷
  ¬(∃ *E1 E2*. *openin* (*top_of_set S*) *E1* ∧
    openin (*top_of_set S*) *E2* ∧
    E1 ∪ *E2* = *S* ∧ *E1* ∩ *E2* = {} ∧
    E1 ≠ {} ∧ *E2* ≠ {})
 unfolding *connected_openin*
 by (*metis* (*no_types*, *lifting*) *Un_subset_iff openin_imp_subset subset_antisym*)

lemma *connected_closedin*:
  connected S ⟷
  (∄ *E1 E2*.
   closedin (*top_of_set S*) *E1* ∧
   closedin (*top_of_set S*) *E2* ∧
   S ⊆ *E1* ∪ *E2* ∧ *E1* ∩ *E2* = {} ∧ *E1* ≠ {} ∧ *E2* ≠ {})
  (**is** *?lhs* = *?rhs*)
proof
 assume *?lhs*
 then show *?rhs*
  by (*auto simp add*: *connected_closed closedin_closed*)
next
 assume *R*: *?rhs*
 then show *?lhs*
 proof (*clarsimp simp add*: *connected_closed closedin_closed*)
  fix *A B*
  assume *s_sub*: *S* ⊆ *A* ∪ *B B* ∩ *S* ≠ {}
   and *disj*: *A* ∩ *B* ∩ *S* = {}
   and *cl*: *closed A closed B*
  have *S* − *A* = *B* ∩ *S*
   using *Diff_subset_conv Un_Diff_Int disj s_sub*(*1*) **by** *auto*
  then show *A* ∩ *S* = {}
  by (*metis Int_Diff_Un Int_Diff_disjoint R cl closedin_closed_Int dual_order.refl*
inf_commute s_sub(*2*))
 qed
qed

lemma *connected_closedin_eq*:
  connected S ⟷
    ¬(∃ *E1 E2*.
      closedin (*top_of_set S*) *E1* ∧
      closedin (*top_of_set S*) *E2* ∧
      E1 ∪ *E2* = *S* ∧ *E1* ∩ *E2* = {} ∧
      E1 ≠ {} ∧ *E2* ≠ {})
 unfolding *connected_closedin*
 by (*metis Un_subset_iff closedin_imp_subset subset_antisym*)

These "transitivity" results are handy too

lemma *openin_trans*[*trans*]:

openin (*top_of_set T*) *S* ⟹ *openin* (*top_of_set U*) *T* ⟹
openin (*top_of_set U*) *S*
by (*metis openin_Int_open openin_open*)

lemma *openin_open_trans*: *openin* (*top_of_set T*) *S* ⟹ *open T* ⟹ *open S*
by (*auto simp*: *openin_open intro*: *openin_trans*)

lemma *closedin_trans*[*trans*]:
closedin (*top_of_set T*) *S* ⟹ *closedin* (*top_of_set U*) *T* ⟹
closedin (*top_of_set U*) *S*
by (*auto simp*: *closedin_closed closed_Inter Int_assoc*)

lemma *closedin_closed_trans*: *closedin* (*top_of_set T*) *S* ⟹ *closed T* ⟹ *closed S*
by (*auto simp*: *closedin_closed intro*: *closedin_trans*)

lemma *openin_subtopology_Int_subset*:
⟦*openin* (*top_of_set u*) (*u* ∩ *S*); *v* ⊆ *u*⟧ ⟹ *openin* (*top_of_set v*) (*v* ∩ *S*)
by (*auto simp*: *openin_subtopology*)

lemma *openin_open_eq*: *open s* ⟹ (*openin* (*top_of_set s*) *t* ⟷ *open t* ∧ *t* ⊆ *s*)
using *open_subset openin_open_trans openin_subset* **by** *fastforce*

1.13.6 Derived set (set of limit points)

definition *derived_set_of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixl** ‹*derived'_set'_of*› *80*)
where *X derived_set_of S* ≡
{*x* ∈ *topspace X*.
(∀ *T*. *x* ∈ *T* ∧ *openin X T* ⟶ (∃ *y*≠*x*. *y* ∈ *S* ∧ *y* ∈ *T*))}

lemma *derived_set_of_restrict* [*simp*]:
X derived_set_of (*topspace X* ∩ *S*) = *X derived_set_of S*
by (*simp add*: *derived_set_of_def*) (*metis openin_subset subset_iff*)

lemma *in_derived_set_of*:
x ∈ *X derived_set_of S* ⟷ *x* ∈ *topspace X* ∧ (∀ *T*. *x* ∈ *T* ∧ *openin X T* ⟶ (∃ *y*≠*x*. *y* ∈ *S* ∧ *y* ∈ *T*))
by (*simp add*: *derived_set_of_def*)

lemma *derived_set_of_subset_topspace*:
X derived_set_of S ⊆ *topspace X*
by (*auto simp add*: *derived_set_of_def*)

lemma *derived_set_of_subtopology*:
(*subtopology X U*) *derived_set_of S* = *U* ∩ (*X derived_set_of* (*U* ∩ *S*))
by (*simp add*: *derived_set_of_def openin_subtopology*) *blast*

lemma *derived_set_of_subset_subtopology*:
(*subtopology X S*) *derived_set_of T* ⊆ *S*
by (*simp add*: *derived_set_of_subtopology*)

lemma *derived_set_of_empty* [*simp*]: *X derived_set_of* {} = {}
by (*auto simp*: *derived_set_of_def*)

lemma *derived_set_of_mono*:
S ⊆ *T* ⟹ *X derived_set_of S* ⊆ *X derived_set_of T*
unfolding *derived_set_of_def* **by** *blast*

lemma *derived_set_of_Un*:
X derived_set_of (*S* ∪ *T*) = *X derived_set_of S* ∪ *X derived_set_of T* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*clarsimp simp*: *in_derived_set_of*) (*metis IntE IntI openin_Int*)
show *?rhs* ⊆ *?lhs*
by (*simp add*: *derived_set_of_mono*)
qed

lemma *derived_set_of_Union*:
finite $\mathcal{F}$ ⟹ *X derived_set_of* (⋃$\mathcal{F}$) = (⋃*S* ∈ $\mathcal{F}$. *X derived_set_of S*)
proof (*induction* $\mathcal{F}$ *rule*: *finite_induct*)
case (*insert S* $\mathcal{F}$)
then show *?case*
by (*simp add*: *derived_set_of_Un*)
qed *auto*

lemma *derived_set_of_topspace*:
X derived_set_of (*topspace X*) = {*x* ∈ *topspace X*. ¬ *openin X* {*x*}} (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*auto simp*: *in_derived_set_of*)
show *?rhs* ⊆ *?lhs*
by (*clarsimp simp*: *in_derived_set_of*) (*metis openin_closedin_eq openin_subopen singletonD subset_eq*)
qed

lemma *discrete_topology_unique_derived_set*:
discrete_topology U = *X* ⟷ *topspace X* = *U* ∧ *X derived_set_of U* = {}
by (*auto simp*: *discrete_topology_unique derived_set_of_topspace*)

lemma *subtopology_eq_discrete_topology_eq*:
subtopology X U = *discrete_topology U* ⟷ *U* ⊆ *topspace X* ∧ *U* ∩ *X derived_set_of U* = {}
using *discrete_topology_unique_derived_set* [*of U subtopology X U*]
by (*auto simp*: *eq_commute derived_set_of_subtopology*)

lemma *subtopology_eq_discrete_topology*:
  $S \subseteq$ *topspace* $X \wedge S \cap X$ *derived_set_of* $S = \{\}$
    $\Longrightarrow$ *subtopology* $X\ S =$ *discrete_topology* S
 by (*simp add*: *subtopology_eq_discrete_topology_eq*)

lemma *subtopology_eq_discrete_topology_gen*:
 assumes $S \cap X$ *derived_set_of* $S = \{\}$
 shows *subtopology* $X\ S =$ *discrete_topology*(*topspace* $X \cap S$)
proof −
 have *subtopology* $X\ S =$ *subtopology* X (*topspace* $X \cap S$)
   by (*simp add*: *subtopology_restrict*)
 then show *?thesis*
   using *assms* **by** (*simp add*: *inf.assoc subtopology_eq_discrete_topology_eq*)
qed

lemma *subtopology_discrete_topology* [*simp*]:
 subtopology (*discrete_topology* U) $S =$ *discrete_topology*($U \cap S$)
proof −
 have $(\lambda T.\ \exists Sa.\ T = Sa \cap S \wedge Sa \subseteq U) = (\lambda Sa.\ Sa \subseteq U \wedge Sa \subseteq S)$
   by *force*
 then show *?thesis*
   by (*simp add*: *subtopology_def*) (*simp add*: *discrete_topology_def*)
qed

lemma *openin_Int_derived_set_of_subset*:
  openin $X\ S \Longrightarrow S \cap X$ *derived_set_of* $T \subseteq X$ *derived_set_of* $(S \cap T)$
 by (*auto simp*: *derived_set_of_def*)

lemma *openin_Int_derived_set_of_eq*:
 assumes *openin* $X\ S$
 shows $S \cap X$ *derived_set_of* $T = S \cap X$ *derived_set_of* $(S \cap T)$ (**is** *?lhs* = *?rhs*)
proof
 show *?lhs* $\subseteq$ *?rhs*
   by (*simp add*: *assms openin_Int_derived_set_of_subset*)
 show *?rhs* $\subseteq$ *?lhs*
   by (*metis derived_set_of_mono inf_commute inf_le1 inf_mono order_refl*)
qed

1.13.7 Closure with respect to a topological space

definition *closure_of* :: *'a topology* $\Rightarrow$ *'a set* $\Rightarrow$ *'a set* (**infixr** ‹*closure'_of*› *80*)
 where X *closure_of* $S \equiv \{x \in$ *topspace* $X.\ \forall T.\ x \in T \wedge$ *openin* $X\ T \longrightarrow (\exists y \in S.\ y \in T)\}$

lemma *closure_of_restrict*: X *closure_of* $S = X$ *closure_of* (*topspace* $X \cap S$)
 unfolding *closure_of_def*
 using *openin_subset* **by** *blast*

lemma *in_closure_of*:
  $x \in X$ *closure_of* $S \longleftrightarrow$
   $x \in$ *topspace* $X \wedge (\forall\, T.\ x \in T \wedge$ *openin* $X\ T \longrightarrow (\exists\, y.\ y \in S \wedge y \in T))$
 by (*auto simp*: *closure_of_def*)

lemma *closure_of*: X *closure_of* $S =$ *topspace* $X \cap (S \cup X$ *derived_set_of* $S)$
 by (*fastforce simp*: *in_closure_of in_derived_set_of*)

lemma *closure_of_alt*: X *closure_of* $S =$ *topspace* $X \cap S \cup X$ *derived_set_of* S
 using *derived_set_of_subset_topspace* [*of* $X\ S$]
 unfolding *closure_of_def in_derived_set_of*
 by *safe* (*auto simp*: *in_derived_set_of*)

lemma *derived_set_of_subset_closure_of*:
  X *derived_set_of* $S \subseteq X$ *closure_of* S
 by (*fastforce simp*: *closure_of_def in_derived_set_of*)

lemma *closure_of_subtopology*:
 (*subtopology* $X\ U$) *closure_of* $S = U \cap (X$ *closure_of* $(U \cap S))$
 unfolding *closure_of_def topspace_subtopology openin_subtopology*
 by *safe* (*metis* (*full_types*) *IntI Int_iff inf.commute*)+

lemma *closure_of_empty* [*simp*]: X *closure_of* $\{\} = \{\}$
 by (*simp add*: *closure_of_alt*)

lemma *closure_of_topspace* [*simp*]: X *closure_of topspace* $X =$ *topspace* X
 by (*simp add*: *closure_of*)

lemma *closure_of_UNIV* [*simp*]: X *closure_of UNIV* = *topspace* X
 by (*simp add*: *closure_of*)

lemma *closure_of_subset_topspace*: X *closure_of* $S \subseteq$ *topspace* X
 by (*simp add*: *closure_of*)

lemma *closure_of_subset_subtopology*: (*subtopology* $X\ S$) *closure_of* $T \subseteq S$
 by (*simp add*: *closure_of_subtopology*)

lemma *closure_of_mono*: $S \subseteq T \Longrightarrow X$ *closure_of* $S \subseteq X$ *closure_of* T
 by (*fastforce simp add*: *closure_of_def*)

lemma *closure_of_subtopology_subset*:
  (*subtopology* $X\ U$) *closure_of* $S \subseteq (X$ *closure_of* $S)$
 unfolding *closure_of_subtopology*
 by *clarsimp* (*meson closure_of_mono contra_subsetD inf.cobounded2*)

lemma *closure_of_subtopology_mono*:
  $T \subseteq U \Longrightarrow$ (*subtopology* $X\ T$) *closure_of* $S \subseteq$ (*subtopology* $X\ U$) *closure_of* S
 unfolding *closure_of_subtopology*

by *auto* (*meson closure_of_mono inf_mono subset_iff*)

lemma *closure_of_Un* [*simp*]: *X closure_of* (*S* ∪ *T*) = *X closure_of S* ∪ *X closure_of T*
by (*simp add*: *Un_assoc Un_left_commute closure_of_alt derived_set_of_Un inf_sup_distrib1*)

lemma *closure_of_Union*:
finite $\mathcal{F}$ ⟹ *X closure_of* (⋃$\mathcal{F}$) = (⋃*S* ∈ $\mathcal{F}$. *X closure_of S*)
by (*induction* $\mathcal{F}$ *rule*: *finite_induct*) *auto*

lemma *closure_of_subset*: *S* ⊆ *topspace X* ⟹ *S* ⊆ *X closure_of S*
by (*auto simp*: *closure_of_def*)

lemma *closure_of_subset_Int*: *topspace X* ∩ *S* ⊆ *X closure_of S*
by (*auto simp*: *closure_of_def*)

lemma *closure_of_subset_eq*: *S* ⊆ *topspace X* ∧ *X closure_of S* ⊆ *S* ⟷ *closedin X S*
proof −
have *openin X* (*topspace X* − *S*)
if ⋀*x*. ⟦*x* ∈ *topspace X*; ∀ *T*. *x* ∈ *T* ∧ *openin X T* ⟶ *S* ∩ *T* ≠ {}⟧ ⟹ *x* ∈ *S*
apply (*subst openin_subopen*)
by (*metis Diff_iff Diff_mono Diff_triv inf.commute openin_subset order_refl that*)
then show *?thesis*
by (*auto simp*: *closedin_def closure_of_def disjoint_iff_not_equal*)
qed

lemma *closure_of_eq*: *X closure_of S* = *S* ⟷ *closedin X S*
by (*metis closure_of_subset closure_of_subset_eq closure_of_subset_topspace subset_antisym*)

lemma *closedin_contains_derived_set*:
closedin X S ⟷ *X derived_set_of S* ⊆ *S* ∧ *S* ⊆ *topspace X*
proof (*intro iffI conjI*)
show *closedin X S* ⟹ *X derived_set_of S* ⊆ *S*
using *closure_of_eq derived_set_of_subset_closure_of* **by** *fastforce*
show *closedin X S* ⟹ *S* ⊆ *topspace X*
using *closedin_subset* **by** *blast*
show *X derived_set_of S* ⊆ *S* ∧ *S* ⊆ *topspace X* ⟹ *closedin X S*
by (*metis closure_of closure_of_eq inf.absorb_iff2 sup.orderE*)
qed

lemma *derived_set_subset_gen*:
X derived_set_of S ⊆ *S* ⟷ *closedin X* (*topspace X* ∩ *S*)
by (*simp add*: *closedin_contains_derived_set derived_set_of_subset_topspace*)

lemma *derived_set_subset*: *S* ⊆ *topspace X* ⟹ (*X derived_set_of S* ⊆ *S* ⟷

closedin X S)
by (*simp add*: *closedin_contains_derived_set*)

lemma *closedin_derived_set*:
closedin (*subtopology X T*) *S* ⟷
S ⊆ *topspace X* ∧ *S* ⊆ *T* ∧ (∀ *x*. *x* ∈ *X derived_set_of S* ∧ *x* ∈ *T* ⟶ *x* ∈ *S*)
by (*auto simp*: *closedin_contains_derived_set derived_set_of_subtopology Int_absorb1*)

lemma *closedin_Int_closure_of*:
closedin (*subtopology X S*) *T* ⟷ *S* ∩ *X closure_of T* = *T*
by (*metis Int_left_absorb closure_of_eq closure_of_subtopology*)

lemma *closure_of_closedin*: *closedin X S* ⟹ *X closure_of S* = *S*
by (*simp add*: *closure_of_eq*)

lemma *closure_of_eq_diff*: *X closure_of S* = *topspace X* − ⋃{*T*. *openin X T* ∧ *disjnt S T*}
by (*auto simp*: *closure_of_def disjnt_iff*)

lemma *closedin_closure_of* [*simp*]: *closedin X* (*X closure_of S*)
unfolding *closure_of_eq_diff* **by** *blast*

lemma *closure_of_closure_of* [*simp*]: *X closure_of* (*X closure_of S*) = *X closure_of S*
by (*simp add*: *closure_of_eq*)

lemma *closure_of_hull*:
assumes *S* ⊆ *topspace X* **shows** *X closure_of S* = (*closedin X*) *hull S*
by (*metis assms closedin_closure_of closure_of_eq closure_of_mono closure_of_subset hull_unique*)

lemma *closure_of_minimal*:
⟦*S* ⊆ *T*; *closedin X T*⟧ ⟹ (*X closure_of S*) ⊆ *T*
by (*metis closure_of_eq closure_of_mono*)

lemma *closure_of_minimal_eq*:
⟦*S* ⊆ *topspace X*; *closedin X T*⟧ ⟹ (*X closure_of S*) ⊆ *T* ⟷ *S* ⊆ *T*
by (*meson closure_of_minimal closure_of_subset subset_trans*)

lemma *closure_of_unique*:
⟦*S* ⊆ *T*; *closedin X T*;
⋀*T'*. ⟦*S* ⊆ *T'*; *closedin X T'*⟧ ⟹ *T* ⊆ *T'*⟧
⟹ *X closure_of S* = *T*
by (*meson closedin_closure_of closedin_subset closure_of_minimal closure_of_subset eq_iff order.trans*)

lemma *closure_of_eq_empty_gen*: *X closure_of S* = {} ⟷ *disjnt* (*topspace X*) *S*

unfolding *disjnt_def closure_of_restrict* [**where** *S*=*S*]
using *closure_of* **by** *fastforce*

lemma *closure_of_eq_empty*: $S \subseteq$ *topspace* $X \Longrightarrow X$ *closure_of* $S = \{\} \longleftrightarrow S = \{\}$
using *closure_of_subset* **by** *fastforce*

lemma *openin_Int_closure_of_subset*:
assumes *openin* X S
shows $S \cap X$ *closure_of* $T \subseteq X$ *closure_of* $(S \cap T)$
proof −
have $S \cap X$ *derived_set_of* $T = S \cap X$ *derived_set_of* $(S \cap T)$
by (*meson assms openin_Int_derived_set_of_eq*)
moreover have $S \cap (S \cap T) = S \cap T$
by *fastforce*
ultimately show *?thesis*
by (*metis closure_of_alt inf.cobounded2 inf_left_commute inf_sup_distrib1*)
qed

lemma *closure_of_openin_Int_closure_of*:
assumes *openin* X S
shows X *closure_of* $(S \cap X$ *closure_of* $T) = X$ *closure_of* $(S \cap T)$
proof
show X *closure_of* $(S \cap X$ *closure_of* $T) \subseteq X$ *closure_of* $(S \cap T)$
by (*simp add*: *assms closure_of_minimal openin_Int_closure_of_subset*)
next
show X *closure_of* $(S \cap T) \subseteq X$ *closure_of* $(S \cap X$ *closure_of* $T)$
by (*metis Int_subset_iff assms closure_of_alt closure_of_mono inf_mono openin_subset subset_refl sup.coboundedI1*)
qed

lemma *openin_Int_closure_of_eq*:
assumes *openin* X S **shows** $S \cap X$ *closure_of* $T = S \cap X$ *closure_of* $(S \cap T)$ (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
by (*simp add*: *assms openin_Int_closure_of_subset*)
show *?rhs* $\subseteq$ *?lhs*
by (*metis closure_of_mono inf_commute inf_le1 inf_mono order_refl*)
qed

lemma *openin_Int_closure_of_eq_empty*:
assumes *openin* X S **shows** $S \cap X$ *closure_of* $T = \{\} \longleftrightarrow S \cap T = \{\}$ (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
unfolding *disjoint_iff*
by (*meson assms in_closure_of in_mono openin_subset*)
show *?rhs* $\Longrightarrow$ *?lhs*

by (*simp add*: *assms openin_Int_closure_of_eq*)
qed

lemma *closure_of_openin_Int_superset*:
openin X S ∧ *S* ⊆ *X closure_of T*
⟹ *X closure_of* (*S* ∩ *T*) = *X closure_of S*
by (*metis closure_of_openin_Int_closure_of inf.orderE*)

lemma *closure_of_openin_subtopology_Int_closure_of*:
assumes *S*: *openin* (*subtopology X U*) *S* **and** *T* ⊆ *U*
shows *X closure_of* (*S* ∩ *X closure_of T*) = *X closure_of* (*S* ∩ *T*) (**is** *?lhs* = *?rhs*)
proof
obtain *S0* **where** *S0*: *openin X S0 S* = *S0* ∩ *U*
using *assms* **by** (*auto simp*: *openin_subtopology*)
then show *?lhs* ⊆ *?rhs*
proof −
have *S0* ∩ *X closure_of T* = *S0* ∩ *X closure_of* (*S0* ∩ *T*)
by (*meson S0*(*1*) *openin_Int_closure_of_eq*)
moreover have *S0* ∩ *T* = *S0* ∩ *U* ∩ *T*
using ‹*T* ⊆ *U*› **by** *fastforce*
ultimately have *S* ∩ *X closure_of T* ⊆ *X closure_of* (*S* ∩ *T*)
using *S0*(*2*) **by** *auto*
then show *?thesis*
by (*meson closedin_closure_of closure_of_minimal*)
qed
next
show *?rhs* ⊆ *?lhs*
proof −
have *T* ∩ *S* ⊆ *T* ∪ *X derived_set_of T*
by *force*
then show *?thesis*
by (*smt* (*verit*, *del_insts*) *Int_iff in_closure_of inf.orderE openin_subset subsetI*)
qed
qed

lemma *closure_of_subtopology_open*:
openin X U ∨ *S* ⊆ *U* ⟹ (*subtopology X U*) *closure_of S* = *U* ∩ *X closure_of S*
by (*metis closure_of_subtopology inf_absorb2 openin_Int_closure_of_eq*)

lemma *discrete_topology_closure_of*:
(*discrete_topology U*) *closure_of S* = *U* ∩ *S*
by (*metis closedin_discrete_topology closure_of_restrict closure_of_unique discrete_topology_unique inf_sup_ord*(*1*) *order_refl*)

Interior with respect to a topological space.

definition *interior_of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixr** ‹*interior'_of*› *80*)

where X *interior_of* $S \equiv \{x.\ \exists T.\ \mathit{openin}\ X\ T \wedge x \in T \wedge T \subseteq S\}$

lemma *interior_of_restrict*:
X *interior_of* $S = X$ *interior_of* $(\mathit{topspace}\ X \cap S)$
using *openin_subset* **by** (*auto simp*: *interior_of_def*)

lemma *interior_of_eq*: $(X$ *interior_of* $S = S) \longleftrightarrow \mathit{openin}\ X\ S$
unfolding *interior_of_def* **using** *openin_subopen* **by** *blast*

lemma *interior_of_openin*: $\mathit{openin}\ X\ S \Longrightarrow X$ *interior_of* $S = S$
by (*simp add*: *interior_of_eq*)

lemma *interior_of_empty* [*simp*]: X *interior_of* $\{\} = \{\}$
by (*simp add*: *interior_of_eq*)

lemma *interior_of_topspace* [*simp*]: X *interior_of* $(\mathit{topspace}\ X) = \mathit{topspace}\ X$
by (*simp add*: *interior_of_eq*)

lemma *openin_interior_of* [*simp*]: *openin* X $(X$ *interior_of* $S)$
unfolding *interior_of_def*
using *openin_subopen* **by** *fastforce*

lemma *interior_of_interior_of* [*simp*]:
X *interior_of* X *interior_of* $S = X$ *interior_of* S
by (*simp add*: *interior_of_eq*)

lemma *interior_of_subset*: X *interior_of* $S \subseteq S$
by (*auto simp*: *interior_of_def*)

lemma *interior_of_subset_closure_of*: X *interior_of* $S \subseteq X$ *closure_of* S
by (*metis closure_of_subset_Int dual_order.trans interior_of_restrict interior_of_subset*)

lemma *subset_interior_of_eq*: $S \subseteq X$ *interior_of* $S \longleftrightarrow \mathit{openin}\ X\ S$
by (*metis interior_of_eq interior_of_subset subset_antisym*)

lemma *interior_of_mono*: $S \subseteq T \Longrightarrow X$ *interior_of* $S \subseteq X$ *interior_of* T
by (*auto simp*: *interior_of_def*)

lemma *interior_of_maximal*: $[\![T \subseteq S;\ \mathit{openin}\ X\ T]\!] \Longrightarrow T \subseteq X$ *interior_of* S
by (*auto simp*: *interior_of_def*)

lemma *interior_of_maximal_eq*: $\mathit{openin}\ X\ T \Longrightarrow T \subseteq X$ *interior_of* $S \longleftrightarrow T \subseteq S$
by (*meson interior_of_maximal interior_of_subset order_trans*)

lemma *interior_of_unique*:
$[\![T \subseteq S;\ \mathit{openin}\ X\ T;\ \bigwedge T'.\ [\![T' \subseteq S;\ \mathit{openin}\ X\ T']\!] \Longrightarrow T' \subseteq T]\!] \Longrightarrow X$ *interior_of* $S = T$
by (*simp add*: *interior_of_maximal_eq interior_of_subset subset_antisym*)

lemma *interior_of_subset_topspace*: X *interior_of* $S \subseteq$ *topspace* X
by (*simp add*: *openin_subset*)

lemma *interior_of_subset_subtopology*: (*subtopology* X S) *interior_of* $T \subseteq S$
by (*meson openin_imp_subset openin_interior_of*)

lemma *interior_of_Int*: X *interior_of* $(S \cap T) = X$ *interior_of* $S \cap X$ *interior_of* T (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
by (*simp add*: *interior_of_mono*)
show *?rhs* $\subseteq$ *?lhs*
by (*meson inf_mono interior_of_maximal interior_of_subset openin_Int openin_interior_of*)
qed

lemma *interior_of_Inter_subset*: X *interior_of* $(\bigcap \mathcal{F}) \subseteq (\bigcap S \in \mathcal{F}.\ X$ *interior_of* $S)$
by (*simp add*: *INT_greatest Inf_lower interior_of_mono*)

lemma *union_interior_of_subset*:
X *interior_of* $S \cup X$ *interior_of* $T \subseteq X$ *interior_of* $(S \cup T)$
by (*simp add*: *interior_of_mono*)

lemma *interior_of_eq_empty*:
X *interior_of* $S = \{\} \longleftrightarrow (\forall T.$ *openin* X $T \wedge T \subseteq S \longrightarrow T = \{\})$
by (*metis bot.extremum_uniqueI interior_of_maximal interior_of_subset openin_interior_of*)

lemma *interior_of_eq_empty_alt*:
X *interior_of* $S = \{\} \longleftrightarrow (\forall T.$ *openin* X $T \wedge T \neq \{\} \longrightarrow T - S \neq \{\})$
by (*auto simp*: *interior_of_eq_empty*)

lemma *interior_of_Union_openin_subsets*:
$\bigcup \{T.$ *openin* X $T \wedge T \subseteq S\} = X$ *interior_of* S
by (*rule interior_of_unique* [*symmetric*]) *auto*

lemma *interior_of_complement*:
X *interior_of* (*topspace* $X - S$) = *topspace* $X - X$ *closure_of* S
by (*auto simp*: *interior_of_def closure_of_def*)

lemma *interior_of_closure_of*:
X *interior_of* S = *topspace* $X - X$ *closure_of* (*topspace* $X - S$)
unfolding *interior_of_complement* [*symmetric*]
by (*metis Diff_Diff_Int interior_of_restrict*)

lemma *closure_of_interior_of*:
X *closure_of* S = *topspace* $X - X$ *interior_of* (*topspace* $X - S$)
by (*simp add*: *interior_of_complement Diff_Diff_Int closure_of*)

lemma *closure_of_complement*: *X closure_of* (*topspace X* − *S*) = *topspace X* − *X interior_of S*
unfolding *interior_of_def closure_of_def*
by (*blast dest*: *openin_subset*)

lemma *interior_of_eq_empty_complement*:
X interior_of S = {} ⟷ *X closure_of* (*topspace X* − *S*) = *topspace X*
using *interior_of_subset_topspace* [*of X S*] *closure_of_complement* **by** *fastforce*

lemma *closure_of_eq_topspace*:
X closure_of S = *topspace X* ⟷ *X interior_of* (*topspace X* − *S*) = {}
using *closure_of_subset_topspace* [*of X S*] *interior_of_complement* **by** *fastforce*

lemma *interior_of_subtopology_subset*:
U ∩ *X interior_of S* ⊆ (*subtopology X U*) *interior_of S*
by (*auto simp*: *interior_of_def openin_subtopology*)

lemma *interior_of_subtopology_subsets*:
T ⊆ *U* ⟹ *T* ∩ (*subtopology X U*) *interior_of S* ⊆ (*subtopology X T*) *interior_of S*
by (*metis inf.absorb_iff2 interior_of_subtopology_subset subtopology_subtopology*)

lemma *interior_of_subtopology_mono*:
⟦*S* ⊆ *T*; *T* ⊆ *U*⟧ ⟹ (*subtopology X U*) *interior_of S* ⊆ (*subtopology X T*) *interior_of S*
by (*metis dual_order.trans inf.orderE inf_commute interior_of_subset interior_of_subtopology_subsets*)

lemma *interior_of_subtopology_open*:
assumes *openin X U*
shows (*subtopology X U*) *interior_of S* = *U* ∩ *X interior_of S* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*meson assms interior_of_maximal interior_of_subset le_infI openin_interior_of openin_open_subtopology*)
show *?rhs* ⊆ *?lhs*
by (*simp add*: *interior_of_subtopology_subset*)
qed

lemma *dense_intersects_open*:
X closure_of S = *topspace X* ⟷ (∀ *T*. *openin X T* ∧ *T* ≠ {} ⟶ *S* ∩ *T* ≠ {})
proof −
have *X closure_of S* = *topspace X* ⟷ (*topspace X* − *X interior_of* (*topspace X* − *S*) = *topspace X*)
by (*simp add*: *closure_of_interior_of*)
also have ... ⟷ *X interior_of* (*topspace X* − *S*) = {}
by (*simp add*: *closure_of_complement interior_of_eq_empty_complement*)

also have ... $\longleftrightarrow$ $(\forall\, T.\ \mathit{openin}\ X\ T \wedge T \neq \{\} \longrightarrow S \cap T \neq \{\})$
unfolding *interior_of_eq_empty_alt*
using *openin_subset* **by** *fastforce*
finally show *?thesis* **.**
qed

lemma *interior_of_closedin_union_empty_interior_of*:
assumes *closedin X S* **and** *disj*: *X interior_of T* = {}
shows *X interior_of* (*S* ∪ *T*) = *X interior_of S*
proof −
have *X closure_of* (*topspace X* − *T*) = *topspace X*
by (*metis Diff_Diff_Int disj closure_of_eq_topspace closure_of_restrict interior_of_closure_of*)
then show *?thesis*
unfolding *interior_of_closure_of*
by (*metis Diff_Un Diff_subset assms*(*1*) *closedin_def closure_of_openin_Int_superset*)
qed

lemma *interior_of_union_eq_empty*:
closedin X S
$\Longrightarrow$ (*X interior_of* (*S* ∪ *T*) = {} $\longleftrightarrow$
X interior_of S = {} ∧ *X interior_of T* = {})
by (*metis interior_of_closedin_union_empty_interior_of le_sup_iff subset_empty union_interior_of_subset*)

lemma *discrete_topology_interior_of* [*simp*]:
(*discrete_topology U*) *interior_of S* = *U* ∩ *S*
by (*simp add*: *interior_of_restrict* [*of* _ *S*] *interior_of_eq*)

1.13.8 Frontier with respect to topological space

definition *frontier_of* :: *'a topology* ⇒ *'a set* ⇒ *'a set* (**infixr** ‹*frontier'_of*› *80*)
where *X frontier_of S* ≡ *X closure_of S* − *X interior_of S*

lemma *frontier_of_closures*:
X frontier_of S = *X closure_of S* ∩ *X closure_of* (*topspace X* − *S*)
by (*metis Diff_Diff_Int closure_of_complement closure_of_subset_topspace double_diff frontier_of_def interior_of_subset_closure_of*)

lemma *interior_of_union_frontier_of* [*simp*]:
X interior_of S ∪ *X frontier_of S* = *X closure_of S*
by (*simp add*: *frontier_of_def interior_of_subset_closure_of subset_antisym*)

lemma *frontier_of_restrict*: *X frontier_of S* = *X frontier_of* (*topspace X* ∩ *S*)
by (*metis closure_of_restrict frontier_of_def interior_of_restrict*)

lemma *closedin_frontier_of*: *closedin X* (*X frontier_of S*)
by (*simp add*: *closedin_Int frontier_of_closures*)

lemma *frontier_of_subset_topspace*: *X frontier_of S ⊆ topspace X*
by (*simp add*: *closedin_frontier_of closedin_subset*)

lemma *frontier_of_subset_subtopology*: (*subtopology X S*) *frontier_of T ⊆ S*
by (*metis* (*no_types*) *closedin_derived_set closedin_frontier_of*)

lemma *frontier_of_subtopology_subset*:
U ∩ (*subtopology X U*) *frontier_of S ⊆* (*X frontier_of S*)
proof −
have *U ∩ X interior_of S − subtopology X U interior_of S* = {}
by (*simp add*: *interior_of_subtopology_subset*)
moreover have *X closure_of S ∩ subtopology X U closure_of S* = *subtopology X U closure_of S*
by (*meson closure_of_subtopology_subset inf.absorb_iff2*)
ultimately show *?thesis*
unfolding *frontier_of_def*
by *blast*
qed

lemma *frontier_of_subtopology_mono*:
⟦*S ⊆ T*; *T ⊆ U*⟧ *⟹* (*subtopology X T*) *frontier_of S ⊆* (*subtopology X U*) *frontier_of S*
by (*simp add*: *frontier_of_def Diff_mono closure_of_subtopology_mono interior_of_subtopology_mono*)

lemma *clopenin_eq_frontier_of*:
closedin X S ∧ openin X S ⟷ S ⊆ topspace X ∧ X frontier_of S = {}
proof (*cases S ⊆ topspace X*)
case *True*
then show *?thesis*
by (*metis Diff_eq_empty_iff closure_of_eq closure_of_subset_eq frontier_of_def interior_of_eq interior_of_subset interior_of_union_frontier_of sup_bot_right*)
next
case *False*
then show *?thesis*
by (*simp add*: *frontier_of_closures openin_closedin_eq*)
qed

lemma *frontier_of_eq_empty*:
S ⊆ topspace X ⟹ (*X frontier_of S* = {} *⟷ closedin X S ∧ openin X S*)
by (*simp add*: *clopenin_eq_frontier_of*)

lemma *frontier_of_openin*:
openin X S ⟹ X frontier_of S = *X closure_of S − S*
by (*metis* (*no_types*) *frontier_of_def interior_of_eq*)

lemma *frontier_of_openin_straddle_Int*:
assumes *openin X U U ∩ X frontier_of S ≠* {}
shows *U ∩ S ≠* {} *U − S ≠* {}

proof −
have $U \cap (X\ closure\_of\ S \cap X\ closure\_of\ (topspace\ X - S)) \neq \{\}$
using *assms* **by** (*simp add*: *frontier_of_closures*)
then show $U \cap S \neq \{\}$
using *assms openin_Int_closure_of_eq_empty* **by** *fastforce*
show $U - S \neq \{\}$
proof −
have $\exists A.\ X\ closure\_of\ (A - S) \cap U \neq \{\}$
using ‹$U \cap (X\ closure\_of\ S \cap X\ closure\_of\ (topspace\ X - S)) \neq \{\}$› **by** *blast*
then have $\neg\ U \subseteq S$
by (*metis Diff_disjoint Diff_eq_empty_iff Int_Diff assms*(*1*) *inf_commute openin_Int_closure_of_eq_empty*)
then show *?thesis*
by *blast*
qed
qed

lemma *frontier_of_subset_closedin*: *closedin X S* $\Longrightarrow$ $(X\ frontier\_of\ S) \subseteq S$
using *closure_of_eq frontier_of_def* **by** *fastforce*

lemma *frontier_of_empty* [*simp*]: $X\ frontier\_of\ \{\} = \{\}$
by (*simp add*: *frontier_of_def*)

lemma *frontier_of_topspace* [*simp*]: $X\ frontier\_of\ topspace\ X = \{\}$
by (*simp add*: *frontier_of_def*)

lemma *frontier_of_subset_eq*:
assumes $S \subseteq topspace\ X$
shows $(X\ frontier\_of\ S) \subseteq S \longleftrightarrow closedin\ X\ S$
proof
show $X\ frontier\_of\ S \subseteq S \Longrightarrow closedin\ X\ S$
by (*metis assms closure_of_subset_eq interior_of_subset interior_of_union_frontier_of le_sup_iff*)
show *closedin X S* $\Longrightarrow$ $X\ frontier\_of\ S \subseteq S$
by (*simp add*: *frontier_of_subset_closedin*)
qed

lemma *frontier_of_complement*: $X\ frontier\_of\ (topspace\ X - S) = X\ frontier\_of\ S$
by (*metis Diff_Diff_Int closure_of_restrict frontier_of_closures inf_commute*)

lemma *frontier_of_disjoint_eq*:
assumes $S \subseteq topspace\ X$
shows $((X\ frontier\_of\ S) \cap S = \{\} \longleftrightarrow openin\ X\ S)$
proof
assume $X\ frontier\_of\ S \cap S = \{\}$
then have $closedin\ X\ (topspace\ X - S)$
using *assms closure_of_subset frontier_of_def interior_of_eq interior_of_subset*

by *fastforce*
then show *openin X S*
using *assms* **by** (*simp add*: *openin_closedin*)
next
show *openin X S* $\Longrightarrow$ *X frontier_of S* $\cap$ *S* = {}
by (*simp add*: *Diff_Diff_Int closedin_def frontier_of_openin inf.absorb_iff2 inf_commute*)
qed

lemma *frontier_of_disjoint_eq_alt*:
S $\subseteq$ (*topspace X* $-$ *X frontier_of S*) $\longleftrightarrow$ *openin X S*
proof (*cases S* $\subseteq$ *topspace X*)
case *True*
show *?thesis*
using *True frontier_of_disjoint_eq* **by** *auto*
next
case *False*
then show *?thesis*
by (*meson Diff_subset openin_subset subset_trans*)
qed

lemma *frontier_of_Int*:
X frontier_of (*S* $\cap$ *T*) =
X closure_of (*S* $\cap$ *T*) $\cap$ (*X frontier_of S* $\cup$ *X frontier_of T*)
proof $-$
have $*$: *U* $\subseteq$ *S* $\wedge$ *U* $\subseteq$ *T* $\Longrightarrow$ *U* $\cap$ (*S* $\cap$ *A* $\cup$ *T* $\cap$ *B*) = *U* $\cap$ (*A* $\cup$ *B*) **for** *U S T A B* :: *'a set*
by *blast*
show *?thesis*
by (*simp add*: *frontier_of_closures closure_of_mono Diff_Int* $*$ *flip*: *closure_of_Un*)
qed

lemma *frontier_of_Int_subset*: *X frontier_of* (*S* $\cap$ *T*) $\subseteq$ *X frontier_of S* $\cup$ *X frontier_of T*
by (*simp add*: *frontier_of_Int*)

lemma *frontier_of_Int_closedin*:
assumes *closedin X S closedin X T*
shows *X frontier_of*(*S* $\cap$ *T*) = *X frontier_of S* $\cap$ *T* $\cup$ *S* $\cap$ *X frontier_of T* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
using *assms* **by** (*force simp add*: *frontier_of_Int closedin_Int closure_of_closedin*)
show *?rhs* $\subseteq$ *?lhs*
using *assms frontier_of_subset_closedin*
by (*auto simp add*: *frontier_of_Int closedin_Int closure_of_closedin*)
qed

lemma *frontier_of_Un_subset*: *X frontier_of*$(S \cup T) \subseteq X$ *frontier_of* $S \cup X$ *frontier_of* T
by (*metis Diff_Un frontier_of_Int_subset frontier_of_complement*)

lemma *frontier_of_Union_subset*:
finite $\mathcal{F} \Longrightarrow X$ *frontier_of* $(\bigcup \mathcal{F}) \subseteq (\bigcup T \in \mathcal{F}.\ X$ *frontier_of* $T)$
proof (*induction* $\mathcal{F}$ *rule*: *finite_induct*)
case (*insert A* $\mathcal{F}$)
then show *?case*
using *frontier_of_Un_subset* **by** *fastforce*
qed *simp*

lemma *frontier_of_frontier_of_subset*:
X *frontier_of* (X *frontier_of* $S) \subseteq X$ *frontier_of* S
by (*simp add*: *closedin_frontier_of frontier_of_subset_closedin*)

lemma *frontier_of_subtopology_open*:
openin $X\ U \Longrightarrow$ (*subtopology* $X\ U$) *frontier_of* $S = U \cap X$ *frontier_of* S
by (*simp add*: *Diff_Int_distrib closure_of_subtopology_open frontier_of_def interior_of_subtopology_open*)

lemma *discrete_topology_frontier_of* [*simp*]:
(*discrete_topology* U) *frontier_of* $S = \{\}$
by (*simp add*: *Diff_eq discrete_topology_closure_of frontier_of_closures*)

lemma *openin_subset_topspace_eq*:
assumes *disjnt* S (X *frontier_of* U)
shows *openin* (*subtopology* $X\ U$) $S \longleftrightarrow$ *openin* $X\ S \wedge S \subseteq U$
proof
assume S: *openin* (*subtopology* $X\ U$) S
show *openin* $X\ S \wedge S \subseteq U$
proof
show $S \subseteq U$
using S *openin_imp_subset* **by** *blast*
have *disjnt* S (X *frontier_of* (*topspace* $X \cap U$))
by (*metis assms frontier_of_restrict*)
moreover
have *openin* (*subtopology* X (*topspace* $X \cap U$)) S
by (*simp add*: S *subtopology_restrict*)
moreover
have *openin* $X\ S$
if *dis*: *disjnt* S (X *frontier_of* U) **and** *ope*: *openin* (*subtopology* $X\ U$) S **and** $U \subseteq$ *topspace* X
for $S\ U$
proof −
obtain T **where** T: *openin* $X\ T\ S = T \cap U$
using *ope* **by** (*auto simp*: *openin_subtopology*)
have $T \cap U = T \cap X$ *interior_of* U
using *that T interior_of_subset in_closure_of* **by** (*fastforce simp*: *disjnt_iff*

frontier_of_def)
then show *?thesis*
using ‹$S = T \cap U$› ‹*openin X T*› **by** *auto*
qed
ultimately show *openin X S*
by *blast*
qed
qed (*metis inf.absorb_iff1 openin_subtopology_Int*)

1.13.9 Locally finite collections

definition *locally_finite_in*
where
locally_finite_in X $\mathcal{A} \longleftrightarrow$
$(\bigcup \mathcal{A} \subseteq$ *topspace X*$) \wedge$
$(\forall x \in$ *topspace X*$.\ \exists V.$ *openin X V* $\wedge\ x \in V \wedge$ *finite* $\{U \in \mathcal{A}.\ U \cap V \neq \{\}\})$

lemma *finite_imp_locally_finite_in*:
$[\![$*finite* $\mathcal{A}$; $\bigcup \mathcal{A} \subseteq$ *topspace X*$]\!] \Longrightarrow$ *locally_finite_in X* $\mathcal{A}$
by (*auto simp*: *locally_finite_in_def*)

lemma *locally_finite_in_subset*:
assumes *locally_finite_in X* $\mathcal{A}$ $\mathcal{B} \subseteq \mathcal{A}$
shows *locally_finite_in X* $\mathcal{B}$
proof −
have *finite* $(\mathcal{A} \cap \{U.\ U \cap V \neq \{\}\}) \Longrightarrow$ *finite* $(\mathcal{B} \cap \{U.\ U \cap V \neq \{\}\})$ **for** V
by (*meson* ‹$\mathcal{B} \subseteq \mathcal{A}$› *finite_subset inf_le1 inf_le2 le_inf_iff subset_trans*)
then show *?thesis*
using *assms* **unfolding** *locally_finite_in_def Int_def* **by** *fastforce*
qed

lemma *locally_finite_in_refinement*:
assumes $\mathcal{A}$: *locally_finite_in X* $\mathcal{A}$ **and** f: $\bigwedge S.\ S \in \mathcal{A} \Longrightarrow f\ S \subseteq S$
shows *locally_finite_in X* $(f\ `\ \mathcal{A})$
proof −
show *?thesis*
unfolding *locally_finite_in_def*
proof *safe*
fix x
assume $x \in$ *topspace X*
then obtain V **where** *openin X V* $x \in V$ *finite* $\{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$
using $\mathcal{A}$ **unfolding** *locally_finite_in_def* **by** *blast*
moreover have $\{U \in \mathcal{A}.\ f\ U \cap V \neq \{\}\} \subseteq \{U \in \mathcal{A}.\ U \cap V \neq \{\}\}$ **for** V
using f **by** *blast*
ultimately have *finite* $\{U \in \mathcal{A}.\ f\ U \cap V \neq \{\}\}$
using *finite_subset* **by** *blast*
moreover have $f\ `\ \{U \in \mathcal{A}.\ f\ U \cap V \neq \{\}\} = \{U \in f\ `\ \mathcal{A}.\ U \cap V \neq \{\}\}$
by *blast*

```
    ultimately have finite {U ∈ f ` 𝒜. U ∩ V ≠ {}}
      by (metis (no_types, lifting) finite_imageI)
    then show ∃ V. openin X V ∧ x ∈ V ∧ finite {U ∈ f ` 𝒜. U ∩ V ≠ {}}
      using ‹openin X V› ‹x ∈ V› by blast
  next
    show ⋀x xa. ⟦xa ∈ 𝒜; x ∈ f xa⟧ ⟹ x ∈ topspace X
      by (meson Sup_upper 𝒜 f locally_finite_in_def subset_iff)
  qed
qed

lemma locally_finite_in_subtopology:
  assumes 𝒜: locally_finite_in X 𝒜 ⋃𝒜 ⊆ S
  shows locally_finite_in (subtopology X S) 𝒜
  unfolding locally_finite_in_def
proof safe
  fix x
  assume x: x ∈ topspace (subtopology X S)
  then obtain V where openin X V x ∈ V and fin: finite {U ∈ 𝒜. U ∩ V ≠
{}}
    using 𝒜 unfolding locally_finite_in_def topspace_subtopology by blast
  show ∃ V. openin (subtopology X S) V ∧ x ∈ V ∧ finite {U ∈ 𝒜. U ∩ V ≠ {}}
  proof (intro exI conjI)
    show openin (subtopology X S) (S ∩ V)
      by (simp add: ‹openin X V› openin_subtopology_Int2)
    have {U ∈ 𝒜. U ∩ (S ∩ V) ≠ {}} ⊆ {U ∈ 𝒜. U ∩ V ≠ {}}
      by auto
    with fin show finite {U ∈ 𝒜. U ∩ (S ∩ V) ≠ {}}
      using finite_subset by auto
    show x ∈ S ∩ V
      using x ‹x ∈ V› by (simp)
  qed
next
  show ⋀x A. ⟦x ∈ A; A ∈ 𝒜⟧ ⟹ x ∈ topspace (subtopology X S)
    using assms unfolding locally_finite_in_def topspace_subtopology by blast
qed


lemma closedin_locally_finite_Union:
  assumes clo: ⋀S. S ∈ 𝒜 ⟹ closedin X S and 𝒜: locally_finite_in X 𝒜
  shows closedin X (⋃𝒜)
  using 𝒜 unfolding locally_finite_in_def closedin_def
proof clarify
  show openin X (topspace X − ⋃𝒜)
  proof (subst openin_subopen, clarify)
    fix x
    assume x ∈ topspace X and x ∉ ⋃𝒜
    then obtain V where openin X V x ∈ V and fin: finite {U ∈ 𝒜. U ∩ V ≠
{}}
      using 𝒜 unfolding locally_finite_in_def by blast
```

```
    let ?T = V − ⋃{S ∈ 𝒜. S ∩ V ≠ {}}
    show ∃ T. openin X T ∧ x ∈ T ∧ T ⊆ topspace X − ⋃𝒜
    proof (intro exI conjI)
      show openin X ?T
      by (metis (no_types, lifting) fin ‹openin X V› clo closedin_Union mem_Collect_eq
openin_diff)
      show x ∈ ?T
        using ‹x ∉ ⋃𝒜› ‹x ∈ V› by auto
      show ?T ⊆ topspace X − ⋃𝒜
        using ‹openin X V› openin_subset by auto
    qed
  qed
qed

lemma locally_finite_in_closure:
  assumes 𝒜: locally_finite_in X 𝒜
  shows locally_finite_in X ((λS. X closure_of S) ` 𝒜)
  using 𝒜 unfolding locally_finite_in_def
proof (intro conjI; clarsimp)
  fix x A
  assume x ∈ X closure_of A
  then show x ∈ topspace X
    by (meson in_closure_of)
next
  fix x
  assume x ∈ topspace X and ⋃𝒜 ⊆ topspace X
  then obtain V where V: openin X V x ∈ V and fin: finite {U ∈ 𝒜. U ∩ V
≠ {}}
    using 𝒜 unfolding locally_finite_in_def by blast
  have eq: {y ∈ f ` 𝒜. Q y} = f ` {x. x ∈ 𝒜 ∧ Q(f x)} for f and Q :: 'a set ⇒
bool
    by blast
  have eq2: {A ∈ 𝒜. X closure_of A ∩ V ≠ {}} = {A ∈ 𝒜. A ∩ V ≠ {}}
    using openin_Int_closure_of_eq_empty V by blast
  have finite {U ∈ (closure_of) X ` 𝒜. U ∩ V ≠ {}}
    by (simp add: eq eq2 fin)
  with V show ∃ V. openin X V ∧ x ∈ V ∧ finite {U ∈ (closure_of) X ` 𝒜. U
∩ V ≠ {}}
    by blast
qed

lemma closedin_Union_locally_finite_closure:
   locally_finite_in X 𝒜 ⟹ closedin X (⋃((λS. X closure_of S) ` 𝒜))
  by (metis (mono_tags) closedin_closure_of closedin_locally_finite_Union im-
ageE locally_finite_in_closure)

lemma closure_of_Union_subset: ⋃((λS. X closure_of S) ` 𝒜) ⊆ X closure_of
(⋃𝒜)
  by (simp add: SUP_le_iff Sup_upper closure_of_mono)
```

lemma *closure_of_locally_finite_Union*:
assumes *locally_finite_in X* $\mathcal{A}$
shows *X closure_of* $(\bigcup \mathcal{A}) = \bigcup ((\lambda S.\ X\ closure\_of\ S)\ `\ \mathcal{A})$
proof (*rule closure_of_unique*)
show $\bigcup\ \mathcal{A} \subseteq \bigcup\ ((closure\_of)\ X\ `\ \mathcal{A})$
using *assms* **by** (*simp add*: *SUP_upper2 Sup_le_iff closure_of_subset locally_finite_in_def*)
show *closedin X* $(\bigcup\ ((closure\_of)\ X\ `\ \mathcal{A}))$
using *assms* **by** (*simp add*: *closedin_Union_locally_finite_closure*)
show $\bigwedge T'.\ [\![\bigcup\ \mathcal{A} \subseteq T';\ closedin\ X\ T]\!] \Longrightarrow \bigcup\ ((closure\_of)\ X\ `\ \mathcal{A}) \subseteq T'$
by (*simp add*: *Sup_le_iff closure_of_minimal*)
qed

1.13.10 Continuous maps

We will need to deal with continuous maps in terms of topologies and not in terms of type classes, as defined below.

definition *continuous_map* **where**
continuous_map X Y f $\equiv$
$f \in topspace\ X \to topspace\ Y \wedge$
$(\forall U.\ openin\ Y\ U \longrightarrow openin\ X\ \{x \in topspace\ X.\ f\ x \in U\})$

lemma *continuous_map*:
continuous_map X Y f $\longleftrightarrow$
$f\ `\ (topspace\ X) \subseteq topspace\ Y \wedge (\forall U.\ openin\ Y\ U \longrightarrow openin\ X\ \{x \in topspace\ X.\ f\ x \in U\})$
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_image_subset_topspace*:
continuous_map X Y f $\Longrightarrow f\ `\ (topspace\ X) \subseteq topspace\ Y$
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_funspace*:
continuous_map X Y f $\Longrightarrow f \in topspace\ X \to topspace\ Y$
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_on_empty* [*simp*]: *continuous_map trivial_topology Y f*
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_on_empty2* [*simp*]: *continuous_map X trivial_topology* $f \longleftrightarrow X = trivial\_topology$
using *continuous_map_image_subset_topspace* **by** *fastforce*

lemma *continuous_map_closedin*:
continuous_map X Y f $\longleftrightarrow$
$f \in topspace\ X \to topspace\ Y \wedge$
$(\forall C.\ closedin\ Y\ C \longrightarrow closedin\ X\ \{x \in topspace\ X.\ f\ x \in C\})$
proof −

```
  have (∀ U. openin Y U ⟶ openin X {x ∈ topspace X. f x ∈ U}) =
        (∀ C. closedin Y C ⟶ closedin X {x ∈ topspace X. f x ∈ C})
    if f ∈ topspace X → topspace Y
  proof −
    have eq: {x ∈ topspace X. f x ∈ topspace Y ∧ f x ∉ C} = (topspace X − {x
∈ topspace X. f x ∈ C}) for C
      using that by blast
    show ?thesis
    proof (intro iffI allI impI)
      fix C
       assume ∀ U. openin Y U ⟶ openin X {x ∈ topspace X. f x ∈ U} and
closedin Y C
      then show closedin X {x ∈ topspace X. f x ∈ C}
        by (auto simp add: closedin_def eq)
    next
      fix U
       assume ∀ C. closedin Y C ⟶ closedin X {x ∈ topspace X. f x ∈ C} and
openin Y U
      then show openin X {x ∈ topspace X. f x ∈ U}
        by (auto simp add: openin_closedin_eq eq)
    qed
  qed
  then show ?thesis
    by (auto simp: continuous_map_def)
qed

lemma openin_continuous_map_preimage:
  ⟦continuous_map X Y f; openin Y U⟧ ⟹ openin X {x ∈ topspace X. f x ∈ U}
  by (simp add: continuous_map_def)

lemma closedin_continuous_map_preimage:
  ⟦continuous_map X Y f; closedin Y C⟧ ⟹ closedin X {x ∈ topspace X. f x ∈
C}
  by (simp add: continuous_map_closedin)

lemma openin_continuous_map_preimage_gen:
  assumes continuous_map X Y f openin X U openin Y V
  shows openin X {x ∈ U. f x ∈ V}
proof −
  have eq: {x ∈ U. f x ∈ V} = U ∩ {x ∈ topspace X. f x ∈ V}
    using assms(2) openin_closedin_eq by fastforce
  show ?thesis
    unfolding eq
    using assms openin_continuous_map_preimage by fastforce
qed

lemma closedin_continuous_map_preimage_gen:
  assumes continuous_map X Y f closedin X U closedin Y V
  shows closedin X {x ∈ U. f x ∈ V}
```

```
proof −
  have eq: {x ∈ U. f x ∈ V} = U ∩ {x ∈ topspace X. f x ∈ V}
    using assms(2) closedin_def by fastforce
  show ?thesis
    unfolding eq
    using assms closedin_continuous_map_preimage by fastforce
qed

lemma continuous_map_image_closure_subset:
  assumes continuous_map X Y f
  shows f ‘ (X closure_of S) ⊆ Y closure_of f ‘ S
proof −
  have ∗: f ‘ (topspace X) ⊆ topspace Y
    by (meson assms continuous_map)
  have X closure_of T ⊆ {x ∈ X closure_of T. f x ∈ Y closure_of (f ‘ T)}
    if T ⊆ topspace X for T
  proof (rule closure_of_minimal)
    show T ⊆ {x ∈ X closure_of T. f x ∈ Y closure_of f ‘ T}
      using closure_of_subset ∗ that by (fastforce simp: in_closure_of)
  next
    show closedin X {x ∈ X closure_of T. f x ∈ Y closure_of f ‘ T}
      using assms closedin_continuous_map_preimage_gen by fastforce
  qed
  then show ?thesis
   by (smt (verit, ccfv_threshold) assms continuous_map image_eqI image_subset_iff
in_closure_of mem_Collect_eq)
qed

lemma continuous_map_subset_aux1: continuous_map X Y f ⟹
      (∀ S. f ‘ (X closure_of S) ⊆ Y closure_of f ‘ S)
  using continuous_map_image_closure_subset by blast

lemma continuous_map_subset_aux2:
  assumes ∀ S. S ⊆ topspace X ⟶ f ‘ (X closure_of S) ⊆ Y closure_of f ‘ S
  shows continuous_map X Y f
  unfolding continuous_map_closedin
proof (intro conjI ballI allI impI)
  show f ∈ topspace X → topspace Y
    using assms closure_of_subset_topspace by fastforce
next
  fix C
  assume closedin Y C
  then show closedin X {x ∈ topspace X. f x ∈ C}
  proof (clarsimp simp flip: closure_of_subset_eq, intro conjI)
    fix x
    assume x: x ∈ X closure_of {x ∈ topspace X. f x ∈ C}
      and C ⊆ topspace Y and Y closure_of C ⊆ C
    show x ∈ topspace X
      by (meson x in_closure_of)
```

```
    have {a ∈ topspace X. f a ∈ C} ⊆ topspace X
      by simp
    moreover have Y closure_of f ‘ {a ∈ topspace X. f a ∈ C} ⊆ C
      by (simp add: ‹closedin Y C› closure_of_minimal image_subset_iff)
    ultimately show f x ∈ C
      using x assms by blast
  qed
qed

lemma continuous_map_eq_image_closure_subset:
    continuous_map X Y f ⟷ (∀ S. f ‘ (X closure_of S) ⊆ Y closure_of f ‘ S)
  using continuous_map_subset_aux1 continuous_map_subset_aux2 by metis

lemma continuous_map_eq_image_closure_subset_alt:
    continuous_map X Y f ⟷ (∀ S. S ⊆ topspace X ⟶ f ‘ (X closure_of S) ⊆
Y closure_of f ‘ S)
  using continuous_map_subset_aux1 continuous_map_subset_aux2 by metis

lemma continuous_map_eq_image_closure_subset_gen:
    continuous_map X Y f ⟷
      f ∈ topspace X → topspace Y ∧
      (∀ S. f ‘ (X closure_of S) ⊆ Y closure_of f ‘ S)
  by (meson Pi_iff continuous_map continuous_map_eq_image_closure_subset
image_subset_iff)

lemma continuous_map_closure_preimage_subset:
   continuous_map X Y f
      ⟹ X closure_of {x ∈ topspace X. f x ∈ T}
         ⊆ {x ∈ topspace X. f x ∈ Y closure_of T}
  unfolding continuous_map_closedin
  by (rule closure_of_minimal) (use in_closure_of in ‹fastforce+›)


lemma continuous_map_frontier_frontier_preimage_subset:
  assumes continuous_map X Y f
  shows X frontier_of {x ∈ topspace X. f x ∈ T} ⊆ {x ∈ topspace X. f x ∈ Y
frontier_of T}
proof −
  have eq: topspace X − {x ∈ topspace X. f x ∈ T} = {x ∈ topspace X. f x ∈
topspace Y − T}
    using assms unfolding continuous_map_def by blast
  have X closure_of {x ∈ topspace X. f x ∈ T} ⊆ {x ∈ topspace X. f x ∈ Y
closure_of T}
    by (simp add: assms continuous_map_closure_preimage_subset)
  moreover
  have X closure_of (topspace X − {x ∈ topspace X. f x ∈ T}) ⊆ {x ∈ topspace
X. f x ∈ Y closure_of (topspace Y − T)}
    using continuous_map_closure_preimage_subset [OF assms] eq by presburger
  ultimately show ?thesis
```

by (*auto simp*: *frontier_of_closures*)
qed

lemma *topology_finer_continuous_id*:
assumes *topspace X* = *topspace Y*
shows (∀ *S*. *openin X S* ⟶ *openin Y S*) ⟷ *continuous_map Y X id* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
unfolding *continuous_map_def*
using *assms openin_subopen openin_subset* **by** *fastforce*
show *?rhs* ⟹ *?lhs*
unfolding *continuous_map_def*
using *assms openin_subopen topspace_def* **by** *fastforce*
qed

lemma *continuous_map_const* [*simp*]:
continuous_map X Y (λ*x*. *C*) ⟷ *X* = *trivial_topology* ∨ *C* ∈ *topspace Y*
proof (*cases X* = *trivial_topology*)
case *nontriv*: *False*
show *?thesis*
proof (*cases C* ∈ *topspace Y*)
case *True*
with *openin_subopen* **show** *?thesis*
by (*auto simp*: *continuous_map_def*)
next
case *False*
with *nontriv* **show** *?thesis*
using *continuous_map_image_subset_topspace discrete_topology_unique image_subset_iff* **by** *fastforce*
qed
qed *auto*

declare *continuous_map_const* [*THEN iffD2*, *continuous_intros*]

lemma *continuous_map_compose* [*continuous_intros*]:
assumes *f*: *continuous_map X X' f* **and** *g*: *continuous_map X' X'' g*
shows *continuous_map X X''* (*g* ∘ *f*)
unfolding *continuous_map_def*
proof (*intro conjI ballI allI impI*)
show *g* ∘ *f* ∈ *topspace X* → *topspace X''*
using *assms* **unfolding** *continuous_map_def* **by** *force*
next
fix *U*
assume *openin X'' U*
have *eq*: {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* ∈ *U*} = {*x* ∈ *topspace X*. *f x* ∈ {*y*. *y* ∈ *topspace X'* ∧ *g y* ∈ *U*}}
using *continuous_map_image_subset_topspace f* **by** *force*
show *openin X* {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* ∈ *U*}

unfolding *eq*
using *assms* **unfolding** *continuous_map_def*
using ‹*openin X'' U*› **by** *blast*
qed

lemma *continuous_map_eq*:
assumes *continuous_map X X' f* **and** $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *f x* = *g x*
shows *continuous_map X X' g*
proof −
have *eq*: $\{x \in$ *topspace X*. *f x* $\in$ *U*$\}$ = $\{x \in$ *topspace X*. *g x* $\in$ *U*$\}$ **for** *U*
using *assms* **by** *auto*
show *?thesis*
using *assms* **by** (*force simp add*: *continuous_map_def eq*)
qed

lemma *restrict_continuous_map* [*simp*]:
topspace X $\subseteq$ *S* $\Longrightarrow$ *continuous_map X X'* (*restrict f S*) $\longleftrightarrow$ *continuous_map X X' f*
by (*auto simp*: *elim*!: *continuous_map_eq*)

lemma *continuous_map_in_subtopology*:
continuous_map X (*subtopology X' S*) *f* $\longleftrightarrow$ *continuous_map X X' f* $\wedge$ *f* ‘ (*topspace X*) $\subseteq$ *S*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof −
have $\bigwedge A.$ *f* ‘ (*X closure_of A*) $\subseteq$ *subtopology X' S closure_of f* ‘ *A*
by (*meson L continuous_map_image_closure_subset*)
then show *?thesis*
by (*metis* (*no_types*) *closure_of_subset_subtopology closure_of_subtopology_subset closure_of_topspace continuous_map_eq_image_closure_subset order.trans*)
qed
next
assume *R*: *?rhs*
then have *eq*: $\{x \in$ *topspace X*. *f x* $\in$ *U*$\}$ = $\{x \in$ *topspace X*. *f x* $\in$ *U* $\wedge$ *f x* $\in$ *S*$\}$ **for** *U*
by *auto*
show *?lhs*
using *R*
unfolding *continuous_map*
by (*auto simp*: *openin_subtopology eq*)
qed

lemma *continuous_map_from_subtopology*:
continuous_map X Y f $\Longrightarrow$ *continuous_map* (*subtopology X S*) *Y f*
by (*auto simp*: *continuous_map openin_subtopology*)

lemma *continuous_map_into_fulltopology*:
continuous_map X (*subtopology Y T*) *f* ⟹ *continuous_map X Y f*
by (*auto simp*: *continuous_map_in_subtopology*)

lemma *continuous_map_into_subtopology*:
⟦*continuous_map X Y f*; *f* ∈ *topspace X* → *T*⟧ ⟹ *continuous_map X* (*subtopology Y T*) *f*
by (*auto simp*: *continuous_map_in_subtopology*)

lemma *continuous_map_from_subtopology_mono*:
⟦*continuous_map* (*subtopology X T*) *Y f*; *S* ⊆ *T*⟧
⟹ *continuous_map* (*subtopology X S*) *Y f*
by (*metis inf.absorb_iff2 continuous_map_from_subtopology subtopology_subtopology*)

lemma *continuous_map_from_discrete_topology* [*simp*]:
continuous_map (*discrete_topology U*) *X f* ⟷ *f* ∈ *U* → *topspace X*
by (*auto simp*: *continuous_map_def*)

lemma *continuous_map_iff_continuous* [*simp*]: *continuous_map* (*top_of_set S*) *euclidean g* = *continuous_on S g*
by (*fastforce simp add*: *continuous_map openin_subtopology continuous_on_open_invariant*)

lemma *continuous_map_iff_continuous2* [*simp*]: *continuous_map euclidean euclidean g* = *continuous_on UNIV g*
by (*metis continuous_map_iff_continuous subtopology_UNIV*)

lemma *continuous_map_openin_preimage_eq*:
continuous_map X Y f ⟷
f ∈ (*topspace X*) → *topspace Y* ∧ (∀ *U*. *openin Y U* ⟶ *openin X* (*topspace X* ∩ *f* −‘ *U*))
by (*auto simp*: *continuous_map_def vimage_def Int_def*)

lemma *continuous_map_closedin_preimage_eq*:
continuous_map X Y f ⟷
f ∈ (*topspace X*) → *topspace Y* ∧ (∀ *U*. *closedin Y U* ⟶ *closedin X* (*topspace X* ∩ *f* −‘ *U*))
by (*auto simp*: *continuous_map_closedin vimage_def Int_def*)

lemma *continuous_map_square_root*: *continuous_map euclideanreal euclideanreal sqrt*
by (*simp add*: *continuous_at_imp_continuous_on isCont_real_sqrt*)

lemma *continuous_map_sqrt* [*continuous_intros*]:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *sqrt*(*f x*))
by (*meson continuous_map_compose continuous_map_eq continuous_map_square_root o_apply*)

lemma *continuous_map_id* [*simp*, *continuous_intros*]: *continuous_map X X id*
unfolding *continuous_map_def* **using** *openin_subopen topspace_def* **by** *fastforce*

declare *continuous_map_id* [*unfolded id_def*, *simp*, *continuous_intros*]

lemma *continuous_map_id_subt* [*simp*]: *continuous_map* (*subtopology X S*) *X id*
by (*simp add*: *continuous_map_from_subtopology*)

declare *continuous_map_id_subt* [*unfolded id_def*, *simp*]

lemma *continuous_map_alt*:
continuous_map T1 T2 f
= ((∀ *U*. *openin T2 U* ⟶ *openin T1* (*f* −‘ *U* ∩ *topspace T1*)) ∧ *f* ∈ *topspace T1* → *topspace T2*)
by (*auto simp*: *continuous_map_def vimage_def image_def Collect_conj_eq inf_commute*)

lemma *continuous_map_open* [*intro*]:
continuous_map T1 T2 f ⟹ *openin T2 U* ⟹ *openin T1* (*f*−‘*U* ∩ *topspace*(*T1*))
unfolding *continuous_map_alt* **by** *auto*

lemma *continuous_map_preimage_topspace* [*intro*]:
assumes *continuous_map T1 T2 f*
shows *f*−‘(*topspace T2*) ∩ *topspace T1* = *topspace T1*
using *assms* **unfolding** *continuous_map_def* **by** *auto*

1.13.11 Open and closed maps (not a priori assumed continuous)

definition *open_map* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *open_map X1 X2 f* ≡ ∀ *U*. *openin X1 U* ⟶ *openin X2* (*f* ‘ *U*)

definition *closed_map* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *closed_map X1 X2 f* ≡ ∀ *U*. *closedin X1 U* ⟶ *closedin X2* (*f* ‘ *U*)

lemma *open_map_imp_subset_topspace*:
open_map X1 X2 f ⟹ *f* ∈ (*topspace X1*) → *topspace X2*
unfolding *open_map_def* **using** *openin_subset* **by** *fastforce*

lemma *open_map_on_empty* [*simp*]: *open_map trivial_topology Y f*
by (*simp add*: *open_map_def*)

lemma *closed_map_on_empty*:
closed_map trivial_topology Y f
by (*simp add*: *closed_map_def closedin_topspace_empty*)

lemma *closed_map_const*:

closed_map X Y (λ*x*. *c*) ⟷ *X* = *trivial_topology* ∨ *closedin Y* {*c*}
by (*metis closed_map_def closed_map_on_empty closedin_topspace discrete_topology_unique equals0D image_constant_conv*)

lemma *open_map_imp_subset*:
⟦*open_map X1 X2 f*; *S* ⊆ *topspace X1*⟧ ⟹ *f* ∈ *S* → *topspace X2*
using *open_map_imp_subset_topspace* **by** *fastforce*

lemma *topology_finer_open_id*:
(∀ *S*. *openin X S* ⟶ *openin X′ S*) ⟷ *open_map X X′ id*
unfolding *open_map_def* **by** *auto*

lemma *open_map_id*: *open_map X X id*
unfolding *open_map_def* **by** *auto*

lemma *open_map_eq*:
⟦*open_map X X′ f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *open_map X X′ g*
unfolding *open_map_def*
by (*metis image_cong openin_subset subset_iff*)

lemma *open_map_inclusion_eq*:
open_map (*subtopology X S*) *X id* ⟷ *openin X* (*topspace X* ∩ *S*)
by (*metis openin_topspace openin_trans_full subtopology_restrict topology_finer_open_id topspace_subtopology*)

lemma *open_map_inclusion*:
openin X S ⟹ *open_map* (*subtopology X S*) *X id*
by (*simp add*: *open_map_inclusion_eq openin_Int*)

lemma *open_map_compose*:
⟦*open_map X X′ f*; *open_map X′ X″ g*⟧ ⟹ *open_map X X″* (*g* ∘ *f*)
by (*metis* (*no_types*, *lifting*) *image_comp open_map_def*)

lemma *closed_map_imp_subset_topspace*:
closed_map X1 X2 f ⟹ *f* ∈ (*topspace X1*) → *topspace X2*
by (*simp add*: *closed_map_def closedin_def image_subset_iff_funcset*)

lemma *closed_map_imp_subset*:
⟦*closed_map X1 X2 f*; *S* ⊆ *topspace X1*⟧ ⟹ *f* ∈ *S* → *topspace X2*
using *closed_map_imp_subset_topspace* **by** *blast*

lemma *topology_finer_closed_id*:
(∀ *S*. *closedin X S* ⟶ *closedin X′ S*) ⟷ *closed_map X X′ id*
by (*simp add*: *closed_map_def*)

lemma *closed_map_id*: *closed_map X X id*
by (*simp add*: *closed_map_def*)

lemma *closed_map_eq*:

⟦*closed_map X X' f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *closed_map X X' g*
unfolding *closed_map_def*
by (*metis image_cong closedin_subset subset_iff*)

lemma *closed_map_compose*:
⟦*closed_map X X' f*; *closed_map X' X'' g*⟧ ⟹ *closed_map X X''* (*g* ∘ *f*)
by (*metis* (*no_types*, *lifting*) *closed_map_def image_comp*)

lemma *closed_map_inclusion_eq*:
closed_map (*subtopology X S*) *X id* ⟷
closedin X (*topspace X* ∩ *S*)
proof −
have ∗: *closedin X* (*T* ∩ *S*) **if** *closedin X* (*S* ∩ *topspace X*) *closedin X T* **for** *T*
by (*smt* (*verit*, *best*) *closedin_Int closure_of_subset_eq inf_sup_aci le_iff_inf that*)
then show *?thesis*
by (*fastforce simp add*: *closed_map_def Int_commute closedin_subtopology_alt intro*: ∗)
qed

lemma *closed_map_inclusion*: *closedin X S* ⟹ *closed_map* (*subtopology X S*) *X id*
by (*simp add*: *closed_map_inclusion_eq closedin_Int*)

lemma *open_map_into_subtopology*:
⟦*open_map X X' f*; *f* ∈ *topspace X* → *S*⟧ ⟹ *open_map X* (*subtopology X' S*) *f*
unfolding *open_map_def openin_subtopology*
using *openin_subset* **by** *fastforce*

lemma *closed_map_into_subtopology*:
⟦*closed_map X X' f*; *f* ∈ *topspace X* → *S*⟧ ⟹ *closed_map X* (*subtopology X' S*) *f*
unfolding *closed_map_def closedin_subtopology*
using *closedin_subset* **by** *fastforce*

lemma *open_map_into_discrete_topology*:
open_map X (*discrete_topology U*) *f* ⟷ *f* ∈ (*topspace X*) → *U*
unfolding *open_map_def openin_discrete_topology* **using** *openin_subset* **by** *blast*

lemma *closed_map_into_discrete_topology*:
closed_map X (*discrete_topology U*) *f* ⟷ *f* ∈ (*topspace X*) → *U*
unfolding *closed_map_def closedin_discrete_topology* **using** *closedin_subset* **by** *blast*

lemma *bijective_open_imp_closed_map*:
⟦*open_map X X' f*; *f* ‘ (*topspace X*) = *topspace X'*; *inj_on f* (*topspace X*)⟧

$\Longrightarrow$ *closed_map X X′ f*
unfolding *open_map_def closed_map_def closedin_def*
by *auto* (*metis Diff_subset inj_on_image_set_diff*)

lemma *bijective_closed_imp_open_map*:
⟦*closed_map X X′ f*; *f ‘ (topspace X) = topspace X′*; *inj_on f (topspace X)*⟧
$\Longrightarrow$ *open_map X X′ f*
unfolding *closed_map_def open_map_def openin_closedin_eq*
by *auto* (*metis Diff_subset inj_on_image_set_diff*)

lemma *open_map_from_subtopology*:
⟦*open_map X X′ f*; *openin X U*⟧ $\Longrightarrow$ *open_map* (*subtopology X U*) *X′ f*
unfolding *open_map_def openin_subtopology_alt* **by** *blast*

lemma *closed_map_from_subtopology*:
⟦*closed_map X X′ f*; *closedin X U*⟧ $\Longrightarrow$ *closed_map* (*subtopology X U*) *X′ f*
unfolding *closed_map_def closedin_subtopology_alt* **by** *blast*

lemma *open_map_restriction*:
assumes *f*: *open_map X X′ f* **and** *U*: $\{x \in$ *topspace X*. *f x* $\in V\} = U$
shows *open_map* (*subtopology X U*) (*subtopology X′ V*) *f*
unfolding *open_map_def*
proof *clarsimp*
fix *W*
assume *openin* (*subtopology X U*) *W*
then obtain *T* **where** *openin X T W* = *T* $\cap$ *U*
by (*meson openin_subtopology*)
with *f U* **have** *f ‘ W* = (*f ‘ T*) $\cap$ *V*
unfolding *open_map_def openin_closedin_eq* **by** *auto*
then show *openin* (*subtopology X′ V*) (*f ‘ W*)
by (*metis* ‹*openin X T*› *f open_map_def openin_subtopology_Int*)
qed

lemma *closed_map_restriction*:
assumes *f*: *closed_map X X′ f* **and** *U*: $\{x \in$ *topspace X*. *f x* $\in V\} = U$
shows *closed_map* (*subtopology X U*) (*subtopology X′ V*) *f*
unfolding *closed_map_def*
proof *clarsimp*
fix *W*
assume *closedin* (*subtopology X U*) *W*
then obtain *T* **where** *closedin X T W* = *T* $\cap$ *U*
by (*meson closedin_subtopology*)
with *f U* **have** *f ‘ W* = (*f ‘ T*) $\cap$ *V*
unfolding *closed_map_def closedin_def* **by** *auto*
then show *closedin* (*subtopology X′ V*) (*f ‘ W*)
by (*metis* ‹*closedin X T*› *closed_map_def closedin_subtopology f*)
qed

lemma *closed_map_closure_of_image*:

closed_map X Y f ⟷
f ∈ topspace X → topspace Y ∧
(∀ S. S ⊆ topspace X ⟶ Y closure_of (f ‘ S) ⊆ f ‘ (X closure_of S)) (**is** *?lhs=?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*simp add*: *closed_map_def closed_map_imp_subset_topspace closure_of_minimal closure_of_subset image_mono*)
next
assume *?rhs*
then show *?lhs*
by (*metis closed_map_def closed_map_into_discrete_topology closure_of_eq*

closure_of_subset_eq topspace_discrete_topology)
qed

lemma *open_map_interior_of_image_subset*:
open_map X Y f ⟷ (∀ S. image f (X interior_of S) ⊆ Y interior_of (f ‘ S))
by (*metis image_mono interior_of_eq interior_of_maximal interior_of_subset open_map_def openin_interior_of subset_antisym*)

lemma *open_map_interior_of_image_subset_alt*:
open_map X Y f ⟷ (∀ S⊆topspace X. f ‘ (X interior_of S) ⊆ Y interior_of f ‘ S)
by (*metis interior_of_eq open_map_def open_map_interior_of_image_subset openin_subset subset_interior_of_eq*)

lemma *open_map_interior_of_image_subset_gen*:
open_map X Y f ⟷
(f ∈ topspace X → topspace Y ∧ (∀ S. f ‘ (X interior_of S) ⊆ Y interior_of f ‘ S))
by (*metis open_map_imp_subset_topspace open_map_interior_of_image_subset*)

lemma *open_map_preimage_neighbourhood*:
open_map X Y f ⟷
(f ∈ topspace X → topspace Y ∧
(∀ U T. closedin X U ∧ T ⊆ topspace Y ∧
{x ∈ topspace X. f x ∈ T} ⊆ U ⟶
(∃ V. closedin Y V ∧ T ⊆ V ∧ {x ∈ topspace X. f x ∈ V} ⊆ U))) (**is** *?lhs=?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof (*intro conjI strip*)
show *f ∈ topspace X → topspace Y*
by (*simp add*: ‹*open_map X Y f*› *open_map_imp_subset_topspace*)
next
fix *U T*

```
    assume UT: closedin X U ∧ T ⊆ topspace Y ∧ {x ∈ topspace X. f x ∈ T} ⊆
U
    have closedin Y (topspace Y − f ‘ (topspace X − U))
    by (meson UT L open_map_def openin_closedin_eq openin_diff openin_topspace)
    with UT
    show ∃ V. closedin Y V ∧ T ⊆ V ∧ {x ∈ topspace X. f x ∈ V} ⊆ U
      using image_iff by auto
  qed
next
  assume R: ?rhs
  show ?lhs
    unfolding open_map_def
  proof (intro strip)
    fix U assume openin X U
    have {x ∈ topspace X. f x ∈ topspace Y − f ‘ U} ⊆ topspace X − U
      by blast
    then obtain V where V: closedin Y V
      and sub: topspace Y − f ‘ U ⊆ V {x ∈ topspace X. f x ∈ V} ⊆ topspace X
− U
      using R ‹openin X U› by (meson Diff_subset openin_closedin_eq)
    then have f ‘ U ⊆ topspace Y − V
      using R ‹openin X U› openin_subset by fastforce
    with sub have f ‘ U = topspace Y − V
      by auto
    then show openin Y (f ‘ U)
      using V(1) by auto
  qed
qed


lemma closed_map_preimage_neighbourhood:
  closed_map X Y f ⟷
      f ∈ topspace X → topspace Y ∧
      (∀ U T. openin X U ∧ T ⊆ topspace Y ∧
            {x ∈ topspace X. f x ∈ T} ⊆ U
            ⟶ (∃ V. openin Y V ∧ T ⊆ V ∧
                  {x ∈ topspace X. f x ∈ V} ⊆ U)) (is ?lhs=?rhs)
proof
  assume L: ?lhs
  show ?rhs
  proof (intro conjI strip)
    show f ∈ topspace X → topspace Y
      by (simp add: L closed_map_imp_subset_topspace)
  next
    fix U T
    assume UT: openin X U ∧ T ⊆ topspace Y ∧ {x ∈ topspace X. f x ∈ T} ⊆ U
    then have openin Y (topspace Y − f ‘ (topspace X − U))
      by (meson L closed_map_def closedin_def closedin_diff closedin_topspace)
    then show ∃ V. openin Y V ∧ T ⊆ V ∧ {x ∈ topspace X. f x ∈ V} ⊆ U
```

```
    using UT image_iff by auto
  qed
next
  assume R: ?rhs
  show ?lhs
    unfolding closed_map_def
  proof (intro strip)
    fix U assume closedin X U
    have {x ∈ topspace X. f x ∈ topspace Y − f ‘ U} ⊆ topspace X − U
      by blast
    then obtain V where V: openin Y V
      and sub: topspace Y − f ‘ U ⊆ V {x ∈ topspace X. f x ∈ V} ⊆ topspace X
− U
      using R Diff_subset ‹closedin X U› unfolding closedin_def
      by (smt (verit, ccfv_threshold) Collect_mem_eq Collect_mono_iff)
    then have f ‘ U ⊆ topspace Y − V
      using R ‹closedin X U› closedin_subset by fastforce
    with sub have f ‘ U = topspace Y − V
      by auto
    with V show closedin Y (f ‘ U)
      by auto
  qed
qed

lemma closed_map_fibre_neighbourhood:
  closed_map X Y f ⟷
    f ∈ (topspace X) → topspace Y ∧
    (∀ U y. openin X U ∧ y ∈ topspace Y ∧ {x ∈ topspace X. f x = y} ⊆ U
    ⟶ (∃ V. openin Y V ∧ y ∈ V ∧ {x ∈ topspace X. f x ∈ V} ⊆ U))
  unfolding closed_map_preimage_neighbourhood
proof (intro conj_cong refl all_cong1)
  fix U
  assume f ∈ topspace X → topspace Y
  show (∀ T. openin X U ∧ T ⊆ topspace Y ∧ {x ∈ topspace X. f x ∈ T} ⊆ U
        ⟶ (∃ V. openin Y V ∧ T ⊆ V ∧ {x ∈ topspace X. f x ∈ V} ⊆ U))
      = (∀ y. openin X U ∧ y ∈ topspace Y ∧ {x ∈ topspace X. f x = y} ⊆ U
        ⟶ (∃ V. openin Y V ∧ y ∈ V ∧ {x ∈ topspace X. f x ∈ V} ⊆ U))
    (is (∀ T. ?P T) ⟷ (∀ y. ?Q y))
  proof
    assume L [rule_format]: ∀ T. ?P T
    show ∀ y. ?Q y
    proof
      fix y show ?Q y
        using L [of {y}] by blast
    qed
  next
    assume R: ∀ y. ?Q y
    show ∀ T. ?P T
    proof (cases openin X U)
```

```
      case True
      note [[unify_search_bound=3]]
      obtain V where V: ⋀y. ⟦y ∈ topspace Y; {x ∈ topspace X. f x = y} ⊆ U⟧
⟹
                        openin Y (V y) ∧ y ∈ V y ∧ {x ∈ topspace X. f x ∈ V y} ⊆ U
        using R by (simp add: True) meson
      show ?thesis
      proof clarify
        fix T
        assume openin X U and T ⊆ topspace Y and {x ∈ topspace X. f x ∈ T}
⊆ U
        with V show ∃ V. openin Y V ∧ T ⊆ V ∧ {x ∈ topspace X. f x ∈ V} ⊆
U
          by (rule_tac x=⋃ y∈T. V y in exI) fastforce
      qed
    qed auto
  qed
qed

lemma open_map_in_subtopology:
   openin Y S
       ⟹ open_map X (subtopology Y S) f ⟷ open_map X Y f ∧ f ∈ topspace
X → S
  by (metis image_subset_iff_funcset open_map_def open_map_into_subtopology
openin_imp_subset openin_topspace openin_trans_full)

lemma open_map_from_open_subtopology:
   ⟦openin Y S; open_map X (subtopology Y S) f⟧ ⟹ open_map X Y f
  using open_map_in_subtopology by blast

lemma closed_map_in_subtopology:
   closedin Y S
         ⟹ closed_map X (subtopology Y S) f ⟷ (closed_map X Y f ∧ f ∈
topspace X → S)
  by (metis closed_map_def closed_map_imp_subset_topspace closed_map_into_subtopology

        closedin_closed_subtopology closedin_subset topspace_subtopology_subset)

lemma closed_map_from_closed_subtopology:
   ⟦closedin Y S; closed_map X (subtopology Y S) f⟧ ⟹ closed_map X Y f
  using closed_map_in_subtopology by blast

lemma closed_map_from_composition_left:
  assumes cmf: closed_map X Z (g ∘ f) and contf: continuous_map X Y f and
fim: f ‘ topspace X = topspace Y
  shows closed_map Y Z g
  unfolding closed_map_def
proof (intro strip)
  fix U assume closedin Y U
```

then have *closedin X* {$x \in$ *topspace X*. *f x* $\in$ *U*}
using *contf closedin_continuous_map_preimage* **by** *blast*
then have *closedin Z* ((*g* $\circ$ *f*) ‘ {$x \in$ *topspace X*. *f x* $\in$ *U*})
using *cmf closed_map_def* **by** *blast*
moreover
have $\bigwedge y.$ *y* $\in$ *U* $\Longrightarrow$ $\exists x \in$ *topspace X*. *f x* $\in$ *U* $\wedge$ *g y* = *g* (*f x*)
by (*smt* (*verit*, *ccfv_SIG*) ‹*closedin Y U*› *closedin_subset fim image_iff subsetD*)
then have (*g* $\circ$ *f*) ‘ {$x \in$ *topspace X*. *f x* $\in$ *U*} = *g*‘*U* **by** *auto*
ultimately show *closedin Z* (*g* ‘ *U*)
by *metis*
qed

identical proof as the above

lemma *open_map_from_composition_left*:
assumes *cmf*: *open_map X Z* (*g* $\circ$ *f*) **and** *contf*: *continuous_map X Y f* **and** *fim*: *f* ‘ *topspace X* = *topspace Y*
shows *open_map Y Z g*
unfolding *open_map_def*
proof (*intro strip*)
fix *U* **assume** *openin Y U*
then have *openin X* {$x \in$ *topspace X*. *f x* $\in$ *U*}
using *contf openin_continuous_map_preimage* **by** *blast*
then have *openin Z* ((*g* $\circ$ *f*) ‘ {$x \in$ *topspace X*. *f x* $\in$ *U*})
using *cmf open_map_def* **by** *blast*
moreover
have $\bigwedge y.$ *y* $\in$ *U* $\Longrightarrow$ $\exists x \in$ *topspace X*. *f x* $\in$ *U* $\wedge$ *g y* = *g* (*f x*)
by (*smt* (*verit*, *ccfv_SIG*) ‹*openin Y U*› *openin_subset fim image_iff subsetD*)
then have (*g* $\circ$ *f*) ‘ {$x \in$ *topspace X*. *f x* $\in$ *U*} = *g*‘*U* **by** *auto*
ultimately show *openin Z* (*g* ‘ *U*)
by *metis*
qed

lemma *closed_map_from_composition_right*:
assumes *cmf*: *closed_map X Z* (*g* $\circ$ *f*) *f* $\in$ *topspace X* $\to$ *topspace Y continuous_map Y Z g inj_on g* (*topspace Y*)
shows *closed_map X Y f*
unfolding *closed_map_def*
proof (*intro strip*)
fix *C* **assume** *closedin X C*
have $\bigwedge y\ c.$ ⟦*y* $\in$ *topspace Y*; *g y* = *g* (*f c*); *c* $\in$ *C*⟧ $\Longrightarrow$ *y* $\in$ *f* ‘ *C*
using ‹*closedin X C*› *assms closedin_subset inj_onD* **by** *fastforce*
then
have *f* ‘ *C* = {$x \in$ *topspace Y*. *g x* $\in$ (*g* $\circ$ *f*) ‘ *C*}
using ‹*closedin X C*› *assms*(2) *closedin_subset* **by** *fastforce*
moreover have *closedin Z* ((*g* $\circ$ *f*) ‘ *C*)
using ‹*closedin X C*› *cmf closed_map_def* **by** *blast*
ultimately show *closedin Y* (*f* ‘ *C*)
using *assms*(3) *closedin_continuous_map_preimage* **by** *fastforce*

qed

identical proof as the above

lemma *open_map_from_composition_right*:
assumes *open_map X Z* $(g \circ f)$ $f \in$ *topspace X* $\to$ *topspace Y continuous_map Y Z g inj_on g* (*topspace Y*)
shows *open_map X Y f*
unfolding *open_map_def*
proof (*intro strip*)
fix *C* **assume** *openin X C*
have $\bigwedge y\ c.\ [\![y \in$ *topspace Y*; *g y* $=$ *g* (*f c*); $c \in C]\!] \Longrightarrow y \in f$ ‘ *C*
using ‹*openin X C*› *assms openin_subset inj_onD* **by** *fastforce*
then
have *f* ‘ *C* $= \{x \in$ *topspace Y*. *g x* $\in (g \circ f)$ ‘ *C*$\}$
using ‹*openin X C*› *assms*(*2*) *openin_subset* **by** *fastforce*
moreover have *openin Z* ((*g* $\circ$ *f*) ‘ *C*)
using ‹*openin X C*› *assms*(*1*) *open_map_def* **by** *blast*
ultimately show *openin Y* (*f* ‘ *C*)
using *assms*(*3*) *openin_continuous_map_preimage* **by** *fastforce*
qed

1.13.12 Quotient maps

definition *quotient_map* **where**
quotient_map X X′ f $\longleftrightarrow$
f ‘ (*topspace X*) $=$ *topspace X′* $\wedge$
$(\forall\ U.\ U \subseteq$ *topspace X′* $\longrightarrow$ (*openin X* $\{x.\ x \in$ *topspace X* $\wedge$ *f x* $\in U\} \longleftrightarrow$ *openin X′ U*))

lemma *quotient_map_eq*:
assumes *quotient_map X X′ f* $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *f x* $=$ *g x*
shows *quotient_map X X′ g*
by (*smt* (*verit*) *Collect_cong assms image_cong quotient_map_def*)

lemma *quotient_map_compose*:
assumes *f*: *quotient_map X X′ f* **and** *g*: *quotient_map X′ X″ g*
shows *quotient_map X X″* $(g \circ f)$
unfolding *quotient_map_def*
proof (*intro conjI allI impI*)
show $(g \circ f)$ ‘ *topspace X* $=$ *topspace X″*
using *assms* **by** (*simp only*: *image_comp* [*symmetric*]) (*simp add*: *quotient_map_def*)
next
fix *U″*
assume *U″*: *U″* $\subseteq$ *topspace X″*
define *U′* **where** *U′* $\equiv \{y \in$ *topspace X′*. *g y* $\in$ *U″*$\}$
have *U′* $\subseteq$ *topspace X′*
by (*auto simp add*: *U′_def*)
then have *U′*: *openin X* $\{x \in$ *topspace X*. *f x* $\in$ *U′*$\}$ $=$ *openin X′ U′*
using *assms* **unfolding** *quotient_map_def* **by** *simp*

have $\{x \in topspace\ X.\ f\ x \in topspace\ X' \wedge g\ (f\ x) \in U''\} = \{x \in topspace\ X.\ (g \circ f)\ x \in U''\}$
using *f quotient_map_def* **by** *fastforce*
then show *openin X* $\{x \in topspace\ X.\ (g \circ f)\ x \in U''\}$ = *openin X'' U''*
by (*smt* (*verit*, *best*) *Collect_cong U' U'_def U'' g mem_Collect_eq quotient_map_def*)
qed

lemma *quotient_map_from_composition*:
assumes *f*: *continuous_map X X' f* **and** *g*: *continuous_map X' X'' g* **and** *gf*: *quotient_map X X'' (g ∘ f)*
shows *quotient_map X' X'' g*
unfolding *quotient_map_def*
proof (*intro conjI allI impI*)
show *g ‘ topspace X' = topspace X''*
using *assms* **unfolding** *continuous_map_def quotient_map_def* **by** *fastforce*
next
fix *U''* :: *'c set*
assume *U''*: *U'' ⊆ topspace X''*
have *eq*: $\{x \in topspace\ X.\ g\ (f\ x) \in U''\} = \{x \in topspace\ X.\ f\ x \in \{y.\ y \in topspace\ X' \wedge g\ y \in U''\}\}$
using *continuous_map_def f* **by** *fastforce*
show *openin X'* $\{x \in topspace\ X'.\ g\ x \in U''\}$ = *openin X'' U''*
using *assms* **unfolding** *continuous_map_def quotient_map_def*
by (*metis* (*mono_tags*, *lifting*) *Collect_cong U'' comp_apply eq*)
qed

lemma *quotient_imp_continuous_map*:
quotient_map X X' f ⟹ *continuous_map X X' f*
by (*simp add*: *continuous_map openin_subset quotient_map_def*)

lemma *quotient_imp_surjective_map*:
quotient_map X X' f ⟹ *f ‘ (topspace X) = topspace X'*
by (*simp add*: *quotient_map_def*)

lemma *quotient_map_closedin*:
quotient_map X X' f ⟷
f ‘ (topspace X) = topspace X' ∧
(∀ *U*. *U ⊆ topspace X'* ⟶ (*closedin X* $\{x.\ x \in topspace\ X \wedge f\ x \in U\}$ ⟷ *closedin X' U*))
proof −
have *eq*: (*topspace X* − $\{x \in topspace\ X.\ f\ x \in U'\}$) = $\{x \in topspace\ X.\ f\ x \in topspace\ X' \wedge f\ x \notin U'\}$
if *f ‘ topspace X = topspace X' U' ⊆ topspace X'* **for** *U'*
using *that* **by** *auto*
have (∀ *U*⊆*topspace X'*. *openin X* $\{x \in topspace\ X.\ f\ x \in U\}$ = *openin X' U*) =
(∀ *U*⊆*topspace X'*. *closedin X* $\{x \in topspace\ X.\ f\ x \in U\}$ = *closedin X' U*)

```
    if f ‘ topspace X = topspace X'
  proof (rule iffI; intro allI impI subsetI)
    fix U'
     assume *[rule_format]: ∀ U⊆topspace X'. openin X {x ∈ topspace X. f x ∈
U} = openin X' U
      and U': U' ⊆ topspace X'
    show closedin X {x ∈ topspace X. f x ∈ U'} = closedin X' U'
     using U' by (auto simp add: closedin_def simp flip: * [of topspace X' − U']
eq [OF that])
  next
    fix U' :: 'b set
    assume *[rule_format]: ∀ U⊆topspace X'. closedin X {x ∈ topspace X. f x ∈
U} = closedin X' U
      and U': U' ⊆ topspace X'
    show openin X {x ∈ topspace X. f x ∈ U'} = openin X' U'
     using U' by (auto simp add: openin_closedin_eq simp flip: * [of topspace
X' − U'] eq [OF that])
  qed
  then show ?thesis
    unfolding quotient_map_def by force
qed

lemma continuous_open_imp_quotient_map:
   assumes continuous_map X X' f and om: open_map X X' f and feq: f ‘
(topspace X) = topspace X'
  shows quotient_map X X' f
proof −
  { fix U
    assume U: U ⊆ topspace X' and openin X {x ∈ topspace X. f x ∈ U}
    then have ope: openin X' (f ‘ {x ∈ topspace X. f x ∈ U})
      using om unfolding open_map_def by blast
    then have openin X' U
      using U feq by (subst openin_subopen) force
  }
  moreover have openin X {x ∈ topspace X. f x ∈ U} if U ⊆ topspace X' and
openin X' U for U
    using that assms unfolding continuous_map_def by blast
  ultimately show ?thesis
    unfolding quotient_map_def using assms by blast
qed

lemma continuous_closed_imp_quotient_map:
   assumes continuous_map X X' f and om: closed_map X X' f and feq: f ‘
(topspace X) = topspace X'
  shows quotient_map X X' f
proof −
  have f ‘ {x ∈ topspace X. f x ∈ U} = U if U ⊆ topspace X' for U
    using that feq by auto
  with assms show ?thesis
```

unfolding *quotient_map_closedin closed_map_def continuous_map_closedin*
by *auto*
qed

lemma *continuous_open_quotient_map*:
⟦*continuous_map X X' f*; *open_map X X' f*⟧ ⟹ *quotient_map X X' f* ⟷ *f* ‘ (*topspace X*) = *topspace X'*
by (*meson continuous_open_imp_quotient_map quotient_map_def*)

lemma *continuous_closed_quotient_map*:
⟦*continuous_map X X' f*; *closed_map X X' f*⟧ ⟹ *quotient_map X X' f* ⟷ *f* ‘ (*topspace X*) = *topspace X'*
by (*meson continuous_closed_imp_quotient_map quotient_map_def*)

lemma *injective_quotient_map*:
assumes *inj_on f* (*topspace X*)
shows *quotient_map X X' f* ⟷
continuous_map X X' f ∧ *open_map X X' f* ∧ *closed_map X X' f* ∧ *f* ‘ (*topspace X*) = *topspace X'*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
have *om*: *open_map X X' f*
proof (*clarsimp simp add*: *open_map_def*)
fix *U*
assume *openin X U*
then have *U* ⊆ *topspace X*
by (*simp add*: *openin_subset*)
moreover have {*x* ∈ *topspace X*. *f x* ∈ *f* ‘ *U*} = *U*
using ‹*U* ⊆ *topspace X*› *assms inj_onD* **by** *fastforce*
ultimately show *openin X'* (*f* ‘ *U*)
using *L* **unfolding** *quotient_map_def*
by (*metis* (*no_types*, *lifting*) *Collect_cong* ‹*openin X U*› *image_mono*)
qed
then have *closed_map X X' f*
by (*simp add*: *L assms bijective_open_imp_closed_map quotient_imp_surjective_map*)
then show *?rhs*
using *L om* **by** (*simp add*: *quotient_imp_continuous_map quotient_imp_surjective_map*)
next
assume *?rhs*
then show *?lhs*
by (*simp add*: *continuous_closed_imp_quotient_map*)
qed

lemma *continuous_compose_quotient_map*:
assumes *f*: *quotient_map X X' f* **and** *g*: *continuous_map X X''* (*g* ∘ *f*)
shows *continuous_map X' X'' g*
unfolding *quotient_map_def continuous_map_def*
proof (*intro conjI ballI allI impI*)

show $g \in$ *topspace* $X' \to$ *topspace* X''
using *assms* **unfolding** *quotient_map_def Pi_iff*
by (*metis* (*no_types*, *opaque_lifting*) *continuous_map_image_subset_topspace image_comp image_subset_iff*)
next
fix U'' :: $'c$ *set*
assume U'': *openin* X'' U''
have f ' *topspace* X = *topspace* X'
by (*simp add*: f *quotient_imp_surjective_map*)
then have *eq*: $\{x \in$ *topspace* X. $f\,x \in$ *topspace* $X' \wedge g\,(f\,x) \in U\} = \{x \in$ *topspace* X. $g\,(f\,x) \in U\}$ **for** U
by *auto*
have *openin* X $\{x \in$ *topspace* X. $f\,x \in$ *topspace* $X' \wedge g\,(f\,x) \in U''\}$
unfolding *eq* **using** U'' g *openin_continuous_map_preimage* **by** *fastforce*
then have $*$: *openin* X $\{x \in$ *topspace* X. $f\,x \in \{x \in$ *topspace* X'. $g\,x \in U''\}\}$
by *auto*
show *openin* X' $\{x \in$ *topspace* X'. $g\,x \in U''\}$
using f **unfolding** *quotient_map_def*
by (*metis* (*no_types*) *Collect_subset* $*$)
qed

lemma *continuous_compose_quotient_map_eq*:
quotient_map X X' $f \Longrightarrow$ *continuous_map* X X'' $(g \circ f) \longleftrightarrow$ *continuous_map* X' X'' g
using *continuous_compose_quotient_map continuous_map_compose quotient_imp_continuous_map*
by *blast*

lemma *quotient_map_compose_eq*:
quotient_map X X' $f \Longrightarrow$ *quotient_map* X X'' $(g \circ f) \longleftrightarrow$ *quotient_map* X' X'' g
by (*meson continuous_compose_quotient_map_eq quotient_imp_continuous_map quotient_map_compose quotient_map_from_composition*)

lemma *quotient_map_restriction*:
assumes *quo*: *quotient_map* X Y f **and** U: $\{x \in$ *topspace* X. $f\,x \in V\} = U$ **and** *disj*: *openin* Y $V \vee$ *closedin* Y V
shows *quotient_map* (*subtopology* X U) (*subtopology* Y V) f
using *disj*
proof
assume V: *openin* Y V
with U **have** *sub*: $U \subseteq$ *topspace* X $V \subseteq$ *topspace* Y
by (*auto simp*: *openin_subset*)
have *fim*: f ' *topspace* X = *topspace* Y
and Y: $\bigwedge U$. $U \subseteq$ *topspace* $Y \Longrightarrow$ *openin* X $\{x \in$ *topspace* X. $f\,x \in U\} =$ *openin* Y U
using *quo* **unfolding** *quotient_map_def* **by** *auto*
have *openin* X U
using U V Y *sub*(*2*) **by** *blast*
show *?thesis*

```
    unfolding quotient_map_def
  proof (intro conjI allI impI)
    show f ‘ topspace (subtopology X U) = topspace (subtopology Y V)
      using sub U fim by (auto)
  next
    fix Y' :: 'b set
    assume Y' ⊆ topspace (subtopology Y V)
    then have Y' ⊆ topspace Y Y' ⊆ V
      by (simp_all)
    then have eq: {x ∈ topspace X. x ∈ U ∧ f x ∈ Y'} = {x ∈ topspace X. f x ∈
Y'}
      using U by blast
    then show openin (subtopology X U) {x ∈ topspace (subtopology X U). f x ∈
Y'} = openin (subtopology Y V) Y'
      using U V Y ‹openin X U› ‹Y' ⊆ topspace Y› ‹Y' ⊆ V›
     by (simp add: openin_open_subtopology eq) (auto simp: openin_closedin_eq)
  qed
next
  assume V: closedin Y V
  with U have sub: U ⊆ topspace X V ⊆ topspace Y
    by (auto simp: closedin_subset)
  have fim: f ‘ topspace X = topspace Y
     and Y: ⋀U. U ⊆ topspace Y ⟹ closedin X {x ∈ topspace X. f x ∈ U} =
closedin Y U
    using quo unfolding quotient_map_closedin by auto
  have closedin X U
    using U V Y sub(2) by blast
  show ?thesis
    unfolding quotient_map_closedin
  proof (intro conjI allI impI)
    show f ‘ topspace (subtopology X U) = topspace (subtopology Y V)
      using sub U fim by (auto)
  next
    fix Y' :: 'b set
    assume Y' ⊆ topspace (subtopology Y V)
    then have Y' ⊆ topspace Y Y' ⊆ V
      by (simp_all)
    then have eq: {x ∈ topspace X. x ∈ U ∧ f x ∈ Y'} = {x ∈ topspace X. f x ∈
Y'}
      using U by blast
    then show closedin (subtopology X U) {x ∈ topspace (subtopology X U). f x ∈
Y'} = closedin (subtopology Y V) Y'
      using U V Y ‹closedin X U› ‹Y' ⊆ topspace Y› ‹Y' ⊆ V›
      by (simp add: closedin_closed_subtopology eq) (auto simp: closedin_def)
  qed
qed

lemma quotient_map_saturated_open:
    quotient_map X Y f ⟷
```

```
      continuous_map X Y f ∧ f ‘ (topspace X) = topspace Y ∧
      (∀ U. openin X U ∧ {x ∈ topspace X. f x ∈ f ‘ U} ⊆ U ⟶ openin Y (f ‘
U))
    (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  then have fim: f ‘ topspace X = topspace Y
    and Y: ⋀U. U ⊆ topspace Y ⟹ openin Y U = openin X {x ∈ topspace X. f
x ∈ U}
    unfolding quotient_map_def by auto
  show ?rhs
  proof (intro conjI allI impI)
    show continuous_map X Y f
      by (simp add: L quotient_imp_continuous_map)
    show f ‘ topspace X = topspace Y
      by (simp add: fim)
  next
    fix U :: 'a set
    assume U: openin X U ∧ {x ∈ topspace X. f x ∈ f ‘ U} ⊆ U
    then have sub: f ‘ U ⊆ topspace Y and eq: {x ∈ topspace X. f x ∈ f ‘ U} =
U
      using fim openin_subset by fastforce+
    show openin Y (f ‘ U)
      by (simp add: sub Y eq U)
  qed
next
  assume ?rhs
  then have YX: ⋀U. openin Y U ⟹ openin X {x ∈ topspace X. f x ∈ U}
      and fim: f ‘ topspace X = topspace Y
      and XY: ⋀U. ⟦openin X U; {x ∈ topspace X. f x ∈ f ‘ U} ⊆ U⟧ ⟹ openin
Y (f ‘ U)
    by (auto simp: quotient_map_def continuous_map_def)
  show ?lhs
  proof (simp add: quotient_map_def fim, intro allI impI iffI)
    fix U :: 'b set
    assume U ⊆ topspace Y and X: openin X {x ∈ topspace X. f x ∈ U}
    have feq: f ‘ {x ∈ topspace X. f x ∈ U} = U
      using ‹U ⊆ topspace Y› fim by auto
    show openin Y U
      using XY [OF X] by (simp add: feq)
  next
    fix U :: 'b set
    assume U ⊆ topspace Y and Y: openin Y U
    show openin X {x ∈ topspace X. f x ∈ U}
      by (metis YX [OF Y])
  qed
qed

lemma quotient_map_saturated_closed:
```

```
    quotient_map X Y f ⟷
      continuous_map X Y f ∧ f ‘ (topspace X) = topspace Y ∧
      (∀ U. closedin X U ∧ {x ∈ topspace X. f x ∈ f ‘ U} ⊆ U ⟶ closedin Y (f
‘ U))
    (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  then obtain fim: f ‘ topspace X = topspace Y
    and Y: ⋀U. U ⊆ topspace Y ⟹ closedin Y U = closedin X {x ∈ topspace
X. f x ∈ U}
    by (simp add: quotient_map_closedin)
  show ?rhs
  proof (intro conjI allI impI)
    show continuous_map X Y f
      by (simp add: L quotient_imp_continuous_map)
    show f ‘ topspace X = topspace Y
      by (simp add: fim)
  next
    fix U :: 'a set
    assume U: closedin X U ∧ {x ∈ topspace X. f x ∈ f ‘ U} ⊆ U
    then have sub: f ‘ U ⊆ topspace Y and eq: {x ∈ topspace X. f x ∈ f ‘ U} =
U
      using fim closedin_subset by fastforce+
    show closedin Y (f ‘ U)
      by (simp add: sub Y eq U)
  qed
next
  assume ?rhs
  then obtain YX: ⋀U. closedin Y U ⟹ closedin X {x ∈ topspace X. f x ∈ U}
    and fim: f ‘ topspace X = topspace Y
    and XY: ⋀U. ⟦closedin X U; {x ∈ topspace X. f x ∈ f ‘ U} ⊆ U⟧ ⟹ closedin
Y (f ‘ U)
    by (simp add: continuous_map_closedin)
  show ?lhs
  proof (simp add: quotient_map_closedin fim, intro allI impI iffI)
    fix U :: 'b set
    assume U ⊆ topspace Y and X: closedin X {x ∈ topspace X. f x ∈ U}
    have feq: f ‘ {x ∈ topspace X. f x ∈ U} = U
      using ‹U ⊆ topspace Y› fim by auto
    show closedin Y U
      using XY [OF X] by (simp add: feq)
  next
    fix U :: 'b set
    assume U ⊆ topspace Y and Y: closedin Y U
    show closedin X {x ∈ topspace X. f x ∈ U}
      by (metis YX [OF Y])
  qed
qed
```

lemma *quotient_map_onto_image*:
assumes *f ‘ topspace X ⊆ topspace Y* **and** *U*: ⋀*U*. *U ⊆ topspace Y ⟹ openin X {x ∈ topspace X*. *f x ∈ U} = openin Y U*
shows *quotient_map X (subtopology Y (f ‘ topspace X)) f*
unfolding *quotient_map_def topspace_subtopology*
proof (*intro conjI strip*)
fix *U*
assume *U ⊆ topspace Y ∩ f ‘ topspace X*
with *U* **have** *openin X {x ∈ topspace X*. *f x ∈ U} ⟹ ∃*x*. *openin Y x ∧ U = f ‘ topspace X ∩ x*
by *fastforce*
moreover have ∃*x*. *openin Y x ∧ U = f ‘ topspace X ∩ x ⟹ openin X {x ∈ topspace X*. *f x ∈ U}*
by (*metis* (*mono_tags*, *lifting*) *Collect_cong IntE IntI U image_eqI openin_subset*)
ultimately show *openin X {x ∈ topspace X*. *f x ∈ U} = openin (subtopology Y (f ‘ topspace X)) U*
by (*force simp*: *openin_subtopology_alt image_iff*)
qed (*use assms* **in** *auto*)

lemma *quotient_map_lift_exists*:
assumes *f*: *quotient_map X Y f* **and** *h*: *continuous_map X Z h*
and ⋀*x y*. ⟦*x ∈ topspace X*; *y ∈ topspace X*; *f x = f y*⟧ ⟹ *h x = h y*
obtains *g* **where** *continuous_map Y Z g g ‘ topspace Y = h ‘ topspace X*
⋀*x*. *x ∈ topspace X ⟹ g(f x) = h x*
proof −
obtain *g* **where** *g*: ⋀*x*. *x ∈ topspace X ⟹ h x = g(f x)*
using *function_factors_left_gen*[*of* λ*x*. *x ∈ topspace X f h*] *assms* **by** *blast*
show *?thesis*
proof
show *g ‘ topspace Y = h ‘ topspace X*
using *f g* **by** (*force dest*!: *quotient_imp_surjective_map*)
show *continuous_map Y Z g*
by (*smt* (*verit*) *f g h continuous_compose_quotient_map_eq continuous_map_eq o_def*)
qed (*simp add*: *g*)
qed

1.13.13 Separated Sets

definition *separatedin* :: *'a topology ⇒ 'a set ⇒ 'a set ⇒ bool*
where *separatedin X S T ≡*
S ⊆ topspace X ∧ T ⊆ topspace X ∧
S ∩ X closure_of T = {} ∧ T ∩ X closure_of S = {}

lemma *separatedin_empty* [*simp*]:
separatedin X S {} ⟷ S ⊆ topspace X
separatedin X {} S ⟷ S ⊆ topspace X
by (*simp_all add*: *separatedin_def*)

lemma *separatedin_refl* [*simp*]:
  separatedin X S S ⟷ *S* = {}
 by (*metis closure_of_subset empty_subsetI inf.orderE separatedin_def*)

lemma *separatedin_sym*:
  separatedin X S T ⟷ *separatedin X T S*
 by (*auto simp*: *separatedin_def*)

lemma *separatedin_imp_disjoint*:
  separatedin X S T ⟹ *disjnt S T*
 by (*meson closure_of_subset disjnt_def disjnt_subset2 separatedin_def*)

lemma *separatedin_mono*:
  ⟦*separatedin X S T*; *S'* ⊆ *S*; *T'* ⊆ *T*⟧ ⟹ *separatedin X S' T'*
 unfolding *separatedin_def*
 using *closure_of_mono* **by** *blast*

lemma *separatedin_open_sets*:
  ⟦*openin X S*; *openin X T*⟧ ⟹ *separatedin X S T* ⟷ *disjnt S T*
 unfolding *disjnt_def separatedin_def*
 by (*auto simp*: *openin_Int_closure_of_eq_empty openin_subset*)

lemma *separatedin_closed_sets*:
  ⟦*closedin X S*; *closedin X T*⟧ ⟹ *separatedin X S T* ⟷ *disjnt S T*
 unfolding *closure_of_eq disjnt_def separatedin_def*
 by (*metis closedin_def closure_of_eq inf_commute*)

lemma *separatedin_subtopology*:
  separatedin (*subtopology X U*) *S T* ⟷ *S* ⊆ *U* ∧ *T* ⊆ *U* ∧ *separatedin X S T*
 by (*auto simp*: *separatedin_def closure_of_subtopology Int_ac disjoint_iff elim*!:
inf.orderE)

lemma *separatedin_discrete_topology*:
  separatedin (*discrete_topology U*) *S T* ⟷ *S* ⊆ *U* ∧ *T* ⊆ *U* ∧ *disjnt S T*
 by (*metis openin_discrete_topology separatedin_def separatedin_open_sets topspace_discrete_topology*)

lemma *separated_eq_distinguishable*:
  separatedin X {*x*} {*y*} ⟷
    x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧
    (∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ (*y* ∉ *U*)) ∧
    (∃ *v*. *openin X v* ∧ *y* ∈ *v* ∧ (*x* ∉ *v*))
 by (*force simp*: *separatedin_def closure_of_def*)

lemma *separatedin_Un* [*simp*]:
  separatedin X S (*T* ∪ *U*) ⟷ *separatedin X S T* ∧ *separatedin X S U*
  separatedin X (*S* ∪ *T*) *U* ⟷ *separatedin X S U* ∧ *separatedin X T U*
 by (*auto simp*: *separatedin_def*)

lemma *separatedin_Union*:

finite $\mathcal{F} \Longrightarrow$ *separatedin* $X\ S\ (\bigcup \mathcal{F}) \longleftrightarrow S \subseteq$ *topspace* $X \wedge (\forall\ T \in \mathcal{F}.$ *separatedin* $X\ S\ T)$
finite $\mathcal{F} \Longrightarrow$ *separatedin* $X\ (\bigcup \mathcal{F})\ S \longleftrightarrow (\forall\ T \in \mathcal{F}.$ *separatedin* $X\ S\ T) \wedge S \subseteq$ *topspace* X
by (*auto simp*: *separatedin_def closure_of_Union*)

lemma *separatedin_openin_diff*:
$[\![$*openin* $X\ S$; *openin* $X\ T]\!] \Longrightarrow$ *separatedin* $X\ (S - T)\ (T - S)$
unfolding *separatedin_def*
by (*metis Diff_Int_distrib2 Diff_disjoint Diff_empty Diff_mono empty_Diff empty_subsetI openin_Int_closure_of_eq_empty openin_subset*)

lemma *separatedin_closedin_diff*:
assumes *closedin* $X\ S$ *closedin* $X\ T$
shows *separatedin* $X\ (S - T)\ (T - S)$
proof −
have $S - T \subseteq$ *topspace* $X\ T - S \subseteq$ *topspace* X
using *assms closedin_subset* **by** *auto*
with *assms* **show** *?thesis*
by (*simp add*: *separatedin_def Diff_Int_distrib2 closure_of_minimal inf_absorb2*)
qed

lemma *separation_closedin_Un_gen*:
separatedin $X\ S\ T \longleftrightarrow$
$S \subseteq$ *topspace* $X \wedge T \subseteq$ *topspace* $X \wedge$ *disjnt* $S\ T \wedge$
closedin (*subtopology* $X\ (S \cup T))\ S \wedge$
closedin (*subtopology* $X\ (S \cup T))\ T$
by (*auto simp add*: *separatedin_def closedin_Int_closure_of disjnt_iff dest*: *closure_of_subset*)

lemma *separation_openin_Un_gen*:
separatedin $X\ S\ T \longleftrightarrow$
$S \subseteq$ *topspace* $X \wedge T \subseteq$ *topspace* $X \wedge$ *disjnt* $S\ T \wedge$
openin (*subtopology* $X\ (S \cup T))\ S \wedge$
openin (*subtopology* $X\ (S \cup T))\ T$
unfolding *openin_closedin_eq topspace_subtopology separation_closedin_Un_gen disjnt_def*
by (*auto simp*: *Diff_triv Int_commute Un_Diff inf_absorb1 topspace_def*)

lemma *separatedin_full*:
$S \cup T =$ *topspace* X
$\Longrightarrow$ *separatedin* $X\ S\ T \longleftrightarrow$ *disjnt* $S\ T \wedge$ *closedin* $X\ S \wedge$ *openin* $X\ S \wedge$ *closedin* $X\ T \wedge$ *openin* $X\ T$
by (*metis separatedin_open_sets separation_closedin_Un_gen separation_openin_Un_gen subtopology_topspace*)

1.13.14 Homeomorphisms

(1-way and 2-way versions may be useful in places)

definition *homeomorphic_map* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where
homeomorphic_map X Y f ≡ *quotient_map X Y f* ∧ *inj_on f* (*topspace X*)

definition *homeomorphic_maps* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) ⇒ *bool*
where
homeomorphic_maps X Y f g ≡
continuous_map X Y f ∧ *continuous_map Y X g* ∧
(∀ *x* ∈ *topspace X*. *g*(*f x*) = *x*) ∧ (∀ *y* ∈ *topspace Y*. *f*(*g y*) = *y*)

lemma *homeomorphic_map_eq*:
⟦*homeomorphic_map X Y f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *homeomorphic_map X Y g*
by (*meson homeomorphic_map_def inj_on_cong quotient_map_eq*)

lemma *homeomorphic_maps_eq*:
⟦*homeomorphic_maps X Y f g*;
⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *f' x*; ⋀*y*. *y* ∈ *topspace Y* ⟹ *g y* = *g' y*⟧
⟹ *homeomorphic_maps X Y f' g'*
unfolding *homeomorphic_maps_def*
by (*metis continuous_map_eq continuous_map_image_subset_topspace image_subset_iff*)

lemma *homeomorphic_maps_sym*:
homeomorphic_maps X Y f g ⟷ *homeomorphic_maps Y X g f*
by (*auto simp*: *homeomorphic_maps_def*)

lemma *homeomorphic_maps_id*:
homeomorphic_maps X Y id id ⟷ *Y* = *X* (**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have *topspace X* = *topspace Y*
by (*auto simp*: *homeomorphic_maps_def continuous_map_def*)
with *L* **show** *?rhs*
unfolding *homeomorphic_maps_def*
by (*metis topology_finer_continuous_id topology_eq*)
next
assume *?rhs*
then show *?lhs*
unfolding *homeomorphic_maps_def* **by** *auto*
qed

lemma *homeomorphic_map_id* [*simp*]: *homeomorphic_map X Y id* ⟷ *Y* = *X*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have *eq*: *topspace X* = *topspace Y*

```
  by (auto simp: homeomorphic_map_def continuous_map_def quotient_map_def)
 then have ⋀S. openin X S ⟶ openin Y S
  by (meson L homeomorphic_map_def injective_quotient_map topology_finer_open_id)
 then show ?rhs
  using L unfolding homeomorphic_map_def
  by (metis eq quotient_imp_continuous_map topology_eq topology_finer_continuous_id)
next
 assume ?rhs
 then show ?lhs
  unfolding homeomorphic_map_def
  by (simp add: closed_map_id continuous_closed_imp_quotient_map)
qed

lemma homeomorphic_map_compose:
 assumes homeomorphic_map X Y f homeomorphic_map Y X'' g
 shows homeomorphic_map X X'' (g ∘ f)
proof −
 have inj_on g (f ‘ topspace X)
  by (metis (no_types) assms homeomorphic_map_def quotient_imp_surjective_map)
 then show ?thesis
  using assms by (meson comp_inj_on homeomorphic_map_def quotient_map_compose_eq)
qed

lemma homeomorphic_maps_compose:
  homeomorphic_maps X Y f h ∧
      homeomorphic_maps Y X'' g k
      ⟹ homeomorphic_maps X X'' (g ∘ f) (h ∘ k)
 unfolding homeomorphic_maps_def
 by (auto simp: continuous_map_compose; simp add: continuous_map_def Pi_iff)

lemma homeomorphic_eq_everything_map:
  homeomorphic_map X Y f ⟷
      continuous_map X Y f ∧ open_map X Y f ∧ closed_map X Y f ∧
      f ‘ (topspace X) = topspace Y ∧ inj_on f (topspace X)
 unfolding homeomorphic_map_def
 by (force simp: injective_quotient_map intro: injective_quotient_map)

lemma homeomorphic_imp_continuous_map:
    homeomorphic_map X Y f ⟹ continuous_map X Y f
 by (simp add: homeomorphic_eq_everything_map)

lemma homeomorphic_imp_open_map:
  homeomorphic_map X Y f ⟹ open_map X Y f
 by (simp add: homeomorphic_eq_everything_map)

lemma homeomorphic_imp_closed_map:
  homeomorphic_map X Y f ⟹ closed_map X Y f
 by (simp add: homeomorphic_eq_everything_map)
```

lemma *homeomorphic_imp_surjective_map*:
homeomorphic_map X Y f ⟹ *f ‘ topspace X = topspace Y*
using *homeomorphic_eq_everything_map* **by** *fastforce*

lemma *homeomorphic_imp_injective_map*:
homeomorphic_map X Y f ⟹ *inj_on f* (*topspace X*)
by (*simp add*: *homeomorphic_eq_everything_map*)

lemma *bijective_open_imp_homeomorphic_map*:
⟦*continuous_map X Y f*; *open_map X Y f*; *f ‘* (*topspace X*) = *topspace Y*; *inj_on f* (*topspace X*)⟧
⟹ *homeomorphic_map X Y f*
by (*simp add*: *homeomorphic_map_def continuous_open_imp_quotient_map*)

lemma *bijective_closed_imp_homeomorphic_map*:
⟦*continuous_map X Y f*; *closed_map X Y f*; *f ‘* (*topspace X*) = *topspace Y*; *inj_on f* (*topspace X*)⟧
⟹ *homeomorphic_map X Y f*
by (*simp add*: *continuous_closed_quotient_map homeomorphic_map_def*)

lemma *open_eq_continuous_inverse_map*:
assumes *X*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ∈ *topspace Y* ∧ *g*(*f x*) = *x*
and *Y*: ⋀*y*. *y* ∈ *topspace Y* ⟹ *g y* ∈ *topspace X* ∧ *f*(*g y*) = *y*
shows *open_map X Y f* ⟷ *continuous_map Y X g*
proof −
have *eq*: {*x* ∈ *topspace Y*. *g x* ∈ *U*} = *f ‘ U* **if** *openin X U* **for** *U*
using *openin_subset* [*OF that*] **by** (*force simp*: *X Y image_iff*)
show *?thesis*
by (*auto simp*: *Y open_map_def continuous_map_def eq*)
qed

lemma *closed_eq_continuous_inverse_map*:
assumes *X*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ∈ *topspace Y* ∧ *g*(*f x*) = *x*
and *Y*: ⋀*y*. *y* ∈ *topspace Y* ⟹ *g y* ∈ *topspace X* ∧ *f*(*g y*) = *y*
shows *closed_map X Y f* ⟷ *continuous_map Y X g*
proof −
have *eq*: {*x* ∈ *topspace Y*. *g x* ∈ *U*} = *f ‘ U* **if** *closedin X U* **for** *U*
using *closedin_subset* [*OF that*] **by** (*force simp*: *X Y image_iff*)
show *?thesis*
by (*auto simp*: *Y closed_map_def continuous_map_closedin eq*)
qed

lemma *homeomorphic_maps_map*:
homeomorphic_maps X Y f g ⟷
homeomorphic_map X Y f ∧ *homeomorphic_map Y X g* ∧
(∀ *x* ∈ *topspace X*. *g*(*f x*) = *x*) ∧ (∀ *y* ∈ *topspace Y*. *f*(*g y*) = *y*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*

```
  then have L: continuous_map X Y f continuous_map Y X g ∀x∈topspace X. g
(f x) = x ∀x'∈topspace Y. f (g x') = x'
    by (auto simp: homeomorphic_maps_def)
  show ?rhs
  proof (intro conjI bijective_open_imp_homeomorphic_map L)
    show open_map X Y f
      using L using open_eq_continuous_inverse_map [of concl: X Y f g]
      by (simp add: continuous_map_def Pi_iff)
    show open_map Y X g
      using L using open_eq_continuous_inverse_map [of concl: Y X g f]
      by (simp add: continuous_map_def Pi_iff)
    show f ‘ topspace X = topspace Y g ‘ topspace Y = topspace X
      using L by (force simp: continuous_map_closedin Pi_iff)+
    show inj_on f (topspace X) inj_on g (topspace Y)
      using L unfolding inj_on_def by metis+
  qed
next
  assume ?rhs
  then show ?lhs
   by (auto simp: homeomorphic_maps_def homeomorphic_imp_continuous_map)
qed

lemma homeomorphic_maps_imp_map:
    homeomorphic_maps X Y f g ⟹ homeomorphic_map X Y f
  using homeomorphic_maps_map by blast

lemma homeomorphic_map_maps:
     homeomorphic_map X Y f ⟷ (∃g. homeomorphic_maps X Y f g)
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have L: continuous_map X Y f open_map X Y f closed_map X Y f
    f ‘ (topspace X) = topspace Y inj_on f (topspace X)
    by (auto simp: homeomorphic_eq_everything_map)
  have X: ⋀x. x ∈ topspace X ⟹ f x ∈ topspace Y ∧ inv_into (topspace X) f (f
x) = x
    using L by auto
  have Y: ⋀y. y ∈ topspace Y ⟹ inv_into (topspace X) f y ∈ topspace X ∧ f
(inv_into (topspace X) f y) = y
    by (simp add: L f_inv_into_f inv_into_into)
  have homeomorphic_maps X Y f (inv_into (topspace X) f)
    unfolding homeomorphic_maps_def
  proof (intro conjI L)
    show continuous_map Y X (inv_into (topspace X) f)
      by (simp add: L X Y flip: open_eq_continuous_inverse_map [where f=f])
  next
    show ∀x∈topspace X. inv_into (topspace X) f (f x) = x
         ∀y∈topspace Y. f (inv_into (topspace X) f y) = y
      using X Y by auto
```

```
  qed
  then show ?rhs
    by metis
next
  assume ?rhs
  then show ?lhs
    using homeomorphic_maps_map by blast
qed

lemma homeomorphic_maps_involution:
   ⟦continuous_map X X f; ⋀x. x ∈ topspace X ⟹ f(f x) = x⟧ ⟹ homeomor-
phic_maps X X f f
  by (auto simp: homeomorphic_maps_def)

lemma homeomorphic_map_involution:
   ⟦continuous_map X X f; ⋀x. x ∈ topspace X ⟹ f(f x) = x⟧ ⟹ homeomor-
phic_map X X f
  using homeomorphic_maps_involution homeomorphic_maps_map by blast

lemma homeomorphic_map_openness:
  assumes hom: homeomorphic_map X Y f and U: U ⊆ topspace X
  shows openin Y (f ‘ U) ⟷ openin X U
proof −
  obtain g where homeomorphic_maps X Y f g
    using assms by (auto simp: homeomorphic_map_maps)
  then have g: homeomorphic_map Y X g and gf: ⋀x. x ∈ topspace X ⟹ g(f
x) = x
    by (auto simp: homeomorphic_maps_map)
  then have openin X U ⟹ openin Y (f ‘ U)
    using hom homeomorphic_imp_open_map open_map_def by blast
  show openin Y (f ‘ U) = openin X U
  proof
    assume L: openin Y (f ‘ U)
    have U = g ‘ (f ‘ U)
      using U gf by force
    then show openin X U
      by (metis L homeomorphic_imp_open_map open_map_def g)
  next
    assume openin X U
    then show openin Y (f ‘ U)
      using hom homeomorphic_imp_open_map open_map_def by blast
  qed
qed


lemma homeomorphic_map_closedness:
  assumes hom: homeomorphic_map X Y f and U: U ⊆ topspace X
  shows closedin Y (f ‘ U) ⟷ closedin X U
proof −
```

obtain g **where** *homeomorphic_maps X Y f g*
using *assms* **by** (*auto simp*: *homeomorphic_map_maps*)
then have *g*: *homeomorphic_map Y X g* **and** *gf*: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *g*(*f x*) = *x*
by (*auto simp*: *homeomorphic_maps_map*)
then have *closedin X U* $\Longrightarrow$ *closedin Y* (*f* ‘ *U*)
using *hom homeomorphic_imp_closed_map closed_map_def* **by** *blast*
show *closedin Y* (*f* ‘ *U*) = *closedin X U*
proof
assume *L*: *closedin Y* (*f* ‘ *U*)
have *U* = *g* ‘ (*f* ‘ *U*)
using *U gf* **by** *force*
then show *closedin X U*
by (*metis L homeomorphic_imp_closed_map closed_map_def g*)
next
assume *closedin X U*
then show *closedin Y* (*f* ‘ *U*)
using *hom homeomorphic_imp_closed_map closed_map_def* **by** *blast*
qed
qed

lemma *homeomorphic_map_openness_eq*:
homeomorphic_map X Y f $\Longrightarrow$ *openin X U* $\longleftrightarrow$ *U* $\subseteq$ *topspace X* $\wedge$ *openin Y* (*f* ‘ *U*)
by (*meson homeomorphic_map_openness openin_closedin_eq*)

lemma *homeomorphic_map_closedness_eq*:
homeomorphic_map X Y f $\Longrightarrow$ *closedin X U* $\longleftrightarrow$ *U* $\subseteq$ *topspace X* $\wedge$ *closedin Y* (*f* ‘ *U*)
by (*meson closedin_subset homeomorphic_map_closedness*)

lemma *all_openin_homeomorphic_image*:
assumes *homeomorphic_map X Y f*
shows ($\forall$ *V*. *openin Y V* $\longrightarrow$ *P V*) $\longleftrightarrow$ ($\forall$ *U*. *openin X U* $\longrightarrow$ *P*(*f* ‘ *U*))
by (*metis* (*no_types*, *lifting*) *assms homeomorphic_eq_everything_map homeomorphic_map_openness openin_subset subset_image_iff*)

lemma *all_closedin_homeomorphic_image*:
assumes *homeomorphic_map X Y f*
shows ($\forall$ *V*. *closedin Y V* $\longrightarrow$ *P V*) $\longleftrightarrow$ ($\forall$ *U*. *closedin X U* $\longrightarrow$ *P*(*f* ‘ *U*))
by (*metis* (*no_types*, *lifting*) *assms closedin_subset homeomorphic_eq_everything_map homeomorphic_map_closedness subset_image_iff*)

lemma *homeomorphic_map_derived_set_of*:
assumes *hom*: *homeomorphic_map X Y f* **and** *S*: *S* $\subseteq$ *topspace X*
shows *Y derived_set_of* (*f* ‘ *S*) = *f* ‘ (*X derived_set_of S*)
proof −
have *fim*: *f* ‘ (*topspace X*) = *topspace Y* **and** *inj*: *inj_on f* (*topspace X*)

```
    using hom by (auto simp: homeomorphic_eq_everything_map)
  have iff: (∀ T. x ∈ T ∧ openin X T ⟶ (∃ y. y ≠ x ∧ y ∈ S ∧ y ∈ T)) =
          (∀ T. T ⊆ topspace Y ⟶ f x ∈ T ⟶ openin Y T ⟶ (∃ y. y ≠ f x ∧
y ∈ f ‘ S ∧ y ∈ T))
    if x ∈ topspace X for x
  proof −
    have §: (x ∈ T ∧ openin X T) = (T ⊆ topspace X ∧ f x ∈ f ‘ T ∧ openin Y
(f ‘ T)) for T
     by (meson hom homeomorphic_map_openness_eq inj inj_on_image_mem_iff
that)
    moreover have (∃ y. y ≠ x ∧ y ∈ S ∧ y ∈ T) = (∃ y. y ≠ f x ∧ y ∈ f ‘ S ∧
y ∈ f ‘ T)  (is ?lhs = ?rhs)
      if T ⊆ topspace X ∧ f x ∈ f ‘ T ∧ openin Y (f ‘ T) for T
       by (smt (verit, del_insts) S ‹x ∈ topspace X› image_iff inj inj_on_def
subsetD that)
    ultimately show ?thesis
      by (auto simp flip: fim simp: all_subset_image)
  qed
  have ∗: ⟦T = f ‘ S; ⋀x. x ∈ S ⟹ P x ⟷ Q(f x)⟧ ⟹ {y. y ∈ T ∧ Q y} = f
‘ {x ∈ S. P x} for T S P Q
    by auto
  show ?thesis
    unfolding derived_set_of_def
    by (rule ∗) (use fim iff openin_subset in force)+
qed


lemma homeomorphic_map_closure_of:
  assumes hom: homeomorphic_map X Y f and S: S ⊆ topspace X
  shows Y closure_of (f ‘ S) = f ‘ (X closure_of S)
  unfolding closure_of
  using homeomorphic_imp_surjective_map [OF hom] S
 by (auto simp: in_derived_set_of homeomorphic_map_derived_set_of [OF assms])

lemma homeomorphic_map_interior_of:
  assumes hom: homeomorphic_map X Y f and S: S ⊆ topspace X
  shows Y interior_of (f ‘ S) = f ‘ (X interior_of S)
proof −
  { fix y
    assume y ∈ topspace Y and y ∉ Y closure_of (topspace Y − f ‘ S)
    then have y ∈ f ‘ (topspace X − X closure_of (topspace X − S))
      using homeomorphic_eq_everything_map [THEN iffD1, OF hom] homeo-
morphic_map_closure_of [OF hom]
    by (metis DiffI Diff_subset S closure_of_subset_topspace inj_on_image_set_diff)
}
  moreover
  { fix x
    assume x ∈ topspace X
    then have f x ∈ topspace Y
```

```
      using hom homeomorphic_imp_surjective_map by blast }
  moreover
  { fix x
    assume x ∈ topspace X and x ∉ X closure_of (topspace X − S) and f x ∈ Y
closure_of (topspace Y − f ‘ S)
    then have False
      using homeomorphic_map_closure_of [OF hom] hom
      unfolding homeomorphic_eq_everything_map
     by (metis Diff_subset S closure_of_subset_topspace inj_on_image_mem_iff
inj_on_image_set_diff)
  }
  ultimately show ?thesis
    by (auto simp: interior_of_closure_of)
qed

lemma homeomorphic_map_frontier_of:
  assumes hom: homeomorphic_map X Y f and S: S ⊆ topspace X
  shows Y frontier_of (f ‘ S) = f ‘ (X frontier_of S)
  unfolding frontier_of_def
proof (intro equalityI subsetI DiffI)
  fix y
  assume y ∈ Y closure_of f ‘ S − Y interior_of f ‘ S
  then show y ∈ f ‘ (X closure_of S − X interior_of S)
   using S hom homeomorphic_map_closure_of homeomorphic_map_interior_of
by fastforce
next
  fix y
  assume y ∈ f ‘ (X closure_of S − X interior_of S)
  then show y ∈ Y closure_of f ‘ S
    using S hom homeomorphic_map_closure_of by fastforce
next
  fix x
  assume x ∈ f ‘ (X closure_of S − X interior_of S)
  then obtain y where y: x = f y y ∈ X closure_of S y ∉ X interior_of S
    by blast
  then show x ∉ Y interior_of f ‘ S
    using S hom homeomorphic_map_interior_of y(1)
    unfolding homeomorphic_map_def
   by (smt (verit, ccfv_SIG) in_closure_of inj_on_image_mem_iff interior_of_subset_topspace)

qed

lemma homeomorphic_maps_subtopologies:
   ⟦homeomorphic_maps X Y f g;  f ‘ (topspace X ∩ S) = topspace Y ∩ T⟧
        ⟹ homeomorphic_maps (subtopology X S) (subtopology Y T) f g
  unfolding homeomorphic_maps_def
 by (force simp: continuous_map_from_subtopology continuous_map_in_subtopology)

lemma homeomorphic_maps_subtopologies_alt:
```

⟦*homeomorphic_maps X Y f g*; *f ‘ (topspace X ∩ S) ⊆ T*; *g ‘ (topspace Y ∩ T) ⊆ S*⟧
⟹ *homeomorphic_maps (subtopology X S) (subtopology Y T) f g*
unfolding *homeomorphic_maps_def*
by (*force simp*: *continuous_map_from_subtopology continuous_map_in_subtopology*)

lemma *homeomorphic_map_subtopologies*:
⟦*homeomorphic_map X Y f*; *f ‘ (topspace X ∩ S) = topspace Y ∩ T*⟧
⟹ *homeomorphic_map (subtopology X S) (subtopology Y T) f*
by (*meson homeomorphic_map_maps homeomorphic_maps_subtopologies*)

lemma *homeomorphic_map_subtopologies_alt*:
assumes *hom*: *homeomorphic_map X Y f*
and *S*: ⋀*x*. ⟦*x ∈ topspace X*; *f x ∈ topspace Y*⟧ ⟹ *f x ∈ T ⟷ x ∈ S*
shows *homeomorphic_map (subtopology X S) (subtopology Y T) f*
proof −
have *homeomorphic_maps (subtopology X S) (subtopology Y T) f g*
if *homeomorphic_maps X Y f g* **for** *g*
proof (*rule homeomorphic_maps_subtopologies* [*OF that*])
have *f ‘ (topspace X ∩ S) ⊆ topspace Y ∩ T*
using *S hom homeomorphic_imp_surjective_map* **by** *fastforce*
then show *f ‘ (topspace X ∩ S) = topspace Y ∩ T*
using *that* **unfolding** *homeomorphic_maps_def continuous_map_def Pi_iff*
by (*smt* (*verit*, *del_insts*) *Int_iff S image_iff subsetI subset_antisym*)
qed
then show *?thesis*
using *hom* **by** (*meson homeomorphic_map_maps*)
qed

1.13.15 Relation of homeomorphism between topological spaces

definition *homeomorphic_space* (**infixr** ‹*homeomorphic′_space*› *50*)
where *X homeomorphic_space Y* ≡ ∃*f g*. *homeomorphic_maps X Y f g*

lemma *homeomorphic_space_refl* [*iff*]: *X homeomorphic_space X*
by (*meson homeomorphic_maps_id homeomorphic_space_def*)

lemma *homeomorphic_space_sym*:
X homeomorphic_space Y ⟷ Y homeomorphic_space X
unfolding *homeomorphic_space_def* **by** (*metis homeomorphic_maps_sym*)

lemma *homeomorphic_space_trans* [*trans*]:
⟦*X1 homeomorphic_space X2*; *X2 homeomorphic_space X3*⟧ ⟹ *X1 homeomorphic_space X3*
unfolding *homeomorphic_space_def* **by** (*metis homeomorphic_maps_compose*)

lemma *homeomorphic_space*:
X homeomorphic_space Y ⟷ (∃*f*. *homeomorphic_map X Y f*)
by (*simp add*: *homeomorphic_map_maps homeomorphic_space_def*)

lemma *homeomorphic_maps_imp_homeomorphic_space*:
homeomorphic_maps X Y f g $\Longrightarrow$ *X homeomorphic_space Y*
unfolding *homeomorphic_space_def* **by** *metis*

lemma *homeomorphic_map_imp_homeomorphic_space*:
homeomorphic_map X Y f $\Longrightarrow$ *X homeomorphic_space Y*
unfolding *homeomorphic_map_maps*
using *homeomorphic_space_def* **by** *blast*

lemma *homeomorphic_empty_space*:
X homeomorphic_space Y $\Longrightarrow$ *X* = *trivial_topology* $\longleftrightarrow$ *Y* = *trivial_topology*
by (*meson continuous_map_on_empty2 homeomorphic_maps_def homeomorphic_space_def*)

lemma *homeomorphic_empty_space_eq*:
assumes *X* = *trivial_topology*
shows *X homeomorphic_space Y* $\longleftrightarrow$ *Y* = *trivial_topology*
using *assms funcset_mem*
by (*fastforce simp*: *homeomorphic_maps_def homeomorphic_space_def continuous_map_def*)

lemma *homeomorphic_space_unfold*:
assumes *X homeomorphic_space Y*
obtains *f g* **where** *homeomorphic_map X Y f homeomorphic_map Y X g*
and $\bigwedge x.$ *x* $\in$ *topspace X* $\Longrightarrow$ *g*(*f x*) = *x* $\bigwedge y.$ *y* $\in$ *topspace Y* $\Longrightarrow$ *f*(*g y*) = *y*
and *f* $\in$ *topspace X* $\to$ *topspace Y g* $\in$ *topspace Y* $\to$ *topspace X*
using *assms* **unfolding** *homeomorphic_space_def homeomorphic_maps_map*
by (*smt* (*verit*, *best*) *Pi_I homeomorphic_imp_surjective_map image_eqI*)

1.13.16 Connected topological spaces

definition *connected_space* :: *'a topology* $\Rightarrow$ *bool* **where**
connected_space X $\equiv$
$\neg$($\exists$ *E1 E2*. *openin X E1* $\wedge$ *openin X E2* $\wedge$
topspace X $\subseteq$ *E1* $\cup$ *E2* $\wedge$ *E1* $\cap$ *E2* = {} $\wedge$ *E1* $\neq$ {} $\wedge$ *E2* $\neq$ {})

definition *connectedin* :: *'a topology* $\Rightarrow$ *'a set* $\Rightarrow$ *bool* **where**
connectedin X S $\equiv$ *S* $\subseteq$ *topspace X* $\wedge$ *connected_space* (*subtopology X S*)

lemma *connected_spaceD*:
$[\![$*connected_space X*;
openin X U; *openin X V*; *topspace X* $\subseteq$ *U* $\cup$ *V*; *U* $\cap$ *V* = {}; *U* $\neq$ {}; *V* $\neq$ {}$]\!] \Longrightarrow$ *False*
by (*auto simp*: *connected_space_def*)

lemma *connectedin_subset_topspace*: *connectedin X S* $\Longrightarrow$ *S* $\subseteq$ *topspace X*
by (*simp add*: *connectedin_def*)

lemma *connectedin_topspace*:
connectedin X (topspace X) ⟷ connected_space X
by (*simp add*: *connectedin_def*)

lemma *connected_space_subtopology*:
connectedin X S ⟹ connected_space (subtopology X S)
by (*simp add*: *connectedin_def*)

lemma *connectedin_subtopology*:
connectedin (subtopology X S) T ⟷ connectedin X T ∧ T ⊆ S
by (*force simp*: *connectedin_def subtopology_subtopology inf_absorb2*)

lemma *connected_space_eq*:
connected_space X ⟷
(∄E1 E2. openin X E1 ∧ openin X E2 ∧ E1 ∪ E2 = topspace X ∧ E1 ∩ E2 = {} ∧ E1 ≠ {} ∧ E2 ≠ {})
unfolding *connected_space_def*
by (*metis openin_Un openin_subset subset_antisym*)

lemma *connected_space_closedin*:
connected_space X ⟷
(∄E1 E2. closedin X E1 ∧ closedin X E2 ∧ topspace X ⊆ E1 ∪ E2 ∧
*E1 ∩ E2 = {} ∧ E1 ≠ {} ∧ E2 ≠ {}) (***is** *?lhs = ?rhs*)
proof
assume *?lhs*
then have *⋀E1 E2. ⟦openin X E1; E1 ∩ E2 = {}; topspace X ⊆ E1 ∪ E2; openin X E2⟧ ⟹ E1 = {} ∨ E2 = {}*
by (*simp add*: *connected_space_def*)
then show *?rhs*
unfolding *connected_space_def*
by (*metis disjnt_def separatedin_closed_sets separation_openin_Un_gen subtopology_superset*)
next
assume *R*: *?rhs*
then show *?lhs*
unfolding *connected_space_def*
by (*metis Diff_triv Int_commute separatedin_openin_diff separation_closedin_Un_gen subtopology_superset*)
qed

lemma *connected_space_closedin_eq*:
connected_space X ⟷
(∄E1 E2. closedin X E1 ∧ closedin X E2 ∧
E1 ∪ E2 = topspace X ∧ E1 ∩ E2 = {} ∧ E1 ≠ {} ∧ E2 ≠ {})
by (*metis closedin_Un closedin_def connected_space_closedin subset_antisym*)

lemma *connected_space_clopen_in*:
connected_space X ⟷
(∀T. openin X T ∧ closedin X T ⟶ T = {} ∨ T = topspace X)

proof −
have *eq*: *openin X E1* ∧ *openin X E2* ∧ *E1* ∪ *E2* = *topspace X* ∧ *E1* ∩ *E2* = {} ∧ *P*
⟷ *E2* = *topspace X* − *E1* ∧ *openin X E1* ∧ *openin X E2* ∧ *P* **for** *E1 E2 P*
using *openin_subset* **by** *blast*
show *?thesis*
unfolding *connected_space_eq eq closedin_def*
by (*auto simp*: *openin_closedin_eq*)
qed

lemma *connectedin*:
connectedin X S ⟷
S ⊆ *topspace X* ∧
(∄*E1 E2*.
openin X E1 ∧ *openin X E2* ∧
S ⊆ *E1* ∪ *E2* ∧ *E1* ∩ *E2* ∩ *S* = {} ∧ *E1* ∩ *S* ≠ {} ∧ *E2* ∩ *S* ≠ {})
(**is** *?lhs* = *?rhs*)
proof −
have ∗: (∃*E1*:: *'a set*. ∃*E2*:: *'a set*. (∃*T1*:: *'a set*. *P1 T1* ∧ *E1* = *f1 T1*) ∧
(∃*T2*:: *'a set*. *P2 T2* ∧ *E2* = *f2 T2*) ∧
R E1 E2) ⟷ (∃*T1 T2*. *P1 T1* ∧ *P2 T2* ∧ *R*(*f1 T1*) (*f2 T2*)) **for** *P1 f1 P2 f2 R*
by *auto*
show *?thesis*
unfolding *connectedin_def connected_space_def openin_subtopology topspace_subtopology*
∗
by (*intro conj_cong arg_cong* [**where** *f*=*Not*] *ex_cong1*; *blast dest*!: *openin_subset*)
qed

lemma *connectedinD*:
⟦*connectedin X S*; *openin X E1*; *openin X E2*; *S* ⊆ *E1* ∪ *E2*; *E1* ∩ *E2* ∩ *S* = {}; *E1* ∩ *S* ≠ {}; *E2* ∩ *S* ≠ {}⟧ ⟹ *False*
by (*meson connectedin*)

lemma *connectedin_iff_connected* [*simp*]: *connectedin euclidean S* ⟷ *connected S*
by (*simp add*: *connected_def connectedin*)

lemma *connectedin_closedin*:
connectedin X S ⟷
S ⊆ *topspace X* ∧
¬(∃*E1 E2*. *closedin X E1* ∧ *closedin X E2* ∧
S ⊆ (*E1* ∪ *E2*) ∧
(*E1* ∩ *E2* ∩ *S* = {}) ∧
¬(*E1* ∩ *S* = {}) ∧ ¬(*E2* ∩ *S* = {}))
proof −
have ∗: (∃*E1*:: *'a set*. ∃*E2*:: *'a set*. (∃*T1*:: *'a set*. *P1 T1* ∧ *E1* = *f1 T1*) ∧
(∃*T2*:: *'a set*. *P2 T2* ∧ *E2* = *f2 T2*) ∧

```
        R E1 E2) ⟷ (∃ T1 T2. P1 T1 ∧ P2 T2 ∧ R(f1 T1) (f2 T2)) for P1
f1 P2 f2 R
    by auto
  show ?thesis
   unfolding connectedin_def connected_space_closedin closedin_subtopology topspace_subtopology
*
   by (intro conj_cong arg_cong [where f=Not] ex_cong1; blast dest!: openin_subset)
qed

lemma connectedin_empty [simp]: connectedin X {}
  by (simp add: connectedin)

lemma connected_space_trivial_topology [simp]: connected_space trivial_topology
  using connectedin_topspace by fastforce

lemma connectedin_sing [simp]: connectedin X {a} ⟷ a ∈ topspace X
  by (simp add: connectedin)

lemma connectedin_absolute [simp]:
  connectedin (subtopology X S) S ⟷ connectedin X S
  by (simp add: connectedin_subtopology)

lemma connectedin_Union:
  assumes 𝒰: ⋀S. S ∈ 𝒰 ⟹ connectedin X S and ne: ⋂𝒰 ≠ {}
  shows connectedin X (⋃𝒰)
proof −
  have ⋃𝒰 ⊆ topspace X
    using 𝒰 by (simp add: Union_least connectedin_def)
  moreover have False
    if openin X E1 openin X E2 and cover: ⋃𝒰 ⊆ E1 ∪ E2 and disj: E1 ∩ E2
∩ ⋃𝒰 = {}
      and overlap1: E1 ∩ ⋃𝒰 ≠ {} and overlap2: E2 ∩ ⋃𝒰 ≠ {}
      for E1 E2
  proof −
    have disjS: E1 ∩ E2 ∩ S = {} if S ∈ 𝒰 for S
      using Diff_triv that disj by auto
    have coverS: S ⊆ E1 ∪ E2 if S ∈ 𝒰 for S
      using that cover by blast
    have 𝒰 ≠ {}
      using overlap1 by blast
    obtain a where a: ⋀U. U ∈ 𝒰 ⟹ a ∈ U
      using ne by force
    with ‹𝒰 ≠ {}› have a ∈ ⋃𝒰
      by blast
    then consider a ∈ E1 | a ∈ E2
      using ‹⋃𝒰 ⊆ E1 ∪ E2› by auto
    then show False
    proof cases
      case 1
```

```
      then obtain b S where b ∈ E2 b ∈ S S ∈ 𝒰
        using overlap2 by blast
      then show ?thesis
        using 1 ‹openin X E1› ‹openin X E2› disjS coverS a [OF ‹S ∈ 𝒰›] 𝒰[OF
‹S ∈ 𝒰›]
        unfolding connectedin
        by (meson disjoint_iff_not_equal)
    next
      case 2
      then obtain b S where b ∈ E1 b ∈ S S ∈ 𝒰
        using overlap1 by blast
      then show ?thesis
        using 2 ‹openin X E1› ‹openin X E2› disjS coverS a [OF ‹S ∈ 𝒰›] 𝒰[OF
‹S ∈ 𝒰›]
        unfolding connectedin
        by (meson disjoint_iff_not_equal)
    qed
  qed
  ultimately show ?thesis
    unfolding connectedin by blast
qed

lemma connectedin_Un:
    ⟦connectedin X S; connectedin X T; S ∩ T ≠ {}⟧ ⟹ connectedin X (S ∪ T)
  using connectedin_Union [of {S,T}] by auto

lemma connected_space_subconnected:
  connected_space X ⟷ (∀ x ∈ topspace X. ∀ y ∈ topspace X. ∃ S. connectedin X
S ∧ x ∈ S ∧ y ∈ S) (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    using connectedin_topspace by blast
next
  assume R [rule_format]: ?rhs
  have False if openin X U openin X V and disj: U ∩ V = {} and cover: topspace
X ⊆ U ∪ V
    and U ≠ {} V ≠ {} for U V
  proof −
    obtain u v where u ∈ U v ∈ V
      using ‹U ≠ {}› ‹V ≠ {}› by auto
    then obtain T where u ∈ T v ∈ T and T: connectedin X T
      using R [of u v] that
      by (meson ‹openin X U› ‹openin X V› subsetD openin_subset)
    then show False
      using that unfolding connectedin
      by (metis IntI ‹u ∈ U› ‹v ∈ V› empty_iff inf_bot_left subset_trans)
  qed
  then show ?lhs
```

by (*auto simp*: *connected_space_def*)
qed

lemma *connectedin_intermediate_closure_of*:
assumes *connectedin X S S* ⊆ *T T* ⊆ *X closure_of S*
shows *connectedin X T*
proof −
have *S*: *S* ⊆ *topspace X* **and** *T*: *T* ⊆ *topspace X*
using *assms* **by** (*meson closure_of_subset_topspace dual_order.trans*)+
have §: ⋀*E1 E2*. ⟦*openin X E1*; *openin X E2*; *E1* ∩ *S* = {} ∨ *E2* ∩ *S* = {}⟧
⟹ *E1* ∩ *T* = {} ∨ *E2* ∩ *T* = {}
using *assms* **unfolding** *disjoint_iff* **by** (*meson in_closure_of subsetD*)
then show *?thesis*
using *assms*
unfolding *connectedin closure_of_subset_topspace S T*
by (*metis Int_empty_right T dual_order.trans inf.orderE inf_left_commute*)
qed

lemma *connectedin_closure_of*:
connectedin X S ⟹ *connectedin X* (*X closure_of S*)
by (*meson closure_of_subset connectedin_def connectedin_intermediate_closure_of
subset_refl*)

lemma *connectedin_separation*:
connectedin X S ⟷
S ⊆ *topspace X* ∧
(∄*C1 C2*. *C1* ∪ *C2* = *S* ∧ *C1* ≠ {} ∧ *C2* ≠ {} ∧ *C1* ∩ *X closure_of C2*
= {} ∧ *C2* ∩ *X closure_of C1* = {})
unfolding *connectedin_def connected_space_closedin_eq closedin_Int_closure_of
topspace_subtopology*
apply (*intro conj_cong refl arg_cong* [**where** *f*=*Not*])
apply (*intro ex_cong1 iffI*, *blast*)
using *closure_of_subset_Int* **by** *force*

lemma *connectedin_eq_not_separated*:
connectedin X S ⟷
S ⊆ *topspace X* ∧
(∄*C1 C2*. *C1* ∪ *C2* = *S* ∧ *C1* ≠ {} ∧ *C2* ≠ {} ∧ *separatedin X C1 C2*)
unfolding *separatedin_def* **by** (*metis connectedin_separation sup.boundedE*)

lemma *connectedin_eq_not_separated_subset*:
connectedin X S ⟷
S ⊆ *topspace X* ∧ (∄*C1 C2*. *S* ⊆ *C1* ∪ *C2* ∧ *S* ∩ *C1* ≠ {} ∧ *S* ∩ *C2* ≠ {}
∧ *separatedin X C1 C2*)
proof −
have ∀*C1 C2*. *S* ⊆ *C1* ∪ *C2* ⟶ *S* ∩ *C1* = {} ∨ *S* ∩ *C2* = {} ∨ ¬ *separatedin
X C1 C2*
if ⋀*C1 C2*. *C1* ∪ *C2* = *S* ⟶ *C1* = {} ∨ *C2* = {} ∨ ¬ *separatedin X C1 C2*
proof (*intro allI*)

fix *C1 C2*
show $S \subseteq C1 \cup C2 \longrightarrow S \cap C1 = \{\} \vee S \cap C2 = \{\} \vee \neg$ *separatedin X C1 C2*
using *that* [*of* $S \cap C1$ $S \cap C2$]
by (*auto simp*: *separatedin_mono*)
qed
then show *?thesis*
by (*metis Un_Int_eq*(*1*) *Un_Int_eq*(*2*) *connectedin_eq_not_separated order_refl*)
qed

lemma *connected_space_eq_not_separated*:
connected_space X $\longleftrightarrow$
$(\nexists C1\ C2.\ C1 \cup C2 = topspace\ X \wedge C1 \neq \{\} \wedge C2 \neq \{\} \wedge$ *separatedin X C1 C2*)
by (*simp add*: *connectedin_eq_not_separated flip*: *connectedin_topspace*)

lemma *connected_space_eq_not_separated_subset*:
connected_space X $\longleftrightarrow$
$(\nexists C1\ C2.\ topspace\ X \subseteq C1 \cup C2 \wedge C1 \neq \{\} \wedge C2 \neq \{\} \wedge$ *separatedin X C1 C2*)
by (*metis connected_space_eq_not_separated le_sup_iff separatedin_def subset_antisym*)

lemma *connectedin_subset_separated_union*:
$[\![$*connectedin X C*; *separatedin X S T*; $C \subseteq S \cup T]\!] \Longrightarrow C \subseteq S \vee C \subseteq T$
unfolding *connectedin_eq_not_separated_subset* **by** *blast*

lemma *connectedin_nonseparated_union*:
assumes *connectedin X S connectedin X T* ¬*separatedin X S T*
shows *connectedin X* $(S \cup T)$
proof −
have $\bigwedge C1\ C2.\ [\![T \subseteq C1 \cup C2; S \subseteq C1 \cup C2]\!] \Longrightarrow$
$S \cap C1 = \{\} \wedge T \cap C1 = \{\} \vee S \cap C2 = \{\} \wedge T \cap C2 = \{\} \vee \neg$ *separatedin X C1 C2*
using *assms*
unfolding *connectedin_eq_not_separated_subset*
by (*metis* (*no_types*, *lifting*) *assms connectedin_subset_separated_union inf.orderE separatedin_empty*(*1*) *separatedin_mono separatedin_sym*)
then show *?thesis*
unfolding *connectedin_eq_not_separated_subset*
by (*simp add*: *assms connectedin_subset_topspace Int_Un_distrib2*)
qed

lemma *connected_space_closures*:
connected_space X $\longleftrightarrow$
$(\nexists e1\ e2.\ e1 \cup e2 = topspace\ X \wedge X$ *closure_of e1* $\cap X$ *closure_of e2* $= \{\} \wedge e1 \neq \{\} \wedge e2 \neq \{\})$
(**is** *?lhs* = *?rhs*)

proof
assume *?lhs*
then show *?rhs*
unfolding *connected_space_closedin_eq*
by (*metis Un_upper1 Un_upper2 closedin_closure_of closure_of_Un closure_of_eq_empty closure_of_topspace*)
next
assume *?rhs*
then show *?lhs*
unfolding *connected_space_closedin_eq*
by (*metis closure_of_eq*)
qed

lemma *connectedin_Int_frontier_of*:
assumes *connectedin X S S* ∩ *T* ≠ {} *S* − *T* ≠ {}
shows *S* ∩ *X frontier_of T* ≠ {}
proof −
have *S* ⊆ *topspace X* **and** ∗:
⋀*E1 E2*. *openin X E1* ⟶ *openin X E2* ⟶ *E1* ∩ *E2* ∩ *S* = {} ⟶ *S* ⊆ *E1* ∪ *E2* ⟶ *E1* ∩ *S* = {} ∨ *E2* ∩ *S* = {}
using ‹*connectedin X S*› **by** (*auto simp*: *connectedin*)
moreover
have *S* − (*topspace X* ∩ *T*) ≠ {}
using *assms*(*3*) **by** *blast*
moreover
have *S* ∩ *topspace X* ∩ *T* ≠ {}
using *assms connectedin* **by** *fastforce*
moreover
have *False* **if** *S* ∩ *T* ≠ {} *S* − *T* ≠ {} *T* ⊆ *topspace X S* ∩ *X frontier_of T* = {} **for** *T*
proof −
have *null*: *S* ∩ (*X closure_of T* − *X interior_of T*) = {}
using *that* **unfolding** *frontier_of_def* **by** *blast*
have *X interior_of T* ∩ (*topspace X* − *X closure_of T*) ∩ *S* = {}
by (*metis Diff_disjoint inf_bot_left interior_of_Int interior_of_complement interior_of_empty*)
moreover have *S* ⊆ *X interior_of T* ∪ (*topspace X* − *X closure_of T*)
using *that* ‹*S* ⊆ *topspace X*› *null* **by** *auto*
moreover have *S* ∩ *X interior_of T* ≠ {}
using *closure_of_subset that*(*1*) *that*(*3*) *null* **by** *fastforce*
ultimately have *S* ∩ *X interior_of* (*topspace X* − *T*) = {}
by (*metis* ∗ *inf_commute interior_of_complement openin_interior_of*)
then have *topspace* (*subtopology X S*) ∩ *X interior_of T* = *S*
using ‹*S* ⊆ *topspace X*› *interior_of_complement null* **by** *fastforce*
then show *?thesis*
using *that* **by** (*metis Diff_eq_empty_iff inf_le2 interior_of_subset subset_trans*)
qed
ultimately show *?thesis*

by (*metis Int_lower1 frontier_of_restrict inf_assoc*)
qed

lemma *connectedin_continuous_map_image*:
assumes *f*: *continuous_map X Y f* **and** *connectedin X S*
shows *connectedin Y* (*f ‘ S*)
proof −
have *S* ⊆ *topspace X* **and** ∗:
⋀*E1 E2*. *openin X E1* ⟶ *openin X E2* ⟶ *E1* ∩ *E2* ∩ *S* = {} ⟶ *S* ⊆ *E1* ∪ *E2* ⟶ *E1* ∩ *S* = {} ∨ *E2* ∩ *S* = {}
using ‹*connectedin X S*› **by** (*auto simp*: *connectedin*)
show *?thesis*
unfolding *connectedin connected_space_def*
proof (*intro conjI notI*; *clarify*)
show *f x* ∈ *topspace Y* **if** *x* ∈ *S* **for** *x*
using ‹*S* ⊆ *topspace X*› *continuous_map_image_subset_topspace f that* **by** *blast*
next
fix *U V*
let *?U* = {*x* ∈ *topspace X*. *f x* ∈ *U*}
let *?V* = {*x* ∈ *topspace X*. *f x* ∈ *V*}
assume *UV*: *openin Y U openin Y V f ‘ S* ⊆ *U* ∪ *V U* ∩ *V* ∩ *f ‘ S* = {} *U* ∩ *f ‘ S* ≠ {} *V* ∩ *f ‘ S* ≠ {}
then have *1*: *?U* ∩ *?V* ∩ *S* = {}
by *auto*
have *2*: *openin X ?U openin X ?V*
using ‹*openin Y U*› ‹*openin Y V*› *continuous_map f* **by** *fastforce*+
show *False*
using ∗ [*of ?U ?V*] *UV* ‹*S* ⊆ *topspace X*›
by (*auto simp*: *1 2*)
qed
qed

lemma *connected_space_quotient_map_image*:
⟦*quotient_map X X′ q*; *connected_space X*⟧ ⟹ *connected_space X′*
by (*metis connectedin_continuous_map_image connectedin_topspace quotient_imp_continuous_map quotient_imp_surjective_map*)

lemma *homeomorphic_connected_space*:
X homeomorphic_space Y ⟹ *connected_space X* ⟷ *connected_space Y*
unfolding *homeomorphic_space_def homeomorphic_maps_def*
by (*metis connected_space_subconnected connectedin_continuous_map_image connectedin_topspace continuous_map_image_subset_topspace image_eqI image_subset_iff*)

lemma *homeomorphic_map_connectedness*:
assumes *f*: *homeomorphic_map X Y f* **and** *U*: *U* ⊆ *topspace X*
shows *connectedin Y* (*f ‘ U*) ⟷ *connectedin X U*
proof −
have *1*: *f ‘ U* ⊆ *topspace Y* ⟷ *U* ⊆ *topspace X*

```
    using U f homeomorphic_imp_surjective_map by blast
  moreover have connected_space (subtopology Y (f ‘ U)) ⟷ connected_space
(subtopology X U)
  proof (rule homeomorphic_connected_space)
    have f ‘ U ⊆ topspace Y
      by (simp add: U 1)
    then have topspace Y ∩ f ‘ U = f ‘ U
      by (simp add: subset_antisym)
    then show subtopology Y (f ‘ U) homeomorphic_space subtopology X U
      by (metis U f homeomorphic_map_imp_homeomorphic_space homeomor-
phic_map_subtopologies homeomorphic_space_sym inf.absorb_iff2)
  qed
  ultimately show ?thesis
    by (auto simp: connectedin_def)
qed

lemma homeomorphic_map_connectedness_eq:
   homeomorphic_map X Y f
       ⟹ connectedin X U ⟷
            U ⊆ topspace X ∧ connectedin Y (f ‘ U)
 using homeomorphic_map_connectedness connectedin_subset_topspace by metis

lemma connectedin_discrete_topology:
   connectedin (discrete_topology U) S ⟷ S ⊆ U ∧ (∃ a. S ⊆ {a})
proof (cases S ⊆ U)
  case True
  show ?thesis
  proof (cases S = {})
    case False
    moreover have connectedin (discrete_topology U) S ⟷ (∃ a. S = {a})
    proof
      show connectedin (discrete_topology U) S ⟹ ∃ a. S = {a}
        using False connectedin_Int_frontier_of insert_Diff by fastforce
    qed (use True in auto)
    ultimately show ?thesis
      by auto
  qed simp
next
  case False
  then show ?thesis
    by (simp add: connectedin_def)
qed

lemma connected_space_discrete_topology:
     connected_space (discrete_topology U) ⟷ (∃ a. U ⊆ {a})
 by (metis connectedin_discrete_topology connectedin_topspace order_refl topspace_discrete_topology)
```

1.13.17 Compact sets

definition *compactin* **where**
compactin X S $\longleftrightarrow$
$S \subseteq$ *topspace X* $\wedge$
$(\forall \mathcal{U}.\ (\forall\, U \in \mathcal{U}.$ *openin X U*$) \wedge S \subseteq \bigcup \mathcal{U}$
$\longrightarrow (\exists \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge S \subseteq \bigcup \mathcal{F}))$

definition *compact_space* **where**
compact_space X $\equiv$ *compactin X* (*topspace X*)

lemma *compact_space_alt*:
compact_space X $\longleftrightarrow$
$(\forall \mathcal{U}.\ (\forall\, U \in \mathcal{U}.$ *openin X U*$) \wedge$ *topspace X* $\subseteq \bigcup \mathcal{U}$
$\longrightarrow (\exists \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge$ *topspace X* $\subseteq \bigcup \mathcal{F}))$
by (*simp add*: *compact_space_def compactin_def*)

lemma *compact_space*:
compact_space X $\longleftrightarrow$
$(\forall \mathcal{U}.\ (\forall\, U \in \mathcal{U}.$ *openin X U*$) \wedge \bigcup \mathcal{U} =$ *topspace X*
$\longrightarrow (\exists \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge \bigcup \mathcal{F} =$ *topspace X*$))$
unfolding *compact_space_alt*
using *openin_subset* **by** *fastforce*

lemma *compactinD*:
$[\![$*compactin X S*; $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow$ *openin X U*; $S \subseteq \bigcup \mathcal{U}]\!] \Longrightarrow \exists \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge S \subseteq \bigcup \mathcal{F}$
by (*auto simp*: *compactin_def*)

lemma *compactin_euclidean_iff* [*simp*]: *compactin euclidean S* $\longleftrightarrow$ *compact S*
by (*simp add*: *compact_eq_Heine_Borel compactin_def*) *meson*

lemma *compactin_absolute* [*simp*]:
compactin (*subtopology X S*) *S* $\longleftrightarrow$ *compactin X S*
proof −
have *eq*: $(\forall\, U \in \mathcal{U}.\ \exists\, Y.$ *openin X Y* $\wedge\ U = Y \cap S) \longleftrightarrow \mathcal{U} \subseteq (\lambda Y.\ Y \cap S)$ ‘ $\{y.$ *openin X y*$\}$ **for** $\mathcal{U}$
by *auto*
show *?thesis*
by (*auto simp*: *compactin_def openin_subtopology eq imp_conjL all_subset_image ex_finite_subset_image*)
qed

lemma *compactin_subspace*: *compactin X S* $\longleftrightarrow$ *S* $\subseteq$ *topspace X* $\wedge$ *compact_space* (*subtopology X S*)
unfolding *compact_space_def topspace_subtopology*
by (*metis compactin_absolute compactin_def inf.absorb2*)

lemma *compact_space_subtopology*: *compactin X S* $\Longrightarrow$ *compact_space* (*subtopology X S*)

by (*simp add*: *compactin_subspace*)

lemma *compactin_subtopology*: *compactin* (*subtopology X S*) *T* ⟷ *compactin X T* ∧ *T* ⊆ *S*
by (*metis compactin_subspace inf.absorb_iff2 le_inf_iff subtopology_subtopology topspace_subtopology*)

lemma *compactin_subset_topspace*: *compactin X S* ⟹ *S* ⊆ *topspace X*
by (*simp add*: *compactin_subspace*)

lemma *compactin_contractive*:
⟦*compactin X' S*; *topspace X' = topspace X*;
⋀*U*. *openin X U* ⟹ *openin X' U*⟧ ⟹ *compactin X S*
by (*simp add*: *compactin_def*)

lemma *finite_imp_compactin*:
⟦*S* ⊆ *topspace X*; *finite S*⟧ ⟹ *compactin X S*
by (*metis compactin_subspace compact_space finite_UnionD inf.absorb_iff2 order_refl topspace_subtopology*)

lemma *compactin_empty* [*iff*]: *compactin X* {}
by (*simp add*: *finite_imp_compactin*)

lemma *compact_space_trivial_topology* [*simp*]: *compact_space trivial_topology*
by (*simp add*: *compact_space_def*)

lemma *finite_imp_compactin_eq*:
finite S ⟹ (*compactin X S* ⟷ *S* ⊆ *topspace X*)
using *compactin_subset_topspace finite_imp_compactin* **by** *blast*

lemma *compactin_sing* [*simp*]: *compactin X* {*a*} ⟷ *a* ∈ *topspace X*
by (*simp add*: *finite_imp_compactin_eq*)

lemma *closed_compactin*:
assumes *XK*: *compactin X K* **and** *C* ⊆ *K* **and** *XC*: *closedin X C*
shows *compactin X C*
unfolding *compactin_def*
proof (*intro conjI allI impI*)
show *C* ⊆ *topspace X*
by (*simp add*: *XC closedin_subset*)
next
fix *𝒰* :: *'a set set*
assume *𝒰*: *Ball 𝒰* (*openin X*) ∧ *C* ⊆ ⋃*𝒰*
have (∀ *U*∈*insert* (*topspace X* − *C*) *𝒰*. *openin X U*)
using *XC 𝒰* **by** *blast*
moreover have *K* ⊆ ⋃(*insert* (*topspace X* − *C*) *𝒰*)
using *𝒰 XK compactin_subset_topspace* **by** *fastforce*
ultimately obtain *ℱ* **where** *finite ℱ ℱ* ⊆ *insert* (*topspace X* − *C*) *𝒰 K* ⊆ ⋃*ℱ*
using *assms* **unfolding** *compactin_def* **by** *metis*

moreover have *openin X* (*topspace X* − *C*)
using *XC* **by** *auto*
ultimately show ∃ℱ. *finite* ℱ ∧ ℱ ⊆ 𝒰 ∧ *C* ⊆ ⋃ℱ
using ‹*C* ⊆ *K*›
by (*rule_tac x*=ℱ − {*topspace X* − *C*} **in** *exI*) *auto*
qed

lemma *closed_compactin_Inter*:
⟦*compactin X K*; *K* ∈ 𝒦; ⋀*K*. *K* ∈ 𝒦 ⟹ *closedin X K*⟧ ⟹ *compactin X* (⋂𝒦)
by (*metis Inf_lower closed_compactin closedin_Inter empty_iff*)

lemma *closedin_subtopology_Int_subset*:
⟦*closedin* (*subtopology X U*) (*U* ∩ *S*); *V* ⊆ *U*⟧ ⟹ *closedin* (*subtopology X V*) (*V* ∩ *S*)
by (*smt* (*verit*, *ccfv_SIG*) *closedin_subtopology inf.left_commute inf.orderE inf_commute*)

lemma *closedin_subtopology_Int_closedin*:
⟦*closedin* (*subtopology X U*) *S*; *closedin X T*⟧ ⟹ *closedin* (*subtopology X U*) (*S* ∩ *T*)
by (*smt* (*verit*, *best*) *closedin_Int closedin_subtopology inf_assoc inf_commute*)

lemma *closedin_compact_space*:
⟦*compact_space X*; *closedin X S*⟧ ⟹ *compactin X S*
by (*simp add*: *closed_compactin closedin_subset compact_space_def*)

lemma *compact_Int_closedin*:
assumes *compactin X S closedin X T* **shows** *compactin X* (*S* ∩ *T*)
proof −
have *compactin* (*subtopology X S*) (*S* ∩ *T*)
by (*metis assms closedin_compact_space closedin_subtopology compactin_subspace inf_commute*)
then show *?thesis*
by (*simp add*: *compactin_subtopology*)
qed

lemma *closed_Int_compactin*: ⟦*closedin X S*; *compactin X T*⟧ ⟹ *compactin X* (*S* ∩ *T*)
by (*metis compact_Int_closedin inf_commute*)

lemma *compactin_Un*:
assumes *S*: *compactin X S* **and** *T*: *compactin X T* **shows** *compactin X* (*S* ∪ *T*)
unfolding *compactin_def*
proof (*intro conjI allI impI*)
show *S* ∪ *T* ⊆ *topspace X*
using *assms* **by** (*auto simp*: *compactin_def*)
next
fix 𝒰 :: *'a set set*

assume $\mathcal{U}$: *Ball* $\mathcal{U}$ (*openin* X) $\wedge$ $S \cup T \subseteq \bigcup \mathcal{U}$
with S **obtain** $\mathcal{F}$ **where** $\mathcal{V}$: *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{U}$ $S \subseteq \bigcup \mathcal{F}$
unfolding *compactin_def* **by** (*meson sup.bounded_iff*)
obtain $\mathcal{W}$ **where** *finite* $\mathcal{W}$ $\mathcal{W} \subseteq \mathcal{U}$ $T \subseteq \bigcup \mathcal{W}$
using $\mathcal{U}$ T
unfolding *compactin_def* **by** (*meson sup.bounded_iff*)
with $\mathcal{V}$ **show** $\exists \mathcal{V}$. *finite* $\mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{U} \wedge S \cup T \subseteq \bigcup \mathcal{V}$
by (*rule_tac x*=$\mathcal{F} \cup \mathcal{W}$ **in** *exI*) *auto*
qed

lemma *compactin_Union*:
⟦*finite* $\mathcal{F}$; $\bigwedge S$. $S \in \mathcal{F} \Longrightarrow$ *compactin* X S⟧ $\Longrightarrow$ *compactin* X ($\bigcup \mathcal{F}$)
by (*induction rule*: *finite_induct*) (*simp_all add*: *compactin_Un*)

lemma *compactin_subtopology_imp_compact*:
assumes *compactin* (*subtopology* X S) K **shows** *compactin* X K
using *assms*
proof (*clarsimp simp add*: *compactin_def*)
fix $\mathcal{U}$
define $\mathcal{V}$ **where** $\mathcal{V} \equiv (\lambda U.\ U \cap S)$ ‘ $\mathcal{U}$
assume $K \subseteq$ *topspace* X **and** $K \subseteq S$ **and** $\forall x \in \mathcal{U}$. *openin* X x **and** $K \subseteq \bigcup \mathcal{U}$
then have $\forall V \in \mathcal{V}$. *openin* (*subtopology* X S) V $K \subseteq \bigcup \mathcal{V}$
unfolding $\mathcal{V}$*_def* **by** (*auto simp*: *openin_subtopology*)
moreover
assume $\forall \mathcal{U}$. ($\forall x \in \mathcal{U}$. *openin* (*subtopology* X S) x) $\wedge$ $K \subseteq \bigcup \mathcal{U} \longrightarrow$ ($\exists \mathcal{F}$. *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge K \subseteq \bigcup \mathcal{F}$)
ultimately obtain $\mathcal{F}$ **where** *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{V}$ $K \subseteq \bigcup \mathcal{F}$
by *meson*
then have $\mathcal{F}$: $\exists U.\ U \in \mathcal{U} \wedge V = U \cap S$ **if** $V \in \mathcal{F}$ **for** V
unfolding $\mathcal{V}$*_def* **using** *that* **by** *blast*
let *?*$\mathcal{F}$ = ($\lambda F.$ @$U.\ U \in \mathcal{U} \wedge F = U \cap S$) ‘ $\mathcal{F}$
show $\exists \mathcal{F}$. *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge K \subseteq \bigcup \mathcal{F}$
proof (*intro exI conjI*)
show *finite ?*$\mathcal{F}$
using ‹*finite* $\mathcal{F}$› **by** *blast*
show *?*$\mathcal{F} \subseteq \mathcal{U}$
using *someI_ex* [*OF* $\mathcal{F}$] **by** *blast*
show $K \subseteq \bigcup$ *?*$\mathcal{F}$
proof *clarsimp*
fix x
assume $x \in K$
then show $\exists V \in \mathcal{F}.\ x \in$ (*SOME* $U.\ U \in \mathcal{U} \wedge V = U \cap S$)
using ‹$K \subseteq \bigcup \mathcal{F}$› *someI_ex* [*OF* $\mathcal{F}$]
by (*metis* (*no_types*, *lifting*) *IntD1 Union_iff subsetCE*)
qed
qed
qed

lemma *compact_imp_compactin_subtopology*:

assumes *compactin X K K* $\subseteq$ *S* **shows** *compactin* (*subtopology X S*) *K*
using *assms*
proof (*clarsimp simp add*: *compactin_def*)
fix $\mathcal{U}$ :: *'a set set*
define $\mathcal{V}$ **where** $\mathcal{V} \equiv \{V.\ \textit{openin}\ X\ V \wedge (\exists\, U \in \mathcal{U}.\ U = V \cap S)\}$
assume $K \subseteq S$ **and** $K \subseteq \textit{topspace}\ X$ **and** $\forall\, U{\in}\mathcal{U}.\ \textit{openin}\ (\textit{subtopology}\ X\ S)\ U$ **and** $K \subseteq \bigcup\mathcal{U}$
then have $\forall\, V \in \mathcal{V}.\ \textit{openin}\ X\ V\ K \subseteq \bigcup\mathcal{V}$
unfolding $\mathcal{V}$*_def* **by** (*fastforce simp*: *subset_eq openin_subtopology*)+
moreover
assume $\forall\mathcal{U}.\ (\forall\, U{\in}\mathcal{U}.\ \textit{openin}\ X\ U) \wedge K \subseteq \bigcup\mathcal{U} \longrightarrow (\exists\mathcal{F}.\ \textit{finite}\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge K \subseteq \bigcup\mathcal{F})$
ultimately obtain $\mathcal{F}$ **where** *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{V}$ $K \subseteq \bigcup\mathcal{F}$
by *meson*
let *?*$\mathcal{F}$ = $(\lambda F.\ F \cap S)$ ` $\mathcal{F}$
show $\exists\mathcal{F}.\ \textit{finite}\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge K \subseteq \bigcup\mathcal{F}$
proof (*intro exI conjI*)
show *finite ?*$\mathcal{F}$
using ‹*finite* $\mathcal{F}$› **by** *blast*
show *?*$\mathcal{F} \subseteq \mathcal{U}$
using $\mathcal{V}$*_def* ‹$\mathcal{F} \subseteq \mathcal{V}$› **by** *blast*
show $K \subseteq \bigcup$ *?*$\mathcal{F}$
using ‹$K \subseteq \bigcup\mathcal{F}$› *assms*(*2*) **by** *auto*
qed
qed

proposition *compact_space_fip*:
compact_space X $\longleftrightarrow$
$(\forall\mathcal{U}.\ (\forall\, C{\in}\mathcal{U}.\ \textit{closedin}\ X\ C) \wedge (\forall\mathcal{F}.\ \textit{finite}\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap\mathcal{F} \neq \{\}) \longrightarrow \bigcap\mathcal{U} \neq \{\})$
(**is** _ = *?rhs*)
proof (*cases X = trivial_topology*)
case *True*
then show *?thesis*
by (*metis Pow_iff closedin_topspace_empty compact_space_trivial_topology finite.emptyI finite_Pow_iff finite_subset subsetI*)
next
case *False*
show *?thesis*
proof *safe*
fix $\mathcal{U}$ :: *'a set set*
assume $*$ [*rule_format*]: $\forall\mathcal{F}.\ \textit{finite}\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap\mathcal{F} \neq \{\}$
define $\mathcal{V}$ **where** $\mathcal{V} \equiv (\lambda S.\ \textit{topspace}\ X - S)$ ` $\mathcal{U}$
assume *clo*: $\forall\, C{\in}\mathcal{U}.\ \textit{closedin}\ X\ C$ **and** [*simp*]: $\bigcap\mathcal{U} = \{\}$
then have $\forall\, V \in \mathcal{V}.\ \textit{openin}\ X\ V$ *topspace* $X \subseteq \bigcup\mathcal{V}$
by (*auto simp*: $\mathcal{V}$*_def*)
moreover assume [*unfolded compact_space_alt*, *rule_format*, *of* $\mathcal{V}$]: *compact_space X*

ultimately obtain $\mathcal{F}$ **where** $\mathcal{F}$: *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{U}$ *topspace* $X \subseteq$ *topspace* $X - \bigcap \mathcal{F}$
by (*auto simp*: *ex_finite_subset_image* $\mathcal{V}$_*def*)
moreover have $\mathcal{F} \neq \{\}$
using $\mathcal{F}$ *False* **by** *force*
ultimately show *False*
using $*$ [*of* $\mathcal{F}$]
by *auto* (*metis Diff_iff Inter_iff clo closedin_def subsetD*)
next
assume *R* [*rule_format*]: *?rhs*
show *compact_space X*
unfolding *compact_space_alt*
proof *clarify*
fix $\mathcal{U}$ :: *'a set set*
define $\mathcal{V}$ **where** $\mathcal{V} \equiv (\lambda S.$ *topspace* $X - S)$ ' $\mathcal{U}$
assume $\forall\, C{\in}\mathcal{U}.$ *openin* X C **and** *topspace* $X \subseteq \bigcup \mathcal{U}$
with *False* **have** $*$: $\forall\, V \in \mathcal{V}.$ *closedin* X V $\mathcal{U} \neq \{\}$
by (*auto simp*: $\mathcal{V}$_*def*)
show $\exists \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge$ *topspace* $X \subseteq \bigcup \mathcal{F}$
proof (*rule ccontr*; *simp*)
assume $\forall \mathcal{F}{\subseteq}\mathcal{U}.$ *finite* $\mathcal{F} \longrightarrow \neg$ *topspace* $X \subseteq \bigcup \mathcal{F}$
then have $\forall \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{V} \longrightarrow \bigcap \mathcal{F} \neq \{\}$
by (*simp add*: $\mathcal{V}$_*def all_finite_subset_image*)
with ‹*topspace* $X \subseteq \bigcup \mathcal{U}$› **show** *False*
using *R* [*of* $\mathcal{V}$] $*$ **by** (*simp add*: $\mathcal{V}$_*def*)
qed
qed
qed
qed

corollary *compactin_fip*:
compactin X $S \longleftrightarrow$
$S \subseteq$ *topspace* $X \wedge$
$(\forall \mathcal{U}.\ (\forall\, C{\in}\mathcal{U}.$ *closedin* X $C) \wedge (\forall \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow S \cap \bigcap \mathcal{F} \neq \{\}) \longrightarrow S \cap \bigcap \mathcal{U} \neq \{\})$
proof (*cases* $S = \{\}$)
case *False*
show *?thesis*
proof (*cases* $S \subseteq$ *topspace* X)
case *True*
then have *compactin* X $S \longleftrightarrow$
$(\forall \mathcal{U}.\ \mathcal{U} \subseteq (\lambda T.\ S \cap T)$ ' $\{T.$ *closedin* X $T\} \longrightarrow$
$(\forall \mathcal{F}.$ *finite* $\mathcal{F} \longrightarrow \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap \mathcal{F} \neq \{\}) \longrightarrow \bigcap \mathcal{U} \neq \{\})$
by (*simp add*: *compact_space_fip compactin_subspace closedin_subtopology image_def subset_eq Int_commute imp_conjL*)
also have $\ldots = (\forall \mathcal{U}{\subseteq}$*Collect* (*closedin* X). $(\forall \mathcal{F}.$ *finite* $\mathcal{F} \longrightarrow \mathcal{F} \subseteq (\cap)$ S ' $\mathcal{U} \longrightarrow \bigcap \mathcal{F} \neq \{\}) \longrightarrow \bigcap$ $((\cap)$ S ' $\mathcal{U}) \neq \{\})$
by (*simp add*: *all_subset_image*)
also have $\ldots = (\forall \mathcal{U}.\ (\forall\, C{\in}\mathcal{U}.$ *closedin* X $C) \wedge (\forall \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow$

```
S ∩ ⋂ℱ ≠ {}) ⟶ S ∩ ⋂𝒰 ≠ {})
    proof −
      have eq: ((∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ⟶ ⋂ ((∩) S ' ℱ) ≠ {}) ⟶ ⋂ ((∩) S '
𝒰) ≠ {}) ⟷
                ((∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ⟶ S ∩ ⋂ℱ ≠ {}) ⟶ S ∩ ⋂𝒰 ≠ {})  for
𝒰
        by simp (use ‹S ≠ {}› in blast)
      show ?thesis
        unfolding imp_conjL [symmetric] all_finite_subset_image eq by blast
    qed
    finally show ?thesis
      using True by simp
  qed (simp add: compactin_subspace)
qed force

corollary compact_space_imp_nest:
  fixes C :: nat ⇒ 'a set
  assumes compact_space X and clo: ⋀n. closedin X (C n)
    and ne: ⋀n. C n ≠ {} and dec: decseq C
  shows (⋂n. C n) ≠ {}
proof −
  let ?𝒰 = range (λn. ⋂m ≤ n. C m)
  have closedin X A if A ∈ ?𝒰 for A
    using that clo by auto
  moreover have (⋂n∈K. ⋂m ≤ n. C m) ≠ {} if finite K for K
  proof −
    obtain n where ⋀k. k ∈ K ⟹ k ≤ n
      using Max.coboundedI ‹finite K› by blast
    with dec have C n ⊆ (⋂n∈K. ⋂m ≤ n. C m)
      unfolding decseq_def by blast
  with ne [of n] show ?thesis
    by blast
  qed
  ultimately show ?thesis
    using ‹compact_space X› [unfolded compact_space_fip, rule_format, of ?𝒰]
   by (simp add: all_finite_subset_image INT_extend_simps UN_atMost_UNIV
del: INT_simps)
qed

lemma compactin_discrete_topology:
   compactin (discrete_topology X) S ⟷ S ⊆ X ∧ finite S (is ?lhs = ?rhs)
proof (intro iffI conjI)
  assume L: ?lhs
  then show S ⊆ X
    by (auto simp: compactin_def)
  have ∗: ⋀𝒰. Ball 𝒰 (openin (discrete_topology X)) ∧ S ⊆ ⋃𝒰 ⟹
        (∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ S ⊆ ⋃ℱ)
    using L by (auto simp: compactin_def)
  show finite S
```

using ∗ [*of* (λ*x*. {*x*}) ‘ *X*] ‹*S* ⊆ *X*›
by *clarsimp* (*metis UN_singleton finite_subset_image infinite_super*)
next
assume *?rhs*
then show *?lhs*
by (*simp add*: *finite_imp_compactin*)
qed

lemma *compact_space_discrete_topology*: *compact_space*(*discrete_topology X*) ⟷ *finite X*
by (*simp add*: *compactin_discrete_topology compact_space_def*)

lemma *compact_space_imp_Bolzano_Weierstrass*:
assumes *compact_space X infinite S S* ⊆ *topspace X*
shows *X derived_set_of S* ≠ {}
proof
assume *X*: *X derived_set_of S* = {}
then have *closedin X S*
by (*simp add*: *closedin_contains_derived_set assms*)
then have *compactin X S*
by (*rule closedin_compact_space* [*OF* ‹*compact_space X*›])
with *X* **show** *False*
by (*metis* ‹*infinite S*› *compactin_subspace compact_space_discrete_topology inf_bot_right subtopology_eq_discrete_topology_eq*)
qed

lemma *compactin_imp_Bolzano_Weierstrass*:
⟦*compactin X S*; *infinite T* ∧ *T* ⊆ *S*⟧ ⟹ *S* ∩ *X derived_set_of T* ≠ {}
using *compact_space_imp_Bolzano_Weierstrass* [*of subtopology X S*]
by (*simp add*: *compactin_subspace derived_set_of_subtopology inf_absorb2*)

lemma *compact_closure_of_imp_Bolzano_Weierstrass*:
⟦*compactin X* (*X closure_of S*); *infinite T*; *T* ⊆ *S*; *T* ⊆ *topspace X*⟧ ⟹ *X derived_set_of T* ≠ {}
using *closure_of_mono closure_of_subset compactin_imp_Bolzano_Weierstrass*
by *fastforce*

lemma *discrete_compactin_eq_finite*:
S ∩ *X derived_set_of S* = {} ⟹ *compactin X S* ⟷ *S* ⊆ *topspace X* ∧ *finite S*
by (*meson compactin_imp_Bolzano_Weierstrass finite_imp_compactin_eq order_refl*)

lemma *discrete_compact_space_eq_finite*:
X derived_set_of (*topspace X*) = {} ⟹ (*compact_space X* ⟷ *finite*(*topspace X*))
by (*metis compact_space_discrete_topology discrete_topology_unique_derived_set*)

lemma *image_compactin*:

assumes *cpt*: *compactin X S* **and** *cont*: *continuous_map X Y f*
shows *compactin Y* (*f ‘ S*)
unfolding *compactin_def*
proof (*intro conjI allI impI*)
show *f ‘ S* ⊆ *topspace Y*
using *compactin_subset_topspace cont continuous_map_image_subset_topspace cpt* **by** *blast*
next
fix $\mathcal{U}$ :: *'b set set*
assume $\mathcal{U}$: *Ball* $\mathcal{U}$ (*openin Y*) ∧ *f ‘ S* ⊆ ⋃$\mathcal{U}$
define $\mathcal{V}$ **where** $\mathcal{V} \equiv (\lambda U.\ \{x \in \textit{topspace } X.\ f\ x \in U\})\ `\mathcal{U}$
have *S* ⊆ *topspace X*
and ∗: ⋀$\mathcal{U}$. ⟦∀ *U*∈$\mathcal{U}$. *openin X U*; *S* ⊆ ⋃$\mathcal{U}$⟧ ⟹ ∃$\mathcal{F}$. *finite* $\mathcal{F}$ ∧ $\mathcal{F}$ ⊆ $\mathcal{U}$ ∧ *S* ⊆ ⋃$\mathcal{F}$
using *cpt* **by** (*auto simp*: *compactin_def*)
obtain $\mathcal{F}$ **where** $\mathcal{F}$: *finite* $\mathcal{F}$ $\mathcal{F}$ ⊆ $\mathcal{V}$ *S* ⊆ ⋃$\mathcal{F}$
proof −
have *1*: ∀ *U*∈$\mathcal{V}$. *openin X U*
unfolding $\mathcal{V}$*_def* **using** $\mathcal{U}$ *cont*[*unfolded continuous_map*] **by** *blast*
have *2*: *S* ⊆ ⋃$\mathcal{V}$
unfolding $\mathcal{V}$*_def* **using** *compactin_subset_topspace cpt* $\mathcal{U}$ **by** *fastforce*
show *thesis*
using ∗ [*OF 1 2*] *that* **by** *metis*
qed
have ∀ *v* ∈ $\mathcal{V}$. ∃ *U*. *U* ∈ $\mathcal{U}$ ∧ *v* = {*x* ∈ *topspace X*. *f x* ∈ *U*}
using $\mathcal{V}$*_def* **by** *blast*
then obtain *U* **where** *U*: ∀ *v* ∈ $\mathcal{V}$. *U v* ∈ $\mathcal{U}$ ∧ *v* = {*x* ∈ *topspace X*. *f x* ∈ *U v*}
by *metis*
show ∃$\mathcal{F}$. *finite* $\mathcal{F}$ ∧ $\mathcal{F}$ ⊆ $\mathcal{U}$ ∧ *f ‘ S* ⊆ ⋃$\mathcal{F}$
proof (*intro conjI exI*)
show *finite* (*U ‘* $\mathcal{F}$)
by (*simp add*: ‹*finite* $\mathcal{F}$›)
next
show *U ‘* $\mathcal{F}$ ⊆ $\mathcal{U}$
using ‹$\mathcal{F}$ ⊆ $\mathcal{V}$› *U* **by** *auto*
next
show *f ‘ S* ⊆ ⋃ (*U ‘* $\mathcal{F}$)
using $\mathcal{F}$*(2−3) U UnionE subset_eq U* **by** *fastforce*
qed
qed

lemma *homeomorphic_compact_space*:
assumes *X homeomorphic_space Y*
shows *compact_space X* ⟷ *compact_space Y*
using *homeomorphic_space_sym*
by (*metis assms compact_space_def homeomorphic_eq_everything_map homeomorphic_space image_compactin*)

lemma *homeomorphic_map_compactness*:
assumes *hom*: *homeomorphic_map X Y f* **and** *U*: *U* ⊆ *topspace X*
shows *compactin Y* (*f* ‘ *U*) ⟷ *compactin X U*
proof −
have *f* ‘ *U* ⊆ *topspace Y*
using *hom U homeomorphic_imp_surjective_map* **by** *blast*
moreover have *homeomorphic_map* (*subtopology X U*) (*subtopology Y* (*f* ‘ *U*)) *f*
using *U hom homeomorphic_imp_surjective_map* **by** (*blast intro*: *homeomorphic_map_subtopologies*)
then have *compact_space* (*subtopology Y* (*f* ‘ *U*)) = *compact_space* (*subtopology X U*)
using *homeomorphic_compact_space homeomorphic_map_imp_homeomorphic_space* **by** *blast*
ultimately show *?thesis*
by (*simp add*: *compactin_subspace U*)
qed

lemma *homeomorphic_map_compactness_eq*:
homeomorphic_map X Y f
⟹ *compactin X U* ⟷ *U* ⊆ *topspace X* ∧ *compactin Y* (*f* ‘ *U*)
by (*meson compactin_subset_topspace homeomorphic_map_compactness*)

1.13.18 Embedding maps

definition *embedding_map*
where *embedding_map X Y f* ≡ *homeomorphic_map X* (*subtopology Y* (*f* ‘ (*topspace X*))) *f*

lemma *embedding_map_eq*:
⟦*embedding_map X Y f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *embedding_map X Y g*
unfolding *embedding_map_def*
by (*metis homeomorphic_map_eq image_cong*)

lemma *embedding_map_compose*:
assumes *embedding_map X X′ f embedding_map X′ X″ g*
shows *embedding_map X X″* (*g* ∘ *f*)
proof −
have *hm*: *homeomorphic_map X* (*subtopology X′* (*f* ‘ *topspace X*)) *f homeomorphic_map X′* (*subtopology X″* (*g* ‘ *topspace X′*)) *g*
using *assms* **by** (*auto simp*: *embedding_map_def*)
then obtain *C* **where** *g* ‘ *topspace X′* ∩ *C* = (*g* ∘ *f*) ‘ *topspace X*
using *continuous_map_image_subset_topspace homeomorphic_imp_continuous_map* **by** *fastforce*
then have *homeomorphic_map* (*subtopology X′* (*f* ‘ *topspace X*)) (*subtopology X″* ((*g* ∘ *f*) ‘ *topspace X*)) *g*
by (*metis hm homeomorphic_eq_everything_map homeomorphic_map_subtopologies image_comp subtopology_subtopology topspace_subtopology*)

then show *?thesis*
unfolding *embedding_map_def*
using *hm(1) homeomorphic_map_compose* **by** *blast*
qed

lemma *surjective_embedding_map*:
embedding_map X Y f ∧ f ‘ (topspace X) = topspace Y ⟷ homeomorphic_map X Y f
by (*force simp*: *embedding_map_def homeomorphic_eq_everything_map*)

lemma *embedding_map_in_subtopology*:
embedding_map X (subtopology Y S) f ⟷ embedding_map X Y f ∧ f ‘ (topspace X) ⊆ S (**is** *?lhs = ?rhs*)
proof
show *?lhs ⟹ ?rhs*
unfolding *embedding_map_def*
by (*metis continuous_map_in_subtopology homeomorphic_imp_continuous_map inf_absorb2 subtopology_subtopology*)
qed (*simp add*: *embedding_map_def inf.absorb_iff2 subtopology_subtopology*)

lemma *injective_open_imp_embedding_map*:
⟦*continuous_map X Y f*; *open_map X Y f*; *inj_on f (topspace X)*⟧ *⟹ embedding_map X Y f*
unfolding *embedding_map_def homeomorphic_map_def*
by (*simp add*: *image_subset_iff_funcset continuous_map_in_subtopology continuous_open_quotient_map eq_iff*
open_map_imp_subset open_map_into_subtopology)

lemma *injective_closed_imp_embedding_map*:
⟦*continuous_map X Y f*; *closed_map X Y f*; *inj_on f (topspace X)*⟧ *⟹ embedding_map X Y f*
unfolding *embedding_map_def homeomorphic_map_def*
by (*simp add*: *closed_map_imp_subset closed_map_into_subtopology continuous_closed_quotient_map*
continuous_map_in_subtopology dual_order.eq_iff image_subset_iff_funcset)

lemma *embedding_map_imp_homeomorphic_space*:
embedding_map X Y f ⟹ X homeomorphic_space (subtopology Y (f ‘ (topspace X)))
unfolding *embedding_map_def*
using *homeomorphic_space* **by** *blast*

lemma *embedding_imp_closed_map*:
⟦*embedding_map X Y f*; *closedin Y (f ‘ topspace X)*⟧ *⟹ closed_map X Y f*
unfolding *closed_map_def*
by (*auto simp*: *closedin_closed_subtopology embedding_map_def homeomorphic_map_closedness_eq*)

lemma *embedding_imp_closed_map_eq*:
embedding_map X Y f ⟹ (closed_map X Y f ⟷ closedin Y (f ‘ topspace

X))
using *closed_map_def embedding_imp_closed_map* **by** *blast*

1.13.19 Retraction and section maps

definition *retraction_maps* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) ⇒ *bool*
where *retraction_maps X Y f g* ≡
continuous_map X Y f ∧ *continuous_map Y X g* ∧ (∀ *x* ∈ *topspace Y*. *f*(*g x*) = *x*)

definition *section_map* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *section_map X Y f* ≡ ∃ *g*. *retraction_maps Y X g f*

definition *retraction_map* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* ⇒ *'b*) ⇒ *bool*
where *retraction_map X Y f* ≡ ∃ *g*. *retraction_maps X Y f g*

lemma *retraction_maps_eq*:
⟦*retraction_maps X Y f g*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *f' x*; ⋀*x*. *x* ∈ *topspace Y* ⟹ *g x* = *g' x*⟧
⟹ *retraction_maps X Y f' g'*
unfolding *retraction_maps_def*
by (*metis continuous_map_eq continuous_map_image_subset_topspace image_subset_iff*)

lemma *section_map_eq*:
⟦*section_map X Y f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *section_map X Y g*
unfolding *section_map_def* **using** *retraction_maps_eq* **by** *blast*

lemma *retraction_map_eq*:
⟦*retraction_map X Y f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧ ⟹ *retraction_map X Y g*
unfolding *retraction_map_def* **using** *retraction_maps_eq* **by** *blast*

lemma *homeomorphic_imp_retraction_maps*:
homeomorphic_maps X Y f g ⟹ *retraction_maps X Y f g*
by (*simp add*: *homeomorphic_maps_def retraction_maps_def*)

lemma *section_and_retraction_eq_homeomorphic_map*:
section_map X Y f ∧ *retraction_map X Y f* ⟷ *homeomorphic_map X Y f* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *g* **where** *homeomorphic_maps X Y f g*
unfolding *homeomorphic_maps_def retraction_map_def section_map_def*
by (*smt* (*verit*) *Pi_iff continuous_map_def retraction_maps_def*)
then show *?rhs*
using *homeomorphic_map_maps* **by** *blast*
next

assume *?rhs*
then show *?lhs*
unfolding *retraction_map_def section_map_def*
by (*meson homeomorphic_imp_retraction_maps homeomorphic_map_maps homeomorphic_maps_sym*)
qed

lemma *section_imp_embedding_map*:
section_map X Y f ⟹ *embedding_map X Y f*
unfolding *section_map_def embedding_map_def homeomorphic_map_maps retraction_maps_def homeomorphic_maps_def*
by (*force simp*: *continuous_map_in_subtopology continuous_map_from_subtopology*)

lemma *retraction_imp_quotient_map*:
assumes *retraction_map X Y f*
shows *quotient_map X Y f*
unfolding *quotient_map_def*
proof (*intro conjI subsetI allI impI*)
show *f ' topspace X = topspace Y*
using *assms* **by** (*force simp*: *retraction_map_def retraction_maps_def continuous_map_def Pi_iff*)
next
fix *U*
assume *U*: *U* ⊆ *topspace Y*
have *openin Y U*
if ∀ *x*∈*topspace Y*. *g x* ∈ *topspace X* ∀ *x*∈*topspace Y*. *f* (*g x*) = *x*
openin Y {*x* ∈ *topspace Y*. *g x* ∈ {*x* ∈ *topspace X*. *f x* ∈ *U*}} **for** *g*
using *openin_subopen U that* **by** *fastforce*
then show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*} = *openin Y U*
using *assms* **by** (*auto simp*: *retraction_map_def retraction_maps_def continuous_map_def Pi_iff*)
qed

lemma *retraction_maps_compose*:
⟦*retraction_maps X Y f f'*; *retraction_maps Y Z g g'*⟧ ⟹ *retraction_maps X Z* (*g* ∘ *f*) (*f'* ∘ *g'*)
unfolding *retraction_maps_def*
by (*smt* (*verit*, *ccfv_threshold*) *comp_apply continuous_map_compose continuous_map_image_subset_topspace image_subset_iff*)

lemma *retraction_map_compose*:
⟦*retraction_map X Y f*; *retraction_map Y Z g*⟧ ⟹ *retraction_map X Z* (*g* ∘ *f*)
by (*meson retraction_map_def retraction_maps_compose*)

lemma *section_map_compose*:
⟦*section_map X Y f*; *section_map Y Z g*⟧ ⟹ *section_map X Z* (*g* ∘ *f*)
by (*meson retraction_maps_compose section_map_def*)

lemma *surjective_section_eq_homeomorphic_map*:
section_map X Y f ∧ f ‘ (topspace X) = topspace Y ⟷ homeomorphic_map X Y f
by (*meson section_and_retraction_eq_homeomorphic_map section_imp_embedding_map surjective_embedding_map*)

lemma *surjective_retraction_or_section_map*:
f ‘ (topspace X) = topspace Y ⟹ retraction_map X Y f ∨ section_map X Y f ⟷ retraction_map X Y f
using *section_and_retraction_eq_homeomorphic_map surjective_section_eq_homeomorphic_map*
by *fastforce*

lemma *retraction_imp_surjective_map*:
retraction_map X Y f ⟹ f ‘ (topspace X) = topspace Y
by (*simp add*: *retraction_imp_quotient_map quotient_imp_surjective_map*)

lemma *section_imp_injective_map*:
⟦*section_map X Y f*; *x ∈ topspace X*; *y ∈ topspace X*⟧ *⟹ f x = f y ⟷ x = y*
by (*metis* (*mono_tags*, *opaque_lifting*) *retraction_maps_def section_map_def*)

lemma *retraction_maps_to_retract_maps*:
retraction_maps X Y r s
⟹ retraction_maps X (*subtopology X* (*s ‘* (*topspace Y*))) (*s ∘ r*) *id*
unfolding *retraction_maps_def*
by (*auto simp*: *continuous_map_compose continuous_map_into_subtopology continuous_map_from_subtopology*)

1.13.20 Continuity

lemma *continuous_on_open*:
continuous_on S f ⟷
(∀ *T*. *openin* (*top_of_set* (*f ‘ S*)) *T ⟶*
openin (*top_of_set S*) (*S ∩ f −‘ T*))
unfolding *continuous_on_open_invariant openin_open Int_def vimage_def Int_commute*
by (*simp add*: *imp_ex imageI conj_commute eq_commute cong*: *conj_cong*)

lemma *continuous_on_closed*:
continuous_on S f ⟷
(∀ *T*. *closedin* (*top_of_set* (*f ‘ S*)) *T ⟶*
closedin (*top_of_set S*) (*S ∩ f −‘ T*))
unfolding *continuous_on_closed_invariant closedin_closed Int_def vimage_def Int_commute*
by (*simp add*: *imp_ex imageI conj_commute eq_commute cong*: *conj_cong*)

lemma *continuous_on_imp_closedin*:
assumes *continuous_on S f closedin* (*top_of_set* (*f ‘ S*)) *T*
shows *closedin* (*top_of_set S*) (*S ∩ f −‘ T*)
using *assms continuous_on_closed* **by** *blast*

lemma *continuous_map_subtopology_eu* [*simp*]:
continuous_map (*top_of_set S*) (*subtopology euclidean T*) *h* ⟷ *continuous_on S h* ∧ *h* ‘ *S* ⊆ *T*
by (*simp add*: *continuous_map_in_subtopology*)

lemma *continuous_map_euclidean_top_of_set*:
assumes *eq*: *f* −‘ *S* = *UNIV* **and** *cont*: *continuous_on UNIV f*
shows *continuous_map euclidean* (*top_of_set S*) *f*
by (*simp add*: *cont continuous_map_in_subtopology eq image_subset_iff_subset_vimage*)

1.13.21 Half-global and completely global cases

lemma *continuous_openin_preimage_gen*:
assumes *continuous_on S f open T*
shows *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*)
proof −
have ∗: (*S* ∩ *f* −‘ *T*) = (*S* ∩ *f* −‘ (*T* ∩ *f* ‘ *S*))
by *auto*
have *openin* (*top_of_set* (*f* ‘ *S*)) (*T* ∩ *f* ‘ *S*)
using *openin_open_Int*[*of T f* ‘ *S*, *OF assms*(*2*)] **unfolding** *openin_open* **by** *auto*
then show *?thesis*
using *assms*(*1*)[*unfolded continuous_on_open*, *THEN spec*[**where** *x*=*T* ∩ *f* ‘ *S*]]
using ∗ **by** *auto*
qed

lemma *continuous_closedin_preimage*:
assumes *continuous_on S f* **and** *closed T*
shows *closedin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*)
proof −
have ∗: (*S* ∩ *f* −‘ *T*) = (*S* ∩ *f* −‘ (*T* ∩ *f* ‘ *S*))
by *auto*
have *closedin* (*top_of_set* (*f* ‘ *S*)) (*T* ∩ *f* ‘ *S*)
using *closedin_closed_Int*[*of T f* ‘ *S*, *OF assms*(*2*)]
by (*simp add*: *Int_commute*)
then show *?thesis*
using *assms*(*1*)[*unfolded continuous_on_closed*, *THEN spec*[**where** *x*=*T* ∩ *f* ‘ *S*]]
using ∗ **by** *auto*
qed

lemma *continuous_openin_preimage_eq*:
continuous_on S f ⟷ (∀ *T*. *open T* ⟶ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*))
by (*metis Int_commute continuous_on_open_invariant open_openin openin_subtopology*)

lemma *continuous_closedin_preimage_eq*:
continuous_on S f ⟷
(∀ *T*. *closed T* ⟶ *closedin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*))

by (*metis Int_commute closedin_closed continuous_on_closed_invariant*)

lemma *continuous_open_preimage*:
assumes *contf*: *continuous_on S f* **and** *open S open T*
shows *open* (*S* ∩ *f* −‘ *T*)
proof−
obtain *U* **where** *open U* (*S* ∩ *f* −‘ *T*) = *S* ∩ *U*
using *continuous_openin_preimage_gen*[*OF contf* ‹*open T*›]
unfolding *openin_open* **by** *auto*
then show *?thesis*
using *open_Int*[*of S U*, *OF* ‹*open S*›] **by** *auto*
qed

lemma *continuous_closed_preimage*:
assumes *contf*: *continuous_on S f* **and** *closed S closed T*
shows *closed* (*S* ∩ *f* −‘ *T*)
proof−
obtain *U* **where** *closed U* (*S* ∩ *f* −‘ *T*) = *S* ∩ *U*
using *continuous_closedin_preimage*[*OF contf* ‹*closed T*›]
unfolding *closedin_closed* **by** *auto*
then show *?thesis* **using** *closed_Int*[*of S U*, *OF* ‹*closed S*›] **by** *auto*
qed

lemma *continuous_open_vimage*: *open S* ⟹ (⋀*x*. *continuous* (*at x*) *f*) ⟹ *open* (*f* −‘ *S*)
by (*metis continuous_on_eq_continuous_within open_vimage*)

lemma *continuous_closed_vimage*: *closed S* ⟹ (⋀*x*. *continuous* (*at x*) *f*) ⟹ *closed* (*f* −‘ *S*)
by (*simp add*: *closed_vimage continuous_on_eq_continuous_within*)

lemma *Times_in_interior_subtopology*:
assumes (*x*, *y*) ∈ *U openin* (*top_of_set* (*S* × *T*)) *U*
obtains *V W* **where** *openin* (*top_of_set S*) *V x* ∈ *V*
openin (*top_of_set T*) *W y* ∈ *W* (*V* × *W*) ⊆ *U*
proof −
from *assms* **obtain** *E* **where** *open E U* = *S* × *T* ∩ *E* (*x*, *y*) ∈ *E x* ∈ *S y* ∈ *T*
by (*auto simp*: *openin_open*)
from *open_prod_elim*[*OF* ‹*open E*› ‹(*x*, *y*) ∈ *E*›]
obtain *E1 E2* **where** *open E1 open E2* (*x*, *y*) ∈ *E1* × *E2 E1* × *E2* ⊆ *E*
by *blast*
show *?thesis*
proof
show *openin* (*top_of_set S*) (*E1* ∩ *S*) *openin* (*top_of_set T*) (*E2* ∩ *T*)
using ‹*open E1*› ‹*open E2*› **by** (*auto simp*: *openin_open*)
show *x* ∈ *E1* ∩ *S y* ∈ *E2* ∩ *T*
using ‹(*x*, *y*) ∈ *E1* × *E2*› ‹*x* ∈ *S*› ‹*y* ∈ *T*› **by** *auto*
show (*E1* ∩ *S*) × (*E2* ∩ *T*) ⊆ *U*
using ‹*E1* × *E2* ⊆ *E*› ‹*U* = _› **by** *auto*

```
  qed
qed

lemma closedin_Times:
  closedin (top_of_set S) S' ⟹ closedin (top_of_set T) T' ⟹
    closedin (top_of_set (S × T)) (S' × T')
  unfolding closedin_closed using closed_Times by blast

lemma openin_Times:
  openin (top_of_set S) S' ⟹ openin (top_of_set T) T' ⟹
    openin (top_of_set (S × T)) (S' × T')
  unfolding openin_open using open_Times by blast

lemma openin_Times_eq:
  fixes S :: 'a::topological_space set and T :: 'b::topological_space set
  shows
    openin (top_of_set (S × T)) (S' × T') ⟷
      S' = {} ∨ T' = {} ∨ openin (top_of_set S) S' ∧ openin (top_of_set T) T'
    (is ?lhs = ?rhs)
proof (cases S' = {} ∨ T' = {})
  case True
  then show ?thesis by auto
next
  case False
  then obtain x y where x ∈ S' y ∈ T'
    by blast
  show ?thesis
  proof
    assume ?lhs
    have openin (top_of_set S) S'
    proof (subst openin_subopen, clarify)
      show ∃ U. openin (top_of_set S) U ∧ x ∈ U ∧ U ⊆ S' if x ∈ S' for x
        using that ‹y ∈ T'› Times_in_interior_subtopology [OF _ ‹?lhs›, of x y]
        by simp (metis mem_Sigma_iff subsetD subsetI)
    qed
    moreover have openin (top_of_set T) T'
    proof (subst openin_subopen, clarify)
      show ∃ U. openin (top_of_set T) U ∧ y ∈ U ∧ U ⊆ T' if y ∈ T' for y
        using that ‹x ∈ S'› Times_in_interior_subtopology [OF _ ‹?lhs›, of x y]
        by simp (metis mem_Sigma_iff subsetD subsetI)
    qed
    ultimately show ?rhs
      by simp
  next
    assume ?rhs
    with False show ?lhs
      by (simp add: openin_Times)
  qed
qed
```

lemma *Lim_transform_within_openin*:
assumes *f*: (*f* ⟶ *l*) (*at a within T*)
and *openin* (*top_of_set T*) *S a* ∈ *S*
and *eq*: ⋀*x*. ⟦*x* ∈ *S*; *x* ≠ *a*⟧ ⟹ *f x* = *g x*
shows (*g* ⟶ *l*) (*at a within T*)
proof −
have ∀ *F x in at a within T*. *x* ∈ *T* ∧ *x* ≠ *a*
by (*simp add*: *eventually_at_filter*)
moreover
from ‹*openin* _ _› **obtain** *U* **where** *open U S* = *T* ∩ *U*
by (*auto simp*: *openin_open*)
then have *a* ∈ *U* **using** ‹*a* ∈ *S*› **by** *auto*
from *topological_tendstoD*[*OF tendsto_ident_at* ‹*open U*› ‹*a* ∈ *U*›]
have ∀ *F x in at a within T*. *x* ∈ *U* **by** *auto*
ultimately
have ∀ *F x in at a within T*. *f x* = *g x*
by *eventually_elim* (*auto simp*: ‹*S* = _› *eq*)
with *f* **show** *?thesis*
by (*rule Lim_transform_eventually*)
qed

lemma *continuous_on_open_gen*:
assumes *f* ∈ *S* → *T*
shows *continuous_on S f* ⟷
(∀ *U*. *openin* (*top_of_set T*) *U*
⟶ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
with *assms* **show** *?rhs*
apply (*simp add*: *Pi_iff continuous_openin_preimage_eq openin_open*)
by (*metis inf.orderE inf_assoc subsetI vimageI vimage_Int*)
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp add*: *continuous_openin_preimage_eq*)
show ⋀*T*. *open T* ⟹ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*)
by (*metis* (*no_types*) *R assms image_subset_iff_funcset image_subset_iff_subset_vimage inf.orderE inf_assoc openin_open_Int vimage_Int*)
qed
qed

lemma *continuous_openin_preimage*:
⟦*continuous_on S f*; *f* ∈ *S* → *T*; *openin* (*top_of_set T*) *U*⟧
⟹ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*)
by (*simp add*: *continuous_on_open_gen*)

lemma *continuous_on_closed_gen*:

assumes $f \in S \to T$
shows *continuous_on* $S\ f \longleftrightarrow$
$(\forall U.\ \mathit{closedin}\ (\mathit{top\_of\_set}\ T)\ U$
$\longrightarrow \mathit{closedin}\ (\mathit{top\_of\_set}\ S)\ (S \cap f\ -`\ U))$
proof −
have $*$: $U \subseteq T \Longrightarrow S \cap f\ -`\ (T - U) = S - (S \cap f\ -`\ U)$ **for** U
using *assms* **by** *blast*
then show *?thesis*
unfolding *continuous_on_open_gen* [*OF assms*]
by (*metis closedin_def inf.cobounded1 openin_closedin_eq topspace_euclidean_subtopology*)
qed

lemma *continuous_closedin_preimage_gen*:
assumes *continuous_on* $S\ f$ $f \in S \to T$ *closedin* (*top_of_set* T) U
shows *closedin* (*top_of_set* S) $(S \cap f\ -`\ U)$
using *assms continuous_on_closed_gen* **by** *blast*

lemma *continuous_transform_within_openin*:
assumes *continuous* (*at a within* T) f
and *openin* (*top_of_set* T) $S\ a \in S$
and *eq*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows *continuous* (*at a within* T) g
using *assms* **by** (*simp add*: *Lim_transform_within_openin continuous_within*)

1.13.22 The topology generated by some (open) subsets

In the definition below of a generated topology, the *Empty* case is not necessary, as it follows from *UN* taking for K the empty set. However, it is convenient to have, and is never a problem in proofs, so I prefer to write it down explicitly.

We do not require *UNIV* to be an open set, as this will not be the case in applications. (We are thinking of a topology on a subset of *UNIV*, the remaining part of *UNIV* being irrelevant.)

inductive *generate_topology_on* **for** S **where**
Empty: *generate_topology_on* S $\{\}$
| *Int*: *generate_topology_on* $S\ a \Longrightarrow$ *generate_topology_on* $S\ b \Longrightarrow$ *generate_topology_on* S $(a \cap b)$
| *UN*: $(\bigwedge k.\ k \in K \Longrightarrow$ *generate_topology_on* $S\ k) \Longrightarrow$ *generate_topology_on* S $(\bigcup K)$
| *Basis*: $s \in S \Longrightarrow$ *generate_topology_on* $S\ s$

lemma *istopology_generate_topology_on*:
istopology (*generate_topology_on* S)
unfolding *istopology_def* **by** (*auto intro*: *generate_topology_on.intros*)

The basic property of the topology generated by a set S is that it is the smallest topology containing all the elements of S:

lemma *generate_topology_on_coarsest*:

assumes T: *istopology* T $\bigwedge s.\ s \in S \Longrightarrow T\ s$
and *gen*: *generate_topology_on* S *s0*
shows T *s0*
using *gen*
by (*induct rule*: *generate_topology_on.induct*) (*use* T **in** ‹*auto simp*: *istopology_def*›)

abbreviation *topology_generated_by*::('*a set set*) ⇒ ('*a topology*)
where *topology_generated_by* $S \equiv$ *topology* (*generate_topology_on* S)

lemma *openin_topology_generated_by_iff*:
openin (*topology_generated_by* S) $s \longleftrightarrow$ *generate_topology_on* S s
using *topology_inverse'*[*OF istopology_generate_topology_on*[*of* S]] **by** *simp*

lemma *openin_topology_generated_by*:
openin (*topology_generated_by* S) $s \Longrightarrow$ *generate_topology_on* S s
using *openin_topology_generated_by_iff* **by** *auto*

lemma *topology_generated_by_topspace* [*simp*]:
topspace (*topology_generated_by* S) = $(\bigcup S)$
proof
{
fix s **assume** *openin* (*topology_generated_by* S) s
then have *generate_topology_on* S s **by** (*rule openin_topology_generated_by*)
then have $s \subseteq (\bigcup S)$ **by** (*induct*, *auto*)
}
then show *topspace* (*topology_generated_by* S) $\subseteq (\bigcup S)$
unfolding *topspace_def* **by** *auto*
next
have *generate_topology_on* S $(\bigcup S)$
using *generate_topology_on.UN*[*OF generate_topology_on.Basis*, *of* S S] **by** *simp*
then show $(\bigcup S) \subseteq$ *topspace* (*topology_generated_by* S)
unfolding *topspace_def* **using** *openin_topology_generated_by_iff* **by** *auto*
qed

lemma *topology_generated_by_Basis*:
$s \in S \Longrightarrow$ *openin* (*topology_generated_by* S) s
by (*simp add*: *Basis openin_topology_generated_by_iff*)

lemma *generate_topology_on_Inter*:
⟦*finite* $\mathcal{F}$; $\bigwedge K.\ K \in \mathcal{F} \Longrightarrow$ *generate_topology_on* $\mathcal{S}$ K; $\mathcal{F} \neq \{\}$⟧ ⟹ *generate_topology_on* $\mathcal{S}$ $(\bigcap \mathcal{F})$
by (*induction* $\mathcal{F}$ *rule*: *finite_induct*; *force intro*: *generate_topology_on.intros*)

1.13.23 Topology bases and sub-bases

lemma *istopology_base_alt*:
istopology (*arbitrary union_of* P) $\longleftrightarrow$
$(\forall S\ T.$ (*arbitrary union_of* P) $S \wedge$ (*arbitrary union_of* P) T

$\longrightarrow$ (*arbitrary union_of P*) (*S* $\cap$ *T*))
by (*simp add*: *istopology_def*) (*blast intro*: *arbitrary_union_of_Union*)

lemma *istopology_base_eq*:
istopology (*arbitrary union_of P*) $\longleftrightarrow$
($\forall$ *S T*. *P S* $\wedge$ *P T* $\longrightarrow$ (*arbitrary union_of P*) (*S* $\cap$ *T*))
by (*simp add*: *istopology_base_alt arbitrary_union_of_Int_eq*)

lemma *istopology_base*:
($\bigwedge$*S T*. $[\![$*P S*; *P T*$]\!] \Longrightarrow$ *P*(*S* $\cap$ *T*)) $\Longrightarrow$ *istopology* (*arbitrary union_of P*)
by (*simp add*: *arbitrary_def istopology_base_eq union_of_inc*)

lemma *openin_topology_base_unique*:
openin X = *arbitrary union_of P* $\longleftrightarrow$
($\forall$ *V*. *P V* $\longrightarrow$ *openin X V*) $\wedge$ ($\forall$ *U x*. *openin X U* $\wedge$ *x* $\in$ *U* $\longrightarrow$ ($\exists$ *V*. *P V* $\wedge$ *x* $\in$ *V* $\wedge$ *V* $\subseteq$ *U*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*auto simp*: *union_of_def arbitrary_def*)
next
assume *R*: *?rhs*
then have $*$: $\exists\mathcal{U}\subseteq$*Collect P*. $\bigcup\mathcal{U}$ = *S* **if** *openin X S* **for** *S*
using *that* **by** (*rule_tac x*={*V*. *P V* $\wedge$ *V* $\subseteq$ *S*} **in** *exI*) *fastforce*
from *R* **show** *?lhs*
by (*fastforce simp add*: *union_of_def arbitrary_def intro*: $*$)
qed

lemma *topology_base_unique*:
assumes $\bigwedge$*S*. *P S* $\Longrightarrow$ *openin X S*
$\bigwedge$*U x*. $[\![$*openin X U*; *x* $\in$ *U*$]\!] \Longrightarrow \exists$ *B*. *P B* $\wedge$ *x* $\in$ *B* $\wedge$ *B* $\subseteq$ *U*
shows *topology* (*arbitrary union_of P*) = *X*
proof −
have *X* = *topology* (*openin X*)
by (*simp add*: *openin_inverse*)
also from *assms* **have** *openin X* = *arbitrary union_of P*
by (*subst openin_topology_base_unique*) *auto*
finally show *?thesis* **..**
qed

lemma *topology_bases_eq_aux*:
$[\![$(*arbitrary union_of P*) *S*;
$\bigwedge$*U x*. $[\![$*P U*; *x* $\in$ *U*$]\!] \Longrightarrow \exists$ *V*. *Q V* $\wedge$ *x* $\in$ *V* $\wedge$ *V* $\subseteq$ *U*$]\!]$
$\Longrightarrow$ (*arbitrary union_of Q*) *S*
by (*metis arbitrary_union_of_alt arbitrary_union_of_idempot*)

lemma *topology_bases_eq*:
$[\![\bigwedge$*U x*. $[\![$*P U*; *x* $\in$ *U*$]\!] \Longrightarrow \exists$ *V*. *Q V* $\wedge$ *x* $\in$ *V* $\wedge$ *V* $\subseteq$ *U*;

$\bigwedge V\ x.\ [\![Q\ V;\ x \in V]\!] \Longrightarrow \exists U.\ P\ U \wedge x \in U \wedge U \subseteq V]\!]$
$\Longrightarrow$ *topology* (*arbitrary union_of P*) =
topology (*arbitrary union_of Q*)
by (*fastforce intro*: *arg_cong* [**where** *f*=*topology*] *elim*: *topology_bases_eq_aux*)

lemma *istopology_subbase*:
istopology (*arbitrary union_of* (*finite intersection_of P relative_to S*))
by (*simp add*: *finite_intersection_of_Int istopology_base relative_to_Int*)

lemma *openin_subbase*:
openin (*topology* (*arbitrary union_of* (*finite intersection_of B relative_to U*))) *S*
$\longleftrightarrow$ (*arbitrary union_of* (*finite intersection_of B relative_to U*)) *S*
by (*simp add*: *istopology_subbase topology_inverse'*)

lemma *topspace_subbase* [*simp*]:
topspace(*topology* (*arbitrary union_of* (*finite intersection_of B relative_to U*))) = *U* (**is** *?lhs* = _)
proof
show *?lhs* $\subseteq$ *U*
by (*metis arbitrary_union_of_relative_to openin_subbase openin_topspace relative_to_imp_subset*)
show *U* $\subseteq$ *?lhs*
by (*metis arbitrary_union_of_inc finite_intersection_of_empty inf.orderE istopology_subbase*
openin_subset relative_to_inc subset_UNIV topology_inverse')
qed

lemma *minimal_topology_subbase*:
assumes *X*: $\bigwedge S.$ *P S* $\Longrightarrow$ *openin X S* **and** *openin X U*
and *S*: *openin*(*topology*(*arbitrary union_of* (*finite intersection_of P relative_to U*))) *S*
shows *openin X S*
proof −
have (*arbitrary union_of* (*finite intersection_of P relative_to U*)) *S*
using *S openin_subbase* **by** *blast*
with *X* ‹*openin X U*› **show** *?thesis*
by (*force simp add*: *union_of_def intersection_of_def relative_to_def intro*: *openin_Int_Inter*)
qed

lemma *istopology_subbase_UNIV*:
istopology (*arbitrary union_of* (*finite intersection_of P*))
by (*simp add*: *istopology_base finite_intersection_of_Int*)

lemma *generate_topology_on_eq*:
generate_topology_on S = *arbitrary union_of finite' intersection_of* ($\lambda x.\ x \in S$) (**is** *?lhs* = *?rhs*)

proof (*intro ext iffI*)
fix *A*
assume *?lhs A*
then show *?rhs A*
proof *induction*
case (*Int a b*)
then show *?case*
by (*metis* (*mono_tags*, *lifting*) *istopology_base_alt finite′_intersection_of_Int istopology_base*)
next
case (*UN K*)
then show *?case*
by (*simp add*: *arbitrary_union_of_Union*)
next
case (*Basis s*)
then show *?case*
by (*simp add*: *Sup_upper arbitrary_union_of_inc finite′_intersection_of_inc relative_to_subset_inc*)
qed *auto*
next
fix *A*
assume *?rhs A*
then obtain $\mathcal{U}$ **where** $\mathcal{U}$: $\bigwedge T.\ T \in \mathcal{U} \Longrightarrow \exists \mathcal{F}.\ \mathit{finite}'\ \mathcal{F} \wedge \mathcal{F} \subseteq S \wedge \bigcap \mathcal{F} = T$ **and** *eq*: $A = \bigcup \mathcal{U}$
unfolding *union_of_def intersection_of_def* **by** *auto*
show *?lhs A*
unfolding *eq*
proof (*rule generate_topology_on.UN*)
fix *T*
assume $T \in \mathcal{U}$
with $\mathcal{U}$ **obtain** $\mathcal{F}$ **where** *finite′* $\mathcal{F}$ $\mathcal{F} \subseteq S$ $\bigcap \mathcal{F} = T$
by *blast*
have *generate_topology_on S* $(\bigcap \mathcal{F})$
proof (*rule generate_topology_on_Inter*)
show *finite* $\mathcal{F}$ $\mathcal{F} \neq \{\}$
by (*auto simp*: ‹*finite′* $\mathcal{F}$›)
show $\bigwedge K.\ K \in \mathcal{F} \Longrightarrow$ *generate_topology_on S K*
by (*metis* ‹$\mathcal{F} \subseteq S$› *generate_topology_on.simps subset_iff*)
qed
then show *generate_topology_on S T*
using ‹$\bigcap \mathcal{F} = T$› **by** *blast*
qed
qed

lemma *continuous_on_generated_topo_iff*:
continuous_map T1 (*topology_generated_by S*) *f* $\longleftrightarrow$
$((\forall U.\ U \in S \longrightarrow \mathit{openin}\ T1\ (f - `U \cap \mathit{topspace}(T1))) \wedge (f`(\mathit{topspace}\ T1) \subseteq (\bigcup S)))$
unfolding *continuous_map_alt topology_generated_by_topspace*

proof (*auto simp add*: *topology_generated_by_Basis*)
assume *H*: ∀ *U*. *U* ∈ *S* ⟶ *openin T1* (*f* −‘ *U* ∩ *topspace T1*)
fix *U* **assume** *openin* (*topology_generated_by S*) *U*
then have *generate_topology_on S U* **by** (*rule openin_topology_generated_by*)
then show *openin T1* (*f* −‘ *U* ∩ *topspace T1*)
proof (*induct*)
fix *a b*
assume *H*: *openin T1* (*f* −‘ *a* ∩ *topspace T1*) *openin T1* (*f* −‘ *b* ∩ *topspace T1*)
have *f* −‘ (*a* ∩ *b*) ∩ *topspace T1* = (*f*−‘*a* ∩ *topspace T1*) ∩ (*f*−‘*b* ∩ *topspace T1*)
by *auto*
then show *openin T1* (*f* −‘ (*a* ∩ *b*) ∩ *topspace T1*) **using** *H* **by** *auto*
next
fix *K*
assume *H*: *openin T1* (*f* −‘ *k* ∩ *topspace T1*) **if** *k*∈ *K* **for** *k*
define *L* **where** *L* = {*f* −‘ *k* ∩ *topspace T1*|*k*. *k* ∈ *K*}
have ∗: *openin T1 l* **if** *l* ∈*L* **for** *l* **using** *that H* **unfolding** *L_def* **by** *auto*
have *openin T1* (⋃ *L*) **using** *openin_Union*[*OF* ∗] **by** *simp*
moreover have (⋃ *L*) = (*f* −‘ ⋃ *K* ∩ *topspace T1*) **unfolding** *L_def* **by** *auto*
ultimately show *openin T1* (*f* −‘ ⋃ *K* ∩ *topspace T1*) **by** *simp*
qed (*auto simp add*: *H*)
qed

lemma *continuous_on_generated_topo*:
assumes ⋀*U*. *U* ∈*S* ⟹ *openin T1* (*f*−‘*U* ∩ *topspace*(*T1*))
f‘(*topspace T1*) ⊆ (⋃ *S*)
shows *continuous_map T1* (*topology_generated_by S*) *f*
using *assms continuous_on_generated_topo_iff* **by** *blast*

1.13.24 Continuity via bases/subbases, hence upper and lower semicontinuity

lemma *continuous_map_into_topology_base*:
assumes *P*: *openin Y* = *arbitrary union_of P*
and *f*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ∈ *topspace Y*
and *ope*: ⋀*U*. *P U* ⟹ *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
shows *continuous_map X Y f*
proof −
have ∗: ⋀*U*. (⋀*t*. *t* ∈ *U* ⟹ *P t*) ⟹ *openin X* {*x* ∈ *topspace X*. ∃ *U*∈*U*. *f x* ∈ *U*}
by (*smt* (*verit*) *Ball_Collect ope mem_Collect_eq openin_subopen*)
show *?thesis*
using *P* **by** (*auto simp*: *continuous_map_def arbitrary_def union_of_def intro*!: *f* ∗)
qed

lemma *continuous_map_into_topology_base_eq*:
assumes *P*: *openin Y* = *arbitrary union_of P*

```
  shows
   continuous_map X Y f ⟷
    f ∈ topspace X → topspace Y ∧ (∀ U. P U ⟶ openin X {x ∈ topspace X. f x
∈ U})
 (is ?lhs=?rhs)
proof
  assume L: ?lhs
  then have f ∈ topspace X → topspace Y
    by (simp add: continuous_map_funspace)
  moreover have ⋀U. P U ⟹ openin X {x ∈ topspace X. f x ∈ U}
    using L assms continuous_map openin_topology_base_unique by fastforce
  ultimately show ?rhs by auto
qed (simp add: assms Pi_iff continuous_map_into_topology_base)

lemma continuous_map_into_topology_subbase:
  fixes U P
  defines Y ≡ topology(arbitrary union_of (finite intersection_of P relative_to
U))
  assumes f: ⋀x. x ∈ topspace X ⟹ f x ∈ topspace Y
    and ope: ⋀U. P U ⟹ openin X {x ∈ topspace X. f x ∈ U}
  shows continuous_map X Y f
proof (intro continuous_map_into_topology_base)
  show openin Y = arbitrary union_of (finite intersection_of P relative_to U)
    unfolding Y_def using istopology_subbase topology_inverse' by blast
  show openin X {x ∈ topspace X. f x ∈ V}
    if §: (finite intersection_of P relative_to U) V for V
  proof −
    define finv where finv ≡ λV. {x ∈ topspace X. f x ∈ V}
    obtain 𝒰 where 𝒰: finite 𝒰 ⋀V. V ∈ 𝒰 ⟹ P V
                    {x ∈ topspace X. f x ∈ V} = (⋂ V ∈ insert U 𝒰. finv V)
      using § by (fastforce simp: finv_def intersection_of_def relative_to_def)
    show ?thesis
      unfolding 𝒰
    proof (intro openin_Inter ope)
      have U: U = topspace Y
        unfolding Y_def using topspace_subbase by fastforce
      fix V
      assume V: V ∈ finv ‘ insert U 𝒰
      with U f have openin X {x ∈ topspace X. f x ∈ U}
        by (auto simp: openin_subopen [of X Collect _])
      then show openin X V
        using V 𝒰(2) ope by (fastforce simp: finv_def)
    qed (use ‹finite 𝒰› in auto)
  qed
qed (use f in auto)

lemma continuous_map_into_topology_subbase_eq:
  assumes Y = topology(arbitrary union_of (finite intersection_of P relative_to
U))
```

```
  shows
   continuous_map X Y f ⟷
    f ∈ topspace X → topspace Y ∧ (∀ U. P U ⟶ openin X {x ∈ topspace X. f x
∈ U})
   (is ?lhs=?rhs)
proof
  assume L: ?lhs
  show ?rhs
  proof (intro conjI strip)
    show f ∈ topspace X → topspace Y
      using L continuous_map_def by fastforce
    fix V
    assume P V
    have U = topspace Y
      unfolding assms using topspace_subbase by fastforce
    then have eq: {x ∈ topspace X. f x ∈ V} = {x ∈ topspace X. f x ∈ U ∩ V}
      using L by (auto simp: continuous_map)
    have openin Y (U ∩ V)
      unfolding assms openin_subbase
       by (meson ‹P V› arbitrary_union_of_inc finite_intersection_of_inc rela-
tive_to_inc)
    show openin X {x ∈ topspace X. f x ∈ V}
      using L ‹openin Y (U ∩ V)› continuous_map eq by fastforce
  qed
next
  show ?rhs ⟹ ?lhs
    unfolding assms
    by (intro continuous_map_into_topology_subbase) auto
qed

lemma subbase_subtopology_euclidean:
  fixes U :: 'a::order_topology set
  shows
  topology
    (arbitrary union_of
       (finite intersection_of (λx. x ∈ range greaterThan ∪ range lessThan) rela-
tive_to U))
 = subtopology euclidean U
proof −
   have ∃ V. (finite intersection_of (λx. x ∈ range greaterThan ∨ x ∈ range
lessThan)) V ∧ x ∈ V ∧ V ⊆ W
    if open W x ∈ W for W and x::'a
    using ‹open W› [unfolded open_generated_order] ‹x ∈ W›
  proof (induct rule: generate_topology.induct)
    case UNIV
    then show ?case
      using finite_intersection_of_empty by blast
  next
    case (Int a b)
```

```
    then show ?case
      by (meson Int_iff finite_intersection_of_Int inf_mono)
  next
    case (UN K)
    then show ?case
      by (meson Union_iff subset_iff)
  next
    case (Basis s)
    then show ?case
      by (metis (no_types, lifting) Un_iff finite_intersection_of_inc order_refl)
  qed
  moreover
  have ⋀V::'a set. (finite intersection_of (λx. x ∈ range greaterThan ∨ x ∈ range
lessThan)) V ⟹ open V
    by (force simp: intersection_of_def subset_iff)
  ultimately have *: openin (euclidean::'a topology) =
        (arbitrary union_of (finite intersection_of (λx. x ∈ range greaterThan ∨
x ∈ range lessThan)))
    by (smt (verit, best) openin_topology_base_unique open_openin)
  show ?thesis
    unfolding subtopology_def arbitrary_union_of_relative_to [symmetric]
    apply (simp add: relative_to_def flip: *)
    by (metis Int_commute)
qed

lemma continuous_map_upper_lower_semicontinuous_lt_gen:
  fixes U :: 'a::order_topology set
  shows continuous_map X (subtopology euclidean U) f ⟷
        (∀x ∈ topspace X. f x ∈ U) ∧
        (∀a. openin X {x ∈ topspace X. f x > a}) ∧
        (∀a. openin X {x ∈ topspace X. f x < a})
  by (auto simp: continuous_map_into_topology_subbase_eq [OF subbase_subtopology_euclidean
[symmetric, of U]]
          greaterThan_def lessThan_def image_iff   simp flip: all_simps)

lemma continuous_map_upper_lower_semicontinuous_lt:
  fixes f :: 'a ⇒ 'b::order_topology
  shows continuous_map X euclidean f ⟷
        (∀a. openin X {x ∈ topspace X. f x > a}) ∧
        (∀a. openin X {x ∈ topspace X. f x < a})
  using continuous_map_upper_lower_semicontinuous_lt_gen [where U=UNIV]
  by auto

lemma Int_Collect_imp_eq: A ∩ {x. x∈A ⟶ P x} = {x∈A. P x}
  by blast

lemma continuous_map_upper_lower_semicontinuous_le_gen:
  shows
    continuous_map X (subtopology euclideanreal U) f ⟷
```

$(\forall x \in topspace\ X.\ f\ x \in U) \wedge$
$(\forall a.\ closedin\ X\ \{x \in topspace\ X.\ f\ x \geq a\}) \wedge$
$(\forall a.\ closedin\ X\ \{x \in topspace\ X.\ f\ x \leq a\})$
unfolding *continuous_map_upper_lower_semicontinuous_lt_gen*
by (*auto simp*: *closedin_def Diff_eq Compl_eq not_le Int_Collect_imp_eq*)

lemma *continuous_map_upper_lower_semicontinuous_le*:
continuous_map X euclideanreal f $\longleftrightarrow$
$(\forall a.\ closedin\ X\ \{x \in topspace\ X.\ f\ x \geq a\}) \wedge$
$(\forall a.\ closedin\ X\ \{x \in topspace\ X.\ f\ x \leq a\})$
using *continuous_map_upper_lower_semicontinuous_le_gen* [**where** $U{=}UNIV$]
by *auto*

lemma *continuous_map_upper_lower_semicontinuous_lte_gen*:
continuous_map X (*subtopology euclideanreal U*) *f* $\longleftrightarrow$
$(\forall x \in topspace\ X.\ f\ x \in U) \wedge$
$(\forall a.\ openin\ X\ \{x \in topspace\ X.\ f\ x < a\}) \wedge$
$(\forall a.\ closedin\ X\ \{x \in topspace\ X.\ f\ x \leq a\})$
unfolding *continuous_map_upper_lower_semicontinuous_lt_gen*
by (*auto simp*: *closedin_def Diff_eq Compl_eq not_le Int_Collect_imp_eq*)

lemma *continuous_map_upper_lower_semicontinuous_lte*:
continuous_map X euclideanreal f $\longleftrightarrow$
$(\forall a.\ openin\ X\ \{x \in topspace\ X.\ f\ x < a\}) \wedge$
$(\forall a.\ closedin\ X\ \{x \in topspace\ X.\ f\ x \leq a\})$
using *continuous_map_upper_lower_semicontinuous_lte_gen* [**where** $U{=}UNIV$]
by *auto*

1.13.25 Pullback topology

Pulling back a topology by map gives again a topology. *subtopology* is a special case of this notion, pulling back by the identity. We introduce the general notion as we will need it to define the strong operator topology on the space of continuous linear operators, by pulling back the product topology on the space of all functions.

pullback_topology A f T is the pullback of the topology T by the map f on the set A.

definition *pullback_topology*::(*'a set*) $\Rightarrow$ (*'a* $\Rightarrow$ *'b*) $\Rightarrow$ (*'b topology*) $\Rightarrow$ (*'a topology*)
where *pullback_topology A f T* = *topology* ($\lambda S.\ \exists U.\ openin\ T\ U \wedge S = f{-}`U \cap A$)

lemma *istopology_pullback_topology*:
istopology ($\lambda S.\ \exists U.\ openin\ T\ U \wedge S = f{-}`U \cap A$)
unfolding *istopology_def* **proof** (*auto*)
fix *K* **assume** $\forall S{\in}K.\ \exists U.\ openin\ T\ U \wedge S = f\ -`\ U \cap A$
then have $\exists U.\ \forall S{\in}K.\ openin\ T\ (U\ S) \wedge S = f{-}`(U\ S) \cap A$
by (*rule bchoice*)

then obtain U **where** U: $\forall S{\in}K.\ openin\ T\ (U\ S) \wedge S = f{-}`(U\ S) \cap A$
by *blast*
define V **where** $V = (\bigcup S{\in}K.\ U\ S)$
have $openin\ T\ V \bigcup K = f\ -`\ V \cap A$ **unfolding** V_def **using** U **by** *auto*
then show $\exists V.\ openin\ T\ V \wedge \bigcup K = f\ -`\ V \cap A$ **by** *auto*
qed

lemma *openin_pullback_topology*:
$openin\ (pullback\_topology\ A\ f\ T)\ S \longleftrightarrow (\exists U.\ openin\ T\ U \wedge S = f{-}`U \cap A)$
unfolding *pullback_topology_def topology_inverse'*[*OF istopology_pullback_topology*]
by *auto*

lemma *topspace_pullback_topology*:
$topspace\ (pullback\_topology\ A\ f\ T) = f{-}`(topspace\ T) \cap A$
by (*auto simp add*: *topspace_def openin_pullback_topology*)

proposition *continuous_map_pullback* [*intro*]:
assumes *continuous_map T1 T2 g*
shows *continuous_map* (*pullback_topology A f T1*) *T2* (*g o f*)
unfolding *continuous_map_alt*
proof (*auto*)
fix U::$'b\ set$ **assume** *openin T2 U*
then have *openin T1* (*g*−`*U* ∩ *topspace T1*)
using *assms* **unfolding** *continuous_map_alt* **by** *auto*
have (*g o f*)−`*U* ∩ *topspace* (*pullback_topology A f T1*) = (*g o f*)−`*U* ∩ *A* ∩ *f*−`(*topspace T1*)
unfolding *topspace_pullback_topology* **by** *auto*
also have ... = *f*−`(*g*−`*U* ∩ *topspace T1*) ∩ *A*
by *auto*
also have *openin* (*pullback_topology A f T1*) (...)
unfolding *openin_pullback_topology* **using** ‹*openin T1* (*g*−`*U* ∩ *topspace T1*)›
by *auto*
finally show *openin* (*pullback_topology A f T1*) ((*g* ∘ *f*) −` *U* ∩ *topspace* (*pullback_topology A f T1*))
by *auto*
next
fix *x* **assume** *x* ∈ *topspace* (*pullback_topology A f T1*)
then have *f x* ∈ *topspace T1*
unfolding *topspace_pullback_topology* **by** *auto*
then show *g* (*f x*) ∈ *topspace T2*
using *assms* **unfolding** *continuous_map_def* **by** *auto*
qed

proposition *continuous_map_pullback'* [*intro*]:
assumes *continuous_map T1 T2* (*f o g*) *topspace T1* ⊆ *g*−`*A*
shows *continuous_map T1* (*pullback_topology A f T2*) *g*
unfolding *continuous_map_alt*
proof (*auto*)
fix *U* **assume** *openin* (*pullback_topology A f T2*) *U*

```
  then have ∃ V. openin T2 V ∧ U = f−‘V ∩ A
    unfolding openin_pullback_topology by auto
  then obtain V where openin T2 V U = f−‘V ∩ A
    by blast
  then have g −‘ U ∩ topspace T1 = g−‘(f−‘V ∩ A) ∩ topspace T1
    by blast
  also have ... = (f o g)−‘V ∩ (g−‘A ∩ topspace T1)
    by auto
  also have ... = (f o g)−‘V ∩ topspace T1
    using assms(2) by auto
  also have openin T1 (...)
    using assms(1) ‹openin T2 V› by auto
  finally show openin T1 (g −‘ U ∩ topspace T1) by simp
next
  fix x assume x ∈ topspace T1
  have (f o g) x ∈ topspace T2
    using assms(1) ‹x ∈ topspace T1› unfolding continuous_map_def by auto
  then have g x ∈ f−‘(topspace T2)
    unfolding comp_def by blast
  moreover have g x ∈ A using assms(2) ‹x ∈ topspace T1› by blast
  ultimately show g x ∈ topspace (pullback_topology A f T2)
    unfolding topspace_pullback_topology by blast
qed
```

1.13.26 Proper maps (not a priori assumed continuous)

```
definition proper_map
  where
 proper_map X Y f ≡
       closed_map X Y f ∧ (∀ y ∈ topspace Y. compactin X {x ∈ topspace X. f x
= y})

lemma proper_imp_closed_map:
   proper_map X Y f ⟹ closed_map X Y f
  by (simp add: proper_map_def)

lemma proper_map_imp_subset_topspace:
   proper_map X Y f ⟹ f ∈ (topspace X) → topspace Y
  by (simp add: closed_map_imp_subset_topspace proper_map_def)

lemma proper_map_restriction:
  assumes proper_map X Y f {x ∈ topspace X. f x ∈ V} = U
  shows proper_map (subtopology X U) (subtopology Y V) f
proof −
  have [simp]: {x ∈ topspace X. f x ∈ V ∧ f x = y} = {x ∈ topspace X. f x = y}
    if y ∈ V for y
    using that by auto
  show ?thesis
    using assms unfolding proper_map_def using closed_map_restriction
```

```
    by (force simp: compactin_subtopology)
qed

lemma closed_injective_imp_proper_map:
  assumes f: closed_map X Y f and inj: inj_on f (topspace X)
  shows proper_map X Y f
  unfolding proper_map_def
proof (clarsimp simp: f)
  show compactin X {x ∈ topspace X. f x = y}
    if y ∈ topspace Y for y
using inj_on_eq_iff [OF inj] that
  proof −
    have {x ∈ topspace X. f x = y} = {} ∨ (∃ a ∈ topspace X. {x ∈ topspace X. f
x = y} = {a})
      using inj_on_eq_iff [OF inj] by auto
    then show ?thesis
      using that by (metis (no_types, lifting) compactin_empty compactin_sing)
  qed
qed

lemma injective_imp_proper_eq_closed_map:
   inj_on f (topspace X) ⟹ (proper_map X Y f ⟷ closed_map X Y f)
  using closed_injective_imp_proper_map proper_imp_closed_map by blast

lemma homeomorphic_imp_proper_map:
   homeomorphic_map X Y f ⟹ proper_map X Y f
 by (simp add: closed_injective_imp_proper_map homeomorphic_eq_everything_map)

lemma compactin_proper_map_preimage:
  assumes f: proper_map X Y f and compactin Y K
  shows compactin X {x. x ∈ topspace X ∧ f x ∈ K}
proof −
  have f ∈ (topspace X) → topspace Y
    by (simp add: f proper_map_imp_subset_topspace)
  have ∗: ⋀y. y ∈ topspace Y ⟹ compactin X {x ∈ topspace X. f x = y}
    using f by (auto simp: proper_map_def)
  show ?thesis
    unfolding compactin_def
  proof clarsimp
    show ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ {x ∈ topspace X. f x ∈ K} ⊆ ⋃ℱ
      if 𝒰: ∀ U∈𝒰. openin X U and sub: {x ∈ topspace X. f x ∈ K} ⊆ ⋃𝒰
      for 𝒰
    proof −
      have ∀ y ∈ K. ∃𝒱. finite 𝒱 ∧ 𝒱 ⊆ 𝒰  ∧ {x ∈ topspace X. f x = y} ⊆ ⋃𝒱
      proof
        fix y
        assume y ∈ K
        then have compactin X {x ∈ topspace X. f x = y}
          by (metis ∗ ‹compactin Y K› compactin_subspace subsetD)
```

```
      with ‹y ∈ K› show ∃𝒱. finite 𝒱 ∧ 𝒱 ⊆ 𝒰 ∧ {x ∈ topspace X. f x = y} ⊆
⋃𝒱
        unfolding compactin_def using 𝒰 sub by fastforce
    qed
    then obtain 𝒱 where 𝒱: ⋀y. y ∈ K ⟹ finite (𝒱 y) ∧ 𝒱 y ⊆ 𝒰 ∧ {x ∈
topspace X. f x = y} ⊆ ⋃(𝒱 y)
      by (metis (full_types))
    define F where F ≡ λy. topspace Y − f ‘ (topspace X − ⋃(𝒱 y))
    have ∃ℱ. finite ℱ ∧ ℱ ⊆ F ‘ K ∧ K ⊆ ⋃ℱ
    proof (rule compactinD [OF ‹compactin Y K›])
      have ⋀x. x ∈ K ⟹ closedin Y (f ‘ (topspace X − ⋃(𝒱 x)))
        using f unfolding proper_map_def closed_map_def
        by (meson 𝒰 𝒱 openin_Union openin_closedin_eq subsetD)
      then show openin Y U if U ∈ F ‘ K for U
        using that by (auto simp: F_def)
      show K ⊆ ⋃(F ‘ K)
        using 𝒱 ‹compactin Y K› unfolding F_def compactin_def by fastforce
    qed
    then obtain J where finite J J ⊆ K and J: K ⊆ ⋃(F ‘ J)
      by (auto simp: ex_finite_subset_image)
    show ?thesis
      unfolding F_def
    proof (intro exI conjI)
      show finite (⋃(𝒱 ‘ J))
        using 𝒱 ‹J ⊆ K› ‹finite J› by blast
      show ⋃(𝒱 ‘ J) ⊆ 𝒰
        using 𝒱 ‹J ⊆ K› by blast
      show {x ∈ topspace X. f x ∈ K} ⊆ ⋃(⋃(𝒱 ‘ J))
        using J ‹J ⊆ K› unfolding F_def by auto
    qed
  qed
 qed
qed


lemma compact_space_proper_map_preimage:
  assumes f: proper_map X Y f and fim: f ‘ (topspace X) = topspace Y and
compact_space Y
  shows compact_space X
proof −
  have eq: topspace X = {x ∈ topspace X. f x ∈ topspace Y}
    using fim by blast
  moreover have compactin Y (topspace Y)
    using ‹compact_space Y› compact_space_def by auto
  ultimately show ?thesis
    unfolding compact_space_def
    using eq f compactin_proper_map_preimage by fastforce
qed
```

lemma *proper_map_alt*:
  proper_map X Y f ⟷
   closed_map X Y f ∧ (∀ *K*. *compactin Y K* ⟶ *compactin X* {*x*. *x* ∈ *topspace X* ∧ *f x* ∈ *K*})
 proof (*intro iffI conjI allI impI*)
 show *compactin X* {*x* ∈ *topspace X*. *f x* ∈ *K*}
  if *proper_map X Y f* **and** *compactin Y K* **for** *K*
  using *that* **by** (*simp add*: *compactin_proper_map_preimage*)
 show *proper_map X Y f*
  if *f*: *closed_map X Y f* ∧ (∀ *K*. *compactin Y K* ⟶ *compactin X* {*x* ∈ *topspace X*. *f x* ∈ *K*})
 proof −
  have *compactin X* {*x* ∈ *topspace X*. *f x* = *y*} **if** *y* ∈ *topspace Y* **for** *y*
  proof −
   have *compactin X* {*x* ∈ *topspace X*. *f x* ∈ {*y*}}
    using *f compactin_sing that* **by** *fastforce*
   then show *?thesis*
    by *auto*
  qed
  with *f* **show** *?thesis*
   by (*auto simp*: *proper_map_def*)
 qed
qed (*simp add*: *proper_imp_closed_map*)

lemma *proper_map_on_empty* [*simp*]: *proper_map trivial_topology Y f*
 by (*auto simp*: *proper_map_def closed_map_on_empty*)

lemma *proper_map_id* [*simp*]:
  proper_map X X id
proof (*clarsimp simp*: *proper_map_alt closed_map_id*)
 fix *K*
 assume *K*: *compactin X K*
 then have {*a* ∈ *topspace X*. *a* ∈ *K*} = *K*
  by (*simp add*: *compactin_subspace subset_antisym subset_iff*)
 then show *compactin X* {*a* ∈ *topspace X*. *a* ∈ *K*}
  using *K* **by** *auto*
qed

lemma *proper_map_compose*:
 assumes *proper_map X Y f proper_map Y Z g*
 shows *proper_map X Z* (*g* ∘ *f*)
proof −
  have *closed_map X Y f* **and** *f*: ⋀*K*. *compactin Y K* ⟹ *compactin X* {*x* ∈ *topspace X*. *f x* ∈ *K*}
   and *closed_map Y Z g* **and** *g*: ⋀*K*. *compactin Z K* ⟹ *compactin Y* {*x* ∈ *topspace Y*. *g x* ∈ *K*}
  using *assms* **by** (*auto simp*: *proper_map_alt*)
 show *?thesis*
  unfolding *proper_map_alt*

```
  proof (intro conjI allI impI)
    show closed_map X Z (g ∘ f)
      using ‹closed_map X Y f› ‹closed_map Y Z g› closed_map_compose by
blast
    have {x ∈ topspace X. g (f x) ∈ K} = {x ∈ topspace X. f x ∈ {b ∈ topspace
Y. g b ∈ K}} for K
      using ‹closed_map X Y f› closed_map_imp_subset_topspace by blast
    then show compactin X {x ∈ topspace X. (g ∘ f) x ∈ K}
      if compactin Z K for K
      using f [OF g [OF that]] by auto
  qed
qed

lemma proper_map_const:
   proper_map X Y (λx. c) ⟷ compact_space X ∧ (X = trivial_topology ∨
closedin Y {c})
proof (cases X = trivial_topology)
  case True
  then show ?thesis
    by simp
next
  case False
  have ∗: compactin X {x ∈ topspace X. c = y} if compact_space X for y
    using that unfolding compact_space_def
   by (metis (mono_tags, lifting) compactin_empty empty_subsetI mem_Collect_eq
subsetI subset_antisym)
  then show ?thesis
    using closed_compactin closedin_subset
    by (force simp: False proper_map_def closed_map_const compact_space_def)
qed

lemma proper_map_inclusion:
   S ⊆ topspace X ⟹ proper_map (subtopology X S) X id ⟷ closedin X S ∧
(∀ k. compactin X k ⟶ compactin X (S ∩ k))
 by (metis closed_Int_compactin closed_map_inclusion_eq inf.absorb_iff2 inj_on_id
injective_imp_proper_eq_closed_map)

lemma proper_map_into_subtopology:
   ⟦proper_map X Y f; f ∈ topspace X → C⟧ ⟹ proper_map X (subtopology Y
C) f
 by (simp add: closed_map_into_subtopology compactin_subtopology proper_map_alt)

lemma proper_map_from_composition_left:
  assumes gf: proper_map X Z (g ∘ f) and contf: continuous_map X Y f and
fim: f ‘ topspace X = topspace Y
  shows proper_map Y Z g
  unfolding proper_map_def
proof (intro strip conjI)
  show closed_map Y Z g
```

by (*meson closed_map_from_composition_left gf contf fim proper_imp_closed_map*)
fix *z* **assume** *z* ∈ *topspace Z*
have *eq*: {*y* ∈ *topspace Y*. *g y* = *z*} = *f* ‘ {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* = *z*}
using *fim* **by** *force*
show *compactin Y* {*x* ∈ *topspace Y*. *g x* = *z*}
unfolding *eq*
proof (*rule image_compactin* [*OF* _ *contf*])
show *compactin X* {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* = *z*}
using ‹*z* ∈ *topspace Z*› *gf proper_map_def* **by** *fastforce*
qed
qed

lemma *proper_map_from_composition_right_inj*:
assumes *gf*: *proper_map X Z* (*g* ∘ *f*) **and** *fim*: *f* ∈ *topspace X* → *topspace Y*
and *contf*: *continuous_map Y Z g* **and** *inj*: *inj_on g* (*topspace Y*)
shows *proper_map X Y f*
unfolding *proper_map_def*
proof (*intro strip conjI*)
show *closed_map X Y f*
by (*meson closed_map_from_composition_right assms proper_imp_closed_map*)
fix *y*
assume *y* ∈ *topspace Y*
with *fim inj* **have** *eq*: {*x* ∈ *topspace X*. *f x* = *y*} = {*x* ∈ *topspace X*. (*g* ∘ *f*) *x* = *g y*}
by (*auto simp*: *Pi_iff inj_onD*)
show *compactin X* {*x* ∈ *topspace X*. *f x* = *y*}
using *contf gf* ‹*y* ∈ *topspace Y*›
unfolding *eq continuous_map_def proper_map_def*
by *blast*
qed

1.13.27 Perfect maps (proper, continuous and surjective)

definition *perfect_map*
where *perfect_map X Y f* ≡ *continuous_map X Y f* ∧ *proper_map X Y f* ∧ *f* ‘ (*topspace X*) = *topspace Y*

lemma *homeomorphic_imp_perfect_map*:
homeomorphic_map X Y f ⟹ *perfect_map X Y f*
by (*simp add*: *homeomorphic_eq_everything_map homeomorphic_imp_proper_map perfect_map_def*)

lemma *perfect_imp_quotient_map*:
perfect_map X Y f ⟹ *quotient_map X Y f*
by (*simp add*: *continuous_closed_imp_quotient_map perfect_map_def proper_map_def*)

lemma *homeomorphic_eq_injective_perfect_map*:
homeomorphic_map X Y f ⟷ *perfect_map X Y f* ∧ *inj_on f* (*topspace X*)
using *homeomorphic_imp_perfect_map homeomorphic_map_def perfect_imp_quotient_map*

by *blast*

lemma *perfect_injective_eq_homeomorphic_map*:
perfect_map X Y f ∧ *inj_on f* (*topspace X*) ⟷ *homeomorphic_map X Y f*
by (*simp add*: *homeomorphic_eq_injective_perfect_map*)

lemma *perfect_map_id* [*simp*]: *perfect_map X X id*
by (*simp add*: *homeomorphic_imp_perfect_map*)

lemma *perfect_map_compose*:
⟦*perfect_map X Y f*; *perfect_map Y Z g*⟧ ⟹ *perfect_map X Z* (*g* ∘ *f*)
by (*meson continuous_map_compose perfect_imp_quotient_map perfect_map_def proper_map_compose quotient_map_compose_eq quotient_map_def*)

lemma *perfect_imp_continuous_map*:
perfect_map X Y f ⟹ *continuous_map X Y f*
using *perfect_map_def* **by** *blast*

lemma *perfect_imp_closed_map*:
perfect_map X Y f ⟹ *closed_map X Y f*
by (*simp add*: *perfect_map_def proper_map_def*)

lemma *perfect_imp_proper_map*:
perfect_map X Y f ⟹ *proper_map X Y f*
by (*simp add*: *perfect_map_def*)

lemma *perfect_imp_surjective_map*:
perfect_map X Y f ⟹ *f* ‘ (*topspace X*) = *topspace Y*
by (*simp add*: *perfect_map_def*)

lemma *perfect_map_from_composition_left*:
assumes *perfect_map X Z* (*g* ∘ *f*) **and** *continuous_map X Y f*
and *continuous_map Y Z g* **and** *f* ‘ *topspace X* = *topspace Y*
shows *perfect_map Y Z g*
using *assms* **unfolding** *perfect_map_def*
by (*metis image_comp proper_map_from_composition_left*)

end

1.14 *F*-Sigma and *G*-Delta sets in a Topological Space

An *F*-sigma set is a countable union of closed sets; a *G*-delta set is the dual.

theory *FSigma*
imports *Abstract_Topology*
begin

definition *fsigma_in*
where *fsigma_in X ≡ countable union_of closedin X*

definition *gdelta_in*
where *gdelta_in X ≡ (countable intersection_of openin X) relative_to topspace X*

lemma *fsigma_in_ascending*:
fsigma_in X S ⟷ (∃ C. (∀ n. closedin X (C n)) ∧ (∀ n. C n ⊆ C(Suc n)) ∧ ⋃ (range C) = S)
unfolding *fsigma_in_def*
by (*metis closedin_Un countable_union_of_ascending closedin_empty*)

lemma *gdelta_in_alt*:
gdelta_in X S ⟷ S ⊆ topspace X ∧ (countable intersection_of openin X) S
proof −
have (*countable intersection_of openin X*) (*topspace X*)
by (*simp add*: *countable_intersection_of_inc*)
then show *?thesis*
unfolding *gdelta_in_def*
by (*metis countable_intersection_of_inter relative_to_def relative_to_imp_subset relative_to_subset_inc*)
qed

lemma *fsigma_in_subset*: *fsigma_in X S ⟹ S ⊆ topspace X*
using *closedin_subset* **by** (*fastforce simp*: *fsigma_in_def union_of_def subset_iff*)

lemma *gdelta_in_subset*: *gdelta_in X S ⟹ S ⊆ topspace X*
by (*simp add*: *gdelta_in_alt*)

lemma *closed_imp_fsigma_in*: *closedin X S ⟹ fsigma_in X S*
by (*simp add*: *countable_union_of_inc fsigma_in_def*)

lemma *open_imp_gdelta_in*: *openin X S ⟹ gdelta_in X S*
by (*simp add*: *countable_intersection_of_inc gdelta_in_alt openin_subset*)

lemma *fsigma_in_empty* [*iff*]: *fsigma_in X {}*
by (*simp add*: *closed_imp_fsigma_in*)

lemma *gdelta_in_empty* [*iff*]: *gdelta_in X {}*
by (*simp add*: *open_imp_gdelta_in*)

lemma *fsigma_in_topspace* [*iff*]: *fsigma_in X (topspace X)*
by (*simp add*: *closed_imp_fsigma_in*)

lemma *gdelta_in_topspace* [*iff*]: *gdelta_in X (topspace X)*
by (*simp add*: *open_imp_gdelta_in*)

lemma *fsigma_in_Union*:
  ⟦*countable T*; ⋀*S*. *S* ∈ *T* ⟹ *fsigma_in X S*⟧ ⟹ *fsigma_in X* (⋃ *T*)
  by (*simp add*: *countable_union_of_Union fsigma_in_def*)

lemma *fsigma_in_Un*:
  ⟦*fsigma_in X S*; *fsigma_in X T*⟧ ⟹ *fsigma_in X* (*S* ∪ *T*)
  by (*simp add*: *countable_union_of_Un fsigma_in_def*)

lemma *fsigma_in_Int*:
  ⟦*fsigma_in X S*; *fsigma_in X T*⟧ ⟹ *fsigma_in X* (*S* ∩ *T*)
  by (*simp add*: *closedin_Int countable_union_of_Int fsigma_in_def*)

lemma *gdelta_in_Inter*:
  ⟦*countable T*; *T*≠{}; ⋀*S*. *S* ∈ *T* ⟹ *gdelta_in X S*⟧ ⟹ *gdelta_in X* (⋂ *T*)
  by (*simp add*: *Inf_less_eq countable_intersection_of_Inter gdelta_in_alt*)

lemma *gdelta_in_Int*:
  ⟦*gdelta_in X S*; *gdelta_in X T*⟧ ⟹ *gdelta_in X* (*S* ∩ *T*)
  by (*simp add*: *countable_intersection_of_inter gdelta_in_alt le_infI2*)

lemma *gdelta_in_Un*:
  ⟦*gdelta_in X S*; *gdelta_in X T*⟧ ⟹ *gdelta_in X* (*S* ∪ *T*)
  by (*simp add*: *countable_intersection_of_union gdelta_in_alt openin_Un*)

lemma *fsigma_in_diff*:
  assumes *fsigma_in X S gdelta_in X T*
  shows *fsigma_in X* (*S* − *T*)
proof −
  have [*simp*]: *S* − (*S* ∩ *T*) = *S* − *T* **for** *S T* :: *'a set*
    by *auto*
  have [*simp*]: *topspace X* − ⋂*𝒯* = (⋃ *T*∈*𝒯*. *topspace X* − *T*) **for** *𝒯*
    by *auto*
  have ⋀*𝒯*. ⟦*countable 𝒯*; *𝒯* ⊆ *Collect* (*openin X*)⟧ ⟹
          (*countable union_of closedin X*) (⋃ ((−) (*topspace X*) ‘ *𝒯*))
   by (*metis Ball_Collect countable_union_of_UN countable_union_of_inc openin_closedin_eq*)
  then have ∀ *S*. *gdelta_in X S* ⟶ *fsigma_in X* (*topspace X* − *S*)
   by (*simp add*: *fsigma_in_def gdelta_in_def all_relative_to all_intersection_of*
del: *UN_simps*)
  then show *?thesis*
   by (*metis Diff_Int2 Diff_Int_distrib2 assms fsigma_in_Int fsigma_in_subset*
inf.absorb_iff2)
qed

lemma *gdelta_in_diff*:
  assumes *gdelta_in X S fsigma_in X T*
  shows *gdelta_in X* (*S* − *T*)
proof −
  have [*simp*]: *topspace X* − ⋃*𝒯* = *topspace X* ∩ (⋂ *T*∈*𝒯*. *topspace X* − *T*) **for**
𝒯

by *auto*
have $\bigwedge\mathcal{T}$. $[\![countable\ \mathcal{T};\ \mathcal{T} \subseteq Collect\ (closedin\ X)]\!]$
$\Longrightarrow$ (*countable intersection_of openin X relative_to topspace X*)
$(topspace\ X \cap \bigcap\ ((-)\ (topspace\ X)\ `\ \mathcal{T}))$
by (*metis Ball_Collect closedin_def countable_intersection_of_INT countable_intersection_of_inc relative_to_inc*)
then have $\forall S.$ *fsigma_in X S* $\longrightarrow$ *gdelta_in X* (*topspace X* $-$ *S*)
by (*simp add*: *fsigma_in_def gdelta_in_def all_union_of del*: *INT_simps*)
then show *?thesis*
by (*metis Diff_Int2 Diff_Int_distrib2 assms gdelta_in_Int gdelta_in_alt inf.orderE inf_commute*)
qed

lemma *gdelta_in_fsigma_in*:
gdelta_in X S $\longleftrightarrow$ $S \subseteq$ *topspace X* $\wedge$ *fsigma_in X* (*topspace X* $-$ *S*)
by (*metis double_diff fsigma_in_diff fsigma_in_topspace gdelta_in_alt gdelta_in_diff gdelta_in_topspace*)

lemma *fsigma_in_gdelta_in*:
fsigma_in X S $\longleftrightarrow$ $S \subseteq$ *topspace X* $\wedge$ *gdelta_in X* (*topspace X* $-$ *S*)
by (*metis Diff_Diff_Int fsigma_in_subset gdelta_in_fsigma_in inf.absorb_iff2*)

lemma *gdelta_in_descending*:
gdelta_in X S $\longleftrightarrow$ $(\exists C.\ (\forall n.\ openin\ X\ (C\ n)) \wedge (\forall n.\ C(Suc\ n) \subseteq C\ n) \wedge \bigcap(range\ C) = S)$ (**is** *?lhs=?rhs*)
proof
assume *?lhs*
then obtain *C* **where** *C*: $S \subseteq$ *topspace X* $\bigwedge n.$ *closedin X* (*C n*)
$\bigwedge n.\ C\ n \subseteq C(Suc\ n) \bigcup(range\ C) = topspace\ X - S$
by (*meson fsigma_in_ascending gdelta_in_fsigma_in*)
define *D* **where** $D \equiv \lambda n.\ topspace\ X - C\ n$
have $\bigwedge n.\ openin\ X\ (D\ n) \wedge D\ (Suc\ n) \subseteq D\ n$
by (*simp add*: *Diff_mono C D_def openin_diff*)
moreover have $\bigcap(range\ D) = S$
by (*simp add*: *Diff_Diff_Int Int_absorb1 C D_def*)
ultimately show *?rhs*
by *metis*
next
assume *?rhs*
then obtain *C* **where** $S \subseteq$ *topspace X*
and *C*: $\bigwedge n.$ *openin X* (*C n*) $\bigwedge n.\ C(Suc\ n) \subseteq C\ n \bigcap(range\ C) = S$
using *openin_subset* **by** *fastforce*
define *D* **where** $D \equiv \lambda n.\ topspace\ X - C\ n$
have $\bigwedge n.\ closedin\ X\ (D\ n) \wedge D\ n \subseteq D(Suc\ n)$
by (*simp add*: *Diff_mono C closedin_diff D_def*)
moreover have $\bigcup(range\ D) = topspace\ X - S$
using *C D_def* **by** *blast*
ultimately show *?lhs*
by (*metis* ‹$S \subseteq$ *topspace X*› *fsigma_in_ascending gdelta_in_fsigma_in*)

qed

lemma *homeomorphic_map_fsigmaness*:
assumes f: *homeomorphic_map X Y f* **and** $U \subseteq$ *topspace X*
shows *fsigma_in Y* $(f\ `\ U) \longleftrightarrow$ *fsigma_in X U* (**is** *?lhs=?rhs*)
proof −
obtain g **where** g: *homeomorphic_maps X Y f g* **and** g: *homeomorphic_map Y X g*
and *1*: $(\forall x \in$ *topspace X*. $g(f\ x) = x)$ **and** *2*: $(\forall y \in$ *topspace Y*. $f(g\ y) = y)$
using *assms homeomorphic_map_maps* **by** (*metis homeomorphic_maps_map*)
show *?thesis*
proof
assume *?lhs*
then obtain $\mathcal{V}$ **where** *countable* $\mathcal{V}$ **and** $\mathcal{V}$: $\mathcal{V} \subseteq$ *Collect* (*closedin Y*) $\bigcup\mathcal{V} =$ *f`U*
by (*force simp*: *fsigma_in_def union_of_def*)
define $\mathcal{U}$ **where** $\mathcal{U} \equiv$ *image* (*image g*) $\mathcal{V}$
have *countable* $\mathcal{U}$
using $\mathcal{U}$*_def* ‹*countable* $\mathcal{V}$› **by** *blast*
moreover have $\mathcal{U} \subseteq$ *Collect* (*closedin X*)
using *f g homeomorphic_map_closedness_eq* $\mathcal{V}$ **unfolding** $\mathcal{U}$*_def* **by** *blast*
moreover have $\bigcup\mathcal{U} \subseteq U$
unfolding $\mathcal{U}$*_def* **by** (*smt* (*verit*) *assms 1* $\mathcal{V}$ *image_Union image_iff in_mono subsetI*)
moreover have $U \subseteq \bigcup\mathcal{U}$
unfolding $\mathcal{U}$*_def* **using** *assms* $\mathcal{V}$
by (*smt* (*verit*, *del_insts*) *1 imageI image_Union subset_iff*)
ultimately show *?rhs*
by (*metis fsigma_in_def subset_antisym union_of_def*)
next
assume *?rhs*
then obtain $\mathcal{V}$ **where** *countable* $\mathcal{V}$ **and** $\mathcal{V}$: $\mathcal{V} \subseteq$ *Collect* (*closedin X*) $\bigcup\mathcal{V} = U$
by (*auto simp*: *fsigma_in_def union_of_def*)
define $\mathcal{U}$ **where** $\mathcal{U} \equiv$ *image* (*image f*) $\mathcal{V}$
have *countable* $\mathcal{U}$
using $\mathcal{U}$*_def* ‹*countable* $\mathcal{V}$› **by** *blast*
moreover have $\mathcal{U} \subseteq$ *Collect* (*closedin Y*)
using *f g homeomorphic_map_closedness_eq* $\mathcal{V}$ **unfolding** $\mathcal{U}$*_def* **by** *blast*
moreover have $\bigcup\mathcal{U} =$ *f`U*
unfolding $\mathcal{U}$*_def* **using** $\mathcal{V}$ **by** *blast*
ultimately show *?lhs*
by (*metis fsigma_in_def union_of_def*)
qed
qed

lemma *homeomorphic_map_fsigmaness_eq*:
homeomorphic_map X Y f
$\Longrightarrow$ (*fsigma_in X U* $\longleftrightarrow$ $U \subseteq$ *topspace X* $\wedge$ *fsigma_in Y* $(f\ `\ U)$)
by (*metis fsigma_in_subset homeomorphic_map_fsigmaness*)

lemma *homeomorphic_map_gdeltaness*:
assumes *homeomorphic_map X Y f U* ⊆ *topspace X*
shows *gdelta_in Y* (*f* ‘ *U*) ⟷ *gdelta_in X U*
proof −
have *topspace Y* − *f* ‘ *U* = *f* ‘ (*topspace X* − *U*)
by (*metis* (*no_types*, *lifting*) *Diff_subset assms homeomorphic_eq_everything_map inj_on_image_set_diff*)
then show *?thesis*
using *assms homeomorphic_imp_surjective_map*
by (*fastforce simp*: *gdelta_in_fsigma_in homeomorphic_map_fsigmaness_eq*)
qed

lemma *homeomorphic_map_gdeltaness_eq*:
homeomorphic_map X Y f
⟹ *gdelta_in X U* ⟷ *U* ⊆ *topspace X* ∧ *gdelta_in Y* (*f* ‘ *U*)
by (*meson gdelta_in_subset homeomorphic_map_gdeltaness*)

lemma *fsigma_in_relative_to*:
(*fsigma_in X relative_to S*) = *fsigma_in* (*subtopology X S*)
by (*simp add*: *fsigma_in_def countable_union_of_relative_to closedin_relative_to*)

lemma *fsigma_in_subtopology*:
fsigma_in (*subtopology X U*) *S* ⟷ (∃ *T*. *fsigma_in X T* ∧ *S* = *T* ∩ *U*)
by (*metis fsigma_in_relative_to inf_commute relative_to_def*)

lemma *gdelta_in_relative_to*:
(*gdelta_in X relative_to S*) = *gdelta_in* (*subtopology X S*)
apply (*simp add*: *gdelta_in_def*)
by (*metis countable_intersection_of_relative_to openin_relative_to subtopology_restrict*)

lemma *gdelta_in_subtopology*:
gdelta_in (*subtopology X U*) *S* ⟷ (∃ *T*. *gdelta_in X T* ∧ *S* = *T* ∩ *U*)
by (*metis gdelta_in_relative_to inf_commute relative_to_def*)

lemma *fsigma_in_fsigma_subtopology*:
fsigma_in X S ⟹ (*fsigma_in* (*subtopology X S*) *T* ⟷ *fsigma_in X T* ∧ *T* ⊆ *S*)
by (*metis fsigma_in_Int fsigma_in_gdelta_in fsigma_in_subtopology inf.orderE topspace_subtopology_subset*)

lemma *gdelta_in_gdelta_subtopology*:
gdelta_in X S ⟹ (*gdelta_in* (*subtopology X S*) *T* ⟷ *gdelta_in X T* ∧ *T* ⊆ *S*)
by (*metis gdelta_in_Int gdelta_in_subset gdelta_in_subtopology inf.orderE topspace_subtopology_sub*

end

1.15 Disjoint sum of arbitarily many spaces

theory *Sum_Topology*
imports *Abstract_Topology*
begin

definition *sum_topology* :: ('*a* ⇒ '*b topology*) ⇒ '*a set* ⇒ ('*a* × '*b*) *topology* **where**
sum_topology X I ≡
topology (λ*U*. *U* ⊆ *Sigma I* (*topspace* ∘ *X*) ∧ (∀ *i* ∈ *I*. *openin* (*X i*) {*x*. (*i*,*x*) ∈ *U*}))

lemma *is_sum_topology*: *istopology* (λ*U*. *U* ⊆ *Sigma I* (*topspace* ∘ *X*) ∧ (∀ *i*∈*I*. *openin* (*X i*) {*x*. (*i*, *x*) ∈ *U*}))
proof −
have *1*: {*x*. (*i*, *x*) ∈ *S* ∩ *T*} = {*x*. (*i*, *x*) ∈ *S*} ∩ {*x*::'*b*. (*i*, *x*) ∈ *T*} **for** *S T* **and** *i*::'*a*
by *auto*
have *2*: {*x*. (*i*, *x*) ∈ ⋃ *K*} = (⋃*K*∈*K*. {*x*::'*b*. (*i*, *x*) ∈ *K*}) **for** *K* **and** *i*::'*a*
by *auto*
show *?thesis*
unfolding *istopology_def 1 2* **by** *blast*
qed

lemma *openin_sum_topology*:
openin (*sum_topology X I*) *U* ⟷
U ⊆ *Sigma I* (*topspace* ∘ *X*) ∧ (∀ *i* ∈ *I*. *openin* (*X i*) {*x*. (*i*,*x*) ∈ *U*})
by (*auto simp*: *sum_topology_def is_sum_topology*)

lemma *openin_disjoint_union*:
openin (*sum_topology X I*) (*Sigma I U*) ⟷ (∀ *i* ∈ *I*. *openin* (*X i*) (*U i*))
using *openin_subset* **by** (*force simp*: *openin_sum_topology*)

lemma *topspace_sum_topology* [*simp*]:
topspace(*sum_topology X I*) = *Sigma I* (*topspace* ∘ *X*)
by (*metis comp_apply openin_disjoint_union openin_subset openin_sum_topology openin_topspace subset_antisym*)

lemma *openin_sum_topology_alt*:
openin (*sum_topology X I*) *U* ⟷ (∃ *T*. *U* = *Sigma I T* ∧ (∀ *i* ∈ *I*. *openin* (*X i*) (*T i*)))
by (*bestsimp simp*: *openin_sum_topology dest*: *openin_subset*)

lemma *forall_openin_sum_topology*:
(∀ *U*. *openin* (*sum_topology X I*) *U* ⟶ *P U*) ⟷ (∀ *T*. (∀ *i* ∈ *I*. *openin* (*X i*) (*T i*)) ⟶ *P*(*Sigma I T*))
by (*auto simp*: *openin_sum_topology_alt*)

lemma *exists_openin_sum_topology*:

(∃ *U*. *openin* (*sum_topology X I*) *U* ∧ *P U*) ⟷
(∃ *T*. (∀ *i* ∈ *I*. *openin* (*X i*) (*T i*)) ∧ *P*(*Sigma I T*))
by (*auto simp*: *openin_sum_topology_alt*)

lemma *closedin_sum_topology*:
closedin (*sum_topology X I*) *U* ⟷ *U* ⊆ *Sigma I* (*topspace* ∘ *X*) ∧ (∀ *i* ∈ *I*. *closedin* (*X i*) {*x*. (*i*,*x*) ∈ *U*})
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have *U*: *U* ⊆ *Sigma I* (*topspace* ∘ *X*)
using *closedin_subset* **by** *fastforce*
then have ∀ *i*∈*I*. {*x*. (*i*, *x*) ∈ *U*} ⊆ *topspace* (*X i*)
by *fastforce*
moreover have *openin* (*X i*) (*topspace* (*X i*) − {*x*. (*i*, *x*) ∈ *U*}) **if** *i*∈*I* **for** *i*
apply (*subst openin_subopen*)
using *L that* **unfolding** *closedin_def openin_sum_topology*
by (*bestsimp simp*: *o_def subset_iff*)
ultimately show *?rhs*
by (*simp add*: *U closedin_def*)
next
assume *R*: *?rhs*
then have *openin* (*X i*) {*x*. (*i*, *x*) ∈ *topspace* (*sum_topology X I*) − *U*} **if** *i*∈*I* **for** *i*
apply (*subst openin_subopen*)
using *that* **unfolding** *closedin_def openin_sum_topology*
by (*bestsimp simp*: *o_def subset_iff*)
then show *?lhs*
by (*simp add*: *R closedin_def openin_sum_topology*)
qed

lemma *closedin_disjoint_union*:
closedin (*sum_topology X I*) (*Sigma I U*) ⟷ (∀ *i* ∈ *I*. *closedin* (*X i*) (*U i*))
using *closedin_subset* **by** (*force simp*: *closedin_sum_topology*)

lemma *closedin_sum_topology_alt*:
closedin (*sum_topology X I*) *U* ⟷ (∃ *T*. *U* = *Sigma I T* ∧ (∀ *i* ∈ *I*. *closedin* (*X i*) (*T i*)))
using *closedin_subset* **unfolding** *closedin_sum_topology* **by** *auto blast*

lemma *forall_closedin_sum_topology*:
(∀ *U*. *closedin* (*sum_topology X I*) *U* ⟶ *P U*) ⟷
(∀ *t*. (∀ *i* ∈ *I*. *closedin* (*X i*) (*t i*)) ⟶ *P*(*Sigma I t*))
by (*metis closedin_sum_topology_alt*)

lemma *exists_closedin_sum_topology*:
(∃ *U*. *closedin* (*sum_topology X I*) *U* ∧ *P U*) ⟷
(∃ *T*. (∀ *i* ∈ *I*. *closedin* (*X i*) (*T i*)) ∧ *P*(*Sigma I T*))
by (*simp add*: *closedin_sum_topology_alt*) *blast*

lemma *open_map_component_injection*:
  $i \in I \Longrightarrow$ *open_map* (*X i*) (*sum_topology X I*) ($\lambda x.$ (*i*,*x*))
  unfolding *open_map_def openin_sum_topology*
  using *openin_subset openin_subopen* **by** (*force simp*: *image_iff*)

lemma *closed_map_component_injection*:
  assumes $i \in I$
  shows *closed_map*(*X i*) (*sum_topology X I*) ($\lambda x.$ (*i*,*x*))
proof −
  have *closedin* (*X j*) {*x*. $j = i \wedge x \in U$}
    if $\bigwedge U\ S.$ *closedin U S* $\Longrightarrow S \subseteq$ *topspace U* **and** *closedin* (*X i*) *U* **and** $j \in I$
for *U j*
    using *that* **by** (*cases j*=*i*) *auto*
  then show *?thesis*
   using *closedin_subset assms* **by** (*force simp*: *closed_map_def closedin_sum_topology image_iff*)
qed

lemma *continuous_map_component_injection*:
  $i \in I \Longrightarrow$ *continuous_map*(*X i*) (*sum_topology X I*) ($\lambda x.$ (*i*,*x*))
  apply (*clarsimp simp*: *continuous_map_def openin_sum_topology*)
  by (*smt* (*verit*, *best*) *Collect_cong mem_Collect_eq openin_subset subsetD*)

lemma *subtopology_sum_topology*:
  subtopology (*sum_topology X I*) (*Sigma I S*) =
        sum_topology ($\lambda i.$ *subtopology* (*X i*) (*S i*)) *I*
  unfolding *topology_eq*
proof (*intro strip iffI*)
  fix *T*
  assume $*$: *openin* (*subtopology* (*sum_topology X I*) (*Sigma I S*)) *T*
  then obtain *U* **where** *U*: $\forall i{\in}I.$ *openin* (*X i*) (*U i*) $\wedge$ *T* = *Sigma I U* $\cap$ *Sigma I S*
    by (*auto simp*: *openin_subtopology openin_sum_topology_alt*)
  have *T* = (*SIGMA i*:*I*. *U i* $\cap$ *S i*)
  proof
    show *T* $\subseteq$ (*SIGMA i*:*I*. *U i* $\cap$ *S i*)
      by (*metis* $*$ *SigmaE Sigma_Int_distrib2 U openin_imp_subset subset_iff*)
    show (*SIGMA i*:*I*. *U i* $\cap$ *S i*) $\subseteq$ *T*
      using *U* **by** *fastforce*
  qed
  then show *openin* (*sum_topology* ($\lambda i.$ *subtopology* (*X i*) (*S i*)) *I*) *T*
    by (*simp add*: *U openin_disjoint_union openin_subtopology_Int*)
next
  fix *T*
  assume $*$: *openin* (*sum_topology* ($\lambda i.$ *subtopology* (*X i*) (*S i*)) *I*) *T*
  then obtain *U* **where** $\forall i{\in}I.\ \exists T.$ *openin* (*X i*) *T* $\wedge$ *U i* = *T* $\cap$ *S i* **and** *Teq*: *T* = *Sigma I U*
    by (*auto simp*: *openin_subtopology openin_sum_topology_alt*)

```
  then obtain B where ⋀i. i∈I ⟹ openin (X i) (B i) ∧ U i = B i ∩ S i
    by metis
  then show openin (subtopology (sum_topology X I) (Sigma I S)) T
    by (auto simp: openin_subtopology openin_sum_topology_alt Teq)
qed

lemma embedding_map_component_injection:
   i ∈ I ⟹ embedding_map (X i) (sum_topology X I) (λx. (i,x))
 by (metis injective_open_imp_embedding_map continuous_map_component_injection
          open_map_component_injection inj_onI prod.inject)

lemma topological_property_of_sum_component:
  assumes major: P (sum_topology X I)
    and minor: ⋀X S. ⟦P X; closedin X S; openin X S⟧ ⟹ P(subtopology X S)
    and PQ:  ⋀X Y. X homeomorphic_space Y ⟹ (P X ⟷ Q Y)
  shows (∀ i ∈ I. Q(X i))
proof −
  have Q(X i) if i ∈ I for i
  proof −
    have P(subtopology (sum_topology X I) (Pair i ` topspace (X i)))
    by (meson closed_map_component_injection closed_map_def closedin_topspace
major minor open_map_component_injection open_map_def openin_topspace that)
    then show ?thesis
    by (metis PQ embedding_map_component_injection embedding_map_imp_homeomorphic_space
homeomorphic_space_sym that)
  qed
  then show ?thesis by metis
qed

end
```

Chapter 2

Topology

theory *Elementary_Topology*
imports
HOL−Library.Set_Idioms
HOL−Library.Disjoint_Sets
Product_Vector
begin

2.1 Elementary Topology

Affine transformations of intervals

lemma *real_affinity_le*: $0 < m \Longrightarrow m * x + c \leq y \longleftrightarrow x \leq inverse\ m * y + -(c\ /\ m)$
for $m :: 'a$::*linordered_field*
by (*simp add*: *field_simps*)

lemma *real_le_affinity*: $0 < m \Longrightarrow y \leq m * x + c \longleftrightarrow inverse\ m * y + -(c\ /\ m) \leq x$
for $m :: 'a$::*linordered_field*
by (*simp add*: *field_simps*)

lemma *real_affinity_lt*: $0 < m \Longrightarrow m * x + c < y \longleftrightarrow x < inverse\ m * y + -(c\ /\ m)$
for $m :: 'a$::*linordered_field*
by (*simp add*: *field_simps*)

lemma *real_lt_affinity*: $0 < m \Longrightarrow y < m * x + c \longleftrightarrow inverse\ m * y + -(c\ /\ m) < x$
for $m :: 'a$::*linordered_field*
by (*simp add*: *field_simps*)

lemma *real_affinity_eq*: $m \neq 0 \Longrightarrow m * x + c = y \longleftrightarrow x = inverse\ m * y + -(c\ /\ m)$
for $m :: 'a$::*linordered_field*

by (*simp add*: *field_simps*)

lemma *real_eq_affinity*: $m \neq 0 \Longrightarrow y = m * x + c \longleftrightarrow \mathit{inverse}\ m * y + - (c / m) = x$
for $m :: 'a$::*linordered_field*
by (*simp add*: *field_simps*)

2.1.1 Topological Basis

context *topological_space*
begin

definition *topological_basis* $B \longleftrightarrow$
$(\forall\, b{\in}B.\ \mathit{open}\ b) \wedge (\forall\, x.\ \mathit{open}\ x \longrightarrow (\exists\, B'.\ B' \subseteq B \wedge \bigcup B' = x))$

lemma *topological_basis*:
topological_basis $B \longleftrightarrow (\forall\, x.\ \mathit{open}\ x \longleftrightarrow (\exists\, B'.\ B' \subseteq B \wedge \bigcup B' = x))$
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*meson local.open_Union subsetD topological_basis_def*)
show *?rhs* $\Longrightarrow$ *?lhs*
unfolding *topological_basis_def*
by (*metis cSup_singleton empty_subsetI insert_subset*)
qed

lemma *topological_basis_iff*:
assumes $\bigwedge B'.\ B' \in B \Longrightarrow \mathit{open}\ B'$
shows *topological_basis* $B \longleftrightarrow (\forall\, O'.\ \mathit{open}\ O' \longrightarrow (\forall\, x{\in}O'.\ \exists\, B'{\in}B.\ x \in B' \wedge B' \subseteq O'))$
(**is** _ $\longleftrightarrow$ *?rhs*)
proof *safe*
fix O' **and** $x::'a$
assume H: *topological_basis* B *open* O' $x \in O'$
then have $(\exists\, B'{\subseteq}B.\ \bigcup B' = O')$ **by** (*simp add*: *topological_basis_def*)
then obtain B' **where** $B' \subseteq B$ $O' = \bigcup B'$ **by** *auto*
then show $\exists\, B'{\in}B.\ x \in B' \wedge B' \subseteq O'$ **using** H **by** *auto*
next
assume H: *?rhs*
show *topological_basis* B
using *assms* **unfolding** *topological_basis_def*
proof *safe*
fix $O' :: 'a\ \mathit{set}$
assume *open* O'
with H **obtain** f **where** $\forall\, x{\in}O'.\ f\ x \in B \wedge x \in f\ x \wedge f\ x \subseteq O'$
by (*force intro*: *bchoice simp*: *Bex_def*)
then show $\exists\, B'{\subseteq}B.\ \bigcup B' = O'$
by (*auto intro*: *exI*[**where** $x{=}\{f\ x\ |x.\ x \in O'\}$])
qed

qed

lemma *topological_basisI*:
assumes $\bigwedge B'.\ B' \in B \Longrightarrow \mathit{open}\ B'$
and $\bigwedge O'\ x.\ \mathit{open}\ O' \Longrightarrow x \in O' \Longrightarrow \exists\, B' \in B.\ x \in B' \wedge B' \subseteq O'$
shows *topological_basis B*
by (*simp add*: *assms topological_basis_iff*)

lemma *topological_basisE*:
fixes O'
assumes *topological_basis B*
and *open* O'
and $x \in O'$
obtains B' **where** $B' \in B\ x \in B'\ B' \subseteq O'$
by (*metis assms topological_basis_def topological_basis_iff*)

lemma *topological_basis_open*:
assumes *topological_basis B* **and** $X \in B$
shows *open X*
using *assms* **by** (*simp add*: *topological_basis_def*)

lemma *topological_basis_imp_subbasis*:
assumes *B*: *topological_basis B*
shows *open* = *generate_topology B*
proof (*intro ext iffI*)
fix $S :: 'a\ \mathit{set}$
assume *open S*
with *B* **obtain** B' **where** $B' \subseteq B\ S = \bigcup B'$
unfolding *topological_basis_def* **by** *blast*
then show *generate_topology B S*
by (*auto intro*: *generate_topology.intros dest*: *topological_basis_open*)
next
fix $S :: 'a\ \mathit{set}$
assume *generate_topology B S*
then show *open S*
by *induct* (*auto dest*: *topological_basis_open*[*OF B*])
qed

lemma *basis_dense*:
fixes $B :: 'a\ \mathit{set}\ \mathit{set}$
and $f :: 'a\ \mathit{set} \Rightarrow 'a$
assumes *topological_basis B* **and** $\bigwedge B'.\ B' \neq \{\} \Longrightarrow f\ B' \in B'$
shows $\forall X.\ \mathit{open}\ X \longrightarrow X \neq \{\} \longrightarrow (\exists\, B' \in B.\ f\ B' \in X)$
by (*metis assms equals0D in_mono topological_basisE*)

end

lemma *topological_basis_prod*:
assumes *A*: *topological_basis A*

```
    and B: topological_basis B
  shows topological_basis ((λ(a, b). a × b) ‘ (A × B))
proof −
  have ∃ X⊆A × B. (⋃(a,b)∈X. a × b) = S if open S for S
  proof −
    have (x, y) ∈ (⋃(a, b)∈{X ∈ A × B. fst X × snd X ⊆ S}. a × b)
      if xy: (x, y) ∈ S for x y
    proof −
      obtain a b where a: open ax ∈ a and b: open b y ∈ b and a × b ⊆ S
        by (metis open_prod_elim[OF ‹open S›] xy mem_Sigma_iff)
      moreover obtain A0 where A0 ∈ A x ∈ A0 A0 ⊆ a
        using A a b topological_basisE by blast
      moreover
      from B b obtain B0 where B0 ∈ B y ∈ B0 B0 ⊆ b
        by (rule topological_basisE)
      ultimately show ?thesis
        by (intro UN_I[of (A0, B0)]) auto
    qed
    then have (⋃(a, b)∈{x ∈ A × B. fst x × snd x ⊆ S}. a × b) = S
      by force
    then show ?thesis
      using subset_eq by force
  qed
  with A B show ?thesis
    unfolding topological_basis_def
    by (smt (verit) SigmaE imageE image_mono open_Times case_prod_conv)
qed
```

2.1.2 Countable Basis

```
locale countable_basis = topological_space p for p::'a set ⇒ bool +
  fixes B :: 'a set set
  assumes is_basis: topological_basis B
    and countable_basis: countable B
begin

lemma open_countable_basis_ex:
  assumes p X
  shows ∃ B' ⊆ B. X = ⋃ B'
  using assms countable_basis is_basis
  unfolding topological_basis_def by blast

lemma open_countable_basisE:
  assumes p X
  obtains B' where B' ⊆ B X = ⋃ B'
  using assms open_countable_basis_ex by auto

lemma countable_dense_exists:
  ∃ D::'a set. countable D ∧ (∀ X. p X ⟶ X ≠ {} ⟶ (∃ d ∈ D. d ∈ X))
```

proof −
let *?f* = (λ*B'*. *SOME x*. *x* ∈ *B'*)
have *countable* (*?f* ‘ *B*) **using** *countable_basis* **by** *simp*
with *basis_dense*[*OF is_basis*, *of ?f*] **show** *?thesis*
by (*intro exI*[**where** *x*=*?f* ‘ *B*]) (*metis* (*mono_tags*) *all_not_in_conv imageI someI*)
qed

lemma *countable_dense_setE*:
obtains *D* :: *'a set*
where *countable D* ⋀*X*. *p X* ⟹ *X* ≠ {} ⟹ ∃ *d* ∈ *D*. *d* ∈ *X*
using *countable_dense_exists* **by** *blast*

end

lemma *countable_basis_openI*: *countable_basis open B*
if *countable B topological_basis B*
using *that*
by *unfold_locales*
(*simp_all add*: *topological_basis topological_space.topological_basis topological_space_axioms*)

lemma (**in** *first_countable_topology*) *first_countable_basisE*:
fixes *x* :: *'a*
obtains 𝒜 **where** *countable* 𝒜 ⋀*A*. *A* ∈ 𝒜 ⟹ *x* ∈ *A* ⋀*A*. *A* ∈ 𝒜 ⟹ *open A*
⋀*S*. *open S* ⟹ *x* ∈ *S* ⟹ (∃ *A*∈𝒜. *A* ⊆ *S*)
proof −
obtain 𝒜 **where** 𝒜: (∀ *i*::*nat*. *x* ∈ 𝒜 *i* ∧ *open* (𝒜 *i*)) (∀ *S*. *open S* ∧ *x* ∈ *S* ⟶ (∃ *i*. 𝒜 *i* ⊆ *S*))
using *first_countable_basis*[*of x*] **by** *metis*
moreover have *countable* (*range* 𝒜)
by *simp*
ultimately show *thesis*
by (*smt* (*verit*, *best*) *imageE rangeI that*)
qed

lemma (**in** *first_countable_topology*) *first_countable_basis_Int_stableE*:
obtains 𝒜 **where** *countable* 𝒜 ⋀*A*. *A* ∈ 𝒜 ⟹ *x* ∈ *A* ⋀*A*. *A* ∈ 𝒜 ⟹ *open A*
⋀*S*. *open S* ⟹ *x* ∈ *S* ⟹ (∃ *A*∈𝒜. *A* ⊆ *S*)
⋀*A B*. *A* ∈ 𝒜 ⟹ *B* ∈ 𝒜 ⟹ *A* ∩ *B* ∈ 𝒜
proof *atomize_elim*
obtain ℬ **where** ℬ:
countable ℬ
⋀*B*. *B* ∈ ℬ ⟹ *x* ∈ *B*
⋀*B*. *B* ∈ ℬ ⟹ *open B*
⋀*S*. *open S* ⟹ *x* ∈ *S* ⟹ ∃ *B*∈ℬ. *B* ⊆ *S*
by (*rule first_countable_basisE*) *blast*
define 𝒜 **where** [*abs_def*]:
𝒜 = (λ*N*. ⋂((λ*n*. *from_nat_into* ℬ *n*) ‘ *N*)) ‘ (*Collect finite*::*nat set set*)

```
  then show ∃ 𝒜. countable 𝒜 ∧ (∀ A. A ∈ 𝒜 ⟶ x ∈ A) ∧ (∀ A. A ∈ 𝒜 ⟶ open
A) ∧
       (∀ S. open S ⟶ x ∈ S ⟶ (∃ A∈𝒜. A ⊆ S)) ∧ (∀ A B. A ∈ 𝒜 ⟶ B ∈ 𝒜
⟶ A ∩ B ∈ 𝒜)
  proof (safe intro!: exI[where x=𝒜])
    show countable 𝒜
      unfolding 𝒜_def by (intro countable_image countable_Collect_finite)
    fix A
    assume A ∈ 𝒜
    then show x ∈ A open A
      using ℬ(4)[OF open_UNIV] by (auto simp: 𝒜_def intro: ℬ from_nat_into)
  next
    let ?int = λN. ⋂(from_nat_into ℬ ` N)
    fix A B
    assume A ∈ 𝒜 B ∈ 𝒜
    then obtain N M where A = ?int N B = ?int M finite (N ∪ M)
      by (auto simp: 𝒜_def)
    then show A ∩ B ∈ 𝒜
      by (auto simp: 𝒜_def intro!: image_eqI[where x=N ∪ M])
  next
    fix S
    assume open S x ∈ S
    then obtain a where a: a∈ℬ a ⊆ S using ℬ by blast
    moreover have a∈𝒜
      unfolding 𝒜_def
    proof (rule image_eqI)
      show a = ⋂ (from_nat_into ℬ ` {to_nat_on ℬ a})
        by (simp add: ℬ a)
    qed auto
    ultimately show ∃ a∈𝒜. a ⊆ S
      by blast
  qed
qed

lemma (in topological_space) first_countableI:
  assumes countable 𝒜
    and 1: ⋀A. A ∈ 𝒜 ⟹ x ∈ A ⋀A. A ∈ 𝒜 ⟹ open A
    and 2: ⋀S. open S ⟹ x ∈ S ⟹ ∃ A∈𝒜. A ⊆ S
  shows ∃ 𝒜::nat ⇒ 'a set. (∀ i. x ∈ 𝒜 i ∧ open (𝒜 i)) ∧ (∀ S. open S ∧ x ∈ S
⟶ (∃ i. 𝒜 i ⊆ S))
proof (safe intro!: exI[of _ from_nat_into 𝒜])
  fix i
  have 𝒜 ≠ {} using 2[of UNIV] by auto
  show x ∈ from_nat_into 𝒜 i open (from_nat_into 𝒜 i)
    using range_from_nat_into_subset[OF ‹𝒜 ≠ {}›] 1 by auto
next
  fix S
  assume open S x∈S
  then show ∃ i. from_nat_into 𝒜 i ⊆ S
```

by (*metis 2* ‹*countable* $\mathcal{A}$› *from_nat_into_surj*)
qed

instance *prod* :: (*first_countable_topology*, *first_countable_topology*) *first_countable_topology*
proof
fix x :: $'a \times 'b$
obtain $\mathcal{A}$ **where** $\mathcal{A}$:
countable $\mathcal{A}$
$\bigwedge a.\ a \in \mathcal{A} \Longrightarrow fst\ x \in a$
$\bigwedge a.\ a \in \mathcal{A} \Longrightarrow open\ a$
$\bigwedge S.\ open\ S \Longrightarrow fst\ x \in S \Longrightarrow \exists\, a{\in}\mathcal{A}.\ a \subseteq S$
by (*rule first_countable_basisE*[*of fst x*]) *blast*
obtain B **where** B:
countable B
$\bigwedge a.\ a \in B \Longrightarrow snd\ x \in a$
$\bigwedge a.\ a \in B \Longrightarrow open\ a$
$\bigwedge S.\ open\ S \Longrightarrow snd\ x \in S \Longrightarrow \exists\, a{\in}B.\ a \subseteq S$
by (*rule first_countable_basisE*[*of snd x*]) *blast*
show $\exists\, \mathcal{A}{::}nat \Rightarrow ('a \times 'b)\ set.$
$(\forall\, i.\ x \in \mathcal{A}\ i \wedge open\ (\mathcal{A}\ i)) \wedge (\forall\, S.\ open\ S \wedge x \in S \longrightarrow (\exists\, i.\ \mathcal{A}\ i \subseteq S))$
proof (*rule first_countableI*[*of* $(\lambda(a,\ b).\ a \times b)$ ‘ $(\mathcal{A} \times B)$], *safe*)
fix $a\ b$
assume x: $a \in \mathcal{A}\ b \in B$
show $x \in a \times b$
by (*simp add*: $\mathcal{A}(2)\ B(2)$ *mem_Times_iff x*)
show *open* $(a \times b)$
by (*simp add*: $\mathcal{A}(3)\ B(3)$ *open_Times x*)
next
fix S
assume *open* $S\ x \in S$
then obtain $a'\ b'$ **where** $a'b'$: *open* a' *open* $b'\ x \in a' \times b'\ a' \times b' \subseteq S$
by (*rule open_prod_elim*)
moreover
obtain $a\ b$ **where** $a \in \mathcal{A}\ a \subseteq a'\ b \in B\ b \subseteq b'$
by (*meson* $B(4)\ \mathcal{A}(4)\ a'b'$ *mem_Times_iff*)
ultimately
show $\exists\, a{\in}(\lambda(a,\ b).\ a \times b)$ ‘ $(\mathcal{A} \times B).\ a \subseteq S$
by (*auto intro*!: *bexI*[*of* _ $a \times b$] *bexI*[*of* _ a] *bexI*[*of* _ b])
qed (*simp add*: $\mathcal{A}\ B$)
qed

class *second_countable_topology* = *topological_space* +
assumes *ex_countable_subbasis*:
$\exists\, B{::}'a\ set\ set.\ countable\ B \wedge open = generate\_topology\ B$
begin

lemma *ex_countable_basis*: $\exists\, B{::}'a\ set\ set.\ countable\ B \wedge topological\_basis\ B$
proof −
from *ex_countable_subbasis* **obtain** B **where** B: *countable* B *open* = *gener-*

```
ate_topology B
    by blast
  let ?B = Inter ` {b. finite b ∧ b ⊆ B }

  show ?thesis
  proof (intro exI conjI)
    show countable ?B
      by (intro countable_image countable_Collect_finite_subset B)
    {
      fix S
      assume open S
      then have ∃ B'⊆{b. finite b ∧ b ⊆ B}. (⋃ b∈B'. ⋂ b) = S
        unfolding B
      proof induct
        case UNIV
        show ?case by (intro exI[of _ {{}}]) simp
      next
        case (Int a b)
       then obtain x y where x: a = ⋃(Inter ` x) ⋀i. i ∈ x ⟹ finite i ∧ i ⊆ B
          and y: b = ⋃(Inter ` y) ⋀i. i ∈ y ⟹ finite i ∧ i ⊆ B
          by blast
        show ?case
          unfolding x y Int_UN_distrib2
          by (intro exI[of _ {i ∪ j| i j.  i ∈ x ∧ j ∈ y}]) (auto dest: x(2) y(2))
      next
        case (UN K)
       then have ∀ k∈K. ∃ B'⊆{b. finite b ∧ b ⊆ B}. ⋃ (Inter ` B') = k by auto
       then obtain k where
            ∀ ka∈K. k ka ⊆ {b. finite b ∧ b ⊆ B} ∧ ⋃(Inter ` (k ka)) = ka
          unfolding bchoice_iff ..
       then show ∃ B'⊆{b. finite b ∧ b ⊆ B}. ⋃ (Inter ` B') = ⋃ K
          by (intro exI[of _ ⋃(k ` K)]) auto
      next
        case (Basis S)
        then show ?case
          by (intro exI[of _ {{S}}]) auto
      qed
      then have (∃ B'⊆Inter ` {b. finite b ∧ b ⊆ B}. ⋃ B' = S)
        unfolding subset_image_iff by blast }
    then show topological_basis ?B
      unfolding topological_basis_def
      by (safe intro!: open_Inter)
         (simp_all add: B generate_topology.Basis subset_eq)
  qed
qed


end
```

lemma *univ_second_countable*:
obtains $\mathcal{B}$ :: *'a*::*second_countable_topology set set*
where *countable* $\mathcal{B}$ $\bigwedge C.\ C \in \mathcal{B} \Longrightarrow$ *open* C
$\bigwedge S.$ *open* $S \Longrightarrow \exists U.\ U \subseteq \mathcal{B} \wedge S = \bigcup U$
by (*metis ex_countable_basis topological_basis_def*)

proposition *Lindelof*:
fixes $\mathcal{F}$ :: *'a*::*second_countable_topology set set*
assumes $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *open* S
obtains $\mathcal{F}'$ **where** $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$
proof −
obtain $\mathcal{B}$ :: *'a set set*
where *countable* $\mathcal{B}$ $\bigwedge C.\ C \in \mathcal{B} \Longrightarrow$ *open* C
and $\mathcal{B}$: $\bigwedge S.$ *open* $S \Longrightarrow \exists U.\ U \subseteq \mathcal{B} \wedge S = \bigcup U$
using *univ_second_countable* **by** *blast*
define $\mathcal{D}$ **where** $\mathcal{D} \equiv \{S.\ S \in \mathcal{B} \wedge (\exists U.\ U \in \mathcal{F} \wedge S \subseteq U)\}$
have *countable* $\mathcal{D}$
by (*simp add*: $\mathcal{D}$_*def* ‹*countable* $\mathcal{B}$›)
have $\bigwedge S.\ \exists U.\ S \in \mathcal{D} \longrightarrow U \in \mathcal{F} \wedge S \subseteq U$
by (*simp add*: $\mathcal{D}$_*def*)
then obtain G **where** G: $\bigwedge S.\ S \in \mathcal{D} \longrightarrow G\ S \in \mathcal{F} \wedge S \subseteq G\ S$
by *metis*
have $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{D}$
unfolding $\mathcal{D}$_*def* **by** (*blast dest*: $\mathcal{F}$ $\mathcal{B}$)
moreover have $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{F}$
using $\mathcal{D}$_*def* **by** *blast*
ultimately have *eq1*: $\bigcup \mathcal{F} = \bigcup \mathcal{D}$ **..**
moreover have $\bigcup \mathcal{D} = \bigcup\ (G\ `\ \mathcal{D})$
using G *eq1* **by** *auto*
ultimately show *?thesis*
by (*metis* G ‹*countable* $\mathcal{D}$› *countable_image image_subset_iff that*)
qed

lemma *countable_disjoint_open_subsets*:
fixes $\mathcal{F}$ :: *'a*::*second_countable_topology set set*
assumes $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *open* S **and** *pw*: *pairwise disjnt* $\mathcal{F}$
shows *countable* $\mathcal{F}$
proof −
obtain $\mathcal{F}'$ **where** $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$
by (*meson assms Lindelof*)
with *pw* **have** $\mathcal{F} \subseteq$ *insert* {} $\mathcal{F}'$
by (*fastforce simp add*: *pairwise_def disjnt_iff*)
then show *?thesis*
by (*simp add*: ‹*countable* $\mathcal{F}'$› *countable_subset*)
qed

sublocale *second_countable_topology* <
countable_basis open SOME B. *countable B* ∧ *topological_basis B*
using *someI_ex*[*OF ex_countable_basis*]

by *unfold_locales safe*

instance *prod* :: (*second_countable_topology*, *second_countable_topology*) *second_countable_topology*
proof
obtain *A* :: *'a set set* **where** *countable A topological_basis A*
using *ex_countable_basis* **by** *auto*
moreover
obtain *B* :: *'b set set* **where** *countable B topological_basis B*
using *ex_countable_basis* **by** *auto*
ultimately show ∃ *B*::(*'a* × *'b*) *set set*. *countable B* ∧ *open* = *generate_topology B*
by (*auto intro*!: *exI*[*of* _ (λ(*a*, *b*). *a* × *b*) ‘ (*A* × *B*)] *topological_basis_prod topological_basis_imp_subbasis*)
qed

instance *second_countable_topology* ⊆ *first_countable_topology*
proof
fix *x* :: *'a*
define *B* :: *'a set set* **where** *B* = (*SOME B*. *countable B* ∧ *topological_basis B*)
then have *B*: *countable B topological_basis B*
using *countable_basis is_basis*
by (*auto simp*: *countable_basis is_basis*)
then show ∃ *A*::*nat* ⇒ *'a set*.
(∀ *i*. *x* ∈ *A i* ∧ *open* (*A i*)) ∧ (∀ *S*. *open S* ∧ *x* ∈ *S* ⟶ (∃ *i*. *A i* ⊆ *S*))
by (*intro first_countableI*[*of* {*b*∈*B*. *x* ∈ *b*}])
(*fastforce simp*: *topological_space_class.topological_basis_def*)+
qed

instance *nat* :: *second_countable_topology*
proof
show ∃ *B*::*nat set set*. *countable B* ∧ *open* = *generate_topology B*
by (*intro exI*[*of* _ *range lessThan* ∪ *range greaterThan*]) (*auto simp*: *open_nat_def*)
qed

lemma *countable_separating_set_linorder1*:
shows ∃ *B*::(*'a*::{*linorder_topology*, *second_countable_topology*} *set*). *countable B* ∧ (∀ *x y*. *x* < *y* ⟶ (∃ *b* ∈ *B*. *x* < *b* ∧ *b* ≤ *y*))
proof −
obtain *A*::*'a set set* **where** *countable A topological_basis A* **using** *ex_countable_basis* **by** *auto*
define *B1* **where** *B1* = {(*LEAST x*. *x* ∈ *U*)| *U*. *U* ∈ *A*}
then have *countable B1* **using** ‹*countable A*› **by** (*simp add*: *Setcompr_eq_image*)
define *B2* **where** *B2* = {(*SOME x*. *x* ∈ *U*)| *U*. *U* ∈ *A*}
then have *countable B2* **using** ‹*countable A*› **by** (*simp add*: *Setcompr_eq_image*)
have ∃ *b* ∈ *B1* ∪ *B2*. *x* < *b* ∧ *b* ≤ *y* **if** *x* < *y* **for** *x y*
proof (*cases*)
assume ∃ *z*. *x* < *z* ∧ *z* < *y*
then obtain *z* **where** *z*: *x* < *z* ∧ *z* < *y* **by** *auto*

```
    define U where U = {x<..<y}
    then have open U by simp
    moreover have z ∈ U using z U_def by simp
    ultimately obtain V where V ∈ A z ∈ V V ⊆ U
      using topological_basisE[OF ‹topological_basis A›] by auto
    define w where w = (SOME x. x ∈ V)
    then have w ∈ V using ‹z ∈ V› by (metis someI2)
    then have x < w ∧ w ≤ y using ‹w ∈ V› ‹V ⊆ U› U_def by fastforce
    moreover have w ∈ B1 ∪ B2 using w_def B2_def ‹V ∈ A› by auto
    ultimately show ?thesis by auto
  next
    assume ¬(∃ z. x < z ∧ z < y)
    then have ∗: ⋀z. z > x ⟹ z ≥ y by auto
    define U where U = {x<..}
    then have open U by simp
    moreover have y ∈ U using ‹x < y› U_def by simp
    ultimately obtain V where V ∈ A y ∈ V V ⊆ U
      using topological_basisE[OF ‹topological_basis A›] by auto
    have U = {y..} unfolding U_def using ∗ ‹x < y› by auto
    then have V ⊆ {y..} using ‹V ⊆ U› by simp
    then have (LEAST w. w ∈ V) = y using ‹y ∈ V› by (meson Least_equality
atLeast_iff subsetCE)
    then have y ∈ B1 ∪ B2 using ‹V ∈ A› B1_def by auto
    moreover have x < y ∧ y ≤ y using ‹x < y› by simp
    ultimately show ?thesis by auto
  qed
  moreover have countable (B1 ∪ B2) using ‹countable B1› ‹countable B2› by
simp
  ultimately show ?thesis by auto
qed

lemma countable_separating_set_linorder2:
  shows ∃ B::('a::{linorder_topology, second_countable_topology} set). countable
B ∧ (∀ x y. x < y ⟶ (∃ b ∈ B. x ≤ b ∧ b < y))
proof −
 obtain A::'a set set where countable A topological_basis A using ex_countable_basis
by auto
  define B1 where B1 = {(GREATEST x. x ∈ U) | U. U ∈ A}
 then have countable B1 using ‹countable A› by (simp add: Setcompr_eq_image)
  define B2 where B2 = {(SOME x. x ∈ U)| U. U ∈ A}
 then have countable B2 using ‹countable A› by (simp add: Setcompr_eq_image)
  have ∃ b ∈ B1 ∪ B2. x ≤ b ∧ b < y if x < y for x y
  proof (cases)
    assume ∃ z. x < z ∧ z < y
    then obtain z where z: x < z ∧ z < y by auto
    define U where U = {x<..<y}
    then have open U by simp
    moreover have z ∈ U using z U_def by simp
    ultimately obtain V where V ∈ A z ∈ V V ⊆ U
```

using *topological_basisE*[*OF* ‹*topological_basis A*›] **by** *auto*
define w **where** $w = (SOME\ x.\ x \in V)$
then have $w \in V$ **using** ‹$z \in V$› **by** (*metis someI2*)
then have $x \leq w \wedge w < y$ **using** ‹$w \in V$› ‹$V \subseteq U$› *U_def* **by** *fastforce*
moreover have $w \in B1 \cup B2$ **using** *w_def B2_def* ‹$V \in A$› **by** *auto*
ultimately show *?thesis* **by** *auto*
next
assume $\neg(\exists z.\ x < z \wedge z < y)$
then have $*$: $\bigwedge z.\ z < y \Longrightarrow z \leq x$ **using** *leI* **by** *blast*
define U **where** $U = \{..<y\}$
then have *open U* **by** *simp*
moreover have $x \in U$ **using** ‹$x < y$› *U_def* **by** *simp*
ultimately obtain V **where** $V \in A\ x \in V\ V \subseteq U$
using *topological_basisE*[*OF* ‹*topological_basis A*›] **by** *auto*
have $U = \{..x\}$ **unfolding** *U_def* **using** $*$ ‹$x < y$› **by** *auto*
then have $V \subseteq \{..x\}$ **using** ‹$V \subseteq U$› **by** *simp*
then have (*GREATEST x. x ∈ V*) = x **using** ‹$x \in V$› **by** (*meson Greatest_equality atMost_iff subsetCE*)
then have $x \in B1 \cup B2$ **using** ‹$V \in A$› *B1_def* **by** *auto*
moreover have $x \leq x \wedge x < y$ **using** ‹$x < y$› **by** *simp*
ultimately show *?thesis* **by** *auto*
qed
moreover have *countable* (*B1 ∪ B2*) **using** ‹*countable B1*› ‹*countable B2*› **by** *simp*
ultimately show *?thesis* **by** *auto*
qed

lemma *countable_separating_set_dense_linorder*:
shows ∃ *B*::('*a*::{*linorder_topology*, *dense_linorder*, *second_countable_topology*} *set*). *countable B* ∧ (∀ *x y*. *x* < *y* ⟶ (∃ *b* ∈ *B*. *x* < *b* ∧ *b* < *y*))
proof −
obtain *B*::'*a set* **where** *B*: *countable B* $\bigwedge x\ y.\ x < y \Longrightarrow (\exists\, b \in B.\ x < b \wedge b \leq y)$
using *countable_separating_set_linorder1* **by** *auto*
have ∃ *b* ∈ *B*. *x* < *b* ∧ *b* < *y* **if** *x* < *y* **for** *x y*
proof −
obtain *z* **where** *x* < *z z* < *y* **using** ‹*x* < *y*› *dense* **by** *blast*
then obtain *b* **where** *b* ∈ *B x* < *b* ∧ *b* ≤ *z* **using** *B*(*2*) **by** *auto*
then have *x* < *b* ∧ *b* < *y* **using** ‹*z* < *y*› **by** *auto*
then show *?thesis* **using** ‹*b* ∈ *B*› **by** *auto*
qed
then show *?thesis* **using** *B*(*1*) **by** *auto*
qed

2.1.3 Polish spaces

Textbooks define Polish spaces as completely metrizable. We assume the topology to be complete for a given metric.

class *polish_space* = *complete_space* + *second_countable_topology*

2.1.4 Limit Points

definition (**in** *topological_space*) *islimpt*:: *'a* ⇒ *'a set* ⇒ *bool* (**infixr** ‹*islimpt*› *60*)
where *x islimpt S* ⟷ (∀ *T*. *x*∈*T* ⟶ *open T* ⟶ (∃ *y*∈*S*. *y*∈*T* ∧ *y*≠*x*))

lemma *islimptI*:
assumes ⋀*T*. *x* ∈ *T* ⟹ *open T* ⟹ ∃ *y*∈*S*. *y* ∈ *T* ∧ *y* ≠ *x*
shows *x islimpt S*
using *assms* **unfolding** *islimpt_def* **by** *auto*

lemma *islimptE*:
assumes *x islimpt S* **and** *x* ∈ *T* **and** *open T*
obtains *y* **where** *y* ∈ *S* **and** *y* ∈ *T* **and** *y* ≠ *x*
using *assms* **unfolding** *islimpt_def* **by** *auto*

lemma *islimpt_iff_eventually*: *x islimpt S* ⟷ ¬ *eventually* (λ*y*. *y* ∉ *S*) (*at x*)
unfolding *islimpt_def eventually_at_topological* **by** *auto*

lemma *islimpt_subset*: *x islimpt S* ⟹ *S* ⊆ *T* ⟹ *x islimpt T*
unfolding *islimpt_def* **by** *fast*

lemma *islimpt_UNIV_iff*: *x islimpt UNIV* ⟷ ¬ *open* {*x*}
unfolding *islimpt_def* **by** (*safe*, *fast*, *case_tac T* = {*x*}, *fast*, *fast*)

lemma *islimpt_punctured*: *x islimpt S* = *x islimpt* (*S*−{*x*})
unfolding *islimpt_def* **by** *blast*

A perfect space has no isolated points.

lemma *islimpt_UNIV* [*simp*, *intro*]: *x islimpt UNIV*
for *x* :: *'a*::*perfect_space*
unfolding *islimpt_UNIV_iff* **by** (*rule not_open_singleton*)

lemma *closed_limpt*: *closed S* ⟷ (∀ *x*. *x islimpt S* ⟶ *x* ∈ *S*)
unfolding *closed_def open_subopen* [*of* −*S*]
by (*metis Compl_iff islimptE islimptI open_subopen subsetI*)

lemma *islimpt_EMPTY*[*simp*]: ¬ *x islimpt* {}
by (*auto simp*: *islimpt_def*)

lemma *islimpt_Un*: *x islimpt* (*S* ∪ *T*) ⟷ *x islimpt S* ∨ *x islimpt T*
by (*simp add*: *islimpt_iff_eventually eventually_conj_iff*)

lemma *islimpt_finite_union_iff*:
assumes *finite A*
shows *z islimpt* (⋃*x*∈*A*. *B x*) ⟷ (∃ *x*∈*A*. *z islimpt B x*)
using *assms* **by** (*induction rule*: *finite_induct*) (*simp_all add*: *islimpt_Un*)

lemma *islimpt_insert*:
fixes *x* :: *'a*::*t1_space*

shows *x islimpt* (*insert a S*) ⟷ *x islimpt S*
proof
assume *x islimpt* (*insert a S*)
then show *x islimpt S*
by (*metis closed_limpt closed_singleton empty_iff insert_iff insert_is_Un islimpt_Un islimpt_def*)
next
assume *x islimpt S*
then show *x islimpt* (*insert a S*)
by (*rule islimpt_subset*) *auto*
qed

lemma *islimpt_finite*:
fixes *x* :: *'a*::*t1_space*
shows *finite S* ⟹ ¬ *x islimpt S*
by (*induct set*: *finite*) (*simp_all add*: *islimpt_insert*)

lemma *islimpt_Un_finite*:
fixes *x* :: *'a*::*t1_space*
shows *finite S* ⟹ *x islimpt* (*S* ∪ *T*) ⟷ *x islimpt T*
by (*simp add*: *islimpt_Un islimpt_finite*)

lemma *islimpt_eq_acc_point*:
fixes *l* :: *'a* :: *t1_space*
shows *l islimpt S* ⟷ (∀ *U*. *l*∈*U* ⟶ *open U* ⟶ *infinite* (*U* ∩ *S*))
proof (*safe intro*!: *islimptI*)
fix *U*
assume *l islimpt S l* ∈ *U open U finite* (*U* ∩ *S*)
then have *l islimpt S l* ∈ (*U* − (*U* ∩ *S* − {*l*})) *open* (*U* − (*U* ∩ *S* − {*l*}))
by (*auto intro*: *finite_imp_closed*)
then show *False*
by (*rule islimptE*) *auto*
next
fix *T*
assume ∗: ∀ *U*. *l*∈*U* ⟶ *open U* ⟶ *infinite* (*U* ∩ *S*) *l* ∈ *T open T*
then have ∃ *x*. *x* ∈ (*T* ∩ *S* − {*l*})
by (*metis ex_in_conv finite.emptyI infinite_remove*)
then show ∃ *y*∈*S*. *y* ∈ *T* ∧ *y* ≠ *l*
by *auto*
qed

lemma *acc_point_range_imp_convergent_subsequence*:
fixes *l* :: *'a* :: *first_countable_topology*
assumes *l*: ∀ *U*. *l*∈*U* ⟶ *open U* ⟶ *infinite* (*U* ∩ *range f*)
shows ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*f* ∘ *r*) ⟶ *l*
proof −
from *countable_basis_at_decseq*[*of l*]
obtain *A* **where** *A*:
⋀*i*. *open* (*A i*)

```
    ⋀i. l ∈ A i
    ⋀S. open S ⟹ l ∈ S ⟹ eventually (λi. A i ⊆ S) sequentially
   by blast
 define s where s n i = (SOME j. i < j ∧ f j ∈ A (Suc n)) for n i
 {
   fix n i
   have infinite (A (Suc n) ∩ range f − f‘{.. i})
     using l A by auto
   then have ∃x. x ∈ A (Suc n) ∩ range f − f‘{.. i}
     by (metis all_not_in_conv finite.emptyI)
   then have ∃a. i < a ∧ f a ∈ A (Suc n)
     by (force simp: linorder_not_le)
   then have i < s n i f (s n i) ∈ A (Suc n)
     unfolding s_def by (auto intro: someI2_ex)
 }
 note s = this
 define r where r = rec_nat (s 0 0) s
 have strict_mono r
   by (auto simp: r_def s strict_mono_Suc_iff)
 moreover
 have (λn. f (r n)) ⟶ l
 proof (rule topological_tendstoI)
   fix S
   assume open S l ∈ S
   with A(3) have eventually (λi. A i ⊆ S) sequentially
     by auto
   moreover
   {
     fix i
     assume Suc 0 ≤ i
     then have f (r i) ∈ A i
       by (cases i) (simp_all add: r_def s)
   }
   then have eventually (λi. f (r i) ∈ A i) sequentially
     by (auto simp: eventually_sequentially)
   ultimately show eventually (λi. f (r i) ∈ S) sequentially
     by eventually_elim auto
 qed
 ultimately show ∃r::nat⇒nat. strict_mono r ∧ (f ∘ r) ⟶ l
   by (auto simp: convergent_def comp_def)
qed

lemma islimpt_range_imp_convergent_subsequence:
 fixes l :: 'a :: {t1_space, first_countable_topology}
 assumes l: l islimpt (range f)
 shows ∃r::nat⇒nat. strict_mono r ∧ (f ∘ r) ⟶ l
 using l unfolding islimpt_eq_acc_point
 by (rule acc_point_range_imp_convergent_subsequence)
```

lemma *sequence_unique_limpt*:
fixes f :: $nat \Rightarrow$ 'a::*t2_space*
assumes f: $(f \longrightarrow l)$ *sequentially*
and l' *islimpt* (*range* f)
shows $l' = l$
proof (*rule ccontr*)
assume $l' \neq l$
obtain s t **where** *open* s *open* t $l' \in s$ $l \in t$ $s \cap t = \{\}$
using *hausdorff* [*OF* ‹$l' \neq l$›] **by** *auto*
then obtain N **where** $\forall n \geq N.\ f\ n \in t$
by (*meson f lim_explicit*)

have *UNIV* = $\{..<N\} \cup \{N..\}$
by *auto*
then have l' *islimpt* (f ‘ ($\{..<N\} \cup \{N..\}$))
using *assms*(*2*) **by** *simp*
then have l' *islimpt* (f ‘ $\{..<N\} \cup f$ ‘ $\{N..\}$)
by (*simp add*: *image_Un*)
then have l' *islimpt* (f ‘ $\{N..\}$)
by (*simp add*: *islimpt_Un_finite*)
then obtain y **where** $y \in f$ ‘ $\{N..\}$ $y \in s$ $y \neq l'$
using ‹$l' \in s$› ‹*open* s› **by** (*rule islimptE*)
then obtain n **where** $N \leq n$ $f\ n \in s$ $f\ n \neq l'$
by *auto*
with ‹$\forall n \geq N.\ f\ n \in t$› ‹$s \cap t = \{\}$› **show** *False*
by *blast*
qed

lemma *islimpt_sequential*:
fixes x :: 'a::*first_countable_topology*
shows x *islimpt* $S \longleftrightarrow (\exists f.\ (\forall n{::}nat.\ f\ n \in S - \{x\}) \wedge (f \longrightarrow x)\ sequentially)$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
from *countable_basis_at_decseq*[*of* x] **obtain** A **where** A:
$\bigwedge i.$ *open* (A i)
$\bigwedge i.$ $x \in A$ i
$\bigwedge S.$ *open* $S \Longrightarrow x \in S \Longrightarrow$ *eventually* ($\lambda i.$ A $i \subseteq S$) *sequentially*
by *blast*
define f **where** $f\ n = (SOME\ y.\ y \in S \wedge y \in A\ n \wedge x \neq y)$ **for** n
{
fix n
from ‹*?lhs*› **have** $\exists y.\ y \in S \wedge y \in A\ n \wedge x \neq y$
unfolding *islimpt_def* **using** $A(1,2)$[*of* n] **by** *auto*
then have $f\ n \in S \wedge f\ n \in A\ n \wedge x \neq f\ n$
unfolding *f_def* **by** (*rule someI_ex*)
then have $f\ n \in S$ $f\ n \in A$ n $x \neq f\ n$ **by** *auto*
}

```
  then have ∀ n. f n ∈ S − {x} by auto
  moreover have (λn. f n) ⟶ x
  proof (rule topological_tendstoI)
    fix S
    assume open S x ∈ S
    from A(3)[OF this] ‹⋀n. f n ∈ A n›
    show eventually (λx. f x ∈ S) sequentially
      by (auto elim!: eventually_mono)
  qed
  ultimately show ?rhs by fast
next
  assume ?rhs
  then obtain f :: nat ⇒ 'a where f: ⋀n. f n ∈ S − {x} and lim: f ⟶ x
    by auto
  show ?lhs
    unfolding islimpt_def
  proof safe
    fix T
    assume open T x ∈ T
    from lim[THEN topological_tendstoD, OF this] f
    show ∃ y∈S. y ∈ T ∧ y ≠ x
      unfolding eventually_sequentially by auto
  qed
qed

lemma islimpt_isCont_image:
  fixes f :: 'a :: {first_countable_topology, t2_space} ⇒ 'b :: {first_countable_topology,
t2_space}
  assumes x islimpt A and isCont f x and ev: eventually (λy. f y ≠ f x) (at x)
  shows   f x islimpt f ‘ A
proof −
  from assms(1) obtain g where g: g ⟶ x range g ⊆ A − {x}
    unfolding islimpt_sequential by blast
  have filterlim g (at x) sequentially
    using g by (auto simp: filterlim_at intro!: always_eventually)
  then obtain N where N: ⋀n. n ≥ N ⟹ f (g n) ≠ f x
    by (metis (mono_tags, lifting) ev eventually_at_top_linorder filterlim_iff)
  have (λx. g (x + N)) ⟶ x
    using g(1) by (rule LIMSEQ_ignore_initial_segment)
  hence (λx. f (g (x + N))) ⟶ f x
    using assms(2) isCont_tendsto_compose by blast
  moreover have range (λx. f (g (x + N))) ⊆ f ‘ A − {f x}
    using g(2) N by auto
  ultimately show ?thesis
    unfolding islimpt_sequential by (intro exI[of _ λx. f (g (x + N))]) auto
qed

lemma islimpt_image:
  assumes z islimpt g −‘ A ∩ B g z ∉ A z ∈ B continuous_on B g
```

shows *g z islimpt A*
by (*smt* (*verit*, *best*) *IntD1 assms continuous_on_topological inf_le2 islimpt_def subset_eq vimageE*)

2.1.5 Interior of a Set

definition *interior* :: ('*a*::*topological_space*) *set* ⇒ '*a set* **where**
interior S = ⋃{*T*. *open T* ∧ *T* ⊆ *S*}

lemma *interiorI* [*intro?*]:
assumes *open T* **and** *x* ∈ *T* **and** *T* ⊆ *S*
shows *x* ∈ *interior S*
using *assms* **unfolding** *interior_def* **by** *fast*

lemma *interiorE* [*elim?*]:
assumes *x* ∈ *interior S*
obtains *T* **where** *open T* **and** *x* ∈ *T* **and** *T* ⊆ *S*
using *assms* **unfolding** *interior_def* **by** *fast*

lemma *open_interior* [*simp*, *intro*]: *open* (*interior S*)
by (*simp add*: *interior_def open_Union*)

lemma *interior_subset*: *interior S* ⊆ *S*
by (*auto simp*: *interior_def*)

lemma *interior_maximal*: *T* ⊆ *S* ⟹ *open T* ⟹ *T* ⊆ *interior S*
by (*auto simp*: *interior_def*)

lemma *interior_open*: *open S* ⟹ *interior S* = *S*
by (*intro equalityI interior_subset interior_maximal subset_refl*)

lemma *interior_eq*: *interior S* = *S* ⟷ *open S*
by (*metis open_interior interior_open*)

lemma *open_subset_interior*: *open S* ⟹ *S* ⊆ *interior T* ⟷ *S* ⊆ *T*
by (*metis interior_maximal interior_subset subset_trans*)

lemma *interior_empty* [*simp*]: *interior* {} = {}
using *open_empty* **by** (*rule interior_open*)

lemma *interior_UNIV* [*simp*]: *interior UNIV* = *UNIV*
using *open_UNIV* **by** (*rule interior_open*)

lemma *interior_interior* [*simp*]: *interior* (*interior S*) = *interior S*
using *open_interior* **by** (*rule interior_open*)

lemma *interior_mono*: *S* ⊆ *T* ⟹ *interior S* ⊆ *interior T*
by (*auto simp*: *interior_def*)

lemma *interior_unique*:
assumes $T \subseteq S$ **and** *open* T
assumes $\bigwedge T'.\ T' \subseteq S \Longrightarrow$ *open* $T' \Longrightarrow T' \subseteq T$
shows *interior* $S = T$
by (*intro equalityI assms interior_subset open_interior interior_maximal*)

lemma *interior_singleton* [*simp*]: *interior* $\{a\} = \{\}$
for a :: $'a$::*perfect_space*
by (*meson interior_eq interior_subset not_open_singleton subset_singletonD*)

lemma *interior_Int* [*simp*]: *interior* $(S \cap T) =$ *interior* $S \cap$ *interior* T
by (*meson Int_mono Int_subset_iff antisym_conv interior_maximal interior_subset open_Int open_interior*)

lemma *eventually_nhds_in_nhd*: $x \in$ *interior* $s \Longrightarrow$ *eventually* $(\lambda y.\ y \in s)$ (*nhds* x)
using *interior_subset*[*of s*] **by** (*subst eventually_nhds*) *blast*

lemma *interior_limit_point* [*intro*]:
fixes x :: $'a$::*perfect_space*
assumes x: $x \in$ *interior* S
shows x *islimpt* S
proof −
obtain T **where** $x \in T$ $T \subseteq S$ *open* T
using *interior_subset* x **by** *blast*
with x *islimpt_UNIV* [*of x*] **show** *?thesis*
unfolding *islimpt_def* **by** (*metis* (*full_types*) *Int_iff open_Int subsetD*)
qed

lemma *open_imp_islimpt*:
fixes x::$'a$:: *perfect_space*
assumes *open* S $x \in S$
shows x *islimpt* S
using *assms interior_eq interior_limit_point* **by** *auto*

lemma *islimpt_Int_eventually*:
assumes x *islimpt* A *eventually* $(\lambda y.\ y \in B)$ (*at* x)
shows x *islimpt* $A \cap B$
using *assms* **unfolding** *islimpt_def eventually_at_filter eventually_nhds*
by (*metis Int_iff UNIV_I open_Int*)

lemma *islimpt_conv_frequently_at*:
x *islimpt* $A \longleftrightarrow$ *frequently* $(\lambda y.\ y \in A)$ (*at* x)
by (*simp add*: *frequently_def islimpt_iff_eventually*)

lemma *frequently_at_imp_islimpt*:
assumes *frequently* $(\lambda y.\ y \in A)$ (*at* x)
shows x *islimpt* A
by (*simp add*: *assms islimpt_conv_frequently_at*)

lemma *interior_closed_Un_empty_interior*:
assumes *cS*: *closed S*
and *iT*: *interior T* = {}
shows *interior* (*S* ∪ *T*) = *interior S*
proof
show *interior S* ⊆ *interior* (*S* ∪ *T*)
by (*rule interior_mono*) (*rule Un_upper1*)
show *interior* (*S* ∪ *T*) ⊆ *interior S*
proof
fix *x*
assume *x* ∈ *interior* (*S* ∪ *T*)
then obtain *R* **where** *open R x* ∈ *R R* ⊆ *S* ∪ *T* **..**
show *x* ∈ *interior S*
proof (*rule ccontr*)
assume *x* ∉ *interior S*
with ‹*x* ∈ *R*› ‹*open R*› **obtain** *y* **where** *y* ∈ *R* − *S*
unfolding *interior_def* **by** *fast*
then show *False*
by (*metis Diff_subset_conv* ‹*R* ⊆ *S* ∪ *T*› ‹*open R*› *cS empty_iff iT interiorI open_Diff*)
qed
qed
qed

lemma *interior_Times*: *interior* (*A* × *B*) = *interior A* × *interior B*
proof (*rule interior_unique*)
show *interior A* × *interior B* ⊆ *A* × *B*
by (*intro Sigma_mono interior_subset*)
show *open* (*interior A* × *interior B*)
by (*intro open_Times open_interior*)
fix *T*
assume *T* ⊆ *A* × *B* **and** *open T*
then show *T* ⊆ *interior A* × *interior B*
proof *safe*
fix *x y*
assume (*x*, *y*) ∈ *T*
then obtain *C D* **where** *open C open D C* × *D* ⊆ *T x* ∈ *C y* ∈ *D*
using ‹*open T*› **unfolding** *open_prod_def* **by** *fast*
then have *open C open D C* ⊆ *A D* ⊆ *B x* ∈ *C y* ∈ *D*
using ‹*T* ⊆ *A* × *B*› **by** *auto*
then show *x* ∈ *interior A* **and** *y* ∈ *interior B*
by (*auto intro*: *interiorI*)
qed
qed

lemma *interior_Ici*:
fixes *x* :: *'a* :: {*dense_linorder*,*linorder_topology*}
assumes *b* < *x*

```
  shows interior {x ..} = {x <..}
proof (rule interior_unique)
  fix T
  assume T ⊆ {x ..} open T
  moreover have x ∉ T
  proof
    assume x ∈ T
    obtain y where y < x {y <.. x} ⊆ T
      using open_left[OF ‹open T› ‹x ∈ T› ‹b < x›] by auto
    with dense[OF ‹y < x›] obtain z where z ∈ T z < x
      by (auto simp: subset_eq Ball_def)
    with ‹T ⊆ {x ..}› show False by auto
  qed
  ultimately show T ⊆ {x <..}
    by (auto simp: subset_eq less_le)
qed auto

lemma interior_Iic:
  fixes x :: 'a ::{dense_linorder,linorder_topology}
  assumes x < b
  shows interior {.. x} = {..< x}
proof (rule interior_unique)
  fix T
  assume T ⊆ {.. x} open T
  moreover have x ∉ T
  proof
    assume x ∈ T
    obtain y where x < y {x ..< y} ⊆ T
      using open_right[OF ‹open T› ‹x ∈ T› ‹x < b›] by auto
    with dense[OF ‹x < y›] obtain z where z ∈ T x < z
      by (auto simp: subset_eq Ball_def less_le)
    with ‹T ⊆ {.. x}› show False by auto
  qed
  ultimately show T ⊆ {..< x}
    by (auto simp: subset_eq less_le)
qed auto

lemma countable_disjoint_nonempty_interior_subsets:
  fixes ℱ :: 'a::second_countable_topology set set
  assumes pw: pairwise disjnt ℱ and int: ⋀S. ⟦S ∈ ℱ; interior S = {}⟧ ⟹ S =
{}
  shows countable ℱ
proof (rule countable_image_inj_on)
  have disjoint (interior ` ℱ)
    using pw by (simp add: disjoint_image_subset interior_subset)
  then show countable (interior ` ℱ)
    by (auto intro: countable_disjoint_open_subsets)
  show inj_on interior ℱ
    using pw apply (clarsimp simp: inj_on_def pairwise_def)
```

apply (*metis disjnt_def disjnt_subset1 inf.orderE int interior_subset*)
done
qed

2.1.6 Closure of a Set

definition *closure* :: ('*a*::*topological_space*) *set* ⇒ '*a set* **where**
closure S = *S* ∪ {*x* . *x islimpt S*}

lemma *interior_closure*: *interior S* = − (*closure* (− *S*))
by (*auto simp*: *interior_def closure_def islimpt_def*)

lemma *closure_interior*: *closure S* = − *interior* (− *S*)
by (*simp add*: *interior_closure*)

lemma *closed_closure*[*simp*, *intro*]: *closed* (*closure S*)
by (*simp add*: *closure_interior closed_Compl*)

lemma *closure_subset*: *S* ⊆ *closure S*
by (*simp add*: *closure_def*)

lemma *closure_hull*: *closure S* = *closed hull S*
by (*auto simp*: *hull_def closure_interior interior_def*)

lemma *closure_eq*: *closure S* = *S* ⟷ *closed S*
unfolding *closure_hull* **using** *closed_Inter* **by** (*rule hull_eq*)

lemma *closure_closed* [*simp*]: *closed S* ⟹ *closure S* = *S*
by (*simp only*: *closure_eq*)

lemma *closure_closure* [*simp*]: *closure* (*closure S*) = *closure S*
unfolding *closure_hull* **by** (*rule hull_hull*)

lemma *closure_mono*: *S* ⊆ *T* ⟹ *closure S* ⊆ *closure T*
unfolding *closure_hull* **by** (*rule hull_mono*)

lemma *closure_minimal*: *S* ⊆ *T* ⟹ *closed T* ⟹ *closure S* ⊆ *T*
unfolding *closure_hull* **by** (*rule hull_minimal*)

lemma *closure_unique*:
assumes *S* ⊆ *T*
and *closed T*
and ⋀*T'*. *S* ⊆ *T'* ⟹ *closed T'* ⟹ *T* ⊆ *T'*
shows *closure S* = *T*
using *assms* **unfolding** *closure_hull* **by** (*rule hull_unique*)

lemma *closure_empty* [*simp*]: *closure* {} = {}
using *closed_empty* **by** (*rule closure_closed*)

lemma *closure_UNIV* [*simp*]: *closure UNIV* = *UNIV*
using *closed_UNIV* **by** (*rule closure_closed*)

lemma *closure_Un* [*simp*]: *closure* (*S* ∪ *T*) = *closure S* ∪ *closure T*
by (*simp add*: *closure_interior*)

lemma *closure_eq_empty* [*iff*]: *closure S* = {} ⟷ *S* = {}
using *closure_empty closure_subset*[*of S*] **by** *blast*

lemma *closure_subset_eq*: *closure S* ⊆ *S* ⟷ *closed S*
using *closure_eq*[*of S*] *closure_subset*[*of S*] **by** *simp*

lemma *open_Int_closure_eq_empty*: *open S* ⟹ (*S* ∩ *closure T*) = {} ⟷ *S* ∩ *T* = {}
using *open_subset_interior*[*of S* − *T*]
using *interior_subset*[*of* − *T*]
by (*auto simp*: *closure_interior*)

lemma *open_Int_closure_subset*: *open S* ⟹ *S* ∩ *closure T* ⊆ *closure* (*S* ∩ *T*)
proof
fix *x*
assume ∗: *open S x* ∈ *S* ∩ *closure T*
then have *x islimpt* (*S* ∩ *T*) **if** *x islimpt T*
by (*metis IntD1 eventually_at_in_open′ inf_commute islimpt_Int_eventually that*)
with ∗ **show** *x* ∈ *closure* (*S* ∩ *T*)
unfolding *closure_def* **by** *blast*
qed

lemma *closure_complement*: *closure* (− *S*) = − *interior S*
by (*simp add*: *closure_interior*)

lemma *interior_complement*: *interior* (− *S*) = − *closure S*
by (*simp add*: *closure_interior*)

lemma *interior_diff*: *interior*(*S* − *T*) = *interior S* − *closure T*
by (*simp add*: *Diff_eq interior_complement*)

lemma *closure_Times*: *closure* (*A* × *B*) = *closure A* × *closure B*
proof (*rule closure_unique*)
show *A* × *B* ⊆ *closure A* × *closure B*
by (*intro Sigma_mono closure_subset*)
show *closed* (*closure A* × *closure B*)
by (*intro closed_Times closed_closure*)
fix *T*
assume *A* × *B* ⊆ *T* **and** *closed T*
then show *closure A* × *closure B* ⊆ *T*
apply (*simp add*: *closed_def open_prod_def*, *clarify*)
apply (*rule ccontr*)

apply (*drule_tac x*=(*a*, *b*) **in** *bspec*, *simp*, *clarify*, *rename_tac C D*)
apply (*simp add*: *closure_interior interior_def*)
apply (*drule_tac x*=*C* **in** *spec*)
apply (*drule_tac x*=*D* **in** *spec*, *auto*)
done
qed

lemma *closure_open_Int_superset*:
assumes *open S S* ⊆ *closure T*
shows *closure*(*S* ∩ *T*) = *closure S*
by (*metis Int_Un_distrib Un_Int_eq*(*4*) *assms closure_Un closure_closure open_Int_closure_subset sup.orderE*)

lemma *closure_Int*: *closure* (⋂ *I*) ⊆ ⋂ {*closure S* |*S*. *S* ∈ *I*}
by (*simp add*: *INF_greatest Inter_lower Setcompr_eq_image closure_mono*)

lemma *islimpt_in_closure*: (*x islimpt S*) = (*x*∈*closure*(*S*−{*x*}))
unfolding *closure_def* **using** *islimpt_punctured* **by** *blast*

lemma *connected_imp_connected_closure*: *connected S* ⟹ *connected* (*closure S*)
by (*rule connectedI*) (*meson closure_subset open_Int open_Int_closure_eq_empty subset_trans connectedD*)

lemma *bdd_below_closure*:
fixes *A* :: *real set*
assumes *bdd_below A*
shows *bdd_below* (*closure A*)
proof −
from *assms* **obtain** *m* **where** ⋀*x*. *x* ∈ *A* ⟹ *m* ≤ *x*
by (*auto simp*: *bdd_below_def*)
then have *A* ⊆ {*m*..} **by** *auto*
then have *closure A* ⊆ {*m*..}
using *closed_real_atLeast* **by** (*rule closure_minimal*)
then show *?thesis*
by (*auto simp*: *bdd_below_def*)
qed

2.1.7 Frontier (also known as boundary)

definition *frontier* :: ('*a*::*topological_space*) *set* ⇒ '*a set* **where**
frontier S = *closure S* − *interior S*

lemma *frontier_closed* [*iff*]: *closed* (*frontier S*)
by (*simp add*: *frontier_def closed_Diff*)

lemma *frontier_closures*: *frontier S* = *closure S* ∩ *closure* (− *S*)
by (*auto simp*: *frontier_def interior_closure*)

lemma *frontier_Int*: *frontier*(*S* ∩ *T*) = *closure*(*S* ∩ *T*) ∩ (*frontier S* ∪ *frontier*

T)
proof −
have *closure* $(S \cap T) \subseteq$ *closure* S *closure* $(S \cap T) \subseteq$ *closure* T
by (*simp_all add*: *closure_mono*)
then show *?thesis*
by (*auto simp*: *frontier_closures*)
qed

lemma *frontier_Int_subset*: *frontier*$(S \cap T) \subseteq$ *frontier* $S \cup$ *frontier* T
by (*auto simp*: *frontier_Int*)

lemma *frontier_Int_closed*:
assumes *closed* S *closed* T
shows *frontier*$(S \cap T) =$ (*frontier* $S \cap T) \cup (S \cap$ *frontier* T)
by (*smt* (*verit*, *best*) *Diff_Int Int_Diff assms closed_Int closure_eq frontier_def inf_commute interior_Int*)

lemma *frontier_subset_closed*: *closed* $S \Longrightarrow$ *frontier* $S \subseteq S$
by (*metis frontier_def closure_closed Diff_subset*)

lemma *frontier_empty* [*simp*]: *frontier* $\{\} = \{\}$
by (*simp add*: *frontier_def*)

lemma *frontier_subset_eq*: *frontier* $S \subseteq S \longleftrightarrow$ *closed* S
by (*metis Diff_subset_conv closure_subset_eq frontier_def interior_subset subset_Un_eq*)

lemma *frontier_complement* [*simp*]: *frontier* $(-S) =$ *frontier* S
by (*auto simp*: *frontier_def closure_complement interior_complement*)

lemma *frontier_Un_subset*: *frontier*$(S \cup T) \subseteq$ *frontier* $S \cup$ *frontier* T
by (*metis compl_sup frontier_Int_subset frontier_complement*)

lemma *frontier_disjoint_eq*: *frontier* $S \cap S = \{\} \longleftrightarrow$ *open* S
using *frontier_complement frontier_subset_eq*[*of* $-S$]
unfolding *open_closed* **by** *auto*

lemma *frontier_UNIV* [*simp*]: *frontier UNIV* $= \{\}$
using *frontier_complement frontier_empty* **by** *fastforce*

lemma *frontier_interiors*: *frontier* $s = -$ *interior*$(s) -$ *interior*$(-s)$
by (*simp add*: *Int_commute frontier_def interior_closure*)

lemma *frontier_interior_subset*: *frontier*(*interior* $S) \subseteq$ *frontier* S
by (*simp add*: *Diff_mono frontier_interiors interior_mono interior_subset*)

lemma *closure_Un_frontier*: *closure* $S = S \cup$ *frontier* S
by (*simp add*: *closure_def frontier_closures sup_inf_distrib1*)

2.1.8 Filters and the "eventually true" quantifier

Identify Trivial limits, where we can't approach arbitrarily closely.

lemma *trivial_limit_within*: *trivial_limit* (*at a within S*) ⟷ ¬ *a islimpt S*
unfolding *trivial_limit_def eventually_at_topological islimpt_def*
by *blast*

lemma *trivial_limit_at_iff*: *trivial_limit* (*at a*) ⟷ ¬ *a islimpt UNIV*
using *trivial_limit_within* [*of a UNIV*] **by** *simp*

lemma *trivial_limit_at*: ¬ *trivial_limit* (*at a*)
for *a* :: *'a*::*perfect_space*
by (*rule at_neq_bot*)

lemma *not_trivial_limit_within*: ¬ *trivial_limit* (*at x within S*) = (*x* ∈ *closure* (*S* − {*x*}))
using *islimpt_in_closure* **by** (*metis trivial_limit_within*)

lemma *not_in_closure_trivial_limitI*:
x ∉ *closure S* ⟹ *trivial_limit* (*at x within S*)
using *not_trivial_limit_within*[*of x S*]
by *safe* (*metis Diff_empty Diff_insert0 closure_subset contra_subsetD*)

lemma *filterlim_at_within_closure_implies_filterlim*: *filterlim f l* (*at x within S*)
if *x* ∈ *closure S* ⟹ *filterlim f l* (*at x within S*)
by (*metis bot.extremum filterlim_iff_le_filtercomap not_in_closure_trivial_limitI that*)

lemma *at_within_eq_bot_iff*: *at c within A* = *bot* ⟷ *c* ∉ *closure* (*A* − {*c*})
using *not_trivial_limit_within*[*of c A*] **by** *blast*

2.1.9 Limits

The expected monotonicity property.

lemma *Lim_Un*:
assumes (*f* ⟶ *l*) (*at x within S*) (*f* ⟶ *l*) (*at x within T*)
shows (*f* ⟶ *l*) (*at x within* (*S* ∪ *T*))
using *assms* **unfolding** *at_within_union* **by** (*rule filterlim_sup*)

lemma *Lim_Un_univ*:
(*f* ⟶ *l*) (*at x within S*) ⟹ (*f* ⟶ *l*) (*at x within T*) ⟹
S ∪ *T* = *UNIV* ⟹ (*f* ⟶ *l*) (*at x*)
by (*metis Lim_Un*)

Interrelations between restricted and unrestricted limits.

lemma *Lim_at_imp_Lim_at_within*: (*f* ⟶ *l*) (*at x*) ⟹ (*f* ⟶ *l*) (*at x within S*)
by (*metis order_refl filterlim_mono subset_UNIV at_le*)

lemma *eventually_within_interior*:
assumes $x \in$ *interior* S
shows *eventually* P (*at* x *within* S) $\longleftrightarrow$ *eventually* P (*at* x)
by (*metis assms at_within_open_subset interior_subset open_interior*)

lemma *at_within_interior*: $x \in$ *interior* $S \Longrightarrow$ *at* x *within* $S =$ *at* x
unfolding *filter_eq_iff* **by** (*intro allI eventually_within_interior*)

lemma *Lim_within_LIMSEQ*:
fixes a :: $'a$::*first_countable_topology*
assumes $\forall S.\ (\forall n.\ S\ n \neq a \wedge S\ n \in T) \wedge S \longrightarrow a \longrightarrow (\lambda n.\ X\ (S\ n)) \longrightarrow L$
shows ($X \longrightarrow L$) (*at* a *within* T)
using *assms* **unfolding** *tendsto_def* [**where** l=L]
by (*simp add*: *sequentially_imp_eventually_within*)

lemma *Lim_right_bound*:
fixes f :: $'a$ :: {*linorder_topology*, *conditionally_complete_linorder*, *no_top*} $\Rightarrow$
$'b$::{*linorder_topology*, *conditionally_complete_linorder*}
assumes *mono*: $\bigwedge a\ b.\ a \in I \Longrightarrow b \in I \Longrightarrow x < a \Longrightarrow a \leq b \Longrightarrow f\ a \leq f\ b$
and *bnd*: $\bigwedge a.\ a \in I \Longrightarrow x < a \Longrightarrow K \leq f\ a$
shows ($f \longrightarrow$ *Inf* (f ‘ ({x<..} $\cap$ I))) (*at* x *within* ({x<..} $\cap$ I))
proof (*cases* {x<..} $\cap$ I = {})
case *True*
then show *?thesis* **by** *simp*
next
case *False*
show *?thesis*
proof (*rule order_tendstoI*)
fix a
assume a: $a <$ *Inf* (f ‘ ({x<..} $\cap$ I))
{
fix y
assume $y \in$ {x<..} $\cap$ I
with *False bnd* **have** *Inf* (f ‘ ({x<..} $\cap$ I)) $\leq f\ y$
by (*auto intro*!: *cInf_lower bdd_belowI2*)
with a **have** $a < f\ y$
by (*blast intro*: *less_le_trans*)
}
then show *eventually* ($\lambda x.\ a < f\ x$) (*at* x *within* ({x<..} $\cap$ I))
by (*auto simp*: *eventually_at_filter intro*: *exI*[*of* _ *1*] *zero_less_one*)
next
fix a
assume *Inf* (f ‘ ({x<..} $\cap$ I)) $< a$
from *cInf_lessD*[*OF* _ *this*] *False* **obtain** y **where** y: $x < y$ $y \in I$ $f\ y < a$
by *auto*
then have *eventually* ($\lambda x.\ x \in I \longrightarrow f\ x < a$) (*at_right* x)
unfolding *eventually_at_right*[*OF* ‹$x < y$›] **by** (*metis less_imp_le le_less_trans*

mono)
  then show *eventually* (λ*x*. *f x* < *a*) (*at x within* ({*x*<..} ∩ *I*))
   unfolding *eventually_at_filter* **by** *eventually_elim simp*
 qed
qed

These are special for limits out of the same topological space.

lemma *Lim_within_id*: (*id* ⟶ *a*) (*at a within s*)
 unfolding *id_def* **by** (*rule tendsto_ident_at*)

lemma *Lim_at_id*: (*id* ⟶ *a*) (*at a*)
 unfolding *id_def* **by** (*rule tendsto_ident_at*)

It's also sometimes useful to extract the limit point from the filter.

abbreviation *netlimit* :: *'a*::*t2_space filter* ⇒ *'a*
 where *netlimit F* ≡ *Lim F* (λ*x*. *x*)

lemma *netlimit_at* [*simp*]:
 fixes *a* :: *'a*::{*perfect_space*,*t2_space*}
 shows *netlimit* (*at a*) = *a*
 using *Lim_ident_at* [*of a UNIV*] **by** *simp*

lemma *lim_within_interior*:
 x ∈ *interior S* ⟹ (*f* ⟶ *l*) (*at x within S*) ⟷ (*f* ⟶ *l*) (*at x*)
 by (*metis at_within_interior*)

lemma *netlimit_within_interior*:
 fixes *x* :: *'a*::{*t2_space*,*perfect_space*}
 assumes *x* ∈ *interior S*
 shows *netlimit* (*at x within S*) = *x*
 using *assms* **by** (*metis at_within_interior netlimit_at*)

Useful lemmas on closure and set of possible sequential limits.

lemma *closure_sequential*:
 fixes *l* :: *'a*::*first_countable_topology*
 shows *l* ∈ *closure S* ⟷ (∃ *x*. (∀ *n*. *x n* ∈ *S*) ∧ (*x* ⟶ *l*) *sequentially*)
 by (*metis Diff_empty Diff_insert0 Un_iff closure_def islimpt_sequential mem_Collect_eq tendsto_const*)

lemma *closed_sequential_limits*:
 fixes *S* :: *'a*::*first_countable_topology set*
 shows *closed S* ⟷ (∀ *x l*. (∀ *n*. *x n* ∈ *S*) ∧ (*x* ⟶ *l*) *sequentially* ⟶ *l* ∈ *S*)
by (*metis closure_sequential closure_subset_eq subset_iff*)

lemma *tendsto_If_within_closures*:
 assumes *f*: *x* ∈ *S* ∪ (*closure S* ∩ *closure T*) ⟹
   (*f* ⟶ *l x*) (*at x within S* ∪ (*closure S* ∩ *closure T*))
 assumes *g*: *x* ∈ *T* ∪ (*closure S* ∩ *closure T*) ⟹
   (*g* ⟶ *l x*) (*at x within T* ∪ (*closure S* ∩ *closure T*))

assumes $x \in S \cup T$
shows $((\lambda x.\ \mathit{if}\ x \in S\ \mathit{then}\ f\ x\ \mathit{else}\ g\ x) \longrightarrow l\ x)\ (\mathit{at}\ x\ \mathit{within}\ S \cup T)$
proof −
have ∗: $(S \cup T) \cap \{x.\ x \in S\} = S\ (S \cup T) \cap \{x.\ x \notin S\} = T - S$
by *auto*
have $(f \longrightarrow l\ x)\ (\mathit{at}\ x\ \mathit{within}\ S)$
by (*rule filterlim_at_within_closure_implies_filterlim*)
(*use* ‹$x \in \_$› **in** ‹*auto simp*: *inf_commute closure_def intro*: *tendsto_within_subset*[*OF f*]›)
moreover
have $(g \longrightarrow l\ x)\ (\mathit{at}\ x\ \mathit{within}\ T - S)$
by (*rule filterlim_at_within_closure_implies_filterlim*)
(*use* ‹$x \in \_$› **in**
‹*auto intro*!: *tendsto_within_subset*[*OF g*] *simp*: *closure_def intro*: *islimpt_subset*›)
ultimately show *?thesis*
by (*intro filterlim_at_within_If*) (*simp_all only*: ∗)
qed

2.1.10 Compactness

lemma *brouwer_compactness_lemma*:
fixes f :: $'a$::*topological_space* ⇒ $'b$::*real_normed_vector*
assumes *compact S*
and *continuous_on S f*
and ¬ $(\exists x{\in}S.\ f\ x = 0)$
obtains d **where** $0 < d$ **and** $\forall x{\in}S.\ d \le \mathit{norm}\ (f\ x)$
proof (*cases* $S = \{\}$)
case *True*
show *thesis*
by (*rule that* [*of 1*]) (*auto simp*: *True*)
next
case *False*
have *continuous_on S* (*norm* ∘ *f*)
by (*rule continuous_intros continuous_on_norm assms*(*2*))+
with *False* **obtain** x **where** x: $x \in S\ \forall y{\in}S.\ (\mathit{norm} \circ f)\ x \le (\mathit{norm} \circ f)\ y$
using *continuous_attains_inf*[*OF assms*(*1*), *of norm* ∘ *f*]
unfolding *o_def*
by *auto*
then show *?thesis*
by (*metis assms*(*3*) *that comp_apply zero_less_norm_iff*)
qed

Bolzano-Weierstrass property

proposition *Heine_Borel_imp_Bolzano_Weierstrass*:
assumes *compact S*
and *infinite T*
and $T \subseteq S$
shows $\exists x \in S.\ x\ \mathit{islimpt}\ T$

proof (*rule ccontr*)
assume ¬ (∃ *x* ∈ *S*. *x islimpt T*)
then obtain *f* **where** *f*: ∀ *x*∈*S*. *x* ∈ *f x* ∧ *open* (*f x*) ∧ (∀ *y*∈*T*. *y* ∈ *f x* ⟶ *y* = *x*)
unfolding *islimpt_def* **by** *metis*
obtain *g* **where** *g*: *g* ⊆ {*T*. ∃ *x*. *x* ∈ *S* ∧ *T* = *f x*} *finite g S* ⊆ ⋃ *g*
using *assms*(*1*)[*unfolded compact_eq_Heine_Borel*, *THEN spec*[**where** *x*={*T*. ∃ *x*. *x*∈*S* ∧ *T* = *f x*}]]
using *f* **by** *auto*
then have *g'*: ∀ *x*∈*g*. ∃ *y* ∈ *S*. *x* = *f y*
by *auto*
have *inj_on f T*
by (*smt* (*verit*, *best*) *assms*(*3*) *f inj_onI subsetD*)
then have *infinite* (*f* ‘ *T*)
using *assms*(*2*) **using** *finite_imageD* **by** *auto*
moreover
have *False* **if** *x* ∈ *T f x* ∉ *g* **for** *x*
by (*smt* (*verit*) *UnionE assms*(*3*) *f g' g*(*3*) *subsetD that*)
then have *f* ‘ *T* ⊆ *g* **by** *auto*
ultimately show *False*
using *g*(*2*) **using** *finite_subset* **by** *auto*
qed

lemma *sequence_infinite_lemma*:
fixes *f* :: *nat* ⇒ *'a*::*t1_space*
assumes ∀ *n*. *f n* ≠ *l*
and (*f* ⟶⟶ *l*) *sequentially*
shows *infinite* (*range f*)
proof
assume *finite* (*range f*)
then have *l* ∉ *range f* ∧ *closed* (*range f*)
using ‹*finite* (*range f*)› *assms*(*1*) *finite_imp_closed* **by** *blast*
then have *eventually* (λ*n*. *f n* ∈ − *range f*) *sequentially*
by (*metis Compl_iff assms*(*2*) *open_Compl topological_tendstoD*)
then show *False*
unfolding *eventually_sequentially* **by** *auto*
qed

lemma *Bolzano_Weierstrass_imp_closed*:
fixes *S* :: *'a*::{*first_countable_topology*,*t2_space*} *set*
assumes ∀ *T*. *infinite T* ∧ *T* ⊆ *S* −−> (∃ *x* ∈ *S*. *x islimpt T*)
shows *closed S*
proof −
{
fix *x l*
assume *as*: ∀ *n*::*nat*. *x n* ∈ *S* (*x* ⟶⟶ *l*) *sequentially*
have *l* ∈ *S*
proof (*cases* ∀ *n*. *x n* ≠ *l*)
case *False*

```
    then show l∈S using as(1) by auto
   next
    case True
    with as(2) have infinite (range x)
     using sequence_infinite_lemma[of x l] by auto
    then obtain l' where l'∈S l' islimpt (range x)
     using as(1) assms by auto
    then show l∈S using sequence_unique_limpt as True by auto
   qed
  }
  then show ?thesis
   unfolding closed_sequential_limits by fast
qed

lemma closure_insert:
  fixes x :: 'a::t1_space
  shows closure (insert x S) = insert x (closure S)
  by (metis closed_singleton closure_Un closure_closed insert_is_Un)

lemma finite_not_islimpt_in_compact:
  assumes compact A ⋀z. z ∈ A ⟹ ¬z islimpt B
  shows   finite (A ∩ B)
   by (meson Heine_Borel_imp_Bolzano_Weierstrass assms inf_le1 inf_le2 is-
limpt_subset)
```

In particular, some common special cases.

```
lemma compact_Un [intro]:
  assumes compact S
    and compact T
  shows  compact (S ∪ T)
proof (rule compactI)
  fix f
  assume ∗: Ball f open S ∪ T ⊆ ⋃f
  from ∗ ‹compact S› obtain s' where s' ⊆ f ∧ finite s' ∧ S ⊆ ⋃s'
   unfolding compact_eq_Heine_Borel by (auto elim!: allE[of _ f])
  moreover
  from ∗ ‹compact T› obtain t' where t' ⊆ f ∧ finite t' ∧ T ⊆ ⋃t'
   unfolding compact_eq_Heine_Borel by (auto elim!: allE[of _ f])
  ultimately show ∃f'⊆f. finite f' ∧ S ∪ T ⊆ ⋃f'
   by (auto intro!: exI[of _ s' ∪ t'])
qed

lemma compact_Union [intro]: finite S ⟹ (⋀T. T ∈ S ⟹ compact T) ⟹
compact (⋃S)
  by (induct set: finite) auto

lemma compact_UN [intro]:
  finite A ⟹ (⋀x. x ∈ A ⟹ compact (B x)) ⟹ compact (⋃x∈A. B x)
  by blast
```

lemma *closed_Int_compact* [*intro*]:
assumes *closed S* **and** *compact T*
shows *compact* (*S* ∩ *T*)
using *compact_Int_closed* [*of T S*] *assms*
by (*simp add*: *Int_commute*)

lemma *compact_Int* [*intro*]:
fixes *S T* :: *'a* :: *t2_space set*
assumes *compact S* **and** *compact T*
shows *compact* (*S* ∩ *T*)
using *assms* **by** (*intro compact_Int_closed compact_imp_closed*)

lemma *compact_sing* [*simp*]: *compact* {*a*}
unfolding *compact_eq_Heine_Borel* **by** *auto*

lemma *compact_insert* [*simp*]:
assumes *compact S*
shows *compact* (*insert x S*)
by (*metis assms compact_Un compact_sing insert_is_Un*)

lemma *finite_imp_compact*: *finite S* ⟹ *compact S*
by (*induct set*: *finite*) *simp_all*

lemma *open_delete*:
fixes *S* :: *'a*::*t1_space set*
shows *open S* ⟹ *open* (*S* − {*x*})
by (*simp add*: *open_Diff*)

Compactness expressed with filters

lemma *closure_iff_nhds_not_empty*:
x ∈ *closure X* ⟷ (∀ *A*. ∀ *S*⊆*A*. *open S* ⟶ *x* ∈ *S* ⟶ *X* ∩ *A* ≠ {})
proof *safe*
assume *x*: *x* ∈ *closure X*
fix *S A*
assume §: *open S x* ∈ *S X* ∩ *A* = {} *S* ⊆ *A*
then have *x* ∉ *closure* (−*S*)
by (*simp add*: *closed_open*)
with *x* **have** *x* ∈ *closure X* − *closure* (−*S*)
by *auto*
with § **show** *False*
by (*metis Compl_iff Diff_iff closure_mono disjoint_iff subsetD subsetI*)
next
assume ∀ *A S*. *S* ⊆ *A* ⟶ *open S* ⟶ *x* ∈ *S* ⟶ *X* ∩ *A* ≠ {}
then show *x* ∈ *closure X*
by (*metis ComplI Compl_disjoint closure_interior interior_subset open_interior*)
qed

lemma *compact_filter*:

compact U ⟷ (∀ *F*. *F* ≠ *bot* ⟶ *eventually* (λ*x*. *x* ∈ *U*) *F* ⟶ (∃ *x*∈*U*. *inf* (*nhds x*) *F* ≠ *bot*))
proof (*intro allI iffI impI compact_fip*[*THEN iffD2*] *notI*)
fix *F*
assume *compact U*
assume *F*: *F* ≠ *bot eventually* (λ*x*. *x* ∈ *U*) *F*
then have *U* ≠ {}
by (*auto simp*: *eventually_False*)

define *Z* **where** *Z* = *closure* ‘ {*A*. *eventually* (λ*x*. *x* ∈ *A*) *F*}
then have ∀ *z*∈*Z*. *closed z*
by *auto*
moreover
have *ev_Z*: ⋀*z*. *z* ∈ *Z* ⟹ *eventually* (λ*x*. *x* ∈ *z*) *F*
unfolding *Z_def* **by** (*auto elim*: *eventually_mono intro*: *subsetD*[*OF closure_subset*])
have (∀ *B* ⊆ *Z*. *finite B* ⟶ *U* ∩ ⋂ *B* ≠ {})
proof (*intro allI impI*)
fix *B* **assume** *finite B B* ⊆ *Z*
with ‹*finite B*› *ev_Z F*(*2*) **have** *eventually* (λ*x*. *x* ∈ *U* ∩ (⋂ *B*)) *F*
by (*auto simp*: *eventually_ball_finite_distrib eventually_conj_iff*)
with *F* **show** *U* ∩ ⋂ *B* ≠ {}
by (*intro notI*) (*simp add*: *eventually_False*)
qed
ultimately have *U* ∩ ⋂ *Z* ≠ {}
using ‹*compact U*› **unfolding** *compact_fip* **by** *blast*
then obtain *x* **where** *x* ∈ *U* **and** *x*: ⋀*z*. *z* ∈ *Z* ⟹ *x* ∈ *z*
by *auto*

have ⋀*P*. *eventually P* (*inf* (*nhds x*) *F*) ⟹ *P* ≠ *bot*
unfolding *eventually_inf eventually_nhds*
proof *safe*
fix *P Q R S*
assume *eventually R F open S x* ∈ *S*
with *open_Int_closure_eq_empty*[*of S* {*x*. *R x*}] *x*
have *S* ∩ {*x*. *R x*} ≠ {} **by** (*auto simp*: *Z_def*)
moreover assume *Ball S Q* ∀ *x*. *Q x* ∧ *R x* ⟶ *bot x*
ultimately show *False* **by** (*auto simp*: *set_eq_iff*)
qed
with ‹*x* ∈ *U*› **show** ∃ *x*∈*U*. *inf* (*nhds x*) *F* ≠ *bot*
by (*metis eventually_bot*)
next
fix *A*
assume *A*: ∀ *a*∈*A*. *closed a* ∀ *B*⊆*A*. *finite B* ⟶ *U* ∩ ⋂ *B* ≠ {} *U* ∩ ⋂ *A* = {}
define *F* **where** *F* = (*INF a*∈*insert U A*. *principal a*)
have *F* ≠ *bot*
unfolding *F_def*
proof (*rule INF_filter_not_bot*)
fix *X*

```
    assume X: X ⊆ insert U A finite X
    with A(2)[THEN spec, of X − {U}] have U ∩ ⋂(X − {U}) ≠ {}
      by auto
    with X show (INF a∈X. principal a) ≠ bot
      by (auto simp: INF_principal_finite principal_eq_bot_iff)
  qed
  moreover
  have F ≤ principal U
    unfolding F_def by auto
  then have eventually (λx. x ∈ U) F
    by (auto simp: le_filter_def eventually_principal)
  moreover
  assume ∀F. F ≠ bot ⟶ eventually (λx. x ∈ U) F ⟶ (∃x∈U. inf (nhds x)
F ≠ bot)
  ultimately obtain x where x ∈ U and x: inf (nhds x) F ≠ bot
    by auto

  { fix V assume V ∈ A
    then have F ≤ principal V
      unfolding F_def by (intro INF_lower2[of V]) auto
    then have V: eventually (λx. x ∈ V) F
      by (auto simp: le_filter_def eventually_principal)
    have x ∈ closure V
      unfolding closure_iff_nhds_not_empty
    proof (intro impI allI)
      fix S A
      assume open S x ∈ S S ⊆ A
      then have eventually (λx. x ∈ A) (nhds x)
        by (auto simp: eventually_nhds)
      with V have eventually (λx. x ∈ V ∩ A) (inf (nhds x) F)
        by (auto simp: eventually_inf)
      with x show V ∩ A ≠ {}
        by (auto simp del: Int_iff simp add: trivial_limit_def)
    qed
    then have x ∈ V
      using ‹V ∈ A› A(1) by simp
  }
  with ‹x∈U› have x ∈ U ∩ ⋂A by auto
  with ‹U ∩ ⋂A = {}› show False by auto
qed

definition countably_compact :: ('a::topological_space) set ⇒ bool where
countably_compact U ⟷
  (∀A. countable A ⟶ (∀a∈A. open a) ⟶ U ⊆ ⋃A
    ⟶ (∃T⊆A. finite T ∧ U ⊆ ⋃T))

lemma countably_compactE:
  assumes countably_compact s and ∀t∈C. open t and s ⊆ ⋃C countable C
  obtains C' where C' ⊆ C and finite C' and s ⊆ ⋃C'
```

using *assms* **unfolding** *countably_compact_def* **by** *metis*

lemma *countably_compactI*:
assumes $\bigwedge C.\ \forall\, t{\in}C.\ \mathit{open}\ t \Longrightarrow s \subseteq \bigcup C \Longrightarrow \mathit{countable}\ C \Longrightarrow (\exists\, C'{\subseteq}C.\ \mathit{finite}\ C' \wedge s \subseteq \bigcup C')$
shows *countably_compact s*
using *assms* **unfolding** *countably_compact_def* **by** *metis*

lemma *compact_imp_countably_compact*: *compact U* $\Longrightarrow$ *countably_compact U*
by (*auto simp*: *compact_eq_Heine_Borel countably_compact_def*)

lemma *countably_compact_imp_compact*:
assumes *countably_compact U*
and *ccover*: *countable B* $\forall\, b{\in}B.\ \mathit{open}\ b$
and *basis*: $\bigwedge T\ x.\ \mathit{open}\ T \Longrightarrow x \in T \Longrightarrow x \in U \Longrightarrow \exists\, b{\in}B.\ x \in b \wedge b \cap U \subseteq T$
shows *compact U*
using ‹*countably_compact U*›
unfolding *compact_eq_Heine_Borel countably_compact_def*
proof *safe*
fix *A*
assume *A*: $\forall\, a{\in}A.\ \mathit{open}\ a\ U \subseteq \bigcup A$
assume $*$: $\forall\, A.\ \mathit{countable}\ A \longrightarrow (\forall\, a{\in}A.\ \mathit{open}\ a) \longrightarrow U \subseteq \bigcup A \longrightarrow (\exists\, T{\subseteq}A.\ \mathit{finite}\ T \wedge U \subseteq \bigcup T)$
moreover define *C* **where** $C = \{b{\in}B.\ \exists\, a{\in}A.\ b \cap U \subseteq a\}$
ultimately have *countable C* $\forall\, a{\in}C.\ \mathit{open}\ a$
unfolding *C_def* **using** *ccover* **by** *auto*
moreover
have $\bigcup A \cap U \subseteq \bigcup C$
proof *safe*
fix *x a*
assume $x \in U\ x \in a\ a \in A$
with *basis*[*of a x*] *A* **obtain** *b* **where** $b \in B\ x \in b\ b \cap U \subseteq a$
by *blast*
with ‹$a \in A$› **show** $x \in \bigcup C$
unfolding *C_def* **by** *auto*
qed
then have $U \subseteq \bigcup C$ **using** ‹$U \subseteq \bigcup A$› **by** *auto*
ultimately obtain *T* **where** *T*: $T{\subseteq}C$ *finite T* $U \subseteq \bigcup T$
using $*$ **by** *metis*
then have $\forall\, t{\in}T.\ \exists\, a{\in}A.\ t \cap U \subseteq a$
by (*auto simp*: *C_def*)
then obtain *f* **where** $\forall\, t{\in}T.\ f\ t \in A \wedge t \cap U \subseteq f\ t$
unfolding *bchoice_iff Bex_def* **..**
with *T* **show** $\exists\, T{\subseteq}A.\ \mathit{finite}\ T \wedge U \subseteq \bigcup T$
unfolding *C_def* **by** (*intro exI*[*of* _ *f*‘*T*]) *fastforce*
qed

proposition *countably_compact_imp_compact_second_countable*:

countably_compact U $\Longrightarrow$ *compact* (*U* :: *'a* :: *second_countable_topology set*)
proof (*rule countably_compact_imp_compact*)
fix *T* **and** *x* :: *'a*
assume *open T x* $\in$ *T*
from *topological_basisE*[*OF is_basis this*] **obtain** *b* **where**
b $\in$ (*SOME B. countable B* $\wedge$ *topological_basis B*) *x* $\in$ *b b* $\subseteq$ *T* **.**
then show $\exists$ *b*$\in$*SOME B. countable B* $\wedge$ *topological_basis B. x* $\in$ *b* $\wedge$ *b* $\cap$ *U* $\subseteq$ *T*
by *blast*
qed (*insert countable_basis topological_basis_open*[*OF is_basis*], *auto*)

lemma *countably_compact_eq_compact*:
countably_compact U $\longleftrightarrow$ *compact* (*U* :: *'a* :: *second_countable_topology set*)
using *countably_compact_imp_compact_second_countable compact_imp_countably_compact*
by *blast*

Sequential compactness

definition *seq_compact* :: *'a*::*topological_space set* $\Rightarrow$ *bool* **where**
seq_compact S $\longleftrightarrow$
(∀*f*. (∀ *n*. *f n* $\in$ *S*) $\longrightarrow$ (∃ *l*$\in$*S*. ∃ *r*::*nat*$\Rightarrow$*nat*. *strict_mono r* $\wedge$ (*f* $\circ$ *r*) $\longrightarrow$ *l*))

lemma *seq_compactI*:
assumes $\bigwedge$*f*. ∀ *n*. *f n* $\in$ *S* $\Longrightarrow$ ∃ *l*$\in$*S*. ∃ *r*::*nat*$\Rightarrow$*nat*. *strict_mono r* $\wedge$ (*f* $\circ$ *r*) $\longrightarrow$ *l*
shows *seq_compact S*
unfolding *seq_compact_def* **using** *assms* **by** *fast*

lemma *seq_compactE*:
assumes *seq_compact S* ∀ *n*. *f n* $\in$ *S*
obtains *l r* **where** *l* $\in$ *S strict_mono* (*r* :: *nat* $\Rightarrow$ *nat*) (*f* $\circ$ *r*) $\longrightarrow$ *l*
using *assms* **unfolding** *seq_compact_def* **by** *fast*

lemma *seq_compact_Int_closed*:
assumes *seq_compact S* **and** *closed T*
shows *seq_compact* (*S* $\cap$ *T*)
proof (*rule seq_compactI*)
fix *f* **assume** ∀ *n*::*nat*. *f n* $\in$ *S* $\cap$ *T*
hence ∀ *n*. *f n* $\in$ *S* **and** ∀ *n*. *f n* $\in$ *T*
by *simp_all*
from ‹*seq_compact S*› **and** ‹∀ *n*. *f n* $\in$ *S*›
obtain *l r* **where** *l* $\in$ *S* **and** *r*: *strict_mono r* **and** *l*: (*f* $\circ$ *r*) $\longrightarrow$ *l*
by (*rule seq_compactE*)
from ‹∀ *n*. *f n* $\in$ *T*› **have** ∀ *n*. (*f* $\circ$ *r*) *n* $\in$ *T*
by *simp*
with ‹*l* $\in$ *S*› **and** *r* **and** *l* **show** ∃ *l*$\in$*S* $\cap$ *T*. ∃ *r*. *strict_mono r* $\wedge$ (*f* $\circ$ *r*) $\longrightarrow$ *l*
by (*metis Int_iff* ‹*closed T*› *closed_sequentially*)

qed

lemma *seq_compact_closed_subset*:
assumes *closed S* **and** $S \subseteq T$ **and** *seq_compact T*
shows *seq_compact S*
using *assms seq_compact_Int_closed* [*of T S*] **by** (*simp add*: *Int_absorb1*)

lemma *seq_compact_imp_countably_compact*:
fixes *U* :: *'a* :: *first_countable_topology set*
assumes *seq_compact U*
shows *countably_compact U*
proof (*safe intro*!: *countably_compactI*)
fix *A*
assume *A*: $\forall a{\in}A.$ *open a* $U \subseteq \bigcup A$ *countable A*
have *subseq*: $\bigwedge X.$ *range* $X \subseteq U \Longrightarrow \exists r\, x.\ x \in U \wedge$ *strict_mono* $(r :: nat \Rightarrow nat) \wedge (X \circ r) \longrightarrow x$
using ‹*seq_compact U*› **by** (*fastforce simp*: *seq_compact_def subset_eq*)
show $\exists T{\subseteq}A.$ *finite* $T \wedge U \subseteq \bigcup T$
proof *cases*
assume *finite A*
with *A* **show** *?thesis* **by** *auto*
next
assume *infinite A*
then have $A \neq \{\}$ **by** *auto*
show *?thesis*
proof (*rule ccontr*)
assume $\neg$ $(\exists T{\subseteq}A.$ *finite* $T \wedge U \subseteq \bigcup T)$
then have $\forall T.\ \exists x.\ T \subseteq A \wedge$ *finite* $T \longrightarrow (x \in U - \bigcup T)$
by *auto*
then obtain *X'* **where** *T*: $\bigwedge T.\ T \subseteq A \Longrightarrow$ *finite* $T \Longrightarrow X'\ T \in U - \bigcup T$
by *metis*
define *X* **where** $X\ n = X'$ (*from_nat_into A* ‘ $\{..\ n\}$) **for** *n*
have *X*: $\bigwedge n.\ X\ n \in U - (\bigcup i{\leq}n.$ *from_nat_into A i*)
using ‹$A \neq \{\}$› **unfolding** *X_def* **by** (*intro T*) (*auto intro*: *from_nat_into*)
then have *range* $X \subseteq U$
by *auto*
with *subseq*[*of X*] **obtain** *r x* **where** $x \in U$ **and** *r*: *strict_mono r* $(X \circ r) \longrightarrow x$
by *auto*
from ‹$x{\in}U$› ‹$U \subseteq \bigcup A$› *from_nat_into_surj*[*OF* ‹*countable A*›]
obtain *n* **where** $x \in$ *from_nat_into A n* **by** *auto*
with *r*(*2*) *A*(*1*) *from_nat_into*[*OF* ‹$A \neq \{\}$›]
have *eventually* $(\lambda i.\ X\ (r\ i) \in$ *from_nat_into A n*) *sequentially*
unfolding *tendsto_def* **by** *fastforce*
then obtain *N* **where** $\bigwedge i.\ N \leq i \Longrightarrow X\ (r\ i) \in$ *from_nat_into A n*
by (*auto simp*: *eventually_sequentially*)
moreover from *X* **have** $\bigwedge i.\ n \leq r\ i \Longrightarrow X\ (r\ i) \notin$ *from_nat_into A n*
by *auto*
moreover from ‹*strict_mono r*›[*THEN seq_suble*, *of max n N*] **have** $\exists i.\ n$

```
≤ r i ∧ N ≤ i
        by (auto intro!: exI[of _ max n N])
      ultimately show False
        by auto
    qed
  qed
qed

lemma compact_imp_seq_compact:
  fixes U :: 'a :: first_countable_topology set
  assumes compact U
  shows seq_compact U
  unfolding seq_compact_def
proof safe
  fix X :: nat ⇒ 'a
  assume ∀ n. X n ∈ U
  then have eventually (λx. x ∈ U) (filtermap X sequentially)
    by (auto simp: eventually_filtermap)
  moreover
  have filtermap X sequentially ≠ bot
    by (simp add: trivial_limit_def eventually_filtermap)
  ultimately
  obtain x where x ∈ U and x: inf (nhds x) (filtermap X sequentially) ≠ bot (is
?F ≠ _)
    using ‹compact U› by (auto simp: compact_filter)

  from countable_basis_at_decseq[of x]
  obtain A where A:
      ⋀i. open (A i)
      ⋀i. x ∈ A i
      ⋀S. open S ⟹ x ∈ S ⟹ eventually (λi. A i ⊆ S) sequentially
    by blast
  define s where s n i = (SOME j. i < j ∧ X j ∈ A (Suc n)) for n i
  {
    fix n i
    have ∃ a. i < a ∧ X a ∈ A (Suc n)
    proof (rule ccontr)
      assume ¬ (∃ a>i. X a ∈ A (Suc n))
      then have ⋀a. Suc i ≤ a ⟹ X a ∉ A (Suc n)
        by auto
      then have eventually (λx. x ∉ A (Suc n)) (filtermap X sequentially)
        by (auto simp: eventually_filtermap eventually_sequentially)
      moreover have eventually (λx. x ∈ A (Suc n)) (nhds x)
        using A(1,2)[of Suc n] by (auto simp: eventually_nhds)
      ultimately have eventually (λx. False) ?F
        by (auto simp: eventually_inf)
      with x show False
        by (simp add: eventually_False)
    qed
```

```
    then have i < s n i X (s n i) ∈ A (Suc n)
      unfolding s_def by (auto intro: someI2_ex)
  }
  note s = this
  define r where r = rec_nat (s 0 0) s
  have strict_mono r
    by (auto simp: r_def s strict_mono_Suc_iff)
  moreover
  have (λn. X (r n)) ⟶ x
  proof (rule topological_tendstoI)
    fix S
    assume open S x ∈ S
    with A(3) have eventually (λi. A i ⊆ S) sequentially
      by auto
    moreover
    {
      fix i
      assume Suc 0 ≤ i
      then have X (r i) ∈ A i
        by (cases i) (simp_all add: r_def s)
    }
    then have eventually (λi. X (r i) ∈ A i) sequentially
      by (auto simp: eventually_sequentially)
    ultimately show eventually (λi. X (r i) ∈ S) sequentially
      by eventually_elim auto
  qed
  ultimately show ∃ x ∈ U. ∃ r. strict_mono r ∧ (X ∘ r) ⟶ x
    using ‹x ∈ U› by (auto simp: convergent_def comp_def)
qed

lemma countably_compact_imp_acc_point:
  assumes countably_compact S
    and countable T
    and infinite T
    and T ⊆ S
  shows ∃ x∈S. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ T)
proof (rule ccontr)
  define C where C = (λF. interior (F ∪ (- T))) ` {F. finite F ∧ F ⊆ T }
  note ‹countably_compact S›
  moreover have ∀ T∈C. open T
    by (auto simp: C_def)
  moreover
  assume ¬ (∃ x∈S. ∀ U. x∈U ∧ open U ⟶ infinite (U ∩ T))
  then have S: ⋀x. x ∈ S ⟹ ∃ U. x∈U ∧ open U ∧ finite (U ∩ T) by metis
  have S ⊆ ⋃ C
    using ‹T ⊆ S›
    unfolding C_def
    apply (safe dest!: S)
    apply (rule_tac a=U ∩ T in UN_I)
```

```
    apply (auto intro!: interiorI simp add: finite_subset)
    done
  moreover
  from ‹countable T› have countable C
    unfolding C_def by (auto intro: countable_Collect_finite_subset)
  ultimately
  obtain D where D ⊆ C finite D S ⊆ ⋃ D
    by (rule countably_compactE)
  then obtain E where E: E ⊆ {F. finite F ∧ F ⊆ T } finite E
    and S: S ⊆ (⋃ F∈E. interior (F ∪ (- T)))
    by (metis (lifting) finite_subset_image C_def)
  from S ‹T ⊆ S› have T ⊆ ⋃ E
    using interior_subset by blast
  moreover have finite (⋃ E)
    using E by auto
  ultimately show False using ‹infinite T›
    by (auto simp: finite_subset)
qed

lemma countable_acc_point_imp_seq_compact:
  fixes S :: 'a::first_countable_topology set
  assumes ⋀T. ⟦infinite T; countable T; T ⊆ S⟧ ⟹ ∃ x∈S. ∀ U. x∈U ∧ open U
⟶ infinite (U ∩ T)
  shows seq_compact S
  unfolding seq_compact_def
proof (intro strip)
  fix f :: nat ⇒ 'a
  assume f: ∀ n. f n ∈ S
  show ∃ l∈S. ∃ r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
  proof (cases finite (range f))
    case True
    obtain l where infinite {n. f n = f l}
      using pigeonhole_infinite[OF _ True] by auto
    then obtain r :: nat ⇒ nat where strict_mono  r and fr: ∀ n. f (r n) = f l
      using infinite_enumerate by blast
    then have strict_mono r ∧ (f ∘ r) ⟶ f l
      by (simp add: fr o_def)
    with f show ∃ l∈S. ∃ r. strict_mono  r ∧ (f ∘ r) ⟶ l
      by auto
  next
    case False
    with f assms obtain l where l ∈ S ∀ U. l∈U ∧ open U ⟶ infinite (U ∩
range f)
      by (metis image_subset_iff uncountable_def)
    with ‹l ∈ S› show ∃ l∈S. ∃ r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
      by (meson acc_point_range_imp_convergent_subsequence)
  qed
qed
```

lemma *seq_compact_eq_countably_compact*:
fixes $U :: 'a :: first\_countable\_topology\ set$
shows $seq\_compact\ U \longleftrightarrow countably\_compact\ U$
by (*metis countable_acc_point_imp_seq_compact countably_compact_imp_acc_point seq_compact_imp_countably_compact*)

lemma *seq_compact_eq_acc_point*:
fixes $S :: 'a :: first\_countable\_topology\ set$
shows $seq\_compact\ S \longleftrightarrow$
$(\forall T.\ infinite\ T \wedge countable\ T \wedge T \subseteq S \longrightarrow (\exists x{\in}S.\ \forall U.\ x{\in}U \wedge open\ U \longrightarrow infinite\ (U \cap T)))$
by (*metis countable_acc_point_imp_seq_compact countably_compact_imp_acc_point seq_compact_imp_countably_compact*)

lemma *seq_compact_eq_compact*:
fixes $U :: 'a :: second\_countable\_topology\ set$
shows $seq\_compact\ U \longleftrightarrow compact\ U$
using *seq_compact_eq_countably_compact countably_compact_eq_compact* **by** *blast*

proposition *Bolzano_Weierstrass_imp_seq_compact*:
fixes $S :: 'a{::}\{t1\_space, first\_countable\_topology\}\ set$
shows $(\bigwedge T.\ [\![infinite\ T;\ T \subseteq S]\!] \Longrightarrow \exists x \in S.\ x\ islimpt\ T) \Longrightarrow seq\_compact\ S$
by (*rule countable_acc_point_imp_seq_compact*) (*metis islimpt_eq_acc_point*)

2.1.11 Cartesian products

lemma *seq_compact_Times*:
assumes *seq_compact S seq_compact T*
shows $seq\_compact\ (S \times T)$
unfolding *seq_compact_def*
proof *clarify*
fix $h :: nat \Rightarrow 'a \times 'b$
assume $\forall n.\ h\ n \in S \times T$
then have $*$: $\bigwedge n.\ (fst \circ h)\ n \in S\ \bigwedge n.\ (snd \circ h)\ n \in T$
by (*simp_all add*: *mem_Times_iff*)
then obtain *lS* **and** *rS* :: *nat⇒nat*
where *lS∈S strict_mono rS* **and** *lS*: $(fst \circ h \circ rS) \longrightarrow lS$
using *assms seq_compact_def* **by** *metis*
then obtain *lT* **and** *rT* :: *nat⇒nat*
where *lT∈T strict_mono rT* **and** *lT*: $(snd \circ h \circ rS \circ rT) \longrightarrow lT$
using *assms seq_compact_def* $*$
by (*metis* (*mono_tags*, *lifting*) *comp_apply*)
have *strict_mono* $(rS \circ rT)$
by (*simp add*: ‹*strict_mono rS*› ‹*strict_mono rT*› *strict_mono_o*)
moreover have $(h \circ (rS \circ rT)) \longrightarrow (lS,lT)$
using *tendsto_Pair* [*OF LIMSEQ_subseq_LIMSEQ* [*OF lS* ‹*strict_mono rT*›] *lT*]
by (*simp add*: *o_def*)

ultimately show $\exists\, l{\in}S \times T.\ \exists\, r.\ \mathit{strict\_mono}\ r \wedge (h \circ r) \longrightarrow l$
using ‹$lS \in S$› ‹$lT \in T$› **by** *blast*
qed

lemma *compact_Times*:
assumes *compact S compact T*
shows *compact* $(S \times T)$
proof (*rule compactI*)
fix $\mathcal{C}$
assume C: $\forall\, T{\in}\mathcal{C}.\ \mathit{open}\ T\ S \times T \subseteq \bigcup \mathcal{C}$
have $\forall\, x{\in}S.\ \exists\, A.\ \mathit{open}\ A \wedge x \in A \wedge (\exists\, D{\subseteq}\mathcal{C}.\ \mathit{finite}\ D \wedge A \times T \subseteq \bigcup D)$
proof
fix x
assume $x \in S$
have $\forall\, y{\in}T.\ \exists\, A\ B\ C.\ C \in \mathcal{C} \wedge \mathit{open}\ A \wedge \mathit{open}\ B \wedge x \in A \wedge y \in B \wedge A \times B \subseteq C$
by (*smt* (*verit*, *ccfv_threshold*) *C UnionE* ‹$x \in S$› *mem_Sigma_iff open_prod_def subsetD*)
then obtain $a\ b\ c$ **where** b: $\bigwedge y.\ y \in T \Longrightarrow \mathit{open}\ (b\ y)$
and c: $\bigwedge y.\ y \in T \Longrightarrow c\ y \in \mathcal{C} \wedge \mathit{open}\ (a\ y) \wedge \mathit{open}\ (b\ y) \wedge x \in a\ y \wedge y \in b\ y \wedge a\ y \times b\ y \subseteq c\ y$
by *metis*
then have $\forall\, y{\in}T.\ \mathit{open}\ (b\ y)\ T \subseteq (\bigcup y{\in}T.\ b\ y)$ **by** *auto*
with *compactE_image*[*OF* ‹*compact T*›] **obtain** D **where** D: $D \subseteq T$ *finite* D $T \subseteq (\bigcup y{\in}D.\ b\ y)$
by *metis*
moreover from $D\ c$ **have** $(\bigcap y{\in}D.\ a\ y) \times T \subseteq (\bigcup y{\in}D.\ c\ y)$
by (*fastforce simp*: *subset_eq*)
ultimately show $\exists\, a.\ \mathit{open}\ a \wedge x \in a \wedge (\exists\, d{\subseteq}\mathcal{C}.\ \mathit{finite}\ d \wedge a \times T \subseteq \bigcup d)$
using *c* **by** (*intro exI*[*of* _ *c*‘*D*] *exI*[*of* _ $\bigcap$(*a*‘*D*)] *conjI*) (*auto intro*!: *open_INT*)
qed
then obtain $a\ d$ **where** a: $\bigwedge x.\ x{\in}S \Longrightarrow \mathit{open}\ (a\ x)\ S \subseteq (\bigcup x{\in}S.\ a\ x)$
and d: $\bigwedge x.\ x \in S \Longrightarrow d\ x \subseteq \mathcal{C} \wedge \mathit{finite}\ (d\ x) \wedge a\ x \times T \subseteq \bigcup (d\ x)$
unfolding *subset_eq UN_iff* **by** *metis*
moreover
from *compactE_image*[*OF* ‹*compact S*› *a*]
obtain e **where** e: $e \subseteq S$ *finite* e **and** S: $S \subseteq (\bigcup x{\in}e.\ a\ x)$
by *auto*
moreover
have $S \times T \subseteq (\bigcup x{\in}e.\ \bigcup (d\ x))$
by (*smt* (*verit*, *del_insts*) *S SigmaE UN_iff d e*(1) *mem_Sigma_iff subset_eq*)
ultimately show $\exists\, C'{\subseteq}\mathcal{C}.\ \mathit{finite}\ C' \wedge S \times T \subseteq \bigcup C'$
by (*intro exI*[*of* _ $(\bigcup x{\in}e.\ d\ x)$]) (*auto simp*: *subset_eq*)
qed

lemma *tube_lemma*:
assumes *compact K*

```
  assumes open W
  assumes {x0} × K ⊆ W
  shows ∃X0. x0 ∈ X0 ∧ open X0 ∧ X0 × K ⊆ W
proof −
  {
    fix y assume y ∈ K
    then have (x0, y) ∈ W using assms by auto
    with ‹open W› 
    have ∃X0 Y. open X0 ∧ open Y ∧ x0 ∈ X0 ∧ y ∈ Y ∧ X0 × Y ⊆ W
      by (rule open_prod_elim) blast
  }
  then obtain X0 Y where
    ∗: ∀y ∈ K. open (X0 y) ∧ open (Y y) ∧ x0 ∈ X0 y ∧ y ∈ Y y ∧ X0 y × Y y
⊆ W
    by metis
  from ∗ have ∀t∈Y ‘ K. open t K ⊆ ⋃(Y ‘ K) by auto
  with ‹compact K› obtain CC where CC: CC ⊆ Y ‘ K finite CC K ⊆ ⋃CC
    by (meson compactE)
  then obtain c where c: ⋀C. C ∈ CC ⟹ c C ∈ K ∧ C = Y (c C)
    by (force intro!: choice)
  with ∗ CC show ?thesis
    by (force intro!: exI[where x=⋂C∈CC. X0 (c C)])
qed

lemma continuous_on_prod_compactE:
  fixes fx::'a::topological_space × 'b::topological_space ⇒ 'c::metric_space
    and e::real
  assumes cont_fx: continuous_on (U × C) fx
  assumes compact C
  assumes [intro]: x0 ∈ U
  notes [continuous_intros] = continuous_on_compose2[OF cont_fx]
  assumes e > 0
  obtains X0 where x0 ∈ X0 open X0
    ∀x∈X0 ∩ U. ∀t ∈ C. dist (fx (x, t)) (fx (x0, t)) ≤ e
proof −
  define psi where psi = (λ(x, t). dist (fx (x, t)) (fx (x0, t)))
  define W0 where W0 = {(x, t) ∈ U × C. psi (x, t) < e}
  have W0_eq: W0 = psi −‘ {..<e} ∩ U × C
    by (auto simp: vimage_def W0_def)
  have open {..<e} by simp
  have continuous_on (U × C) psi
    by (auto intro!: continuous_intros simp: psi_def split_beta')
  then obtain W where W: open W W ∩ U × C = W0 ∩ U × C
    unfolding W0_eq
    by (metis ‹open {..<e}› continuous_on_open_invariant inf_right_idem)
  have {x0} × C ⊆ W ∩ U × C
    unfolding W
    by (auto simp: W0_def psi_def ‹0 < e›)
  then have {x0} × C ⊆ W by blast
```

from *tube_lemma*[*OF* ‹*compact C*› ‹*open W*› *this*]
obtain *X0* **where** *X0*: *x0* ∈ *X0 open X0 X0* × *C* ⊆ *W*
by *blast*

have ∀ *x*∈*X0* ∩ *U*. ∀ *t* ∈ *C*. *dist* (*fx* (*x*, *t*)) (*fx* (*x0*, *t*)) ≤ *e*
proof *safe*
fix *x* **assume** *x*: *x* ∈ *X0 x* ∈ *U*
fix *t* **assume** *t*: *t* ∈ *C*
have *dist* (*fx* (*x*, *t*)) (*fx* (*x0*, *t*)) = *psi* (*x*, *t*)
by (*auto simp*: *psi_def*)
also have *psi* (*x*, *t*) < *e*
using *W*(*2*) *W0_def X0*(*3*) *t x* **by** *fastforce*
finally show *dist* (*fx* (*x*,*t*)) (*fx* (*x0*,*t*)) ≤ *e* **by** *simp*
qed
from *X0*(*1*,*2*) *this* **show** *?thesis* **..**
qed

2.1.12 Continuity

lemma *continuous_at_imp_continuous_within*:
continuous (*at x*) *f* ⟹ *continuous* (*at x within s*) *f*
unfolding *continuous_within continuous_at* **using** *Lim_at_imp_Lim_at_within*
by *auto*

lemma *Lim_trivial_limit*: *trivial_limit net* ⟹ (*f* ⟶ *l*) *net*
by *simp*

lemmas *continuous_on* = *continuous_on_def* — legacy theorem name

lemma *continuous_within_subset*:
continuous (*at x within S*) *f* ⟹ *t* ⊆ *S* ⟹ *continuous* (*at x within t*) *f*
unfolding *continuous_within* **by**(*metis tendsto_within_subset*)

lemma *continuous_on_interior*:
continuous_on S f ⟹ *x* ∈ *interior S* ⟹ *continuous* (*at x*) *f*
by (*metis continuous_on_eq_continuous_at continuous_on_subset interiorE*)

lemma *continuous_on_eq*:
⟦*continuous_on S f*; ⋀*x*. *x* ∈ *S* ⟹ *f x* = *g x*⟧ ⟹ *continuous_on S g*
unfolding *continuous_on_def tendsto_def eventually_at_topological*
by *simp*

Characterization of various kinds of continuity in terms of sequences.

lemma *continuous_within_sequentiallyI*:
fixes *f* :: *'a*::{*first_countable_topology*, *t2_space*} ⇒ *'b*::*topological_space*
assumes ⋀*u*::*nat* ⇒ *'a*. *u* ⟶ *a* ⟹ (∀ *n*. *u n* ∈ *S*) ⟹ (λ*n*. *f* (*u n*)) ⟶ *f a*
shows *continuous* (*at a within S*) *f*
using *assms* **unfolding** *continuous_within tendsto_def*[**where** *l* = *f a*]

by (*auto intro*!: *sequentially_imp_eventually_within*)

lemma *continuous_within_tendsto_compose*:
fixes *f*::'*a*::*t2_space* ⇒ '*b*::*topological_space*
assumes *f*: *continuous* (*at a within S*) *f*
and *eventually* (λ*n*. *x n* ∈ *S*) *F* (*x* ⟶ *a*) *F*
shows ((λ*n*. *f* (*x n*)) ⟶ *f a*) *F*
proof −
have ∗: *filterlim x* (*inf* (*nhds a*) (*principal S*)) *F*
by (*simp add*: *assms filterlim_inf filterlim_principal*)
show *?thesis*
using ∗ *f continuous_within filterlim_compose tendsto_at_within_iff_tendsto_nhds*
by *blast*
qed

lemma *continuous_within_tendsto_compose'*:
fixes *f*::'*a*::*t2_space* ⇒ '*b*::*topological_space*
assumes *continuous* (*at a within S*) *f*
⋀*n*. *x n* ∈ *S*
(*x* ⟶ *a*) *F*
shows ((λ*n*. *f* (*x n*)) ⟶ *f a*) *F*
using *always_eventually assms continuous_within_tendsto_compose* **by** *blast*

lemma *continuous_within_sequentially*:
fixes *f* :: '*a*::{*first_countable_topology*, *t2_space*} ⇒ '*b*::*topological_space*
shows *continuous* (*at a within S*) *f* ⟷
(∀ *x*. (∀ *n*::*nat*. *x n* ∈ *S*) ∧ (*x* ⟶ *a*) *sequentially*
⟶ ((*f* ∘ *x*) ⟶ *f a*) *sequentially*)
using *continuous_within_tendsto_compose'*[*of a S f* _ *sequentially*]
continuous_within_sequentiallyI[*of a S f*]
by (*auto simp*: *o_def*)

lemma *continuous_at_sequentiallyI*:
fixes *f* :: '*a*::{*first_countable_topology*, *t2_space*} ⇒ '*b*::*topological_space*
assumes ⋀*u*. *u* ⟶ *a* ⟹ (λ*n*. *f* (*u n*)) ⟶ *f a*
shows *continuous* (*at a*) *f*
using *continuous_within_sequentiallyI*[*of a UNIV f*] *assms* **by** *auto*

lemma *continuous_at_sequentially*:
fixes *f* :: '*a*::*metric_space* ⇒ '*b*::*topological_space*
shows *continuous* (*at a*) *f* ⟷
(∀ *x*. (*x* ⟶ *a*) *sequentially* −−> ((*f* ∘ *x*) ⟶ *f a*) *sequentially*)
using *continuous_within_sequentially*[*of a UNIV f*] **by** *simp*

lemma *continuous_on_sequentiallyI*:
fixes *f* :: '*a*::{*first_countable_topology*, *t2_space*} ⇒ '*b*::*topological_space*
assumes ⋀*u a*. (∀ *n*. *u n* ∈ *S*) ⟹ *a* ∈ *S* ⟹ *u* ⟶ *a* ⟹ (λ*n*. *f* (*u n*))
⟶ *f a*
shows *continuous_on S f*

using *assms* **unfolding** *continuous_on_eq_continuous_within*
using *continuous_within_sequentiallyI*[*of* _ *S f*] **by** *auto*

lemma *continuous_on_sequentially*:
fixes *f* :: *'a*::{*first_countable_topology*, *t2_space*} ⇒ *'b*::*topological_space*
shows *continuous_on S f* ⟷
(∀ *x*. ∀ *a* ∈ *S*. (∀ *n*. *x*(*n*) ∈ *S*) ∧ (*x* ⟶ *a*) *sequentially*
⟶ ((*f* ∘ *x*) ⟶ *f a*) *sequentially*)
by (*meson continuous_on_eq_continuous_within continuous_within_sequentially*)

Continuity in terms of open preimages.

lemma *continuous_at_open*:
continuous (*at x*) *f* ⟷ (∀ *t*. *open t* ∧ *f x* ∈ *t* ⟶ (∃ *S*. *open S* ∧ *x* ∈ *S* ∧ (∀ *x'* ∈ *S*. (*f x'*) ∈ *t*)))
by (*metis UNIV_I continuous_within_topological*)

lemma *continuous_imp_tendsto*:
assumes *continuous* (*at x0*) *f* **and** *x* ⟶ *x0*
shows (*f* ∘ *x*) ⟶ (*f x0*)
proof (*rule topological_tendstoI*)
fix *S*
assume *open S f x0* ∈ *S*
then obtain *T* **where** *T_def*: *open T x0* ∈ *T* ∀ *x*∈*T*. *f x* ∈ *S*
using *assms continuous_at_open* **by** *metis*
then have *eventually* (λ*n*. *x n* ∈ *T*) *sequentially*
using *assms T_def* **by** (*auto simp*: *tendsto_def*)
then show *eventually* (λ*n*. (*f* ∘ *x*) *n* ∈ *S*) *sequentially*
using *T_def* **by** (*auto elim*!: *eventually_mono*)
qed

2.1.13 Homeomorphisms

definition *homeomorphism S T f g* ⟷
(∀ *x*∈*S*. (*g*(*f x*) = *x*)) ∧ (*f* ‘ *S* = *T*) ∧ *continuous_on S f* ∧
(∀ *y*∈*T*. (*f*(*g y*) = *y*)) ∧ (*g* ‘ *T* = *S*) ∧ *continuous_on T g*

lemma *homeomorphismI* [*intro?*]:
assumes *continuous_on S f continuous_on T g*
f ‘ *S* ⊆ *T g* ‘ *T* ⊆ *S* ⋀*x*. *x* ∈ *S* ⟹ *g*(*f x*) = *x* ⋀*y*. *y* ∈ *T* ⟹ *f*(*g y*) = *y*
shows *homeomorphism S T f g*
using *assms* **by** (*force simp*: *homeomorphism_def*)

lemma *homeomorphism_translation*:
fixes *a* :: *'a* :: *real_normed_vector*
shows *homeomorphism* ((+) *a* ‘ *S*) *S* ((+) (− *a*)) ((+) *a*)
unfolding *homeomorphism_def* **by** (*auto simp*: *algebra_simps continuous_intros*)

lemma *homeomorphism_ident*: *homeomorphism T T* (λ*a*. *a*) (λ*a*. *a*)
by (*rule homeomorphismI*) *auto*

lemma *homeomorphism_compose*:
assumes *homeomorphism S T f g homeomorphism T U h k*
shows *homeomorphism S U* (*h o f*) (*g o k*)
using *assms*
unfolding *homeomorphism_def*
by (*intro conjI ballI continuous_on_compose*) (*auto simp*: *image_iff*)

lemma *homeomorphism_cong*:
homeomorphism X' Y' f' g'
if *homeomorphism X Y f g X' = X Y' = Y* ⋀*x*. *x* ∈ *X* ⟹ *f' x* = *f x* ⋀*y*. *y* ∈ *Y* ⟹ *g' y* = *g y*
using *that* **by** (*auto simp add*: *homeomorphism_def*)

lemma *homeomorphism_empty* [*simp*]:
homeomorphism {} {} *f g*
unfolding *homeomorphism_def* **by** *auto*

lemma *homeomorphism_symD*: *homeomorphism S t f g* ⟹ *homeomorphism t S g f*
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_sym*: *homeomorphism S t f g* = *homeomorphism t S g f*
by (*force simp*: *homeomorphism_def*)

lemma *continuous_on_translation_eq*:
fixes *g* :: *'a* :: *real_normed_vector* ⇒ *'b* :: *real_normed_vector*
shows *continuous_on A* ((+) *a* ∘ *g*) = *continuous_on A g*
proof −
have *g*: *g* = (λ*x*. −*a* + *x*) ∘ ((λ*x*. *a* + *x*) ∘ *g*)
by (*rule ext*) *simp*
show *?thesis*
by (*metis* (*no_types*, *opaque_lifting*) *g continuous_on_compose homeomorphism_def homeomorphism_translation*)
qed

definition *homeomorphic* :: *'a*::*topological_space set* ⇒ *'b*::*topological_space set* ⇒ *bool*
(**infixr** ‹*homeomorphic*› *60*)
where *s homeomorphic t* ≡ (∃*f g*. *homeomorphism s t f g*)

lemma *homeomorphic_empty* [*iff*]:
S homeomorphic {} ⟷ *S* = {} {} *homeomorphic S* ⟷ *S* = {}
by (*auto simp*: *homeomorphic_def homeomorphism_def*)

lemma *homeomorphic_refl*: *S homeomorphic S*
using *homeomorphic_def homeomorphism_ident* **by** *fastforce*

lemma *homeomorphic_sym*: *S homeomorphic T* ⟷ *T homeomorphic S*

unfolding *homeomorphic_def homeomorphism_def*
by *blast*

lemma *homeomorphic_trans* [*trans*]:
assumes *S homeomorphic T* **and** *T homeomorphic U*
shows *S homeomorphic U*
using *assms* **unfolding** *homeomorphic_def*
by (*metis homeomorphism_compose*)

lemma *homeomorphic_minimal*:
S homeomorphic T $\longleftrightarrow$
$(\exists f\ g.\ (\forall\, x{\in}S.\ f(x) \in T \wedge (g(f(x)) = x)) \wedge$
$(\forall\, y{\in}T.\ g(y) \in S \wedge (f(g(y)) = y)) \wedge$
continuous_on S f $\wedge$ *continuous_on T g*)
by (*smt* (*verit*, *ccfv_threshold*) *homeomorphic_def homeomorphismI homeomorphism_def image_eqI image_subset_iff*)

lemma *homeomorphicI* [*intro?*]:
$[\![f\ `\ S = T;\ g\ `\ T = S;$
continuous_on S f; *continuous_on T g*;
$\bigwedge x.\ x \in S \Longrightarrow g(f(x)) = x;$
$\bigwedge y.\ y \in T \Longrightarrow f(g(y)) = y]\!] \Longrightarrow$ *S homeomorphic T*
unfolding *homeomorphic_def homeomorphism_def* **by** *metis*

lemma *homeomorphism_of_subsets*:
$[\![$*homeomorphism S T f g*; $S' \subseteq S$; $T'' \subseteq T$; $f\ `\ S' = T']\!]$
$\Longrightarrow$ *homeomorphism S' T' f g*
by (*smt* (*verit*, *del_insts*) *continuous_on_subset homeomorphismI homeomorphism_def imageE subset_eq*)

lemma *homeomorphism_apply1*: $[\![$*homeomorphism S T f g*; $x \in S]\!] \Longrightarrow g(f\ x) = x$
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_apply2*: $[\![$*homeomorphism S T f g*; $x \in T]\!] \Longrightarrow f(g\ x) = x$
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_image1*: *homeomorphism S T f g* $\Longrightarrow$ *f ` S = T*
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_image2*: *homeomorphism S T f g* $\Longrightarrow$ *g ` T = S*
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_cont1*: *homeomorphism S T f g* $\Longrightarrow$ *continuous_on S f*
by (*simp add*: *homeomorphism_def*)

lemma *homeomorphism_cont2*: *homeomorphism S T f g* $\Longrightarrow$ *continuous_on T g*
by (*simp add*: *homeomorphism_def*)

lemma *continuous_on_no_limpt*:
(⋀*x*. ¬ *x islimpt S*) ⟹ *continuous_on S f*
unfolding *continuous_on_def*
by (*metis UNIV_I empty_iff eventually_at_topological islimptE open_UNIV tendsto_def trivial_limit_within*)

lemma *continuous_on_finite*:
fixes *S* :: *'a*::*t1_space set*
shows *finite S* ⟹ *continuous_on S f*
by (*metis continuous_on_no_limpt islimpt_finite*)

lemma *homeomorphic_finite*:
fixes *S* :: *'a*::*t1_space set* **and** *T* :: *'b*::*t1_space set*
assumes *finite T*
shows *S homeomorphic T* ⟷ *finite S* ∧ *finite T* ∧ *card S* = *card T* (**is** *?lhs* = *?rhs*)
proof
assume *S homeomorphic T*
with *assms* **show** *?rhs*
by (*metis* (*full_types*) *card_image_le finite_imageI homeomorphic_def homeomorphism_def le_antisym*)
next
assume *R*: *?rhs*
with *finite_same_card_bij* **obtain** *h* **where** *bij_betw h S T*
by *auto*
with *R* **show** *?lhs*
apply (*simp only*: *homeomorphic_def homeomorphism_def continuous_on_finite*)
by (*smt* (*verit*, *ccfv_SIG*) *bij_betw_imp_surj_on bij_betw_inv_into bij_betw_inv_into_left bij_betw_inv_into_right*)
qed

Relatively weak hypotheses if a set is compact.

lemma *homeomorphism_compact*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*t2_space*
assumes *compact S continuous_on S f f* ‘ *S* = *T inj_on f S*
shows ∃ *g*. *homeomorphism S T f g*
proof −
obtain *g* **where** *g*: ∀ *x*∈*S*. *g* (*f x*) = *x* ∀ *x*∈*T*. *f* (*g x*) = *x g* ‘ *T* = *S*
using *assms the_inv_into_f_f* **by** *fastforce*
with *assms* **show** *?thesis*
unfolding *homeomorphism_def homeomorphic_def* **by** (*metis continuous_on_inv*)
qed

lemma *homeomorphic_compact*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*t2_space*
shows *compact S* ⟹ *continuous_on S f* ⟹ (*f* ‘ *S* = *T*) ⟹ *inj_on f S* ⟹ *S homeomorphic T*
unfolding *homeomorphic_def* **by** (*metis homeomorphism_compact*)

Preservation of topological properties.

lemma *homeomorphic_compactness*: *S homeomorphic T* $\Longrightarrow$ (*compact S* $\longleftrightarrow$ *compact T*)
unfolding *homeomorphic_def homeomorphism_def*
by (*metis compact_continuous_image*)

2.1.14 On Linorder Topologies

lemma *islimpt_greaterThanLessThan1*:
fixes *a b*::'*a*::{*linorder_topology*, *dense_order*}
assumes $a < b$
shows *a islimpt* {*a*<..<*b*}
proof (*rule islimptI*)
fix *T*
assume *open T a* $\in$ *T*
then obtain *c* **where** *c*: $a < c$ {*a*..<*c*} $\subseteq$ *T*
by (*meson assms open_right*)
with *assms dense*[*of a min c b*]
show $\exists\, y{\in}${*a*<..<*b*}. $y \in T \wedge y \neq a$
by (*metis atLeastLessThan_iff greaterThanLessThan_iff min_less_iff_conj not_le order.strict_implies_order subset_eq*)
qed

lemma *islimpt_greaterThanLessThan2*:
fixes *a b*::'*a*::{*linorder_topology*, *dense_order*}
assumes $a < b$
shows *b islimpt* {*a*<..<*b*}
proof (*rule islimptI*)
fix *T*
assume *open T b* $\in$ *T*
from *open_left*[*OF this* ‹$a < b$›]
obtain *c* **where** *c*: $c < b$ {*c*<..*b*} $\subseteq$ *T* **by** *auto*
with *assms dense*[*of max a c b*]
show $\exists\, y{\in}${*a*<..<*b*}. $y \in T \wedge y \neq b$
by (*metis greaterThanAtMost_iff greaterThanLessThan_iff max_less_iff_conj not_le order.strict_implies_order subset_eq*)
qed

lemma *closure_greaterThanLessThan*[*simp*]:
fixes *a b*::'*a*::{*linorder_topology*, *dense_order*}
shows $a < b \Longrightarrow$ *closure* {*a* <..< *b*} = {*a* .. *b*} (**is** _ $\Longrightarrow$ *?l* = *?r*)
proof
have *?l* $\subseteq$ *closure ?r*
by (*rule closure_mono*) *auto*
thus *closure* {*a*<..<*b*} $\subseteq$ {*a*..*b*} **by** *simp*
qed (*auto simp*: *closure_def order.order_iff_strict islimpt_greaterThanLessThan1 islimpt_greaterThanLessThan2*)

lemma *closure_greaterThan*[*simp*]:
fixes *a b*::'*a*::{*no_top*, *linorder_topology*, *dense_order*}

```
  shows closure {a<..} = {a..}
proof −
  from gt_ex obtain b where a < b by auto
  hence {a<..} = {a<..<b} ∪ {b..} by auto
  also have closure ... = {a..} using ‹a < b› unfolding closure_Un
    by auto
  finally show ?thesis .
qed

lemma closure_lessThan[simp]:
  fixes b::'a::{no_bot, linorder_topology, dense_order}
  shows closure {..<b} = {..b}
proof −
  from lt_ex obtain a where a < b by auto
  hence {..<b} = {a<..<b} ∪ {..a} by auto
  also have closure ... = {..b} using ‹a < b› unfolding closure_Un
    by auto
  finally show ?thesis .
qed

lemma closure_atLeastLessThan[simp]:
  fixes a b::'a::{linorder_topology, dense_order}
  assumes a < b
  shows closure {a ..< b} = {a .. b}
proof −
  from assms have {a ..< b} = {a} ∪ {a <..< b} by auto
  also have closure ... = {a .. b} unfolding closure_Un
    by (auto simp: assms less_imp_le)
  finally show ?thesis .
qed

lemma closure_greaterThanAtMost[simp]:
  fixes a b::'a::{linorder_topology, dense_order}
  assumes a < b
  shows closure {a <.. b} = {a .. b}
proof −
  from assms have {a <.. b} = {b} ∪ {a <..< b} by auto
  also have closure ... = {a .. b} unfolding closure_Un
    by (auto simp: assms less_imp_le)
  finally show ?thesis .
qed

end
theory Abstract_Limits
  imports
    Abstract_Topology
begin
```

2.1.15 nhdsin and atin

definition *nhdsin* :: *'a topology* ⇒ *'a* ⇒ *'a filter*
where *nhdsin X a* =
(*if a* ∈ *topspace X then* (*INF S*∈{*S*. *openin X S* ∧ *a* ∈ *S*}. *principal S*)
else bot)

definition *atin_within* :: [*'a topology*, *'a*, *'a set*] ⇒ *'a filter*
where *atin_within X a S* = *inf* (*nhdsin X a*) (*principal* (*topspace X* ∩ *S* − {*a*}))

abbreviation *atin* :: *'a topology* ⇒ *'a* ⇒ *'a filter*
where *atin X a* ≡ *atin_within X a UNIV*

lemma *atin_def*: *atin X a* = *inf* (*nhdsin X a*) (*principal* (*topspace X* − {*a*}))
by (*simp add*: *atin_within_def*)

lemma *nhdsin_degenerate* [*simp*]: *a* ∉ *topspace X* ⟹ *nhdsin X a* = *bot*
and *atin_degenerate* [*simp*]: *a* ∉ *topspace X* ⟹ *atin X a* = *bot*
by (*simp_all add*: *nhdsin_def atin_def*)

lemma *eventually_nhdsin*:
eventually P (*nhdsin X a*) ⟷ *a* ∉ *topspace X* ∨ (∃ *S*. *openin X S* ∧ *a* ∈ *S* ∧
(∀ *x*∈*S*. *P x*))
proof (*cases a* ∈ *topspace X*)
case *True*
hence *nhdsin X a* = (*INF S*∈{*S*. *openin X S* ∧ *a* ∈ *S*}. *principal S*)
by (*simp add*: *nhdsin_def*)
also have *eventually P* ... ⟷ (∃ *S*. *openin X S* ∧ *a* ∈ *S* ∧ (∀ *x*∈*S*. *P x*))
using *True* **by** (*subst eventually_INF_base*) (*auto simp*: *eventually_principal*)
finally show *?thesis* **using** *True* **by** *simp*
qed *auto*

lemma *eventually_atin_within*:
eventually P (*atin_within X a S*) ⟷ *a* ∉ *topspace X* ∨ (∃ *T*. *openin X T* ∧ *a*
∈ *T* ∧ (∀ *x*∈*T*. *x* ∈ *S* ∧ *x* ≠ *a* ⟶ *P x*))
proof (*cases a* ∈ *topspace X*)
case *True*
hence *eventually P* (*atin_within X a S*) ⟷
(∃ *T*. *openin X T* ∧ *a* ∈ *T* ∧
(∀ *x*∈*T*. *x* ∈ *topspace X* ∧ *x* ∈ *S* ∧ *x* ≠ *a* ⟶ *P x*))
by (*simp add*: *atin_within_def eventually_inf_principal eventually_nhdsin*)
also have ... ⟷ (∃ *T*. *openin X T* ∧ *a* ∈ *T* ∧ (∀ *x*∈*T*. *x* ∈ *S* ∧ *x* ≠ *a* ⟶ *P*
x))
using *openin_subset* **by** (*intro ex_cong*) *auto*
finally show *?thesis* **by** (*simp add*: *True*)
qed (*simp add*: *atin_within_def*)

lemma *eventually_atin*:
eventually P (*atin X a*) ⟷ *a* ∉ *topspace X* ∨
(∃ *U*. *openin X U* ∧ *a* ∈ *U* ∧ (∀ *x* ∈ *U* − {*a*}. *P x*))

by (*auto simp add*: *eventually_atin_within*)

lemma *nontrivial_limit_atin*:
atin X a ≠ *bot* ⟷ *a* ∈ *X derived_set_of topspace X*
proof
assume *L*: *atin X a* ≠ *bot*
then have *a* ∈ *topspace X*
by (*meson atin_degenerate*)
moreover have ¬ *openin X* {*a*}
using *L* **by** (*auto simp*: *eventually_atin trivial_limit_eq*)
ultimately
show *a* ∈ *X derived_set_of topspace X*
by (*auto simp*: *derived_set_of_topspace*)
next
assume *a*: *a* ∈ *X derived_set_of topspace X*
show *atin X a* ≠ *bot*
proof
assume *atin X a* = *bot*
then have *eventually* (λ_. *False*) (*atin X a*)
by *simp*
then show *False*
by (*smt* (*verit*, *best*) *a eventually_atin in_derived_set_of insertE insert_Diff*)
qed
qed

lemma *eventually_atin_subtopology*:
assumes *a* ∈ *topspace X*
shows *eventually P* (*atin* (*subtopology X S*) *a*) ⟷
(*a* ∈ *S* ⟶ (∃ *U*. *openin* (*subtopology X S*) *U* ∧ *a* ∈ *U* ∧ (∀ *x*∈*U* − {*a*}. *P x*)))
using *assms* **by** (*simp add*: *eventually_atin*)

lemma *eventually_within_imp*:
eventually P (*atin_within X a S*) ⟷ *eventually* (λ*x*. *x* ∈ *S* ⟶ *P x*) (*atin X a*)
by (*auto simp add*: *eventually_atin_within eventually_atin*)

lemma *atin_subtopology_within*:
assumes *a* ∈ *S*
shows *atin* (*subtopology X S*) *a* = *atin_within X a S*
proof −
have *eventually P* (*atin* (*subtopology X S*) *a*) ⟷ *eventually P* (*atin_within X a S*) **for** *P*
unfolding *eventually_atin eventually_atin_within openin_subtopology*
using *assms* **by** *auto*
then show *?thesis*
by (*meson le_filter_def order.eq_iff*)
qed

lemma *atin_subtopology_within_if*:

shows *atin* (*subtopology X S*) *a* = (*if a* ∈ *S then atin_within X a S else bot*)
by (*simp add*: *atin_subtopology_within*)

lemma *trivial_limit_atpointof_within*:
trivial_limit(*atin_within X a S*) ⟷
(*a* ∉ *X derived_set_of S*)
by (*auto simp*: *trivial_limit_def eventually_atin_within in_derived_set_of*)

lemma *derived_set_of_trivial_limit*:
a ∈ *X derived_set_of S* ⟷ ¬ *trivial_limit*(*atin_within X a S*)
by (*simp add*: *trivial_limit_atpointof_within*)

2.1.16 Limits in a topological space

definition *limitin* :: *'a topology* ⇒ (*'b* ⇒ *'a*) ⇒ *'a* ⇒ *'b filter* ⇒ *bool* **where**
limitin X f l F ≡ *l* ∈ *topspace X* ∧ (∀ *U*. *openin X U* ∧ *l* ∈ *U* ⟶ *eventually* (λ*x*. *f x* ∈ *U*) *F*)

lemma *limit_within_subset*:
⟦*limitin X f l* (*atin_within Y a S*); *T* ⊆ *S*⟧ ⟹ *limitin X f l* (*atin_within Y a T*)
by (*smt* (*verit*) *eventually_atin_within limitin_def subset_eq*)

lemma *limitinD*: ⟦*limitin X f l F*; *openin X U*; *l* ∈ *U*⟧ ⟹ *eventually* (λ*x*. *f x* ∈ *U*) *F*
by (*simp add*: *limitin_def*)

lemma *limitin_canonical_iff* [*simp*]: *limitin euclidean f l F* ⟷ (*f* ⟶⟶ *l*) *F*
by (*auto simp*: *limitin_def tendsto_def*)

lemma *limitin_topspace*: *limitin X f l F* ⟹ *l* ∈ *topspace X*
by (*simp add*: *limitin_def*)

lemma *limitin_const_iff* [*simp*]: *limitin X* (λ*a*. *l*) *l F* ⟷ *l* ∈ *topspace X*
by (*simp add*: *limitin_def*)

lemma *limitin_const*: *limitin euclidean* (λ*a*. *l*) *l F*
by *simp*

lemma *limitin_eventually*:
⟦*l* ∈ *topspace X*; *eventually* (λ*x*. *f x* = *l*) *F*⟧ ⟹ *limitin X f l F*
by (*auto simp*: *limitin_def eventually_mono*)

lemma *limitin_subsequence*:
⟦*strict_mono r*; *limitin X f l sequentially*⟧ ⟹ *limitin X* (*f* ∘ *r*) *l sequentially*
unfolding *limitin_def* **using** *eventually_subseq* **by** *fastforce*

lemma *limitin_subtopology*:
limitin (*subtopology X S*) *f l F*

```
  ⟷ l ∈ S ∧ eventually (λa. f a ∈ S) F ∧ limitin X f l F  (is ?lhs = ?rhs)
proof (cases l ∈ S ∩ topspace X)
  case True
  show ?thesis
  proof
    assume L: ?lhs
    with True
    have ∀\_F b in F. f b ∈ topspace X ∩ S
      by (metis (no_types) limitin_def openin_topspace topspace_subtopology)
    with L show ?rhs
    apply (clarsimp simp add: limitin_def eventually_mono openin_subtopology_alt)
      apply (drule_tac x=S ∩ U in spec, force simp: elim: eventually_mono)
      done
  next
    assume ?rhs
    then show ?lhs
      using eventually_elim2
      by (fastforce simp add: limitin_def openin_subtopology_alt)
  qed
qed (auto simp: limitin_def)


lemma limitin_canonical_iff_gen [simp]:
  assumes open S
  shows limitin (top_of_set S) f l F ⟷ (f ⟶ l) F ∧ l ∈ S
  using assms by (auto simp: limitin_subtopology tendsto_def)

lemma limitin_sequentially:
   limitin X S l sequentially ⟷
    l ∈ topspace X ∧ (∀ U. openin X U ∧ l ∈ U ⟶ (∃ N. ∀ n. N ≤ n ⟶ S n ∈
U))
  by (simp add: limitin_def eventually_sequentially)

lemma limitin_sequentially_offset:
   limitin X f l sequentially ⟹ limitin X (λi. f (i + k)) l sequentially
  unfolding limitin_sequentially
  by (metis add.commute le_add2 order_trans)

lemma limitin_sequentially_offset_rev:
  assumes limitin X (λi. f (i + k)) l sequentially
  shows limitin X f l sequentially
proof −
  have ∃ N. ∀ n≥N. f n ∈ U if U: openin X U l ∈ U for U
  proof −
    obtain N where ⋀n. n≥N ⟹ f (n + k) ∈ U
      using assms U unfolding limitin_sequentially by blast
    then have ∀ n≥N+k. f n ∈ U
    by (metis add_leD2 le_add_diff_inverse ordered_cancel_comm_monoid_diff_class.le_diff_conv2
add.commute)
```

```
    then show ?thesis ..
  qed
  with assms show ?thesis
    unfolding limitin_sequentially
    by simp
qed

lemma limitin_atin:
   limitin Y f y (atin X x) ⟷
        y ∈ topspace Y ∧
        (x ∈ topspace X
        ⟶ (∀ V. openin Y V ∧ y ∈ V
                ⟶ (∃ U. openin X U ∧ x ∈ U ∧ f ‘ (U − {x}) ⊆ V)))
  by (auto simp: limitin_def eventually_atin image_subset_iff)

lemma limitin_atin_self:
   limitin Y f (f a) (atin X a) ⟷
        f a ∈ topspace Y ∧
        (a ∈ topspace X
         ⟶ (∀ V. openin Y V ∧ f a ∈ V
                 ⟶ (∃ U. openin X U ∧ a ∈ U ∧ f ‘ U ⊆ V)))
  unfolding limitin_atin by fastforce

lemma limitin_trivial:
   ⟦trivial_limit F; y ∈ topspace X⟧ ⟹ limitin X f y F
  by (simp add: limitin_def)

lemma limitin_transform_eventually:
   ⟦eventually (λx. f x = g x) F; limitin X f l F⟧ ⟹ limitin X g l F
  unfolding limitin_def using eventually_elim2 by fastforce

lemma continuous_map_limit:
  assumes continuous_map X Y g and f: limitin X f l F
  shows limitin Y (g ∘ f) (g l) F
proof −
  have g l ∈ topspace Y
    by (meson assms continuous_map f image_eqI in_mono limitin_def)
  moreover
  have ⋀U. ⟦∀ V. openin X V ∧ l ∈ V ⟶ (∀⇩F x in F. f x ∈ V); openin Y U; g
l ∈ U⟧
          ⟹ ∀⇩F x in F. g (f x) ∈ U
    using assms eventually_mono
    by (fastforce simp: limitin_def dest!: openin_continuous_map_preimage)
  ultimately show ?thesis
    using f by (fastforce simp add: limitin_def)
qed
```

2.1.17 Pointwise continuity in topological spaces

definition *topcontinuous_at* **where**
topcontinuous_at X Y f x ⟷
x ∈ *topspace X* ∧
f ∈ *topspace X* → *topspace Y* ∧
(∀ *V*. *openin Y V* ∧ *f x* ∈ *V*
⟶ (∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ (∀ *y* ∈ *U*. *f y* ∈ *V*)))

lemma *topcontinuous_at_atin*:
topcontinuous_at X Y f x ⟷
x ∈ *topspace X* ∧
f ∈ *topspace X* → *topspace Y* ∧
limitin Y f (*f x*) (*atin X x*)
unfolding *topcontinuous_at_def*
by (*fastforce simp add*: *limitin_atin*)+

lemma *continuous_map_eq_topcontinuous_at*:
continuous_map X Y f ⟷ (∀ *x* ∈ *topspace X*. *topcontinuous_at X Y f x*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*auto simp*: *continuous_map_def topcontinuous_at_def*)
next
assume *R*: *?rhs*
then show *?lhs*
apply (*auto simp*: *continuous_map_def topcontinuous_at_def*)
by (*smt* (*verit*, *del_insts*) *mem_Collect_eq openin_subopen openin_subset subset_eq*)
qed

lemma *continuous_map_atin*:
continuous_map X Y f ⟷ (∀ *x* ∈ *topspace X*. *limitin Y f* (*f x*) (*atin X x*))
by (*auto simp*: *limitin_def topcontinuous_at_atin continuous_map_eq_topcontinuous_at*)

lemma *limitin_continuous_map*:
⟦*continuous_map X Y f*; *a* ∈ *topspace X*; *f a* = *b*⟧ ⟹ *limitin Y f b* (*atin X a*)
by (*auto simp*: *continuous_map_atin*)

lemma *limit_continuous_map_within*:
⟦*continuous_map* (*subtopology X S*) *Y f*; *a* ∈ *S*; *a* ∈ *topspace X*⟧
⟹ *limitin Y f* (*f a*) (*atin_within X a S*)
by (*metis Int_iff atin_subtopology_within continuous_map_atin topspace_subtopology*)

2.1.18 Combining theorems for continuous functions into the reals

lemma *continuous_map_canonical_const* [*continuous_intros*]:
continuous_map X euclidean (λ*x*. *c*)

by *simp*

lemma *continuous_map_real_mult* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *f x* ∗ *g x*)
by (*simp add*: *continuous_map_atin tendsto_mult*)

lemma *continuous_map_real_pow* [*continuous_intros*]:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *f x* ^ *n*)
by (*induction n*) (*auto simp*: *continuous_map_real_mult*)

lemma *continuous_map_real_mult_left*:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *c* ∗ *f x*)
by (*simp add*: *continuous_map_atin tendsto_mult*)

lemma *continuous_map_real_mult_left_eq*:
continuous_map X euclideanreal (λ*x*. *c* ∗ *f x*) ⟷ *c* = *0* ∨ *continuous_map X euclideanreal f*
proof (*cases c* = *0*)
case *False*
have *continuous_map X euclideanreal* (λ*x*. *c* ∗ *f x*) ⟹ *continuous_map X euclideanreal f*
apply (*frule continuous_map_real_mult_left* [**where** *c*=*inverse c*])
apply (*simp add*: *field_simps False*)
done
with *False* **show** *?thesis*
using *continuous_map_real_mult_left* **by** *blast*
qed *simp*

lemma *continuous_map_real_mult_right*:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* (λ*x*. *f x* ∗ *c*)
by (*simp add*: *continuous_map_atin tendsto_mult*)

lemma *continuous_map_real_mult_right_eq*:
continuous_map X euclideanreal (λ*x*. *f x* ∗ *c*) ⟷ *c* = *0* ∨ *continuous_map X euclideanreal f*
by (*simp add*: *mult.commute flip*: *continuous_map_real_mult_left_eq*)

lemma *continuous_map_minus* [*continuous_intros*]:
fixes *f* :: *'a*⇒*'b*::*real_normed_vector*
shows *continuous_map X euclidean f* ⟹ *continuous_map X euclidean* (λ*x*. − *f x*)
by (*simp add*: *continuous_map_atin tendsto_minus*)

lemma *continuous_map_minus_eq* [*simp*]:
fixes *f* :: *'a*⇒*'b*::*real_normed_vector*

shows *continuous_map X euclidean* $(\lambda x. -f\,x) \longleftrightarrow$ *continuous_map X euclidean f*
using *continuous_map_minus add.inverse_inverse continuous_map_eq* **by** *fastforce*

lemma *continuous_map_add* [*continuous_intros*]:
fixes $f :: 'a \Rightarrow 'b$::*real_normed_vector*
shows ⟦*continuous_map X euclidean f*; *continuous_map X euclidean g*⟧ ⟹ *continuous_map X euclidean* $(\lambda x.\ f\,x + g\,x)$
by (*simp add*: *continuous_map_atin tendsto_add*)

lemma *continuous_map_diff* [*continuous_intros*]:
fixes $f :: 'a \Rightarrow 'b$::*real_normed_vector*
shows ⟦*continuous_map X euclidean f*; *continuous_map X euclidean g*⟧ ⟹ *continuous_map X euclidean* $(\lambda x.\ f\,x - g\,x)$
by (*simp add*: *continuous_map_atin tendsto_diff*)

lemma *continuous_map_norm* [*continuous_intros*]:
fixes $f :: 'a \Rightarrow 'b$::*real_normed_vector*
shows *continuous_map X euclidean f* ⟹ *continuous_map X euclidean* $(\lambda x.\ norm(f\,x))$
by (*simp add*: *continuous_map_atin tendsto_norm*)

lemma *continuous_map_real_abs* [*continuous_intros*]:
continuous_map X euclideanreal f ⟹ *continuous_map X euclideanreal* $(\lambda x.\ abs(f\,x))$
by (*simp add*: *continuous_map_atin tendsto_rabs*)

lemma *continuous_map_real_max* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*⟧
⟹ *continuous_map X euclideanreal* $(\lambda x.\ max\ (f\,x)\ (g\,x))$
by (*simp add*: *continuous_map_atin tendsto_max*)

lemma *continuous_map_real_min* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*⟧
⟹ *continuous_map X euclideanreal* $(\lambda x.\ min\ (f\,x)\ (g\,x))$
by (*simp add*: *continuous_map_atin tendsto_min*)

lemma *continuous_map_sum* [*continuous_intros*]:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c$::*real_normed_vector*
shows ⟦*finite I*; $\bigwedge i.\ i \in I \Longrightarrow$ *continuous_map X euclidean* $(\lambda x.\ f\,x\,i)$⟧
⟹ *continuous_map X euclidean* $(\lambda x.\ sum\ (f\,x)\ I)$
by (*simp add*: *continuous_map_atin tendsto_sum*)

lemma *continuous_map_prod* [*continuous_intros*]:
⟦*finite I*;
$\bigwedge i.\ i \in I \Longrightarrow$ *continuous_map X euclideanreal* $(\lambda x.\ f\,x\,i)$⟧
⟹ *continuous_map X euclideanreal* $(\lambda x.\ prod\ (f\,x)\ I)$
by (*simp add*: *continuous_map_atin tendsto_prod*)

lemma *continuous_map_real_inverse* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* ≠ *0*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *inverse*(*f x*))
by (*simp add*: *continuous_map_atin tendsto_inverse*)

lemma *continuous_map_real_divide* [*continuous_intros*]:
⟦*continuous_map X euclideanreal f*; *continuous_map X euclideanreal g*; ⋀*x*. *x* ∈ *topspace X* ⟹ *g x* ≠ *0*⟧
⟹ *continuous_map X euclideanreal* (λ*x*. *f x* / *g x*)
by (*simp add*: *continuous_map_atin tendsto_divide*)

end

2.2 Non-Denumerability of the Continuum

theory *Continuum_Not_Denumerable*
imports
Complex_Main
HOL−Library.Countable_Set
begin

2.2.1 Abstract

The following document presents a proof that the Continuum is uncountable. It is formalised in the Isabelle/Isar theorem proving system.

Theorem: The Continuum ℝ is not denumerable. In other words, there does not exist a function *f*: ℕ ⇒ ℝ such that *f* is surjective.

Outline: An elegant informal proof of this result uses Cantor's Diagonalisation argument. The proof presented here is not this one.

First we formalise some properties of closed intervals, then we prove the Nested Interval Property. This property relies on the completeness of the Real numbers and is the foundation for our argument. Informally it states that an intersection of countable closed intervals (where each successive interval is a subset of the last) is non-empty. We then assume a surjective function *f*: ℕ ⇒ ℝ exists and find a real *x* such that *x* is not in the range of *f* by generating a sequence of closed intervals then using the Nested Interval Property.

theorem *real_non_denum*: ∄*f* :: *nat* ⇒ *real*. *surj f*
proof
assume ∃*f*::*nat* ⇒ *real*. *surj f*
then obtain *f* :: *nat* ⇒ *real* **where** *surj f* **..**

First we construct a sequence of nested intervals, ignoring *range f*.

have *a* < *b* ⟹ ∃*ka kb*. *ka* < *kb* ∧ {*ka*..*kb*} ⊆ {*a*..*b*} ∧ *c* ∉ {*ka*..*kb*} **for** *a b c* :: *real*

```
    by (auto simp add: not_le cong: conj_cong)
      (metis dense le_less_linear less_linear less_trans order_refl)
  then obtain i j where ij:
    a < b ⟹ i a b c < j a b c
      a < b ⟹ {i a b c .. j a b c} ⊆ {a .. b}
      a < b ⟹ c ∉ {i a b c .. j a b c}
    for a b c :: real
    by metis

  define ivl where ivl =
    rec_nat (f 0 + 1, f 0 + 2) (λn x. (i (fst x) (snd x) (f n), j (fst x) (snd x) (f
n)))
  define I where I n = {fst (ivl n) .. snd (ivl n)} for n

  have ivl [simp]:
    ivl 0 = (f 0 + 1, f 0 + 2)
    ⋀n. ivl (Suc n) = (i (fst (ivl n)) (snd (ivl n)) (f n), j (fst (ivl n)) (snd (ivl n))
(f n))
    unfolding ivl_def by simp_all
```

This is a decreasing sequence of non-empty intervals.

```
  have less: fst (ivl n) < snd (ivl n) for n
    by (induct n) (auto intro!: ij)

  have decseq I
    unfolding I_def decseq_Suc_iff ivl fst_conv snd_conv
    by (intro ij allI less)
```

Now we apply the finite intersection property of compact sets.

```
  have I 0 ∩ (⋂ i. I i) ≠ {}
  proof (rule compact_imp_fip_image)
    fix S :: nat set
    assume fin: finite S
    have {} ⊂ I (Max (insert 0 S))
      unfolding I_def using less[of Max (insert 0 S)] by auto
    also have I (Max (insert 0 S)) ⊆ (⋂ i∈insert 0 S. I i)
      using fin decseqD[OF ‹decseq I›, of _ Max (insert 0 S)]
      by (auto simp: Max_ge_iff)
    also have (⋂ i∈insert 0 S. I i) = I 0 ∩ (⋂ i∈S. I i)
      by auto
    finally show I 0 ∩ (⋂ i∈S. I i) ≠ {}
      by auto
  qed (auto simp: I_def)
  then obtain x where x ∈ I n for n
    by blast
  moreover from ‹surj f› obtain j where x = f j
    by blast
  ultimately have f j ∈ I (Suc j)
    by blast
```

```
  with ij(3)[OF less] show False
    unfolding I_def ivl fst_conv snd_conv by auto
qed

lemma uncountable_UNIV_real: uncountable (UNIV :: real set)
  using real_non_denum unfolding uncountable_def by auto

corollary complex_non_denum: ∄f :: nat ⇒ complex. surj f
 by (metis (full_types) Re_complex_of_real comp_surj real_non_denum surj_def)

lemma uncountable_UNIV_complex: uncountable (UNIV :: complex set)
  using complex_non_denum unfolding uncountable_def by auto

lemma bij_betw_open_intervals:
  fixes a b c d :: real
  assumes a < b c < d
  shows ∃f. bij_betw f {a<..<b} {c<..<d}
proof −
  define f where f a b c d x = (d − c)/(b − a) ∗ (x − a) + c for a b c d x :: real
  {
    fix a b c d x :: real
    assume ∗: a < b c < d a < x x < b
    moreover from ∗ have (d − c) ∗ (x − a) < (d − c) ∗ (b − a)
      by (intro mult_strict_left_mono) simp_all
    moreover from ∗ have 0 < (d − c) ∗ (x − a) / (b − a)
      by simp
    ultimately have f a b c d x < d c < f a b c d x
      by (simp_all add: f_def field_simps)
  }
  with assms have bij_betw (f a b c d) {a<..<b} {c<..<d}
    by (intro bij_betw_byWitness[where f'=f c d a b]) (auto simp: f_def)
  then show ?thesis by auto
qed

lemma bij_betw_tan: bij_betw tan {−pi/2<..<pi/2} UNIV
  using arctan_ubound by (intro bij_betw_byWitness[where f'=arctan]) (auto
simp: arctan arctan_tan)

lemma uncountable_open_interval: uncountable {a<..<b} ⟷ a < b for a b ::
real
proof
  show a < b if uncountable {a<..<b}
    using uncountable_def that by force
  show uncountable {a<..<b} if a < b
  proof −
    obtain f where bij_betw f {a <..< b} {−pi/2<..<pi/2}
      using bij_betw_open_intervals[OF ‹a < b›, of −pi/2 pi/2] by auto
    then show ?thesis
      by (metis bij_betw_tan uncountable_bij_betw uncountable_UNIV_real)
```

```
  qed
qed

lemma uncountable_half_open_interval_1: uncountable {a..<b} ⟷ a < b for
a b :: real
  apply auto
  using atLeastLessThan_empty_iff
  apply fastforce
  using uncountable_open_interval [of a b]
  apply (metis countable_Un_iff ivl_disj_un_singleton(3))
  done

lemma uncountable_half_open_interval_2: uncountable {a<..b} ⟷ a < b for
a b :: real
  apply auto
  using atLeastLessThan_empty_iff
  apply fastforce
  using uncountable_open_interval [of a b]
  apply (metis countable_Un_iff ivl_disj_un_singleton(4))
  done

lemma real_interval_avoid_countable_set:
  fixes a b :: real and A :: real set
  assumes a < b and countable A
  shows ∃x∈{a<..<b}. x ∉ A
proof −
  from ‹countable A› have ∗: countable (A ∩ {a<..<b})
    by auto
  with ‹a < b› have ¬ countable {a<..<b}
    by (simp add: uncountable_open_interval)
  with ∗ have A ∩ {a<..<b} ≠ {a<..<b}
    by auto
  then have A ∩ {a<..<b} ⊂ {a<..<b}
    by (intro psubsetI) auto
  then have ∃x. x ∈ {a<..<b} − A ∩ {a<..<b}
    by (rule psubset_imp_ex_mem)
  then show ?thesis
    by auto
qed

lemma uncountable_closed_interval: uncountable {a..b} ⟷ a < b for a b :: real
 using infinite_Icc_iff by (fastforce dest: countable_finite real_interval_avoid_countable_set)

lemma open_minus_countable:
  fixes S A :: real set
  assumes countable A S ≠ {} open S
  shows ∃x∈S. x ∉ A
proof −
  obtain x where x ∈ S
```

```
    using ‹S ≠ {}› by auto
  then obtain e where 0 < e {y. dist y x < e} ⊆ S
    using ‹open S› by (auto simp: open_dist subset_eq)
  moreover have {y. dist y x < e} = {x - e <..< x + e}
    by (auto simp: dist_real_def)
  ultimately have uncountable (S - A)
    using uncountable_open_interval[of x - e x + e] ‹countable A›
    by (intro uncountable_minus_countable) (auto dest: countable_subset)
  then show ?thesis
    unfolding uncountable_def by auto
qed

end
```

2.3 Abstract Topology 2

```
theory Abstract_Topology_2
  imports
    Elementary_Topology Abstract_Topology Continuum_Not_Denumerable
    HOL−Library.Indicator_Function
    HOL−Library.Equipollence
begin
```

Combination of Elementary and Abstract Topology

```
lemma approachable_lt_le2:
  (∃(d::real) > 0. ∀x. Q x ⟶ f x < d ⟶ P x) ⟷ (∃d>0. ∀x. f x ≤ d ⟶ Q
x ⟶ P x)
  by (meson dense less_eq_real_def order_le_less_trans)

lemma triangle_lemma:
  fixes x y z :: real
  assumes x: 0 ≤ x
    and y: 0 ≤ y
    and z: 0 ≤ z
    and xy: x² ≤ y² + z²
  shows x ≤ y + z
proof −
  have y² + z² ≤ y² + 2 * y * z + z²
    using z y by simp
  with xy have th: x² ≤ (y + z)²
    by (simp add: power2_eq_square field_simps)
  from y z have yz: y + z ≥ 0
    by arith
  from power2_le_imp_le[OF th yz] show ?thesis .
qed

lemma isCont_indicator:
  fixes x :: 'a::t2_space
  shows isCont (indicator A :: 'a ⇒ real) x = (x ∉ frontier A)
```

proof *auto*
fix *x*
assume *cts_at*: *isCont* (*indicator A* :: *'a* ⇒ *real*) *x* **and** *fr*: *x* ∈ *frontier A*
with *continuous_at_open* **have** *1*: ∀ *V*::*real set*. *open V* ∧ *indicator A x* ∈ *V* ⟶
(∃ *U*::*'a set*. *open U* ∧ *x* ∈ *U* ∧ (∀ *y*∈*U*. *indicator A y* ∈ *V*)) **by** *auto*
show *False*
proof (*cases x* ∈ *A*)
assume *x*: *x* ∈ *A*
hence *indicator A x* ∈ ({*0*<..<*2*} :: *real set*) **by** *simp*
with *1* **obtain** *U* **where** *U*: *open U x* ∈ *U* ∀ *y*∈*U*. *indicator A y* ∈ ({*0*<..<*2*} :: *real set*)
using *open_greaterThanLessThan* **by** *metis*
hence ∀ *y*∈*U*. *indicator A y* > (*0*::*real*)
unfolding *greaterThanLessThan_def* **by** *auto*
hence *U* ⊆ *A* **using** *indicator_eq_0_iff* **by** *force*
hence *x* ∈ *interior A* **using** *U interiorI* **by** *auto*
thus *?thesis* **using** *fr* **unfolding** *frontier_def* **by** *simp*
next
assume *x*: *x* ∉ *A*
hence *indicator A x* ∈ ({*−1*<..<*1*} :: *real set*) **by** *simp*
with *1* **obtain** *U* **where** *U*: *open U x* ∈ *U* ∀ *y*∈*U*. *indicator A y* ∈ ({*−1*<..<*1*} :: *real set*)
using *1 open_greaterThanLessThan* **by** *metis*
hence ∀ *y*∈*U*. *indicator A y* < (*1*::*real*)
unfolding *greaterThanLessThan_def* **by** *auto*
hence *U* ⊆ −*A* **by** *auto*
hence *x* ∈ *interior* (−*A*) **using** *U interiorI* **by** *auto*
thus *?thesis* **using** *fr interior_complement* **unfolding** *frontier_def* **by** *auto*
qed
next
assume *nfr*: *x* ∉ *frontier A*
hence *x* ∈ *interior A* ∨ *x* ∈ *interior* (−*A*)
by (*auto simp*: *frontier_def closure_interior*)
thus *isCont* ((*indicator A*)::*'a* ⇒ *real*) *x*
proof
assume *int*: *x* ∈ *interior A*
then obtain *U* **where** *U*: *open U x* ∈ *U U* ⊆ *A* **unfolding** *interior_def* **by** *auto*
hence ∀ *y*∈*U*. *indicator A y* = (*1*::*real*) **unfolding** *indicator_def* **by** *auto*
hence *continuous_on U* (*indicator A*) **by** (*simp add*: *indicator_eq_1_iff*)
thus *?thesis* **using** *U continuous_on_eq_continuous_at* **by** *auto*
next
assume *ext*: *x* ∈ *interior* (−*A*)
then obtain *U* **where** *U*: *open U x* ∈ *U U* ⊆ −*A* **unfolding** *interior_def* **by** *auto*
then have *continuous_on U* (*indicator A*)
using *continuous_on_topological* **by** (*auto simp*: *subset_iff*)
thus *?thesis* **using** *U continuous_on_eq_continuous_at* **by** *auto*

qed
qed

lemma *islimpt_closure*:
$[\![S \subseteq T; \bigwedge x.\ [\![x\ islimpt\ S;\ x \in T]\!] \Longrightarrow x \in S]\!] \Longrightarrow S = T \cap closure\ S$
using *closure_def* **by** *fastforce*

lemma *closedin_limpt*:
$closedin\ (top\_of\_set\ T)\ S \longleftrightarrow S \subseteq T \wedge (\forall x.\ x\ islimpt\ S \wedge x \in T \longrightarrow x \in S)$
proof −
have $\bigwedge U\ x.\ [\![closed\ U;\ S = T \cap U;\ x\ islimpt\ S;\ x \in T]\!] \Longrightarrow x \in S$
by (*meson IntI closed_limpt inf_le2 islimpt_subset*)
then show *?thesis*
by (*metis closed_closure closedin_closed closedin_imp_subset islimpt_closure*)
qed

lemma *closedin_closed_eq*: $closed\ S \Longrightarrow closedin\ (top\_of\_set\ S)\ T \longleftrightarrow closed\ T \wedge T \subseteq S$
by (*meson closedin_limpt closed_subset closedin_closed_trans*)

lemma *connected_closed_set*:
$closed\ S$
$\Longrightarrow connected\ S \longleftrightarrow (\nexists A\ B.\ closed\ A \wedge closed\ B \wedge A \neq \{\} \wedge B \neq \{\} \wedge A \cup B = S \wedge A \cap B = \{\})$
unfolding *connected_closedin_eq closedin_closed_eq connected_closedin_eq* **by** *blast*

If a connnected set is written as the union of two nonempty closed sets, then these sets have to intersect.

lemma *connected_as_closed_union*:
assumes $connected\ C\ C = A \cup B\ closed\ A\ closed\ B\ A \neq \{\}\ B \neq \{\}$
shows $A \cap B \neq \{\}$
by (*metis assms closed_Un connected_closed_set*)

lemma *closedin_subset_trans*:
$closedin\ (top\_of\_set\ U)\ S \Longrightarrow S \subseteq T \Longrightarrow T \subseteq U \Longrightarrow$
$closedin\ (top\_of\_set\ T)\ S$
by (*meson closedin_limpt subset_iff*)

lemma *openin_subset_trans*:
$openin\ (top\_of\_set\ U)\ S \Longrightarrow S \subseteq T \Longrightarrow T \subseteq U \Longrightarrow$
$openin\ (top\_of\_set\ T)\ S$
by (*auto simp*: *openin_open*)

lemma *closedin_compact*:
$[\![compact\ S;\ closedin\ (top\_of\_set\ S)\ T]\!] \Longrightarrow compact\ T$
by (*metis closedin_closed compact_Int_closed*)

lemma *closedin_compact_eq*:

fixes $S :: 'a::t2\_space\ set$
shows $compact\ S \Longrightarrow (closedin\ (top\_of\_set\ S)\ T \longleftrightarrow compact\ T \wedge T \subseteq S)$
by (*metis closedin_imp_subset closedin_compact closed_subset compact_imp_closed*)

2.3.1 Closure

lemma *euclidean_closure_of* [*simp*]: $euclidean\ closure\_of\ S = closure\ S$
by (*auto simp*: *closure_of_def closure_def islimpt_def*)

lemma *closure_openin_Int_closure*:
assumes *ope*: $openin\ (top\_of\_set\ U)\ S$ **and** $T \subseteq U$
shows $closure(S \cap closure\ T) = closure(S \cap T)$
proof
obtain V **where** $open\ V$ **and** S: $S = U \cap V$
using *ope* **using** *openin_open* **by** *metis*
show $closure\ (S \cap closure\ T) \subseteq closure\ (S \cap T)$
proof (*clarsimp simp*: S)
fix x
assume $x \in closure\ (U \cap V \cap closure\ T)$
then have $V \cap closure\ T \subseteq A \Longrightarrow x \in closure\ A$ **for** A
by (*metis closure_mono subsetD inf.coboundedI2 inf_assoc*)
then have $x \in closure\ (T \cap V)$
by (*metis ‹open V› closure_closure inf_commute open_Int_closure_subset*)
then show $x \in closure\ (U \cap V \cap T)$
by (*metis ‹T ⊆ U› inf.absorb_iff2 inf_assoc inf_commute*)
qed
next
show $closure\ (S \cap T) \subseteq closure\ (S \cap closure\ T)$
by (*meson Int_mono closure_mono closure_subset order_refl*)
qed

corollary *infinite_openin*:
fixes $S :: 'a :: t1\_space\ set$
shows $\llbracket openin\ (top\_of\_set\ U)\ S;\ x \in S;\ x\ islimpt\ U \rrbracket \Longrightarrow infinite\ S$
by (*clarsimp simp add*: *openin_open islimpt_eq_acc_point inf_commute*)

lemma *closure_Int_ballI*:
assumes $\bigwedge U.\ \llbracket openin\ (top\_of\_set\ S)\ U;\ U \neq \{\} \rrbracket \Longrightarrow T \cap U \neq \{\}$
shows $S \subseteq closure\ T$
proof (*clarsimp simp*: *closure_iff_nhds_not_empty*)
fix x **and** A **and** V
assume $x \in S\ V \subseteq A\ open\ V\ x \in V\ T \cap A = \{\}$
then have $openin\ (top\_of\_set\ S)\ (A \cap V \cap S)$
by (*simp add*: *inf_absorb2 openin_subtopology_Int*)
moreover have $A \cap V \cap S \neq \{\}$ **using** ‹$x \in V$› ‹$V \subseteq A$› ‹$x \in S$›
by *auto*
ultimately show *False*
using ‹$T \cap A = \{\}$› *assms* **by** *fastforce*
qed

2.3.2 Frontier

lemma *euclidean_interior_of* [*simp*]: *euclidean interior_of S = interior S*
by (*auto simp*: *interior_of_def interior_def*)

lemma *euclidean_frontier_of* [*simp*]: *euclidean frontier_of S = frontier S*
by (*auto simp*: *frontier_of_def frontier_def*)

lemma *connected_Int_frontier*:
⟦*connected S*; *S* ∩ *T* ≠ {}; *S* − *T* ≠ {}⟧ ⟹ *S* ∩ *frontier T* ≠ {}
apply (*simp add*: *frontier_interiors connected_openin*, *safe*)
apply (*drule_tac x*=*S* ∩ *interior T* **in** *spec*, *safe*)
apply (*drule_tac* [*2*] *x*=*S* ∩ *interior* (−*T*) **in** *spec*)
apply (*auto simp*: *disjoint_eq_subset_Compl dest*: *interior_subset* [*THEN subsetD*])
done

2.3.3 Compactness

lemma *openin_delete*:
fixes *a* :: *'a* :: *t1_space*
shows *openin* (*top_of_set u*) *S* ⟹ *openin* (*top_of_set u*) (*S* − {*a*})
by (*metis Int_Diff open_delete openin_open*)

lemma *compact_eq_openin_cover*:
compact S ⟷
(∀ *C*. (∀ *c*∈*C*. *openin* (*top_of_set S*) *c*) ∧ *S* ⊆ ⋃ *C* ⟶
(∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D*))
proof *safe*
fix *C*
assume *compact S* **and** ∀ *c*∈*C*. *openin* (*top_of_set S*) *c* **and** *S* ⊆ ⋃ *C*
then have ∀ *c*∈{*T*. *open T* ∧ *S* ∩ *T* ∈ *C*}. *open c* **and** *S* ⊆ ⋃{*T*. *open T* ∧ *S* ∩ *T* ∈ *C*}
unfolding *openin_open* **by** *force*+
with ‹*compact S*› **obtain** *D* **where** *D* ⊆ {*T*. *open T* ∧ *S* ∩ *T* ∈ *C*} **and** *finite D* **and** *S* ⊆ ⋃ *D*
by (*meson compactE*)
then have *image* (λ*T*. *S* ∩ *T*) *D* ⊆ *C* ∧ *finite* (*image* (λ*T*. *S* ∩ *T*) *D*) ∧ *S* ⊆ ⋃(*image* (λ*T*. *S* ∩ *T*) *D*)
by *auto*
then show ∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D* **..**
next
assume *1*: ∀ *C*. (∀ *c*∈*C*. *openin* (*top_of_set S*) *c*) ∧ *S* ⊆ ⋃ *C* ⟶
(∃ *D*⊆*C*. *finite D* ∧ *S* ⊆ ⋃ *D*)
show *compact S*
proof (*rule compactI*)
fix *C*
let *?C* = *image* (λ*T*. *S* ∩ *T*) *C*
assume ∀ *t*∈*C*. *open t* **and** *S* ⊆ ⋃ *C*
then have (∀ *c*∈*?C*. *openin* (*top_of_set S*) *c*) ∧ *S* ⊆ ⋃ *?C*

```
    unfolding openin_open by auto
  with 1 obtain D where D ⊆ ?C and finite D and S ⊆ ⋃ D
    by metis
  let ?D = inv_into C (λT. S ∩ T) ‘ D
  have ?D ⊆ C ∧ finite ?D ∧ S ⊆ ⋃ ?D
  proof (intro conjI)
    from ‹D ⊆ ?C› show ?D ⊆ C
      by (fast intro: inv_into_into)
    from ‹finite D› show finite ?D
      by (rule finite_imageI)
    from ‹S ⊆ ⋃ D› show S ⊆ ⋃ ?D
      by (metis ‹D ⊆ (∩) S ‘ C› image_inv_into_cancel inf_Sup le_infE)
  qed
  then show ∃D⊆C. finite D ∧ S ⊆ ⋃ D ..
 qed
qed
```

2.3.4 Continuity

```
lemma interior_image_subset:
 assumes inj f ⋀x. continuous (at x) f
 shows interior (f ‘ S) ⊆ f ‘ (interior S)
proof
 fix x assume x ∈ interior (f ‘ S)
 then obtain T where as: open T x ∈ T T ⊆ f ‘ S ..
 then have x ∈ f ‘ S by auto
 then obtain y where y: y ∈ S x = f y by auto
 have open (f −‘ T)
    using assms ‹open T› by (simp add: continuous_at_imp_continuous_on
open_vimage)
 moreover have y ∈ vimage f T
   using ‹x = f y› ‹x ∈ T› by simp
 moreover have vimage f T ⊆ S
   using ‹T ⊆ image f S› ‹inj f› unfolding inj_on_def subset_eq by auto
 ultimately have y ∈ interior S ..
 with ‹x = f y› show x ∈ f ‘ interior S ..
qed
```

2.3.5 Equality of continuous functions on closure and related results

```
lemma continuous_closedin_preimage_constant:
 fixes f :: _ ⇒ 'b::t1_space
 shows continuous_on S f ⟹ closedin (top_of_set S) {x ∈ S. f x = a}
 using continuous_closedin_preimage[of S f {a}] by (simp add: vimage_def Col-
lect_conj_eq)

lemma continuous_closed_preimage_constant:
 fixes f :: _ ⇒ 'b::t1_space
```

shows *continuous_on S f* $\Longrightarrow$ *closed S* $\Longrightarrow$ *closed* $\{x \in S.\ f\ x = a\}$
using *continuous_closed_preimage*[*of S f* $\{a\}$] **by** (*simp add*: *vimage_def Collect_conj_eq*)

lemma *continuous_constant_on_closure*:
fixes *f* :: _ $\Rightarrow$ *'b*::*t1_space*
assumes *continuous_on* (*closure S*) *f*
and $\bigwedge x.\ x \in S \Longrightarrow f\ x = a$
and $x \in$ *closure S*
shows *f x* = *a*
using *continuous_closed_preimage_constant*[*of closure S f a*]
assms closure_minimal[*of S* $\{x \in$ *closure S*. *f x* = *a*$\}$] *closure_subset*
unfolding *subset_eq*
by *auto*

lemma *image_closure_subset*:
assumes *contf*: *continuous_on* (*closure S*) *f*
and *closed T*
and (*f* ‘ *S*) $\subseteq$ *T*
shows *f* ‘ (*closure S*) $\subseteq$ *T*
proof −
have *S* $\subseteq$ $\{x \in$ *closure S*. *f x* $\in$ *T*$\}$
using *assms*(*3*) *closure_subset* **by** *auto*
moreover have *closed* (*closure S* $\cap$ *f* −‘ *T*)
using *continuous_closed_preimage*[*OF contf*] ‹*closed T*› **by** *auto*
ultimately have *closure S* = (*closure S* $\cap$ *f* −‘ *T*)
using *closure_minimal*[*of S* (*closure S* $\cap$ *f* −‘ *T*)] **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *continuous_image_closure_subset*:
assumes *continuous_on A f closure B* $\subseteq$ *A*
shows *f* ‘ *closure B* $\subseteq$ *closure* (*f* ‘ *B*)
using *assms* **by** (*meson closed_closure closure_subset continuous_on_subset image_closure_subset*)

2.3.6 A function constant on a set

definition *constant_on* (**infixl** ‹(*constant'_on*)› *50*)
where *f constant_on A* $\equiv \exists y.\ \forall x{\in}A.\ f\ x = y$

lemma *constant_on_subset*: $[\![f$ *constant_on A*; *B* $\subseteq$ *A*$]\!] \Longrightarrow f$ *constant_on B*
unfolding *constant_on_def* **by** *blast*

lemma *injective_not_constant*:
fixes *S* :: *'a*::{*perfect_space*} *set*
shows $[\![$*open S*; *inj_on f S*; *f constant_on S*$]\!] \Longrightarrow S = \{\}$
unfolding *constant_on_def*
by (*metis equals0I inj_on_contraD islimpt_UNIV islimpt_def*)

lemma *constant_on_compose*:
assumes *f constant_on A*
shows *g ∘ f constant_on A*
using *assms* **by** (*auto simp*: *constant_on_def*)

lemma *not_constant_onI*:
f x ≠ f y ⟹ x ∈ A ⟹ y ∈ A ⟹ ¬f constant_on A
unfolding *constant_on_def* **by** *metis*

lemma *constant_onE*:
assumes *f constant_on S* **and** *⋀x. x∈S ⟹ f x = g x*
shows *g constant_on S*
using *assms* **unfolding** *constant_on_def* **by** *simp*

lemma *constant_on_closureI*:
fixes *f :: _ ⇒ 'b::t1_space*
assumes *cof*: *f constant_on S* **and** *contf*: *continuous_on* (*closure S*) *f*
shows *f constant_on* (*closure S*)
using *continuous_constant_on_closure* [*OF contf*] *cof* **unfolding** *constant_on_def*
by *metis*

2.3.7 Continuity relative to a union.

lemma *continuous_on_Un_local*:
⟦*closedin* (*top_of_set* (*S ∪ T*)) *S*; *closedin* (*top_of_set* (*S ∪ T*)) *T*;
continuous_on S f; *continuous_on T f*⟧
⟹ continuous_on (*S ∪ T*) *f*
unfolding *continuous_on closedin_limpt*
by (*metis Lim_trivial_limit Lim_within_Un Un_iff trivial_limit_within*)

lemma *continuous_on_cases_local*:
⟦*closedin* (*top_of_set* (*S ∪ T*)) *S*; *closedin* (*top_of_set* (*S ∪ T*)) *T*;
continuous_on S f; *continuous_on T g*;
⋀x. ⟦*x ∈ S ∧ ¬P x ∨ x ∈ T ∧ P x*⟧ *⟹ f x = g x*⟧
⟹ continuous_on (*S ∪ T*) (*λx. if P x then f x else g x*)
by (*rule continuous_on_Un_local*) (*auto intro*: *continuous_on_eq*)

lemma *continuous_on_cases_le*:
fixes *h :: 'a :: topological_space ⇒ real*
assumes *continuous_on* {*x ∈ S. h x ≤ a*} *f*
and *continuous_on* {*x ∈ S. a ≤ h x*} *g*
and *h*: *continuous_on S h*
and *⋀x.* ⟦*x ∈ S*; *h x = a*⟧ *⟹ f x = g x*
shows *continuous_on S* (*λx. if h x ≤ a then f*(*x*) *else g*(*x*))
proof −
have *S*: *S* = (*S ∩ h −‘ atMost a*) ∪ (*S ∩ h −‘ atLeast a*)
by *force*
have *1*: *closedin* (*top_of_set S*) (*S ∩ h −‘ atMost a*)

```
    by (rule continuous_closedin_preimage [OF h closed_atMost])
  have 2: closedin (top_of_set S) (S ∩ h -‘ atLeast a)
    by (rule continuous_closedin_preimage [OF h closed_atLeast])
  have [simp]: S ∩ h -‘ {..a} = {x ∈ S. h x ≤ a} S ∩ h -‘ {a..} = {x ∈ S. a ≤
h x}
    by auto
  have continuous_on (S ∩ h -‘ {..a} ∪ S ∩ h -‘ {a..}) (λx. if h x ≤ a then f x
else g x)
    by (intro continuous_on_cases_local) (use 1 2 S assms in auto)
  then show ?thesis
    using S by force
qed

lemma continuous_on_cases_1:
  fixes S :: real set
  assumes continuous_on {t ∈ S. t ≤ a} f
      and continuous_on {t ∈ S. a ≤ t} g
      and a ∈ S ⟹ f a = g a
    shows continuous_on S (λt. if t ≤ a then f(t) else g(t))
using assms
by (auto intro: continuous_on_cases_le [where h = id, simplified])
```

2.3.8 Inverse function property for open/closed maps

```
lemma continuous_on_inverse_open_map:
  assumes contf: continuous_on S f
    and imf: f ‘ S = T
    and injf: ⋀x. x ∈ S ⟹ g (f x) = x
    and oo: ⋀U. openin (top_of_set S) U ⟹ openin (top_of_set T) (f ‘ U)
  shows continuous_on T g
proof −
  from imf injf have gTS: g ‘ T = S
    by force
  from imf injf have fU: U ⊆ S ⟹ (f ‘ U) = T ∩ g -‘ U for U
    by force
  show ?thesis
    by (simp add: continuous_on_open [of T g] gTS) (metis openin_imp_subset
fU oo)
qed

lemma continuous_on_inverse_closed_map:
  assumes contf: continuous_on S f
    and imf: f ‘ S = T
    and injf: ⋀x. x ∈ S ⟹ g(f x) = x
    and oo: ⋀U. closedin (top_of_set S) U ⟹ closedin (top_of_set T) (f ‘ U)
  shows continuous_on T g
proof −
  from imf injf have gTS: g ‘ T = S
    by force
```

from *imf injf* **have** *fU*: $U \subseteq S \Longrightarrow (f \mathbin{`} U) = T \cap g -` U$ **for** *U*
by *force*
show *?thesis*
by (*simp add*: *continuous_on_closed* [*of T g*] *gTS*) (*metis closedin_imp_subset fU oo*)
qed

lemma *homeomorphism_injective_open_map*:
assumes *contf*: *continuous_on S f*
and *imf*: $f \mathbin{`} S = T$
and *injf*: *inj_on f S*
and *oo*: $\bigwedge U$. *openin* (*top_of_set S*) *U* $\Longrightarrow$ *openin* (*top_of_set T*) ($f \mathbin{`} U$)
obtains *g* **where** *homeomorphism S T f g*
proof
have *continuous_on T* (*inv_into S f*)
by (*metis contf continuous_on_inverse_open_map imf injf inv_into_f_f oo*)
with *imf injf contf* **show** *homeomorphism S T f* (*inv_into S f*)
by (*auto simp*: *homeomorphism_def*)
qed

lemma *homeomorphism_injective_closed_map*:
assumes *contf*: *continuous_on S f*
and *imf*: $f \mathbin{`} S = T$
and *injf*: *inj_on f S*
and *oo*: $\bigwedge U$. *closedin* (*top_of_set S*) *U* $\Longrightarrow$ *closedin* (*top_of_set T*) ($f \mathbin{`} U$)
obtains *g* **where** *homeomorphism S T f g*
proof
have *continuous_on T* (*inv_into S f*)
by (*metis contf continuous_on_inverse_closed_map imf injf inv_into_f_f oo*)
with *imf injf contf* **show** *homeomorphism S T f* (*inv_into S f*)
by (*auto simp*: *homeomorphism_def*)
qed

lemma *homeomorphism_imp_open_map*:
assumes *hom*: *homeomorphism S T f g*
and *oo*: *openin* (*top_of_set S*) *U*
shows *openin* (*top_of_set T*) ($f \mathbin{`} U$)
proof −
from *hom oo* **have** [*simp*]: $f \mathbin{`} U = T \cap g -` U$
using *openin_subset* **by** (*fastforce simp*: *homeomorphism_def rev_image_eqI*)
from *hom* **have** *continuous_on T g*
unfolding *homeomorphism_def* **by** *blast*
moreover have $g \mathbin{`} T = S$
by (*metis hom homeomorphism_def*)
ultimately show *?thesis*
by (*simp add*: *continuous_on_open oo*)
qed

lemma *homeomorphism_imp_closed_map*:

assumes *hom*: *homeomorphism S T f g*
and *oo*: *closedin* (*top_of_set S*) *U*
shows *closedin* (*top_of_set T*) (*f ‘ U*)
proof −
from *hom oo* **have** [*simp*]: *f ‘ U* = *T* ∩ *g* −‘ *U*
using *closedin_subset* **by** (*fastforce simp*: *homeomorphism_def rev_image_eqI*)
from *hom* **have** *continuous_on T g*
unfolding *homeomorphism_def* **by** *blast*
moreover have *g ‘ T* = *S*
by (*metis hom homeomorphism_def*)
ultimately show *?thesis*
by (*simp add*: *continuous_on_closed oo*)
qed

2.3.9 Seperability

lemma *subset_second_countable*:
obtains $\mathcal{B}$ :: *'a*:: *second_countable_topology set set*
where *countable* $\mathcal{B}$
$\{\} \notin \mathcal{B}$
$\bigwedge C.\ C \in \mathcal{B} \Longrightarrow$ *openin*(*top_of_set S*) *C*
$\bigwedge T.$ *openin*(*top_of_set S*) *T* $\Longrightarrow \exists \mathcal{U}.\ \mathcal{U} \subseteq \mathcal{B} \wedge T = \bigcup \mathcal{U}$
proof −
obtain $\mathcal{B}$ :: *'a set set*
where *countable* $\mathcal{B}$
and *opeB*: $\bigwedge C.\ C \in \mathcal{B} \Longrightarrow$ *openin*(*top_of_set S*) *C*
and $\mathcal{B}$: $\bigwedge T.$ *openin*(*top_of_set S*) *T* $\Longrightarrow \exists \mathcal{U}.\ \mathcal{U} \subseteq \mathcal{B} \wedge T = \bigcup \mathcal{U}$
proof −
obtain $\mathcal{C}$ :: *'a set set*
where *countable* $\mathcal{C}$ **and** *ope*: $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow$ *open C*
and $\mathcal{C}$: $\bigwedge S.$ *open S* $\Longrightarrow \exists\, U.\ U \subseteq \mathcal{C} \wedge S = \bigcup U$
by (*metis univ_second_countable that*)
show *?thesis*
proof
show *countable* ((λ*C*. *S* ∩ *C*) ‘ $\mathcal{C}$)
by (*simp add*: ‹*countable* $\mathcal{C}$›)
show $\bigwedge C.\ C \in (\cap)\ S$ ‘ $\mathcal{C} \Longrightarrow$ *openin* (*top_of_set S*) *C*
using *ope* **by** *auto*
show $\bigwedge T.$ *openin* (*top_of_set S*) *T* $\Longrightarrow \exists \mathcal{U} \subseteq (\cap)\ S$ ‘ $\mathcal{C}.\ T = \bigcup \mathcal{U}$
by (*metis* $\mathcal{C}$ *image_mono inf_Sup openin_open*)
qed
qed
show *?thesis*
proof
show *countable* ($\mathcal{B} - \{\{\}\}$)
using ‹*countable* $\mathcal{B}$› **by** *blast*
show $\bigwedge C.\ [\![C \in \mathcal{B} - \{\{\}\}]\!] \Longrightarrow$ *openin* (*top_of_set S*) *C*
by (*simp add*: ‹$\bigwedge C.\ C \in \mathcal{B} \Longrightarrow$ *openin* (*top_of_set S*) *C*›)
show $\exists \mathcal{U} \subseteq \mathcal{B} - \{\{\}\}.\ T = \bigcup \mathcal{U}$ **if** *openin* (*top_of_set S*) *T* **for** *T*

```
    using \mathcal{B} [OF that]
    apply clarify
    by (rule_tac x=\mathcal{U} - {{}} in exI, auto)
  qed auto
qed

lemma Lindelof_openin:
  fixes \mathcal{F} :: 'a::second_countable_topology set set
  assumes \bigwedgeS. S \in \mathcal{F} \Longrightarrow openin (top_of_set U) S
  obtains \mathcal{F}' where \mathcal{F}' \subseteq \mathcal{F} countable \mathcal{F}' \bigcup\mathcal{F}' = \bigcup\mathcal{F}
proof -
  have \bigwedgeS. S \in \mathcal{F} \Longrightarrow \exists T. open T \wedge S = U \cap T
    using assms by (simp add: openin_open)
  then obtain tf where tf: \bigwedgeS. S \in \mathcal{F} \Longrightarrow open (tf S) \wedge (S = U \cap tf S)
    by metis
  have [simp]: \bigwedge\mathcal{F}'. \mathcal{F}' \subseteq \mathcal{F} \Longrightarrow \bigcup\mathcal{F}' = U \cap \bigcup(tf ` \mathcal{F}')
    using tf by fastforce
  obtain \mathcal{G} where countable \mathcal{G} \wedge \mathcal{G} \subseteq tf ` \mathcal{F} \bigcup\mathcal{G} = \bigcup(tf ` \mathcal{F})
    using tf by (force intro: Lindelof [of tf ` \mathcal{F}])
  then obtain \mathcal{F}' where \mathcal{F}': \mathcal{F}' \subseteq \mathcal{F} countable \mathcal{F}' \bigcup\mathcal{F}' = \bigcup\mathcal{F}
    by (clarsimp simp add: countable_subset_image)
  then show ?thesis ..
qed
```

2.3.10 Closed Maps

```
lemma continuous_imp_closed_map:
  fixes f :: 'a::t2_space \Rightarrow 'b::t2_space
  assumes closedin (top_of_set S) U
          continuous_on S f f ` S = T compact S
    shows closedin (top_of_set T) (f ` U)
 by (metis assms closedin_compact_eq compact_continuous_image continuous_on_subset
subset_image_iff)

lemma closed_map_restrict:
  assumes cloU: closedin (top_of_set (S \cap f -` T')) U
    and cc: \bigwedgeU. closedin (top_of_set S) U \Longrightarrow closedin (top_of_set T) (f ` U)
    and T' \subseteq T
  shows closedin (top_of_set T') (f ` U)
proof -
  obtain V where closed V U = S \cap f -` T' \cap V
    using cloU by (auto simp: closedin_closed)
  with cc [of S \cap V] ‹T' \subseteq T› show ?thesis
    by (fastforce simp add: closedin_closed)
qed
```

2.3.11 Open Maps

```
lemma open_map_restrict:
  assumes opeU: openin (top_of_set (S \cap f -` T')) U
```

and *oo*: $\bigwedge U.$ *openin* (*top_of_set S*) $U \Longrightarrow$ *openin* (*top_of_set T*) (*f* ‘ *U*)
and $T' \subseteq T$
shows *openin* (*top_of_set T′*) (*f* ‘ *U*)
proof −
obtain *V* **where** *open V U* = *S* ∩ *f* −‘ *T′* ∩ *V*
using *opeU* **by** (*auto simp*: *openin_open*)
with *oo* [*of S* ∩ *V*] ‹*T′* ⊆ *T*› **show** *?thesis*
by (*fastforce simp add*: *openin_open*)
qed

2.3.12 Quotient maps

lemma *quotient_map_imp_continuous_open*:
assumes *T*: *f* ∈ *S* → *T*
and *ope*: $\bigwedge U.$ *U* ⊆ *T*
⟹ (*openin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*) ⟷
openin (*top_of_set T*) *U*)
shows *continuous_on S f*
proof −
have [*simp*]: *S* ∩ *f* −‘ *f* ‘ *S* = *S* **by** *auto*
show *?thesis*
by (*meson T continuous_on_open_gen ope openin_imp_subset*)
qed

lemma *quotient_map_imp_continuous_closed*:
assumes *T*: *f* ∈ *S* → *T*
and *ope*: $\bigwedge U.$ *U* ⊆ *T*
⟹ (*closedin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*) ⟷
closedin (*top_of_set T*) *U*)
shows *continuous_on S f*
proof −
have [*simp*]: *S* ∩ *f* −‘ *f* ‘ *S* = *S* **by** *auto*
show *?thesis*
by (*meson T closedin_imp_subset continuous_on_closed_gen ope*)
qed

lemma *open_map_imp_quotient_map*:
assumes *contf*: *continuous_on S f*
and *T*: *T* ⊆ *f* ‘ *S*
and *ope*: $\bigwedge T.$ *openin* (*top_of_set S*) *T*
⟹ *openin* (*top_of_set* (*f* ‘ *S*)) (*f* ‘ *T*)
shows *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*) =
openin (*top_of_set* (*f* ‘ *S*)) *T*
proof −
have *T* = *f* ‘ (*S* ∩ *f* −‘ *T*)
using *T* **by** *blast*
then show *?thesis*
using *ope contf continuous_on_open* **by** *metis*
qed

lemma *closed_map_imp_quotient_map*:
assumes *contf*: *continuous_on S f*
and *T*: $T \subseteq f$ ‘ S
and *ope*: $\bigwedge T$. *closedin* (*top_of_set S*) T
$\Longrightarrow$ *closedin* (*top_of_set* (f ‘ S)) (f ‘ T)
shows *openin* (*top_of_set S*) ($S \cap f$ −‘ T) $\longleftrightarrow$
openin (*top_of_set* (f ‘ S)) T
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have ∗: *closedin* (*top_of_set S*) (S − ($S \cap f$ −‘ T))
using *closedin_diff* **by** *fastforce*
have [*simp*]: (f ‘ S − f ‘ (S − ($S \cap f$ −‘ T))) = T
using T **by** *blast*
show *?rhs*
using *ope* [*OF* ∗, *unfolded closedin_def*] **by** *auto*
next
assume *?rhs*
with *contf* **show** *?lhs*
by (*auto simp*: *continuous_on_open*)
qed

lemma *continuous_right_inverse_imp_quotient_map*:
assumes *contf*: *continuous_on S f* **and** *imf*: $f \in S \to T$
and *contg*: *continuous_on T g* **and** *img*: $g \in T \to S$
and *fg* [*simp*]: $\bigwedge y$. $y \in T \Longrightarrow f(g\ y) = y$
and *U*: $U \subseteq T$
shows *openin* (*top_of_set S*) ($S \cap f$ −‘ U) $\longleftrightarrow$
openin (*top_of_set T*) U
(**is** *?lhs* = *?rhs*)
proof −
have *f*: $\bigwedge Z$. *openin* (*top_of_set* (f ‘ S)) $Z \Longrightarrow$
openin (*top_of_set S*) ($S \cap f$ −‘ Z)
and *g*: $\bigwedge Z$. *openin* (*top_of_set* (g ‘ T)) $Z \Longrightarrow$
openin (*top_of_set T*) ($T \cap g$ −‘ Z)
using *contf contg* **by** (*auto simp*: *continuous_on_open*)
show *?thesis*
proof
have $T \cap g$ −‘ (g ‘ $T \cap$ ($S \cap f$ −‘ U)) = {$x \in T$. f ($g\ x$) $\in U$}
using *imf img* **by** *blast*
also have ... = U
using U **by** *auto*
finally have *eq*: $T \cap g$ −‘ (g ‘ $T \cap$ ($S \cap f$ −‘ U)) = U **.**
assume *?lhs*
then have ∗: *openin* (*top_of_set* (g ‘ T)) (g ‘ $T \cap$ ($S \cap f$ −‘ U))
by (*meson img openin_Int openin_subtopology_Int_subset openin_subtopology_self image_subset_iff_funcset*)
show *?rhs*

```
    using g [OF *] eq by auto
  next
    assume rhs: ?rhs
    show ?lhs
      using assms continuous_openin_preimage rhs by blast
  qed
qed

lemma continuous_left_inverse_imp_quotient_map:
  assumes continuous_on S f
      and continuous_on (f ' S) g
      and  ⋀x. x ∈ S ⟹ g(f x) = x
      and U ⊆ f ' S
    shows openin (top_of_set S) (S ∩ f -' U) ⟷
          openin (top_of_set (f ' S)) U
  using assms
  by (intro continuous_right_inverse_imp_quotient_map) auto

lemma continuous_imp_quotient_map:
  fixes f :: 'a::t2_space ⇒ 'b::t2_space
  assumes continuous_on S f f ' S = T compact S U ⊆ T
    shows openin (top_of_set S) (S ∩ f -' U) ⟷
          openin (top_of_set T) U
 by (simp add: assms closed_map_imp_quotient_map continuous_imp_closed_map)
```

2.3.13 Pasting lemmas for functions, for of casewise definitions

on open sets

```
lemma pasting_lemma:
  assumes ope: ⋀i. i ∈ I ⟹ openin X (T i)
      and cont: ⋀i. i ∈ I ⟹ continuous_map(subtopology X (T i)) Y (f i)
      and f: ⋀i j x. ⟦i ∈ I; j ∈ I; x ∈ topspace X ∩ T i ∩ T j⟧ ⟹ f i x = f j x
      and g: ⋀x. x ∈ topspace X ⟹ ∃j. j ∈ I ∧ x ∈ T j ∧ g x = f j x
    shows continuous_map X Y g
  unfolding continuous_map_openin_preimage_eq
proof (intro conjI allI impI)
  show g ∈ topspace X → topspace Y
    using g cont continuous_map_image_subset_topspace by fastforce
next
  fix U
  assume Y: openin Y U
  have T: T i ⊆ topspace X if i ∈ I for i
    using ope by (simp add: openin_subset that)
  have *: topspace X ∩ g -' U = (⋃ i ∈ I. T i ∩ f i -' U)
    using f g T by fastforce
  have ⋀i. i ∈ I ⟹ openin X (T i ∩ f i -' U)
    using cont unfolding continuous_map_openin_preimage_eq
   by (metis Y T inf.commute inf_absorb1 ope topspace_subtopology openin_trans_full)
```

```
  then show openin X (topspace X ∩ g −‘ U)
    by (auto simp: ∗)
qed

lemma pasting_lemma_exists:
  assumes X: topspace X ⊆ (⋃ i ∈ I. T i)
      and ope: ⋀i. i ∈ I ⟹ openin X (T i)
      and cont: ⋀i. i ∈ I ⟹ continuous_map (subtopology X (T i)) Y (f i)
      and f: ⋀i j x. ⟦i ∈ I; j ∈ I; x ∈ topspace X ∩ T i ∩ T j⟧ ⟹ f i x = f j x
    obtains g where continuous_map X Y g ⋀x i. ⟦i ∈ I; x ∈ topspace X ∩ T i⟧
⟹ g x = f i x
proof
  let ?h = λx. f (SOME i. i ∈ I ∧ x ∈ T i) x
  show continuous_map X Y ?h
    apply (rule pasting_lemma [OF ope cont])
     apply (blast intro: f)+
    by (metis (no_types, lifting) UN_E X subsetD someI_ex)
  show f (SOME i. i ∈ I ∧ x ∈ T i) x = f i x if i ∈ I x ∈ topspace X ∩ T i for
i x
    by (metis (no_types, lifting) IntD2 IntI f someI_ex that)
qed

lemma pasting_lemma_locally_finite:
  assumes fin: ⋀x. x ∈ topspace X ⟹ ∃ V. openin X V ∧ x ∈ V ∧ finite {i ∈
I. T i ∩ V ≠ {}}
    and clo: ⋀i. i ∈ I ⟹ closedin X (T i)
    and cont: ⋀i. i ∈ I ⟹ continuous_map(subtopology X (T i)) Y (f i)
    and f: ⋀i j x. ⟦i ∈ I; j ∈ I; x ∈ topspace X ∩ T i ∩ T j⟧ ⟹ f i x = f j x
    and g: ⋀x. x ∈ topspace X ⟹ ∃j. j ∈ I ∧ x ∈ T j ∧ g x = f j x
  shows continuous_map X Y g
  unfolding continuous_map_closedin_preimage_eq
proof (intro conjI allI impI)
  show g ∈ topspace X → topspace Y
    using g cont continuous_map_image_subset_topspace by fastforce
next
  fix U
  assume Y: closedin Y U
  have T: T i ⊆ topspace X if i ∈ I for i
    using clo by (simp add: closedin_subset that)
  have ∗: topspace X ∩ g −‘ U = (⋃ i ∈ I. T i ∩ f i −‘ U)
    using f g T by fastforce
  have cTf: ⋀i. i ∈ I ⟹ closedin X (T i ∩ f i −‘ U)
   using cont unfolding continuous_map_closedin_preimage_eq topspace_subtopology
    by (simp add: Int_absorb1 T Y clo closedin_closed_subtopology)
  have sub: {Z ∈ (λi. T i ∩ f i −‘ U) ‘ I. Z ∩ V ≠ {}}
        ⊆ (λi. T i ∩ f i −‘ U) ‘ {i ∈ I. T i ∩ V ≠ {}} for V
    by auto
  have 1: (⋃ i∈I. T i ∩ f i −‘ U) ⊆ topspace X
    using T by blast
```

then have *locally_finite_in X* ((λ*i*. *T i* ∩ *f i* −‘ *U*) ‘ *I*)
unfolding *locally_finite_in_def*
using *finite_subset* [*OF sub*] *fin* **by** *force*
then show *closedin X* (*topspace X* ∩ *g* −‘ *U*)
by (*smt* (*verit*, *best*) ∗ *cTf closedin_locally_finite_Union image_iff*)
qed

Likewise on closed sets, with a finiteness assumption

lemma *pasting_lemma_closed*:
assumes *fin*: *finite I*
and *clo*: ⋀*i*. *i* ∈ *I* ⟹ *closedin X* (*T i*)
and *cont*: ⋀*i*. *i* ∈ *I* ⟹ *continuous_map*(*subtopology X* (*T i*)) *Y* (*f i*)
and *f*: ⋀*i j x*. ⟦*i* ∈ *I*; *j* ∈ *I*; *x* ∈ *topspace X* ∩ *T i* ∩ *T j*⟧ ⟹ *f i x* = *f j x*
and *g*: ⋀*x*. *x* ∈ *topspace X* ⟹ ∃*j*. *j* ∈ *I* ∧ *x* ∈ *T j* ∧ *g x* = *f j x*
shows *continuous_map X Y g*
using *pasting_lemma_locally_finite* [*OF _ clo cont f g*] *fin* **by** *auto*

lemma *pasting_lemma_exists_locally_finite*:
assumes *fin*: ⋀*x*. *x* ∈ *topspace X* ⟹ ∃ *V*. *openin X V* ∧ *x* ∈ *V* ∧ *finite* {*i* ∈ *I*. *T i* ∩ *V* ≠ {}}
and *X*: *topspace X* ⊆ ⋃(*T* ‘ *I*)
and *clo*: ⋀*i*. *i* ∈ *I* ⟹ *closedin X* (*T i*)
and *cont*: ⋀*i*. *i* ∈ *I* ⟹ *continuous_map*(*subtopology X* (*T i*)) *Y* (*f i*)
and *f*: ⋀*i j x*. ⟦*i* ∈ *I*; *j* ∈ *I*; *x* ∈ *topspace X* ∩ *T i* ∩ *T j*⟧ ⟹ *f i x* = *f j x*
and *g*: ⋀*x*. *x* ∈ *topspace X* ⟹ ∃*j*. *j* ∈ *I* ∧ *x* ∈ *T j* ∧ *g x* = *f j x*
obtains *g* **where** *continuous_map X Y g* ⋀*x i*. ⟦*i* ∈ *I*; *x* ∈ *topspace X* ∩ *T i*⟧ ⟹ *g x* = *f i x*
proof
show *continuous_map X Y* (λ*x*. *f*(@*i*. *i* ∈ *I* ∧ *x* ∈ *T i*) *x*)
apply (*rule pasting_lemma_locally_finite* [*OF fin*])
apply (*blast intro*: *assms*)+
by (*metis* (*no_types*, *lifting*) *UN_E X set_rev_mp someI_ex*)
next
fix *x i*
assume *i* ∈ *I* **and** *x* ∈ *topspace X* ∩ *T i*
then show *f* (*SOME i*. *i* ∈ *I* ∧ *x* ∈ *T i*) *x* = *f i x*
by (*metis* (*mono_tags*, *lifting*) *IntE IntI f someI2*)
qed

lemma *pasting_lemma_exists_closed*:
assumes *fin*: *finite I*
and *X*: *topspace X* ⊆ ⋃(*T* ‘ *I*)
and *clo*: ⋀*i*. *i* ∈ *I* ⟹ *closedin X* (*T i*)
and *cont*: ⋀*i*. *i* ∈ *I* ⟹ *continuous_map*(*subtopology X* (*T i*)) *Y* (*f i*)
and *f*: ⋀*i j x*. ⟦*i* ∈ *I*; *j* ∈ *I*; *x* ∈ *topspace X* ∩ *T i* ∩ *T j*⟧ ⟹ *f i x* = *f j x*
obtains *g* **where** *continuous_map X Y g* ⋀*x i*. ⟦*i* ∈ *I*; *x* ∈ *topspace X* ∩ *T i*⟧ ⟹ *g x* = *f i x*
proof

```
  show continuous_map X Y (λx. f (SOME i. i ∈ I ∧ x ∈ T i) x)
    apply (rule pasting_lemma_closed [OF ‹finite I› clo cont])
     apply (blast intro: f)+
    by (metis (mono_tags, lifting) UN_iff X someI_ex subset_iff)
next
  fix x i
  assume i ∈ I x ∈ topspace X ∩ T i
  then show f (SOME i. i ∈ I ∧ x ∈ T i) x = f i x
    by (metis (no_types, lifting) IntD2 IntI f someI_ex)
qed

lemma continuous_map_cases:
  assumes f: continuous_map (subtopology X (X closure_of {x. P x})) Y f
      and g: continuous_map (subtopology X (X closure_of {x. ¬ P x})) Y g
      and fg: ⋀x. x ∈ X frontier_of {x. P x} ⟹ f x = g x
  shows continuous_map X Y (λx. if P x then f x else g x)
proof (rule pasting_lemma_closed)
  let ?f = λb. if b then f else g
  let ?g = λx. if P x then f x else g x
  let ?T = λb. if b then X closure_of {x. P x} else X closure_of {x. ~P x}
  show finite {True,False} by auto
  have eq: topspace X − Collect P = topspace X ∩ {x. ¬ P x}
    by blast
  show ?f i x = ?f j x
    if i ∈ {True,False} j ∈ {True,False} and x: x ∈ topspace X ∩ ?T i ∩ ?T j for
i j x
  proof −
    have f x = g x if i ¬ j
     by (smt (verit, best) Diff_Diff_Int closure_of_interior_of closure_of_restrict
eq fg
          frontier_of_closures interior_of_complement that x)
    moreover
    have g x = f x
      if x ∈ X closure_of {x. ¬ P x} x ∈ X closure_of Collect P ¬ i j for x
      by (metis IntI closure_of_restrict eq fg frontier_of_closures that)
    ultimately show ?thesis
      using that by (auto simp flip: closure_of_restrict)
  qed
  show ∃j. j ∈ {True,False} ∧ x ∈ ?T j ∧ (if P x then f x else g x) = ?f j x
    if x ∈ topspace X for x
    by simp (metis in_closure_of mem_Collect_eq that)
qed (auto simp: f g)

lemma continuous_map_cases_alt:
  assumes f: continuous_map (subtopology X (X closure_of {x ∈ topspace X. P
x})) Y f
      and g: continuous_map (subtopology X (X closure_of {x ∈ topspace X. ~P
x})) Y g
      and fg: ⋀x. x ∈ X frontier_of {x ∈ topspace X. P x} ⟹ f x = g x
```

shows *continuous_map X Y* (λ*x*. *if P x then f x else g x*)
apply (*rule continuous_map_cases*)
using *assms*
apply (*simp_all add*: *Collect_conj_eq closure_of_restrict* [*symmetric*] *frontier_of_restrict* [*symmetric*])
done

lemma *continuous_map_cases_function*:
assumes *contp*: *continuous_map X Z p*
and *contf*: *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *p x* ∈ *Z closure_of U*}) *Y f*
and *contg*: *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *p x* ∈ *Z closure_of* (*topspace Z* − *U*)}) *Y g*
and *fg*: ⋀*x*. ⟦*x* ∈ *topspace X*; *p x* ∈ *Z frontier_of U*⟧ ⟹ *f x* = *g x*
shows *continuous_map X Y* (λ*x*. *if p x* ∈ *U then f x else g x*)
proof (*rule continuous_map_cases_alt*)
show *continuous_map* (*subtopology X* (*X closure_of* {*x* ∈ *topspace X*. *p x* ∈ *U*})) *Y f*
proof (*rule continuous_map_from_subtopology_mono*)
let *?T* = {*x* ∈ *topspace X*. *p x* ∈ *Z closure_of U*}
show *continuous_map* (*subtopology X ?T*) *Y f*
by (*simp add*: *contf*)
show *X closure_of* {*x* ∈ *topspace X*. *p x* ∈ *U*} ⊆ *?T*
by (*rule continuous_map_closure_preimage_subset* [*OF contp*])
qed
show *continuous_map* (*subtopology X* (*X closure_of* {*x* ∈ *topspace X*. *p x* ∉ *U*})) *Y g*
proof (*rule continuous_map_from_subtopology_mono*)
let *?T* = {*x* ∈ *topspace X*. *p x* ∈ *Z closure_of* (*topspace Z* − *U*)}
show *continuous_map* (*subtopology X ?T*) *Y g*
by (*simp add*: *contg*)
have *X closure_of* {*x* ∈ *topspace X*. *p x* ∉ *U*} ⊆ *X closure_of* {*x* ∈ *topspace X*. *p x* ∈ *topspace Z* − *U*}
by (*smt* (*verit*) *Collect_mono_iff DiffI closure_of_mono continuous_map contp image_subset_iff*)
then show *X closure_of* {*x* ∈ *topspace X*. *p x* ∉ *U*} ⊆ *?T*
by (*rule order_trans* [*OF _ continuous_map_closure_preimage_subset* [*OF contp*]])
qed
next
show *f x* = *g x* **if** *x* ∈ *X frontier_of* {*x* ∈ *topspace X*. *p x* ∈ *U*} **for** *x*
using *that continuous_map_frontier_frontier_preimage_subset* [*OF contp*, *of U*] *fg* **by** *blast*
qed

2.3.14 Retractions

definition *retraction* :: ('*a*::*topological_space*) *set* ⇒ '*a set* ⇒ ('*a* ⇒ '*a*) ⇒ *bool*
where *retraction S T r* ⟷

$T \subseteq S \wedge$ *continuous_on* $S\ r \wedge r \in S \to T \wedge (\forall x{\in}T.\ r\ x = x)$

definition *retract_of* (**infixl** ‹*retract'_of*› 50) **where**
T *retract_of* $S \longleftrightarrow (\exists r.$ *retraction* $S\ T\ r)$

lemma *retraction_idempotent*: *retraction* $S\ T\ r \Longrightarrow x \in S \Longrightarrow\ r\ (r\ x) = r\ x$
unfolding *retraction_def* **by** *auto*

Preservation of fixpoints under (more general notion of) retraction

lemma *invertible_fixpoint_property*:
fixes S :: *'a::topological_space set*
and T :: *'b::topological_space set*
assumes *contt*: *continuous_on* $T\ i$
and $i \in T \to S$
and *contr*: *continuous_on* $S\ r$
and $r \in S \to T$
and *ri*: $\bigwedge y.\ y \in T \Longrightarrow r\ (i\ y) = y$
and *FP*: $\bigwedge f.\ [\![$*continuous_on* $S\ f$; $f \in S \to S]\!] \Longrightarrow \exists x{\in}S.\ f\ x = x$
and *contg*: *continuous_on* $T\ g$
and $g \in T \to T$
obtains y **where** $y \in T$ **and** $g\ y = y$
proof −
have $\exists x{\in}S.\ (i \circ g \circ r)\ x = x$
proof (*rule FP*)
show *continuous_on* $S\ (i \circ g \circ r)$
by (*metis assms*(4) *assms*(8) *contg continuous_on_compose continuous_on_subset contr contt funcset_image*)
show $(i \circ g \circ r) \in S \to S$
using *assms*(2,4,8) **by** *force*
qed
then obtain x **where** x: $x \in S\ (i \circ g \circ r)\ x = x$ **..**
then have $*$: $g\ (r\ x) \in T$
using *assms*(4,8) **by** *auto*
have $r\ ((i \circ g \circ r)\ x) = r\ x$
using x **by** *auto*
then show *?thesis*
using $*$ *ri that* **by** *auto*
qed

lemma *homeomorphic_fixpoint_property*:
fixes S :: *'a::topological_space set*
and T :: *'b::topological_space set*
assumes S *homeomorphic* T
shows $(\forall f.$ *continuous_on* $S\ f \wedge f \in S \to S \longrightarrow (\exists x{\in}S.\ f\ x = x)) \longleftrightarrow$
$(\forall g.$ *continuous_on* $T\ g \wedge g \in T \to T \longrightarrow (\exists y{\in}T.\ g\ y = y))$
(**is** *?lhs* = *?rhs*)
proof −
obtain $r\ i$ **where** r:
$\forall x{\in}S.\ i\ (r\ x) = x\ r\ `\ S = T$ *continuous_on* $S\ r$

```
    ∀ y∈T. r (i y) = y i ‘ T = S continuous_on T i
   using assms unfolding homeomorphic_def homeomorphism_def by blast
  show ?thesis
  proof
   assume ?lhs
   with r show ?rhs
    by (smt (verit, del_insts) Pi_iff image_eqI invertible_fixpoint_property[of T
i S r])
  next
   assume ?rhs
   with r show ?lhs
    by (smt (verit, del_insts) Pi_iff image_eqI invertible_fixpoint_property[of S
r T i])
  qed
qed

lemma retract_fixpoint_property:
  fixes f :: 'a::topological_space ⇒ 'b::topological_space
   and S :: 'a set
  assumes T retract_of S
   and FP: ⋀f. ⟦continuous_on S f; f ∈ S → S⟧ ⟹ ∃ x∈S. f x = x
   and contg: continuous_on T g
   and g ∈ T → T
  obtains y where y ∈ T and g y = y
proof −
  obtain h where retraction S T h
   using assms(1) unfolding retract_of_def ..
  then show ?thesis
   unfolding retraction_def
   using invertible_fixpoint_property[OF continuous_on_id _ _ _ _ FP]
   by (smt (verit, del_insts) Pi_iff assms(4) contg subsetD that)
qed

lemma retraction:
  retraction S T r ⟷ T ⊆ S ∧ continuous_on S r ∧ r ‘ S = T ∧ (∀ x ∈ T. r x
= x)
  by (force simp: retraction_def simp flip: image_subset_iff_funcset)

lemma retractionE: — yields properties normalized wrt. simp – less likely to loop
  assumes retraction S T r
  obtains T = r ‘ S r ∈ S → S continuous_on S r ⋀x. x ∈ S ⟹ r (r x) = r x
proof (rule that)
  from retraction [of S T r] assms
  have T ⊆ S continuous_on S r r ‘ S = T and ∀ x ∈ T. r x = x
   by simp_all
  then show  r ∈ S → S continuous_on S r
   by auto
  then show T = r ‘ S
   using ‹r ‘ S = T› by blast
```

```
  from ‹∀x∈T. r x = x› have r x = x if x ∈ T for x
    using that by simp
  with ‹r ‘ S = T› show r (r x) = r x if x ∈ S for x
    using that by auto
qed

lemma retract_ofE: — yields properties normalized wrt. simp – less likely to loop
  assumes T retract_of S
  obtains r where T = r ‘ S r ∈ S → S continuous_on S r ⋀x. x ∈ S ⟹ r (r
x) = r x
proof −
  from assms obtain r where retraction S T r
    by (auto simp add: retract_of_def)
  with that show thesis
    by (auto elim: retractionE)
qed

lemma retract_of_imp_extensible:
  assumes S retract_of T and continuous_on S f and f ∈ S → U
  obtains g where continuous_on T g g ∈ T → U ⋀x. x ∈ S ⟹ g x = f x
proof −
  from ‹S retract_of T› obtain r where retraction T S r
    by (auto simp add: retract_of_def)
  then have continuous_on T (f ∘ r)
     by (metis assms(2) continuous_on_compose retraction)
  then show thesis
    by (smt (verit, best) Pi_iff ‹retraction T S r› assms(3) comp_apply retrac-
tion_def that)
qed

lemma idempotent_imp_retraction:
  assumes continuous_on S f and f ∈ S → S and ⋀x. x ∈ S ⟹ f(f x) = f x
    shows retraction S (f ‘ S) f
  by (simp add: assms funcset_image retraction)

lemma retraction_subset:
  assumes retraction S T r and T ⊆ S′ and S′ ⊆ S
  shows retraction S′ T r
  unfolding retraction_def
  by (metis assms continuous_on_subset image_mono image_subset_iff_funcset
retraction)

lemma retract_of_subset:
  assumes T retract_of S and T ⊆ S′ and S′ ⊆ S
    shows T retract_of S′
by (meson assms retract_of_def retraction_subset)

lemma retraction_refl [simp]: retraction S S (λx. x)
by (simp add: retraction)
```

lemma *retract_of_refl* [*iff*]: *S retract_of S*
unfolding *retract_of_def retraction_def*
using *continuous_on_id* **by** *blast*

lemma *retract_of_imp_subset*:
S retract_of T $\Longrightarrow$ *S* $\subseteq$ *T*
by (*simp add*: *retract_of_def retraction_def*)

lemma *retract_of_empty* [*simp*]:
({} *retract_of S*) $\longleftrightarrow$ *S* = {} (*S retract_of* {}) $\longleftrightarrow$ *S* = {}
by (*auto simp*: *retract_of_def retraction_def*)

lemma *retract_of_singleton* [*iff*]: ({*x*} *retract_of S*) $\longleftrightarrow$ *x* $\in$ *S*
unfolding *retract_of_def retraction_def* **by** *force*

lemma *retraction_comp*:
$[\![$*retraction S T f*; *retraction T U g*$]\!] \Longrightarrow$ *retraction S U* (*g* $\circ$ *f*)
by (*smt* (*verit*, *best*) *comp_apply continuous_on_compose image_comp retraction subset_iff*)

lemma *retract_of_trans* [*trans*]:
assumes *S retract_of T* **and** *T retract_of U*
shows *S retract_of U*
using *assms* **by** (*auto simp*: *retract_of_def intro*: *retraction_comp*)

lemma *closedin_retract*:
fixes *S* :: *'a* :: *t2_space set*
assumes *S retract_of T*
shows *closedin* (*top_of_set T*) *S*
proof −
obtain *r* **where** *r*: *S* $\subseteq$ *T continuous_on T r r* $\in$ *T* $\to$ *S* $\bigwedge$*x*. *x* $\in$ *S* $\Longrightarrow$ *r x* = *x*
using *assms* **by** (*auto simp*: *retract_of_def retraction_def*)
have *S* = {*x*$\in$*T*. *x* = *r x*}
using *r* **by** *force*
also have ... = *T* $\cap$ ((*λx*. (*x*, *r x*)) −‘ ({*y*. $\exists$ *x*. *y* = (*x*, *x*)}))
unfolding *vimage_def mem_Times_iff fst_conv snd_conv*
using *r*
by *auto*
also have *closedin* (*top_of_set T*) ...
by (*rule continuous_closedin_preimage*) (*auto intro*!: *closed_diagonal continuous_on_Pair r*)
finally show *?thesis* **.**
qed

lemma *closedin_self* [*simp*]: *closedin* (*top_of_set S*) *S*
by *simp*

lemma *retract_of_closed*:
  fixes $S :: 'a :: t2\_space\ set$
  shows $[\![closed\ T;\ S\ retract\_of\ T]\!] \Longrightarrow closed\ S$
 by (*metis closedin_retract closedin_closed_eq*)

lemma *retract_of_compact*:
  $[\![compact\ T;\ S\ retract\_of\ T]\!] \Longrightarrow compact\ S$
 by (*metis compact_continuous_image retract_of_def retraction*)

lemma *retract_of_connected*:
  $[\![connected\ T;\ S\ retract\_of\ T]\!] \Longrightarrow connected\ S$
 by (*metis Topological_Spaces.connected_continuous_image retract_of_def retraction*)

lemma *retraction_openin_vimage_iff*:
 assumes *r*: *retraction S T r* **and** $U \subseteq T$
 shows *openin* (*top_of_set S*) ($S \cap r -` U$) $\longleftrightarrow$ *openin* (*top_of_set T*) *U* (**is** *?lhs = ?rhs*)
proof
 show *?lhs* $\Longrightarrow$ *?rhs*
  using *r*
  by (*smt* (*verit*, *del_insts*) *assms*(*2*) *continuous_right_inverse_imp_quotient_map image_subset_iff_funcset r retractionE retraction_def retraction_subset*)
 show *?rhs* $\Longrightarrow$ *?lhs*
  by (*metis continuous_on_open r retraction*)
qed

lemma *retract_of_Times*:
  $[\![S\ retract\_of\ S';\ T\ retract\_of\ T']\!] \Longrightarrow (S \times T)\ retract\_of\ (S' \times T')$
apply (*simp add*: *retract_of_def retraction_def Sigma_mono*, *clarify*)
apply (*rename_tac f g*)
apply (*rule_tac x*=$\lambda z.\ ((f \circ fst)\ z, (g \circ snd)\ z)$ **in** *exI*)
apply (*rule conjI continuous_intros* | *erule continuous_on_subset* | *force*)+
done

2.3.15 Retractions on a topological space

definition *retract_of_space* :: $'a\ set \Rightarrow 'a\ topology \Rightarrow bool$ (**infix** ‹*retract'_of'_space*› *50*)
 where $S\ retract\_of\_space\ X \equiv S \subseteq topspace\ X \wedge (\exists r.\ continuous\_map\ X\ (subtopology\ X\ S)\ r \wedge (\forall x \in S.\ r\ x = x))$

lemma *retract_of_space_retraction_maps*:
 $S\ retract\_of\_space\ X \longleftrightarrow S \subseteq topspace\ X \wedge (\exists r.\ retraction\_maps\ X\ (subtopology\ X\ S)\ r\ id)$
 by (*auto simp*: *retract_of_space_def retraction_maps_def*)

lemma *retract_of_space_section_map*:

S retract_of_space X ⟷ S ⊆ topspace X ∧ section_map (subtopology X S) X id
unfolding *retract_of_space_def retraction_maps_def section_map_def*
by (*auto simp*: *continuous_map_from_subtopology*)

lemma *retract_of_space_imp_subset*:
S retract_of_space X ⟹ S ⊆ topspace X
by (*simp add*: *retract_of_space_def*)

lemma *retract_of_space_topspace*:
topspace X retract_of_space X
using *retract_of_space_def* **by** *force*

lemma *retract_of_space_empty* [*simp*]:
{} retract_of_space X ⟷ X = trivial_topology
by (*auto simp*: *retract_of_space_def*)

lemma *retract_of_space_singleton* [*simp*]:
{a} retract_of_space X ⟷ a ∈ topspace X
proof −
have *continuous_map X (subtopology X {a}) (λx. a) ∧ (λx. a) a = a* **if** *a ∈ topspace X*
using *that* **by** *simp*
then show *?thesis*
by (*force simp*: *retract_of_space_def*)
qed

lemma *retract_of_space_trans*:
assumes *S retract_of_space X T retract_of_space subtopology X S*
shows *T retract_of_space X*
using *assms*
apply (*simp add*: *retract_of_space_retraction_maps*)
by (*metis id_comp inf.absorb_iff2 retraction_maps_compose subtopology_subtopology*)

lemma *retract_of_space_subtopology*:
assumes *S retract_of_space X S ⊆ U*
shows *S retract_of_space subtopology X U*
using *assms*
apply (*clarsimp simp*: *retract_of_space_def*)
by (*metis continuous_map_from_subtopology inf.absorb2 subtopology_subtopology*)

lemma *retract_of_space_clopen*:
assumes *openin X S closedin X S S = {} ⟹ X = trivial_topology*
shows *S retract_of_space X*
proof (*cases S = {}*)
case *False*
then obtain *a* **where** *a ∈ S*
by *blast*
show *?thesis*

```
    unfolding retract_of_space_def
  proof (intro exI conjI)
    show S ⊆ topspace X
      by (simp add: assms closedin_subset)
    have continuous_map X X (λx. if x ∈ S then x else a)
    proof (rule continuous_map_cases)
      show continuous_map (subtopology X (X closure_of {x. x ∈ S})) X (λx. x)
        by (simp add: continuous_map_from_subtopology)
      show continuous_map (subtopology X (X closure_of {x. x ∉ S})) X (λx. a)
        using ‹S ⊆ topspace X› ‹a ∈ S› by force
      show x = a if x ∈ X frontier_of {x. x ∈ S} for x
        using assms that clopenin_eq_frontier_of by fastforce
    qed
    then show continuous_map X (subtopology X S) (λx. if x ∈ S then x else a)
    using ‹S ⊆ topspace X› ‹a ∈ S› by (auto simp: continuous_map_in_subtopology)
  qed auto
qed (use assms in auto)

lemma retract_of_space_disjoint_union:
  assumes openin X S openin X T and ST: disjnt S T S ∪ T = topspace X and
S = {} ⟹ X = trivial_topology
  shows S retract_of_space X
proof (rule retract_of_space_clopen)
  have S ∩ T = {}
    by (meson ST disjnt_def)
  then have S = topspace X - T
    using ST by auto
  then show closedin X S
    using ‹openin X T› by blast
qed (auto simp: assms)

lemma retraction_maps_section_image1:
  assumes retraction_maps X Y r s
  shows s ‘ (topspace Y) retract_of_space X
  unfolding retract_of_space_section_map
proof
  show s ‘ topspace Y ⊆ topspace X
     using assms continuous_map_image_subset_topspace retraction_maps_def
by blast
  show section_map (subtopology X (s ‘ topspace Y)) X id
    unfolding section_map_def
    using assms retraction_maps_to_retract_maps by blast
qed

lemma retraction_maps_section_image2:
   retraction_maps X Y r s
       ⟹ subtopology X (s ‘ (topspace Y)) homeomorphic_space Y
   using embedding_map_imp_homeomorphic_space homeomorphic_space_sym
section_imp_embedding_map
```

section_map_def **by** *blast*

lemma *hereditary_imp_retractive_property*:
assumes $\bigwedge X\ S.\ P\ X \Longrightarrow P(\mathit{subtopology}\ X\ S)$
$\bigwedge X\ X'.\ X\ \mathit{homeomorphic\_space}\ X' \Longrightarrow (P\ X \longleftrightarrow Q\ X')$
assumes *retraction_map X X' r P X*
shows $Q\ X'$
by (*meson assms retraction_map_def retraction_maps_section_image2*)

2.3.16 Paths and path-connectedness

definition *pathin* :: *'a topology* ⇒ (*real* ⇒ *'a*) ⇒ *bool* **where**
pathin X g ≡ *continuous_map* (*subtopology euclideanreal* {*0..1*}) *X g*

lemma *pathin_compose*:
⟦*pathin X g*; *continuous_map X Y f*⟧ ⟹ *pathin Y* (*f* ∘ *g*)
by (*simp add*: *continuous_map_compose pathin_def*)

lemma *pathin_subtopology*:
pathin (*subtopology X S*) *g* ⟷ *pathin X g* ∧ (∀ *x* ∈ {*0..1*}. *g x* ∈ *S*)
by (*auto simp*: *pathin_def continuous_map_in_subtopology*)

lemma *pathin_const* [*simp*]: *pathin X* (λ*x*. *a*) ⟷ *a* ∈ *topspace X*
using *pathin_subtopology* **by** (*fastforce simp add*: *pathin_def*)

lemma *path_start_in_topspace*: *pathin X g* ⟹ *g 0* ∈ *topspace X*
by (*force simp*: *pathin_def continuous_map*)

lemma *path_finish_in_topspace*: *pathin X g* ⟹ *g 1* ∈ *topspace X*
by (*force simp*: *pathin_def continuous_map*)

lemma *path_image_subset_topspace*: *pathin X g* ⟹ *g* ∈ ({*0..1*}) → *topspace X*
by (*force simp*: *pathin_def continuous_map*)

definition *path_connected_space* :: *'a topology* ⇒ *bool*
where *path_connected_space X* ≡ ∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. ∃ *g*. *pathin X g* ∧ *g 0* = *x* ∧ *g 1* = *y*

definition *path_connectedin* :: *'a topology* ⇒ *'a set* ⇒ *bool*
where *path_connectedin X S* ≡ *S* ⊆ *topspace X* ∧ *path_connected_space*(*subtopology X S*)

lemma *path_connectedin_absolute* [*simp*]:
path_connectedin (*subtopology X S*) *S* ⟷ *path_connectedin X S*
by (*simp add*: *path_connectedin_def subtopology_subtopology*)

lemma *path_connectedin_subset_topspace*:
path_connectedin X S ⟹ *S* ⊆ *topspace X*
by (*simp add*: *path_connectedin_def*)

lemma *path_connectedin_subtopology*:
  path_connectedin (*subtopology X S*) *T* ⟷ *path_connectedin X T* ∧ *T* ⊆ *S*
 by (*auto simp*: *path_connectedin_def subtopology_subtopology inf.absorb2*)

lemma *path_connectedin*:
  path_connectedin X S ⟷
    S ⊆ *topspace X* ∧
    (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. ∃ *g*. *pathin X g* ∧ *g* ∈ {*0..1*} → *S* ∧ *g 0* = *x* ∧ *g 1* = *y*)
 unfolding *path_connectedin_def path_connected_space_def pathin_def continuous_map_in_subtopology*
  by (*intro conj_cong refl ball_cong*) (*simp_all add*: *inf.absorb_iff2 flip*: *image_subset_iff_funcset*)

lemma *path_connectedin_topspace*:
  path_connectedin X (*topspace X*) ⟷ *path_connected_space X*
 by (*simp add*: *path_connectedin_def*)

lemma *path_connected_imp_connected_space*:
 assumes *path_connected_space X*
 shows *connected_space X*
proof −
 have ∗: ∃ *S*. *connectedin X S* ∧ *g 0* ∈ *S* ∧ *g 1* ∈ *S* **if** *pathin X g* **for** *g*
 proof (*intro exI conjI*)
   have *continuous_map* (*subtopology euclideanreal* {*0..1*}) *X g*
     using *connectedin_absolute that* **by** (*simp add*: *pathin_def*)
   then show *connectedin X* (*g* ‘ {*0..1*})
     by (*rule connectedin_continuous_map_image*) *auto*
 qed *auto*
 show *?thesis*
   using *assms*
  by (*auto intro*: ∗ *simp add*: *path_connected_space_def connected_space_subconnected Ball_def*)
qed

lemma *path_connectedin_imp_connectedin*:
  path_connectedin X S ⟹ *connectedin X S*
 by (*simp add*: *connectedin_def path_connected_imp_connected_space path_connectedin_def*)

lemma *path_connected_space_topspace_empty*:
  path_connected_space trivial_topology
 by (*simp add*: *path_connected_space_def*)

lemma *path_connectedin_empty* [*simp*]: *path_connectedin X* {}
 by (*simp add*: *path_connectedin*)

lemma *path_connectedin_singleton* [*simp*]: *path_connectedin X* {*a*} ⟷ *a* ∈ *topspace X*
proof

```
  show path_connectedin X {a} ⟹ a ∈ topspace X
    by (simp add: path_connectedin)
  show a ∈ topspace X ⟹ path_connectedin X {a}
    unfolding path_connectedin
    using pathin_const by fastforce
qed

lemma path_connectedin_continuous_map_image:
  assumes f: continuous_map X Y f and S: path_connectedin X S
  shows path_connectedin Y (f ‘ S)
proof −
  have fX: f ∈ (topspace X) → topspace Y
    using continuous_map_def f by fastforce
  show ?thesis
    unfolding path_connectedin
  proof (intro conjI ballI; clarify?)
    fix x
    assume x ∈ S
    show f x ∈ topspace Y
      using S ‹x ∈ S› fX path_connectedin_subset_topspace by fastforce
  next
    fix x y
    assume x ∈ S and y ∈ S
    then obtain g where g: pathin X g g ∈ {0..1} → S g 0 = x g 1 = y
      using S  by (force simp: path_connectedin)
    show ∃ g. pathin Y g ∧ g ∈ {0..1} → f ‘ S ∧ g 0 = f x ∧ g 1 = f y
    proof (intro exI conjI)
      show pathin Y (f ∘ g)
        using ‹pathin X g› f pathin_compose by auto
    qed (use g in auto)
  qed
qed

lemma path_connectedin_discrete_topology:
  path_connectedin (discrete_topology U) S ⟷ S ⊆ U ∧ (∃ a. S ⊆ {a}) (is ?lhs
= ?rhs)
proof
  show ?lhs ⟹ ?rhs
    by (meson connectedin_discrete_topology path_connectedin_imp_connectedin)
  show ?rhs ⟹ ?lhs
    using subset_singletonD by fastforce
qed

lemma path_connected_space_discrete_topology:
   path_connected_space (discrete_topology U) ⟷ (∃ a. U ⊆ {a})
 by (metis path_connectedin_discrete_topology path_connectedin_topspace path_connected_space_tops
          subset_singletonD topspace_discrete_topology)
```

lemma *homeomorphic_path_connected_space_imp*:
⟦*path_connected_space X*; *X homeomorphic_space Y*⟧ ⟹ *path_connected_space Y*
unfolding *homeomorphic_space_def homeomorphic_maps_def*
by (*smt* (*verit*, *ccfv_threshold*) *homeomorphic_imp_surjective_map homeomorphic_maps_def homeomorphic_maps_map path_connectedin_continuous_map_image path_connectedin_topspace*)

lemma *homeomorphic_path_connected_space*:
X homeomorphic_space Y ⟹ *path_connected_space X* ⟷ *path_connected_space Y*
by (*meson homeomorphic_path_connected_space_imp homeomorphic_space_sym*)

lemma *homeomorphic_map_path_connectedness*:
assumes *homeomorphic_map X Y f U* ⊆ *topspace X*
shows *path_connectedin Y* (*f* ‘ *U*) ⟷ *path_connectedin X U*
unfolding *path_connectedin_def*
proof (*intro conj_cong homeomorphic_path_connected_space*)
show *f* ‘ *U* ⊆ *topspace Y* ⟷ (*U* ⊆ *topspace X*)
using *assms homeomorphic_imp_surjective_map* **by** *blast*
next
assume *U* ⊆ *topspace X*
show *subtopology Y* (*f* ‘ *U*) *homeomorphic_space subtopology X U*
using *assms* **unfolding** *homeomorphic_eq_everything_map*
by (*metis* (*no_types*, *opaque_lifting*) *assms homeomorphic_map_subtopologies homeomorphic_space homeomorphic_space_sym image_mono inf.absorb_iff2*)
qed

lemma *homeomorphic_map_path_connectedness_eq*:
homeomorphic_map X Y f ⟹ *path_connectedin X U* ⟷ *U* ⊆ *topspace X* ∧ *path_connectedin Y* (*f* ‘ *U*)
by (*meson homeomorphic_map_path_connectedness path_connectedin_def*)

2.3.17 Connected components

definition *connected_component_of* :: *'a topology* ⇒ *'a* ⇒ *'a* ⇒ *bool*
where *connected_component_of X x y* ≡
∃ *T*. *connectedin X T* ∧ *x* ∈ *T* ∧ *y* ∈ *T*

abbreviation *connected_component_of_set*
where *connected_component_of_set X x* ≡ *Collect* (*connected_component_of X x*)

definition *connected_components_of* :: *'a topology* ⇒ (*'a set*) *set*
where *connected_components_of X* ≡ *connected_component_of_set X* ‘ *topspace X*

lemma *connected_component_in_topspace*:
connected_component_of X x y ⟹ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X*

by (*meson connected_component_of_def connectedin_subset_topspace in_mono*)

lemma *connected_component_of_refl*:
connected_component_of X x x ⟷ *x* ∈ *topspace X*
by (*meson connected_component_in_topspace connected_component_of_def connectedin_sing insertI1*)

lemma *connected_component_of_sym*:
connected_component_of X x y ⟷ *connected_component_of X y x*
by (*meson connected_component_of_def*)

lemma *connected_component_of_trans*:
⟦*connected_component_of X x y*; *connected_component_of X y z*⟧
⟹ *connected_component_of X x z*
unfolding *connected_component_of_def*
using *connectedin_Un* **by** *blast*

lemma *connected_component_of_mono*:
⟦*connected_component_of* (*subtopology X S*) *x y*; *S* ⊆ *T*⟧
⟹ *connected_component_of* (*subtopology X T*) *x y*
by (*metis connected_component_of_def connectedin_subtopology inf.absorb_iff2 subtopology_subtopology*)

lemma *connected_component_of_set*:
connected_component_of_set X x = {*y*. ∃ *T*. *connectedin X T* ∧ *x* ∈ *T* ∧ *y* ∈ *T*}
by (*meson connected_component_of_def*)

lemma *connected_component_of_subset_topspace*:
connected_component_of_set X x ⊆ *topspace X*
using *connected_component_in_topspace* **by** *force*

lemma *connected_component_of_eq_empty*:
connected_component_of_set X x = {} ⟷ (*x* ∉ *topspace X*)
using *connected_component_in_topspace connected_component_of_refl* **by** *fastforce*

lemma *connected_space_iff_connected_component*:
connected_space X ⟷ (∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. *connected_component_of X x y*)
by (*simp add*: *connected_component_of_def connected_space_subconnected*)

lemma *connected_space_imp_connected_component_of*:
⟦*connected_space X*; *a* ∈ *topspace X*; *b* ∈ *topspace X*⟧
⟹ *connected_component_of X a b*
by (*simp add*: *connected_space_iff_connected_component*)

lemma *connected_space_connected_component_set*:
connected_space X ⟷ (∀ *x* ∈ *topspace X*. *connected_component_of_set X x*

= *topspace X*)
using *connected_component_of_subset_topspace connected_space_iff_connected_component*
by *fastforce*

lemma *connected_component_of_maximal*:
⟦*connectedin X S*; *x* ∈ *S*⟧ ⟹ *S* ⊆ *connected_component_of_set X x*
by (*meson Ball_Collect connected_component_of_def*)

lemma *connected_component_of_equiv*:
connected_component_of X x y ⟷
x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *connected_component_of X x* = *connected_component_of X y*
apply (*simp add*: *connected_component_in_topspace fun_eq_iff*)
by (*meson connected_component_of_refl connected_component_of_sym connected_component_of_trans*)

lemma *connected_component_of_disjoint*:
disjnt (*connected_component_of_set X x*) (*connected_component_of_set X y*)
⟷ ~(*connected_component_of X x y*)
using *connected_component_of_equiv* **unfolding** *disjnt_iff* **by** *force*

lemma *connected_component_of_eq*:
connected_component_of X x = *connected_component_of X y* ⟷
(*x* ∉ *topspace X*) ∧ (*y* ∉ *topspace X*) ∨
x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧
connected_component_of X x y
by (*metis Collect_empty_eq_bot connected_component_of_eq_empty connected_component_of_equiv*)

lemma *connectedin_connected_component_of*:
connectedin X (*connected_component_of_set X x*)
proof −
have *connected_component_of_set X x* = ⋃ {*T*. *connectedin X T* ∧ *x* ∈ *T*}
by (*auto simp*: *connected_component_of_def*)
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *InterI connectedin_Union emptyE mem_Collect_eq*)
qed

lemma *Union_connected_components_of*:
⋃(*connected_components_of X*) = *topspace X*
unfolding *connected_components_of_def*
using *connected_component_in_topspace connected_component_of_refl* **by** *fastforce*

lemma *connected_components_of_maximal*:
⟦*C* ∈ *connected_components_of X*; *connectedin X S*; ~*disjnt C S*⟧ ⟹ *S* ⊆ *C*
unfolding *connected_components_of_def disjnt_def*
apply *clarify*
by (*metis Int_emptyI connected_component_of_def connected_component_of_trans*

mem_Collect_eq)

lemma *pairwise_disjoint_connected_components_of*:
pairwise disjnt (*connected_components_of X*)
unfolding *connected_components_of_def pairwise_def*
by (*smt* (*verit*, *best*) *connected_component_of_disjoint connected_component_of_eq imageE*)

lemma *complement_connected_components_of_Union*:
C ∈ *connected_components_of X*
⟹ *topspace X* − *C* = ⋃ (*connected_components_of X* − {*C*})
by (*metis Union_connected_components_of bot.extremum ccpo_Sup_singleton diff_Union_pairwise_disjoint*
insert_subset pairwise_disjoint_connected_components_of)

lemma *nonempty_connected_components_of*:
C ∈ *connected_components_of X* ⟹ *C* ≠ {}
unfolding *connected_components_of_def*
by (*metis* (*no_types*, *lifting*) *connected_component_of_eq_empty imageE*)

lemma *connected_components_of_subset*:
C ∈ *connected_components_of X* ⟹ *C* ⊆ *topspace X*
using *Union_connected_components_of* **by** *fastforce*

lemma *connectedin_connected_components_of*:
assumes *C* ∈ *connected_components_of X*
shows *connectedin X C*
proof −
have *C* ∈ *connected_component_of_set X* ‘ *topspace X*
using *assms connected_components_of_def* **by** *blast*
then show *?thesis*
using *connectedin_connected_component_of* **by** *fastforce*
qed

lemma *connected_component_in_connected_components_of*:
connected_component_of_set X a ∈ *connected_components_of X* ⟷ *a* ∈ *topspace X*
by (*metis* (*no_types*, *lifting*) *connected_component_of_eq_empty connected_components_of_def image_iff*)

lemma *connected_space_iff_components_eq*:
connected_space X ⟷ (∀ *C* ∈ *connected_components_of X*. ∀ *C′* ∈ *connected_components_of X*. *C* = *C′*)
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*simp add*: *connected_components_of_def connected_space_connected_component_set*)
show *?rhs* ⟹ *?lhs*
by (*metis Union_connected_components_of Union_iff connected_space_subconnected*

connectedin_connected_components_of)
qed

lemma *connected_components_of_eq_empty*:
connected_components_of X = {} ⟷ X = trivial_topology
by (*simp add*: *connected_components_of_def*)

lemma *connected_components_of_empty_space*:
connected_components_of trivial_topology = {}
by (*simp add*: *connected_components_of_eq_empty*)

lemma *connected_components_of_subset_sing*:
connected_components_of X ⊆ {S} ⟷ connected_space X ∧ (X = trivial_topology ∨ topspace X = S)
proof (*cases X = trivial_topology*)
case *True*
then show *?thesis*
by (*simp add*: *connected_components_of_empty_space*)
next
case *False*
then have *⟦connected_components_of X ⊆ {S}⟧ ⟹ topspace X = S*
using *Union_connected_components_of connected_components_of_eq_empty*
by *fastforce*
with *False* **show** *?thesis*
unfolding *connected_components_of_def*
by (*metis connected_space_connected_component_set empty_iff image_subset_iff insert_iff*)
qed

lemma *connected_space_iff_components_subset_singleton*:
connected_space X ⟷ (∃ a. connected_components_of X ⊆ {a})
by (*simp add*: *connected_components_of_subset_sing*)

lemma *connected_components_of_eq_singleton*:
connected_components_of X = {S} ⟷ connected_space X ∧ X ≠ trivial_topology ∧ S = topspace X
by (*metis connected_components_of_eq_empty connected_components_of_subset_sing insert_not_empty subset_singleton_iff*)

lemma *connected_components_of_connected_space*:
connected_space X ⟹ connected_components_of X = (if X = trivial_topology then {} else {topspace X})
by (*simp add*: *connected_components_of_eq_empty connected_components_of_eq_singleton*)

lemma *exists_connected_component_of_superset*:
assumes *connectedin X S* **and** *ne*: *X ≠ trivial_topology*
shows *∃ C. C ∈ connected_components_of X ∧ S ⊆ C*
proof (*cases S = {}*)
case *True*

then show *?thesis*
using *ne connected_components_of_eq_empty* **by** *fastforce*
next
case *False*
then show *?thesis*
by (*meson all_not_in_conv assms*(*1*) *connected_component_in_connected_components_of connected_component_of_maximal connectedin_subset_topspace in_mono*)
qed

lemma *closedin_connected_components_of*:
assumes $C \in$ *connected_components_of* X
shows *closedin* X C
proof −
obtain x **where** $x \in$ *topspace* X **and** x: $C =$ *connected_component_of_set* X x
using *assms* **by** (*auto simp*: *connected_components_of_def*)
have *connected_component_of_set* X $x \subseteq$ *topspace* X
by (*simp add*: *connected_component_of_subset_topspace*)
moreover have X *closure_of connected_component_of_set* X $x \subseteq$ *connected_component_of_set* X x
proof (*rule connected_component_of_maximal*)
show *connectedin* X (X *closure_of connected_component_of_set* X x)
by (*simp add*: *connectedin_closure_of connectedin_connected_component_of*)
show $x \in X$ *closure_of connected_component_of_set* X x
by (*simp add*: ‹$x \in$ *topspace* X› *closure_of connected_component_of_refl*)
qed
ultimately
show *?thesis*
using *closure_of_subset_eq* x **by** *auto*
qed

lemma *closedin_connected_component_of*:
closedin X (*connected_component_of_set* X x)
by (*metis closedin_connected_components_of closedin_empty connected_component_in_connected_co connected_component_of_eq_empty*)

lemma *connected_component_of_eq_overlap*:
connected_component_of_set X x = *connected_component_of_set* X y ⟷
($x \notin$ *topspace* X) ∧ ($y \notin$ *topspace* X) ∨
~(*connected_component_of_set* X x ∩ *connected_component_of_set* X y = {})
using *connected_component_of_equiv* **by** *fastforce*

lemma *connected_component_of_nonoverlap*:
connected_component_of_set X x ∩ *connected_component_of_set* X y = {} ⟷
($x \notin$ *topspace* X) ∨ ($y \notin$ *topspace* X) ∨
~(*connected_component_of_set* X x = *connected_component_of_set* X y)
by (*metis connected_component_of_eq_empty connected_component_of_eq_overlap inf.idem*)

lemma *connected_component_of_overlap*:
  ~(*connected_component_of_set X x* ∩ *connected_component_of_set X y* = {})
⟷
    x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧
    connected_component_of_set X x = *connected_component_of_set X y*
  by (*meson connected_component_of_nonoverlap*)

2.3.18 Combining theorems for continuous functions into the reals

The homeomorphism between the real line and the open interval $(-1, 1)$

lemma *continuous_map_real_shrink*:
  continuous_map euclideanreal (*top_of_set* {−*1*<..<*1*}) (λ*x*. *x* / (*1* + |*x*|))
proof −
  have *continuous_on UNIV* (λ*x*::*real*. *x* / (*1* + |*x*|))
    by (*intro continuous_intros*) *auto*
  then show *?thesis*
    by (*auto simp*: *continuous_map_in_subtopology divide_simps*)
qed

lemma *continuous_on_real_grow*:
  continuous_on {−*1*<..<*1*} (λ*x*::*real*. *x* / (*1* − |*x*|))
  by (*intro continuous_intros*) *auto*

lemma *real_grow_shrink*:
  fixes *x*::*real*
  shows *x* / (*1* + |*x*|) / (*1* − |*x* / (*1* + |*x*|)|) = *x*
  by (*force simp*: *divide_simps*)

lemma *homeomorphic_maps_real_shrink*:
  homeomorphic_maps euclideanreal (*subtopology euclideanreal* {−*1*<..<*1*})
    (λ*x*. *x* / (*1* + |*x*|)) (λ*y*. *y* / (*1* − |*y*|))
  by (*force simp*: *homeomorphic_maps_def continuous_map_real_shrink continuous_on_real_grow divide_simps*)

lemma *real_shrink_Galois*:
  fixes *x*::*real*
  shows (*x* / (*1* + |*x*|) = *y*) ⟷ (|*y*| < *1* ∧ *y* / (*1* − |*y*|) = *x*)
  using *real_grow_shrink* **by** (*fastforce simp add*: *distrib_left*)

lemma *real_shrink_eq*:
  fixes *x y*::*real*
  shows (*x* / (*1* + |*x*|) = *y* / (*1* + |*y*|)) ⟷ *x* = *y*
  by (*metis real_shrink_Galois*)

lemma *real_shrink_lt*:
  fixes *x*::*real*
  shows *x* / (*1* + |*x*|) < *y* / (*1* + |*y*|) ⟷ *x* < *y*

using *zero_less_mult_iff* [*of x y*] **by** (*auto simp*: *field_simps abs_if not_less*)

lemma *real_shrink_le*:
fixes *x*::*real*
shows *x* / (*1* + |*x*|) ≤ *y* / (*1* + |*y*|) ⟷ *x* ≤ *y*
by (*meson linorder_not_le real_shrink_lt*)

lemma *real_shrink_grow*:
fixes *x*::*real*
shows |*x*| < *1* ⟹ *x* / (*1* − |*x*|) / (*1* + |*x* / (*1* − |*x*|)|) = *x*
using *real_shrink_Galois* **by** *blast*

lemma *continuous_shrink*:
continuous_on UNIV (λ*x*::*real*. *x* / (*1* + |*x*|))
by (*intro continuous_intros*) *auto*

lemma *strict_mono_shrink*:
strict_mono (λ*x*::*real*. *x* / (*1* + |*x*|))
by (*simp add*: *monotoneI real_shrink_lt*)

lemma *shrink_range*: (λ*x*::*real*. *x* / (*1* + |*x*|)) ‘ *S* ⊆ {−*1*<..<*1*}
by (*auto simp*: *divide_simps*)

Note: connected sets of real numbers are the same thing as intervals

lemma *connected_shrink*:
fixes *S* :: *real set*
shows *connected* ((λ*x*. *x* / (*1* + |*x*|)) ‘ *S*) ⟷ *connected S* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have *connected* ((λ*x*. *x* / (*1* − |*x*|)) ‘ (λ*x*. *x* / (*1* + |*x*|)) ‘ *S*)
by (*metis continuous_on_real_grow shrink_range connected_continuous_image*
continuous_on_subset)
then show *?rhs*
using *real_grow_shrink* **by** (*force simp add*: *image_comp*)
next
assume *?rhs*
then show *?lhs*
using *connected_continuous_image*
by (*metis continuous_on_subset continuous_shrink subset_UNIV*)
qed

2.3.19 A few cardinality results

lemma *eqpoll_real_subset*:
fixes *a b*::*real*
assumes *a* < *b* **and** *S*: ⋀*x*. ⟦*a* < *x*; *x* < *b*⟧ ⟹ *x* ∈ *S*
shows *S* ≈ (*UNIV*::*real set*)
proof (*rule lepoll_antisym*)

```
  show S ≲ (UNIV::real set)
    by (simp add: subset_imp_lepoll)
  define f where f ≡ λx. (a+b) / 2 + (b−a) / 2 * (x / (1 + |x|))
  show (UNIV::real set) ≲ S
    unfolding lepoll_def
  proof (intro exI conjI)
    show inj f
      unfolding inj_on_def f_def
    by (smt (verit, ccfv_SIG) real_shrink_eq ‹a<b› divide_eq_0_iff mult_cancel_left
times_divide_eq_right)
    have pos: (b−a) / 2 > 0
      using ‹a<b› by auto
    have *: a < (a + b) / 2 + (b − a) / 2 * x ⟷ (b − a) > (b − a) * −x
          (a + b) / 2 + (b − a) / 2 * x < b ⟷ (b − a) * x < (b − a) for x
      by (auto simp: field_simps)
    show range f ⊆ S
      using shrink_range [of UNIV] ‹a < b›
      unfolding subset_iff f_def greaterThanLessThan_iff image_iff
      by (smt (verit, best) S * mult_less_cancel_left2 mult_minus_right)
  qed
qed

lemma reals01_lepoll_nat_sets: {0..<1::real} ≲ (UNIV::nat set set)
proof −
  define nxt where nxt ≡ λx::real. if x < 1/2 then (True, 2*x) else (False, 2*x
− 1)
  have nxt_fun: nxt ∈ {0..<1} → UNIV × {0..<1}
    by (simp add: nxt_def Pi_iff)
  define σ where σ ≡ λx. rec_nat (True, x) (λn (b,y). nxt y)
  have σSuc [simp]: σ x (Suc k) = nxt (snd (σ x k)) for k x
    by (simp add: σ_def case_prod_beta')
  have σ01: x ∈ {0..<1} ⟹ σ x n ∈ UNIV × {0..<1} for x n
  proof (induction n)
    case 0
    then show ?case
      by (simp add: σ_def)
   next
    case (Suc n)
    with nxt_fun show ?case
      by (force simp add: Pi_iff split: prod.split)
  qed
  define f where f ≡ λx. {n. fst (σ x (Suc n))}
  have snd_nxt: snd (nxt y) − snd (nxt x) = 2 * (y−x)
    if fst (nxt x) = fst (nxt y) for x y
    using that by (simp add: nxt_def split: if_split_asm)
  have False if f x = f y x < y 0 ≤ x x < 1 0 ≤ y y < 1 for x y :: real
  proof −
    have ⋀k. fst (σ x (Suc k)) = fst (σ y (Suc k))
      using that by (force simp add: f_def)
```

```
    then have eq: ⋀k. fst (nxt (snd (σ x k))) = fst (nxt (snd (σ y k)))
      by (metis σ_def case_prod_beta' rec_nat_Suc_imp)
    have ∗: snd (σ y k) − snd (σ x k) = 2 ^ k ∗ (y−x) for k
    proof (induction k)
      case 0
      then show ?case
        by (simp add: σ_def)
    next
      case (Suc k)
      then show ?case
        by (simp add: eq snd_nxt)
    qed
    define n where n ≡ nat (⌈log 2 (1 / (y − x))⌉)
    have 2^n ≥ 1 / (y − x)
      by (simp add: n_def power_of_nat_log_ge)
    then have 2^n ∗ (y−x) ≥ 1
      using ‹x < y› by (simp add: n_def field_simps)
    with ∗ have snd (σ y n) − snd (σ x n) ≥ 1
      by presburger
    moreover have snd (σ x n) ∈ {0..<1} snd (σ y n) ∈ {0..<1}
      using that by (meson σ01 atLeastLessThan_iff mem_Times_iff)+
    ultimately show False by simp
  qed
  then have inj_on f {0..<1}
    by (meson atLeastLessThan_iff linorder_inj_onI')
  then show ?thesis
    unfolding lepoll_def by blast
qed

lemma nat_sets_lepoll_reals01: (UNIV::nat set set) ≲ {0..<1::real}
proof −
  define F where F ≡ λS i. if i∈S then (inverse 3::real) ^ i else 0
  have Fge0: F S i ≥ 0 for S i
    by (simp add: F_def)
  have F: summable (F S) for S
    unfolding F_def by (force intro: summable_comparison_test_ev [where g =
power (inverse 3)])
  have sum (F S) {..<n} ≤ 3/2 for n S
  proof (cases n)
    case (Suc n')
    have sum (F S) {..<n} ≤ (∑ i<n. inverse 3 ^ i)
      by (simp add: F_def sum_mono)
    also have ... = (∑ i=0..n'. inverse 3 ^ i)
      using Suc atLeast0AtMost lessThan_Suc_atMost by presburger
    also have ... = (3/2) ∗ (1 − inverse 3 ^ n)
        using sum_gp_multiplied [of 0 n' inverse (3::real)] by (simp add: Suc
field_simps)
    also have ... ≤ 3/2
      by (simp add: field_simps)
```

```
    finally show ?thesis .
  qed auto
  then have F32: suminf (F S) ≤ 3/2 for S
    using F suminf_le_const by blast
  define f where f ≡ λS. suminf (F S) / 2
  have monoF: F S n ≤ F T n if S ⊆ T for S T n
    using F_def that by auto
  then have monof: f S ≤ f T if S ⊆ T for S T
    using that F by (simp add: f_def suminf_le)
  have f S ∈ {0..<1::real} for S
  proof −
    have 0 ≤ suminf (F S)
      using F by (simp add: F_def suminf_nonneg)
    with F32[of S] show ?thesis
      by (auto simp: f_def)
  qed
  moreover have inj f
  proof
    fix S T
    assume f S = f T
    show S = T
    proof (rule ccontr)
      assume S ≠ T
      then have ST_ne: (S−T) ∪ (T−S) ≠ {}
        by blast
      define n where n ≡ LEAST n. n ∈ (S−T) ∪ (T−S)
      have sum_split: suminf (F U) = sum (F U) {..<Suc n} + (∑k. F U (k +
Suc n)) for U
        by (metis F add.commute suminf_split_initial_segment)
      have yes: f U ≥ (sum (F U) {..<n} + (inverse 3::real) ^ n) / 2
        if n ∈ U for U
      proof −
        have 0 ≤ (∑k. F U (k + Suc n))
          by (metis F Fge0 suminf_nonneg summable_iff_shift)
        moreover have F U n = (1/3) ^ n
          by (simp add: F_def that)
        ultimately show ?thesis
          by (simp add: sum_split f_def)
      qed
      have ∗: (∑k. F UNIV (k + n)) = (∑k. F UNIV k) ∗ (inverse 3::real) ^ n
for n
        by (simp add: F_def power_add suminf_mult2)
      have no: f U < (sum (F U) {..<n} + (inverse 3::real) ^ n) / 2
        if n ∉ U for U
      proof −
        have [simp]: F U n = 0
          by (simp add: F_def that)
        have (∑k. F U (k + Suc n)) ≤ (∑k. F UNIV (k + Suc n))
       by (metis F monoF subset_UNIV suminf_le summable_ignore_initial_segment)
```

```
      then have suminf (F U) ≤ (∑k. F UNIV (k + Suc n)) + (∑i<n. F U i)
        by (simp add: sum_split)
       also have ... < (inverse 3::real) ^ n + (∑i<n. F U i)
        unfolding * using F32[of UNIV] by simp
       finally have suminf (F U) < inverse 3 ^ n + sum (F U) {..<n} .
       then show ?thesis
        by (simp add: f_def)
     qed
     have S ∩ {..<n} = T ∩ {..<n}
      using not_less_Least by (fastforce simp add: n_def)
     then have sum (F S) {..<n} = sum (F T) {..<n}
      by (metis (no_types, lifting) F_def Int_iff sum.cong)
     moreover consider n ∈ S−T | n ∈ T−S
      by (metis LeastI_ex ST_ne UnE ex_in_conv n_def)
     ultimately show False
      by (smt (verit, best) Diff_iff ‹f S = f T› yes no)
   qed
  qed
  ultimately show ?thesis
   by (meson image_subsetI lepoll_def)
qed

lemma open_interval_eqpoll_reals:
  fixes a b::real
  shows {a<..<b} ≈ (UNIV::real set) ⟷ a<b
  using bij_betw_tan bij_betw_open_intervals eqpoll_def
 by (smt (verit, best) UNIV_I eqpoll_real_subset eqpoll_iff_bijections greaterThanLessThan_iff)

lemma closed_interval_eqpoll_reals:
  fixes a b::real
  shows {a..b} ≈ (UNIV::real set) ⟷ a < b
proof
  show {a..b} ≈ (UNIV::real set) ⟹ a < b
   using eqpoll_finite_iff infinite_Icc_iff infinite_UNIV_char_0 by blast
qed (auto simp: eqpoll_real_subset)


lemma reals_lepoll_reals01: (UNIV::real set) ≲ {0..<1::real}
proof −
  have (UNIV::real set) ≈ {0<..<1::real}
   by (simp add: open_interval_eqpoll_reals eqpoll_sym)
  also have ... ≲ {0..<1::real}
   by (simp add: greaterThanLessThan_subseteq_atLeastLessThan_iff subset_imp_lepoll)
  finally show ?thesis .
qed

lemma nat_sets_eqpoll_reals: (UNIV::nat set set) ≈ (UNIV::real set)
  by (metis (mono_tags, opaque_lifting) reals_lepoll_reals01 lepoll_antisym lepoll_trans
```

nat_sets_lepoll_reals01 reals01_lepoll_nat_sets subset_UNIV subset_imp_lepoll)

end

2.4 Connected Components

theory *Connected*
imports
Abstract_Topology_2
begin

2.4.1 Connectedness

lemma *connected_local*:
connected S ⟷
¬ (∃ *e1 e2*.
openin (*top_of_set S*) *e1* ∧
openin (*top_of_set S*) *e2* ∧
S ⊆ *e1* ∪ *e2* ∧
e1 ∩ *e2* = {} ∧
e1 ≠ {} ∧
e2 ≠ {})
using *connected_openin* **by** *blast*

lemma *exists_diff*:
fixes *P* :: *'a set* ⇒ *bool*
shows (∃ *S*. *P* (− *S*)) ⟷ (∃ *S*. *P S*)
by (*metis boolean_algebra_class.boolean_algebra.double_compl*)

lemma *connected_clopen*: *connected S* ⟷
(∀ *T*. *openin* (*top_of_set S*) *T* ∧
closedin (*top_of_set S*) *T* ⟶ *T* = {} ∨ *T* = *S*) (**is** *?lhs* ⟷ *?rhs*)
proof −
have ¬ *connected S* ⟷
(∃ *e1 e2*. *open e1* ∧ *open* (− *e2*) ∧ *S* ⊆ *e1* ∪ (− *e2*) ∧ *e1* ∩ (− *e2*) ∩ *S* = {}
∧ *e1* ∩ *S* ≠ {} ∧ (− *e2*) ∩ *S* ≠ {})
unfolding *connected_def openin_open closedin_closed*
by (*metis double_complement*)
then have *th0*: *connected S* ⟷
¬ (∃ *e2 e1*. *closed e2* ∧ *open e1* ∧ *S* ⊆ *e1* ∪ (− *e2*) ∧ *e1* ∩ (− *e2*) ∩ *S* = {}
∧ *e1* ∩ *S* ≠ {} ∧ (− *e2*) ∩ *S* ≠ {})
(**is** _ ⟷ ¬ (∃ *e2 e1*. *?P e2 e1*))
by (*simp add*: *closed_def*) *metis*
have *th1*: *?rhs* ⟷ ¬ (∃ *t' t*. *closed t'*∧*t* = *S*∩*t'* ∧ *t*≠{} ∧ *t*≠*S* ∧ (∃ *t'*. *open t'*
∧ *t* = *S* ∩ *t'*))
(**is** _ ⟷ ¬ (∃ *t' t*. *?Q t' t*))
unfolding *connected_def openin_open closedin_closed* **by** *auto*
have (∃ *e1*. *?P e2 e1*) ⟷ (∃ *t*. *?Q e2 t*) **for** *e2*
proof −

have *?P e2 e1* $\longleftrightarrow$ ($\exists$ *t. closed e2* $\wedge$ *t* = *S*$\cap$*e2* $\wedge$ *open e1* $\wedge$ *t* = *S*$\cap$*e1* $\wedge$ *t*$\neq${} $\wedge$ *t* $\neq$ *S*) **for** *e1*
by *auto*
then show *?thesis*
by *metis*
qed
then have $\forall$ *e2*. ($\exists$ *e1*. *?P e2 e1*) $\longleftrightarrow$ ($\exists$ *t*. *?Q e2 t*)
by *blast*
then show *?thesis*
by (*simp add*: *th0 th1*)
qed

2.4.2 Connected components, considered as a connectedness relation or a set

definition *connected_component S x y* $\equiv$ $\exists$ *T*. *connected T* $\wedge$ *T* $\subseteq$ *S* $\wedge$ *x* $\in$ *T* $\wedge$ *y* $\in$ *T*

abbreviation *connected_component_set S x* $\equiv$ *Collect* (*connected_component S x*)

lemma *connected_componentI*:
connected T $\Longrightarrow$ *T* $\subseteq$ *S* $\Longrightarrow$ *x* $\in$ *T* $\Longrightarrow$ *y* $\in$ *T* $\Longrightarrow$ *connected_component S x y*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_in*: *connected_component S x y* $\Longrightarrow$ *x* $\in$ *S* $\wedge$ *y* $\in$ *S*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_refl*: *x* $\in$ *S* $\Longrightarrow$ *connected_component S x x*
using *connected_component_def connected_sing* **by** *blast*

lemma *connected_component_refl_eq* [*simp*]: *connected_component S x x* $\longleftrightarrow$ *x* $\in$ *S*
using *connected_component_in connected_component_refl* **by** *blast*

lemma *connected_component_sym*: *connected_component S x y* $\Longrightarrow$ *connected_component S y x*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_trans*:
connected_component S x y $\Longrightarrow$ *connected_component S y z* $\Longrightarrow$ *connected_component S x z*
unfolding *connected_component_def*
by (*metis Int_iff Un_iff Un_subset_iff equals0D connected_Un*)

lemma *connected_component_of_subset*:
connected_component S x y $\Longrightarrow$ *S* $\subseteq$ *T* $\Longrightarrow$ *connected_component T x y*
by (*auto simp*: *connected_component_def*)

lemma *connected_component_Union*: *connected_component_set S x* = $\bigcup$ {*T*. *connected T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*}
by (*auto simp*: *connected_component_def*)

lemma *connected_connected_component* [*iff*]: *connected* (*connected_component_set S x*)
by (*auto simp*: *connected_component_Union intro*: *connected_Union*)

lemma *connected_iff_eq_connected_component_set*:
connected S ⟷ (∀ *x* ∈ *S*. *connected_component_set S x* = *S*)
by (*metis connected_component_def connected_component_in connected_connected_component mem_Collect_eq subsetI subset_antisym*)

lemma *connected_component_subset*: *connected_component_set S x* ⊆ *S*
using *connected_component_in* **by** *blast*

lemma *connected_component_eq_self*: *connected S* ⟹ *x* ∈ *S* ⟹ *connected_component_set S x* = *S*
by (*simp add*: *connected_iff_eq_connected_component_set*)

lemma *connected_iff_connected_component*:
connected S ⟷ (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *connected_component S x y*)
using *connected_component_in* **by** (*auto simp*: *connected_iff_eq_connected_component_set*)

lemma *connected_component_maximal*:
x ∈ *T* ⟹ *connected T* ⟹ *T* ⊆ *S* ⟹ *T* ⊆ (*connected_component_set S x*)
using *connected_component_eq_self connected_component_of_subset* **by** *blast*

lemma *connected_component_mono*:
S ⊆ *T* ⟹ *connected_component_set S x* ⊆ *connected_component_set T x*
by (*simp add*: *Collect_mono connected_component_of_subset*)

lemma *connected_component_eq_empty* [*simp*]: *connected_component_set S x* = {} ⟷ *x* ∉ *S*
using *connected_component_refl* **by** (*fastforce simp*: *connected_component_in*)

lemma *connected_component_set_empty* [*simp*]: *connected_component_set* {} *x* = {}
using *connected_component_eq_empty* **by** *blast*

lemma *connected_component_eq*:
y ∈ *connected_component_set S x* ⟹ (*connected_component_set S y* = *connected_component_set S x*)
by (*metis* (*no_types*, *lifting*)
Collect_cong connected_component_sym connected_component_trans mem_Collect_eq)

lemma *closed_connected_component*:
assumes *S*: *closed S*

shows *closed* (*connected_component_set S x*)
proof (*cases* $x \in S$)
case *False*
then show *?thesis*
by (*metis connected_component_eq_empty closed_empty*)
next
case *True*
show *?thesis*
unfolding *closure_eq* [*symmetric*]
proof
show *closure* (*connected_component_set S x*) $\subseteq$ *connected_component_set S x*
proof (*rule connected_component_maximal*)
show $x \in$ *closure* (*connected_component_set S x*)
by (*simp add*: *closure_def True*)
show *connected* (*closure* (*connected_component_set S x*))
by (*simp add*: *connected_imp_connected_closure*)
show *closure* (*connected_component_set S x*) $\subseteq S$
by (*simp add*: *S closure_minimal connected_component_subset*)
qed
qed (*simp add*: *closure_subset*)
qed

lemma *connected_component_disjoint*:
connected_component_set S a $\cap$ *connected_component_set S b* $= \{\} \longleftrightarrow$
$a \notin$ *connected_component_set S b*
by (*smt* (*verit*) *connected_component_eq connected_component_eq_empty connected_component_refl_eq*
disjoint_iff_not_equal mem_Collect_eq)

lemma *connected_component_nonoverlap*:
connected_component_set S a $\cap$ *connected_component_set S b* $= \{\} \longleftrightarrow$
$a \notin S \vee b \notin S \vee$ *connected_component_set S a* $\neq$ *connected_component_set S b*
by (*metis connected_component_disjoint connected_component_eq connected_component_eq_empty inf.idem*)

lemma *connected_component_overlap*:
connected_component_set S a $\cap$ *connected_component_set S b* $\neq \{\} \longleftrightarrow$
$a \in S \wedge b \in S \wedge$ *connected_component_set S a* $=$ *connected_component_set S b*
by (*auto simp*: *connected_component_nonoverlap*)

lemma *connected_component_sym_eq*: *connected_component S x y* $\longleftrightarrow$ *connected_component S y x*
using *connected_component_sym* **by** *blast*

lemma *connected_component_eq_eq*:
connected_component_set S x $=$ *connected_component_set S y* $\longleftrightarrow$

$x \notin S \wedge y \notin S \vee x \in S \wedge y \in S \wedge$ *connected_component S x y*
by (*metis connected_component_eq connected_component_eq_empty connected_component_refl mem_Collect_eq*)

lemma *connected_iff_connected_component_eq*:
connected S $\longleftrightarrow$ $(\forall x \in S.\ \forall y \in S.$ *connected_component_set S x = connected_component_set S y*)
by (*simp add*: *connected_component_eq_eq connected_iff_connected_component*)

lemma *connected_component_idemp*:
connected_component_set (*connected_component_set S x*) *x* = *connected_component_set S x*
by (*metis Int_absorb connected_component_disjoint connected_component_eq_empty connected_component_eq_self connected_connected_component*)

lemma *connected_component_unique*:
$[\![x \in c;\ c \subseteq S;$ *connected c*;
$\bigwedge c'.\ [\![x \in c';\ c' \subseteq S;$ *connected c*$']\!] \Longrightarrow c' \subseteq c]\!]$
$\Longrightarrow$ *connected_component_set S x = c*
by (*meson connected_component_maximal connected_component_subset connected_connected_component subsetD subset_antisym*)

lemma *joinable_connected_component_eq*:
$[\![$*connected T*; $T \subseteq S$;
connected_component_set S x $\cap$ $T \neq \{\}$;
connected_component_set S y $\cap$ $T \neq \{\}]\!]$
$\Longrightarrow$ *connected_component_set S x = connected_component_set S y*
by (*metis* (*full_types*) *subsetD connected_component_eq connected_component_maximal disjoint_iff_not_equal*)

lemma *Union_connected_component*: $\bigcup$(*connected_component_set S ' S*) = *S*
proof
show $\bigcup$(*connected_component_set S ' S*) $\subseteq$ *S*
by (*simp add*: *SUP_least connected_component_subset*)
qed (*use connected_component_refl_eq* **in** *force*)

lemma *complement_connected_component_unions*:
S $-$ *connected_component_set S x* =
$\bigcup$(*connected_component_set S ' S* $-$ {*connected_component_set S x*})
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
by (*metis Diff_subset Diff_subset_conv Sup_insert Union_connected_component insert_Diff_single*)
show *?rhs* $\subseteq$ *?lhs*
by *clarsimp* (*metis connected_component_eq_eq connected_component_in*)
qed

lemma *connected_component_intermediate_subset*:
  ⟦*connected_component_set U a* ⊆ *T*; *T* ⊆ *U*⟧
  ⟹ *connected_component_set T a* = *connected_component_set U a*
 by (*metis connected_component_idemp connected_component_mono subset_antisym*)

lemma *connected_component_homeomorphismI*:
 assumes *homeomorphism A B f g connected_component A x y*
 shows *connected_component B* (*f x*) (*f y*)
proof −
 from *assms* **obtain** *T* **where** *T*: *connected T T* ⊆ *A x* ∈ *T y* ∈ *T*
  unfolding *connected_component_def* **by** *blast*
 have *connected* (*f* ‘ *T*) *f* ‘ *T* ⊆ *B f x* ∈ *f* ‘ *T f y* ∈ *f* ‘ *T*
  using *assms T continuous_on_subset*[*of A f T*]
  by (*auto intro*!: *connected_continuous_image simp*: *homeomorphism_def*)
 thus *?thesis*
  unfolding *connected_component_def* **by** *blast*
qed

lemma *connected_component_homeomorphism_iff*:
 assumes *homeomorphism A B f g*
 shows *connected_component A x y* ⟷ *x* ∈ *A* ∧ *y* ∈ *A* ∧ *connected_component*
B (*f x*) (*f y*)
 by (*metis assms connected_component_homeomorphismI connected_component_in*
homeomorphism_apply1 homeomorphism_sym)

lemma *connected_component_set_homeomorphism*:
 assumes *homeomorphism A B f g x* ∈ *A*
 shows *connected_component_set B* (*f x*) = *f* ‘ *connected_component_set A x*
(**is** *?lhs* = *?rhs*)
proof −
 have *y* ∈ *?lhs* ⟷ *y* ∈ *?rhs* **for** *y*
  by (*smt* (*verit*, *best*) *assms connected_component_homeomorphism_iff homeomorphism_def image_iff mem_Collect_eq*)
 thus *?thesis*
  by *blast*
qed

2.4.3 The set of connected components of a set

definition *components*:: *'a*::*topological_space set* ⇒ *'a set set*
 where *components S* ≡ *connected_component_set S* ‘ *S*

lemma *components_iff*: *S* ∈ *components U* ⟷ (∃ *x*. *x* ∈ *U* ∧ *S* = *connected_component_set*
U x)
 by (*auto simp*: *components_def*)

lemma *componentsI*: *x* ∈ *U* ⟹ *connected_component_set U x* ∈ *components U*
 by (*auto simp*: *components_def*)

lemma *componentsE*:
assumes $S \in$ *components* U
obtains x **where** $x \in U$ $S =$ *connected_component_set* U x
using *assms* **by** (*auto simp*: *components_def*)

lemma *Union_components* [*simp*]: $\bigcup$(*components* U) $= U$
by (*simp add*: *Union_connected_component components_def*)

lemma *pairwise_disjoint_components*: *pairwise* ($\lambda X\ Y.\ X \cap Y = \{\}$) (*components* U)
unfolding *pairwise_def*
by (*metis* (*full_types*) *components_iff connected_component_nonoverlap*)

lemma *in_components_nonempty*: $C \in$ *components* $S \Longrightarrow C \neq \{\}$
by (*metis components_iff connected_component_eq_empty*)

lemma *in_components_subset*: $C \in$ *components* $S \Longrightarrow C \subseteq S$
using *Union_components* **by** *blast*

lemma *in_components_connected*: $C \in$ *components* $S \Longrightarrow$ *connected* C
by (*metis components_iff connected_connected_component*)

lemma *in_components_maximal*:
$C \in$ *components* $S \longleftrightarrow$
$C \neq \{\} \wedge C \subseteq S \wedge$ *connected* $C \wedge (\forall d.\ d \neq \{\} \wedge C \subseteq d \wedge d \subseteq S \wedge$ *connected* $d \longrightarrow d = C)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *L*: *?lhs*
then have $C \subseteq S$ *connected* C
by (*simp_all add*: *in_components_subset in_components_connected*)
then show *?rhs*
by (*metis* (*full_types*) *L components_iff connected_component_maximal connected_component_refl empty_iff mem_Collect_eq subsetD subset_antisym*)
next
show *?rhs* $\Longrightarrow$ *?lhs*
by (*metis bot.extremum_uniqueI components_iff connected_component_eq_empty connected_component_maximal connected_component_subset connected_connected_component subset_emptyI*)
qed

lemma *joinable_components_eq*:
connected $T \wedge T \subseteq S \wedge c1 \in$ *components* $S \wedge c2 \in$ *components* $S \wedge c1 \cap T \neq \{\} \wedge c2 \cap T \neq \{\} \Longrightarrow c1 = c2$
by (*metis* (*full_types*) *components_iff joinable_connected_component_eq*)

lemma *closed_components*: $[\![$*closed* S; $C \in$ *components* $S]\!] \Longrightarrow$ *closed* C
by (*metis closed_connected_component components_iff*)

lemma *components_nonoverlap*:
$[\![C \in \mathit{components}\ S;\ C' \in \mathit{components}\ S]\!] \Longrightarrow (C \cap C' = \{\}) \longleftrightarrow (C \neq C')$
by (*metis* (*full_types*) *components_iff connected_component_nonoverlap*)

lemma *components_eq*: $[\![C \in \mathit{components}\ S;\ C' \in \mathit{components}\ S]\!] \Longrightarrow (C = C' \longleftrightarrow C \cap C' \neq \{\})$
by (*metis components_nonoverlap*)

lemma *components_eq_empty* [*simp*]: *components* $S = \{\} \longleftrightarrow S = \{\}$
by (*simp add*: *components_def*)

lemma *components_empty* [*simp*]: *components* $\{\} = \{\}$
by *simp*

lemma *connected_eq_connected_components_eq*: *connected* $S \longleftrightarrow (\forall C \in \mathit{components}\ S.\ \forall C' \in \mathit{components}\ S.\ C = C')$
by (*metis* (*no_types*, *opaque_lifting*) *components_iff connected_component_eq_eq connected_iff_connected_component*)

lemma *components_eq_sing_iff*: *components* $S = \{S\} \longleftrightarrow$ *connected* $S \wedge S \neq \{\}$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?rhs* $\Longrightarrow$ *?lhs*
by (*metis components_iff connected_component_eq_self equals0I insert_iff mk_disjoint_insert*)
qed (*use in_components_connected* **in** *fastforce*)

lemma *components_eq_sing_exists*: $(\exists a.$ *components* $S = \{a\}) \longleftrightarrow$ *connected* $S \wedge S \neq \{\}$
by (*metis Union_components ccpo_Sup_singleton components_eq_sing_iff*)

lemma *connected_eq_components_subset_sing*: *connected* $S \longleftrightarrow$ *components* $S \subseteq \{S\}$
by (*metis components_eq_empty components_eq_sing_iff connected_empty subset_singleton_iff*)

lemma *connected_eq_components_subset_sing_exists*: *connected* $S \longleftrightarrow (\exists a.$ *components* $S \subseteq \{a\})$
by (*metis components_eq_sing_exists connected_eq_components_subset_sing subset_singleton_iff*)

lemma *in_components_self*: $S \in$ *components* $S \longleftrightarrow$ *connected* $S \wedge S \neq \{\}$
by (*metis components_empty components_eq_sing_iff empty_iff in_components_connected insertI1*)

lemma *components_maximal*: $[\![C \in \mathit{components}\ S;$ *connected* $T;\ T \subseteq S;\ C \cap T \neq \{\}]\!] \Longrightarrow T \subseteq C$
by (*smt* (*verit*, *best*) *Int_Un_eq*(*4*) *Un_upper1 bot_eq_sup_iff connected_Un*

in_components_maximal inf.orderE sup.mono sup.orderI)

lemma *exists_component_superset*: $[\![T \subseteq S; S \neq \{\}; \textit{connected } T]\!] \Longrightarrow \exists C.\ C \in \textit{components } S \wedge T \subseteq C$
by (*meson componentsI connected_component_maximal equals0I subset_eq*)

lemma *components_intermediate_subset*: $[\![S \in \textit{components } U; S \subseteq T; T \subseteq U]\!] \Longrightarrow S \in \textit{components } T$
by (*smt* (*verit*, *best*) *dual_order.trans in_components_maximal*)

lemma *in_components_unions_complement*: $C \in \textit{components } S \Longrightarrow S - C = \bigcup(\textit{components } S - \{C\})$
by (*metis complement_connected_component_unions components_def components_iff*)

lemma *connected_intermediate_closure*:
assumes *cs*: *connected S* **and** *st*: $S \subseteq T$ **and** *ts*: $T \subseteq \textit{closure } S$
shows *connected T*
using *assms* **unfolding** *connected_def*
by (*smt* (*verit*) *Int_assoc inf.absorb_iff2 inf_bot_left open_Int_closure_eq_empty*)

lemma *closedin_connected_component*: *closedin* (*top_of_set S*) (*connected_component_set S x*)
proof (*cases connected_component_set S x* = {})
case *True*
then show *?thesis*
by (*metis closedin_empty*)
next
case *False*
then obtain *y* **where** *y*: *connected_component S x y* **and** $x \in S$
using *connected_component_eq_empty* **by** *blast*
have $*$: *connected_component_set S x* $\subseteq S \cap$ *closure* (*connected_component_set S x*)
by (*auto simp*: *closure_def connected_component_in*)
have $**$: $x \in$ *closure* (*connected_component_set S x*)
by (*simp add*: ‹$x \in S$› *closure_def*)
have $S \cap$ *closure* (*connected_component_set S x*) $\subseteq$ *connected_component_set S x* **if** *connected_component S x y*
proof (*rule connected_component_maximal*)
show *connected* ($S \cap$ *closure* (*connected_component_set S x*))
using $*$ *connected_intermediate_closure* **by** *blast*
qed (*use* ‹$x \in S$› $**$ **in** *auto*)
with *y* $*$ **show** *?thesis*
by (*auto simp*: *closedin_closed*)
qed

lemma *closedin_component*:
$C \in \textit{components } S \Longrightarrow \textit{closedin } (\textit{top\_of\_set } S)\ C$
using *closedin_connected_component componentsE* **by** *blast*

2.4.4 Proving a function is constant on a connected set by proving that a level set is open

lemma *continuous_levelset_openin_cases*:
fixes $f :: \_ \Rightarrow 'b$::*t1_space*
shows *connected* $S \Longrightarrow$ *continuous_on* $S\ f \Longrightarrow$
openin (*top_of_set* S) $\{x \in S.\ f\ x = a\}$
$\Longrightarrow (\forall x \in S.\ f\ x \neq a) \vee (\forall x \in S.\ f\ x = a)$
unfolding *connected_clopen*
using *continuous_closedin_preimage_constant* **by** *auto*

lemma *continuous_levelset_openin*:
fixes $f :: \_ \Rightarrow 'b$::*t1_space*
shows *connected* $S \Longrightarrow$ *continuous_on* $S\ f \Longrightarrow$
openin (*top_of_set* S) $\{x \in S.\ f\ x = a\} \Longrightarrow$
$(\exists x \in S.\ f\ x = a) \Longrightarrow (\forall x \in S.\ f\ x = a)$
using *continuous_levelset_openin_cases*[*of* $S\ f$]
by *meson*

lemma *continuous_levelset_open*:
fixes $f :: \_ \Rightarrow 'b$::*t1_space*
assumes S: *connected* S *continuous_on* $S\ f$
and a: *open* $\{x \in S.\ f\ x = a\}$ $\exists x \in S.\ \ f\ x = a$
shows $\forall x \in S.\ f\ x = a$
using a *continuous_levelset_openin*[*OF* S, *of* a, *unfolded openin_open*]
by *fast*

2.4.5 Preservation of Connectedness

lemma *homeomorphic_connectedness*:
assumes S *homeomorphic* T
shows *connected* $S \longleftrightarrow$ *connected* T
using *assms* **unfolding** *homeomorphic_def homeomorphism_def* **by** (*metis connected_continuous_image*)

lemma *connected_monotone_quotient_preimage*:
assumes *connected* T
and *contf*: *continuous_on* $S\ f$ **and** *fim*: $f\ `\ S = T$
and *opT*: $\bigwedge U.\ U \subseteq T$
$\Longrightarrow$ *openin* (*top_of_set* S) $(S \cap f\ -`\ U) \longleftrightarrow$
openin (*top_of_set* T) U
and *connT*: $\bigwedge y.\ y \in T \Longrightarrow$ *connected* $(S \cap f\ -`\ \{y\})$
shows *connected* S
proof (*rule connectedI*)
fix $U\ V$
assume *open* U **and** *open* V **and** $U \cap S \neq \{\}$ **and** $V \cap S \neq \{\}$
and $U \cap V \cap S = \{\}$ **and** $S \subseteq U \cup V$
moreover
have *disjoint*: $f\ `\ (S \cap U) \cap f\ `\ (S \cap V) = \{\}$
proof −

```
    have False if y ∈ f ‘ (S ∩ U) ∩ f ‘ (S ∩ V) for y
    proof −
      have y ∈ T
        using fim that by blast
      show ?thesis
        using connectedD [OF connT [OF ‹y ∈ T›] ‹open U› ‹open V›]
            ‹S ⊆ U ∪ V› ‹U ∩ V ∩ S = {}› that by fastforce
    qed
    then show ?thesis by blast
  qed
  ultimately have UU: (S ∩ f −‘ f ‘ (S ∩ U)) = S ∩ U and VV: (S ∩ f −‘ f ‘
(S ∩ V)) = S ∩ V
    by auto
  have opeU: openin (top_of_set T) (f ‘ (S ∩ U))
   by (metis UU ‹open U› fim image_Int_subset le_inf_iff opT openin_open_Int)
  have opeV: openin (top_of_set T) (f ‘ (S ∩ V))
   by (metis opT fim VV ‹open V› openin_open_Int image_Int_subset inf.bounded_iff)
  have T ⊆ f ‘ (S ∩ U) ∪ f ‘ (S ∩ V)
    using ‹S ⊆ U ∪ V› fim by auto
  then show False
    using ‹connected T› disjoint opeU opeV ‹U ∩ S ≠ {}› ‹V ∩ S ≠ {}›
    by (auto simp: connected_openin)
qed

lemma connected_open_monotone_preimage:
  assumes contf: continuous_on S f and fim: f ‘ S = T
    and ST: ⋀C. openin (top_of_set S) C ⟹ openin (top_of_set T) (f ‘ C)
    and connT: ⋀y. y ∈ T ⟹ connected (S ∩ f −‘ {y})
    and connected C C ⊆ T
  shows connected (S ∩ f −‘ C)
proof −
  have contf′: continuous_on (S ∩ f −‘ C) f
    by (meson contf continuous_on_subset inf_le1)
  have eqC: f ‘ (S ∩ f −‘ C) = C
    using ‹C ⊆ T› fim by blast
  show ?thesis
  proof (rule connected_monotone_quotient_preimage [OF ‹connected C› contf′
eqC])
    show connected (S ∩ f −‘ C ∩ f −‘ {y}) if y ∈ C for y
      by (metis Int_assoc Int_empty_right Int_insert_right_if1 assms(6) connT
in_mono that vimage_Int)
    have ⋀U. openin (top_of_set (S ∩ f −‘ C)) U
          ⟹ openin (top_of_set C) (f ‘ U)
      using open_map_restrict [OF _ ST ‹C ⊆ T›] by metis
    then show ⋀D. D ⊆ C
        ⟹ openin (top_of_set (S ∩ f −‘ C)) (S ∩ f −‘ C ∩ f −‘ D) =
           openin (top_of_set C) D
      using open_map_imp_quotient_map [of (S ∩ f −‘ C) f] contf′ by (simp
add: eqC)
```

```
  qed
qed


lemma connected_closed_monotone_preimage:
  assumes contf: continuous_on S f and fim: f ‘ S = T
    and ST: ⋀C. closedin (top_of_set S) C ⟹ closedin (top_of_set T) (f ‘ C)
    and connT: ⋀y. y ∈ T ⟹ connected (S ∩ f −‘ {y})
    and connected C C ⊆ T
  shows connected (S ∩ f −‘ C)
proof −
  have contf': continuous_on (S ∩ f −‘ C) f
    by (meson contf continuous_on_subset inf_le1)
  have eqC: f ‘ (S ∩ f −‘ C) = C
    using ‹C ⊆ T› fim by blast
  show ?thesis
  proof (rule connected_monotone_quotient_preimage [OF ‹connected C› contf'
eqC])
    show connected (S ∩ f −‘ C ∩ f −‘ {y}) if y ∈ C for y
      by (metis Int_assoc Int_empty_right Int_insert_right_if1 ‹C ⊆ T› connT
subsetD that vimage_Int)
    have ⋀U. closedin (top_of_set (S ∩ f −‘ C)) U
            ⟹ closedin (top_of_set C) (f ‘ U)
      using closed_map_restrict [OF _ ST ‹C ⊆ T›] by metis
    then show ⋀D. D ⊆ C
        ⟹ openin (top_of_set (S ∩ f −‘ C)) (S ∩ f −‘ C ∩ f −‘ D) =
           openin (top_of_set C) D
      using closed_map_imp_quotient_map [of (S ∩ f −‘ C) f] contf' by (simp
add: eqC)
  qed
qed
```

2.4.6 Lemmas about components

See Newman IV, 3.3 and 3.4.

```
lemma connected_Un_clopen_in_complement:
  fixes S U :: 'a::metric_space set
  assumes connected S connected U S ⊆ U
     and opeT: openin (top_of_set (U − S)) T
     and cloT: closedin (top_of_set (U − S)) T
    shows connected (S ∪ T)
proof −
  have ∗: ⟦⋀x y. P x y ⟷ P y x; ⋀x y. P x y ⟹ S ⊆ x ∨ S ⊆ y;
           ⋀x y. ⟦P x y; S ⊆ x⟧ ⟹ False⟧ ⟹ ¬(∃x y. (P x y)) for P
    by metis
  show ?thesis
    unfolding connected_closedin_eq
  proof (rule ∗)
    fix H1 H2
```

assume *H*: *closedin* (*top_of_set* (*S* ∪ *T*)) *H1* ∧
closedin (*top_of_set* (*S* ∪ *T*)) *H2* ∧
H1 ∪ *H2* = *S* ∪ *T* ∧ *H1* ∩ *H2* = {} ∧ *H1* ≠ {} ∧ *H2* ≠ {}
then have *clo*: *closedin* (*top_of_set S*) (*S* ∩ *H1*)
closedin (*top_of_set S*) (*S* ∩ *H2*)
by (*metis Un_upper1 closedin_closed_subset inf_commute*)+
moreover have *S* ∩ ((*S* ∪ *T*) ∩ *H1*) ∪ *S* ∩ ((*S* ∪ *T*) ∩ *H2*) = *S*
using *H* **by** *blast*
moreover have *H1* ∩ (*S* ∩ ((*S* ∪ *T*) ∩ *H2*)) = {}
using *H* **by** *blast*
ultimately have *S* ∩ *H1* = {} ∨ *S* ∩ *H2* = {}
by (*smt* (*verit*) *Int_assoc* ‹*connected S*› *connected_closedin_eq inf_commute inf_sup_absorb*)
then show *S* ⊆ *H1* ∨ *S* ⊆ *H2*
using *H* ‹*connected S*› **unfolding** *connected_closedin* **by** *blast*
next
fix *H1 H2*
assume *H*: *closedin* (*top_of_set* (*S* ∪ *T*)) *H1* ∧
closedin (*top_of_set* (*S* ∪ *T*)) *H2* ∧
H1 ∪ *H2* = *S* ∪ *T* ∧ *H1* ∩ *H2* = {} ∧ *H1* ≠ {} ∧ *H2* ≠ {}
and *S* ⊆ *H1*
then have *H2T*: *H2* ⊆ *T*
by *auto*
have *T* ⊆ *U*
using *Diff_iff opeT openin_imp_subset* **by** *auto*
with ‹*S* ⊆ *U*› **have** *Ueq*: *U* = (*U* − *S*) ∪ (*S* ∪ *T*)
by *auto*
have *openin* (*top_of_set* ((*U* − *S*) ∪ (*S* ∪ *T*))) *H2*
proof (*rule openin_subtopology_Un*)
show *openin* (*top_of_set* (*S* ∪ *T*)) *H2*
by (*metis Diff_cancel H Un_Diff Un_Diff_Int closedin_subset openin_closedin_eq topspace_euclidean_subtopology*)
then show *openin* (*top_of_set* (*U* − *S*)) *H2*
by (*meson H2T Un_upper2 opeT openin_subset_trans openin_trans*)
qed
moreover have *closedin* (*top_of_set* ((*U* − *S*) ∪ (*S* ∪ *T*))) *H2*
proof (*rule closedin_subtopology_Un*)
show *closedin* (*top_of_set* (*U* − *S*)) *H2*
using *H H2T cloT closedin_subset_trans*
by (*blast intro*: *closedin_subtopology_Un closedin_trans*)
qed (*simp add*: *H*)
ultimately have *H2*: *H2* = {} ∨ *H2* = *U*
using *Ueq* ‹*connected U*› **unfolding** *connected_clopen* **by** *metis*
then have *H2* ⊆ *S*
by (*metis Diff_partition H Un_Diff_cancel Un_subset_iff* ‹*H2* ⊆ *T*› *assms*(*3*) *inf.orderE opeT openin_imp_subset*)
moreover have *T* ⊆ *H2* − *S*
by (*metis* (*no_types*) *H2 H opeT openin_closedin_eq topspace_euclidean_subtopology*)
ultimately show *False*

```
    using H ‹S ⊆ H1› by blast
  qed blast
qed


proposition component_diff_connected:
  fixes S :: 'a::metric_space set
  assumes connected S connected U S ⊆ U and C: C ∈ components (U − S)
  shows connected(U − C)
  using ‹connected S› unfolding connected_closedin_eq not_ex de_Morgan_conj
proof clarify
  fix H3 H4
  assume clo3: closedin (top_of_set (U − C)) H3
    and clo4: closedin (top_of_set (U − C)) H4
    and H34: H3 ∪ H4 = U − C H3 ∩ H4 = {} and H3 ≠ {} and H4 ≠ {}
    and ∗ [rule_format]:
    ∀ H1 H2. ¬ closedin (top_of_set S) H1 ∨
                  ¬ closedin (top_of_set S) H2 ∨
                  H1 ∪ H2 ≠ S ∨ H1 ∩ H2 ≠ {} ∨ ¬ H1 ≠ {} ∨ ¬ H2 ≠ {}
  then have H3 ⊆ U−C and ope3: openin (top_of_set (U − C)) (U − C −
H3)
    and H4 ⊆ U−C and ope4: openin (top_of_set (U − C)) (U − C − H4)
    by (auto simp: closedin_def)
  have C ≠ {} C ⊆ U−S connected C
   using C in_components_nonempty in_components_subset in_components_maximal
by blast+
  have cCH3: connected (C ∪ H3)
   proof (rule connected_Un_clopen_in_complement [OF ‹connected C› ‹con-
nected U› _ _ clo3])
    show openin (top_of_set (U − C)) H3
     by (metis Diff_cancel Un_Diff Un_Diff_Int ‹H3 ∩ H4 = {}› ‹H3 ∪ H4 =
U − C› ope4)
  qed (use clo3 ‹C ⊆ U − S› in auto)
  have cCH4: connected (C ∪ H4)
   proof (rule connected_Un_clopen_in_complement [OF ‹connected C› ‹con-
nected U› _ _ clo4])
    show openin (top_of_set (U − C)) H4
     by (metis Diff_cancel Diff_triv Int_Un_eq(2) Un_Diff H34 inf_commute
ope3)
  qed (use clo4 ‹C ⊆ U − S› in auto)
  have closedin (top_of_set S) (S ∩ H3) closedin (top_of_set S) (S ∩ H4)
    using clo3 clo4 ‹S ⊆ U› ‹C ⊆ U − S› by (auto simp: closedin_closed)
  moreover have S ∩ H3 ≠ {}
    using components_maximal [OF C cCH3] ‹C ≠ {}› ‹C ⊆ U − S› ‹H3 ≠ {}›
‹H3 ⊆ U − C› by auto
  moreover have S ∩ H4 ≠ {}
    using components_maximal [OF C cCH4] ‹C ≠ {}› ‹C ⊆ U − S› ‹H4 ≠ {}›
‹H4 ⊆ U − C› by auto
  ultimately show False
```

using ∗ [*of* $S \cap H3\ S \cap H4$] ‹$H3 \cap H4 = \{\}$› ‹$C \subseteq U - S$› ‹$H3 \cup H4 = U - C$› ‹$S \subseteq U$›
by *auto*
qed

2.4.7 Constancy of a function from a connected set into a finite, disconnected or discrete set

Still missing: versions for a set that is smaller than R, or countable.

lemma *continuous_disconnected_range_constant*:
assumes *S*: *connected S*
and *conf*: *continuous_on S f*
and *fim*: $f \in S \to T$
and *cct*: $\bigwedge y.\ y \in T \Longrightarrow$ *connected_component_set T y* $= \{y\}$
shows *f constant_on S*
proof (*cases* $S = \{\}$)
case *True* **then show** *?thesis*
by (*simp add*: *constant_on_def*)
next
case *False*
then have f ‘ $S \subseteq \{f\ x\}$ **if** $x \in S$ **for** x
by (*metis PiE S cct connected_component_maximal connected_continuous_image* [*OF conf*] *fim image_eqI*
image_subset_iff that)
with *False* **show** *?thesis*
unfolding *constant_on_def* **by** *blast*
qed

This proof requires the existence of two separate values of the range type.

lemma *finite_range_constant_imp_connected*:
assumes $\bigwedge f$::'*a*::*topological_space* ⇒ '*b*::*real_normed_algebra_1*.
⟦*continuous_on S f*; *finite*(*f* ‘ *S*)⟧ ⟹ *f constant_on S*
shows *connected S*
proof −
{ fix *T U*
assume *clt*: *closedin* (*top_of_set S*) *T*
and *clu*: *closedin* (*top_of_set S*) *U*
and *tue*: $T \cap U = \{\}$ **and** *tus*: $T \cup U = S$
have *continuous_on* ($T \cup U$) ($\lambda x.$ *if* $x \in T$ *then 0 else 1*)
using *clt clu tue* **by** (*intro continuous_on_cases_local*) (*auto simp*: *tus*)
then have *conif*: *continuous_on S* ($\lambda x.$ *if* $x \in T$ *then 0 else 1*)
using *tus* **by** *blast*
have *fi*: *finite* (($\lambda x.$ *if* $x \in T$ *then 0 else 1*) ‘ *S*)
by (*rule finite_subset* [*of* _ {*0*,*1*}]) *auto*
have $T = \{\} \vee U = \{\}$
using *assms* [*OF conif fi*] *tus* [*symmetric*]
by (*auto simp*: *Ball_def constant_on_def*) (*metis IntI empty_iff one_neq_zero tue*)

```
  }
  then show ?thesis
    by (simp add: connected_closedin_eq)
qed

end


theory Function_Topology
  imports
    Elementary_Topology
    Abstract_Limits
    Connected
begin
```

2.5 Function Topology

We want to define the general product topology.

The product topology on a product of topological spaces is generated by the sets which are products of open sets along finitely many coordinates, and the whole space along the other coordinates. This is the coarsest topology for which the projection to each factor is continuous.

To form a product of objects in Isabelle/HOL, all these objects should be subsets of a common type 'a. The product is then $Pi_E\ I\ X$, the set of elements from $'i$ to $'a$ such that the i-th coordinate belongs to $X\ i$ for all $i \in I$.

Hence, to form a product of topological spaces, all these spaces should be subsets of a common type. This means that type classes can not be used to define such a product if one wants to take the product of different topological spaces (as the type 'a can only be given one structure of topological space using type classes). On the other hand, one can define different topologies (as introduced in *thy*) on one type, and these topologies do not need to share the same maximal open set. Hence, one can form a product of topologies in this sense, and this works well. The big caveat is that it does not interact well with the main body of topology in Isabelle/HOL defined in terms of type classes... For instance, continuity of maps is not defined in this setting.

As the product of different topological spaces is very important in several areas of mathematics (for instance adeles), I introduce below the product topology in terms of topologies, and reformulate afterwards the consequences in terms of type classes (which are of course very handy for applications).

Given this limitation, it looks to me that it would be very beneficial to revamp the theory of topological spaces in Isabelle/HOL in terms of topologies, and keep the statements involving type classes as consequences of more general statements in terms of topologies (but I am probably too naive here).

Here is an example of a reformulation using topologies. Let

continuous_map T1 T2 f =
  ((∀ *U*. *openin T2 U* ⟶ *openin T1* (*f*−‘*U* ∩ *topspace*(*T1*)))
    ∧ (*f*‘(*topspace T1*) ⊆ (*topspace T2*)))

be the natural continuity definition of a map from the topology *T1* to the topology *T2*. Then the current *continuous_on* (with type classes) can be redefined as

continuous_on s f =
  continuous_map (*top_of_set s*) (*topology euclidean*) *f*

In fact, I need *continuous_map* to express the continuity of the projection on subfactors for the product topology, in Lemma *continuous_on_restrict_product_topology*, and I show the above equivalence in Lemma *continuous_map_iff_continuous*.

I only develop the basics of the product topology in this theory. The most important missing piece is Tychonov theorem, stating that a product of compact spaces is always compact for the product topology, even when the product is not finite (or even countable).

I realized afterwards that this theory has a lot in common with `~~/src/HOL/Library/Finite_Map.thy`.

2.5.1 The product topology

We can now define the product topology, as generated by the sets which are products of open sets along finitely many coordinates, and the whole space along the other coordinates. Equivalently, it is generated by sets which are one open set along one single coordinate, and the whole space along other coordinates. In fact, this is only equivalent for nonempty products, but for the empty product the first formulation is better (the second one gives an empty product space, while an empty product should have exactly one point, equal to *undefined* along all coordinates.

So, we use the first formulation, which moreover seems to give rise to more straightforward proofs.

definition *product_topology*::(*'i* ⇒ (*'a topology*)) ⇒ (*'i set*) ⇒ ((*'i* ⇒ *'a*) *topology*)
  where *product_topology T I* =
    topology_generated_by {(Π$_E$ *i*∈*I*. *X i*) |*X*. (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)}}

abbreviation *powertop_real* :: *'a set* ⇒ (*'a* ⇒ *real*) *topology*
  where *powertop_real* ≡ *product_topology* (λ*i*. *euclideanreal*)

The total set of the product topology is the product of the total sets along each coordinate.

proposition *product_topology*:
product_topology X I =
topology
(*arbitrary union_of*
((*finite intersection_of*
(λ*F*. ∃ *i U*. *F* = {*f*. *f i* ∈ *U*} ∧ *i* ∈ *I* ∧ *openin* (*X i*) *U*))
relative_to (Π$_E$ *i*∈*I*. *topspace* (*X i*))))
(**is** _ = *topology* (_ *union_of* ((_ *intersection_of* *?*Ψ) *relative_to* *?TOP*)))
proof −
let *?*Ω = (λ*F*. ∃ *Y*. *F* = *Pi*$_E$ *I Y* ∧ (∀ *i*. *openin* (*X i*) (*Y i*)) ∧ *finite* {*i*. *Y i* ≠ *topspace* (*X i*)})
have ∗: (*finite*′ *intersection_of* *?*Ω) *A* = (*finite intersection_of* *?*Ψ *relative_to* *?TOP*) *A* **for** *A*
proof −
have *1*: ∃ *U*. (∃$\mathcal{U}$. *finite* $\mathcal{U}$ ∧ $\mathcal{U}$ ⊆ *Collect* *?*Ψ ∧ ⋂$\mathcal{U}$ = *U*) ∧ *?TOP* ∩ *U* = ⋂$\mathcal{U}$
if $\mathcal{U}$: $\mathcal{U}$ ⊆ *Collect* *?*Ω **and** *finite*′ $\mathcal{U}$ *A* = ⋂$\mathcal{U}$ $\mathcal{U}$ ≠ {} **for** $\mathcal{U}$
proof −
have ∀ *U* ∈ $\mathcal{U}$. ∃ *Y*. *U* = *Pi*$_E$ *I Y* ∧ (∀ *i*. *openin* (*X i*) (*Y i*)) ∧ *finite* {*i*. *Y i* ≠ *topspace* (*X i*)}
using $\mathcal{U}$ **by** *auto*
then obtain *Y* **where** *Y*: ⋀*U*. *U* ∈ $\mathcal{U}$ ⟹ *U* = *Pi*$_E$ *I* (*Y U*) ∧ (∀ *i*. *openin* (*X i*) (*Y U i*)) ∧ *finite* {*i*. (*Y U*) *i* ≠ *topspace* (*X i*)}
by *metis*
obtain *U* **where** *U* ∈ $\mathcal{U}$
using ‹$\mathcal{U}$ ≠ {}› **by** *blast*
let *?F* = λ*U*. (λ*i*. {*f*. *f i* ∈ *Y U i*}) ‘ {*i* ∈ *I*. *Y U i* ≠ *topspace* (*X i*)}
show *?thesis*
proof (*intro conjI exI*)
show *finite* (⋃ *U*∈$\mathcal{U}$. *?F U*)
using *Y* ‹*finite*′ $\mathcal{U}$› **by** *auto*
show *?TOP* ∩ ⋂(⋃ *U*∈$\mathcal{U}$. *?F U*) = ⋂$\mathcal{U}$
proof
have ∗: *f* ∈ *U*
if *U* ∈ $\mathcal{U}$ **and** ∀ *V*∈$\mathcal{U}$. ∀ *i*. *i* ∈ *I* ∧ *Y V i* ≠ *topspace* (*X i*) ⟶ *f i* ∈ *Y V i*
and ∀ *i*∈*I*. *f i* ∈ *topspace* (*X i*) **and** *f* ∈ *extensional I* **for** *f U*
by (*smt* (*verit*) *PiE_iff Y that*)
show *?TOP* ∩ ⋂(⋃ *U*∈$\mathcal{U}$. *?F U*) ⊆ ⋂$\mathcal{U}$
by (*auto simp*: *PiE_iff* ∗)
show ⋂$\mathcal{U}$ ⊆ *?TOP* ∩ ⋂(⋃ *U*∈$\mathcal{U}$. *?F U*)
using *Y openin_subset* ‹*finite*′ $\mathcal{U}$› **by** *fastforce*
qed
qed (*use Y openin_subset* **in** ‹*blast*+›)
qed
have *2*: ∃$\mathcal{U}$′. *finite*′ $\mathcal{U}$′ ∧ $\mathcal{U}$′ ⊆ *Collect* *?*Ω ∧ ⋂$\mathcal{U}$′ = *?TOP* ∩ ⋂$\mathcal{U}$
if $\mathcal{U}$: $\mathcal{U}$ ⊆ *Collect* *?*Ψ **and** *finite* $\mathcal{U}$ **for** $\mathcal{U}$
proof (*cases* $\mathcal{U}$={})
case *True*

```
      then show ?thesis
        apply (rule_tac x={?TOP} in exI, simp)
        apply (rule_tac x=λi. topspace (X i) in exI)
        apply force
        done
    next
      case False
      then obtain U where U ∈ 𝒰
        by blast
      have ∀ U ∈ 𝒰. ∃ i Y. U = {f. f i ∈ Y} ∧ i ∈ I ∧ openin (X i) Y
        using 𝒰 by auto
      then obtain J Y where
        Y: ⋀U. U ∈ 𝒰 ⟹ U = {f. f (J U) ∈ Y U} ∧ J U ∈ I ∧ openin (X (J
U)) (Y U)
        by metis
      let ?G = λU. Π_E i∈I. if i = J U then Y U else topspace (X i)
      show ?thesis
      proof (intro conjI exI)
        show finite (?G ` 𝒰) ?G ` 𝒰 ≠ {}
          using ‹finite 𝒰› ‹U ∈ 𝒰› by blast+
        have ∗: ⋀U. U ∈ 𝒰 ⟹ openin (X (J U)) (Y U)
          using Y by force
        show ?G ` 𝒰 ⊆ {Pi_E I Y |Y. (∀ i. openin (X i) (Y i)) ∧ finite {i. Y i ≠
topspace (X i)}}
          apply clarsimp
          apply (rule_tac x= (λi. if i = J U then Y U else topspace (X i)) in exI)
          apply (auto simp: ∗)
          done
      next
        show (⋂ U∈𝒰. ?G U) = ?TOP ∩ ⋂𝒰
        proof
          have (Π_E i∈I. if i = J U then Y U else topspace (X i)) ⊆ (Π_E i∈I.
topspace (X i))
            by (simp add: PiE_mono Y ‹U ∈ 𝒰› openin_subset)
          then have (⋂ U∈𝒰. ?G U) ⊆ ?TOP
            using ‹U ∈ 𝒰› by fastforce
          moreover have (⋂ U∈𝒰. ?G U) ⊆ ⋂𝒰
            using PiE_mem Y by fastforce
          ultimately show (⋂ U∈𝒰. ?G U) ⊆ ?TOP ∩ ⋂𝒰
            by auto
        qed (use Y in fastforce)
      qed
    qed
    show ?thesis
      unfolding relative_to_def intersection_of_def
      by (safe; blast dest!: 1 2)
  qed
  show ?thesis
    unfolding product_topology_def generate_topology_on_eq
```

apply (*rule arg_cong* [**where** f = *topology*])
apply (*rule arg_cong* [**where** f = (*union_of*)*arbitrary*])
apply (*force simp*: $*$)
done
qed

lemma *topspace_product_topology* [*simp*]:
topspace (*product_topology* T I) = (Π_E i∈I. *topspace*(T i))
proof
show *topspace* (*product_topology* T I) ⊆ (Π_E i∈I. *topspace* (T i))
unfolding *product_topology_def topology_generated_by_topspace*
unfolding *topspace_def* **by** *auto*
have (Π_E i∈I. *topspace* (T i)) ∈ {(Π_E i∈I. X i) |X. (∀ i. *openin* (T i) (X i)) ∧ *finite* {i. X i ≠ *topspace* (T i)}}
using *openin_topspace not_finite_existsD* **by** *auto*
then show (Π_E i∈I. *topspace* (T i)) ⊆ *topspace* (*product_topology* T I)
unfolding *product_topology_def* **using** *PiE_def* **by** (*auto*)
qed

lemma *product_topology_trivial_iff*:
product_topology X I = *trivial_topology* ⟷ (∃ i ∈ I. X i = *trivial_topology*)
by (*auto simp*: *PiE_eq_empty_iff simp flip*: *null_topspace_iff_trivial*)

lemma *topspace_product_topology_alt*:
topspace (*product_topology* X I) = {x ∈ *extensional* I. ∀ i ∈ I. x i ∈ *topspace*(X i)}
by (*fastforce simp*: *PiE_iff*)

lemma *product_topology_basis*:
assumes ⋀i. *openin* (T i) (X i) *finite* {i. X i ≠ *topspace* (T i)}
shows *openin* (*product_topology* T I) (Π_E i∈I. X i)
unfolding *product_topology_def*
by (*rule topology_generated_by_Basis*) (*use assms* **in** *auto*)

proposition *product_topology_open_contains_basis*:
assumes *openin* (*product_topology* T I) U x ∈ U
shows ∃ X. x ∈ (Π_E i∈I. X i) ∧ (∀ i. *openin* (T i) (X i)) ∧ *finite* {i. X i ≠ *topspace* (T i)} ∧ (Π_E i∈I. X i) ⊆ U
proof −
define IT **where** IT ≡ λX. {i. X i ≠ *topspace* (T i)}
have *generate_topology_on* {(Π_E i∈I. X i) |X. (∀ i. *openin* (T i) (X i)) ∧ *finite* (IT X)} U
using *assms* **unfolding** *product_topology_def IT_def* **by** (*intro openin_topology_generated_by*) *auto*
then have ⋀x. x∈U ⟹ ∃ X. x ∈ (Π_E i∈I. X i) ∧ (∀ i. *openin* (T i) (X i)) ∧ *finite* (IT X) ∧ (Π_E i∈I. X i) ⊆ U
proof *induction*
case (*Int* U V x)
then obtain XU XV **where** H:

```
    x ∈ Pi_E I XU ⋀i. openin (T i) (XU i) finite (IT XU) Pi_E I XU ⊆ U
    x ∈ Pi_E I XV ⋀i. openin (T i) (XV i) finite (IT XV) Pi_E I XV ⊆ V
    by (meson Int_iff)
  define X where X = (λi. XU i ∩ XV i)
  have Pi_E I X ⊆ Pi_E I XU ∩ Pi_E I XV
    by (auto simp add: PiE_iff X_def)
  then have Pi_E I X ⊆ U ∩ V using H by auto
  moreover have ∀ i. openin (T i) (X i)
    unfolding X_def using H by auto
  moreover have finite (IT X)
    apply (rule rev_finite_subset[of IT XU ∪ IT XV])
    using H by (auto simp: X_def IT_def)
  moreover have x ∈ Pi_E I X
    unfolding X_def using H by auto
  ultimately show ?case
    by auto
 next
  case (UN K x)
  then obtain k where k ∈ K x ∈ k by auto
  with ‹k ∈ K› UN show ?case
    by (meson Sup_upper2)
 qed auto
 then show ?thesis using ‹x ∈ U› IT_def by blast
qed

lemma product_topology_empty_discrete:
  product_topology T {} = discrete_topology {(λx. undefined)}
 by (simp add: subtopology_eq_discrete_topology_sing)

lemma openin_product_topology:
  openin (product_topology X I) =
   arbitrary union_of
       ((finite intersection_of (λF. (∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin
(X i) U)))
        relative_to topspace (product_topology X I))
 by (simp add: istopology_subbase product_topology)

lemma subtopology_product_topology:
  subtopology (product_topology X I) (Π_E i∈I. (S i)) = product_topology (λi.
subtopology (X i) (S i)) I
proof −
 let ?P = λF. ∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin (X i) U
 let ?X = Π_E i∈I. topspace (X i)
 have finite intersection_of ?P relative_to ?X ∩ Pi_E I S =
       finite intersection_of (?P relative_to ?X ∩ Pi_E I S) relative_to ?X ∩ Pi_E
I S
   by (rule finite_intersection_of_relative_to)
 also have ... = finite intersection_of
                 ((λF. ∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ (openin (X i) relative_to
```

```
S i) U)
                        relative_to ?X ∩ Pi_E I S)
                    relative_to ?X ∩ Pi_E I S
    apply (rule arg_cong2 [where f = (relative_to)])
     apply (rule arg_cong [where f = (intersection_of)finite])
     apply (rule ext)
     apply (auto simp: relative_to_def intersection_of_def)
    done
  finally
  have finite intersection_of ?P relative_to ?X ∩ Pi_E I S =
        finite intersection_of
          (λF. ∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ (openin (X i) relative_to S i) U)
           relative_to ?X ∩ Pi_E I S
    by (metis finite_intersection_of_relative_to)
  then show ?thesis
  unfolding topology_eq
  apply clarify
  apply (simp add: openin_product_topology flip: openin_relative_to)
  apply (simp add: arbitrary_union_of_relative_to flip: PiE_Int)
  done
qed

lemma product_topology_base_alt:
   finite intersection_of (λF. (∃ i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin (X i) U))
    relative_to (Π_E i∈I. topspace (X i)) =
    (λF. (∃ U. F =  Pi_E I U ∧ finite {i ∈ I. U i ≠ topspace(X i)} ∧ (∀ i ∈ I.
openin (X i) (U i))))
   (is ?lhs = ?rhs)
proof −
  have (∀ F. ?lhs F ⟶ ?rhs F)
    unfolding all_relative_to all_intersection_of topspace_product_topology
  proof clarify
    fix ℱ
    assume finite ℱ and ℱ ⊆ {{f. f i ∈ U} |i U. i ∈ I ∧ openin (X i) U}
    then show ∃ U. (Π_E i∈I. topspace (X i)) ∩ ⋂ℱ = Pi_E I U ∧
          finite {i ∈ I. U i ≠ topspace (X i)} ∧ (∀ i∈I. openin (X i) (U i))
    proof (induction)
      case (insert F ℱ)
      then obtain U where eq: (Π_E i∈I. topspace (X i)) ∩ ⋂ℱ = Pi_E I U
        and fin: finite {i ∈ I. U i ≠ topspace (X i)}
        and ope: ⋀i. i ∈ I ⟹ openin (X i) (U i)
        by auto
      obtain i V where F = {f. f i ∈ V} i ∈ I openin (X i) V
        using insert by auto
      let ?U = λj. U j ∩ (if j = i then V else topspace(X j))
      show ?case
      proof (intro exI conjI)
        show (Π_E i∈I. topspace (X i)) ∩ ⋂(insert F ℱ) = Pi_E I ?U
        using eq  PiE_mem ‹i ∈ I›  by (auto simp: ‹F = {f. f i ∈ V}›) fastforce
```

```
      next
        show finite {i ∈ I. ?U i ≠ topspace (X i)}
          by (rule rev_finite_subset [OF finite.insertI [OF fin]]) auto
      next
        show ∀i∈I. openin (X i) (?U i)
          by (simp add: ‹openin (X i) V› ope openin_Int)
      qed
    qed (auto intro: dest: not_finite_existsD)
  qed
  moreover have (∀F. ?rhs F ⟶ ?lhs F)
  proof clarify
    fix U :: 'a ⇒ 'b set
    assume fin: finite {i ∈ I. U i ≠ topspace (X i)} and ope: ∀i∈I. openin (X i)
(U i)
    let ?U = ⋂i∈{i ∈ I. U i ≠ topspace (X i)}. {x. x i ∈ U i}
    show ?lhs (Pi_E I U)
      unfolding relative_to_def topspace_product_topology
    proof (intro exI conjI)
      show (finite intersection_of (λF. ∃i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin
(X i) U)) ?U
       using fin ope by (intro finite_intersection_of_Inter finite_intersection_of_inc)
auto
      show (Π_E i∈I. topspace (X i)) ∩ ?U = Pi_E I U
        using ope openin_subset by fastforce
    qed
  qed
  ultimately show ?thesis
    by meson
qed

corollary openin_product_topology_alt:
  openin (product_topology X I) S ⟷
    (∀x ∈ S. ∃U. finite {i ∈ I. U i ≠ topspace(X i)} ∧
                 (∀i ∈ I. openin (X i) (U i)) ∧ x ∈ Pi_E I U ∧ Pi_E I U ⊆ S)
 unfolding openin_product_topology arbitrary_union_of_alt product_topology_base_alt
topspace_product_topology
  by (smt (verit, best))

lemma closure_of_product_topology:
  (product_topology X I) closure_of (PiE I S) = PiE I (λi. (X i) closure_of (S
i))
proof −
  have ∗: (∀T. f ∈ T ∧ openin (product_topology X I) T ⟶ (∃y∈Pi_E I S. y ∈
T))
       ⟷ (∀i ∈ I. ∀T. f i ∈ T ∧ openin (X i) T ⟶ S i ∩ T ≠ {})
    (is ?lhs = ?rhs)
    if top: ⋀i. i ∈ I ⟹ f i ∈ topspace (X i) and ext: f ∈ extensional I for f
  proof
    assume L[rule_format]: ?lhs
```

show *?rhs*
proof *clarify*
fix i T
assume $i \in I$ $f\ i \in T$ *openin* $(X\ i)$ T $S\ i \cap T = \{\}$
then have *openin* $(product\_topology\ X\ I)$ $((\Pi_E\ i{\in}I.\ topspace\ (X\ i)) \cap \{x.\ x\ i \in T\})$
by (*force simp*: *openin_product_topology intro*: *arbitrary_union_of_inc relative_to_inc finite_intersection_of_inc*)
then show *False*
using L [*of topspace* $(product\_topology\ X\ I) \cap \{f.\ f\ i \in T\}$] ‹$S\ i \cap T = \{\}$› ‹$f\ i \in T$› ‹$i \in I$›
by (*auto simp*: *top ext PiE_iff*)
qed
next
assume R [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp*: *openin_product_topology union_of_def arbitrary_def*)
fix $\mathcal{U}$ U
assume
$\mathcal{U}$: $\mathcal{U} \subseteq Collect$
(*finite intersection_of* $(\lambda F.\ \exists i\ U.\ F = \{x.\ x\ i \in U\} \wedge i \in I \wedge openin\ (X\ i)\ U)$ *relative_to*
$(\Pi_E\ i{\in}I.\ topspace\ (X\ i))$) **and**
$f \in U$ $U \in \mathcal{U}$
then have (*finite intersection_of* $(\lambda F.\ \exists i\ U.\ F = \{x.\ x\ i \in U\} \wedge i \in I \wedge openin\ (X\ i)\ U)$
relative_to $(\Pi_E\ i{\in}I.\ topspace\ (X\ i)))\ U$
by *blast*
with ‹$f \in U$› ‹$U \in \mathcal{U}$›
obtain $\mathcal{T}$ **where** *finite* $\mathcal{T}$
and $\mathcal{T}$: $\bigwedge C.\ C \in \mathcal{T} \Longrightarrow \exists i \in I.\ \exists V.\ openin\ (X\ i)\ V \wedge C = \{x.\ x\ i \in V\}$
and $topspace\ (product\_topology\ X\ I) \cap \bigcap \mathcal{T} \subseteq U$ $f \in topspace\ (product\_topology\ X\ I) \cap \bigcap \mathcal{T}$
apply (*clarsimp simp add*: *relative_to_def intersection_of_def*)
apply (*rule that*, *auto dest*!: *subsetD*)
done
then have $f \in PiE\ I\ (topspace \circ X)$ $f \in \bigcap\mathcal{T}$ **and** $subU$: $PiE\ I\ (topspace \circ X) \cap \bigcap\mathcal{T} \subseteq U$
by (*auto simp*: *PiE_iff*)
have $*$: $f\ i \in topspace\ (X\ i) \cap \bigcap\{U.\ openin\ (X\ i)\ U \wedge \{x.\ x\ i \in U\} \in \mathcal{T}\}$
$\wedge\ openin\ (X\ i)\ (topspace\ (X\ i) \cap \bigcap\{U.\ openin\ (X\ i)\ U \wedge \{x.\ x\ i \in U\} \in \mathcal{T}\})$
if $i \in I$ **for** i
proof −
have *finite* $((\lambda U.\ \{x.\ x\ i \in U\}) -` \mathcal{T})$
proof (*rule finite_vimageI* [*OF* ‹*finite* $\mathcal{T}$›])
show *inj* $(\lambda U.\ \{x.\ x\ i \in U\})$
by (*auto simp*: *inj_on_def*)
qed

```
      then have fin: finite {U. openin (X i) U ∧ {x. x i ∈ U} ∈ 𝒯}
        by (rule rev_finite_subset) auto
      have openin (X i) (⋂ (insert (topspace (X i)) {U. openin (X i) U ∧ {x. x
i ∈ U} ∈ 𝒯}))
        by (rule openin_Inter) (auto simp: fin)
      then show ?thesis
        using ‹f ∈ ⋂ 𝒯› by (fastforce simp: that top)
    qed
    define Φ where Φ ≡ λi. topspace (X i) ∩ ⋂{U. openin (X i) U ∧ {f. f i ∈
U} ∈ 𝒯}
    have ∀ i ∈ I. ∃ x. x ∈ S i ∩ Φ i
      using R [OF _ *] unfolding Φ_def by blast
    then obtain ϑ where ϑ [rule_format]: ∀ i ∈ I. ϑ i ∈ S i ∩ Φ i
      by metis
    show ∃ y∈Pi_E I S. ∃ x∈𝒰. y ∈ x
    proof
      show ∃ U ∈ 𝒰. (λi ∈ I. ϑ i) ∈ U
      proof
        have restrict ϑ I ∈ PiE I (topspace ∘ X) ∩ ⋂𝒯
          using 𝒯 by (fastforce simp: Φ_def PiE_def dest: ϑ)
        then show restrict ϑ I ∈ U
          using subU by blast
      qed (rule ‹U ∈ 𝒰›)
    next
      show (λi ∈ I. ϑ i) ∈ Pi_E I S
        using ϑ by simp
    qed
  qed
 qed
 show ?thesis
  apply (simp add: * closure_of_def PiE_iff set_eq_iff cong: conj_cong)
  by metis
qed

corollary closedin_product_topology:
  closedin (product_topology X I) (PiE I S) ⟷ PiE I S = {} ∨ (∀ i ∈ I. closedin
(X i) (S i))
 by (smt (verit, best) PiE_eq closedin_empty closure_of_eq closure_of_product_topology)

corollary closedin_product_topology_singleton:
  f ∈ extensional I ⟹ closedin (product_topology X I) {f} ⟷ (∀ i ∈ I. closedin
(X i) {f i})
 using PiE_singleton closedin_product_topology [of X I]
 by (metis (no_types, lifting) all_not_in_conv insertI1)

lemma product_topology_empty:
  product_topology X {} = topology (λS. S ∈ {{},{λk. undefined}})
 unfolding product_topology union_of_def intersection_of_def arbitrary_def rel-
ative_to_def
```

by (*auto intro*: *arg_cong* [**where** *f*=*topology*])

lemma *openin_product_topology_empty*: *openin* (*product_topology X* {}) *S* ⟷ *S* ∈ {{},{λ*k*. *undefined*}}
unfolding *union_of_def intersection_of_def arbitrary_def relative_to_def openin_product_topology*
by *auto*

The basic property of the product topology is the continuity of projections:

lemma *continuous_map_product_coordinates* [*simp*]:
assumes *i* ∈ *I*
shows *continuous_map* (*product_topology T I*) (*T i*) (λ*x*. *x i*)
proof −
{
fix *U* **assume** *openin* (*T i*) *U*
define *X* **where** *X* = (λ*j*. *if j* = *i then U else topspace* (*T j*))
then have ∗: (λ*x*. *x i*) −‘ *U* ∩ (Π$_E$ *i*∈*I*. *topspace* (*T i*)) = (Π$_E$ *j*∈*I*. *X j*)
unfolding *X_def* **using** *assms openin_subset*[*OF* ‹*openin* (*T i*) *U*›]
by (*auto simp add*: *PiE_iff*, *auto*, *metis subsetCE*)
have ∗∗: (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)}
unfolding *X_def* **using** ‹*openin* (*T i*) *U*› **by** *auto*
have *openin* (*product_topology T I*) ((λ*x*. *x i*) −‘ *U* ∩ (Π$_E$ *i*∈*I*. *topspace* (*T i*)))
unfolding *product_topology_def*
apply (*rule topology_generated_by_Basis*)
apply (*subst* ∗)
using ∗∗ **by** *auto*
}
then show *?thesis* **unfolding** *continuous_map_alt*
by (*auto simp add*: *assms PiE_iff*)
qed

lemma *continuous_map_coordinatewise_then_product* [*intro*]:
assumes ⋀*i*. *i* ∈ *I* ⟹ *continuous_map T1* (*T i*) (λ*x*. *f x i*)
⋀*i x*. *i* ∉ *I* ⟹ *x* ∈ *topspace T1* ⟹ *f x i* = *undefined*
shows *continuous_map T1* (*product_topology T I*) *f*
unfolding *product_topology_def*
proof (*rule continuous_on_generated_topo*)
fix *U* **assume** *U* ∈ {*Pi$_E$ I X* |*X*. (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)}}
then obtain *X* **where** *H*: *U* = *Pi$_E$ I X* ⋀*i*. *openin* (*T i*) (*X i*) *finite* {*i*. *X i* ≠ *topspace* (*T i*)}
by *blast*
define *J* **where** *J* = {*i* ∈ *I*. *X i* ≠ *topspace* (*T i*)}
have *finite J J* ⊆ *I* **unfolding** *J_def* **using** *H*(*3*) **by** *auto*
have (λ*x*. *f x i*)−‘(*topspace*(*T i*)) ∩ *topspace T1* = *topspace T1* **if** *i* ∈ *I* **for** *i*
using *that assms*(*1*) **by** (*simp add*: *continuous_map_preimage_topspace*)
then have ∗: (λ*x*. *f x i*)−‘(*X i*) ∩ *topspace T1* = *topspace T1* **if** *i* ∈ *I*−*J* **for** *i*

using *that* **unfolding** *J_def* **by** *auto*
have $f-`U \cap topspace\ T1 = (\bigcap i{\in}I.\ (\lambda x.\ f\ x\ i)-`(X\ i) \cap topspace\ T1) \cap (topspace\ T1)$
by (*subst H(1)*, *auto simp add*: *PiE_iff assms*)
also have $... = (\bigcap i{\in}J.\ (\lambda x.\ f\ x\ i)-`(X\ i) \cap topspace\ T1) \cap (topspace\ T1)$
using $*$ ‹$J \subseteq I$› **by** *auto*
also have *openin T1* (...)
using *H(2)* ‹$J \subseteq I$› ‹*finite J*› *assms(1)* **by** *blast*
ultimately show *openin T1* $(f-`U \cap topspace\ T1)$ **by** *simp*
next
have $f \in topspace\ T1 \to topspace\ (product\_topology\ T\ I)$
using *assms continuous_map_funspace* **by** (*force simp*: *Pi_iff*)
then show $f\ `topspace\ T1 \subseteq \bigcup \{Pi_E\ I\ X\ |X.\ (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\}\}$
by (*fastforce simp add*: *product_topology_def Pi_iff*)
qed

lemma *continuous_map_product_then_coordinatewise* [*intro*]:
assumes *continuous_map T1* (*product_topology T I*) *f*
shows $\bigwedge i.\ i \in I \Longrightarrow continuous\_map\ T1\ (T\ i)\ (\lambda x.\ f\ x\ i)$
$\bigwedge i\ x.\ i \notin I \Longrightarrow x \in topspace\ T1 \Longrightarrow f\ x\ i = undefined$
proof −
fix *i* **assume** $i \in I$
have $(\lambda x.\ f\ x\ i) = (\lambda y.\ y\ i)\ o\ f$ **by** *auto*
also have *continuous_map T1* (*T i*) (...)
by (*metis* ‹$i \in I$› *assms continuous_map_compose continuous_map_product_coordinates*)
ultimately show *continuous_map T1* (*T i*) $(\lambda x.\ f\ x\ i)$
by *simp*
next
fix *i x* **assume** $i \notin I\ x \in topspace\ T1$
have $f\ x \in topspace\ (product\_topology\ T\ I)$
using *assms* ‹$x \in topspace\ T1$› **unfolding** *continuous_map_def* **by** *auto*
then have $f\ x \in (\Pi_E\ i{\in}I.\ topspace\ (T\ i))$
using *topspace_product_topology* **by** *metis*
then show $f\ x\ i = undefined$
using ‹$i \notin I$› **by** (*auto simp add*: *PiE_iff extensional_def*)
qed

lemma *continuous_on_restrict*:
assumes $J \subseteq I$
shows *continuous_map* (*product_topology T I*) (*product_topology T J*) $(\lambda x.\ restrict\ x\ J)$
proof (*rule continuous_map_coordinatewise_then_product*)
fix *i* **assume** $i \in J$
then have $(\lambda x.\ restrict\ x\ J\ i) = (\lambda x.\ x\ i)$ **unfolding** *restrict_def* **by** *auto*
then show *continuous_map* (*product_topology T I*) (*T i*) $(\lambda x.\ restrict\ x\ J\ i)$
using ‹$i \in J$› ‹$J \subseteq I$› **by** *auto*
next
fix *i* **assume** $i \notin J$

then show *restrict x J i = undefined* **for** *x::'a ⇒ 'b*
unfolding *restrict_def* **by** *auto*
qed

Powers of a single topological space as a topological space, using type classes

instantiation *fun* :: (*type*, *topological_space*) *topological_space*
begin

definition *open_fun_def*:
open U = openin (product_topology (λi. euclidean) UNIV) U

instance proof
have *topspace (product_topology (λ(i::'a). euclidean::('b topology)) UNIV) = UNIV*
unfolding *topspace_product_topology topspace_euclidean* **by** *auto*
then show *open (UNIV::('a ⇒ 'b) set)*
unfolding *open_fun_def* **by** (*metis openin_topspace*)
qed (*auto simp add*: *open_fun_def*)

end

lemma *open_PiE* [*intro?*]:
fixes *X::'i ⇒ ('b::topological_space) set*
assumes *⋀i. open (X i) finite {i. X i ≠ UNIV}*
shows *open (Pi$_E$ UNIV X)*
by (*simp add*: *assms open_fun_def product_topology_basis*)

lemma *euclidean_product_topology*:
product_topology (λi. euclidean::('b::topological_space) topology) UNIV = euclidean
by (*metis open_openin topology_eq open_fun_def*)

proposition *product_topology_basis'*:
fixes *x::'i ⇒ 'a* **and** *U::'i ⇒ ('b::topological_space) set*
assumes *finite I ⋀i. i ∈ I ⟹ open (U i)*
shows *open {f. ∀i∈I. f (x i) ∈ U i}*
proof −
define *V* **where** *V ≡ (λy. if y ∈ x`I then ⋂{U i|i. i∈I ∧ x i = y} else UNIV)*
define *X* **where** *X ≡ (λy. if y ∈ x`I then V y else UNIV)*
have *∗*: *open (X i)* **for** *i*
unfolding *X_def V_def* **using** *assms* **by** *auto*
then have *open (Pi$_E$ UNIV X)*
by (*simp add*: *X_def assms(1) open_PiE*)
moreover have *Pi$_E$ UNIV X = {f. ∀i∈I. f (x i) ∈ U i}*
by (*fastforce simp add*: *PiE_iff X_def V_def split*: *if_split_asm*)
ultimately show *?thesis* **by** *simp*
qed

The results proved in the general situation of products of possibly different spaces have their counterparts in this simpler setting.

lemma *continuous_on_product_coordinates* [*simp*]:
continuous_on UNIV (λ*x*. *x i*::(′*b*::*topological_space*))
using *continuous_map_product_coordinates* [*of _ UNIV* λ*i*. *euclidean*]
by (*metis* (*no_types*) *continuous_map_iff_continuous euclidean_product_topology iso_tuple_UNIV_I subtopology_UNIV*)

lemma *continuous_on_coordinatewise_then_product* [*continuous_intros*]:
fixes *f* :: ′*a*::*topological_space* ⇒ ′*b* ⇒ ′*c*::*topological_space*
assumes ⋀*i*. *continuous_on S* (λ*x*. *f x i*)
shows *continuous_on S f*
by (*metis UNIV_I assms continuous_map_iff_continuous euclidean_product_topology continuous_map_coordinatewise_then_product*)

lemma *continuous_on_product_then_coordinatewise*:
assumes *continuous_on S f*
shows *continuous_on S* (λ*x*. *f x i*)
by (*metis UNIV_I assms continuous_map_iff_continuous continuous_map_product_then_coordinatewise*(*1*) *euclidean_product_topology*)

lemma *continuous_on_coordinatewise_iff*:
fixes *f* :: (′*a* ⇒ *real*) ⇒ ′*b* ⇒ *real*
shows *continuous_on* (*A* ∩ *S*) *f* ⟷ (∀ *i*. *continuous_on* (*A* ∩ *S*) (λ*x*. *f x i*))
by (*auto simp*: *continuous_on_product_then_coordinatewise continuous_on_coordinatewise_then_product*)

lemma *continuous_map_span_sum*:
fixes *B* :: ′*a*::*real_normed_vector set*
assumes *biB*: ⋀*i*. *i* ∈ *I* ⟹ *b i* ∈ *B*
shows *continuous_map euclidean* (*top_of_set* (*span B*)) (λ*x*. ∑*i*∈*I*. *x i* ∗*R b i*)
proof (*rule continuous_map_euclidean_top_of_set*)
show (λ*x*. ∑*i*∈*I*. *x i* ∗*R b i*) −‘ *span B* = *UNIV*
by *auto* (*meson biB lessThan_iff span_base span_scale span_sum*)
show *continuous_on UNIV* (λ*x*. ∑*i*∈ *I*. *x i* ∗*R b i*)
by (*intro continuous_intros*) *auto*
qed

Topological countability for product spaces

The next two lemmas are useful to prove first or second countability of product spaces, but they have more to do with countability and could be put in the corresponding theory.

lemma *countable_nat_product_event_const*:
fixes *F*::′*a set* **and** *a*::′*a*
assumes *a* ∈ *F countable F*
shows *countable* {*x*::(*nat* ⇒ ′*a*). (∀ *i*. *x i* ∈ *F*) ∧ *finite* {*i*. *x i* ≠ *a*}}
proof −

```
  have ∗: {x::(nat ⇒ 'a). (∀ i. x i ∈ F) ∧ finite {i. x i ≠ a}}
                ⊆ (⋃ N. {x. (∀ i. x i ∈ F) ∧ (∀ i≥N. x i = a)})
    using infinite_nat_iff_unbounded_le by fastforce
  have countable {x. (∀ i. x i ∈ F) ∧ (∀ i≥N. x i = a)} for N::nat
  proof (induction N)
    case 0
    have {x. (∀ i. x i ∈ F) ∧ (∀ i≥(0::nat). x i = a)} = {(λi. a)}
      using ‹a ∈ F› by auto
    then show ?case by auto
  next
    case (Suc N)
    define f::((nat ⇒ 'a) × 'a) ⇒ (nat ⇒ 'a)
      where f = (λ(x, b). x(N:=b))
     have {x. (∀ i. x i ∈ F) ∧ (∀ i≥Suc N. x i = a)} ⊆ f‘({x. (∀ i. x i ∈ F) ∧
(∀ i≥N. x i = a)} × F)
    proof (auto)
      fix x assume H: ∀ i::nat. x i ∈ F ∀ i≥Suc N. x i = a
      have f (x(N:=a), x N) = x
        unfolding f_def by auto
      moreover have (x(N:=a), x N) ∈ {x. (∀ i. x i ∈ F) ∧ (∀ i≥N. x i = a)} ×
F
        using H ‹a ∈ F› by auto
      ultimately show x ∈ f ‘ ({x. (∀ i. x i ∈ F) ∧ (∀ i≥N. x i = a)} × F)
        by (metis (no_types, lifting) image_eqI)
    qed
    moreover have countable ({x. (∀ i. x i ∈ F) ∧ (∀ i≥N. x i = a)} × F)
      using Suc.IH assms(2) by auto
    ultimately show ?case
      by (meson countable_image countable_subset)
  qed
  then show ?thesis using countable_subset[OF ∗] by auto
qed

lemma countable_product_event_const:
  fixes F::('a::countable) ⇒ 'b set and b::'b
  assumes ⋀i. countable (F i)
  shows countable {f::('a ⇒ 'b). (∀ i. f i ∈ F i) ∧ (finite {i. f i ≠ b})}
proof −
  define G where G = (⋃ i. F i) ∪ {b}
  have countable G unfolding G_def using assms by auto
  have b ∈ G unfolding G_def by auto
  define pi where pi = (λ(x::(nat ⇒ 'b)). (λ i::'a. x ((to_nat::('a ⇒ nat)) i)))
  have {f::('a ⇒ 'b). (∀ i. f i ∈ F i) ∧ (finite {i. f i ≠ b})}
        ⊆ pi‘{g::(nat ⇒ 'b). (∀ j. g j ∈ G) ∧ (finite {j. g j ≠ b})}
  proof (auto)
    fix f assume H: ∀ i. f i ∈ F i finite {i. f i ≠ b}
    define I where I = {i. f i ≠ b}
    define g where g = (λj. if j ∈ to_nat‘I then f (from_nat j) else b)
    have {j. g j ≠ b} ⊆ to_nat‘I unfolding g_def by auto
```

```
    then have finite {j. g j ≠ b}
      unfolding I_def using H(2) using finite_surj by blast
    moreover have g j ∈ G for j
      unfolding g_def G_def using H by auto
    ultimately have g ∈ {g::(nat ⇒ 'b). (∀ j. g j ∈ G) ∧ (finite {j. g j ≠ b})}
      by auto
    moreover have f = pi g
      unfolding pi_def g_def I_def using H by fastforce
    ultimately show f ∈ pi‘{g. (∀ j. g j ∈ G) ∧ finite {j. g j ≠ b}}
      by auto
  qed
  then show ?thesis
    using countable_nat_product_event_const[OF ‹b ∈ G› ‹countable G›]
    by (meson countable_image countable_subset)
qed

instance fun :: (countable, first_countable_topology) first_countable_topology
proof
  fix x::'a ⇒ 'b
  have ∃ A::('b ⇒ nat ⇒ 'b set). ∀ x. (∀ i. x ∈ A x i ∧ open (A x i)) ∧ (∀ S. open
S ∧ x ∈ S ⟶ (∃ i. A x i ⊆ S))
    apply (rule choice) using first_countable_basis by auto
  then obtain A::('b ⇒ nat ⇒ 'b set) where A: ⋀x i. x ∈ A x i
                                              ⋀x i. open (A x i)
                                              ⋀x S. open S ⟹ x ∈ S ⟹ (∃ i. A x i ⊆ S)
    by metis
```

$B\ i$ is a countable basis of neighborhoods of x_i.

```
  define B where B = (λi. (A (x i))‘UNIV ∪ {UNIV})
  have countB: countable (B i) for i unfolding B_def by auto
  have open_B: ⋀X i. X ∈ B i ⟹ open X
    by (auto simp: B_def A)
  define K where K = {Pi_E UNIV X | X. (∀ i. X i ∈ B i) ∧ finite {i. X i ≠
UNIV}}
  have Pi_E UNIV (λi. UNIV) ∈ K
    unfolding K_def B_def by auto
  then have K ≠ {} by auto
  have countable {X. (∀ i. X i ∈ B i) ∧ finite {i. X i ≠ UNIV}}
    by (simp add: countB countable_product_event_const)
  moreover have K = (λX. Pi_E UNIV X)‘{X. (∀ i. X i ∈ B i) ∧ finite {i. X i
≠ UNIV}}
    unfolding K_def by auto
  ultimately have countable K by auto
  have I: x ∈ k if k ∈ K for k
    using that unfolding K_def B_def apply auto using A(1) by auto
  have II: open k if k ∈ K for k
    using that unfolding K_def by (blast intro: open_B open_PiE)
  have Inc: ∃ k∈K. k ⊆ U if open U ∧ x ∈ U for U
  proof −
```

have *openin* (*product_topology* (λ*i*. *euclidean*) *UNIV*) *U x* ∈ *U*
using ‹*open U* ∧ *x* ∈ *U*› **unfolding** *open_fun_def* **by** *auto*
with *product_topology_open_contains_basis*[*OF this*]
have ∃ *X*. *x* ∈ (Π$_E$ *i*∈*UNIV*. *X i*) ∧ (∀ *i*. *open* (*X i*)) ∧ *finite* {*i*. *X i* ≠ *UNIV*} ∧ (Π$_E$ *i*∈*UNIV*. *X i*) ⊆ *U*
by *simp*
then obtain *X* **where** *H*: *x* ∈ (Π$_E$ *i*∈*UNIV*. *X i*)
⋀*i*. *open* (*X i*)
finite {*i*. *X i* ≠ *UNIV*}
(Π$_E$ *i*∈*UNIV*. *X i*) ⊆ *U*
by *auto*
define *I* **where** *I* = {*i*. *X i* ≠ *UNIV*}
define *Y* **where** *Y* = (λ*i*. *if i* ∈ *I then* (*SOME y*. *y* ∈ *B i* ∧ *y* ⊆ *X i*) *else UNIV*)
have ∗: ∃ *y*. *y* ∈ *B i* ∧ *y* ⊆ *X i* **for** *i*
unfolding *B_def* **using** *A*(*3*)[*OF H*(*2*)] *H*(*1*) **by** (*metis PiE_E UNIV_I UnCI image_iff*)
have ∗∗: *Y i* ∈ *B i* ∧ *Y i* ⊆ *X i* **for** *i*
proof (*cases i* ∈ *I*)
case *True*
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) ∗ *Nitpick.Eps_psimp Y_def*)
next
case *False*
then show *?thesis* **by** (*simp add*: *B_def I_def Y_def*)
qed
have {*i*. *Y i* ≠ *UNIV*} ⊆ *I*
unfolding *Y_def* **by** *auto*
with ∗∗ **have** (∀ *i*. *Y i* ∈ *B i*) ∧ *finite* {*i*. *Y i* ≠ *UNIV*}
using *H*(*3*) *I_def finite_subset* **by** *blast*
then have *Pi$_E$ UNIV Y* ∈ *K*
unfolding *K_def* **by** *auto*
have *Y i* ⊆ *X i* **for** *i*
using ∗∗ **by** *auto*
then have *Pi$_E$ UNIV Y* ⊆ *U*
by (*metis H*(*4*) *PiE_mono subset_trans*)
then show *?thesis* **using** ‹*Pi$_E$ UNIV Y* ∈ *K*› **by** *auto*
qed
show ∃ *L*. (∀ (*i*::*nat*). *x* ∈ *L i* ∧ *open* (*L i*)) ∧ (∀ *U*. *open U* ∧ *x* ∈ *U* ⟶ (∃ *i*. *L i* ⊆ *U*))
using ‹*countable K*› *I II Inc* **by** (*simp add*: *first_countableI*)
qed

proposition *product_topology_countable_basis*:
shows ∃ *K*::((*'a*::*countable* ⇒ *'b*::*second_countable_topology*) *set set*).
topological_basis K ∧ *countable K* ∧
(∀ *k*∈*K*. ∃ *X*. (*k* = *Pi$_E$ UNIV X*) ∧ (∀ *i*. *open* (*X i*)) ∧ *finite* {*i*. *X i* ≠ *UNIV*})
proof −

```
  obtain B::'b set set where B: countable B ∧ topological_basis B
    using ex_countable_basis by auto
  then have B ≠ {} by (meson UNIV_I empty_iff open_UNIV topological_basisE)
  define B2 where B2 = B ∪ {UNIV}
  have countable B2
    unfolding B2_def using B by auto
  have open U if U ∈ B2 for U
    using that unfolding B2_def using B topological_basis_open by auto

  define K where K = {Pi_E UNIV X | X. (∀ i::'a. X i ∈ B2) ∧ finite {i. X i ≠
UNIV}}
  have i: ∀ k∈K. ∃ X. (k = Pi_E UNIV X) ∧ (∀ i. open (X i)) ∧ finite {i. X i ≠
UNIV}
    unfolding K_def using ‹⋀U. U ∈ B2 ⟹ open U› by auto

  have countable {X. (∀ (i::'a). X i ∈ B2) ∧ finite {i. X i ≠ UNIV}}
    using ‹countable B2› by (intro countable_product_event_const) auto
  moreover have K = (λX. Pi_E UNIV X)`{X. (∀ i. X i ∈ B2) ∧ finite {i. X i
≠ UNIV}}
    unfolding K_def by auto
  ultimately have ii: countable K by auto

  have iii: topological_basis K
  proof (rule topological_basisI)
    fix U and x::'a⇒'b assume open U x ∈ U
    then have openin (product_topology (λi. euclidean) UNIV) U
      unfolding open_fun_def by auto
    with product_topology_open_contains_basis[OF this ‹x ∈ U›]
    obtain X where H: x ∈ (Π_E i∈UNIV. X i)
                      ⋀i. open (X i)
                      finite {i. X i ≠ UNIV}
                      (Π_E i∈UNIV. X i) ⊆ U
      by auto
    then have x i ∈ X i for i by auto
    define I where I = {i. X i ≠ UNIV}
    define Y where Y = (λi. if i ∈ I then (SOME y. y ∈ B2 ∧ y ⊆ X i ∧ x i ∈
y) else UNIV)
    have ∗: ∃ y. y ∈ B2 ∧ y ⊆ X i ∧ x i ∈ y for i
       unfolding B2_def using B ‹open (X i)› ‹x i ∈ X i› by (meson UnCI
topological_basisE)
    have ∗∗: Y i ∈ B2 ∧ Y i ⊆ X i ∧ x i ∈ Y i for i
      using someI_ex[OF ∗] by (simp add: B2_def I_def Y_def)
    have {i. Y i ≠ UNIV} ⊆ I
      unfolding Y_def by auto
    then have (∀ i. Y i ∈ B2) ∧ finite {i. Y i ≠ UNIV}
      using ∗∗ H(3) I_def finite_subset by blast
    then have Pi_E UNIV Y ∈ K
      unfolding K_def by auto
    then show ∃ V∈K. x ∈ V ∧ V ⊆ U
```

```
    by (meson ** H(4) PiE_I PiE_mono UNIV_I order.trans)
  next
    fix U assume U ∈ K
    show open U
     using ‹U ∈ K› unfolding open_fun_def K_def  by clarify (metis ‹U ∈ K›
i open_PiE open_fun_def)
  qed

  show ?thesis using i ii iii by auto
qed

instance fun :: (countable, second_countable_topology) second_countable_topology
proof
  show ∃B::('a ⇒ 'b) set set. countable B ∧ open = generate_topology B
    using product_topology_countable_basis topological_basis_imp_subbasis
    by auto
qed
```

2.5.2 The Alexander subbase theorem

```
theorem Alexander_subbase:
  assumes X: topology (arbitrary union_of (finite intersection_of (λx. x ∈ ℬ)
relative_to ⋃ℬ)) = X
    and fin: ⋀C. ⟦C ⊆ ℬ; ⋃ C = topspace X⟧ ⟹ ∃ C'. finite C' ∧ C' ⊆ C ∧
⋃ C' = topspace X
  shows compact_space X
proof −
  have Uℬ: ⋃ℬ = topspace X
    by (simp flip: X)
  have False if 𝒰: ∀ U∈𝒰. openin X U and sub: topspace X ⊆ ⋃𝒰
    and neg: ⋀ℱ. ⟦ℱ ⊆ 𝒰; finite ℱ⟧ ⟹ ¬ topspace X ⊆ ⋃ℱ for 𝒰
  proof −
    define 𝒜 where 𝒜 ≡ {𝒞. (∀ U ∈ 𝒞. openin X U) ∧ topspace X ⊆ ⋃𝒞 ∧ (∀ℱ.
finite ℱ ⟶ ℱ ⊆ 𝒞 ⟶ ~(topspace X ⊆ ⋃ℱ))}
    have 1: 𝒜 ≠ {}
      unfolding 𝒜_def using sub 𝒰 neg by force
    have 2: ⋃𝒞 ∈ 𝒜 if 𝒞≠{} and 𝒞: subset.chain 𝒜 𝒞 for 𝒞
      unfolding 𝒜_def
    proof (intro CollectI conjI ballI allI impI notI)
      show openin X U if U: U ∈ ⋃𝒞 for U
        using U 𝒞 unfolding 𝒜_def subset_chain_def by force
      have 𝒞 ⊆ 𝒜
        using subset_chain_def 𝒞 by blast
      with that 𝒜_def show UUC: topspace X ⊆ ⋃(⋃𝒞)
        by blast
      show False if finite ℱ and ℱ ⊆ ⋃𝒞 and topspace X ⊆ ⋃ℱ for ℱ
      proof −
        obtain ℬ where ℬ ∈ 𝒞 ℱ ⊆ ℬ
            by (metis Sup_empty 𝒞 ‹ℱ ⊆ ⋃𝒞› ‹finite ℱ› UUC empty_subsetI
```

finite.emptyI finite_subset_Union_chain neg)
then show *False*
using *A_def* ‹$\mathcal{C} \subseteq \mathcal{A}$› ‹*finite* $\mathcal{F}$› ‹*topspace* $X \subseteq \bigcup \mathcal{F}$› **by** *blast*
qed
qed
obtain $\mathcal{K}$ **where** $\mathcal{K} \in \mathcal{A}$ **and** $\bigwedge X.\ [\![X \in \mathcal{A};\ \mathcal{K} \subseteq X]\!] \Longrightarrow X = \mathcal{K}$
using *subset_Zorn_nonempty* [*OF 1 2*] **by** *metis*
then have $*$: $\bigwedge \mathcal{W}.\ [\![\bigwedge W.\ W \in \mathcal{W} \Longrightarrow$ *openin* X W; *topspace* $X \subseteq \bigcup \mathcal{W}$; $\mathcal{K} \subseteq \mathcal{W}$;
$\bigwedge \mathcal{F}.\ [\![$*finite* $\mathcal{F}$; $\mathcal{F} \subseteq \mathcal{W}$; *topspace* $X \subseteq \bigcup \mathcal{F}]\!] \Longrightarrow$ *False*$]\!]$
$\Longrightarrow \mathcal{W} = \mathcal{K}$
and *ope*: $\forall\, U \in \mathcal{K}.$ *openin* X U **and** *top*: *topspace* $X \subseteq \bigcup \mathcal{K}$
and *non*: $\bigwedge \mathcal{F}.\ [\![$*finite* $\mathcal{F}$; $\mathcal{F} \subseteq \mathcal{K}$; *topspace* $X \subseteq \bigcup \mathcal{F}]\!] \Longrightarrow$ *False*
unfolding *A_def* **by** *simp_all metis+*
then obtain x **where** $x \in$ *topspace* X $x \notin \bigcup (\mathcal{B} \cap \mathcal{K})$
proof −
have $\bigcup (\mathcal{B} \cap \mathcal{K}) \neq \bigcup \mathcal{B}$
by (*metis* ‹$\bigcup \mathcal{B} =$ *topspace* X› *fin inf.bounded_iff non order_refl*)
then have $\exists\, a.\ a \notin \bigcup (\mathcal{B} \cap \mathcal{K}) \wedge a \in \bigcup \mathcal{B}$
by *blast*
then show *?thesis*
using *that* **by** (*metis U*$\mathcal{B}$)
qed
obtain C **where** C: *openin* X C $C \in \mathcal{K}$ $x \in C$
using ‹$x \in$ *topspace* X› *ope top* **by** *auto*
then have $C \subseteq$ *topspace* X
by (*metis openin_subset*)
then have (*arbitrary union_of* (*finite intersection_of* ($\lambda x.\ x \in \mathcal{B}$) *relative_to* $\bigcup \mathcal{B}$)) C
using *openin_subbase* C **unfolding** X [*symmetric*] **by** *blast*
moreover have $C \neq$ *topspace* X
using ‹$\mathcal{K} \in \mathcal{A}$› ‹$C \in \mathcal{K}$› **unfolding** *A_def* **by** *blast*
ultimately obtain $\mathcal{V}$ W **where** W: (*finite intersection_of* ($\lambda x.\ x \in \mathcal{B}$) *relative_to topspace* X) W
and $x \in W$ $W \in \mathcal{V}$ $\bigcup \mathcal{V} \neq$ *topspace* X $C = \bigcup \mathcal{V}$
using C **by** (*auto simp*: *union_of_def U*$\mathcal{B}$)
then have $\bigcup \mathcal{V} \subseteq$ *topspace* X
by (*metis* ‹$C \subseteq$ *topspace* X›)
then have *topspace* $X \notin \mathcal{V}$
using ‹$\bigcup \mathcal{V} \neq$ *topspace* X› **by** *blast*
then obtain $\mathcal{B}'$ **where** $\mathcal{B}'$: *finite* $\mathcal{B}'$ $\mathcal{B}' \subseteq \mathcal{B}$ $x \in \bigcap \mathcal{B}'$ $W =$ *topspace* $X \cap \bigcap \mathcal{B}'$
using W ‹$x \in W$› **unfolding** *relative_to_def intersection_of_def* **by** *auto*
then have $\bigcap \mathcal{B}' \subseteq \bigcup \mathcal{B}$
using ‹$W \in \mathcal{V}$› ‹$\bigcup \mathcal{V} \neq$ *topspace* X› ‹$\bigcup \mathcal{V} \subseteq$ *topspace* X› **by** *blast*
then have $\bigcap \mathcal{B}' \subseteq C$
using *U*$\mathcal{B}$ ‹$C = \bigcup \mathcal{V}$› ‹$W =$ *topspace* $X \cap \bigcap \mathcal{B}'$› ‹$W \in \mathcal{V}$› **by** *auto*
have $\forall\, b \in \mathcal{B}'.\ \exists\, C'.$ *finite* $C' \wedge\ C' \subseteq \mathcal{K} \wedge$ *topspace* $X \subseteq \bigcup$ (*insert* b C')
proof
fix b

```
      assume b ∈ 𝓑'
       have insert b 𝓚 = 𝓚 if neg: ¬ (∃ C'. finite C' ∧ C' ⊆ 𝓚 ∧ topspace X ⊆
⋃(insert b C'))
      proof (rule ∗)
        show openin X W if W ∈ insert b 𝓚 for W
          using that
        proof
          have b ∈ 𝓑
            using ‹b ∈ 𝓑'› ‹𝓑' ⊆ 𝓑› by blast
          then have ∃𝓤. finite 𝓤 ∧ 𝓤 ⊆ 𝓑 ∧ ⋂𝓤 = b
            by (rule_tac x={b} in exI) auto
          moreover have ⋃𝓑 ∩ b = b
            using 𝓑'(2) ‹b ∈ 𝓑'› by auto
          ultimately show openin X W if W = b
            using  that ‹b ∈ 𝓑'›
            apply (simp add: openin_subbase flip: X)
              apply (auto simp: arbitrary_def intersection_of_def relative_to_def
intro!: union_of_inc)
            done
          show openin X W if W ∈ 𝓚
            by (simp add: ‹W ∈ 𝓚› ope)
        qed
      next
        show topspace X ⊆ ⋃ (insert b 𝓚)
          using top by auto
      next
        show False if finite 𝓕 and 𝓕 ⊆ insert b 𝓚 topspace X ⊆ ⋃𝓕 for 𝓕
        proof −
          have insert b (𝓕 ∩ 𝓚) = 𝓕
            using non that by blast
          then show False
            by (metis Int_lower2 finite_insert neg that(1) that(3))
        qed
      qed auto
      then show ∃ C'. finite C' ∧ C' ⊆ 𝓚 ∧ topspace X ⊆ ⋃(insert b C')
        using ‹b ∈ 𝓑'› ‹x ∉ ⋃(𝓑 ∩ 𝓚)› 𝓑'
        by (metis IntI InterE Union_iff subsetD insertI1)
    qed
     then obtain F where F: ∀ b ∈ 𝓑'. finite (F b) ∧ F b ⊆ 𝓚 ∧ topspace X ⊆
⋃(insert b (F b))
      by metis
    let ?𝒟 = insert C (⋃(F ‘ 𝓑'))
    show False
    proof (rule non)
      have topspace X ⊆ (⋂ b ∈ 𝓑'. ⋃(insert b (F b)))
        using F by (simp add: INT_greatest)
      also have ... ⊆ ⋃ ?𝒟
        using ‹⋂𝓑' ⊆ C› by force
      finally show topspace X ⊆ ⋃ ?𝒟 .
```

```
    show ?D ⊆ K
      using ‹C ∈ K› F by auto
    show finite ?D
      using ‹finite B'› F by auto
   qed
  qed
  then show ?thesis
   by (force simp: compact_space_def compactin_def)
qed


corollary Alexander_subbase_alt:
  assumes U ⊆ ⋃B
  and fin: ⋀C. ⟦C ⊆ B; U ⊆ ⋃C⟧ ⟹ ∃C'. finite C' ∧ C' ⊆ C ∧ U ⊆ ⋃C'
   and X: topology
            (arbitrary union_of
                (finite intersection_of (λx. x ∈ B) relative_to U)) = X
 shows compact_space X
proof −
  have topspace X = U
    using X topspace_subbase by fastforce
  have eq: ⋃ (Collect ((λx. x ∈ B) relative_to U)) = U
    unfolding relative_to_def
    using ‹U ⊆ ⋃B› by blast
  have ∗: ∃F. finite F ∧ F ⊆ C ∧ ⋃F = topspace X
    if C ⊆ Collect ((λx. x ∈ B) relative_to topspace X) and UC: ⋃C = topspace
X for C
  proof −
    have C ⊆ (λU. topspace X ∩ U) ‘ B
      using that by (auto simp: relative_to_def)
    then obtain B' where B' ⊆ B and B': C = (∩) (topspace X) ‘ B'
      by (auto simp: subset_image_iff)
    moreover have U ⊆ ⋃B'
      using B' ‹topspace X = U› UC by auto
    ultimately obtain C' where finite C' C' ⊆ B' U ⊆ ⋃C'
      using fin [of B'] ‹topspace X = U› ‹U ⊆ ⋃B› by blast
    then show ?thesis
      unfolding B' ex_finite_subset_image ‹topspace X = U› by auto
  qed
  show ?thesis
    apply (rule Alexander_subbase [where B = Collect ((λx. x ∈ B) relative_to
(topspace X))])
     apply (simp flip: X)
     apply (metis finite_intersection_of_relative_to eq)
    apply (blast intro: ∗)
    done
qed

proposition continuous_map_componentwise:
```

```
   continuous_map X (product_topology Y I) f ⟷
    f ‘ (topspace X) ⊆ extensional I ∧ (∀ k ∈ I. continuous_map X (Y k) (λx. f x
k))
    (is ?lhs ⟷ _ ∧ ?rhs)
proof (cases ∀ x ∈ topspace X. f x ∈ extensional I)
  case True
  then have f ‘ (topspace X) ⊆ extensional I
    by force
  moreover have ?rhs if L: ?lhs
  proof −
    have openin X {x ∈ topspace X. f x k ∈ U} if k ∈ I and openin (Y k) U for
k U
    proof −
      have openin (product_topology Y I) ({Y. Y k ∈ U} ∩ (Π_E i∈I. topspace (Y
i)))
      apply (simp add: openin_product_topology flip: arbitrary_union_of_relative_to)
        apply (simp add: relative_to_def)
      using that apply (blast intro: arbitrary_union_of_inc finite_intersection_of_inc)
        done
      with that have openin X {x ∈ topspace X. f x ∈ ({Y. Y k ∈ U} ∩ (Π_E i∈I.
topspace (Y i)))}
        using L unfolding continuous_map_def by blast
      moreover have {x ∈ topspace X. f x ∈ ({Y. Y k ∈ U} ∩ (Π_E i∈I. topspace
(Y i)))} = {x ∈ topspace X. f x k ∈ U}
        using L by (auto simp: continuous_map_def)
      ultimately show ?thesis
        by metis
    qed
    with that
    show ?thesis
      by (auto simp: continuous_map_def)
  qed
  moreover have ?lhs if ?rhs
  proof −
    have 1: ⋀x. x ∈ topspace X ⟹ f x ∈ (Π_E i∈I. topspace (Y i))
      using that True by (auto simp: continuous_map_def PiE_iff)
    have 2: {x ∈ S. ∃ T∈𝒯. f x ∈ T} = (⋃ T∈𝒯. {x ∈ S. f x ∈ T}) for S 𝒯
      by blast
    have 3: {x ∈ S. ∀ U∈𝒰. f x ∈ U} = (⋂ (insert S ((λU. {x ∈ S. f x ∈ U}) ‘
𝒰))) for S 𝒰
      by blast
    show ?thesis
      unfolding continuous_map_def openin_product_topology arbitrary_def
    proof (clarsimp simp: all_union_of 1 2)
      fix 𝒯
      assume 𝒯: 𝒯 ⊆ Collect (finite intersection_of (λF. ∃ i U. F = {f. f i ∈ U}
∧ i ∈ I ∧ openin (Y i) U)
                    relative_to (Π_E i∈I. topspace (Y i)))
      show openin X (⋃ T∈𝒯. {x ∈ topspace X. f x ∈ T})
```

```
      proof (rule openin_Union; clarify)
        fix S T
        assume T ∈ 𝒯
        obtain 𝒰 where T = (Π_E i∈I. topspace (Y i)) ∩ ⋂𝒰 and finite 𝒰
          𝒰 ⊆ {{f. f i ∈ U} |i U. i ∈ I ∧ openin (Y i) U}
            using subsetD [OF 𝒯 ‹T ∈ 𝒯›] by (auto simp: intersection_of_def
relative_to_def)
        with that show openin X {x ∈ topspace X. f x ∈ T}
          apply (simp add: continuous_map_def 1 cong: conj_cong)
          unfolding 3
          apply (rule openin_Inter; auto)
          done
      qed
    qed
  qed
  ultimately show ?thesis
    by metis
qed (auto simp: continuous_map_def PiE_def)


lemma continuous_map_componentwise_UNIV:
   continuous_map X (product_topology Y UNIV) f ⟷ (∀k. continuous_map X
(Y k) (λx. f x k))
  by (simp add: continuous_map_componentwise)

lemma continuous_map_product_projection [continuous_intros]:
   k ∈ I ⟹ continuous_map (product_topology X I) (X k) (λx. x k)
  using continuous_map_componentwise [of product_topology X I X I id] by simp

declare continuous_map_from_subtopology [OF continuous_map_product_projection,
continuous_intros]

proposition open_map_product_projection:
  assumes i ∈ I
  shows open_map (product_topology Y I) (Y i) (λf. f i)
  unfolding openin_product_topology all_union_of arbitrary_def open_map_def
image_Union
proof clarify
  fix 𝒱
  assume 𝒱: 𝒱 ⊆ Collect
              (finite intersection_of
               (λF. ∃i U. F = {f. f i ∈ U} ∧ i ∈ I ∧ openin (Y i) U) relative_to
                    topspace (product_topology Y I))
  show openin (Y i) (⋃x∈𝒱. (λf. f i) ` x)
  proof (rule openin_Union, clarify)
    fix S V
    assume V ∈ 𝒱
    obtain ℱ where finite ℱ
      and V: V = (Π_E i∈I. topspace (Y i)) ∩ ⋂ℱ
```

```
    and \mathcal{F}: \mathcal{F} \subseteq \{\{f.\ f\ i \in U\}\ |i\ U.\ i \in I \wedge openin\ (Y\ i)\ U\}
    using subsetD [OF \mathcal{V} ‹V \in \mathcal{V}›]
    by (auto simp: intersection_of_def relative_to_def)
  show openin (Y i) ((\lambdaf. f i) ` V)
  proof (subst openin_subopen; clarify)
    fix x f
    assume f \in V
    let ?T = {a \in topspace(Y i).
                (\lambdaj∈I. f j)(i:=a) \in (\Pi_E i∈I. topspace (Y i)) \cap \bigcap\mathcal{F}}
    show \exists T. openin (Y i) T \wedge f i \in T \wedge T \subseteq (\lambdaf. f i) ` V
    proof (intro exI conjI)
      show openin (Y i) ?T
      proof (rule openin_continuous_map_preimage)
        have continuous_map (Y i) (Y k) (\lambdax. if k = i then x else f k) if k \in I
for k
        proof (cases k=i)
          case True
          then show ?thesis
            by (metis (mono_tags) continuous_map_id eq_id_iff)
        next
          case False
          then show ?thesis
            by simp (metis IntD1 PiE_iff V ‹f \in V› that)
        qed
        then show continuous_map (Y i) (product_topology Y I)
              (\lambdax. (\lambdaj∈I. f j)(i:=x))
          by (auto simp: continuous_map_componentwise assms extensional_def
restrict_def)
      next
        have openin (product_topology Y I) (\Pi_E i∈I. topspace (Y i))
          by (metis openin_topspace topspace_product_topology)
        moreover have openin (product_topology Y I) (\bigcap B∈\mathcal{F}. (\Pi_E i∈I. topspace
(Y i)) \cap B)
                      if \mathcal{F} \neq {}
        proof -
          show ?thesis
          proof (rule openin_Inter)
                show \bigwedgeX. X \in (\cap) (\Pi_E i∈I. topspace (Y i)) ` \mathcal{F} \Longrightarrow openin
(product_topology Y I) X
              unfolding openin_product_topology relative_to_def
              apply (clarify intro!: arbitrary_union_of_inc)
              using subsetD [OF \mathcal{F}]
            by (metis (mono_tags, lifting) finite_intersection_of_inc mem_Collect_eq
topspace_product_topology)
          qed (use ‹finite \mathcal{F}› ‹\mathcal{F} \neq {}› in auto)
        qed
        ultimately show openin (product_topology Y I) ((\Pi_E i∈I. topspace (Y
i)) \cap \bigcap\mathcal{F})
          by (auto simp only: Int_Inter_eq split: if_split)
```

```
      qed
    next
      have eqf: (λj∈I. f j)(i:=f i) = f
        using PiE_arb V ‹f ∈ V› by force
      show f i ∈ ?T
        using V assms ‹f ∈ V› by (auto simp: PiE_iff eqf)
    next
      show ?T ⊆ (λf. f i) ‘ V
        unfolding V by (auto simp: intro!: rev_image_eqI)
    qed
  qed
qed

lemma retraction_map_product_projection:
  assumes i ∈ I
  shows (retraction_map (product_topology X I) (X i) (λx. x i) ⟷
        ((product_topology X I) = trivial_topology) ⟶ (X i) = trivial_topology)
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    using retraction_imp_surjective_map
    by (metis image_empty subtopology_eq_discrete_topology_empty)
next
  assume R: ?rhs
  show ?lhs
  proof (cases (product_topology X I) = trivial_topology)
    case True
    then show ?thesis
      using R by (auto simp: retraction_map_def retraction_maps_def)
  next
    case False
    have ∗: ∃ g. continuous_map (X i) (product_topology X I) g ∧ (∀ x∈topspace
(X i). g x i = x)
      if z: z ∈ (Π_E i∈I. topspace (X i)) for z
    proof −
      have cm: continuous_map (X i) (X j) (λx. if j = i then x else z j) if j ∈ I
for j
        using ‹j ∈ I› z  by (case_tac j = i) auto
      show ?thesis
        using ‹i ∈ I› that
        by (rule_tac x=λx j. if j = i then x else z j in exI) (auto simp: continu-
ous_map_componentwise PiE_iff extensional_def cm)
    qed
    with  ‹i ∈ I› False assms show ?thesis
    by (auto simp: retraction_map_def retraction_maps_def simp flip: null_topspace_iff_trivial)
  qed
qed
```

2.5.3 Open Pi-sets in the product topology

proposition *openin_PiE_gen*:
openin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨
finite {*i* ∈ *I*. *S i* ≠ *topspace* (*X i*)} ∧ (∀ *i* ∈ *I*. *openin* (*X i*) (*S i*))
(**is** *?lhs* ⟷ _ ∨ *?rhs*)
proof (*cases PiE I S* = {})
case *False*
moreover have *?lhs* = *?rhs*
proof
assume *L*: *?lhs*
moreover
obtain *z* **where** *z*: *z* ∈ *PiE I S*
using *False* **by** *blast*
ultimately obtain *U* **where** *fin*: *finite* {*i* ∈ *I*. *U i* ≠ *topspace* (*X i*)}
and Pi_E *I U* ≠ {}
and *sub*: Pi_E *I U* ⊆ Pi_E *I S*
by (*fastforce simp add*: *openin_product_topology_alt*)
then have ∗: ⋀*i*. *i* ∈ *I* ⟹ *U i* ⊆ *S i*
by (*simp add*: *subset_PiE*)
show *?rhs*
proof (*intro conjI ballI*)
show *finite* {*i* ∈ *I*. *S i* ≠ *topspace* (*X i*)}
apply (*rule finite_subset* [*OF* _ *fin*], *clarify*)
using ∗
by (*metis False L openin_subset topspace_product_topology subset_PiE subset_antisym*)
next
fix *i* :: *'a*
assume *i* ∈ *I*
then show *openin* (*X i*) (*S i*)
using *open_map_product_projection* [*of i I X*] *L*
apply (*simp add*: *open_map_def*)
apply (*drule_tac x=PiE I S* **in** *spec*)
apply (*simp add*: *False image_projection_PiE split*: *if_split_asm*)
done
qed
next
assume *?rhs*
then show *?lhs*
unfolding *openin_product_topology*
by (*intro arbitrary_union_of_inc*) (*auto simp*: *product_topology_base_alt*)
qed
ultimately show *?thesis*
by *simp*
qed *simp*

corollary *openin_PiE*:

finite I $\Longrightarrow$ *openin* (*product_topology X I*) (*PiE I S*) $\longleftrightarrow$ *PiE I S* = {} $\vee$ ($\forall$ *i* $\in$ *I*. *openin* (*X i*) (*S i*))
by (*simp add*: *openin_PiE_gen*)

proposition *compact_space_product_topology*:
compact_space(*product_topology X I*) $\longleftrightarrow$
(*product_topology X I*) = *trivial_topology* $\vee$ ($\forall$ *i* $\in$ *I*. *compact_space*(*X i*))
(**is** *?lhs* = *?rhs*)
proof (*cases* (*product_topology X I*) = *trivial_topology*)
case *False*
then obtain *z* **where** *z*: *z* $\in$ (Π_E *i*$\in$*I*. *topspace*(*X i*))
by (*auto simp flip*: *null_topspace_iff_trivial*)
show *?thesis*
proof
assume *L*: *?lhs*
show *?rhs*
proof (*clarsimp simp add*: *False compact_space_def*)
fix *i*
assume *i* $\in$ *I*
with *L* **have** *continuous_map* (*product_topology X I*) (*X i*) (λ*f*. *f i*)
by (*simp add*: *continuous_map_product_projection*)
moreover
have $\bigwedge$*x*. *x* $\in$ *topspace* (*X i*) $\Longrightarrow$ *x* $\in$ (λ*f*. *f i*) ` (Π_E *i*$\in$*I*. *topspace* (*X i*))
using ‹*i* $\in$ *I*› *z* **by** (*rule_tac x*=*z*(*i*:=*x*) **in** *image_eqI*) *auto*
then have (λ*f*. *f i*) ` (Π_E *i*$\in$*I*. *topspace* (*X i*)) = *topspace* (*X i*)
using ‹*i* $\in$ *I*› *z* **by** *auto*
ultimately show *compactin* (*X i*) (*topspace* (*X i*))
by (*metis L compact_space_def image_compactin topspace_product_topology*)
qed
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases I* = {})
case *True*
with *R* **show** *?thesis*
by (*simp add*: *compact_space_def*)
next
case *False*
then obtain *i* **where** *i* $\in$ *I*
by *blast*
show *?thesis*
using *R*
proof
assume *com* [*rule_format*]: $\forall$ *i*$\in$*I*. *compact_space* (*X i*)
let *?C* = {{*f*. *f i* $\in$ *U*} |*i U*. *i* $\in$ *I* $\wedge$ *openin* (*X i*) *U*}
show *compact_space* (*product_topology X I*)
proof (*rule Alexander_subbase_alt*)
show *topspace* (*product_topology X I*) $\subseteq$ $\bigcup$ *?C*
unfolding *topspace_product_topology* **using** ‹*i* $\in$ *I*› **by** *blast*

```
      next
        fix C
        assume Csub: C ⊆ ?𝒞 and UC: topspace (product_topology X I) ⊆ ⋃ C
        define 𝒟 where 𝒟 ≡ λi. {U. openin (X i) U ∧ {f. f i ∈ U} ∈ C}
       show ∃ C'. finite C' ∧ C' ⊆ C ∧ topspace (product_topology X I) ⊆ ⋃ C'
        proof (cases ∃ i. i ∈ I ∧ topspace (X i) ⊆ ⋃(𝒟 i))
          case True
          then obtain i where i ∈ I
                and i: topspace (X i) ⊆ ⋃(𝒟 i)
            unfolding 𝒟_def by blast
          then have ∗: ⋀𝒰. ⟦Ball 𝒰 (openin (X i)); topspace (X i) ⊆ ⋃𝒰⟧ ⟹
                       ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ topspace (X i) ⊆ ⋃ℱ
              using com [OF ‹i ∈ I›] by (auto simp: compact_space_def com-
pactin_def)
          have topspace (X i) ⊆ ⋃(𝒟 i)
            using i by auto
          with ∗ obtain ℱ where finite ℱ ∧ ℱ ⊆ (𝒟 i) ∧ topspace (X i) ⊆ ⋃ℱ
            unfolding 𝒟_def by fastforce
          with ‹i ∈ I› show ?thesis
            unfolding 𝒟_def
            by (rule_tac x=(λU. {x. x i ∈ U}) ` ℱ in exI) auto
        next
          case False
          then have ∀ i ∈ I. ∃ y. y ∈ topspace (X i) ∧ y ∉ ⋃(𝒟 i)
            by force
           then obtain g where g: ⋀i. i ∈ I ⟹ g i ∈ topspace (X i) ∧ g i ∉
⋃(𝒟 i)
            by metis
         then have (λi. if i ∈ I then g i else undefined) ∈ topspace (product_topology
X I)
            by (simp add: PiE_I)
          moreover have (λi. if i ∈ I then g i else undefined) ∉ ⋃ C
            using Csub g unfolding 𝒟_def by force
          ultimately show ?thesis
            using UC by blast
        qed
       qed (simp add: product_topology)
     qed simp
   qed
  qed
qed auto

corollary compactin_PiE:
   compactin (product_topology X I) (PiE I S) ⟷
        PiE I S = {} ∨ (∀ i ∈ I. compactin (X i) (S i))
  by (fastforce simp add: compactin_subspace subtopology_product_topology com-
pact_space_product_topology
      subset_PiE product_topology_trivial_iff subtopology_trivial_iff)
```

lemma *in_product_topology_closure_of*:
  $z \in$ (*product_topology X I*) *closure_of S*
    $\Longrightarrow i \in I \Longrightarrow z\ i \in$ ((*X i*) *closure_of* ((λ*x*. *x i*) ‘ *S*))
 using *continuous_map_product_projection*
 by (*force simp*: *continuous_map_eq_image_closure_subset image_subset_iff*)

lemma *homeomorphic_space_singleton_product*:
  product_topology X {*k*} *homeomorphic_space* (*X k*)
 unfolding *homeomorphic_space*
 apply (*rule_tac x*=λ*x*. *x k* **in** *exI*)
 apply (*rule bijective_open_imp_homeomorphic_map*)
  apply (*simp_all add*: *continuous_map_product_projection open_map_product_projection*)
 unfolding *PiE_over_singleton_iff*
  apply (*auto simp*: *image_iff inj_on_def*)
 done

2.5.4 Relationship with connected spaces, paths, etc.

proposition *connected_space_product_topology*:
  connected_space(*product_topology X I*) ⟷
   (∃ *i* ∈ *I*. *X i* = *trivial_topology*) ∨ (∀ *i* ∈ *I*. *connected_space*(*X i*))
 (**is** *?lhs* ⟷ *?eq* ∨ *?rhs*)
proof (*cases ?eq*)
 case *False*
 moreover have *?lhs* = *?rhs*
 proof
  assume *?lhs*
  moreover
  have *connectedin*(*X i*) (*topspace*(*X i*))
   if *i* ∈ *I* **and** *ci*: *connectedin*(*product_topology X I*) (*topspace*(*product_topology X I*)) **for** *i*
  proof −
   have *cm*: *continuous_map* (*product_topology X I*) (*X i*) (λ*f*. *f i*)
    by (*simp add*: ‹*i* ∈ *I*› *continuous_map_product_projection*)
   show *?thesis*
    using *connectedin_continuous_map_image* [*OF cm ci*] ‹*i* ∈ *I*›
    by (*simp add*: *False image_projection_PiE PiE_eq_empty_iff*)
  qed
  ultimately show *?rhs*
   by (*meson connectedin_topspace*)
 next
  assume *cs* [*rule_format*]: *?rhs*
  have *False*
   if *disj*: *U* ∩ *V* = {} **and** *subUV*: (Π$_E$ *i*∈*I*. *topspace* (*X i*)) ⊆ *U* ∪ *V*
    and *U*: *openin* (*product_topology X I*) *U*
    and *V*: *openin* (*product_topology X I*) *V*
    and *U* ≠ {} *V* ≠ {}
   for *U V*
  proof −

```
      obtain f where f ∈ U
        using ‹U ≠ {}› by blast
      then have f: f ∈ (Π_E i∈I. topspace (X i))
        using U openin_subset by fastforce
      have U ⊆ topspace(product_topology X I) V ⊆ topspace(product_topology X
I)
        using U V openin_subset by blast+
      moreover have (Π_E i∈I. topspace (X i)) ⊆ U
      proof −
        obtain C where (finite intersection_of (λF. ∃ i U. F = {x. x i ∈ U} ∧ i
∈ I ∧ openin (X i) U) relative_to
            (Π_E i∈I. topspace (X i))) C C ⊆ U f ∈ C
          using U ‹f ∈ U› unfolding openin_product_topology union_of_def by
auto
        then obtain T where finite T
         and t: ⋀C. C ∈ T ⟹ ∃ i u. (i ∈ I ∧ openin (X i) u) ∧ C = {x. x i ∈ u}
          and subU: topspace (product_topology X I) ∩ ⋂T ⊆ U
          and ftop: f ∈ topspace (product_topology X I)
          and fint: f ∈ ⋂ T
          by (fastforce simp: relative_to_def intersection_of_def subset_iff)
        let ?L = ⋃ C∈T. {i. (λx. x i) ‘ C ⊂ topspace (X i)}
        obtain L where finite L
          and L: ⋀i U. ⟦i ∈ I; openin (X i) U; U ⊂ topspace(X i); {x. x i ∈ U} ∈
T⟧ ⟹ i ∈ L
        proof
          show finite ?L
          proof (rule finite_Union)
            fix M
            assume M ∈ (λC. {i. (λx. x i) ‘ C ⊂ topspace (X i)}) ‘ T
           then obtain C where C ∈ T and C: M = {i. (λx. x i) ‘ C ⊂ topspace
(X i)}
              by blast
            then obtain j V where j ∈ I and ope: openin (X j) V and Ceq: C =
{x. x j ∈ V}
              using t by meson
            then have f j ∈ V
              using ‹C ∈ T› fint by force
            then have (λx. x k) ‘ {x. x j ∈ V} = UNIV if k ≠ j for k
              using that
              apply (clarsimp simp add: set_eq_iff)
              apply (rule_tac x=f(k:=x) in image_eqI, auto)
              done
            then have {i. (λx. x i) ‘ C ⊂ topspace (X i)} ⊆ {j}
              using Ceq by auto
            then show finite M
              using C finite_subset by fastforce
          qed (use ‹finite T› in blast)
        next
          fix i U
```

```
        assume  i ∈ I and ope: openin (X i) U and psub: U ⊂ topspace (X i)
and int: {x. x i ∈ U} ∈ 𝒯
        then show i ∈ ?L
          by (rule_tac a={x. x i ∈ U} in UN_I) (force+)
      qed
      show ?thesis
      proof
        fix h
        assume h: h ∈ (Π_E i∈I. topspace (X i))
        define g where g ≡ λi. if i ∈ L then f i else h i
        have gin: g ∈ (Π_E i∈I. topspace (X i))
          unfolding g_def using f h by auto
        moreover have g ∈ X if X ∈ 𝒯 for X
          using fint openin_subset t [OF that] L g_def h that by fastforce
        ultimately have g ∈ U
          using subU by auto
        have h ∈ U if finite M h ∈ PiE I (topspace ∘ X) {i ∈ I. h i ≠ g i} ⊆ M
for M h
          using that
        proof (induction arbitrary: h)
          case empty
          then show ?case
            using PiE_ext ‹g ∈ U› gin by force
        next
          case (insert i M)
          define f where f ≡ h(i:=g i)
          have fin: f ∈ PiE I (topspace ∘ X)
            unfolding f_def using gin insert.prems(1) by auto
          have subM: {j ∈ I. f j ≠ g j} ⊆ M
            unfolding f_def using insert.prems(2) by auto
          have f ∈ U
            using insert.IH [OF fin subM] .
          show ?case
          proof (cases h ∈ V)
            case True
            show ?thesis
            proof (cases i ∈ I)
              case True
              let ?U = {x ∈ topspace(X i). h(i:=x) ∈ U}
              let ?V = {x ∈ topspace(X i). h(i:=x) ∈ V}
              have False
              proof (rule connected_spaceD [OF cs [OF ‹i ∈ I›]])
               have ⋀k. k ∈ I ⟹ continuous_map (X i) (X k) (λx. if k = i then
x else h k)
                    using continuous_map_eq_topcontinuous_at insert.prems(1)
topcontinuous_at_def by fastforce
               then have cm: continuous_map (X i) (product_topology X I) (λx.
h(i:=x))
                 using ‹i ∈ I› insert.prems(1)
```

```
                by (auto simp: continuous_map_componentwise extensional_def)
              show openin (X i) ?U
                by (rule openin_continuous_map_preimage [OF cm U])
              show openin (X i) ?V
                by (rule openin_continuous_map_preimage [OF cm V])
              show topspace (X i) ⊆ ?U ∪ ?V
              proof clarsimp
                fix x
                assume x ∈ topspace (X i) and h(i:=x) ∉ V
                with True subUV ‹h ∈ Pi_E I (topspace ∘ X)›
                show h(i:=x) ∈ U
                  by (force dest: subsetD [where c=h(i:=x)])
              qed
              show ?U ∩ ?V = {}
                using disj by blast
              show ?U ≠ {}
                using True ‹f ∈ U› f_def gin by auto
              show ?V ≠ {}
                using True ‹h ∈ V› V openin_subset by fastforce
            qed
            then show ?thesis ..
          next
            case False
            show ?thesis
              using insert.prems(1) by (metis False gin PiE_E ‹f ∈ U› f_def
fun_upd_triv)
          qed
        next
          case False
          then show ?thesis
            using subUV insert.prems(1) by auto
        qed
      qed
      then show h ∈ U
        unfolding g_def using PiE_iff ‹finite L› h by fastforce
    qed
  qed
  ultimately show ?thesis
    using disj inf_absorb2 ‹V ≠ {}› by fastforce
 qed
 then show ?lhs
  unfolding connected_space_def
  by auto
qed
ultimately show ?thesis
 by simp
qed (metis connected_space_trivial_topology product_topology_trivial_iff)
```

lemma *connectedin_PiE*:
connectedin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨ (∀ *i* ∈ *I*. *connectedin* (*X i*) (*S i*))
by (*auto simp*: *connectedin_def subtopology_product_topology connected_space_product_topology subset_PiE*
PiE_eq_empty_iff subtopology_trivial_iff)

lemma *path_connected_space_product_topology*:
path_connected_space(*product_topology X I*) ⟷
topspace(*product_topology X I*) = {} ∨ (∀ *i* ∈ *I*. *path_connected_space*(*X i*))
(**is** *?lhs* ⟷ *?eq* ∨ *?rhs*)
proof (*cases ?eq*)
case *False*
moreover have *?lhs* = *?rhs*
proof
assume *L*: *?lhs*
show *?rhs*
proof (*clarsimp simp flip*: *path_connectedin_topspace*)
fix *i* :: *'a*
assume *i* ∈ *I*
have *cm*: *continuous_map* (*product_topology X I*) (*X i*) (λ*f*. *f i*)
by (*simp add*: ‹*i* ∈ *I*› *continuous_map_product_projection*)
show *path_connectedin* (*X i*) (*topspace* (*X i*))
using *path_connectedin_continuous_map_image* [*OF cm L* [*unfolded path_connectedin_topspace* [*symmetric*]]]
by (*metis* ‹*i* ∈ *I*› *False retraction_imp_surjective_map retraction_map_product_projection topspace_discrete_topology*)
qed
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
unfolding *path_connected_space_def topspace_product_topology*
proof *clarify*
fix *x y*
assume *x*: *x* ∈ (Π$_E$ *i*∈*I*. *topspace* (*X i*)) **and** *y*: *y* ∈ (Π$_E$ *i*∈*I*. *topspace* (*X i*))
have ∀ *i*. ∃ *g*. *i* ∈ *I* ⟶ *pathin* (*X i*) *g* ∧ *g* 0 = *x i* ∧ *g* 1 = *y i*
using *PiE_mem R path_connected_space_def x y* **by** *force*
then obtain *g* **where** *g*: ⋀*i*. *i* ∈ *I* ⟹ *pathin* (*X i*) (*g i*) ∧ *g i* 0 = *x i* ∧ *g i* 1 = *y i*
by *metis*
with *x y* **show** ∃ *g*. *pathin* (*product_topology X I*) *g* ∧ *g* 0 = *x* ∧ *g* 1 = *y*
apply (*rule_tac x*=λ*a*. λ*i* ∈ *I*. *g i a* **in** *exI*)
apply (*force simp*: *pathin_def continuous_map_componentwise*)
done
qed
qed
ultimately show *?thesis*
by *simp*

next
qed (*force simp*: *path_connected_space_topspace_empty iff*: *null_topspace_iff_trivial*)

lemma *path_connectedin_PiE*:
path_connectedin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨ (∀ *i* ∈ *I*. *path_connectedin* (*X i*) (*S i*))
by (*fastforce simp add*: *path_connectedin_def subtopology_product_topology path_connected_space_pro*
subset_PiE PiE_eq_empty_iff topspace_subtopology_subset)

2.5.5 Projections from a function topology to a component

lemma *quotient_map_product_projection*:
assumes *i* ∈ *I*
shows *quotient_map*(*product_topology X I*) (*X i*) (λ*x*. *x i*) ⟷
((*product_topology X I*) = *trivial_topology* ⟶ (*X i*) = *trivial_topology*)
by (*metis* (*no_types*) *assms image_is_empty null_topspace_iff_trivial quotient_imp_surjective_map*

retraction_imp_quotient_map retraction_map_product_projection)

lemma *product_topology_homeomorphic_component*:
assumes *i* ∈ *I* ⋀*j*. ⟦*j* ∈ *I*; *j* ≠ *i*⟧ ⟹ ∃ *a*. *topspace*(*X j*) = {*a*}
shows *product_topology X I homeomorphic_space* (*X i*)
proof −
have *quotient_map* (*product_topology X I*) (*X i*) (λ*x*. *x i*)
using *assms* **by** (*metis* (*full_types*) *discrete_topology_unique empty_not_insert*

product_topology_trivial_iff quotient_map_product_projection)
moreover
have *inj_on* (λ*x*. *x i*) (Π$_E$ *i*∈*I*. *topspace* (*X i*))
using *assms* **by** (*auto simp*: *inj_on_def PiE_iff*) (*metis extensionalityI singletonD*)
ultimately show *?thesis*
unfolding *homeomorphic_space_def*
by (*rule_tac x*=λ*x*. *x i* **in** *exI*) (*simp add*: *homeomorphic_map_def flip*: *homeomorphic_map_maps*)
qed

lemma *topological_property_of_product_component*:
assumes *major*: *P* (*product_topology X I*)
and *minor*: ⋀*z i*. ⟦*z* ∈ (Π$_E$ *i*∈*I*. *topspace*(*X i*)); *P*(*product_topology X I*); *i* ∈ *I*⟧
⟹ *P*(*subtopology* (*product_topology X I*) (*PiE I* (λ*j*. *if j* = *i* *then topspace*(*X i*) *else* {*z j*})))
(**is** ⋀*z i*. ⟦_; _; _⟧ ⟹ *P* (*?SX z i*))
and *PQ*: ⋀*X X*′. *X homeomorphic_space X*′ ⟹ (*P X* ⟷ *Q X*′)
shows (∃ *i*∈*I*. (*X i*) = *trivial_topology*) ∨ (∀ *i* ∈ *I*. *Q*(*X i*))
proof −
have *Q*(*X i*) **if** ∀ *i*∈*I*. (*X i*) ≠ *trivial_topology i* ∈ *I* **for** *i*
proof −

```
    from that obtain f where f: f ∈ (Π_E i∈I. topspace (X i))
      by (meson null_topspace_iff_trivial PiE_eq_empty_iff ex_in_conv)
    have ?SX f i homeomorphic_space X i
      using f product_topology_homeomorphic_component [OF ‹i ∈ I›, of λj.
subtopology (X j) (if j = i then topspace (X i) else {f j})]
      by (force simp add: subtopology_product_topology)
    then show ?thesis
      using minor [OF f major ‹i ∈ I›] PQ by auto
  qed
  then show ?thesis by metis
qed
```

2.5.6 Limits

The original HOL Light proof was a mess, yuk

```
lemma limitin_componentwise:
   limitin (product_topology X I) f l F ⟷
        l ∈ extensional I ∧
        eventually (λa. f a ∈ topspace(product_topology X I)) F ∧
        (∀ i ∈ I. limitin (X i) (λc. f c i) (l i) F)
    (is ?L ⟷ _ ∧ ?R1 ∧ ?R2)
proof (cases l ∈ extensional I)
  case l: True
  show ?thesis
  proof (cases ∀ i∈I. l i ∈ topspace (X i))
    case True
    have ?R1 if ?L
      by (metis limitin_subtopology subtopology_topspace that)
    moreover
    have ?R2 if ?L
      unfolding limitin_def
    proof (intro conjI strip)
      fix i U
      assume i ∈ I and U: openin (X i) U ∧ l i ∈ U
    then have openin (product_topology X I) ({y. y i ∈ U} ∩ topspace (product_topology
X I))
      unfolding openin_product_topology arbitrary_union_of_relative_to [symmetric]
        apply (simp add: relative_to_def topspace_product_topology_alt)
      by (smt (verit, del_insts) Collect_cong arbitrary_union_of_inc finite_intersection_of_inc
inf_commute)
      moreover have l ∈ {y. y i ∈ U} ∩ topspace (product_topology X I)
        using U True l by (auto simp: extensional_def)
    ultimately have eventually (λx. f x ∈ {y. y i ∈ U} ∩ topspace (product_topology
X I)) F
        by (metis limitin_def that)
      then show ∀_F x in F. f x i ∈ U
        by (simp add: eventually_conj_iff)
    qed (use True in auto)
    moreover
```

have *?L* **if** *R1*: *?R1* **and** *R2*: *?R2*
unfolding *limitin_def openin_product_topology all_union_of imp_conjL arbitrary_def*
proof (*intro conjI strip*)
show *l*: $l \in$ *topspace* (*product_topology X I*)
by (*simp add*: *PiE_iff True l*)
fix $\mathcal{V}$
assume $\mathcal{V} \subseteq$ *Collect* (*finite intersection_of* ($\lambda F.\ \exists i\ U.\ F = \{f.\ f\ i \in U\} \wedge i \in I \wedge$ *openin* (*X i*) *U*) *relative_to topspace* (*product_topology X I*))
and $l \in \bigcup\ \mathcal{V}$
then obtain $\mathcal{W}$ **where** *finite* $\mathcal{W}$ **and** $\mathcal{W}X$: $\forall\, X{\in}\mathcal{W}.\ l \in X$
and $\mathcal{W}$: $\bigwedge C.\ C \in \mathcal{W} \Longrightarrow C \in \{\{x.\ x\ i \in U\}\ |i\ U.\ i \in I \wedge$ *openin* (*X i*) *U*$\}$
and $\mathcal{W}\mathcal{V}$: *topspace* (*product_topology X I*) $\cap \bigcap\ \mathcal{W} \in \mathcal{V}$
by (*fastforce simp*: *intersection_of_def relative_to_def subset_eq*)
have $\forall_F\ x$ *in F*. *f x* $\in$ *topspace* (*product_topology X I*) $\cap \bigcap \mathcal{W}$
proof −
have $\bigwedge W.\ W \in \{\{x.\ x\ i \in U\} \mid i\ U.\ i \in I \wedge$ *openin* (*X i*) *U*$\} \Longrightarrow W \in \mathcal{W} \Longrightarrow \forall_F\ x$ *in F*. *f x* $\in$ *W*
using $\mathcal{W}X$ *R2* **by** (*auto simp*: *limitin_def*)
with $\mathcal{W}$ **have** $\forall_F\ x$ *in F*. $\forall\, W{\in}\mathcal{W}.$ *f x* $\in$ *W*
by (*simp add*: ‹*finite* $\mathcal{W}$› *eventually_ball_finite*)
with *R1* **show** *?thesis*
by (*simp add*: *eventually_conj_iff*)
qed
then show $\forall_F\ x$ *in F*. *f x* $\in \bigcup \mathcal{V}$
by (*smt* (*verit*, *ccfv_threshold*) $\mathcal{W}\mathcal{V}$ *UnionI eventually_mono*)
qed
ultimately show *?thesis*
using *l* **by** *blast*
next
case *False*
then show *?thesis*
by (*metis PiE_iff limitin_def topspace_product_topology*)
qed
next
case *False*
then show *?thesis*
by (*simp add*: *limitin_def PiE_iff*)
qed

end

2.6 The binary product topology

theory *Product_Topology*
imports *Function_Topology*
begin

2.7 Product Topology

2.7.1 Definition

definition *prod_topology* :: *'a topology* ⇒ *'b topology* ⇒ (*'a* × *'b*) *topology* **where**
prod_topology X Y ≡ *topology* (*arbitrary union_of* (λ*U*. *U* ∈ {*S* × *T* |*S T*. *openin X S* ∧ *openin Y T*}))

lemma *open_product_open*:
assumes *open A*
shows ∃*𝒰*. *𝒰* ⊆ {*S* × *T* |*S T*. *open S* ∧ *open T*} ∧ ⋃ *𝒰* = *A*
proof −
obtain *f g* **where** ∗: ⋀*u*. *u* ∈ *A* ⟹ *open* (*f u*) ∧ *open* (*g u*) ∧ *u* ∈ (*f u*) × (*g u*) ∧ (*f u*) × (*g u*) ⊆ *A*
using *open_prod_def* [*of A*] *assms* **by** *metis*
let *?𝒰* = (λ*u*. *f u* × *g u*) ‘ *A*
show *?thesis*
by (*rule_tac x*=*?𝒰* **in** *exI*) (*auto simp*: *dest*: ∗)
qed

lemma *open_product_open_eq*: (*arbitrary union_of* (λ*U*. ∃*S T*. *U* = *S* × *T* ∧ *open S* ∧ *open T*)) = *open*
by (*force simp*: *union_of_def arbitrary_def intro*: *open_product_open open_Times*)

lemma *openin_prod_topology*:
openin (*prod_topology X Y*) = *arbitrary union_of* (λ*U*. *U* ∈ {*S* × *T* |*S T*. *openin X S* ∧ *openin Y T*})
unfolding *prod_topology_def*
proof (*rule topology_inverse'*)
show *istopology* (*arbitrary union_of* (λ*U*. *U* ∈ {*S* × *T* |*S T*. *openin X S* ∧ *openin Y T*}))
apply (*rule istopology_base*, *simp*)
by (*metis openin_Int Times_Int_Times*)
qed

lemma *topspace_prod_topology* [*simp*]:
topspace (*prod_topology X Y*) = *topspace X* × *topspace Y*
proof −
have *topspace*(*prod_topology X Y*) = ⋃ (*Collect* (*openin* (*prod_topology X Y*))) (**is** *_* = *?Z*)
unfolding *topspace_def* **..**
also have ... = *topspace X* × *topspace Y*
proof
show *?Z* ⊆ *topspace X* × *topspace Y*
apply (*auto simp*: *openin_prod_topology union_of_def arbitrary_def*)
using *openin_subset* **by** *force*+
next
have ∗: ∃*A B*. *topspace X* × *topspace Y* = *A* × *B* ∧ *openin X A* ∧ *openin Y B*
by *blast*

show *topspace X* × *topspace Y* ⊆ *?Z*
apply (*rule Union_upper*)
using ∗ **by** (*simp add*: *openin_prod_topology arbitrary_union_of_inc*)
qed
finally show *?thesis* **.**
qed

lemma *prod_topology_trivial_iff* [*simp*]:
prod_topology X Y = *trivial_topology* ⟷ *X* = *trivial_topology* ∨ *Y* = *trivial_topology*
by (*metis* (*full_types*) *Sigma_empty1 null_topspace_iff_trivial subset_empty times_subset_iff topspace_prod_topology*)

lemma *subtopology_Times*:
shows *subtopology* (*prod_topology X Y*) (*S* × *T*) = *prod_topology* (*subtopology X S*) (*subtopology Y T*)
proof −
have ((λ*U*. ∃ *S T*. *U* = *S* × *T* ∧ *openin X S* ∧ *openin Y T*) *relative_to S* × *T*) =
(λ*U*. ∃ *S′ T′*. *U* = *S′* × *T′* ∧ (*openin X relative_to S*) *S′* ∧ (*openin Y relative_to T*) *T′*)
by (*auto simp*: *relative_to_def Times_Int_Times fun_eq_iff*) *metis*
then show *?thesis*
by (*simp add*: *topology_eq openin_prod_topology arbitrary_union_of_relative_to flip*: *openin_relative_to*)
qed

lemma *prod_topology_subtopology*:
prod_topology (*subtopology X S*) *Y* = *subtopology* (*prod_topology X Y*) (*S* × *topspace Y*)
prod_topology X (*subtopology Y T*) = *subtopology* (*prod_topology X Y*) (*topspace X* × *T*)
by (*auto simp*: *subtopology_Times*)

lemma *prod_topology_discrete_topology*:
discrete_topology (*S* × *T*) = *prod_topology* (*discrete_topology S*) (*discrete_topology T*)
by (*auto simp*: *discrete_topology_unique openin_prod_topology intro*: *arbitrary_union_of_inc*)

lemma *prod_topology_euclidean* [*simp*]: *prod_topology euclidean euclidean* = *euclidean*
by (*simp add*: *prod_topology_def open_product_open_eq*)

lemma *prod_topology_subtopology_eu* [*simp*]:
prod_topology (*subtopology euclidean S*) (*subtopology euclidean T*) = *subtopology euclidean* (*S* × *T*)
by (*simp add*: *prod_topology_subtopology subtopology_subtopology Times_Int_Times*)

lemma *openin_prod_topology_alt*:

openin (*prod_topology X Y*) *S* ⟷
(∀ *x y*. (*x*,*y*) ∈ *S* ⟶ (∃ *U V*. *openin X U* ∧ *openin Y V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *U* × *V* ⊆ *S*))
apply (*auto simp*: *openin_prod_topology arbitrary_union_of_alt*, *fastforce*)
by (*metis mem_Sigma_iff*)

lemma *open_map_fst*: *open_map* (*prod_topology X Y*) *X fst*
unfolding *open_map_def openin_prod_topology_alt*
by (*force simp*: *openin_subopen* [*of X fst* ‘ _] *intro*: *subset_fst_imageI*)

lemma *open_map_snd*: *open_map* (*prod_topology X Y*) *Y snd*
unfolding *open_map_def openin_prod_topology_alt*
by (*force simp*: *openin_subopen* [*of Y snd* ‘ _] *intro*: *subset_snd_imageI*)

lemma *openin_prod_Times_iff*:
openin (*prod_topology X Y*) (*S* × *T*) ⟷ *S* = {} ∨ *T* = {} ∨ *openin X S* ∧ *openin Y T*
proof (*cases S* = {} ∨ *T* = {})
case *False*
then show *?thesis*
apply (*simp add*: *openin_prod_topology_alt openin_subopen* [*of X S*] *openin_subopen* [*of Y T*] *times_subset_iff*, *safe*)
apply (*meson*|*force*)+
done
qed *force*

lemma *closure_of_Times*:
(*prod_topology X Y*) *closure_of* (*S* × *T*) = (*X closure_of S*) × (*Y closure_of T*) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*clarsimp simp*: *closure_of_def openin_prod_topology_alt*) *blast*
show *?rhs* ⊆ *?lhs*
by (*clarsimp simp*: *closure_of_def openin_prod_topology_alt*) (*meson SigmaI subsetD*)
qed

lemma *closedin_prod_Times_iff*:
closedin (*prod_topology X Y*) (*S* × *T*) ⟷ *S* = {} ∨ *T* = {} ∨ *closedin X S* ∧ *closedin Y T*
by (*auto simp*: *closure_of_Times times_eq_iff simp flip*: *closure_of_eq*)

lemma *interior_of_Times*: (*prod_topology X Y*) *interior_of* (*S* × *T*) = (*X interior_of S*) × (*Y interior_of T*)
proof (*rule interior_of_unique*)
show (*X interior_of S*) × *Y interior_of T* ⊆ *S* × *T*
by (*simp add*: *Sigma_mono interior_of_subset*)
show *openin* (*prod_topology X Y*) ((*X interior_of S*) × *Y interior_of T*)

```
    by (simp add: openin_prod_Times_iff)
next
  show T' ⊆ (X interior_of S) × Y interior_of T if T' ⊆ S × T openin
(prod_topology X Y) T' for T'
  proof (clarsimp; intro conjI)
    fix a :: 'a and b :: 'b
    assume (a, b) ∈ T'
    with that obtain U V where UV: openin X U openin Y V a ∈ U b ∈ V U ×
V ⊆ T'
      by (metis openin_prod_topology_alt)
    then show a ∈ X interior_of S
      using interior_of_maximal_eq that(1) by fastforce
    show b ∈ Y interior_of T
      using UV interior_of_maximal_eq that(1)
      by (metis SigmaI mem_Sigma_iff subset_eq)
  qed
qed
```

Missing the opposite direction. Does it hold? A converse is proved for proper maps, a stronger condition

```
lemma closed_map_prod:
  assumes closed_map (prod_topology X Y) (prod_topology X' Y') (λ(x,y). (f x,
g y))
  shows (prod_topology X Y) = trivial_topology ∨ closed_map X X' f ∧ closed_map
Y Y' g
proof (cases (prod_topology X Y) = trivial_topology)
  case False
  then have ne: topspace X ≠ {} topspace Y ≠ {}
    by (auto simp flip: null_topspace_iff_trivial)
  have closed_map X X' f
    unfolding closed_map_def
  proof (intro strip)
    fix C
    assume closedin X C
    show closedin X' (f ‘ C)
    proof (cases C={})
      case False
      with assms have closedin (prod_topology X' Y') ((λ(x,y). (f x, g y)) ‘ (C ×
topspace Y))
        by (simp add: ‹closedin X C› closed_map_def closedin_prod_Times_iff)
      with False ne show ?thesis
      by (simp add: image_paired_Times closedin_Times closedin_prod_Times_iff)
    qed auto
  qed
  moreover
  have closed_map Y Y' g
    unfolding closed_map_def
  proof (intro strip)
    fix C
```

```
    assume closedin Y C
    show closedin Y' (g ‘ C)
    proof (cases C={})
      case False
        with assms have closedin (prod_topology X' Y') ((λ(x,y). (f x, g y)) ‘
(topspace X × C))
        by (simp add: ‹closedin Y C› closed_map_def closedin_prod_Times_iff)
      with False ne show ?thesis
      by (simp add: image_paired_Times closedin_Times closedin_prod_Times_iff)
    qed auto
  qed
  ultimately show ?thesis
    by (auto simp: False)
qed auto
```

2.7.2 Continuity

```
lemma continuous_map_pairwise:
   continuous_map Z (prod_topology X Y) f ⟷ continuous_map Z X (fst ∘ f)
∧ continuous_map Z Y (snd ∘ f)
   (is ?lhs = ?rhs)
proof −
  let ?g = fst ∘ f and ?h = snd ∘ f
  have f: f x = (?g x, ?h x) for x
    by auto
  show ?thesis
  proof (cases ?g ∈ topspace Z → topspace X ∧ ?h ∈ topspace Z → topspace Y)
    case True
    show ?thesis
    proof safe
      assume continuous_map Z (prod_topology X Y) f
      then have openin Z {x ∈ topspace Z. fst (f x) ∈ U} if openin X U for U
        unfolding continuous_map_def using True that
        apply clarify
        apply (drule_tac x=U × topspace Y in spec)
           by (auto simp: openin_prod_Times_iff mem_Times_iff Pi_iff cong:
conj_cong)
      with True show continuous_map Z X (fst ∘ f)
        by (auto simp: continuous_map_def)
    next
      assume continuous_map Z (prod_topology X Y) f
      then have openin Z {x ∈ topspace Z. snd (f x) ∈ V} if openin Y V for V
        unfolding continuous_map_def using True that
        apply clarify
        apply (drule_tac x=topspace X × V in spec)
      by (simp add: openin_prod_Times_iff mem_Times_iff Pi_iff cong: conj_cong)
      with True show continuous_map Z Y (snd ∘ f)
        by (auto simp: continuous_map_def)
    next
```

assume *Z*: *continuous_map Z X* (*fst* ∘ *f*) *continuous_map Z Y* (*snd* ∘ *f*)
have ∗: *openin Z* {*x* ∈ *topspace Z*. *f x* ∈ *W*}
if ⋀*w*. *w* ∈ *W* ⟹ ∃ *U V*. *openin X U* ∧ *openin Y V* ∧ *w* ∈ *U* × *V* ∧ *U* × *V* ⊆ *W* **for** *W*
proof (*subst openin_subopen*, *clarify*)
fix *x* :: *'a*
assume *x* ∈ *topspace Z* **and** *f x* ∈ *W*
with *that* [*OF* ‹*f x* ∈ *W*›]
obtain *U V* **where** *UV*: *openin X U openin Y V f x* ∈ *U* × *V U* × *V* ⊆ *W*
by *auto*
with *Z UV* **show** ∃ *T*. *openin Z T* ∧ *x* ∈ *T* ∧ *T* ⊆ {*x* ∈ *topspace Z*. *f x* ∈ *W*}
apply (*rule_tac x*={*x* ∈ *topspace Z*. *?g x* ∈ *U*} ∩ {*x* ∈ *topspace Z*. *?h x* ∈ *V*} **in** *exI*)
apply (*auto simp*: ‹*x* ∈ *topspace Z*› *continuous_map_def*)
done
qed
show *continuous_map Z* (*prod_topology X Y*) *f*
using *True* **by** (*force simp*: *continuous_map_def openin_prod_topology_alt mem_Times_iff* ∗)
qed
qed (*force simp*: *continuous_map_def*)
qed

lemma *continuous_map_paired*:
continuous_map Z (*prod_topology X Y*) (λ*x*. (*f x*,*g x*)) ⟷ *continuous_map Z X f* ∧ *continuous_map Z Y g*
by (*simp add*: *continuous_map_pairwise o_def*)

lemma *continuous_map_pairedI* [*continuous_intros*]:
⟦*continuous_map Z X f*; *continuous_map Z Y g*⟧ ⟹ *continuous_map Z* (*prod_topology X Y*) (λ*x*. (*f x*,*g x*))
by (*simp add*: *continuous_map_pairwise o_def*)

lemma *continuous_map_fst* [*continuous_intros*]: *continuous_map* (*prod_topology X Y*) *X fst*
using *continuous_map_pairwise* [*of prod_topology X Y X Y id*]
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_snd* [*continuous_intros*]: *continuous_map* (*prod_topology X Y*) *Y snd*
using *continuous_map_pairwise* [*of prod_topology X Y X Y id*]
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_fst_of* [*continuous_intros*]:
continuous_map Z (*prod_topology X Y*) *f* ⟹ *continuous_map Z X* (*fst* ∘ *f*)
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_snd_of* [*continuous_intros*]:

continuous_map Z (prod_topology X Y) f ⟹ continuous_map Z Y (snd ∘ f)
by (*simp add*: *continuous_map_pairwise*)

lemma *continuous_map_prod_fst*:
i ∈ I ⟹ continuous_map (prod_topology (product_topology (λi. Y) I) X) Y (λx. fst x i)
using *continuous_map_componentwise_UNIV continuous_map_fst* **by** *fastforce*

lemma *continuous_map_prod_snd*:
i ∈ I ⟹ continuous_map (prod_topology X (product_topology (λi. Y) I)) Y (λx. snd x i)
using *continuous_map_componentwise_UNIV continuous_map_snd* **by** *fastforce*

lemma *continuous_map_if_iff* [*simp*]: *continuous_map X Y (λx. if P then f x else g x) ⟷ continuous_map X Y (if P then f else g)*
by *simp*

lemma *continuous_map_if* [*continuous_intros*]: *⟦P ⟹ continuous_map X Y f; ~P ⟹ continuous_map X Y g⟧*
⟹ continuous_map X Y (λx. if P then f x else g x)
by *simp*

lemma *prod_topology_trivial1* [*simp*]: *prod_topology trivial_topology Y = trivial_topology*
using *continuous_map_fst continuous_map_on_empty2* **by** *blast*

lemma *prod_topology_trivial2* [*simp*]: *prod_topology X trivial_topology = trivial_topology*
using *continuous_map_snd continuous_map_on_empty2* **by** *blast*

lemma *continuous_map_subtopology_fst* [*continuous_intros*]: *continuous_map (subtopology (prod_topology X Y) Z) X fst*
using *continuous_map_from_subtopology continuous_map_fst* **by** *force*

lemma *continuous_map_subtopology_snd* [*continuous_intros*]: *continuous_map (subtopology (prod_topology X Y) Z) Y snd*
using *continuous_map_from_subtopology continuous_map_snd* **by** *force*

lemma *quotient_map_fst* [*simp*]:
quotient_map(prod_topology X Y) X fst ⟷ (Y = trivial_topology ⟶ X = trivial_topology)
apply (*simp add*: *continuous_open_quotient_map open_map_fst continuous_map_fst*)
by (*metis null_topspace_iff_trivial*)

lemma *quotient_map_snd* [*simp*]:
quotient_map(prod_topology X Y) Y snd ⟷ (X = trivial_topology ⟶ Y = trivial_topology)
apply (*simp add*: *continuous_open_quotient_map open_map_snd continuous_map_snd*)

by (*metis null_topspace_iff_trivial*)

lemma *retraction_map_fst*:
retraction_map (*prod_topology X Y*) *X fst* ⟷ (*Y* = *trivial_topology* ⟶ *X* = *trivial_topology*)
proof (*cases Y* = *trivial_topology*)
case *True*
then show *?thesis*
using *continuous_map_image_subset_topspace*
by (*auto simp*: *retraction_map_def retraction_maps_def continuous_map_pairwise*)
next
case *False*
have ∃ *g*. *continuous_map X* (*prod_topology X Y*) *g* ∧ (∀ *x*∈*topspace X*. *fst* (*g x*) = *x*)
if *y*: *y* ∈ *topspace Y* **for** *y*
by (*rule_tac x*=λ*x*. (*x*,*y*) **in** *exI*) (*auto simp*: *y continuous_map_paired*)
with *False* **have** *retraction_map* (*prod_topology X Y*) *X fst*
by (*fastforce simp*: *retraction_map_def retraction_maps_def continuous_map_fst*)
with *False* **show** *?thesis*
by *simp*
qed

lemma *retraction_map_snd*:
retraction_map (*prod_topology X Y*) *Y snd* ⟷ (*X* = *trivial_topology* ⟶ *Y* = *trivial_topology*)
proof (*cases X* = *trivial_topology*)
case *True*
then show *?thesis*
using *continuous_map_image_subset_topspace*
by (*fastforce simp*: *retraction_map_def retraction_maps_def continuous_map_fst*)
next
case *False*
have ∃ *g*. *continuous_map Y* (*prod_topology X Y*) *g* ∧ (∀ *y*∈*topspace Y*. *snd* (*g y*) = *y*)
if *x*: *x* ∈ *topspace X* **for** *x*
by (*rule_tac x*=λ*y*. (*x*,*y*) **in** *exI*) (*auto simp*: *x continuous_map_paired*)
with *False* **have** *retraction_map* (*prod_topology X Y*) *Y snd*
by (*fastforce simp*: *retraction_map_def retraction_maps_def continuous_map_snd*)
with *False* **show** *?thesis*
by *simp*
qed

lemma *continuous_map_of_fst*:
continuous_map (*prod_topology X Y*) *Z* (*f* ∘ *fst*) ⟷ *Y* = *trivial_topology* ∨ *continuous_map X Z f*
proof (*cases Y* = *trivial_topology*)
case *True*
then show *?thesis*

by (*simp add*: *continuous_map_on_empty*)
next
case *False*
then show *?thesis*
by (*simp add*: *continuous_compose_quotient_map_eq*)
qed

lemma *continuous_map_of_snd*:
continuous_map (*prod_topology X Y*) *Z* (*f* ∘ *snd*) ⟷ *X* = *trivial_topology* ∨ *continuous_map Y Z f*
proof (*cases X* = *trivial_topology*)
case *True*
then show *?thesis*
by (*simp add*: *continuous_map_on_empty*)
next
case *False*
then show *?thesis*
by (*simp add*: *continuous_compose_quotient_map_eq*)
qed

lemma *continuous_map_prod_top*:
continuous_map (*prod_topology X Y*) (*prod_topology X′ Y′*) (λ(*x*,*y*). (*f x*, *g y*)) ⟷
(*prod_topology X Y*) = *trivial_topology* ∨ *continuous_map X X′ f* ∧ *continuous_map Y Y′ g*
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
case *False*
then show *?thesis*
by (*auto simp*: *continuous_map_paired case_prod_unfold*
continuous_map_of_fst [*unfolded o_def*] *continuous_map_of_snd* [*unfolded o_def*])
qed *auto*

lemma *in_prod_topology_closure_of*:
assumes *z* ∈ (*prod_topology X Y*) *closure_of S*
shows *fst z* ∈ *X closure_of* (*fst* ‘ *S*) *snd z* ∈ *Y closure_of* (*snd* ‘ *S*)
using *assms continuous_map_eq_image_closure_subset continuous_map_fst*
apply *fastforce*
using *assms continuous_map_eq_image_closure_subset continuous_map_snd*
apply *fastforce*
done

proposition *compact_space_prod_topology*:
compact_space(*prod_topology X Y*) ⟷ (*prod_topology X Y*) = *trivial_topology* ∨ *compact_space X* ∧ *compact_space Y*
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
case *True*
then show *?thesis*

```
    by fastforce
next
  case False
  then have non_mt: topspace X ≠ {} topspace Y ≠ {}
    by auto
  have compact_space X compact_space Y if compact_space(prod_topology X Y)
  proof −
    have compactin X (fst ‘ (topspace X × topspace Y))
      by (metis compact_space_def continuous_map_fst image_compactin that
topspace_prod_topology)
    moreover
    have compactin Y (snd ‘ (topspace X × topspace Y))
      by (metis compact_space_def continuous_map_snd image_compactin that
topspace_prod_topology)
    ultimately show compact_space X compact_space Y
      using non_mt by (auto simp: compact_space_def)
  qed
  moreover
  define 𝒳 where 𝒳 ≡ (λV. topspace X × V) ‘ Collect (openin Y)
  define 𝒴 where 𝒴 ≡ (λU. U × topspace Y) ‘ Collect (openin X)
  have compact_space(prod_topology X Y) if compact_space X compact_space Y
  proof (rule Alexander_subbase_alt)
    show toptop: topspace X × topspace Y ⊆ ⋃(𝒳 ∪ 𝒴)
      unfolding 𝒳_def 𝒴_def by auto
    fix 𝒞 :: ('a × 'b) set set
    assume 𝒞: 𝒞 ⊆ 𝒳 ∪ 𝒴 topspace X × topspace Y ⊆ ⋃𝒞
    then obtain 𝒳' 𝒴' where XY: 𝒳' ⊆ 𝒳 𝒴' ⊆ 𝒴 and 𝒞eq: 𝒞 = 𝒳' ∪ 𝒴'
      using subset_UnE by metis
    then have sub: topspace X × topspace Y ⊆ ⋃(𝒳' ∪ 𝒴')
      using 𝒞 by simp
    obtain 𝒰 𝒱 where 𝒰: ⋀U. U ∈ 𝒰 ⟹ openin X U 𝒴' = (λU. U × topspace
Y) ‘ 𝒰
      and 𝒱: ⋀V. V ∈ 𝒱 ⟹ openin Y V 𝒳' = (λV. topspace X × V) ‘ 𝒱
      using XY by (clarsimp simp add: 𝒳_def 𝒴_def subset_image_iff) (force
simp: subset_iff)
    have ∃𝒟. finite 𝒟 ∧ 𝒟 ⊆ 𝒳' ∪ 𝒴' ∧ topspace X × topspace Y ⊆ ⋃ 𝒟
    proof −
      have topspace X ⊆ ⋃𝒰 ∨ topspace Y ⊆ ⋃𝒱
        using 𝒰 𝒱 𝒞 𝒞eq by auto
      then have ∗: ∃𝒟. finite 𝒟 ∧
            (∀x ∈ 𝒟. x ∈ (λV. topspace X × V) ‘ 𝒱 ∨ x ∈ (λU. U × topspace
Y) ‘ 𝒰) ∧
            (topspace X × topspace Y ⊆ ⋃𝒟)
      proof
        assume topspace X ⊆ ⋃𝒰
        with ‹compact_space X› 𝒰 obtain ℱ where finite ℱ ℱ ⊆ 𝒰 topspace X
⊆ ⋃ℱ
          by (meson compact_space_alt)
        with that show ?thesis
```

```
        by (rule_tac x=(λD. D × topspace Y) ` ℱ in exI) auto
    next
      assume topspace Y ⊆ ⋃𝒱
       with ‹compact_space Y› 𝒱 obtain ℱ where finite ℱ ℱ ⊆ 𝒱 topspace Y
⊆ ⋃ℱ
        by (meson compact_space_alt)
      with that show ?thesis
        by (rule_tac x=(λC. topspace X × C) ` ℱ in exI) auto
    qed
    then show ?thesis
      using that 𝒰 𝒱 by blast
  qed
  then show ∃𝒟. finite 𝒟 ∧ 𝒟 ⊆ 𝒞 ∧ topspace X × topspace Y ⊆ ⋃ 𝒟
    using 𝒞 𝒞eq by blast
next
  have (finite intersection_of (λx. x ∈ 𝒳 ∨ x ∈ 𝒴) relative_to topspace X ×
topspace Y)
        = (λU. ∃S T. U = S × T ∧ openin X S ∧ openin Y T)
    (is ?lhs = ?rhs)
  proof −
    have ?rhs U if ?lhs U for U
    proof −
     have topspace X × topspace Y ∩ ⋂ T ∈ {A × B |A B. A ∈ Collect (openin
X) ∧ B ∈ Collect (openin Y)}
       if finite T T ⊆ 𝒳 ∪ 𝒴 for T
       using that
      proof induction
        case (insert B ℬ)
        then show ?case
          unfolding 𝒳_def 𝒴_def
          apply (simp add: Int_ac subset_eq image_def)
        apply (metis (no_types) openin_Int openin_topspace Times_Int_Times)
          done
      qed auto
      then show ?thesis
        using that
        by (auto simp: subset_eq  elim!: relative_toE intersection_ofE)
    qed
    moreover
    have ?lhs Z if Z: ?rhs Z for Z
    proof −
      obtain U V where Z = U × V openin X U openin Y V
        using Z by blast
      then have UV: U × V = (topspace X × topspace Y) ∩ (U × V)
        by (simp add: Sigma_mono inf_absorb2 openin_subset)
      moreover
      have ?lhs ((topspace X × topspace Y) ∩ (U × V))
      proof (rule relative_to_inc)
        show (finite intersection_of (λx. x ∈ 𝒳 ∨ x ∈ 𝒴)) (U × V)
```

apply (*simp add*: *intersection_of_def* $\mathcal{X}$*_def* $\mathcal{Y}$*_def*)
apply (*rule_tac x*={(*U* × *topspace Y*),(*topspace X* × *V*)} **in** *exI*)
using ‹*openin X U*› ‹*openin Y V*› *openin_subset UV* **apply** (*fastforce simp*:)
done
qed
ultimately show *?thesis*
using ‹*Z* = *U* × *V*› **by** *auto*
qed
ultimately show *?thesis*
by *meson*
qed
then show *topology* (*arbitrary union_of* (*finite intersection_of* (λ*x*. *x* ∈ $\mathcal{X}$ ∪ $\mathcal{Y}$)
relative_to (*topspace X* × *topspace Y*))) =
prod_topology X Y
by (*simp add*: *prod_topology_def*)
qed
ultimately show *?thesis*
using *False* **by** *blast*
qed

lemma *compactin_Times*:
compactin (*prod_topology X Y*) (*S* × *T*) ⟷ *S* = {} ∨ *T* = {} ∨ *compactin X S* ∧ *compactin Y T*
by (*auto simp*: *compactin_subspace subtopology_Times compact_space_prod_topology subtopology_trivial_iff*)

2.7.3 Homeomorphic maps

lemma *homeomorphic_maps_prod*:
homeomorphic_maps (*prod_topology X Y*) (*prod_topology X′ Y′*) (λ(*x*,*y*). (*f x*, *g y*)) (λ(*x*,*y*). (*f′ x*, *g′ y*)) ⟷
(*prod_topology X Y*) = *trivial_topology* ∧ (*prod_topology X′ Y′*) = *trivial_topology*
∨ *homeomorphic_maps X X′ f f′* ∧ *homeomorphic_maps Y Y′ g g′* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*fastforce simp*: *homeomorphic_maps_def continuous_map_prod_top ball_conj_distrib*)
next
show *?rhs* ⟹ *?lhs*
by (*auto simp*: *homeomorphic_maps_def continuous_map_prod_top*)
qed

lemma *homeomorphic_maps_swap*:
homeomorphic_maps (*prod_topology X Y*) (*prod_topology Y X*) (λ(*x*,*y*). (*y*,*x*)) (λ(*y*,*x*). (*x*,*y*))

by (*auto simp*: *homeomorphic_maps_def case_prod_unfold continuous_map_fst continuous_map_pairedI continuous_map_snd*)

lemma *homeomorphic_map_swap*:
homeomorphic_map (*prod_topology X Y*) (*prod_topology Y X*) (λ(*x*,*y*). (*y*,*x*))
using *homeomorphic_map_maps homeomorphic_maps_swap* **by** *metis*

lemma *homeomorphic_space_prod_topology_swap*:
(*prod_topology X Y*) *homeomorphic_space* (*prod_topology Y X*)
using *homeomorphic_map_swap homeomorphic_space* **by** *blast*

lemma *embedding_map_graph*:
embedding_map X (*prod_topology X Y*) (λ*x*. (*x*, *f x*)) ⟷ *continuous_map X Y f*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
have *snd* ∘ (λ*x*. (*x*, *f x*)) = *f*
by *force*
moreover have *continuous_map X Y* (*snd* ∘ (λ*x*. (*x*, *f x*)))
using *L* **unfolding** *embedding_map_def*
by (*meson continuous_map_in_subtopology continuous_map_snd_of homeomorphic_imp_continuous_map*)
ultimately show *?rhs*
by *simp*
next
assume *R*: *?rhs*
then show *?lhs*
unfolding *homeomorphic_map_maps embedding_map_def homeomorphic_maps_def*
by (*rule_tac x*=*fst* **in** *exI*)
(*auto simp*: *continuous_map_in_subtopology continuous_map_paired continuous_map_from_subtopology*
continuous_map_fst)
qed

lemma *homeomorphic_space_prod_topology*:
⟦*X homeomorphic_space X''*; *Y homeomorphic_space Y'*⟧
⟹ *prod_topology X Y homeomorphic_space prod_topology X'' Y'*
using *homeomorphic_maps_prod* **unfolding** *homeomorphic_space_def* **by** *blast*

lemma *prod_topology_homeomorphic_space_left*:
Y = *discrete_topology* {*b*} ⟹ *prod_topology X Y homeomorphic_space X*
unfolding *homeomorphic_space_def*
apply (*rule_tac x*=*fst* **in** *exI*)
apply (*simp add*: *homeomorphic_map_def inj_on_def discrete_topology_unique flip*: *homeomorphic_map_maps*)
done

lemma *prod_topology_homeomorphic_space_right*:

$X = \mathit{discrete\_topology}\ \{a\} \Longrightarrow \mathit{prod\_topology}\ X\ Y\ \mathit{homeomorphic\_space}\ Y$
unfolding *homeomorphic_space_def*
by (*meson homeomorphic_space_def homeomorphic_space_prod_topology_swap homeomorphic_space_trans prod_topology_homeomorphic_space_left*)

lemma *homeomorphic_space_prod_topology_sing1*:
$b \in \mathit{topspace}\ Y \Longrightarrow X\ \mathit{homeomorphic\_space}\ (\mathit{prod\_topology}\ X\ (\mathit{subtopology}\ Y\ \{b\}))$
by (*metis empty_subsetI homeomorphic_space_sym insert_subset prod_topology_homeomorphic_space subtopology_eq_discrete_topology_sing topspace_subtopology_subset*)

lemma *homeomorphic_space_prod_topology_sing2*:
$a \in \mathit{topspace}\ X \Longrightarrow Y\ \mathit{homeomorphic\_space}\ (\mathit{prod\_topology}\ (\mathit{subtopology}\ X\ \{a\})\ Y)$
by (*metis empty_subsetI homeomorphic_space_sym insert_subset prod_topology_homeomorphic_space subtopology_eq_discrete_topology_sing topspace_subtopology_subset*)

lemma *topological_property_of_prod_component*:
assumes *major*: $P(\mathit{prod\_topology}\ X\ Y)$
and X: $\bigwedge x.\ [\![x \in \mathit{topspace}\ X;\ P(\mathit{prod\_topology}\ X\ Y)]\!] \Longrightarrow P(\mathit{subtopology}\ (\mathit{prod\_topology}\ X\ Y)\ (\{x\} \times \mathit{topspace}\ Y))$
and Y: $\bigwedge y.\ [\![y \in \mathit{topspace}\ Y;\ P(\mathit{prod\_topology}\ X\ Y)]\!] \Longrightarrow P(\mathit{subtopology}\ (\mathit{prod\_topology}\ X\ Y)\ (\mathit{topspace}\ X \times \{y\}))$
and *PQ*: $\bigwedge X\ X'.\ X\ \mathit{homeomorphic\_space}\ X' \Longrightarrow (P\ X \longleftrightarrow Q\ X')$
and *PR*: $\bigwedge X\ X'.\ X\ \mathit{homeomorphic\_space}\ X' \Longrightarrow (P\ X \longleftrightarrow R\ X')$
shows $(\mathit{prod\_topology}\ X\ Y) = \mathit{trivial\_topology} \vee Q\ X \wedge R\ Y$
proof −
have $Q\ X \wedge R\ Y$ **if** $\mathit{topspace}(\mathit{prod\_topology}\ X\ Y) \neq \{\}$
proof −
from *that* **obtain** $a\ b$ **where** a: $a \in \mathit{topspace}\ X$ **and** b: $b \in \mathit{topspace}\ Y$
by *force*
show *?thesis*
using X [*OF a major*] **and** Y [*OF b major*] *homeomorphic_space_prod_topology_sing1* [*OF b, of X*] *homeomorphic_space_prod_topology_sing2* [*OF a, of Y*]
by (*simp add*: *subtopology_Times*) (*meson PQ PR homeomorphic_space_prod_topology_sing2 homeomorphic_space_sym*)
qed
then show *?thesis* **by** *force*
qed

lemma *limitin_pairwise*:
$\mathit{limitin}\ (\mathit{prod\_topology}\ X\ Y)\ f\ l\ F \longleftrightarrow \mathit{limitin}\ X\ (\mathit{fst} \circ f)\ (\mathit{fst}\ l)\ F \wedge \mathit{limitin}\ Y\ (\mathit{snd} \circ f)\ (\mathit{snd}\ l)\ F$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain $a\ b$ **where** *ev*: $\bigwedge U.\ [\![(a,b) \in U;\ \mathit{openin}\ (\mathit{prod\_topology}\ X\ Y)\ U]\!] \Longrightarrow \forall_F\ x\ \mathit{in}\ F.\ f\ x \in U$

```
                    and a: a ∈ topspace X and b: b ∈ topspace Y and l: l = (a,b)
    by (auto simp: limitin_def)
  moreover have ∀⇩F x in F. fst (f x) ∈ U if openin X U a ∈ U for U
  proof −
    have ∀⇩F c in F. f c ∈ U × topspace Y
     using b that ev [of U × topspace Y] by (auto simp: openin_prod_topology_alt)
    then show ?thesis
     by (rule eventually_mono) (metis (mono_tags, lifting) SigmaE2 prod.collapse)
  qed
  moreover have ∀⇩F x in F. snd (f x) ∈ U if openin Y U b ∈ U for U
  proof −
    have ∀⇩F c in F. f c ∈ topspace X × U
     using a that ev [of topspace X × U] by (auto simp: openin_prod_topology_alt)
    then show ?thesis
     by (rule eventually_mono) (metis (mono_tags, lifting) SigmaE2 prod.collapse)
  qed
  ultimately show ?rhs
    by (simp add: limitin_def)
next
  have limitin (prod_topology X Y) f (a,b) F
    if limitin X (fst ∘ f) a F limitin Y (snd ∘ f) b F for a b
    using that
  proof (clarsimp simp: limitin_def)
    fix Z :: ('a × 'b) set
    assume a: a ∈ topspace X ∀ U. openin X U ∧ a ∈ U ⟶ (∀⇩F x in F. fst (f
x) ∈ U)
       and b: b ∈ topspace Y ∀ U. openin Y U ∧ b ∈ U ⟶ (∀⇩F x in F. snd (f x)
∈ U)
       and Z: openin (prod_topology X Y) Z (a, b) ∈ Z
    then obtain U V where openin X U openin Y V a ∈ U b ∈ V U × V ⊆ Z
      using Z by (force simp: openin_prod_topology_alt)
    then have ∀⇩F x in F. fst (f x) ∈ U ∀⇩F x in F. snd (f x) ∈ V
      by (simp_all add: a b)
    then show ∀⇩F x in F. f x ∈ Z
      by (rule eventually_elim2) (use ‹U × V ⊆ Z› subsetD in auto)
  qed
  then show ?rhs ⟹ ?lhs
    by (metis prod.collapse)
qed

proposition connected_space_prod_topology:
   connected_space(prod_topology X Y) ⟷
   (prod_topology X Y) = trivial_topology ∨ connected_space X ∧ connected_space
Y (is ?lhs=?rhs)
proof (cases (prod_topology X Y) = trivial_topology)
  case True
  then show ?thesis
    by auto
next
```

```
  case False
  then have nonempty: topspace X ≠ {} topspace Y ≠ {}
    by force+
  show ?thesis
  proof
    assume ?lhs
    then show ?rhs
     by (metis connected_space_quotient_map_image nonempty quotient_map_fst
quotient_map_snd
         subtopology_eq_discrete_topology_empty)
  next
    assume ?rhs
    then have conX: connected_space X and conY: connected_space Y
      using False by blast+
    have False
      if openin (prod_topology X Y) U and openin (prod_topology X Y) V
        and UV: topspace X × topspace Y ⊆ U ∪ V U ∩ V = {}
        and U ≠ {} and V ≠ {}
      for U V
    proof −
      have Usub: U ⊆ topspace X × topspace Y and Vsub: V ⊆ topspace X ×
topspace Y
        using that by (metis openin_subset topspace_prod_topology)+
      obtain a b where ab: (a,b) ∈ U and a: a ∈ topspace X and b: b ∈ topspace
Y
        using ‹U ≠ {}› Usub by auto
      have ¬ topspace X × topspace Y ⊆ U
        using Usub Vsub ‹U ∩ V = {}› ‹V ≠ {}› by auto
      then obtain x y where x: x ∈ topspace X and y: y ∈ topspace Y and (x,y)
∉ U
        by blast
      have oX: openin X {x ∈ topspace X. (x,y) ∈ U} openin X {x ∈ topspace X.
(x,y) ∈ V}
       and oY: openin Y {y ∈ topspace Y. (a,y) ∈ U} openin Y {y ∈ topspace Y.
(a,y) ∈ V}
      by (force intro: openin_continuous_map_preimage [where Y = prod_topology
X Y]
           simp: that continuous_map_pairwise o_def x y a)+
      have 1: topspace Y ⊆ {y ∈ topspace Y. (a,y) ∈ U} ∪ {y ∈ topspace Y. (a,y)
∈ V}
        using a that(3) by auto
      have 2: {y ∈ topspace Y. (a,y) ∈ U} ∩ {y ∈ topspace Y. (a,y) ∈ V} = {}
        using that(4) by auto
      have 3: {y ∈ topspace Y. (a,y) ∈ U} ≠ {}
        using ab b by auto
      have 4: {y ∈ topspace Y. (a,y) ∈ V} ≠ {}
      proof −
        show ?thesis
          using connected_spaceD [OF conX oX] UV ‹(x,y) ∉ U› a x y
```

disjoint_iff_not_equal **by** *blast*
qed
show *?thesis*
using *connected_spaceD* [*OF conY oY 1 2 3 4*] **by** *auto*
qed
then show *?lhs*
unfolding *connected_space_def topspace_prod_topology* **by** *blast*
qed
qed

lemma *connectedin_Times*:
connectedin (*prod_topology X Y*) (*S* × *T*) ⟷
S = {} ∨ *T* = {} ∨ *connectedin X S* ∧ *connectedin Y T*
by (*auto simp*: *connectedin_def subtopology_Times connected_space_prod_topology subtopology_trivial_iff*)

end

2.8 T1 and Hausdorff spaces

theory *T1_Spaces*
imports *Product_Topology*
begin

2.9 T1 spaces with equivalences to many naturally "nice" properties.

definition *t1_space* **where**
t1_space X ≡ ∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. *x*≠*y* ⟶ (∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ *y* ∉ *U*)

lemma *t1_space_expansive*:
⟦*topspace Y* = *topspace X*; ⋀*U*. *openin X U* ⟹ *openin Y U*⟧ ⟹ *t1_space X* ⟹ *t1_space Y*
by (*metis t1_space_def*)

lemma *t1_space_alt*:
t1_space X ⟷ (∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. *x*≠*y* ⟶ (∃ *U*. *closedin X U* ∧ *x* ∈ *U* ∧ *y* ∉ *U*))
by (*metis DiffE DiffI closedin_def openin_closedin_eq t1_space_def*)

lemma *t1_space_empty* [*iff*]: *t1_space trivial_topology*
by (*simp add*: *t1_space_def*)

lemma *t1_space_derived_set_of_singleton*:
t1_space X ⟷ (∀ *x* ∈ *topspace X*. *X derived_set_of* {*x*} = {})
apply (*simp add*: *t1_space_def derived_set_of_def*, *safe*)
apply (*metis openin_topspace*)

by *force*

lemma *t1_space_derived_set_of_finite*:
t1_space X $\longleftrightarrow$ ($\forall$ *S*. *finite S* $\longrightarrow$ *X derived_set_of S* = {})
proof (*intro iffI allI impI*)
fix *S* :: *'a set*
assume *finite S*
then have *fin*: *finite* ((*λx*. {*x*}) ' (*topspace X* ∩ *S*))
by *blast*
assume *t1_space X*
then have *X derived_set_of* ($\bigcup$ *x* ∈ *topspace X* ∩ *S*. {*x*}) = {}
unfolding *derived_set_of_Union* [*OF fin*]
by (*auto simp*: *t1_space_derived_set_of_singleton*)
then have *X derived_set_of* (*topspace X* ∩ *S*) = {}
by *simp*
then show *X derived_set_of S* = {}
by *simp*
qed (*auto simp*: *t1_space_derived_set_of_singleton*)

lemma *t1_space_closedin_singleton*:
t1_space X $\longleftrightarrow$ ($\forall$ *x* ∈ *topspace X*. *closedin X* {*x*})
apply (*rule iffI*)
apply (*simp add*: *closedin_contains_derived_set t1_space_derived_set_of_singleton*)
using *t1_space_alt* **by** *auto*

lemma *continuous_closed_imp_proper_map*:
⟦*compact_space X*; *t1_space Y*; *continuous_map X Y f*; *closed_map X Y f*⟧
$\Longrightarrow$ *proper_map X Y f*
unfolding *proper_map_def*
by (*smt* (*verit*) *closedin_compact_space closedin_continuous_map_preimage*
Collect_cong singleton_iff t1_space_closedin_singleton)

lemma *t1_space_euclidean*: *t1_space* (*euclidean* :: *'a*::*metric_space topology*)
by (*simp add*: *t1_space_closedin_singleton*)

lemma *closedin_t1_singleton*:
⟦*t1_space X*; *a* ∈ *topspace X*⟧ $\Longrightarrow$ *closedin X* {*a*}
by (*simp add*: *t1_space_closedin_singleton*)

lemma *t1_space_closedin_finite*:
t1_space X $\longleftrightarrow$ ($\forall$ *S*. *finite S* ∧ *S* ⊆ *topspace X* $\longrightarrow$ *closedin X S*)
apply (*rule iffI*)
apply (*simp add*: *closedin_contains_derived_set t1_space_derived_set_of_finite*)
by (*simp add*: *t1_space_closedin_singleton*)

lemma *closure_of_singleton*:
t1_space X $\Longrightarrow$ *X closure_of* {*a*} = (*if a* ∈ *topspace X then* {*a*} *else* {})
by (*simp add*: *closure_of_eq t1_space_closedin_singleton closure_of_eq_empty_gen*)

lemma *separated_in_singleton*:
assumes *t1_space X*
shows *separatedin X {a} S ⟷ a ∈ topspace X ∧ S ⊆ topspace X ∧ (a ∉ X closure_of S)*
separatedin X S {a} ⟷ a ∈ topspace X ∧ S ⊆ topspace X ∧ (a ∉ X closure_of S)
unfolding *separatedin_def*
using *assms closure_of closure_of_singleton* **by** *fastforce+*

lemma *t1_space_openin_delete*:
t1_space X ⟷ (∀ U x. openin X U ∧ x ∈ U ⟶ openin X (U − {x}))
apply (*rule iffI*)
apply (*meson closedin_t1_singleton in_mono openin_diff openin_subset*)
by (*simp add*: *closedin_def t1_space_closedin_singleton*)

lemma *t1_space_openin_delete_alt*:
t1_space X ⟷ (∀ U x. openin X U ⟶ openin X (U − {x}))
by (*metis Diff_empty Diff_insert0 t1_space_openin_delete*)

lemma *t1_space_singleton_Inter_open*:
t1_space X ⟷ (∀ x ∈ topspace X. ⋂{U. openin X U ∧ x ∈ U} = {x}) (**is** *?P=?Q*)
and *t1_space_Inter_open_supersets*:
t1_space X ⟷ (∀ S. S ⊆ topspace X ⟶ ⋂{U. openin X U ∧ S ⊆ U} = S) (**is** *?P=?R*)
proof −
have *?R ⟹ ?Q*
apply *clarify*
apply (*drule_tac x={x}* **in** *spec*, *simp*)
done
moreover have *?Q ⟹ ?P*
apply (*clarsimp simp add*: *t1_space_def*)
apply (*drule_tac x=x* **in** *bspec*)
apply (*simp_all add*: *set_eq_iff*)
by (*metis* (*no_types*, *lifting*))
moreover have *?P ⟹ ?R*
proof (*clarsimp simp add*: *t1_space_closedin_singleton*, *rule subset_antisym*)
fix *S*
assume *S*: *∀ x∈topspace X. closedin X {x} S ⊆ topspace X*
then show *⋂ {U. openin X U ∧ S ⊆ U} ⊆ S*
apply *clarsimp*
by (*metis Diff_insert_absorb Set.set_insert closedin_def openin_topspace subset_insert*)
qed *force*
ultimately show *?P=?Q ?P=?R*
by *auto*
qed

lemma *t1_space_derived_set_of_infinite_openin*:
t1_space X ⟷
(∀ *S*. *X derived_set_of S* =
{*x* ∈ *topspace X*. ∀ *U*. *x* ∈ *U* ∧ *openin X U* ⟶ *infinite*(*S* ∩ *U*)})
(**is** _ = *?rhs*)
proof
assume *t1_space X*
show *?rhs*
proof *safe*
fix *S x U*
assume *x* ∈ *X derived_set_of S x* ∈ *U openin X U finite* (*S* ∩ *U*)
with ‹*t1_space X*› **show** *False*
apply (*simp add*: *t1_space_derived_set_of_finite*)
by (*metis IntI empty_iff empty_subsetI inf_commute openin_Int_derived_set_of_subset subset_antisym*)
next
fix *S x*
have *eq*: (∃ *y*. (*y* ≠ *x*) ∧ *y* ∈ *S* ∧ *y* ∈ *T*) ⟷ ~((*S* ∩ *T*) ⊆ {*x*}) **for** *x S T*
by *blast*
assume *x* ∈ *topspace X* ∀ *U*. *x* ∈ *U* ∧ *openin X U* ⟶ *infinite* (*S* ∩ *U*)
then show *x* ∈ *X derived_set_of S*
apply (*clarsimp simp add*: *derived_set_of_def eq*)
by (*meson finite.emptyI finite.insertI finite_subset*)
qed (*auto simp*: *in_derived_set_of*)
qed (*auto simp*: *t1_space_derived_set_of_singleton*)

lemma *finite_t1_space_imp_discrete_topology*:
⟦*topspace X* = *U*; *finite U*; *t1_space X*⟧ ⟹ *X* = *discrete_topology U*
by (*metis discrete_topology_unique_derived_set t1_space_derived_set_of_finite*)

lemma *t1_space_subtopology*: *t1_space X* ⟹ *t1_space*(*subtopology X U*)
by (*simp add*: *derived_set_of_subtopology t1_space_derived_set_of_finite*)

lemma *closedin_derived_set_of_gen*:
t1_space X ⟹ *closedin X* (*X derived_set_of S*)
apply (*clarsimp simp add*: *in_derived_set_of closedin_contains_derived_set derived_set_of_subset_topspace*)
by (*metis DiffD2 insert_Diff insert_iff t1_space_openin_delete*)

lemma *derived_set_of_derived_set_subset_gen*:
t1_space X ⟹ *X derived_set_of* (*X derived_set_of S*) ⊆ *X derived_set_of S*
by (*meson closedin_contains_derived_set closedin_derived_set_of_gen*)

lemma *subtopology_eq_discrete_topology_gen_finite*:
⟦*t1_space X*; *finite S*⟧ ⟹ *subtopology X S* = *discrete_topology*(*topspace X* ∩ *S*)
by (*simp add*: *subtopology_eq_discrete_topology_gen t1_space_derived_set_of_finite*)

lemma *subtopology_eq_discrete_topology_finite*:

```
  ⟦t1_space X; S ⊆ topspace X; finite S⟧
      ⟹ subtopology X S = discrete_topology S
 by (simp add: subtopology_eq_discrete_topology_eq t1_space_derived_set_of_finite)

lemma t1_space_closed_map_image:
  ⟦closed_map X Y f; f ‘ (topspace X) = topspace Y; t1_space X⟧ ⟹ t1_space
Y
 by (metis closed_map_def finite_subset_image t1_space_closedin_finite)

lemma homeomorphic_t1_space: X homeomorphic_space Y ⟹ (t1_space X ⟷
t1_space Y)
 apply (clarsimp simp add: homeomorphic_space_def)
 by (meson homeomorphic_eq_everything_map homeomorphic_maps_map t1_space_closed_map_image)

proposition t1_space_product_topology:
  t1_space (product_topology X I)
⟷ (product_topology X I) = trivial_topology ∨ (∀ i ∈ I. t1_space (X i))
proof (cases (product_topology X I) = trivial_topology)
 case True
 then show ?thesis
   using True t1_space_empty by force
next
 case False
 then obtain f where f: f ∈ (Π_E i∈I. topspace(X i))
   using discrete_topology_unique by (fastforce iff: null_topspace_iff_trivial)
 have t1_space (product_topology X I) ⟷ (∀ i∈I. t1_space (X i))
 proof (intro iffI ballI)
   show t1_space (X i) if t1_space (product_topology X I) and i ∈ I for i
   proof −
     have clo: ⋀h. h ∈ (Π_E i∈I. topspace (X i)) ⟹ closedin (product_topology
X I) {h}
       using that by (simp add: t1_space_closedin_singleton)
     show ?thesis
       unfolding t1_space_closedin_singleton
     proof clarify
       show closedin (X i) {xi} if xi ∈ topspace (X i) for xi
         using clo [of λj ∈ I. if i=j then xi else f j] f that ‹i ∈ I›
         by (fastforce simp add: closedin_product_topology_singleton)
     qed
   qed
 next
 next
   show t1_space (product_topology X I) if ∀ i∈I. t1_space (X i)
     using that
    by (simp add: t1_space_closedin_singleton Ball_def PiE_iff closedin_product_topology_singleton)
 qed
 then show ?thesis
   using False by blast
qed
```

lemma *t1_space_prod_topology*:
  t1_space(*prod_topology X Y*) ⟷ (*prod_topology X Y*) = *trivial_topology* ∨ *t1_space X* ∧ *t1_space Y*
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
 case *True* **then show** *?thesis*
 by *auto*
next
 case *False*
 have *eq*: {(*x*,*y*)} = {*x*} × {*y*} **for** *x*::'*a* **and** *y*::'*b*
  by *simp*
 have *t1_space* (*prod_topology X Y*) ⟷ (*t1_space X* ∧ *t1_space Y*)
  using *False*
  apply(*simp add*: *t1_space_closedin_singleton closedin_prod_Times_iff eq*
        del: *insert_Times_insert flip*: *null_topspace_iff_trivial ex_in_conv*)
  by *blast*
 with *False* **show** *?thesis*
  by *simp*
qed

2.9.1 Hausdorff Spaces

definition *Hausdorff_space*
 where
Hausdorff_space X ≡
    ∀ *x y*. *x* ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ (*x* ≠ *y*)
      ⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *disjnt U V*)

lemma *Hausdorff_space_expansive*:
  ⟦*Hausdorff_space X*; *topspace X* = *topspace Y*; ⋀*U*. *openin X U* ⟹ *openin Y U*⟧ ⟹ *Hausdorff_space Y*
 by (*metis Hausdorff_space_def*)

lemma *Hausdorff_space_topspace_empty* [*iff*]: *Hausdorff_space trivial_topology*
 by (*simp add*: *Hausdorff_space_def*)

lemma *Hausdorff_imp_t1_space*:
  Hausdorff_space X ⟹ *t1_space X*
 by (*metis Hausdorff_space_def disjnt_iff t1_space_def*)

lemma *closedin_derived_set_of*:
  Hausdorff_space X ⟹ *closedin X* (*X derived_set_of S*)
 by (*simp add*: *Hausdorff_imp_t1_space closedin_derived_set_of_gen*)

lemma *t1_or_Hausdorff_space*:
  t1_space X ∨ *Hausdorff_space X* ⟷ *t1_space X*
 using *Hausdorff_imp_t1_space* **by** *blast*

lemma *Hausdorff_space_sing_Inter_opens*:
  ⟦*Hausdorff_space X*; *a* ∈ *topspace X*⟧ ⟹ ⋂{*u*. *openin X u* ∧ *a* ∈ *u*} = {*a*}
  using *Hausdorff_imp_t1_space t1_space_singleton_Inter_open* **by** *force*

lemma *Hausdorff_space_subtopology*:
  assumes *Hausdorff_space X* **shows** *Hausdorff_space*(*subtopology X S*)
proof −
  have ∗: *disjnt U V* ⟹ *disjnt* (*S* ∩ *U*) (*S* ∩ *V*) **for** *U V*
    by (*simp add*: *disjnt_iff*)
  from *assms* **show** *?thesis*
    apply (*simp add*: *Hausdorff_space_def openin_subtopology_alt*)
    apply (*fast intro*: ∗ *elim*!: *all_forward*)
    done
qed

lemma *Hausdorff_space_compact_separation*:
  assumes *X*: *Hausdorff_space X* **and** *S*: *compactin X S* **and** *T*: *compactin X T*
and *disjnt S T*
  obtains *U V* **where** *openin X U openin X V S* ⊆ *U T* ⊆ *V disjnt U V*
proof (*cases S* = {})
  case *True*
  then show *thesis*
   by (*metis* ‹*compactin X T*› *compactin_subset_topspace disjnt_empty1 empty_subsetI*
openin_empty openin_topspace that)
next
  case *False*
  have ∀ *x* ∈ *S*. ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *x* ∈ *U* ∧ *T* ⊆ *V* ∧ *disjnt U*
V
  proof
    fix *a*
    assume *a* ∈ *S*
    then have *a* ∉ *T*
      by (*meson assms*(*4*) *disjnt_iff*)
    have *a*: *a* ∈ *topspace X*
      using *S* ‹*a* ∈ *S*› *compactin_subset_topspace* **by** *blast*
    show ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *a* ∈ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*
    proof (*cases T* = {})
      case *True*
      then show *?thesis*
        using *a disjnt_empty2 openin_empty* **by** *blast*
    next
      case *False*
      have ∀ *x* ∈ *topspace X* − {*a*}. ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *x* ∈ *U* ∧
a ∈ *V* ∧ *disjnt U V*
        using *X a* **by** (*simp add*: *Hausdorff_space_def*)
      then obtain *U V* **where** *UV*: ∀ *x* ∈ *topspace X* − {*a*}. *openin X* (*U x*) ∧
openin X (*V x*) ∧ *x* ∈ *U x* ∧ *a* ∈ *V x* ∧ *disjnt* (*U x*) (*V x*)
        by *metis*
      with ‹*a* ∉ *T*› *compactin_subset_topspace* [*OF T*]

```
      have Topen: ∀ W ∈ U ‘ T. openin X W and Tsub: T ⊆ ⋃ (U ‘ T)
        by auto
      then obtain ℱ where ℱ: finite ℱ ℱ ⊆ U ‘ T and T ⊆ ⋃ℱ
        using T unfolding compactin_def by meson
      then obtain F where F: finite F F ⊆ T ℱ = U ‘ F and SUF: T ⊆ ⋃(U ‘
F) and a ∉ F
        using finite_subset_image [OF ℱ] ‹a ∉ T› by (metis subsetD)
      have U: ⋀x. ⟦x ∈ topspace X; x ≠ a⟧ ⟹ openin X (U x)
        and V: ⋀x. ⟦x ∈ topspace X; x ≠ a⟧ ⟹ openin X (V x)
        and disj: ⋀x. ⟦x ∈ topspace X; x ≠ a⟧ ⟹ disjnt (U x) (V x)
        using UV by blast+
      show ?thesis
      proof (intro exI conjI)
        have F ≠ {}
          using False SUF by blast
        with ‹a ∉ F› show openin X (⋂(V ‘ F))
          using F compactin_subset_topspace [OF T] by (force intro: V)
        show openin X (⋃(U ‘ F))
          using F Topen Tsub by (force intro: U)
        show disjnt (⋂(V ‘ F)) (⋃(U ‘ F))
          using disj
          apply (auto simp: disjnt_def)
          using ‹F ⊆ T› ‹a ∉ F› compactin_subset_topspace [OF T] by blast
        show a ∈ (⋂(V ‘ F))
          using ‹F ⊆ T› T UV ‹a ∉ T› compactin_subset_topspace by blast
      qed (auto simp: SUF)
    qed
  qed
  then obtain U V where UV: ∀x ∈ S. openin X (U x) ∧ openin X (V x) ∧ x
∈ U x ∧ T ⊆ V x ∧ disjnt (U x) (V x)
    by metis
  then have S ⊆ ⋃ (U ‘ S)
    by auto
  moreover have ∀ W ∈ U ‘ S. openin X W
    using UV by blast
  ultimately obtain I where I: S ⊆ ⋃ (U ‘ I) I ⊆ S finite I
    by (metis S compactin_def finite_subset_image)
  show thesis
  proof
    show openin X (⋃(U ‘ I))
      using ‹I ⊆ S› UV by blast
    show openin X (⋂ (V ‘ I))
      using False UV ‹I ⊆ S› ‹S ⊆ ⋃ (U ‘ I)› ‹finite I› by blast
    show disjnt (⋃(U ‘ I)) (⋂ (V ‘ I))
      by simp (meson UV ‹I ⊆ S› disjnt_subset2 in_mono le_INF_iff order_refl)
  qed (use UV I in auto)
qed
```

lemma *Hausdorff_space_compact_sets*:
Hausdorff_space X ⟷
(∀ *S T*. *compactin X S* ∧ *compactin X T* ∧ *disjnt S T*
⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*meson Hausdorff_space_compact_separation*)
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp add*: *Hausdorff_space_def*)
fix *x y*
assume *x* ∈ *topspace X y* ∈ *topspace X x* ≠ *y*
then show ∃ *U*. *openin X U* ∧ (∃ *V*. *openin X V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *disjnt U V*)
using *R* [*of* {*x*} {*y*}] **by** *auto*
qed
qed

lemma *compactin_imp_closedin*:
assumes *X*: *Hausdorff_space X* **and** *S*: *compactin X S* **shows** *closedin X S*
proof −
have *S* ⊆ *topspace X*
by (*simp add*: *assms compactin_subset_topspace*)
moreover
have ∃ *T*. *openin X T* ∧ *x* ∈ *T* ∧ *T* ⊆ *topspace X* − *S* **if** *x* ∈ *topspace X x* ∉ *S*
for *x*
using *Hausdorff_space_compact_separation* [*OF X _ S*, *of* {*x*}] *that*
apply (*simp add*: *disjnt_def*)
by (*metis Diff_mono Diff_triv openin_subset*)
ultimately show *?thesis*
using *closedin_def openin_subopen* **by** *force*
qed

lemma *closedin_Hausdorff_singleton*:
⟦*Hausdorff_space X*; *x* ∈ *topspace X*⟧ ⟹ *closedin X* {*x*}
by (*simp add*: *Hausdorff_imp_t1_space closedin_t1_singleton*)

lemma *closedin_Hausdorff_sing_eq*:
Hausdorff_space X ⟹ *closedin X* {*x*} ⟷ *x* ∈ *topspace X*
by (*meson closedin_Hausdorff_singleton closedin_subset insert_subset*)

lemma *Hausdorff_space_discrete_topology* [*simp*]:
Hausdorff_space (*discrete_topology U*)
unfolding *Hausdorff_space_def*
by (*metis Hausdorff_space_compact_sets Hausdorff_space_def compactin_discrete_topology*

equalityE openin_discrete_topology)

lemma *compactin_Int*:
  ⟦*Hausdorff_space X*; *compactin X S*; *compactin X T*⟧ ⟹ *compactin X* (*S* ∩ *T*)
 by (*simp add*: *closed_Int_compactin compactin_imp_closedin*)

lemma *finite_topspace_imp_discrete_topology*:
  ⟦*topspace X* = *U*; *finite U*; *Hausdorff_space X*⟧ ⟹ *X* = *discrete_topology U*
 using *Hausdorff_imp_t1_space finite_t1_space_imp_discrete_topology* **by** *blast*

lemma *derived_set_of_finite*:
  ⟦*Hausdorff_space X*; *finite S*⟧ ⟹ *X derived_set_of S* = {}
 using *Hausdorff_imp_t1_space t1_space_derived_set_of_finite* **by** *auto*

lemma *infinite_perfect_set*:
  ⟦*Hausdorff_space X*; *S* ⊆ *X derived_set_of S*; *S* ≠ {}⟧ ⟹ *infinite S*
 using *derived_set_of_finite* **by** *blast*

lemma *derived_set_of_singleton*:
  Hausdorff_space X ⟹ *X derived_set_of* {*x*} = {}
 by (*simp add*: *derived_set_of_finite*)

lemma *closedin_Hausdorff_finite*:
  ⟦*Hausdorff_space X*; *S* ⊆ *topspace X*; *finite S*⟧ ⟹ *closedin X S*
 by (*simp add*: *compactin_imp_closedin finite_imp_compactin_eq*)

lemma *open_in_Hausdorff_delete*:
  ⟦*Hausdorff_space X*; *openin X S*⟧ ⟹ *openin X* (*S* − {*x*})
 using *Hausdorff_imp_t1_space t1_space_openin_delete_alt* **by** *auto*

lemma *closedin_Hausdorff_finite_eq*:
  ⟦*Hausdorff_space X*; *finite S*⟧ ⟹ *closedin X S* ⟷ *S* ⊆ *topspace X*
 by (*meson closedin_Hausdorff_finite closedin_def*)

lemma *derived_set_of_infinite_openin*:
  Hausdorff_space X
    ⟹ *X derived_set_of S* =
       {*x* ∈ *topspace X*. ∀ *U*. *x* ∈ *U* ∧ *openin X U* ⟶ *infinite*(*S* ∩ *U*)}
 using *Hausdorff_imp_t1_space t1_space_derived_set_of_infinite_openin* **by** *fastforce*

lemma *Hausdorff_space_discrete_compactin*:
  Hausdorff_space X
    ⟹ *S* ∩ *X derived_set_of S* = {} ∧ *compactin X S* ⟷ *S* ⊆ *topspace X* ∧ *finite S*
 using *derived_set_of_finite discrete_compactin_eq_finite* **by** *fastforce*

lemma *Hausdorff_space_finite_topspace*:
  Hausdorff_space X ⟹ *X derived_set_of* (*topspace X*) = {} ∧ *compact_space*

$X \longleftrightarrow$ *finite*(*topspace X*)
using *derived_set_of_finite discrete_compact_space_eq_finite* **by** *auto*

lemma *derived_set_of_derived_set_subset*:
Hausdorff_space X $\Longrightarrow$ *X derived_set_of* (*X derived_set_of S*) $\subseteq$ *X derived_set_of S*
by (*simp add*: *Hausdorff_imp_t1_space derived_set_of_derived_set_subset_gen*)

lemma *Hausdorff_space_injective_preimage*:
assumes *Hausdorff_space Y* **and** *cmf*: *continuous_map X Y f* **and** *inj_on f* (*topspace X*)
shows *Hausdorff_space X*
unfolding *Hausdorff_space_def*
proof *clarify*
fix *x y*
assume *x*: $x \in$ *topspace X* **and** *y*: $y \in$ *topspace X* **and** $x \neq y$
then obtain *U V* **where** *openin Y U openin Y V f x* $\in$ *U f y* $\in$ *V disjnt U V*
using *assms*
by (*smt* (*verit*, *ccfv_threshold*) *Hausdorff_space_def continuous_map image_subset_iff inj_on_def*)
show $\exists$ *U V*. *openin X U* $\wedge$ *openin X V* $\wedge$ *x* $\in$ *U* $\wedge$ *y* $\in$ *V* $\wedge$ *disjnt U V*
proof (*intro exI conjI*)
show *openin X* {*x* $\in$ *topspace X*. *f x* $\in$ *U*}
using ‹*openin Y U*› *cmf continuous_map* **by** *fastforce*
show *openin X* {*x* $\in$ *topspace X*. *f x* $\in$ *V*}
using ‹*openin Y V*› *cmf openin_continuous_map_preimage* **by** *blast*
show *disjnt* {*x* $\in$ *topspace X*. *f x* $\in$ *U*} {*x* $\in$ *topspace X*. *f x* $\in$ *V*}
using ‹*disjnt U V*› **by** (*auto simp add*: *disjnt_def*)
qed (*use x* ‹*f x* $\in$ *U*› *y* ‹*f y* $\in$ *V*› **in** *auto*)
qed

lemma *homeomorphic_Hausdorff_space*:
X homeomorphic_space Y $\Longrightarrow$ *Hausdorff_space X* $\longleftrightarrow$ *Hausdorff_space Y*
unfolding *homeomorphic_space_def homeomorphic_maps_map*
by (*auto simp*: *homeomorphic_eq_everything_map Hausdorff_space_injective_preimage*)

lemma *Hausdorff_space_retraction_map_image*:
⟦*retraction_map X Y r*; *Hausdorff_space X*⟧ $\Longrightarrow$ *Hausdorff_space Y*
unfolding *retraction_map_def*
using *Hausdorff_space_subtopology homeomorphic_Hausdorff_space retraction_maps_section_image2*
by *blast*

lemma *compact_Hausdorff_space_optimal*:
assumes *eq*: *topspace Y* = *topspace X* **and** *XY*: $\bigwedge$*U*. *openin X U* $\Longrightarrow$ *openin Y U*
and *Hausdorff_space X compact_space Y*
shows *Y* = *X*
proof −

```
  have ⋀U. closedin X U ⟹ closedin Y U
    using XY using topology_finer_closedin [OF eq]
    by metis
  have openin Y S = openin X S for S
   by (metis XY assms(3) assms(4) closedin_compact_space compactin_contractive
compactin_imp_closedin eq openin_closedin_eq)
  then show ?thesis
    by (simp add: topology_eq)
qed

lemma continuous_map_imp_closed_graph:
  assumes f: continuous_map X Y f and Y: Hausdorff_space Y
  shows closedin (prod_topology X Y) ((λx. (x,f x)) ` topspace X)
  unfolding closedin_def
proof
  show (λx. (x, f x)) ` topspace X ⊆ topspace (prod_topology X Y)
    using continuous_map_def f by fastforce
  show openin (prod_topology X Y) (topspace (prod_topology X Y) − (λx. (x, f
x)) ` topspace X)
    unfolding openin_prod_topology_alt
  proof (intro allI impI)
   show ∃ U V. openin X U ∧ openin Y V ∧ x ∈ U ∧ y ∈ V ∧ U × V ⊆ topspace
(prod_topology X Y) − (λx. (x, f x)) ` topspace X
      if (x,y) ∈ topspace (prod_topology X Y) − (λx. (x, f x)) ` topspace X
      for x y
    proof −
      have x ∈ topspace X and y: y ∈ topspace Y y ≠ f x
        using that by auto
      then have f x ∈ topspace Y
        using continuous_map_image_subset_topspace f by blast
      then obtain U V where UV: openin Y U openin Y V f x ∈ U y ∈ V disjnt
U V
        using Y y Hausdorff_space_def by metis
      show ?thesis
      proof (intro exI conjI)
        show openin X {x ∈ topspace X. f x ∈ U}
          using ‹openin Y U› f openin_continuous_map_preimage by blast
        show {x ∈ topspace X. f x ∈ U} × V ⊆ topspace (prod_topology X Y) −
(λx. (x, f x)) ` topspace X
          using UV by (auto simp: disjnt_iff dest: openin_subset)
      qed (use UV ‹x ∈ topspace X› in auto)
    qed
  qed
qed

lemma continuous_imp_closed_map:
  ⟦continuous_map X Y f; compact_space X; Hausdorff_space Y⟧ ⟹ closed_map
X Y f
   by (meson closed_map_def closedin_compact_space compactin_imp_closedin
```

image_compactin)

lemma *continuous_imp_quotient_map*:
⟦*continuous_map X Y f*; *compact_space X*; *Hausdorff_space Y*; *f ‘ (topspace X) = topspace Y*⟧
⟹ *quotient_map X Y f*
by (*simp add*: *continuous_imp_closed_map continuous_closed_imp_quotient_map*)

lemma *continuous_imp_homeomorphic_map*:
⟦*continuous_map X Y f*; *compact_space X*; *Hausdorff_space Y*;
f ‘ (topspace X) = topspace Y; *inj_on f (topspace X)*⟧
⟹ *homeomorphic_map X Y f*
by (*simp add*: *continuous_imp_closed_map bijective_closed_imp_homeomorphic_map*)

lemma *continuous_imp_embedding_map*:
⟦*continuous_map X Y f*; *compact_space X*; *Hausdorff_space Y*; *inj_on f (topspace X)*⟧
⟹ *embedding_map X Y f*
by (*simp add*: *continuous_imp_closed_map injective_closed_imp_embedding_map*)

lemma *continuous_inverse_map*:
assumes *compact_space X Hausdorff_space Y*
and *cmf*: *continuous_map X Y f* **and** *gf*: ⋀*x*. *x ∈ topspace X* ⟹ *g*(*f x*) = *x*
and *Sf*: *S ⊆ f ‘ (topspace X)*
shows *continuous_map* (*subtopology Y S*) *X g*
proof (*rule continuous_map_from_subtopology_mono* [*OF _ ‹S ⊆ f ‘ (topspace X)›*])
show *continuous_map* (*subtopology Y* (*f ‘ (topspace X)*)) *X g*
unfolding *continuous_map_closedin*
proof (*intro conjI ballI allI impI*)
show *g ∈ topspace* (*subtopology Y* (*f ‘ topspace X*)) → *topspace X*
using *gf* **by** *auto*
next
fix *C*
assume *C*: *closedin X C*
show *closedin* (*subtopology Y* (*f ‘ topspace X*))
{*x ∈ topspace* (*subtopology Y* (*f ‘ topspace X*)). *g x ∈ C*}
proof (*rule compactin_imp_closedin*)
show *Hausdorff_space* (*subtopology Y* (*f ‘ topspace X*))
using *Hausdorff_space_subtopology* [*OF ‹Hausdorff_space Y›*] **by** *blast*
have *compactin Y* (*f ‘ C*)
using *C cmf image_compactin closedin_compact_space* [*OF ‹compact_space X›*] **by** *blast*
moreover have {*x ∈ topspace Y*. *x ∈ f ‘ topspace X ∧ g x ∈ C*} = *f ‘ C*
using *closedin_subset* [*OF C*] *cmf* **by** (*auto simp*: *gf continuous_map_def*)
ultimately have *compactin Y* {*x ∈ topspace Y*. *x ∈ f ‘ topspace X ∧ g x ∈ C*}
by *simp*
then show *compactin* (*subtopology Y* (*f ‘ topspace X*))

```
        {x ∈ topspace (subtopology Y (f ‘ topspace X)). g x ∈ C}
    by (auto simp add: compactin_subtopology)
  qed
 qed
qed

lemma closed_map_paired_continuous_map_right:
 ⟦continuous_map X Y f; Hausdorff_space Y⟧ ⟹ closed_map X (prod_topology
X Y) (λx. (x,f x))
 by (simp add: continuous_map_imp_closed_graph embedding_map_graph em-
bedding_imp_closed_map)

lemma closed_map_paired_continuous_map_left:
 assumes f: continuous_map X Y f and Y: Hausdorff_space Y
 shows closed_map X (prod_topology Y X) (λx. (f x,x))
proof −
 have eq: (λx. (f x,x)) = (λ(a,b). (b,a)) ∘ (λx. (x,f x))
  by auto
 show ?thesis
  unfolding eq
 proof (rule closed_map_compose)
  show closed_map X (prod_topology X Y) (λx. (x, f x))
   using Y closed_map_paired_continuous_map_right f by blast
  show closed_map (prod_topology X Y) (prod_topology Y X) (λ(a, b). (b, a))
   by (metis homeomorphic_map_swap homeomorphic_imp_closed_map)
 qed
qed

lemma proper_map_paired_continuous_map_right:
 ⟦continuous_map X Y f; Hausdorff_space Y⟧
     ⟹ proper_map X (prod_topology X Y) (λx. (x,f x))
 using closed_injective_imp_proper_map closed_map_paired_continuous_map_right
 by (metis (mono_tags, lifting) Pair_inject inj_onI)

lemma proper_map_paired_continuous_map_left:
 ⟦continuous_map X Y f; Hausdorff_space Y⟧
     ⟹ proper_map X (prod_topology Y X) (λx. (f x,x))
 using closed_injective_imp_proper_map closed_map_paired_continuous_map_left
 by (metis (mono_tags, lifting) Pair_inject inj_onI)

lemma Hausdorff_space_prod_topology:
 Hausdorff_space(prod_topology X Y) ⟷ (prod_topology X Y) = trivial_topology
∨ Hausdorff_space X ∧ Hausdorff_space Y
 (is ?lhs = ?rhs)
proof
 assume ?lhs
 then show ?rhs
  by (rule topological_property_of_prod_component) (auto simp: Hausdorff_space_subtopology
homeomorphic_Hausdorff_space)
```

```
next
  assume R: ?rhs
  show ?lhs
  proof (cases (topspace X × topspace Y) = {})
    case False
    with R have ne: topspace X ≠ {} topspace Y ≠ {} and X: Hausdorff_space
X and Y: Hausdorff_space Y
      by auto
    show ?thesis
      unfolding Hausdorff_space_def
    proof clarify
      fix x y x' y'
      assume xy: (x, y) ∈ topspace (prod_topology X Y)
        and xy': (x',y') ∈ topspace (prod_topology X Y)
        and ∗: ∄U V. openin (prod_topology X Y) U ∧ openin (prod_topology X
Y) V
             ∧ (x, y) ∈ U ∧ (x', y') ∈ V ∧ disjnt U V
      have False if x ≠ x' ∨ y ≠ y'
        using that
      proof
        assume x ≠ x'
        then obtain U V where openin X U openin X V x ∈ U x' ∈ V disjnt U V
          by (metis Hausdorff_space_def X mem_Sigma_iff topspace_prod_topology
xy xy')
        let ?U = U × topspace Y
        let ?V = V × topspace Y
        have openin (prod_topology X Y) ?U openin (prod_topology X Y) ?V
          by (simp_all add: openin_prod_Times_iff ‹openin X U› ‹openin X V›)
        moreover have disjnt ?U ?V
          by (simp add: ‹disjnt U V›)
        ultimately show False
       using ∗ ‹x ∈ U› ‹x' ∈ V› xy xy' by (metis SigmaD2 SigmaI topspace_prod_topology)
      next
        assume y ≠ y'
        then obtain U V where openin Y U openin Y V y ∈ U y' ∈ V disjnt U V
          by (metis Hausdorff_space_def Y mem_Sigma_iff topspace_prod_topology
xy xy')
        let ?U = topspace X × U
        let ?V = topspace X × V
        have openin (prod_topology X Y) ?U openin (prod_topology X Y) ?V
          by (simp_all add: openin_prod_Times_iff ‹openin Y U› ‹openin Y V›)
        moreover have disjnt ?U ?V
          by (simp add: ‹disjnt U V›)
        ultimately show False
       using ∗ ‹y ∈ U› ‹y' ∈ V› xy xy' by (metis SigmaD1 SigmaI topspace_prod_topology)
      qed
      then show x = x' ∧ y = y'
        by blast
    qed
```

qed *force*
qed

lemma *Hausdorff_space_product_topology*:
Hausdorff_space (*product_topology X I*) $\longleftrightarrow$ (Π_E *i*$\in$*I*. *topspace*(*X i*)) = {} $\vee$ ($\forall$ *i* $\in$ *I*. *Hausdorff_space* (*X i*))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*simp add*: *Hausdorff_space_subtopology PiE_eq_empty_iff homeomorphic_Hausdorff_space*
topological_property_of_product_component)
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases* (Π_E *i*$\in$*I*. *topspace*(*X i*)) = {})
case *True*
then show *?thesis*
by (*simp add*: *Hausdorff_space_def*)
next
case *False*
have $\exists$ *U V*. *openin* (*product_topology X I*) *U* $\wedge$ *openin* (*product_topology X I*) *V* $\wedge$ *f* $\in$ *U* $\wedge$ *g* $\in$ *V* $\wedge$ *disjnt U V*
if *f*: *f* $\in$ (Π_E *i*$\in$*I*. *topspace* (*X i*)) **and** *g*: *g* $\in$ (Π_E *i*$\in$*I*. *topspace* (*X i*)) **and** *f* $\neq$ *g*
for *f g* :: *'a* $\Rightarrow$ *'b*
proof −
obtain *m* **where** *f m* $\neq$ *g m*
using ‹*f* $\neq$ *g*› **by** *blast*
then have *m* $\in$ *I*
using *f g* **by** *fastforce*
then have *Hausdorff_space* (*X m*)
using *False that R* **by** *blast*
then obtain *U V* **where** *U*: *openin* (*X m*) *U* **and** *V*: *openin* (*X m*) *V* **and** *f m* $\in$ *U g m* $\in$ *V disjnt U V*
by (*metis Hausdorff_space_def PiE_mem* ‹*f m* $\neq$ *g m*› ‹*m* $\in$ *I*› *f g*)
show *?thesis*
proof (*intro exI conjI*)
let *?U* = (Π_E *i*$\in$*I*. *topspace*(*X i*)) $\cap$ {*x*. *x m* $\in$ *U*}
let *?V* = (Π_E *i*$\in$*I*. *topspace*(*X i*)) $\cap$ {*x*. *x m* $\in$ *V*}
show *openin* (*product_topology X I*) *?U openin* (*product_topology X I*) *?V*
using ‹*m* $\in$ *I*› *U V*
by (*force simp add*: *openin_product_topology intro*: *arbitrary_union_of_inc relative_to_inc finite_intersection_of_inc*)+
show *f* $\in$ *?U*
using ‹*f m* $\in$ *U*› *f* **by** *blast*
show *g* $\in$ *?V*

```
        using ‹g m ∈ V› g by blast
      show disjnt ?U ?V
        using ‹disjnt U V› by (auto simp: PiE_def Pi_def disjnt_def)
      qed
    qed
    then show ?thesis
      by (simp add: Hausdorff_space_def)
  qed
qed

lemma Hausdorff_space_closed_neighbourhood:
   Hausdorff_space X ⟷
   (∀x ∈ topspace X. ∃U C. openin X U ∧ closedin X C ∧
                  Hausdorff_space(subtopology X C) ∧ x ∈ U ∧ U ⊆ C) (is _ =
?rhs)
proof
  assume R: ?rhs
  show Hausdorff_space X
    unfolding Hausdorff_space_def
  proof clarify
    fix x y
    assume x: x ∈ topspace X and y: y ∈ topspace X and x ≠ y
    obtain T C where ∗: openin X T closedin X C x ∈ T T ⊆ C
                and C: Hausdorff_space (subtopology X C)
      by (meson R ‹x ∈ topspace X›)
    show ∃U V. openin X U ∧ openin X V ∧ x ∈ U ∧ y ∈ V ∧ disjnt U V
    proof (cases y ∈ C)
      case True
      with ∗ C obtain U V where U: openin (subtopology X C) U
                           and V: openin (subtopology X C) V
                           and x ∈ U y ∈ V disjnt U V
        unfolding Hausdorff_space_def
      by (smt (verit, best) ‹x ≠ y› closedin_subset subsetD topspace_subtopology_subset)
       then obtain U' V' where UV': U = U' ∩ C openin X U' V = V' ∩ C
openin X V'
        by (meson openin_subtopology)
      have disjnt (T ∩ U') V'
        using ‹disjnt U V› UV' ‹T ⊆ C› by (force simp: disjnt_iff)
      with ‹T ⊆ C› have disjnt (T ∩ U') (V' ∪ (topspace X − C))
        unfolding disjnt_def by blast
      moreover
      have openin X (T ∩ U')
        by (simp add: ‹openin X T› ‹openin X U'› openin_Int)
      moreover have openin X (V' ∪ (topspace X − C))
        using ‹closedin X C› ‹openin X V'› by auto
      ultimately show ?thesis
        using UV' ‹x ∈ T› ‹x ∈ U› ‹y ∈ V› by blast
    next
      case False
```

with ∗ *y* **show** *?thesis*
by (*force simp*: *closedin_def disjnt_def*)
qed
qed
qed *fastforce*

end

2.10 Lindelöf spaces

theory *Lindelof_Spaces*
imports *T1_Spaces*
begin

definition *Lindelof_space* **where**
Lindelof_space X ≡
∀$\mathcal{U}$. (∀ *U* ∈ $\mathcal{U}$. *openin X U*) ∧ ⋃$\mathcal{U}$ = *topspace X*
⟶ (∃$\mathcal{V}$. *countable* $\mathcal{V}$ ∧ $\mathcal{V}$ ⊆ $\mathcal{U}$ ∧ ⋃$\mathcal{V}$ = *topspace X*)

lemma *Lindelof_spaceD*:
⟦*Lindelof_space X*; ⋀*U*. *U* ∈ $\mathcal{U}$ ⟹ *openin X U*; ⋃$\mathcal{U}$ = *topspace X*⟧
⟹ ∃$\mathcal{V}$. *countable* $\mathcal{V}$ ∧ $\mathcal{V}$ ⊆ $\mathcal{U}$ ∧ ⋃$\mathcal{V}$ = *topspace X*
by (*auto simp*: *Lindelof_space_def*)

lemma *Lindelof_space_alt*:
Lindelof_space X ⟷
(∀$\mathcal{U}$. (∀ *U* ∈ $\mathcal{U}$. *openin X U*) ∧ *topspace X* ⊆ ⋃$\mathcal{U}$
⟶ (∃$\mathcal{V}$. *countable* $\mathcal{V}$ ∧ $\mathcal{V}$ ⊆ $\mathcal{U}$ ∧ *topspace X* ⊆ ⋃$\mathcal{V}$))
unfolding *Lindelof_space_def*
using *openin_subset* **by** *fastforce*

lemma *compact_imp_Lindelof_space*:
compact_space X ⟹ *Lindelof_space X*
unfolding *Lindelof_space_def compact_space*
by (*meson uncountable_infinite*)

lemma *Lindelof_space_topspace_empty*:
topspace X = {} ⟹ *Lindelof_space X*
using *compact_imp_Lindelof_space compact_space_trivial_topology* **by** *force*

lemma *Lindelof_space_Union*:
assumes $\mathcal{U}$: *countable* $\mathcal{U}$ **and** *lin*: ⋀*U*. *U* ∈ $\mathcal{U}$ ⟹ *Lindelof_space* (*subtopology X U*)
shows *Lindelof_space* (*subtopology X* (⋃$\mathcal{U}$))
proof −
have ∃$\mathcal{V}$. *countable* $\mathcal{V}$ ∧ $\mathcal{V}$ ⊆ $\mathcal{F}$ ∧ ⋃$\mathcal{U}$ ∩ ⋃$\mathcal{V}$ = *topspace X* ∩ ⋃$\mathcal{U}$
if $\mathcal{F}$: $\mathcal{F}$ ⊆ *Collect* (*openin X*) **and** *UF*: ⋃$\mathcal{U}$ ∩ ⋃$\mathcal{F}$ = *topspace X* ∩ ⋃$\mathcal{U}$
for $\mathcal{F}$
proof −

```
    have ⋀U. ⟦U ∈ 𝒰; U ∩ ⋃ℱ = topspace X ∩ U⟧
          ⟹ ∃𝒱. countable 𝒱 ∧ 𝒱 ⊆ ℱ ∧ U ∩ ⋃𝒱 = topspace X ∩ U
      using lin ℱ
       unfolding Lindelof_space_def openin_subtopology_alt Ball_def subset_iff
[symmetric]
      by (simp add: all_subset_image imp_conjL ex_countable_subset_image)
    then obtain g where g: ⋀U. ⟦U ∈ 𝒰; U ∩ ⋃ℱ = topspace X ∩ U⟧
                           ⟹ countable (g U) ∧ (g U) ⊆ ℱ ∧ U ∩ ⋃(g U) =
topspace X ∩ U
      by metis
    show ?thesis
    proof (intro exI conjI)
      show countable (⋃(g ` 𝒰))
        using Int_commute UF g  by (fastforce intro: countable_UN [OF 𝒰])
      show ⋃(g ` 𝒰) ⊆ ℱ
        using g UF by blast
      show ⋃𝒰 ∩ ⋃(⋃(g ` 𝒰)) = topspace X ∩ ⋃𝒰
      proof
        show ⋃𝒰 ∩ ⋃(⋃(g ` 𝒰)) ⊆ topspace X ∩ ⋃𝒰
          using g UF by blast
        show topspace X ∩ ⋃𝒰 ⊆ ⋃𝒰 ∩ ⋃(⋃(g ` 𝒰))
        proof clarsimp
          show ∃y∈𝒰. ∃W∈g y. x ∈ W
            if x ∈ topspace X x ∈ V V ∈ 𝒰 for x V
          proof −
            have V ∩ ⋃ℱ = topspace X ∩ V
              using UF ‹V ∈ 𝒰› by blast
            with that g [OF ‹V ∈ 𝒰›]  show ?thesis by blast
          qed
        qed
      qed
    qed
  qed
  then show ?thesis
       unfolding Lindelof_space_def openin_subtopology_alt Ball_def subset_iff
[symmetric]
      by (simp add: all_subset_image imp_conjL ex_countable_subset_image)
qed

lemma countable_imp_Lindelof_space:
  assumes countable(topspace X)
  shows Lindelof_space X
proof −
  have Lindelof_space (subtopology X (⋃x ∈ topspace X. {x}))
  proof (rule Lindelof_space_Union)
    show countable ((λx. {x}) ` topspace X)
      using assms by blast
    show Lindelof_space (subtopology X U)
      if U ∈ (λx. {x}) ` topspace X for U
```

```
    proof −
      have compactin X U
        using that by force
      then show ?thesis
        by (meson compact_imp_Lindelof_space compact_space_subtopology)
    qed
  qed
  then show ?thesis
    by simp
qed
lemma Lindelof_space_subtopology:
   Lindelof_space(subtopology X S) ⟷
        (∀𝒰. (∀ U ∈ 𝒰. openin X U) ∧ topspace X ∩ S ⊆ ⋃𝒰
            ⟶ (∃ V. countable V ∧ V ⊆ 𝒰 ∧ topspace X ∩ S ⊆ ⋃ V))
proof −
  have ∗: (S ∩ ⋃𝒰 = topspace X ∩ S) = (topspace X ∩ S ⊆ ⋃𝒰)
    if ⋀x. x ∈ 𝒰 ⟹ openin X x for 𝒰
    by (blast dest: openin_subset [OF that])
  moreover have (𝒱 ⊆ 𝒰 ∧ S ∩ ⋃𝒱 = topspace X ∩ S) = (𝒱 ⊆ 𝒰 ∧ topspace X
∩ S ⊆ ⋃𝒱)
    if ∀ x. x ∈ 𝒰 ⟶ openin X x topspace X ∩ S ⊆ ⋃𝒰 countable 𝒱 for 𝒰 𝒱
    using that ∗ by blast
  ultimately show ?thesis
    unfolding Lindelof_space_def openin_subtopology_alt Ball_def
     apply (simp add: all_subset_image imp_conjL ex_countable_subset_image
flip: subset_iff)
    apply (intro all_cong1 imp_cong ex_cong, auto)
    done
qed

lemma Lindelof_space_subtopology_subset:
   S ⊆ topspace X
       ⟹ (Lindelof_space(subtopology X S) ⟷
           (∀𝒰. (∀ U ∈ 𝒰. openin X U) ∧ S ⊆ ⋃𝒰
               ⟶ (∃ V. countable V ∧ V ⊆ 𝒰 ∧ S ⊆ ⋃ V)))
 by (metis Lindelof_space_subtopology topspace_subtopology topspace_subtopology_subset)

lemma Lindelof_space_closedin_subtopology:
  assumes X: Lindelof_space X and clo: closedin X S
  shows Lindelof_space (subtopology X S)
proof −
  have S ⊆ topspace X
    by (simp add: clo closedin_subset)
  then show ?thesis
  proof (clarsimp simp add: Lindelof_space_subtopology_subset)
    show ∃ V. countable V ∧ V ⊆ ℱ ∧ S ⊆ ⋃ V
      if ∀ U∈ℱ. openin X U and S ⊆ ⋃ℱ for ℱ
    proof −
      have ∃𝒱. countable 𝒱 ∧ 𝒱 ⊆ insert (topspace X − S) ℱ ∧ ⋃𝒱 = topspace X
```

```
    proof (rule Lindelof_spaceD [OF X, of insert (topspace X - S) ℱ])
      show openin X U
        if U ∈ insert (topspace X - S) ℱ for U
        using that ‹∀U∈ℱ. openin X U› clo by blast
      show ⋃(insert (topspace X - S) ℱ) = topspace X
        apply auto
        apply (meson in_mono openin_closedin_eq that(1))
        using UnionE ‹S ⊆ ⋃ℱ› by auto
    qed
    then obtain 𝒱 where countable 𝒱 𝒱 ⊆ insert (topspace X - S) ℱ ⋃𝒱 =
topspace X
      by metis
    with ‹S ⊆ topspace X›
    show ?thesis
      by (rule_tac x=(𝒱 - {topspace X - S}) in exI) auto
  qed
 qed
qed

lemma Lindelof_space_continuous_map_image:
  assumes X: Lindelof_space X and f: continuous_map X Y f and fim: f ‘
(topspace X) = topspace Y
 shows Lindelof_space Y
proof -
 have ∃𝒱. countable 𝒱 ∧ 𝒱 ⊆ 𝒰 ∧ ⋃𝒱 = topspace Y
   if 𝒰: ⋀U. U ∈ 𝒰 ⟹ openin Y U and UU: ⋃𝒰 = topspace Y for 𝒰
 proof -
   define 𝒱 where 𝒱 ≡ (λU. {x ∈ topspace X. f x ∈ U}) ‘𝒰
   have ⋀V. V ∈ 𝒱 ⟹ openin X V
     unfolding 𝒱_def using 𝒰 continuous_map f by fastforce
   moreover have ⋃𝒱 = topspace X
     unfolding 𝒱_def using UU fim by fastforce
   ultimately have ∃𝒲. countable 𝒲 ∧ 𝒲 ⊆ 𝒱 ∧ ⋃𝒲 = topspace X
     using X by (simp add: Lindelof_space_def)
   then obtain 𝒞 where countable 𝒞 𝒞 ⊆ 𝒰 and 𝒞: (⋃U∈𝒞. {x ∈ topspace X. f
x ∈ U}) = topspace X
     by (metis (no_types, lifting) 𝒱_def countable_subset_image)
   moreover have ⋃𝒞 = topspace Y
   proof
     show ⋃𝒞 ⊆ topspace Y
       using UU 𝒞 ‹𝒞 ⊆ 𝒰› by fastforce
     have y ∈ ⋃𝒞 if y ∈ topspace Y for y
     proof -
       obtain x where x ∈ topspace X y = f x
         using that fim by (metis ‹y ∈ topspace Y› imageE)
       with 𝒞 show ?thesis by auto
     qed
     then show topspace Y ⊆ ⋃𝒞 by blast
   qed
```

```
    ultimately show ?thesis
      by blast
  qed
  then show ?thesis
    unfolding Lindelof_space_def
    by auto
qed

lemma Lindelof_space_quotient_map_image:
   ⟦quotient_map X Y q; Lindelof_space X⟧ ⟹ Lindelof_space Y
  by (meson Lindelof_space_continuous_map_image quotient_imp_continuous_map
quotient_imp_surjective_map)

lemma Lindelof_space_retraction_map_image:
   ⟦retraction_map X Y r; Lindelof_space X⟧ ⟹ Lindelof_space Y
  using Abstract_Topology.retraction_imp_quotient_map Lindelof_space_quotient_map_image
by blast

lemma locally_finite_cover_of_Lindelof_space:
  assumes X: Lindelof_space X and UU: topspace X ⊆ ⋃𝒰 and fin: locally_finite_in
X 𝒰
  shows countable 𝒰
proof −
  have UU_eq: ⋃𝒰 = topspace X
    by (meson UU fin locally_finite_in_def subset_antisym)
  obtain T where T: ⋀x. x ∈ topspace X ⟹ openin X (T x) ∧ x ∈ T x ∧ finite
{U ∈ 𝒰. U ∩ T x ≠ {}}
    using fin unfolding locally_finite_in_def by meson
  then obtain I where countable I I ⊆ topspace X and I: topspace X ⊆ ⋃(T ‘
I)
    using X unfolding Lindelof_space_alt
   by (drule_tac x=image T (topspace X) in spec) (auto simp: ex_countable_subset_image)
  show ?thesis
  proof (rule countable_subset)
    have ⋀i. i ∈ I ⟹ countable {U ∈ 𝒰. U ∩ T i ≠ {}}
      using T
      by (meson ‹I ⊆ topspace X› in_mono uncountable_infinite)
    then show countable (insert {} (⋃i∈I. {U ∈ 𝒰. U ∩ T i ≠ {}}))
      by (simp add: ‹countable I›)
  qed (use UU_eq I in auto)
qed


lemma Lindelof_space_proper_map_preimage:
  assumes f: proper_map X Y f and Y: Lindelof_space Y
  shows Lindelof_space X
proof (clarsimp simp: Lindelof_space_alt)
  show ∃𝒱. countable 𝒱 ∧ 𝒱 ⊆ 𝒰 ∧ topspace X ⊆ ⋃𝒱
    if 𝒰: ∀U∈𝒰. openin X U and sub_UU: topspace X ⊆ ⋃𝒰 for 𝒰
```

```
  proof −
    have ∃𝒱. finite 𝒱 ∧ 𝒱 ⊆ 𝒰 ∧ {x ∈ topspace X. f x = y} ⊆ ⋃𝒱 if y ∈ topspace
Y for y
    proof (rule compactinD)
      show compactin X {x ∈ topspace X. f x = y}
        using f proper_map_def that by fastforce
    qed (use sub_UU 𝒰 in auto)
    then obtain 𝒱 where 𝒱: ⋀y. y ∈ topspace Y ⟹ finite (𝒱 y) ∧ 𝒱 y ⊆ 𝒰 ∧
{x ∈ topspace X. f x = y} ⊆ ⋃(𝒱 y)
      by meson
    define 𝒲 where 𝒲 ≡ (λy. topspace Y − image f (topspace X − ⋃(𝒱 y))) ‘
topspace Y
    have ∀ U ∈ 𝒲. openin Y U
      using f 𝒰 𝒱 unfolding 𝒲_def proper_map_def closed_map_def
      by (simp add: closedin_diff openin_Union openin_diff subset_iff)
    moreover have topspace Y ⊆ ⋃𝒲
      using 𝒱 unfolding 𝒲_def by clarsimp fastforce
    ultimately have ∃𝒱. countable 𝒱 ∧ 𝒱 ⊆ 𝒲 ∧ topspace Y ⊆ ⋃𝒱
      using Y by (simp add: Lindelof_space_alt)
    then obtain I where countable I I ⊆ topspace Y
      and I: topspace Y ⊆ (⋃i∈I. topspace Y − f ‘ (topspace X − ⋃(𝒱 i)))
      unfolding 𝒲_def ex_countable_subset_image by metis
    show ?thesis
    proof (intro exI conjI)
      have ⋀i. i ∈ I ⟹ countable (𝒱 i)
        by (meson 𝒱 ‹I ⊆ topspace Y› in_mono uncountable_infinite)
      with ‹countable I› show countable (⋃(𝒱 ‘ I))
        by auto
      show ⋃(𝒱 ‘ I) ⊆ 𝒰
        using 𝒱 ‹I ⊆ topspace Y› by fastforce
      show topspace X ⊆ ⋃(⋃(𝒱 ‘ I))
      proof
        show x ∈ ⋃ (⋃ (𝒱 ‘ I)) if x ∈ topspace X for x
        proof −
          have f x ∈ topspace Y
            using f proper_map_imp_subset_topspace that by fastforce
          then show ?thesis
            using that I by auto
        qed
      qed
    qed
  qed
qed

lemma Lindelof_space_perfect_map_image:
  ⟦Lindelof_space X; perfect_map X Y f⟧ ⟹ Lindelof_space Y
  using Lindelof_space_quotient_map_image perfect_imp_quotient_map by blast

lemma Lindelof_space_perfect_map_image_eq:
```

perfect_map X Y f $\Longrightarrow$ *Lindelof_space X* $\longleftrightarrow$ *Lindelof_space Y*
using *Lindelof_space_perfect_map_image Lindelof_space_proper_map_preimage perfect_map_def* **by** *blast*

end

Chapter 3

Functional Analysis

3.1 A decision procedure for metric spaces

theory *Metric_Arith*
  imports *HOL.Real_Vector_Spaces*
begin

A port of the decision procedure "Formalization of metric spaces in HOL Light" [6] for the type class *metric_space*, with the *Argo* solver as backend.

named_theorems *metric_prenex*
named_theorems *metric_nnf*
named_theorems *metric_unfold*
named_theorems *metric_pre_arith*

lemmas *pre_arith_simps* =
  max.bounded_iff max_less_iff_conj
  le_max_iff_disj less_max_iff_disj
  simp_thms HOL.eq_commute
declare *pre_arith_simps* [*metric_pre_arith*]

lemmas *unfold_simps* =
  Un_iff subset_iff disjoint_iff_not_equal
  Ball_def Bex_def
declare *unfold_simps* [*metric_unfold*]

declare *HOL.nnf_simps*(*4*) [*metric_prenex*]

lemma *imp_prenex* [*metric_prenex*]:
  $\bigwedge P\ Q.\ (\exists x.\ P\ x) \longrightarrow Q \equiv \forall x.\ (P\ x \longrightarrow Q)$
  $\bigwedge P\ Q.\ P \longrightarrow (\exists x.\ Q\ x) \equiv \exists x.\ (P \longrightarrow Q\ x)$
  $\bigwedge P\ Q.\ (\forall x.\ P\ x) \longrightarrow Q \equiv \exists x.\ (P\ x \longrightarrow Q)$
  $\bigwedge P\ Q.\ P \longrightarrow (\forall x.\ Q\ x) \equiv \forall x.\ (P \longrightarrow Q\ x)$
  by *simp_all*

lemma *ex_prenex* [*metric_prenex*]:

$\bigwedge P\ Q.\ (\exists\, x.\ P\ x) \wedge Q \equiv \exists\, x.\ (P\ x \wedge Q)$
$\bigwedge P\ Q.\ P \wedge (\exists\, x.\ Q\ x) \equiv \exists\, x.\ (P \wedge Q\ x)$
$\bigwedge P\ Q.\ (\exists\, x.\ P\ x) \vee Q \equiv \exists\, x.\ (P\ x \vee Q)$
$\bigwedge P\ Q.\ P \vee (\exists\, x.\ Q\ x) \equiv \exists\, x.\ (P \vee Q\ x)$
$\bigwedge P.\ \neg(\exists\, x.\ P\ x) \equiv \forall\, x.\ \neg P\ x$
by *simp_all*

lemma *all_prenex* [*metric_prenex*]:
$\bigwedge P\ Q.\ (\forall\, x.\ P\ x) \wedge Q \equiv \forall\, x.\ (P\ x \wedge Q)$
$\bigwedge P\ Q.\ P \wedge (\forall\, x.\ Q\ x) \equiv \forall\, x.\ (P \wedge Q\ x)$
$\bigwedge P\ Q.\ (\forall\, x.\ P\ x) \vee Q \equiv \forall\, x.\ (P\ x \vee Q)$
$\bigwedge P\ Q.\ P \vee (\forall\, x.\ Q\ x) \equiv \forall\, x.\ (P \vee Q\ x)$
$\bigwedge P.\ \neg(\forall\, x.\ P\ x) \equiv \exists\, x.\ \neg P\ x$
by *simp_all*

lemma *nnf_thms* [*metric_nnf*]:
$(\neg\ (P \wedge Q)) = (\neg\ P \vee \neg\ Q)$
$(\neg\ (P \vee Q)) = (\neg\ P \wedge \neg\ Q)$
$(P \longrightarrow Q) = (\neg\ P \vee Q)$
$(P = Q) \longleftrightarrow (P \vee \neg\ Q) \wedge (\neg\ P \vee Q)$
$(\neg\ (P = Q)) \longleftrightarrow (\neg\ P \vee \neg\ Q) \wedge (P \vee Q)$
$(\neg\ \neg\ P) = P$
by *blast+*

lemmas *nnf_simps* = *nnf_thms linorder_not_less linorder_not_le*
declare *nnf_simps*[*metric_nnf*]

lemma *ball_insert*: $(\forall\, x{\in}\mathit{insert}\ a\ B.\ P\ x) = (P\ a \wedge (\forall\, x{\in}B.\ P\ x))$
by *blast*

lemma *Sup_insert_insert*:
fixes *a*::*real*
shows *Sup* (*insert a* (*insert b s*)) = *Sup* (*insert* (*max a b*) *s*)
by (*simp add*: *Sup_real_def*)

lemma *real_abs_dist*: $|\mathit{dist}\ x\ y| = \mathit{dist}\ x\ y$
by *simp*

lemma *maxdist_thm* [*THEN HOL.eq_reflection*]:
assumes *finite s x* ∈ *s y* ∈ *s*
defines $\bigwedge a.\ f\ a \equiv |\mathit{dist}\ x\ a - \mathit{dist}\ a\ y|$
shows *dist x y* = *Sup* (*f* ‘ *s*)
proof −
have *dist x y* ≤ *Sup* (*f* ‘ *s*)
proof −
have *finite* (*f* ‘ *s*)
by (*simp add*: ‹*finite s*›)
moreover have $|\mathit{dist}\ x\ y - \mathit{dist}\ y\ y| \in f$ ‘ *s*
by (*metis* ‹*y* ∈ *s*› *f_def imageI*)

```
    ultimately show ?thesis
      using le_cSup_finite by simp
  qed
  also have Sup (f ‘ s) ≤ dist x y
    using ‹x ∈ s› cSUP_least[of s f] abs_dist_diff_le
    unfolding f_def
    by blast
  finally show ?thesis .
qed

theorem metric_eq_thm [THEN HOL.eq_reflection]:
  x ∈ s ⟹ y ∈ s ⟹ x = y ⟷ (∀ a∈s. dist x a = dist y a)
  by auto

ML_file ‹metric_arith.ML›

method_setup metric = ‹
  Scan.succeed (SIMPLE_METHOD' o Metric_Arith.metric_arith_tac)
› prove simple linear statements in metric spaces (∀∃₍p₎ fragment)

end
```

Chapter 4

Elementary Metric Spaces

theory *Elementary_Metric_Spaces*
imports
Abstract_Topology_2
Metric_Arith
begin

4.1 Open and closed balls

definition *ball* :: *'a*::*metric_space* ⇒ *real* ⇒ *'a set*
where *ball x e* = {*y*. *dist x y* < *e*}

definition *cball* :: *'a*::*metric_space* ⇒ *real* ⇒ *'a set*
where *cball x e* = {*y*. *dist x y* ≤ *e*}

definition *sphere* :: *'a*::*metric_space* ⇒ *real* ⇒ *'a set*
where *sphere x e* = {*y*. *dist x y* = *e*}

lemma *mem_ball* [*simp*, *metric_unfold*]: *y* ∈ *ball x e* ⟷ *dist x y* < *e*
by (*simp add*: *ball_def*)

lemma *mem_cball* [*simp*, *metric_unfold*]: *y* ∈ *cball x e* ⟷ *dist x y* ≤ *e*
by (*simp add*: *cball_def*)

lemma *mem_sphere* [*simp*]: *y* ∈ *sphere x e* ⟷ *dist x y* = *e*
by (*simp add*: *sphere_def*)

lemma *ball_trivial* [*simp*]: *ball x* 0 = {}
by *auto*

lemma *cball_trivial* [*simp*]: *cball x* 0 = {*x*}
by *auto*

lemma *sphere_trivial* [*simp*]: *sphere x* 0 = {*x*}
by *auto*

lemma *disjoint_ballI*: *dist x y* $\geq$ *r+s* $\Longrightarrow$ *ball x r* $\cap$ *ball y s* = {}
using *dist_triangle_less_add not_le* **by** *fastforce*

lemma *disjoint_cballI*: *dist x y* > *r* + *s* $\Longrightarrow$ *cball x r* $\cap$ *cball y s* = {}
by (*metis add_mono disjoint_iff_not_equal dist_triangle2 dual_order.trans leD mem_cball*)

lemma *sphere_empty* [*simp*]: *r* < *0* $\Longrightarrow$ *sphere a r* = {}
for *a* :: *'a*::*metric_space*
by *auto*

lemma *centre_in_ball* [*simp*]: *x* $\in$ *ball x e* $\longleftrightarrow$ *0* < *e*
by *simp*

lemma *centre_in_cball* [*simp*]: *x* $\in$ *cball x e* $\longleftrightarrow$ *0* $\leq$ *e*
by *simp*

lemma *ball_subset_cball* [*simp*, *intro*]: *ball x e* $\subseteq$ *cball x e*
by (*simp add*: *subset_eq*)

lemma *mem_ball_imp_mem_cball*: *x* $\in$ *ball y e* $\Longrightarrow$ *x* $\in$ *cball y e*
by *auto*

lemma *sphere_cball* [*simp*,*intro*]: *sphere z r* $\subseteq$ *cball z r*
by *force*

lemma *cball_diff_sphere*: *cball a r* $-$ *sphere a r* = *ball a r*
by *auto*

lemma *subset_ball*[*intro*]: *d* $\leq$ *e* $\Longrightarrow$ *ball x d* $\subseteq$ *ball x e*
by *auto*

lemma *subset_cball*[*intro*]: *d* $\leq$ *e* $\Longrightarrow$ *cball x d* $\subseteq$ *cball x e*
by *auto*

lemma *mem_ball_leI*: *x* $\in$ *ball y e* $\Longrightarrow$ *e* $\leq$ *f* $\Longrightarrow$ *x* $\in$ *ball y f*
by *auto*

lemma *mem_cball_leI*: *x* $\in$ *cball y e* $\Longrightarrow$ *e* $\leq$ *f* $\Longrightarrow$ *x* $\in$ *cball y f*
by *auto*

lemma *cball_trans*: *y* $\in$ *cball z b* $\Longrightarrow$ *x* $\in$ *cball y a* $\Longrightarrow$ *x* $\in$ *cball z* (*b* + *a*)
by *metric*

lemma *ball_max_Un*: *ball a* (*max r s*) = *ball a r* $\cup$ *ball a s*
by *auto*

lemma *ball_min_Int*: *ball a* (*min r s*) = *ball a r* $\cap$ *ball a s*

by *auto*

lemma *cball_max_Un*: *cball a* (*max r s*) = *cball a r* ∪ *cball a s*
by *auto*

lemma *cball_min_Int*: *cball a* (*min r s*) = *cball a r* ∩ *cball a s*
by *auto*

lemma *cball_diff_eq_sphere*: *cball a r* − *ball a r* = *sphere a r*
by *auto*

lemma *open_ball* [*intro*, *simp*]: *open* (*ball x e*)
proof −
have *open* (*dist x* −‘ {..<*e*})
by (*intro open_vimage open_lessThan continuous_intros*)
also have *dist x* −‘ {..<*e*} = *ball x e*
by *auto*
finally show *?thesis* .
qed

lemma *open_contains_ball*: *open S* ⟷ (∀ *x*∈*S*. ∃ *e*>*0*. *ball x e* ⊆ *S*)
by (*simp add*: *open_dist subset_eq Ball_def dist_commute*)

lemma *openI* [*intro?*]: (⋀*x*. *x*∈*S* ⟹ ∃ *e*>*0*. *ball x e* ⊆ *S*) ⟹ *open S*
by (*auto simp*: *open_contains_ball*)

lemma *openE*[*elim?*]:
assumes *open S x*∈*S*
obtains *e* **where** *e*>*0 ball x e* ⊆ *S*
using *assms* **unfolding** *open_contains_ball* **by** *auto*

lemma *open_contains_ball_eq*: *open S* ⟹ *x*∈*S* ⟷ (∃ *e*>*0*. *ball x e* ⊆ *S*)
by (*metis open_contains_ball subset_eq centre_in_ball*)

lemma *ball_eq_empty*[*simp*]: *ball x e* = {} ⟷ *e* ≤ *0*
unfolding *mem_ball set_eq_iff*
by (*simp add*: *not_less*) *metric*

lemma *ball_empty*: *e* ≤ *0* ⟹ *ball x e* = {}
by *simp*

lemma *closed_cball* [*iff*]: *closed* (*cball x e*)
proof −
have *closed* (*dist x* −‘ {..*e*})
by (*intro closed_vimage closed_atMost continuous_intros*)
also have *dist x* −‘ {..*e*} = *cball x e*
by *auto*
finally show *?thesis* .
qed

lemma *open_contains_cball*: *open S* $\longleftrightarrow$ ($\forall$ *x*$\in$*S*. $\exists$ *e*>*0*. *cball x e* $\subseteq$ *S*)
proof −
{
fix *x* **and** *e*::*real*
assume *x*$\in$*S e*>*0 ball x e* $\subseteq$ *S*
then have $\exists$ *d*>*0*. *cball x d* $\subseteq$ *S*
unfolding *subset_eq* **by** (*rule_tac x*=*e*/*2* **in** *exI*, *auto*)
}
moreover
{
fix *x* **and** *e*::*real*
assume *x*$\in$*S e*>*0 cball x e* $\subseteq$ *S*
then have $\exists$ *d*>*0*. *ball x d* $\subseteq$ *S*
using *mem_ball_imp_mem_cball* **by** *blast*
}
ultimately show *?thesis*
unfolding *open_contains_ball* **by** *auto*
qed

lemma *open_contains_cball_eq*: *open S* $\Longrightarrow$ ($\forall$ *x*. *x* $\in$ *S* $\longleftrightarrow$ ($\exists$ *e*>*0*. *cball x e* $\subseteq$ *S*))
by (*metis open_contains_cball subset_eq order_less_imp_le centre_in_cball*)

lemma *eventually_nhds_ball*: *d* > *0* $\Longrightarrow$ *eventually* (λ*x*. *x* $\in$ *ball z d*) (*nhds z*)
by (*rule eventually_nhds_in_open*) *simp_all*

lemma *eventually_at_ball*: *d* > *0* $\Longrightarrow$ *eventually* (λ*t*. *t* $\in$ *ball z d* $\wedge$ *t* $\in$ *A*) (*at z within A*)
unfolding *eventually_at* **by** (*intro exI*[*of* _ *d*]) (*simp_all add*: *dist_commute*)

lemma *eventually_at_ball'*: *d* > *0* $\Longrightarrow$ *eventually* (λ*t*. *t* $\in$ *ball z d* $\wedge$ *t* $\neq$ *z* $\wedge$ *t* $\in$ *A*) (*at z within A*)
unfolding *eventually_at* **by** (*intro exI*[*of* _ *d*]) (*simp_all add*: *dist_commute*)

lemma *at_within_ball*: *e* > *0* $\Longrightarrow$ *dist x y* < *e* $\Longrightarrow$ *at y within ball x e* = *at y*
by (*subst at_within_open*) *auto*

lemma *atLeastAtMost_eq_cball*:
fixes *a b*::*real*
shows {*a* .. *b*} = *cball* ((*a* + *b*)/*2*) ((*b* − *a*)/*2*)
by (*auto simp*: *dist_real_def field_simps*)

lemma *cball_eq_atLeastAtMost*:
fixes *a b*::*real*
shows *cball a b* = {*a* − *b* .. *a* + *b*}
by (*auto simp*: *dist_real_def*)

lemma *greaterThanLessThan_eq_ball*:

fixes *a b*::*real*
shows $\{a <..< b\} = ball\ ((a + b)/2)\ ((b - a)/2)$
by (*auto simp*: *dist_real_def field_simps*)

lemma *ball_eq_greaterThanLessThan*:
fixes *a b*::*real*
shows $ball\ a\ b = \{a - b <..< a + b\}$
by (*auto simp*: *dist_real_def*)

lemma *interior_ball* [*simp*]: $interior\ (ball\ x\ e) = ball\ x\ e$
by (*simp add*: *interior_open*)

lemma *cball_eq_empty* [*simp*]: $cball\ x\ e = \{\} \longleftrightarrow e < 0$
by (*smt* (*verit*, *best*) *Diff_empty ball_eq_empty cball_diff_sphere centre_in_ball centre_in_cball sphere_empty*)

lemma *cball_empty* [*simp*]: $e < 0 \Longrightarrow cball\ x\ e = \{\}$
by *simp*

lemma *cball_sing*:
fixes $x :: 'a$::*metric_space*
shows $e = 0 \Longrightarrow cball\ x\ e = \{x\}$
by *simp*

lemma *ball_divide_subset*: $d \geq 1 \Longrightarrow ball\ x\ (e/d) \subseteq ball\ x\ e$
by (*metis ball_eq_empty div_by_1 frac_le linear subset_ball zero_less_one*)

lemma *ball_divide_subset_numeral*: $ball\ x\ (e\ /\ numeral\ w) \subseteq ball\ x\ e$
using *ball_divide_subset one_le_numeral* **by** *blast*

lemma *cball_divide_subset*: $d \geq 1 \Longrightarrow cball\ x\ (e/d) \subseteq cball\ x\ e$
by (*smt* (*verit*, *best*) *cball_empty div_by_1 frac_le subset_cball zero_le_divide_iff*)

lemma *cball_divide_subset_numeral*: $cball\ x\ (e\ /\ numeral\ w) \subseteq cball\ x\ e$
using *cball_divide_subset one_le_numeral* **by** *blast*

lemma *cball_scale*:
assumes $a \neq 0$
shows $(\lambda x.\ a *_R x)\ `\ cball\ c\ r = cball\ (a *_R c :: 'a :: real\_normed\_vector)\ (|a| * r)$
proof −
have *1*: $(\lambda x.\ a *_R x)\ `\ cball\ c\ r \subseteq cball\ (a *_R c)\ (|a| * r)$ **if** $a \neq 0$ **for** *a r* **and** $c :: 'a$
proof *safe*
fix *x*
assume *x*: $x \in cball\ c\ r$
have $dist\ (a *_R c)\ (a *_R x) = norm\ (a *_R c - a *_R x)$
by (*auto simp*: *dist_norm*)
also have $a *_R c - a *_R x = a *_R (c - x)$

```
    by (simp add: algebra_simps)
  finally show a *R x ∈ cball (a *R c) (|a| * r)
    using that x by (auto simp: dist_norm)
 qed

 have cball (a *R c) (|a| * r) = (λx. a *R x) ‘ (λx. inverse a *R x) ‘ cball (a *R
c) (|a| * r)
   unfolding image_image using assms by simp
 also have ... ⊆ (λx. a *R x) ‘ cball (inverse a *R (a *R c)) (|inverse a| * (|a|
* r))
   using assms by (intro image_mono 1) auto
 also have ... = (λx. a *R x) ‘ cball c r
   using assms by (simp add: algebra_simps)
 finally have cball (a *R c) (|a| * r) ⊆ (λx. a *R x) ‘ cball c r .
 moreover from assms have (λx. a *R x) ‘ cball c r ⊆ cball (a *R c) (|a| * r)
   by (intro 1) auto
 ultimately show ?thesis by blast
qed

lemma ball_scale:
 assumes a ≠ 0
 shows   (λx. a *R x) ‘ ball c r = ball (a *R c :: 'a :: real_normed_vector) (|a|
* r)
proof −
 have 1: (λx. a *R x) ‘ ball c r ⊆ ball (a *R c) (|a| * r) if a ≠ 0 for a r and c
:: 'a
 proof safe
   fix x
   assume x: x ∈ ball c r
   have dist (a *R c) (a *R x) = norm (a *R c − a *R x)
     by (auto simp: dist_norm)
   also have a *R c − a *R x = a *R (c − x)
     by (simp add: algebra_simps)
   finally show a *R x ∈ ball (a *R c) (|a| * r)
     using that x by (auto simp: dist_norm)
 qed

 have ball (a *R c) (|a| * r) = (λx. a *R x) ‘ (λx. inverse a *R x) ‘ ball (a *R c)
(|a| * r)
   unfolding image_image using assms by simp
 also have ... ⊆ (λx. a *R x) ‘ ball (inverse a *R (a *R c)) (|inverse a| * (|a| *
r))
   using assms by (intro image_mono 1) auto
 also have ... = (λx. a *R x) ‘ ball c r
   using assms by (simp add: algebra_simps)
 finally have ball (a *R c) (|a| * r) ⊆ (λx. a *R x) ‘ ball c r .
 moreover from assms have (λx. a *R x) ‘ ball c r ⊆ ball (a *R c) (|a| * r)
   by (intro 1) auto
 ultimately show ?thesis by blast
```

qed

lemma *frequently_atE*:
fixes $x :: 'a :: metric\_space$
assumes *frequently P* (*at x within s*)
shows $\exists f.$ *filterlim f* (*at x within s*) *sequentially* $\wedge$ $(\forall n.$ *P* (*f n*))
proof −
have $\exists y.$ $y \in s \cap$ (*ball x* (*1 / real* (*Suc n*)) − $\{x\}$) $\wedge$ *P y* **for** *n*
proof −
have $\exists z{\in}s.$ $z \neq x \wedge$ *dist z x* $<$ (*1 / real* (*Suc n*)) $\wedge$ *P z*
by (*metis assms divide_pos_pos frequently_at of_nat_0_less_iff zero_less_Suc zero_less_one*)
then show *?thesis*
by (*auto simp*: *dist_commute conj_commute*)
qed
then obtain *f* **where** *f*: $\bigwedge n.$ *f n* $\in s \cap$ (*ball x* (*1 / real* (*Suc n*)) − $\{x\}$) $\wedge$ *P* (*f n*)
by *metis*
have *filterlim f* (*nhds x*) *sequentially*
unfolding *tendsto_iff*
proof *clarify*
fix *e* :: *real*
assume *e*: $e > 0$
then obtain *n* **where** *n*: $Suc\ n > 1 / e$
by (*meson le_nat_floor lessI not_le*)
have *dist* (*f k*) *x* $< e$ **if** $k \geq n$ **for** *k*
proof −
have *dist* (*f k*) *x* $<$ *1 / real* (*Suc k*)
using *f*[*of k*] **by** (*auto simp*: *dist_commute*)
also have $\ldots \leq$ *1 / real* (*Suc n*)
using *that* **by** (*intro divide_left_mono*) *auto*
also have $\ldots < e$
using *n e* **by** (*simp add*: *field_simps*)
finally show *?thesis* **.**
qed
thus $\forall_F$ *k in sequentially*. *dist* (*f k*) *x* $< e$
unfolding *eventually_at_top_linorder* **by** *blast*
qed
moreover have *f n* $\neq x$ **for** *n*
using *f*[*of n*] **by** *auto*
ultimately have *filterlim f* (*at x within s*) *sequentially*
using *f* **by** (*auto simp*: *filterlim_at*)
with *f* **show** *?thesis*
by *blast*
qed

4.2 Limit Points

lemma *islimpt_approachable*:

fixes $x :: 'a::metric\_space$
shows x *islimpt* $S \longleftrightarrow (\forall\, e{>}0.\ \exists\, x'{\in}S.\ x' \neq x \wedge dist\ x'\ x < e)$
unfolding *islimpt_iff_eventually eventually_at* **by** *fast*

lemma *islimpt_approachable_le*: x *islimpt* $S \longleftrightarrow (\forall\, e{>}0.\ \exists\, x'{\in}\ S.\ x' \neq x \wedge dist\ x'\ x \leq e)$
for $x :: 'a::metric\_space$
unfolding *islimpt_approachable*
using *approachable_lt_le2* [**where** $f{=}\lambda y.\ dist\ y\ x$ **and** $P{=}\lambda y.\ y \notin S \vee y = x$ **and** $Q{=}\lambda x.\ True$]
by *auto*

lemma *limpt_of_limpts*: x *islimpt* $\{y.\ y\ islimpt\ S\} \Longrightarrow x$ *islimpt* S
for $x :: 'a::metric\_space$
by (*metis islimpt_def islimpt_eq_acc_point mem_Collect_eq*)

lemma *closed_limpts*: *closed* $\{x::'a::metric\_space.\ x\ islimpt\ S\}$
using *closed_limpt limpt_of_limpts* **by** *blast*

lemma *limpt_of_closure*: x *islimpt closure* $S \longleftrightarrow x$ *islimpt* S
for $x :: 'a::metric\_space$
by (*auto simp*: *closure_def islimpt_Un dest*: *limpt_of_limpts*)

lemma *islimpt_eq_infinite_ball*: x *islimpt* $S \longleftrightarrow (\forall\, e{>}0.\ infinite(S \cap ball\ x\ e))$
unfolding *islimpt_eq_acc_point*
by (*metis open_ball Int_commute Int_mono finite_subset open_contains_ball_eq subset_eq*)

lemma *islimpt_eq_infinite_cball*: x *islimpt* $S \longleftrightarrow (\forall\, e{>}0.\ infinite(S \cap cball\ x\ e))$
unfolding *islimpt_eq_infinite_ball*
by (*metis open_ball ball_subset_cball centre_in_ball finite_Int inf.absorb_iff2 inf_assoc open_contains_cball_eq*)

4.3 Perfect Metric Spaces

lemma *perfect_choose_dist*: $0 < r \Longrightarrow \exists\, a.\ a \neq x \wedge dist\ a\ x < r$
for $x :: 'a::\{perfect\_space,metric\_space\}$
using *islimpt_UNIV* [*of* x] **by** (*simp add*: *islimpt_approachable*)

lemma *cball_eq_sing*:
fixes $x :: 'a::\{metric\_space,perfect\_space\}$
shows $cball\ x\ e = \{x\} \longleftrightarrow e = 0$
by (*smt* (*verit*, *best*) *open_ball ball_eq_empty ball_subset_cball cball_empty cball_trivial*
not_open_singleton subset_singleton_iff)

4.4 Finite and discrete

lemma *finite_ball_include*:
fixes a :: $'a$::*metric_space*
assumes *finite* S
shows $\exists\, e{>}0.\ S \subseteq \mathit{ball}\ a\ e$
using *assms*
proof *induction*
case (*insert* x S)
then obtain *e0* **where** *e0*$>$*0* **and** *e0*:$S \subseteq \mathit{ball}\ a\ e0$ **by** *auto*
define e **where** $e = \mathit{max}\ e0\ (2 * \mathit{dist}\ a\ x)$
have *e*$>$*0* **unfolding** *e_def* **using** ‹*e0*$>$*0*› **by** *auto*
moreover have $\mathit{insert}\ x\ S \subseteq \mathit{ball}\ a\ e$
using *e0* ‹*e*$>$*0*› **unfolding** *e_def* **by** *auto*
ultimately show *?case* **by** *auto*
qed (*auto intro*: *zero_less_one*)

lemma *finite_set_avoid*:
fixes a :: $'a$::*metric_space*
assumes *finite* S
shows $\exists\, d{>}0.\ \forall\, x{\in}S.\ x \neq a \longrightarrow d \leq \mathit{dist}\ a\ x$
using *assms*
proof *induction*
case (*insert* x S)
then obtain d **where** $d > 0$ **and** d: $\forall\, x{\in}S.\ x \neq a \longrightarrow d \leq \mathit{dist}\ a\ x$
by *blast*
show *?case*
by (*smt* (*verit*, *del_insts*) *dist_pos_lt insert.IH insert_iff*)
qed (*auto intro*: *zero_less_one*)

lemma *discrete_imp_closed*:
fixes S :: $'a$::*metric_space set*
assumes e: $0 < e$
and d: $\forall\, x \in S.\ \forall\, y \in S.\ \mathit{dist}\ y\ x < e \longrightarrow y = x$
shows *closed* S
proof −
have *False* **if** C: $\bigwedge e.\ e{>}0 \Longrightarrow \exists\, x'{\in}S.\ x' \neq x \wedge \mathit{dist}\ x'\ x < e$ **for** x
proof −
from e **have** *e2*: $e/2 > 0$ **by** *arith*
from C[*OF e2*] **obtain** y **where** y: $y \in S$ $y \neq x$ $\mathit{dist}\ y\ x < e/2$
by *blast*
from *e2* *y*(*2*) **have** *mp*: $\mathit{min}\ (e/2)\ (\mathit{dist}\ x\ y) > 0$
by *simp*
from *d* *y* C[*OF mp*] **show** *?thesis*
by *metric*
qed
then show *?thesis*
by (*metis islimpt_approachable closed_limpt* [**where** $'a{=}'a$])
qed

lemma *discrete_imp_not_islimpt*:
  assumes *e*: $0 < e$
    and *d*: $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow dist\ y\ x < e \Longrightarrow y = x$
  shows $\neg\ x\ islimpt\ S$
proof
  assume $x\ islimpt\ S$
  hence $x\ islimpt\ S - \{x\}$
    by (*meson islimpt_punctured*)
  moreover from *assms* **have** *closed* $(S - \{x\})$
    by (*intro discrete_imp_closed*) *auto*
  ultimately show *False*
    unfolding *closed_limpt* **by** *blast*
qed

4.5 Interior

lemma *mem_interior*: $x \in interior\ S \longleftrightarrow (\exists\, e{>}0.\ ball\ x\ e \subseteq S)$
  using *open_contains_ball_eq* [**where** *S*=*interior S*]
  by (*simp add*: *open_subset_interior*)

lemma *mem_interior_cball*: $x \in interior\ S \longleftrightarrow (\exists\, e{>}0.\ cball\ x\ e \subseteq S)$
  by (*meson ball_subset_cball interior_subset mem_interior open_contains_cball open_interior*
      subset_trans)

lemma *ball_iff_cball*: $(\exists\, r{>}0.\ ball\ x\ r \subseteq U) \longleftrightarrow (\exists\, r{>}0.\ cball\ x\ r \subseteq U)$
  by (*meson mem_interior mem_interior_cball*)

4.6 Frontier

lemma *frontier_straddle*:
  fixes *a* :: *'a*::*metric_space*
  shows $a \in frontier\ S \longleftrightarrow (\forall\, e{>}0.\ (\exists\, x{\in}S.\ dist\ a\ x < e) \wedge (\exists x.\ x \notin S \wedge dist\ a\ x < e))$
  unfolding *frontier_def closure_interior*
  by (*auto simp*: *mem_interior subset_eq ball_def*)

4.7 Limits

proposition *Lim*: $(f \longrightarrow l)\ net \longleftrightarrow trivial\_limit\ net \vee (\forall\, e{>}0.\ eventually\ (\lambda x.\ dist\ (f\ x)\ l < e)\ net)$
  by (*auto simp*: *tendsto_iff trivial_limit_eq*)

Show that they yield usual definitions in the various cases.

proposition *Lim_within_le*: $(f \longrightarrow l)(at\ a\ within\ S) \longleftrightarrow$
    $(\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x{\in}S.\ 0 < dist\ x\ a \wedge dist\ x\ a \leq d \longrightarrow dist\ (f\ x)\ l < e)$

by (*auto simp*: *tendsto_iff eventually_at_le*)

proposition *Lim_within*: $(f \longrightarrow l)\ (at\ a\ within\ S) \longleftrightarrow$
$(\forall e>0.\ \exists d>0.\ \forall x \in S.\ 0 < dist\ x\ a \wedge dist\ x\ a < d \longrightarrow dist\ (f\ x)\ l < e)$
by (*auto simp*: *tendsto_iff eventually_at*)

corollary *Lim_withinI* [*intro?*]:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists d>0.\ \forall x \in S.\ 0 < dist\ x\ a \wedge dist\ x\ a < d \longrightarrow dist\ (f\ x)\ l \leq e$
shows $(f \longrightarrow l)\ (at\ a\ within\ S)$
unfolding *Lim_within* **by** (*smt* (*verit*, *ccfv_SIG*) *assms zero_less_dist_iff*)

proposition *Lim_at*: $(f \longrightarrow l)\ (at\ a) \longleftrightarrow$
$(\forall e>0.\ \exists d>0.\ \forall x.\ 0 < dist\ x\ a \wedge dist\ x\ a < d \longrightarrow dist\ (f\ x)\ l < e)$
by (*auto simp*: *tendsto_iff eventually_at*)

lemma *Lim_transform_within_set*:
fixes $a :: 'a$::*metric_space* **and** $l :: 'b$::*metric_space*
shows $[\![(f \longrightarrow l)\ (at\ a\ within\ S);\ eventually\ (\lambda x.\ x \in S \longleftrightarrow x \in T)\ (at\ a)]\!]$
$\Longrightarrow (f \longrightarrow l)\ (at\ a\ within\ T)$
by (*simp add*: *eventually_at Lim_within*) (*smt* (*verit*, *best*))

Another limit point characterization.

lemma *limpt_sequential_inj*:
fixes $x :: 'a$::*metric_space*
shows $x\ islimpt\ S \longleftrightarrow$
$(\exists f.\ (\forall n$::*nat*$.\ f\ n \in S - \{x\}) \wedge inj\ f \wedge (f \longrightarrow x)\ sequentially)$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have $\forall e>0.\ \exists x' \in S.\ x' \neq x \wedge dist\ x'\ x < e$
by (*force simp*: *islimpt_approachable*)
then obtain y **where** y: $\bigwedge e.\ e>0 \Longrightarrow y\ e \in S \wedge y\ e \neq x \wedge dist\ (y\ e)\ x < e$
by *metis*
define f **where** $f \equiv rec\_nat\ (y\ 1)\ (\lambda n\ fn.\ y\ (min\ (inverse(2\ \hat{}\ (Suc\ n)))\ (dist\ fn\ x)))$
have [*simp*]: $f\ 0 = y\ 1$
and *fSuc*: $f(Suc\ n) = y\ (min\ (inverse(2\ \hat{}\ (Suc\ n)))\ (dist\ (f\ n)\ x))$ **for** n
by (*simp_all add*: *f_def*)
have f: $f\ n \in S \wedge (f\ n \neq x) \wedge dist\ (f\ n)\ x < inverse(2\ \hat{}\ n)$ **for** n
proof (*induction n*)
case *0* **show** *?case*
by (*simp add*: *y*)
next
case (*Suc n*) **then show** *?case*
by (*smt* (*verit*, *best*) *fSuc dist_pos_lt inverse_positive_iff_positive y zero_less_power*)
qed
show *?rhs*
proof (*intro exI conjI allI*)

```
    show ⋀n. f n ∈ S − {x}
      using f by blast
    have dist (f n) x < dist (f m) x if m < n for m n
    using that
    proof (induction n)
      case 0 then show ?case by simp
    next
      case (Suc n)
      then consider m < n | m = n using less_Suc_eq by blast
      then show ?case
      proof cases
        assume m < n
       have dist (f(Suc n)) x = dist (y (min (inverse(2 ^(Suc n))) (dist (f n) x)))
x
          by (simp add: fSuc)
        also have ... < dist (f n) x
          by (metis dist_pos_lt f min.strict_order_iff min_less_iff_conj y)
        also have ... < dist (f m) x
          using Suc.IH ‹m < n› by blast
        finally show ?thesis .
      next
        assume m = n then show ?case
          by (smt (verit, best) dist_pos_lt f fSuc y)
      qed
    qed
    then show inj f
      by (metis less_irrefl linorder_injI)
    have ⋀e n. ⟦0 < e; nat ⌈1 / e⌉ ≤ n⟧ ⟹ dist (f n) x < e
      apply (rule less_trans [OF f [THEN conjunct2, THEN conjunct2]])
      by (simp add: divide_simps order_le_less_trans)
    then show f ⟶ x
      using lim_sequentially by blast
  qed
next
  assume ?rhs
  then show ?lhs
    by (fastforce simp add: islimpt_approachable lim_sequentially)
qed

lemma Lim_dist_ubound:
  assumes ¬(trivial_limit net)
    and (f ⟶ l) net
    and eventually (λx. dist a (f x) ≤ e) net
  shows dist a l ≤ e
  using assms by (fast intro: tendsto_le tendsto_intros)
```

4.8 Continuity

Derive the epsilon-delta forms, which we often use as "definitions"

proposition *continuous_within_eps_delta*:
continuous (*at x within s*) *f* ⟷ (∀ *e*>*0*. ∃ *d*>*0*. ∀ *x'*∈ *s*. *dist x' x* < *d* −−> *dist* (*f x'*) (*f x*) < *e*)
unfolding *continuous_within* **and** *Lim_within* **by** *fastforce*

corollary *continuous_at_eps_delta*:
continuous (*at x*) *f* ⟷ (∀ *e* > *0*. ∃ *d* > *0*. ∀ *x'*. *dist x' x* < *d* ⟶ *dist* (*f x'*) (*f x*) < *e*)
using *continuous_within_eps_delta* [*of x UNIV f*] **by** *simp*

lemma *continuous_at_right_real_increasing*:
fixes *f* :: *real* ⇒ *real*
assumes *nondecF*: ⋀*x y*. *x* ≤ *y* ⟹ *f x* ≤ *f y*
shows *continuous* (*at_right a*) *f* ⟷ (∀ *e*>*0*. ∃ *d*>*0*. *f* (*a* + *d*) − *f a* < *e*)
apply (*simp add*: *greaterThan_def dist_real_def continuous_within Lim_within_le*)
apply (*intro all_cong ex_cong*)
by (*smt* (*verit*, *best*) *nondecF*)

lemma *continuous_at_left_real_increasing*:
assumes *nondecF*: ⋀ *x y*. *x* ≤ *y* ⟹ *f x* ≤ ((*f y*) :: *real*)
shows (*continuous* (*at_left* (*a* :: *real*)) *f*) ⟷ (∀ *e* > *0*. ∃ *delta* > *0*. *f a* − *f* (*a* − *delta*) < *e*)
apply (*simp add*: *lessThan_def dist_real_def continuous_within Lim_within_le*)
apply (*intro all_cong ex_cong*)
by (*smt* (*verit*) *nondecF*)

Versions in terms of open balls.

lemma *continuous_within_ball*:
continuous (*at x within S*) *f* ⟷
(∀ *e* > *0*. ∃ *d* > *0*. *f* ‘ (*ball x d* ∩ *S*) ⊆ *ball* (*f x*) *e*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
{
fix *e* :: *real*
assume *e* > *0*
then obtain *d* **where** *d*>*0* **and** *d*: ∀ *y*∈*S*. *0* < *dist y x* ∧ *dist y x* < *d* ⟶ *dist* (*f y*) (*f x*) < *e*
using ‹*?lhs*›[*unfolded continuous_within Lim_within*] **by** *auto*
{ fix *y*
assume *y* ∈ *f* ‘ (*ball x d* ∩ *S*) **then have** *y* ∈ *ball* (*f x*) *e*
using *d* ‹*e* > *0*› **by** (*auto simp*: *dist_commute*)
}
then have ∃ *d*>*0*. *f* ‘ (*ball x d* ∩ *S*) ⊆ *ball* (*f x*) *e*
using ‹*d* > *0*› **by** *blast*
}

then show *?rhs* **by** *auto*
next
assume *?rhs*
then show *?lhs*
apply (*simp add*: *continuous_within Lim_within ball_def subset_eq*)
by (*metis* (*mono_tags*, *lifting*) *Int_iff dist_commute mem_Collect_eq*)
qed

lemma *continuous_at_ball*:
continuous (*at x*) *f* ⟷ (∀ *e*>0. ∃ *d*>0. *f* ‘ (*ball x d*) ⊆ *ball* (*f x*) *e*)
apply (*simp add*: *continuous_at Lim_at subset_eq Ball_def Bex_def image_iff*)
by (*smt* (*verit*, *ccfv_threshold*) *dist_commute dist_self*)

Define setwise continuity in terms of limits within the set.

lemma *continuous_on_iff*:
continuous_on s f ⟷
(∀ *x*∈*s*. ∀ *e*>0. ∃ *d*>0. ∀ *x*′∈*s*. *dist x*′ *x* < *d* ⟶ *dist* (*f x*′) (*f x*) < *e*)
unfolding *continuous_on_def Lim_within*
by (*metis dist_pos_lt dist_self*)

lemma *continuous_within_E*:
assumes *continuous* (*at x within S*) *f e*>0
obtains *d* **where** *d*>0 ⋀*x*′. ⟦*x*′∈ *S*; *dist x*′ *x* ≤ *d*⟧ ⟹ *dist* (*f x*′) (*f x*) < *e*
using *assms* **unfolding** *continuous_within_eps_delta*
by (*metis dense order_le_less_trans*)

lemma *continuous_onI* [*intro?*]:
assumes ⋀*x e*. ⟦*e* > 0; *x* ∈ *S*⟧ ⟹ ∃ *d*>0. ∀ *x*′∈*S*. *dist x*′ *x* < *d* ⟶ *dist* (*f x*′) (*f x*) ≤ *e*
shows *continuous_on S f*
apply (*simp add*: *continuous_on_iff*, *clarify*)
apply (*rule ex_forward* [*OF assms* [*OF half_gt_zero*]], *auto*)
done

Some simple consequential lemmas.

lemma *continuous_onE*:
assumes *continuous_on s f x*∈*s e*>0
obtains *d* **where** *d*>0 ⋀*x*′. ⟦*x*′ ∈ *s*; *dist x*′ *x* ≤ *d*⟧ ⟹ *dist* (*f x*′) (*f x*) < *e*
using *assms*
unfolding *continuous_on_iff* **by** (*metis dense order_le_less_trans*)

The usual transformation theorems.

lemma *continuous_transform_within*:
fixes *f g* :: ′*a*::*metric_space* ⇒ ′*b*::*topological_space*
assumes *continuous* (*at x within s*) *f*
and 0 < *d*
and *x* ∈ *s*
and ⋀*x*′. ⟦*x*′ ∈ *s*; *dist x*′ *x* < *d*⟧ ⟹ *f x*′ = *g x*′
shows *continuous* (*at x within s*) *g*

using *assms*
unfolding *continuous_within* **by** (*force intro*: *Lim_transform_within*)

4.9 Closure and Limit Characterization

lemma *closure_approachable*:
fixes S :: *'a*::*metric_space set*
shows $x \in$ *closure* $S \longleftrightarrow (\forall\, e{>}0.\ \exists\, y{\in}S.\ dist\ y\ x < e)$
using *dist_self* **by** (*force simp*: *closure_def islimpt_approachable*)

lemma *closure_approachable_le*:
fixes S :: *'a*::*metric_space set*
shows $x \in$ *closure* $S \longleftrightarrow (\forall\, e{>}0.\ \exists\, y{\in}S.\ dist\ y\ x \leq e)$
unfolding *closure_approachable*
using *dense* **by** *force*

lemma *closure_approachableD*:
assumes $x \in$ *closure* S $e{>}0$
shows $\exists\, y{\in}S.\ dist\ x\ y < e$
using *assms* **unfolding** *closure_approachable* **by** (*auto simp*: *dist_commute*)

lemma *closed_approachable*:
fixes S :: *'a*::*metric_space set*
shows *closed* $S \Longrightarrow (\forall\, e{>}0.\ \exists\, y{\in}S.\ dist\ y\ x < e) \longleftrightarrow x \in S$
by (*metis closure_closed closure_approachable*)

lemma *closure_contains_Inf*:
fixes S :: *real set*
assumes $S \neq \{\}$ *bdd_below* S
shows *Inf* $S \in$ *closure* S
proof −
have ∗: $\forall\, x{\in}S.\ Inf\ S \leq x$
using *cInf_lower*[*of* _ S] *assms* **by** *metis*
{ fix e :: *real*
assume $e > 0$
then have *Inf* $S <$ *Inf* $S + e$ **by** *simp*
with *assms* **obtain** x **where** $x \in S$ $x <$ *Inf* $S + e$
using *cInf_lessD* **by** *blast*
with ∗ **have** $\exists\, x{\in}S.\ dist\ x\ (Inf\ S) < e$
using *dist_real_def* **by** *force*
}
then show *?thesis* **unfolding** *closure_approachable* **by** *auto*
qed

lemma *closure_contains_Sup*:
fixes S :: *real set*
assumes $S \neq \{\}$ *bdd_above* S
shows *Sup* $S \in$ *closure* S
proof −

```
  have ∗: ∀ x∈S. x ≤ Sup S
    using cSup_upper[of _ S] assms by metis
  {
    fix e :: real
    assume e > 0
    then have Sup S − e < Sup S by simp
    with assms obtain x where x ∈ S Sup S − e < x
      using less_cSupE by blast
    with ∗ have ∃ x∈S. dist x (Sup S) < e
      using dist_real_def by force
  }
  then show ?thesis unfolding closure_approachable by auto
qed

lemma not_trivial_limit_within_ball:
  ¬ trivial_limit (at x within S) ⟷ (∀ e>0. S ∩ ball x e − {x} ≠ {})
  (is ?lhs ⟷ ?rhs)
proof
  show ?rhs if ?lhs
  proof −
    { fix e :: real
      assume e > 0
      then obtain y where y ∈ S − {x} and dist y x < e
        using ‹?lhs› not_trivial_limit_within[of x S] closure_approachable[of x S
− {x}]
        by auto
      then have y ∈ S ∩ ball x e − {x}
        unfolding ball_def by (simp add: dist_commute)
      then have S ∩ ball x e − {x} ≠ {} by blast
    }
    then show ?thesis by auto
  qed
  show ?lhs if ?rhs
  proof −
    { fix e :: real
      assume e > 0
      then obtain y where y ∈ S ∩ ball x e − {x}
        using ‹?rhs› by blast
      then have y ∈ S − {x} and dist y x < e
        unfolding ball_def by (simp_all add: dist_commute)
      then have ∃ y ∈ S − {x}. dist y x < e
        by auto
    }
    then show ?thesis
      using not_trivial_limit_within[of x S] closure_approachable[of x S − {x}]
      by auto
  qed
qed
```

4.10 Boundedness

definition (**in** *metric_space*) *bounded* :: *'a set* ⇒ *bool*
where *bounded S* ⟷ (∃ *x e*. ∀ *y*∈*S*. *dist x y* ≤ *e*)

lemma *bounded_subset_cball*: *bounded S* ⟷ (∃ *e x*. *S* ⊆ *cball x e* ∧ *0* ≤ *e*)
unfolding *bounded_def subset_eq* **by** *auto* (*meson order_trans zero_le_dist*)

lemma *bounded_any_center*: *bounded S* ⟷ (∃ *e*. ∀ *y*∈*S*. *dist a y* ≤ *e*)
unfolding *bounded_def*
by *auto* (*metis add.commute add_le_cancel_right dist_commute dist_triangle_le*)

lemma *bounded_iff*: *bounded S* ⟷ (∃ *a*. ∀ *x*∈*S*. *norm x* ≤ *a*)
unfolding *bounded_any_center* [**where** *a*=*0*]
by (*simp add*: *dist_norm*)

lemma *bdd_above_norm*: *bdd_above* (*norm* ‘ *X*) ⟷ *bounded X*
by (*simp add*: *bounded_iff bdd_above_def*)

lemma *bounded_norm_comp*: *bounded* ((λ*x*. *norm* (*f x*)) ‘ *S*) = *bounded* (*f* ‘ *S*)
by (*simp add*: *bounded_iff*)

lemma *boundedI*:
assumes ⋀*x*. *x* ∈ *S* ⟹ *norm x* ≤ *B*
shows *bounded S*
using *assms bounded_iff* **by** *blast*

lemma *bounded_empty* [*simp*]: *bounded* {}
by (*simp add*: *bounded_def*)

lemma *bounded_subset*: *bounded T* ⟹ *S* ⊆ *T* ⟹ *bounded S*
by (*metis bounded_def subset_eq*)

lemma *bounded_interior*[*intro*]: *bounded S* ⟹ *bounded*(*interior S*)
by (*metis bounded_subset interior_subset*)

lemma *bounded_closure*[*intro*]:
assumes *bounded S*
shows *bounded* (*closure S*)
proof −
from *assms* **obtain** *x* **and** *a* **where** *a*: ∀ *y*∈*S*. *dist x y* ≤ *a*
unfolding *bounded_def* **by** *auto*
{ fix *y*
assume *y* ∈ *closure S*
then obtain *f* **where** *f*: ∀ *n*. *f n* ∈ *S* (*f* ⟶ *y*) *sequentially*
unfolding *closure_sequential* **by** *auto*
have ∀ *n*. *f n* ∈ *S* ⟶ *dist x* (*f n*) ≤ *a* **using** *a* **by** *simp*
then have *eventually* (λ*n*. *dist x* (*f n*) ≤ *a*) *sequentially*
by (*simp add*: *f*(*1*))

```
    then have dist x y ≤ a
      using Lim_dist_ubound f(2) trivial_limit_sequentially by blast
  }
  then show ?thesis
    unfolding bounded_def by auto
qed

lemma bounded_closure_image: bounded (f ‘ closure S) ⟹ bounded (f ‘ S)
  by (simp add: bounded_subset closure_subset image_mono)

lemma bounded_cball[simp,intro]: bounded (cball x e)
  unfolding bounded_def  using mem_cball by blast

lemma bounded_ball[simp,intro]: bounded (ball x e)
  by (metis ball_subset_cball bounded_cball bounded_subset)

lemma bounded_Un[simp]: bounded (S ∪ T) ⟷ bounded S ∧ bounded T
 by (auto simp: bounded_def) (metis Un_iff bounded_any_center le_max_iff_disj)

lemma bounded_Union[intro]: finite F ⟹ ∀ S∈F. bounded S ⟹ bounded (⋃ F)
  by (induct rule: finite_induct[of F]) auto

lemma bounded_UN [intro]: finite A ⟹ ∀ x∈A. bounded (B x) ⟹ bounded
(⋃ x∈A. B x)
  by auto

lemma bounded_insert [simp]: bounded (insert x S) ⟷ bounded S
proof −
  have ∀ y∈{x}. dist x y ≤ 0
    by simp
  then have bounded {x}
    unfolding bounded_def by fast
  then show ?thesis
    by (metis insert_is_Un bounded_Un)
qed

lemma bounded_subset_ballI: S ⊆ ball x r ⟹ bounded S
  by (meson bounded_ball bounded_subset)

lemma bounded_subset_ballD:
  assumes bounded S shows ∃ r. 0 < r ∧ S ⊆ ball x r
proof −
  obtain e::real and y where S ⊆ cball y e 0 ≤ e
    using assms by (auto simp: bounded_subset_cball)
  then show ?thesis
    by (intro exI[where x=dist x y + e + 1]) metric
qed

lemma finite_imp_bounded [intro]: finite S ⟹ bounded S
```

by (*induct set*: *finite*) *simp_all*

lemma *bounded_Int*[*intro*]: *bounded S* ∨ *bounded T* ⟹ *bounded* (*S* ∩ *T*)
by (*metis Int_lower1 Int_lower2 bounded_subset*)

lemma *bounded_diff*[*intro*]: *bounded S* ⟹ *bounded* (*S* − *T*)
by (*metis Diff_subset bounded_subset*)

lemma *bounded_dist_comp*:
assumes *bounded* (*f* ‘ *S*) *bounded* (*g* ‘ *S*)
shows *bounded* ((λ*x*. *dist* (*f x*) (*g x*)) ‘ *S*)
proof −
from *assms* **obtain** *M1 M2* **where** ∗: *dist* (*f x*) *undefined* ≤ *M1 dist undefined* (*g x*) ≤ *M2* **if** *x* ∈ *S* **for** *x*
by (*auto simp*: *bounded_any_center*[*of* _ *undefined*] *dist_commute*)
have *dist* (*f x*) (*g x*) ≤ *M1* + *M2* **if** *x* ∈ *S* **for** *x*
using ∗[*OF that*]
by *metric*
then show *?thesis*
by (*auto intro*!: *boundedI*)
qed

lemma *bounded_Times*:
assumes *bounded s bounded t*
shows *bounded* (*s* × *t*)
proof −
obtain *x y a b* **where** ∀ *z*∈*s*. *dist x z* ≤ *a* ∀ *z*∈*t*. *dist y z* ≤ *b*
using *assms* [*unfolded bounded_def*] **by** *auto*
then have ∀ *z*∈*s* × *t*. *dist* (*x*, *y*) *z* ≤ *sqrt* (*a*^{2} + *b*^{2})
by (*auto simp*: *dist_Pair_Pair real_sqrt_le_mono add_mono power_mono*)
then show *?thesis* **unfolding** *bounded_any_center* [**where** *a*=(*x*, *y*)] **by** *auto*
qed

4.11 Compactness

lemma *compact_imp_bounded*:
assumes *compact U*
shows *bounded U*
proof −
have *compact U* ∀ *x*∈*U*. *open* (*ball x 1*) *U* ⊆ (⋃ *x*∈*U*. *ball x 1*)
using *assms* **by** *auto*
then obtain *D* **where** *D*: *D* ⊆ *U finite D U* ⊆ (⋃ *x*∈*D*. *ball x 1*)
by (*metis compactE_image*)
from ‹*finite D*› **have** *bounded* (⋃ *x*∈*D*. *ball x 1*)
by (*simp add*: *bounded_UN*)
then show *bounded U* **using** ‹*U* ⊆ (⋃ *x*∈*D*. *ball x 1*)›
by (*rule bounded_subset*)
qed

lemma *continuous_on_compact_bound*:
assumes *compact A continuous_on A f*
obtains *B* **where** $B \geq 0$ $\bigwedge x.\ x \in A \Longrightarrow norm\ (f\ x) \leq B$
proof −
have *compact* (*f* ‘ *A*) **by** (*metis assms compact_continuous_image*)
then obtain *B* **where** $\forall\, x{\in}A.\ norm\ (f\ x) \leq B$
by (*auto dest*!: *compact_imp_bounded simp*: *bounded_iff*)
hence *max B 0* ≥ 0 **and** $\forall\, x{\in}A.\ norm\ (f\ x) \leq max\ B\ 0$ **by** *auto*
thus *?thesis* **using** *that* **by** *blast*
qed

lemma *closure_Int_ball_not_empty*:
assumes $S \subseteq closure\ T\ x \in S\ r > 0$
shows $T \cap ball\ x\ r \neq \{\}$
using *assms centre_in_ball closure_iff_nhds_not_empty* **by** *blast*

lemma *compact_sup_maxdistance*:
fixes *S* :: *'a*::*metric_space set*
assumes *compact S*
and $S \neq \{\}$
shows $\exists\, x{\in}S.\ \exists\, y{\in}S.\ \forall\, u{\in}S.\ \forall\, v{\in}S.\ dist\ u\ v \leq dist\ x\ y$
proof −
have *compact* (*S* × *S*)
using ‹*compact S*› **by** (*intro compact_Times*)
moreover have $S \times S \neq \{\}$
using ‹$S \neq \{\}$› **by** *auto*
moreover have *continuous_on* (*S* × *S*) (λ*x*. *dist* (*fst x*) (*snd x*))
by (*intro continuous_at_imp_continuous_on ballI continuous_intros*)
ultimately show *?thesis*
using *continuous_attains_sup*[*of S* × *S* λ*x*. *dist* (*fst x*) (*snd x*)] **by** *auto*
qed

If A is a compact subset of an open set B in a metric space, then there exists an $\varepsilon > 0$ such that the Minkowski sum of A with an open ball of radius ε is also a subset of B.

lemma *compact_subset_open_imp_ball_epsilon_subset*:
assumes *compact A open B A* ⊆ *B*
obtains *e* **where** $e > 0$ $(\bigcup x{\in}A.\ ball\ x\ e) \subseteq B$
proof −
have $\forall\, x{\in}A.\ \exists\, e.\ e > 0 \wedge ball\ x\ e \subseteq B$
using *assms* **unfolding** *open_contains_ball* **by** *blast*
then obtain *e* **where** *e*: $\bigwedge x.\ x \in A \Longrightarrow e\ x > 0\ \bigwedge x.\ x \in A \Longrightarrow ball\ x\ (e\ x) \subseteq B$
by *metis*
define *C* **where** *C* = *e* ‘ *A*
obtain *X* **where** *X*: *X* ⊆ *A finite X A* ⊆ $(\bigcup c{\in}X.\ ball\ c\ (e\ c\ /\ 2))$
using *assms*(*1*)
proof (*rule compactE_image*)
show *open* (*ball x* (*e x* / *2*)) **if** $x \in A$ **for** *x*

```
    by simp
   show A ⊆ (⋃c∈A. ball c (e c / 2))
    using e by auto
  qed auto

  define e' where e' = Min (insert 1 ((λx. e x / 2) ` X))
  have e' > 0
   unfolding e'_def using e X by (subst Min_gr_iff) auto
  have e': e' ≤ e x / 2 if x ∈ X for x
   using that X unfolding e'_def by (intro Min.coboundedI) auto

  show ?thesis
  proof
   show e' > 0
    by fact
  next
   show (⋃x∈A. ball x e') ⊆ B
   proof clarify
    fix x y assume xy: x ∈ A y ∈ ball x e'
    from xy(1) X obtain z where z: z ∈ X x ∈ ball z (e z / 2)
     by auto
    have dist y z ≤ dist x y + dist z x
     by (metis dist_commute dist_triangle)
    also have dist z x < e z / 2
     using xy z by auto
    also have dist x y < e'
     using xy by auto
    also have ... ≤ e z / 2
     using z by (intro e') auto
    finally have y ∈ ball z (e z)
     by (simp add: dist_commute)
    also have ... ⊆ B
     using z X by (intro e) auto
    finally show y ∈ B .
   qed
  qed
qed

lemma compact_subset_open_imp_cball_epsilon_subset:
  assumes compact A open B A ⊆ B
  obtains e where e > 0 (⋃x∈A. cball x e) ⊆ B
proof −
  obtain e where e > 0 and e: (⋃x∈A. ball x e) ⊆ B
   using compact_subset_open_imp_ball_epsilon_subset [OF assms] by blast
  then have (⋃x∈A. cball x (e / 2)) ⊆ (⋃x∈A. ball x e)
   by auto
  with ‹0 < e› that show ?thesis
   by (metis e half_gt_zero_iff order_trans)
qed
```

Totally bounded

proposition *seq_compact_imp_totally_bounded*:
assumes *seq_compact S*
shows $\forall e{>}0.\ \exists k.\ \mathit{finite}\ k \wedge k \subseteq S \wedge S \subseteq (\bigcup x{\in}k.\ \mathit{ball}\ x\ e)$
proof −
{ fix *e*::*real* **assume** $e > 0$ **assume** ∗: $\bigwedge k.\ \mathit{finite}\ k \Longrightarrow k \subseteq S \Longrightarrow \neg\ S \subseteq (\bigcup x{\in}k.\ \mathit{ball}\ x\ e)$
let *?Q* = $\lambda x\ n\ r.\ r \in S \wedge (\forall m < (n{::}\mathit{nat}).\ \neg\ (\mathit{dist}\ (x\ m)\ r < e))$
have $\exists x.\ \forall n{::}\mathit{nat}.\ ?Q\ x\ n\ (x\ n)$
proof (*rule dependent_wellorder_choice*)
fix *n x* **assume** $\bigwedge y.\ y < n \Longrightarrow ?Q\ x\ y\ (x\ y)$
then have $\neg\ S \subseteq (\bigcup x{\in}x\ `\ \{0..{<}n\}.\ \mathit{ball}\ x\ e)$
using ∗[*of x* ` {*0* ..< *n*}] **by** (*auto simp*: *subset_eq*)
then obtain *z* **where** $z{:}z{\in}S\ z \notin (\bigcup x{\in}x\ `\ \{0..{<}n\}.\ \mathit{ball}\ x\ e)$
unfolding *subset_eq* **by** *auto*
show $\exists r.\ ?Q\ x\ n\ r$
using *z* **by** *auto*
qed *simp*
then obtain *x* **where** $\forall n{::}\mathit{nat}.\ x\ n \in S$ **and** $x{:}\bigwedge n\ m.\ m < n \Longrightarrow \neg\ (\mathit{dist}\ (x\ m)\ (x\ n) < e)$
by *blast*
then obtain *l r* **where** $l \in S$ **and** *r*:*strict_mono r* **and** $((x \circ r) \longrightarrow l)$ *sequentially*
using *assms* **by** (*metis seq_compact_def*)
then have *Cauchy* $(x \circ r)$
using *LIMSEQ_imp_Cauchy* **by** *auto*
then obtain *N*::*nat* **where** $\bigwedge m\ n.\ N \leq m \Longrightarrow N \leq n \Longrightarrow \mathit{dist}\ ((x \circ r)\ m)\ ((x \circ r)\ n) < e$
unfolding *Cauchy_def* **using** ‹$e > 0$› **by** *blast*
then have *False*
using *x*[*of r N r* (*N*+*1*)] *r* **by** (*auto simp*: *strict_mono_def*) **}**
then show *?thesis*
by *metis*
qed

Heine-Borel theorem

proposition *seq_compact_imp_Heine_Borel*:
fixes *S* :: *'a* :: *metric_space set*
assumes *seq_compact S*
shows *compact S*
proof −
from *seq_compact_imp_totally_bounded*[*OF* ‹*seq_compact S*›]
obtain *f* **where** *f*: $\forall e{>}0.\ \mathit{finite}\ (f\ e) \wedge f\ e \subseteq S \wedge S \subseteq (\bigcup x{\in}f\ e.\ \mathit{ball}\ x\ e)$
unfolding *choice_iff'* **..**
define *K* **where** $K = (\lambda(x,\ r).\ \mathit{ball}\ x\ r)\ `\ ((\bigcup e \in \mathbb{Q} \cap \{0\ <..\}.\ f\ e) \times \mathbb{Q})$
have *countably_compact S*
using ‹*seq_compact S*› **by** (*rule seq_compact_imp_countably_compact*)
then show *compact S*

```
  proof (rule countably_compact_imp_compact)
    show countable K
      unfolding K_def using f
      by (auto intro: countable_finite countable_subset countable_rat
            intro!: countable_image countable_SIGMA countable_UN)
    show ∀b∈K. open b by (auto simp: K_def)
  next
    fix T x
    assume T: open T x ∈ T and x: x ∈ S
    from openE[OF T] obtain e where 0 < e ball x e ⊆ T
      by auto
    then have 0 < e/2 ball x (e/2) ⊆ T
      by auto
    from Rats_dense_in_real[OF ‹0 < e/2›] obtain r where r ∈ ℚ 0 < r r <
e/2
      by auto
    from f[rule_format, of r] ‹0 < r› ‹x ∈ S› obtain k where k ∈ f r x ∈ ball k
r
      by auto
    from ‹r ∈ ℚ› ‹0 < r› ‹k ∈ f r› have ball k r ∈ K
      by (auto simp: K_def)
    then show ∃b∈K. x ∈ b ∧ b ∩ S ⊆ T
    proof (rule bexI[rotated], safe)
      fix y
      assume y ∈ ball k r
      with ‹r < e/2› ‹x ∈ ball k r› have dist x y < e
        by (intro dist_triangle_half_r [of k _ e]) (auto simp: dist_commute)
      with ‹ball x e ⊆ T› show y ∈ T
        by auto
    next
      show x ∈ ball k r by fact
    qed
  qed
qed

proposition compact_eq_seq_compact_metric:
  compact (S :: 'a::metric_space set) ⟷ seq_compact S
  using compact_imp_seq_compact seq_compact_imp_Heine_Borel by blast

proposition compact_def: — this is the definition of compactness in HOL Light
  compact (S :: 'a::metric_space set) ⟷
   (∀f. (∀n. f n ∈ S) ⟶ (∃l∈S. ∃r::nat⇒nat. strict_mono r ∧ (f ∘ r) ⟶⟶
l))
  unfolding compact_eq_seq_compact_metric seq_compact_def by auto
```

Complete the chain of compactness variants

```
proposition compact_eq_Bolzano_Weierstrass:
  fixes S :: 'a::metric_space set
```

shows *compact S* $\longleftrightarrow$ ($\forall T.$ *infinite* $T \wedge T \subseteq S \longrightarrow (\exists x \in S.$ *x islimpt T*))
by (*meson Bolzano_Weierstrass_imp_seq_compact Heine_Borel_imp_Bolzano_Weierstrass seq_compact_imp_Heine_Borel*)

proposition *Bolzano_Weierstrass_imp_bounded*:
($\bigwedge T.$ $[\![$*infinite T*; $T \subseteq S]\!] \Longrightarrow (\exists x \in S.$ *x islimpt T*)) $\Longrightarrow$ *bounded S*
using *compact_imp_bounded* **unfolding** *compact_eq_Bolzano_Weierstrass* **by** *metis*

4.12 Banach fixed point theorem

theorem *banach_fix*:— TODO: rename to *Banach_fix*
assumes *s*: *complete s s* $\neq \{\}$
and *c*: $0 \leq c$ $c < 1$
and *f*: *f ' s* $\subseteq s$
and *lipschitz*: $\forall x{\in}s.\ \forall y{\in}s.$ *dist* (*f x*) (*f y*) $\leq c *$ *dist x y*
shows $\exists!x{\in}s.$ *f x* = *x*
proof −
from *c* **have** $1 - c > 0$ **by** *simp*

from *s*(*2*) **obtain** *z0* **where** *z0*: *z0* $\in$ *s* **by** *blast*
define *z* **where** *z n* = (*f* ^^ *n*) *z0* **for** *n*
with *f z0* **have** *z_in_s*: *z n* $\in$ *s* **for** *n* :: *nat*
by (*induct n*) *auto*
define *d* **where** *d* = *dist* (*z 0*) (*z 1*)

have *fzn*: *f* (*z n*) = *z* (*Suc n*) **for** *n*
by (*simp add*: *z_def*)
have *cf_z*: *dist* (*z n*) (*z* (*Suc n*)) $\leq$ (*c* ^ *n*) $*$ *d* **for** *n* :: *nat*
proof (*induct n*)
case *0*
then show *?case*
by (*simp add*: *d_def*)
next
case (*Suc m*)
with ‹*0* $\leq$ *c*› **have** *c* $*$ *dist* (*z m*) (*z* (*Suc m*)) $\leq$ *c* ^ *Suc m* $*$ *d*
using *mult_left_mono*[*of dist* (*z m*) (*z* (*Suc m*)) *c* ^ *m* $*$ *d c*] **by** *simp*
then show *?case*
using *lipschitz*[*THEN bspec*[**where** *x*=*z m*], *OF z_in_s*, *THEN bspec*[**where** *x*=*z* (*Suc m*)], *OF z_in_s*]
by (*simp add*: *fzn mult_le_cancel_left*)
qed

have *cf_z2*: (*1* − *c*) $*$ *dist* (*z m*) (*z* (*m* + *n*)) $\leq$ (*c* ^ *m*) $*$ *d* $*$ (*1* − *c* ^ *n*) **for** *n m* :: *nat*
proof (*induct n*)
case *0*
show *?case* **by** *simp*
next

```
    case (Suc k)
    from c have (1 - c) * dist (z m) (z (m + Suc k)) ≤
        (1 - c) * (dist (z m) (z (m + k)) + dist (z (m + k)) (z (Suc (m + k))))
      by (simp add: dist_triangle)
    also from c cf_z[of m + k] have ... ≤ (1 - c) * (dist (z m) (z (m + k)) +
c ^ (m + k) * d)
      by simp
    also from Suc have ... ≤ c ^ m * d * (1 - c ^ k) + (1 - c) * c ^ (m + k)
* d
      by (simp add: field_simps)
    also have ... = (c ^ m) * (d * (1 - c ^ k) + (1 - c) * c ^ k * d)
      by (simp add: power_add field_simps)
    also from c have ... ≤ (c ^ m) * d * (1 - c ^ Suc k)
      by (simp add: field_simps)
    finally show ?case by simp
  qed

  have ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (z m) (z n) < e if e > 0 for e
  proof (cases d = 0)
    case True
    from ‹1 - c > 0› have (1 - c) * x ≤ 0 ⟷ x ≤ 0 for x
      by (simp add: mult_le_0_iff)
    with c cf_z2[of 0] True have z n = z0 for n
      by (simp add: z_def)
    with ‹e > 0› show ?thesis by simp
  next
    case False
    with zero_le_dist[of z 0 z 1] have d > 0
      by (metis d_def less_le)
    with ‹1 - c > 0› ‹e > 0› have 0 < e * (1 - c) / d
      by simp
    with c obtain N where N: c ^ N < e * (1 - c) / d
      using real_arch_pow_inv[of e * (1 - c) / d c] by auto
    have *: dist (z m) (z n) < e if m > n and as: m ≥ N n ≥ N for m n :: nat
    proof -
      from c ‹n ≥ N› have *: c ^ n ≤ c ^ N
        using power_decreasing[OF ‹n≥N›, of c] by simp
      from c ‹m > n› have 1 - c ^ (m - n) > 0
        using power_strict_mono[of c 1 m - n] by simp
      with ‹d > 0› ‹0 < 1 - c› have **: d * (1 - c ^ (m - n)) / (1 - c) > 0
        by simp
      from cf_z2[of n m - n] ‹m > n›
      have dist (z m) (z n) ≤ c ^ n * d * (1 - c ^ (m - n)) / (1 - c)
      by (simp add: pos_le_divide_eq[OF ‹1 - c > 0›] mult.commute dist_commute)
      also have ... ≤ c ^ N * d * (1 - c ^ (m - n)) / (1 - c)
        using mult_right_mono[OF * order_less_imp_le[OF **]]
        by (simp add: mult.assoc)
      also have ... < (e * (1 - c) / d) * d * (1 - c ^ (m - n)) / (1 - c)
        using mult_strict_right_mono[OF N **] by (auto simp: mult.assoc)
```

```
    also from c ‹d > 0› ‹1 − c > 0› have ... = e * (1 − c ^(m − n))
      by simp
    also from c ‹1 − c ^(m − n) > 0› ‹e > 0› have ... ≤ e
      using mult_right_le_one_le[of e 1 − c ^(m − n)] by auto
    finally show ?thesis by simp
  qed
  have dist (z n) (z m) < e if N ≤ m N ≤ n for m n :: nat
  proof (cases n = m)
    case True
    with ‹e > 0› show ?thesis by simp
  next
    case False
    with *[of n m] *[of m n] and that show ?thesis
      by (auto simp: dist_commute nat_neq_iff)
  qed
  then show ?thesis by auto
 qed
 then have Cauchy z
  by (metis metric_CauchyI)
 then obtain x where x∈s and x:(z ⟶ x) sequentially
  using s(1)[unfolded compact_def complete_def, THEN spec[where x=z]] and
z_in_s by auto

 define e where e = dist (f x) x
 have e = 0
 proof (rule ccontr)
  assume e ≠ 0
  then have e > 0
   unfolding e_def using zero_le_dist[of f x x]
   by (metis dist_eq_0_iff dist_nz e_def)
  then obtain N where N:∀ n≥N. dist (z n) x < e/2
   using x[unfolded lim_sequentially, THEN spec[where x=e/2]] by auto
  then have N':dist (z N) x < e/2 by auto
  have *: c * dist (z N) x ≤ dist (z N) x
   unfolding mult_le_cancel_right2
   using zero_le_dist[of z N x] and c
   by (metis dist_eq_0_iff dist_nz order_less_asym less_le)
  have dist (f (z N)) (f x) ≤ c * dist (z N) x
   using lipschitz[THEN bspec[where x=z N], THEN bspec[where x=x]]
   using z_in_s[of N] ‹x∈s›
   using c
   by auto
  also have ... < e/2
   using N' and c using * by auto
  finally show False
   unfolding fzn
   using N[THEN spec[where x=Suc N]] and dist_triangle_half_r[of z (Suc
N) f x e x]
   unfolding e_def
```

```
    by auto
  qed
  then have f x = x by (auto simp: e_def)
  moreover have y = x if f y = y y ∈ s for y
  proof −
    from ‹x ∈ s› ‹f x = x› that have dist x y ≤ c ∗ dist x y
      using lipschitz[THEN bspec[where x=x], THEN bspec[where x=y]] by simp
    with c and zero_le_dist[of x y] have dist x y = 0
      by (simp add: mult_le_cancel_right1)
    then show ?thesis by simp
  qed
  ultimately show ?thesis
    using ‹x∈s› by blast
qed
```

4.13 Edelstein fixed point theorem

```
theorem Edelstein_fix:
  fixes S :: 'a::metric_space set
  assumes S: compact S S ≠ {}
    and gs: (g ‘ S) ⊆ S
    and dist: ∀ x∈S. ∀ y∈S. x ≠ y ⟶ dist (g x) (g y) < dist x y
  shows ∃!x∈S. g x = x
proof −
  let ?D = (λx. (x, x)) ‘ S
  have D: compact ?D ?D ≠ {}
    by (rule compact_continuous_image)
     (auto intro!: S continuous_Pair continuous_ident simp: continuous_on_eq_continuous_within)

  have ⋀x y e. x ∈ S ⟹ y ∈ S ⟹ 0 < e ⟹ dist y x < e ⟹ dist (g y) (g x)
< e
    using dist by fastforce
  then have continuous_on S g
    by (auto simp: continuous_on_iff)
  then have cont: continuous_on ?D (λx. dist ((g ∘ fst) x) (snd x))
    unfolding continuous_on_eq_continuous_within
    by (intro continuous_dist ballI continuous_within_compose)
       (auto intro!: continuous_fst continuous_snd continuous_ident simp: im-
age_image)

  obtain a where a ∈ S and le: ⋀x. x ∈ S ⟹ dist (g a) a ≤ dist (g x) x
    using continuous_attains_inf[OF D cont] by auto

  have g a = a
  proof (rule ccontr)
    assume g a ≠ a
    with ‹a ∈ S› gs have dist (g (g a)) (g a) < dist (g a) a
      by (intro dist[rule_format]) auto
    moreover have dist (g a) a ≤ dist (g (g a)) (g a)
```

```
    using ‹a ∈ S› gs by (intro le) auto
   ultimately show False by auto
 qed
 moreover have ⋀x. x ∈ S ⟹ g x = x ⟹ x = a
   using dist[THEN bspec[where x=a]] ‹g a = a› and ‹a∈S› by auto
 ultimately show ∃!x∈S. g x = x
   using ‹a ∈ S› by blast
qed
```

4.14 The diameter of a set

```
definition diameter :: 'a::metric_space set ⇒ real where
 diameter S = (if S = {} then 0 else SUP (x,y)∈S×S. dist x y)

lemma diameter_empty [simp]: diameter{} = 0
 by (auto simp: diameter_def)

lemma diameter_singleton [simp]: diameter{x} = 0
 by (auto simp: diameter_def)

lemma diameter_le:
 assumes S ≠ {} ∨ 0 ≤ d
   and no: ⋀x y. ⟦x ∈ S; y ∈ S⟧ ⟹ norm(x - y) ≤ d
 shows diameter S ≤ d
 using assms
 by (auto simp: dist_norm diameter_def intro: cSUP_least)

lemma diameter_bounded_bound:
 fixes S :: 'a :: metric_space set
 assumes S: bounded S x ∈ S y ∈ S
 shows dist x y ≤ diameter S
proof -
 from S obtain z d where z: ⋀x. x ∈ S ⟹ dist z x ≤ d
   unfolding bounded_def by auto
 have bdd_above (case_prod dist ` (S×S))
 proof (intro bdd_aboveI, safe)
   fix a b
   assume a ∈ S b ∈ S
   with z[of a] z[of b] dist_triangle[of a b z]
   show dist a b ≤ 2 * d
     by (simp add: dist_commute)
 qed
 moreover have (x,y) ∈ S×S using S by auto
 ultimately have dist x y ≤ (SUP (x,y)∈S×S. dist x y)
   by (rule cSUP_upper2) simp
 with ‹x ∈ S› show ?thesis
   by (auto simp: diameter_def)
qed
```

lemma *diameter_lower_bounded*:
fixes $S :: 'a :: metric\_space\ set$
assumes S: *bounded* S
and d: $0 < d$ $d < diameter\ S$
shows $\exists\, x{\in}S.\ \exists\, y{\in}S.\ d < dist\ x\ y$
proof (*rule ccontr*)
assume *contr*: $\neg$ *?thesis*
moreover have $S \neq \{\}$
using d **by** (*auto simp*: *diameter_def*)
ultimately have $diameter\ S \leq d$
by (*auto simp*: *not_less diameter_def intro*!: *cSUP_least*)
with ‹$d < diameter\ S$› **show** *False* **by** *auto*
qed

lemma *diameter_bounded*:
assumes *bounded* S
shows $\forall\, x{\in}S.\ \forall\, y{\in}S.\ dist\ x\ y \leq diameter\ S$
and $\forall\, d{>}0.\ d < diameter\ S \longrightarrow (\exists\, x{\in}S.\ \exists\, y{\in}S.\ dist\ x\ y > d)$
using *diameter_bounded_bound*[*of* S] *diameter_lower_bounded*[*of* S] *assms*
by *auto*

lemma *bounded_two_points*: *bounded* $S \longleftrightarrow (\exists\, e.\ \forall\, x{\in}S.\ \forall\, y{\in}S.\ dist\ x\ y \leq e)$
by (*meson bounded_def diameter_bounded*(*1*))

lemma *diameter_compact_attained*:
assumes *compact* S
and $S \neq \{\}$
shows $\exists\, x{\in}S.\ \exists\, y{\in}S.\ dist\ x\ y = diameter\ S$
proof −
have *b*: *bounded* S **using** *assms*(*1*)
by (*rule compact_imp_bounded*)
then obtain $x\ y$ **where** *xys*: $x{\in}S$ $y{\in}S$
and *xy*: $\forall\, u{\in}S.\ \forall\, v{\in}S.\ dist\ u\ v \leq dist\ x\ y$
using *compact_sup_maxdistance*[*OF assms*] **by** *auto*
then have $diameter\ S \leq dist\ x\ y$
unfolding *diameter_def* **by** (*force intro*!: *cSUP_least*)
then show *?thesis*
by (*metis b diameter_bounded_bound order_antisym xys*)
qed

lemma *diameter_ge_0*:
assumes *bounded* S **shows** $0 \leq diameter\ S$
by (*metis all_not_in_conv assms diameter_bounded_bound diameter_empty dist_self order_refl*)

lemma *diameter_subset*:
assumes $S \subseteq T$ *bounded* T
shows $diameter\ S \leq diameter\ T$
proof (*cases* $S = \{\} \vee T = \{\}$)

```
  case True
  with assms show ?thesis
    by (force simp: diameter_ge_0)
next
  case False
  then have bdd_above ((λx. case x of (x, xa) ⇒ dist x xa) ` (T × T))
    using ‹bounded T› diameter_bounded_bound by (force simp: bdd_above_def)
  with False ‹S ⊆ T› show ?thesis
    apply (simp add: diameter_def)
    apply (rule cSUP_subset_mono, auto)
    done
qed

lemma diameter_closure:
  assumes bounded S
  shows diameter(closure S) = diameter S
proof (rule order_antisym)
  have False if d_less_d: diameter S < diameter (closure S)
  proof −
    define d where d = diameter(closure S) − diameter(S)
    have d > 0
      using that by (simp add: d_def)
    then have dle: diameter(closure(S)) − d / 2 < diameter(closure(S))
      by simp
   have dd: diameter (closure S) − d / 2 = (diameter(closure(S)) + diameter(S))
/ 2
      by (simp add: d_def field_split_simps)
     have bocl: bounded (closure S)
      using assms by blast
    moreover have 0 ≤ diameter S
      using assms diameter_ge_0 by blast
    ultimately obtain x y where x ∈ closure S y ∈ closure S and xy: diame-
ter(closure(S)) − d / 2 < dist x y
      by (smt (verit) dle d_less_d d_def dd diameter_lower_bounded)
    then obtain x′ y′ where x′y′: x′ ∈ S dist x′ x < d/4 y′ ∈ S dist y′ y < d/4
    by (metis ‹0 < d› zero_less_divide_iff zero_less_numeral closure_approachable)
    then have dist x′ y′ ≤ diameter S
      using assms diameter_bounded_bound by blast
    with x′y′ have dist x y ≤ d / 4 + diameter S + d / 4
        by (meson add_mono dist_triangle dist_triangle3 less_eq_real_def or-
der_trans)
    then show ?thesis
      using xy d_def by linarith
  qed
  then show diameter (closure S) ≤ diameter S
    by fastforce
  next
    show diameter S ≤ diameter (closure S)
      by (simp add: assms bounded_closure closure_subset diameter_subset)
```

qed

proposition *Lebesgue_number_lemma*:
assumes *compact* S $\mathcal{C} \neq \{\}$ $S \subseteq \bigcup \mathcal{C}$ **and** *ope*: $\bigwedge B.\ B \in \mathcal{C} \Longrightarrow$ *open* B
obtains δ **where** $0 < \delta$ $\bigwedge T.\ [\![T \subseteq S;$ *diameter* $T < \delta]\!] \Longrightarrow \exists B \in \mathcal{C}.\ T \subseteq B$
proof (*cases* $S = \{\}$)
case *True*
then show *?thesis*
by (*metis* ‹$\mathcal{C} \neq \{\}$› *zero_less_one empty_subsetI equals0I subset_trans that*)
next
case *False*
{ fix x **assume** $x \in S$
then obtain C **where** C: $x \in C$ $C \in \mathcal{C}$
using ‹$S \subseteq \bigcup \mathcal{C}$› **by** *blast*
then obtain r **where** r: $r>0$ *ball* x $(2{*}r) \subseteq C$
by (*metis mult.commute mult_2_right not_le ope openE field_sum_of_halves zero_le_numeral zero_less_mult_iff*)
then have $\exists r\ C.\ r > 0 \wedge$ *ball* x $(2{*}r) \subseteq C \wedge C \in \mathcal{C}$
using C **by** *blast*
}
then obtain r **where** r: $\bigwedge x.\ x \in S \Longrightarrow r\ x > 0 \wedge (\exists C \in \mathcal{C}.$ *ball* x $(2{*}r\ x) \subseteq C)$
by *metis*
then have $S \subseteq (\bigcup x \in S.$ *ball* x $(r\ x))$
by *auto*
then obtain $\mathcal{T}$ **where** *finite* $\mathcal{T}$ $S \subseteq \bigcup \mathcal{T}$ **and** $\mathcal{T}$: $\mathcal{T} \subseteq (\lambda x.$ *ball* x $(r\ x))$ ‘ S
by (*rule compactE* [*OF* ‹*compact S*›]) *auto*
then obtain *S0* **where** *S0* $\subseteq S$ *finite S0* **and** *S0*: $\mathcal{T} = (\lambda x.$ *ball* x $(r\ x))$ ‘ *S0*
by (*meson finite_subset_image*)
then have *S0* $\neq \{\}$
using *False* ‹$S \subseteq \bigcup \mathcal{T}$› **by** *auto*
define δ **where** $\delta =$ *Inf* $(r$ ‘ *S0*$)$
have $\delta > 0$
using ‹*finite S0*› ‹*S0* $\subseteq S$› ‹*S0* $\neq \{\}$› r **by** (*auto simp*: δ*_def finite_less_Inf_iff*)
show *?thesis*
proof
show $0 < \delta$
by (*simp add*: ‹$0 < \delta$›)
show $\exists B \in \mathcal{C}.\ T \subseteq B$ **if** $T \subseteq S$ **and** *dia*: *diameter* $T < \delta$ **for** T
proof (*cases* $T = \{\}$)
case *True*
then show *?thesis*
using ‹$\mathcal{C} \neq \{\}$› **by** *blast*
next
case *False*
then obtain y **where** $y \in T$ **by** *blast*
then have $y \in S$
using ‹$T \subseteq S$› **by** *auto*
then obtain x **where** $x \in S0$ **and** x: $y \in$ *ball* x $(r\ x)$

using ‹$S \subseteq \bigcup \mathcal{T}$› *S0 that* **by** *blast*
have *ball y δ ⊆ ball y (r x)*
by (*metis δ_def* ‹*S0 ≠ {}*› ‹*finite S0*› ‹*x ∈ S0*› *empty_is_image finite_imageI finite_less_Inf_iff imageI less_irrefl not_le subset_ball*)
also have *... ⊆ ball x (2∗r x)*
using *x* **by** *metric*
finally obtain *C* **where** *C ∈ 𝒞 ball y δ ⊆ C*
by (*meson r* ‹*S0 ⊆ S*› ‹*x ∈ S0*› *dual_order.trans subsetCE*)
have *bounded T*
using ‹*compact S*› *bounded_subset compact_imp_bounded* ‹*T ⊆ S*› **by** *blast*
then have *T ⊆ ball y δ*
using ‹*y ∈ T*› *dia diameter_bounded_bound* **by** *fastforce*
then show *?thesis*
by (*meson* ‹*C ∈ 𝒞*› ‹*ball y δ ⊆ C*› *subset_eq*)
qed
qed
qed

4.15 Metric spaces with the Heine-Borel property

A metric space (or topological vector space) is said to have the Heine-Borel property if every closed and bounded subset is compact.

class *heine_borel = metric_space +*
assumes *bounded_imp_convergent_subsequence*:
bounded (range f) ⟹ ∃ l r. strict_mono (r::nat⇒nat) ∧ ((f ∘ r) ⟶ l) sequentially

proposition *bounded_closed_imp_seq_compact*:
fixes *S::'a::heine_borel set*
assumes *bounded S*
and *closed S*
shows *seq_compact S*
proof (*unfold seq_compact_def*, *clarify*)
fix *f :: nat ⇒ 'a*
assume *f*: *∀ n. f n ∈ S*
with ‹*bounded S*› **have** *bounded (range f)*
by (*auto intro*: *bounded_subset*)
obtain *l r* **where** *r*: *strict_mono (r :: nat ⇒ nat)* **and** *l*: *((f ∘ r) ⟶ l) sequentially*
using *bounded_imp_convergent_subsequence* [*OF* ‹*bounded (range f)*›] **by** *auto*
from *f* **have** *fr*: *∀ n. (f ∘ r) n ∈ S*
by *simp*
show *∃ l∈S. ∃ r. strict_mono r ∧ (f ∘ r) ⟶ l*
using *assms(2) closed_sequentially fr l r* **by** *blast*
qed

lemma *compact_eq_bounded_closed*:
fixes *S :: 'a::heine_borel set*

shows *compact S* $\longleftrightarrow$ *bounded S* $\wedge$ *closed S*
using *bounded_closed_imp_seq_compact compact_eq_seq_compact_metric compact_imp_bounded compact_imp_closed*
by *auto*

lemma *bounded_infinite_imp_islimpt*:
fixes *S* :: *'a*::*heine_borel set*
assumes *T* $\subseteq$ *S bounded S infinite T*
obtains *x* **where** *x islimpt S*
by (*meson assms closed_limpt compact_eq_Bolzano_Weierstrass compact_eq_bounded_closed islimpt_subset*)

lemma *compact_Inter*:
fixes $\mathcal{F}$ :: *'a* :: *heine_borel set set*
assumes *com*: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *compact S* **and** $\mathcal{F} \neq \{\}$
shows *compact*($\bigcap\ \mathcal{F}$)
using *assms*
by (*meson Inf_lower all_not_in_conv bounded_subset closed_Inter compact_eq_bounded_closed*)

lemma *compact_closure* [*simp*]:
fixes *S* :: *'a*::*heine_borel set*
shows *compact*(*closure S*) $\longleftrightarrow$ *bounded S*
by (*meson bounded_closure bounded_subset closed_closure closure_subset compact_eq_bounded_closed*)

instance *real* :: *heine_borel*
proof
fix *f* :: *nat* $\Rightarrow$ *real*
assume *f*: *bounded* (*range f*)
obtain *r* :: *nat* $\Rightarrow$ *nat* **where** *r*: *strict_mono r monoseq* (*f* $\circ$ *r*)
unfolding *comp_def* **by** (*metis seq_monosub*)
then have *Bseq* (*f* $\circ$ *r*)
unfolding *Bseq_eq_bounded* **by** (*metis f BseqI' bounded_iff comp_apply rangeI*)
with *r* **show** $\exists\, l\ r.$ *strict_mono r* $\wedge$ (*f* $\circ$ *r*) $\longrightarrow$ *l*
using *Bseq_monoseq_convergent*[*of f* $\circ$ *r*] **by** (*auto simp*: *convergent_def*)
qed

lemma *compact_lemma_general*:
fixes *f* :: *nat* $\Rightarrow$ *'a*
fixes *proj*::*'a* $\Rightarrow$ *'b* $\Rightarrow$ *'c*::*heine_borel* (**infixl** ‹*proj*› *60*)
fixes *unproj*:: (*'b* $\Rightarrow$ *'c*) $\Rightarrow$ *'a*
assumes *finite_basis*: *finite basis*
assumes *bounded_proj*: $\bigwedge k.\ k \in$ *basis* $\Longrightarrow$ *bounded* ((*λx. x proj k*) ‘ *range f*)
assumes *proj_unproj*: $\bigwedge e\ k.\ k \in$ *basis* $\Longrightarrow$ (*unproj e*) *proj k* = *e k*
assumes *unproj_proj*: $\bigwedge x.$ *unproj* (*λk. x proj k*) = *x*
shows $\forall\, d \subseteq$ *basis*. $\exists\, l$::*'a*. $\exists$ *r*::*nat*$\Rightarrow$*nat*.
strict_mono r $\wedge$ ($\forall\, e{>}0.$ *eventually* (*λn.* $\forall\, i \in d.$ *dist* (*f* (*r n*) *proj i*) (*l proj i*) $<$ *e*) *sequentially*)

```
proof safe
  fix d :: 'b set
  assume d: d ⊆ basis
  with finite_basis have finite d
    by (blast intro: finite_subset)
  from this d show ∃ l::'a. ∃ r::nat⇒nat. strict_mono r ∧
    (∀ e>0. eventually (λn. ∀ i∈d. dist (f (r n) proj i) (l proj i) < e) sequentially)
  proof (induct d)
    case empty
    then show ?case
      unfolding strict_mono_def by auto
  next
    case (insert k d)
    have k[intro]: k ∈ basis
      using insert by auto
    have s': bounded ((λx. x proj k) ` range f)
      using k
      by (rule bounded_proj)
    obtain l1::'a and r1 where r1: strict_mono r1
      and lr1: ∀ e > 0. ∀ F n in sequentially. ∀ i∈d. dist (f (r1 n) proj i) (l1 proj
i) < e
      using insert by auto
    have f': ∀ n. f (r1 n) proj k ∈ (λx. x proj k) ` range f
      by simp
    have bounded (range (λi. f (r1 i) proj k))
      by (metis (lifting) bounded_subset f' image_subsetI s')
    then obtain l2 r2 where r2: strict_mono r2 and lr2: (λi. f (r1 (r2 i)) proj
k) ⟶ l2
      using bounded_imp_convergent_subsequence[of λi. f (r1 i) proj k]
      by (auto simp: o_def)
    define r where r = r1 ∘ r2
    have r: strict_mono r
      using r1 and r2 unfolding r_def o_def strict_mono_def by auto
    moreover
    define l where l = unproj (λi. if i = k then l2 else l1 proj i)
    { fix e::real
      assume e > 0
      from lr1 ‹e > 0› have N1: ∀ F n in sequentially. ∀ i∈d. dist (f (r1 n) proj
i) (l1 proj i) < e
        by blast
      from lr2 ‹e > 0› have N2: ∀ F n in sequentially. dist (f (r1 (r2 n)) proj k)
l2 < e
        by (rule tendstoD)
      from r2 N1 have N1': ∀ F n in sequentially. ∀ i∈d. dist (f (r1 (r2 n)) proj
i) (l1 proj i) < e
        by (rule eventually_subseq)
      have ∀ F n in sequentially. ∀ i∈insert k d. dist (f (r n) proj i) (l proj i) < e
        using N1' N2
          by eventually_elim (use insert.prems in ‹auto simp: l_def r_def o_def
```

```
proj_unproj›)
    }
    ultimately show ?case by auto
  qed
qed

lemma bounded_fst: bounded s ⟹ bounded (fst ‘ s)
  unfolding bounded_def
  by (metis (erased, opaque_lifting) dist_fst_le image_iff order_trans)

lemma bounded_snd: bounded s ⟹ bounded (snd ‘ s)
  unfolding bounded_def
  by (metis (no_types, opaque_lifting) dist_snd_le image_iff order.trans)

instance prod :: (heine_borel, heine_borel) heine_borel
proof
  fix f :: nat ⇒ 'a × 'b
  assume f: bounded (range f)
  then have bounded (fst ‘ range f)
    by (rule bounded_fst)
  then have s1: bounded (range (fst ∘ f))
    by (simp add: image_comp)
  obtain l1 r1 where r1: strict_mono r1 and l1: (λn. fst (f (r1 n))) ⟶ l1
   using bounded_imp_convergent_subsequence [OF s1] unfolding o_def by fast
  from f have s2: bounded (range (snd ∘ f ∘ r1))
    by (auto simp: image_comp intro: bounded_snd bounded_subset)
  obtain l2 r2 where r2: strict_mono r2 and l2: (λn. snd (f (r1 (r2 n)))) ⟶
l2
    using bounded_imp_convergent_subsequence [OF s2]
    unfolding o_def by fast
  have l1': ((λn. fst (f (r1 (r2 n)))) ⟶ l1) sequentially
    using LIMSEQ_subseq_LIMSEQ [OF l1 r2] unfolding o_def .
  have l: ((f ∘ (r1 ∘ r2)) ⟶ (l1, l2)) sequentially
    using tendsto_Pair [OF l1' l2] unfolding o_def by simp
  have r: strict_mono (r1 ∘ r2)
    using r1 r2 unfolding strict_mono_def by simp
  show ∃l r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
    using l r by fast
qed
```

4.16 Completeness

```
proposition (in metric_space) completeI:
  assumes ⋀f. ∀n. f n ∈ s ⟹ Cauchy f ⟹ ∃l∈s. f ⟶ l
  shows complete s
  using assms unfolding complete_def by fast

proposition (in metric_space) completeE:
  assumes complete s and ∀n. f n ∈ s and Cauchy f
```

```
  obtains l where l ∈ s and f ——⟶ l
  using assms unfolding complete_def by fast


lemma compact_imp_complete:
  fixes s :: 'a::metric_space set
  assumes compact s
  shows complete s
proof −
  {
    fix f
    assume as: (∀ n::nat. f n ∈ s) Cauchy f
    from as(1) obtain l r where lr: l∈s strict_mono r (f ∘ r) ——⟶ l
      using assms unfolding compact_def by blast

    note lr' = seq_suble [OF lr(2)]
    {
      fix e :: real
      assume e > 0
      from as(2) obtain N where N:∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (f m) (f
n) < e/2
        unfolding Cauchy_def using ‹e > 0› by (meson half_gt_zero)
      then obtain M where M:∀ n≥M. dist ((f ∘ r) n) l < e/2
        by (metis dist_self lim_sequentially lr(3))
      {
        fix n :: nat
        assume n: n ≥ max N M
        have dist ((f ∘ r) n) l < e/2
          using n M by auto
        moreover have r n ≥ N
          using lr'[of n] n by auto
        then have dist (f n) ((f ∘ r) n) < e/2
          using N and n by auto
        ultimately have dist (f n) l < e using n M
          by metric
      }
      then have ∃ N. ∀ n≥N. dist (f n) l < e by blast
    }
    then have ∃ l∈s. (f ——⟶ l) sequentially using ‹l∈s›
      unfolding lim_sequentially by auto
  }
  then show ?thesis unfolding complete_def by auto
qed

proposition compact_eq_totally_bounded:
  compact s ⟷ complete s ∧ (∀ e>0. ∃ k. finite k ∧ s ⊆ (⋃x∈k. ball x e))
    (is _ ⟷ ?rhs)
proof
  assume assms: ?rhs
```

```
  then obtain k where k: ⋀e. 0 < e ⟹ finite (k e) ⋀e. 0 < e ⟹ s ⊆ (⋃x∈k
e. ball x e)
    by (auto simp: choice_iff')

  show compact s
  proof cases
    assume s = {}
    then show compact s by (simp add: compact_def)
  next
    assume s ≠ {}
    show ?thesis
      unfolding compact_def
    proof safe
      fix f :: nat ⇒ 'a
      assume f: ∀n. f n ∈ s

      define e where e n = 1 / (2 * Suc n) for n
      then have [simp]: ⋀n. 0 < e n by auto
      define B where B n U = (SOME b. infinite {n. f n ∈ b} ∧ (∃x. b ⊆ ball x
(e n) ∩ U)) for n U
      {
        fix n U
        assume infinite {n. f n ∈ U}
        then have ∃b∈k (e n). infinite {i∈{n. f n ∈ U}. f i ∈ ball b (e n)}
          using k f by (intro pigeonhole_infinite_rel) (auto simp: subset_eq)
        then obtain a where
          a ∈ k (e n)
          infinite {i ∈ {n. f n ∈ U}. f i ∈ ball a (e n)} ..
        then have ∃b. infinite {i. f i ∈ b} ∧ (∃x. b ⊆ ball x (e n) ∩ U)
          by (intro exI[of _ ball a (e n) ∩ U] exI[of _ a]) (auto simp: ac_simps)
        from someI_ex[OF this]
        have infinite {i. f i ∈ B n U} ∃x. B n U ⊆ ball x (e n) ∩ U
          unfolding B_def by auto
      }
      note B = this

      define F where F = rec_nat (B 0 UNIV) B
      {
        fix n
        have infinite {i. f i ∈ F n}
          by (induct n) (auto simp: F_def B)
      }
      then have F: ⋀n. ∃x. F (Suc n) ⊆ ball x (e n) ∩ F n
        using B by (simp add: F_def)
      then have F_dec: ⋀m n. m ≤ n ⟹ F n ⊆ F m
        using decseq_SucI[of F] by (auto simp: decseq_def)

      obtain sel where sel: ⋀k i. i < sel k i ⋀k i. f (sel k i) ∈ F k
      proof (atomize_elim, unfold all_conj_distrib[symmetric], intro choice allI)
```

```
      fix k i
      have infinite ({n. f n ∈ F k} − {.. i})
        using ‹infinite {n. f n ∈ F k}› by auto
      from infinite_imp_nonempty[OF this]
      show ∃ x>i. f x ∈ F k
        by (simp add: set_eq_iff not_le conj_commute)
    qed

    define t where t = rec_nat (sel 0 0) (λn i. sel (Suc n) i)
    have strict_mono t
      unfolding strict_mono_Suc_iff by (simp add: t_def sel)
    moreover have ∀ i. (f ∘ t) i ∈ s
      using f by auto
    moreover
    have t: (f ∘ t) n ∈ F n for n
      by (cases n) (simp_all add: t_def sel)

    have Cauchy (f ∘ t)
    proof (safe intro!: metric_CauchyI exI elim!: nat_approx_posE)
      fix r :: real and N n m
      assume 1 / Suc N < r Suc N ≤ n Suc N ≤ m
      then have (f ∘ t) n ∈ F (Suc N) (f ∘ t) m ∈ F (Suc N) 2 * e N < r
        using F_dec t by (auto simp: e_def field_simps)
      with F[of N] obtain x where dist x ((f ∘ t) n) < e N dist x ((f ∘ t) m)
< e N
        by (auto simp: subset_eq)
      with ‹2 * e N < r› show dist ((f ∘ t) m) ((f ∘ t) n) < r
        by metric
    qed

    ultimately show ∃ l∈s. ∃ r. strict_mono r ∧ (f ∘ r) ⟶ l
      using assms unfolding complete_def by blast
  qed
 qed
qed (metis compact_imp_complete compact_imp_seq_compact seq_compact_imp_totally_bounded)

lemma cauchy_imp_bounded:
 assumes Cauchy s
 shows bounded (range s)
proof −
 from assms obtain N :: nat where ∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (s m) (s
n) < 1
   by (meson Cauchy_def zero_less_one)
 then have N:∀ n. N ≤ n ⟶ dist (s N) (s n) < 1 by auto
 moreover
 have bounded (s ‘ {0..N})
   using finite_imp_bounded[of s ‘ {1..N}] by auto
 then obtain a where a:∀ x∈s ‘ {0..N}. dist (s N) x ≤ a
   unfolding bounded_any_center [where a=s N] by auto
```

```
  ultimately show ?thesis
    unfolding bounded_any_center [where a=s N]
    apply (rule_tac x=max a 1 in exI, auto)
    apply (erule_tac x=y in allE)
    apply (erule_tac x=y in ballE, auto)
    done
qed

instance heine_borel < complete_space
proof
  fix f :: nat ⇒ 'a assume Cauchy f
  then show convergent f
    unfolding convergent_def
   using Cauchy_converges_subseq cauchy_imp_bounded bounded_imp_convergent_subsequence
by blast
qed

lemma complete_UNIV: complete (UNIV :: ('a::complete_space) set)
proof (rule completeI)
  fix f :: nat ⇒ 'a assume Cauchy f
  then show ∃l∈UNIV. f ⟶ l
    using Cauchy_convergent convergent_def by blast
qed

lemma complete_imp_closed:
  fixes S :: 'a::metric_space set
  shows complete S ⟹ closed S
 by (metis LIMSEQ_imp_Cauchy LIMSEQ_unique closed_sequential_limits com-
pleteE)

lemma complete_Int_closed:
  fixes S :: 'a::metric_space set
  assumes complete S and closed t
  shows complete (S ∩ t)
  by (metis Int_iff assms closed_sequentially completeE completeI)

lemma complete_closed_subset:
  fixes S :: 'a::metric_space set
  assumes closed S and S ⊆ t and complete t
  shows complete S
  using assms complete_Int_closed [of t S] by (simp add: Int_absorb1)

lemma complete_eq_closed:
  fixes S :: ('a::complete_space) set
  shows complete S ⟷ closed S
proof
  assume closed S then show complete S
    using subset_UNIV complete_UNIV by (rule complete_closed_subset)
next
```

assume *complete S* **then show** *closed S*
by (*rule complete_imp_closed*)
qed

lemma *convergent_eq_Cauchy*:
fixes *S* :: *nat* ⇒ *'a*::*complete_space*
shows (∃ *l*. (*S* ⟶ *l*) *sequentially*) ⟷ *Cauchy S*
unfolding *Cauchy_convergent_iff convergent_def* **..**

lemma *convergent_imp_bounded*:
fixes *S* :: *nat* ⇒ *'a*::*metric_space*
shows (*S* ⟶ *l*) *sequentially* ⟹ *bounded* (*range S*)
by (*intro cauchy_imp_bounded LIMSEQ_imp_Cauchy*)

lemma *frontier_subset_compact*:
fixes *S* :: *'a*::*heine_borel set*
shows *compact S* ⟹ *frontier S* ⊆ *S*
using *frontier_subset_closed compact_eq_bounded_closed*
by *blast*

lemma *banach_fix_type*:
fixes *f*::*'a*::*complete_space*⇒*'a*
assumes *c*:*0* ≤ *c c* < *1*
and *lipschitz*:∀ *x*. ∀ *y*. *dist* (*f x*) (*f y*) ≤ *c* ∗ *dist x y*
shows ∃ !*x*. (*f x* = *x*)
using *assms banach_fix*[*OF complete_UNIV UNIV_not_empty assms*(*1*,*2*) *subset_UNIV*, *of f*]
by *auto*

4.17 Cauchy continuity

definition *Cauchy_continuous_on* **where**
Cauchy_continuous_on ≡ λ*S f*. ∀ σ. *Cauchy* σ ⟶ *range* σ ⊆ *S* ⟶ *Cauchy* (*f* ∘ σ)

lemma *continuous_closed_imp_Cauchy_continuous*:
fixes *S* :: (*'a*::*complete_space*) *set*
shows ⟦*continuous_on S f*; *closed S*⟧ ⟹ *Cauchy_continuous_on S f*
unfolding *Cauchy_continuous_on_def*
by (*metis LIMSEQ_imp_Cauchy completeE complete_eq_closed continuous_on_sequentially range_subsetD*)

lemma *uniformly_continuous_imp_Cauchy_continuous*:
fixes *f* :: *'a*::*metric_space* ⇒ *'b*::*metric_space*
shows *uniformly_continuous_on S f* ⟹ *Cauchy_continuous_on S f*
by (*simp add*: *uniformly_continuous_on_def Cauchy_continuous_on_def Cauchy_def image_subset_iff*) *metis*

lemma *Cauchy_continuous_on_imp_continuous*:

fixes f :: $'a$::*metric_space* ⇒ $'b$::*metric_space*
assumes *Cauchy_continuous_on S f*
shows *continuous_on S f*
proof −
have *False* **if** x: ∀ n. ∃ x'∈S. *dist* x' x < *inverse*(*Suc* n) ∧ ¬ *dist* (*f* x') (*f* x) < ε ε>*0* x ∈ S **for** x **and** ε::*real*
proof −
obtain ϱ **where** ϱ: ∀ n. ϱ n ∈ S **and** dx: ∀ n. *dist* (ϱ n) x < *inverse*(*Suc* n) **and** dfx: ∀ n. ¬ *dist* (*f* (ϱ n)) (*f* x) < ε
using x **by** *metis*
define σ **where** $\sigma \equiv \lambda n.$ *if even* n *then* ϱ n *else* x
with ϱ ‹x ∈ S› **have** *range* σ ⊆ S
by *auto*
have σ ⟶ x
unfolding *tendsto_iff*
proof (*intro strip*)
fix e :: *real*
assume e>*0*
then obtain N **where** *inverse* (*Suc* N) < e
using *reals_Archimedean* **by** *blast*
then have ∀ n. N ≤ n ⟶ *dist* (ϱ n) x < e
by (*smt* (*verit*, *ccfv_SIG*) dx *inverse_Suc inverse_less_iff_less inverse_positive_iff_positive of_nat_Suc of_nat_mono*)
with ‹e>*0*› **show** ∀$_F$ n *in sequentially*. *dist* (σ n) x < e
by (*auto simp add*: *eventually_sequentially* σ_*def*)
qed
then have *Cauchy* σ
by (*intro LIMSEQ_imp_Cauchy*)
then have *Cf*: *Cauchy* (*f* ∘ σ)
by (*meson Cauchy_continuous_on_def* ‹*range* σ ⊆ S› *assms*)
have (*f* ∘ σ) ⟶ *f* x
unfolding *tendsto_iff*
proof (*intro strip*)
fix e :: *real*
assume e>*0*
then obtain N **where** N: ∀ m≥N. ∀ n≥N. *dist* ((*f* ∘ σ) m) ((*f* ∘ σ) n) < e
using *Cf* **unfolding** *Cauchy_def* **by** *presburger*
moreover have (*f* ∘ σ) (*Suc*(N+N)) = *f* x
by (*simp add*: σ_*def*)
ultimately have ∀ n≥N. *dist* ((*f* ∘ σ) n) (*f* x) < e
by (*metis add_Suc le_add2*)
then show ∀$_F$ n *in sequentially*. *dist* ((*f* ∘ σ) n) (*f* x) < e
using *eventually_sequentially* **by** *blast*
qed
moreover have ⋀n. ¬ *dist* (*f* (σ (*2*∗n))) (*f* x) < ε
using dfx **by** (*simp add*: σ_*def*)
ultimately show *False*
using ‹ε>*0*› **by** (*fastforce simp*: *mult_2 nat_le_iff_add tendsto_iff eventually_sequentially*)

qed
then show *?thesis*
unfolding *continuous_on_iff* **by** (*meson inverse_Suc*)
qed

4.18 Finite intersection property

Also developed in HOL's toplogical spaces theory, but the Heine-Borel type class isn't available there.

lemma *closed_imp_fip*:
fixes S :: *'a*::*heine_borel set*
assumes *closed S*
and T: $T \in \mathcal{F}$ *bounded T*
and *clof*: $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *closed T*
and $\bigwedge \mathcal{F}'.\ [\![\mathit{finite}\ \mathcal{F}';\ \mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow S \cap \bigcap \mathcal{F}' \neq \{\}$
shows $S \cap \bigcap \mathcal{F} \neq \{\}$
proof −
have *compact* $(S \cap T)$
using ‹*closed S*› *clof compact_eq_bounded_closed T* **by** *blast*
then have $(S \cap T) \cap \bigcap \mathcal{F} \neq \{\}$
by (*smt* (*verit*, *best*) *Inf_insert Int_assoc assms compact_imp_fip finite_insert insert_subset*)
then show *?thesis* **by** *blast*
qed

lemma *closed_imp_fip_compact*:
fixes S :: *'a*::*heine_borel set*
shows
$[\![\mathit{closed}\ S;\ \bigwedge T.\ T \in \mathcal{F} \Longrightarrow \mathit{compact}\ T;$
$\bigwedge \mathcal{F}'.\ [\![\mathit{finite}\ \mathcal{F}';\ \mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow S \cap \bigcap \mathcal{F}' \neq \{\}]\!]$
$\Longrightarrow S \cap \bigcap \mathcal{F} \neq \{\}$
by (*metis closed_imp_fip compact_eq_bounded_closed equals0I finite.emptyI order.refl*)

lemma *closed_fip_Heine_Borel*:
fixes $\mathcal{F}$ :: *'a*::*heine_borel set set*
assumes $T \in \mathcal{F}$ *bounded T*
and $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *closed T*
and $\bigwedge \mathcal{F}'.\ [\![\mathit{finite}\ \mathcal{F}';\ \mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow \bigcap \mathcal{F}' \neq \{\}$
shows $\bigcap \mathcal{F} \neq \{\}$
using *closed_imp_fip* [*OF closed_UNIV*] *assms* **by** *auto*

lemma *compact_fip_Heine_Borel*:
fixes $\mathcal{F}$ :: *'a*::*heine_borel set set*
assumes *clof*: $\bigwedge T.\ T \in \mathcal{F} \Longrightarrow$ *compact T*
and *none*: $\bigwedge \mathcal{F}'.\ [\![\mathit{finite}\ \mathcal{F}';\ \mathcal{F}' \subseteq \mathcal{F}]\!] \Longrightarrow \bigcap \mathcal{F}' \neq \{\}$
shows $\bigcap \mathcal{F} \neq \{\}$
by (*metis InterI clof closed_fip_Heine_Borel compact_eq_bounded_closed equals0D*

none)

lemma *compact_sequence_with_limit*:
fixes $f :: nat \Rightarrow 'a::heine\_borel$
shows $f \longrightarrow l \Longrightarrow compact\ (insert\ l\ (range\ f))$
by (*simp add*: *closed_limpt compact_eq_bounded_closed convergent_imp_bounded islimpt_insert sequence_unique_limpt*)

4.19 Properties of Balls and Spheres

lemma *compact_cball*[*simp*]:
fixes $x :: 'a::heine\_borel$
shows *compact* (*cball x e*)
using *compact_eq_bounded_closed bounded_cball closed_cball* **by** *blast*

lemma *compact_frontier_bounded*[*intro*]:
fixes $S :: 'a::heine\_borel\ set$
shows *bounded S* $\Longrightarrow$ *compact* (*frontier S*)
unfolding *frontier_def*
using *compact_eq_bounded_closed* **by** *blast*

lemma *compact_frontier*[*intro*]:
fixes $S :: 'a::heine\_borel\ set$
shows *compact S* $\Longrightarrow$ *compact* (*frontier S*)
using *compact_eq_bounded_closed compact_frontier_bounded* **by** *blast*

lemma *no_limpt_imp_countable*:
assumes $\bigwedge z.\ \neg z\ islimpt\ (X :: 'a :: \{real\_normed\_vector, heine\_borel\}\ set)$
shows *countable X*
proof −
have *countable* $(\bigcup n.\ cball\ 0\ (real\ n) \cap X)$
proof (*intro countable_UN*[*OF _ countable_finite*])
fix $n :: nat$
show *finite* $(cball\ 0\ (real\ n) \cap X)$
using *assms* **by** (*intro finite_not_islimpt_in_compact*) *auto*
qed *auto*
also have $(\bigcup n.\ cball\ 0\ (real\ n)) = (UNIV :: 'a\ set)$
by (*force simp*: *real_arch_simple*)
hence $(\bigcup n.\ cball\ 0\ (real\ n) \cap X) = X$
by *blast*
finally show *countable X* **.**
qed

4.20 Distance from a Set

lemma *distance_attains_sup*:
assumes *compact s* $s \neq \{\}$
shows $\exists x{\in}s.\ \forall y{\in}s.\ dist\ a\ y \leq dist\ a\ x$

proof (*rule continuous_attains_sup* [*OF assms*])
  {
    fix *x*
    assume *x*∈*s*
    have (*dist a* ⟶ *dist a x*) (*at x within s*)
      by (*intro tendsto_dist tendsto_const tendsto_ident_at*)
  }
  then show *continuous_on s* (*dist a*)
    unfolding *continuous_on* **..**
qed

For *minimal* distance, we only need closure, not compactness.

lemma *distance_attains_inf*:
  fixes *a* :: *'a*::*heine_borel*
  assumes *closed s* **and** *s* ≠ {}
  obtains *x* **where** *x*∈*s* ⋀*y*. *y* ∈ *s* ⟹ *dist a x* ≤ *dist a y*
proof −
  from *assms* **obtain** *b* **where** *b* ∈ *s* **by** *auto*
  let *?B* = *s* ∩ *cball a* (*dist b a*)
  have *?B* ≠ {} **using** ‹*b* ∈ *s*›
    by (*auto simp*: *dist_commute*)
  moreover have *continuous_on ?B* (*dist a*)
    by (*auto intro*!: *continuous_at_imp_continuous_on continuous_dist continuous_ident continuous_const*)
  moreover have *compact ?B*
    by (*intro closed_Int_compact* ‹*closed s*› *compact_cball*)
  ultimately obtain *x* **where** *x* ∈ *?B* ∀ *y*∈*?B*. *dist a x* ≤ *dist a y*
    by (*metis continuous_attains_inf*)
  with *that* **show** *?thesis* **by** *fastforce*
qed

4.21 Infimum Distance

definition *infdist x A* = (*if A* = {} *then 0 else INF a*∈*A*. *dist x a*)

lemma *bdd_below_image_dist*[*intro*, *simp*]: *bdd_below* (*dist x* ‘ *A*)
  by (*auto intro*!: *zero_le_dist*)

lemma *infdist_notempty*: *A* ≠ {} ⟹ *infdist x A* = (*INF a*∈*A*. *dist x a*)
  by (*simp add*: *infdist_def*)

lemma *infdist_nonneg*: *0* ≤ *infdist x A*
  by (*auto simp*: *infdist_def intro*: *cINF_greatest*)

lemma *infdist_le*: *a* ∈ *A* ⟹ *infdist x A* ≤ *dist x a*
  by (*auto intro*: *cINF_lower simp add*: *infdist_def*)

lemma *infdist_le2*: *a* ∈ *A* ⟹ *dist x a* ≤ *d* ⟹ *infdist x A* ≤ *d*
  by (*auto intro*!: *cINF_lower2 simp add*: *infdist_def*)

lemma *infdist_zero*[*simp*]: $a \in A \Longrightarrow$ *infdist a A* $= 0$
by (*auto intro*!: *antisym infdist_nonneg infdist_le2*)

lemma *infdist_Un_min*:
assumes $A \neq \{\}$ $B \neq \{\}$
shows *infdist x* $(A \cup B) =$ *min* (*infdist x A*) (*infdist x B*)
using *assms* **by** (*simp add*: *infdist_def cINF_union inf_real_def*)

lemma *infdist_triangle*: *infdist x A* $\leq$ *infdist y A* $+$ *dist x y*
proof (*cases* $A = \{\}$)
case *True*
then show *?thesis* **by** (*simp add*: *infdist_def*)
next
case *False*
then obtain *a* **where** $a \in A$ **by** *auto*
have *infdist x A* $\leq$ *Inf* $\{$*dist x y* $+$ *dist y a* $|a.\ a \in A\}$
proof (*rule cInf_greatest*)
from ‹$A \neq \{\}$› **show** $\{$*dist x y* $+$ *dist y a* $|a.\ a \in A\} \neq \{\}$
by *simp*
fix *d*
assume $d \in \{$*dist x y* $+$ *dist y a* $|a.\ a \in A\}$
then obtain *a* **where** *d*: $d =$ *dist x y* $+$ *dist y a* $a \in A$
by *auto*
show *infdist x A* $\leq d$
unfolding *infdist_notempty*[*OF* ‹$A \neq \{\}$›]
proof (*rule cINF_lower2*)
show $a \in A$ **by** *fact*
show *dist x a* $\leq d$
unfolding *d* **by** (*rule dist_triangle*)
qed *simp*
qed
also have $\ldots =$ *dist x y* $+$ *infdist y A*
proof (*rule cInf_eq*, *safe*)
fix *a*
assume $a \in A$
then show *dist x y* $+$ *infdist y A* $\leq$ *dist x y* $+$ *dist y a*
by (*auto intro*: *infdist_le*)
next
fix *i*
assume *inf*: $\bigwedge d.\ d \in \{$*dist x y* $+$ *dist y a* $|a.\ a \in A\} \Longrightarrow i \leq d$
then have *i* $-$ *dist x y* $\leq$ *infdist y A*
unfolding *infdist_notempty*[*OF* ‹$A \neq \{\}$›] **using** ‹$a \in A$›
by (*intro cINF_greatest*) (*auto simp*: *field_simps*)
then show *i* $\leq$ *dist x y* $+$ *infdist y A*
by *simp*
qed
finally show *?thesis* **by** *simp*
qed

lemma *infdist_triangle_abs*: $|infdist\ x\ A - infdist\ y\ A| \leq dist\ x\ y$
by (*metis* (*full_types*) *abs_diff_le_iff diff_le_eq dist_commute infdist_triangle*)

lemma *in_closure_iff_infdist_zero*:
assumes $A \neq \{\}$
shows $x \in closure\ A \longleftrightarrow infdist\ x\ A = 0$
proof
assume $x \in closure\ A$
show $infdist\ x\ A = 0$
proof (*rule ccontr*)
assume $infdist\ x\ A \neq 0$
with *infdist_nonneg*[*of x A*] **have** $infdist\ x\ A > 0$
by *auto*
then have $ball\ x\ (infdist\ x\ A) \cap closure\ A = \{\}$
by (*smt* (*verit*, *best*) ‹$x \in closure\ A$› *closure_approachableD infdist_le*)
then have $x \notin closure\ A$
by (*metis* ‹$0 < infdist\ x\ A$› *centre_in_ball disjoint_iff_not_equal*)
then show *False* **using** ‹$x \in closure\ A$› **by** *simp*
qed
next
assume *x*: $infdist\ x\ A = 0$
then obtain *a* **where** $a \in A$
by *atomize_elim* (*metis all_not_in_conv assms*)
have *False* **if** $e > 0$ $\neg\ (\exists\, y{\in}A.\ dist\ y\ x < e)$ **for** *e*
proof −
have $infdist\ x\ A \geq e$ **using** ‹$a \in A$›
unfolding *infdist_def* **using** *that*
by (*force simp*: *dist_commute intro*: *cINF_greatest*)
with *x* ‹$e > 0$› **show** *False* **by** *auto*
qed
then show $x \in closure\ A$
using *closure_approachable* **by** *blast*
qed

lemma *in_closed_iff_infdist_zero*:
assumes *closed A* $A \neq \{\}$
shows $x \in A \longleftrightarrow infdist\ x\ A = 0$
proof −
have $x \in closure\ A \longleftrightarrow infdist\ x\ A = 0$
by (*simp add*: ‹$A \neq \{\}$› *in_closure_iff_infdist_zero*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *infdist_pos_not_in_closed*:
assumes *closed S* $S \neq \{\}$ $x \notin S$
shows $infdist\ x\ S > 0$
by (*smt* (*verit*, *best*) *assms in_closed_iff_infdist_zero infdist_nonneg*)

```
lemma
  infdist_attains_inf:
  fixes X::'a::heine_borel set
  assumes closed X
  assumes X ≠ {}
  obtains x where x ∈ X infdist y X = dist y x
proof −
  have bdd_below (dist y ‘ X)
    by auto
  from distance_attains_inf[OF assms, of y]
  obtain x where x ∈ X ⋀z. z ∈ X ⟹ dist y x ≤ dist y z by auto
  then have infdist y X = dist y x
    by (metis antisym assms(2) cINF_greatest infdist_def infdist_le)
  with ‹x ∈ X› show ?thesis ..
qed
```

Every metric space is a T4 space:

```
instance metric_space ⊆ t4_space
proof
  fix S T::'a set assume H: closed S closed T S ∩ T = {}
  consider S = {} | T = {} | S ≠ {} ∧ T ≠ {} by auto
  then show ∃ U V. open U ∧ open V ∧ S ⊆ U ∧ T ⊆ V ∧ U ∩ V = {}
  proof (cases)
    case 1 then show ?thesis by blast
  next
    case 2 then show ?thesis by blast
  next
    case 3
    define U where U = (⋃x∈S. ball x ((infdist x T)/2))
    have A: open U unfolding U_def by auto
    have infdist x T > 0 if x ∈ S for x
      using H that 3 by (auto intro!: infdist_pos_not_in_closed)
    then have B: S ⊆ U unfolding U_def by auto
    define V where V = (⋃x∈T. ball x ((infdist x S)/2))
    have C: open V unfolding V_def by auto
    have infdist x S > 0 if x ∈ T for x
      using H that 3 by (auto intro!: infdist_pos_not_in_closed)
    then have D: T ⊆ V unfolding V_def by auto

    have (ball x ((infdist x T)/2)) ∩ (ball y ((infdist y S)/2)) = {} if x ∈ S y ∈
T for x y
    proof auto
      fix z assume H: dist x z * 2 < infdist x T dist y z * 2 < infdist y S
      have 2 * dist x y ≤ 2 * dist x z + 2 * dist y z
        by metric
      also have ... < infdist x T + infdist y S
        using H by auto
      finally have dist x y < infdist x T ∨ dist x y < infdist y S
        by auto
```

```
    then show False
      using infdist_le[OF ‹x ∈ S›, of y] infdist_le[OF ‹y ∈ T›, of x] by (auto
simp add: dist_commute)
   qed
   then have E: U ∩ V = {}
    unfolding U_def V_def by auto
   show ?thesis
    using A B C D E by blast
 qed
qed

lemma tendsto_infdist [tendsto_intros]:
 assumes f: (f ⟶ l) F
 shows ((λx. infdist (f x) A) ⟶ infdist l A) F
proof (rule tendstoI)
 fix e ::real
 assume e > 0
 from tendstoD[OF f this]
 show eventually (λx. dist (infdist (f x) A) (infdist l A) < e) F
 proof (eventually_elim)
  fix x
  from infdist_triangle[of l A f x] infdist_triangle[of f x A l]
  have dist (infdist (f x) A) (infdist l A) ≤ dist (f x) l
   by (simp add: dist_commute dist_real_def)
  also assume dist (f x) l < e
  finally show dist (infdist (f x) A) (infdist l A) < e .
 qed
qed

lemma continuous_infdist[continuous_intros]:
 assumes continuous F f
 shows continuous F (λx. infdist (f x) A)
 using assms unfolding continuous_def by (rule tendsto_infdist)

lemma continuous_on_infdist [continuous_intros]:
 assumes continuous_on S f
 shows continuous_on S (λx. infdist (f x) A)
using assms unfolding continuous_on by (auto intro: tendsto_infdist)

lemma compact_infdist_le:
 fixes A::'a::heine_borel set
 assumes A ≠ {}
 assumes compact A
 assumes e > 0
 shows compact {x. infdist x A ≤ e}
proof −
 from continuous_closed_vimage[of {0..e} λx. infdist x A]
   continuous_infdist[OF continuous_ident, of _ UNIV A]
 have closed {x. infdist x A ≤ e} by (auto simp: vimage_def infdist_nonneg)
```

```
  moreover
  from assms obtain x0 b where b: ⋀x. x ∈ A ⟹ dist x0 x ≤ b closed A
    by (auto simp: compact_eq_bounded_closed bounded_def)
  {
    fix y
    assume infdist y A ≤ e
    moreover
    from infdist_attains_inf[OF ‹closed A› ‹A ≠ {}›, of y]
    obtain z where z ∈ A infdist y A = dist y z by blast
    ultimately
    have dist x0 y ≤ b + e using b by metric
  } then
  have bounded {x. infdist x A ≤ e}
     by (auto simp: bounded_any_center[where a=x0] intro!: exI[where x=b +
e])
  ultimately show compact {x. infdist x A ≤ e}
    by (simp add: compact_eq_bounded_closed)
qed
```

4.22 Separation between Points and Sets

```
proposition separate_point_closed:
  fixes S :: 'a::heine_borel set
  assumes closed S and a ∉ S
  shows ∃ d>0. ∀ x∈S. d ≤ dist a x
  by (metis assms distance_attains_inf empty_iff gt_ex zero_less_dist_iff)

proposition separate_compact_closed:
  fixes S T :: 'a::heine_borel set
  assumes compact S
    and T: closed T S ∩ T = {}
  shows ∃ d>0. ∀ x∈S. ∀ y∈T. d ≤ dist x y
proof cases
  assume S ≠ {} ∧ T ≠ {}
  then have S ≠ {} T ≠ {} by auto
  let ?inf = λx. infdist x T
  have continuous_on S ?inf
    by (auto intro!: continuous_at_imp_continuous_on continuous_infdist contin-
uous_ident)
  then obtain x where x: x ∈ S ∀ y∈S. ?inf x ≤ ?inf y
    using continuous_attains_inf[OF ‹compact S› ‹S ≠ {}›] by auto
  then have 0 < ?inf x
   using T ‹T ≠ {}› in_closed_iff_infdist_zero by (auto simp: less_le infdist_nonneg)
  moreover have ∀ x'∈S. ∀ y∈T. ?inf x ≤ dist x' y
    using x by (auto intro: order_trans infdist_le)
  ultimately show ?thesis by auto
qed (auto intro!: exI[of _ 1])

proposition separate_closed_compact:
```

fixes S T :: $'a$::*heine_borel set*
assumes S: *closed* S
and T: *compact* T
and *dis*: $S \cap T = \{\}$
shows $\exists\, d{>}0.\ \forall\, x{\in}S.\ \forall\, y{\in}T.\ d \leq dist\ x\ y$
by (*metis separate_compact_closed*[*OF* T S] *dis dist_commute inf_commute*)

proposition *compact_in_open_separated*:
fixes A::$'a$::*heine_borel set*
assumes A: $A \neq \{\}$ *compact* A
assumes *open* B
assumes $A \subseteq B$
obtains e **where** $e > 0$ $\{x.\ infdist\ x\ A \leq e\} \subseteq B$
proof *atomize_elim*
have *closed* $(-B)$ *compact* $A - B \cap A = \{\}$
using *assms* **by** (*auto simp*: *open_Diff compact_eq_bounded_closed*)
from *separate_closed_compact*[*OF this*]
obtain d'::*real* **where** d': $d'{>}0$ $\bigwedge x\ y.\ x \notin B \Longrightarrow y \in A \Longrightarrow d' \leq dist\ x\ y$
by *auto*
define d **where** $d = d'\ /\ 2$
hence $d{>}0$ $d < d'$ **using** d' **by** *auto*
with d' **have** d: $\bigwedge x\ y.\ x \notin B \Longrightarrow y \in A \Longrightarrow d < dist\ x\ y$
by *force*
show $\exists\, e{>}0.\ \{x.\ infdist\ x\ A \leq e\} \subseteq B$
proof (*rule ccontr*)
assume $\nexists\, e.\ 0 < e \wedge \{x.\ infdist\ x\ A \leq e\} \subseteq B$
with ‹$d > 0$› **obtain** x **where** x: $infdist\ x\ A \leq d$ $x \notin B$
by *auto*
then show *False*
by (*metis A compact_eq_bounded_closed infdist_attains_inf x d linorder_not_less*)
qed
qed

4.23 Uniform Continuity

lemma *uniformly_continuous_onE*:
assumes *uniformly_continuous_on s f* $0 < e$
obtains d **where** $d{>}0$ $\bigwedge x\ x'.\ [\![x{\in}s;\ x'{\in}s;\ dist\ x'\ x < d]\!] \Longrightarrow dist\ (f\ x')\ (f\ x) < e$
using *assms*
by (*auto simp*: *uniformly_continuous_on_def*)

lemma *uniformly_continuous_on_sequentially*:
uniformly_continuous_on s f $\longleftrightarrow$ $(\forall\, x\ y.\ (\forall\, n.\ x\ n \in s) \wedge (\forall\, n.\ y\ n \in s) \wedge$
$(\lambda n.\ dist\ (x\ n)\ (y\ n)) \longrightarrow 0 \longrightarrow (\lambda n.\ dist\ (f(x\ n))\ (f(y\ n))) \longrightarrow 0)$ (**is** *?lhs = ?rhs*)
proof
assume *?lhs*
{
fix x y

```
    assume x: ∀n. x n ∈ s
      and y: ∀n. y n ∈ s
      and xy: ((λn. dist (x n) (y n)) ⟶ 0) sequentially
    {
      fix e :: real
      assume e > 0
      then obtain d where d > 0 and d: ∀x∈s. ∀x'∈s. dist x' x < d ⟶ dist (f
x') (f x) < e
        by (metis ‹?lhs› uniformly_continuous_onE)
      obtain N where N: ∀n≥N. dist (x n) (y n) < d
        using xy[unfolded lim_sequentially dist_norm] and ‹d>0› by auto
      then have ∃N. ∀n≥N. dist (f (x n)) (f (y n)) < e
        using d x y by blast
    }
    then have ((λn. dist (f(x n)) (f(y n))) ⟶ 0) sequentially
      unfolding lim_sequentially and dist_real_def by auto
  }
  then show ?rhs by auto
next
  assume ?rhs
  {
    assume ¬ ?lhs
    then obtain e where e > 0 ∀d>0. ∃x∈s. ∃x'∈s. dist x' x < d ∧ ¬ dist (f
x') (f x) < e
      unfolding uniformly_continuous_on_def by auto
    then obtain fa where fa:
      ∀x. 0 < x ⟶ fst (fa x) ∈ s ∧ snd (fa x) ∈ s ∧ dist (fst (fa x)) (snd (fa x))
< x ∧ ¬ dist (f (fst (fa x))) (f (snd (fa x))) < e
      using choice[of λd x. d>0 ⟶ fst x ∈ s ∧ snd x ∈ s ∧ dist (snd x) (fst x)
< d ∧ ¬ dist (f (snd x)) (f (fst x)) < e]
      by (auto simp: Bex_def dist_commute)
    define x where x n = fst (fa (inverse (real n + 1))) for n
    define y where y n = snd (fa (inverse (real n + 1))) for n
    have xyn: ∀n. x n ∈ s ∧ y n ∈ s
      and xy0: ∀n. dist (x n) (y n) < inverse (real n + 1)
      and fxy:∀n. ¬ dist (f (x n)) (f (y n)) < e
      unfolding x_def and y_def using fa
      by auto
    {
      fix e :: real
      assume e > 0
      then obtain N :: nat where N ≠ 0 and N: 0 < inverse (real N) ∧ inverse
(real N) < e
        unfolding real_arch_inverse[of e] by auto
      then have ∃N. ∀n≥N. dist (x n) (y n) < e
      by (smt (verit, ccfv_SIG) inverse_le_imp_le nat_le_real_less of_nat_le_0_iff
xy0)
    }
    then have ∀e>0. ∃N. ∀n≥N. dist (f (x n)) (f (y n)) < e
```

```
    using ‹?rhs›[THEN spec[where x=x], THEN spec[where x=y]] and xyn
    unfolding lim_sequentially dist_real_def by auto
  then have False using fxy and ‹e>0› by auto
 }
 then show ?lhs
  unfolding uniformly_continuous_on_def by blast
qed
```

4.24 Continuity on a Compact Domain Implies Uniform Continuity

From the proof of the Heine-Borel theorem: Lemma 2 in section 3.7, page 69 of J. C. Burkill and H. Burkill. A Second Course in Mathematical Analysis (CUP, 2002)

```
lemma Heine_Borel_lemma:
 assumes compact S and Ssub: S ⊆ ⋃𝒢 and opn: ⋀G. G ∈ 𝒢 ⟹ open G
 obtains e where 0 < e ⋀x. x ∈ S ⟹ ∃ G ∈ 𝒢. ball x e ⊆ G
proof −
 have False if neg: ⋀e. 0 < e ⟹ ∃ x ∈ S. ∀ G ∈ 𝒢. ¬ ball x e ⊆ G
 proof −
  have ∃ x ∈ S. ∀ G ∈ 𝒢. ¬ ball x (1 / Suc n) ⊆ G for n
   using neg by simp
  then obtain f where ⋀n. f n ∈ S and fG: ⋀G n. G ∈ 𝒢 ⟹ ¬ ball (f n) (1
/ Suc n) ⊆ G
   by metis
  then obtain l r where l ∈ S strict_mono r and to_l: (f ∘ r) ⟶ l
   using ‹compact S› compact_def that by metis
  then obtain G where l ∈ G G ∈ 𝒢
   using Ssub by auto
  then obtain e where 0 < e and e: ⋀z. dist z l < e ⟹ z ∈ G
   using opn open_dist by blast
  obtain N1 where N1: ⋀n. n ≥ N1 ⟹ dist (f (r n)) l < e/2
   using to_l apply (simp add: lim_sequentially)
   using ‹0 < e› half_gt_zero that by blast
  obtain N2 where N2: of_nat N2 > 2/e
   using reals_Archimedean2 by blast
  obtain x where x ∈ ball (f (r (max N1 N2))) (1 / real (Suc (r (max N1
N2)))) and x ∉ G
   using fG [OF ‹G ∈ 𝒢›, of r (max N1 N2)] by blast
  then have dist (f (r (max N1 N2))) x < 1 / real (Suc (r (max N1 N2)))
   by simp
  also have ... ≤ 1 / real (Suc (max N1 N2))
  by (meson Suc_le_mono ‹strict_mono r› inverse_of_nat_le nat.discI seq_suble)
  also have ... ≤ 1 / real (Suc N2)
   by (simp add: field_simps)
  also have ... < e/2
   using N2 ‹0 < e› by (simp add: field_simps)
```

```
    finally have dist (f (r (max N1 N2))) x < e/2 .
    moreover have dist (f (r (max N1 N2))) l < e/2
      using N1 max.cobounded1 by blast
    ultimately have dist x l < e
      by metric
    then show ?thesis
      using e ‹x ∉ G› by blast
  qed
  then show ?thesis
    by (meson that)
qed

lemma compact_uniformly_equicontinuous:
  assumes compact S
      and cont: ⋀x e. ⟦x ∈ S; 0 < e⟧
                      ⟹ ∃d. 0 < d ∧
                           (∀f ∈ ℱ. ∀x′ ∈ S. dist x′ x < d ⟶ dist (f x′) (f x) < e)
      and 0 < e
  obtains d where 0 < d
                  ⋀f x x′. ⟦f ∈ ℱ; x ∈ S; x′ ∈ S; dist x′ x < d⟧ ⟹ dist (f x′) (f x)
< e
proof −
  obtain d where d_pos: ⋀x e. ⟦x ∈ S; 0 < e⟧ ⟹ 0 < d x e
     and d_dist : ⋀x x′ e f. ⟦dist x′ x < d x e; x ∈ S; x′ ∈ S; 0 < e; f ∈ ℱ⟧ ⟹
dist (f x′) (f x) < e
    using cont by metis
  let ?𝒢 = ((λx. ball x (d x (e/2))) ‘ S)
  have Ssub: S ⊆ ⋃ ?𝒢
    using ‹0 < e› d_pos by auto
  then obtain k where 0 < k and k: ⋀x. x ∈ S ⟹ ∃G ∈ ?𝒢. ball x k ⊆ G
    by (rule Heine_Borel_lemma [OF ‹compact S›]) auto
  moreover have dist (f v) (f u) < e if f ∈ ℱ u ∈ S v ∈ S dist v u < k for f u v
  proof −
    obtain G where G ∈ ?𝒢 u ∈ G v ∈ G
      using k that
     by (metis ‹dist v u < k› ‹u ∈ S› ‹0 < k› centre_in_ball subsetD dist_commute
mem_ball)
    then obtain w where w: dist w u < d w (e/2) dist w v < d w (e/2) w ∈ S
      by auto
    with that d_dist have dist (f w) (f v) < e/2
      by (metis ‹0 < e› dist_commute half_gt_zero)
    moreover
    have dist (f w) (f u) < e/2
        using that d_dist w by (metis ‹0 < e› dist_commute divide_pos_pos
zero_less_numeral)
    ultimately show ?thesis
      using dist_triangle_half_r by blast
  qed
  ultimately show ?thesis using that by blast
```

qed

corollary *compact_uniformly_continuous*:
fixes $f :: 'a :: metric\_space \Rightarrow 'b :: metric\_space$
assumes f: *continuous_on S f* **and** S: *compact S*
shows *uniformly_continuous_on S f*
using f
unfolding *continuous_on_iff uniformly_continuous_on_def*
by (*force intro*: *compact_uniformly_equicontinuous* [*OF S*, *of* {*f*}])

4.25 Theorems relating continuity and uniform continuity to closures

lemma *continuous_on_closure*:
continuous_on (*closure S*) *f* $\longleftrightarrow$
$(\forall x\ e.\ x \in closure\ S \wedge 0 < e$
$\longrightarrow (\exists d.\ 0 < d \wedge (\forall y.\ y \in S \wedge dist\ y\ x < d \longrightarrow dist\ (f\ y)\ (f\ x) < e)))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
unfolding *continuous_on_iff* **by** (*metis Un_iff closure_def*)
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof
fix x **and** e::*real*
assume $0 < e$ **and** x: $x \in closure\ S$
obtain δ::*real* **where** $\delta > 0$
and δ: $\bigwedge y.\ [\![y \in S;\ dist\ y\ x < \delta]\!] \Longrightarrow dist\ (f\ y)\ (f\ x) < e/2$
using *R* [*of x e/2*] ‹*0 < e*› *x* **by** *auto*
have *dist* (*f y*) (*f x*) $\leq e$ **if** *y*: $y \in closure\ S$ **and** *dyx*: *dist y x* $< \delta/2$ **for** *y*
proof −
obtain δ'::*real* **where** $\delta' > 0$
and δ': $\bigwedge z.\ [\![z \in S;\ dist\ z\ y < \delta']\!] \Longrightarrow dist\ (f\ z)\ (f\ y) < e/2$
using *R* [*of y e/2*] ‹*0 < e*› *y* **by** *auto*
obtain *z* **where** $z \in S$ **and** *z*: *dist z y* $< min\ \delta'\ \delta\ /\ 2$
using *closure_approachable y*
by (*metis* ‹*0 < δ'*› ‹*0 < δ*› *divide_pos_pos min_less_iff_conj zero_less_numeral*)
have *dist* (*f z*) (*f y*) $< e/2$
using δ' [*OF* ‹*z ∈ S*›] *z* ‹*0 < δ'*› **by** *metric*
moreover have *dist* (*f z*) (*f x*) $< e/2$
using δ[*OF* ‹*z ∈ S*›] *z dyx* **by** *metric*
ultimately show *?thesis*
by *metric*
qed
then show $\exists d{>}0.\ \forall x'{\in}closure\ S.\ dist\ x'\ x < d \longrightarrow dist\ (f\ x')\ (f\ x) \leq e$
by (*rule_tac x*=$\delta/2$ **in** *exI*) (*simp add*: ‹$\delta > 0$›)
qed

qed

lemma *continuous_on_closure_sequentially*:
fixes f :: $'a$::*metric_space* ⇒ $'b$ :: *metric_space*
shows
continuous_on (*closure* S) f ⟷
($\forall x\ a.\ a \in$ *closure* $S \wedge (\forall n.\ x\ n \in S) \wedge x \longrightarrow a \longrightarrow (f \circ x) \longrightarrow f\ a$)
(**is** *?lhs* = *?rhs*)
proof −
have *continuous_on* (*closure* S) f ⟷
($\forall x \in$ *closure* S. *continuous* (*at* x *within* S) f)
by (*force simp*: *continuous_on_closure continuous_within_eps_delta*)
also have ... = *?rhs*
by (*force simp*: *continuous_within_sequentially*)
finally show *?thesis* **.**
qed

lemma *uniformly_continuous_on_closure*:
fixes f :: $'a$::*metric_space* ⇒ $'b$::*metric_space*
assumes *ucont*: *uniformly_continuous_on* S f
and *cont*: *continuous_on* (*closure* S) f
shows *uniformly_continuous_on* (*closure* S) f
unfolding *uniformly_continuous_on_def*
proof (*intro allI impI*)
fix *e*::*real*
assume $0 < e$
then obtain *d*::*real*
where $d{>}0$
and d: $\bigwedge x\ x'.\ [\![x{\in}S;\ x'{\in}S;\ dist\ x'\ x < d]\!] \Longrightarrow dist\ (f\ x')\ (f\ x) < e/3$
using *ucont* [*unfolded uniformly_continuous_on_def*, *rule_format*, *of* $e/3$] **by**
auto
show $\exists\, d{>}0.\ \forall x{\in}$*closure* $S.\ \forall x'{\in}$*closure* $S.\ dist\ x'\ x < d \longrightarrow dist\ (f\ x')\ (f\ x) <$
e
proof (*rule exI* [**where** $x{=}d/3$], *clarsimp simp*: ‹$d > 0$›)
fix $x\ y$
assume x: $x \in$ *closure* S **and** y: $y \in$ *closure* S **and** *dyx*: $dist\ y\ x * 3 < d$
obtain *d1*::*real* **where** $d1 > 0$
and $d1$: $\bigwedge w.\ [\![w \in$ *closure* S; $dist\ w\ x < d1]\!] \Longrightarrow dist\ (f\ w)\ (f\ x) < e/3$
using *cont* [*unfolded continuous_on_iff*, *rule_format*, *of* x $e/3$] ‹$0 < e$› x
by *auto*
obtain x' **where** $x' \in S$ **and** x': $dist\ x'\ x < min\ d1\ (d\ /\ 3)$
using *closure_approachable* [*of* x S]
by (*metis* ‹$0 < d1$› ‹$0 < d$› *divide_pos_pos min_less_iff_conj x zero_less_numeral*)
obtain *d2*::*real* **where** $d2 > 0$
and $d2$: $\forall w \in$ *closure* $S.\ dist\ w\ y < d2 \longrightarrow dist\ (f\ w)\ (f\ y) < e/3$
using *cont* [*unfolded continuous_on_iff*, *rule_format*, *of* y $e/3$] ‹$0 < e$› y
by *auto*
obtain y' **where** $y' \in S$ **and** y': $dist\ y'\ y < min\ d2\ (d\ /\ 3)$
using *closure_approachable* [*of* y S]

```
    by (metis ‹0 < d2› ‹0 < d› divide_pos_pos min_less_iff_conj y zero_less_numeral)
    have dist x' x < d/3 using x' by auto
    then have dist x' y' < d
      using dyx y' by metric
    then have dist (f x') (f y') < e/3
      by (rule d [OF ‹y' ∈ S› ‹x' ∈ S›])
    moreover have dist (f x') (f x) < e/3 using ‹x' ∈ S› closure_subset x' d1
      by (simp add: closure_def)
    moreover have dist (f y') (f y) < e/3 using ‹y' ∈ S› closure_subset y' d2
      by (simp add: closure_def)
    ultimately show dist (f y) (f x) < e by metric
  qed
qed

lemma uniformly_continuous_on_extension_at_closure:
  fixes f::'a::metric_space ⇒ 'b::complete_space
  assumes uc: uniformly_continuous_on X f
  assumes x ∈ closure X
  obtains l where (f ⟶ l) (at x within X)
proof −
  from assms obtain xs where xs: xs ⟶ x ⋀n. xs n ∈ X
    by (auto simp: closure_sequential)

  from uniformly_continuous_on_Cauchy[OF uc LIMSEQ_imp_Cauchy, OF xs]
  obtain l where l: (λn. f (xs n)) ⟶ l
    by atomize_elim (simp only: convergent_eq_Cauchy)

  have (f ⟶ l) (at x within X)
  proof (safe intro!: Lim_within_LIMSEQ)
    fix xs'
    assume ∀n. xs' n ≠ x ∧ xs' n ∈ X
      and xs': xs' ⟶ x
    then have xs' n ≠ x xs' n ∈ X for n by auto

    from uniformly_continuous_on_Cauchy[OF uc LIMSEQ_imp_Cauchy, OF
‹xs' ⟶ x› ‹xs' _ ∈ X›]
    obtain l' where l': (λn. f (xs' n)) ⟶ l'
      by atomize_elim (simp only: convergent_eq_Cauchy)

    show (λn. f (xs' n)) ⟶ l
    proof (rule tendstoI)
      fix e::real assume e > 0
      define e' where e' ≡ e/2
      have e' > 0 using ‹e > 0› by (simp add: e'_def)

      have ∀F n in sequentially. dist (f (xs n)) l < e'
        by (simp add: ‹0 < e'› l tendstoD)
      moreover
      from uc[unfolded uniformly_continuous_on_def, rule_format, OF ‹e' > 0›]
```

```
      obtain d where d: d > 0 ⋀x x'. x ∈ X ⟹ x' ∈ X ⟹ dist x x' < d ⟹
dist (f x) (f x') < e'
        by auto
      have ∀F n in sequentially. dist (xs n) (xs' n) < d
        by (auto intro!: ‹0 < d› order_tendstoD tendsto_eq_intros xs xs')
      ultimately
      show ∀F n in sequentially. dist (f (xs' n)) l < e
      proof eventually_elim
        case (elim n)
        have dist (f (xs' n)) l ≤ dist (f (xs n)) (f (xs' n)) + dist (f (xs n)) l
          by metric
        also have dist (f (xs n)) (f (xs' n)) < e'
          by (auto intro!: d xs ‹xs' _ ∈ _› elim)
        also note ‹dist (f (xs n)) l < e'›
        also have e' + e' = e by (simp add: e'_def)
        finally show ?case by simp
      qed
    qed
  qed
  thus ?thesis ..
qed

lemma uniformly_continuous_on_extension_on_closure:
  fixes f::'a::metric_space ⇒ 'b::complete_space
  assumes uc: uniformly_continuous_on X f
  obtains g where uniformly_continuous_on (closure X) g ⋀x. x ∈ X ⟹ f x =
g x
    ⋀Y h x. X ⊆ Y ⟹ Y ⊆ closure X ⟹ continuous_on Y h ⟹ (⋀x. x ∈ X
⟹ f x = h x) ⟹ x ∈ Y ⟹ h x = g x
proof −
  from uc have cont_f: continuous_on X f
    by (simp add: uniformly_continuous_imp_continuous)
  obtain y where y: (f ⟶ y x) (at x within X) if x ∈ closure X for x
    using uniformly_continuous_on_extension_at_closure[OF assms] by meson
  let ?g = λx. if x ∈ X then f x else y x

  have uniformly_continuous_on (closure X) ?g
    unfolding uniformly_continuous_on_def
  proof safe
    fix e::real assume e > 0
    define e' where e' ≡ e / 3
    have e' > 0 using ‹e > 0› by (simp add: e'_def)
    from uc[unfolded uniformly_continuous_on_def, rule_format, OF ‹0 < e'›]
    obtain d where d > 0 and d: ⋀x x'. x ∈ X ⟹ x' ∈ X ⟹ dist x' x < d
⟹ dist (f x') (f x) < e'
      by auto
    define d' where d' = d / 3
    have d' > 0 using ‹d > 0› by (simp add: d'_def)
    show ∃ d>0. ∀ x∈closure X. ∀ x'∈closure X. dist x' x < d ⟶ dist (?g x') (?g
```

```
x) < e
    proof (safe intro!: exI[where x=d'] ‹d' > 0›)
      fix x x' assume x: x ∈ closure X and x': x' ∈ closure X and dist: dist x' x
< d'
      then obtain xs xs' where xs: xs ⟶ x ⋀n. xs n ∈ X
        and xs': xs' ⟶ x' ⋀n. xs' n ∈ X
        by (auto simp: closure_sequential)
      have ∀⇩F n in sequentially. dist (xs' n) x' < d'
        and ∀⇩F n in sequentially. dist (xs n) x < d'
        by (auto intro!: ‹0 < d'› order_tendstoD tendsto_eq_intros xs xs')
      moreover
      have (λx. f (xs x)) ⟶ y x if x ∈ closure X x ∉ X xs ⟶ x ⋀n. xs n
∈ X for xs x
        using that not_eventuallyD
        by (force intro!: filterlim_compose[OF y[OF ‹x ∈ closure X›]] simp: filter-
lim_at)
      then have (λx. f (xs' x)) ⟶ ?g x' (λx. f (xs x)) ⟶ ?g x
        using x x'
        by (auto intro!: continuous_on_tendsto_compose[OF cont_f] simp: xs' xs)
      then have ∀⇩F n in sequentially. dist (f (xs' n)) (?g x') < e'
        ∀⇩F n in sequentially. dist (f (xs n)) (?g x) < e'
        by (auto intro!: ‹0 < e'› order_tendstoD tendsto_eq_intros)
      ultimately
      have ∀⇩F n in sequentially. dist (?g x') (?g x) < e
      proof eventually_elim
        case (elim n)
        have dist (?g x') (?g x) ≤
          dist (f (xs' n)) (?g x') + dist (f (xs' n)) (f (xs n)) + dist (f (xs n)) (?g x)
            by (metis add.commute add_le_cancel_left dist_commute dist_triangle
dist_triangle_le)
        also
        from ‹dist (xs' n) x' < d'› ‹dist x' x < d'› ‹dist (xs n) x < d'›
        have dist (xs' n) (xs n) < d unfolding d'_def by metric
        with ‹xs _ ∈ X› ‹xs' _ ∈ X› have dist (f (xs' n)) (f (xs n)) < e'
          by (rule d)
        also note ‹dist (f (xs' n)) (?g x') < e'›
        also note ‹dist (f (xs n)) (?g x) < e'›
        finally show ?case by (simp add: e'_def)
      qed
      then show dist (?g x') (?g x) < e by simp
    qed
  qed
  moreover have f x = ?g x if x ∈ X for x using that by simp
  moreover
  {
    fix Y h x
    assume Y: x ∈ Y X ⊆ Y Y ⊆ closure X and cont_h: continuous_on Y h
      and extension: (⋀x. x ∈ X ⟹ f x = h x)
    {
```

```
    assume x ∉ X
    have x ∈ closure X using Y by auto
    then obtain xs where xs: xs ⟶ x ⋀n. xs n ∈ X
      by (auto simp: closure_sequential)
    from continuous_on_tendsto_compose[OF cont_h xs(1)] xs(2) Y
    have hx: (λx. f (xs x)) ⟶ h x
      by (auto simp: subsetD extension)
    then have (λx. f (xs x)) ⟶ y x
      using ‹x ∉ X› not_eventuallyD xs(2)
      by (force intro!: filterlim_compose[OF y[OF ‹x ∈ closure X›]] simp: filter-
lim_at xs)
    with hx have h x = y x by (rule LIMSEQ_unique)
   } then
   have h x = ?g x
    using extension by auto
  }
  ultimately show ?thesis ..
qed

lemma bounded_uniformly_continuous_image:
  fixes f :: 'a :: heine_borel ⇒ 'b :: heine_borel
  assumes uniformly_continuous_on S f bounded S
  shows bounded(f ‘ S)
   by (metis (no_types, lifting) assms bounded_closure_image compact_closure
compact_continuous_image compact_eq_bounded_closed image_cong uniformly_continuous_imp_continuous
uniformly_continuous_on_extension_on_closure)
```

4.26 With Abstract Topology (TODO: move and remove dependency?)

```
lemma openin_contains_ball:
    openin (top_of_set T) S ⟷
     S ⊆ T ∧ (∀x ∈ S. ∃e. 0 < e ∧ ball x e ∩ T ⊆ S)
    (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (metis IntD2 Int_commute Int_lower1 Int_mono inf.idem openE openin_open)
next
  assume ?rhs
  then show ?lhs
   by (smt (verit) open_ball Int_commute Int_iff centre_in_ball in_mono openin_open_Int
openin_subopen)
qed

lemma openin_contains_cball:
   openin (top_of_set T) S ⟷
       S ⊆ T ∧ (∀x ∈ S. ∃e. 0 < e ∧ cball x e ∩ T ⊆ S)
```

(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*force simp add*: *openin_contains_ball intro*: *exI* [**where** *x*=*_/2*])
next
assume *?rhs*
then show *?lhs*
by (*force simp add*: *openin_contains_ball*)
qed

4.27 Closed Nest

Bounded closed nest property (proof does not use Heine-Borel)

lemma *bounded_closed_nest*:
fixes *S* :: *nat* ⇒ ('*a*::*heine_borel*) *set*
assumes ⋀*n*. *closed* (*S n*)
and ⋀*n*. *S n* ≠ {}
and ⋀*m n*. *m* ≤ *n* ⟹ *S n* ⊆ *S m*
and *bounded* (*S 0*)
obtains *a* **where** ⋀*n*. *a* ∈ *S n*
proof −
from *assms*(*2*) **obtain** *x* **where** *x*: ∀ *n*. *x n* ∈ *S n*
using *choice*[*of* λ*n x*. *x* ∈ *S n*] **by** *auto*
from *assms*(*4*,*1*) **have** *seq_compact* (*S 0*)
by (*simp add*: *bounded_closed_imp_seq_compact*)
then obtain *l r* **where** *lr*: *l* ∈ *S 0 strict_mono r* (*x* ∘ *r*) ⟶ *l*
using *x* **and** *assms*(*3*) **unfolding** *seq_compact_def* **by** *blast*
have ∀ *n*. *l* ∈ *S n*
proof
fix *n* :: *nat*
have *closed* (*S n*)
using *assms*(*1*) **by** *simp*
moreover have ∀ *i*. (*x* ∘ *r*) *i* ∈ *S i*
using *x* **and** *assms*(*3*) **and** *lr*(*2*) [*THEN seq_suble*] **by** *auto*
then have ∀ *i*. (*x* ∘ *r*) (*i* + *n*) ∈ *S n*
using *assms*(*3*) **by** (*fast intro*!: *le_add2*)
moreover have (λ*i*. (*x* ∘ *r*) (*i* + *n*)) ⟶ *l*
using *lr*(*3*) **by** (*rule LIMSEQ_ignore_initial_segment*)
ultimately show *l* ∈ *S n*
by (*metis closed_sequentially*)
qed
then show *?thesis*
using *that* **by** *blast*
qed

Decreasing case does not even need compactness, just completeness.

lemma *decreasing_closed_nest*:

```
  fixes S :: nat ⇒ ('a::complete_space) set
  assumes ⋀n. closed (S n)
          ⋀n. S n ≠ {}
          ⋀m n. m ≤ n ⟹ S n ⊆ S m
          ⋀e. e>0 ⟹ ∃n. ∀x∈S n. ∀y∈S n. dist x y < e
  obtains a where ⋀n. a ∈ S n
proof −
  obtain t where t: ∀n. t n ∈ S n
    by (meson assms(2) equals0I)
  {
    fix e :: real
    assume e > 0
    then obtain N where N: ∀x∈S N. ∀y∈S N. dist x y < e
      using assms(4) by blast
    {
      fix m n :: nat
      assume N ≤ m ∧ N ≤ n
      then have t m ∈ S N t n ∈ S N
        using assms(3) t unfolding  subset_eq t by blast+
      then have dist (t m) (t n) < e
        using N by auto
    }
    then have ∃N. ∀m n. N ≤ m ∧ N ≤ n ⟶ dist (t m) (t n) < e
      by auto
  }
  then have Cauchy t
    by (metis metric_CauchyI)
  then obtain l where l:(t ⟶ l) sequentially
    using complete_UNIV unfolding complete_def by auto
  { fix n :: nat
    { fix e :: real
      assume e > 0
      then obtain N :: nat where N: ∀n≥N. dist (t n) l < e
        using l[unfolded lim_sequentially] by auto
      have t (max n N) ∈ S n
        by (meson assms(3) contra_subsetD max.cobounded1 t)
      then have ∃y∈S n. dist y l < e
        using N max.cobounded2 by blast
    }
    then have l ∈ S n
      using closed_approachable[of S n l] assms(1) by auto
  }
  then show ?thesis
    using that by blast
qed
```

Strengthen it to the intersection actually being a singleton.

```
lemma decreasing_closed_nest_sing:
  fixes S :: nat ⇒ 'a::complete_space set
```

assumes $\bigwedge n.\ closed(S\ n)$
$\bigwedge n.\ S\ n \neq \{\}$
$\bigwedge m\ n.\ m \leq n \Longrightarrow S\ n \subseteq S\ m$
$\bigwedge e.\ e{>}0 \Longrightarrow \exists\, n.\ \forall\, x \in (S\ n).\ \forall\ y{\in}(S\ n).\ dist\ x\ y < e$
shows $\exists\, a.\ \bigcap(range\ S) = \{a\}$
proof −
obtain a **where** a: $\forall\, n.\ a \in S\ n$
using *decreasing_closed_nest*[*of* S] **using** *assms* **by** *auto*
{ fix b
assume b: $b \in \bigcap(range\ S)$
{ fix e :: *real*
assume $e > 0$
then have $dist\ a\ b < e$
using *assms*(*4*) **and** b **and** a **by** *blast*
}
then have $dist\ a\ b = 0$
by (*metis dist_eq_0_iff dist_nz less_le*)
}
with a **have** $\bigcap(range\ S) = \{a\}$
unfolding *image_def* **by** *auto*
then show *?thesis* **..**
qed

4.28 Making a continuous function avoid some value in a neighbourhood

lemma *continuous_within_avoid*:
fixes f :: *'a*::*metric_space* ⇒ *'b*::*t1_space*
assumes *continuous* (*at* x *within* s) f
and $f\ x \neq a$
shows $\exists\, e{>}0.\ \forall\, y \in s.\ dist\ x\ y < e \longrightarrow f\ y \neq a$
proof −
obtain U **where** *open* U **and** $f\ x \in U$ **and** $a \notin U$
using *t1_space* [*OF* ‹$f\ x \neq a$›] **by** *fast*
have ($f \longrightarrow f\ x$) (*at* x *within* s)
using *assms*(*1*) **by** (*simp add*: *continuous_within*)
then have *eventually* ($\lambda y.\ f\ y \in U$) (*at* x *within* s)
using ‹*open* U› **and** ‹$f\ x \in U$›
unfolding *tendsto_def* **by** *fast*
then have *eventually* ($\lambda y.\ f\ y \neq a$) (*at* x *within* s)
using ‹$a \notin U$› **by** (*fast elim*: *eventually_mono*)
then show *?thesis*
using ‹$f\ x \neq a$› **by** (*auto simp*: *dist_commute eventually_at*)
qed

lemma *continuous_at_avoid*:
fixes f :: *'a*::*metric_space* ⇒ *'b*::*t1_space*
assumes *continuous* (*at* x) f

and $f\ x \neq a$
shows $\exists\, e{>}0.\ \forall\, y.\ dist\ x\ y < e \longrightarrow f\ y \neq a$
using *assms continuous_within_avoid*[*of x UNIV f a*] **by** *simp*

lemma *continuous_on_avoid*:
fixes $f :: 'a{::}metric\_space \Rightarrow 'b{::}t1\_space$
assumes *continuous_on s f*
and $x \in s$
and $f\ x \neq a$
shows $\exists\, e{>}0.\ \forall\, y \in s.\ dist\ x\ y < e \longrightarrow f\ y \neq a$
using *assms*(*1*)[*unfolded continuous_on_eq_continuous_within*, *THEN bspec*[**where** *x*=*x*],
OF assms(*2*)] *continuous_within_avoid*[*of x s f a*]
using *assms*(*3*)
by *auto*

lemma *continuous_on_open_avoid*:
fixes $f :: 'a{::}metric\_space \Rightarrow 'b{::}t1\_space$
assumes *continuous_on s f*
and *open s*
and $x \in s$
and $f\ x \neq a$
shows $\exists\, e{>}0.\ \forall\, y.\ dist\ x\ y < e \longrightarrow f\ y \neq a$
using *assms*(*1*)[*unfolded continuous_on_eq_continuous_at*[*OF assms*(*2*)], *THEN bspec*[**where** *x*=*x*], *OF assms*(*3*)]
using *continuous_at_avoid*[*of x f a*] *assms*(*4*)
by *auto*

4.29 Consequences for Real Numbers

lemma *closed_contains_Inf*:
fixes $S :: real\ set$
shows $S \neq \{\} \Longrightarrow bdd\_below\ S \Longrightarrow closed\ S \Longrightarrow Inf\ S \in S$
by (*metis closure_contains_Inf closure_closed*)

lemma *closed_subset_contains_Inf*:
fixes $A\ C :: real\ set$
shows $closed\ C \Longrightarrow A \subseteq C \Longrightarrow A \neq \{\} \Longrightarrow bdd\_below\ A \Longrightarrow Inf\ A \in C$
by (*metis closure_contains_Inf closure_minimal subset_eq*)

lemma *closed_contains_Sup*:
fixes $S :: real\ set$
shows $S \neq \{\} \Longrightarrow bdd\_above\ S \Longrightarrow closed\ S \Longrightarrow Sup\ S \in S$
by (*subst closure_closed*[*symmetric*], *assumption*, *rule closure_contains_Sup*)

lemma *closed_subset_contains_Sup*:
fixes $A\ C :: real\ set$
shows $closed\ C \Longrightarrow A \subseteq C \Longrightarrow A \neq \{\} \Longrightarrow bdd\_above\ A \Longrightarrow Sup\ A \in C$
by (*metis closure_contains_Sup closure_minimal subset_eq*)

lemma *atLeastAtMost_subset_contains_Inf*:
fixes A :: *real set* **and** a b :: *real*
shows $A \neq \{\} \Longrightarrow a \leq b \Longrightarrow A \subseteq \{a..b\} \Longrightarrow Inf\ A \in \{a..b\}$
by (*rule closed_subset_contains_Inf*)
(*auto intro*: *closed_real_atLeastAtMost intro*!: *bdd_belowI*[*of A a*])

lemma *bounded_real*: *bounded* (S::*real set*) $\longleftrightarrow$ ($\exists\, a.\ \forall\, x{\in}S.\ |x| \leq a$)
by (*simp add*: *bounded_iff*)

lemma *bounded_imp_bdd_above*: *bounded* $S \Longrightarrow$ *bdd_above* (S :: *real set*)
by (*auto simp*: *bounded_def bdd_above_def dist_real_def*)
(*metis abs_le_D1 abs_minus_commute diff_le_eq*)

lemma *bounded_imp_bdd_below*: *bounded* $S \Longrightarrow$ *bdd_below* (S :: *real set*)
by (*auto simp*: *bounded_def bdd_below_def dist_real_def*)
(*metis abs_le_D1 add.commute diff_le_eq*)

lemma *bounded_norm_le_SUP_norm*:
bounded (*range* f) $\Longrightarrow$ *norm* ($f\ x$) $\leq$ (*SUP* x. *norm* ($f\ x$))
by (*auto intro*!: *cSUP_upper bounded_imp_bdd_above simp*: *bounded_norm_comp*)

lemma *bounded_has_Sup*:
fixes S :: *real set*
assumes *bounded* S
and $S \neq \{\}$
shows $\forall\, x{\in}S.\ x \leq Sup\ S$
and $\forall\, b.\ (\forall\, x{\in}S.\ x \leq b) \longrightarrow Sup\ S \leq b$
proof
show $\forall\, b.\ (\forall\, x{\in}S.\ x \leq b) \longrightarrow Sup\ S \leq b$
using *assms* **by** (*metis cSup_least*)
qed (*metis cSup_upper assms*(*1*) *bounded_imp_bdd_above*)

lemma *Sup_insert*:
fixes S :: *real set*
shows *bounded* $S \Longrightarrow$ *Sup* (*insert* x S) = (*if* $S = \{\}$ *then* x *else max* x (*Sup* S))
by (*auto simp*: *bounded_imp_bdd_above sup_max cSup_insert_If*)

lemma *bounded_has_Inf*:
fixes S :: *real set*
assumes *bounded* S
and $S \neq \{\}$
shows $\forall\, x{\in}S.\ x \geq Inf\ S$
and $\forall\, b.\ (\forall\, x{\in}S.\ x \geq b) \longrightarrow Inf\ S \geq b$
proof
show $\forall\, b.\ (\forall\, x{\in}S.\ x \geq b) \longrightarrow Inf\ S \geq b$
using *assms* **by** (*metis cInf_greatest*)
qed (*metis cInf_lower assms*(*1*) *bounded_imp_bdd_below*)

lemma *Inf_insert*:
fixes S :: *real set*
shows *bounded* $S \Longrightarrow$ *Inf* (*insert* x S) = (*if* $S = \{\}$ *then* x *else min* x (*Inf* S))
by (*auto simp*: *bounded_imp_bdd_below inf_min cInf_insert_If*)

lemma *open_real*:
fixes s :: *real set*
shows *open* $s \longleftrightarrow$ ($\forall x \in s.\ \exists e{>}0.\ \forall x'.\ |x' - x| < e \longrightarrow x' \in s$)
unfolding *open_dist dist_norm* **by** *simp*

lemma *islimpt_approachable_real*:
fixes s :: *real set*
shows x *islimpt* $s \longleftrightarrow$ ($\forall e{>}0.\ \exists x'{\in}\ s.\ x' \neq x \wedge |x' - x| < e$)
unfolding *islimpt_approachable dist_norm* **by** *simp*

lemma *closed_real*:
fixes s :: *real set*
shows *closed* $s \longleftrightarrow$ ($\forall x.\ (\forall e{>}0.\ \exists x' \in s.\ x' \neq x \wedge |x' - x| < e) \longrightarrow x \in s$)
unfolding *closed_limpt islimpt_approachable dist_norm* **by** *simp*

lemma *continuous_at_real_range*:
fixes f :: *'a::real_normed_vector* $\Rightarrow$ *real*
shows *continuous* (*at* x) $f \longleftrightarrow$ ($\forall e{>}0.\ \exists d{>}0.\ \forall x'.\ norm(x' - x) < d \longrightarrow |f\ x' - f\ x| < e$)
by (*metis* (*mono_tags*, *opaque_lifting*) *LIM_eq continuous_within norm_eq_zero real_norm_def right_minus_eq*)

lemma *continuous_on_real_range*:
fixes f :: *'a::real_normed_vector* $\Rightarrow$ *real*
shows *continuous_on* s $f \longleftrightarrow$
($\forall x \in s.\ \forall e{>}0.\ \exists d{>}0.\ (\forall x' \in s.\ norm(x' - x) < d \longrightarrow |f\ x' - f\ x| < e)$)
unfolding *continuous_on_iff dist_norm* **by** *simp*

lemma *continuous_on_closed_Collect_le*:
fixes $f\ g$ :: *'a::topological_space* $\Rightarrow$ *real*
assumes f: *continuous_on* s f **and** g: *continuous_on* s g **and** s: *closed* s
shows *closed* $\{x \in s.\ f\ x \leq g\ x\}$
proof −
have *closed* ((λx. $g\ x - f\ x$) −‘ $\{0..\} \cap s$)
using *closed_real_atLeast continuous_on_diff* [*OF g f*]
by (*simp add*: *continuous_on_closed_vimage* [*OF s*])
also have ((λx. $g\ x - f\ x$) −‘ $\{0..\} \cap s$) = $\{x{\in}s.\ f\ x \leq g\ x\}$
by *auto*
finally show *?thesis* **.**
qed

lemma *continuous_le_on_closure*:
fixes a::*real*
assumes f: *continuous_on* (*closure* s) f

and x: $x \in closure(s)$
and xlo: $\bigwedge x.\ x \in s \Longrightarrow f(x) \leq a$
shows $f(x) \leq a$
using *image_closure_subset* [*OF f*, **where** $T = \{x.\ x \leq a\}$] *assms*
continuous_on_closed_Collect_le[*of UNIV* $\lambda x.\ x$ $\lambda x.\ a$]
by *auto*

lemma *continuous_ge_on_closure*:
fixes *a*::*real*
assumes *f*: *continuous_on* (*closure s*) *f*
and x: $x \in closure(s)$
and xlo: $\bigwedge x.\ x \in s \Longrightarrow f(x) \geq a$
shows $f(x) \geq a$
using *image_closure_subset* [*OF f*, **where** $T = \{x.\ a \leq x\}$] *assms*
continuous_on_closed_Collect_le[*of UNIV* $\lambda x.\ a$ $\lambda x.\ x$]
by *auto*

4.30 The infimum of the distance between two sets

definition *setdist* :: *'a*::*metric_space set* ⇒ *'a set* ⇒ *real* **where**
setdist s t ≡
(*if* $s = \{\} \vee t = \{\}$ *then 0*
else Inf $\{dist\ x\ y |\ x\ y.\ x \in s \wedge y \in t\}$)

lemma *setdist_empty1* [*simp*]: *setdist* $\{\}$ $t = 0$
by (*simp add*: *setdist_def*)

lemma *setdist_empty2* [*simp*]: *setdist t* $\{\} = 0$
by (*simp add*: *setdist_def*)

lemma *setdist_pos_le* [*simp*]: $0 \leq setdist\ s\ t$
by (*auto simp*: *setdist_def ex_in_conv* [*symmetric*] *intro*: *cInf_greatest*)

lemma *le_setdistI*:
assumes $s \neq \{\}$ $t \neq \{\}$ $\bigwedge x\ y.\ [\![x \in s;\ y \in t]\!] \Longrightarrow d \leq dist\ x\ y$
shows $d \leq setdist\ s\ t$
using *assms*
by (*auto simp*: *setdist_def Set.ex_in_conv* [*symmetric*] *intro*: *cInf_greatest*)

lemma *setdist_le_dist*: $[\![x \in s;\ y \in t]\!] \Longrightarrow setdist\ s\ t \leq dist\ x\ y$
unfolding *setdist_def*
by (*auto intro*!: *bdd_belowI* [**where** *m*=*0*] *cInf_lower*)

lemma *le_setdist_iff*:
$d \leq setdist\ S\ T \longleftrightarrow$
$(\forall x \in S.\ \forall y \in T.\ d \leq dist\ x\ y) \wedge (S = \{\} \vee T = \{\} \longrightarrow d \leq 0)$
by (*smt* (*verit*) *le_setdistI setdist_def setdist_le_dist*)

lemma *setdist_ltE*:

assumes *setdist S T < b S ≠ {} T ≠ {}*
obtains *x y* **where** *x ∈ S y ∈ T dist x y < b*
using *assms*
by (*auto simp*: *not_le* [*symmetric*] *le_setdist_iff*)

lemma *setdist_refl*: *setdist S S = 0*
by (*metis dist_eq_0_iff ex_in_conv order_antisym setdist_def setdist_le_dist setdist_pos_le*)

lemma *setdist_sym*: *setdist S T = setdist T S*
by (*force simp*: *setdist_def dist_commute intro*!: *arg_cong* [**where** *f=Inf*])

lemma *setdist_triangle*: *setdist S T ≤ setdist S {a} + setdist {a} T*
proof (*cases S = {} ∨ T = {}*)
case *True* **then show** *?thesis*
using *setdist_pos_le* **by** *fastforce*
next
case *False*
then have *⋀x. x ∈ S ⟹ setdist S T − dist x a ≤ setdist {a} T*
using *dist_self dist_triangle3 empty_not_insert le_setdist_iff setdist_le_dist singleton_iff*
by (*smt* (*verit*, *best*) *dist_self dist_triangle3 empty_not_insert le_setdist_iff setdist_le_dist singleton_iff*)
then have *setdist S T − setdist {a} T ≤ setdist S {a}*
using *False* **by** (*fastforce intro*: *le_setdistI*)
then show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *setdist_singletons* [*simp*]: *setdist {x} {y} = dist x y*
by (*simp add*: *setdist_def*)

lemma *setdist_Lipschitz*: *|setdist {x} S − setdist {y} S| ≤ dist x y*
apply (*subst setdist_singletons* [*symmetric*])
by (*metis abs_diff_le_iff diff_le_eq setdist_triangle setdist_sym*)

lemma *continuous_at_setdist* [*continuous_intros*]: *continuous* (*at x*) (*λy.* (*setdist {y} S*))
by (*force simp*: *continuous_at_eps_delta dist_real_def intro*: *le_less_trans* [*OF setdist_Lipschitz*])

lemma *continuous_on_setdist* [*continuous_intros*]: *continuous_on T* (*λy.* (*setdist {y} S*))
by (*metis continuous_at_setdist continuous_at_imp_continuous_on*)

lemma *uniformly_continuous_on_setdist*: *uniformly_continuous_on T* (*λy.* (*setdist {y} S*))
by (*force simp*: *uniformly_continuous_on_def dist_real_def intro*: *le_less_trans* [*OF setdist_Lipschitz*])

lemma *setdist_subset_right*: $[\![T \neq \{\};\ T \subseteq u]\!] \Longrightarrow setdist\ S\ u \leq setdist\ S\ T$
by (*smt* (*verit*, *best*) *in_mono le_setdist_iff*)

lemma *setdist_subset_left*: $[\![S \neq \{\};\ S \subseteq T]\!] \Longrightarrow setdist\ T\ u \leq setdist\ S\ u$
by (*metis setdist_subset_right setdist_sym*)

lemma *setdist_closure_1* [*simp*]: *setdist* (*closure S*) *T* = *setdist S T*
proof (*cases* $S = \{\} \vee T = \{\}$)
case *True* **then show** *?thesis* **by** *force*
next
case *False*
{ fix *y*
assume $y \in T$
have *continuous_on* (*closure S*) ($\lambda a.$ *dist a y*)
by (*auto simp*: *continuous_intros dist_norm*)
then have $*$: $\bigwedge x.\ x \in closure\ S \Longrightarrow setdist\ S\ T \leq dist\ x\ y$
by (*fast intro*: *setdist_le_dist* ‹$y \in T$› *continuous_ge_on_closure*)
} then
show *?thesis*
by (*metis False antisym closure_eq_empty closure_subset le_setdist_iff setdist_subset_left*)
qed

lemma *setdist_closure_2* [*simp*]: *setdist T* (*closure S*) = *setdist T S*
by (*metis setdist_closure_1 setdist_sym*)

lemma *setdist_eq_0I*: $[\![x \in S;\ x \in T]\!] \Longrightarrow setdist\ S\ T = 0$
by (*metis antisym dist_self setdist_le_dist setdist_pos_le*)

lemma *setdist_unique*:
$[\![a \in S;\ b \in T;\ \bigwedge x\ y.\ x \in S \wedge y \in T \Longrightarrow dist\ a\ b \leq dist\ x\ y]\!]$
$\Longrightarrow setdist\ S\ T = dist\ a\ b$
by (*force simp add*: *setdist_le_dist le_setdist_iff intro*: *antisym*)

lemma *setdist_le_sing*: $x \in S \Longrightarrow setdist\ S\ T \leq setdist\ \{x\}\ T$
using *setdist_subset_left* **by** *auto*

lemma *infdist_eq_setdist*: *infdist x A* = *setdist* $\{x\}$ *A*
by (*simp add*: *infdist_def setdist_def Setcompr_eq_image*)

lemma *setdist_eq_infdist*: *setdist A B* = (*if A* = $\{\}$ *then 0 else INF a*$\in$*A. infdist a B*)
proof −
have *Inf* $\{dist\ x\ y \mid x\ y.\ x \in A \wedge y \in B\}$ = (*INF x*$\in$*A. Inf* (*dist x* ‘ *B*))
if $b \in B$ $a \in A$ **for** *a b*
proof (*rule order_antisym*)
have *Inf* $\{dist\ x\ y \mid x\ y.\ x \in A \wedge y \in B\} \leq$ *Inf* (*dist x* ‘ *B*)
if $b \in B$ $a \in A$ $x \in A$ **for** *x*

```
    proof −
      have ⋀b'. b' ∈ B ⟹ Inf {dist x y |x y. x ∈ A ∧ y ∈ B} ≤ dist x b'
          by (metis (mono_tags, lifting) ex_in_conv setdist_def setdist_le_dist
that(3))
      then show ?thesis
        by (smt (verit) cINF_greatest ex_in_conv ‹b ∈ B›)
    qed
    then show Inf {dist x y |x y. x ∈ A ∧ y ∈ B} ≤ (INF x∈A. Inf (dist x ‘ B))
      by (metis (mono_tags, lifting) cINF_greatest emptyE that)
  next
    have ∗: ⋀x y. ⟦b ∈ B; a ∈ A; x ∈ A; y ∈ B⟧ ⟹ ∃a∈A. Inf (dist a ‘ B) ≤
dist x y
      by (meson bdd_below_image_dist cINF_lower)
    show (INF x∈A. Inf (dist x ‘ B)) ≤ Inf {dist x y |x y. x ∈ A ∧ y ∈ B}
    proof (rule conditionally_complete_lattice_class.cInf_mono)
      show bdd_below ((λx. Inf (dist x ‘ B)) ‘ A)
      by (metis (no_types, lifting) bdd_belowI2 ex_in_conv infdist_def infdist_nonneg
that(1))
    qed (use that in ‹auto simp: ∗›)
  qed
  then show ?thesis
    by (auto simp: setdist_def infdist_def)
qed

lemma infdist_mono:
  assumes A ⊆ B A ≠ {}
  shows infdist x B ≤ infdist x A
  by (simp add: assms infdist_eq_setdist setdist_subset_right)

lemma infdist_singleton [simp]: infdist x {y} = dist x y
  by (simp add: infdist_eq_setdist)

proposition setdist_attains_inf:
  assumes compact B B ≠ {}
  obtains y where y ∈ B setdist A B = infdist y A
proof (cases A = {})
  case True
  then show thesis
    by (metis assms diameter_compact_attained infdist_def setdist_def that)
next
  case False
  obtain y where y ∈ B and min: ⋀y'. y' ∈ B ⟹ infdist y A ≤ infdist y' A
    by (metis continuous_attains_inf [OF assms continuous_on_infdist] continu-
ous_on_id)
  show thesis
  proof
    have setdist A B = (INF y∈B. infdist y A)
      by (metis ‹B ≠ {}› setdist_eq_infdist setdist_sym)
    also have ... = infdist y A
```

```
    proof (rule order_antisym)
      show (INF y∈B. infdist y A) ≤ infdist y A
        by (meson ‹y ∈ B› bdd_belowI2 cInf_lower image_eqI infdist_nonneg)
    next
      show infdist y A ≤ (INF y∈B. infdist y A)
        by (simp add: ‹B ≠ {}› cINF_greatest min)
    qed
    finally show setdist A B = infdist y A .
  qed (fact ‹y ∈ B›)
qed
```

4.31 Diameter Lemma

taken from the AFP entry Martingales, by Ata Keskin, TU München

```
lemma diameter_comp_strict_mono:
  fixes s :: nat ⇒ 'a :: metric_space
  assumes strict_mono r bounded {s i |i. r n ≤ i}
  shows diameter {s (r i) | i. n ≤ i} ≤ diameter {s i | i. r n ≤ i}
proof (rule diameter_subset)
   show {s (r i) | i. n ≤ i} ⊆ {s i | i. r n ≤ i} using assms(1) monotoneD
strict_mono_mono by fastforce
qed (intro assms(2))

lemma diameter_bounded_bound':
  fixes S :: 'a :: metric_space set
  assumes S: bdd_above (case_prod dist ` (S×S)) and x ∈ S y ∈ S
  shows dist x y ≤ diameter S
proof −
  have (x,y) ∈ S×S using assms by auto
  then have dist x y ≤ (SUP (x,y)∈S×S. dist x y)
    by (metis S cSUP_upper case_prod_conv)
  with ‹x ∈ S› show ?thesis by (auto simp: diameter_def)
qed

lemma bounded_imp_dist_bounded:
  assumes bounded (range s)
  shows bounded ((λ(i, j). dist (s i) (s j)) ` ({n..} × {n..}))
 using bounded_dist_comp[OF bounded_fst bounded_snd, OF bounded_Times(1,1)[OF
assms(1,1)]] by (rule bounded_subset, force)
```

A sequence is Cauchy, if and only if it is bounded and its diameter tends to zero. The diameter is well-defined only if the sequence is bounded.

```
lemma cauchy_iff_diameter_tends_to_zero_and_bounded:
  fixes s :: nat ⇒ 'a :: metric_space
  shows Cauchy s ⟷ ((λn. diameter {s i | i. i ≥ n}) ⟶ 0 ∧ bounded (range
s))
proof −
  have {s i |i. N ≤ i} ≠ {} for N by blast
```

hence *diameter_SUP*: *diameter* $\{s\ i\ |i.\ N \leq i\} = (SUP\ (i, j) \in \{N..\} \times \{N..\}.\ dist\ (s\ i)\ (s\ j))$ **for** *N* **unfolding** *diameter_def* **by** (*auto intro*!: *arg_cong*[*of* _ _ *Sup*])
show *?thesis*
proof (*intro iffI*)
assume *asm*: *Cauchy s*
have $\exists N.\ \forall n{\geq}N.\ norm\ (diameter\ \{s\ i\ |i.\ n \leq i\}) < e$ **if** *e_pos*: $e > 0$ **for** *e*
proof −
obtain *N* **where** *dist_less*: $dist\ (s\ n)\ (s\ m) < (1/2) * e$ **if** $n \geq N\ m \geq N$ **for** *n m* **using** *asm e_pos* **by** (*metis Cauchy_def mult_pos_pos zero_less_divide_iff zero_less_numeral zero_less_one*)
{
fix *r* **assume** $r \geq N$
hence $dist\ (s\ n)\ (s\ m) < (1/2) * e$ **if** $n \geq r\ m \geq r$ **for** *n m* **using** *dist_less that* **by** *simp*
hence $(SUP\ (i, j) \in \{r..\} \times \{r..\}.\ dist\ (s\ i)\ (s\ j)) \leq (1/2) * e$ **by** (*intro cSup_least*) *fastforce*+
also have ... $< e$ **using** *e_pos* **by** *simp*
finally have $diameter\ \{s\ i\ |i.\ r \leq i\} < e$ **using** *diameter_SUP* **by** *presburger*
}
moreover have $diameter\ \{s\ i\ |i.\ r \leq i\} \geq 0$ **for** *r* **unfolding** *diameter_SUP* **using** *bounded_imp_dist_bounded*[*OF cauchy_imp_bounded*, *THEN bounded_imp_bdd_above*, *OF asm*] **by** (*intro cSup_upper2*, *auto*)
ultimately show *?thesis* **by** *auto*
qed
thus $(\lambda n.\ diameter\ \{s\ i\ |i.\ n \leq i\}) \longrightarrow 0 \wedge bounded\ (range\ s)$ **using** *cauchy_imp_bounded*[*OF asm*] **by** (*simp add*: *LIMSEQ_iff*)
next
assume *asm*: $(\lambda n.\ diameter\ \{s\ i\ |i.\ n \leq i\}) \longrightarrow 0 \wedge bounded\ (range\ s)$
have $\exists N.\ \forall n{\geq}N.\ \forall m{\geq}N.\ dist\ (s\ n)\ (s\ m) < e$ **if** *e_pos*: $e > 0$ **for** *e*
proof −
obtain *N* **where** *diam_less*: $diameter\ \{s\ i\ |i.\ r \leq i\} < e$ **if** $r \geq N$ **for** *r* **using** *LIMSEQ_D asm*(*1*) *e_pos* **by** *fastforce*
{
fix *n m* **assume** $n \geq N\ m \geq N$
hence $dist\ (s\ n)\ (s\ m) < e$ **using** *cSUP_lessD*[*OF bounded_imp_dist_bounded*[*THEN bounded_imp_bdd_above*], *OF* _ *diam_less*[*unfolded diameter_SUP*]] *asm* **by** *auto*
}
thus *?thesis* **by** *blast*
qed
then show *Cauchy s* **by** (*simp add*: *Cauchy_def*)
qed
qed

end

4.32 Elementary Normed Vector Spaces

theory *Elementary_Normed_Spaces*

imports
HOL−Library.FuncSet
Elementary_Metric_Spaces Cartesian_Space
Connected
begin

4.32.1 Orthogonal Transformation of Balls

4.32.2 Various Lemmas Combining Imports

lemma *open_sums*:
fixes $T :: ('b::real\_normed\_vector)\ set$
assumes *open* $S \vee$ *open* T
shows *open* $(\bigcup x{\in}\ S.\ \bigcup y \in T.\ \{x + y\})$
using *assms*
proof
assume S: *open* S
show *?thesis*
proof (*clarsimp simp*: *open_dist*)
fix $x\ y$
assume $x \in S\ y \in T$
with S **obtain** e **where** $e > 0$ **and** e: $\bigwedge x'.\ dist\ x'\ x < e \Longrightarrow x' \in S$
by (*auto simp*: *open_dist*)
then have $\bigwedge z.\ dist\ z\ (x + y) < e \Longrightarrow \exists\, x{\in}S.\ \exists\, y{\in}T.\ z = x + y$
by (*metis ‹y ∈ T› diff_add_cancel dist_add_cancel2*)
then show $\exists\, e{>}0.\ \forall\, z.\ dist\ z\ (x + y) < e \longrightarrow (\exists\, x{\in}S.\ \exists\, y{\in}T.\ z = x + y)$
using *‹0 < e› ‹x ∈ S›* **by** *blast*
qed
next
assume T: *open* T
show *?thesis*
proof (*clarsimp simp*: *open_dist*)
fix $x\ y$
assume $x \in S\ y \in T$
with T **obtain** e **where** $e > 0$ **and** e: $\bigwedge x'.\ dist\ x'\ y < e \Longrightarrow x' \in T$
by (*auto simp*: *open_dist*)
then have $\bigwedge z.\ dist\ z\ (x + y) < e \Longrightarrow \exists\, x{\in}S.\ \exists\, y{\in}T.\ z = x + y$
by (*metis ‹x ∈ S› add_diff_cancel_left' add_diff_eq diff_diff_add dist_norm*)
then show $\exists\, e{>}0.\ \forall\, z.\ dist\ z\ (x + y) < e \longrightarrow (\exists\, x{\in}S.\ \exists\, y{\in}T.\ z = x + y)$
using *‹0 < e› ‹y ∈ T›* **by** *blast*
qed
qed

lemma *image_orthogonal_transformation_ball*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'a$
assumes *orthogonal_transformation f*
shows $f\ `\ ball\ x\ r = ball\ (f\ x)\ r$
proof (*intro equalityI subsetI*)
fix y **assume** $y \in f\ `\ ball\ x\ r$
with *assms* **show** $y \in ball\ (f\ x)\ r$

by (*auto simp*: *orthogonal_transformation_isometry*)
next
fix *y* **assume** *y*: *y* ∈ *ball* (*f x*) *r*
then obtain *z* **where** *z*: *y* = *f z*
using *assms orthogonal_transformation_surj* **by** *blast*
with *y assms* **show** *y* ∈ *f* ‘ *ball x r*
by (*auto simp*: *orthogonal_transformation_isometry*)
qed

lemma *image_orthogonal_transformation_cball*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*
assumes *orthogonal_transformation f*
shows *f* ‘ *cball x r* = *cball* (*f x*) *r*
proof (*intro equalityI subsetI*)
fix *y* **assume** *y* ∈ *f* ‘ *cball x r*
with *assms* **show** *y* ∈ *cball* (*f x*) *r*
by (*auto simp*: *orthogonal_transformation_isometry*)
next
fix *y* **assume** *y*: *y* ∈ *cball* (*f x*) *r*
then obtain *z* **where** *z*: *y* = *f z*
using *assms orthogonal_transformation_surj* **by** *blast*
with *y assms* **show** *y* ∈ *f* ‘ *cball x r*
by (*auto simp*: *orthogonal_transformation_isometry*)
qed

4.32.3 Support

definition (**in** *monoid_add*) *support_on* :: *'b set* ⇒ (*'b* ⇒ *'a*) ⇒ *'b set*
where *support_on S f* = {*x*∈*S*. *f x* ≠ *0*}

lemma *in_support_on*: *x* ∈ *support_on S f* ⟷ *x* ∈ *S* ∧ *f x* ≠ *0*
by (*simp add*: *support_on_def*)

lemma *support_on_simps*[*simp*]:
support_on {} *f* = {}
support_on (*insert x S*) *f* =
(*if f x* = *0 then support_on S f else insert x* (*support_on S f*))
support_on (*S* ∪ *T*) *f* = *support_on S f* ∪ *support_on T f*
support_on (*S* ∩ *T*) *f* = *support_on S f* ∩ *support_on T f*
support_on (*S* − *T*) *f* = *support_on S f* − *support_on T f*
support_on (*f* ‘ *S*) *g* = *f* ‘ (*support_on S* (*g* ∘ *f*))
unfolding *support_on_def* **by** *auto*

lemma *support_on_cong*:
(⋀*x*. *x* ∈ *S* ⟹ *f x* = *0* ⟷ *g x* = *0*) ⟹ *support_on S f* = *support_on S g*
by (*auto simp*: *support_on_def*)

lemma *support_on_if*: *a* ≠ *0* ⟹ *support_on A* (λ*x*. *if P x then a else 0*) = {*x*∈*A*. *P x*}

by (*auto simp*: *support_on_def*)

lemma *support_on_if_subset*: *support_on A* (λ*x*. *if P x then a else 0*) ⊆ {*x* ∈ *A*. *P x*}
by (*auto simp*: *support_on_def*)

lemma *finite_support*[*intro*]: *finite S* ⟹ *finite* (*support_on S f*)
unfolding *support_on_def* **by** *auto*

definition (**in** *comm_monoid_add*) *supp_sum* :: ('*b* ⇒ '*a*) ⇒ '*b set* ⇒ '*a*
where *supp_sum f S* = (∑ *x*∈*support_on S f*. *f x*)

lemma *supp_sum_empty*[*simp*]: *supp_sum f* {} = *0*
unfolding *supp_sum_def* **by** *auto*

lemma *supp_sum_insert*[*simp*]:
finite (*support_on S f*) ⟹
supp_sum f (*insert x S*) = (*if x* ∈ *S then supp_sum f S else f x* + *supp_sum f S*)
by (*simp add*: *supp_sum_def in_support_on insert_absorb*)

lemma *supp_sum_divide_distrib*: *supp_sum f A* / (*r*::'*a*::*field*) = *supp_sum* (λ*n*. *f n* / *r*) *A*
by (*cases r* = *0*)
(*auto simp*: *supp_sum_def sum_divide_distrib intro*!: *sum.cong support_on_cong*)

4.32.4 Intervals

lemma *image_affinity_interval*:
fixes *c* :: '*a*::*ordered_real_vector*
shows ((λ*x*. *m* ∗*R x* + *c*) ' {*a*..*b*}) =
(*if* {*a*..*b*}={} *then* {}
else if 0 ≤ *m then* {*m* ∗*R a* + *c* .. *m* ∗*R b* + *c*}
else {*m* ∗*R b* + *c* .. *m* ∗*R a* + *c*})
(**is** *?lhs* = *?rhs*)
proof (*cases m*=*0*)
case *True*
then show *?thesis*
by *force*
next
case *False*
show *?thesis*
proof
show *?lhs* ⊆ *?rhs*
by (*auto simp*: *scaleR_left_mono scaleR_left_mono_neg*)
show *?rhs* ⊆ *?lhs*
proof (*clarsimp*, *intro conjI impI subsetI*)
show ⟦0 ≤ *m*; *a* ≤ *b*; *x* ∈ {*m* ∗*R a* + *c*..*m* ∗*R b* + *c*}⟧

```
        ⟹ x ∈ (λx. m *R x + c) ` {a..b} for x
      using False
      by (rule_tac x=inverse m *R (x−c) in image_eqI)
       (auto simp: pos_le_divideR_eq pos_divideR_le_eq le_diff_eq diff_le_eq)
    show ⟦¬ 0 ≤ m; a ≤ b; x ∈ {m *R b + c..m *R a + c}⟧
        ⟹ x ∈ (λx. m *R x + c) ` {a..b} for x
      by (rule_tac x=inverse m *R (x−c) in image_eqI)
          (auto simp add: neg_le_divideR_eq neg_divideR_le_eq le_diff_eq
diff_le_eq)
   qed
  qed
qed
```

4.32.5 Limit Points

```
lemma islimpt_ball:
  fixes x y :: 'a::{real_normed_vector,perfect_space}
  shows y islimpt ball x e ⟷ 0 < e ∧ y ∈ cball x e
  (is ?lhs ⟷ ?rhs)
proof
  show ?rhs if ?lhs
  proof
    {
      assume e ≤ 0
      then have *: ball x e = {}
        using ball_eq_empty[of x e] by auto
      have False using ‹?lhs›
        unfolding * using islimpt_EMPTY[of y] by auto
    }
    then show e > 0 by (metis not_less)
    show y ∈ cball x e
      using closed_cball[of x e] islimpt_subset[of y ball x e cball x e]
        ball_subset_cball[of x e] ‹?lhs›
      unfolding closed_limpt by auto
  qed
  show ?lhs if ?rhs
  proof −
    from that have e > 0 by auto
    {
      fix d :: real
      assume d > 0
      have ∃x'∈ball x e. x' ≠ y ∧ dist x' y < d
      proof (cases d ≤ dist x y)
        case True
        then show ?thesis
        proof (cases x = y)
          case True
          then have False
            using ‹d ≤ dist x y› ‹d>0› by auto
```

```
      then show ?thesis
        by auto
    next
      case False
      have dist x (y − (d / (2 ∗ dist y x)) ∗R (y − x)) =
        norm (x − y + (d / (2 ∗ norm (y − x))) ∗R (y − x))
     unfolding mem_cball mem_ball dist_norm diff_diff_eq2 diff_add_eq[symmetric]
        by auto
      also have ... = |− 1 + d / (2 ∗ norm (x − y))| ∗ norm (x − y)
       using scaleR_left_distrib[of − 1 d / (2 ∗ norm (y − x)), symmetric, of
y − x]
        unfolding scaleR_minus_left scaleR_one
        by (auto simp: norm_minus_commute)
      also have ... = |− norm (x − y) + d / 2|
       unfolding abs_mult_pos[of norm (x − y), OF norm_ge_zero[of x − y]]
        unfolding distrib_right using ‹x≠y› by auto
      also have ... ≤ e − d/2 using ‹d ≤ dist x y› and ‹d>0› and ‹?rhs›
        by (auto simp: dist_norm)
      finally have y − (d / (2 ∗ dist y x)) ∗R (y − x) ∈ ball x e using ‹d>0›
        by auto
      moreover
      have (d / (2∗dist y x)) ∗R (y − x) ≠ 0
        using ‹x≠y›[unfolded dist_nz] ‹d>0› unfolding scaleR_eq_0_iff
        by (auto simp: dist_commute)
      moreover
      have dist (y − (d / (2 ∗ dist y x)) ∗R (y − x)) y < d
        using ‹0 < d› by (fastforce simp: dist_norm)
      ultimately show ?thesis
        by (rule_tac x = y − (d / (2∗dist y x)) ∗R (y − x) in bexI) auto
    qed
  next
    case False
    then have d > dist x y by auto
    show ∃x′ ∈ ball x e. x′ ≠ y ∧ dist x′ y < d
    proof (cases x = y)
      case True
      obtain z where z: z ≠ y dist z y < min e d
        using perfect_choose_dist[of min e d y]
        using ‹d > 0› ‹e>0› by auto
      show ?thesis
        by (metis True z dist_commute mem_ball min_less_iff_conj)
    next
      case False
      then show ?thesis
        using ‹d>0› ‹d > dist x y› ‹?rhs› by force
    qed
  qed
}
then show ?thesis
```

```
    unfolding mem_cball islimpt_approachable mem_ball by auto
  qed
qed

lemma closure_ball_lemma:
  fixes x y :: 'a::real_normed_vector
  assumes x ≠ y
  shows y islimpt ball x (dist x y)
proof (rule islimptI)
  fix T
  assume y ∈ T open T
  then obtain r where 0 < r ∀ z. dist z y < r ⟶ z ∈ T
    unfolding open_dist by fast
  — choose point between x and y, within distance r of y.
  define k where k = min 1 (r / (2 * dist x y))
  define z where z = y + scaleR k (x − y)
  have z_def2: z = x + scaleR (1 − k) (y − x)
    unfolding z_def by (simp add: algebra_simps)
  have dist z y < r
    unfolding z_def k_def using ‹0 < r›
    by (simp add: dist_norm min_def)
  then have z ∈ T
    using ‹∀ z. dist z y < r ⟶ z ∈ T› by simp
  have dist x z < dist x y
   using ‹0 < r› assms by (simp add: z_def2 k_def dist_norm norm_minus_commute)

  then have z ∈ ball x (dist x y)
    by simp
  have z ≠ y
    unfolding z_def k_def using ‹x ≠ y› ‹0 < r›
    by (simp add: min_def)
  show ∃ z∈ball x (dist x y). z ∈ T ∧ z ≠ y
    using ‹z ∈ ball x (dist x y)› ‹z ∈ T› ‹z ≠ y›
    by fast
qed
```

4.32.6 Balls and Spheres in Normed Spaces

```
lemma mem_ball_0 [simp]: x ∈ ball 0 e ⟷ norm x < e
  for x :: 'a::real_normed_vector
  by simp

lemma mem_cball_0 [simp]: x ∈ cball 0 e ⟷ norm x ≤ e
  for x :: 'a::real_normed_vector
  by simp

lemma closure_ball [simp]:
  fixes x :: 'a::real_normed_vector
  assumes 0 < e
```

```
  shows closure (ball x e) = cball x e
proof
  show closure (ball x e) ⊆ cball x e
    using closed_cball closure_minimal by blast
  have ⋀y. dist x y < e ∨ dist x y = e ⟹ y ∈ closure (ball x e)
   by (metis Un_iff assms closure_ball_lemma closure_def dist_eq_0_iff mem_Collect_eq
mem_ball)
  then show cball x e ⊆ closure (ball x e)
    by force
qed

lemma mem_sphere_0 [simp]: x ∈ sphere 0 e ⟷ norm x = e
  for x :: 'a::real_normed_vector
  by simp


lemma interior_cball [simp]:
  fixes x :: 'a::{real_normed_vector, perfect_space}
  shows interior (cball x e) = ball x e
proof (cases e ≥ 0)
  case False note cs = this
  from cs have null: ball x e = {}
    using ball_empty[of e x] by auto
  moreover
  have cball x e = {}
  proof (rule equals0I)
    fix y
    assume y ∈ cball x e
    then show False
     by (metis ball_eq_empty null cs dist_eq_0_iff dist_le_zero_iff empty_subsetI
mem_cball
          subset_antisym subset_ball)
  qed
  then have interior (cball x e) = {}
    using interior_empty by auto
  ultimately show ?thesis by blast
next
  case True note cs = this
  have ball x e ⊆ cball x e
    using ball_subset_cball by auto
  moreover
  {
    fix S y
    assume as: S ⊆ cball x e open S y∈S
    then obtain d where d>0 and d: ∀ x'. dist x' y < d ⟶ x' ∈ S
      unfolding open_dist by blast
    then obtain xa where xa_y: xa ≠ y and xa: dist xa y < d
      using perfect_choose_dist [of d] by auto
    have xa ∈ S
```

```
      using d[THEN spec[where x = xa]]
      using xa by (auto simp: dist_commute)
    then have xa_cball: xa ∈ cball x e
      using as(1) by auto
    then have y ∈ ball x e
    proof (cases x = y)
      case True
     then have e > 0 using cs order.order_iff_strict xa_cball xa_y by fastforce
      then show y ∈ ball x e
        using ‹x = y › by simp
    next
      case False
      have dist (y + (d / 2 / dist y x) *R (y − x)) y < d
        unfolding dist_norm
        using ‹d>0› norm_ge_zero[of y − x] ‹x ≠ y› by auto
      then have ∗: y + (d / 2 / dist y x) *R (y − x) ∈ cball x e
        using d as(1)[unfolded subset_eq] by blast
      have y − x ≠ 0 using ‹x ≠ y› by auto
      hence ∗∗:d / (2 ∗ norm (y − x)) > 0
        unfolding zero_less_norm_iff[symmetric] using ‹d>0› by auto
      have dist (y + (d / 2 / dist y x) *R (y − x)) x =
        norm (y + (d / (2 ∗ norm (y − x))) *R y − (d / (2 ∗ norm (y − x))) *R
x − x)
        by (auto simp: dist_norm algebra_simps)
      also have ... = norm ((1 + d / (2 ∗ norm (y − x))) *R (y − x))
        by (auto simp: algebra_simps)
      also have ... = |1 + d / (2 ∗ norm (y − x))| ∗ norm (y − x)
        using ∗∗ by auto
      also have ... = (dist y x) + d/2
        using ∗∗ by (auto simp: distrib_right dist_norm)
      finally have e ≥ dist x y +d/2
        using ∗[unfolded mem_cball] by (auto simp: dist_commute)
      then show y ∈ ball x e
        unfolding mem_ball using ‹d>0› by auto
    qed
  }
  then have ∀ S ⊆ cball x e. open S ⟶ S ⊆ ball x e
    by auto
  ultimately show ?thesis
    using interior_unique[of ball x e cball x e]
    using open_ball[of x e]
    by auto
qed

lemma frontier_ball [simp]:
  fixes a :: 'a::real_normed_vector
  shows 0 < e ⟹ frontier (ball a e) = sphere a e
  by (force simp: frontier_def)
```

lemma *frontier_cball* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*, *perfect_space*}
shows *frontier* (*cball a e*) = *sphere a e*
by (*force simp*: *frontier_def*)

corollary *compact_sphere* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
shows *compact* (*sphere a r*)
using *compact_frontier* [*of cball a r*] **by** *simp*

corollary *bounded_sphere* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
shows *bounded* (*sphere a r*)
by (*simp add*: *compact_imp_bounded*)

corollary *closed_sphere* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
shows *closed* (*sphere a r*)
by (*simp add*: *compact_imp_closed*)

lemma *image_add_ball* [*simp*]:
fixes *a* :: *'a*::*real_normed_vector*
shows (+) *b* ‘ *ball a r* = *ball* (*a*+*b*) *r*
proof −
{ fix *x* :: *'a*
assume *dist* (*a* + *b*) *x* < *r*
moreover
have *b* + (*x* − *b*) = *x*
by *simp*
ultimately have *x* ∈ (+) *b* ‘ *ball a r*
by (*metis add.commute dist_add_cancel image_eqI mem_ball*) **}**
then show *?thesis*
by (*auto simp*: *add.commute*)
qed

lemma *image_add_cball* [*simp*]:
fixes *a* :: *'a*::*real_normed_vector*
shows (+) *b* ‘ *cball a r* = *cball* (*a*+*b*) *r*
proof −
have ⋀*x*. *dist* (*a* + *b*) *x* ≤ *r* ⟹ ∃ *y*∈*cball a r*. *x* = *b* + *y*
by (*metis* (*no_types*) *add.commute diff_add_cancel dist_add_cancel2 mem_cball*)
then show *?thesis*
by (*force simp*: *add.commute*)
qed

4.32.7 Various Lemmas on Normed Algebras

lemma *closed_of_nat_image*: *closed* (*of_nat* ‘ *A* :: *'a*::*real_normed_algebra_1 set*)

by (*rule discrete_imp_closed*[*of 1*]) (*auto simp*: *dist_of_nat*)

lemma *closed_of_int_image*: *closed* (*of_int* ‘ *A* :: *'a*::*real_normed_algebra_1 set*)
by (*rule discrete_imp_closed*[*of 1*]) (*auto simp*: *dist_of_int*)

lemma *closed_Nats* [*simp*]: *closed* (ℕ :: *'a* :: *real_normed_algebra_1 set*)
unfolding *Nats_def* **by** (*rule closed_of_nat_image*)

lemma *closed_Ints* [*simp*]: *closed* (ℤ :: *'a* :: *real_normed_algebra_1 set*)
unfolding *Ints_def* **by** (*rule closed_of_int_image*)

lemma *closed_subset_Ints*:
fixes *A* :: *'a* :: *real_normed_algebra_1 set*
assumes *A* ⊆ ℤ
shows *closed A*
proof (*intro discrete_imp_closed*[*OF zero_less_one*] *ballI impI*, *goal_cases*)
case (*1 x y*)
with *assms* **have** *x* ∈ ℤ **and** *y* ∈ ℤ **by** *auto*
with ‹*dist y x* < *1*› **show** *y* = *x*
by (*auto elim*!: *Ints_cases simp*: *dist_of_int*)
qed

4.32.8 Filters

definition *indirection* :: *'a*::*real_normed_vector* ⇒ *'a* ⇒ *'a filter* (**infixr** ‹*indirection*› *70*)
where *a indirection v* = *at a within* {*b*. ∃ *c*≥*0*. *b* − *a* = *scaleR c v*}

4.32.9 Trivial Limits

lemma *trivial_limit_at_infinity*:
¬ *trivial_limit* (*at_infinity* :: (*'a*::{*real_normed_vector*,*perfect_space*}) *filter*)
proof −
obtain *x*::*'a* **where** *x* ≠ *0*
by (*meson perfect_choose_dist zero_less_one*)
then have *b* ≤ *norm* ((*b* / *norm x*) *∗R x*) **for** *b*
by *simp*
then show *?thesis*
unfolding *trivial_limit_def eventually_at_infinity*
by *blast*
qed

lemma *at_within_ball_bot_iff*:
fixes *x y* :: *'a*::{*real_normed_vector*,*perfect_space*}
shows *at x within ball y r* = *bot* ⟷ (*r*=*0* ∨ *x* ∉ *cball y r*)
unfolding *trivial_limit_within*
by (*metis* (*no_types*) *cball_empty equals0D islimpt_ball less_linear*)

4.32.10 Limits

proposition *Lim_at_infinity*: $(f \longrightarrow l)$ *at_infinity* $\longleftrightarrow$ $(\forall e{>}0.\ \exists b.\ \forall x.\ norm\ x \geq b \longrightarrow dist\ (f\ x)\ l < e)$
by (*auto simp*: *tendsto_iff eventually_at_infinity*)

corollary *Lim_at_infinityI* [*intro?*]:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists B.\ \forall x.\ norm\ x \geq B \longrightarrow dist\ (f\ x)\ l \leq e$
shows $(f \longrightarrow l)$ *at_infinity*
proof −
have $\bigwedge e.\ e > 0 \Longrightarrow \exists B.\ \forall x.\ norm\ x \geq B \longrightarrow dist\ (f\ x)\ l < e$
by (*meson assms dense le_less_trans*)
then show *?thesis*
using *Lim_at_infinity* **by** *blast*
qed

lemma *Lim_transform_within_set_eq*:
fixes *a* :: *'a*::*metric_space* **and** *l* :: *'b*::*metric_space*
shows *eventually* $(\lambda x.\ x \in S \longleftrightarrow x \in T)$ $(at\ a)$
$\Longrightarrow ((f \longrightarrow l)\ (at\ a\ within\ S) \longleftrightarrow (f \longrightarrow l)\ (at\ a\ within\ T))$
by (*force intro*: *Lim_transform_within_set elim*: *eventually_mono*)

lemma *Lim_null*:
fixes *f* :: *'a* ⇒ *'b*::*real_normed_vector*
shows $(f \longrightarrow l)$ *net* $\longleftrightarrow$ $((\lambda x.\ f(x) - l) \longrightarrow 0)$ *net*
by (*simp add*: *Lim dist_norm*)

lemma *Lim_null_comparison*:
fixes *f* :: *'a* ⇒ *'b*::*real_normed_vector*
assumes *eventually* $(\lambda x.\ norm\ (f\ x) \leq g\ x)$ *net* $(g \longrightarrow 0)$ *net*
shows $(f \longrightarrow 0)$ *net*
using *assms*(*2*)
proof (*rule metric_tendsto_imp_tendsto*)
show *eventually* $(\lambda x.\ dist\ (f\ x)\ 0 \leq dist\ (g\ x)\ 0)$ *net*
using *assms*(*1*) **by** (*rule eventually_mono*) (*simp add*: *dist_norm*)
qed

lemma *Lim_transform_bound*:
fixes *f* :: *'a* ⇒ *'b*::*real_normed_vector*
and *g* :: *'a* ⇒ *'c*::*real_normed_vector*
assumes *eventually* $(\lambda n.\ norm\ (f\ n) \leq norm\ (g\ n))$ *net*
and $(g \longrightarrow 0)$ *net*
shows $(f \longrightarrow 0)$ *net*
using *assms*(*1*) *tendsto_norm_zero* [*OF assms*(*2*)]
by (*rule Lim_null_comparison*)

lemma *lim_null_mult_right_bounded*:
fixes *f* :: *'a* ⇒ *'b*::*real_normed_div_algebra*
assumes *f*: $(f \longrightarrow 0)$ *F* **and** *g*: *eventually* $(\lambda x.\ norm(g\ x) \leq B)$ *F*
shows $((\lambda z.\ f\ z * g\ z) \longrightarrow 0)$ *F*

proof −
have $((\lambda x.\ norm\ (f\ x) * norm\ (g\ x)) \longrightarrow 0)\ F$
proof (*rule Lim_null_comparison*)
show $\forall_F\ x\ in\ F.\ norm\ (norm\ (f\ x) * norm\ (g\ x)) \leq norm\ (f\ x) * B$
by (*simp add*: *eventually_mono* [*OF g*] *mult_left_mono*)
show $((\lambda x.\ norm\ (f\ x) * B) \longrightarrow 0)\ F$
by (*simp add*: *f tendsto_mult_left_zero tendsto_norm_zero*)
qed
then show *?thesis*
by (*subst tendsto_norm_zero_iff* [*symmetric*]) (*simp add*: *norm_mult*)
qed

lemma *lim_null_mult_left_bounded*:
fixes $f :: 'a \Rightarrow 'b$::*real_normed_div_algebra*
assumes *g*: *eventually* $(\lambda x.\ norm(g\ x) \leq B)\ F$ **and** *f*: $(f \longrightarrow 0)\ F$
shows $((\lambda z.\ g\ z * f\ z) \longrightarrow 0)\ F$
proof −
have $((\lambda x.\ norm\ (g\ x) * norm\ (f\ x)) \longrightarrow 0)\ F$
proof (*rule Lim_null_comparison*)
show $\forall_F\ x\ in\ F.\ norm\ (norm\ (g\ x) * norm\ (f\ x)) \leq B * norm\ (f\ x)$
by (*simp add*: *eventually_mono* [*OF g*] *mult_right_mono*)
show $((\lambda x.\ B * norm\ (f\ x)) \longrightarrow 0)\ F$
by (*simp add*: *f tendsto_mult_right_zero tendsto_norm_zero*)
qed
then show *?thesis*
by (*subst tendsto_norm_zero_iff* [*symmetric*]) (*simp add*: *norm_mult*)
qed

lemma *lim_null_scaleR_bounded*:
assumes *f*: $(f \longrightarrow 0)$ *net* **and** *gB*: *eventually* $(\lambda a.\ f\ a = 0 \vee norm(g\ a) \leq B)$ *net*
shows $((\lambda n.\ f\ n *_R g\ n) \longrightarrow 0)$ *net*
proof
fix ε::*real*
assume $0 < \varepsilon$
then have *B*: $0 < \varepsilon\ /\ (abs\ B + 1)$ **by** *simp*
have $*$: $|f\ x| * norm\ (g\ x) < \varepsilon$ **if** *f*: $|f\ x| * (|B| + 1) < \varepsilon$ **and** *g*: $norm\ (g\ x) \leq B$ **for** *x*
proof −
have $|f\ x| * norm\ (g\ x) \leq |f\ x| * B$
by (*simp add*: *mult_left_mono g*)
also have $\ldots \leq |f\ x| * (|B| + 1)$
by (*simp add*: *mult_left_mono*)
also have $\ldots < \varepsilon$
by (*rule f*)
finally show *?thesis* **.**
qed
have $\bigwedge x.\ [\![|f\ x| < \varepsilon\ /\ (|B| + 1);\ norm\ (g\ x) \leq B]\!] \Longrightarrow |f\ x| * norm\ (g\ x) < \varepsilon$
by (*simp add*: $*$ *pos_less_divide_eq*)

then show $\forall_F\ x$ *in net. dist* $(f\ x *_R g\ x)\ 0 < \varepsilon$
using ‹$0 < \varepsilon$› **by** (*auto intro*: *eventually_mono* [*OF eventually_conj* [*OF tendstoD* [*OF f B*] *gB*]])
qed

lemma *Lim_norm_ubound*:
fixes $f :: 'a \Rightarrow 'b$::*real_normed_vector*
assumes $\neg$(*trivial_limit net*) $(f \longrightarrow l)$ *net eventually* $(\lambda x.\ norm(f\ x) \leq e)$ *net*
shows $norm(l) \leq e$
using *assms* **by** (*fast intro*: *tendsto_le tendsto_intros*)

lemma *Lim_norm_lbound*:
fixes $f :: 'a \Rightarrow 'b$::*real_normed_vector*
assumes $\neg$ *trivial_limit net*
and $(f \longrightarrow l)$ *net*
and *eventually* $(\lambda x.\ e \leq norm\ (f\ x))$ *net*
shows $e \leq norm\ l$
using *assms* **by** (*fast intro*: *tendsto_le tendsto_intros*)

Limit under bilinear function

lemma *Lim_bilinear*:
assumes $(f \longrightarrow l)$ *net*
and $(g \longrightarrow m)$ *net*
and *bounded_bilinear h*
shows $((\lambda x.\ h\ (f\ x)\ (g\ x)) \longrightarrow (h\ l\ m))$ *net*
using ‹*bounded_bilinear h*› ‹$(f \longrightarrow l)$ *net*› ‹$(g \longrightarrow m)$ *net*›
by (*rule bounded_bilinear.tendsto*)

lemma *Lim_at_zero*:
fixes $a :: 'a$::*real_normed_vector*
and $l :: 'b$::*topological_space*
shows $(f \longrightarrow l)\ (at\ a) \longleftrightarrow ((\lambda x.\ f(a + x)) \longrightarrow l)\ (at\ 0)$
using *LIM_offset_zero LIM_offset_zero_cancel* **..**

4.32.11 Limit Point of Filter

lemma *netlimit_at_vector*:
fixes $a :: 'a$::*real_normed_vector*
shows *netlimit* $(at\ a) = a$
proof (*cases* $\exists x.\ x \neq a$)
case *True* **then obtain** x **where** x: $x \neq a$ **..**
have $\bigwedge d.\ 0 < d \Longrightarrow \exists x.\ x \neq a \wedge norm\ (x - a) < d$
by (*rule_tac x*=$a + scaleR\ (d\ /\ 2)\ (sgn\ (x - a))$ **in** *exI*) (*simp add*: *norm_sgn sgn_zero_iff x*)
then have $\neg$ *trivial_limit* $(at\ a)$
by (*auto simp*: *trivial_limit_def eventually_at dist_norm*)
then show *?thesis*
by (*rule Lim_ident_at* [*of a UNIV*])
qed *simp*

4.32.12 Boundedness

lemma *continuous_on_closure_norm_le*:
fixes f :: *'a::metric_space* ⇒ *'b::real_normed_vector*
assumes *continuous_on* (*closure s*) f
and ∀ $y \in s$. *norm*(f y) ≤ b
and $x \in$ (*closure s*)
shows *norm* (f x) ≤ b
proof −
have ∗: f ‘ $s \subseteq$ *cball 0 b*
using *assms*(*2*)[*unfolded mem_cball_0*[*symmetric*]] **by** *auto*
show *?thesis*
by (*meson* ∗ *assms*(*1*) *assms*(*3*) *closed_cball image_closure_subset image_subset_iff mem_cball_0*)
qed

lemma *bounded_pos*: *bounded* S ⟷ (∃ b>*0*. ∀ x∈ S. *norm* x ≤ b)
unfolding *bounded_iff*
by (*meson less_imp_le not_le order_trans zero_less_one*)

lemma *bounded_pos_less*: *bounded* S ⟷ (∃ b>*0*. ∀ x∈ S. *norm* x < b)
by (*metis bounded_pos le_less_trans less_imp_le linordered_field_no_ub*)

lemma *bounded_normE*:
assumes *bounded* A
obtains B **where** $B > 0$ ⋀z. $z \in A$ ⟹ *norm* $z \leq B$
by (*meson assms bounded_pos*)

lemma *bounded_normE_less*:
assumes *bounded* A
obtains B **where** $B > 0$ ⋀z. $z \in A$ ⟹ *norm* $z < B$
by (*meson assms bounded_pos_less*)

lemma *Bseq_eq_bounded*:
fixes f :: *nat* ⇒ *'a::real_normed_vector*
shows *Bseq* f ⟷ *bounded* (*range* f)
unfolding *Bseq_def bounded_pos* **by** *auto*

lemma *bounded_linear_image*:
assumes *bounded* S
and *bounded_linear* f
shows *bounded* (f ‘ S)
proof −
from *assms*(*1*) **obtain** b **where** $b > 0$ **and** b: ∀ x∈S. *norm* $x \leq b$
unfolding *bounded_pos* **by** *auto*
from *assms*(*2*) **obtain** B **where** B: $B > 0$ ∀ x. *norm* (f x) ≤ B ∗ *norm* x
using *bounded_linear.pos_bounded* **by** (*auto simp*: *ac_simps*)
show *?thesis*
unfolding *bounded_pos*
proof (*intro exI*, *safe*)

```
    show norm (f x) ≤ B * b if x ∈ S for x
      by (meson B b less_imp_le mult_left_mono order_trans that)
  qed (use ‹b > 0› ‹B > 0› in auto)
qed

lemma bounded_scaling:
  fixes S :: 'a::real_normed_vector set
  shows bounded S ⟹ bounded ((λx. c *R x) ` S)
  by (simp add: bounded_linear_image bounded_linear_scaleR_right)

lemma bounded_scaleR_comp:
  fixes f :: 'a ⇒ 'b::real_normed_vector
  assumes bounded (f ` S)
  shows bounded ((λx. r *R f x) ` S)
  using bounded_scaling[of f ` S r] assms
  by (auto simp: image_image)

lemma bounded_translation:
  fixes S :: 'a::real_normed_vector set
  assumes bounded S
  shows bounded ((λx. a + x) ` S)
proof −
  from assms obtain b where b: b > 0 ∀x∈S. norm x ≤ b
    unfolding bounded_pos by auto
  {
    fix x
    assume x ∈ S
    then have norm (a + x) ≤ b + norm a
      using norm_triangle_ineq[of a x] b by auto
  }
  then show ?thesis
    unfolding bounded_pos
    using norm_ge_zero[of a] b(1) and add_strict_increasing[of b 0 norm a]
    by (auto intro!: exI[of _ b + norm a])
qed

lemma bounded_translation_minus:
  fixes S :: 'a::real_normed_vector set
  shows bounded S ⟹ bounded ((λx. x − a) ` S)
using bounded_translation [of S −a] by simp

lemma bounded_uminus [simp]:
  fixes X :: 'a::real_normed_vector set
  shows bounded (uminus ` X) ⟷ bounded X
by (auto simp: bounded_def dist_norm; rule_tac x=−x in exI; force simp: add.commute
norm_minus_commute)

lemma uminus_bounded_comp [simp]:
  fixes f :: 'a ⇒ 'b::real_normed_vector
```

shows *bounded* ((λ*x*. − *f x*) ‘ *S*) ⟷ *bounded* (*f* ‘ *S*)
using *bounded_uminus*[*of f* ‘ *S*]
by (*auto simp*: *image_image*)

lemma *bounded_plus_comp*:
fixes *f g*::′*a* ⇒ ′*b*::*real_normed_vector*
assumes *bounded* (*f* ‘ *S*)
assumes *bounded* (*g* ‘ *S*)
shows *bounded* ((λ*x*. *f x* + *g x*) ‘ *S*)
proof −
{
fix *B C*
assume ⋀*x*. *x*∈*S* ⟹ *norm* (*f x*) ≤ *B* ⋀*x*. *x*∈*S* ⟹ *norm* (*g x*) ≤ *C*
then have ⋀*x*. *x* ∈ *S* ⟹ *norm* (*f x* + *g x*) ≤ *B* + *C*
by (*auto intro*!: *norm_triangle_le add_mono*)
} then show *?thesis*
using *assms* **by** (*fastforce simp*: *bounded_iff*)
qed

lemma *bounded_plus*:
fixes *S* ::′*a*::*real_normed_vector set*
assumes *bounded S bounded T*
shows *bounded* ((λ(*x*,*y*). *x* + *y*) ‘ (*S* × *T*))
using *bounded_plus_comp* [*of fst S* × *T snd*] *assms*
by (*auto simp*: *split_def split*: *if_split_asm*)

lemma *bounded_minus_comp*:
bounded (*f* ‘ *S*) ⟹ *bounded* (*g* ‘ *S*) ⟹ *bounded* ((λ*x*. *f x* − *g x*) ‘ *S*)
for *f g*::′*a* ⇒ ′*b*::*real_normed_vector*
using *bounded_plus_comp*[*of f S* λ*x*. − *g x*]
by *auto*

lemma *bounded_minus*:
fixes *S* ::′*a*::*real_normed_vector set*
assumes *bounded S bounded T*
shows *bounded* ((λ(*x*,*y*). *x* − *y*) ‘ (*S* × *T*))
using *bounded_minus_comp* [*of fst S* × *T snd*] *assms*
by (*auto simp*: *split_def split*: *if_split_asm*)

lemma *bounded_sums*:
fixes *S* :: ′*a*::*real_normed_vector set*
assumes *bounded S* **and** *bounded T*
shows *bounded* (⋃*x*∈ *S*. ⋃*y* ∈ *T*. {*x* + *y*})
using *assms* **by** (*simp add*: *bounded_iff*) (*meson norm_triangle_mono*)

lemma *bounded_differences*:
fixes *S* :: ′*a*::*real_normed_vector set*
assumes *bounded S* **and** *bounded T*
shows *bounded* (⋃*x*∈ *S*. ⋃*y* ∈ *T*. {*x* − *y*})

using *assms* **by** (*simp add*: *bounded_iff*) (*meson add_mono norm_triangle_le_diff*)

lemma *not_bounded_UNIV*[*simp*]:
 ¬ *bounded* (*UNIV* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*)
proof (*auto simp*: *bounded_pos not_le*)
 obtain *x* :: *'a* **where** *x* ≠ *0*
  using *perfect_choose_dist* [*OF zero_less_one*] **by** *fast*
 fix *b* :: *real*
 assume *b*: *b* >*0*
 have *b1*: *b* +*1* ≥ *0*
  using *b* **by** *simp*
 with ‹*x* ≠ *0*› **have** *b* < *norm* (*scaleR* (*b* + *1*) (*sgn x*))
  by (*simp add*: *norm_sgn*)
 then show ∃ *x*::*'a*. *b* < *norm x* **..**
qed

corollary *cobounded_imp_unbounded*:
  fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
  shows *bounded* (− *S*) ⟹ ¬ *bounded S*
 using *bounded_Un* [*of S* −*S*] **by** (*simp*)

4.32.13 Relations among convergence and absolute convergence for power series

lemma *summable_imp_bounded*:
 fixes *f* :: *nat* ⇒ *'a*::*real_normed_vector*
 shows *summable f* ⟹ *bounded* (*range f*)
by (*frule summable_LIMSEQ_zero*) (*simp add*: *convergent_imp_bounded*)

lemma *summable_imp_sums_bounded*:
  summable f ⟹ *bounded* (*range* (λ*n*. *sum f* {..<*n*}))
by (*auto simp*: *summable_def sums_def dest*: *convergent_imp_bounded*)

lemma *power_series_conv_imp_absconv_weak*:
 fixes *a*:: *nat* ⇒ *'a*::{*real_normed_div_algebra*,*banach*} **and** *w* :: *'a*
 assumes *sum*: *summable* (λ*n*. *a n* ∗ *z* ^ *n*) **and** *no*: *norm w* < *norm z*
  shows *summable* (λ*n*. *of_real*(*norm*(*a n*)) ∗ *w* ^ *n*)
proof −
 obtain *M* **where** *M*: ⋀*x*. *norm* (*a x* ∗ *z* ^ *x*) ≤ *M*
  using *summable_imp_bounded* [*OF sum*] **by** (*force simp*: *bounded_iff*)
 show *?thesis*
 proof (*rule series_comparison_complex*)
  have ⋀*n*. *norm* (*a n*) ∗ *norm z* ^ *n* ≤ *M*
   by (*metis* (*no_types*) *M norm_mult norm_power*)
  then show *summable* (λ*n*. *complex_of_real* (*norm* (*a n*) ∗ *norm w* ^ *n*))
   using *Abel_lemma no norm_ge_zero summable_of_real* **by** *blast*
 qed (*auto simp*: *norm_mult norm_power*)
qed

4.32.14 Normed spaces with the Heine-Borel property

lemma *not_compact_UNIV*[*simp*]:
fixes *s* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*} *set*
shows ¬ *compact* (*UNIV*::*'a set*)
by (*simp add*: *compact_eq_bounded_closed*)

lemma *not_compact_space_euclideanreal* [*simp*]: ¬ *compact_space euclideanreal*
by (*simp add*: *compact_space_def*)

Representing sets as the union of a chain of compact sets.

lemma *closed_Union_compact_subsets*:
fixes *S* :: *'a*::{*heine_borel*,*real_normed_vector*} *set*
assumes *closed S*
obtains *F* **where** ⋀*n*. *compact*(*F n*) ⋀*n*. *F n* ⊆ *S* ⋀*n*. *F n* ⊆ *F*(*Suc n*)
(⋃ *n*. *F n*) = *S* ⋀*K*. ⟦*compact K*; *K* ⊆ *S*⟧ ⟹ ∃ *N*. ∀ *n* ≥ *N*. *K* ⊆ *F n*
proof
show *compact* (*S* ∩ *cball 0* (*of_nat n*)) **for** *n*
using *assms compact_eq_bounded_closed* **by** *auto*
next
show (⋃ *n*. *S* ∩ *cball 0* (*real n*)) = *S*
by (*auto simp*: *real_arch_simple*)
next
fix *K* :: *'a set*
assume *compact K K* ⊆ *S*
then obtain *N* **where** *K* ⊆ *cball 0 N*
by (*meson bounded_pos mem_cball_0 compact_imp_bounded subsetI*)
then show ∃ *N*. ∀ *n*≥*N*. *K* ⊆ *S* ∩ *cball 0* (*real n*)
by (*metis of_nat_le_iff Int_subset_iff* ‹*K* ⊆ *S*› *real_arch_simple subset_cball subset_trans*)
qed *auto*

4.32.15 Intersecting chains of compact sets and the Baire property

proposition *bounded_closed_chain*:
fixes *F* :: *'a*::*heine_borel set set*
assumes *B* ∈ *F bounded B* **and** *F*: ⋀*S*. *S* ∈ *F* ⟹ *closed S* **and** {} ∉ *F*
and *chain*: ⋀*S T*. *S* ∈ *F* ∧ *T* ∈ *F* ⟹ *S* ⊆ *T* ∨ *T* ⊆ *S*
shows ⋂*F* ≠ {}
proof −
have *B* ∩ ⋂*F* ≠ {}
proof (*rule compact_imp_fip*)
show *compact B* ⋀*T*. *T* ∈ *F* ⟹ *closed T*
by (*simp_all add*: *assms compact_eq_bounded_closed*)
show ⟦*finite* *G*; *G* ⊆ *F*⟧ ⟹ *B* ∩ ⋂*G* ≠ {} **for** *G*
proof (*induction G rule*: *finite_induct*)
case *empty*
with *assms* **show** *?case* **by** *force*

```
    next
      case (insert U 𝒢)
      then have U ∈ ℱ and ne: B ∩ ⋂𝒢 ≠ {} by auto
      then consider B ⊆ U | U ⊆ B
          using ‹B ∈ ℱ› chain by blast
        then show ?case
        proof cases
          case 1
          then show ?thesis
            using Int_left_commute ne by auto
        next
          case 2
          have U ≠ {}
            using ‹U ∈ ℱ› ‹{} ∉ ℱ› by blast
          moreover
          have False if ⋀x. x ∈ U ⟹ ∃ Y∈𝒢. x ∉ Y
          proof −
            have ⋀x. x ∈ U ⟹ ∃ Y∈𝒢. Y ⊆ U
              by (metis chain contra_subsetD insert.prems insert_subset that)
            then obtain Y where Y ∈ 𝒢 Y ⊆ U
              by (metis all_not_in_conv ‹U ≠ {}›)
            moreover obtain x where x ∈ ⋂𝒢
              by (metis Int_emptyI ne)
            ultimately show ?thesis
              by (metis Inf_lower subset_eq that)
          qed
          with 2 show ?thesis
            by blast
        qed
      qed
    qed
    then show ?thesis by blast
qed

corollary compact_chain:
  fixes ℱ :: 'a::heine_borel set set
  assumes ⋀S. S ∈ ℱ ⟹ compact S {} ∉ ℱ
          ⋀S T. S ∈ ℱ ∧ T ∈ ℱ ⟹ S ⊆ T ∨ T ⊆ S
    shows ⋂ ℱ ≠ {}
proof (cases ℱ = {})
  case True
  then show ?thesis by auto
next
  case False
  show ?thesis
   by (metis False all_not_in_conv assms compact_imp_bounded compact_imp_closed
bounded_closed_chain)
qed
```

lemma *compact_nest*:
fixes $F :: 'a::linorder \Rightarrow 'b::heine\_borel\ set$
assumes F: $\bigwedge n.\ compact(F\ n)$ $\bigwedge n.\ F\ n \neq \{\}$ **and** *mono*: $\bigwedge m\ n.\ m \leq n \Longrightarrow F\ n \subseteq F\ m$
shows $\bigcap(range\ F) \neq \{\}$
proof −
have $*$: $\bigwedge S\ T.\ S \in range\ F \wedge T \in range\ F \Longrightarrow S \subseteq T \vee T \subseteq S$
by (*metis mono image_iff le_cases*)
show *?thesis*
using F **by** (*intro compact_chain* [*OF* _ _ $*$]; *blast dest*: $*$)
qed

The Baire property of dense sets

theorem *Baire*:
fixes $S::'a::\{real\_normed\_vector,heine\_borel\}\ set$
assumes *closed S countable* $\mathcal{G}$
and *ope*: $\bigwedge T.\ T \in \mathcal{G} \Longrightarrow openin\ (top\_of\_set\ S)\ T \wedge S \subseteq closure\ T$
shows $S \subseteq closure(\bigcap \mathcal{G})$
proof (*cases* $\mathcal{G} = \{\}$)
case *True*
then show *?thesis*
using *closure_subset* **by** *auto*
next
let *?g* = *from_nat_into* $\mathcal{G}$
case *False*
then have *gin*: *?g n* $\in \mathcal{G}$ **for** *n*
by (*simp add*: *from_nat_into*)
show *?thesis*
proof (*clarsimp simp*: *closure_approachable*)
fix *x* **and** *e*::*real*
assume $x \in S\ 0 < e$
obtain *TF* **where** *openF*: $\bigwedge n.\ openin\ (top\_of\_set\ S)\ (TF\ n)$
and *ne*: $\bigwedge n.\ TF\ n \neq \{\}$
and *subg*: $\bigwedge n.\ S \cap closure(TF\ n) \subseteq$ *?g n*
and *subball*: $\bigwedge n.\ closure(TF\ n) \subseteq ball\ x\ e$
and *decr*: $\bigwedge n.\ TF(Suc\ n) \subseteq TF\ n$
proof −
have $*$: $\exists Y.\ (openin\ (top\_of\_set\ S)\ Y \wedge Y \neq \{\} \wedge$
$S \cap closure\ Y \subseteq$ *?g n* $\wedge\ closure\ Y \subseteq ball\ x\ e) \wedge Y \subseteq U$
if *openU*: $openin\ (top\_of\_set\ S)\ U$ **and** $U \neq \{\}$ **and** *cloU*: $closure\ U \subseteq ball\ x\ e$ **for** *U n*
proof −
obtain *T* **where** *T*: *open T U* = $T \cap S$
using ‹*openin* (*top_of_set S*) *U*› **by** (*auto simp*: *openin_subtopology*)
with ‹$U \neq \{\}$› **have** $T \cap closure\ (?g\ n) \neq \{\}$
using *gin ope* **by** *fastforce*
then have $T \cap$ *?g n* $\neq \{\}$
using ‹*open T*› *open_Int_closure_eq_empty* **by** *blast*
then obtain *y* **where** $y \in U\ y \in$ *?g n*

```
      using T ope [of ?g n, OF gin] by (blast dest: openin_imp_subset)
    moreover have openin (top_of_set S) (U ∩ ?g n)
      using gin ope opeU by blast
    ultimately obtain d where U: U ∩ ?g n ⊆ S and d > 0 and d: ball y d
∩ S ⊆ U ∩ ?g n
      by (force simp: openin_contains_ball)
    show ?thesis
    proof (intro exI conjI)
      show openin (top_of_set S) (S ∩ ball y (d/2))
        by (simp add: openin_open_Int)
      show S ∩ ball y (d/2) ≠ {}
        using ‹0 < d› ‹y ∈ U› opeU openin_imp_subset by fastforce
      have S ∩ closure (S ∩ ball y (d/2)) ⊆ S ∩ closure (ball y (d/2))
        using closure_mono by blast
      also have ... ⊆ ?g n
        using ‹d > 0› d by force
      finally show S ∩ closure (S ∩ ball y (d/2)) ⊆ ?g n .
      have closure (S ∩ ball y (d/2)) ⊆ S ∩ ball y d
      proof −
        have closure (ball y (d/2)) ⊆ ball y d
          using ‹d > 0› by auto
        then have closure (S ∩ ball y (d/2)) ⊆ ball y d
          by (meson closure_mono inf.cobounded2 subset_trans)
        then show ?thesis
          by (simp add: ‹closed S› closure_minimal)
      qed
      also have ... ⊆ ball x e
        using cloU closure_subset d by blast
      finally show closure (S ∩ ball y (d/2)) ⊆ ball x e .
      show S ∩ ball y (d/2) ⊆ U
        using ball_divide_subset_numeral d by blast
    qed
  qed
  let ?Φ = λn X. openin (top_of_set S) X ∧ X ≠ {} ∧
                S ∩ closure X ⊆ ?g n ∧ closure X ⊆ ball x e
  have closure (S ∩ ball x (e/2)) ⊆ closure(ball x (e/2))
    by (simp add: closure_mono)
  also have ... ⊆ ball x e
    using ‹e > 0› by auto
  finally have closure (S ∩ ball x (e/2)) ⊆ ball x e .
  moreover haveopenin (top_of_set S) (S ∩ ball x (e/2)) S ∩ ball x (e/2) ≠
{}
    using ‹0 < e› ‹x ∈ S› by auto
  ultimately obtain Y where Y: ?Φ 0 Y ∧ Y ⊆ S ∩ ball x (e/2)
      using ∗ [of S ∩ ball x (e/2) 0] by metis
  show thesis
  proof (rule exE [OF dependent_nat_choice])
    show ∃x. ?Φ 0 x
      using Y by auto
```

```
      show ∃ Y. ?Φ (Suc n) Y ∧ Y ⊆ X if ?Φ n X for X n
        using that by (blast intro: ∗)
    qed (use that in metis)
  qed
  have (⋂ n. S ∩ closure (TF n)) ≠ {}
  proof (rule compact_nest)
    show ⋀n. compact (S ∩ closure (TF n))
    by (metis closed_closure subball bounded_subset_ballI compact_eq_bounded_closed
closed_Int_compact [OF ‹closed S›])
    show ⋀n. S ∩ closure (TF n) ≠ {}
      by (metis Int_absorb1 opeF ‹closed S› closure_eq_empty closure_minimal
ne openin_imp_subset)
    show ⋀m n. m ≤ n ⟹ S ∩ closure (TF n) ⊆ S ∩ closure (TF m)
    by (meson closure_mono decr dual_order.refl inf_mono lift_Suc_antimono_le)
  qed
  moreover have (⋂ n. S ∩ closure (TF n)) ⊆ {y ∈ ⋂ 𝒢. dist y x < e}
  proof (clarsimp, intro conjI)
    fix y
    assume y ∈ S and y: ∀ n. y ∈ closure (TF n)
    then show ∀ T∈𝒢. y ∈ T
      by (metis Int_iff from_nat_into_surj [OF ‹countable 𝒢›] subsetD subg)
    show dist y x < e
      by (metis y dist_commute mem_ball subball subsetCE)
  qed
  ultimately show ∃ y ∈ ⋂ 𝒢. dist y x < e
    by auto
 qed
qed
```

4.32.16 Continuity

Structural rules for uniform continuity

```
lemma (in bounded_linear) uniformly_continuous_on[continuous_intros]:
  fixes g :: _::metric_space ⇒ _
  assumes uniformly_continuous_on s g
  shows uniformly_continuous_on s (λx. f (g x))
  using assms unfolding uniformly_continuous_on_sequentially
  unfolding dist_norm tendsto_norm_zero_iff diff[symmetric]
  by (auto intro: tendsto_zero)

lemma uniformly_continuous_on_dist[continuous_intros]:
  fixes f g :: 'a::metric_space ⇒ 'b::metric_space
  assumes uniformly_continuous_on s f
    and uniformly_continuous_on s g
  shows uniformly_continuous_on s (λx. dist (f x) (g x))
proof −
  {
    fix a b c d :: 'b
    have |dist a b − dist c d| ≤ dist a c + dist b d
```

```
    using dist_triangle2 [of a b c] dist_triangle2 [of b c d]
    using dist_triangle3 [of c d a] dist_triangle [of a d b]
    by arith
  } note le = this
  {
   fix x y
   assume f: (λn. dist (f (x n)) (f (y n))) ⟶ 0
   assume g: (λn. dist (g (x n)) (g (y n))) ⟶ 0
   have (λn. |dist (f (x n)) (g (x n)) − dist (f (y n)) (g (y n))|) ⟶ 0
    by (rule Lim_transform_bound [OF _ tendsto_add_zero [OF f g]],
      simp add: le)
  }
  then show ?thesis
   using assms unfolding uniformly_continuous_on_sequentially
   unfolding dist_real_def by simp
qed

lemma uniformly_continuous_on_cmul_right [continuous_intros]:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_algebra
  shows uniformly_continuous_on s f ⟹ uniformly_continuous_on s (λx. f x *
c)
  using bounded_linear.uniformly_continuous_on[OF bounded_linear_mult_left]
.

lemma uniformly_continuous_on_cmul_left[continuous_intros]:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_algebra
  assumes uniformly_continuous_on s f
   shows uniformly_continuous_on s (λx. c * f x)
by (metis assms bounded_linear.uniformly_continuous_on bounded_linear_mult_right)

lemma uniformly_continuous_on_norm[continuous_intros]:
  fixes f :: 'a :: metric_space ⇒ 'b :: real_normed_vector
  assumes uniformly_continuous_on s f
  shows uniformly_continuous_on s (λx. norm (f x))
  unfolding norm_conv_dist using assms
  by (intro uniformly_continuous_on_dist uniformly_continuous_on_const)

lemma uniformly_continuous_on_cmul[continuous_intros]:
  fixes f :: 'a::metric_space ⇒ 'b::real_normed_vector
  assumes uniformly_continuous_on s f
  shows uniformly_continuous_on s (λx. c *R f(x))
  using bounded_linear_scaleR_right assms
  by (rule bounded_linear.uniformly_continuous_on)

lemma dist_minus:
  fixes x y :: 'a::real_normed_vector
  shows dist (− x) (− y) = dist x y
  unfolding dist_norm minus_diff_minus norm_minus_cancel ..
```

lemma *uniformly_continuous_on_minus*[*continuous_intros*]:
fixes $f :: 'a::metric\_space \Rightarrow 'b::real\_normed\_vector$
shows *uniformly_continuous_on s f* $\Longrightarrow$ *uniformly_continuous_on s* $(\lambda x. - f\ x)$
unfolding *uniformly_continuous_on_def dist_minus* .

lemma *uniformly_continuous_on_add*[*continuous_intros*]:
fixes $f\ g :: 'a::metric\_space \Rightarrow 'b::real\_normed\_vector$
assumes *uniformly_continuous_on s f*
and *uniformly_continuous_on s g*
shows *uniformly_continuous_on s* $(\lambda x.\ f\ x + g\ x)$
using *assms*
unfolding *uniformly_continuous_on_sequentially*
unfolding *dist_norm tendsto_norm_zero_iff add_diff_add*
by (*auto intro*: *tendsto_add_zero*)

lemma *uniformly_continuous_on_diff*[*continuous_intros*]:
fixes $f :: 'a::metric\_space \Rightarrow 'b::real\_normed\_vector$
assumes *uniformly_continuous_on s f*
and *uniformly_continuous_on s g*
shows *uniformly_continuous_on s* $(\lambda x.\ f\ x - g\ x)$
using *assms uniformly_continuous_on_add* [*of s f* $-$ *g*]
by (*simp add*: *fun_Compl_def uniformly_continuous_on_minus*)

lemma *uniformly_continuous_on_sum* [*continuous_intros*]:
fixes $f :: 'a \Rightarrow 'b::metric\_space \Rightarrow 'c::real\_normed\_vector$
shows $(\bigwedge i.\ i \in I \Longrightarrow$ *uniformly_continuous_on S* $(f\ i)) \Longrightarrow$ *uniformly_continuous_on S* $(\lambda x.\ \sum i{\in}I.\ f\ i\ x)$
by (*induction I rule*: *infinite_finite_induct*)
(*auto simp*: *uniformly_continuous_on_add uniformly_continuous_on_const*)

4.32.17 Arithmetic Preserves Topological Properties

lemma *open_scaling*[*intro*]:
fixes $s :: 'a::real\_normed\_vector\ set$
assumes $c \neq 0$
and *open s*
shows *open*$((\lambda x.\ c *_R x)\ `\ s)$
proof −
{
fix x
assume $x \in s$
then obtain e **where** $e{>}0$
and $e{:}\forall x'.\ dist\ x'\ x < e \longrightarrow x' \in s$ **using** *assms*(*2*)[*unfolded open_dist*, *THEN bspec*[**where** $x{=}x$]]
by *auto*
have $e * |c| > 0$
using *assms*(*1*)[*unfolded zero_less_abs_iff*[*symmetric*]] ‹$e{>}0$› **by** *auto*
moreover

```
    {
      fix y
      assume dist y (c *R x) < e * |c|
      then have norm (c *R ((1 / c) *R y − x)) < e * norm c
        by (simp add: ‹c ≠ 0› dist_norm scale_right_diff_distrib)
      then have norm ((1 / c) *R y − x) < e
        by (simp add: ‹c ≠ 0›)
      then have y ∈ (*R) c ‘ s
        using rev_image_eqI[of (1 / c) *R y s y (*R) c]
        by (simp add: ‹c ≠ 0› dist_norm e)
    }
    ultimately have ∃ e>0. ∀ x'. dist x' (c *R x) < e ⟶ x' ∈ (*R) c ‘ s
      by (rule_tac x=e * |c| in exI, auto)
  }
  then show ?thesis unfolding open_dist by auto
qed

lemma minus_image_eq_vimage:
  fixes A :: 'a::ab_group_add set
  shows (λx. − x) ‘ A = (λx. − x) −‘ A
  by (auto intro!: image_eqI [where f=λx. − x])

lemma open_negations:
  fixes S :: 'a::real_normed_vector set
  shows open S ⟹ open ((λx. − x) ‘ S)
  using open_scaling [of − 1 S] by simp

lemma open_translation:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open((λx. a + x) ‘ S)
proof −
  {
    fix x
    have continuous (at x) (λx. x − a)
      by (intro continuous_diff continuous_ident continuous_const)
  }
  moreover have {x. x − a ∈ S} = (+) a ‘ S
    by force
  ultimately show ?thesis
    by (metis assms continuous_open_vimage vimage_def)
qed

lemma open_translation_subtract:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open ((λx. x − a) ‘ S)
  using assms open_translation [of S − a] by (simp cong: image_cong_simp)
```

```
lemma open_neg_translation:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open((λx. a − x) ‘ S)
  using open_translation[OF open_negations[OF assms], of a]
  by (auto simp: image_image)

lemma open_affinity:
  fixes S :: 'a::real_normed_vector set
  assumes open S  c ≠ 0
  shows open ((λx. a + c *R x) ‘ S)
proof −
  have ∗: (λx. a + c *R x) = (λx. a + x) ∘ (λx. c *R x)
    unfolding o_def ..
  have (+) a ‘ (*R) c ‘ S = ((+) a ∘ (*R) c) ‘ S
    by auto
  then show ?thesis
    using assms open_translation[of (*R) c ‘ S a]
    unfolding ∗
    by auto
qed

lemma interior_translation:
  interior ((+) a ‘ S) = (+) a ‘ (interior S) for S :: 'a::real_normed_vector set
proof (rule set_eqI, rule)
  fix x
  assume x ∈ interior ((+) a ‘ S)
  then obtain e where e > 0 and e: ball x e ⊆ (+) a ‘ S
    unfolding mem_interior by auto
  then have ball (x − a) e ⊆ S
    unfolding subset_eq Ball_def mem_ball dist_norm
    by (auto simp: diff_diff_eq)
  then show x ∈ (+) a ‘ interior S
    unfolding image_iff
    by (metis ‹0 < e› add.commute diff_add_cancel mem_interior)
next
  fix x
  assume x ∈ (+) a ‘ interior S
  then obtain y e where e > 0 and e: ball y e ⊆ S and y: x = a + y
    unfolding image_iff Bex_def mem_interior by auto
  {
    fix z
    have ∗: a + y − z = y + a − z by auto
    assume z ∈ ball x e
    then have z − a ∈ S
      using e[unfolded subset_eq, THEN bspec[where x=z − a]]
      unfolding mem_ball dist_norm y group_add_class.diff_diff_eq2 ∗
      by auto
    then have z ∈ (+) a ‘ S
```

```
    unfolding image_iff by (auto intro!: bexI[where x=z - a])
  }
  then have ball x e ⊆ (+) a ‘ S
    unfolding subset_eq by auto
  then show x ∈ interior ((+) a ‘ S)
    unfolding mem_interior using ‹e > 0› by auto
qed

lemma interior_translation_subtract:
  interior ((λx. x - a) ‘ S) = (λx. x - a) ‘ interior S for S :: 'a::real_normed_vector
set
  using interior_translation [of - a] by (simp cong: image_cong_simp)


lemma compact_scaling:
  fixes s :: 'a::real_normed_vector set
  assumes compact s
  shows compact ((λx. c *R x) ‘ s)
proof -
  let ?f = λx. scaleR c x
  have *: bounded_linear ?f by (rule bounded_linear_scaleR_right)
  show ?thesis
   using compact_continuous_image[of s ?f] continuous_at_imp_continuous_on[of
s ?f]
    using linear_continuous_at[OF *] assms
    by auto
qed

lemma compact_negations:
  fixes s :: 'a::real_normed_vector set
  assumes compact s
  shows compact ((λx. - x) ‘ s)
  using compact_scaling [OF assms, of - 1] by auto

lemma compact_sums:
  fixes s t :: 'a::real_normed_vector set
  assumes compact s
    and compact t
  shows compact {x + y | x y. x ∈ s ∧ y ∈ t}
proof -
  have *: {x + y | x y. x ∈ s ∧ y ∈ t} = (λz. fst z + snd z) ‘ (s × t)
    by (fastforce simp: image_iff)
  have continuous_on (s × t) (λz. fst z + snd z)
    unfolding continuous_on by (rule ballI) (intro tendsto_intros)
  then show ?thesis
    unfolding * using compact_continuous_image compact_Times [OF assms]
by auto
qed
```

lemma *compact_differences*:
fixes $s\ t :: 'a{::}real\_normed\_vector\ set$
assumes *compact s*
and *compact t*
shows *compact* $\{x - y \mid x\ y.\ x \in s \wedge y \in t\}$
proof−
have $\{x - y \mid x\ y.\ x{\in}s \wedge y \in t\} = \{x + y \mid x\ y.\ x \in s \wedge y \in (uminus\ `\ t)\}$
using *diff_conv_add_uminus* **by** *force*
then show *?thesis*
using *compact_sums*[*OF assms*(*1*) *compact_negations*[*OF assms*(*2*)]] **by** *auto*
qed

lemma *compact_sums'*:
fixes $S :: 'a{::}real\_normed\_vector\ set$
assumes *compact S* **and** *compact T*
shows *compact* $(\bigcup x{\in}\ S.\ \bigcup y \in T.\ \{x + y\})$
proof −
have $(\bigcup x{\in}S.\ \bigcup y{\in}T.\ \{x + y\}) = \{x + y\ |x\ y.\ x \in S \wedge y \in T\}$
by *blast*
then show *?thesis*
using *compact_sums* [*OF assms*] **by** *simp*
qed

lemma *compact_differences'*:
fixes $S :: 'a{::}real\_normed\_vector\ set$
assumes *compact S* **and** *compact T*
shows *compact* $(\bigcup x{\in}\ S.\ \bigcup y \in T.\ \{x - y\})$
proof −
have $(\bigcup x{\in}S.\ \bigcup y{\in}T.\ \{x - y\}) = \{x - y\ |x\ y.\ x \in S \wedge y \in T\}$
by *blast*
then show *?thesis*
using *compact_differences* [*OF assms*] **by** *simp*
qed

lemma *compact_translation*:
compact $((+)\ a\ `\ s)$ **if** *compact s* **for** $s :: 'a{::}real\_normed\_vector\ set$
proof −
have $\{x + y\ |x\ y.\ x \in s \wedge y \in \{a\}\} = (\lambda x.\ a + x)\ `\ s$
by *auto*
then show *?thesis*
using *compact_sums* [*OF that compact_sing* [*of a*]] **by** *auto*
qed

lemma *compact_translation_subtract*:
compact $((\lambda x.\ x - a)\ `\ s)$ **if** *compact s* **for** $s :: 'a{::}real\_normed\_vector\ set$
using *that compact_translation* [*of s* − *a*] **by** (*simp cong*: *image_cong_simp*)

lemma *compact_affinity*:
fixes $s :: 'a{::}real\_normed\_vector\ set$

assumes *compact s*
shows *compact* ((λ*x*. *a* + *c* ∗$_R$ *x*) ‘ *s*)
proof −
have (+) *a* ‘ (∗$_R$) *c* ‘ *s* = (λ*x*. *a* + *c* ∗$_R$ *x*) ‘ *s*
by *auto*
then show *?thesis*
using *compact_translation*[*OF compact_scaling*[*OF assms*], *of a c*] **by** *auto*
qed

lemma *closed_scaling*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *closed S*
shows *closed* ((λ*x*. *c* ∗$_R$ *x*) ‘ *S*)
proof (*cases c* = *0*)
case *True* **then show** *?thesis*
by (*auto simp*: *image_constant_conv*)
next
case *False*
from *assms* **have** *closed* ((λ*x*. *inverse c* ∗$_R$ *x*) −‘ *S*)
by (*simp add*: *continuous_closed_vimage*)
also have (λ*x*. *inverse c* ∗$_R$ *x*) −‘ *S* = (λ*x*. *c* ∗$_R$ *x*) ‘ *S*
using ‹*c* ≠ *0*› **by** (*auto elim*: *image_eqI* [*rotated*])
finally show *?thesis* **.**
qed

lemma *closed_negations*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *closed S*
shows *closed* ((λ*x*. −*x*) ‘ *S*)
using *closed_scaling*[*OF assms*, *of* − *1*] **by** *simp*

lemma *compact_closed_sums*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *compact S* **and** *closed T*
shows *closed* (⋃*x*∈ *S*. ⋃*y* ∈ *T*. {*x* + *y*})
proof −
let *?S* = {*x* + *y* |*x y*. *x* ∈ *S* ∧ *y* ∈ *T*}
{
fix *x l*
assume *as*: ∀ *n*. *x n* ∈ *?S* (*x* ⟶ *l*) *sequentially*
from *as*(*1*) **obtain** *f* **where** *f*: ∀ *n*. *x n* = *fst* (*f n*) + *snd* (*f n*) ∀ *n*. *fst* (*f n*) ∈ *S* ∀ *n*. *snd* (*f n*) ∈ *T*
using *choice*[*of* λ*n y*. *x n* = (*fst y*) + (*snd y*) ∧ *fst y* ∈ *S* ∧ *snd y* ∈ *T*] **by** *auto*
obtain *l'* *r* **where** *l'*∈*S* **and** *r*: *strict_mono r* **and** *lr*: (((λ*n*. *fst* (*f n*)) ∘ *r*) ⟶ *l'*) *sequentially*
using *assms*(*1*)[*unfolded compact_def*, *THEN spec*[**where** *x*=λ *n*. *fst* (*f n*)]]
using *f*(*2*) **by** *auto*
have ((λ*n*. *snd* (*f* (*r n*))) ⟶ *l* − *l'*) *sequentially*

```
    using tendsto_diff[OF LIMSEQ_subseq_LIMSEQ[OF as(2) r] lr] and f(1)
    unfolding o_def
    by auto
  then have l - l' ∈ T
    using assms(2)[unfolded closed_sequential_limits,
      THEN spec[where x=λ n. snd (f (r n))],
      THEN spec[where x=l - l']]
    using f(3)
    by auto
  then have l ∈ ?S
    using ‹l' ∈ S› by force
 }
 moreover have ?S = (⋃x∈ S. ⋃y ∈ T. {x + y})
  by force
 ultimately show ?thesis
  unfolding closed_sequential_limits
  by (metis (no_types, lifting))
qed

lemma closed_compact_sums:
 fixes S T :: 'a::real_normed_vector set
 assumes closed S compact T
 shows closed (⋃x∈ S. ⋃y ∈ T. {x + y})
proof -
 have (⋃x∈ T. ⋃y ∈ S. {x + y}) = (⋃x∈ S. ⋃y ∈ T. {x + y})
  by auto
 then show ?thesis
  using compact_closed_sums[OF assms(2,1)] by simp
qed

lemma compact_closed_differences:
 fixes S T :: 'a::real_normed_vector set
 assumes compact S closed T
 shows closed (⋃x∈ S. ⋃y ∈ T. {x - y})
proof -
 have (⋃x∈ S. ⋃y ∈ uminus ‘ T. {x + y}) = (⋃x∈ S. ⋃y ∈ T. {x - y})
  by force
 then show ?thesis
  by (metis assms closed_negations compact_closed_sums)
qed

lemma closed_compact_differences:
 fixes S T :: 'a::real_normed_vector set
 assumes closed S compact T
 shows closed (⋃x∈ S. ⋃y ∈ T. {x - y})
proof -
 have (⋃x∈ S. ⋃y ∈ uminus ‘ T. {x + y}) = {x - y |x y. x ∈ S ∧ y ∈ T}
  by auto
 then show ?thesis
```

using *closed_compact_sums*[*OF assms*(*1*) *compact_negations*[*OF assms*(*2*)]] **by** *simp*
qed

lemma *closed_translation*:
closed ((+) *a* ‘ *S*) **if** *closed S* **for** *a* :: *'a*::*real_normed_vector*
proof −
have (⋃*x*∈ {*a*}. ⋃*y* ∈ *S*. {*x* + *y*}) = ((+) *a* ‘ *S*) **by** *auto*
then show *?thesis*
using *compact_closed_sums* [*OF compact_sing* [*of a*] *that*] **by** *auto*
qed

lemma *closed_translation_subtract*:
closed ((λ*x*. *x* − *a*) ‘ *S*) **if** *closed S* **for** *a* :: *'a*::*real_normed_vector*
using *that closed_translation* [*of S* − *a*] **by** (*simp cong*: *image_cong_simp*)

lemma *closure_translation*:
closure ((+) *a* ‘ *s*) = (+) *a* ‘ *closure s* **for** *a* :: *'a*::*real_normed_vector*
proof −
have ∗: (+) *a* ‘ (− *s*) = − (+) *a* ‘ *s*
by (*auto intro*!: *image_eqI* [**where** *x* = *x* − *a* **for** *x*])
show *?thesis*
using *interior_translation* [*of a* − *s*, *symmetric*]
by (*simp add*: *closure_interior translation_Compl* ∗)
qed

lemma *closure_translation_subtract*:
closure ((λ*x*. *x* − *a*) ‘ *s*) = (λ*x*. *x* − *a*) ‘ *closure s* **for** *a* :: *'a*::*real_normed_vector*
using *closure_translation* [*of* − *a s*] **by** (*simp cong*: *image_cong_simp*)

lemma *frontier_translation*:
frontier ((+) *a* ‘ *s*) = (+) *a* ‘ *frontier s* **for** *a* :: *'a*::*real_normed_vector*
by (*auto simp add*: *frontier_def translation_diff interior_translation closure_translation*)

lemma *frontier_translation_subtract*:
frontier ((+) *a* ‘ *s*) = (+) *a* ‘ *frontier s* **for** *a* :: *'a*::*real_normed_vector*
by (*auto simp add*: *frontier_def translation_diff interior_translation closure_translation*)

lemma *sphere_translation*:
sphere (*a* + *c*) *r* = (+) *a* ‘ *sphere c r* **for** *a* :: *'n*::*real_normed_vector*
by (*auto simp*: *dist_norm algebra_simps intro*!: *image_eqI* [**where** *x* = *x* − *a* **for** *x*])

lemma *sphere_translation_subtract*:
sphere (*c* − *a*) *r* = (λ*x*. *x* − *a*) ‘ *sphere c r* **for** *a* :: *'n*::*real_normed_vector*
using *sphere_translation* [*of* − *a c*] **by** (*simp cong*: *image_cong_simp*)

lemma *cball_translation*:
cball (*a* + *c*) *r* = (+) *a* ‘ *cball c r* **for** *a* :: *'n*::*real_normed_vector*

by (*auto simp*: *dist_norm algebra_simps intro*!: *image_eqI* [**where** $x = x - a$ **for** x])

lemma *cball_translation_subtract*:
cball $(c - a)$ $r = (\lambda x.\ x - a)$ ‘ *cball* c r **for** a :: $'n$::*real_normed_vector*
using *cball_translation* [*of* $-$ a c] **by** (*simp cong*: *image_cong_simp*)

lemma *ball_translation*:
ball $(a + c)$ $r = (+)$ a ‘ *ball* c r **for** a :: $'n$::*real_normed_vector*
by (*auto simp*: *dist_norm algebra_simps intro*!: *image_eqI* [**where** $x = x - a$ **for** x])

lemma *ball_translation_subtract*:
ball $(c - a)$ $r = (\lambda x.\ x - a)$ ‘ *ball* c r **for** a :: $'n$::*real_normed_vector*
using *ball_translation* [*of* $-$ a c] **by** (*simp cong*: *image_cong_simp*)

4.32.18 Homeomorphisms

lemma *homeomorphic_scaling*:
fixes S :: $'a$::*real_normed_vector set*
assumes $c \neq 0$
shows S *homeomorphic* $((\lambda x.\ c *_R x)$ ‘ $S)$
unfolding *homeomorphic_minimal*
apply (*rule_tac* x=$\lambda x.\ c *_R x$ **in** *exI*)
apply (*rule_tac* x=$\lambda x.\ (1 / c) *_R x$ **in** *exI*)
using *assms* **by** (*auto simp*: *continuous_intros*)

lemma *homeomorphic_translation*:
fixes S :: $'a$::*real_normed_vector set*
shows S *homeomorphic* $((\lambda x.\ a + x)$ ‘ $S)$
unfolding *homeomorphic_minimal*
apply (*rule_tac* x=$\lambda x.\ a + x$ **in** *exI*)
apply (*rule_tac* x=$\lambda x.\ -a + x$ **in** *exI*)
by (*auto simp*: *continuous_intros*)

lemma *homeomorphic_affinity*:
fixes S :: $'a$::*real_normed_vector set*
assumes $c \neq 0$
shows S *homeomorphic* $((\lambda x.\ a + c *_R x)$ ‘ $S)$
proof −
have $*$: $(+)$ a ‘ $(*_R)$ c ‘ $S = (\lambda x.\ a + c *_R x)$ ‘ S **by** *auto*
show *?thesis*
by (*metis* $*$ *assms homeomorphic_scaling homeomorphic_trans homeomorphic_translation*)
qed

lemma *homeomorphic_balls*:
fixes a b :: $'a$::*real_normed_vector*
assumes $0 < d$ $0 < e$

```
  shows (ball a d) homeomorphic  (ball b e) (is ?th)
    and (cball a d) homeomorphic (cball b e) (is ?cth)
proof −
  show ?th unfolding homeomorphic_minimal
    apply(rule_tac x=λx. b + (e/d) *R (x − a) in exI)
    apply(rule_tac x=λx. a + (d/e) *R (x − b) in exI)
    using assms
   by (auto intro!: continuous_intros simp: dist_commute dist_norm pos_divide_less_eq)
  show ?cth unfolding homeomorphic_minimal
    apply(rule_tac x=λx. b + (e/d) *R (x − a) in exI)
    apply(rule_tac x=λx. a + (d/e) *R (x − b) in exI)
    using assms
   by (auto intro!: continuous_intros simp: dist_commute dist_norm pos_divide_le_eq)
qed

lemma homeomorphic_spheres:
  fixes a b ::'a::real_normed_vector
  assumes 0 < d  0 < e
  shows (sphere a d) homeomorphic (sphere b e)
unfolding homeomorphic_minimal
    apply(rule_tac x=λx. b + (e/d) *R (x − a) in exI)
    apply(rule_tac x=λx. a + (d/e) *R (x − b) in exI)
    using assms
   by (auto intro!: continuous_intros simp: dist_commute dist_norm pos_divide_less_eq)

lemma homeomorphic_ball01_UNIV:
  ball (0::'a::real_normed_vector) 1 homeomorphic (UNIV:: 'a set)
  (is ?B homeomorphic ?U)
proof
  have x ∈ (λz. z /R (1 − norm z)) ‘ ball 0 1 for x::'a
    apply (rule_tac x=x /R (1 + norm x) in image_eqI)
     apply (auto simp: field_split_simps)
    using norm_ge_zero [of x] apply linarith+
    done
  then show (λz::'a. z /R (1 − norm z)) ‘ ?B = ?U
    by blast
  have x ∈ range (λz. (1 / (1 + norm z)) *R z) if norm x < 1 for x::'a
    using that
   by (rule_tac x=x /R (1 − norm x) in image_eqI) (auto simp: field_split_simps)
  then show (λz::'a. z /R (1 + norm z)) ‘ ?U = ?B
    by (force simp: field_split_simps dest: add_less_zeroD)
  show continuous_on (ball 0 1) (λz. z /R (1 − norm z))
    by (rule continuous_intros | force)+
  have 0: ⋀z. 1 + norm z ≠ 0
    by (metis (no_types) le_add_same_cancel1 norm_ge_zero not_one_le_zero)
  then show continuous_on UNIV (λz. z /R (1 + norm z))
    by (auto intro!: continuous_intros)
  show ⋀x. x ∈ ball 0 1 ⟹
        x /R (1 − norm x) /R (1 + norm (x /R (1 − norm x))) = x
```

by (*auto simp*: *field_split_simps*)
show $\bigwedge y.\ y\ /_R\ (1 + norm\ y)\ /_R\ (1 - norm\ (y\ /_R\ (1 + norm\ y))) = y$
using *0* **by** (*auto simp*: *field_split_simps*)
qed

proposition *homeomorphic_ball_UNIV*:
fixes *a* ::'*a*::*real_normed_vector*
assumes $0 < r$ **shows** *ball a r homeomorphic* (*UNIV*:: '*a set*)
using *assms homeomorphic_ball01_UNIV homeomorphic_balls*(*1*) *homeomorphic_trans zero_less_one* **by** *blast*

4.32.19 Discrete

lemma *finite_implies_discrete*:
fixes S :: '*a*::*topological_space set*
assumes *finite* (f ' S)
shows $(\forall x \in S.\ \exists e{>}0.\ \forall y.\ y \in S \wedge f\ y \neq f\ x \longrightarrow e \leq norm\ (f\ y - f\ x))$
proof −
have $\exists e{>}0.\ \forall y.\ y \in S \wedge f\ y \neq f\ x \longrightarrow e \leq norm\ (f\ y - f\ x)$ **if** $x \in S$ **for** x
proof (*cases* f ' $S - \{f\ x\} = \{\}$)
case *True*
with *zero_less_numeral* **show** *?thesis*
by (*fastforce simp add*: *Set.image_subset_iff cong*: *conj_cong*)
next
case *False*
then obtain z **where** $z \in S\ f\ z \neq f\ x$
by *blast*
moreover have *finn*: *finite* $\{norm\ (z - f\ x)\ |z.\ z \in f$ ' $S - \{f\ x\}\}$
using *assms* **by** *simp*
ultimately have $*$: $0 < Inf\{norm(z - f\ x) \mid z.\ z \in f$ ' $S - \{f\ x\}\}$
by (*force intro*: *finite_imp_less_Inf*)
show *?thesis*
by (*force intro*!: $*$ *cInf_le_finite* [*OF finn*])
qed
with *assms* **show** *?thesis*
by *blast*
qed

4.32.20 Completeness of "Isometry" (up to constant bounds)

lemma *cauchy_isometric*:— TODO: rename lemma to *Cauchy_isometric*
assumes e: $e > 0$
and s: *subspace s*
and f: *bounded_linear f*
and *normf*: $\forall x{\in}s.\ norm\ (f\ x) \geq e * norm\ x$
and *xs*: $\forall n.\ x\ n \in s$
and *cf*: *Cauchy* $(f \circ x)$
shows *Cauchy x*
proof −
interpret f: *bounded_linear f* **by** *fact*

```
  have ∃ N. ∀ n≥N. norm (x n − x N) < d if d > 0 for d :: real
  proof −
    from that obtain N where N: ∀ n≥N. norm (f (x n) − f (x N)) < e ∗ d
      using cf[unfolded Cauchy_def o_def dist_norm, THEN spec[where x=e∗d]]
e
      by auto
    have norm (x n − x N) < d if n ≥ N for n
    proof −
      have e ∗ norm (x n − x N) ≤ norm (f (x n − x N))
        using subspace_diff[OF s, of x n x N]
        using xs[THEN spec[where x=N]] and xs[THEN spec[where x=n]]
        using normf[THEN bspec[where x=x n − x N]]
        by auto
      also have norm (f (x n − x N)) < e ∗ d
        using ‹N ≤ n› N unfolding f.diff[symmetric] by auto
      finally show ?thesis
        using ‹e>0› by simp
    qed
    then show ?thesis by auto
  qed
  then show ?thesis
    by (simp add: Cauchy_altdef2 dist_norm)
qed

lemma complete_isometric_image:
  assumes 0 < e
    and s: subspace s
    and f: bounded_linear f
    and normf: ∀ x∈s. norm(f x) ≥ e ∗ norm(x)
    and cs: complete s
  shows complete (f ‘ s)
proof −
  have ∃ l∈f ‘ s. (g ⟶ l) sequentially
    if as:∀ n::nat. g n ∈ f ‘ s and cfg:Cauchy g for g
  proof −
    from that obtain x where ∀ n. x n ∈ s ∧ g n = f (x n)
      using choice[of λ n xa. xa ∈ s ∧ g n = f xa] by auto
    then have x: ∀ n. x n ∈ s ∀ n. g n = f (x n) by auto
    then have f ∘ x = g by (simp add: fun_eq_iff)
    then obtain l where l∈s and l:(x ⟶ l) sequentially
      using cs[unfolded complete_def, THEN spec[where x=x]]
      using cauchy_isometric[OF ‹0 < e› s f normf] and cfg and x(1)
      by auto
    then show ?thesis
        using linear_continuous_at[OF f, unfolded continuous_at_sequentially,
THEN spec[where x=x], of l]
      by (auto simp: ‹f ∘ x = g›)
  qed
  then show ?thesis
```

unfolding *complete_def* **by** *auto*
qed

4.32.21 Connected Normed Spaces

lemma *compact_components*:
fixes *s* :: *'a*::*heine_borel set*
shows ⟦*compact s*; *c* ∈ *components s*⟧ ⟹ *compact c*
by (*meson bounded_subset closed_components in_components_subset compact_eq_bounded_closed*)

lemma *discrete_subset_disconnected*:
fixes *S* :: *'a*::*topological_space set*
fixes *t* :: *'b*::*real_normed_vector set*
assumes *conf*: *continuous_on S f*
and *no*: ⋀*x*. *x* ∈ *S* ⟹ ∃ *e*>*0*. ∀ *y*. *y* ∈ *S* ∧ *f y* ≠ *f x* ⟶ *e* ≤ *norm* (*f y* − *f x*)
shows *f* ‘ *S* ⊆ {*y*. *connected_component_set* (*f* ‘ *S*) *y* = {*y*}}
proof −
{ fix *x* **assume** *x*: *x* ∈ *S*
then obtain *e* **where** *e*>*0* **and** *ele*: ⋀*y*. ⟦*y* ∈ *S*; *f y* ≠ *f x*⟧ ⟹ *e* ≤ *norm* (*f y* − *f x*)
using *conf no* [*OF x*] **by** *auto*
then have *e2*: *0* ≤ *e*/*2*
by *simp*
define *F* **where** *F* ≡ *connected_component_set* (*f* ‘ *S*) (*f x*)
have *False* **if** *y* ∈ *S* **and** *ccs*: *f y* ∈ *F* **and** *not*: *f y* ≠ *f x* **for** *y*
proof −
define *C* **where** *C* ≡ *cball* (*f x*) (*e*/*2*)
define *D* **where** *D* ≡ − *ball* (*f x*) *e*
have *disj*: *C* ∩ *D* = {}
unfolding *C_def D_def* **using** ‹*0* < *e*› **by** *fastforce*
moreover have *FCD*: *F* ⊆ *C* ∪ *D*
proof −
have *t* ∈ *C* ∨ *t* ∈ *D* **if** *t* ∈ *F* **for** *t*
proof −
obtain *y* **where** *y* ∈ *S t* = *f y*
using *F_def* ‹*t* ∈ *F*› *connected_component_in* **by** *blast*
then show *?thesis*
by (*metis C_def ComplI D_def centre_in_cball dist_norm e2 ele mem_ball norm_minus_commute not_le*)
qed
then show *?thesis*
by *auto*
qed
ultimately have *C* ∩ *F* = {} ∨ *D* ∩ *F* = {}
using *connected_closed* [*of F*] ‹*e*>*0*› *not*
unfolding *C_def D_def*
by (*metis Elementary_Metric_Spaces.open_ball F_def closed_cball connected_connected_component inf_bot_left open_closed*)

moreover have $C \cap F \neq \{\}$
unfolding *disjoint_iff*
by (*metis FCD ComplD image_eqI mem_Collect_eq subsetD x D_def F_def Un_iff ‹0 < e› centre_in_ball connected_component_refl_eq*)
moreover have $D \cap F \neq \{\}$
unfolding *disjoint_iff*
by (*metis ComplI D_def ccs dist_norm ele mem_ball norm_minus_commute not not_le that(1)*)
ultimately show *?thesis* **by** *metis*
qed
moreover have *connected_component_set* $(f \mathbin{`} S)$ $(f\ x) \subseteq f \mathbin{`} S$
by (*auto simp*: *connected_component_in*)
ultimately have *connected_component_set* $(f \mathbin{`} S)$ $(f\ x) = \{f\ x\}$
by (*auto simp*: *x F_def*)
}
with *assms* **show** *?thesis*
by *blast*
qed

lemma *continuous_disconnected_range_constant_eq*:
(*connected S* $\longleftrightarrow$
$(\forall f::'a::topological\_space \Rightarrow 'b::real\_normed\_algebra\_1.$
$\forall t.$ *continuous_on S f* $\wedge$ $f \mathbin{`} S \subseteq t \wedge (\forall y \in t.$ *connected_component_set t y* $= \{y\})$
$\longrightarrow$ *f constant_on S*)) (**is** *?thesis1*)
and *continuous_discrete_range_constant_eq*:
(*connected S* $\longleftrightarrow$
$(\forall f::'a::topological\_space \Rightarrow 'b::real\_normed\_algebra\_1.$
continuous_on S f $\wedge$
$(\forall x \in S.\ \exists e.\ 0 < e \wedge (\forall y.\ y \in S \wedge (f\ y \neq f\ x) \longrightarrow e \leq norm(f\ y - f\ x)))$
$\longrightarrow$ *f constant_on S*)) (**is** *?thesis2*)
and *continuous_finite_range_constant_eq*:
(*connected S* $\longleftrightarrow$
$(\forall f::'a::topological\_space \Rightarrow 'b::real\_normed\_algebra\_1.$
continuous_on S f $\wedge$ *finite* $(f \mathbin{`} S)$
$\longrightarrow$ *f constant_on S*)) (**is** *?thesis3*)
proof −
have $*$: $\bigwedge s\ t\ u\ v.\ [\![s \Longrightarrow t;\ t \Longrightarrow u;\ u \Longrightarrow v;\ v \Longrightarrow s]\!]$
$\Longrightarrow (s \longleftrightarrow t) \wedge (s \longleftrightarrow u) \wedge (s \longleftrightarrow v)$
by *blast*
have *?thesis1* $\wedge$ *?thesis2* $\wedge$ *?thesis3*
apply (*rule* $*$)
using *continuous_disconnected_range_constant*
apply (*metis image_subset_iff_funcset*)
apply (*smt* (*verit*, *best*) *discrete_subset_disconnected mem_Collect_eq subsetD subsetI*)
apply (*blast dest*: *finite_implies_discrete*)
apply (*blast intro*!: *finite_range_constant_imp_connected*)
done

then show *?thesis1 ?thesis2 ?thesis3*
by *blast+*
qed

lemma *continuous_discrete_range_constant*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*real_normed_algebra_1*
assumes *S*: *connected S*
and *continuous_on S f*
and ⋀*x*. *x* ∈ *S* ⟹ ∃ *e*>*0*. ∀ *y*. *y* ∈ *S* ∧ *f y* ≠ *f x* ⟶ *e* ≤ *norm* (*f y* − *f x*)
shows *f constant_on S*
using *continuous_discrete_range_constant_eq* [*THEN iffD1*, *OF S*] *assms* **by** *blast*

lemma *continuous_finite_range_constant*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*real_normed_algebra_1*
assumes *connected S*
and *continuous_on S f*
and *finite* (*f* ‘ *S*)
shows *f constant_on S*
using *assms continuous_finite_range_constant_eq* **by** *blast*

end

4.33 Linear Decision Procedure for Normed Spaces

theory *Norm_Arith*
imports *HOL−Library.Sum_of_Squares*
begin

lemma *sum_sqs_eq*:
fixes *x*::*'a*::*idom* **shows** *x* ∗ *x* + *y* ∗ *y* = *x* ∗ (*y* ∗ *2*) ⟹ *y* = *x*
by *algebra*

lemma *norm_cmul_rule_thm*:
fixes *x* :: *'a*::*real_normed_vector*
shows *b* ≥ *norm x* ⟹ |*c*| ∗ *b* ≥ *norm* (*scaleR c x*)
unfolding *norm_scaleR*
apply (*erule mult_left_mono*)
apply *simp*
done

lemma *norm_add_rule_thm*:
fixes *x1 x2* :: *'a*::*real_normed_vector*
shows *norm x1* ≤ *b1* ⟹ *norm x2* ≤ *b2* ⟹ *norm* (*x1* + *x2*) ≤ *b1* + *b2*
by (*rule order_trans* [*OF norm_triangle_ineq add_mono*])

lemma *ge_iff_diff_ge_0*:

fixes $a :: 'a::linordered\_ring$
shows $a \geq b \equiv a - b \geq 0$
by (*simp add*: *field_simps*)

lemma *pth_1*:
fixes $x :: 'a::real\_normed\_vector$
shows $x \equiv scaleR\ 1\ x$ **by** *simp*

lemma *pth_2*:
fixes $x :: 'a::real\_normed\_vector$
shows $x - y \equiv x + -y$
by (*atomize* (*full*)) *simp*

lemma *pth_3*:
fixes $x :: 'a::real\_normed\_vector$
shows $- x \equiv scaleR\ (-1)\ x$
by *simp*

lemma *pth_4*:
fixes $x :: 'a::real\_normed\_vector$
shows $scaleR\ 0\ x \equiv 0$
and $scaleR\ c\ 0 = (0::'a)$
by *simp_all*

lemma *pth_5*:
fixes $x :: 'a::real\_normed\_vector$
shows $scaleR\ c\ (scaleR\ d\ x) \equiv scaleR\ (c * d)\ x$
by *simp*

lemma *pth_6*:
fixes $x :: 'a::real\_normed\_vector$
shows $scaleR\ c\ (x + y) \equiv scaleR\ c\ x + scaleR\ c\ y$
by (*simp add*: *scaleR_right_distrib*)

lemma *pth_7*:
fixes $x :: 'a::real\_normed\_vector$
shows $0 + x \equiv x$
and $x + 0 \equiv x$
by *simp_all*

lemma *pth_8*:
fixes $x :: 'a::real\_normed\_vector$
shows $scaleR\ c\ x + scaleR\ d\ x \equiv scaleR\ (c + d)\ x$
by (*simp add*: *scaleR_left_distrib*)

lemma *pth_9*:
fixes $x :: 'a::real\_normed\_vector$
shows $(scaleR\ c\ x + z) + scaleR\ d\ x \equiv scaleR\ (c + d)\ x + z$
and $scaleR\ c\ x + (scaleR\ d\ x + z) \equiv scaleR\ (c + d)\ x + z$

and $(scaleR\ c\ x + w) + (scaleR\ d\ x + z) \equiv scaleR\ (c + d)\ x + (w + z)$
by (*simp_all add*: *algebra_simps*)

lemma *pth_a*:
fixes $x :: 'a$::*real_normed_vector*
shows $scaleR\ 0\ x + y \equiv y$
by *simp*

lemma *pth_b*:
fixes $x :: 'a$::*real_normed_vector*
shows $scaleR\ c\ x + scaleR\ d\ y \equiv scaleR\ c\ x + scaleR\ d\ y$
and $(scaleR\ c\ x + z) + scaleR\ d\ y \equiv scaleR\ c\ x + (z + scaleR\ d\ y)$
and $scaleR\ c\ x + (scaleR\ d\ y + z) \equiv scaleR\ c\ x + (scaleR\ d\ y + z)$
and $(scaleR\ c\ x + w) + (scaleR\ d\ y + z) \equiv scaleR\ c\ x + (w + (scaleR\ d\ y + z))$
by (*simp_all add*: *algebra_simps*)

lemma *pth_c*:
fixes $x :: 'a$::*real_normed_vector*
shows $scaleR\ c\ x + scaleR\ d\ y \equiv scaleR\ d\ y + scaleR\ c\ x$
and $(scaleR\ c\ x + z) + scaleR\ d\ y \equiv scaleR\ d\ y + (scaleR\ c\ x + z)$
and $scaleR\ c\ x + (scaleR\ d\ y + z) \equiv scaleR\ d\ y + (scaleR\ c\ x + z)$
and $(scaleR\ c\ x + w) + (scaleR\ d\ y + z) \equiv scaleR\ d\ y + ((scaleR\ c\ x + w) + z)$
by (*simp_all add*: *algebra_simps*)

lemma *pth_d*:
fixes $x :: 'a$::*real_normed_vector*
shows $x + 0 \equiv x$
by *simp*

lemma *norm_imp_pos_and_ge*:
fixes $x :: 'a$::*real_normed_vector*
shows $norm\ x \equiv n \Longrightarrow norm\ x \geq 0 \wedge n \geq norm\ x$
by *atomize auto*

lemma *real_eq_0_iff_le_ge_0*:
fixes $x :: real$
shows $x = 0 \equiv x \geq 0 \wedge - x \geq 0$
by *arith*

lemma *norm_pths*:
fixes $x :: 'a$::*real_normed_vector*
shows $x = y \longleftrightarrow norm\ (x - y) \leq 0$
and $x \neq y \longleftrightarrow \neg\ (norm\ (x - y) \leq 0)$
using *norm_ge_zero*[*of* $x - y$] **by** *auto*

lemmas *arithmetic_simps* =
arith_simps

add_numeral_special
add_neg_numeral_special
mult_1_left
mult_1_right

ML_file ‹*normarith.ML*›

method_setup *norm* = ‹
Scan.succeed (*SIMPLE_METHOD'* o *NormArith.norm_arith_tac*)
› *prove simple linear statements about vector norms*

Hence more metric properties.

proposition *dist_triangle_add*:
fixes $x\ y\ x'\ y'$ :: *'a*::*real_normed_vector*
shows $dist\ (x + y)\ (x' + y') \leq dist\ x\ x' + dist\ y\ y'$
by *norm*

lemma *dist_triangle_add_half*:
fixes $x\ x'\ y\ y'$ :: *'a*::*real_normed_vector*
shows $dist\ x\ x' < e\ /\ 2 \Longrightarrow dist\ y\ y' < e\ /\ 2 \Longrightarrow dist(x + y)\ (x' + y') < e$
by *norm*

end

Chapter 5

Vector Analysis

theory *Topology_Euclidean_Space*
imports
Elementary_Normed_Spaces
Linear_Algebra
Norm_Arith
begin

5.1 Elementary Topology in Euclidean Space

lemma *euclidean_dist_l2*:
fixes x y :: *'a* :: *euclidean_space*
shows *dist* x y = *L2_set* (λi. *dist* ($x \cdot i$) ($y \cdot i$)) *Basis*
unfolding *dist_norm norm_eq_sqrt_inner L2_set_def*
by (*subst euclidean_inner*) (*simp add*: *power2_eq_square inner_diff_left*)

lemma *norm_nth_le*: *norm* ($x \cdot i$) $\leq$ *norm* x **if** $i \in$ *Basis*
proof −
have $(x \cdot i)^2 = (\sum i{\in}\{i\}.\ (x \cdot i)^2)$
by *simp*
also have ... $\leq (\sum i{\in}$*Basis*$.\ (x \cdot i)^2)$
by (*intro sum_mono2*) (*auto simp*: *that*)
finally show *?thesis*
unfolding *norm_conv_dist euclidean_dist_l2*[*of x*] *L2_set_def*
by (*auto intro*!: *real_le_rsqrt*)
qed

5.1.1 Continuity of the representation WRT an orthogonal basis

lemma *orthogonal_Basis*: *pairwise orthogonal Basis*
by (*simp add*: *inner_not_same_Basis orthogonal_def pairwise_def*)

lemma *representation_bound*:
fixes B :: *'N*::*real_inner set*

assumes *finite B independent B b ∈ B* **and** *orth*: *pairwise orthogonal B*
obtains *m* **where** *m > 0* $\bigwedge x.\ x \in span\ B \Longrightarrow |representation\ B\ x\ b| \leq m * norm\ x$
proof
fix *x*
assume *x*: *x ∈ span B*
have *b ≠ 0*
using ‹*independent B*› ‹*b ∈ B*› *dependent_zero* **by** *blast*
have [*simp*]: $b \cdot b' = (if\ b' = b\ then\ (norm\ b)^2\ else\ 0)$
if *b ∈ B b′ ∈ B* **for** *b b′*
using *orth* **by** (*simp add*: *orthogonal_def pairwise_def norm_eq_sqrt_inner that*)
have $norm\ x = norm\ (\sum b{\in}B.\ representation\ B\ x\ b *_R b)$
using *real_vector.sum_representation_eq* [*OF* ‹*independent B*› *x* ‹*finite B*› *order_refl*]
by *simp*
also have $\ldots = sqrt\ ((\sum b{\in}B.\ representation\ B\ x\ b *_R b) \cdot (\sum b{\in}B.\ representation\ B\ x\ b *_R b))$
by (*simp add*: *norm_eq_sqrt_inner*)
also have $\ldots = sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b *_R b) \cdot (representation\ B\ x\ b *_R b))$
using ‹*finite B*›
by (*simp add*: *inner_sum_left inner_sum_right if_distrib* [*of* $\lambda x.\ \_ * x$] *cong*: *if_cong sum.cong_simp*)
also have $\ldots = sqrt\ (\sum b{\in}B.\ (norm\ (representation\ B\ x\ b *_R b))^2)$
by (*simp add*: *mult.commute mult.left_commute power2_eq_square*)
also have $\ldots = sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$
by (*simp add*: *norm_mult power_mult_distrib*)
finally have $norm\ x = sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$.
moreover
have $sqrt\ ((representation\ B\ x\ b)^2 * (norm\ b)^2) \leq sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$
using ‹*b ∈ B*› ‹*finite B*› **by** (*auto intro*: *member_le_sum*)
then have $|representation\ B\ x\ b| \leq (1\ /\ norm\ b) * sqrt\ (\sum b{\in}B.\ (representation\ B\ x\ b)^2 * (norm\ b)^2)$
using ‹*b ≠ 0*› **by** (*simp add*: *field_split_simps real_sqrt_mult del*: *real_sqrt_le_iff*)
ultimately show $|representation\ B\ x\ b| \leq (1\ /\ norm\ b) * norm\ x$
by *simp*
next
show *0 < 1 / norm b*
using ‹*independent B*› ‹*b ∈ B*› *dependent_zero* **by** *auto*
qed

lemma *continuous_on_representation*:
fixes *B* :: *'N*::*euclidean_space set*
assumes *finite B independent B b ∈ B pairwise orthogonal B*
shows *continuous_on* (*span B*) ($\lambda x.$ *representation B x b*)
proof
show $\exists\, d{>}0.\ \forall\, x'{\in}span\ B.\ dist\ x'\ x < d \longrightarrow dist\ (representation\ B\ x'\ b)\ (representation$

```
B x b) ≤ e
    if e > 0 x ∈ span B for x e
  proof −
    obtain m where m > 0 and m: ⋀x. x ∈ span B ⟹ |representation B x b| ≤
m * norm x
      using assms representation_bound by blast
    show ?thesis
      unfolding dist_norm
    proof (intro exI conjI ballI impI)
      show e/m > 0
        by (simp add: ‹e > 0› ‹m > 0›)
      show norm (representation B x' b − representation B x b) ≤ e
        if x': x' ∈ span B and less: norm (x'−x) < e/m for x'
      proof −
        have |representation B (x'−x) b| ≤ m * norm (x'−x)
          using m [of x'−x] ‹x ∈ span B› span_diff x' by blast
        also have ... < e
          by (metis ‹m > 0› less mult.commute pos_less_divide_eq)
        finally have |representation B (x'−x) b| ≤ e by simp
        then show ?thesis
          by (simp add: ‹x ∈ span B› ‹independent B› representation_diff x')
      qed
    qed
  qed
qed
```

5.1.2 Balls in Euclidean Space

```
lemma cball_subset_cball_iff:
  fixes a :: 'a :: euclidean_space
  shows cball a r ⊆ cball a' r' ⟷ dist a a' + r ≤ r' ∨ r < 0
    (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  then show ?rhs
  proof (cases r < 0)
    case True
    then show ?rhs by simp
  next
    case False
    then have [simp]: r ≥ 0 by simp
    have norm (a − a') + r ≤ r'
    proof (cases a = a')
      case True
      then show ?thesis
        using subsetD [where c = a + r *R (SOME i. i ∈ Basis), OF ‹?lhs›]
subsetD [where c = a, OF ‹?lhs›]
        by (force simp: SOME_Basis dist_norm)
    next
```

```
      case False
      have norm (a' - (a + (r / norm (a - a')) *R (a - a'))) = norm ((-1 -
(r / norm (a - a'))) *R (a - a'))
        by (simp add: algebra_simps)
      also from ‹a ≠ a'› have ... = |- norm (a - a') - r|
        by (simp add: divide_simps)
      finally have [simp]: norm (a' - (a + (r / norm (a - a')) *R (a - a'))) =
|norm (a - a') + r|
        by linarith
      from ‹a ≠ a'› show ?thesis
        using subsetD [where c = a' + (1 + r / norm(a - a')) *R (a - a'), OF
‹?lhs›]
        by (simp add: dist_norm scaleR_add_left)
    qed
    then show ?rhs
      by (simp add: dist_norm)
  qed
qed metric

lemma cball_subset_ball_iff: cball a r ⊆ ball a' r' ⟷ dist a a' + r < r' ∨ r <
0
  (is ?lhs ⟷ ?rhs)
  for a :: 'a::euclidean_space
proof
  assume ?lhs
  then show ?rhs
  proof (cases r < 0)
    case True then
    show ?rhs by simp
  next
    case False
    then have [simp]: r ≥ 0 by simp
    have norm (a - a') + r < r'
    proof (cases a = a')
      case True
      then show ?thesis
        using subsetD [where c = a + r *R (SOME i. i ∈ Basis), OF ‹?lhs›]
subsetD [where c = a, OF ‹?lhs›]
        by (force simp: SOME_Basis dist_norm)
    next
      case False
      have False if norm (a - a') + r ≥ r'
      proof -
        from that have |r' - norm (a - a')| ≤ r
        by (smt (verit, best) ‹0 ≤ r› ‹?lhs› ball_subset_cball cball_subset_cball_iff
dist_norm order_trans)
        then show ?thesis
          using subsetD [where c = a + (r' / norm(a - a') - 1) *R (a - a'), OF
‹?lhs›] ‹a ≠ a'›
```

```
        apply (simp add: dist_norm)
        apply (simp add: scaleR_left_diff_distrib)
        apply (simp add: field_simps)
        done
    qed
    then show ?thesis by force
  qed
  then show ?rhs by (simp add: dist_norm)
 qed
next
 assume ?rhs
 then show ?lhs
  by metric
qed

lemma ball_subset_cball_iff: ball a r ⊆ cball a' r' ⟷ dist a a' + r ≤ r' ∨ r ≤
0
 (is ?lhs = ?rhs)
 for a :: 'a::euclidean_space
proof (cases r ≤ 0)
 case True
 then show ?thesis
  by metric
next
 case False
 show ?thesis
 proof
  assume ?lhs
  then have (cball a r ⊆ cball a' r')
  by (metis False closed_cball closure_ball closure_closed closure_mono not_less)
  with False show ?rhs
   by (fastforce iff: cball_subset_cball_iff)
 next
  assume ?rhs
  with False show ?lhs
   by metric
 qed
qed

lemma ball_subset_ball_iff:
 fixes a :: 'a :: euclidean_space
 shows ball a r ⊆ ball a' r' ⟷ dist a a' + r ≤ r' ∨ r ≤ 0
     (is ?lhs = ?rhs)
proof (cases r ≤ 0)
 case True then show ?thesis
  by metric
next
 case False show ?thesis
 proof
```

assume *?lhs*
then have $0 < r'$
using *False* **by** *metric*
then have *cball a r* $\subseteq$ *cball a' r'*
by (*metis False* ‹*?lhs*› *closure_ball closure_mono not_less*)
then show *?rhs*
using *False cball_subset_cball_iff* **by** *fastforce*
qed *metric*
qed

lemma *ball_eq_ball_iff*:
fixes $x :: 'a :: euclidean\_space$
shows *ball x d* = *ball y e* $\longleftrightarrow$ $d \leq 0 \wedge e \leq 0 \vee x{=}y \wedge d{=}e$
by (*smt* (*verit*, *del_insts*) *ball_empty ball_subset_cball_iff dist_norm norm_pths*(*2*))

lemma *cball_eq_cball_iff*:
fixes $x :: 'a :: euclidean\_space$
shows *cball x d* = *cball y e* $\longleftrightarrow$ $d < 0 \wedge e < 0 \vee x{=}y \wedge d{=}e$
by (*smt* (*verit*, *ccfv_SIG*) *cball_empty cball_subset_cball_iff dist_norm norm_pths*(*2*) *zero_le_dist*)

lemma *ball_eq_cball_iff*:
fixes $x :: 'a :: euclidean\_space$
shows *ball x d* = *cball y e* $\longleftrightarrow$ $d \leq 0 \wedge e < 0$ (**is** *?lhs* = *?rhs*)
by (*smt* (*verit*) *ball_eq_empty ball_subset_cball_iff cball_eq_empty cball_subset_ball_iff order.refl*)

lemma *cball_eq_ball_iff*:
fixes $x :: 'a :: euclidean\_space$
shows *cball x d* = *ball y e* $\longleftrightarrow$ $d < 0 \wedge e \leq 0$
using *ball_eq_cball_iff* **by** *blast*

lemma *finite_ball_avoid*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *open S finite X p* $\in S$
shows $\exists\, e{>}0.\ \forall\, w{\in}ball\ p\ e.\ w{\in}S \wedge (w{\neq}p \longrightarrow w{\notin}X)$
proof −
obtain *e1* **where** $0 < e1$ **and** *e1_b*:*ball p e1* $\subseteq S$
using *open_contains_ball_eq*[*OF* ‹*open S*›] *assms* **by** *auto*
obtain *e2* **where** $0 < e2$ **and** $\forall\, x{\in}X.\ x \neq p \longrightarrow e2 \leq dist\ p\ x$
using *finite_set_avoid*[*OF* ‹*finite X*›,*of p*] **by** *auto*
hence $\forall\, w{\in}ball\ p\ (min\ e1\ e2).\ w{\in}S \wedge (w{\neq}p \longrightarrow w{\notin}X)$ **using** *e1_b* **by** *auto*
thus $\exists\, e{>}0.\ \forall\, w{\in}ball\ p\ e.\ w \in S \wedge (w \neq p \longrightarrow w \notin X)$
using ‹*e2*>*0*› ‹*e1*>*0*› **by** (*rule_tac x*=*min e1 e2* **in** *exI*) *auto*
qed

lemma *finite_cball_avoid*:
fixes $S :: 'a :: euclidean\_space\ set$

assumes *open S finite X* $p \in S$
shows $\exists e{>}0.\ \forall w{\in}cball\ p\ e.\ w{\in}S \wedge (w{\neq}p \longrightarrow w{\notin}X)$
proof −
obtain *e1* **where** $e1{>}0$ **and** $e1{:}\ \forall w{\in}ball\ p\ e1.\ w{\in}S \wedge (w{\neq}p \longrightarrow w{\notin}X)$
using *finite_ball_avoid*[*OF assms*] **by** *auto*
define *e2* **where** $e2 \equiv e1/2$
have $e2{>}0$ **and** $e2 < e1$ **unfolding** *e2_def* **using** ‹$e1{>}0$› **by** *auto*
then have *cball p e2* $\subseteq$ *ball p e1* **by** (*subst cball_subset_ball_iff*,*auto*)
then show $\exists e{>}0.\ \forall w{\in}cball\ p\ e.\ w \in S \wedge (w \neq p \longrightarrow w \notin X)$ **using** ‹$e2{>}0$› *e1* **by** *auto*
qed

lemma *dim_cball*:
assumes $e > 0$
shows *dim* (*cball* (0 :: 'n::*euclidean_space*) *e*) = *DIM*('n)
proof −
{
fix *x* :: 'n::*euclidean_space*
define *y* **where** $y = (e\ /\ norm\ x) *_R x$
then have $y \in cball\ 0\ e$
using *assms* **by** *auto*
moreover have ∗: $x = (norm\ x\ /\ e) *_R y$
using *y_def assms* **by** *simp*
moreover from ∗ **have** $x = (norm\ x/e) *_R y$
by *auto*
ultimately have $x \in span\ (cball\ 0\ e)$
using *span_scale*[*of y cball 0 e norm x*/*e*]
span_superset[*of cball 0 e*]
by (*simp add*: *span_base*)
}
then have *span* (*cball 0 e*) = (*UNIV* :: 'n::*euclidean_space set*)
by *auto*
then show *?thesis*
using *dim_span*[*of cball* (0 :: 'n::*euclidean_space*) *e*] **by** (*auto*)
qed

5.1.3 Boxes

abbreviation *One* :: 'a::*euclidean_space* **where**
$One \equiv \sum Basis$

lemma *One_non_0*: **assumes** *One* = (0::'a::*euclidean_space*) **shows** *False*
proof −
have *dependent* (*Basis* :: 'a *set*)
apply (*simp add*: *dependent_finite*)
apply (*rule_tac* $x{=}\lambda i.\ 1$ **in** *exI*)
using *SOME_Basis* **apply** (*auto simp*: *assms*)
done
with *independent_Basis* **show** *False* **by** *force*

qed

corollary *One_neq_0*[*iff*]: $One \neq 0$
by (*metis One_non_0*)

corollary *Zero_neq_One*[*iff*]: $0 \neq One$
by (*metis One_non_0*)

definition (**in** *euclidean_space*) *eucl_less* (**infix** ‹<e› *50*) **where**
$eucl\_less\ a\ b \longleftrightarrow (\forall i{\in}Basis.\ a \cdot i < b \cdot i)$

definition *box_eucl_less*: $box\ a\ b = \{x.\ a <e\ x \wedge x <e\ b\}$
definition $cbox\ a\ b = \{x.\ \forall i{\in}Basis.\ a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i\}$

lemma *box_def*: $box\ a\ b = \{x.\ \forall i{\in}Basis.\ a \cdot i < x \cdot i \wedge x \cdot i < b \cdot i\}$
and *in_box_eucl_less*: $x \in box\ a\ b \longleftrightarrow a <e\ x \wedge x <e\ b$
and *mem_box*: $x \in box\ a\ b \longleftrightarrow (\forall i{\in}Basis.\ a \cdot i < x \cdot i \wedge x \cdot i < b \cdot i)$
$x \in cbox\ a\ b \longleftrightarrow (\forall i{\in}Basis.\ a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i)$
by (*auto simp*: *box_eucl_less eucl_less_def cbox_def*)

lemma *cbox_Pair_eq*: $cbox\ (a,\ c)\ (b,\ d) = cbox\ a\ b \times cbox\ c\ d$
by (*force simp*: *cbox_def Basis_prod_def*)

lemma *cbox_Pair_iff* [*iff*]: $(x,\ y) \in cbox\ (a,\ c)\ (b,\ d) \longleftrightarrow x \in cbox\ a\ b \wedge y \in cbox\ c\ d$
by (*force simp*: *cbox_Pair_eq*)

lemma *cbox_Complex_eq*: $cbox\ (Complex\ a\ c)\ (Complex\ b\ d) = (\lambda(x,y).\ Complex\ x\ y)\ `\ (cbox\ a\ b \times cbox\ c\ d)$
by (*force simp*: *cbox_def Basis_complex_def*)

lemma *cbox_Pair_eq_0*: $cbox\ (a,\ c)\ (b,\ d) = \{\} \longleftrightarrow cbox\ a\ b = \{\} \vee cbox\ c\ d = \{\}$
by (*force simp*: *cbox_Pair_eq*)

lemma *swap_cbox_Pair* [*simp*]: $prod.swap\ `\ cbox\ (c,\ a)\ (d,\ b) = cbox\ (a,c)\ (b,d)$
by *auto*

lemma *mem_box_real*[*simp*]:
$(x{::}real) \in box\ a\ b \longleftrightarrow a < x \wedge x < b$
$(x{::}real) \in cbox\ a\ b \longleftrightarrow a \leq x \wedge x \leq b$
by (*auto simp*: *mem_box*)

lemma *box_real*[*simp*]:
fixes *a b*:: *real*
shows $box\ a\ b = \{a <..< b\}\ cbox\ a\ b = \{a\ ..\ b\}$
by *auto*

lemma *box_Int_box*:

```
  fixes a :: 'a::euclidean_space
  shows box a b ∩ box c d =
    box (∑ i∈Basis. max (a•i) (c•i) *R i) (∑ i∈Basis. min (b•i) (d•i) *R i)
  unfolding set_eq_iff and Int_iff and mem_box by auto

lemma rational_boxes:
  fixes x :: 'a::euclidean_space
  assumes e > 0
  shows ∃ a b. (∀ i∈Basis. a • i ∈ ℚ ∧ b • i ∈ ℚ) ∧ x ∈ box a b ∧ box a b ⊆ ball x
e
proof −
  define e' where e' = e / (2 * sqrt (real (DIM ('a))))
  then have e: e' > 0
    using assms by (auto)
  have ∃ y. y ∈ ℚ ∧ y < x • i ∧ x • i − y < e' for i
    using Rats_dense_in_real[of x • i − e' x • i] e by force
  then obtain a where
    a: ⋀u. a u ∈ ℚ ∧ a u < x • u ∧ x • u − a u < e' by metis
  have ∃ y. y ∈ ℚ ∧ x • i < y ∧ y − x • i < e' for i
    using Rats_dense_in_real[of x • i x • i + e'] e by force
  then obtain b where
    b: ⋀u. b u ∈ ℚ ∧ x • u < b u ∧ b u − x • u < e' by metis
  let ?a = ∑ i∈Basis. a i *R i and ?b = ∑ i∈Basis. b i *R i
  show ?thesis
  proof (rule exI[of _ ?a], rule exI[of _ ?b], safe)
    fix y :: 'a
    assume *: y ∈ box ?a ?b
    have dist x y = sqrt (∑ i∈Basis. (dist (x • i) (y • i))^2)
      unfolding L2_set_def[symmetric] by (rule euclidean_dist_l2)
    also have ... < sqrt (∑ (i::'a)∈Basis. e^2 / real (DIM('a)))
    proof (rule real_sqrt_less_mono, rule sum_strict_mono)
      fix i :: 'a
      assume i: i ∈ Basis
      have a i < y•i ∧ y•i < b i
        using * i by (auto simp: box_def)
      moreover have a i < x•i x•i − a i < e' x•i < b i b i − x•i < e'
        using a b by auto
      ultimately have |x•i − y•i| < 2 * e'
        by auto
      then have dist (x • i) (y • i) < e/sqrt (real (DIM('a)))
        unfolding e'_def by (auto simp: dist_real_def)
      then have (dist (x • i) (y • i))^2 < (e/sqrt (real (DIM('a))))^2
        by (rule power_strict_mono) auto
      then show (dist (x • i) (y • i))^2 < e^2 / real DIM('a)
        by (simp add: power_divide)
    qed auto
    also have ... = e
      using ‹0 < e› by simp
    finally show y ∈ ball x e
```

```
    by (auto simp: ball_def)
  qed (use a b in ‹auto simp: box_def›)
qed

lemma open_UNION_box:
  fixes M :: 'a::euclidean_space set
  assumes open M
  defines a' ≡ λf :: 'a ⇒ real × real. (∑(i::'a)∈Basis. fst (f i) *R i)
  defines b' ≡ λf :: 'a ⇒ real × real. (∑(i::'a)∈Basis. snd (f i) *R i)
  defines I ≡ {f∈Basis →E ℚ × ℚ. box (a' f) (b' f) ⊆ M}
  shows M = (⋃f∈I. box (a' f) (b' f))
proof −
  have x ∈ (⋃f∈I. box (a' f) (b' f)) if x ∈ M for x
  proof −
    obtain e where e: e > 0 ball x e ⊆ M
      using openE[OF ‹open M› ‹x ∈ M›] by auto
    moreover obtain a b where ab:
      x ∈ box a b
      ∀ i ∈ Basis. a · i ∈ ℚ
      ∀ i∈Basis. b · i ∈ ℚ
      box a b ⊆ ball x e
      using rational_boxes[OF e(1)] by metis
    ultimately show ?thesis
      by (intro UN_I[of λi∈Basis. (a · i, b · i)])
        (auto simp: euclidean_representation I_def a'_def b'_def)
  qed
  then show ?thesis by (auto simp: I_def)
qed

corollary open_countable_Union_open_box:
  fixes S :: 'a :: euclidean_space set
  assumes open S
  obtains 𝒟 where countable 𝒟 𝒟 ⊆ Pow S ⋀X. X ∈ 𝒟 ⟹ ∃ a b. X = box a b
⋃𝒟 = S
proof −
  let ?a = λf. (∑(i::'a)∈Basis. fst (f i) *R i)
  let ?b = λf. (∑(i::'a)∈Basis. snd (f i) *R i)
  let ?I = {f∈Basis →E ℚ × ℚ. box (?a f) (?b f) ⊆ S}
  let ?𝒟 = (λf. box (?a f) (?b f)) ‘ ?I
  show ?thesis
  proof
    have countable ?I
      by (simp add: countable_PiE countable_rat)
    then show countable ?𝒟
      by blast
    show ⋃?𝒟 = S
      using open_UNION_box [OF assms] by metis
  qed auto
qed
```

lemma *rational_cboxes*:
fixes $x :: 'a::euclidean\_space$
assumes $e > 0$
shows $\exists a\ b.\ (\forall i{\in}Basis.\ a \cdot i \in \mathbb{Q} \wedge b \cdot i \in \mathbb{Q}) \wedge x \in cbox\ a\ b \wedge cbox\ a\ b \subseteq ball\ x\ e$
proof −
define e' **where** $e' = e\ /\ (2 * sqrt\ (real\ (DIM\ ('a))))$
then have e': $e' > 0$
using *assms* **by** *auto*
have $\exists y.\ y \in \mathbb{Q} \wedge y < x \cdot i \wedge x \cdot i - y < e'$ **for** i
using *Rats_dense_in_real*[*of* $x \cdot i - e'\ x \cdot i$] e **by** *force*
then obtain a **where**
a: $\forall u.\ a\ u \in \mathbb{Q} \wedge a\ u < x \cdot u \wedge x \cdot u - a\ u < e'$ **by** *metis*
have $\exists y.\ y \in \mathbb{Q} \wedge x \cdot i < y \wedge y - x \cdot i < e'$ **for** i
using *Rats_dense_in_real*[*of* $x \cdot i\ x \cdot i + e'$] e **by** *force*
then obtain b **where**
b: $\forall u.\ b\ u \in \mathbb{Q} \wedge x \cdot u < b\ u \wedge b\ u - x \cdot u < e'$ **by** *metis*
let $?a = \sum i{\in}Basis.\ a\ i *_R i$ **and** $?b = \sum i{\in}Basis.\ b\ i *_R i$
show *?thesis*
proof (*rule exI*[*of* _ *?a*], *rule exI*[*of* _ *?b*], *safe*)
fix $y :: 'a$
assume $*$: $y \in cbox\ ?a\ ?b$
have $dist\ x\ y = sqrt\ (\sum i{\in}Basis.\ (dist\ (x \cdot i)\ (y \cdot i))^2)$
unfolding *L2_set_def*[*symmetric*] **by** (*rule euclidean_dist_l2*)
also have $\ldots < sqrt\ (\sum (i::'a){\in}Basis.\ e\hat{}2\ /\ real\ (DIM('a)))$
proof (*rule real_sqrt_less_mono*, *rule sum_strict_mono*)
fix $i :: 'a$
assume i: $i \in Basis$
have $a\ i \le y{\cdot}i \wedge y{\cdot}i \le b\ i$
using $*$ i **by** (*auto simp*: *cbox_def*)
moreover have $a\ i < x{\cdot}i\ x{\cdot}i - a\ i < e'\ x{\cdot}i < b\ i\ b\ i - x{\cdot}i < e'$
using a b **by** *auto*
ultimately have $|x{\cdot}i - y{\cdot}i| < 2 * e'$
by *auto*
then have $dist\ (x \cdot i)\ (y \cdot i) < e/sqrt\ (real\ (DIM('a)))$
unfolding e'*_def* **by** (*auto simp*: *dist_real_def*)
then have $(dist\ (x \cdot i)\ (y \cdot i))^2 < (e/sqrt\ (real\ (DIM('a))))^2$
by (*rule power_strict_mono*) *auto*
then show $(dist\ (x \cdot i)\ (y \cdot i))^2 < e^2\ /\ real\ DIM('a)$
by (*simp add*: *power_divide*)
qed *auto*
also have $\ldots = e$
using ‹$0 < e$› **by** *simp*
finally show $y \in ball\ x\ e$
by (*auto simp*: *ball_def*)
next
show $x \in cbox\ (\sum i{\in}Basis.\ a\ i *_R i)\ (\sum i{\in}Basis.\ b\ i *_R i)$
using a b *less_imp_le* **by** (*auto simp*: *cbox_def*)

qed (*use a b cbox_def* **in** *auto*)
qed

lemma *open_UNION_cbox*:
fixes $M :: 'a::euclidean\_space\ set$
assumes *open M*
defines $a' \equiv \lambda f.\ (\sum (i::'a)\in Basis.\ fst\ (f\ i) *_R i)$
defines $b' \equiv \lambda f.\ (\sum (i::'a)\in Basis.\ snd\ (f\ i) *_R i)$
defines $I \equiv \{f\in Basis \to_E \mathbb{Q} \times \mathbb{Q}.\ cbox\ (a'\ f)\ (b'\ f) \subseteq M\}$
shows $M = (\bigcup f\in I.\ cbox\ (a'\ f)\ (b'\ f))$
proof −
have $x \in (\bigcup f\in I.\ cbox\ (a'\ f)\ (b'\ f))$ **if** $x \in M$ **for** x
proof −
obtain e **where** e: $e > 0$ *ball x e* $\subseteq M$
using *openE*[*OF* ‹*open M*› ‹$x \in M$›] **by** *auto*
moreover obtain $a\ b$ **where** ab: $x \in cbox\ a\ b\ \forall i \in Basis.\ a \cdot i \in \mathbb{Q}$
$\forall i \in Basis.\ b \cdot i \in \mathbb{Q}\ cbox\ a\ b \subseteq ball\ x\ e$
using *rational_cboxes*[*OF e*(*1*)] **by** *metis*
ultimately show *?thesis*
by (*intro UN_I*[*of* $\lambda i\in Basis.\ (a \cdot i,\ b \cdot i)$])
(*auto simp*: *euclidean_representation I_def a'_def b'_def*)
qed
then show *?thesis* **by** (*auto simp*: *I_def*)
qed

corollary *open_countable_Union_open_cbox*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *open S*
obtains $\mathcal{D}$ **where** *countable* $\mathcal{D}$ $\mathcal{D} \subseteq Pow\ S$ $\bigwedge X.\ X \in \mathcal{D} \Longrightarrow \exists a\ b.\ X = cbox\ a\ b$ $\bigcup \mathcal{D} = S$
proof −
let $?a = \lambda f.\ (\sum (i::'a)\in Basis.\ fst\ (f\ i) *_R i)$
let $?b = \lambda f.\ (\sum (i::'a)\in Basis.\ snd\ (f\ i) *_R i)$
let $?I = \{f\in Basis \to_E \mathbb{Q} \times \mathbb{Q}.\ cbox\ (?a\ f)\ (?b\ f) \subseteq S\}$
let $?\mathcal{D} = (\lambda f.\ cbox\ (?a\ f)\ (?b\ f))\ `\ ?I$
show *?thesis*
proof
have *countable ?I*
by (*simp add*: *countable_PiE countable_rat*)
then show *countable ?*$\mathcal{D}$
by *blast*
show $\bigcup ?\mathcal{D} = S$
using *open_UNION_cbox* [*OF assms*] **by** *metis*
qed *auto*
qed

lemma *box_eq_empty*:
fixes $a :: 'a::euclidean\_space$
shows (*box a b* $= \{\} \longleftrightarrow (\exists i\in Basis.\ b{\cdot}i \le a{\cdot}i)$) (**is** *?th1*)

```
    and (cbox a b = {} ⟷ (∃ i∈Basis. b•i < a•i)) (is ?th2)
proof −
  have False if i ∈ Basis and b•i ≤ a•i and x ∈ box a b for i x
    by (smt (verit, ccfv_SIG) mem_box(1) that)
  moreover
  { assume as: ∀ i∈Basis. ¬ (b•i ≤ a•i)
    let ?x = (1/2) *R (a + b)
    { fix i :: 'a
      assume i: i ∈ Basis
      have a•i < b•i
        using as i by fastforce
      then have a•i < ((1/2) *R (a+b)) • i ((1/2) *R (a+b)) • i < b•i
        by (auto simp: inner_add_left)
    }
    then have box a b ≠ {}
      by (metis (no_types, opaque_lifting) emptyE mem_box(1))
  }
  ultimately show ?th1 by blast

  have False if i∈Basis and b•i < a•i and x ∈ cbox a b for i x
    using mem_box(2) that by force
  moreover
  have cbox a b ≠ {} if ∀ i∈Basis. ¬ (b•i < a•i)
    by (metis emptyE linorder_linear mem_box(2) order.strict_iff_not that)
  ultimately show ?th2 by blast
qed

lemma box_ne_empty:
  fixes a :: 'a::euclidean_space
  shows cbox a b ≠ {} ⟷ (∀ i∈Basis. a•i ≤ b•i)
  and box a b ≠ {} ⟷ (∀ i∈Basis. a•i < b•i)
  unfolding box_eq_empty[of a b] by fastforce+

lemma
  fixes a :: 'a::euclidean_space
  shows cbox_idem [simp]: cbox a a = {a}
    and box_idem [simp]: box a a = {}
  unfolding set_eq_iff mem_box eq_iff [symmetric] using euclidean_eq_iff by
fastforce+

lemma subset_box_imp:
  fixes a :: 'a::euclidean_space
  shows (∀ i∈Basis. a•i ≤ c•i ∧ d•i ≤ b•i) ⟹ cbox c d ⊆ cbox a b
    and (∀ i∈Basis. a•i < c•i ∧ d•i < b•i) ⟹ cbox c d ⊆ box a b
    and (∀ i∈Basis. a•i ≤ c•i ∧ d•i ≤ b•i) ⟹ box c d ⊆ cbox a b
     and (∀ i∈Basis. a•i ≤ c•i ∧ d•i ≤ b•i) ⟹ box c d ⊆ box a b
  unfolding subset_eq[unfolded Ball_def] unfolding mem_box
  by (best intro: order_trans less_le_trans le_less_trans less_imp_le)+
```

lemma *box_subset_cbox*:
fixes *a* :: *'a*::*euclidean_space*
shows *box a b* ⊆ *cbox a b*
unfolding *subset_eq* [*unfolded Ball_def*] *mem_box*
by (*fast intro*: *less_imp_le*)

lemma *subset_box*:
fixes *a* :: *'a*::*euclidean_space*
shows *cbox c d* ⊆ *cbox a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* ≤ *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th1*)
and *cbox c d* ⊆ *box a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* ≤ *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* < *c*•*i* ∧ *d*•*i* < *b*•*i*) (**is** *?th2*)
and *box c d* ⊆ *cbox a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* < *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th3*)
and *box c d* ⊆ *box a b* ⟷ (∀ *i*∈*Basis*. *c*•*i* < *d*•*i*) ⟶ (∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*) (**is** *?th4*)
proof −
let *?lesscd* = ∀ *i*∈*Basis*. *c*•*i* < *d*•*i*
let *?lerhs* = ∀ *i*∈*Basis*. *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*
show *?th1 ?th2*
by (*fastforce simp*: *mem_box*)+
have *acdb*: *a*•*i* ≤ *c*•*i* ∧ *d*•*i* ≤ *b*•*i*
if *i*: *i* ∈ *Basis* **and** *box*: *box c d* ⊆ *cbox a b* **and** *cd*: ⋀*i*. *i* ∈ *Basis* ⟹ *c*•*i* < *d*•*i* **for** *i*
proof −
have *box c d* ≠ {}
using *that*
unfolding *box_eq_empty* **by** *force*
{ let *?x* = (∑ *j*∈*Basis*. (*if j*=*i then* ((*min* (*a*•*j*) (*d*•*j*))+*c*•*j*)/*2 else* (*c*•*j*+*d*•*j*)/*2*) ∗*R* *j*)::*'a*
assume ∗: *a*•*i* > *c*•*i*
then have *c* • *j* < *?x* • *j* ∧ *?x* • *j* < *d* • *j* **if** *j* ∈ *Basis* **for** *j*
using *cd that* **by** (*fastforce simp add*: *i* ∗)
then have *?x* ∈ *box c d*
unfolding *mem_box* **by** *auto*
moreover have *?x* ∉ *cbox a b*
using *i cd* ∗ **by** (*force simp*: *mem_box*)
ultimately have *False* **using** *box* **by** *auto*
}
then have *a*•*i* ≤ *c*•*i* **by** *force*
moreover
{ let *?x* = (∑ *j*∈*Basis*. (*if j*=*i then* ((*max* (*b*•*j*) (*c*•*j*))+*d*•*j*)/*2 else* (*c*•*j*+*d*•*j*)/*2*) ∗*R* *j*)::*'a*
assume ∗: *b*•*i* < *d*•*i*
then have *d* • *j* > *?x* • *j* ∧ *?x* • *j* > *c* • *j* **if** *j* ∈ *Basis* **for** *j*
using *cd that* **by** (*fastforce simp add*: *i* ∗)
then have *?x* ∈ *box c d*
unfolding *mem_box* **by** *auto*
moreover have *?x* ∉ *cbox a b*

```
      using i cd * by (force simp: mem_box)
    ultimately have False using box by auto
  }
  then have b•i ≥ d•i by (rule ccontr) auto
  ultimately show ?thesis by auto
 qed
 show ?th3
  using acdb by (fastforce simp add: mem_box)
 have acdb': a•i ≤ c•i ∧ d•i ≤ b•i
  if i ∈ Basis box c d ⊆ box a b ⋀i. i ∈ Basis ⟹ c•i < d•i for i
   using box_subset_cbox[of a b] that acdb by auto
 show ?th4
  using acdb' by (fastforce simp add: mem_box)
qed

lemma eq_cbox: cbox a b = cbox c d ⟷ cbox a b = {} ∧ cbox c d = {} ∨ a =
c ∧ b = d
    (is ?lhs = ?rhs)
proof
 assume ?lhs
 then have cbox a b ⊆ cbox c d cbox c d ⊆ cbox a b
  by auto
 then show ?rhs
  by (force simp: subset_box box_eq_empty intro: antisym euclidean_eqI)
qed auto

lemma eq_cbox_box [simp]: cbox a b = box c d ⟷ cbox a b = {} ∧ box c d =
{}
 (is ?lhs ⟷ ?rhs)
proof
 assume L: ?lhs
 then have cbox a b ⊆ box c d box c d ⊆ cbox a b
  by auto
 with L subset_box show ?rhs
  by (smt (verit) SOME_Basis box_ne_empty(1))
qed force

lemma eq_box_cbox [simp]: box a b = cbox c d ⟷ box a b = {} ∧ cbox c d =
{}
 by (metis eq_cbox_box)

lemma eq_box: box a b = box c d ⟷ box a b = {} ∧ box c d = {} ∨ a = c ∧ b
= d
 (is ?lhs ⟷ ?rhs)
proof
 assume L: ?lhs
 then have box a b ⊆ box c d box c d ⊆ box a b
  by auto
 then show ?rhs
```

unfolding *subset_box* **by** (*smt* (*verit*) *box_ne_empty*(2) *euclidean_eq_iff*)+
qed *force*

lemma *subset_box_complex*:
cbox a b ⊆ *cbox c d* ⟷
(*Re a* ≤ *Re b* ∧ *Im a* ≤ *Im b*) ⟶ *Re a* ≥ *Re c* ∧ *Im a* ≥ *Im c* ∧ *Re b* ≤ *Re d* ∧ *Im b* ≤ *Im d*
cbox a b ⊆ *box c d* ⟷
(*Re a* ≤ *Re b* ∧ *Im a* ≤ *Im b*) ⟶ *Re a* > *Re c* ∧ *Im a* > *Im c* ∧ *Re b* < *Re d* ∧ *Im b* < *Im d*
box a b ⊆ *cbox c d* ⟷
(*Re a* < *Re b* ∧ *Im a* < *Im b*) ⟶ *Re a* ≥ *Re c* ∧ *Im a* ≥ *Im c* ∧ *Re b* ≤ *Re d* ∧ *Im b* ≤ *Im d*
box a b ⊆ *box c d* ⟷
(*Re a* < *Re b* ∧ *Im a* < *Im b*) ⟶ *Re a* ≥ *Re c* ∧ *Im a* ≥ *Im c* ∧ *Re b* ≤ *Re d* ∧ *Im b* ≤ *Im d*
by (*subst subset_box*; *force simp*: *Basis_complex_def*)+

lemma *in_cbox_complex_iff*:
x ∈ *cbox a b* ⟷ *Re x* ∈ {*Re a*..*Re b*} ∧ *Im x* ∈ {*Im a*..*Im b*}
by (*cases x*; *cases a*; *cases b*) (*auto simp*: *cbox_Complex_eq*)

lemma *cbox_complex_of_real*: *cbox* (*complex_of_real x*) (*complex_of_real y*) = *complex_of_real* ‘ {*x*..*y*}
proof −
have (*x* ≤ *Re z* ∧ *Re z* ≤ *y* ∧ *Im z* = *0*) = (*z* ∈ *complex_of_real* ‘ {*x*..*y*}) **for** *z*
by (*cases z*) (*simp add*: *complex_eq_cancel_iff2 image_iff*)
then show *?thesis*
by (*auto simp*: *in_cbox_complex_iff*)
qed

lemma *box_Complex_eq*:
box (*Complex a c*) (*Complex b d*) = (λ(*x*,*y*). *Complex x y*) ‘ (*box a b* × *box c d*)
by (*auto simp*: *box_def Basis_complex_def image_iff complex_eq_iff*)

lemma *in_box_complex_iff*:
x ∈ *box a b* ⟷ *Re x* ∈ {*Re a*<..<*Re b*} ∧ *Im x* ∈ {*Im a*<..<*Im b*}
by (*cases x*; *cases a*; *cases b*) (*auto simp*: *box_Complex_eq*)

lemma *box_complex_of_real* [*simp*]: *box* (*complex_of_real x*) (*complex_of_real y*) = {}
by (*auto simp*: *in_box_complex_iff*)

lemma *Int_interval*:
fixes *a* :: *'a*::*euclidean_space*
shows *cbox a b* ∩ *cbox c d* =
cbox (∑ *i*∈*Basis*. *max* (*a*•*i*) (*c*•*i*) ∗$_R$ *i*) (∑ *i*∈*Basis*. *min* (*b*•*i*) (*d*•*i*) ∗$_R$ *i*)
unfolding *set_eq_iff* **and** *Int_iff* **and** *mem_box*
by *auto*

lemma *disjoint_interval*:
fixes *a*::'*a*::*euclidean_space*
shows *cbox a b* ∩ *cbox c d* = {} ⟷ (∃ *i*∈*Basis*. (*b*•*i* < *a*•*i* ∨ *d*•*i* < *c*•*i* ∨ *b*•*i* < *c*•*i* ∨ *d*•*i* < *a*•*i*)) (**is** *?th1*)
and *cbox a b* ∩ *box c d* = {} ⟷ (∃ *i*∈*Basis*. (*b*•*i* < *a*•*i* ∨ *d*•*i* ≤ *c*•*i* ∨ *b*•*i* ≤ *c*•*i* ∨ *d*•*i* ≤ *a*•*i*)) (**is** *?th2*)
and *box a b* ∩ *cbox c d* = {} ⟷ (∃ *i*∈*Basis*. (*b*•*i* ≤ *a*•*i* ∨ *d*•*i* < *c*•*i* ∨ *b*•*i* ≤ *c*•*i* ∨ *d*•*i* ≤ *a*•*i*)) (**is** *?th3*)
and *box a b* ∩ *box c d* = {} ⟷ (∃ *i*∈*Basis*. (*b*•*i* ≤ *a*•*i* ∨ *d*•*i* ≤ *c*•*i* ∨ *b*•*i* ≤ *c*•*i* ∨ *d*•*i* ≤ *a*•*i*)) (**is** *?th4*)
proof −
let *?z* = (∑ *i*∈*Basis*. (((*max* (*a*•*i*) (*c*•*i*)) + (*min* (*b*•*i*) (*d*•*i*))) / *2*) *R* *i*)::'*a*
have **: ⋀*P Q*. (⋀*i* :: '*a*. *i* ∈ *Basis* ⟹ *Q ?z i* ⟹ *P i*) ⟹
(⋀*i x* :: '*a*. *i* ∈ *Basis* ⟹ *P i* ⟹ *Q x i*) ⟹ (∀ *x*. ∃ *i*∈*Basis*. *Q x i*) ⟷ (∃ *i*∈*Basis*. *P i*)
by *blast*
note * = *set_eq_iff Int_iff empty_iff mem_box ball_conj_distrib*[*symmetric*] *eq_False ball_simps*(*10*)
show *?th1* **unfolding** * **by** (*intro* **) *auto*
show *?th2* **unfolding** * **by** (*intro* **) *auto*
show *?th3* **unfolding** * **by** (*intro* **) *auto*
show *?th4* **unfolding** * **by** (*intro* **) *auto*
qed

lemma *UN_box_eq_UNIV*: (⋃ *i*::*nat*. *box* (− (*real i R One*)) (*real i R One*)) = *UNIV*
proof −
have |*x* • *b*| < *real_of_int* (⌈*Max* ((λ*b*. |*x* • *b*|) ‘*Basis*)⌉ + *1*)
if [*simp*]: *b* ∈ *Basis* **for** *x b* :: '*a*
proof −
have |*x* • *b*| ≤ *real_of_int* ⌈|*x* • *b*|⌉
by (*rule le_of_int_ceiling*)
also have ... ≤ *real_of_int* ⌈*Max* ((λ*b*. |*x* • *b*|) ‘*Basis*)⌉
by (*auto intro*!: *ceiling_mono*)
also have ... < *real_of_int* (⌈*Max* ((λ*b*. |*x* • *b*|) ‘*Basis*)⌉ + *1*)
by *simp*
finally show *?thesis* .
qed
then have ∃ *n*::*nat*. ∀ *b*∈*Basis*. |*x* • *b*| < *real n* **for** *x* :: '*a*
by (*metis order.strict_trans reals_Archimedean2*)
moreover have ⋀*x b*::'*a*. ⋀*n*::*nat*. |*x* • *b*| < *real n* ⟷ − *real n* < *x* • *b* ∧ *x* • *b* < *real n*
by *auto*
ultimately show *?thesis*
by (*auto simp*: *box_def inner_sum_left inner_Basis sum.If_cases*)
qed

lemma *image_affinity_cbox*: **fixes** *m*::*real*

```
  fixes a b c :: 'a::euclidean_space
  shows (λx. m *R x + c) ‘ cbox a b =
    (if cbox a b = {} then {}
     else (if 0 ≤ m then cbox (m *R a + c) (m *R b + c)
     else cbox (m *R b + c) (m *R a + c)))
proof (cases m = 0)
  case True
  {
    fix x
    assume ∀ i∈Basis. x · i ≤ c · i ∀ i∈Basis. c · i ≤ x · i
    then have x = c
      by (simp add: dual_order.antisym euclidean_eqI)
  }
  moreover have c ∈ cbox (m *R a + c) (m *R b + c)
    unfolding True by auto
  ultimately show ?thesis using True by (auto simp: cbox_def)
next
  case False
  {
    fix y
    assume ∀ i∈Basis. a · i ≤ y · i ∀ i∈Basis. y · i ≤ b · i m > 0
    then have ∀ i∈Basis. (m *R a + c) · i ≤ (m *R y + c) · i
          and ∀ i∈Basis. (m *R y + c) · i ≤ (m *R b + c) · i
      by (auto simp: inner_distrib)
  }
  moreover
  {
    fix y
    assume ∀ i∈Basis. a · i ≤ y · i ∀ i∈Basis. y · i ≤ b · i m < 0
    then have ∀ i∈Basis. (m *R b + c) · i ≤ (m *R y + c) · i
        and  ∀ i∈Basis. (m *R y + c) · i ≤ (m *R a + c) · i
      by (auto simp: mult_left_mono_neg inner_distrib)
  }
  moreover
  {
    fix y
    assume m > 0 and ∀ i∈Basis. (m *R a + c) · i ≤ y · i
      and  ∀ i∈Basis. y · i ≤ (m *R b + c) · i
    then have y ∈ (λx. m *R x + c) ‘ cbox a b
      unfolding image_iff Bex_def mem_box
      apply (intro exI[where x=(1 / m) *R (y − c)])
       apply (auto simp: pos_le_divide_eq pos_divide_le_eq mult.commute in-
ner_distrib inner_diff_left)
      done
  }
  moreover
  {
    fix y
    assume ∀ i∈Basis. (m *R b + c) · i ≤ y · i ∀ i∈Basis. y · i ≤ (m *R a + c) ·
```

```
i m < 0
    then have y ∈ (λx. m *R x + c) ‘ cbox a b
      unfolding image_iff Bex_def mem_box
      apply (intro exI[where x=(1 / m) *R (y − c)])
       apply (auto simp: neg_le_divide_eq neg_divide_le_eq mult.commute inner_distrib inner_diff_left)
      done
  }
  ultimately show ?thesis using False by (auto simp: cbox_def)
qed

lemma image_smult_cbox:(λx. m *R (x::_::euclidean_space)) ‘ cbox a b =
  (if cbox a b = {} then {} else if 0 ≤ m then cbox (m *R a) (m *R b) else cbox (m *R b) (m *R a))
  using image_affinity_cbox[of m 0 a b] by auto

lemma swap_continuous:
  assumes continuous_on (cbox (a,c) (b,d)) (λ(x,y). f x y)
    shows continuous_on (cbox (c,a) (d,b)) (λ(x, y). f y x)
proof −
  have (λ(x, y). f y x) = (λ(x, y). f x y) ∘ prod.swap
    by auto
  then show ?thesis
   by (metis assms continuous_on_compose continuous_on_swap swap_cbox_Pair)
qed

lemma open_contains_cbox:
  fixes x :: 'a :: euclidean_space
  assumes open A x ∈ A
  obtains a b where cbox a b ⊆ A x ∈ box a b ∀ i∈Basis. a • i < b • i
proof −
  from assms obtain R where R: R > 0 ball x R ⊆ A
    by (auto simp: open_contains_ball)
  define r :: real where r = R / (2 * sqrt DIM('a))
  from ‹R > 0› have [simp]: r > 0 by (auto simp: r_def)
  define d :: 'a where d = r *R Topology_Euclidean_Space.One
  have cbox (x − d) (x + d) ⊆ A
  proof safe
    fix y assume y: y ∈ cbox (x − d) (x + d)
    have dist x y = sqrt (∑ i∈Basis. (dist (x • i) (y • i))^2)
      by (subst euclidean_dist_l2) (auto simp: L2_set_def)
   also from y have sqrt (∑ i∈Basis. (dist (x • i) (y • i))^2) ≤ sqrt (∑ i∈(Basis::'a set). r^2)
      by (intro real_sqrt_le_mono sum_mono power_mono)
         (auto simp: dist_norm d_def cbox_def algebra_simps)
    also have ... = sqrt (DIM('a) * r^2) by simp
    also have DIM('a) * r^2 = (R / 2) ^ 2
      by (simp add: r_def power_divide)
    also have sqrt ... = R / 2
```

```
    using ‹R > 0› by simp
  also from ‹R > 0› have ... < R by simp
  finally have y ∈ ball x R by simp
  with R show y ∈ A by blast
 qed
 thus ?thesis
  using that[of x − d x + d] by (auto simp: algebra_simps d_def box_def)
qed

lemma open_contains_box:
 fixes x :: 'a :: euclidean_space
 assumes open A x ∈ A
 obtains a b where box a b ⊆ A x ∈ box a b ∀ i∈Basis. a • i < b • i
 by (meson assms box_subset_cbox dual_order.trans open_contains_cbox)

lemma inner_image_box:
 assumes (i :: 'a :: euclidean_space) ∈ Basis
 assumes ∀ i∈Basis. a • i < b • i
 shows   (λx. x • i) ‘ box a b = {a • i<..<b • i}
proof safe
 fix x assume x: x ∈ {a • i<..<b • i}
 let ?y = (∑ j∈Basis. (if i = j then x else (a + b) • j / 2) *R j)
 from x assms have ?y • i ∈ (λx. x • i) ‘ box a b
  by (intro imageI) (auto simp: box_def algebra_simps)
 also have ?y • i = (∑ j∈Basis. (if i = j then x else (a + b) • j / 2) * (j • i))
  by (simp add: inner_sum_left)
 also have ... = (∑ j∈Basis. if i = j then x else 0)
  by (intro sum.cong) (auto simp: inner_not_same_Basis assms)
 also have ... = x using assms by simp
 finally show x ∈ (λx. x • i) ‘ box a b  .
qed (insert assms, auto simp: box_def)

lemma inner_image_cbox:
 assumes (i :: 'a :: euclidean_space) ∈ Basis
 assumes ∀ i∈Basis. a • i ≤ b • i
 shows   (λx. x • i) ‘ cbox a b = {a • i..b • i}
proof safe
 fix x assume x: x ∈ {a • i..b • i}
 let ?y = (∑ j∈Basis. (if i = j then x else a • j) *R j)
 from x assms have ?y • i ∈ (λx. x • i) ‘ cbox a b
  by (intro imageI) (auto simp: cbox_def)
 also have ?y • i = (∑ j∈Basis. (if i = j then x else a • j) * (j • i))
  by (simp add: inner_sum_left)
 also have ... = (∑ j∈Basis. if i = j then x else 0)
  by (intro sum.cong) (auto simp: inner_not_same_Basis assms)
 also have ... = x using assms by simp
 finally show x ∈ (λx. x • i) ‘ cbox a b  .
qed (insert assms, auto simp: cbox_def)
```

5.1.4 General Intervals

definition *is_interval* (*s*::('*a*::*euclidean_space*) *set*) ⟷
(∀ *a*∈*s*. ∀ *b*∈*s*. ∀ *x*. (∀ *i*∈*Basis*. ((*a*•*i* ≤ *x*•*i* ∧ *x*•*i* ≤ *b*•*i*) ∨ (*b*•*i* ≤ *x*•*i* ∧ *x*•*i* ≤ *a*•*i*))) ⟶ *x* ∈ *s*)

lemma *is_interval_1*:
is_interval (*s*::*real set*) ⟷ (∀ *a*∈*s*. ∀ *b*∈*s*. ∀ *x*. *a* ≤ *x* ∧ *x* ≤ *b* ⟶ *x* ∈ *s*)
unfolding *is_interval_def* **by** *auto*

lemma *is_interval_Int*: *is_interval X* ⟹ *is_interval Y* ⟹ *is_interval* (*X* ∩ *Y*)
unfolding *is_interval_def*
by *blast*

lemma *is_interval_cbox* [*simp*]: *is_interval* (*cbox a* (*b*::'*a*::*euclidean_space*)) (**is** *?th1*)
and *is_interval_box* [*simp*]: *is_interval* (*box a b*) (**is** *?th2*)
unfolding *is_interval_def mem_box Ball_def atLeastAtMost_iff*
by (*meson order_trans le_less_trans less_le_trans less_trans*)+

lemma *is_interval_empty* [*iff*]: *is_interval* {}
unfolding *is_interval_def* **by** *simp*

lemma *is_interval_univ* [*iff*]: *is_interval UNIV*
unfolding *is_interval_def* **by** *simp*

lemma *mem_is_intervalI*:
assumes *is_interval S*
and *a* ∈ *S b* ∈ *S*
and ⋀*i*. *i* ∈ *Basis* ⟹ *a* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *b* • *i* ∨ *b* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *a* • *i*
shows *x* ∈ *S*
using *assms is_interval_def* **by** *force*

lemma *interval_subst*:
fixes *S*::'*a*::*euclidean_space set*
assumes *is_interval S*
and *x* ∈ *S y j* ∈ *S*
and *j* ∈ *Basis*
shows (∑ *i*∈*Basis*. (*if i* = *j then y i* • *i else x* • *i*) ∗$_R$ *i*) ∈ *S*
by (*rule mem_is_intervalI*[*OF assms*(1,2)]) (*auto simp*: *assms*)

lemma *mem_box_componentwiseI*:
fixes *S*::'*a*::*euclidean_space set*
assumes *is_interval S*
assumes ⋀*i*. *i* ∈ *Basis* ⟹ *x* • *i* ∈ ((λ*x*. *x* • *i*) ‘ *S*)
shows *x* ∈ *S*
proof −
from *assms* **have** ∀ *i* ∈ *Basis*. ∃ *s* ∈ *S*. *x* • *i* = *s* • *i*

```
    by auto
  with finite_Basis obtain s and bs::'a list
    where s: ⋀i. i ∈ Basis ⟹ x • i = s i • i ⋀i. i ∈ Basis ⟹ s i ∈ S
      and bs: set bs = Basis distinct bs
    by (metis finite_distinct_list)
  from nonempty_Basis s obtain j where j: j ∈ Basis s j ∈ S
    by blast
  define y where
    y = rec_list (s j) (λj _ Y. (∑ i∈Basis. (if i = j then s i • i else Y • i) *R i))
  have x = (∑ i∈Basis. (if i ∈ set bs then s i • i else s j • i) *R i)
    using bs by (auto simp: s(1)[symmetric] euclidean_representation)
  also have [symmetric]: y bs = ...
    using bs(2) bs(1)[THEN equalityD1]
   by (induct bs) (auto simp: y_def euclidean_representation intro!: euclidean_eqI[where
'a='a])
  also have y bs ∈ S
    using bs(1)[THEN equalityD1]
  proof (induction bs)
    case Nil
    then show ?case
      by (simp add: j y_def)
  next
    case (Cons a bs)
    then show ?case
      using interval_subst[OF assms(1)] s by (simp add: y_def)
  qed
  finally show ?thesis .
qed

lemma cbox01_nonempty [simp]: cbox 0 One ≠ {}
  by (simp add: box_ne_empty inner_Basis inner_sum_left sum_nonneg)

lemma box01_nonempty [simp]: box 0 One ≠ {}
  by (simp add: box_ne_empty inner_Basis inner_sum_left)

lemma empty_as_interval: {} = cbox One (0::'a::euclidean_space)
   using nonempty_Basis box01_nonempty box_eq_empty(1) box_ne_empty(1)
by blast

lemma interval_subset_is_interval:
  assumes is_interval S
  shows cbox a b ⊆ S ⟷ cbox a b = {} ∨ a ∈ S ∧ b ∈ S (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs  using box_ne_empty(1) mem_box(2) by fastforce
next
  assume ?rhs
  have cbox a b ⊆ S if a ∈ S b ∈ S
    using assms that
```

```
    by (force simp: mem_box intro: mem_is_intervalI)
  with ‹?rhs› show ?lhs
    by blast
qed

lemma is_real_interval_union:
  is_interval (X ∪ Y)
  if X: is_interval X and Y: is_interval Y and I: (X ≠ {} ⟹ Y ≠ {} ⟹ X
∩ Y ≠ {})
  for X Y::real set
proof −
  consider X ≠ {} Y ≠ {} | X = {} | Y = {} by blast
  then show ?thesis
  proof cases
    case 1
    then obtain r where r ∈ X ∨ X ∩ Y = {} r ∈ Y ∨ X ∩ Y = {}
      by blast
    then show ?thesis
      using I 1 X Y unfolding is_interval_1
      by (metis (full_types) Un_iff le_cases)
  qed (use that in auto)
qed

lemma is_interval_translationI:
  assumes is_interval X
  shows is_interval ((+) x ‘ X)
  unfolding is_interval_def
proof safe
  fix b d e
  assume b ∈ X d ∈ X
    ∀ i∈Basis. (x + b) · i ≤ e · i ∧ e · i ≤ (x + d) · i ∨
       (x + d) · i ≤ e · i ∧ e · i ≤ (x + b) · i
  hence e − x ∈ X
    by (intro mem_is_intervalI[OF assms ‹b ∈ X› ‹d ∈ X›, of e − x])
      (auto simp: algebra_simps)
  thus e ∈ (+) x ‘ X by force
qed

lemma is_interval_uminusI:
  assumes is_interval X
  shows is_interval (uminus ‘ X)
  unfolding is_interval_def
proof safe
  fix b d e
  assume b ∈ X d ∈ X
    ∀ i∈Basis. (− b) · i ≤ e · i ∧ e · i ≤ (− d) · i ∨
       (− d) · i ≤ e · i ∧ e · i ≤ (− b) · i
  hence − e ∈ X
    by (smt (verit, ccfv_threshold) assms inner_minus_left mem_is_intervalI)
```

thus $e \in \mathit{uminus}\ `\ X$ **by** *force*
qed

lemma *is_interval_uminus*[*simp*]: *is_interval* (*uminus* ` *x*) = *is_interval x*
using *is_interval_uminusI*[*of x*] *is_interval_uminusI*[*of uminus* ` *x*]
by (*auto simp*: *image_image*)

lemma *is_interval_neg_translationI*:
assumes *is_interval X*
shows *is_interval* ((−) *x* ` *X*)
proof −
have (−) *x* ` *X* = (+) *x* ` *uminus* ` *X*
by (*force simp*: *algebra_simps*)
also have *is_interval* ...
by (*metis is_interval_uminusI is_interval_translationI assms*)
finally show *?thesis* **.**
qed

lemma *is_interval_translation*[*simp*]:
is_interval ((+) *x* ` *X*) = *is_interval X*
using *is_interval_neg_translationI*[*of* (+) *x* ` *X x*]
by (*auto intro*!: *is_interval_translationI simp*: *image_image*)

lemma *is_interval_minus_translation*[*simp*]:
shows *is_interval* ((−) *x* ` *X*) = *is_interval X*
proof −
have (−) *x* ` *X* = (+) *x* ` *uminus* ` *X*
by (*force simp*: *algebra_simps*)
also have *is_interval* ... = *is_interval X*
by *simp*
finally show *?thesis* **.**
qed

lemma *is_interval_minus_translation'*[*simp*]:
shows *is_interval* ((λ*x*. *x* − *c*) ` *X*) = *is_interval X*
using *is_interval_translation*[*of* −*c X*]
by (*metis image_cong uminus_add_conv_diff*)

lemma *is_interval_cball_1*[*intro*, *simp*]: *is_interval* (*cball a b*) **for** *a b*::*real*
by (*simp add*: *cball_eq_atLeastAtMost is_interval_def*)

lemma *is_interval_ball_real*: *is_interval* (*ball a b*) **for** *a b*::*real*
by (*simp add*: *ball_eq_greaterThanLessThan is_interval_def*)

5.1.5 Bounded Projections

lemma *bounded_inner_imp_bdd_above*:
assumes *bounded s*
shows *bdd_above* ((λ*x*. *x* · *a*) ` *s*)

by (*simp add*: *assms bounded_imp_bdd_above bounded_linear_image bounded_linear_inner_left*)

lemma *bounded_inner_imp_bdd_below*:
assumes *bounded s*
shows *bdd_below* ((λ*x*. *x* • *a*) ‘ *s*)
by (*simp add*: *assms bounded_imp_bdd_below bounded_linear_image bounded_linear_inner_left*)

5.1.6 Structural rules for pointwise continuity

lemma *continuous_infnorm*[*continuous_intros*]:
continuous F f ⟹ *continuous F* (λ*x*. *infnorm* (*f x*))
unfolding *continuous_def* **by** (*rule tendsto_infnorm*)

lemma *continuous_inner*[*continuous_intros*]:
assumes *continuous F f*
and *continuous F g*
shows *continuous F* (λ*x*. *inner* (*f x*) (*g x*))
using *assms* **unfolding** *continuous_def* **by** (*rule tendsto_inner*)

5.1.7 Structural rules for setwise continuity

lemma *continuous_on_infnorm*[*continuous_intros*]:
continuous_on s f ⟹ *continuous_on s* (λ*x*. *infnorm* (*f x*))
unfolding *continuous_on* **by** (*fast intro*: *tendsto_infnorm*)

lemma *continuous_on_inner*[*continuous_intros*]:
fixes *g* :: *'a*::*topological_space* ⇒ *'b*::*real_inner*
assumes *continuous_on s f*
and *continuous_on s g*
shows *continuous_on s* (λ*x*. *inner* (*f x*) (*g x*))
using *bounded_bilinear_inner assms*
by (*rule bounded_bilinear.continuous_on*)

5.1.8 Openness of halfspaces.

lemma *open_halfspace_lt*: *open* {*x*. *inner a x* < *b*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *open_halfspace_gt*: *open* {*x*. *inner a x* > *b*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *open_halfspace_component_lt*: *open* {*x*::*'a*::*euclidean_space*. *x*•*i* < *a*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *open_halfspace_component_gt*: *open* {*x*::*'a*::*euclidean_space*. *x*•*i* > *a*}
by (*simp add*: *open_Collect_less continuous_on_inner*)

lemma *eucl_less_eq_halfspaces*:
fixes *a* :: *'a*::*euclidean_space*
shows {*x*. *x* <*e* *a*} = (⋂ *i*∈*Basis*. {*x*. *x* • *i* < *a* • *i*})

$\{x.\ a <e\ x\} = (\bigcap i{\in}Basis.\ \{x.\ a \cdot i < x \cdot i\})$
by (*auto simp*: *eucl_less_def*)

lemma *open_Collect_eucl_less*[*simp*, *intro*]:
fixes $a :: 'a{::}euclidean\_space$
shows *open* $\{x.\ x <e\ a\}$ *open* $\{x.\ a <e\ x\}$
by (*auto simp*: *eucl_less_eq_halfspaces open_halfspace_component_lt open_halfspace_component_gt*)

5.1.9 Closure and Interior of halfspaces and hyperplanes

lemma *continuous_at_inner*: *continuous* (*at* x) (*inner* a)
unfolding *continuous_at* **by** (*intro tendsto_intros*)

lemma *closed_halfspace_le*: *closed* $\{x.\ inner\ a\ x \leq b\}$
by (*simp add*: *closed_Collect_le continuous_on_inner*)

lemma *closed_halfspace_ge*: *closed* $\{x.\ inner\ a\ x \geq b\}$
by (*simp add*: *closed_Collect_le continuous_on_inner*)

lemma *closed_hyperplane*: *closed* $\{x.\ inner\ a\ x = b\}$
by (*simp add*: *closed_Collect_eq continuous_on_inner*)

lemma *closed_halfspace_component_le*: *closed* $\{x{::}'a{::}euclidean\_space.\ x{\cdot}i \leq a\}$
by (*simp add*: *closed_Collect_le continuous_on_inner*)

lemma *closed_halfspace_component_ge*: *closed* $\{x{::}'a{::}euclidean\_space.\ x{\cdot}i \geq a\}$
by (*simp add*: *closed_Collect_le continuous_on_inner*)

lemma *closed_interval_left*:
fixes $b :: 'a{::}euclidean\_space$
shows *closed* $\{x{::}'a.\ \forall i{\in}Basis.\ x{\cdot}i \leq b{\cdot}i\}$
by (*simp add*: *Collect_ball_eq closed_INT closed_Collect_le continuous_on_inner*)

lemma *closed_interval_right*:
fixes $a :: 'a{::}euclidean\_space$
shows *closed* $\{x{::}'a.\ \forall i{\in}Basis.\ a{\cdot}i \leq x{\cdot}i\}$
by (*simp add*: *Collect_ball_eq closed_INT closed_Collect_le continuous_on_inner*)

lemma *interior_halfspace_le* [*simp*]:
assumes $a \neq 0$
shows *interior* $\{x.\ a \cdot x \leq b\} = \{x.\ a \cdot x < b\}$
proof −
have $*$: $a \cdot x < b$ **if** x: $x \in S$ **and** S: $S \subseteq \{x.\ a \cdot x \leq b\}$ **and** *open* S **for** $S\ x$
proof −
obtain e **where** $e{>}0$ **and** e: *cball* $x\ e \subseteq S$
using ‹*open* S› *open_contains_cball* x **by** *blast*
then have $x + (e\ /\ norm\ a) *_R\ a \in cball\ x\ e$
by (*simp add*: *dist_norm*)
then have $x + (e\ /\ norm\ a) *_R\ a \in S$

```
    using e by blast
  then have x + (e / norm a) *R a ∈ {x. a • x ≤ b}
    using S by blast
  moreover have e * (a • a) / norm a > 0
    by (simp add: ‹0 < e› assms)
  ultimately show ?thesis
    by (simp add: algebra_simps)
 qed
 show ?thesis
  by (rule interior_unique) (auto simp: open_halfspace_lt *)
qed

lemma interior_halfspace_ge [simp]:
  a ≠ 0 ⟹ interior {x. a • x ≥ b} = {x. a • x > b}
using interior_halfspace_le [of −a −b] by simp

lemma closure_halfspace_lt [simp]:
 assumes a ≠ 0
  shows closure {x. a • x < b} = {x. a • x ≤ b}
proof −
 have [simp]: −{x. a • x < b} = {x. a • x ≥ b}
  by force
 then show ?thesis
  using interior_halfspace_ge [of a b] assms
  by (force simp: closure_interior)
qed

lemma closure_halfspace_gt [simp]:
  a ≠ 0 ⟹ closure {x. a • x > b} = {x. a • x ≥ b}
using closure_halfspace_lt [of −a −b] by simp

lemma interior_hyperplane [simp]:
 assumes a ≠ 0
  shows interior {x. a • x = b} = {}
proof −
 have [simp]: {x. a • x = b} = {x. a • x ≤ b} ∩ {x. a • x ≥ b}
  by force
 then show ?thesis
  by (auto simp: assms)
qed

lemma frontier_halfspace_le:
 assumes a ≠ 0 ∨ b ≠ 0
  shows frontier {x. a • x ≤ b} = {x. a • x = b}
proof (cases a = 0)
 case True with assms show ?thesis by simp
next
 case False then show ?thesis
  by (force simp: frontier_def closed_halfspace_le)
```

qed

lemma *frontier_halfspace_ge*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x \geq b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier_def closed_halfspace_ge*)
qed

lemma *frontier_halfspace_lt*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x < b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier_def interior_open open_halfspace_lt*)
qed

lemma *frontier_halfspace_gt*:
assumes $a \neq 0 \vee b \neq 0$
shows *frontier* $\{x.\ a \cdot x > b\} = \{x.\ a \cdot x = b\}$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis* **by** *simp*
next
case *False* **then show** *?thesis*
by (*force simp*: *frontier_def interior_open open_halfspace_gt*)
qed

5.1.10 Some more convenient intermediate-value theorem formulations

lemma *connected_ivt_hyperplane*:
assumes *connected* S **and** *xy*: $x \in S$ $y \in S$ **and** b: *inner* a $x \leq b$ $b \leq$ *inner* a y
shows $\exists z \in S.$ *inner* a $z = b$
proof (*rule ccontr*)
assume *as*:$\neg$ $(\exists z{\in}S.$ *inner* a $z = b)$
let *?A* $= \{x.$ *inner* a $x < b\}$
let *?B* $= \{x.$ *inner* a $x > b\}$
have *open ?A open ?B*
using *open_halfspace_lt* **and** *open_halfspace_gt* **by** *auto*
moreover have *?A* $\cap$ *?B* $= \{\}$ **by** *auto*
moreover have $S \subseteq$ *?A* $\cup$ *?B* **using** *as* **by** *auto*
ultimately show *False*
using ‹*connected S*› **unfolding** *connected_def*
by (*smt* (*verit*, *del_insts*) *as b disjoint_iff empty_iff mem_Collect_eq xy*)

qed

lemma *connected_ivt_component*:
fixes *x*::*'a*::*euclidean_space*
shows *connected S* $\Longrightarrow$ $x \in S \Longrightarrow y \in S \Longrightarrow x \cdot k \leq a \Longrightarrow a \leq y \cdot k \Longrightarrow (\exists z{\in}S.\ z \cdot k = a)$
using *connected_ivt_hyperplane*[*of S x y k*::*'a a*]
by (*auto simp*: *inner_commute*)

5.1.11 Limit Component Bounds

lemma *Lim_component_le*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::*euclidean_space*
assumes ($f \longrightarrow l$) *net*
and $\neg$ (*trivial_limit net*)
and *eventually* ($\lambda x.\ f(x) \cdot i \leq b$) *net*
shows $l \cdot i \leq b$
by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_const tendsto_inner*[*OF assms*(*1*) *tendsto_const*] *assms*(*3*)])

lemma *Lim_component_ge*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::*euclidean_space*
assumes ($f \longrightarrow l$) *net*
and $\neg$ (*trivial_limit net*)
and *eventually* ($\lambda x.\ b \leq (f\ x) \cdot i$) *net*
shows $b \leq l \cdot i$
by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_inner*[*OF assms*(*1*) *tendsto_const*] *tendsto_const assms*(*3*)])

lemma *Lim_component_eq*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::*euclidean_space*
assumes *net*: ($f \longrightarrow l$) *net* $\neg$ *trivial_limit net*
and *ev*:*eventually* ($\lambda x.\ f(x) \cdot i = b$) *net*
shows $l \cdot i = b$
using *ev*[*unfolded order_eq_iff eventually_conj_iff*]
using *Lim_component_ge*[*OF net*, *of b i*]
using *Lim_component_le*[*OF net*, *of i b*]
by *auto*

lemma *open_box*[*intro*]: *open* (*box a b*)
proof −
have *open* ($\bigcap i{\in}Basis.\ ((\cdot)\ i) -` \{a \cdot i <..< b \cdot i\}$)
by (*auto intro*!: *continuous_open_vimage continuous_inner continuous_ident continuous_const*)
also have ($\bigcap i{\in}Basis.\ ((\cdot)\ i) -` \{a \cdot i <..< b \cdot i\}$) = *box a b*
by (*auto simp*: *box_def inner_commute*)
finally show *?thesis* .
qed

lemma *closed_cbox*[*intro*]:
fixes *a b* :: *'a*::*euclidean_space*
shows *closed* (*cbox a b*)
proof −
have *closed* (⋂ *i*∈*Basis*. (λ*x*. *x*•*i*) −‘ {*a*•*i* .. *b*•*i*})
by (*intro closed_INT ballI continuous_closed_vimage allI*
linear_continuous_at closed_real_atLeastAtMost finite_Basis bounded_linear_inner_left)
also have (⋂ *i*∈*Basis*. (λ*x*. *x*•*i*) −‘ {*a*•*i* .. *b*•*i*}) = *cbox a b*
by (*auto simp*: *cbox_def*)
finally show *closed* (*cbox a b*) **.**
qed

lemma *interior_cbox* [*simp*]:
fixes *a b* :: *'a*::*euclidean_space*
shows *interior* (*cbox a b*) = *box a b* (**is** *?L* = *?R*)
proof(*rule subset_antisym*)
show *?R* ⊆ *?L*
using *box_subset_cbox open_box*
by (*rule interior_maximal*)
{
fix *x*
assume *x* ∈ *interior* (*cbox a b*)
then obtain *s* **where** *s*: *open s x* ∈ *s s* ⊆ *cbox a b* **..**
then obtain *e* **where** *e*>*0* **and** *e*:∀ *x'*. *dist x' x* < *e* ⟶ *x'* ∈ *cbox a b*
unfolding *open_dist* **and** *subset_eq* **by** *auto*
{
fix *i* :: *'a*
assume *i*: *i* ∈ *Basis*
have *dist* (*x* − (*e* / *2*) ∗*R i*) *x* < *e*
and *dist* (*x* + (*e* / *2*) ∗*R i*) *x* < *e*
using *norm_Basis*[*OF i*] ‹*e*>*0*› **by** (*auto simp*: *dist_norm*)
then have *a* • *i* ≤ (*x* − (*e* / *2*) ∗*R i*) • *i* **and** (*x* + (*e* / *2*) ∗*R i*) • *i* ≤ *b* • *i*
using *e*[*THEN spec*[**where** *x*=*x* − (*e*/*2*) ∗*R i*]]
and *e*[*THEN spec*[**where** *x*=*x* + (*e*/*2*) ∗*R i*]]
unfolding *mem_box* **using** *i* **by** *blast*+
then have *a* • *i* < *x* • *i* **and** *x* • *i* < *b* • *i*
using ‹*e*>*0*› *i*
by (*auto simp*: *inner_diff_left inner_Basis inner_add_left*)
}
then have *x* ∈ *box a b*
unfolding *mem_box* **by** *auto*
}
then show *?L* ⊆ *?R* **..**
qed

lemma *bounded_cbox* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows *bounded* (*cbox a b*)
proof −

```
  let ?b = ∑ i∈Basis. |a•i| + |b•i|
  {
    fix x :: 'a
    assume ⋀i. i∈Basis ⟹ a • i ≤ x • i ∧ x • i ≤ b • i
    then have (∑ i∈Basis. |x • i|) ≤ ?b
      by (force simp: intro!: sum_mono)
    then have norm x ≤ ?b
      using norm_le_l1[of x] by auto
  }
  then show ?thesis
    unfolding cbox_def bounded_iff by force
qed

lemma bounded_box [simp]:
  fixes a :: 'a::euclidean_space
  shows bounded (box a b)
  by (metis bounded_cbox bounded_interior interior_cbox)

lemma not_interval_UNIV [simp]:
  fixes a :: 'a::euclidean_space
  shows cbox a b ≠ UNIV box a b ≠ UNIV
  using bounded_box[of a b] bounded_cbox[of a b] by force+

lemma not_interval_UNIV2 [simp]:
  fixes a :: 'a::euclidean_space
  shows UNIV ≠ cbox a b UNIV ≠ box a b
  using bounded_box[of a b] bounded_cbox[of a b] by force+

lemma box_midpoint:
  fixes a :: 'a::euclidean_space
  assumes box a b ≠ {}
  shows ((1/2) *R (a + b)) ∈ box a b
proof −
  have a • i < ((1 / 2) *R (a + b)) • i ∧ ((1 / 2) *R (a + b)) • i < b • i if i ∈
Basis for i
    using assms that by (auto simp: inner_add_left box_ne_empty)
  then show ?thesis unfolding mem_box by auto
qed

lemma open_cbox_convex:
  fixes x :: 'a::euclidean_space
  assumes x: x ∈ box a b
    and y: y ∈ cbox a b
    and e: 0 < e e ≤ 1
  shows (e *R x + (1 − e) *R y) ∈ box a b
proof −
  {
    fix i :: 'a
    assume i: i ∈ Basis
```

```
    have a · i = e * (a · i) + (1 − e) * (a · i)
      unfolding left_diff_distrib by simp
    also have ... < e * (x · i) + (1 − e) * (y · i)
      by (smt (verit, best) e i mem_box mult_le_cancel_left_pos mult_left_mono
x y)
    finally have a · i < (e *R x + (1 − e) *R y) · i
      unfolding inner_simps by auto
    moreover
    {
      have b · i = e * (b·i) + (1 − e) * (b·i)
        unfolding left_diff_distrib by simp
      also have ... > e * (x · i) + (1 − e) * (y · i)
       by (smt (verit, best) e i mem_box mult_le_cancel_left_pos mult_left_mono
x y)
      finally have (e *R x + (1 − e) *R y) · i < b · i
        unfolding inner_simps by auto
    }
    ultimately have a · i < (e *R x + (1 − e) *R y) · i ∧ (e *R x + (1 − e) *R
y) · i < b · i
      by auto
  }
  then show ?thesis
    unfolding mem_box by auto
qed

lemma closure_cbox [simp]: closure (cbox a b) = cbox a b
  by (simp add: closed_cbox)

lemma closure_box [simp]:
  fixes a :: 'a::euclidean_space
   assumes box a b ≠ {}
  shows closure (box a b) = cbox a b
proof −
  have ab: a <e b
    using assms by (simp add: eucl_less_def box_ne_empty)
  let ?c = (1 / 2) *R (a + b)
  {
    fix x
    assume as: x ∈ cbox a b
    define f where [abs_def]: f n = x + (inverse (real n + 1)) *R (?c − x) for n
    {
      fix n
      assume fn: f n <e b ⟶ a <e f n ⟶ f n = x and xc: x ≠ ?c
      have *: 0 < inverse (real n + 1) inverse (real n + 1) ≤ 1
        unfolding inverse_le_1_iff by auto
      have (inverse (real n + 1)) *R ((1 / 2) *R (a + b)) + (1 − inverse (real n
+ 1)) *R x =
        x + (inverse (real n + 1)) *R (((1 / 2) *R (a + b)) − x)
        by (auto simp: algebra_simps)
```

```
    then have f n <e b and a <e f n
      using open_cbox_convex[OF box_midpoint[OF assms] as *]
      unfolding f_def by (auto simp: box_def eucl_less_def)
    then have False
      using fn unfolding f_def using xc by auto
  }
  moreover
  {
    have ∃N::nat. ∀n≥N. inverse (real n + 1) < ε if ε > 0 for ε
      using reals_Archimedean [of ε] that
      by (metis inverse_inverse_eq inverse_less_imp_less nat_le_real_less
order_less_trans
          reals_Archimedean2)
    then have (λn. inverse (real n + 1)) ⟶ 0
      unfolding lim_sequentially by(auto simp: dist_norm)
    then have f ⟶ x
      unfolding f_def
      using tendsto_add[OF tendsto_const, of λn. (inverse (real n + 1)) *R ((1
/ 2) *R (a + b) − x) 0 sequentially x]
      using tendsto_scaleR [OF _ tendsto_const, of λn. inverse (real n + 1) 0
sequentially ((1 / 2) *R (a + b) − x)]
      by auto
  }
  ultimately have x ∈ closure (box a b)
    using as box_midpoint[OF assms]
    unfolding closure_def islimpt_sequential
    by (cases x=?c) (auto simp: in_box_eucl_less)
 }
 then show ?thesis
  using closure_minimal[OF box_subset_cbox, of a b] by blast
qed

lemma bounded_subset_box_symmetric:
 fixes S :: ('a::euclidean_space) set
 assumes bounded S
 obtains a where S ⊆ box (−a) a
proof −
 obtain b where b>0 and b: ∀x∈S. norm x ≤ b
  using assms[unfolded bounded_pos] by auto
 define a :: 'a where a = (∑ i∈Basis. (b + 1) *R i)
 have (−a)•i < x•i and x•i < a•i if x ∈ S and i: i ∈ Basis for x i
  using b Basis_le_norm[OF i, of x] that by (auto simp: a_def)
 then have S ⊆ box (−a) a
  by (auto simp: simp add: box_def)
 then show ?thesis ..
qed

lemma bounded_subset_cbox_symmetric:
 fixes S :: ('a::euclidean_space) set
```

assumes *bounded S*
obtains *a* **where** $S \subseteq$ *cbox* $(-a)$ *a*
by (*meson assms bounded_subset_box_symmetric box_subset_cbox order.trans*)

lemma *frontier_cbox*:
fixes *a b* :: *'a*::*euclidean_space*
shows *frontier* (*cbox a b*) = *cbox a b* − *box a b*
unfolding *frontier_def* **unfolding** *interior_cbox* **and** *closure_closed*[*OF closed_cbox*]
..

lemma *frontier_box*:
fixes *a b* :: *'a*::*euclidean_space*
shows *frontier* (*box a b*) = (*if box a b* = {} *then* {} *else cbox a b* − *box a b*)
by (*simp add*: *frontier_def interior_open open_box*)

lemma *Int_interval_mixed_eq_empty*:
fixes *a* :: *'a*::*euclidean_space*
assumes *box c d* ≠ {}
shows *box a b* ∩ *cbox c d* = {} ⟷ *box a b* ∩ *box c d* = {}
unfolding *closure_box*[*OF assms*, *symmetric*]
unfolding *open_Int_closure_eq_empty*[*OF open_box*] **..**

5.1.12 Class Instances

lemma *compact_lemma*:
fixes *f* :: *nat* ⇒ *'a*::*euclidean_space*
assumes *bounded* (*range f*)
shows ∀ *d*⊆*Basis*. ∃ *l*::*'a*. ∃ *r*.
strict_mono r ∧ (∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) · *i*) (*l* · *i*) < *e*) *sequentially*)
by (*rule compact_lemma_general*[**where** *unproj*=λ*e*. $\sum$ *i*∈*Basis*. *e i* ∗$_R$ *i*])
(*auto intro*!: *assms bounded_linear_inner_left bounded_linear_image*
simp: *euclidean_representation*)

instance *euclidean_space* ⊆ *heine_borel*
proof
fix *f* :: *nat* ⇒ *'a*
assume *f*: *bounded* (*range f*)
then obtain *l*::*'a* **and** *r* **where** *r*: *strict_mono r*
and *l*: ∀ *e*>*0*. *eventually* (λ*n*. ∀ *i*∈*Basis*. *dist* (*f* (*r n*) · *i*) (*l* · *i*) < *e*) *sequentially*
using *compact_lemma* [*OF f*] **by** *blast*
{
fix *e*::*real*
assume *e* > *0*
hence *e* / *real_of_nat DIM*(*'a*) > *0* **by** (*simp*)
with *l* **have** *eventually* (λ*n*. ∀ *i*∈*Basis*. *dist* (*f* (*r n*) · *i*) (*l* · *i*) < *e* / (*real_of_nat DIM*(*'a*))) *sequentially*
by *simp*
moreover

```
    { fix n
      assume n: ∀i∈Basis. dist (f (r n) • i) (l • i) < e / (real_of_nat DIM('a))
      have dist (f (r n)) l ≤ (∑i∈Basis. dist (f (r n) • i) (l • i))
        using L2_set_le_sum [OF zero_le_dist] by (subst euclidean_dist_l2)
      also have ... < (∑i∈(Basis::'a set). e / (real_of_nat DIM('a)))
        by (meson eucl.finite_Basis n nonempty_Basis sum_strict_mono)
      finally have dist (f (r n)) l < e
        by auto
    }
    ultimately have ∀F n in sequentially. dist (f (r n)) l < e
      by (rule eventually_mono)
  }
  then have *: (f ∘ r) ⟶ l
    unfolding o_def tendsto_iff by simp
  with r show ∃l r. strict_mono r ∧ (f ∘ r) ⟶ l
    by auto
qed

instance euclidean_space ⊆ banach ..

instance euclidean_space ⊆ second_countable_topology
proof
  define a where a f = (∑i∈Basis. fst (f i) *R i) for f :: 'a ⇒ real × real
  then have a: ⋀f. (∑i∈Basis. fst (f i) *R i) = a f
    by simp
  define b where b f = (∑i∈Basis. snd (f i) *R i) for f :: 'a ⇒ real × real
  then have b: ⋀f. (∑i∈Basis. snd (f i) *R i) = b f
    by simp
  define B where B = (λf. box (a f) (b f)) ` (Basis →E (ℚ × ℚ))

  have Ball B open by (simp add: B_def open_box)
  moreover have (∀A. open A ⟶ (∃B'⊆B. ⋃B' = A))
  proof safe
    fix A::'a set
    assume open A
    show ∃B'⊆B. ⋃B' = A
      using open_UNION_box[OF ‹open A›]
     by (smt (verit, ccfv_threshold) B_def a b image_iff mem_Collect_eq subsetI)
  qed
  ultimately
  have topological_basis B
    unfolding topological_basis_def by blast
  moreover
  have countable B
    unfolding B_def
   by (intro countable_image countable_PiE finite_Basis countable_SIGMA countable_rat)
  ultimately show ∃B::'a set set. countable B ∧ open = generate_topology B
    by (blast intro: topological_basis_imp_subbasis)
```

qed

instance *euclidean_space* ⊆ *polish_space* **..**

5.1.13 Compact Boxes

lemma *compact_cbox* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows *compact* (*cbox a b*)
using *bounded_closed_imp_seq_compact*[*of cbox a b*] **using** *bounded_cbox*[*of a b*]
by (*auto simp*: *compact_eq_seq_compact_metric*)

proposition *is_interval_compact*:
is_interval S ∧ *compact S* ⟷ (∃ *a b*. *S* = *cbox a b*) (**is** *?lhs* = *?rhs*)
proof (*cases S* = {})
case *True*
with *empty_as_interval* **show** *?thesis* **by** *auto*
next
case *False*
show *?thesis*
proof
assume *L*: *?lhs*
then have *is_interval S compact S* **by** *auto*
define *a* **where** *a* ≡ ∑ *i*∈*Basis*. (*INF x*∈*S*. *x* ∙ *i*) *∗*$_R$ *i*
define *b* **where** *b* ≡ ∑ *i*∈*Basis*. (*SUP x*∈*S*. *x* ∙ *i*) *∗*$_R$ *i*
have *1*: ⋀*x i*. ⟦*x* ∈ *S*; *i* ∈ *Basis*⟧ ⟹ (*INF x*∈*S*. *x* ∙ *i*) ≤ *x* ∙ *i*
by (*simp add*: *cInf_lower bounded_inner_imp_bdd_below compact_imp_bounded L*)
have *2*: ⋀*x i*. ⟦*x* ∈ *S*; *i* ∈ *Basis*⟧ ⟹ *x* ∙ *i* ≤ (*SUP x*∈*S*. *x* ∙ *i*)
by (*simp add*: *cSup_upper bounded_inner_imp_bdd_above compact_imp_bounded L*)
have *3*: *x* ∈ *S* **if** *inf*: ⋀*i*. *i* ∈ *Basis* ⟹ (*INF x*∈*S*. *x* ∙ *i*) ≤ *x* ∙ *i*
and *sup*: ⋀*i*. *i* ∈ *Basis* ⟹ *x* ∙ *i* ≤ (*SUP x*∈*S*. *x* ∙ *i*) **for** *x*
proof (*rule mem_box_componentwiseI* [*OF* ‹*is_interval S*›])
fix *i*::*'a*
assume *i*: *i* ∈ *Basis*
have *cont*: *continuous_on S* (λ*x*. *x* ∙ *i*)
by (*intro continuous_intros*)
obtain *a* **where** *a* ∈ *S* **and** *a*: ⋀*y*. *y*∈*S* ⟹ *a* ∙ *i* ≤ *y* ∙ *i*
using *continuous_attains_inf* [*OF* ‹*compact S*› *False cont*] **by** *blast*
obtain *b* **where** *b* ∈ *S* **and** *b*: ⋀*y*. *y*∈*S* ⟹ *y* ∙ *i* ≤ *b* ∙ *i*
using *continuous_attains_sup* [*OF* ‹*compact S*› *False cont*] **by** *blast*
have *a* ∙ *i* ≤ (*INF x*∈*S*. *x* ∙ *i*)
by (*simp add*: *False a cINF_greatest*)
also have ... ≤ *x* ∙ *i*
by (*simp add*: *i inf*)
finally have *ai*: *a* ∙ *i* ≤ *x* ∙ *i* **.**
have *x* ∙ *i* ≤ (*SUP x*∈*S*. *x* ∙ *i*)

```
      by (simp add: i sup)
    also have (SUP x∈S. x · i) ≤ b · i
      by (simp add: False b cSUP_least)
    finally have bi: x · i ≤ b · i .
    show x · i ∈ (λx. x · i) ‘ S
      apply (rule_tac x=∑j∈Basis. (((·)a)(i := x · j))j *R j in image_eqI)
      apply (simp add: i)
      apply (rule mem_is_intervalI [OF ‹is_interval S› ‹a ∈ S› ‹b ∈ S›])
      using i ai bi
      apply force
      done
  qed
  have S = cbox a b
    by (auto simp: a_def b_def mem_box intro: 1 2 3)
  then show ?rhs
    by blast
 next
  assume R: ?rhs
  then show ?lhs
    using compact_cbox is_interval_cbox by blast
 qed
qed
```

5.1.14 Componentwise limits and continuity

But is the premise really necessary? Need to generalise *dist ?x ?y = L2_set (λi. dist (?x · i) (?y · i)) Basis*

```
lemma Euclidean_dist_upper: i ∈ Basis ⟹ dist (x · i) (y · i) ≤ dist x y
 by (metis (no_types) member_le_L2_set euclidean_dist_l2 finite_Basis)
```

But is the premise $i \in Basis$ really necessary?

```
lemma open_preimage_inner:
 assumes open S i ∈ Basis
   shows open {x. x · i ∈ S}
proof (rule openI, simp)
 fix x
 assume x: x · i ∈ S
 with assms obtain e where 0 < e and e: ball (x · i) e ⊆ S
   by (auto simp: open_contains_ball_eq)
 have ∃e>0. ball (y · i) e ⊆ S if dxy: dist x y < e / 2 for y
 proof (intro exI conjI)
   have dist (x · i) (y · i) < e / 2
     by (meson ‹i ∈ Basis› dual_order.trans Euclidean_dist_upper not_le that)
   then have dist (x · i) z < e if dist (y · i) z < e / 2 for z
     by (metis dist_commute dist_triangle_half_l that)
   then have ball (y · i) (e / 2) ⊆ ball (x · i) e
     using mem_ball by blast
     with e show ball (y · i) (e / 2) ⊆ S
```

```
      by (metis order_trans)
  qed (simp add: ‹0 < e›)
  then show ∃ e>0. ball x e ⊆ {s. s · i ∈ S}
   by (metis (no_types, lifting) ‹0 < e› ‹open S› half_gt_zero_iff mem_Collect_eq
mem_ball open_contains_ball_eq subsetI)
qed

proposition tendsto_componentwise_iff:
  fixes f :: _ ⇒ 'b::euclidean_space
  shows (f ⟶ l) F ⟷ (∀ i ∈ Basis. ((λx. (f x · i)) ⟶ (l · i)) F)
       (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    unfolding tendsto_def
    by (smt (verit) eventually_elim2 mem_Collect_eq open_preimage_inner)
next
  assume R: ?rhs
  then have ⋀e. e > 0 ⟹ ∀ i∈Basis. ∀ F x in F. dist (f x · i) (l · i) < e
    unfolding tendsto_iff by blast
  then have R': ⋀e. e > 0 ⟹ ∀ F x in F. ∀ i∈Basis. dist (f x · i) (l · i) < e
     by (simp add: eventually_ball_finite_distrib [symmetric])
  show ?lhs
  unfolding tendsto_iff
  proof clarify
    fix e::real
    assume 0 < e
    have ∗: L2_set (λi. dist (f x · i) (l · i)) Basis < e
           if ∀ i∈Basis. dist (f x · i) (l · i) < e / real DIM('b) for x
    proof −
      have L2_set (λi. dist (f x · i) (l · i)) Basis ≤ sum (λi. dist (f x · i) (l · i))
Basis
        by (simp add: L2_set_le_sum)
      also have ... < DIM('b) ∗ (e / real DIM('b))
        by (meson DIM_positive sum_bounded_above_strict that)
      also have ... = e
        by (simp add: field_simps)
      finally show L2_set (λi. dist (f x · i) (l · i)) Basis < e .
    qed
    have ∀ F x in F. ∀ i∈Basis. dist (f x · i) (l · i) < e / DIM('b)
      by (simp add: R' ‹0 < e›)
    then show ∀ F x in F. dist (f x) l < e
      by eventually_elim (metis (full_types) ∗ euclidean_dist_l2)
  qed
qed


corollary continuous_componentwise:
   continuous F f ⟷ (∀ i ∈ Basis. continuous F (λx. (f x · i)))
```

by (*simp add*: *continuous_def tendsto_componentwise_iff* [*symmetric*])

corollary *continuous_on_componentwise*:
fixes $S :: 'a :: t2\_space\ set$
shows *continuous_on S f* $\longleftrightarrow$ $(\forall i \in Basis.$ *continuous_on S* $(\lambda x.\ (f\ x \cdot i)))$
by (*metis continuous_componentwise continuous_on_eq_continuous_within*)

lemma *linear_componentwise_iff*:
linear f' $\longleftrightarrow$ $(\forall i{\in}Basis.$ *linear* $(\lambda x.\ f'\ x \cdot i))$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*simp add*: *Real_Vector_Spaces.linear_iff inner_left_distrib*)
show *?rhs* $\Longrightarrow$ *?lhs*
by (*simp add*: *linear_iff*) (*metis euclidean_eqI inner_left_distrib inner_scaleR_left*)
qed

lemma *bounded_linear_componentwise_iff*:
(*bounded_linear f'*) $\longleftrightarrow$ $(\forall i{\in}Basis.$ *bounded_linear* $(\lambda x.\ f'\ x \cdot i))$
(**is** *?lhs* = *?rhs*)
proof
assume *?rhs*
then have $(\forall i{\in}Basis.\ \exists K.\ \forall x.\ |f'\ x \cdot i| \le norm\ x * K)$ *linear f'*
by (*auto simp*: *bounded_linear_def bounded_linear_axioms_def linear_componentwise_iff* [*symmetric*] *ball_conj_distrib*)
then obtain *F* **where** *F*: $\bigwedge i\ x.\ i \in Basis \Longrightarrow |f'\ x \cdot i| \le norm\ x * F\ i$
by *metis*
have *norm* $(f'\ x) \le norm\ x * sum\ F\ Basis$ **for** *x*
proof −
have *norm* $(f'\ x) \le (\sum i{\in}Basis.\ |f'\ x \cdot i|)$
by (*rule norm_le_l1*)
also have ... $\le (\sum i{\in}Basis.\ norm\ x * F\ i)$
by (*metis F sum_mono*)
also have ... = *norm x* $*$ *sum F Basis*
by (*simp add*: *sum_distrib_left*)
finally show *?thesis* **.**
qed
then show *?lhs*
by (*force simp*: *bounded_linear_def bounded_linear_axioms_def* ‹*linear f'*›)
qed (*simp add*: *bounded_linear_inner_left_comp*)

5.1.15 Continuous Extension

definition *clamp* :: $'a{::}euclidean\_space \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$ **where**
clamp a b x = (*if* $(\forall i{\in}Basis.\ a \cdot i \le b \cdot i)$
then $(\sum i{\in}Basis.$ (*if* $x{\cdot}i < a{\cdot}i$ *then* $a{\cdot}i$ *else if* $x{\cdot}i \le b{\cdot}i$ *then* $x{\cdot}i$ *else* $b{\cdot}i) *_R\ i)$
else a)

lemma *clamp_in_interval*[*simp*]:
assumes $\bigwedge i.\ i \in Basis \Longrightarrow a \cdot i \le b \cdot i$

```
  shows clamp a b x ∈ cbox a b
  unfolding clamp_def
  using box_ne_empty(1)[of a b] assms by (auto simp: cbox_def)

lemma clamp_cancel_cbox[simp]:
  fixes x a b :: 'a::euclidean_space
  assumes x: x ∈ cbox a b
  shows clamp a b x = x
  using assms
  by (auto simp: clamp_def mem_box intro!: euclidean_eqI[where 'a='a])

lemma clamp_empty_interval:
  assumes i ∈ Basis a • i > b • i
  shows clamp a b = (λ_. a)
  using assms
  by (force simp: clamp_def[abs_def] split: if_splits intro!: ext)

lemma dist_clamps_le_dist_args:
  fixes x :: 'a::euclidean_space
  shows dist (clamp a b y) (clamp a b x) ≤ dist y x
proof cases
  assume le: (∀ i∈Basis. a • i ≤ b • i)
  then have (∑ i∈Basis. (dist (clamp a b y • i) (clamp a b x • i))^2) ≤
    (∑ i∈Basis. (dist (y • i) (x • i))^2)
   by (auto intro!: sum_mono simp: clamp_def dist_real_def abs_le_square_iff[symmetric])
  then show ?thesis
    by (auto intro: real_sqrt_le_mono
      simp: euclidean_dist_l2[where y=x] euclidean_dist_l2[where y=clamp a b
x] L2_set_def)
qed (auto simp: clamp_def)

lemma clamp_continuous_at:
  fixes f :: 'a::euclidean_space ⇒ 'b::metric_space
    and x :: 'a
  assumes f_cont: continuous_on (cbox a b) f
  shows continuous (at x) (λx. f (clamp a b x))
proof cases
  assume le: (∀ i∈Basis. a • i ≤ b • i)
  show ?thesis
    unfolding continuous_at_eps_delta
  proof safe
    fix x :: 'a
    fix e :: real
    assume e > 0
    moreover have clamp a b x ∈ cbox a b
      by (simp add: le)
    moreover note f_cont[simplified continuous_on_iff]
    ultimately
    obtain d where d: 0 < d
```

⋀x'. $x' \in$ *cbox a b* ⟹ *dist x'* (*clamp a b x*) < *d* ⟹ *dist* (*f x'*) (*f* (*clamp a b x*)) < *e*
by *force*
show ∃ *d*>*0*. ∀ *x'*. *dist x' x* < *d* ⟶ *dist* (*f* (*clamp a b x'*)) (*f* (*clamp a b x*)) < *e*
using *le*
by (*auto intro*!: *d clamp_in_interval dist_clamps_le_dist_args*[*THEN le_less_trans*])
qed
qed (*auto simp*: *clamp_empty_interval*)

lemma *clamp_continuous_on*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*metric_space*
assumes *f_cont*: *continuous_on* (*cbox a b*) *f*
shows *continuous_on S* (λ*x*. *f* (*clamp a b x*))
using *assms*
by (*auto intro*: *continuous_at_imp_continuous_on clamp_continuous_at*)

lemma *clamp_bounded*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*metric_space*
assumes *bounded*: *bounded* (*f* ‘ (*cbox a b*))
shows *bounded* (*range* (λ*x*. *f* (*clamp a b x*)))
proof *cases*
assume *le*: (∀ *i*∈*Basis*. *a* ∙ *i* ≤ *b* ∙ *i*)
from *bounded* **obtain** *c* **where** *f_bound*: ∀ *x*∈*f* ‘ *cbox a b*. *dist undefined x* ≤ *c*
by (*auto simp*: *bounded_any_center*[**where** *a*=*undefined*])
then show *?thesis*
by (*metis bounded bounded_subset clamp_in_interval image_mono image_subsetI le range_composition*)
qed (*auto simp*: *clamp_empty_interval image_def*)

definition *ext_cont* :: (*'a*::*euclidean_space* ⇒ *'b*::*metric_space*) ⇒ *'a* ⇒ *'a* ⇒ *'a* ⇒ *'b*
where *ext_cont f a b* = (λ*x*. *f* (*clamp a b x*))

lemma *ext_cont_cancel_cbox*[*simp*]:
fixes *x a b* :: *'a*::*euclidean_space*
assumes *x*: *x* ∈ *cbox a b*
shows *ext_cont f a b x* = *f x*
using *assms* **by** (*simp add*: *ext_cont_def*)

lemma *continuous_on_ext_cont*[*continuous_intros*]:
continuous_on (*cbox a b*) *f* ⟹ *continuous_on S* (*ext_cont f a b*)
by (*auto intro*!: *clamp_continuous_on simp*: *ext_cont_def*)

5.1.16 Separability

lemma *univ_second_countable_sequence*:
obtains *B* :: *nat* ⇒ *'a*::*euclidean_space set*

```
    where inj B ⋀n. open(B n) ⋀S. open S ⟹ ∃k. S = ⋃{B n |n. n ∈ k}
proof −
  obtain ℬ :: 'a set set
  where countable ℬ
    and opn: ⋀C. C ∈ ℬ ⟹ open C
    and Un: ⋀S. open S ⟹ ∃U. U ⊆ ℬ ∧ S = ⋃U
    using univ_second_countable by blast
  have ∗: infinite (range (λn. ball (0::'a) (inverse(Suc n))))
    by (simp add: inj_on_def ball_eq_ball_iff Infinite_Set.range_inj_infinite)
  have infinite ℬ
  proof
    assume finite ℬ
    then have finite (Union ‘ (Pow ℬ))
      by simp
    moreover have range (λn. ball 0 (inverse (real (Suc n)))) ⊆ ⋃ ‘ Pow ℬ
      by (metis (no_types, lifting) PowI image_eqI image_subset_iff Un [OF
open_ball])
    ultimately show False
      by (metis finite_subset ∗)
  qed
  obtain f :: nat ⇒ 'a set where ℬ = range f inj f
    by (blast intro: countable_as_injective_image [OF ‹countable ℬ› ‹infinite ℬ›])
  have ∗: ∃k. S = ⋃{f n |n. n ∈ k} if open S for S
    using Un [OF that]
    apply clarify
    apply (rule_tac x=f−‘U in exI)
    using ‹inj f› ‹ℬ = range f› apply force
    done
  show ?thesis
    using ∗ ‹ℬ = range f› ‹inj f› opn that by force
qed

proposition separable:
  fixes S :: 'a::{metric_space, second_countable_topology} set
  obtains T where countable T T ⊆ S S ⊆ closure T
proof −
  obtain ℬ :: 'a set set
    where countable ℬ
      and {} ∉ ℬ
      and ope: ⋀C. C ∈ ℬ ⟹ openin(top_of_set S) C
      and if_ope: ⋀T. openin(top_of_set S) T ⟹ ∃𝒰. 𝒰 ⊆ ℬ ∧ T = ⋃𝒰
    by (meson subset_second_countable)
  then obtain f where f: ⋀C. C ∈ ℬ ⟹ f C ∈ C
    by (metis equals0I)
  show ?thesis
  proof
    show countable (f ‘ ℬ)
      by (simp add: ‹countable ℬ›)
    show f ‘ ℬ ⊆ S
```

```
      using ope f openin_imp_subset by blast
    show S ⊆ closure (f ` ℬ)
    proof (clarsimp simp: closure_approachable)
      fix x and e::real
      assume x ∈ S 0 < e
      have openin (top_of_set S) (S ∩ ball x e)
        by (simp add: openin_Int_open)
      with if_ope obtain 𝒰 where 𝒰: 𝒰 ⊆ ℬ S ∩ ball x e = ⋃𝒰
        by meson
      show ∃ C ∈ ℬ. dist (f C) x < e
      proof (cases 𝒰 = {})
        case True
        then show ?thesis
          using ‹0 < e› 𝒰 ‹x ∈ S› by auto
      next
        case False
        then show ?thesis
          by (metis IntI Union_iff 𝒰 ‹0 < e› ‹x ∈ S› dist_commute dist_self f
inf_le2 mem_ball subset_eq)
      qed
    qed
  qed
qed
```

5.1.17 Diameter

```
lemma diameter_cball [simp]:
  fixes a :: 'a::euclidean_space
  shows diameter(cball a r) = (if r < 0 then 0 else 2*r)
proof −
  have diameter(cball a r) = 2*r if r ≥ 0
  proof (rule order_antisym)
    show diameter (cball a r) ≤ 2*r
    proof (rule diameter_le)
      fix x y assume x ∈ cball a r y ∈ cball a r
      then have norm (x − a) ≤ r norm (a − y) ≤ r
        by (auto simp: dist_norm norm_minus_commute)
      then have norm (x − y) ≤ r+r
        using norm_diff_triangle_le by blast
      then show norm (x − y) ≤ 2*r by simp
    qed (simp add: that)
    have 2*r = dist (a + r *R (SOME i. i ∈ Basis)) (a − r *R (SOME i. i ∈
Basis))
      using ‹0 ≤ r› that by (simp add: dist_norm flip: scaleR_2)
    also have ... ≤ diameter (cball a r)
      apply (rule diameter_bounded_bound)
      using that by (auto simp: dist_norm)
    finally show 2*r ≤ diameter (cball a r) .
  qed
```

then show *?thesis* **by** *simp*
qed

lemma *diameter_ball* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows *diameter*(*ball a r*) = (*if r* < *0 then 0 else 2*∗*r*)
proof −
have *diameter*(*ball a r*) = *2*∗*r* **if** *r* > *0*
by (*metis bounded_ball diameter_closure closure_ball diameter_cball less_eq_real_def linorder_not_less that*)
then show *?thesis*
by (*simp add*: *diameter_def*)
qed

lemma *diameter_closed_interval* [*simp*]: *diameter* {*a*..*b*} = (*if b* < *a then 0 else b*−*a*)
proof −
have {*a*..*b*} = *cball* ((*a*+*b*)/*2*) ((*b*−*a*)/*2*)
using *atLeastAtMost_eq_cball* **by** *blast*
then show *?thesis*
by *simp*
qed

lemma *diameter_open_interval* [*simp*]: *diameter* {*a*<..<*b*} = (*if b* < *a then 0 else b*−*a*)
proof −
have {*a* <..< *b*} = *ball* ((*a*+*b*)/*2*) ((*b*−*a*)/*2*)
using *greaterThanLessThan_eq_ball* **by** *blast*
then show *?thesis*
by *simp*
qed

lemma *diameter_cbox*:
fixes *a b*::*'a*::*euclidean_space*
shows (∀ *i* ∈ *Basis*. *a* · *i* ≤ *b* · *i*) ⟹ *diameter* (*cbox a b*) = *dist a b*
by (*force simp*: *diameter_def intro*!: *cSup_eq_maximum L2_set_mono simp*: *euclidean_dist_l2*[**where** *'a*=*'a*] *cbox_def dist_norm*)

5.1.18 Relating linear images to open/closed/interior/closure/connected

proposition *open_surjective_linear_image*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *'b*::*euclidean_space*
assumes *open A linear f surj f*
shows *open*(*f* ‘ *A*)
unfolding *open_dist*
proof *clarify*
fix *x*
assume *x* ∈ *A*

```
  have bounded (inv f ‘ Basis)
    by (simp add: finite_imp_bounded)
  with bounded_pos obtain B where B > 0 and B: ⋀x. x ∈ inv f ‘ Basis ⟹
norm x ≤ B
    by metis
  obtain e where e > 0 and e: ⋀z. dist z x < e ⟹ z ∈ A
    by (metis open_dist ‹x ∈ A› ‹open A›)
  define δ where δ ≡ e / B / DIM('b)
  show ∃e>0. ∀y. dist y (f x) < e ⟶ y ∈ f ‘ A
  proof (intro exI conjI)
    show δ > 0
      using ‹e > 0› ‹B > 0›  by (simp add: δ_def field_split_simps)
    have y ∈ f ‘ A if dist y (f x) * (B * real DIM('b)) < e for y
    proof −
      define u where u ≡ y − f x
      show ?thesis
      proof (rule image_eqI)
        show y = f (x + (∑i∈Basis. (u • i) *R inv f i))
       apply (simp add: linear_add linear_sum linear.scaleR ‹linear f› surj_f_inv_f
‹surj f›)
          apply (simp add: euclidean_representation u_def)
          done
        have dist (x + (∑i∈Basis. (u • i) *R inv f i)) x ≤ (∑i∈Basis. norm ((u
• i) *R inv f i))
          by (simp add: dist_norm sum_norm_le)
        also have ... = (∑i∈Basis. |u • i| * norm (inv f i))
          by simp
        also have ... ≤ (∑i∈Basis. |u • i|) * B
          by (simp add: B sum_distrib_right sum_mono mult_left_mono)
        also have ... ≤ DIM('b) * dist y (f x) * B
          apply (rule mult_right_mono [OF sum_bounded_above])
          using ‹0 < B› by (auto simp: Basis_le_norm dist_norm u_def)
        also have ... < e
          by (metis mult.commute mult.left_commute that)
        finally show x + (∑i∈Basis. (u • i) *R inv f i) ∈ A
          by (rule e)
      qed
    qed
    then show ∀y. dist y (f x) < δ ⟶ y ∈ f ‘ A
      using ‹e > 0› ‹B > 0›
      by (auto simp: δ_def field_split_simps)
  qed
qed

corollary open_bijective_linear_image_eq:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f bij f
    shows open(f ‘ A) ⟷ open A
proof
```

assume *open*(*f* ‘ *A*)
then show *open A*
by (*metis assms bij_is_inj continuous_open_vimage inj_vimage_image_eq linear_continuous_at linear_linear*)
next
assume *open A*
then show *open*(*f* ‘ *A*)
by (*simp add*: *assms bij_is_surj open_surjective_linear_image*)
qed

corollary *interior_bijective_linear_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f bij f*
shows *interior* (*f* ‘ *S*) = *f* ‘ *interior S*
by (*smt* (*verit*) *assms bij_is_inj inj_image_subset_iff interior_maximal interior_subset*
open_bijective_linear_image_eq open_interior subset_antisym subset_imageE)

lemma *interior_injective_linear_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*::*euclidean_space*
assumes *linear f inj f*
shows *interior*(*f* ‘ *S*) = *f* ‘ (*interior S*)
by (*simp add*: *linear_injective_imp_surjective assms bijI interior_bijective_linear_image*)

lemma *interior_surjective_linear_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*::*euclidean_space*
assumes *linear f surj f*
shows *interior*(*f* ‘ *S*) = *f* ‘ (*interior S*)
by (*simp add*: *assms interior_injective_linear_image linear_surjective_imp_injective*)

lemma *interior_negations*:
fixes *S* :: *'a*::*euclidean_space set*
shows *interior*(*uminus* ‘ *S*) = *image uminus* (*interior S*)
by (*simp add*: *bij_uminus interior_bijective_linear_image linear_uminus*)

lemma *connected_linear_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes *linear f* **and** *connected s*
shows *connected* (*f* ‘ *s*)
using *connected_continuous_image assms linear_continuous_on linear_conv_bounded_linear*
by *blast*

5.1.19 "Isometry" (up to constant bounds) of Injective Linear Map

proposition *injective_imp_isometric*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *s*: *closed s subspace s*
and *f*: *bounded_linear f* ∀ *x*∈*s*. *f x* = *0* ⟶ *x* = *0*

```
  shows ∃ e>0. ∀ x∈s. norm (f x) ≥ e * norm x
proof (cases s ⊆ {0::'a})
  case True
  have norm x ≤ norm (f x) if x ∈ s for x
  proof −
    from True that have x = 0 by auto
    then show ?thesis by simp
  qed
  then show ?thesis
    by (auto intro!: exI[where x=1])
next
  case False
  interpret f: bounded_linear f by fact
  from False obtain a where a: a ≠ 0 a ∈ s
    by auto
  from False have s ≠ {}
    by auto
  let ?S = {f x| x. x ∈ s ∧ norm x = norm a}
  let ?S' = {x::'a. x∈s ∧ norm x = norm a}
  let ?S'' = {x::'a. norm x = norm a}

  have ?S'' = frontier (cball 0 (norm a))
    by (simp add: sphere_def dist_norm)
  then have compact ?S'' by (metis compact_cball compact_frontier)
  moreover have ?S' = s ∩ ?S'' by auto
  ultimately have compact ?S'
    using closed_Int_compact[of s ?S''] using s(1) by auto
  moreover have *:f ` ?S' = ?S by auto
  ultimately have compact ?S
    using compact_continuous_image[OF linear_continuous_on[OF f(1)], of ?S']
by auto
  then have closed ?S
    using compact_imp_closed by auto
  moreover from a have ?S ≠ {} by auto
  ultimately obtain b' where b'∈?S ∀ y∈?S. norm b' ≤ norm y
    using distance_attains_inf[of ?S 0] unfolding dist_0_norm by auto
  then obtain b where b∈s
    and ba: norm b = norm a
    and b: ∀ x∈{x ∈ s. norm x = norm a}. norm (f b) ≤ norm (f x)
    unfolding *[symmetric] unfolding image_iff by auto

  let ?e = norm (f b) / norm b
  have norm b > 0
    using ba and a and norm_ge_zero by auto
  moreover have norm (f b) > 0
    using f(2)[THEN bspec[where x=b], OF ‹b∈s›]
    using ‹norm b >0› by simp
  ultimately have 0 < norm (f b) / norm b by simp
  moreover
```

```
  have norm (f b) / norm b * norm x ≤ norm (f x) if x∈s for x
  proof (cases x = 0)
    case True
    then show norm (f b) / norm b * norm x ≤ norm (f x)
      by auto
  next
    case False
    with ‹a ≠ 0› have *: 0 < norm a / norm x
      unfolding zero_less_norm_iff[symmetric] by simp
    have ∀ x∈s. c *R x ∈ s for c
      using s[unfolded subspace_def] by simp
   with ‹x ∈ s› ‹x ≠ 0› have (norm a / norm x) *R x ∈ {x ∈ s. norm x = norm
a}
      by simp
    with ‹x ≠ 0› ‹a ≠ 0› show norm (f b) / norm b * norm x ≤ norm (f x)
      using b[THEN bspec[where x=(norm a / norm x) *R x]]
      unfolding f.scaleR and ba
      by (auto simp: mult.commute pos_le_divide_eq pos_divide_le_eq)
  qed
  ultimately show ?thesis by auto
qed

proposition closed_injective_image_subspace:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes subspace s bounded_linear f ∀ x∈s. f x = 0 ⟶ x = 0 closed s
  shows closed(f ‘ s)
proof −
  obtain e where e > 0 and e: ∀ x∈s. e * norm x ≤ norm (f x)
    using assms injective_imp_isometric by blast
  with assms show ?thesis
    by (meson complete_eq_closed complete_isometric_image)
qed


lemma closure_bounded_linear_image_subset:
  assumes f: bounded_linear f
  shows f ‘ closure S ⊆ closure (f ‘ S)
  using linear_continuous_on [OF f] closed_closure closure_subset
  by (rule image_closure_subset)

lemma closure_linear_image_subset:
  fixes f :: 'm::euclidean_space ⇒ 'n::real_normed_vector
  assumes linear f
  shows f ‘ (closure S) ⊆ closure (f ‘ S)
  using assms unfolding linear_conv_bounded_linear
  by (rule closure_bounded_linear_image_subset)

lemma closed_injective_linear_image:
    fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
```

```
    assumes S: closed S and f: linear f inj f
    shows closed (f ‘ S)
proof −
  obtain g where g: linear g g ∘ f = id
    using linear_injective_left_inverse [OF f] by blast
  then have confg: continuous_on (range f) g
    using linear_continuous_on linear_conv_bounded_linear by blast
  have [simp]: g ‘ f ‘ S = S
    using g by (simp add: image_comp)
  have cgf: closed (g ‘ f ‘ S)
    by (simp add: ‹g ∘ f = id› S image_comp)
  have [simp]: (range f ∩ g −‘ S) = f ‘ S
    using g unfolding o_def id_def image_def by auto metis+
  show ?thesis
  proof (rule closedin_closed_trans [of range f])
    show closedin (top_of_set (range f)) (f ‘ S)
      using continuous_closedin_preimage [OF confg cgf] by simp
    show closed (range f)
      using closed_injective_image_subspace f linear_conv_bounded_linear
        linear_injective_0 subspace_UNIV by blast
  qed
qed

lemma closed_injective_linear_image_eq:
    fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    assumes f: linear f inj f
      shows (closed(image f s) ⟷ closed s)
 by (metis closed_injective_linear_image closure_eq closure_linear_image_subset
closure_subset_eq f(1) f(2) inj_image_subset_iff)

lemma closure_injective_linear_image:
    fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    shows ⟦linear f; inj f⟧ ⟹ f ‘ (closure S) = closure (f ‘ S)
  by (simp add: closed_injective_linear_image closure_linear_image_subset
       closure_minimal closure_subset image_mono subset_antisym)

lemma closure_bounded_linear_image:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f bounded S
    shows f ‘ (closure S) = closure (f ‘ S)  (is ?lhs = ?rhs)
proof
  show ?lhs ⊆ ?rhs
    using assms closure_linear_image_subset by blast
  show ?rhs ⊆ ?lhs
   using assms by (meson closure_minimal closure_subset compact_closure com-
pact_eq_bounded_closed
                  compact_continuous_image image_mono linear_continuous_on
linear_linear)
qed
```

lemma *closure_scaleR*:
fixes $S :: 'a::real\_normed\_vector\ set$
shows $((*_R)\ c)\ `\ (closure\ S) = closure\ (((*_R)\ c)\ `\ S)$ (**is** *?lhs = ?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
using *bounded_linear_scaleR_right* **by** (*rule closure_bounded_linear_image_subset*)
show *?rhs* $\subseteq$ *?lhs*
by (*intro closure_minimal image_mono closure_subset closed_scaling closed_closure*)
qed

5.1.20 Some properties of a canonical subspace

lemma *closed_substandard*: *closed* $\{x::'a::euclidean\_space.\ \forall i{\in}Basis.\ P\ i \longrightarrow x{\cdot}i = 0\}$
(**is** *closed ?A*)
proof −
let *?D* $= \{i{\in}Basis.\ P\ i\}$
have *closed* $(\bigcap i{\in}?D.\ \{x::'a.\ x{\cdot}i = 0\})$
by (*simp add*: *closed_INT closed_Collect_eq continuous_on_inner*)
also have $(\bigcap i{\in}?D.\ \{x::'a.\ x{\cdot}i = 0\}) = ?A$
by *auto*
finally show *closed ?A* **.**
qed

lemma *closed_subspace*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *subspace S*
shows *closed S*
proof −
have *dim S* $\leq$ *card* $(Basis :: 'a\ set)$
using *dim_subset_UNIV* **by** *auto*
with *obtain_subset_with_card_n*
obtain $d :: 'a\ set$ **where** *cd*: *card d* $=$ *dim S* **and** *d*: $d \subseteq Basis$
by *metis*
let *?t* $= \{x::'a.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x{\cdot}i = 0\}$
have $\exists f.\ linear\ f \wedge f\ `\ \{x::'a.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\} = S \wedge$
inj_on f $\{x::'a.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\}$
using *dim_substandard*[*of d*] *cd d assms*
by (*intro subspace_isomorphism*[*OF subspace_substandard*[*of* $\lambda i.\ i \notin d$]]) (*auto simp*: *inner_Basis*)
then obtain *f* **where** *f*:
linear f
$f\ `\ \{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\} = S$
inj_on f $\{x.\ \forall i{\in}Basis.\ i \notin d \longrightarrow x \cdot i = 0\}$
by *blast*
interpret *f*: *bounded_linear f*
using *f* **by** (*simp add*: *linear_conv_bounded_linear*)
have $x \in ?t \Longrightarrow f\ x = 0 \Longrightarrow x = 0$ **for** *x*

using *f.zero d f(3)*[*THEN inj_onD*, *of x 0*] **by** *auto*
then show *?thesis*
using *closed_injective_image_subspace*[*of ?t f*] *closed_substandard subspace_substandard*
using *f(2) f.bounded_linear_axioms* **by** *force*
qed

lemma *complete_subspace*: *subspace S* $\Longrightarrow$ *complete S*
for *S* :: *'a*::*euclidean_space set*
using *complete_eq_closed closed_subspace* **by** *auto*

lemma *closed_span* [*iff*]: *closed* (*span S*)
for *S* :: *'a*::*euclidean_space set*
by (*simp add*: *closed_subspace*)

lemma *dim_closure* [*simp*]: *dim* (*closure S*) = *dim S* (**is** *?dc* = *?d*)
for *S* :: *'a*::*euclidean_space set*
by (*metis closed_span closure_minimal closure_subset dim_eq_span span_eq_dim span_superset subset_le_dim*)

5.1.21 Set Distance

lemma *setdist_compact_closed*:
fixes *A* :: *'a*::*heine_borel set*
assumes *compact A closed B*
and *A* $\neq$ {} *B* $\neq$ {}
shows $\exists x \in A.\ \exists y \in B.$ *dist x y* = *setdist A B*
by (*metis assms infdist_attains_inf setdist_attains_inf setdist_sym*)

lemma *setdist_closed_compact*:
fixes *S* :: *'a*::*heine_borel set*
assumes *S*: *closed S* **and** *T*: *compact T*
and *S* $\neq$ {} *T* $\neq$ {}
shows $\exists x \in S.\ \exists y \in T.$ *dist x y* = *setdist S T*
using *setdist_compact_closed* [*OF T S* ‹*T* $\neq$ {}› ‹*S* $\neq$ {}›]
by (*metis dist_commute setdist_sym*)

lemma *setdist_eq_0_compact_closed*:
assumes *S*: *compact S* **and** *T*: *closed T*
shows *setdist S T* = *0* $\longleftrightarrow$ *S* = {} $\vee$ *T* = {} $\vee$ *S* $\cap$ *T* $\neq$ {}
proof (*cases S* = {} $\vee$ *T* = {})
case *False*
then show *?thesis*
by (*metis S T disjoint_iff in_closed_iff_infdist_zero setdist_attains_inf setdist_eq_0I setdist_sym*)
qed *auto*

corollary *setdist_gt_0_compact_closed*:
assumes *S*: *compact S* **and** *T*: *closed T*
shows *setdist S T* > *0* $\longleftrightarrow$ (*S* $\neq$ {} $\wedge$ *T* $\neq$ {} $\wedge$ *S* $\cap$ *T* = {})

using *setdist_pos_le* [*of S T*] *setdist_eq_0_compact_closed* [*OF assms*] **by** *linarith*

lemma *setdist_eq_0_closed_compact*:
assumes *S*: *closed S* **and** *T*: *compact T*
shows *setdist S T = 0* $\longleftrightarrow$ *S = {} ∨ T = {} ∨ S ∩ T ≠ {}*
using *setdist_eq_0_compact_closed* [*OF T S*]
by (*metis Int_commute setdist_sym*)

lemma *setdist_eq_0_bounded*:
fixes *S* :: *'a*::*heine_borel set*
assumes *bounded S ∨ bounded T*
shows *setdist S T = 0* $\longleftrightarrow$ *S = {} ∨ T = {} ∨ closure S ∩ closure T ≠ {}*
proof (*cases S = {} ∨ T = {}*)
case *False*
then show *?thesis*
using *setdist_eq_0_compact_closed* [*of closure S closure T*]
setdist_eq_0_closed_compact [*of closure S closure T*] *assms*
by (*force simp*: *bounded_closure compact_eq_bounded_closed*)
qed *force*

lemma *setdist_eq_0_sing_1*:
setdist {x} S = 0 $\longleftrightarrow$ *S = {} ∨ x ∈ closure S*
by (*metis in_closure_iff_infdist_zero infdist_def infdist_eq_setdist*)

lemma *setdist_eq_0_sing_2*:
setdist S {x} = 0 $\longleftrightarrow$ *S = {} ∨ x ∈ closure S*
by (*metis setdist_eq_0_sing_1 setdist_sym*)

lemma *setdist_neq_0_sing_1*:
⟦*setdist {x} S = a*; *a ≠ 0*⟧ $\Longrightarrow$ *S ≠ {} ∧ x ∉ closure S*
by (*metis setdist_closure_2 setdist_empty2 setdist_eq_0I singletonI*)

lemma *setdist_neq_0_sing_2*:
⟦*setdist S {x} = a*; *a ≠ 0*⟧ $\Longrightarrow$ *S ≠ {} ∧ x ∉ closure S*
by (*simp add*: *setdist_neq_0_sing_1 setdist_sym*)

lemma *setdist_sing_in_set*:
x ∈ S $\Longrightarrow$ *setdist {x} S = 0*
by (*simp add*: *setdist_eq_0I*)

lemma *setdist_eq_0_closed*:
closed S $\Longrightarrow$ (*setdist {x} S = 0* $\longleftrightarrow$ *S = {} ∨ x ∈ S*)
by (*simp add*: *setdist_eq_0_sing_1*)

lemma *setdist_eq_0_closedin*:
shows ⟦*closedin* (*top_of_set U*) *S*; *x ∈ U*⟧
$\Longrightarrow$ (*setdist {x} S = 0* $\longleftrightarrow$ *S = {} ∨ x ∈ S*)
by (*auto simp*: *closedin_limpt setdist_eq_0_sing_1 closure_def*)

lemma *setdist_gt_0_closedin*:
shows ⟦*closedin* (*top_of_set U*) *S*; *x* ∈ *U*; *S* ≠ {}; *x* ∉ *S*⟧
⟹ *setdist* {*x*} *S* > *0*
using *less_eq_real_def setdist_eq_0_closedin* **by** *fastforce*

A consequence of the results above

lemma *compact_minkowski_sum_cball*:
fixes *A* :: *'a* :: *heine_borel set*
assumes *compact A*
shows *compact* (⋃*x*∈*A*. *cball x r*)
proof (*cases A* = {})
case *False*
show *?thesis*
unfolding *compact_eq_bounded_closed*
proof *safe*
have *open* (−(⋃*x*∈*A*. *cball x r*))
unfolding *open_dist*
proof *safe*
fix *x* **assume** *x*: *x* ∉ (⋃*x*∈*A*. *cball x r*)
have ∃ *x'*∈{*x*}. ∃ *y*∈*A*. *dist x' y* = *setdist* {*x*} *A*
using ‹*A* ≠ {}› *assms*
by (*intro setdist_compact_closed*) (*auto simp*: *compact_imp_closed*)
then obtain *y* **where** *y*: *y* ∈ *A dist x y* = *setdist* {*x*} *A*
by *blast*
with *x* **have** *setdist* {*x*} *A* > *r*
by (*auto simp*: *dist_commute*)
moreover have *False* **if** *dist z x* < *setdist* {*x*} *A* − *r u* ∈ *A z* ∈ *cball u r* **for** *z u*
by (*smt* (*verit*, *del_insts*) *mem_cball setdist_Lipschitz setdist_sing_in_set that*)
ultimately
show ∃ *e*>*0*. ∀ *y*. *dist y x* < *e* ⟶ *y* ∈ − (⋃*x*∈*A*. *cball x r*)
by (*force intro*!: *exI*[*of* _ *setdist* {*x*} *A* − *r*])
qed
thus *closed* (⋃*x*∈*A*. *cball x r*)
using *closed_open* **by** *blast*
next
from *assms* **have** *bounded A*
by (*simp add*: *compact_imp_bounded*)
then obtain *x c* **where** *c*: ⋀*y*. *y* ∈ *A* ⟹ *dist x y* ≤ *c*
unfolding *bounded_def* **by** *blast*
have ∀ *y*∈(⋃*x*∈*A*. *cball x r*). *dist x y* ≤ *c* + *r*
proof *safe*
fix *y z* **assume** ∗: *y* ∈ *A z* ∈ *cball y r*
thus *dist x z* ≤ *c* + *r*
using ∗ *c*[*of y*] *cball_trans mem_cball* **by** *blast*
qed
thus *bounded* (⋃*x*∈*A*. *cball x r*)

unfolding *bounded_def* **by** *blast*
qed
qed *auto*

no_notation *eucl_less* (**infix** ‹<e› *50*)

end

5.2 Line Segment

theory *Line_Segment*
imports
Convex
Topology_Euclidean_Space
begin

5.2.1 Topological Properties of Convex Sets, Metric Spaces and Functions

lemma *convex_supp_sum*:
assumes *convex S* **and** *1*: *supp_sum u I = 1*
and $\bigwedge i.\ i \in I \Longrightarrow 0 \leq u\ i \wedge (u\ i = 0 \vee f\ i \in S)$
shows *supp_sum* ($\lambda i.\ u\ i *_R f\ i$) $I \in S$
proof −
have *fin*: *finite* $\{i \in I.\ u\ i \neq 0\}$
using *1 sum.infinite* **by** (*force simp*: *supp_sum_def support_on_def*)
then have *supp_sum* ($\lambda i.\ u\ i *_R f\ i$) I = *sum* ($\lambda i.\ u\ i *_R f\ i$) $\{i \in I.\ u\ i \neq 0\}$
by (*force intro*: *sum.mono_neutral_left simp*: *supp_sum_def support_on_def*)
also have ... ∈ *S*
using *1 assms* **by** (*force simp*: *supp_sum_def support_on_def intro*: *convex_sum* [*OF fin* ‹*convex S*›])
finally show *?thesis* **.**
qed

lemma *sphere_eq_empty* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*, *perfect_space*}
shows *sphere a r* = {} ⟷ *r* < *0*
by (*metis empty_iff linorder_not_less mem_sphere sphere_empty vector_choose_dist*)

lemma *cone_closure*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *cone S*
shows *cone* (*closure S*)
by (*metis UnCI assms closure_Un_frontier closure_eq_empty closure_scaleR cone_iff*)

corollary *component_complement_connected*:
fixes *S* :: *'a*::*real_normed_vector set*

assumes *connected S C* ∈ *components* (−*S*)
shows *connected*(−*C*)
using *component_diff_connected* [*of S UNIV*] *assms*
by (*auto simp*: *Compl_eq_Diff_UNIV*)

proposition *clopen*:
fixes *S* :: *'a* :: *real_normed_vector set*
shows *closed S* ∧ *open S* ⟷ *S* = {} ∨ *S* = *UNIV*
using *connected_UNIV* **by** (*force simp add*: *connected_clopen*)

corollary *compact_open*:
fixes *S* :: *'a* :: *euclidean_space set*
shows *compact S* ∧ *open S* ⟷ *S* = {}
by (*auto simp*: *compact_eq_bounded_closed clopen*)

corollary *finite_imp_not_open*:
fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
shows ⟦*finite S*; *open S*⟧ ⟹ *S*={}
using *clopen* [*of S*] *finite_imp_closed not_bounded_UNIV* **by** *blast*

corollary *empty_interior_finite*:
fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
shows *finite S* ⟹ *interior S* = {}
by (*metis interior_subset finite_subset open_interior* [*of S*] *finite_imp_not_open*)

Balls, being convex, are connected.

lemma *convex_local_global_minimum*:
fixes *s* :: *'a*::*real_normed_vector set*
assumes *e* > *0*
and *convex_on s f*
and *ball x e* ⊆ *s*
and ∀ *y*∈*ball x e*. *f x* ≤ *f y*
shows ∀ *y*∈*s*. *f x* ≤ *f y*
proof (*rule ccontr*)
have *x* ∈ *s* **using** *assms*(*1*,*3*) **by** *auto*
assume ¬ *?thesis*
then obtain *y* **where** *y*∈*s* **and** *y*: *f x* > *f y* **by** *auto*
then have *xy*: *0* < *dist x y* **by** *auto*
then obtain *u* **where** *0* < *u u* ≤ *1* **and** *u*: *u* < *e* / *dist x y*
using *field_lbound_gt_zero*[*of 1 e* / *dist x y*] *xy* ‹*e*>*0*› **by** *auto*
then have *f* ((*1*−*u*) ∗*R x* + *u* ∗*R y*) ≤ (*1*−*u*) ∗ *f x* + *u* ∗ *f y*
using ‹*x*∈*s*› ‹*y*∈*s*› **by** (*smt* (*verit*) *assms*(*2*) *convex_on_def*)
moreover
have ∗: *x* − ((*1* − *u*) ∗*R x* + *u* ∗*R y*) = *u* ∗*R* (*x* − *y*)
by (*simp add*: *algebra_simps*)
have (*1* − *u*) ∗*R x* + *u* ∗*R y* ∈ *ball x e*
by (*smt* (*verit*) ∗ ‹*0* < *u*› *dist_norm mem_ball norm_scaleR pos_less_divide_eq*
u xy)
then have *f x* ≤ *f* ((*1* − *u*) ∗*R x* + *u* ∗*R y*)

```
    using assms(4) by auto
  ultimately show False
    using mult_strict_left_mono[OF y ‹u>0›]
    unfolding left_diff_distrib
    by auto
qed

lemma convex_ball [iff]:
  fixes x :: 'a::real_normed_vector
  shows convex (ball x e)
proof (auto simp: convex_def)
  fix y z
  assume yz: dist x y < e dist x z < e
  fix u v :: real
  assume uv: 0 ≤ u 0 ≤ v u + v = 1
  have dist x (u *R y + v *R z) ≤ u * dist x y + v * dist x z
    using uv yz by (meson UNIV_I convex_def convex_on_def convex_on_dist)
  then show dist x (u *R y + v *R z) < e
    using convex_bound_lt[OF yz uv] by auto
qed

lemma convex_cball [iff]:
  fixes x :: 'a::real_normed_vector
  shows convex (cball x e)
proof −
  {
    fix y z
    assume yz: dist x y ≤ e dist x z ≤ e
    fix u v :: real
    assume uv: 0 ≤ u 0 ≤ v u + v = 1
    have dist x (u *R y + v *R z) ≤ u * dist x y + v * dist x z
     using uv yz by (meson UNIV_I convex_def convex_on_def convex_on_dist)
    then have dist x (u *R y + v *R z) ≤ e
      using convex_bound_le[OF yz uv] by auto
  }
  then show ?thesis by (auto simp: convex_def Ball_def)
qed

lemma connected_ball [iff]:
  fixes x :: 'a::real_normed_vector
  shows connected (ball x e)
  using convex_connected convex_ball by auto

lemma connected_cball [iff]:
  fixes x :: 'a::real_normed_vector
  shows connected (cball x e)
  using convex_connected convex_cball by auto

lemma bounded_convex_hull:
```

fixes s :: $'a$::*real_normed_vector set*
assumes *bounded* s
shows *bounded* (*convex hull* s)
proof −
from *assms* **obtain** B **where** B: $\forall x{\in}s.$ *norm* $x \leq B$
unfolding *bounded_iff* **by** *auto*
show *?thesis*
by (*simp add*: *bounded_subset*[*OF bounded_cball*, *of* _ *0 B*] *B subsetI subset_hull*)
qed

lemma *finite_imp_bounded_convex_hull*:
fixes s :: $'a$::*real_normed_vector set*
shows *finite* $s \Longrightarrow$ *bounded* (*convex hull* s)
using *bounded_convex_hull finite_imp_bounded*
by *auto*

5.2.2 Midpoint

definition *midpoint* :: $'a$::*real_vector* ⇒ $'a$ ⇒ $'a$
where *midpoint* a b = (*inverse* (*2*::*real*)) $*_R$ (a + b)

lemma *midpoint_idem* [*simp*]: *midpoint* x x = x
unfolding *midpoint_def* **by** *simp*

lemma *midpoint_sym*: *midpoint* a b = *midpoint* b a
unfolding *midpoint_def* **by** (*auto simp add*: *scaleR_right_distrib*)

lemma *midpoint_eq_iff*: *midpoint* a b = c ⟷ a + b = c + c
proof −
have *midpoint* a b = c ⟷ *scaleR* *2* (*midpoint* a b) = *scaleR* *2* c
by *simp*
then show *?thesis*
unfolding *midpoint_def scaleR_2* [*symmetric*] **by** *simp*
qed

lemma
fixes a::*real*
assumes $a \leq b$ **shows** *ge_midpoint_1*: $a \leq$ *midpoint* a b
and *le_midpoint_1*: *midpoint* a $b \leq b$
by (*simp_all add*: *midpoint_def assms*)

lemma *dist_midpoint*:
fixes a b :: $'a$::*real_normed_vector* **shows**
dist a (*midpoint* a b) = (*dist* a b) / *2* (**is** *?t1*)
dist b (*midpoint* a b) = (*dist* a b) / *2* (**is** *?t2*)
dist (*midpoint* a b) a = (*dist* a b) / *2* (**is** *?t3*)
dist (*midpoint* a b) b = (*dist* a b) / *2* (**is** *?t4*)
proof −

```
  have *: ⋀x y::'a. 2 *R x = - y ⟹ norm x = (norm y) / 2
    unfolding equation_minus_iff by auto
  have **: ⋀x y::'a. 2 *R x =   y ⟹ norm x = (norm y) / 2
    by auto
  note scaleR_right_distrib [simp]
  show ?t1
    unfolding midpoint_def dist_norm
    apply (rule **)
    apply (simp add: scaleR_right_diff_distrib)
    apply (simp add: scaleR_2)
    done
  show ?t2
    unfolding midpoint_def dist_norm
    apply (rule *)
    apply (simp add: scaleR_right_diff_distrib)
    apply (simp add: scaleR_2)
    done
  show ?t3
    unfolding midpoint_def dist_norm
    apply (rule *)
    apply (simp add: scaleR_right_diff_distrib)
    apply (simp add: scaleR_2)
    done
  show ?t4
    unfolding midpoint_def dist_norm
    apply (rule **)
    apply (simp add: scaleR_right_diff_distrib)
    apply (simp add: scaleR_2)
    done
qed

lemma midpoint_eq_endpoint [simp]:
  midpoint a b = a ⟷ a = b
  midpoint a b = b ⟷ a = b
  unfolding midpoint_eq_iff by auto

lemma midpoint_plus_self [simp]: midpoint a b + midpoint a b = a + b
  using midpoint_eq_iff by metis

lemma midpoint_linear_image:
  linear f ⟹ midpoint(f a)(f b) = f(midpoint a b)
  by (simp add: linear_iff midpoint_def)
```

5.2.3 Open and closed segments

```
definition closed_segment :: 'a::real_vector ⇒ 'a ⇒ 'a set
  where closed_segment a b = {(1 - u) *R a + u *R b | u::real. 0 ≤ u ∧ u ≤ 1}

definition open_segment :: 'a::real_vector ⇒ 'a ⇒ 'a set where
```

open_segment a b ≡ closed_segment a b − {a,b}

lemmas *segment = open_segment_def closed_segment_def*

lemma *in_segment*:
*x ∈ closed_segment a b ⟷ (∃ u. 0 ≤ u ∧ u ≤ 1 ∧ x = (1 − u) *$_R$ a + u *$_R$ b)*
*x ∈ open_segment a b ⟷ a ≠ b ∧ (∃ u. 0 < u ∧ u < 1 ∧ x = (1 − u) *$_R$ a + u *$_R$ b)*
using *less_eq_real_def* **by** (*auto simp*: *segment algebra_simps*)

lemma *closed_segment_linear_image*:
closed_segment (f a) (f b) = f ‘ (closed_segment a b) **if** *linear f*
proof −
interpret *linear f* **by** *fact*
show *?thesis*
by (*force simp add*: *in_segment add scale*)
qed

lemma *open_segment_linear_image*:
⟦linear f; inj f⟧ ⟹ open_segment (f a) (f b) = f ‘ (open_segment a b)
by (*force simp*: *open_segment_def closed_segment_linear_image inj_on_def*)

lemma *closed_segment_translation*:
closed_segment (c + a) (c + b) = (λx. c + x) ‘ (closed_segment a b) (**is** *?L = _ ‘ ?R*)
proof −
have *⋀x. x ∈ ?L ⟹ x − c ∈ ?R ⋀x. ⟦x ∈ ?R⟧ ⟹ c + x ∈ ?L*
by (*auto simp*: *in_segment algebra_simps*)
then show *?thesis* **by** *force*
qed

lemma *open_segment_translation*:
open_segment (c + a) (c + b) = image (λx. c + x) (open_segment a b)
by (*simp add*: *open_segment_def closed_segment_translation translation_diff*)

lemma *closed_segment_of_real*:
closed_segment (of_real x) (of_real y) = of_real ‘ closed_segment x y
by (*simp add*: *closed_segment_linear_image linearI scaleR_conv_of_real*)

lemma *open_segment_of_real*:
open_segment (of_real x) (of_real y) = of_real ‘ open_segment x y
by (*simp add*: *closed_segment_of_real image_set_diff inj_of_real open_segment_def*)

lemma *closed_segment_Reals*:
⟦x ∈ Reals; y ∈ Reals⟧ ⟹ closed_segment x y = of_real ‘ closed_segment (Re x) (Re y)
by (*metis closed_segment_of_real of_real_Re*)

lemma *open_segment_Reals*:
  $[\![x \in \mathit{Reals};\ y \in \mathit{Reals}]\!] \Longrightarrow \mathit{open\_segment}\ x\ y = \mathit{of\_real}\ `\ \mathit{open\_segment}\ (\mathit{Re}\ x)\ (\mathit{Re}\ y)$
  by (*metis open_segment_of_real of_real_Re*)

lemma *open_segment_PairD*:
  $(x,\ x') \in \mathit{open\_segment}\ (a,\ a')\ (b,\ b')$
    $\Longrightarrow (x \in \mathit{open\_segment}\ a\ b \lor a = b) \land (x' \in \mathit{open\_segment}\ a'\ b' \lor a' = b')$
  by (*auto simp*: *in_segment*)

lemma *closed_segment_PairD*:
  $(x,\ x') \in \mathit{closed\_segment}\ (a,\ a')\ (b,\ b') \Longrightarrow x \in \mathit{closed\_segment}\ a\ b \land x' \in \mathit{closed\_segment}\ a'\ b'$
  by (*auto simp*: *closed_segment_def*)

lemma *closed_segment_translation_eq* [*simp*]:
  $d + x \in \mathit{closed\_segment}\ (d + a)\ (d + b) \longleftrightarrow x \in \mathit{closed\_segment}\ a\ b$
proof −
  have $*$: $\bigwedge d\ x\ a\ b.\ x \in \mathit{closed\_segment}\ a\ b \Longrightarrow d + x \in \mathit{closed\_segment}\ (d + a)\ (d + b)$
    using *closed_segment_translation* **by** *blast*
  show *?thesis*
    using $*$ [**where** $d = -d$] $*$ **by** *fastforce*
qed

lemma *open_segment_translation_eq* [*simp*]:
  $d + x \in \mathit{open\_segment}\ (d + a)\ (d + b) \longleftrightarrow x \in \mathit{open\_segment}\ a\ b$
  by (*simp add*: *open_segment_def*)

lemma *of_real_closed_segment* [*simp*]:
  $\mathit{of\_real}\ x \in \mathit{closed\_segment}\ (\mathit{of\_real}\ a)\ (\mathit{of\_real}\ b) \longleftrightarrow x \in \mathit{closed\_segment}\ a\ b$
  by (*simp add*: *closed_segment_of_real image_iff*)

lemma *of_real_open_segment* [*simp*]:
  $\mathit{of\_real}\ x \in \mathit{open\_segment}\ (\mathit{of\_real}\ a)\ (\mathit{of\_real}\ b) \longleftrightarrow x \in \mathit{open\_segment}\ a\ b$
  by (*simp add*: *image_iff open_segment_of_real*)

lemma *convex_contains_segment*:
  $\mathit{convex}\ S \longleftrightarrow (\forall\, a{\in}S.\ \forall\, b{\in}S.\ \mathit{closed\_segment}\ a\ b \subseteq S)$
  unfolding *convex_alt closed_segment_def* **by** *auto*

lemma *closed_segment_in_Reals*:
  $[\![x \in \mathit{closed\_segment}\ a\ b;\ a \in \mathit{Reals};\ b \in \mathit{Reals}]\!] \Longrightarrow x \in \mathit{Reals}$
  by (*meson subsetD convex_Reals convex_contains_segment*)

lemma *open_segment_in_Reals*:
  $[\![x \in \mathit{open\_segment}\ a\ b;\ a \in \mathit{Reals};\ b \in \mathit{Reals}]\!] \Longrightarrow x \in \mathit{Reals}$
  by (*metis Diff_iff closed_segment_in_Reals open_segment_def*)

lemma *closed_segment_subset*: $[\![x \in S; y \in S; \mathit{convex}\ S]\!] \Longrightarrow \mathit{closed\_segment}\ x\ y \subseteq S$
by (*simp add*: *convex_contains_segment*)

lemma *closed_segment_subset_convex_hull*:
$[\![x \in \mathit{convex\ hull}\ S; y \in \mathit{convex\ hull}\ S]\!] \Longrightarrow \mathit{closed\_segment}\ x\ y \subseteq \mathit{convex\ hull}\ S$
using *convex_contains_segment* **by** *blast*

lemma *segment_convex_hull*:
closed_segment a b = *convex hull* {*a*,*b*}
proof −
have ∗: $\bigwedge x.\ \{x\} \neq \{\}$ **by** *auto*
show *?thesis*
unfolding *segment convex_hull_insert*[*OF* ∗] *convex_hull_singleton*
by (*safe*; *rule_tac x*=*1* − *u* **in** *exI*; *force*)
qed

lemma *open_closed_segment*: $u \in \mathit{open\_segment}\ w\ z \Longrightarrow u \in \mathit{closed\_segment}\ w\ z$
by (*auto simp add*: *closed_segment_def open_segment_def*)

lemma *segment_open_subset_closed*:
open_segment a b ⊆ *closed_segment a b*
by (*simp add*: *open_closed_segment subsetI*)

lemma *bounded_closed_segment*:
fixes *a* :: *'a*::*real_normed_vector* **shows** *bounded* (*closed_segment a b*)
by (*simp add*: *bounded_convex_hull segment_convex_hull*)

lemma *bounded_open_segment*:
fixes *a* :: *'a*::*real_normed_vector* **shows** *bounded* (*open_segment a b*)
by (*rule bounded_subset* [*OF bounded_closed_segment segment_open_subset_closed*])

lemmas *bounded_segment* = *bounded_closed_segment open_closed_segment*

lemma *ends_in_segment* [*iff*]: *a* ∈ *closed_segment a b b* ∈ *closed_segment a b*
by (*simp_all add*: *hull_inc segment_convex_hull*)

lemma *eventually_closed_segment*:
fixes *x0*::*'a*::*real_normed_vector*
assumes *open X0 x0* ∈ *X0*
shows $\forall_F\ x\ \mathit{in\ at\ x0\ within\ U}.\ \mathit{closed\_segment\ x0\ x} \subseteq \mathit{X0}$
proof −
from *openE*[*OF assms*]
obtain *e* **where** *e*: *0* < *e ball x0 e* ⊆ *X0* **.**
then have $\forall_F\ x\ \mathit{in\ at\ x0\ within\ U}.\ x \in \mathit{ball\ x0\ e}$
by (*auto simp*: *dist_commute eventually_at*)
then show *?thesis*

```
  proof eventually_elim
    case (elim x)
    have x0 ∈ ball x0 e using ‹e > 0› by simp
    then have closed_segment x0 x ⊆ ball x0 e
      using closed_segment_subset elim by blast
    then show ?case
      using e(2) by auto
  qed
qed

lemma closed_segment_commute: closed_segment a b = closed_segment b a
proof −
  have {a, b} = {b, a} by auto
  thus ?thesis
    by (simp add: segment_convex_hull)
qed

lemma segment_bound1:
  assumes x ∈ closed_segment a b
  shows norm (x − a) ≤ norm (b − a)
proof −
  obtain u where u: x = (1 − u) *R a + u *R b 0 ≤ u u ≤ 1
    using assms by (auto simp add: closed_segment_def)
  then have norm (u *R b − u *R a) ≤ norm (b − a)
    by (simp add: mult_left_le_one_le flip: scaleR_diff_right)
  with u show ?thesis
    by (metis add_diff_cancel_left scaleR_collapse)
qed

lemma segment_bound:
  assumes x ∈ closed_segment a b
  shows norm (x − a) ≤ norm (b − a) norm (x − b) ≤ norm (b − a)
 by (metis assms closed_segment_commute dist_commute dist_norm segment_bound1 )+

lemma open_segment_bound1:
  assumes x ∈ open_segment a b
  shows norm (x − a) < norm (b − a)
proof −
  obtain u where u: x = (1 − u) *R a + u *R b 0 < u u < 1
    by (meson assms in_segment)
  then have norm (u *R b − u *R a) < norm (b − a)
   using assms in_segment(2) less_eq_real_def by (fastforce simp flip: scaleR_diff_right)
  with u show ?thesis
    by (metis add_diff_cancel_left scaleR_collapse)
qed

lemma open_segment_commute: open_segment a b = open_segment b a
  by (simp add: closed_segment_commute insert_commute open_segment_def)
```

lemma *closed_segment_idem* [*simp*]: *closed_segment a a* = {*a*}
unfolding *segment* **by** (*auto simp add*: *algebra_simps*)

lemma *open_segment_idem* [*simp*]: *open_segment a a* = {}
by (*simp add*: *open_segment_def*)

lemma *closed_segment_eq_open*: *closed_segment a b* = *open_segment a b* ∪ {*a*,*b*}
using *open_segment_def* **by** *auto*

lemma *convex_contains_open_segment*:
convex s ⟷ (∀ *a*∈*s*. ∀ *b*∈*s*. *open_segment a b* ⊆ *s*)
by (*simp add*: *convex_contains_segment closed_segment_eq_open*)

lemma *closed_segment_eq_real_ivl1*:
fixes *a b*::*real*
assumes *a* ≤ *b*
shows *closed_segment a b* = {*a* .. *b*}
proof *safe*
fix *x*
assume *x* ∈ *closed_segment a b*
then obtain *u* **where** *u*: *0* ≤ *u u* ≤ *1* **and** *x_def*: *x* = (*1* − *u*) ∗ *a* + *u* ∗ *b*
by (*auto simp*: *closed_segment_def*)
have *u* ∗ *a* ≤ *u* ∗ *b* (*1* − *u*) ∗ *a* ≤ (*1* − *u*) ∗ *b*
by (*auto intro*!: *mult_left_mono u assms*)
then show *x* ∈ {*a* .. *b*}
unfolding *x_def* **by** (*auto simp*: *algebra_simps*)
next
show ⋀*x*. *x* ∈ {*a*..*b*} ⟹ *x* ∈ *closed_segment a b*
by (*force simp*: *closed_segment_def divide_simps algebra_simps*
intro: *exI*[**where** *x*=(*x* − *a*) / (*b* − *a*) **for** *x*])
qed

lemma *closed_segment_eq_real_ivl*:
fixes *a b*::*real*
shows *closed_segment a b* = (*if a* ≤ *b then* {*a* .. *b*} *else* {*b* .. *a*})
by (*metis closed_segment_commute closed_segment_eq_real_ivl1 nle_le*)

lemma *open_segment_eq_real_ivl*:
fixes *a b*::*real*
shows *open_segment a b* = (*if a* ≤ *b then* {*a*<..<*b*} *else* {*b*<..<*a*})
by (*auto simp*: *closed_segment_eq_real_ivl open_segment_def split*: *if_split_asm*)

lemma *closed_segment_real_eq*:
fixes *u*::*real* **shows** *closed_segment u v* = (λ*x*. (*v* − *u*) ∗ *x* + *u*) ‘ {*0*..*1*}
by (*simp add*: *closed_segment_eq_real_ivl image_affinity_atLeastAtMost*)

lemma *closed_segment_same_Re*:
assumes *Re a* = *Re b*
shows *closed_segment a b* = {*z*. *Re z* = *Re a* ∧ *Im z* ∈ *closed_segment* (*Im*

a) (*Im b*)}
proof *safe*
fix *z* **assume** *z* ∈ *closed_segment a b*
then obtain *u* **where** *u*: *u* ∈ {*0..1*} *z* = *a* + *of_real u* ∗ (*b* − *a*)
by (*auto simp*: *closed_segment_def scaleR_conv_of_real algebra_simps*)
from *assms* **show** *Re z* = *Re a* **by** (*auto simp*: *u*)
from *u*(*1*) **show** *Im z* ∈ *closed_segment* (*Im a*) (*Im b*)
by (*force simp*: *u closed_segment_def algebra_simps*)
next
fix *z* **assume** [*simp*]: *Re z* = *Re a* **and** *Im z* ∈ *closed_segment* (*Im a*) (*Im b*)
then obtain *u* **where** *u*: *u* ∈ {*0..1*} *Im z* = *Im a* + *of_real u* ∗ (*Im b* − *Im a*)
by (*auto simp*: *closed_segment_def scaleR_conv_of_real algebra_simps*)
from *u*(*1*) **show** *z* ∈ *closed_segment a b* **using** *assms*
by (*force simp*: *u closed_segment_def algebra_simps scaleR_conv_of_real complex_eq_iff*)
qed

lemma *closed_segment_same_Im*:
assumes *Im a* = *Im b*
shows *closed_segment a b* = {*z*. *Im z* = *Im a* ∧ *Re z* ∈ *closed_segment* (*Re a*) (*Re b*)}
proof *safe*
fix *z* **assume** *z* ∈ *closed_segment a b*
then obtain *u* **where** *u*: *u* ∈ {*0..1*} *z* = *a* + *of_real u* ∗ (*b* − *a*)
by (*auto simp*: *closed_segment_def scaleR_conv_of_real algebra_simps*)
from *assms* **show** *Im z* = *Im a* **by** (*auto simp*: *u*)
from *u*(*1*) **show** *Re z* ∈ *closed_segment* (*Re a*) (*Re b*)
by (*force simp*: *u closed_segment_def algebra_simps*)
next
fix *z* **assume** [*simp*]: *Im z* = *Im a* **and** *Re z* ∈ *closed_segment* (*Re a*) (*Re b*)
then obtain *u* **where** *u*: *u* ∈ {*0..1*} *Re z* = *Re a* + *of_real u* ∗ (*Re b* − *Re a*)
by (*auto simp*: *closed_segment_def scaleR_conv_of_real algebra_simps*)
from *u*(*1*) **show** *z* ∈ *closed_segment a b* **using** *assms*
by (*force simp*: *u closed_segment_def algebra_simps scaleR_conv_of_real complex_eq_iff*)
qed

lemma *dist_in_closed_segment*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *x* ∈ *closed_segment a b*
shows *dist x a* ≤ *dist a b* ∧ *dist x b* ≤ *dist a b*
by (*metis assms dist_commute dist_norm segment_bound*(*2*) *segment_bound1*)

lemma *dist_in_open_segment*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *x* ∈ *open_segment a b*
shows *dist x a* < *dist a b* ∧ *dist x b* < *dist a b*
by (*metis assms dist_commute dist_norm open_segment_bound1 open_segment_commute*)

lemma *dist_decreases_open_segment_0*:
fixes $x :: 'a :: euclidean\_space$
assumes $x \in open\_segment\ 0\ b$
shows $dist\ c\ x < dist\ c\ 0 \vee dist\ c\ x < dist\ c\ b$
proof (*rule ccontr*, *clarsimp simp*: *not_less*)
obtain u **where** u: $0 \neq b\ 0 < u\ u < 1$ **and** x: $x = u *_R b$
using *assms* **by** (*auto simp*: *in_segment*)
have *xb*: $x \cdot b < b \cdot b$
using $u\ x$ **by** *auto*
assume $norm\ c \leq dist\ c\ x$
then have $c \cdot c \leq (c - x) \cdot (c - x)$
by (*simp add*: *dist_norm norm_le*)
moreover have $0 < x \cdot b$
using $u\ x$ **by** *auto*
ultimately have *less*: $c \cdot b < x \cdot b$
by (*simp add*: *x algebra_simps inner_commute u*)
assume $dist\ c\ b \leq dist\ c\ x$
then have $(c - b) \cdot (c - b) \leq (c - x) \cdot (c - x)$
by (*simp add*: *dist_norm norm_le*)
then have $(b \cdot b) * (1 - u*u) \leq 2 * (b \cdot c) * (1-u)$
by (*simp add*: *x algebra_simps inner_commute*)
then have $(1+u) * (b \cdot b) * (1-u) \leq 2 * (b \cdot c) * (1-u)$
by (*simp add*: *algebra_simps*)
then have $(1+u) * (b \cdot b) \leq 2 * (b \cdot c)$
using ‹$u < 1$› **by** *auto*
with *xb* **have** $c \cdot b \geq x \cdot b$
by (*auto simp*: *x algebra_simps inner_commute*)
with *less* **show** *False* **by** *auto*
qed

proposition *dist_decreases_open_segment*:
fixes $a :: 'a :: euclidean\_space$
assumes $x \in open\_segment\ a\ b$
shows $dist\ c\ x < dist\ c\ a \vee dist\ c\ x < dist\ c\ b$
proof −
have $*$: $x - a \in open\_segment\ 0\ (b - a)$ **using** *assms*
by (*metis diff_self open_segment_translation_eq uminus_add_conv_diff*)
show *?thesis*
using *dist_decreases_open_segment_0* [*OF* $*$, *of* $c-a$] *assms*
by (*simp add*: *dist_norm*)
qed

corollary *open_segment_furthest_le*:
fixes $a\ b\ x\ y :: 'a::euclidean\_space$
assumes $x \in open\_segment\ a\ b$
shows $norm\ (y - x) < norm\ (y - a) \vee\ \ norm\ (y - x) < norm\ (y - b)$
by (*metis assms dist_decreases_open_segment dist_norm*)

corollary *dist_decreases_closed_segment*:

fixes $a :: 'a :: euclidean\_space$
assumes $x \in closed\_segment\ a\ b$
shows $dist\ c\ x \leq dist\ c\ a \vee dist\ c\ x \leq dist\ c\ b$
by (*smt* (*verit*, *ccfv_threshold*) *Un_iff assms closed_segment_eq_open dist_norm empty_iff insertE open_segment_furthest_le*)

corollary *segment_furthest_le*:
fixes $a\ b\ x\ y :: 'a::euclidean\_space$
assumes $x \in closed\_segment\ a\ b$
shows $norm\ (y - x) \leq norm\ (y - a) \vee\ norm\ (y - x) \leq norm\ (y - b)$
by (*metis assms dist_decreases_closed_segment dist_norm*)

lemma *convex_intermediate_ball*:
fixes $a :: 'a :: euclidean\_space$
shows $[\![ball\ a\ r \subseteq T;\ T \subseteq cball\ a\ r]\!] \Longrightarrow convex\ T$
by (*smt* (*verit*) *convex_contains_open_segment dist_decreases_open_segment mem_ball mem_cball subset_eq*)

lemma *csegment_midpoint_subset*: $closed\_segment\ (midpoint\ a\ b)\ b \subseteq closed\_segment\ a\ b$
apply (*clarsimp simp*: *midpoint_def in_segment*)
apply (*rule_tac x=(1 + u) / 2* **in** *exI*)
apply (*simp add*: *algebra_simps add_divide_distrib diff_divide_distrib*)
by (*metis field_sum_of_halves scaleR_left.add*)

lemma *notin_segment_midpoint*:
fixes $a :: 'a :: euclidean\_space$
shows $a \neq b \Longrightarrow a \notin closed\_segment\ (midpoint\ a\ b)\ b$
by (*auto simp*: *dist_midpoint dest*!: *dist_in_closed_segment*)

More lemmas, especially for working with the underlying formula

lemma *segment_eq_compose*:
fixes $a :: 'a :: real\_vector$
shows $(\lambda u.\ (1 - u) *_R a + u *_R b) = (\lambda x.\ a + x)\ o\ (\lambda u.\ u *_R (b - a))$
by (*simp add*: *o_def algebra_simps*)

lemma *segment_degen_1*:
fixes $a :: 'a :: real\_vector$
shows $(1 - u) *_R a + u *_R b = b \longleftrightarrow a=b \vee u=1$
by (*smt* (*verit*, *best*) *add_right_cancel scaleR_cancel_left scaleR_collapse*)

lemma *segment_degen_0*:
fixes $a :: 'a :: real\_vector$
shows $(1 - u) *_R a + u *_R b = a \longleftrightarrow a=b \vee u=0$
using *segment_degen_1* [*of 1−u b a*] **by** (*auto simp*: *algebra_simps*)

lemma *add_scaleR_degen*:
fixes $a\ b ::'a::real\_vector$

assumes $(u *_R b + v *_R a) = (u *_R a + v *_R b)$ $u \neq v$
shows $a=b$
by (*smt* (*verit*) *add_diff_cancel_left'* *add_diff_eq* *assms* *scaleR_cancel_left* *scaleR_left.diff*)

lemma *closed_segment_image_interval*:
$closed\_segment\ a\ b = (\lambda u.\ (1 - u) *_R a + u *_R b)\ `\ \{0..1\}$
by (*auto simp*: *set_eq_iff* *image_iff* *closed_segment_def*)

lemma *open_segment_image_interval*:
$open\_segment\ a\ b = (if\ a=b\ then\ \{\}\ else\ (\lambda u.\ (1 - u) *_R a + u *_R b)\ `\ \{0<..<1\})$
by (*auto simp*: *open_segment_def* *closed_segment_def* *segment_degen_0* *segment_degen_1*)

lemmas *segment_image_interval* = *closed_segment_image_interval* *open_segment_image_interval*

lemma *closed_segment_neq_empty* [*simp*]: $closed\_segment\ a\ b \neq \{\}$
by *auto*

lemma *open_segment_eq_empty* [*simp*]: $open\_segment\ a\ b = \{\} \longleftrightarrow a = b$
by (*simp add*: *segment_image_interval*(*2*))

lemma *open_segment_eq_empty'* [*simp*]: $\{\} = open\_segment\ a\ b \longleftrightarrow a = b$
using *open_segment_eq_empty* **by** *blast*

lemmas *segment_eq_empty* = *closed_segment_neq_empty* *open_segment_eq_empty*

lemma *inj_segment*:
fixes $a :: 'a :: real\_vector$
assumes $a \neq b$
shows $inj\_on\ (\lambda u.\ (1 - u) *_R a + u *_R b)\ I$
proof
fix $x\ y$
assume $(1 - x) *_R a + x *_R b = (1 - y) *_R a + y *_R b$
then have $x *_R (b - a) = y *_R (b - a)$
by (*simp add*: *algebra_simps*)
with *assms* **show** $x = y$
by (*simp add*: *real_vector.scale_right_imp_eq*)
qed

lemma *finite_closed_segment* [*simp*]: $finite(closed\_segment\ a\ b) \longleftrightarrow a = b$
using *infinite_Icc* [*OF zero_less_one*] *finite_imageD* [*OF _ inj_segment* [*of a b*]]
unfolding *segment_image_interval*
by (*smt* (*verit*, *del_insts*) *finite.emptyI* *finite_insert* *finite_subset* *image_subset_iff* *insertCI* *segment_degen_0*)

lemma *finite_open_segment* [*simp*]: $finite(open\_segment\ a\ b) \longleftrightarrow a = b$

by (*auto simp*: *open_segment_def*)

lemmas *finite_segment* = *finite_closed_segment finite_open_segment*

lemma *closed_segment_eq_sing*: *closed_segment a b* = {*c*} ⟷ *a* = *c* ∧ *b* = *c*
by *auto*

lemma *open_segment_eq_sing*: *open_segment a b* ≠ {*c*}
by (*metis finite_insert finite_open_segment insert_not_empty open_segment_image_interval*)

lemmas *segment_eq_sing* = *closed_segment_eq_sing open_segment_eq_sing*

lemma *compact_segment* [*simp*]:
fixes *a* :: *'a*::*real_normed_vector*
shows *compact* (*closed_segment a b*)
by (*auto simp*: *segment_image_interval intro*!: *compact_continuous_image continuous_intros*)

lemma *closed_segment* [*simp*]:
fixes *a* :: *'a*::*real_normed_vector*
shows *closed* (*closed_segment a b*)
by (*simp add*: *compact_imp_closed*)

lemma *closure_closed_segment* [*simp*]:
fixes *a* :: *'a*::*real_normed_vector*
shows *closure*(*closed_segment a b*) = *closed_segment a b*
by *simp*

lemma *open_segment_bound*:
assumes *x* ∈ *open_segment a b*
shows *norm* (*x* − *a*) < *norm* (*b* − *a*) *norm* (*x* − *b*) < *norm* (*b* − *a*)
by (*metis assms norm_minus_commute open_segment_bound1 open_segment_commute*)+

lemma *closure_open_segment* [*simp*]:
closure (*open_segment a b*) = (*if a* = *b then* {} *else closed_segment a b*)
for *a* :: *'a*::*euclidean_space*
proof (*cases a* = *b*)
case *True*
then show *?thesis*
by *simp*
next
case *False*
have *closure* ((λ*u*. *u* ∗*R* (*b* − *a*)) ‘ {*0*<..<*1*}) = (λ*u*. *u* ∗*R* (*b* − *a*)) ‘ *closure* {*0*<..<*1*}
proof (*rule closure_injective_linear_image* [*symmetric*])
qed (*use False* **in** ‹*auto intro*!: *injI*›)
then have *closure*
((λ*u*. (*1* − *u*) ∗*R* *a* + *u* ∗*R* *b*) ‘ {*0*<..<*1*}) = (λ*x*. (*1* − *x*) ∗*R* *a* + *x* ∗*R* *b*) ‘ *closure* {*0*<..<*1*}

using *closure_translation* [*of a* ((λ*x*. *x* ∗$_R$ *b* − *x* ∗$_R$ *a*) ‘ {*0*<..<*1*})]
by (*simp add*: *segment_eq_compose field_simps scaleR_diff_left scaleR_diff_right image_image*)
then show *?thesis*
by (*simp add*: *segment_image_interval closure_greaterThanLessThan* [*symmetric*] *del*: *closure_greaterThanLessThan*)
qed

lemma *closed_open_segment_iff* [*simp*]:
fixes *a* :: ′*a*::*euclidean_space* **shows** *closed*(*open_segment a b*) ⟷ *a* = *b*
by (*metis open_segment_def DiffE closure_eq closure_open_segment ends_in_segment*(*1*) *insert_iff segment_image_interval*(*2*))

lemma *compact_open_segment_iff* [*simp*]:
fixes *a* :: ′*a*::*euclidean_space* **shows** *compact*(*open_segment a b*) ⟷ *a* = *b*
by (*simp add*: *bounded_open_segment compact_eq_bounded_closed*)

lemma *convex_closed_segment* [*iff*]: *convex* (*closed_segment a b*)
unfolding *segment_convex_hull* **by**(*rule convex_convex_hull*)

lemma *convex_open_segment* [*iff*]: *convex* (*open_segment a b*)
proof −
have *convex* ((λ*u*. *u* ∗$_R$ (*b* − *a*)) ‘ {*0*<..<*1*})
by (*rule convex_linear_image*) *auto*
then have *convex* ((+) *a* ‘ (λ*u*. *u* ∗$_R$ (*b* − *a*)) ‘ {*0*<..<*1*})
by (*rule convex_translation*)
then show *?thesis*
by (*simp add*: *image_image open_segment_image_interval segment_eq_compose field_simps scaleR_diff_left scaleR_diff_right*)
qed

lemmas *convex_segment* = *convex_closed_segment convex_open_segment*

lemma *subset_closed_segment*:
closed_segment a b ⊆ *closed_segment c d* ⟷
a ∈ *closed_segment c d* ∧ *b* ∈ *closed_segment c d*
using *closed_segment_subset convex_closed_segment ends_in_segment in_mono*
by *blast*

lemma *subset_co_segment*:
closed_segment a b ⊆ *open_segment c d* ⟷
a ∈ *open_segment c d* ∧ *b* ∈ *open_segment c d*
using *closed_segment_subset* **by** *blast*

lemma *subset_open_segment*:
fixes *a* :: ′*a*::*euclidean_space*
shows *open_segment a b* ⊆ *open_segment c d* ⟷
a = *b* ∨ *a* ∈ *closed_segment c d* ∧ *b* ∈ *closed_segment c d*
(**is** *?lhs* = *?rhs*)

```
proof (cases a = b)
  case True then show ?thesis by simp
next
  case False show ?thesis
  proof
    assume rhs: ?rhs
    with ‹a ≠ b› have c ≠ d
      using closed_segment_idem singleton_iff by auto
    have ∃uc. (1 − u) *R ((1 − ua) *R c + ua *R d) + u *R ((1 − ub) *R c +
ub *R d) =
              (1 − uc) *R c + uc *R d ∧ 0 < uc ∧ uc < 1
      if neq: (1 − ua) *R c + ua *R d ≠ (1 − ub) *R c + ub *R d c ≠ d
        and a = (1 − ua) *R c + ua *R d b = (1 − ub) *R c + ub *R d
        and u: 0 < u u < 1 and uab: 0 ≤ ua ua ≤ 1 0 ≤ ub ub ≤ 1
      for u ua ub
    proof −
      have ua ≠ ub
        using neq by auto
      moreover have (u − 1) * ua ≤ 0 using u uab
        by (simp add: mult_nonpos_nonneg)
      ultimately have lt: (u − 1) * ua < u * ub using u uab
        by (metis antisym_conv diff_ge_0_iff_ge le_less_trans mult_eq_0_iff
mult_le_0_iff not_less)
      have p * ua + q * ub < p+q if p: 0 < p and  q: 0 < q for p q
      proof −
        have ¬ p ≤ 0 ¬ q ≤ 0
          using p q not_less by blast+
        then show ?thesis
          by (smt (verit) ‹ua ≠ ub› mult_cancel_left1 mult_left_le uab(2) uab(4))
      qed
      then have (1 − u) * ua + u * ub < 1 using u ‹ua ≠ ub›
        by (metis diff_add_cancel diff_gt_0_iff_gt)
      with lt show ?thesis
        by (rule_tac x=ua + u*(ub−ua) in exI) (simp add: algebra_simps)
    qed
    with rhs ‹a ≠ b› ‹c ≠ d› show ?lhs
      unfolding open_segment_image_interval closed_segment_def
      by (fastforce simp add:)
  next
    assume lhs: ?lhs
    with ‹a ≠ b› have c ≠ d
      by (meson finite_open_segment rev_finite_subset)
    have closure (open_segment a b) ⊆ closure (open_segment c d)
      using lhs closure_mono by blast
    then have closed_segment a b ⊆ closed_segment c d
      by (simp add: ‹a ≠ b› ‹c ≠ d›)
    then show ?rhs
      by (force simp: ‹a ≠ b›)
  qed
```

qed

lemma *closed_segment_same_fst*:
fst a = fst b ⟹ closed_segment a b = {fst a} × closed_segment (snd a) (snd b)
by (*auto simp*: *closed_segment_def scaleR_prod_def*)

lemma *closed_segment_same_snd*:
snd a = snd b ⟹ closed_segment a b = closed_segment (fst a) (fst b) × {snd a}
by (*auto simp*: *closed_segment_def scaleR_prod_def*)

lemma *subset_oc_segment*:
fixes *a* :: *'a*::*euclidean_space*
shows *open_segment a b ⊆ closed_segment c d ⟷*
a = b ∨ a ∈ closed_segment c d ∧ b ∈ closed_segment c d
(**is** *?lhs = ?rhs*)
proof
show *?lhs ⟹ ?rhs*
by (*metis closure_closed_segment closure_mono closure_open_segment subset_closed_segment*)
show *?rhs ⟹ ?lhs*
by (*meson dual_order.trans segment_open_subset_closed subset_open_segment*)
qed

lemmas *subset_segment = subset_closed_segment subset_co_segment subset_oc_segment subset_open_segment*

lemma *dist_half_times2*:
fixes *a* :: *'a* :: *real_normed_vector*
shows *dist ((1 / 2) *R (a + b)) x * 2 = dist (a+b) (2 *R x)*
proof −
have *norm ((1 / 2) *R (a + b) − x) * 2 = norm (2 *R ((1 / 2) *R (a + b) − x))*
by *simp*
also have *... = norm ((a + b) − 2 *R x)*
by (*simp add*: *real_vector.scale_right_diff_distrib*)
finally show *?thesis*
by (*simp only*: *dist_norm*)
qed

lemma *closed_segment_as_ball*:
*closed_segment a b = affine hull {a,b} ∩ cball(inverse 2 *R (a + b))(norm(b − a) / 2)*
proof (*cases b = a*)
case *True* **then show** *?thesis* **by** (*auto simp*: *hull_inc*)
next
case *False*
then have *∗*: *((∃ u v. x = u *R a + v *R b ∧ u + v = 1) ∧*
*dist ((1 / 2) *R (a + b)) x * 2 ≤ norm (b − a)) =*

```
                (∃u. x = (1 − u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1) for x
  proof −
    have ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 ≤ norm (b − a)) =
          ((∃u. x = (1 − u) *R a + u *R b) ∧
                dist ((1 / 2) *R (a + b)) x * 2 ≤ norm (b − a))
      unfolding eq_diff_eq [symmetric] by simp
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
                        norm ((a+b) − (2 *R x)) ≤ norm (b − a))
      by (simp add: dist_half_times2) (simp add: dist_norm)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
          norm ((a+b) − (2 *R ((1 − u) *R a + u *R b))) ≤ norm (b − a))
      by auto
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
              norm ((1 − u * 2) *R (b − a)) ≤ norm (b − a))
      by (simp add: algebra_simps scaleR_2)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
                        |1 − u * 2| * norm (b − a) ≤ norm (b − a))
      by simp
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧ |1 − u * 2| ≤ 1)
      by (simp add: mult_le_cancel_right2 False)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1)
      by auto
    finally show ?thesis .
  qed
  show ?thesis
    by (simp add: affine_hull_2 Set.set_eq_iff closed_segment_def ∗)
qed

lemma open_segment_as_ball:
    open_segment a b =
     affine hull {a,b} ∩ ball(inverse 2 *R (a + b))(norm(b − a) / 2)
proof (cases b = a)
  case True then show ?thesis by (auto simp: hull_inc)
next
  case False
  then have ∗: ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b − a)) =
               (∃u. x = (1 − u) *R a + u *R b ∧ 0 < u ∧ u < 1) for x
  proof −
    have ((∃u v. x = u *R a + v *R b ∧ u + v = 1) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b − a)) =
          ((∃u. x = (1 − u) *R a + u *R b) ∧
                dist ((1 / 2) *R (a + b)) x * 2 < norm (b − a))
      unfolding eq_diff_eq [symmetric] by simp
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
                        norm ((a+b) − (2 *R x)) < norm (b − a))
      by (simp add: dist_half_times2) (simp add: dist_norm)
    also have ... = (∃u. x = (1 − u) *R a + u *R b ∧
```

```
        norm ((a+b) - (2 *R ((1 - u) *R a + u *R b))) < norm (b - a))
    by auto
  also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
        norm ((1 - u * 2) *R (b - a)) < norm (b - a))
    by (simp add: algebra_simps scaleR_2)
  also have ... = (∃u. x = (1 - u) *R a + u *R b ∧
                  |1 - u * 2| * norm (b - a) < norm (b - a))
    by simp
  also have ... = (∃u. x = (1 - u) *R a + u *R b ∧ |1 - u * 2| < 1)
    by (simp add: mult_le_cancel_right2 False)
  also have ... = (∃u. x = (1 - u) *R a + u *R b ∧ 0 < u ∧ u < 1)
    by auto
  finally show ?thesis .
 qed
 show ?thesis
  using False by (force simp: affine_hull_2 Set.set_eq_iff open_segment_image_interval
*)
qed

lemmas segment_as_ball = closed_segment_as_ball open_segment_as_ball

lemma connected_segment [iff]:
 fixes x :: 'a :: real_normed_vector
 shows connected (closed_segment x y)
 by (simp add: convex_connected)

lemma is_interval_closed_segment_1[intro, simp]: is_interval (closed_segment
a b) for a b::real
 unfolding closed_segment_eq_real_ivl
 by (auto simp: is_interval_def)

lemma IVT'_closed_segment_real:
 fixes f :: real ⇒ real
 assumes y ∈ closed_segment (f a) (f b)
 assumes continuous_on (closed_segment a b) f
 shows ∃x ∈ closed_segment a b. f x = y
 using IVT'[of f a y b]
   IVT'[of -f a -y b]
   IVT'[of f b y a]
   IVT'[of -f b -y a] assms
 by (cases a ≤ b; cases f b ≥ f a) (auto simp: closed_segment_eq_real_ivl con-
tinuous_on_minus)
```

5.2.4 Betweenness

```
definition between = (λ(a,b) x. x ∈ closed_segment a b)

lemma betweenI:
 assumes 0 ≤ u u ≤ 1 x = (1 - u) *R a + u *R b
```

shows *between* (*a*, *b*) *x*
using *assms* **unfolding** *between_def closed_segment_def* **by** *auto*

lemma *betweenE*:
assumes *between* (*a*, *b*) *x*
obtains *u* **where** *0* ≤ *u u* ≤ *1 x* = (*1* − *u*) *∗*$_R$ *a* + *u ∗*$_R$ *b*
using *assms* **unfolding** *between_def closed_segment_def* **by** *auto*

lemma *between_implies_scaled_diff*:
assumes *between* (*S*, *T*) *X between* (*S*, *T*) *Y S* ≠ *Y*
obtains *c* **where** (*X* − *Y*) = *c ∗*$_R$ (*S* − *Y*)
proof −
from ‹*between* (*S*, *T*) *X*› **obtain** u_X **where** *X*: *X* = u_X *∗*$_R$ *S* + (*1* − u_X) *∗*$_R$ *T*
by (*metis add.commute betweenE eq_diff_eq*)
from ‹*between* (*S*, *T*) *Y*› **obtain** u_Y **where** *Y*: *Y* = u_Y *∗*$_R$ *S* + (*1* − u_Y) *∗*$_R$ *T*
by (*metis add.commute betweenE eq_diff_eq*)
have *X* − *Y* = (u_X − u_Y) *∗*$_R$ (*S* − *T*)
by (*simp add*: *X Y scaleR_left.diff scaleR_right_diff_distrib*)
moreover from *Y* **have** *S* − *Y* = (*1* − u_Y) *∗*$_R$ (*S* − *T*)
by (*simp add*: *real_vector.scale_left_diff_distrib real_vector.scale_right_diff_distrib*)
moreover note ‹*S* ≠ *Y*›
ultimately have (*X* − *Y*) = ((u_X − u_Y) / (*1* − u_Y)) *∗*$_R$ (*S* − *Y*) **by** *auto*
from *this that* **show** *thesis* **by** *blast*
qed

lemma *between_mem_segment*: *between* (*a*,*b*) *x* ⟷ *x* ∈ *closed_segment a b*
unfolding *between_def* **by** *auto*

lemma *between*: *between* (*a*, *b*) (*x*::*'a*::*euclidean_space*) ⟷ *dist a b* = (*dist a x*) + (*dist x b*)
proof (*cases a* = *b*)
case *True*
then show *?thesis*
by (*auto simp add*: *between_def dist_commute*)
next
case *False*
then have *Fal*: *norm* (*a* − *b*) ≠ *0* **and** *Fal2*: *norm* (*a* − *b*) > *0*
by *auto*
have *∗*: ⋀*u*. *a* − ((*1* − *u*) *∗*$_R$ *a* + *u ∗*$_R$ *b*) = *u ∗*$_R$ (*a* − *b*)
by (*auto simp add*: *algebra_simps*)
have *norm* (*a* − *x*) *∗*$_R$ (*x* − *b*) = *norm* (*x* − *b*) *∗*$_R$ (*a* − *x*) **if** *x* = (*1* − *u*) *∗*$_R$ *a* + *u ∗*$_R$ *b 0* ≤ *u u* ≤ *1* **for** *u*
proof −
have *∗*: *a* − *x* = *u ∗*$_R$ (*a* − *b*) *x* − *b* = (*1* − *u*) *∗*$_R$ (*a* − *b*)
unfolding *that*(*1*) **by** (*auto simp add*:*algebra_simps*)
show *norm* (*a* − *x*) *∗*$_R$ (*x* − *b*) = *norm* (*x* − *b*) *∗*$_R$ (*a* − *x*)
unfolding *norm_minus_commute*[*of x a*] *∗* **using** ‹*0* ≤ *u*› ‹*u* ≤ *1*›

```
    by simp
  qed
  moreover have ∃u. x = (1 − u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1 if dist a b
= dist a x + dist x b
  proof −
    let ?β = norm (a − x) / norm (a − b)
    show ∃u. x = (1 − u) *R a + u *R b ∧ 0 ≤ u ∧ u ≤ 1
    proof (intro exI conjI)
      show ?β ≤ 1
        using Fal2 unfolding that[unfolded dist_norm] norm_ge_zero by auto
      show x = (1 − ?β) *R a + (?β) *R b
      proof (subst euclidean_eq_iff; intro ballI)
        fix i :: 'a
        assume i: i ∈ Basis
        have ((1 − ?β) *R a + (?β) *R b) · i
              = ((norm (a − b) − norm (a − x)) * (a · i) + norm (a − x) * (b ·
i)) / norm (a − b)
          using Fal by (auto simp add: field_simps inner_simps)
        also have ... = x·i
          apply (rule divide_eq_imp[OF Fal])
          unfolding that[unfolded dist_norm]
          using that[unfolded dist_triangle_eq] i
          apply (subst (asm) euclidean_eq_iff)
           apply (auto simp add: field_simps inner_simps)
          done
        finally show x · i = ((1 − ?β) *R a + (?β) *R b) · i
          by auto
      qed
    qed (use Fal2 in auto)
  qed
  ultimately show ?thesis
    by (force simp add: between_def closed_segment_def dist_triangle_eq)
qed

lemma between_midpoint:
  fixes a :: 'a::euclidean_space
  shows between (a,b) (midpoint a b) (is ?t1)
    and between (b,a) (midpoint a b) (is ?t2)
proof −
  have *: ⋀x y z. x = (1/2::real) *R z ⟹ y = (1/2) *R z ⟹ norm z = norm
x + norm y
    by auto
  show ?t1 ?t2
    unfolding between midpoint_def dist_norm
    by (auto simp add: field_simps inner_simps euclidean_eq_iff[where 'a='a]
intro!: *)
qed

lemma between_mem_convex_hull:
```

between (*a*,*b*) *x* ⟷ *x* ∈ *convex hull* {*a*,*b*}
unfolding *between_mem_segment segment_convex_hull* **..**

lemma *between_triv_iff* [*simp*]: *between* (*a*,*a*) *b* ⟷ *a*=*b*
by (*auto simp*: *between_def*)

lemma *between_triv1* [*simp*]: *between* (*a*,*b*) *a*
by (*auto simp*: *between_def*)

lemma *between_triv2* [*simp*]: *between* (*a*,*b*) *b*
by (*auto simp*: *between_def*)

lemma *between_commute*:
between (*a*,*b*) = *between* (*b*,*a*)
by (*auto simp*: *between_def closed_segment_commute*)

lemma *between_antisym*:
fixes *a* :: *'a* :: *euclidean_space*
shows ⟦*between* (*b*,*c*) *a*; *between* (*a*,*c*) *b*⟧ ⟹ *a* = *b*
by (*auto simp*: *between dist_commute*)

lemma *between_trans*:
fixes *a* :: *'a* :: *euclidean_space*
shows ⟦*between* (*b*,*c*) *a*; *between* (*a*,*c*) *d*⟧ ⟹ *between* (*b*,*c*) *d*
using *dist_triangle2* [*of b c d*] *dist_triangle3* [*of b d a*]
by (*auto simp*: *between dist_commute*)

lemma *between_norm*:
fixes *a* :: *'a* :: *euclidean_space*
shows *between* (*a*,*b*) *x* ⟷ *norm*(*x* − *a*) ∗$_R$ (*b* − *x*) = *norm*(*b* − *x*) ∗$_R$ (*x* − *a*)
by (*auto simp*: *between dist_triangle_eq norm_minus_commute algebra_simps*)

lemma *between_swap*:
fixes *A B X Y* :: *'a*::*euclidean_space*
assumes *between* (*A*, *B*) *X*
assumes *between* (*A*, *B*) *Y*
shows *between* (*X*, *B*) *Y* ⟷ *between* (*A*, *Y*) *X*
using *assms* **by** (*auto simp add*: *between*)

lemma *between_translation* [*simp*]: *between* (*a* + *y*,*a* + *z*) (*a* + *x*) ⟷ *between* (*y*,*z*) *x*
by (*auto simp*: *between_def*)

lemma *between_trans_2*:
fixes *a* :: *'a* :: *euclidean_space*
shows ⟦*between* (*b*,*c*) *a*; *between* (*a*,*b*) *d*⟧ ⟹ *between* (*c*,*d*) *a*
by (*metis between_commute between_swap between_trans*)

lemma *between_scaleR_lift* [*simp*]:
fixes $v :: 'a::euclidean\_space$
shows $between\ (a *_R v,\ b *_R v)\ (c *_R v) \longleftrightarrow v = 0 \vee between\ (a,\ b)\ c$
by (*simp add*: *between dist_norm flip*: *scaleR_left_diff_distrib distrib_right*)

lemma *between_1*:
fixes $x::real$
shows $between\ (a,b)\ x \longleftrightarrow (a \le x \wedge x \le b) \vee (b \le x \wedge x \le a)$
by (*auto simp*: *between_mem_segment closed_segment_eq_real_ivl*)

end

5.3 Convex Sets and Functions on (Normed) Euclidean Spaces

theory *Convex_Euclidean_Space*
imports
Convex Topology_Euclidean_Space Line_Segment
begin

5.3.1 Topological Properties of Convex Sets and Functions

lemma *aff_dim_cball*:
fixes $a :: 'n::euclidean\_space$
assumes $e > 0$
shows $aff\_dim\ (cball\ a\ e) = int\ (DIM('n))$
proof −
have $(\lambda x.\ a + x)\ `\ (cball\ 0\ e) \subseteq cball\ a\ e$
unfolding *cball_def dist_norm* **by** *auto*
then have $aff\_dim\ (cball\ (0 :: 'n::euclidean\_space)\ e) \le aff\_dim\ (cball\ a\ e)$
using *aff_dim_translation_eq*[*of a cball 0 e*]
aff_dim_subset[*of* (+) *a* ` *cball 0 e cball a e*]
by *auto*
moreover have $aff\_dim\ (cball\ (0 :: 'n::euclidean\_space)\ e) = int\ (DIM('n))$
using *hull_inc*[*of* $(0 :: 'n::euclidean\_space)$ *cball 0 e*]
centre_in_cball[*of* $(0 :: 'n::euclidean\_space)$] *assms*
by (*simp add*: *dim_cball*[*of e*] *aff_dim_zero*[*of cball 0 e*])
ultimately show *?thesis*
using *aff_dim_le_DIM*[*of cball a e*] **by** *auto*
qed

lemma *aff_dim_open*:
fixes $S :: 'n::euclidean\_space\ set$
assumes *open S*
and $S \neq \{\}$
shows $aff\_dim\ S = int\ (DIM('n))$
proof −
obtain x **where** $x \in S$

using *assms* **by** *auto*
then obtain *e* **where** *e*: *e* > *0 cball x e* ⊆ *S*
using *open_contains_cball*[*of S*] *assms* **by** *auto*
then have *aff_dim* (*cball x e*) ≤ *aff_dim S*
using *aff_dim_subset* **by** *auto*
with *e* **show** *?thesis*
using *aff_dim_cball*[*of e x*] *aff_dim_le_DIM*[*of S*] **by** *auto*
qed

lemma *low_dim_interior*:
fixes *S* :: *'n*::*euclidean_space set*
assumes ¬ *aff_dim S* = *int* (*DIM*(*'n*))
shows *interior S* = {}
proof −
have *aff_dim*(*interior S*) ≤ *aff_dim S*
using *interior_subset aff_dim_subset*[*of interior S S*] **by** *auto*
then show *?thesis*
using *aff_dim_open*[*of interior S*] *aff_dim_le_DIM*[*of S*] *assms* **by** *auto*
qed

corollary *empty_interior_lowdim*:
fixes *S* :: *'n*::*euclidean_space set*
shows *dim S* < *DIM* (*'n*) ⟹ *interior S* = {}
by (*metis low_dim_interior affine_hull_UNIV dim_affine_hull less_not_refl dim_UNIV*)

corollary *aff_dim_nonempty_interior*:
fixes *S* :: *'a*::*euclidean_space set*
shows *interior S* ≠ {} ⟹ *aff_dim S* = *DIM*(*'a*)
by (*metis low_dim_interior*)

5.3.2 Relative interior of a set

definition *rel_interior S* =
{*x*. ∃ *T*. *openin* (*top_of_set* (*affine hull S*)) *T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*}

lemma *rel_interior_mono*:
⟦*S* ⊆ *T*; *affine hull S* = *affine hull T*⟧
⟹ (*rel_interior S*) ⊆ (*rel_interior T*)
by (*auto simp*: *rel_interior_def*)

lemma *rel_interior_maximal*:
⟦*T* ⊆ *S*; *openin*(*top_of_set* (*affine hull S*)) *T*⟧ ⟹ *T* ⊆ (*rel_interior S*)
by (*auto simp*: *rel_interior_def*)

lemma *rel_interior*: *rel_interior S* = {*x* ∈ *S*. ∃ *T*. *open T* ∧ *x* ∈ *T* ∧ *T* ∩ *affine hull S* ⊆ *S*}
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*

by (*force simp add*: *rel_interior_def openin_open*)
{ fix *x T*
assume ∗: *x* ∈ *S open T x* ∈ *T T* ∩ *affine hull S* ⊆ *S*
then have ∗∗: *x* ∈ *T* ∩ *affine hull S*
using *hull_inc* **by** *auto*
with ∗ **have** ∃ *Tb*. (∃ *Ta*. *open Ta* ∧ *Tb* = *affine hull S* ∩ *Ta*) ∧ *x* ∈ *Tb* ∧ *Tb* ⊆ *S*
by (*rule_tac x* = *T* ∩ (*affine hull S*) **in** *exI*) *auto*
}
then show *?rhs* ⊆ *?lhs*
by (*force simp add*: *rel_interior_def openin_open*)
qed

lemma *mem_rel_interior*: *x* ∈ *rel_interior S* ⟷ (∃ *T*. *open T* ∧ *x* ∈ *T* ∩ *S* ∧ *T* ∩ *affine hull S* ⊆ *S*)
by (*auto simp*: *rel_interior*)

lemma *mem_rel_interior_ball*:
x ∈ *rel_interior S* ⟷ *x* ∈ *S* ∧ (∃ *e*. *e* > *0* ∧ *ball x e* ∩ *affine hull S* ⊆ *S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?rhs* **then show** *?lhs*
by (*simp add*: *rel_interior*) (*meson Elementary_Metric_Spaces.open_ball centre_in_ball*)
qed (*force simp*: *rel_interior open_contains_ball*)

lemma *rel_interior_ball*:
rel_interior S = {*x* ∈ *S*. ∃ *e*. *e* > *0* ∧ *ball x e* ∩ *affine hull S* ⊆ *S*}
using *mem_rel_interior_ball* [*of* _ *S*] **by** *auto*

lemma *mem_rel_interior_cball*:
x ∈ *rel_interior S* ⟷ *x* ∈ *S* ∧ (∃ *e*. *e* > *0* ∧ *cball x e* ∩ *affine hull S* ⊆ *S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?rhs* **then obtain** *e* **where** *x* ∈ *S e* > *0 cball x e* ∩ *affine hull S* ⊆ *S*
by (*auto simp*: *rel_interior*)
then have *ball x e* ∩ *affine hull S* ⊆ *S*
by *auto*
then show *?lhs*
using ‹*0* < *e*› ‹*x* ∈ *S*› *rel_interior_ball* **by** *auto*
qed (*force simp*: *rel_interior open_contains_cball*)

lemma *rel_interior_cball*:
rel_interior S = {*x* ∈ *S*. ∃ *e*. *e* > *0* ∧ *cball x e* ∩ *affine hull S* ⊆ *S*}
using *mem_rel_interior_cball* [*of* _ *S*] **by** *auto*

lemma *rel_interior_empty* [*simp*]: *rel_interior* {} = {}
by (*auto simp*: *rel_interior_def*)

lemma *affine_hull_sing* [*simp*]: *affine hull* {*a* :: *'n*::*euclidean_space*} = {*a*}
by (*metis affine_hull_eq affine_sing*)

lemma *rel_interior_sing* [*simp*]:
fixes *a* :: *'n*::*euclidean_space* **shows** *rel_interior* {*a*} = {*a*}
proof −
have ∃ *x*::*real*. *0* < *x*
using *zero_less_one* **by** *blast*
then show *?thesis*
by (*auto simp*: *rel_interior_ball*)
qed

lemma *subset_rel_interior*:
fixes *S T* :: *'n*::*euclidean_space set*
assumes *S* ⊆ *T*
and *affine hull S* = *affine hull T*
shows *rel_interior S* ⊆ *rel_interior T*
using *assms* **by** (*auto simp*: *rel_interior_def*)

lemma *rel_interior_subset*: *rel_interior S* ⊆ *S*
by (*auto simp*: *rel_interior_def*)

lemma *rel_interior_subset_closure*: *rel_interior S* ⊆ *closure S*
using *rel_interior_subset* **by** (*auto simp*: *closure_def*)

lemma *interior_subset_rel_interior*: *interior S* ⊆ *rel_interior S*
by (*auto simp*: *rel_interior interior_def*)

lemma *interior_rel_interior*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *aff_dim S* = *int*(*DIM*(*'n*))
shows *rel_interior S* = *interior S*
proof −
have *affine hull S* = *UNIV*
using *assms affine_hull_UNIV*[*of S*] **by** *auto*
then show *?thesis*
unfolding *rel_interior interior_def* **by** *auto*
qed

lemma *rel_interior_interior*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *affine hull S* = *UNIV*
shows *rel_interior S* = *interior S*
using *assms* **unfolding** *rel_interior interior_def* **by** *auto*

lemma *rel_interior_open*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *open S*
shows *rel_interior S* = *S*

by (*metis assms interior_eq interior_subset_rel_interior rel_interior_subset set_eq_subset*)

lemma *interior_rel_interior_gen*:
fixes S :: *'n::euclidean_space set*
shows *interior* S = (*if aff_dim* S = *int*(*DIM*(*'n*)) *then rel_interior* S *else* {})
by (*metis interior_rel_interior low_dim_interior*)

lemma *rel_interior_nonempty_interior*:
fixes S :: *'n::euclidean_space set*
shows *interior* $S \neq \{\} \Longrightarrow$ *rel_interior* S = *interior* S
by (*metis interior_rel_interior_gen*)

lemma *affine_hull_nonempty_interior*:
fixes S :: *'n::euclidean_space set*
shows *interior* $S \neq \{\} \Longrightarrow$ *affine hull* S = *UNIV*
by (*metis affine_hull_UNIV interior_rel_interior_gen*)

lemma *rel_interior_affine_hull* [*simp*]:
fixes S :: *'n::euclidean_space set*
shows *rel_interior* (*affine hull* S) = *affine hull* S
proof −
have ∗: *rel_interior* (*affine hull* S) ⊆ *affine hull* S
using *rel_interior_subset* **by** *auto*
{
fix x
assume x: $x \in$ *affine hull* S
define e :: *real* **where** e = *1*
then have $e > 0$ *ball* x $e \cap$ *affine hull* (*affine hull* S) ⊆ *affine hull* S
using *hull_hull*[*of* _ S] **by** *auto*
then have $x \in$ *rel_interior* (*affine hull* S)
using x *rel_interior_ball*[*of affine hull* S] **by** *auto*
}
then show *?thesis* **using** ∗ **by** *auto*
qed

lemma *rel_interior_UNIV* [*simp*]: *rel_interior* (*UNIV* :: (*'n::euclidean_space*) *set*) = *UNIV*
by (*metis open_UNIV rel_interior_open*)

lemma *rel_interior_convex_shrink*:
fixes S :: *'a::euclidean_space set*
assumes *convex* S
and $c \in$ *rel_interior* S
and $x \in S$
and $0 < e$
and $e \leq 1$
shows $x - e *_R (x - c) \in$ *rel_interior* S
proof −

```
  obtain d where d > 0 and d: ball c d ∩ affine hull S ⊆ S
    using assms(2) unfolding  mem_rel_interior_ball by auto
  {
    fix y
    assume as: dist (x − e *R (x − c)) y < e * d y ∈ affine hull S
    have *: y = (1 − (1 − e)) *R ((1 / e) *R y − ((1 − e) / e) *R x) + (1 − e)
*R x
      using ‹e > 0› by (auto simp: scaleR_left_diff_distrib scaleR_right_diff_distrib)
    have x ∈ affine hull S
      using assms hull_subset[of S] by auto
    moreover have 1 / e + − ((1 − e) / e) = 1
      using ‹e > 0› left_diff_distrib[of 1 (1−e) 1/e] by auto
    ultimately have **: (1 / e) *R y − ((1 − e) / e) *R x ∈ affine hull S
     using as affine_affine_hull[of S] mem_affine[of affine hull S y x (1 / e) −((1
− e) / e)]
      by (simp add: algebra_simps)
    have c − ((1 / e) *R y − ((1 − e) / e) *R x) = (1 / e) *R (e *R c − y + (1
− e) *R x)
      using ‹e > 0›
      by (auto simp: euclidean_eq_iff[where 'a='a] field_simps inner_simps)
    then have dist c ((1 / e) *R y − ((1 − e) / e) *R x) = |1/e| * norm (e *R c
− y + (1 − e) *R x)
      unfolding dist_norm norm_scaleR[symmetric] by auto
    also have ... = |1/e| * norm (x − e *R (x − c) − y)
      by (auto intro!:arg_cong[where f=norm] simp add: algebra_simps)
    also have ... < d
      using as[unfolded dist_norm] and ‹e > 0›
      by (auto simp:pos_divide_less_eq[OF ‹e > 0›] mult.commute)
    finally have (1 / e) *R y − ((1 − e) / e) *R x ∈ S
      using **  d by auto
    then have y ∈ S
      using * convexD [OF ‹convex S›] assms(3−5)
     by (metis diff_add_cancel diff_ge_0_iff_ge le_add_same_cancel1 less_eq_real_def)
  }
  then have ball (x − e *R (x − c)) (e*d) ∩ affine hull S ⊆ S
    by auto
  moreover have e * d > 0
    using ‹e > 0› ‹d > 0› by simp
  moreover have c: c ∈ S
    using assms rel_interior_subset by auto
  moreover from c have x − e *R (x − c) ∈ S
    using convexD_alt[of S x c e] assms
   by (metis  diff_add_eq diff_diff_eq2 less_eq_real_def scaleR_diff_left scaleR_one
scale_right_diff_distrib)
  ultimately show ?thesis
    using mem_rel_interior_ball[of x − e *R (x − c) S] ‹e > 0› by auto
qed

lemma interior_real_atLeast [simp]:
```

```
  fixes a :: real
  shows interior {a..} = {a<..}
proof −
  {
    fix y
    have ball y (y − a) ⊆ {a..}
      by (auto simp: dist_norm)
    moreover assume a < y
    ultimately have y ∈ interior {a..}
      by (force simp add: mem_interior)
  }
  moreover
  {
    fix y
    assume y ∈ interior {a..}
    then obtain e where e: e > 0 cball y e ⊆ {a..}
      using mem_interior_cball[of y {a..}] by auto
    moreover from e have y − e ∈ cball y e
      by (auto simp: cball_def dist_norm)
    ultimately have a ≤ y − e by blast
    then have a < y using e by auto
  }
  ultimately show ?thesis by auto
qed

lemma continuous_ge_on_Ioo:
   assumes continuous_on {c..d} g ⋀x. x ∈ {c<..<d} ⟹ g x ≥ a c < d x ∈
{c..d}
  shows g (x::real) ≥ (a::real)
proof−
 from assms(3) have {c..d} = closure {c<..<d} by (rule closure_greaterThanLessThan[symmetric])
  also from assms(2) have {c<..<d} ⊆ (g −‘ {a..} ∩ {c..d}) by auto
  hence closure {c<..<d} ⊆ closure (g −‘ {a..} ∩ {c..d}) by (rule closure_mono)
  also from assms(1) have closed (g −‘ {a..} ∩ {c..d})
    by (auto simp: continuous_on_closed_vimage)
  hence closure (g −‘ {a..} ∩ {c..d}) = g −‘ {a..} ∩ {c..d} by simp
  finally show ?thesis using ‹x ∈ {c..d}› by auto
qed

lemma interior_real_atMost [simp]:
  fixes a :: real
  shows interior {..a} = {..<a}
proof −
  {
    fix y
    have ball y (a − y) ⊆ {..a}
      by (auto simp: dist_norm)
    moreover assume a > y
    ultimately have y ∈ interior {..a}
```

```
    by (force simp add: mem_interior)
  }
  moreover
  {
    fix y
    assume y ∈ interior {..a}
    then obtain e where e: e > 0 cball y e ⊆ {..a}
      using mem_interior_cball[of y {..a}] by auto
    moreover from e have y + e ∈ cball y e
      by (auto simp: cball_def dist_norm)
    ultimately have a ≥ y + e by auto
    then have a > y using e by auto
  }
  ultimately show ?thesis by auto
qed

lemma interior_atLeastAtMost_real [simp]: interior {a..b} = {a<..<b :: real}
proof−
  have {a..b} = {a..} ∩ {..b} by auto
  also have interior ... = {a<..} ∩ {..<b}
    by (simp)
  also have ... = {a<..<b} by auto
  finally show ?thesis .
qed

lemma interior_atLeastLessThan [simp]:
  fixes a::real shows interior {a..<b} = {a<..<b}
  by (metis atLeastLessThan_def greaterThanLessThan_def interior_atLeastAtMost_real
interior_Int interior_interior interior_real_atLeast)

lemma interior_lessThanAtMost [simp]:
  fixes a::real shows interior {a<..b} = {a<..<b}
  by (metis atLeastAtMost_def greaterThanAtMost_def interior_atLeastAtMost_real
interior_Int
          interior_interior interior_real_atLeast)

lemma interior_greaterThanLessThan_real [simp]: interior {a<..<b} = {a<..<b
:: real}
  by (metis interior_atLeastAtMost_real interior_interior)

lemma frontier_real_atMost [simp]:
  fixes a :: real
  shows frontier {..a} = {a}
  unfolding frontier_def by auto

lemma frontier_real_atLeast [simp]: frontier {a..} = {a::real}
  by (auto simp: frontier_def)

lemma frontier_real_greaterThan [simp]: frontier {a<..} = {a::real}
```

by (*auto simp*: *interior_open frontier_def*)

lemma *frontier_real_lessThan* [*simp*]: *frontier* {..<*a*} = {*a*::*real*}
by (*auto simp*: *interior_open frontier_def*)

lemma *rel_interior_real_box* [*simp*]:
fixes *a b* :: *real*
assumes *a* < *b*
shows *rel_interior* {*a* .. *b*} = {*a* <..< *b*}
proof −
have *box a b* ≠ {}
using *assms*
unfolding *set_eq_iff*
by (*auto intro*!: *exI*[*of* _ (*a* + *b*) / *2*] *simp*: *box_def*)
then show *?thesis*
using *interior_rel_interior_gen*[*of cbox a b*, *symmetric*]
by (*simp split*: *if_split_asm del*: *box_real add*: *box_real*[*symmetric*])
qed

lemma *rel_interior_real_semiline* [*simp*]:
fixes *a* :: *real*
shows *rel_interior* {*a*..} = {*a*<..}
proof −
have ∗: {*a*<..} ≠ {}
unfolding *set_eq_iff* **by** (*auto intro*!: *exI*[*of* _ *a* + *1*])
then show *?thesis* **using** *interior_real_atLeast interior_rel_interior_gen*[*of* {*a*..}]
by (*auto split*: *if_split_asm*)
qed

Relative open sets

definition *rel_open S* ⟷ *rel_interior S* = *S*

lemma *rel_open*: *rel_open S* ⟷ *openin* (*top_of_set* (*affine hull S*)) *S* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *rel_open_def rel_interior_def*
using *openin_subopen*[*of top_of_set* (*affine hull S*) *S*] **by** *auto*
qed (*auto simp*: *rel_open_def rel_interior_def*)

lemma *openin_rel_interior*: *openin* (*top_of_set* (*affine hull S*)) (*rel_interior S*)
using *openin_subopen* **by** (*fastforce simp add*: *rel_interior_def*)

lemma *openin_set_rel_interior*:
openin (*top_of_set S*) (*rel_interior S*)
by (*rule openin_subset_trans* [*OF openin_rel_interior rel_interior_subset hull_subset*])

lemma *affine_rel_open*:
fixes S :: *'n::euclidean_space set*
assumes *affine S*
shows *rel_open S*
unfolding *rel_open_def*
using *assms rel_interior_affine_hull*[*of S*] *affine_hull_eq*[*of S*]
by *metis*

lemma *affine_closed*:
fixes S :: *'n::euclidean_space set*
assumes *affine S*
shows *closed S*
proof −
{
assume $S \neq \{\}$
then obtain L **where** L: *subspace L affine_parallel S L*
using *assms affine_parallel_subspace*[*of S*] **by** *auto*
then obtain a **where** a: $S = ((+)\ a\ `\ L)$
using *affine_parallel_def*[*of L S*] *affine_parallel_commute* **by** *auto*
from L **have** *closed L* **using** *closed_subspace* **by** *auto*
then have *closed S*
using *closed_translation a* **by** *auto*
}
then show *?thesis* **by** *auto*
qed

lemma *closure_affine_hull*:
fixes S :: *'n::euclidean_space set*
shows *closure S* $\subseteq$ *affine hull S*
by (*intro closure_minimal hull_subset affine_closed affine_affine_hull*)

lemma *closed_affine_hull* [*iff*]:
fixes S :: *'n::euclidean_space set*
shows *closed* (*affine hull S*)
by (*metis affine_affine_hull affine_closed*)

lemma *closure_same_affine_hull* [*simp*]:
fixes S :: *'n::euclidean_space set*
shows *affine hull* (*closure S*) = *affine hull S*
proof −
have *affine hull* (*closure S*) $\subseteq$ *affine hull S*
using *hull_mono*[*of closure S affine hull S affine*]
closure_affine_hull[*of S*] *hull_hull*[*of affine S*]
by *auto*
moreover have *affine hull* (*closure S*) $\supseteq$ *affine hull S*
using *hull_mono*[*of S closure S affine*] *closure_subset* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *closure_aff_dim* [*simp*]:
fixes S :: *'n::euclidean_space set*
shows *aff_dim* (*closure* S) = *aff_dim* S
proof −
have *aff_dim* $S \leq$ *aff_dim* (*closure* S)
using *aff_dim_subset closure_subset* **by** *auto*
moreover have *aff_dim* (*closure* S) $\leq$ *aff_dim* (*affine hull* S)
using *aff_dim_subset closure_affine_hull* **by** *blast*
moreover have *aff_dim* (*affine hull* S) = *aff_dim* S
using *aff_dim_affine_hull* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *rel_interior_closure_convex_shrink*:
fixes S :: *_::euclidean_space set*
assumes *convex* S
and $c \in$ *rel_interior* S
and $x \in$ *closure* S
and $e > 0$
and $e \leq 1$
shows $x - e *_R (x - c) \in$ *rel_interior* S
proof −
obtain d **where** $d > 0$ **and** d: *ball* c $d \cap$ *affine hull* $S \subseteq S$
using *assms*(*2*) **unfolding** *mem_rel_interior_ball* **by** *auto*
have $\exists y \in S$. *norm* $(y - x) * (1 - e) < e * d$
proof (*cases* $x \in S$)
case *True*
then show *?thesis* **using** ‹$e > 0$› ‹$d > 0$› **by** *force*
next
case *False*
then have x: x *islimpt* S
using *assms*(*3*)[*unfolded closure_def*] **by** *auto*
show *?thesis*
proof (*cases* $e = 1$)
case *True*
obtain y **where** $y \in S$ $y \neq x$ *dist* y $x < 1$
using x[*unfolded islimpt_approachable*,*THEN spec*[**where** x=*1*]] **by** *auto*
then show *?thesis*
unfolding *True* **using** ‹$d > 0$› **by** (*force simp add*:)
next
case *False*
then have $0 < e * d$ / $(1 - e)$ **and** $*$: $1 - e > 0$
using ‹$e \leq 1$› ‹$e > 0$› ‹$d > 0$› **by** *auto*
then obtain y **where** $y \in S$ $y \neq x$ *dist* y $x < e * d$ / $(1 - e)$
using x[*unfolded islimpt_approachable*,*THEN spec*[**where** x=e∗d / $(1 -$
e)]] **by** *auto*
then show *?thesis*
unfolding *dist_norm* **using** *pos_less_divide_eq*[*OF* $*$] **by** *force*

```
    qed
  qed
  then obtain y where y ∈ S and y: norm (y − x) ∗ (1 − e) < e ∗ d
    by auto
  define z where z = c + ((1 − e) / e) ∗R (x − y)
  have ∗: x − e ∗R (x − c) = y − e ∗R (y − z)
    unfolding z_def using ‹e > 0›
   by (auto simp: scaleR_right_diff_distrib scaleR_right_distrib scaleR_left_diff_distrib)
  have zball: z ∈ ball c d
    using mem_ball z_def dist_norm[of c]
    using y and assms(4,5)
    by (simp add: norm_minus_commute) (simp add: field_simps)
  have x ∈ affine hull S
    using closure_affine_hull assms by auto
  moreover have y ∈ affine hull S
    using ‹y ∈ S› hull_subset[of S] by auto
  moreover have c ∈ affine hull S
    using assms rel_interior_subset hull_subset[of S] by auto
  ultimately have z ∈ affine hull S
    using z_def affine_affine_hull[of S]
      mem_affine_3_minus [of affine hull S c x y (1 − e) / e]
      assms
    by simp
  then have z ∈ S using d zball by auto
  obtain d1 where d1 > 0 and d1: ball z d1 ≤ ball c d
    using zball open_ball[of c d] openE[of ball c d z] by auto
  then have ball z d1 ∩ affine hull S ⊆ ball c d ∩ affine hull S
    by auto
  then have ball z d1 ∩ affine hull S ⊆ S
    using d by auto
  then have z ∈ rel_interior S
    using mem_rel_interior_ball using ‹d1 > 0› ‹z ∈ S› by auto
  then have y − e ∗R (y − z) ∈ rel_interior S
    using rel_interior_convex_shrink[of S z y e] assms ‹y ∈ S› by auto
  then show ?thesis using ∗ by auto
qed

lemma rel_interior_eq:
   rel_interior s = s ⟷ openin(top_of_set (affine hull s)) s
using rel_open rel_open_def by blast

lemma rel_interior_openin:
   openin(top_of_set (affine hull s)) s ⟹ rel_interior s = s
by (simp add: rel_interior_eq)

lemma rel_interior_affine:
  fixes S :: 'n::euclidean_space set
  shows  affine S ⟹ rel_interior S = S
using affine_rel_open rel_open_def by auto
```

lemma *rel_interior_eq_closure*:
fixes S :: *'n::euclidean_space set*
shows *rel_interior* S = *closure* $S \longleftrightarrow$ *affine* S
proof (*cases* $S = \{\}$)
case *True*
then show *?thesis*
by *auto*
next
case *False* **show** *?thesis*
proof
assume *eq*: *rel_interior* S = *closure* S
have *openin* (*top_of_set* (*affine hull* S)) S
by (*metis eq closure_subset openin_rel_interior rel_interior_subset subset_antisym*)
moreover have *closedin* (*top_of_set* (*affine hull* S)) S
by (*metis closed_subset closure_subset_eq eq hull_subset rel_interior_subset*)
ultimately have $S = \{\} \vee S$ = *affine hull* S
using *convex_connected connected_clopen convex_affine_hull* **by** *metis*
with *False* **have** *affine hull* S = S
by *auto*
then show *affine* S
by (*metis affine_hull_eq*)
next
assume *affine* S
then show *rel_interior* S = *closure* S
by (*simp add*: *rel_interior_affine affine_closed*)
qed
qed

Relative interior preserves under linear transformations

lemma *rel_interior_translation_aux*:
fixes a :: *'n::euclidean_space*
shows $((\lambda x.\ a + x)$ ‘ *rel_interior* $S) \subseteq$ *rel_interior* $((\lambda x.\ a + x)$ ‘ $S)$
proof −
{
fix x
assume x: $x \in$ *rel_interior* S
then obtain T **where** *open* T $x \in T \cap S$ $T \cap$ *affine hull* $S \subseteq S$
using *mem_rel_interior*[*of* x S] **by** *auto*
then have *open* $((\lambda x.\ a + x)$ ‘ $T)$
and $a + x \in ((\lambda x.\ a + x)$ ‘ $T) \cap ((\lambda x.\ a + x)$ ‘ $S)$
and $((\lambda x.\ a + x)$ ‘ $T) \cap$ *affine hull* $((\lambda x.\ a + x)$ ‘ $S) \subseteq (\lambda x.\ a + x)$ ‘ S
using *affine_hull_translation*[*of* a S] *open_translation*[*of* T a] x **by** *auto*
then have $a + x \in$ *rel_interior* $((\lambda x.\ a + x)$ ‘ $S)$
using *mem_rel_interior*[*of* a+x $((\lambda x.\ a + x)$ ‘ $S)$] **by** *auto*
}
then show *?thesis* **by** *auto*

qed

lemma *rel_interior_translation*:
fixes *a* :: *'n::euclidean_space*
shows *rel_interior* ((λ*x*. *a* + *x*) ‘ *S*) = (λ*x*. *a* + *x*) ‘ *rel_interior S*
proof −
have (λ*x*. (−*a*) + *x*) ‘ *rel_interior* ((λ*x*. *a* + *x*) ‘ *S*) ⊆ *rel_interior S*
using *rel_interior_translation_aux*[*of* −*a* (λ*x*. *a* + *x*) ‘ *S*]
translation_assoc[*of* −*a a*]
by *auto*
then have ((λ*x*. *a* + *x*) ‘ *rel_interior S*) ⊇ *rel_interior* ((λ*x*. *a* + *x*) ‘ *S*)
using *translation_inverse_subset*[*of a rel_interior* ((+) *a* ‘ *S*) *rel_interior S*]
by *auto*
then show *?thesis*
using *rel_interior_translation_aux*[*of a S*] **by** *auto*
qed

lemma *affine_hull_linear_image*:
assumes *bounded_linear f*
shows *f* ‘ (*affine hull s*) = *affine hull f* ‘ *s*
proof −
interpret *f*: *bounded_linear f* **by** *fact*
have *affine* {*x*. *f x* ∈ *affine hull f* ‘ *s*}
unfolding *affine_def*
by (*auto simp*: *f.scaleR f.add affine_affine_hull*[*unfolded affine_def*, *rule_format*])
moreover have *affine* {*x*. *x* ∈ *f* ‘ (*affine hull s*)}
using *affine_affine_hull*[*unfolded affine_def*, *of s*]
unfolding *affine_def* **by** (*auto simp*: *f.scaleR* [*symmetric*] *f.add* [*symmetric*])
ultimately show *?thesis*
by (*auto simp*: *hull_inc elim*!: *hull_induct*)
qed

lemma *rel_interior_injective_on_span_linear_image*:
fixes *f* :: *'m::euclidean_space* ⇒ *'n::euclidean_space*
and *S* :: *'m::euclidean_space set*
assumes *bounded_linear f*
and *inj_on f* (*span S*)
shows *rel_interior* (*f* ‘ *S*) = *f* ‘ (*rel_interior S*)
proof −
{
fix *z*
assume *z*: *z* ∈ *rel_interior* (*f* ‘ *S*)
then have *z* ∈ *f* ‘ *S*
using *rel_interior_subset*[*of f* ‘ *S*] **by** *auto*
then obtain *x* **where** *x*: *x* ∈ *S f x* = *z* **by** *auto*
obtain *e2* **where** *e2*: *e2* > *0 cball z e2* ∩ *affine hull* (*f* ‘ *S*) ⊆ (*f* ‘ *S*)
using *z rel_interior_cball*[*of f* ‘ *S*] **by** *auto*

```
  obtain K where K: K > 0 ⋀x. norm (f x) ≤ norm x * K
   using assms Real_Vector_Spaces.bounded_linear.pos_bounded[of f] by auto
  define e1 where e1 = 1 / K
  then have e1: e1 > 0 ⋀x. e1 * norm (f x) ≤ norm x
    using K pos_le_divide_eq[of e1] by auto
  define e where e = e1 * e2
  then have e > 0 using e1 e2 by auto
  {
    fix y
    assume y: y ∈ cball x e ∩ affine hull S
    then have h1: f y ∈ affine hull (f ‘ S)
      using affine_hull_linear_image[of f S] assms by auto
    from y have norm (x−y) ≤ e1 * e2
      using cball_def[of x e] dist_norm[of x y] e_def by auto
    moreover have f x − f y = f (x − y)
     using assms linear_diff[of f x y] linear_conv_bounded_linear[of f] by auto
    moreover have e1 * norm (f (x−y)) ≤ norm (x − y)
      using e1 by auto
    ultimately have e1 * norm ((f x)−(f y)) ≤ e1 * e2
      by auto
    then have f y ∈ cball z e2
      using cball_def[of f x e2] dist_norm[of f x f y] e1 x by auto
    then have f y ∈ f ‘ S
      using y e2 h1 by auto
    then have y ∈ S
      using assms y hull_subset[of S] affine_hull_subset_span
        inj_on_image_mem_iff [OF ‹inj_on f (span S)›]
      by (metis Int_iff span_superset subsetCE)
  }
  then have z ∈ f ‘ (rel_interior S)
    using mem_rel_interior_cball[of x S] ‹e > 0› x by auto
}
moreover
{
  fix x
  assume x: x ∈ rel_interior S
  then obtain e2 where e2: e2 > 0 cball x e2 ∩ affine hull S ⊆ S
    using rel_interior_cball[of S] by auto
  have x ∈ S using x rel_interior_subset by auto
  then have ∗: f x ∈ f ‘ S by auto
  have ∀ x∈span S. f x = 0 ⟶ x = 0
    using assms subspace_span linear_conv_bounded_linear[of f]
      linear_injective_on_subspace_0[of f span S]
    by auto
  then obtain e1 where e1: e1 > 0 ∀ x ∈ span S. e1 * norm x ≤ norm (f x)
    using assms injective_imp_isometric[of span S f]
      subspace_span[of S] closed_subspace[of span S]
    by auto
  define e where e = e1 * e2
```

```
    hence e > 0 using e1 e2 by auto
    {
      fix y
      assume y: y ∈ cball (f x) e ∩ affine hull (f ‘ S)
      then have y ∈ f ‘ (affine hull S)
        using affine_hull_linear_image[of f S] assms by auto
      then obtain xy where xy: xy ∈ affine hull S f xy = y by auto
      with y have norm (f x − f xy) ≤ e1 ∗ e2
        using cball_def[of f x e] dist_norm[of f x y] e_def by auto
      moreover have f x − f xy = f (x − xy)
       using assms linear_diff[of f x xy] linear_conv_bounded_linear[of f] by auto
      moreover have ∗: x − xy ∈ span S
        using subspace_diff[of span S x xy] subspace_span ‹x ∈ S› xy
          affine_hull_subset_span[of S] span_superset
        by auto
      moreover from ∗ have e1 ∗ norm (x − xy) ≤ norm (f (x − xy))
        using e1 by auto
      ultimately have e1 ∗ norm (x − xy) ≤ e1 ∗ e2
        by auto
      then have xy ∈ cball x e2
        using cball_def[of x e2] dist_norm[of x xy] e1 by auto
      then have y ∈ f ‘ S
        using xy e2 by auto
    }
    then have f x ∈ rel_interior (f ‘ S)
      using mem_rel_interior_cball[of (f x) (f ‘ S)] ∗ ‹e > 0› by auto
  }
  ultimately show ?thesis by auto
qed

lemma rel_interior_injective_linear_image:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes bounded_linear f
    and inj f
  shows rel_interior (f ‘ S) = f ‘ (rel_interior S)
  using assms rel_interior_injective_on_span_linear_image[of f S]
    subset_inj_on[of f UNIV span S]
  by auto
```

5.3.3 Openness and compactness are preserved by convex hull operation

```
lemma open_convex_hull[intro]:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open (convex hull S)
proof (clarsimp simp: open_contains_cball convex_hull_explicit)
  fix T and u :: 'a⇒real
  assume obt: finite T T⊆S ∀ x∈T. 0 ≤ u x sum u T = 1
```

```
  from assms[unfolded open_contains_cball] obtain b
    where b: ⋀x. x∈S ⟹ 0 < b x ∧ cball x (b x) ⊆ S by metis
  have b ‘ T ≠ {}
    using obt by auto
  define i where i = b ‘ T
  let ?Φ = λy. ∃ F. finite F ∧ F ⊆ S ∧ (∃ u. (∀ x∈F. 0 ≤ u x) ∧ sum u F = 1 ∧
(∑ v∈F. u v *R v) = y)
  let ?a = ∑ v∈T. u v *R v
  show ∃ e > 0. cball ?a e ⊆ {y. ?Φ y}
  proof (intro exI subsetI conjI)
    show 0 < Min i
      unfolding i_def and Min_gr_iff[OF finite_imageI[OF obt(1)] ‹b ‘ T≠{}›]
      using b ‹T⊆S› by auto
  next
    fix y
    assume y ∈ cball ?a (Min i)
    then have y: norm (?a − y) ≤ Min i
      unfolding dist_norm[symmetric] by auto
    { fix x
      assume x ∈ T
      then have Min i ≤ b x
        by (simp add: i_def obt(1))
      then have x + (y − ?a) ∈ cball x (b x)
        using y unfolding mem_cball dist_norm by auto
      moreover have x ∈ S
        using ‹x∈T› ‹T⊆S› by auto
      ultimately have x + (y − ?a) ∈ S
        using y b by blast
    }
    moreover
    have *: inj_on (λv. v + (y − ?a)) T
      unfolding inj_on_def by auto
    have (∑ v∈(λv. v + (y − ?a)) ‘ T. u (v − (y − ?a)) *R v) = y
      unfolding sum.reindex[OF *] o_def using obt(4)
    by (simp add: sum.distrib sum_subtractf scaleR_left.sum[symmetric] scaleR_right_distrib)
    ultimately show y ∈ {y. ?Φ y}
    proof (intro CollectI exI conjI)
      show finite ((λv. v + (y − ?a)) ‘ T)
        by (simp add: obt(1))
      show sum (λv. u (v − (y − ?a))) ((λv. v + (y − ?a)) ‘ T) = 1
        unfolding sum.reindex[OF *] o_def using obt(4) by auto
    qed (use obt(1, 3) in auto)
  qed
qed

lemma compact_convex_combinations:
  fixes S T :: 'a::real_normed_vector set
  assumes compact S compact T
```

```
  shows compact { (1 − u) *R x + u *R y | x y u. 0 ≤ u ∧ u ≤ 1 ∧ x ∈ S ∧ y
∈ T}
proof −
  let ?X = {0..1} × S × T
  let ?h = (λz. (1 − fst z) *R fst (snd z) + fst z *R snd (snd z))
  have *: { (1 − u) *R x + u *R y | x y u. 0 ≤ u ∧ u ≤ 1 ∧ x ∈ S ∧ y ∈ T} =
?h ‘ ?X
    by force
  have continuous_on ?X (λz. (1 − fst z) *R fst (snd z) + fst z *R snd (snd z))
    unfolding continuous_on by (rule ballI) (intro tendsto_intros)
  with assms show ?thesis
    by (simp add: * compact_Times compact_continuous_image)
qed

lemma finite_imp_compact_convex_hull:
  fixes S :: 'a::real_normed_vector set
  assumes finite S
  shows compact (convex hull S)
proof (cases S = {})
  case True
  then show ?thesis by simp
next
  case False
  with assms show ?thesis
  proof (induct rule: finite_ne_induct)
    case (singleton x)
    show ?case by simp
  next
    case (insert x A)
    let ?f = λ(u, y::'a). u *R x + (1 − u) *R y
    let ?T = {0..1::real} × (convex hull A)
    have continuous_on ?T ?f
      unfolding split_def continuous_on by (intro ballI tendsto_intros)
    moreover have compact ?T
      by (intro compact_Times compact_Icc insert)
    ultimately have compact (?f ‘ ?T)
      by (rule compact_continuous_image)
    also have ?f ‘ ?T = convex hull (insert x A)
      unfolding convex_hull_insert [OF ‹A ≠ {}›]
      apply safe
      apply (rule_tac x=a in exI, simp)
      apply (rule_tac x=1 − a in exI, simp, fast)
      apply (rule_tac x=(u, b) in image_eqI, simp_all)
      done
    finally show compact (convex hull (insert x A)) .
  qed
qed

lemma compact_convex_hull:
```

```
  fixes S :: 'a::euclidean_space set
  assumes compact S
  shows compact (convex hull S)
proof (cases S = {})
  case True
  then show ?thesis using compact_empty by simp
next
  case False
  then obtain w where w ∈ S by auto
  show ?thesis
    unfolding caratheodory[of S]
  proof (induct (DIM('a) + 1))
    case 0
    have ∗: {x.∃ sa. finite sa ∧ sa ⊆ S ∧ card sa ≤ 0 ∧ x ∈ convex hull sa} = {}
      using compact_empty by auto
    from 0 show ?case unfolding ∗ by simp
  next
    case (Suc n)
    show ?case
    proof (cases n = 0)
      case True
      have {x. ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T} = S
        unfolding set_eq_iff and mem_Collect_eq
      proof (rule, rule)
        fix x
        assume ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T
        then obtain T where T: finite T T ⊆ S card T ≤ Suc n x ∈ convex hull
T
          by auto
        show x ∈ S
        proof (cases card T = 0)
          case True
          then show ?thesis
            using T(4) unfolding card_0_eq[OF T(1)] by simp
        next
          case False
          then have card T = Suc 0 using T(3) ‹n=0› by auto
          then obtain a where T = {a} unfolding card_Suc_eq by auto
          then show ?thesis using T(2,4) by simp
        qed
      next
        fix x assume x∈S
        then show ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T
          by (rule_tac x={x} in exI) (use convex_hull_singleton in auto)
      qed
      then show ?thesis using assms by simp
    next
      case False
      have {x. ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T} =
```

```
    {(1 − u) *R x + u *R y | x y u.
      0 ≤ u ∧ u ≤ 1 ∧ x ∈ S ∧ y ∈ {x. ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ n
∧ x ∈ convex hull T}}
    unfolding set_eq_iff and mem_Collect_eq
  proof (rule, rule)
    fix x
    assume ∃ u v c. x = (1 − c) *R u + c *R v ∧
      0 ≤ c ∧ c ≤ 1 ∧ u ∈ S ∧ (∃ T. finite T ∧ T ⊆ S ∧ card T ≤ n ∧ v ∈
convex hull T)
    then obtain u v c T where obt: x = (1 − c) *R u + c *R v
      0 ≤ c ∧ c ≤ 1 u ∈ S finite T T ⊆ S card T ≤ n  v ∈ convex hull T
      by auto
    moreover have (1 − c) *R u + c *R v ∈ convex hull insert u T
    by (meson convexD_alt convex_convex_hull hull_inc hull_mono in_mono
insertCI obt(2) obt(7) subset_insertI)
    ultimately show ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex
hull T
      by (rule_tac x=insert u T in exI) (auto simp: card_insert_if)
  next
    fix x
    assume ∃ T. finite T ∧ T ⊆ S ∧ card T ≤ Suc n ∧ x ∈ convex hull T
    then obtain T where T: finite T T ⊆ S card T ≤ Suc n x ∈ convex hull
T
      by auto
    show ∃ u v c. x = (1 − c) *R u + c *R v ∧
      0 ≤ c ∧ c ≤ 1 ∧ u ∈ S ∧ (∃ T. finite T ∧ T ⊆ S ∧ card T ≤ n ∧ v ∈
convex hull T)
    proof (cases card T = Suc n)
      case False
      then have card T ≤ n using T(3) by auto
      then show ?thesis
        using ‹w∈S› and T
        by (rule_tac x=w in exI, rule_tac x=x in exI, rule_tac x=1 in exI)
auto
    next
      case True
      then obtain a u where au: T = insert a u a∉u
        by (metis card_le_Suc_iff order_refl)
      show ?thesis
      proof (cases u = {})
        case True
        then have x = a using T(4)[unfolded au] by auto
        show ?thesis unfolding ‹x = a›
          using T ‹n ≠ 0› unfolding au
          by (rule_tac x=a in exI, rule_tac x=a in exI, rule_tac x=1 in exI)
force
      next
        case False
        obtain ux vx b where obt: ux≥0 vx≥0 ux + vx = 1
```

```
            b ∈ convex hull u x = ux *R a + vx *R b
            using T(4)[unfolded au convex_hull_insert[OF False]]
            by auto
          have *: 1 − vx = ux using obt(3) by auto
          show ?thesis
           using obt T(1−3) card_insert_disjoint[OF _ au(2)] unfolding au *
            by (rule_tac x=a in exI, rule_tac x=b in exI, rule_tac x=vx in exI)
force
        qed
      qed
    qed
    then show ?thesis
      using compact_convex_combinations[OF assms Suc] by simp
  qed
 qed
qed
```

5.3.4 Extremal points of a simplex are some vertices

```
lemma dist_increases_online:
 fixes a b d :: 'a::real_inner
 assumes d ≠ 0
 shows dist a (b + d) > dist a b ∨ dist a (b − d) > dist a b
proof (cases inner a d − inner b d > 0)
 case True
 then have 0 < inner d d + (inner a d * 2 − inner b d * 2)
   using assms
   by (intro add_pos_pos) auto
 then show ?thesis
   unfolding dist_norm and norm_eq_sqrt_inner and real_sqrt_less_iff
   by (simp add: algebra_simps inner_commute)
next
 case False
 then have 0 < inner d d + (inner b d * 2 − inner a d * 2)
   using assms
   by (intro add_pos_nonneg) auto
 then show ?thesis
   unfolding dist_norm and norm_eq_sqrt_inner and real_sqrt_less_iff
   by (simp add: algebra_simps inner_commute)
qed

lemma norm_increases_online:
 fixes d :: 'a::real_inner
 shows d ≠ 0 ⟹ norm (a + d) > norm a ∨ norm(a − d) > norm a
 using dist_increases_online[of d a 0] unfolding dist_norm by auto

lemma simplex_furthest_lt:
 fixes S :: 'a::real_inner set
 assumes finite S
```

```
  shows ∀x ∈ convex hull S.  x ∉ S ⟶ (∃y ∈ convex hull S. norm (x − a) <
norm(y − a))
  using assms
proof induct
  fix x S
  assume as: finite S x∉S ∀x∈convex hull S. x ∉ S ⟶ (∃y∈convex hull S. norm
(x − a) < norm (y − a))
  show ∀xa∈convex hull insert x S. xa ∉ insert x S ⟶
    (∃y∈convex hull insert x S. norm (xa − a) < norm (y − a))
  proof (intro impI ballI, cases S = {})
    case False
    fix y
    assume y: y ∈ convex hull insert x S y ∉ insert x S
    obtain u v b where obt: u≥0 v≥0 u + v = 1 b ∈ convex hull S y = u *R x +
v *R b
      using y(1)[unfolded convex_hull_insert[OF False]] by auto
    show ∃z∈convex hull insert x S. norm (y − a) < norm (z − a)
    proof (cases y ∈ convex hull S)
      case True
      then obtain z where z ∈ convex hull S norm (y − a) < norm (z − a)
        using as(3)[THEN bspec[where x=y]] and y(2) by auto
      then show ?thesis
        by (meson hull_mono subsetD subset_insertI)
    next
      case False
      show ?thesis
      proof (cases u = 0 ∨ v = 0)
        case True
        with False show ?thesis
          using obt y by auto
      next
        case False
        then obtain w where w: w>0 w<u w<v
          using field_lbound_gt_zero[of u v] and obt(1,2) by auto
        have x ≠ b
        proof
          assume x = b
          then have y = b unfolding obt(5)
            using obt(3) by (auto simp: scaleR_left_distrib[symmetric])
          then show False using obt(4) and False
            using ‹x = b› y(2) by blast
        qed
        then have ∗: w *R (x − b) ≠ 0 using w(1) by auto
        show ?thesis
          using dist_increases_online[OF ∗, of a y]
        proof (elim disjE)
          assume dist a y < dist a (y + w *R (x − b))
          then have norm (y − a) < norm ((u + w) *R x + (v − w) *R b − a)
            unfolding dist_commute[of a]
```

```
        unfolding dist_norm obt(5)
        by (simp add: algebra_simps)
      moreover have (u + w) *R x + (v − w) *R b ∈ convex hull insert x S
        unfolding convex_hull_insert[OF ‹S≠{}›]
      proof (intro CollectI conjI exI)
        show u + w ≥ 0 v − w ≥ 0
          using obt(1) w by auto
      qed (use obt in auto)
      ultimately show ?thesis by auto
    next
      assume dist a y < dist a (y − w *R (x − b))
      then have norm (y − a) < norm ((u − w) *R x + (v + w) *R b − a)
        unfolding dist_commute[of a]
        unfolding dist_norm obt(5)
        by (simp add: algebra_simps)
      moreover have (u − w) *R x + (v + w) *R b ∈ convex hull insert x S
        unfolding convex_hull_insert[OF ‹S≠{}›]
      proof (intro CollectI conjI exI)
        show u − w ≥ 0 v + w ≥ 0
          using obt(1) w by auto
      qed (use obt in auto)
      ultimately show ?thesis by auto
    qed
  qed
 qed
qed auto
qed (auto simp: assms)

lemma simplex_furthest_le:
 fixes S :: 'a::real_inner set
 assumes finite S
   and S ≠ {}
 shows ∃ y∈S. ∀ x∈ convex hull S. norm (x − a) ≤ norm (y − a)
proof −
 have convex hull S ≠ {}
   using hull_subset[of S convex] and assms(2) by auto
 then obtain x where x: x ∈ convex hull S ∀ y∈convex hull S. norm (y − a) ≤
norm (x − a)
    using distance_attains_sup[OF finite_imp_compact_convex_hull[OF ‹finite
S›], of a]
   unfolding dist_commute[of a]
   unfolding dist_norm
   by auto
 show ?thesis
 proof (cases x ∈ S)
   case False
   then obtain y where y ∈ convex hull S norm (x − a) < norm (y − a)
    using simplex_furthest_lt[OF assms(1), THEN bspec[where x=x]] and x(1)
     by auto
```

```
    then show ?thesis
      using x(2)[THEN bspec[where x=y]] by auto
  next
    case True
    with x show ?thesis by auto
  qed
qed

lemma simplex_furthest_le_exists:
  fixes S :: ('a::real_inner) set
  shows finite S ⟹ ∀ x∈(convex hull S). ∃ y∈S. norm (x − a) ≤ norm (y − a)
  using simplex_furthest_le[of S] by (cases S = {}) auto

lemma simplex_extremal_le:
  fixes S :: 'a::real_inner set
  assumes finite S
    and S ≠ {}
  shows ∃ u∈S. ∃ v∈S. ∀ x∈convex hull S. ∀ y ∈ convex hull S. norm (x − y) ≤
norm (u − v)
proof −
  have convex hull S ≠ {}
    using hull_subset[of S convex] and assms(2) by auto
  then obtain u v where obt: u ∈ convex hull S v ∈ convex hull S
    ∀ x∈convex hull S. ∀ y∈convex hull S. norm (x − y) ≤ norm (u − v)
      using compact_sup_maxdistance[OF finite_imp_compact_convex_hull[OF
assms(1)]]
    by (auto simp: dist_norm)
  then show ?thesis
  proof (cases u∉S ∨ v∉S, elim disjE)
    assume u ∉ S
    then obtain y where y ∈ convex hull S norm (u − v) < norm (y − v)
        using simplex_furthest_lt[OF assms(1), THEN bspec[where x=u]] and
obt(1)
      by auto
    then show ?thesis
     using obt(3)[THEN bspec[where x=y], THEN bspec[where x=v]] and obt(2)
      by auto
  next
    assume v ∉ S
    then obtain y where y ∈ convex hull S norm (v − u) < norm (y − u)
        using simplex_furthest_lt[OF assms(1), THEN bspec[where x=v]] and
obt(2)
      by auto
    then show ?thesis
     using obt(3)[THEN bspec[where x=u], THEN bspec[where x=y]] and obt(1)
      by (auto simp: norm_minus_commute)
  qed auto
qed
```

lemma *simplex_extremal_le_exists*:
fixes $S :: 'a::real\_inner\ set$
shows $finite\ S \Longrightarrow x \in convex\ hull\ S \Longrightarrow y \in convex\ hull\ S \Longrightarrow$
$\exists u{\in}S.\ \exists v{\in}S.\ norm\ (x - y) \leq norm\ (u - v)$
using *convex_hull_empty simplex_extremal_le*[*of S*]
by(*cases* $S = \{\}$) *auto*

5.3.5 Closest point of a convex set is unique, with a continuous projection

definition $closest\_point :: 'a::\{real\_inner,heine\_borel\}\ set \Rightarrow 'a \Rightarrow 'a$
where $closest\_point\ S\ a = (SOME\ x.\ x \in S \wedge (\forall y{\in}S.\ dist\ a\ x \leq dist\ a\ y))$

lemma *closest_point_exists*:
assumes *closed S*
and $S \neq \{\}$
shows *closest_point_in_set*: $closest\_point\ S\ a \in S$
and $\forall y{\in}S.\ dist\ a\ (closest\_point\ S\ a) \leq dist\ a\ y$
unfolding *closest_point_def*
by (*rule_tac someI2_ex*, *auto intro*: *distance_attains_inf*[*OF assms*(*1*,*2*), *of a*])+

lemma *closest_point_le*: $closed\ S \Longrightarrow x \in S \Longrightarrow dist\ a\ (closest\_point\ S\ a) \leq dist\ a\ x$
using *closest_point_exists*[*of S*] **by** *auto*

lemma *closest_point_self*:
assumes $x \in S$
shows $closest\_point\ S\ x = x$
unfolding *closest_point_def*
by (*rule some1_equality*, *rule ex1I*[*of* _ *x*]) (*use assms* **in** *auto*)

lemma *closest_point_refl*: $closed\ S \Longrightarrow S \neq \{\} \Longrightarrow closest\_point\ S\ x = x \longleftrightarrow x \in S$
using *closest_point_in_set*[*of S x*] *closest_point_self*[*of x S*]
by *auto*

lemma *closer_points_lemma*:
assumes $inner\ y\ z > 0$
shows $\exists u{>}0.\ \forall v{>}0.\ v \leq u \longrightarrow norm(v *_R z - y) < norm\ y$
proof −
have *z*: $inner\ z\ z > 0$
unfolding *inner_gt_zero_iff* **using** *assms* **by** *auto*
have $norm\ (v *_R z - y) < norm\ y$
if $0 < v$ **and** $v \leq inner\ y\ z\ /\ inner\ z\ z$ **for** *v*
unfolding *norm_lt* **using** *z assms that*
by (*simp add*: *field_simps inner_diff inner_commute mult_strict_left_mono*[*OF* _ ‹*0*<*v*›])
then show *?thesis*

using *assms z*
by (*rule_tac x = inner y z / inner z z* **in** *exI*) *auto*
qed

lemma *closer_point_lemma*:
assumes *inner* $(y - x)$ $(z - x) > 0$
shows $\exists u{>}0.\ u \leq 1 \wedge$ *dist* $(x + u *_R (z - x))$ $y <$ *dist x y*
proof −
obtain *u* **where** $u > 0$
and *u*: $\bigwedge v.\ [\![0{<}v;\ v \leq u]\!] \Longrightarrow$ *norm* $(v *_R (z - x) - (y - x)) <$ *norm* $(y - x)$
using *closer_points_lemma*[*OF assms*] **by** *auto*
show *?thesis*
using *u*[*of min u 1*] **and** ‹$u > 0$›
by (*metis diff_diff_add dist_commute dist_norm less_eq_real_def not_less u zero_less_one*)
qed

lemma *any_closest_point_dot*:
assumes *convex S closed S* $x \in S$ $y \in S$ $\forall z{\in}S.$ *dist a x* $\leq$ *dist a z*
shows *inner* $(a - x)$ $(y - x) \leq 0$
proof (*rule ccontr*)
assume ¬ *?thesis*
then obtain *u* **where** *u*: $u{>}0$ $u{\leq}1$ *dist* $(x + u *_R (y - x))$ $a <$ *dist x a*
using *closer_point_lemma*[*of a x y*] **by** *auto*
let *?z* $= (1 - u) *_R x + u *_R y$
have $?z \in S$
using *convexD_alt*[*OF assms*(*1*,*3*,*4*), *of u*] **using** *u* **by** *auto*
then show *False*
using *assms*(*5*)[*THEN bspec*[**where** *x*=*?z*]] **and** *u*(*3*)
by (*auto simp*: *dist_commute algebra_simps*)
qed

lemma *any_closest_point_unique*:
fixes *x* :: *'a*::*real_inner*
assumes *convex S closed S* $x \in S$ $y \in S$
$\forall z{\in}S.$ *dist a x* $\leq$ *dist a z* $\forall z{\in}S.$ *dist a y* $\leq$ *dist a z*
shows $x = y$
using *any_closest_point_dot*[*OF assms*(*1*−*4*,*5*)] **and** *any_closest_point_dot*[*OF assms*(*1*−*2*,*4*,*3*,*6*)]
unfolding *norm_pths*(*1*) **and** *norm_le_square*
by (*auto simp*: *algebra_simps*)

lemma *closest_point_unique*:
assumes *convex S closed S* $x \in S$ $\forall z{\in}S.$ *dist a x* $\leq$ *dist a z*
shows $x =$ *closest_point S a*
using *any_closest_point_unique*[*OF assms*(*1*−*3*) _ *assms*(*4*), *of closest_point S a*]
using *closest_point_exists*[*OF assms*(*2*)] **and** *assms*(*3*) **by** *auto*

lemma *closest_point_dot*:
assumes *convex S closed S x ∈ S*
shows *inner* (*a − closest_point S a*) (*x − closest_point S a*) ≤ *0*
using *any_closest_point_dot*[*OF assms*(*1*,*2*) _ *assms*(*3*)]
by (*metis assms*(*2*) *assms*(*3*) *closest_point_in_set closest_point_le empty_iff*)

lemma *closest_point_lt*:
assumes *convex S closed S x ∈ S x ≠ closest_point S a*
shows *dist a* (*closest_point S a*) < *dist a x*
using *closest_point_unique*[**where** *a*=*a*] *closest_point_le*[**where** *a*=*a*] *assms*
by *fastforce*

lemma *setdist_closest_point*:
⟦*closed S*; *S ≠ {}*⟧ ⟹ *setdist* {*a*} *S* = *dist a* (*closest_point S a*)
by (*metis closest_point_exists*(*2*) *closest_point_in_set emptyE insert_iff setdist_unique*)

lemma *closest_point_lipschitz*:
assumes *convex S*
and *closed S S ≠ {}*
shows *dist* (*closest_point S x*) (*closest_point S y*) ≤ *dist x y*
proof −
have *inner* (*x − closest_point S x*) (*closest_point S y − closest_point S x*) ≤ *0*
and *inner* (*y − closest_point S y*) (*closest_point S x − closest_point S y*) ≤ *0*
by (*simp_all add*: *assms closest_point_dot closest_point_in_set*)
then show *?thesis* **unfolding** *dist_norm* **and** *norm_le*
using *inner_ge_zero*[*of* (*x − closest_point S x*) − (*y − closest_point S y*)]
by (*simp add*: *inner_add inner_diff inner_commute*)
qed

lemma *continuous_at_closest_point*:
assumes *convex S*
and *closed S*
and *S ≠ {}*
shows *continuous* (*at x*) (*closest_point S*)
unfolding *continuous_at_eps_delta*
using *le_less_trans*[*OF closest_point_lipschitz*[*OF assms*]] **by** *auto*

lemma *continuous_on_closest_point*:
assumes *convex S*
and *closed S*
and *S ≠ {}*
shows *continuous_on t* (*closest_point S*)
by (*metis continuous_at_imp_continuous_on continuous_at_closest_point*[*OF assms*])

proposition *closest_point_in_rel_interior*:
assumes *closed S S ≠ {}* **and** *x*: *x ∈ affine hull S*
shows *closest_point S x ∈ rel_interior S* ⟷ *x ∈ rel_interior S*

proof (*cases* $x \in S$)
case *True*
then show *?thesis*
by (*simp add*: *closest_point_self*)
next
case *False*
then have *False* **if** *asm*: *closest_point S x* $\in$ *rel_interior S*
proof −
obtain *e* **where** $e > 0$ **and** *clox*: *closest_point S x* $\in S$
and *e*: *cball* (*closest_point S x*) *e* $\cap$ *affine hull S* $\subseteq S$
using *asm mem_rel_interior_cball* **by** *blast*
then have *clo_notx*: *closest_point S x* $\neq x$
using ‹$x \notin S$› **by** *auto*
define *y* **where** $y \equiv$ *closest_point S x* −
(*min 1* (*e* / *norm*(*closest_point S x* − *x*))) $*_R$ (*closest_point S x* − *x*)
have $x - y = (1 - min\ 1\ (e\ /\ norm\ (closest\_point\ S\ x - x))) *_R (x - closest\_point\ S\ x)$
by (*simp add*: *y_def algebra_simps*)
then have *norm* (*x* − *y*) = *abs* ((*1* − *min 1* (*e* / *norm* (*closest_point S x* − *x*)))) ∗ *norm*(*x* − *closest_point S x*)
by *simp*
also have ... < *norm*(*x* − *closest_point S x*)
using *clo_notx* ‹*e* > *0*›
by (*auto simp*: *mult_less_cancel_right2 field_split_simps*)
finally have *no_less*: *norm* (*x* − *y*) < *norm* (*x* − *closest_point S x*) **.**
have *y* ∈ *affine hull S*
unfolding *y_def*
by (*meson affine_affine_hull clox hull_subset mem_affine_3_minus2 subsetD x*)
moreover have *dist* (*closest_point S x*) *y* ≤ *e*
using ‹*e* > *0*› **by** (*auto simp*: *y_def min_mult_distrib_right*)
ultimately have *y* ∈ *S*
using *subsetD* [*OF e*] **by** *simp*
then have *dist x* (*closest_point S x*) ≤ *dist x y*
by (*simp add*: *closest_point_le* ‹*closed S*›)
with *no_less* **show** *False*
by (*simp add*: *dist_norm*)
qed
moreover have *x* ∉ *rel_interior S*
using *rel_interior_subset False* **by** *blast*
ultimately show *?thesis* **by** *blast*
qed

Various point-to-set separating/supporting hyperplane theorems

lemma *supporting_hyperplane_closed_point*:
fixes *z* :: *'a*::{*real_inner*,*heine_borel*}
assumes *convex S*

```
    and closed S
    and S ≠ {}
    and z ∉ S
  shows ∃ a b. ∃ y∈S. inner a z < b ∧ inner a y = b ∧ (∀ x∈S. inner a x ≥ b)
proof −
  obtain y where y ∈ S and y: ∀ x∈S. dist z y ≤ dist z x
    by (metis distance_attains_inf[OF assms(2−3)])
  show ?thesis
  proof (intro exI bexI conjI ballI)
    show (y − z) · z < (y − z) · y
     by (metis ‹y ∈ S› assms(4) diff_gt_0_iff_gt inner_commute inner_diff_left
inner_gt_zero_iff right_minus_eq)
    show (y − z) · y ≤ (y − z) · x if x ∈ S for x
    proof (rule ccontr)
      have ∗: ⋀u. 0 ≤ u ∧ u ≤ 1 ⟶ dist z y ≤ dist z ((1 − u) ∗R y + u ∗R x)
       using assms(1)[unfolded convex_alt] and y and ‹x∈S› and ‹y∈S› by auto
      assume ¬ (y − z) · y ≤ (y − z) · x
      then obtain v where v > 0 v ≤ 1 dist (y + v ∗R (x − y)) z < dist y z
       using closer_point_lemma[of z y x] by (auto simp: inner_diff)
      then show False
       using ∗[of v] by (auto simp: dist_commute algebra_simps)
    qed
  qed (use ‹y ∈ S› in auto)
qed

lemma separating_hyperplane_closed_point:
  fixes z :: 'a::{real_inner,heine_borel}
  assumes convex S
    and closed S
    and z ∉ S
  shows ∃ a b. inner a z < b ∧ (∀ x∈S. inner a x > b)
proof (cases S = {})
  case True
  then show ?thesis
    by (simp add: gt_ex)
next
  case False
  obtain y where y ∈ S and y: ⋀x. x ∈ S ⟹ dist z y ≤ dist z x
    by (metis distance_attains_inf[OF assms(2) False])
  show ?thesis
  proof (intro exI conjI ballI)
    show (y − z) · z < inner (y − z) z + (norm (y − z))^2 / 2
      using ‹y∈S› ‹z∉S› by auto
  next
    fix x
    assume x ∈ S
    have False if ∗: 0 < inner (z − y) (x − y)
    proof −
      obtain u where u > 0 u ≤ 1 dist (y + u ∗R (x − y)) z < dist y z
```

```
      using * closer_point_lemma by blast
    then show False using y[of y + u *_R (x - y)] convexD_alt [OF ‹convex S›]
      using ‹x∈S› ‹y∈S› by (auto simp: dist_commute algebra_simps)
  qed
  moreover have 0 < (norm (y - z))^2
    using ‹y∈S› ‹z∉S› by auto
  then have 0 < inner (y - z) (y - z)
    unfolding power2_norm_eq_inner by simp
  ultimately show (y - z) · z + (norm (y - z))^2 / 2 < (y - z) · x
      by (force simp: field_simps power2_norm_eq_inner inner_commute in-
ner_diff)
 qed
qed

lemma separating_hyperplane_closed_0:
 assumes convex (S::('a::euclidean_space) set)
   and closed S
   and 0 ∉ S
 shows ∃ a b. a ≠ 0 ∧ 0 < b ∧ (∀ x∈S. inner a x > b)
proof (cases S = {})
 case True
 have (SOME i. i∈Basis) ≠ (0::'a)
   by (metis Basis_zero SOME_Basis)
 then show ?thesis
   using True zero_less_one by blast
next
 case False
 then show ?thesis
   using False using separating_hyperplane_closed_point[OF assms]
  by (metis all_not_in_conv inner_zero_left inner_zero_right less_eq_real_def
not_le)
qed
```

Now set-to-set for closed/compact sets

```
lemma separating_hyperplane_closed_compact:
 fixes S :: 'a::euclidean_space set
 assumes convex S
   and closed S
   and convex T
   and compact T
   and T ≠ {}
   and S ∩ T = {}
 shows ∃ a b. (∀ x∈S. inner a x < b) ∧ (∀ x∈T. inner a x > b)
proof (cases S = {})
 case True
 obtain b where b: b > 0 ∀ x∈T. norm x ≤ b
   using compact_imp_bounded[OF assms(4)] unfolding bounded_pos by auto
 obtain z :: 'a where z: norm z = b + 1
```

```
    using vector_choose_size[of b + 1] and b(1) by auto
  then have z ∉ T using b(2)[THEN bspec[where x=z]] by auto
  then obtain a b where ab: inner a z < b ∀x∈T. b < inner a x
   using separating_hyperplane_closed_point[OF assms(3) compact_imp_closed[OF
assms(4)], of z]
    by auto
  then show ?thesis
    using True by auto
next
  case False
  then obtain y where y ∈ S by auto
  obtain a b where 0 < b and §: ⋀x. x ∈ (⋃x∈ S. ⋃y ∈ T. {x − y}) ⟹ b <
inner a x
   using separating_hyperplane_closed_point[OF convex_differences[OF assms(1,3)],
of 0]
    using closed_compact_differences assms by fastforce
  have ab: b + inner a y < inner a x if x∈S y∈T for x y
    using § [of x−y] that by (auto simp add: inner_diff_right less_diff_eq)
  define k where k = (SUP x∈T. a • x)
  have k + b / 2 < a • x if x ∈ S for x
  proof −
    have k ≤ inner a x − b
      unfolding k_def
      using ‹T ≠ {}› ab that by (fastforce intro: cSUP_least)
    then show ?thesis
      using ‹0 < b› by auto
  qed
  moreover
  have − (k + b / 2) < − a • x if x ∈ T for x
  proof −
    have inner a x − b / 2 < k
      unfolding k_def
    proof (subst less_cSUP_iff)
      show T ≠ {} by fact
      show bdd_above ((•) a ‘ T)
        using ab[rule_format, of y] ‹y ∈ S›
        by (intro bdd_aboveI2[where M=inner a y − b]) (auto simp: field_simps
intro: less_imp_le)
      show ∃y∈T. a • x − b / 2 < a • y
        using ‹0 < b› that by force
    qed
    then show ?thesis
      by auto
  qed
  ultimately show ?thesis
    by (metis inner_minus_left neg_less_iff_less)
qed

lemma separating_hyperplane_compact_closed:
```

fixes $S :: 'a::euclidean\_space\ set$
assumes *convex* S
and *compact* S
and $S \neq \{\}$
and *convex* T
and *closed* T
and $S \cap T = \{\}$
shows $\exists\, a\ b.\ (\forall\, x{\in}S.\ inner\ a\ x < b) \wedge (\forall\, x{\in}T.\ inner\ a\ x > b)$
proof −
obtain $a\ b$ **where** $(\forall\, x{\in}T.\ inner\ a\ x < b) \wedge (\forall\, x{\in}S.\ b < inner\ a\ x)$
by (*metis disjoint_iff_not_equal separating_hyperplane_closed_compact assms*)
then show *?thesis*
by (*metis inner_minus_left neg_less_iff_less*)
qed

General case without assuming closure and getting non-strict separation

lemma *separating_hyperplane_set_0*:
assumes *convex* S $(0::'a::euclidean\_space) \notin S$
shows $\exists\, a.\ a \neq 0 \wedge (\forall\, x{\in}S.\ 0 \leq inner\ a\ x)$
proof −
let *?k* $= \lambda c.\ \{x::'a.\ 0 \leq inner\ c\ x\}$
have $*$: *frontier* $(cball\ 0\ 1) \cap \bigcap f \neq \{\}$ **if** *as*: $f \subseteq$ *?k* ‘ S *finite* f **for** f
proof −
obtain c **where** *c*: $f =$ *?k* ‘ c $c \subseteq S$ *finite* c
using *finite_subset_image*[*OF as*(*2*,*1*)] **by** *auto*
then obtain $a\ b$ **where** *ab*: $a \neq 0$ $0 < b$ $\forall\, x{\in}convex\ hull\ c.\ b < inner\ a\ x$
using *separating_hyperplane_closed_0*[*OF convex_convex_hull*, *of c*]
using *finite_imp_compact_convex_hull*[*OF c*(*3*), *THEN compact_imp_closed*]
and *assms*(*2*)
using *subset_hull*[*of convex*, *OF assms*(*1*), *symmetric*, *of c*]
by *force*
have *norm* $(a\ /_R\ norm\ a) = 1$
by (*simp add*: *ab*(*1*))
moreover have $(\forall\, y{\in}c.\ 0 \leq y \cdot (a\ /_R\ norm\ a))$
using *hull_subset*[*of c convex*] *ab* **by** (*force simp*: *inner_commute*)
ultimately have $\exists\, x.\ norm\ x = 1 \wedge (\forall\, y{\in}c.\ 0 \leq inner\ y\ x)$
by *blast*
then show *frontier* $(cball\ 0\ 1) \cap \bigcap f \neq \{\}$
unfolding *c*(*1*) *frontier_cball sphere_def dist_norm* **by** *auto*
qed
have *frontier* $(cball\ 0\ 1) \cap (\bigcap (?k\ `\ S)) \neq \{\}$
by (*rule compact_imp_fip*) (*use* $*$ *closed_halfspace_ge* **in** *auto*)
then obtain x **where** *norm* $x = 1$ $\forall\, y{\in}S.\ x{\in}$*?k* y
unfolding *frontier_cball dist_norm sphere_def* **by** *auto*
then show *?thesis*
by (*metis inner_commute mem_Collect_eq norm_eq_zero zero_neq_one*)
qed

lemma *separating_hyperplane_sets*:
fixes S T :: *'a::euclidean_space set*
assumes *convex* S
and *convex* T
and $S \neq \{\}$
and $T \neq \{\}$
and $S \cap T = \{\}$
shows $\exists\, a\ b.\ a \neq 0 \wedge (\forall\, x{\in}S.\ \mathit{inner}\ a\ x \leq b) \wedge (\forall\, x{\in}T.\ \mathit{inner}\ a\ x \geq b)$
proof −
from *separating_hyperplane_set_0*[*OF convex_differences*[*OF assms*(*2*,*1*)]]
obtain a **where** $a \neq 0\ \forall\, x{\in}\{x - y\ |x\ y.\ x \in T \wedge y \in S\}.\ 0 \leq \mathit{inner}\ a\ x$
using *assms*(*3*−*5*) **by** *force*
then have $*$: $\bigwedge x\ y.\ x \in T \Longrightarrow y \in S \Longrightarrow \mathit{inner}\ a\ y \leq \mathit{inner}\ a\ x$
by (*force simp*: *inner_diff*)
then have *bdd*: *bdd_above* (((•) *a*)‘*S*)
using ‹$T \neq \{\}$› **by** (*auto intro*: *bdd_aboveI2*[*OF* $*$])
show *?thesis*
using ‹$a{\neq}0$›
by (*intro exI*[*of* _ *a*] *exI*[*of* _ *SUP* $x{\in}S.\ a \cdot x$])
(*auto intro*!: *cSUP_upper bdd cSUP_least* ‹$a \neq 0$› ‹$S \neq \{\}$› $*$)
qed

5.3.6 More convexity generalities

lemma *convex_closure* [*intro*,*simp*]:
fixes S :: *'a::real_normed_vector set*
assumes *convex* S
shows *convex* (*closure* S)
apply (*rule convexI*)
unfolding *closure_sequential*
apply (*elim exE*)
subgoal for x y u v f g
by (*rule_tac* $x{=}\lambda n.\ u *_R f\ n + v *_R g\ n$ **in** *exI*) (*force intro*: *tendsto_intros dest*: *convexD* [*OF assms*])
done

lemma *convex_interior* [*intro*,*simp*]:
fixes S :: *'a::real_normed_vector set*
assumes *convex* S
shows *convex* (*interior* S)
unfolding *convex_alt Ball_def mem_interior*
proof *clarify*
fix x y u
assume *u*: $0 \leq u\ u \leq (1{::}\mathit{real})$
fix e d
assume *ed*: *ball* x $e \subseteq S$ *ball* y $d \subseteq S$ $0{<}d$ $0{<}e$
show $\exists\, e{>}0.$ *ball* $((1 - u) *_R x + u *_R y)$ $e \subseteq S$
proof (*intro exI conjI subsetI*)

```
    fix z
    assume z: z ∈ ball ((1 − u) *R x + u *R y) (min d e)
    have (1− u) *R (z − u *R (y − x)) + u *R (z + (1 − u) *R (y − x)) ∈ S
    proof (rule_tac assms[unfolded convex_alt, rule_format])
      show z − u *R (y − x) ∈ S z + (1 − u) *R (y − x) ∈ S
        using ed z u by (auto simp add: algebra_simps dist_norm)
    qed (use u in auto)
    then show z ∈ S
      using u by (auto simp: algebra_simps)
  qed(use u ed in auto)
qed

lemma convex_hull_eq_empty[simp]: convex hull S = {} ⟷ S = {}
  using hull_subset[of S convex] convex_hull_empty by auto
```

5.3.7 Convex set as intersection of halfspaces

```
lemma convex_halfspace_intersection:
  fixes S :: ('a::euclidean_space) set
  assumes closed S convex S
  shows S = ⋂{h. S ⊆ h ∧ (∃ a b. h = {x. inner a x ≤ b})}
proof −
  { fix z
    assume ∀ T. S ⊆ T ∧ (∃ a b. T = {x. inner a x ≤ b}) ⟶ z ∈ T  z ∉ S
    then have §: ⋀a b. S ⊆ {x. inner a x ≤ b} ⟹ z ∈ {x. inner a x ≤ b}
      by blast
    obtain a b where inner a z < b (∀ x∈S. inner a x > b)
      using ‹z ∉ S› assms separating_hyperplane_closed_point by blast
    then have False
      using § [of −a −b] by fastforce
  }
  then show ?thesis
    by force
qed
```

5.3.8 Convexity of general and special intervals

```
lemma is_interval_convex:
  fixes S :: 'a::euclidean_space set
  assumes is_interval S
  shows convex S
proof (rule convexI)
  fix x y and u v :: real
  assume x ∈ S y ∈ S and uv: 0 ≤ u 0 ≤ v u + v = 1
  then have ∗: u = 1 − v 1 − v ≥ 0 and ∗∗: v = 1 − u 1 − u ≥ 0
    by auto
  {
    fix a b
    assume ¬ b ≤ u ∗ a + v ∗ b
    then have u ∗ a < (1 − v) ∗ b
```

```
    unfolding not_le using ‹0 ≤ v›by (auto simp: field_simps)
  then have a < b
    using ∗(1) less_eq_real_def uv(1) by auto
  then have a ≤ u ∗ a + v ∗ b
    unfolding ∗ using ‹0 ≤ v›
    by (auto simp: field_simps intro!:mult_right_mono)
 }
 moreover
 {
  fix a b
  assume ¬ u ∗ a + v ∗ b ≤ a
  then have v ∗ b > (1 − u) ∗ a
    unfolding not_le using ‹0 ≤ v› by (auto simp: field_simps)
  then have a < b
    unfolding ∗ using ‹0 ≤ v›
    by (rule_tac mult_left_less_imp_less) (auto simp: field_simps)
  then have u ∗ a + v ∗ b ≤ b
    unfolding ∗∗
    using ∗∗(2) ‹0 ≤ u› by (auto simp: algebra_simps mult_right_mono)
 }
 ultimately show u ∗R x + v ∗R y ∈ S
  using DIM_positive[where 'a='a]
   by (intro mem_is_intervalI [OF assms ‹x ∈ S› ‹y ∈ S›]) (auto simp: in-
ner_simps)
qed

lemma is_interval_connected:
 fixes S :: 'a::euclidean_space set
 shows is_interval S ⟹ connected S
 using is_interval_convex convex_connected by auto

lemma convex_box [simp]: convex (cbox a b) convex (box a (b::'a::euclidean_space))
 by (auto simp add: is_interval_convex)
```

A non-singleton connected set is perfect (i.e. has no isolated points).

```
lemma connected_imp_perfect:
 fixes a :: 'a::metric_space
 assumes connected S a ∈ S and S: ⋀x. S ≠ {x}
 shows a islimpt S
proof −
 have False if a ∈ T open T ⋀y. ⟦y ∈ S; y ∈ T⟧ ⟹ y = a for T
 proof −
  obtain e where e > 0 and e: cball a e ⊆ T
   using ‹open T› ‹a ∈ T› by (auto simp: open_contains_cball)
  have openin (top_of_set S) {a}
   unfolding openin_open using that ‹a ∈ S› by blast
  moreover have closedin (top_of_set S) {a}
   by (simp add: assms)
  ultimately show False
```

using ‹*connected S*› *connected_clopen S* **by** *blast*
qed
then show *?thesis*
unfolding *islimpt_def* **by** *blast*
qed

lemma *islimpt_Ioc* [*simp*]:
fixes *a* :: *real*
assumes *a*<*b*
shows *x islimpt* {*a*<..*b*} ⟷ *x* ∈ {*a*..*b*} (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*metis assms closed_atLeastAtMost closed_limpt closure_greaterThanAtMost closure_subset islimpt_subset*)
next
assume *?rhs*
then have *x* ∈ *closure* {*a*<..<*b*}
using *assms closure_greaterThanLessThan* **by** *blast*
then show *?lhs*
by (*metis* (*no_types*) *Diff_empty Diff_insert0 interior_lessThanAtMost interior_limit_point interior_subset islimpt_in_closure islimpt_subset*)
qed

lemma *islimpt_Ico* [*simp*]:
fixes *a* :: *real*
assumes *a*<*b* **shows** *x islimpt* {*a*..<*b*} ⟷ *x* ∈ {*a*..*b*}
by (*metis assms closure_atLeastLessThan closure_greaterThanAtMost islimpt_Ioc limpt_of_closure*)

lemma *islimpt_Icc* [*simp*]:
fixes *a* :: *real*
assumes *a*<*b* **shows** *x islimpt* {*a*..*b*} ⟷ *x* ∈ {*a*..*b*}
by (*metis assms closure_atLeastLessThan islimpt_Ico limpt_of_closure*)

lemma *connected_imp_perfect_aff_dim*:
⟦*connected S*; *aff_dim S* ≠ *0*; *a* ∈ *S*⟧ ⟹ *a islimpt S*
using *aff_dim_sing connected_imp_perfect* **by** *blast*

5.3.9 On *real*, *is_interval*, *convex* and *connected* are all equivalent

lemma *mem_is_interval_1_I*:
fixes *a b c*::*real*
assumes *is_interval S*
assumes *a* ∈ *S c* ∈ *S*
assumes *a* ≤ *b b* ≤ *c*
shows *b* ∈ *S*
using *assms is_interval_1* **by** *blast*

lemma *is_interval_connected_1*:
fixes S :: *real set*
shows *is_interval* $S \longleftrightarrow$ *connected* S
by (*meson connected_iff_interval is_interval_1*)

lemma *is_interval_convex_1*:
fixes S :: *real set*
shows *is_interval* $S \longleftrightarrow$ *convex* S
by (*metis is_interval_convex convex_connected is_interval_connected_1*)

lemma *connected_compact_interval_1*:
$\quad$ *connected* $S \wedge$ *compact* $S \longleftrightarrow (\exists\, a\ b.\ S = \{a..b::real\})$
by (*auto simp*: *is_interval_connected_1* [*symmetric*] *is_interval_compact*)

lemma *connected_convex_1*:
fixes S :: *real set*
shows *connected* $S \longleftrightarrow$ *convex* S
by (*metis is_interval_convex convex_connected is_interval_connected_1*)

lemma *connected_space_iff_is_interval_1* [*iff*]:
fixes S :: *real set*
shows *connected_space* (*top_of_set* S) $\longleftrightarrow$ *is_interval* S
using *connectedin_topspace is_interval_connected_1* **by** *force*

lemma *connected_convex_1_gen*:
fixes S :: $'a$ :: *euclidean_space set*
assumes $DIM('a) = 1$
shows *connected* $S \longleftrightarrow$ *convex* S
proof −
obtain f:: $'a \Rightarrow$ *real* **where** *linf*: *linear* f **and** *inj* f
using *subspace_isomorphism*[*OF subspace_UNIV subspace_UNIV*,
where $'a='a$ **and** $'b$=*real*]
unfolding *Euclidean_Space.dim_UNIV*
by (*auto simp*: *assms*)
then have $f -` (f ` S) = S$
by (*simp add*: *inj_vimage_image_eq*)
then show *?thesis*
by (*metis connected_convex_1 convex_linear_vimage linf convex_connected connected_linear_image*)
qed

lemma [*simp*]:
fixes r s::*real*
shows *is_interval_io*: *is_interval* $\{..<r\}$
and *is_interval_oi*: *is_interval* $\{r<..\}$
and *is_interval_oo*: *is_interval* $\{r<..<s\}$
and *is_interval_oc*: *is_interval* $\{r<..s\}$
and *is_interval_co*: *is_interval* $\{r..<s\}$
by (*simp_all add*: *is_interval_convex_1*)

5.3.10 Another intermediate value theorem formulation

lemma *ivt_increasing_component_on_1*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
assumes $a \leq b$
and *continuous_on* $\{a..b\}$ f
and $(f\ a)\cdot k \leq y$ $y \leq (f\ b)\cdot k$
shows $\exists x{\in}\{a..b\}.\ (f\ x)\cdot k = y$
proof −
have $f\ a \in f\ `\ cbox\ a\ b$ $f\ b \in f\ `\ cbox\ a\ b$
using $\langle a \leq b \rangle$ **by** *auto*
then show *?thesis*
using *connected_ivt_component*[*of* $f\ `\ cbox\ a\ b$ $f\ a$ $f\ b$ k y]
by (*simp add*: *connected_continuous_image assms*)
qed

lemma *ivt_increasing_component_1*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
shows $a \leq b \Longrightarrow \forall x{\in}\{a..b\}.\ continuous\ (at\ x)\ f \Longrightarrow$
$f\ a\cdot k \leq y \Longrightarrow y \leq f\ b\cdot k \Longrightarrow \exists x{\in}\{a..b\}.\ (f\ x)\cdot k = y$
by (*rule ivt_increasing_component_on_1*) (*auto simp*: *continuous_at_imp_continuous_on*)

lemma *ivt_decreasing_component_on_1*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
assumes $a \leq b$
and *continuous_on* $\{a..b\}$ f
and $(f\ b)\cdot k \leq y$
and $y \leq (f\ a)\cdot k$
shows $\exists x{\in}\{a..b\}.\ (f\ x)\cdot k = y$
using *ivt_increasing_component_on_1*[*of* a b $\lambda x.\ -f\ x$ k $-y$] *neg_equal_iff_equal*
using *assms continuous_on_minus* **by** *force*

lemma *ivt_decreasing_component_1*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
shows $a \leq b \Longrightarrow \forall x{\in}\{a..b\}.\ continuous\ (at\ x)\ f \Longrightarrow$
$f\ b\cdot k \leq y \Longrightarrow y \leq f\ a\cdot k \Longrightarrow \exists x{\in}\{a..b\}.\ (f\ x)\cdot k = y$
by (*rule ivt_decreasing_component_on_1*) (*auto simp*: *continuous_at_imp_continuous_on*)

5.3.11 A bound within an interval

lemma *convex_hull_eq_real_cbox*:
fixes $x\ y :: real$ **assumes** $x \leq y$
shows *convex hull* $\{x, y\} = cbox\ x\ y$
proof (*rule hull_unique*)
show $\{x, y\} \subseteq cbox\ x\ y$ **using** $\langle x \leq y \rangle$ **by** *auto*
show *convex* (*cbox x y*)
by (*rule convex_box*)
next
fix S **assume** $\{x, y\} \subseteq S$ **and** *convex* S
then show $cbox\ x\ y \subseteq S$

```
    unfolding is_interval_convex_1 [symmetric] is_interval_def Basis_real_def
    by − (clarify, simp (no_asm_use), fast)
qed

lemma unit_interval_convex_hull:
  cbox (0::'a::euclidean_space) One = convex hull {x. ∀i∈Basis. (x·i = 0) ∨ (x·i
= 1)}
  (is ?int = convex hull ?points)
proof −
  have One[simp]: ⋀i. i ∈ Basis ⟹ One · i = 1
    by (simp add: inner_sum_left sum.If_cases inner_Basis)
  have ?int = {x. ∀i∈Basis. x · i ∈ cbox 0 1}
    by (auto simp: cbox_def)
  also have ... = (∑ i∈Basis. (λx. x *R i) ` cbox 0 1)
    by (simp only: box_eq_set_sum_Basis)
  also have ... = (∑ i∈Basis. (λx. x *R i) ` (convex hull {0, 1}))
    by (simp only: convex_hull_eq_real_cbox zero_le_one)
  also have ... = (∑ i∈Basis. convex hull ((λx. x *R i) ` {0, 1}))
    by (simp add: convex_hull_linear_image)
  also have ... = convex hull (∑ i∈Basis. (λx. x *R i) ` {0, 1})
    by (simp only: convex_hull_set_sum)
  also have ... = convex hull {x. ∀i∈Basis. x·i ∈ {0, 1}}
    by (simp only: box_eq_set_sum_Basis)
  also have convex hull {x. ∀i∈Basis. x·i ∈ {0, 1}} = convex hull ?points
    by simp
  finally show ?thesis .
qed
```

And this is a finite set of vertices.

```
lemma unit_cube_convex_hull:
  obtains S :: 'a::euclidean_space set
  where finite S and cbox 0 (∑ Basis) = convex hull S
proof
  show finite {x::'a. ∀i∈Basis. x · i = 0 ∨ x · i = 1}
  proof (rule finite_subset, clarify)
    show finite ((λS. ∑ i∈Basis. (if i ∈ S then 1 else 0) *R i) ` Pow Basis)
      using finite_Basis by blast
    fix x :: 'a
    assume x: ∀i∈Basis. x · i = 0 ∨ x · i = 1
    show x ∈ (λS. ∑ i∈Basis. (if i∈S then 1 else 0) *R i) ` Pow Basis
      apply (rule image_eqI[where x={i. i ∈ Basis ∧ x·i = 1}])
      using x
      by (subst euclidean_eq_iff, auto)
  qed
  show cbox 0 One = convex hull {x. ∀i∈Basis. x · i = 0 ∨ x · i = 1}
    using unit_interval_convex_hull by blast
qed
```

Hence any cube (could do any nonempty interval).

lemma *cube_convex_hull*:
assumes $d > 0$
obtains $S :: 'a::euclidean\_space\ set$ **where**
finite S **and** $cbox\ (x - (\sum i{\in}Basis.\ d{*_R}i))\ (x + (\sum i{\in}Basis.\ d{*_R}i)) = convex\ hull\ S$
proof −
let $?d = (\sum i{\in}Basis.\ d *_R i)::'a$
have $*$: $cbox\ (x - ?d)\ (x + ?d) = (\lambda y.\ x - ?d + (2 * d) *_R y)\ `\ cbox\ 0\ (\sum Basis)$
proof (*intro set_eqI iffI*)
fix y
assume $y \in cbox\ (x - ?d)\ (x + ?d)$
then have $inverse\ (2 * d) *_R (y - (x - ?d)) \in cbox\ 0\ (\sum Basis)$
using *assms* **by** (*simp add*: *mem_box inner_simps*) (*simp add*: *field_simps*)
with $\langle 0 < d \rangle$ **show** $y \in (\lambda y.\ x - sum\ ((*_R)\ d)\ Basis + (2 * d) *_R y)\ `\ cbox\ 0\ One$
by (*auto intro*: *image_eqI*[**where** $x{=}\ inverse\ (2 * d) *_R (y - (x - ?d))$])
next
fix y
assume $y \in (\lambda y.\ x - ?d + (2 * d) *_R y)\ `\ cbox\ 0\ One$
then obtain z **where** z: $z \in cbox\ 0\ One\ y = x - ?d + (2{*}d) *_R z$
by *auto*
then show $y \in cbox\ (x - ?d)\ (x + ?d)$
using z *assms* **by** (*auto simp*: *mem_box inner_simps*)
qed
obtain S **where** *finite* S $cbox\ 0\ (\sum Basis::'a) = convex\ hull\ S$
using *unit_cube_convex_hull* **by** *auto*
then show *?thesis*
by (*rule_tac that*[*of* $(\lambda y.\ x - ?d + (2 * d) *_R y)`\ S$]) (*auto simp*: *convex_hull_affinity* $*$)
qed

5.3.12 Representation of any interval as a finite convex hull

lemma *image_stretch_interval*:
$(\lambda x.\ \sum k{\in}Basis.\ (m\ k * (x{\cdot}k)) *_R k)\ `\ cbox\ a\ (b::'a::euclidean\_space) =$
(*if* $(cbox\ a\ b) = \{\}$ *then* $\{\}$ *else*
$cbox\ (\sum k{\in}Basis.\ (min\ (m\ k * (a{\cdot}k))\ (m\ k * (b{\cdot}k))) *_R k::'a)$
$(\sum k{\in}Basis.\ (max\ (m\ k * (a{\cdot}k))\ (m\ k * (b{\cdot}k))) *_R k)$)
proof *cases*
assume $*$: $cbox\ a\ b \neq \{\}$
show *?thesis*
unfolding *box_ne_empty if_not_P*[*OF* $*$]
apply (*simp add*: *cbox_def image_Collect set_eq_iff euclidean_eq_iff*[**where** $'a='a$] *ball_conj_distrib*[*symmetric*])
apply (*subst choice_Basis_iff*[*symmetric*])
proof (*intro allI ball_cong refl*)
fix $x\ i :: 'a$ **assume** $i \in Basis$
with $*$ **have** a_le_b: $a \cdot i \leq b \cdot i$
unfolding *box_ne_empty* **by** *auto*

```
    show (∃xa. x · i = m i * xa ∧ a · i ≤ xa ∧ xa ≤ b · i) ⟷
        min (m i * (a · i)) (m i * (b · i)) ≤ x · i ∧ x · i ≤ max (m i * (a · i)) (m
i * (b · i))
    proof (cases m i = 0)
      case True
      with a_le_b show ?thesis by auto
    next
      case False
      then have *: ⋀a b. a = m i * b ⟷ b = a / m i
        by (auto simp: field_simps)
      from False have
          min (m i * (a · i)) (m i * (b · i)) = (if 0 < m i then m i * (a · i) else m
i * (b · i))
          max (m i * (a · i)) (m i * (b · i)) = (if 0 < m i then m i * (b · i) else m
i * (a · i))
        using a_le_b by (auto simp: min_def max_def mult_le_cancel_left)
      with False show ?thesis using a_le_b *
        by (simp add: le_divide_eq divide_le_eq) (simp add: ac_simps)
    qed
  qed
qed simp

lemma interval_image_stretch_interval:
  ∃u v. (λx. ∑k∈Basis. (m k * (x·k))*R k) ' cbox a (b::'a::euclidean_space) =
cbox u (v::'a::euclidean_space)
  unfolding image_stretch_interval by auto

lemma cbox_translation: cbox (c + a) (c + b) = image (λx. c + x) (cbox a b)
  using image_affinity_cbox [of 1 c a b]
  using box_ne_empty [of a+c b+c]  box_ne_empty [of a b]
  by (auto simp: inner_left_distrib add.commute)

lemma cbox_image_unit_interval:
  fixes a :: 'a::euclidean_space
  assumes cbox a b ≠ {}
    shows cbox a b =
          (+) a ' (λx. ∑k∈Basis. ((b · k − a · k) * (x · k)) *R k) ' cbox 0 One
using assms
apply (simp add: box_ne_empty image_stretch_interval cbox_translation [symmetric])
apply (simp add: min_def max_def algebra_simps sum_subtractf euclidean_representation)
done

lemma closed_interval_as_convex_hull:
  fixes a :: 'a::euclidean_space
  obtains S where finite S cbox a b = convex hull S
proof (cases cbox a b = {})
  case True with convex_hull_empty that show ?thesis
    by blast
next
```

case *False*
obtain *S*::*'a set* **where** *finite S* **and** *eq*: *cbox 0 One = convex hull S*
by (*blast intro*: *unit_cube_convex_hull*)
let *?S* = ((+) *a* ‘ (λ*x*. ∑ *k*∈*Basis*. ((*b* · *k* − *a* · *k*) ∗ (*x* · *k*)) ∗$_R$ *k*) ‘ *S*)
show *thesis*
proof
show *finite ?S*
by (*simp add*: ‹*finite S*›)
have *lin*: *linear* (λ*x*. ∑ *k*∈*Basis*. ((*b* · *k* − *a* · *k*) ∗ (*x* · *k*)) ∗$_R$ *k*)
by (*rule linear_compose_sum*) (*auto simp*: *algebra_simps linearI*)
show *cbox a b = convex hull ?S*
using *convex_hull_linear_image* [*OF lin*]
by (*simp add*: *convex_hull_translation eq cbox_image_unit_interval* [*OF False*])
qed
qed

5.3.13 Bounded convex function on open set is continuous

lemma *convex_on_bounded_continuous*:
fixes *S* :: (*'a*::*real_normed_vector*) *set*
assumes *open S*
and *f*: *convex_on S f*
and ∀ *x*∈*S*. |*f x*| ≤ *b*
shows *continuous_on S f*
proof −
have ∃ *d*>*0*. ∀ *x'*. *norm* (*x'* − *x*) < *d* ⟶ |*f x'* − *f x*| < *e* **if** *x* ∈ *S e* > *0* **for** *x* **and** *e* :: *real*
proof −
define *B* **where** *B* = |*b*| + *1*
then have *B*: *0* < *B*⋀*x*. *x*∈*S* ⟹ |*f x*| ≤ *B*
using *assms*(*3*) **by** *auto*
obtain *k* **where** *k* > *0* **and** *k*: *cball x k* ⊆ *S*
using ‹*x* ∈ *S*› *assms*(*1*) *open_contains_cball_eq* **by** *blast*
show ∃ *d*>*0*. ∀ *x'*. *norm* (*x'* − *x*) < *d* ⟶ |*f x'* − *f x*| < *e*
proof (*intro exI conjI allI impI*)
fix *y*
assume *as*: *norm* (*y* − *x*) < *min* (*k* / *2*) (*e* / (*2* ∗ *B*) ∗ *k*)
show |*f y* − *f x*| < *e*
proof (*cases y* = *x*)
case *False*
define *t* **where** *t* = *k* / *norm* (*y* − *x*)
have *2* < *t* *0*<*t*
unfolding *t_def* **using** *as False* **and** ‹*k*>*0*›
by (*auto simp*:*field_simps*)
have *y* ∈ *S*
apply (*rule k*[*THEN subsetD*])
unfolding *mem_cball dist_norm*
apply (*rule order_trans*[*of* _ *2* ∗ *norm* (*x* − *y*)])

```
        using as
        by (auto simp: field_simps norm_minus_commute)
      {
        define w where w = x + t *R (y - x)
        have w ∈ S
          using ‹k>0› by (auto simp: dist_norm t_def w_def k[THEN subsetD])
        have (1 / t) *R x + - x + ((t - 1) / t) *R x = (1 / t - 1 + (t - 1) /
t) *R x
          by (auto simp: algebra_simps)
        also have ... = 0
          using ‹t > 0› by (auto simp:field_simps)
        finally have w: (1 / t) *R w + ((t - 1) / t) *R x = y
          unfolding w_def using False and ‹t > 0›
          by (auto simp: algebra_simps)
        have 2: 2 * B < e * t
          unfolding t_def using ‹0 < e› ‹0 < k› ‹B > 0› and as and False
          by (auto simp:field_simps)
        have f y - f x ≤ (f w - f x) / t
          using convex_onD [OF f, of (t - 1)/t w x] ‹0 < t› ‹2 < t› ‹x ∈ S› ‹w
∈ S›
          by (simp add: w field_simps)
        also have ... < e
          using B(2)[OF ‹w∈S›] and B(2)[OF ‹x∈S›] 2 ‹t > 0› by (auto simp:
field_simps)
        finally have th1: f y - f x < e .
      }
      moreover
      {
        define w where w = x - t *R (y - x)
        have w ∈ S
          using ‹k > 0› by (auto simp: dist_norm t_def w_def k[THEN subsetD])
        have (1 / (1 + t)) *R x + (t / (1 + t)) *R x = (1 / (1 + t) + t / (1 +
t)) *R x
          by (auto simp: algebra_simps)
        also have ... = x
          using ‹t > 0› by (auto simp:field_simps)
        finally have w: (1 / (1+t)) *R w + (t / (1 + t)) *R y = x
          unfolding w_def using False and ‹t > 0›
          by (auto simp: algebra_simps)
        have 2 * B < e * t
          unfolding t_def
          using ‹0 < e› ‹0 < k› ‹B > 0› and as and False
          by (auto simp:field_simps)
        then have *: (f w - f y) / t < e
          using B(2)[OF ‹w∈S›] and B(2)[OF ‹y∈S›]
          using ‹t > 0›
          by (auto simp:field_simps)
        have f x ≤ 1 / (1 + t) * f w + (t / (1 + t)) * f y
          using convex_onD [OF f, of t / (1+t) w y] ‹0 < t› ‹2 < t› ‹y ∈ S› ‹w
```

```
∈ S›
          by (simp add: w field_simps)
        also have ... = (f w + t * f y) / (1 + t)
          using ‹t > 0› by (simp add: add_divide_distrib)
        also have ... < e + f y
          using ‹t > 0› * ‹e > 0› by (auto simp: field_simps)
        finally have f x - f y < e by auto
      }
      ultimately show ?thesis by auto
    qed (use ‹0<e› in auto)
  qed (use ‹0<e› ‹0<k› ‹0<B› in ‹auto simp: field_simps›)
 qed
 then show ?thesis
  by (metis continuous_on_iff dist_norm real_norm_def)
qed
```

5.3.14 Upper bound on a ball implies upper and lower bounds

```
lemma convex_bounds_lemma:
 fixes x :: 'a::real_normed_vector
 assumes f: convex_on (cball x e) f
  and b: ⋀y. y ∈ cball x e ⟹ f y ≤ b and y: y ∈ cball x e
 shows |f y| ≤ b + 2 * |f x|
proof (cases 0 ≤ e)
 case True
 define z where z = 2 *R x - y
 have *: x - (2 *R x - y) = y - x
  by (simp add: scaleR_2)
 have z: z ∈ cball x e
  using y unfolding z_def mem_cball dist_norm * by (auto simp: norm_minus_commute)
 have (1 / 2) *R y + (1 / 2) *R z = x
  unfolding z_def by (auto simp: algebra_simps)
 then show |f y| ≤ b + 2 * |f x|
  using convex_onD [OF f, of 1/2 y z] b[OF y] b y z
  by (fastforce simp add: field_simps)
next
 case False
 have dist x y < 0
  using False y unfolding mem_cball not_le by (auto simp del: dist_not_less_zero)
 then show |f y| ≤ b + 2 * |f x|
  using zero_le_dist[of x y] by auto
qed
```

Hence a convex function on an open set is continuous

```
lemma real_of_nat_ge_one_iff: 1 ≤ real (n::nat) ⟷ 1 ≤ n
 by auto

lemma convex_on_continuous:
 fixes S :: 'a::euclidean_space set
```

```
  assumes open S convex_on S f
  shows continuous_on S f
  unfolding continuous_on_eq_continuous_at[OF ‹open S›]
proof
  note dimge1 = DIM_positive[where 'a='a]
  fix x
  assume x ∈ S
  then obtain e where e: cball x e ⊆ S e > 0
    using assms(1) unfolding open_contains_cball by auto
  define d where d = e / real DIM('a)
  have 0 < d
    unfolding d_def using ‹e > 0› dimge1 by auto
  let ?d = (∑ i∈Basis. d *R i)::'a
  obtain c
    where c: finite c and c1: convex hull c ⊆ cball x e and c2: cball x d ⊆ convex
hull c
  proof
    define c where c = (∑ i∈Basis. (λa. a *R i) ` {x•i - d, x•i + d})
    show finite c
      unfolding c_def by (simp add: finite_set_sum)
    have ⋀i. i ∈ Basis ⟹ convex hull {x • i - d, x • i + d} = cbox (x • i - d)
(x • i + d)
      using ‹0 < d› convex_hull_eq_real_cbox by auto
    then have 1: convex hull c = {a. ∀ i∈Basis. a • i ∈ cbox (x • i - d) (x • i +
d)}
      unfolding box_eq_set_sum_Basis c_def convex_hull_set_sum
      apply (subst convex_hull_linear_image [symmetric])
      by (force simp add: linear_iff scaleR_add_left)+
    then have 2: convex hull c = {a. ∀ i∈Basis. a • i ∈ cball (x • i) d}
      by (simp add: dist_norm abs_le_iff algebra_simps)
    show cball x d ⊆ convex hull c
      unfolding 2
        by (clarsimp simp: dist_norm) (metis inner_commute inner_diff_right
norm_bound_Basis_le)
    have e': e = (∑ (i::'a)∈Basis. d)
      by (simp add: d_def)
    show convex hull c ⊆ cball x e
      unfolding 2
    proof clarsimp
      show dist x y ≤ e if ∀ i∈Basis. dist (x • i) (y • i) ≤ d for y
      proof -
        have ⋀i. i ∈ Basis ⟹ 0 ≤ dist (x • i) (y • i)
          by simp
        have (∑ i∈Basis. dist (x • i) (y • i)) ≤ e
          using e' sum_mono that by fastforce
        then show ?thesis
       by (metis (mono_tags) euclidean_dist_l2 order_trans [OF L2_set_le_sum]
zero_le_dist)
      qed
```

```
    qed
  qed
  define k where k = Max (f ‘ c)
  have convex_on (convex hull c) f
    using assms(2) c1 convex_on_subset e(1) by blast
  then have k: ∀ y∈convex hull c. f y ≤ k
    using c convex_on_convex_hull_bound k_def by fastforce
  have e ≤ e * real DIM('a)
    using e(2) real_of_nat_ge_one_iff by auto
  then have d ≤ e
    by (simp add: d_def field_split_simps)
  then have dsube: cball x d ⊆ cball x e
    by (rule subset_cball)
  have conv: convex_on (cball x d) f
    using ‹convex_on (convex hull c) f› c2 convex_on_subset by blast
  then have ⋀y. y∈cball x d ⟹ |f y| ≤ k + 2 * |f x|
    by (rule convex_bounds_lemma) (use c2 k in blast)
  moreover have convex_on (ball x d) f
    using conv convex_on_subset by fastforce
  ultimately
  have continuous_on (ball x d) f
   by (metis convex_on_bounded_continuous Elementary_Metric_Spaces.open_ball
mem_ball_imp_mem_cball)
  then show continuous (at x) f
    unfolding continuous_on_eq_continuous_at[OF open_ball]
    using ‹d > 0› by auto
qed

end
```

Chapter 6

Unsorted

theory *Starlike*
imports
Convex_Euclidean_Space
Line_Segment
begin

lemma *affine_hull_closed_segment* [*simp*]:
affine hull (*closed_segment a b*) = *affine hull* {*a*,*b*}
by (*simp add*: *segment_convex_hull*)

lemma *affine_hull_open_segment* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows *affine hull* (*open_segment a b*) = (*if a* = *b then* {} *else affine hull* {*a*,*b*})
by (*metis affine_hull_convex_hull affine_hull_empty closure_open_segment closure_same_affine_hull segment_convex_hull*)

lemma *rel_interior_closure_convex_segment*:
fixes *S* :: *_*::*euclidean_space set*
assumes *convex S a* ∈ *rel_interior S b* ∈ *closure S*
shows *open_segment a b* ⊆ *rel_interior S*
proof
fix *x*
have [*simp*]: (*1* − *u*) *∗R a* + *u ∗R b* = *b* − (*1* − *u*) *∗R* (*b* − *a*) **for** *u*
by (*simp add*: *algebra_simps*)
assume *x* ∈ *open_segment a b*
then show *x* ∈ *rel_interior S*
unfolding *closed_segment_def open_segment_def* **using** *assms*
by (*auto intro*: *rel_interior_closure_convex_shrink*)
qed

lemma *convex_hull_insert_segments*:
convex hull (*insert a S*) =
(*if S* = {} *then* {*a*} *else* ⋃*x* ∈ *convex hull S*. *closed_segment a x*)
by (*force simp add*: *convex_hull_insert_alt in_segment*)

lemma *Int_convex_hull_insert_rel_exterior*:
fixes z :: *'a::euclidean_space*
assumes *convex C T* $\subseteq$ *C* **and** *z*: $z \in$ *rel_interior C* **and** *dis*: *disjnt S* (*rel_interior C*)
shows $S \cap$ (*convex hull* (*insert z T*)) = $S \cap$ (*convex hull T*) (**is** *?lhs = ?rhs*)
proof
have $*$: $T = \{\} \Longrightarrow z \notin S$
using *dis z* **by** (*auto simp add*: *disjnt_def*)
{ fix *x y*
assume $x \in S$ **and** *y*: $y \in$ *convex hull T* **and** $x \in$ *closed_segment z y*
have $y \in$ *closure C*
by (*metis y ‹convex C› ‹T ⊆ C› closure_subset contra_subsetD convex_hull_eq hull_mono*)
moreover have $x \notin$ *rel_interior C*
by (*meson ‹x ∈ S› dis disjnt_iff*)
moreover have $x \in$ *open_segment z y* $\cup \{z, y\}$
using *‹x ∈ closed_segment z y› closed_segment_eq_open* **by** *blast*
ultimately have $x \in$ *convex hull T*
using *rel_interior_closure_convex_segment* [*OF ‹convex C› z*]
using *y z* **by** *blast*
}
with $*$ **show** *?lhs* $\subseteq$ *?rhs*
by (*auto simp add*: *convex_hull_insert_segments*)
show *?rhs* $\subseteq$ *?lhs*
by (*meson hull_mono inf_mono subset_insertI subset_refl*)
qed

6.0.1 Shrinking towards the interior of a convex set

lemma *mem_interior_convex_shrink*:
fixes S :: *'a::euclidean_space set*
assumes *convex S*
and $c \in$ *interior S*
and $x \in S$
and $0 < e$
and $e \leq 1$
shows $x - e *_R (x - c) \in$ *interior S*
proof −
obtain d **where** $d > 0$ **and** *d*: *ball c d* $\subseteq S$
using *assms(2)* **unfolding** *mem_interior* **by** *auto*
show *?thesis*
unfolding *mem_interior*
proof (*intro exI subsetI conjI*)
fix *y*
assume $y \in$ *ball* $(x - e *_R (x - c))$ $(e*d)$
then have *as*: *dist* $(x - e *_R (x - c))$ $y < e * d$
by *simp*
have $*$: $y = (1 - (1 - e)) *_R ((1 \,/\, e) *_R y - ((1 - e) \,/\, e) *_R x) + (1 - e)$

```
*R x
    using ‹e > 0› by (auto simp add: scaleR_left_diff_distrib scaleR_right_diff_distrib)
    have c − ((1 / e) *R y − ((1 − e) / e) *R x) = (1 / e) *R (e *R c − y + (1
− e) *R x)
      using ‹e > 0›
      by (auto simp add: euclidean_eq_iff[where 'a='a] field_simps inner_simps)
    then have dist c ((1 / e) *R y − ((1 − e) / e) *R x) = |1/e| * norm (e *R c
− y + (1 − e) *R x)
      by (simp add: dist_norm)
    also have ... = |1/e| * norm (x − e *R (x − c) − y)
      by (auto intro!:arg_cong[where f=norm] simp add: algebra_simps)
    also have ... < d
      using as[unfolded dist_norm] and ‹e > 0›
      by (auto simp add:pos_divide_less_eq[OF ‹e > 0›] mult.commute)
    finally have (1 − (1 − e)) *R ((1 / e) *R y − ((1 − e) / e) *R x) + (1 −
e) *R x ∈ S
      using assms(3−5) d
      by (intro convexD_alt [OF ‹convex S›]) (auto intro: convexD_alt [OF ‹convex
S›])
    with ‹e > 0› show y ∈ S
      by (auto simp add: scaleR_left_diff_distrib scaleR_right_diff_distrib)
  qed (use ‹e>0› ‹d>0› in auto)
qed

lemma mem_interior_closure_convex_shrink:
  fixes S :: 'a::euclidean_space set
  assumes convex S
    and c ∈ interior S
    and x ∈ closure S
    and 0 < e
    and e ≤ 1
  shows x − e *R (x − c) ∈ interior S
proof −
  obtain d where d > 0 and d: ball c d ⊆ S
    using assms(2) unfolding mem_interior by auto
  have ∃ y∈S. norm (y − x) * (1 − e) < e * d
  proof (cases x ∈ S)
    case True
    then show ?thesis
      using ‹e > 0› ‹d > 0› by force
  next
    case False
    then have x: x islimpt S
      using assms(3)[unfolded closure_def] by auto
    show ?thesis
    proof (cases e = 1)
      case True
      obtain y where y ∈ S y ≠ x dist y x < 1
        using x[unfolded islimpt_approachable,THEN spec[where x=1]] by auto
```

```
      then show ?thesis
        using True ‹0 < d› by auto
    next
      case False
      then have 0 < e * d / (1 - e) and *: 1 - e > 0
        using ‹e ≤ 1› ‹e > 0› ‹d > 0› by auto
      then obtain y where y ∈ S y ≠ x dist y x < e * d / (1 - e)
        using islimpt_approachable x by blast
      then have norm (y - x) * (1 - e) < e * d
        by (metis * dist_norm mult_imp_div_pos_le not_less)
      then show ?thesis
        using ‹y ∈ S› by blast
    qed
  qed
  then obtain y where y ∈ S and y: norm (y - x) * (1 - e) < e * d
    by auto
  define z where z = c + ((1 - e) / e) *R (x - y)
  have *: x - e *R (x - c) = y - e *R (y - z)
    unfolding z_def using ‹e > 0›
   by (auto simp add: scaleR_right_diff_distrib scaleR_right_distrib scaleR_left_diff_distrib)
  have (1 - e) * norm (x - y) / e < d
    using y ‹0 < e› by (simp add: field_simps norm_minus_commute)
  then have z ∈ interior (ball c d)
     using ‹0 < e› ‹e ≤ 1› by (simp add: interior_open[OF open_ball] z_def
dist_norm)
  then have z ∈ interior S
    using d interiorI interior_ball by blast
  then show ?thesis
    unfolding * using mem_interior_convex_shrink ‹y ∈ S› assms by blast
qed

lemma in_interior_closure_convex_segment:
  fixes S :: 'a::euclidean_space set
  assumes convex S and a: a ∈ interior S and b: b ∈ closure S
  shows open_segment a b ⊆ interior S
proof −
  { fix u::real
    assume u: 0 < u u < 1
    have (1 - u) *R a + u *R b = b - (1 - u) *R (b - a)
      by (simp add: algebra_simps)
     also have ... ∈ interior S using mem_interior_closure_convex_shrink [OF
assms] u
      by simp
    finally have (1 - u) *R a + u *R b ∈ interior S .
  }
  then show ?thesis
    by (clarsimp simp: in_segment)
qed
```

```
lemma convex_closure_interior:
  fixes S :: 'a::euclidean_space set
  assumes convex S and int: interior S ≠ {}
  shows closure(interior S) = closure S
proof −
  obtain a where a: a ∈ interior S
    using int by auto
  have closure S ⊆ closure(interior S)
  proof
    fix x
    assume x: x ∈ closure S
    show x ∈ closure (interior S)
    proof (cases x=a)
      case True
      then show ?thesis
        using ‹a ∈ interior S› closure_subset by blast
    next
      case False
      { fix e::real
        assume xnotS: x ∉ interior S and 0 < e
        have ∃x'∈interior S. x' ≠ x ∧ dist x' x < e
        proof (intro bexI conjI)
          show x − min (e/2 / norm (x − a)) 1 *R (x − a) ≠ x
            using False ‹0 < e› by (auto simp: algebra_simps min_def)
          show dist (x − min (e/2 / norm (x − a)) 1 *R (x − a)) x < e
            using ‹0 < e› by (auto simp: dist_norm min_def)
          show x − min (e/2 / norm (x − a)) 1 *R (x − a) ∈ interior S
            using ‹0 < e› False
         by (auto simp add: min_def a intro: mem_interior_closure_convex_shrink
[OF ‹convex S› a x])
        qed
      }
      then show ?thesis
        by (auto simp add: closure_def islimpt_approachable)
    qed
  qed
  then show ?thesis
    by (simp add: closure_mono interior_subset subset_antisym)
qed

lemma openin_subset_relative_interior:
  fixes S :: 'a::euclidean_space set
  shows openin (top_of_set (affine hull T)) S ⟹ (S ⊆ rel_interior T) = (S ⊆
T)
  by (meson order.trans rel_interior_maximal rel_interior_subset)

lemma conic_hull_eq_span_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes 0 ∈ rel_interior S
```

```
  shows conic hull S = span S ∧ conic hull S = affine hull S
proof −
  obtain ε where ε>0 and ε: cball 0 ε ∩ affine hull S ⊆ S
    using assms mem_rel_interior_cball by blast
  have ∗: affine hull S = span S
    by (meson affine_hull_span_0 assms hull_inc mem_rel_interior_cball)
  moreover
  have conic hull S ⊆ span S
    by (simp add: hull_minimal span_superset)
  moreover
  { fix x
    assume x ∈ affine hull S
    have x ∈ conic hull S
    proof (cases x=0)
      case True
      then show ?thesis
        using ‹x ∈ affine hull S› by auto
    next
      case False
      then have (ε / norm x) ∗R x ∈ cball 0 ε ∩ affine hull S
        using ‹0 < ε› ‹x ∈ affine hull S› ∗ span_mul by fastforce
      then have (ε / norm x) ∗R x ∈ S
        by (meson ε subsetD)
      then have ∃ c xa. x = c ∗R xa ∧ 0 ≤ c ∧ xa ∈ S
      by (smt (verit, del_insts) ‹0 < ε› divide_nonneg_nonneg eq_vector_fraction_iff
norm_eq_zero norm_ge_zero)
      then show ?thesis
        by (simp add: conic_hull_explicit)
    qed
  }
  then have affine hull S ⊆ conic hull S
    by auto
  ultimately show ?thesis
    by blast
qed

lemma conic_hull_eq_span:
  fixes S :: 'a::euclidean_space set
  assumes 0 ∈ rel_interior S
  shows conic hull S = span S
  by (simp add: assms conic_hull_eq_span_affine_hull)

lemma conic_hull_eq_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes 0 ∈ rel_interior S
  shows conic hull S = affine hull S
  using assms conic_hull_eq_span_affine_hull by blast

lemma conic_hull_eq_span_eq:
```

fixes S :: $'a$::*euclidean_space set*
shows $0 \in$ *rel_interior*(*conic hull* S) $\longleftrightarrow$ *conic hull* S = *span* S (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*metis conic_hull_eq_span conic_span hull_hull hull_minimal hull_subset span_eq*)
show *?rhs* $\Longrightarrow$ *?lhs*
by (*metis rel_interior_affine subspace_affine subspace_span*)
qed

lemma *aff_dim_psubset*:
(*affine hull* S) $\subset$ (*affine hull* T) $\Longrightarrow$ *aff_dim* S < *aff_dim* T
by (*metis aff_dim_affine_hull aff_dim_empty aff_dim_subset affine_affine_hull affine_dim_equal order_less_le*)

lemma *aff_dim_eq_full_gen*:
$S \subseteq T \Longrightarrow$ (*aff_dim* S = *aff_dim* $T \longleftrightarrow$ *affine hull* S = *affine hull* T)
by (*smt* (*verit*, *del_insts*) *aff_dim_affine_hull2 aff_dim_psubset hull_mono psubsetI*)

lemma *aff_dim_eq_full*:
fixes S :: $'n$::*euclidean_space set*
shows *aff_dim* S = (*DIM*($'n$)) $\longleftrightarrow$ *affine hull* S = *UNIV*
by (*metis aff_dim_UNIV aff_dim_affine_hull affine_hull_UNIV*)

lemma *closure_convex_Int_superset*:
fixes S :: $'a$::*euclidean_space set*
assumes *convex* S *interior* $S \neq \{\}$ *interior* $S \subseteq$ *closure* T
shows *closure*($S \cap T$) = *closure* S
proof −
have *closure* $S \subseteq$ *closure*(*interior* S)
by (*simp add*: *convex_closure_interior assms*)
also have ... $\subseteq$ *closure* ($S \cap T$)
using *interior_subset* [*of* S] *assms*
by (*metis* (*no_types*, *lifting*) *Int_assoc Int_lower2 closure_mono closure_open_Int_superset inf.orderE open_interior*)
finally show *?thesis*
by (*simp add*: *closure_mono dual_order.antisym*)
qed

6.0.2 Some obvious but surprisingly hard simplex lemmas

lemma *simplex*:
assumes *finite* S
and $0 \notin S$
shows *convex hull* (*insert* 0 S) = $\{y.\ \exists u.\ (\forall x{\in}S.\ 0 \leq u\ x) \wedge$ *sum* $u\ S \leq 1 \wedge$ *sum* $(\lambda x.\ u\ x *_R x)\ S = y\}$
proof −
{ fix x **and** u :: $'a \Rightarrow$ *real*

```
    assume ∀x∈S. 0 ≤ u x sum u S ≤ 1
    then have ∃v. 0 ≤ v 0 ∧ (∀x∈S. 0 ≤ v x) ∧ v 0 + sum v S = 1 ∧ (∑x∈S.
v x *R x) = (∑x∈S. u x *R x)
      by (rule_tac x=λx. if x = 0 then 1 − sum u S else u x in exI) (auto simp:
sum_delta_notmem assms if_smult)
  }
  then show ?thesis by (auto simp: convex_hull_finite set_eq_iff assms)
qed

lemma substd_simplex:
  assumes d: d ⊆ Basis
  shows convex hull (insert 0 d) =
    {x. (∀i∈Basis. 0 ≤ x•i) ∧ (∑i∈d. x•i) ≤ 1 ∧ (∀i∈Basis. i ∉ d ⟶ x•i = 0)}
  (is convex hull (insert 0 ?p) = ?s)
proof −
  let ?D = d
  have 0 ∉ ?p
    using assms by (auto simp: image_def)
  from d have finite d
    by (blast intro: finite_subset finite_Basis)
  show ?thesis
    unfolding simplex[OF ‹finite d› ‹0 ∉ ?p›]
  proof (intro set_eqI; safe)
    fix u :: 'a ⇒ real
    assume as: ∀x∈?D. 0 ≤ u x sum u ?D ≤ 1
    let ?x = (∑x∈?D. u x *R x)
    have ind: ∀i∈Basis. i ∈ d ⟶ u i = ?x • i
      and notind: (∀i∈Basis. i ∉ d ⟶ ?x • i = 0)
      using substdbasis_expansion_unique[OF assms] by blast+
    then have **: sum u ?D = sum ((•) ?x) ?D
      using assms by (auto intro!: sum.cong)
    show 0 ≤ ?x • i if i ∈ Basis for i
      using as(1) ind notind that by fastforce
    show sum ((•) ?x) ?D ≤ 1
      using ** as(2) by linarith
    show ?x • i = 0 if i ∈ Basis i ∉ d for i
      using notind that by blast
  next
    fix x
    assume ∀i∈Basis. 0 ≤ x • i sum ((•) x) ?D ≤ 1 (∀i∈Basis. i ∉ d ⟶ x • i
= 0)
    with d show ∃u. (∀x∈?D. 0 ≤ u x) ∧ sum u ?D ≤ 1 ∧ (∑x∈?D. u x *R x)
= x
      unfolding substdbasis_expansion_unique[OF assms]
      by (rule_tac x=inner x in exI) auto
  qed
qed

lemma std_simplex:
```

```
  convex hull (insert 0 Basis) =
    {x::'a::euclidean_space. (∀ i∈Basis. 0 ≤ x•i) ∧ sum (λi. x•i) Basis ≤ 1}
  using substd_simplex[of Basis] by auto

lemma interior_std_simplex:
  interior (convex hull (insert 0 Basis)) =
    {x::'a::euclidean_space. (∀ i∈Basis. 0 < x•i) ∧ sum (λi. x•i) Basis < 1}
  unfolding set_eq_iff mem_interior std_simplex
proof (intro allI iffI CollectI; clarify)
  fix x :: 'a
  fix e
  assume e > 0 and as: ball x e ⊆ {x. (∀ i∈Basis. 0 ≤ x • i) ∧ sum ((•) x) Basis
≤ 1}
  show (∀ i∈Basis. 0 < x • i) ∧ sum ((•) x) Basis < 1
  proof safe
    fix i :: 'a
    assume i: i ∈ Basis
    then show 0 < x • i
      using as[THEN subsetD[where c=x − (e/2) *R i]] and ‹e > 0›
      by (force simp add: inner_simps)
  next
    have **: dist x (x + (e/2) *R (SOME i. i∈Basis)) < e using ‹e > 0›
      unfolding dist_norm
      by (auto intro!: mult_strict_left_mono simp: SOME_Basis)
    have ⋀i. i ∈ Basis ⟹ (x + (e/2) *R (SOME i. i∈Basis)) • i =
      x•i + (if i = (SOME i. i∈Basis) then e/2 else 0)
      by (auto simp: SOME_Basis inner_Basis inner_simps)
    then have *: sum ((•) (x + (e/2) *R (SOME i. i∈Basis))) Basis =
      sum (λi. x•i + (if (SOME i. i∈Basis) = i then e/2 else 0)) Basis
      by (auto simp: intro!: sum.cong)
    have sum ((•) x) Basis < sum ((•) (x + (e/2) *R (SOME i. i∈Basis))) Basis
      using ‹e > 0› DIM_positive by (auto simp: SOME_Basis sum.distrib *)
    also have ... ≤ 1
      using ** as by force
    finally show sum ((•) x) Basis < 1 by auto
  qed
next
  fix x :: 'a
  assume as: ∀ i∈Basis. 0 < x • i sum ((•) x) Basis < 1
  obtain a :: 'b where a ∈ UNIV using UNIV_witness ..
  let ?d = (1 − sum ((•) x) Basis) / real (DIM('a))
  show ∃ e>0. ball x e ⊆ {x. (∀ i∈Basis. 0 ≤ x • i) ∧ sum ((•) x) Basis ≤ 1}
   proof (rule_tac x=min (Min (((•) x) ' Basis)) D for D in exI, intro conjI
subsetI CollectI)
    fix y
    assume y: y ∈ ball x (min (Min ((•) x ' Basis)) ?d)
    have sum ((•) y) Basis ≤ sum (λi. x•i + ?d) Basis
    proof (rule sum_mono)
      fix i :: 'a
```

```
    assume i: i ∈ Basis
    have |y·i − x·i| ≤ norm (y − x)
      by (metis Basis_le_norm i inner_commute inner_diff_right)
    also have ... < ?d
      using y by (simp add: dist_norm norm_minus_commute)
    finally have |y·i − x·i| < ?d .
    then show y · i ≤ x · i + ?d by auto
  qed
  also have ... ≤ 1
    unfolding sum.distrib sum_constant
    by (auto simp add: Suc_le_eq)
  finally show sum ((·) y) Basis ≤ 1 .
  show (∀ i∈Basis. 0 ≤ y · i)
  proof safe
    fix i :: 'a
    assume i: i ∈ Basis
    have norm (x − y) < Min (((·) x) ' Basis)
      using y by (auto simp: dist_norm less_eq_real_def)
    also have ... ≤ x·i
      using i by auto
    finally have norm (x − y) < x·i .
    then show 0 ≤ y·i
      using Basis_le_norm[OF i, of x − y] and as(1)[rule_format, OF i]
      by (auto simp: inner_simps)
  qed
 next
  have Min (((·) x) ' Basis) > 0
    using as by simp
  moreover have ?d > 0
    using as by (auto simp: Suc_le_eq)
  ultimately show 0 < min (Min ((·) x ' Basis)) ((1 − sum ((·) x) Basis) /
real DIM('a))
    by linarith
 qed
qed

lemma interior_std_simplex_nonempty:
 obtains a :: 'a::euclidean_space where
  a ∈ interior(convex hull (insert 0 Basis))
proof −
 let ?D = Basis :: 'a set
 let ?a = sum (λb::'a. inverse (2 * real DIM('a)) *R b) Basis
 {
  fix i :: 'a
  assume i: i ∈ Basis
  have ?a · i = inverse (2 * real DIM('a))
    by (rule trans[of _ sum (λj. if i = j then inverse (2 * real DIM('a)) else 0)
?D])
       (simp_all add: sum.If_cases i) }
```

```
  note ** = this
  show ?thesis
  proof
    show ?a ∈ interior(convex hull (insert 0 Basis))
      unfolding interior_std_simplex mem_Collect_eq
    proof safe
      fix i :: 'a
      assume i: i ∈ Basis
      show 0 < ?a • i
        unfolding **[OF i] by (auto simp add: Suc_le_eq)
    next
      have sum ((•) ?a) ?D = sum (λi. inverse (2 * real DIM('a))) ?D
        by (auto intro: sum.cong)
      also have ... < 1
        unfolding sum_constant divide_inverse[symmetric]
        by (auto simp add: field_simps)
      finally show sum ((•) ?a) ?D < 1 by auto
    qed
  qed
qed

lemma rel_interior_substd_simplex:
  assumes D: D ⊆ Basis
  shows rel_interior (convex hull (insert 0 D)) =
        {x::'a::euclidean_space. (∀ i∈D. 0 < x•i) ∧ (∑ i∈D. x•i) < 1 ∧ (∀ i∈Basis.
i ∉ D ⟶ x•i = 0)}
     (is _ = ?s)
proof -
  have finite D
    using D finite_Basis finite_subset by blast
  show ?thesis
  proof (cases D = {})
    case True
    then show ?thesis
      using rel_interior_sing using euclidean_eq_iff[of _ 0] by auto
  next
    case False
    have h0: affine hull (convex hull (insert 0 D)) =
             {x::'a::euclidean_space. (∀ i∈Basis. i ∉ D ⟶ x•i = 0)}
      using affine_hull_convex_hull affine_hull_substd_basis assms by auto
    have aux: ⋀x::'a. ∀ i∈Basis. (∀ i∈D. 0 ≤ x•i) ∧ (∀ i∈Basis. i ∉ D ⟶ x•i =
0) ⟶ 0 ≤ x•i
      by auto
    {
      fix x :: 'a::euclidean_space
      assume x: x ∈ rel_interior (convex hull (insert 0 D))
      then obtain e where e > 0 and
        ball x e ∩ {xa. (∀ i∈Basis. i ∉ D ⟶ xa•i = 0)} ⊆ convex hull (insert 0 D)
        using mem_rel_interior_ball[of x convex hull (insert 0 D)] h0 by auto
```

```
    then have as: ⋀y. ⟦dist x y < e ∧ (∀ i∈Basis. i ∉ D ⟶ y•i = 0)⟧ ⟹
                        (∀ i∈D. 0 ≤ y • i) ∧ sum ((•) y) D ≤ 1
      using assms by (force simp: substd_simplex)
    have x0: (∀ i∈Basis. i ∉ D ⟶ x•i = 0)
      using x rel_interior_subset  substd_simplex[OF assms] by auto
    have (∀ i∈D. 0 < x • i) ∧ sum ((•) x) D < 1 ∧ (∀ i∈Basis. i ∉ D ⟶ x•i =
0)
    proof (intro conjI ballI)
      fix i :: 'a
      assume i ∈ D
      then have ∀ j∈D. 0 ≤ (x − (e/2) *R i) • j
        using D ‹e > 0› x0
          by (intro as[THEN conjunct1]) (force simp: dist_norm inner_simps
inner_Basis)
      then show 0 < x • i
        using ‹e > 0› ‹i ∈ D› D  by (force simp: inner_simps inner_Basis)
    next
      obtain a where a: a ∈ D
        using ‹D ≠ {}› by auto
      then have **: dist x (x + (e/2) *R a) < e
        using ‹e > 0› norm_Basis[of a] D by (auto simp: dist_norm)
       have ⋀i. i ∈ Basis ⟹ (x + (e/2) *R a) • i = x•i + (if i = a then e/2
else 0)
        using a D by (auto simp: inner_simps inner_Basis)
      then have *: sum ((•) (x + (e/2) *R a)) D = sum (λi. x•i + (if a = i then
e/2 else 0)) D
        using D by (intro sum.cong) auto
      have a ∈ Basis
        using ‹a ∈ D› D by auto
      then have h1: (∀ i∈Basis. i ∉ D ⟶ (x + (e/2) *R a) • i = 0)
        using x0 D ‹a∈D› by (auto simp add: inner_add_left inner_Basis)
      have sum ((•) x) D < sum ((•) (x + (e/2) *R a)) D
        using ‹e > 0› ‹a ∈ D› ‹finite D› by (auto simp add: * sum.distrib)
      also have ... ≤ 1
        using ** h1 as[rule_format, of x + (e/2) *R a]
        by auto
      finally show sum ((•) x) D < 1 ⋀i. i∈Basis ⟹ i ∉ D ⟶ x•i = 0
        using x0 by auto
    qed
  }
  moreover
  {
    fix x :: 'a::euclidean_space
    assume as: x ∈ ?s
    have ∀ i. 0 < x•i ∨ 0 = x•i ⟶ 0 ≤ x•i
      by auto
    moreover have ∀ i. i ∈ D ∨ i ∉ D by auto
    ultimately
    have ∀ i. (∀ i∈D. 0 < x•i) ∧ (∀ i. i ∉ D ⟶ x•i = 0) ⟶ 0 ≤ x•i
```

```
      by metis
    then have h2: x ∈ convex hull (insert 0 D)
      using as assms by (force simp add: substd_simplex)
    obtain a where a: a ∈ D
      using ‹D ≠ {}› by auto
    define d where d ≡ (1 - sum ((•) x) D) / real (card D)
     have ∃e>0. ball x e ∩ {x. ∀i∈Basis. i ∉ D ⟶ x • i = 0} ⊆ convex hull
insert 0 D
      unfolding substd_simplex[OF assms]
    proof (intro exI; safe)
      have 0 < card D using ‹D ≠ {}› ‹finite D›
        by (simp add: card_gt_0_iff)
      have Min (((•) x) ‘ D) > 0
        using as ‹D ≠ {}› ‹finite D› by (simp)
      moreover have d > 0
        using as ‹0 < card D› by (auto simp: d_def)
      ultimately show min (Min (((•) x) ‘ D)) d > 0
        by auto
      fix y :: 'a
      assume y2: ∀i∈Basis. i ∉ D ⟶ y•i = 0
      assume y ∈ ball x (min (Min ((•) x ‘ D)) d)
      then have y: dist x y < min (Min ((•) x ‘ D)) d
        by auto
      have sum ((•) y) D ≤ sum (λi. x•i + d) D
      proof (rule sum_mono)
        fix i
        assume i ∈ D
        with D have i: i ∈ Basis
          by auto
        have |y•i - x•i| ≤ norm (y - x)
            by (metis i inner_commute inner_diff_right norm_bound_Basis_le
order_refl)
        also have ... < d
          by (metis dist_norm min_less_iff_conj norm_minus_commute y)
        finally have |y•i - x•i| < d .
        then show y • i ≤ x • i + d by auto
      qed
      also have ... ≤ 1
        unfolding sum.distrib sum_constant d_def using ‹0 < card D›
        by auto
      finally show sum ((•) y) D ≤ 1 .

      fix i :: 'a
      assume i: i ∈ Basis
      then show 0 ≤ y•i
      proof (cases i∈D)
        case True
        have norm (x - y) < x•i
          using y Min_gr_iff[of (•) x ‘ D norm (x - y)] ‹0 < card D› ‹i ∈ D›
```

```
        by (simp add: dist_norm card_gt_0_iff)
      then show 0 ≤ y·i
        using Basis_le_norm[OF i, of x − y] and as(1)[rule_format]
        by (auto simp: inner_simps)
    qed (use y2 in auto)
  qed
  then have x ∈ rel_interior (convex hull (insert 0 D))
    using h0 h2 rel_interior_ball by force
 }
 ultimately have
  ⋀x. x ∈ rel_interior (convex hull insert 0 D) ⟷
   x ∈ {x. (∀ i∈D. 0 < x · i) ∧ sum ((·) x) D < 1 ∧ (∀ i∈Basis. i ∉ D ⟶ x
· i = 0)}
  by blast
 then show ?thesis by (rule set_eqI)
qed
qed

lemma rel_interior_substd_simplex_nonempty:
 assumes D ≠ {}
  and D ⊆ Basis
 obtains a :: 'a::euclidean_space
  where a ∈ rel_interior (convex hull (insert 0 D))
proof −
 let ?a = sum (λb::'a::euclidean_space. inverse (2 * real (card D)) *R b) D
 have finite D
  using assms finite_Basis infinite_super by blast
 then have d1: 0 < real (card D)
  using ‹D ≠ {}› by auto
 {
  fix i
  assume i ∈ D
  have ?a · i = sum (λj. if i = j then inverse (2 * real (card D)) else 0) D
   unfolding inner_sum_left
    using ‹i ∈ D› by (auto simp: inner_Basis subsetD[OF assms(2)] intro:
sum.cong)
  also have ... = inverse (2 * real (card D))
   using ‹i ∈ D› ‹finite D› by auto
  finally have ?a · i = inverse (2 * real (card D)) .
 }
 note ** = this
 show ?thesis
 proof
  show ?a ∈ rel_interior (convex hull (insert 0 D))
   unfolding rel_interior_substd_simplex[OF assms(2)]
  proof safe
   fix i
   assume i ∈ D
   have 0 < inverse (2 * real (card D))
```

```
      using d1 by auto
    also have ... = ?a · i using **[of i] ‹i ∈ D›
      by auto
    finally show 0 < ?a · i by auto
  next
    have sum ((·) ?a) D = sum (λi. inverse (2 * real (card D))) D
      by (rule sum.cong) (rule refl, rule **)
    also have ... < 1
      unfolding sum_constant divide_real_def[symmetric]
      by (auto simp add: field_simps)
    finally show sum ((·) ?a) D < 1 by auto
  next
    fix i
    assume i ∈ Basis and i ∉ D
    have ?a ∈ span D
    proof (rule span_sum[of D (λb. b /R (2 * real (card D))) D])
      {
        fix x :: 'a::euclidean_space
        assume x ∈ D
        then have x ∈ span D
          using span_base[of _ D] by auto
        then have x /R (2 * real (card D)) ∈ span D
          using span_mul[of x D (inverse (real (card D)) / 2)] by auto
      }
      then show ⋀x. x∈D ⟹ x /R (2 * real (card D)) ∈ span D
        by auto
    qed
    then show ?a · i = 0
      using ‹i ∉ D› unfolding span_substd_basis[OF assms(2)] using ‹i ∈
Basis› by auto
  qed
 qed
qed
```

6.0.3 Relative interior of convex set

```
lemma rel_interior_convex_nonempty_aux:
 fixes S :: 'n::euclidean_space set
 assumes convex S
   and 0 ∈ S
 shows rel_interior S ≠ {}
proof (cases S = {0})
 case True
 then show ?thesis using rel_interior_sing by auto
next
 case False
 obtain B where B: independent B ∧ B ≤ S ∧ S ≤ span B ∧ card B = dim S
   using basis_exists[of S] by metis
 then have B ≠ {}
```

```
    using B assms ‹S ≠ {0}› span_empty by auto
  have insert 0 B ≤ span B
    using subspace_span[of B] subspace_0[of span B]
      span_superset by auto
  then have span (insert 0 B) ≤ span B
    using span_span[of B] span_mono[of insert 0 B span B] by blast
  then have convex hull insert 0 B ≤ span B
    using convex_hull_subset_span[of insert 0 B] by auto
  then have span (convex hull insert 0 B) ≤ span B
    using span_span[of B]
      span_mono[of convex hull insert 0 B span B] by blast
  then have ∗: span (convex hull insert 0 B) = span B
    using span_mono[of B convex hull insert 0 B] hull_subset[of insert 0 B] by
auto
  then have span (convex hull insert 0 B) = span S
    using B span_mono[of B S] span_mono[of S span B]
      span_span[of B] by auto
  moreover have 0 ∈ affine hull (convex hull insert 0 B)
    using hull_subset[of convex hull insert 0 B] hull_subset[of insert 0 B] by auto
  ultimately have ∗∗: affine hull (convex hull insert 0 B) = affine hull S
    using affine_hull_span_0[of convex hull insert 0 B] affine_hull_span_0[of S]
      assms hull_subset[of S]
    by auto
  obtain d and f :: 'n ⇒ 'n where
    fd: card d = card B linear f f ‘ B = d
      f ‘ span B = {x. ∀ i∈Basis. i ∉ d ⟶ x · i = (0::real)} ∧ inj_on f (span B)
    and d: d ⊆ Basis
    using basis_to_substdbasis_subspace_isomorphism[of B,OF _ ] B by auto
  then have bounded_linear f
    using linear_conv_bounded_linear by auto
  have d ≠ {}
    using fd B ‹B ≠ {}› by auto
  have insert 0 d = f ‘ (insert 0 B)
    using fd linear_0 by auto
  then have (convex hull (insert 0 d)) = f ‘ (convex hull (insert 0 B))
    using convex_hull_linear_image[of f (insert 0 d)]
      convex_hull_linear_image[of f (insert 0 B)] ‹linear f›
    by auto
  moreover have rel_interior (f ‘ (convex hull insert 0 B)) = f ‘ rel_interior
(convex hull insert 0 B)
  proof (rule rel_interior_injective_on_span_linear_image[OF ‹bounded_linear
f›])
    show inj_on f (span (convex hull insert 0 B))
      using fd ∗ by auto
  qed
  ultimately have rel_interior (convex hull insert 0 B) ≠ {}
    using rel_interior_substd_simplex_nonempty[OF ‹d ≠ {}› d] by fastforce
  moreover have convex hull (insert 0 B) ⊆ S
    using B assms hull_mono[of insert 0 B S convex] convex_hull_eq by auto
```

```
  ultimately show ?thesis
    using subset_rel_interior[of convex hull insert 0 B S] ** by auto
qed

lemma rel_interior_eq_empty:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows rel_interior S = {} ⟷ S = {}
proof −
  {
    assume S ≠ {}
    then obtain a where a ∈ S by auto
    then have 0 ∈ (+) (−a) ` S
      using assms exI[of (λx. x ∈ S ∧ − a + x = 0) a] by auto
    then have rel_interior ((+) (−a) ` S) ≠ {}
      using rel_interior_convex_nonempty_aux[of (+) (−a) ` S]
        convex_translation[of S −a] assms
      by auto
    then have rel_interior S ≠ {}
      using rel_interior_translation [of − a] by simp
  }
  then show ?thesis by auto
qed

lemma interior_simplex_nonempty:
  fixes S :: 'N :: euclidean_space set
  assumes independent S finite S card S = DIM('N)
  obtains a where a ∈ interior (convex hull (insert 0 S))
proof −
  have affine hull (insert 0 S) = UNIV
    by (simp add: hull_inc affine_hull_span_0 dim_eq_full[symmetric]
        assms(1) assms(3) dim_eq_card_independent)
  moreover have rel_interior (convex hull insert 0 S) ≠ {}
    using rel_interior_eq_empty [of convex hull (insert 0 S)] by auto
  ultimately have interior (convex hull insert 0 S) ≠ {}
    by (simp add: rel_interior_interior)
  with that show ?thesis
    by auto
qed

lemma convex_rel_interior:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows convex (rel_interior S)
proof −
  {
    fix x y and u :: real
    assume assm: x ∈ rel_interior S y ∈ rel_interior S 0 ≤ u u ≤ 1
    then have x ∈ S
```

```
    using rel_interior_subset by auto
  have x - u *R (x-y) ∈ rel_interior S
  proof (cases 0 = u)
    case False
    then have 0 < u using assm by auto
    then show ?thesis
      using assm rel_interior_convex_shrink[of S y x u] assms ‹x ∈ S› by auto
  next
    case True
    then show ?thesis using assm by auto
  qed
  then have (1 - u) *R x + u *R y ∈ rel_interior S
    by (simp add: algebra_simps)
 }
 then show ?thesis
  unfolding convex_alt by auto
qed

lemma convex_closure_rel_interior:
 fixes S :: 'n::euclidean_space set
 assumes convex S
 shows closure (rel_interior S) = closure S
proof -
 have h1: closure (rel_interior S) ≤ closure S
  using closure_mono[of rel_interior S S] rel_interior_subset[of S] by auto
 show ?thesis
 proof (cases S = {})
  case False
  then obtain a where a: a ∈ rel_interior S
   using rel_interior_eq_empty assms by auto
  { fix x
   assume x: x ∈ closure S
   {
    assume x = a
    then have x ∈ closure (rel_interior S)
     using a unfolding closure_def by auto
   }
   moreover
   {
    assume x ≠ a
    {
     fix e :: real
     assume e > 0
     define e1 where e1 = min 1 (e/norm (x - a))
     then have e1: e1 > 0 e1 ≤ 1 e1 * norm (x - a) ≤ e
      using ‹x ≠ a› ‹e > 0› le_divide_eq[of e1 e norm (x - a)]
      by simp_all
     then have *: x - e1 *R (x - a) ∈ rel_interior S
     using rel_interior_closure_convex_shrink[of S a x e1] assms x a e1_def
```

```
          by auto
        have ∃ y. y ∈ rel_interior S ∧ y ≠ x ∧ dist y x ≤ e
          using ∗ ‹x ≠ a› e1 by force
      }
      then have x islimpt rel_interior S
        unfolding islimpt_approachable_le by auto
      then have x ∈ closure(rel_interior S)
        unfolding closure_def by auto
    }
    ultimately have x ∈ closure(rel_interior S) by auto
  }
  then show ?thesis using h1 by auto
 qed auto
qed

lemma empty_interior_subset_hyperplane_aux:
 fixes S :: 'a::euclidean_space set
 assumes convex S 0 ∈ S and empty_int: interior S = {}
 shows ∃ a b. a≠0 ∧ S ⊆ {x. a · x = b}
proof −
 have False if ⋀a. a = 0 ∨ (∀ b. ∃ T ∈ S. a · T ≠ b)
 proof −
  have rel_int: rel_interior S ≠ {}
   using assms rel_interior_eq_empty by auto
  moreover
  have dim S ≠ dim (UNIV::'a set)
   by (metis aff_dim_zero affine_hull_UNIV ‹0 ∈ S› dim_UNIV empty_int
hull_inc rel_int rel_interior_interior)
  then obtain a where a ≠ 0 and a: span S ⊆ {x. a · x = 0}
   using lowdim_subset_hyperplane
   by (metis dim_UNIV dim_subset_UNIV order_less_le)
  have span UNIV = span S
   by (metis span_base span_not_UNIV_orthogonal that)
  then have UNIV ⊆ affine hull S
   by (simp add: ‹0 ∈ S› hull_inc affine_hull_span_0)
  ultimately show False
   using ‹rel_interior S ≠ {}› empty_int rel_interior_interior by blast
 qed
 then show ?thesis
  by blast
qed

lemma empty_interior_subset_hyperplane:
 fixes S :: 'a::euclidean_space set
 assumes convex S and int: interior S = {}
 obtains a b where a ≠ 0 S ⊆ {x. a · x = b}
proof (cases S = {})
 case True
 then show ?thesis
```

```
    using that by blast
next
  case False
  then obtain u where u ∈ S
    by blast
  have ∃ a b. a ≠ 0 ∧ (λx. x − u) ‘ S ⊆ {x. a • x = b}
  proof (rule empty_interior_subset_hyperplane_aux)
    show convex ((λx. x − u) ‘ S)
      using ‹convex S› by force
    show 0 ∈ (λx. x − u) ‘ S
      by (simp add: ‹u ∈ S›)
    show interior ((λx. x − u) ‘ S) = {}
      by (simp add: int interior_translation_subtract)
  qed
  then obtain a b where a ≠ 0 and ab: (λx. x − u) ‘ S ⊆ {x. a • x = b}
    by metis
  then have S ⊆ {x. a • x = b + (a • u)}
    using ab by (auto simp: algebra_simps)
  then show ?thesis
    using ‹a ≠ 0› that by auto
qed

lemma rel_interior_same_affine_hull:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows affine hull (rel_interior S) = affine hull S
  by (metis assms closure_same_affine_hull convex_closure_rel_interior)

lemma rel_interior_aff_dim:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows aff_dim (rel_interior S) = aff_dim S
  by (metis aff_dim_affine_hull2 assms rel_interior_same_affine_hull)

lemma rel_interior_rel_interior:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows rel_interior (rel_interior S) = rel_interior S
proof −
  have openin (top_of_set (affine hull (rel_interior S))) (rel_interior S)
    using openin_rel_interior[of S] rel_interior_same_affine_hull[of S] assms by
auto
  then show ?thesis
    using rel_interior_def by auto
qed

lemma rel_interior_rel_open:
  fixes S :: 'n::euclidean_space set
  assumes convex S
```

```
  shows rel_open (rel_interior S)
  unfolding rel_open_def using rel_interior_rel_interior assms by auto

lemma convex_rel_interior_closure_aux:
  fixes x y z :: 'n::euclidean_space
  assumes 0 < a 0 < b (a + b) *R z = a *R x + b *R y
  obtains e where 0 < e e < 1 z = y - e *R (y - x)
proof -
  define e where e = a / (a + b)
  have z = (1 / (a + b)) *R ((a + b) *R z)
    using assms  by (simp add: eq_vector_fraction_iff)
  also have ... = (1 / (a + b)) *R (a *R x + b *R y)
    using assms scaleR_cancel_left[of 1/(a+b) (a + b) *R z a *R x + b *R y]
    by auto
  also have ... = y - e *R (y-x)
    using e_def assms
   by (simp add: divide_simps vector_fraction_eq_iff) (simp add: algebra_simps)
  finally have z = y - e *R (y-x)
    by auto
  moreover have e > 0 e < 1 using e_def assms by auto
  ultimately show ?thesis using that[of e] by auto
qed

lemma convex_rel_interior_closure:
  fixes S :: 'n::euclidean_space set
  assumes convex S
  shows rel_interior (closure S) = rel_interior S
proof (cases S = {})
  case True
  then show ?thesis
    using assms rel_interior_eq_empty by auto
next
  case False
  have rel_interior (closure S) ⊇ rel_interior S
   using subset_rel_interior[of S closure S] closure_same_affine_hull closure_subset
    by auto
  moreover
  {
    fix z
    assume z: z ∈ rel_interior (closure S)
    obtain x where x: x ∈ rel_interior S
      using ‹S ≠ {}› assms rel_interior_eq_empty by auto
    have z ∈ rel_interior S
    proof (cases x = z)
      case True
      then show ?thesis using x by auto
    next
      case False
      obtain e where e: e > 0 cball z e ∩ affine hull closure S ≤ closure S
```

```
      using z rel_interior_cball[of closure S] by auto
    hence ∗: 0 < e/norm(z−x) using e False by auto
    define y where y = z + (e/norm(z−x)) ∗R (z−x)
    have yball: y ∈ cball z e
      using y_def dist_norm[of z y] e by auto
    have x ∈ affine hull closure S
      using x rel_interior_subset_closure hull_inc[of x closure S] by blast
    moreover have z ∈ affine hull closure S
      using z rel_interior_subset hull_subset[of closure S] by blast
    ultimately have y ∈ affine hull closure S
      using y_def affine_affine_hull[of closure S]
       mem_affine_3_minus [of affine hull closure S z z x e/norm(z−x)] by auto
    then have y ∈ closure S using e yball by auto
    have (1 + (e/norm(z−x))) ∗R z = (e/norm(z−x)) ∗R x + y
      using y_def by (simp add: algebra_simps)
    then obtain e1 where 0 < e1 e1 < 1 z = y − e1 ∗R (y − x)
      using ∗ convex_rel_interior_closure_aux[of e / norm (z − x) 1 z x y]
      by (auto simp add: algebra_simps)
    then show ?thesis
      using rel_interior_closure_convex_shrink assms x ‹y ∈ closure S›
      by fastforce
  qed
 }
 ultimately show ?thesis by auto
qed

lemma convex_interior_closure:
 fixes S :: 'n::euclidean_space set
 assumes convex S
 shows interior (closure S) = interior S
 using closure_aff_dim[of S] interior_rel_interior_gen[of S]
   interior_rel_interior_gen[of closure S]
   convex_rel_interior_closure[of S] assms
 by auto

lemma open_subset_closure_of_interval:
 assumes open U is_interval S
 shows U ⊆ closure S ⟷ U ⊆ interior S
 by (metis assms convex_interior_closure is_interval_convex open_subset_interior)

lemma closure_eq_rel_interior_eq:
 fixes S1 S2 :: 'n::euclidean_space set
 assumes convex S1
   and convex S2
 shows closure S1 = closure S2 ⟷ rel_interior S1 = rel_interior S2
 by (metis convex_rel_interior_closure convex_closure_rel_interior assms)

lemma closure_eq_between:
 fixes S1 S2 :: 'n::euclidean_space set
```

```
  assumes convex S1
    and convex S2
  shows closure S1 = closure S2 ⟷ rel_interior S1 ≤ S2 ∧ S2 ⊆ closure S1
  (is ?A ⟷ ?B)
proof
  assume ?A
  then show ?B
   by (metis assms closure_subset convex_rel_interior_closure rel_interior_subset)
next
  assume ?B
  then have closure S1 ⊆ closure S2
    by (metis assms(1) convex_closure_rel_interior closure_mono)
  moreover from ‹?B› have closure S1 ⊇ closure S2
    by (metis closed_closure closure_minimal)
  ultimately show ?A ..
qed

lemma open_inter_closure_rel_interior:
  fixes S A :: 'n::euclidean_space set
  assumes convex S
    and open A
  shows A ∩ closure S = {} ⟷ A ∩ rel_interior S = {}
  by (metis assms convex_closure_rel_interior open_Int_closure_eq_empty)

lemma rel_interior_open_segment:
  fixes a :: 'a :: euclidean_space
  shows rel_interior(open_segment a b) = open_segment a b
proof (cases a = b)
  case True then show ?thesis by auto
next
  case False then
  have open_segment a b = affine hull {a, b} ∩ ball ((a + b) /R 2) (norm (b −
a) / 2)
    by (simp add: open_segment_as_ball)
  then show ?thesis
    unfolding rel_interior_eq openin_open
   by (metis Elementary_Metric_Spaces.open_ball False affine_hull_open_segment)
qed

lemma rel_interior_closed_segment:
  fixes a :: 'a :: euclidean_space
  shows rel_interior(closed_segment a b) =
        (if a = b then {a} else open_segment a b)
proof (cases a = b)
  case True then show ?thesis by auto
next
  case False then show ?thesis
    by simp
     (metis closure_open_segment convex_open_segment convex_rel_interior_closure
```

rel_interior_open_segment)
qed

lemmas *rel_interior_segment* = *rel_interior_closed_segment rel_interior_open_segment*

6.0.4 The relative frontier of a set

definition *rel_frontier S* = *closure S* − *rel_interior S*

lemma *rel_frontier_empty* [*simp*]: *rel_frontier* {} = {}
by (*simp add*: *rel_frontier_def*)

lemma *rel_frontier_eq_empty*:
fixes *S* :: *'n*::*euclidean_space set*
shows *rel_frontier S* = {} ⟷ *affine S*
unfolding *rel_frontier_def*
using *rel_interior_subset_closure* **by** (*auto simp add*: *rel_interior_eq_closure* [*symmetric*])

lemma *rel_frontier_sing* [*simp*]:
fixes *a* :: *'n*::*euclidean_space*
shows *rel_frontier* {*a*} = {}
by (*simp add*: *rel_frontier_def*)

lemma *rel_frontier_affine_hull*:
fixes *S* :: *'a*::*euclidean_space set*
shows *rel_frontier S* ⊆ *affine hull S*
using *closure_affine_hull rel_frontier_def* **by** *fastforce*

lemma *rel_frontier_cball* [*simp*]:
fixes *a* :: *'n*::*euclidean_space*
shows *rel_frontier*(*cball a r*) = (*if r* = *0 then* {} *else sphere a r*)
proof (*cases rule*: *linorder_cases* [*of r 0*])
case *less* **then show** *?thesis*
by (*force simp*: *sphere_def*)
next
case *equal* **then show** *?thesis* **by** *simp*
next
case *greater* **then show** *?thesis*
by *simp* (*metis centre_in_ball empty_iff frontier_cball frontier_def interior_cball interior_rel_interior_gen rel_frontier_def*)
qed

lemma *rel_frontier_translation*:
fixes *a* :: *'a*::*euclidean_space*
shows *rel_frontier*((λ*x*. *a* + *x*) ‘ *S*) = (λ*x*. *a* + *x*) ‘ (*rel_frontier S*)
by (*simp add*: *rel_frontier_def translation_diff rel_interior_translation closure_translation*)

lemma *rel_frontier_nonempty_interior*:
fixes $S :: 'n::euclidean\_space\ set$
shows $interior\ S \neq \{\} \Longrightarrow rel\_frontier\ S = frontier\ S$
by (*metis frontier_def interior_rel_interior_gen rel_frontier_def*)

lemma *rel_frontier_frontier*:
fixes $S :: 'n::euclidean\_space\ set$
shows $affine\ hull\ S = UNIV \Longrightarrow rel\_frontier\ S = frontier\ S$
by (*simp add*: *frontier_def rel_frontier_def rel_interior_interior*)

lemma *closest_point_in_rel_frontier*:
$[\![closed\ S;\ S \neq \{\};\ x \in affine\ hull\ S - rel\_interior\ S]\!]$
$\Longrightarrow closest\_point\ S\ x \in rel\_frontier\ S$
by (*simp add*: *closest_point_in_rel_interior closest_point_in_set rel_frontier_def*)

lemma *closed_rel_frontier* [*iff*]:
fixes $S :: 'n::euclidean\_space\ set$
shows $closed\ (rel\_frontier\ S)$
proof −
have ∗: $closedin\ (top\_of\_set\ (affine\ hull\ S))\ (closure\ S - rel\_interior\ S)$
by (*simp add*: *closed_subset closedin_diff closure_affine_hull openin_rel_interior*)
show *?thesis*
proof (*rule closedin_closed_trans*[*of affine hull S rel_frontier S*])
show $closedin\ (top\_of\_set\ (affine\ hull\ S))\ (rel\_frontier\ S)$
by (*simp add*: ∗ *rel_frontier_def*)
qed *simp*
qed

lemma *closed_rel_boundary*:
fixes $S :: 'n::euclidean\_space\ set$
shows $closed\ S \Longrightarrow closed(S - rel\_interior\ S)$
by (*metis closed_rel_frontier closure_closed rel_frontier_def*)

lemma *compact_rel_boundary*:
fixes $S :: 'n::euclidean\_space\ set$
shows $compact\ S \Longrightarrow compact(S - rel\_interior\ S)$
by (*metis bounded_diff closed_rel_boundary closure_eq compact_closure compact_imp_closed*)

lemma *bounded_rel_frontier*:
fixes $S :: 'n::euclidean\_space\ set$
shows $bounded\ S \Longrightarrow bounded(rel\_frontier\ S)$
by (*simp add*: *bounded_closure bounded_diff rel_frontier_def*)

lemma *compact_rel_frontier_bounded*:
fixes $S :: 'n::euclidean\_space\ set$
shows $bounded\ S \Longrightarrow compact(rel\_frontier\ S)$
using *bounded_rel_frontier closed_rel_frontier compact_eq_bounded_closed* **by** *blast*

lemma *compact_rel_frontier*:
fixes $S :: 'n::euclidean\_space\ set$
shows $compact\ S \Longrightarrow compact(rel\_frontier\ S)$
by (*meson compact_eq_bounded_closed compact_rel_frontier_bounded*)

lemma *convex_same_rel_interior_closure*:
fixes $S :: 'n::euclidean\_space\ set$
shows $[\![convex\ S;\ convex\ T]\!]$
$\Longrightarrow rel\_interior\ S = rel\_interior\ T \longleftrightarrow closure\ S = closure\ T$
by (*simp add*: *closure_eq_rel_interior_eq*)

lemma *convex_same_rel_interior_closure_straddle*:
fixes $S :: 'n::euclidean\_space\ set$
shows $[\![convex\ S;\ convex\ T]\!]$
$\Longrightarrow rel\_interior\ S = rel\_interior\ T \longleftrightarrow$
$rel\_interior\ S \subseteq T \wedge T \subseteq closure\ S$
by (*simp add*: *closure_eq_between convex_same_rel_interior_closure*)

lemma *convex_rel_frontier_aff_dim*:
fixes $S1\ S2 :: 'n::euclidean\_space\ set$
assumes *convex S1*
and *convex S2*
and $S2 \neq \{\}$
and $S1 \leq rel\_frontier\ S2$
shows $aff\_dim\ S1 < aff\_dim\ S2$
proof −
have $S1 \subseteq closure\ S2$
using *assms* **unfolding** *rel_frontier_def* **by** *auto*
then have $*$: $affine\ hull\ S1 \subseteq affine\ hull\ S2$
using *hull_mono*[*of S1 closure S2*] *closure_same_affine_hull*[*of S2*] **by** *blast*
then have $aff\_dim\ S1 \leq aff\_dim\ S2$
using $*$ *aff_dim_affine_hull*[*of S1*] *aff_dim_affine_hull*[*of S2*]
aff_dim_subset[*of affine hull S1 affine hull S2*]
by *auto*
moreover
{
assume *eq*: $aff\_dim\ S1 = aff\_dim\ S2$
then have $S1 \neq \{\}$
using *aff_dim_empty*[*of S1*] *aff_dim_empty*[*of S2*] ‹$S2 \neq \{\}$› **by** *auto*
have $**$: $affine\ hull\ S1 = affine\ hull\ S2$
by (*simp_all add*: $*$ *eq* ‹$S1 \neq \{\}$› *affine_dim_equal*)
obtain *a* **where** *a*: $a \in rel\_interior\ S1$
using ‹$S1 \neq \{\}$› *rel_interior_eq_empty assms* **by** *auto*
obtain *T* **where** *T*: $open\ T\ a \in T \cap S1\ T \cap affine\ hull\ S1 \subseteq S1$
using *mem_rel_interior*[*of a S1*] *a* **by** *auto*
then have $a \in T \cap closure\ S2$
using *a assms* **unfolding** *rel_frontier_def* **by** *auto*
then obtain *b* **where** *b*: $b \in T \cap rel\_interior\ S2$

```
    using open_inter_closure_rel_interior[of S2 T] assms T by auto
  then have b ∈ affine hull S1
    using rel_interior_subset hull_subset[of S2] ** by auto
  then have b ∈ S1
    using T b by auto
  then have False
    using b assms unfolding rel_frontier_def by auto
 }
 ultimately show ?thesis
  using less_le by auto
qed

lemma convex_rel_interior_if:
 fixes S :: 'n::euclidean_space set
 assumes convex S
  and z ∈ rel_interior S
 shows ∀ x∈affine hull S. ∃ m. m > 1 ∧ (∀ e. e > 1 ∧ e ≤ m ⟶ (1 − e) *R x
+ e *R z ∈ S)
proof −
 obtain e1 where e1: e1 > 0 ∧ cball z e1 ∩ affine hull S ⊆ S
  using mem_rel_interior_cball[of z S] assms by auto
 {
  fix x
  assume x: x ∈ affine hull S
  {
   assume x ≠ z
   define m where m = 1 + e1/norm(x−z)
   hence m > 1 using e1 ‹x ≠ z› by auto
   {
    fix e
    assume e: e > 1 ∧ e ≤ m
    have z ∈ affine hull S
     using assms rel_interior_subset hull_subset[of S] by auto
    then have *: (1 − e)*R x + e *R z ∈ affine hull S
     using mem_affine[of affine hull S x z (1−e) e] affine_affine_hull[of S] x
     by auto
    have norm (z + e *R x − (x + e *R z)) = norm ((e − 1) *R (x − z))
     by (simp add: algebra_simps)
    also have ... = (e − 1) * norm (x−z)
     using norm_scaleR e by auto
    also have ... ≤ (m − 1) * norm (x − z)
     using e mult_right_mono[of _ _ norm(x−z)] by auto
    also have ... = (e1 / norm (x − z)) * norm (x − z)
     using m_def by auto
    also have ... = e1
     using ‹x ≠ z› e1 by simp
    finally have **: norm (z + e *R x − (x + e *R z)) ≤ e1
     by auto
    have (1 − e)*R x+ e *R z ∈ cball z e1
```

```
        using m_def **
        unfolding cball_def dist_norm
        by (auto simp add: algebra_simps)
      then have (1 - e) *R x+ e *R z ∈ S
        using e * e1 by auto
    }
    then have ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 - e) *R x + e *R z
∈ S )
      using ‹m> 1 › by auto
  }
  moreover
  {
    assume x = z
    define m where m = 1 + e1
    then have m > 1
      using e1 by auto
    {
      fix e
      assume e: e > 1 ∧ e ≤ m
      then have (1 - e) *R x + e *R z ∈ S
        using e1 x ‹x = z› by (auto simp add: algebra_simps)
      then have (1 - e) *R x + e *R z ∈ S
        using e by auto
    }
    then have ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 - e) *R x + e *R z
∈ S)
      using ‹m > 1› by auto
  }
  ultimately have ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 - e) *R x + e
*R z ∈ S )
    by blast
 }
 then show ?thesis by auto
qed

lemma convex_rel_interior_if2:
 fixes S :: 'n::euclidean_space set
 assumes convex S
 assumes z ∈ rel_interior S
 shows ∀x∈affine hull S. ∃e. e > 1 ∧ (1 - e)*R x + e *R z ∈ S
 using convex_rel_interior_if[of S z] assms by auto

lemma convex_rel_interior_only_if:
 fixes S :: 'n::euclidean_space set
 assumes convex S
   and S ≠ {}
 assumes ∀x∈S. ∃e. e > 1 ∧ (1 - e) *R x + e *R z ∈ S
 shows z ∈ rel_interior S
proof -
```

obtain x **where** x: $x \in$ *rel_interior* S
using *rel_interior_eq_empty assms* **by** *auto*
then have $x \in S$
using *rel_interior_subset* **by** *auto*
then obtain e **where** e: $e > 1 \wedge (1 - e) *_R x + e *_R z \in S$
using *assms* **by** *auto*
define y **where** [*abs_def*]: $y = (1 - e) *_R x + e *_R z$
then have $y \in S$ **using** e **by** *auto*
define *e1* **where** $e1 = 1/e$
then have $0 < e1 \wedge e1 < 1$ **using** e **by** *auto*
then have $z = y - (1 - e1) *_R (y - x)$
using *e1_def y_def* **by** (*auto simp add*: *algebra_simps*)
then show *?thesis*
using *rel_interior_convex_shrink*[*of S x y 1−e1*] ‹$0 < e1 \wedge e1 < 1$› ‹$y \in S$› *x assms*
by *auto*
qed

lemma *convex_rel_interior_iff*:
fixes S :: *'n*::*euclidean_space set*
assumes *convex S*
and $S \neq \{\}$
shows $z \in$ *rel_interior* $S \longleftrightarrow (\forall x{\in}S.\ \exists e.\ e > 1 \wedge (1 - e) *_R x + e *_R z \in S)$
using *assms hull_subset*[*of S affine*]
convex_rel_interior_if[*of S z*] *convex_rel_interior_only_if*[*of S z*]
by *auto*

lemma *convex_rel_interior_iff2*:
fixes S :: *'n*::*euclidean_space set*
assumes *convex S*
and $S \neq \{\}$
shows $z \in$ *rel_interior* $S \longleftrightarrow (\forall x{\in}$*affine hull S*$.\ \exists e.\ e > 1 \wedge (1 - e) *_R x + e *_R z \in S)$
using *assms hull_subset*[*of S*] *convex_rel_interior_if2*[*of S z*] *convex_rel_interior_only_if*[*of S z*]
by *auto*

lemma *convex_interior_iff*:
fixes S :: *'n*::*euclidean_space set*
assumes *convex S*
shows $z \in$ *interior* $S \longleftrightarrow (\forall x.\ \exists e.\ e > 0 \wedge z + e *_R x \in S)$
proof (*cases aff_dim S = int DIM*(*'n*))
case *False*
{ assume $z \in$ *interior S*
then have *False*
using *False interior_rel_interior_gen*[*of S*] **by** *auto* **}**
moreover
{ assume r: $\forall x.\ \exists e.\ e > 0 \wedge z + e *_R x \in S$
{ fix x

```
    obtain e1 where e1: e1 > 0 ∧ z + e1 *R (x − z) ∈ S
      using r by auto
    obtain e2 where e2: e2 > 0 ∧ z + e2 *R (z − x) ∈ S
      using r by auto
    define x1 where [abs_def]: x1 = z + e1 *R (x − z)
    then have x1: x1 ∈ affine hull S
      using e1 hull_subset[of S] by auto
    define x2 where [abs_def]: x2 = z + e2 *R (z − x)
    then have x2: x2 ∈ affine hull S
      using e2 hull_subset[of S] by auto
    have ∗: e1/(e1+e2) + e2/(e1+e2) = 1
      using add_divide_distrib[of e1 e2 e1+e2] e1 e2 by simp
    then have z = (e2/(e1+e2)) *R x1 + (e1/(e1+e2)) *R x2
      by (simp add: x1_def x2_def algebra_simps) (simp add: ∗ pth_8)
    then have z: z ∈ affine hull S
      using mem_affine[of affine hull S x1 x2 e2/(e1+e2) e1/(e1+e2)]
        x1 x2 affine_affine_hull[of S] ∗
      by auto
    have x1 − x2 = (e1 + e2) *R (x − z)
      using x1_def x2_def by (auto simp add: algebra_simps)
    then have x = z+(1/(e1+e2)) *R (x1−x2)
      using e1 e2 by simp
    then have x ∈ affine hull S
      using mem_affine_3_minus[of affine hull S z x1 x2 1/(e1+e2)]
        x1 x2 z affine_affine_hull[of S]
      by auto
  }
  then have affine hull S = UNIV
    by auto
  then have aff_dim S = int DIM('n)
    using aff_dim_affine_hull[of S] by (simp)
  then have False
    using False by auto
 }
 ultimately show ?thesis by auto
next
 case True
 then have S ≠ {}
  using aff_dim_empty[of S] by auto
 have ∗: affine hull S = UNIV
  using True affine_hull_UNIV by auto
 {
  assume z ∈ interior S
  then have z ∈ rel_interior S
    using True interior_rel_interior_gen[of S] by auto
  then have ∗∗: ∀ x. ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S
    using convex_rel_interior_iff2[of S z] assms ‹S ≠ {}› ∗ by auto
  fix x
  obtain e1 where e1: e1 > 1 (1 − e1) *R (z − x) + e1 *R z ∈ S
```

```
    using **[rule_format, of z−x] by auto
  define e where [abs_def]: e = e1 − 1
  then have (1 − e1) *R (z − x) + e1 *R z = z + e *R x
    by (simp add: algebra_simps)
  then have e > 0 z + e *R x ∈ S
    using e1 e_def by auto
  then have ∃ e. e > 0 ∧ z + e *R x ∈ S
    by auto
 }
 moreover
 {
  assume r: ∀ x. ∃ e. e > 0 ∧ z + e *R x ∈ S
  {
   fix x
   obtain e1 where e1: e1 > 0 z + e1 *R (z − x) ∈ S
    using r[rule_format, of z−x] by auto
   define e where e = e1 + 1
   then have z + e1 *R (z − x) = (1 − e) *R x + e *R z
    by (simp add: algebra_simps)
   then have e > 1 (1 − e)*R x + e *R z ∈ S
    using e1 e_def by auto
   then have ∃ e. e > 1 ∧ (1 − e) *R x + e *R z ∈ S by auto
  }
  then have z ∈ rel_interior S
   using convex_rel_interior_iff2[of S z] assms ‹S ≠ {}› by auto
  then have z ∈ interior S
   using True interior_rel_interior_gen[of S] by auto
 }
 ultimately show ?thesis by auto
qed
```

Relative interior and closure under common operations

```
lemma rel_interior_inter_aux: ⋂{rel_interior S |S. S ∈ I} ⊆ ⋂I
proof −
 { fix y
  assume y ∈ ⋂{rel_interior S |S. S ∈ I}
  then have y: ∀ S ∈ I. y ∈ rel_interior S
   by auto
  { fix S
   assume S ∈ I
   then have y ∈ S
    using rel_interior_subset y by auto
  }
  then have y ∈ ⋂I by auto
 }
 then show ?thesis by auto
qed
```

```
lemma convex_closure_rel_interior_Int:
  assumes ⋀S. S∈ℱ ⟹ convex (S :: 'n::euclidean_space set)
    and ⋂(rel_interior ` ℱ) ≠ {}
  shows ⋂(closure ` ℱ) ⊆ closure (⋂(rel_interior ` ℱ))
proof −
  obtain x where x: ∀S∈ℱ. x ∈ rel_interior S
    using assms by auto
  show ?thesis
  proof
    fix y
    assume y: y ∈ ⋂ (closure ` ℱ)
    show y ∈ closure (⋂(rel_interior ` ℱ))
    proof (cases y=x)
      case True
      with closure_subset x show ?thesis
        by fastforce
    next
      case False
      show ?thesis
      proof (clarsimp simp: closure_approachable_le)
        fix ε :: real
        assume e: ε > 0
        define e1 where e1 = min 1 (ε/norm (y − x))
        then have e1: e1 > 0 e1 ≤ 1 e1 ∗ norm (y − x) ≤ ε
          using ‹y ≠ x› ‹ε > 0› le_divide_eq[of e1 ε norm (y − x)]
          by simp_all
        define z where z = y − e1 ∗R (y − x)
        {
          fix S
          assume S ∈ ℱ
          then have z ∈ rel_interior S
              using rel_interior_closure_convex_shrink[of S x y e1] assms x y e1
z_def
            by auto
        }
        then have ∗: z ∈ ⋂(rel_interior ` ℱ)
          by auto
        show ∃x∈⋂ (rel_interior ` ℱ). dist x y ≤ ε
          using ‹y ≠ x› z_def ∗ e1 e dist_norm[of z y]
          by force
      qed
    qed
  qed
qed


lemma closure_Inter_convex:
  fixes ℱ :: 'n::euclidean_space set set
  assumes ⋀S. S ∈ ℱ ⟹ convex S and ⋂(rel_interior ` ℱ) ≠ {}
```

shows *closure*($\bigcap \mathcal{F}$) = $\bigcap$ (*closure* ‘ $\mathcal{F}$)
proof −
have $\bigcap$ (*closure* ‘ $\mathcal{F}$) ≤ *closure* ($\bigcap$ (*rel_interior* ‘ $\mathcal{F}$))
by (*meson assms convex_closure_rel_interior_Int*)
moreover
have *closure* ($\bigcap$ (*rel_interior* ‘ $\mathcal{F}$)) ⊆ *closure* ($\bigcap \mathcal{F}$)
using *rel_interior_inter_aux closure_mono*[*of* $\bigcap$ (*rel_interior* ‘ $\mathcal{F}$) $\bigcap \mathcal{F}$]
by *auto*
ultimately show *?thesis*
using *closure_Int*[*of* $\mathcal{F}$] **by** *blast*
qed

lemma *closure_Inter_convex_open*:
(⋀*S*::′*n*::*euclidean_space set*. $S \in \mathcal{F} \Longrightarrow$ *convex* $S \wedge$ *open* S)
$\Longrightarrow$ *closure*($\bigcap \mathcal{F}$) = (*if* $\bigcap \mathcal{F} = \{\}$ *then* {} *else* $\bigcap$ (*closure* ‘ $\mathcal{F}$))
by (*simp add*: *closure_Inter_convex rel_interior_open*)

lemma *convex_Inter_rel_interior_same_closure*:
fixes $\mathcal{F}$ :: ′*n*::*euclidean_space set set*
assumes ⋀*S*. $S \in \mathcal{F} \Longrightarrow$ *convex* S
and $\bigcap$ (*rel_interior* ‘ $\mathcal{F}$) ≠ {}
shows *closure* ($\bigcap$ (*rel_interior* ‘ $\mathcal{F}$)) = *closure* ($\bigcap \mathcal{F}$)
proof −
have $\bigcap$ (*closure* ‘ $\mathcal{F}$) ⊆ *closure* ($\bigcap$ (*rel_interior* ‘ $\mathcal{F}$))
by (*meson assms convex_closure_rel_interior_Int*)
moreover
have *closure* ($\bigcap$ (*rel_interior* ‘ $\mathcal{F}$)) ⊆ *closure* ($\bigcap \mathcal{F}$)
by (*metis Setcompr_eq_image closure_mono rel_interior_inter_aux*)
ultimately show *?thesis*
by (*simp add*: *assms closure_Inter_convex*)
qed

lemma *convex_rel_interior_Inter*:
fixes $\mathcal{F}$ :: ′*n*::*euclidean_space set set*
assumes ⋀*S*. $S \in \mathcal{F} \Longrightarrow$ *convex* S
and $\bigcap$ (*rel_interior* ‘ $\mathcal{F}$) ≠ {}
shows *rel_interior* ($\bigcap \mathcal{F}$) ⊆ $\bigcap$ (*rel_interior* ‘ $\mathcal{F}$)
proof −
have *convex* ($\bigcap \mathcal{F}$)
using *assms convex_Inter* **by** *auto*
moreover
have *convex* ($\bigcap$ (*rel_interior* ‘ $\mathcal{F}$))
using *assms* **by** (*metis convex_rel_interior convex_INT*)
ultimately
have *rel_interior* ($\bigcap$ (*rel_interior* ‘ $\mathcal{F}$)) = *rel_interior* ($\bigcap \mathcal{F}$)
using *convex_Inter_rel_interior_same_closure assms*
closure_eq_rel_interior_eq[*of* $\bigcap$ (*rel_interior* ‘ $\mathcal{F}$) $\bigcap \mathcal{F}$]
by *blast*
then show *?thesis*

```
    using rel_interior_subset[of ⋂(rel_interior ` F)] by auto
qed

lemma convex_rel_interior_finite_Inter:
  fixes F :: 'n::euclidean_space set set
  assumes ⋀S. S ∈ F ⟹ convex S
    and ⋂(rel_interior ` F) ≠ {}
    and finite F
  shows rel_interior (⋂F) = ⋂(rel_interior ` F)
proof −
  have ⋂F ≠ {}
    using assms rel_interior_inter_aux[of F] by auto
  have convex (⋂F)
    using convex_Inter assms by auto
  show ?thesis
  proof (cases F = {})
    case True
    then show ?thesis
      using Inter_empty rel_interior_UNIV by auto
  next
    case False
    {
      fix z
      assume z: z ∈ ⋂(rel_interior ` F)
      {
        fix x
        assume x: x ∈ ⋂F
        {
          fix S
          assume S: S ∈ F
          then have z ∈ rel_interior S x ∈ S
            using z x by auto
          then have ∃m. m > 1 ∧ (∀e. e > 1 ∧ e ≤ m ⟶ (1 − e)*R x + e *R
z ∈ S)
            using convex_rel_interior_if[of S z] S assms hull_subset[of S] by auto
        }
        then obtain mS where
          mS: ∀S∈F. mS S > 1 ∧ (∀e. e > 1 ∧ e ≤ mS S ⟶ (1 − e) *R x + e
*R z ∈ S) by metis
        define e where e = Min (mS ` F)
        then have e ∈ mS ` F using assms ‹F ≠ {}› by simp
        then have e > 1 using mS by auto
        moreover have ∀S∈F. e ≤ mS S
          using e_def assms by auto
        ultimately have ∃e > 1. (1 − e) *R x + e *R z ∈ ⋂F
          using mS by auto
      }
      then have z ∈ rel_interior (⋂F)
        using convex_rel_interior_iff[of ⋂F z] ‹⋂F ≠ {}› ‹convex (⋂F)› by
```

auto
}
then show *?thesis*
using *convex_rel_interior_Inter*[*of* $\mathcal{F}$] *assms* **by** *auto*
qed
qed

lemma *closure_Int_convex*:
fixes S T :: '*n*::*euclidean_space set*
assumes *convex* S
and *convex* T
assumes *rel_interior* $S \cap$ *rel_interior* $T \neq \{\}$
shows *closure* $(S \cap T) =$ *closure* $S \cap$ *closure* T
using *closure_Inter_convex*[*of* $\{S,T\}$] *assms* **by** *auto*

lemma *convex_rel_interior_inter_two*:
fixes S T :: '*n*::*euclidean_space set*
assumes *convex* S
and *convex* T
and *rel_interior* $S \cap$ *rel_interior* $T \neq \{\}$
shows *rel_interior* $(S \cap T) =$ *rel_interior* $S \cap$ *rel_interior* T
using *convex_rel_interior_finite_Inter*[*of* $\{S,T\}$] *assms* **by** *auto*

lemma *convex_affine_closure_Int*:
fixes S T :: '*n*::*euclidean_space set*
assumes *convex* S
and *affine* T
and *rel_interior* $S \cap T \neq \{\}$
shows *closure* $(S \cap T) =$ *closure* $S \cap T$
by (*metis affine_imp_convex assms closure_Int_convex rel_interior_affine rel_interior_eq_closure*)

lemma *connected_component_1_gen*:
fixes S :: '*a* :: *euclidean_space set*
assumes *DIM*('*a*) = *1*
shows *connected_component* S a $b \longleftrightarrow$ *closed_segment* a $b \subseteq S$
unfolding *connected_component_def*
by (*metis* (*no_types*, *lifting*) *assms subsetD subsetI convex_contains_segment convex_segment*(*1*)
ends_in_segment connected_convex_1_gen)

lemma *connected_component_1*:
fixes S :: *real set*
shows *connected_component* S a $b \longleftrightarrow$ *closed_segment* a $b \subseteq S$
by (*simp add*: *connected_component_1_gen*)

lemma *convex_affine_rel_interior_Int*:
fixes S T :: '*n*::*euclidean_space set*
assumes *convex* S
and *affine* T

and *rel_interior S* ∩ *T* ≠ {}
shows *rel_interior* (*S* ∩ *T*) = *rel_interior S* ∩ *T*
by (*simp add*: *affine_imp_convex assms convex_rel_interior_inter_two rel_interior_affine*)

lemma *convex_affine_rel_frontier_Int*:
fixes *S T* :: *'n*::*euclidean_space set*
assumes *convex S*
and *affine T*
and *interior S* ∩ *T* ≠ {}
shows *rel_frontier*(*S* ∩ *T*) = *frontier S* ∩ *T*
using *assms*
unfolding *rel_frontier_def frontier_def*
using *convex_affine_closure_Int convex_affine_rel_interior_Int rel_interior_nonempty_interior*
by *fastforce*

lemma *rel_interior_convex_Int_affine*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *convex S affine T interior S* ∩ *T* ≠ {}
shows *rel_interior*(*S* ∩ *T*) = *interior S* ∩ *T*
by (*metis Int_emptyI assms convex_affine_rel_interior_Int empty_iff interior_rel_interior_gen*)

lemma *subset_rel_interior_convex*:
fixes *S T* :: *'n*::*euclidean_space set*
assumes *convex S*
and *convex T*
and *S* ≤ *closure T*
and ¬ *S* ⊆ *rel_frontier T*
shows *rel_interior S* ⊆ *rel_interior T*
proof −
have ∗: *S* ∩ *closure T* = *S*
using *assms* **by** *auto*
have ¬ *rel_interior S* ⊆ *rel_frontier T*
using *closure_mono*[*of rel_interior S rel_frontier T*] *closed_rel_frontier*[*of T*]
closure_closed[*of S*] *convex_closure_rel_interior*[*of S*] *closure_subset*[*of S*]
assms
by *auto*
then have *rel_interior S* ∩ *rel_interior* (*closure T*) ≠ {}
using *assms rel_frontier_def*[*of T*] *rel_interior_subset convex_rel_interior_closure*[*of T*]
by *auto*
then have *rel_interior S* ∩ *rel_interior T* = *rel_interior* (*S* ∩ *closure T*)
using *assms convex_closure convex_rel_interior_inter_two*[*of S closure T*]
convex_rel_interior_closure[*of T*]
by *auto*
also have ... = *rel_interior S*
using ∗ **by** *auto*
finally show *?thesis*
by *auto*

qed

lemma *rel_interior_convex_linear_image*:
fixes $f :: 'm::euclidean\_space \Rightarrow 'n::euclidean\_space$
assumes *linear f*
and *convex S*
shows $f \mathbin{`} (rel\_interior\ S) = rel\_interior\ (f \mathbin{`} S)$
proof (*cases* $S = \{\}$)
case *True*
then show *?thesis*
using *assms* **by** *auto*
next
case *False*
interpret *linear f* **by** *fact*
have $*$: $f \mathbin{`} (rel\_interior\ S) \subseteq f \mathbin{`} S$
unfolding *image_mono* **using** *rel_interior_subset* **by** *auto*
have $f \mathbin{`} S \subseteq f \mathbin{`} (closure\ S)$
unfolding *image_mono* **using** *closure_subset* **by** *auto*
also have $\ldots = f \mathbin{`} (closure\ (rel\_interior\ S))$
using *convex_closure_rel_interior assms* **by** *auto*
also have $\ldots \subseteq closure\ (f \mathbin{`} (rel\_interior\ S))$
using *closure_linear_image_subset assms* **by** *auto*
finally have $closure\ (f \mathbin{`} S) = closure\ (f \mathbin{`} rel\_interior\ S)$
using *closure_mono*[*of* $f \mathbin{`} S$ $closure\ (f \mathbin{`} rel\_interior\ S)$] *closure_closure*
closure_mono[*of* $f \mathbin{`} rel\_interior\ S$ $f \mathbin{`} S$] $*$
by *auto*
then have $rel\_interior\ (f \mathbin{`} S) = rel\_interior\ (f \mathbin{`} rel\_interior\ S)$
using *assms convex_rel_interior*
linear_conv_bounded_linear[*of f*] *convex_linear_image*[*of* _ *S*]
convex_linear_image[*of* _ $rel\_interior\ S$]
closure_eq_rel_interior_eq[*of* $f \mathbin{`} S$ $f \mathbin{`} rel\_interior\ S$]
by *auto*
then have $rel\_interior\ (f \mathbin{`} S) \subseteq f \mathbin{`} rel\_interior\ S$
using *rel_interior_subset* **by** *auto*
moreover
{
fix *z*
assume $z \in f \mathbin{`} rel\_interior\ S$
then obtain *z1* **where** *z1*: $z1 \in rel\_interior\ S$ $f\ z1 = z$ **by** *auto*
{
fix *x*
assume $x \in f \mathbin{`} S$
then obtain *x1* **where** *x1*: $x1 \in S$ $f\ x1 = x$ **by** *auto*
then obtain *e* **where** *e*: $e > 1$ $(1 - e) *_R x1 + e *_R z1 \in S$
using *convex_rel_interior_iff*[*of* S *z1*] ‹*convex S*› *x1 z1* **by** *auto*
moreover have $f\ ((1 - e) *_R x1 + e *_R z1) = (1 - e) *_R x + e *_R z$
using *x1 z1* **by** (*simp add*: *linear_add linear_scale* ‹*linear f*›)
ultimately have $(1 - e) *_R x + e *_R z \in f \mathbin{`} S$
using *imageI*[*of* $(1 - e) *_R x1 + e *_R z1$ S f] **by** *auto*

```
    then have ∃e. e > 1 ∧ (1 − e) *R x + e *R z ∈ f ‘ S
      using e by auto
  }
  then have z ∈ rel_interior (f ‘ S)
    using convex_rel_interior_iff[of f ‘ S z] ‹convex S› ‹linear f›
      ‹S ≠ {}› convex_linear_image[of f S]  linear_conv_bounded_linear[of f]
    by auto
 }
 ultimately show ?thesis by auto
qed

lemma rel_interior_convex_linear_preimage:
 fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
 assumes linear f
  and convex S
  and f −‘ (rel_interior S) ≠ {}
 shows rel_interior (f −‘ S) = f −‘ (rel_interior S)
proof −
 interpret linear f by fact
 have S ≠ {}
  using assms by auto
 have nonemp: f −‘ S ≠ {}
  by (metis assms(3) rel_interior_subset subset_empty vimage_mono)
 then have S ∩ (range f) ≠ {}
  by auto
 have conv: convex (f −‘ S)
  using convex_linear_vimage assms by auto
 then have convex (S ∩ range f)
  by (simp add: assms(2) convex_Int convex_linear_image linear_axioms)
 {
  fix z
  assume z ∈ f −‘ (rel_interior S)
  then have z: f z ∈ rel_interior S
   by auto
  {
   fix x
   assume x ∈ f −‘ S
   then have f x ∈ S by auto
   then obtain e where e: e > 1 (1 − e) *R f x + e *R f z ∈ S
    using convex_rel_interior_iff[of S f z] z assms ‹S ≠ {}› by auto
   moreover have (1 − e) *R f x + e *R f z = f ((1 − e) *R x + e *R z)
    using ‹linear f› by (simp add: linear_iff)
   ultimately have ∃e. e > 1 ∧ (1 − e) *R x + e *R z ∈ f −‘ S
    using e by auto
  }
  then have z ∈ rel_interior (f −‘ S)
   using convex_rel_interior_iff[of f −‘ S z] conv nonemp by auto
 }
 moreover
```

```
  {
    fix z
    assume z: z ∈ rel_interior (f −‘ S)
    {
      fix x
      assume x ∈ S ∩ range f
      then obtain y where y: f y = x y ∈ f −‘ S by auto
      then obtain e where e: e > 1 (1 − e) *R y + e *R z ∈ f −‘ S
        using convex_rel_interior_iff[of f −‘ S z] z conv by auto
      moreover have (1 − e) *R x + e *R f z = f ((1 − e) *R y + e *R z)
        using ‹linear f› y by (simp add: linear_iff)
      ultimately have ∃ e. e > 1 ∧ (1 − e) *R x + e *R f z ∈ S ∩ range f
        using e by auto
    }
    then have f z ∈ rel_interior (S ∩ range f)
      using ‹convex (S ∩ (range f))› ‹S ∩ range f ≠ {}›
        convex_rel_interior_iff[of S ∩ (range f) f z]
      by auto
    moreover have affine (range f)
      by (simp add: linear_axioms linear_subspace_image subspace_imp_affine)
    ultimately have f z ∈ rel_interior S
      using convex_affine_rel_interior_Int[of S range f] assms by auto
    then have z ∈ f −‘ (rel_interior S)
      by auto
  }
  ultimately show ?thesis by auto
qed

lemma rel_interior_Times:
  fixes S :: 'n::euclidean_space set
    and T :: 'm::euclidean_space set
  assumes convex S
    and convex T
  shows rel_interior (S × T) = rel_interior S × rel_interior T
proof (cases S = {} ∨ T = {})
  case True
  then show ?thesis
    by auto
next
  case False
  then have S ≠ {} T ≠ {}
    by auto
  then have ri: rel_interior S ≠ {} rel_interior T ≠ {}
    using rel_interior_eq_empty assms by auto
  then have fst −‘ rel_interior S ≠ {}
    using fst_vimage_eq_Times[of rel_interior S] by auto
  then have rel_interior ((fst :: 'n * 'm ⇒ 'n) −‘ S) = fst −‘ rel_interior S
    using linear_fst ‹convex S› rel_interior_convex_linear_preimage[of fst S] by
auto
```

then have *s*: *rel_interior* (*S* × (*UNIV* :: *'m set*)) = *rel_interior S* × *UNIV*
by (*simp add*: *fst_vimage_eq_Times*)
from *ri* **have** *snd* −‘ *rel_interior T* ≠ {}
using *snd_vimage_eq_Times*[*of rel_interior T*] **by** *auto*
then have *rel_interior* ((*snd* :: *'n* ∗ *'m* ⇒ *'m*) −‘ *T*) = *snd* −‘ *rel_interior T*
using *linear_snd* ‹*convex T*› *rel_interior_convex_linear_preimage*[*of snd T*]
by *auto*
then have *t*: *rel_interior* ((*UNIV* :: *'n set*) × *T*) = *UNIV* × *rel_interior T*
by (*simp add*: *snd_vimage_eq_Times*)
from *s t* **have** ∗: *rel_interior* (*S* × (*UNIV* :: *'m set*)) ∩ *rel_interior* ((*UNIV* :: *'n set*) × *T*) =
rel_interior S × *rel_interior T* **by** *auto*
have *S* × *T* = *S* × (*UNIV* :: *'m set*) ∩ (*UNIV* :: *'n set*) × *T*
by *auto*
then have *rel_interior* (*S* × *T*) = *rel_interior* ((*S* × (*UNIV* :: *'m set*)) ∩ ((*UNIV* :: *'n set*) × *T*))
by *auto*
also have ... = *rel_interior* (*S* × (*UNIV* :: *'m set*)) ∩ *rel_interior* ((*UNIV* :: *'n set*) × *T*)
using ∗ *ri assms convex_Times*
by (*subst convex_rel_interior_inter_two*) *auto*
finally show *?thesis* **using** ∗ **by** *auto*
qed

lemma *rel_interior_scaleR*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *c* ≠ *0*
shows ((∗*R*) *c*) ‘ (*rel_interior S*) = *rel_interior* (((∗*R*) *c*) ‘ *S*)
using *rel_interior_injective_linear_image*[*of* ((∗*R*) *c*) *S*]
linear_conv_bounded_linear[*of* (∗*R*) *c*] *linear_scaleR injective_scaleR*[*of c*] *assms*
by *auto*

lemma *rel_interior_convex_scaleR*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *convex S*
shows ((∗*R*) *c*) ‘ (*rel_interior S*) = *rel_interior* (((∗*R*) *c*) ‘ *S*)
by (*metis assms linear_scaleR rel_interior_convex_linear_image*)

lemma *convex_rel_open_scaleR*:
fixes *S* :: *'n*::*euclidean_space set*
assumes *convex S*
and *rel_open S*
shows *convex* (((∗*R*) *c*) ‘ *S*) ∧ *rel_open* (((∗*R*) *c*) ‘ *S*)
by (*metis assms convex_scaling rel_interior_convex_scaleR rel_open_def*)

lemma *convex_rel_open_finite_Inter*:
fixes *F* :: *'n*::*euclidean_space set set*
assumes ⋀*S*. *S* ∈ *F* ⟹ *convex S* ∧ *rel_open S*

```
    and finite ℱ
  shows convex (⋂ℱ) ∧ rel_open (⋂ℱ)
proof (cases ⋂{rel_interior S |S. S ∈ ℱ} = {})
  case True
  then have ⋂ℱ = {}
    using assms unfolding rel_open_def by auto
  then show ?thesis
    unfolding rel_open_def by auto
next
  case False
  then have rel_open (⋂ℱ)
   using assms convex_rel_interior_finite_Inter[of ℱ] by (force simp: rel_open_def)
  then show ?thesis
    using convex_Inter assms by auto
qed

lemma convex_rel_open_linear_image:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
    and convex S
    and rel_open S
  shows convex (f ` S) ∧ rel_open (f ` S)
 by (metis assms convex_linear_image rel_interior_convex_linear_image rel_open_def)

lemma convex_rel_open_linear_preimage:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
    and convex S
    and rel_open S
  shows convex (f -` S) ∧ rel_open (f -` S)
proof (cases f -` (rel_interior S) = {})
  case True
  then have f -` S = {}
    using assms unfolding rel_open_def by auto
  then show ?thesis
    unfolding rel_open_def by auto
next
  case False
  then have rel_open (f -` S)
    using assms unfolding rel_open_def
    using rel_interior_convex_linear_preimage[of f S]
    by auto
  then show ?thesis
    using convex_linear_vimage assms
    by auto
qed

lemma rel_interior_projection:
  fixes S :: ('m::euclidean_space × 'n::euclidean_space) set
```

```
    and f :: 'm::euclidean_space ⇒ 'n::euclidean_space set
  assumes convex S
    and f = (λy. {z. (y, z) ∈ S})
  shows (y, z) ∈ rel_interior S ⟷ (y ∈ rel_interior {y. (f y ≠ {})} ∧ z ∈
rel_interior (f y))
proof −
  {
    fix y
    assume y ∈ {y. f y ≠ {}}
    then obtain z where (y, z) ∈ S
      using assms by auto
    then have ∃x. x ∈ S ∧ y = fst x
      by auto
    then obtain x where x ∈ S y = fst x
      by blast
    then have y ∈ fst ‘ S
      unfolding image_def by auto
  }
  then have fst ‘ S = {y. f y ≠ {}}
    unfolding fst_def using assms by auto
  then have h1: fst ‘ rel_interior S = rel_interior {y. f y ≠ {}}
    using rel_interior_convex_linear_image[of fst S] assms linear_fst by auto
  {
    fix y
    assume y ∈ rel_interior {y. f y ≠ {}}
    then have y ∈ fst ‘ rel_interior S
      using h1 by auto
    then have ∗: rel_interior S ∩ fst −‘ {y} ≠ {}
      by auto
    moreover have aff: affine (fst −‘ {y})
      unfolding affine_alt by (simp add: algebra_simps)
    ultimately have ∗∗: rel_interior (S ∩ fst −‘ {y}) = rel_interior S ∩ fst −‘
{y}
      using convex_affine_rel_interior_Int[of S fst −‘ {y}] assms by auto
    have conv: convex (S ∩ fst −‘ {y})
      using convex_Int assms aff affine_imp_convex by auto
    {
      fix x
      assume x ∈ f y
      then have (y, x) ∈ S ∩ (fst −‘ {y})
        using assms by auto
      moreover have x = snd (y, x) by auto
      ultimately have x ∈ snd ‘ (S ∩ fst −‘ {y})
        by blast
    }
    then have snd ‘ (S ∩ fst −‘ {y}) = f y
      using assms by auto
    then have ∗∗∗: rel_interior (f y) = snd ‘ rel_interior (S ∩ fst −‘ {y})
      using rel_interior_convex_linear_image[of snd S ∩ fst −‘ {y}] linear_snd
```

```
conv
    by auto
  {
    fix z
    assume z ∈ rel_interior (f y)
    then have z ∈ snd ‘ rel_interior (S ∩ fst −‘ {y})
      using ∗∗∗ by auto
    moreover have {y} = fst ‘ rel_interior (S ∩ fst −‘ {y})
      using ∗ ∗∗ rel_interior_subset by auto
    ultimately have (y, z) ∈ rel_interior (S ∩ fst −‘ {y})
      by force
    then have (y,z) ∈ rel_interior S
      using ∗∗ by auto
  }
  moreover
  {
    fix z
    assume (y, z) ∈ rel_interior S
    then have (y, z) ∈ rel_interior (S ∩ fst −‘ {y})
      using ∗∗ by auto
    then have z ∈ snd ‘ rel_interior (S ∩ fst −‘ {y})
      by (metis Range_iff snd_eq_Range)
    then have z ∈ rel_interior (f y)
      using ∗∗∗ by auto
  }
  ultimately have ⋀z. (y, z) ∈ rel_interior S ⟷ z ∈ rel_interior (f y)
    by auto
 }
 then have h2: ⋀y z. y ∈ rel_interior {t. f t ≠ {}} ⟹
  (y, z) ∈ rel_interior S ⟷ z ∈ rel_interior (f y)
  by auto
 {
  fix y z
  assume asm: (y, z) ∈ rel_interior S
  then have y ∈ fst ‘ rel_interior S
    by (metis Domain_iff fst_eq_Domain)
  then have y ∈ rel_interior {t. f t ≠ {}}
    using h1 by auto
  then have y ∈ rel_interior {t. f t ≠ {}} and (z ∈ rel_interior (f y))
    using h2 asm by auto
 }
 then show ?thesis using h2 by blast
qed

lemma rel_frontier_Times:
 fixes S :: 'n::euclidean_space set
   and T :: 'm::euclidean_space set
 assumes convex S
   and convex T
```

shows *rel_frontier S* × *rel_frontier T* ⊆ *rel_frontier* (*S* × *T*)
by (*force simp*: *rel_frontier_def rel_interior_Times assms closure_Times*)

Relative interior of convex cone

lemma *cone_rel_interior*:
fixes *S* :: *'m*::*euclidean_space set*
assumes *cone S*
shows *cone* ({*0*} ∪ *rel_interior S*)
proof (*cases S* = {})
case *True*
then show *?thesis*
by (*simp add*: *cone_0*)
next
case *False*
then have ∗: *0* ∈ *S* ∧ (∀ *c*. *c* > *0* ⟶ (∗$_R$) *c* ‘ *S* = *S*)
using *cone_iff*[*of S*] *assms* **by** *auto*
then have ∗: *0* ∈ ({*0*} ∪ *rel_interior S*)
and ∀ *c*. *c* > *0* ⟶ (∗$_R$) *c* ‘ ({*0*} ∪ *rel_interior S*) = ({*0*} ∪ *rel_interior S*)
by (*auto simp add*: *rel_interior_scaleR*)
then show *?thesis*
using *cone_iff*[*of* {*0*} ∪ *rel_interior S*] **by** *auto*
qed

lemma *rel_interior_convex_cone_aux*:
fixes *S* :: *'m*::*euclidean_space set*
assumes *convex S*
shows (*c*, *x*) ∈ *rel_interior* (*cone hull* ({(*1* :: *real*)} × *S*)) ⟷
c > *0* ∧ *x* ∈ (((∗$_R$) *c*) ‘ (*rel_interior S*))
proof (*cases S* = {})
case *True*
then show *?thesis*
by (*simp add*: *cone_hull_empty*)
next
case *False*
then obtain *s* **where** *s* ∈ *S* **by** *auto*
have *conv*: *convex* ({(*1* :: *real*)} × *S*)
using *convex_Times*[*of* {(*1* :: *real*)} *S*] *assms convex_singleton*[*of 1* :: *real*]
by *auto*
define *f* **where** *f y* = {*z*. (*y*, *z*) ∈ *cone hull* ({*1* :: *real*} × *S*)} **for** *y*
then have ∗: (*c*, *x*) ∈ *rel_interior* (*cone hull* ({(*1* :: *real*)} × *S*)) =
(*c* ∈ *rel_interior* {*y*. *f y* ≠ {}} ∧ *x* ∈ *rel_interior* (*f c*))
using *convex_cone_hull*[*of* {(*1* :: *real*)} × *S*] *conv*
by (*subst rel_interior_projection*) *auto*
{
fix *y* :: *real*
assume *y* ≥ *0*
then have *y* ∗$_R$ (*1*,*s*) ∈ *cone hull* ({*1* :: *real*} × *S*)
using *cone_hull_expl*[*of* {(*1* :: *real*)} × *S*] ‹*s* ∈ *S*› **by** *auto*

```
    then have f y ≠ {}
      using f_def by auto
  }
  then have {y. f y ≠ {}} = {0..}
    using f_def cone_hull_expl[of {1 :: real} × S] by auto
  then have **: rel_interior {y. f y ≠ {}} = {0<..}
    using rel_interior_real_semiline by auto
  {
    fix c :: real
    assume c > 0
    then have f c = ((*R) c ` S)
      using f_def cone_hull_expl[of {1 :: real} × S] by auto
    then have rel_interior (f c) = (*R) c ` rel_interior S
      using rel_interior_convex_scaleR[of S c] assms by auto
  }
  then show ?thesis using * ** by auto
qed

lemma rel_interior_convex_cone:
  fixes S :: 'm::euclidean_space set
  assumes convex S
  shows rel_interior (cone hull ({1 :: real} × S)) =
    {(c, c *R x) | c x. c > 0 ∧ x ∈ rel_interior S}
  (is ?lhs = ?rhs)
proof −
  {
    fix z
    assume z ∈ ?lhs
    have *: z = (fst z, snd z)
      by auto
    then have z ∈ ?rhs
      using rel_interior_convex_cone_aux[of S fst z snd z] assms ‹z ∈ ?lhs› by
fastforce
  }
  moreover
  {
    fix z
    assume z ∈ ?rhs
    then have z ∈ ?lhs
      using rel_interior_convex_cone_aux[of S fst z snd z] assms
      by auto
  }
  ultimately show ?thesis by blast
qed

lemma convex_hull_finite_union:
  assumes finite I
  assumes ∀ i∈I. convex (S i) ∧ (S i) ≠ {}
  shows convex hull (⋃(S ` I)) =
```

```
   {sum (λi. c i *R s i) I | c s. (∀i∈I. c i ≥ 0) ∧ sum c I = 1 ∧ (∀i∈I. s i ∈ S
i)}
  (is ?lhs = ?rhs)
proof −
  have ?lhs ⊇ ?rhs
  proof
    fix x
    assume x ∈ ?rhs
    then obtain c s where *: sum (λi. c i *R s i) I = x sum c I = 1
      (∀i∈I. c i ≥ 0) ∧ (∀i∈I. s i ∈ S i) by auto
    then have ∀i∈I. s i ∈ convex hull (⋃(S ‘ I))
      using hull_subset[of ⋃(S ‘ I) convex] by auto
    then show x ∈ ?lhs
      unfolding *(1)[symmetric]
      using * assms convex_convex_hull
      by (subst convex_sum) auto
  qed
  {
    fix i
    assume i ∈ I
    with assms have ∃p. p ∈ S i by auto
  }
  then obtain p where p: ∀i∈I. p i ∈ S i by metis
  {
    fix i
    assume i ∈ I
    {
      fix x
      assume x ∈ S i
      define c where c j = (if j = i then 1::real else 0) for j
      then have *: sum c I = 1
        using ‹finite I› ‹i ∈ I› sum.delta[of I i λj::'a. 1::real]
        by auto
      define s where s j = (if j = i then x else p j) for j
      then have ∀j. c j *R s j = (if j = i then x else 0)
        using c_def by (auto simp add: algebra_simps)
      then have x = sum (λi. c i *R s i) I
        using s_def c_def ‹finite I› ‹i ∈ I› sum.delta[of I i λj::'a. x]
        by auto
      moreover have (∀i∈I. 0 ≤ c i) ∧ sum c I = 1 ∧ (∀i∈I. s i ∈ S i)
        using * c_def s_def p ‹x ∈ S i› by auto
      ultimately have x ∈ ?rhs
        by force
    }
    then have ?rhs ⊇ S i by auto
  }
  then have *: ?rhs ⊇ ⋃(S ‘ I) by auto

  {
```

```
    fix u v :: real
    assume uv: u ≥ 0 ∧ v ≥ 0 ∧ u + v = 1
    fix x y
    assume xy: x ∈ ?rhs ∧ y ∈ ?rhs
    from xy obtain c s where
      xc: x = sum (λi. c i *R s i) I ∧ (∀i∈I. c i ≥ 0) ∧ sum c I = 1 ∧ (∀i∈I. s
i ∈ S i)
      by auto
    from xy obtain d t where
      yc: y = sum (λi. d i *R t i) I ∧ (∀i∈I. d i ≥ 0) ∧ sum d I = 1 ∧ (∀i∈I. t
i ∈ S i)
      by auto
    define e where e i = u * c i + v * d i for i
    have ge0: ∀i∈I. e i ≥ 0
      using e_def xc yc uv by simp
    have sum (λi. u * c i) I = u * sum c I
      by (simp add: sum_distrib_left)
    moreover have sum (λi. v * d i) I = v * sum d I
      by (simp add: sum_distrib_left)
    ultimately have sum1: sum e I = 1
      using e_def xc yc uv by (simp add: sum.distrib)
    define q where q i = (if e i = 0 then p i else (u * c i / e i) *R s i + (v * d i
/ e i) *R t i)
      for i
    {
      fix i
      assume i: i ∈ I
      have q i ∈ S i
      proof (cases e i = 0)
        case True
        then show ?thesis using i p q_def by auto
      next
        case False
        then show ?thesis
          using mem_convex_alt[of S i s i t i u * (c i) v * (d i)]
            mult_nonneg_nonneg[of u c i] mult_nonneg_nonneg[of v d i]
            assms q_def e_def i False xc yc uv
          by (auto simp del: mult_nonneg_nonneg)
      qed
    }
    then have qs: ∀i∈I. q i ∈ S i by auto
    {
      fix i
      assume i: i ∈ I
      have (u * c i) *R s i + (v * d i) *R t i = e i *R q i
      proof (cases e i = 0)
        case True
        have ge: u * (c i) ≥ 0 ∧ v * d i ≥ 0
          using xc yc uv i by simp
```

```
      moreover from ge have u * c i ≤ 0 ∧ v * d i ≤ 0
        using True e_def i by simp
      ultimately have u * c i = 0 ∧ v * d i = 0 by auto
      with True show ?thesis by auto
    next
      case False
      then have (u * (c i)/(e i))*R (s i)+(v * (d i)/(e i))*R (t i) = q i
        using q_def by auto
      then have e i *R ((u * (c i)/(e i))*R (s i)+(v * (d i)/(e i))*R (t i))
            = (e i) *R (q i) by auto
      with False show ?thesis by (simp add: algebra_simps)
    qed
  }
  then have *: ∀ i∈I. (u * c i) *R s i + (v * d i) *R t i = e i *R q i
    by auto
  have u *R x + v *R y = sum (λi. (u * c i) *R s i + (v * d i) *R t i) I
    using xc yc by (simp add: algebra_simps scaleR_right.sum sum.distrib)
  also have ... = sum (λi. e i *R q i) I
    using * by auto
  finally have u *R x + v *R y = sum (λi. (e i) *R (q i)) I
    by auto
  then have u *R x + v *R y ∈ ?rhs
    using ge0 sum1 qs by auto
 }
 then have convex ?rhs unfolding convex_def by auto
 then show ?thesis
  using ‹?lhs ⊇ ?rhs› * hull_minimal[of ⋃(S ‘ I) ?rhs convex]
  by blast
qed

lemma convex_hull_union_two:
 fixes S T :: 'm::euclidean_space set
 assumes convex S
  and S ≠ {}
  and convex T
  and T ≠ {}
 shows convex hull (S ∪ T) =
  {u *R s + v *R t | u v s t. u ≥ 0 ∧ v ≥ 0 ∧ u + v = 1 ∧ s ∈ S ∧ t ∈ T}
 (is ?lhs = ?rhs)
proof
 define I :: nat set where I = {1, 2}
 define s where s i = (if i = (1::nat) then S else T) for i
 have ⋃(s ‘ I) = S ∪ T
  using s_def I_def by auto
 then have convex hull (⋃(s ‘ I)) = convex hull (S ∪ T)
  by auto
 moreover have convex hull ⋃(s ‘ I) =
  {∑ i∈I. c i *R sa i | c sa. (∀ i∈I. 0 ≤ c i) ∧ sum c I = 1 ∧ (∀ i∈I. sa i ∈ s
i)}
```

```
    using assms s_def I_def
    by (subst convex_hull_finite_union) auto
  moreover have
    {∑ i∈I. c i *R sa i | c sa. (∀ i∈I. 0 ≤ c i) ∧ sum c I = 1 ∧ (∀ i∈I. sa i ∈ s
i)} ≤ ?rhs
    using s_def I_def by auto
  ultimately show ?lhs ⊆ ?rhs by auto
  {
    fix x
    assume x ∈ ?rhs
    then obtain u v s t where *: x = u *R s + v *R t ∧ u ≥ 0 ∧ v ≥ 0 ∧ u +
v = 1 ∧ s ∈ S ∧ t ∈ T
      by auto
    then have x ∈ convex hull {s, t}
      using convex_hull_2[of s t] by auto
    then have x ∈ convex hull (S ∪ T)
      using * hull_mono[of {s, t} S ∪ T] by auto
  }
  then show ?lhs ⊇ ?rhs by blast
qed

proposition ray_to_rel_frontier:
  fixes a :: 'a::real_inner
  assumes bounded S
      and a: a ∈ rel_interior S
      and aff: (a + l) ∈ affine hull S
      and l ≠ 0
  obtains d where 0 < d (a + d *R l) ∈ rel_frontier S
          ⋀e. [[0 ≤ e; e < d]] ⟹ (a + e *R l) ∈ rel_interior S
proof −
  have aaff: a ∈ affine hull S
    by (meson a hull_subset rel_interior_subset rev_subsetD)
  let ?D = {d. 0 < d ∧ a + d *R l ∉ rel_interior S}
  obtain B where B > 0 and B: S ⊆ ball a B
    using bounded_subset_ballD [OF ‹bounded S›] by blast
  have a + (B / norm l) *R l ∉ ball a B
    by (simp add: dist_norm ‹l ≠ 0›)
  with B have a + (B / norm l) *R l ∉ rel_interior S
    using rel_interior_subset subsetCE by blast
  with ‹B > 0› ‹l ≠ 0› have nonMT: ?D ≠ {}
    using divide_pos_pos zero_less_norm_iff by fastforce
  have bdd: bdd_below ?D
    by (metis (no_types, lifting) bdd_belowI le_less mem_Collect_eq)
  have relin_Ex: ⋀x. x ∈ rel_interior S ⟹
                 ∃e>0. ∀x'∈affine hull S. dist x' x < e ⟶ x' ∈ rel_interior S
   using openin_rel_interior [of S] by (simp add: openin_euclidean_subtopology_iff)
  define d where d = Inf ?D
  obtain ε where 0 < ε and ε: ⋀η. [[0 ≤ η; η < ε]] ⟹ (a + η *R l) ∈ rel_interior
S
```

```
  proof −
    obtain e where e>0
          and e: ⋀x'. x' ∈ affine hull S ⟹ dist x' a < e ⟹ x' ∈ rel_interior S
      using relin_Ex a by blast
    show thesis
    proof (rule_tac ε = e / norm l in that)
      show 0 < e / norm l by (simp add: ‹0 < e› ‹l ≠ 0›)
    next
      show a + η *R l ∈ rel_interior S if 0 ≤ η η < e / norm l for η
      proof (rule e)
        show a + η *R l ∈ affine hull S
       by (metis (no_types) add_diff_cancel_left' aff affine_affine_hull mem_affine_3_minus
aaff)
        show dist (a + η *R l) a < e
          using that by (simp add: ‹l ≠ 0› dist_norm pos_less_divide_eq)
      qed
    qed
  qed
  have inint: ⋀e. ⟦0 ≤ e; e < d⟧ ⟹ a + e *R l ∈ rel_interior S
    unfolding d_def using cInf_lower [OF _ bdd]
      by (metis (no_types, lifting) a add.right_neutral le_less mem_Collect_eq
not_less real_vector.scale_zero_left)
  have ε ≤ d
    unfolding d_def
    using ε dual_order.strict_implies_order le_less_linear
    by (blast intro: cInf_greatest [OF nonMT])
  with ‹0 < ε› have 0 < d by simp
  have a + d *R l ∉ rel_interior S
  proof
    assume adl: a + d *R l ∈ rel_interior S
    obtain e where e > 0
     and e: ⋀x'. x' ∈ affine hull S ⟹ dist x' (a + d *R l) < e ⟹ x' ∈ rel_interior
S
      using relin_Ex adl by blast
    have d + e / norm l ≤ x
      if 0 < x and nonrel: a + x *R l ∉ rel_interior S for x
    proof (cases x < d)
      case True with inint nonrel ‹0 < x›
      show ?thesis by auto
    next
      case False
      then have dle: x < d + e / norm l ⟹ dist (a + x *R l) (a + d *R l) < e
        by (simp add: field_simps ‹l ≠ 0›)
      have ain: a + x *R l ∈ affine hull S
      by (metis add_diff_cancel_left' aff affine_affine_hull mem_affine_3_minus
aaff)
      show ?thesis
        using e [OF ain] nonrel dle by force
    qed
```

```
    then
    have d + e / norm l ≤ Inf {d. 0 < d ∧ a + d *R l ∉ rel_interior S}
      by (force simp add: intro: cInf_greatest [OF nonMT])
    then show False
      using ‹0 < e› ‹l ≠ 0› by (simp add: d_def [symmetric] field_simps)
  qed
  moreover
  have ∃y∈S. dist y (a + d *R l) < η if 0 < η for η::real
  proof −
    have 1: a + (d − min d (η / 2 / norm l)) *R l ∈ S
    proof (rule subsetD [OF rel_interior_subset inint])
      show d − min d (η / 2 / norm l) < d
        using ‹l ≠ 0› ‹0 < d› ‹0 < η› by auto
    qed auto
    have norm l * min d (η / (norm l * 2)) ≤ norm l * (η / (norm l * 2))
      by (metis min_def mult_left_mono norm_ge_zero order_refl)
    also have ... < η
      using ‹l ≠ 0› ‹0 < η› by (simp add: field_simps)
    finally have 2: norm l * min d (η / (norm l * 2)) < η .
    show ?thesis
      using 1 2 ‹0 < d› ‹0 < η›
      by (rule_tac x=a + (d − min d (η / 2 / norm l)) *R l in bexI) (auto simp:
algebra_simps)
  qed
  then have a + d *R l ∈ closure S
    by (auto simp: closure_approachable)
  ultimately have infront: a + d *R l ∈ rel_frontier S
    by (simp add: rel_frontier_def)
  show ?thesis
    by (rule that [OF ‹0 < d› infront inint])
qed

corollary ray_to_frontier:
  fixes a :: 'a::euclidean_space
  assumes bounded S
      and a: a ∈ interior S
      and l ≠ 0
  obtains d where 0 < d (a + d *R l) ∈ frontier S
          ⋀e. [[0 ≤ e; e < d]] ⟹ (a + e *R l) ∈ interior S
proof −
  have §: interior S = rel_interior S
    using a rel_interior_nonempty_interior by auto
  then have a ∈ rel_interior S
    using a by simp
  moreover have a + l ∈ affine hull S
    using a affine_hull_nonempty_interior by blast
  ultimately show thesis
   by (metis § ‹bounded S› ‹l ≠ 0› frontier_def ray_to_rel_frontier rel_frontier_def
that)
```

qed

lemma *segment_to_rel_frontier_aux*:
fixes x :: *'a*::*euclidean_space*
assumes *convex S bounded S* **and** x: $x \in$ *rel_interior S* **and** y: $y \in S$ **and** xy: $x \neq y$
obtains z **where** $z \in$ *rel_frontier S* $y \in$ *closed_segment x z*
open_segment x z $\subseteq$ *rel_interior S*
proof −
have $x + (y - x) \in$ *affine hull S*
using *hull_inc* [*OF y*] **by** *auto*
then obtain d **where** $0 < d$ **and** df: $(x + d *_R (y-x)) \in$ *rel_frontier S*
and di: $\bigwedge e.\ [\![0 \leq e;\ e < d]\!] \Longrightarrow (x + e *_R (y-x)) \in$ *rel_interior S*
by (*rule ray_to_rel_frontier* [*OF* ‹*bounded S*› *x*]) (*use xy* **in** *auto*)
show *?thesis*
proof
show $x + d *_R (y - x) \in$ *rel_frontier S*
by (*simp add*: *df*)
next
have *open_segment x y* $\subseteq$ *rel_interior S*
using *rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *x*] *closure_subset y* **by** *blast*
moreover have $x + d *_R (y - x) \in$ *open_segment x y* **if** $d < 1$
using *xy* ‹$0 < d$› *that* **by** (*force simp*: *in_segment algebra_simps*)
ultimately have $1 \leq d$
using *df rel_frontier_def* **by** *fastforce*
moreover have $x = (1\ /\ d) *_R x + ((d - 1)\ /\ d) *_R x$
by (*metis* ‹$0 < d$› *add.commute add_divide_distrib diff_add_cancel divide_self_if less_irrefl scaleR_add_left scaleR_one*)
ultimately show $y \in$ *closed_segment x* $(x + d *_R (y - x))$
unfolding *in_segment*
by (*rule_tac x=1/d* **in** *exI*) (*auto simp*: *algebra_simps*)
next
show *open_segment x* $(x + d *_R (y - x)) \subseteq$ *rel_interior S*
proof (*rule rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *x*])
show $x + d *_R (y - x) \in$ *closure S*
using *df rel_frontier_def* **by** *auto*
qed
qed
qed

lemma *segment_to_rel_frontier*:
fixes x :: *'a*::*euclidean_space*
assumes S: *convex S bounded S* **and** x: $x \in$ *rel_interior S*
and y: $y \in S$ **and** xy: $\neg(x = y \wedge S = \{x\})$
obtains z **where** $z \in$ *rel_frontier S* $y \in$ *closed_segment x z*
open_segment x z $\subseteq$ *rel_interior S*
proof (*cases* x=y)

case *True*
with *xy* **have** $S \neq \{x\}$
by *blast*
with *True* **show** *?thesis*
by (*metis Set.set_insert all_not_in_conv ends_in_segment(1) insert_iff segment_to_rel_frontier_aux*[*OF S x*] *that y*)
next
case *False*
then show *?thesis*
using *segment_to_rel_frontier_aux* [*OF S x y*] *that* **by** *blast*
qed

proposition *rel_frontier_not_sing*:
fixes *a* :: *'a*::*euclidean_space*
assumes *bounded S*
shows *rel_frontier S* $\neq \{a\}$
proof (*cases* $S = \{\}$)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
then obtain *z* **where** $z \in S$
by *blast*
then show *?thesis*
proof (*cases* $S = \{z\}$)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
then obtain *w* **where** $w \in S$ $w \neq z$
using ‹$z \in S$› **by** *blast*
show *?thesis*
proof
assume *rel_frontier S* $= \{a\}$
then consider $w \notin$ *rel_frontier S* | $z \notin$ *rel_frontier S*
using ‹$w \neq z$› **by** *auto*
then show *False*
proof *cases*
case *1*
then have *w*: $w \in$ *rel_interior S*
using ‹$w \in S$› *closure_subset rel_frontier_def* **by** *fastforce*
have $w + (w - z) \in$ *affine hull S*
by (*metis* ‹$w \in S$› ‹$z \in S$› *affine_affine_hull hull_inc mem_affine_3_minus scaleR_one*)
then obtain *e* **where** $0 < e$ $(w + e *_R (w - z)) \in$ *rel_frontier S*
using ‹$w \neq z$› ‹$z \in S$› **by** (*metis assms ray_to_rel_frontier right_minus_eq w*)
moreover obtain *d* **where** $0 < d$ $(w + d *_R (z - w)) \in$ *rel_frontier S*
using *ray_to_rel_frontier* [*OF* ‹*bounded S*› *w*, *of* $1 *_R (z - w)$] ‹$w \neq z$› ‹$z \in S$›
by (*metis add.commute add.right_neutral diff_add_cancel hull_inc*

scaleR_one)
      ultimately have $d *_R (z - w) = e *_R (w - z)$
        using ‹*rel_frontier S = {a}*› **by** *force*
      moreover have $e \neq -d$
        using ‹$0 < e$› ‹$0 < d$› **by** *force*
      ultimately show *False*
        by (*metis* (*no_types*, *lifting*) ‹$w \neq z$› *eq_iff_diff_eq_0 minus_diff_eq real_vector.scale_cancel_right real_vector.scale_minus_right scaleR_left.minus*)
    next
      case *2*
      then have z: $z \in$ *rel_interior S*
        using ‹$z \in S$› *closure_subset rel_frontier_def* **by** *fastforce*
      have $z + (z - w) \in$ *affine hull S*
      by (*metis* ‹$z \in S$› ‹$w \in S$› *affine_affine_hull hull_inc mem_affine_3_minus scaleR_one*)
      then obtain e **where** $0 < e$ $(z + e *_R (z - w)) \in$ *rel_frontier S*
      using ‹$w \neq z$› ‹$w \in S$› **by** (*metis assms ray_to_rel_frontier right_minus_eq z*)
      moreover obtain d **where** $0 < d$ $(z + d *_R (w - z)) \in$ *rel_frontier S*
        using *ray_to_rel_frontier* [*OF* ‹*bounded S*› *z*, *of* $1 *_R (w - z)$] ‹$w \neq z$› ‹$w \in S$›
          by (*metis add.commute add.right_neutral diff_add_cancel hull_inc scaleR_one*)
      ultimately have $d *_R (w - z) = e *_R (z - w)$
        using ‹*rel_frontier S = {a}*› **by** *force*
      moreover have $e \neq -d$
        using ‹$0 < e$› ‹$0 < d$› **by** *force*
      ultimately show *False*
        by (*metis* (*no_types*, *lifting*) ‹$w \neq z$› *eq_iff_diff_eq_0 minus_diff_eq real_vector.scale_cancel_right real_vector.scale_minus_right scaleR_left.minus*)
    qed
  qed
 qed
qed

6.0.5 Convexity on direct sums

lemma *closure_sum*:
 fixes S T :: *'a*::*real_normed_vector set*
 shows *closure S + closure T* $\subseteq$ *closure* $(S + T)$
 unfolding *set_plus_image closure_Times* [*symmetric*] *split_def*
 by (*intro closure_bounded_linear_image_subset bounded_linear_add bounded_linear_fst bounded_linear_snd*)

lemma *fst_snd_linear*: *linear* $(\lambda(x,y).\ x + y)$
 unfolding *linear_iff* **by** (*simp add*: *algebra_simps*)

lemma *rel_interior_sum*:
 fixes S T :: *'n*::*euclidean_space set*

assumes *convex S*
and *convex T*
shows *rel_interior (S + T) = rel_interior S + rel_interior T*
proof −
have *rel_interior S + rel_interior T = (λ(x,y). x + y) ‘ (rel_interior S × rel_interior T)*
by (*simp add*: *set_plus_image*)
also have *... = (λ(x,y). x + y) ‘ rel_interior (S × T)*
using *rel_interior_Times assms* **by** *auto*
also have *... = rel_interior (S + T)*
using *fst_snd_linear convex_Times assms*
rel_interior_convex_linear_image[*of* (*λ*(*x*,*y*). *x* + *y*) *S* × *T*]
by (*auto simp add*: *set_plus_image*)
finally show *?thesis* **..**
qed

lemma *rel_interior_sum_gen*:
fixes *S* :: *'a* ⇒ *'n*::*euclidean_space set*
assumes ⋀*i*. *i*∈*I* ⟹ *convex* (*S i*)
shows *rel_interior* (*sum S I*) = *sum* (*λi*. *rel_interior* (*S i*)) *I*
using *rel_interior_sum rel_interior_sing*[*of 0*] *assms*
by (*subst sum_set_cond_linear*[*of convex*], *auto simp add*: *convex_set_plus*)

lemma *convex_rel_open_direct_sum*:
fixes *S T* :: *'n*::*euclidean_space set*
assumes *convex S*
and *rel_open S*
and *convex T*
and *rel_open T*
shows *convex* (*S* × *T*) ∧ *rel_open* (*S* × *T*)
by (*metis assms convex_Times rel_interior_Times rel_open_def*)

lemma *convex_rel_open_sum*:
fixes *S T* :: *'n*::*euclidean_space set*
assumes *convex S*
and *rel_open S*
and *convex T*
and *rel_open T*
shows *convex* (*S* + *T*) ∧ *rel_open* (*S* + *T*)
by (*metis assms convex_set_plus rel_interior_sum rel_open_def*)

lemma *convex_hull_finite_union_cones*:
assumes *finite I*
and *I* ≠ {}
assumes ⋀*i*. *i*∈*I* ⟹ *convex* (*S i*) ∧ *cone* (*S i*) ∧ *S i* ≠ {}
shows *convex hull* (⋃(*S* ‘ *I*)) = *sum S I*
(**is** *?lhs* = *?rhs*)
proof −
{

```
    fix x
    assume x ∈ ?lhs
    then obtain c xs where
      x: x = sum (λi. c i *R xs i) I ∧ (∀i∈I. c i ≥ 0) ∧ sum c I = 1 ∧ (∀i∈I. xs
i ∈ S i)
      using convex_hull_finite_union[of I S] assms by auto
    define s where s i = c i *R xs i for i
    have ∀i∈I. s i ∈ S i
      using s_def x assms by (simp add: mem_cone)
    moreover have x = sum s I using x s_def by auto
    ultimately have x ∈ ?rhs
      using set_sum_alt[of I S] assms by auto
  }
  moreover
  {
    fix x
    assume x ∈ ?rhs
    then obtain s where x: x = sum s I ∧ (∀i∈I. s i ∈ S i)
      using set_sum_alt[of I S] assms by auto
    define xs where xs i = of_nat(card I) *R s i for i
    then have x = sum (λi. ((1 :: real) / of_nat(card I)) *R xs i) I
      using x assms by auto
    moreover have ∀i∈I. xs i ∈ S i
      using x xs_def assms by (simp add: cone_def)
    moreover have ∀i∈I. (1 :: real) / of_nat (card I) ≥ 0
      by auto
    moreover have sum (λi. (1 :: real) / of_nat (card I)) I = 1
      using assms by auto
    ultimately have x ∈ ?lhs
      using assms
      apply (simp add: convex_hull_finite_union[of I S])
      by (rule_tac x = (λi. 1 / (card I)) in exI) auto
  }
  ultimately show ?thesis by auto
qed

lemma convex_hull_union_cones_two:
  fixes S T :: 'm::euclidean_space set
  assumes convex S
    and cone S
    and S ≠ {}
  assumes convex T
    and cone T
    and T ≠ {}
  shows convex hull (S ∪ T) = S + T
proof −
  define I :: nat set where I = {1, 2}
  define A where A i = (if i = (1::nat) then S else T) for i
  have ⋃(A ‘ I) = S ∪ T
```

```
    using A_def I_def by auto
  then have convex hull (⋃(A ‘ I)) = convex hull (S ∪ T)
    by auto
  moreover have convex hull ⋃(A ‘ I) = sum A I
    using A_def I_def
    by (metis assms convex_hull_finite_union_cones empty_iff finite.emptyI fi-
nite.insertI insertI1)
  moreover have sum A I = S + T
    using A_def I_def by (force simp add: set_plus_def)
  ultimately show ?thesis by auto
qed

lemma rel_interior_convex_hull_union:
  fixes S :: 'a ⇒ 'n::euclidean_space set
  assumes finite I
    and ∀ i∈I. convex (S i) ∧ S i ≠ {}
  shows rel_interior (convex hull (⋃(S ‘ I))) =
    {sum (λi. c i *R s i) I | c s. (∀ i∈I. c i > 0) ∧ sum c I = 1 ∧
      (∀ i∈I. s i ∈ rel_interior(S i))}
  (is ?lhs = ?rhs)
proof (cases I = {})
  case True
  then show ?thesis
    using convex_hull_empty by auto
next
  case False
  define C0 where C0 = convex hull (⋃(S ‘ I))
  have ∀ i∈I. C0 ≥ S i
    unfolding C0_def using hull_subset[of ⋃(S ‘ I)] by auto
  define K0 where K0 = cone hull ({1 :: real} × C0)
  define K where K i = cone hull ({1 :: real} × S i) for i
  have ∀ i∈I. K i ≠ {}
    unfolding K_def using assms
    by (simp add: cone_hull_empty_iff[symmetric])
  have convK: ∀ i∈I. convex (K i)
    unfolding K_def
    by (simp add: assms(2) convex_Times convex_cone_hull)
  have K0 ⊇ K i if  i ∈ I for i
    unfolding K0_def K_def
    by (simp add: Sigma_mono ‹∀ i∈I. S i ⊆ C0› hull_mono that)
  then have K0 ⊇ ⋃(K ‘ I) by auto
  moreover have convex K0
    unfolding K0_def by (simp add: C0_def convex_Times convex_cone_hull)
  ultimately have geq: K0 ⊇ convex hull (⋃(K ‘ I))
    using hull_minimal[of _ K0 convex] by blast
  have ∀ i∈I. K i ⊇ {1 :: real} × S i
    using K_def by (simp add: hull_subset)
  then have ⋃(K ‘ I) ⊇ {1 :: real} × ⋃(S ‘ I)
    by auto
```

```
  then have convex hull ⋃(K ‘ I) ⊇ convex hull ({1 :: real} × ⋃(S ‘ I))
    by (simp add: hull_mono)
  then have convex hull ⋃(K ‘ I) ⊇ {1 :: real} × C0
    unfolding C0_def
    using convex_hull_Times[of {(1 :: real)} ⋃(S ‘ I)] convex_hull_singleton
    by auto
  moreover have cone (convex hull (⋃(K ‘ I)))
    by (simp add: K_def cone_Union cone_cone_hull cone_convex_hull)
  ultimately have convex hull (⋃(K ‘ I)) ⊇ K0
    unfolding K0_def
    using hull_minimal[of _ convex hull (⋃(K ‘ I)) cone]
    by blast
  then have K0 = convex hull (⋃(K ‘ I))
    using geq by auto
  also have ... = sum K I
    using assms False ‹∀ i∈I. K i ≠ {}› cone_hull_eq convK
    by (intro convex_hull_finite_union_cones; fastforce simp: K_def)
  finally have K0 = sum K I by auto
  then have ∗: rel_interior K0 = sum (λi. (rel_interior (K i))) I
    using rel_interior_sum_gen[of I K] convK by auto
  {
    fix x
    assume x ∈ ?lhs
    then have (1::real, x) ∈ rel_interior K0
      using K0_def C0_def rel_interior_convex_cone_aux[of C0 1::real x] con-
vex_convex_hull
      by auto
    then obtain k where k: (1::real, x) = sum k I ∧ (∀ i∈I. k i ∈ rel_interior
(K i))
      using ‹finite I› ∗ set_sum_alt[of I λi. rel_interior (K i)] by auto
    {
      fix i
      assume i ∈ I
      then have convex (S i) ∧ k i ∈ rel_interior (cone hull {1} × S i)
        using k K_def assms by auto
      then have ∃ ci si. k i = (ci, ci ∗R si) ∧ 0 < ci ∧ si ∈ rel_interior (S i)
        using rel_interior_convex_cone[of S i] by auto
    }
    then obtain c s where cs: ∀ i∈I. k i = (c i, c i ∗R s i) ∧ 0 < c i ∧ s i ∈
rel_interior (S i)
      by metis
    then have x = (∑ i∈I. c i ∗R s i) ∧ sum c I = 1
      using k by (simp add: sum_prod)
    then have x ∈ ?rhs
      using k cs by auto
  }
  moreover
  {
    fix x
```

```
    assume x ∈ ?rhs
    then obtain c s where cs: x = sum (λi. c i *R s i) I ∧
        (∀ i∈I. c i > 0) ∧ sum c I = 1 ∧ (∀ i∈I. s i ∈ rel_interior (S i))
      by auto
    define k where k i = (c i, c i *R s i) for i
    {
      fix i assume i ∈ I
      then have k i ∈ rel_interior (K i)
        using k_def K_def assms cs rel_interior_convex_cone[of S i]
        by auto
    }
    then have (1, x) ∈ rel_interior K0
      using * set_sum_alt[of I (λi. rel_interior (K i))] assms cs
      by (simp add: k_def) (metis (mono_tags, lifting) sum_prod)
    then have x ∈ ?lhs
      using K0_def C0_def rel_interior_convex_cone_aux[of C0 1 x]
      by auto
  }
  ultimately show ?thesis by blast
qed


lemma convex_le_Inf_differential:
  fixes f :: real ⇒ real
  assumes convex_on I f
    and x ∈ interior I
    and y ∈ I
  shows f y ≥ f x + Inf ((λt. (f x − f t) / (x − t)) ` ({x<..} ∩ I)) * (y − x)
  (is _ ≥ _ + Inf (?F x) * (y − x))
proof (cases rule: linorder_cases)
  assume x < y
  moreover
  have open (interior I) by auto
  from openE[OF this ‹x ∈ interior I›]
  obtain e where e: 0 < e ball x e ⊆ interior I .
  moreover define t where t = min (x + e / 2) ((x + y) / 2)
  ultimately have x < t t < y t ∈ ball x e
    by (auto simp: dist_real_def field_simps split: split_min)
  with ‹x ∈ interior I› e interior_subset[of I] have t ∈ I x ∈ I by auto

  define K where K = x − e / 2
  with ‹0 < e› have K ∈ ball x e K < x
    by (auto simp: dist_real_def)
  then have K ∈ I
    using ‹interior I ⊆ I› e(2) by blast

  have Inf (?F x) ≤ (f x − f y) / (x − y)
  proof (intro bdd_belowI cInf_lower2)
    show (f x − f t) / (x − t) ∈ ?F x
```

```
      using ‹t ∈ I› ‹x < t› by auto
    show (f x − f t) / (x − t) ≤ (f x − f y) / (x − y)
      using ‹convex_on I f› ‹x ∈ I› ‹y ∈ I› ‹x < t› ‹t < y›
      by (rule convex_on_slope_le)
  next
    fix y
    assume y ∈ ?F x
    with order_trans[OF convex_on_slope_le[OF ‹convex_on I f› ‹K ∈ I› _ ‹K
< x› _]]
    show (f K − f x) / (K − x) ≤ y by auto
  qed
  then show ?thesis
    using ‹x < y› by (simp add: field_simps)
next
  assume y < x
  moreover
  have open (interior I) by auto
  from openE[OF this ‹x ∈ interior I›]
  obtain e where e: 0 < e ball x e ⊆ interior I .
  moreover define t where t = x + e / 2
  ultimately have x < t t ∈ ball x e
    by (auto simp: dist_real_def field_simps)
  with ‹x ∈ interior I› e interior_subset[of I] have t ∈ I x ∈ I by auto

  have (f x − f y) / (x − y) ≤ Inf (?F x)
  proof (rule cInf_greatest)
    have (f x − f y) / (x − y) = (f y − f x) / (y − x)
      using ‹y < x› by (auto simp: field_simps)
    also
    fix z
    assume z ∈ ?F x
    with order_trans[OF convex_on_slope_le[OF ‹convex_on I f› ‹y ∈ I› _ ‹y
< x›]]
    have (f y − f x) / (y − x) ≤ z
      by auto
    finally show (f x − f y) / (x − y) ≤ z .
  next
    have x + e / 2 ∈ ball x e
      using e by (auto simp: dist_real_def)
    with e interior_subset[of I] have x + e / 2 ∈ {x<..} ∩ I
      by auto
    then show ?F x ≠ {}
      by blast
  qed
  then show ?thesis
    using ‹y < x› by (simp add: field_simps)
qed simp
```

6.0.6 Explicit formulas for interior and relative interior of convex hull

lemma *at_within_cbox_finite*:
assumes $x \in box\ a\ b\ x \notin S$ *finite* S
shows $(at\ x\ within\ cbox\ a\ b - S) = at\ x$
proof −
have $interior\ (cbox\ a\ b - S) = box\ a\ b - S$
using ‹*finite* S› **by** (*simp add*: *interior_diff finite_imp_closed*)
then show *?thesis*
using *at_within_interior assms* **by** *fastforce*
qed

lemma *affine_independent_convex_affine_hull*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $\neg\ affine\_dependent\ S\ T \subseteq S$
shows $convex\ hull\ T = affine\ hull\ T \cap convex\ hull\ S$
proof −
have *fin*: *finite* S *finite* T **using** *assms aff_independent_finite finite_subset* **by** *auto*
have $convex\ hull\ T \subseteq affine\ hull\ T$
using *convex_hull_subset_affine_hull* **by** *blast*
moreover have $convex\ hull\ T \subseteq convex\ hull\ S$
using *assms hull_mono* **by** *blast*
moreover have $affine\ hull\ T \cap convex\ hull\ S \subseteq convex\ hull\ T$
proof −
have *0*: $\bigwedge u.\ sum\ u\ S = 0 \Longrightarrow (\forall v{\in}S.\ u\ v = 0) \vee (\sum v{\in}S.\ u\ v *_R v) \neq 0$
using *affine_dependent_explicit_finite assms*(*1*) *fin*(*1*) **by** *auto*
show *?thesis*
proof (*clarsimp simp add*: *affine_hull_finite fin*)
fix u
assume S: $(\sum v{\in}T.\ u\ v *_R v) \in convex\ hull\ S$
and *T1*: $sum\ u\ T = 1$
then obtain v **where** v: $\forall x{\in}S.\ 0 \leq v\ x\ sum\ v\ S = 1\ (\sum x{\in}S.\ v\ x *_R x) = (\sum v{\in}T.\ u\ v *_R v)$
by (*auto simp add*: *convex_hull_finite fin*)
{ fix x
assume$x \in T$
then have S: $S = (S - T) \cup T$ — split into separate cases
using *assms* **by** *auto*
have [*simp*]: $(\sum x{\in}T.\ v\ x *_R x) + (\sum x{\in}S - T.\ v\ x *_R x) = (\sum x{\in}T.\ u\ x *_R x)$
$sum\ v\ T + sum\ v\ (S - T) = 1$
using *v fin S*
by (*auto simp*: *sum.union_disjoint* [*symmetric*] *Un_commute*)
have $(\sum x{\in}S.\ if\ x \in T\ then\ v\ x - u\ x\ else\ v\ x) = 0$
$(\sum x{\in}S.\ (if\ x \in T\ then\ v\ x - u\ x\ else\ v\ x) *_R x) = 0$
using *v fin T1*
by (*subst S*, *subst sum.union_disjoint*, *auto simp*: *algebra_simps sum_subtractf*)+
} note [*simp*] = *this*

```
    have (∀ x∈T. 0 ≤ u x)
     using 0 [of λx. if x ∈ T then v x − u x else v x] ‹T ⊆ S› v(1) by fastforce
    then show (∑ v∈T. u v *R v) ∈ convex hull T
     using 0 [of λx. if x ∈ T then v x − u x else v x] ‹T ⊆ S› T1
     by (fastforce simp add: convex_hull_finite fin)
   qed
  qed
  ultimately show ?thesis
   by blast
qed

lemma affine_independent_span_eq:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S card S = Suc (DIM ('a))
   shows affine hull S = UNIV
proof (cases S = {})
  case True then show ?thesis
   using assms by simp
next
  case False
   then obtain a T where T: a ∉ T S = insert a T
    by blast
   then have fin: finite T using assms
    by (metis finite_insert aff_independent_finite)
   have UNIV ⊆ (+) a ‘ span ((λx. x − a) ‘ T)
   proof (intro card_ge_dim_independent Fun.vimage_subsetD)
    show independent ((λx. x − a) ‘ T)
     using T affine_dependent_iff_dependent assms(1) by auto
    show dim ((+) a −‘ UNIV) ≤ card ((λx. x − a) ‘ T)
     using assms T fin by (auto simp: card_image inj_on_def)
   qed (use surj_plus in auto)
   then show ?thesis
    using T(2) affine_hull_insert_span_gen equalityI by fastforce
qed

lemma affine_independent_span_gt:
  fixes S :: 'a::euclidean_space set
  assumes ind: ¬ affine_dependent S and dim: DIM ('a) < card S
   shows affine hull S = UNIV
proof (intro affine_independent_span_eq [OF ind] antisym)
  show card S ≤ Suc DIM('a)
   using aff_independent_finite affine_dependent_biggerset ind by fastforce
  show Suc DIM('a) ≤ card S
   using Suc_leI dim by blast
qed

lemma empty_interior_affine_hull:
  fixes S :: 'a::euclidean_space set
  assumes finite S and dim: card S ≤ DIM ('a)
```

```
    shows interior(affine hull S) = {}
  using assms
proof (induct S rule: finite_induct)
  case (insert x S)
  then have dim (span ((λy. y − x) ` S)) < DIM('a)
   by (auto simp: Suc_le_lessD card_image_le dual_order.trans intro!: dim_le_card'[THEN
le_less_trans])
  then show ?case
   by (simp add: empty_interior_lowdim affine_hull_insert_span_gen interior_translation)
qed auto

lemma empty_interior_convex_hull:
  fixes S :: 'a::euclidean_space set
  assumes finite S and dim: card S ≤ DIM ('a)
    shows interior(convex hull S) = {}
  by (metis Diff_empty Diff_eq_empty_iff convex_hull_subset_affine_hull
          interior_mono empty_interior_affine_hull [OF assms])

lemma explicit_subset_rel_interior_convex_hull:
  fixes S :: 'a::euclidean_space set
  shows finite S
        ⟹ {y. ∃u. (∀x ∈ S. 0 < u x ∧ u x < 1) ∧ sum u S = 1 ∧ sum (λx. u x
*R x) S = y}
           ⊆ rel_interior (convex hull S)
  by (force simp add:  rel_interior_convex_hull_union [where S=λx. {x} and
I=S, simplified])

lemma explicit_subset_rel_interior_convex_hull_minimal:
  fixes S :: 'a::euclidean_space set
  shows finite S
        ⟹ {y. ∃u. (∀x ∈ S. 0 < u x) ∧ sum u S = 1 ∧ sum (λx. u x *R x) S =
y}
           ⊆ rel_interior (convex hull S)
  by (force simp add:  rel_interior_convex_hull_union [where S=λx. {x} and
I=S, simplified])

lemma rel_interior_convex_hull_explicit:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
  shows rel_interior(convex hull S) =
        {y. ∃u. (∀x ∈ S. 0 < u x) ∧ sum u S = 1 ∧ sum (λx. u x *R x) S = y}
        (is ?lhs = ?rhs)
proof
  show ?rhs ≤ ?lhs
   by (simp add: aff_independent_finite explicit_subset_rel_interior_convex_hull_minimal
assms)
next
  show ?lhs ≤ ?rhs
  proof (cases ∃a. S = {a})
```

```
    case True then show ?lhs ≤ ?rhs
      by force
  next
    case False
    have fs: finite S
      using assms by (simp add: aff_independent_finite)
    { fix a b and d::real
      assume ab: a ∈ S b ∈ S a ≠ b
      then have S: S = (S - {a,b}) ∪ {a,b} — split into separate cases
        by auto
      have (∑ x∈S. if x = a then - d else if x = b then d else 0) = 0
          (∑ x∈S. (if x = a then - d else if x = b then d else 0) *R x) = d *R b
- d *R a
        using ab fs
        by (subst S, subst sum.union_disjoint, auto)+
    } note [simp] = this
    { fix y
      assume y: y ∈ convex hull S y ∉ ?rhs
      have *: False if
        ua: ∀ x∈S. 0 ≤ u x sum u S = 1 ¬ 0 < u a a ∈ S
        and yT: y = (∑ x∈S. u x *R x) y ∈ T open T
        and sb: T ∩ affine hull S ⊆ {w. ∃ u. (∀ x∈S. 0 ≤ u x) ∧ sum u S = 1 ∧
(∑ x∈S. u x *R x) = w}
      for u T a
      proof -
        have ua0: u a = 0
          using ua by auto
        obtain b where b: b∈S a ≠ b
          using ua False by auto
        obtain e where e: 0 < e ball (∑ x∈S. u x *R x) e ⊆ T
          using yT by (auto elim: openE)
        with b obtain d where d: 0 < d norm(d *R (a-b)) < e
          by (auto intro: that [of e / 2 / norm(a-b)])
        have (∑ x∈S. u x *R x) ∈ affine hull S
          using yT y by (metis affine_hull_convex_hull hull_redundant_eq)
        then have (∑ x∈S. u x *R x) - d *R (a - b) ∈ affine hull S
          using ua b by (auto simp: hull_inc intro: mem_affine_3_minus2)
        then have y - d *R (a - b) ∈ T ∩ affine hull S
          using d e yT by auto
        then obtain v where v: ∀ x∈S. 0 ≤ v x
          sum v S = 1
          (∑ x∈S. v x *R x) = (∑ x∈S. u x *R x) - d *R (a - b)
          using subsetD [OF sb] yT
          by auto
        have aff: ⋀u. sum u S = 0 ⟹ (∀ v∈S. u v = 0) ∨ (∑ v∈S. u v *R v) ≠ 0
          using assms by (simp add: affine_dependent_explicit_finite fs)
        show False
          using ua b d v aff [of λx. (v x - u x) - (if x = a then -d else if x = b
then d else 0)]
```

```
        by (auto simp: algebra_simps sum_subtractf sum.distrib)
      qed
      have y ∉ rel_interior (convex hull S)
        using y convex_hull_finite [OF fs] *
        apply simp
      by (metis (no_types, lifting) IntD1 affine_hull_convex_hull mem_rel_interior)
    } with rel_interior_subset show ?lhs ≤ ?rhs
      by blast
  qed
qed

lemma interior_convex_hull_explicit_minimal:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
  shows
   interior(convex hull S) =
            (if card(S) ≤ DIM('a) then {}
             else {y. ∃ u. (∀ x ∈ S. 0 < u x) ∧ sum u S = 1 ∧ (∑ x∈S. u x *R x)
= y})
   (is _ = (if _ then _ else ?rhs))
proof −
  { assume S: ¬ card S ≤ DIM('a)
    have interior (convex hull S) = rel_interior(convex hull S)
    using assms S by (simp add: affine_independent_span_gt rel_interior_interior)
    then have interior(convex hull S) = ?rhs
      by (simp add: assms S rel_interior_convex_hull_explicit)
  }
  then show ?thesis
    by (auto simp: aff_independent_finite empty_interior_convex_hull assms)
qed

lemma interior_convex_hull_explicit:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
  shows
   interior(convex hull S) =
            (if card(S) ≤ DIM('a) then {}
             else {y. ∃ u. (∀ x ∈ S. 0 < u x ∧ u x < 1) ∧ sum u S = 1 ∧ (∑ x∈S.
u x *R x) = y})
proof −
  { fix u :: 'a ⇒ real and a
    assume card Basis < card S and u: ⋀x. x∈S ⟹ 0 < u x sum u S = 1 and
a: a ∈ S
    then have cs: Suc 0 < card S
      by (metis DIM_positive less_trans_Suc)
    obtain b where b: b ∈ S a ≠ b
    proof (cases S ≤ {a})
      case True
      then show thesis
```

```
      using cs subset_singletonD by fastforce
    qed blast
    have u a + u b ≤ sum u {a,b}
      using a b by simp
    also have ... ≤ sum u S
      using a b u
     by (intro Groups_Big.sum_mono2) (auto simp: less_imp_le aff_independent_finite
assms)
    finally have u a < 1
      using ‹b ∈ S› u by fastforce
  } note [simp] = this
  show ?thesis
   using assms by (force simp add: not_le interior_convex_hull_explicit_minimal)
qed

lemma interior_closed_segment_ge2:
  fixes a :: 'a::euclidean_space
  assumes 2 ≤ DIM('a)
    shows  interior(closed_segment a b) = {}
using assms unfolding segment_convex_hull
proof −
  have card {a, b} ≤ DIM('a)
    using assms
    by (simp add: card_insert_if linear not_less_eq_eq numeral_2_eq_2)
  then show interior (convex hull {a, b}) = {}
    by (metis empty_interior_convex_hull finite.insertI finite.emptyI)
qed

lemma interior_open_segment:
  fixes a :: 'a::euclidean_space
  shows  interior(open_segment a b) =
               (if 2 ≤ DIM('a) then {} else open_segment a b)
proof (cases 2 ≤ DIM('a))
  case True
  then have interior (open_segment a b) = {}
   using interior_closed_segment_ge2 interior_mono segment_open_subset_closed
by blast
  with True show ?thesis
    by auto
next
  case ge2: False
  have interior (open_segment a b) = open_segment a b
  proof (cases a = b)
    case True then show ?thesis by auto
  next
    case False
    with ge2 have affine hull (open_segment a b) = UNIV
      by (simp add: False affine_independent_span_gt)
    then show interior (open_segment a b) = open_segment a b
```

```
    using rel_interior_interior rel_interior_open_segment by blast
  qed
  with ge2 show ?thesis
    by auto
qed

lemma interior_closed_segment:
  fixes a :: 'a::euclidean_space
  shows interior(closed_segment a b) =
                (if 2 ≤ DIM('a) then {} else open_segment a b)
proof (cases a = b)
  case True then show ?thesis by simp
next
  case False
  then have closure (open_segment a b) = closed_segment a b
    by simp
  then show ?thesis
    by (metis (no_types) convex_interior_closure convex_open_segment inte-
rior_open_segment)
qed

lemmas interior_segment = interior_closed_segment interior_open_segment

lemma closed_segment_eq [simp]:
  fixes a :: 'a::euclidean_space
  shows closed_segment a b = closed_segment c d ⟷ {a,b} = {c,d}
proof
  assume abcd: closed_segment a b = closed_segment c d
  show {a,b} = {c,d}
  proof (cases a=b ∨ c=d)
    case True with abcd show ?thesis by force
  next
    case False
    then have neq: a ≠ b ∧ c ≠ d by force
    have ∗: closed_segment c d − {a, b} = rel_interior (closed_segment c d)
    using neq abcd by (metis (no_types) open_segment_def rel_interior_closed_segment)
    have b ∈ {c, d}
    proof −
      have insert b (closed_segment c d) = closed_segment c d
        using abcd by blast
      then show ?thesis
      by (metis DiffD2 Diff_insert2 False ∗ insertI1 insert_Diff_if open_segment_def
rel_interior_closed_segment)
    qed
    moreover have a ∈ {c, d}
    by (metis Diff_iff False ∗ abcd ends_in_segment(1) insertI1 open_segment_def
rel_interior_closed_segment)
    ultimately show {a, b} = {c, d}
      using neq by fastforce
```

qed
next
assume $\{a,b\} = \{c,d\}$
then show *closed_segment a b = closed_segment c d*
by (*simp add*: *segment_convex_hull*)
qed

lemma *closed_open_segment_eq* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows *closed_segment a b ≠ open_segment c d*
by (*metis DiffE closed_segment_neq_empty closure_closed_segment closure_open_segment ends_in_segment*(*1*) *insertI1 open_segment_def*)

lemma *open_closed_segment_eq* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows *open_segment a b ≠ closed_segment c d*
using *closed_open_segment_eq* **by** *blast*

lemma *open_segment_eq* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows *open_segment a b = open_segment c d* ⟷ *a = b* ∧ *c = d* ∨ $\{a,b\} = \{c,d\}$
(**is** *?lhs = ?rhs*)
proof
assume *abcd*: *?lhs*
show *?rhs*
proof (*cases a=b* ∨ *c=d*)
case *True* **with** *abcd* **show** *?thesis*
using *finite_open_segment* **by** *fastforce*
next
case *False*
then have *a2*: *a* ≠ *b* ∧ *c* ≠ *d* **by** *force*
with *abcd* **show** *?rhs*
unfolding *open_segment_def*
by (*metis* (*no_types*) *abcd closed_segment_eq closure_open_segment*)
qed
next
assume *?rhs*
then show *?lhs*
by (*metis Diff_cancel convex_hull_singleton insert_absorb2 open_segment_def segment_convex_hull*)
qed

6.0.7 Similar results for closure and (relative or absolute) frontier

lemma *closure_convex_hull* [*simp*]:
fixes *S* :: *'a*::*euclidean_space set*
shows *compact S* ==> *closure*(*convex hull S*) = *convex hull S*

by (*simp add*: *compact_imp_closed compact_convex_hull*)

lemma *rel_frontier_convex_hull_explicit*:
fixes S :: *'a*::*euclidean_space set*
assumes ¬ *affine_dependent S*
shows *rel_frontier*(*convex hull S*) =
$\{y.\ \exists u.\ (\forall x \in S.\ 0 \le u\ x) \wedge (\exists x \in S.\ u\ x = 0) \wedge sum\ u\ S = 1 \wedge sum\ (\lambda x.\ u\ x *_R x)\ S = y\}$
proof −
have *fs*: *finite S*
using *assms* **by** (*simp add*: *aff_independent_finite*)
have $\bigwedge u\ y\ v.$
$[\![y \in S;\ u\ y = 0;\ sum\ u\ S = 1;\ \forall x{\in}S.\ 0 < v\ x;$
$sum\ v\ S = 1;\ (\sum x{\in}S.\ v\ x *_R x) = (\sum x{\in}S.\ u\ x *_R x)]\!]$
$\Longrightarrow \exists u.\ sum\ u\ S = 0 \wedge (\exists v{\in}S.\ u\ v \neq 0) \wedge (\sum v{\in}S.\ u\ v *_R v) = 0$
apply (*rule_tac* $x = \lambda x.\ u\ x - v\ x$ **in** *exI*)
apply (*force simp*: *sum_subtractf scaleR_diff_left*)
done
then show *?thesis*
using *fs assms*
apply (*simp add*: *rel_frontier_def finite_imp_compact rel_interior_convex_hull_explicit*)
apply (*auto simp*: *convex_hull_finite*)
apply (*metis less_eq_real_def*)
by (*simp add*: *affine_dependent_explicit_finite*)
qed

lemma *frontier_convex_hull_explicit*:
fixes S :: *'a*::*euclidean_space set*
assumes ¬ *affine_dependent S*
shows *frontier*(*convex hull S*) =
$\{y.\ \exists u.\ (\forall x \in S.\ 0 \le u\ x) \wedge (DIM\ ('a) < card\ S \longrightarrow (\exists x \in S.\ u\ x = 0)) \wedge$
$sum\ u\ S = 1 \wedge sum\ (\lambda x.\ u\ x *_R x)\ S = y\}$
proof −
have *fs*: *finite S*
using *assms* **by** (*simp add*: *aff_independent_finite*)
show *?thesis*
proof (*cases DIM* (*'a*) < *card S*)
case *True*
with *assms fs* **show** *?thesis*
by (*simp add*: *rel_frontier_def frontier_def rel_frontier_convex_hull_explicit* [*symmetric*]
interior_convex_hull_explicit_minimal rel_interior_convex_hull_explicit)
next
case *False*
then have *card S* ≤ *DIM* (*'a*)
by *linarith*
then show *?thesis*
using *assms fs*

```
    apply (simp add: frontier_def interior_convex_hull_explicit finite_imp_compact)
     apply (simp add: convex_hull_finite)
     done
  qed
qed

lemma rel_frontier_convex_hull_cases:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
  shows rel_frontier(convex hull S) = ⋃{convex hull (S − {x}) |x. x ∈ S}
proof −
  have fs: finite S
    using assms by (simp add: aff_independent_finite)
  { fix u a
  have ∀ x∈S. 0 ≤ u x ⟹ a ∈ S ⟹ u a = 0 ⟹ sum u S = 1 ⟹
          ∃ x v. x ∈ S ∧
               (∀ x∈S − {x}. 0 ≤ v x) ∧
                  sum v (S − {x}) = 1 ∧ (∑ x∈S − {x}. v x *R x) = (∑ x∈S. u
x *R x)
    apply (rule_tac x=a in exI)
    apply (rule_tac x=u in exI)
    apply (simp add: Groups_Big.sum_diff1 fs)
    done }
  moreover
  { fix a u
    have a ∈ S ⟹ ∀ x∈S − {a}. 0 ≤ u x ⟹ sum u (S − {a}) = 1 ⟹
          ∃ v. (∀ x∈S. 0 ≤ v x) ∧
              (∃ x∈S. v x = 0) ∧ sum v S = 1 ∧ (∑ x∈S. v x *R x) = (∑ x∈S −
{a}. u x *R x)
    apply (rule_tac x=λx. if x = a then 0 else u x in exI)
    apply (auto simp: sum.If_cases Diff_eq if_smult fs)
    done }
  ultimately show ?thesis
    using assms
    apply (simp add: rel_frontier_convex_hull_explicit)
    apply (auto simp add: convex_hull_finite fs Union_SetCompr_eq)
    done
qed

lemma frontier_convex_hull_eq_rel_frontier:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
  shows frontier(convex hull S) =
        (if card S ≤ DIM ('a) then convex hull S else rel_frontier(convex hull S))
  using assms
  unfolding rel_frontier_def frontier_def
  by (simp add: affine_independent_span_gt rel_interior_interior
          finite_imp_compact empty_interior_convex_hull aff_independent_finite)
```

lemma *frontier_convex_hull_cases*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $\neg$ *affine_dependent* S
shows *frontier*(*convex hull* S) =
(*if card* $S \leq DIM\ ('a)$ *then convex hull* S *else* $\bigcup \{$*convex hull* $(S - \{x\})$
$|x.\ x \in S\}$)
by (*simp add*: *assms frontier_convex_hull_eq_rel_frontier rel_frontier_convex_hull_cases*)

lemma *in_frontier_convex_hull*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *finite* S *card* $S \leq Suc\ (DIM\ ('a))$ $x \in S$
shows $x \in$ *frontier*(*convex hull* S)
proof (*cases affine_dependent* S)
case *True*
with *assms* **obtain** y **where** $y \in S$ **and** y: $y \in$ *affine hull* $(S - \{y\})$
by (*auto simp*: *affine_dependent_def*)
moreover have $x \in$ *closure* (*convex hull* S)
by (*meson closure_subset hull_inc subset_eq* ‹$x \in S$›)
moreover have $x \notin$ *interior* (*convex hull* S)
using *assms*
by (*metis Suc_mono affine_hull_convex_hull affine_hull_nonempty_interior*
‹$y \in S$› y *card.remove empty_iff empty_interior_affine_hull finite_Diff hull_redundant*
insert_Diff interior_UNIV not_less)
ultimately show *?thesis*
unfolding *frontier_def* **by** *blast*
next
case *False*
{ assume *card* S = *Suc* (*card Basis*)
then have *cs*: *Suc* 0 < *card* S
by (*simp*)
with *subset_singletonD* **have** $\exists y \in S.\ y \neq x$
by (*cases* $S \leq \{x\}$) *fastforce*+
} note [*dest*!] = *this*
show *?thesis* **using** *assms*
unfolding *frontier_convex_hull_cases* [*OF False*] *Union_SetCompr_eq*
by (*auto simp*: *le_Suc_eq hull_inc*)
qed

lemma *not_in_interior_convex_hull*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *finite* S *card* $S \leq Suc\ (DIM\ ('a))$ $x \in S$
shows $x \notin$ *interior*(*convex hull* S)
using *in_frontier_convex_hull* [*OF assms*]
by (*metis Diff_iff frontier_def*)

lemma *interior_convex_hull_eq_empty*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *card* S = *Suc* (*DIM* ('*a*))
shows *interior*(*convex hull* S) = {} $\longleftrightarrow$ *affine_dependent* S

proof
show *affine_dependent S* $\Longrightarrow$ *interior* (*convex hull S*) = {}
proof (*clarsimp simp*: *affine_dependent_def*)
fix *a b*
assume *b* ∈ *S b* ∈ *affine hull* (*S* − {*b*})
then have *interior*(*affine hull S*) = {} **using** *assms*
by (*metis DIM_positive One_nat_def Suc_mono card.remove card.infinite empty_interior_affine_hull eq_iff hull_redundant insert_Diff not_less zero_le_one*)
then show *interior* (*convex hull S*) = {}
using *affine_hull_nonempty_interior* **by** *fastforce*
qed
next
show *interior* (*convex hull S*) = {} $\Longrightarrow$ *affine_dependent S*
by (*metis affine_hull_convex_hull affine_hull_empty affine_independent_span_eq assms convex_convex_hull empty_not_UNIV rel_interior_eq_empty rel_interior_interior*)
qed

6.0.8 Coplanarity, and collinearity in terms of affine hull

definition *coplanar* **where**
coplanar S ≡ ∃ *u v w*. *S* ⊆ *affine hull* {*u*,*v*,*w*}

lemma *collinear_affine_hull*:
collinear S ⟷ (∃ *u v*. *S* ⊆ *affine hull* {*u*,*v*})
proof (*cases S*={})
case *True* **then show** *?thesis*
by *simp*
next
case *False*
then obtain *x* **where** *x*: *x* ∈ *S* **by** *auto*
{ fix *u*
assume ∗: ⋀*x y*. ⟦*x*∈*S*; *y*∈*S*⟧ $\Longrightarrow$ ∃ *c*. *x* − *y* = *c* ∗*R u*
have ⋀*y c*. *x* − *y* = *c* ∗*R u* $\Longrightarrow$ ∃ *a b*. *y* = *a* ∗*R x* + *b* ∗*R* (*x* + *u*) ∧ *a* + *b* = *1*
by (*rule_tac x*=*1*+*c* **in** *exI*, *rule_tac x*=−*c* **in** *exI*, *simp add*: *algebra_simps*)
then have ∃ *u v*. *S* ⊆ {*a* ∗*R u* + *b* ∗*R v* |*a b*. *a* + *b* = *1*}
using ∗ [*OF x*] **by** (*rule_tac x*=*x* **in** *exI*, *rule_tac x*=*x*+*u* **in** *exI*, *force*)
} moreover
{ fix *u v x y*
assume ∗: *S* ⊆ {*a* ∗*R u* + *b* ∗*R v* |*a b*. *a* + *b* = *1*}
have ∃ *c*. *x* − *y* = *c* ∗*R* (*v*−*u*) **if** *x*∈*S y*∈*S*
proof −
obtain *a r* **where** *a* + *r* = *1 x* = *a* ∗*R u* + *r* ∗*R v*
using ∗ ‹*x* ∈ *S*› **by** *blast*
moreover
obtain *b s* **where** *b* + *s* = *1 y* = *b* ∗*R u* + *s* ∗*R v*
using ∗ ‹*y* ∈ *S*› **by** *blast*
ultimately have *x* − *y* = (*r*−*s*) ∗*R* (*v*−*u*)
by (*simp add*: *algebra_simps*) (*metis scaleR_left.add*)
then show *?thesis*

```
      by blast
    qed
  } ultimately
  show ?thesis
  unfolding collinear_def affine_hull_2
    by blast
qed

lemma collinear_closed_segment [simp]: collinear (closed_segment a b)
  by (metis affine_hull_convex_hull collinear_affine_hull hull_subset segment_convex_hull)

lemma collinear_open_segment [simp]: collinear (open_segment a b)
  unfolding open_segment_def
  by (metis convex_hull_subset_affine_hull segment_convex_hull dual_order.trans
      convex_hull_subset_affine_hull Diff_subset collinear_affine_hull)

lemma collinear_between_cases:
  fixes c :: 'a::euclidean_space
  shows collinear {a,b,c} ⟷ between (b,c) a ∨ between (c,a) b ∨ between (a,b) c
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain u v where uv: ⋀x. x ∈ {a, b, c} ⟹ ∃c. x = u + c *R v
    by (auto simp: collinear_alt)
  show ?rhs
    using uv [of a] uv [of b] uv [of c] by (auto simp: between_1)
next
  assume ?rhs
  then show ?lhs
    unfolding between_mem_convex_hull
   by (metis (no_types, opaque_lifting) collinear_closed_segment collinear_subset
hull_redundant hull_subset insert_commute segment_convex_hull)
qed


lemma subset_continuous_image_segment_1:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes continuous_on (closed_segment a b) f
  shows closed_segment (f a) (f b) ⊆ image f (closed_segment a b)
by (metis connected_segment convex_contains_segment ends_in_segment imageI
        is_interval_connected_1 is_interval_convex connected_continuous_image
[OF assms])

lemma continuous_injective_image_segment_1:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes contf: continuous_on (closed_segment a b) f
     and injf: inj_on f (closed_segment a b)
  shows f ‘ (closed_segment a b) = closed_segment (f a) (f b)
proof
```

```
  show closed_segment (f a) (f b) ⊆ f ‘ closed_segment a b
    by (metis subset_continuous_image_segment_1 contf)
  show f ‘ closed_segment a b ⊆ closed_segment (f a) (f b)
  proof (cases a = b)
    case True
    then show ?thesis by auto
  next
    case False
    then have fnot: f a ≠ f b
      using inj_onD injf by fastforce
    moreover
    have f a ∉ open_segment (f c) (f b) if c: c ∈ closed_segment a b for c
    proof (clarsimp simp add: open_segment_def)
      assume fa: f a ∈ closed_segment (f c) (f b)
      moreover have closed_segment (f c) (f b) ⊆ f ‘ closed_segment c b
      by (meson closed_segment_subset contf continuous_on_subset convex_closed_segment
ends_in_segment(2) subset_continuous_image_segment_1 that)
      ultimately have f a ∈ f ‘ closed_segment c b
        by blast
      then have a: a ∈ closed_segment c b
      by (meson ends_in_segment inj_on_image_mem_iff injf subset_closed_segment
that)
      have cb: closed_segment c b ⊆ closed_segment a b
        by (simp add: closed_segment_subset that)
      show f a = f c
      proof (rule between_antisym)
        show between (f c, f b) (f a)
          by (simp add: between_mem_segment fa)
        show between (f a, f b) (f c)
            by (metis a cb between_antisym between_mem_segment between_triv1
subset_iff)
      qed
    qed
    moreover
    have f b ∉ open_segment (f a) (f c) if c: c ∈ closed_segment a b for c
    proof (clarsimp simp add: open_segment_def fnot eq_commute)
      assume fb: f b ∈ closed_segment (f a) (f c)
      moreover have closed_segment (f a) (f c) ⊆ f ‘ closed_segment a c
      by (meson contf continuous_on_subset ends_in_segment(1) subset_closed_segment
subset_continuous_image_segment_1 that)
      ultimately have f b ∈ f ‘ closed_segment a c
        by blast
      then have b: b ∈ closed_segment a c
      by (meson ends_in_segment inj_on_image_mem_iff injf subset_closed_segment
that)
      have ca: closed_segment a c ⊆ closed_segment a b
        by (simp add: closed_segment_subset that)
      show f b = f c
      proof (rule between_antisym)
```

```
      show between (f c, f a) (f b)
        by (simp add: between_commute between_mem_segment fb)
      show between (f b, f a) (f c)
        by (metis b between_antisym between_commute between_mem_segment
between_triv2 that)
    qed
  qed
  ultimately show ?thesis
   by (force simp: closed_segment_eq_real_ivl open_segment_eq_real_ivl split:
if_split_asm)
 qed
qed

lemma continuous_injective_image_open_segment_1:
 fixes f :: 'a::euclidean_space ⇒ real
 assumes contf: continuous_on (closed_segment a b) f
    and injf: inj_on f (closed_segment a b)
  shows f ‘ (open_segment a b) = open_segment (f a) (f b)
proof −
 have f ‘ (open_segment a b) = f ‘ (closed_segment a b) − {f a, f b}
    by (metis (no_types, opaque_lifting) empty_subsetI ends_in_segment im-
age_insert image_is_empty inj_on_image_set_diff injf insert_subset open_segment_def
segment_open_subset_closed)
 also have ... = open_segment (f a) (f b)
  using continuous_injective_image_segment_1 [OF assms]
  by (simp add: open_segment_def inj_on_image_set_diff [OF injf])
 finally show ?thesis .
qed

lemma collinear_imp_coplanar:
 collinear s ==> coplanar s
by (metis collinear_affine_hull coplanar_def insert_absorb2)

lemma collinear_small:
 assumes finite s card s ≤ 2
  shows collinear s
proof −
 have card s = 0 ∨ card s = 1 ∨ card s = 2
  using assms by linarith
 then show ?thesis using assms
  using card_eq_SucD numeral_2_eq_2 by (force simp: card_1_singleton_iff)
qed

lemma coplanar_small:
 assumes finite s card s ≤ 3
  shows coplanar s
proof −
 consider card s ≤ 2 | card s = Suc (Suc (Suc 0))
  using assms by linarith
```

```
  then show ?thesis
  proof cases
    case 1
    then show ?thesis
      by (simp add: ‹finite s› collinear_imp_coplanar collinear_small)
  next
    case 2
    then show ?thesis
      using hull_subset [of {_,_,_}]
      by (fastforce simp: coplanar_def dest!: card_eq_SucD)
  qed
qed

lemma coplanar_empty: coplanar {}
  by (simp add: coplanar_small)

lemma coplanar_sing: coplanar {a}
  by (simp add: coplanar_small)

lemma coplanar_2: coplanar {a,b}
  by (auto simp: card_insert_if coplanar_small)

lemma coplanar_3: coplanar {a,b,c}
  by (auto simp: card_insert_if coplanar_small)

lemma collinear_affine_hull_collinear: collinear(affine hull s) ⟷ collinear s
  unfolding collinear_affine_hull
  by (metis affine_affine_hull subset_hull hull_hull hull_mono)

lemma coplanar_affine_hull_coplanar: coplanar(affine hull s) ⟷ coplanar s
  unfolding coplanar_def
  by (metis affine_affine_hull subset_hull hull_hull hull_mono)

lemma coplanar_linear_image:
  fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes coplanar S linear f shows coplanar(f ‘ S)
proof −
  { fix u v w
    assume S ⊆ affine hull {u, v, w}
    then have f ‘ S ⊆ f ‘ (affine hull {u, v, w})
      by (simp add: image_mono)
    then have f ‘ S ⊆ affine hull (f ‘ {u, v, w})
      by (metis assms(2) linear_conv_bounded_linear affine_hull_linear_image)
  } then
  show ?thesis
    by auto (meson assms(1) coplanar_def)
qed

lemma coplanar_translation_imp:
```

assumes *coplanar S* **shows** *coplanar* ((λ*x*. *a* + *x*) ‘ *S*)
proof −
obtain *u v w* **where** *S* ⊆ *affine hull* {*u*,*v*,*w*}
by (*meson assms coplanar_def*)
then have (+) *a* ‘ *S* ⊆ *affine hull* {*u* + *a*, *v* + *a*, *w* + *a*}
using *affine_hull_translation* [*of a* {*u*,*v*,*w*} **for** *u v w*]
by (*force simp*: *add.commute*)
then show *?thesis*
unfolding *coplanar_def* **by** *blast*
qed

lemma *coplanar_translation_eq*: *coplanar*((λ*x*. *a* + *x*) ‘ *S*) ⟷ *coplanar S*
by (*metis* (*no_types*) *coplanar_translation_imp translation_galois*)

lemma *coplanar_linear_image_eq*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f* **shows** *coplanar*(*f* ‘ *S*) = *coplanar S*
proof
assume *coplanar S*
then show *coplanar* (*f* ‘ *S*)
using *assms*(*1*) *coplanar_linear_image* **by** *blast*
next
obtain *g* **where** *g*: *linear g g* ∘ *f* = *id*
using *linear_injective_left_inverse* [*OF assms*]
by *blast*
assume *coplanar* (*f* ‘ *S*)
then show *coplanar S*
by (*metis coplanar_linear_image g*(*1*) *g*(*2*) *id_apply image_comp image_id*)
qed

lemma *coplanar_subset*: ⟦*coplanar t*; *S* ⊆ *t*⟧ ⟹ *coplanar S*
by (*meson coplanar_def order_trans*)

lemma *affine_hull_3_imp_collinear*: *c* ∈ *affine hull* {*a*,*b*} ⟹ *collinear* {*a*,*b*,*c*}
by (*metis collinear_2 collinear_affine_hull_collinear hull_redundant insert_commute*)

lemma *collinear_3_imp_in_affine_hull*:
assumes *collinear* {*a*,*b*,*c*} *a* ≠ *b* **shows** *c* ∈ *affine hull* {*a*,*b*}
proof −
obtain *u x y* **where** *b* − *a* = *y* ∗*R u c* − *a* = *x* ∗*R u*
using *assms* **unfolding** *collinear_def* **by** *auto*
with ‹*a* ≠ *b*› **have** ∃ *v*. *c* = (*1* − *x* / *y*) ∗*R a* + *v* ∗*R b* ∧ *1* − *x* / *y* + *v* = *1*
by (*simp add*: *algebra_simps*)
then show *?thesis*
by (*simp add*: *hull_inc mem_affine*)
qed

lemma *collinear_3_affine_hull*:
assumes *a* ≠ *b*

shows *collinear* $\{a,b,c\} \longleftrightarrow c \in$ *affine hull* $\{a,b\}$
using *affine_hull_3_imp_collinear assms collinear_3_imp_in_affine_hull* **by** *blast*

lemma *collinear_3_eq_affine_dependent*:
collinear$\{a,b,c\} \longleftrightarrow a = b \vee a = c \vee b = c \vee$ *affine_dependent* $\{a,b,c\}$
proof (*cases* $a = b \vee a = c \vee b = c$)
case *True*
then show *?thesis*
by (*auto simp*: *insert_commute*)
next
case *False*
then have *collinear*$\{a,b,c\}$ **if** *affine_dependent* $\{a,b,c\}$
using *that* **unfolding** *affine_dependent_def*
by (*auto simp*: *insert_Diff_if*; *metis affine_hull_3_imp_collinear insert_commute*)
moreover
have *affine_dependent* $\{a,b,c\}$ **if** *collinear*$\{a,b,c\}$
using *False that* **by** (*auto simp*: *affine_dependent_def collinear_3_affine_hull insert_Diff_if*)
ultimately
show *?thesis*
using *False* **by** *blast*
qed

lemma *affine_dependent_imp_collinear_3*:
affine_dependent $\{a,b,c\} \Longrightarrow$ *collinear*$\{a,b,c\}$
by (*simp add*: *collinear_3_eq_affine_dependent*)

lemma *collinear_3*: *NO_MATCH 0 x* $\Longrightarrow$ *collinear* $\{x,y,z\} \longleftrightarrow$ *collinear* $\{0, x-y, z-y\}$
by (*auto simp add*: *collinear_def*)

lemma *collinear_3_expand*:
collinear$\{a,b,c\} \longleftrightarrow a = c \vee (\exists u.\ b = u *_R a + (1 - u) *_R c)$
proof −
have *collinear*$\{a,b,c\}$ = *collinear*$\{a,c,b\}$
by (*simp add*: *insert_commute*)
also have ... = *collinear* $\{0,\ a - c,\ b - c\}$
by (*simp add*: *collinear_3*)
also have ... $\longleftrightarrow (a = c \vee b = c \vee (\exists ca.\ b - c = ca *_R (a - c)))$
by (*simp add*: *collinear_lemma*)
also have ... $\longleftrightarrow a = c \vee (\exists u.\ b = u *_R a + (1 - u) *_R c)$
by (*cases* $a = c \vee b = c$) (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *collinear_aff_dim*: *collinear S* $\longleftrightarrow$ *aff_dim S* $\leq$ *1*
proof
assume *collinear S*

```
  then obtain u and v :: 'a where aff_dim S ≤ aff_dim {u,v}
   by (metis ‹collinear S› aff_dim_affine_hull aff_dim_subset collinear_affine_hull)
  then show aff_dim S ≤ 1
    using order_trans by fastforce
next
  assume aff_dim S ≤ 1
  then have le1: aff_dim (affine hull S) ≤ 1
    by simp
  obtain B where B ⊆ S and B: ¬ affine_dependent B affine hull S = affine hull
B
    using affine_basis_exists [of S] by auto
  then have finite B card B ≤ 2
    using B le1 by (auto simp: affine_independent_iff_card)
  then have collinear B
    by (rule collinear_small)
  then show collinear S
    by (metis ‹affine hull S = affine hull B› collinear_affine_hull_collinear)
qed

lemma collinear_midpoint: collinear{a, midpoint a b, b}
proof −
  have §: ⟦a ≠ midpoint a b; b − midpoint a b ≠ − 1 *R (a − midpoint a b)⟧ ⟹
b = midpoint a b
    by (simp add: algebra_simps)
  show ?thesis
    by (auto simp: collinear_3 collinear_lemma intro: §)
qed

lemma midpoint_collinear:
  fixes a b c :: 'a::real_normed_vector
  assumes a ≠ c
    shows b = midpoint a c ⟷ collinear{a,b,c} ∧ dist a b = dist b c
proof −
  have *: a − (u *R a + (1 − u) *R c) = (1 − u) *R (a − c)
        u *R a + (1 − u) *R c − c = u *R (a − c)
        |1 − u| = |u| ⟷ u = 1/2 for u::real
    by (auto simp: algebra_simps)
  have b = midpoint a c ⟹ collinear{a,b,c}
    using collinear_midpoint by blast
  moreover have b = midpoint a c ⟷ dist a b = dist b c if collinear{a,b,c}
  proof −
    consider a = c | u where b = u *R a + (1 − u) *R c
      using ‹collinear {a,b,c}› unfolding collinear_3_expand by blast
    then show ?thesis
    proof cases
      case 2
      with assms have dist a b = dist b c ⟹ b = midpoint a c
      by (simp add: dist_norm * midpoint_def scaleR_add_right del: divide_const_simps)
      then show ?thesis
```

```
      by (auto simp: dist_midpoint)
    qed (use assms in auto)
  qed
  ultimately show ?thesis by blast
qed

lemma between_imp_collinear:
  fixes x :: 'a :: euclidean_space
  assumes between (a,b) x
    shows collinear {a,x,b}
proof (cases x = a ∨ x = b ∨ a = b)
  case True with assms show ?thesis
    by (auto simp: dist_commute)
next
  case False
  then have False if ⋀c. b - x ≠ c *R (a - x)
    using that [of -(norm(b - x) / norm(x - a))] assms
   by (simp add: between_norm vector_add_divide_simps flip: real_vector.scale_minus_right)
  then show ?thesis
    by (auto simp: collinear_3 collinear_lemma)
qed

lemma midpoint_between:
  fixes a b :: 'a::euclidean_space
  shows b = midpoint a c ⟷ between (a,c) b ∧ dist a b = dist b c
proof (cases a = c)
  case False
  show ?thesis
    using False between_imp_collinear between_midpoint(1) midpoint_collinear
by blast
qed (auto simp: dist_commute)

lemma collinear_triples:
  assumes a ≠ b
    shows collinear(insert a (insert b S)) ⟷ (∀ x ∈ S. collinear{a,b,x})
          (is ?lhs = ?rhs)
proof safe
  fix x
  assume ?lhs and x ∈ S
  then show collinear {a, b, x}
    using collinear_subset by force
next
  assume ?rhs
  then have ∀ x ∈ S. collinear{a,x,b}
    by (simp add: insert_commute)
  then have *: ∃ u. x = u *R a + (1 - u) *R b if x ∈ insert a (insert b S) for x
    using that assms collinear_3_expand by fastforce+
  have ∃ c. x - y = c *R (b - a)
    if x: x ∈ insert a (insert b S) and y: y ∈ insert a (insert b S) for x y
```

```
  proof −
    obtain u v where x = u *R a + (1 − u) *R b y = v *R a + (1 − v) *R b
      using ∗ x y by presburger
    then have x − y = (v − u) *R (b − a)
      by (simp add: scale_left_diff_distrib scale_right_diff_distrib)
    then show ?thesis ..
  qed
  then show ?lhs
    unfolding collinear_def by metis
qed

lemma collinear_4_3:
  assumes a ≠ b
    shows collinear {a,b,c,d} ⟷ collinear{a,b,c} ∧ collinear{a,b,d}
  using collinear_triples [OF assms, of {c,d}] by (force simp:)

lemma collinear_3_trans:
  assumes collinear{a,b,c} collinear{b,c,d} b ≠ c
    shows collinear{a,b,d}
proof −
  have collinear{b,c,a,d}
    by (metis (full_types) assms collinear_4_3 insert_commute)
  then show ?thesis
    by (simp add: collinear_subset)
qed

lemma affine_hull_2_alt:
  fixes a b :: 'a::real_vector
  shows affine hull {a,b} = range (λu. a + u *R (b − a))
proof −
  have 1: u *R a + v *R b = a + v *R (b − a) if u + v = 1 for u v
    using that
    by (simp add: algebra_simps flip: scaleR_add_left)
  have 2: a + u *R (b − a) = (1 − u) *R a + u *R b for u
    by (auto simp: algebra_simps)
  show ?thesis
    by (force simp add: affine_hull_2 dest: 1 intro!: 2)
qed

lemma interior_convex_hull_3_minimal:
  fixes a :: 'a::euclidean_space
  assumes ¬ collinear{a,b,c} and 2: DIM('a) = 2
  shows interior(convex hull {a,b,c}) =
        {v. ∃x y z. 0 < x ∧ 0 < y ∧ 0 < z ∧ x + y + z = 1 ∧ x *R a + y *R b
+ z *R c = v}
       (is ?lhs = ?rhs)
proof
  have abc: a ≠ b a ≠ c b ≠ c ¬ affine_dependent {a, b, c}
    using assms by (auto simp: collinear_3_eq_affine_dependent)
```

with *2* **show** *?lhs ⊆ ?rhs*
by (*fastforce simp add*: *interior_convex_hull_explicit_minimal*)
show *?rhs ⊆ ?lhs*
using *abc 2*
apply (*clarsimp simp add*: *interior_convex_hull_explicit_minimal*)
subgoal for *x y z*
by (*rule_tac x=λr. (if r=a then x else if r=b then y else if r=c then z else 0)* **in** *exI*) *auto*
done
qed

6.0.9 Basic lemmas about hyperplanes and halfspaces

lemma *halfspace_Int_eq*:
$\{x.\ a \cdot x \leq b\} \cap \{x.\ b \leq a \cdot x\} = \{x.\ a \cdot x = b\}$
$\{x.\ b \leq a \cdot x\} \cap \{x.\ a \cdot x \leq b\} = \{x.\ a \cdot x = b\}$
by *auto*

lemma *hyperplane_eq_Ex*:
assumes $a \neq 0$ **obtains** x **where** $a \cdot x = b$
by (*rule_tac* $x = (b\ /\ (a \cdot a)) *_R a$ **in** *that*) (*simp add*: *assms*)

lemma *hyperplane_eq_empty*:
$\{x.\ a \cdot x = b\} = \{\} \longleftrightarrow a = 0 \wedge b \neq 0$
using *hyperplane_eq_Ex*
by (*metis* (*mono_tags*, *lifting*) *empty_Collect_eq inner_zero_left*)

lemma *hyperplane_eq_UNIV*:
$\{x.\ a \cdot x = b\} = UNIV \longleftrightarrow a = 0 \wedge b = 0$
proof −
have $a = 0 \wedge b = 0$ **if** $UNIV \subseteq \{x.\ a \cdot x = b\}$
using *subsetD* [*OF that*, **where** $c = ((b+1)\ /\ (a \cdot a)) *_R a$]
by (*simp add*: *field_split_simps split*: *if_split_asm*)
then show *?thesis* **by** *force*
qed

lemma *halfspace_eq_empty_lt*:
$\{x.\ a \cdot x < b\} = \{\} \longleftrightarrow a = 0 \wedge b \leq 0$
proof −
have $a = 0 \wedge b \leq 0$ **if** $\{x.\ a \cdot x < b\} \subseteq \{\}$
using *subsetD* [*OF that*, **where** $c = ((b-1)\ /\ (a \cdot a)) *_R a$]
by (*force simp add*: *field_split_simps split*: *if_split_asm*)
then show *?thesis* **by** *force*
qed

lemma *halfspace_eq_empty_gt*:
$\{x.\ a \cdot x > b\} = \{\} \longleftrightarrow a = 0 \wedge b \geq 0$
using *halfspace_eq_empty_lt* [*of* $-a$ $-b$]
by *simp*

lemma *halfspace_eq_empty_le*:
$\{x.\ a \cdot x \leq b\} = \{\} \longleftrightarrow a = 0 \wedge b < 0$
proof −
have $a = 0 \wedge b < 0$ **if** $\{x.\ a \cdot x \leq b\} \subseteq \{\}$
using *subsetD* [*OF that*, **where** $c = ((b-1)\ /\ (a \cdot a)) *_R a$]
by (*force simp add*: *field_split_simps split*: *if_split_asm*)
then show *?thesis* **by** *force*
qed

lemma *halfspace_eq_empty_ge*:
$\{x.\ a \cdot x \geq b\} = \{\} \longleftrightarrow a = 0 \wedge b > 0$
using *halfspace_eq_empty_le* [*of* $-a$ $-b$] **by** *simp*

6.0.10 Use set distance for an easy proof of separation properties

proposition *separation_closures*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $S \cap closure\ T = \{\}$ $T \cap closure\ S = \{\}$
obtains $U\ V$ **where** $U \cap V = \{\}$ *open* U *open* V $S \subseteq U$ $T \subseteq V$
proof (*cases* $S = \{\} \vee T = \{\}$)
case *True* **with** *that* **show** *?thesis* **by** *auto*
next
case *False*
define f **where** $f \equiv \lambda x.\ setdist\ \{x\}\ T - setdist\ \{x\}\ S$
have *contf*: *continuous_on UNIV f*
unfolding *f_def* **by** (*intro continuous_intros continuous_on_setdist*)
show *?thesis*
proof (*rule_tac* $U = \{x.\ f\ x > 0\}$ **and** $V = \{x.\ f\ x < 0\}$ **in** *that*)
show $\{x.\ 0 < f\ x\} \cap \{x.\ f\ x < 0\} = \{\}$
by *auto*
show *open* $\{x.\ 0 < f\ x\}$
by (*simp add*: *open_Collect_less contf*)
show *open* $\{x.\ f\ x < 0\}$
by (*simp add*: *open_Collect_less contf*)
have $\bigwedge x.\ x \in S \Longrightarrow setdist\ \{x\}\ T \neq 0\ \bigwedge x.\ x \in T \Longrightarrow setdist\ \{x\}\ S \neq 0$
by (*meson False assms disjoint_iff setdist_eq_0_sing_1*)+
then show $S \subseteq \{x.\ 0 < f\ x\}$ $T \subseteq \{x.\ f\ x < 0\}$
using *less_eq_real_def* **by** (*fastforce simp add*: *f_def setdist_sing_in_set*)+
qed
qed

lemma *separation_normal*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *closed* S *closed* T $S \cap T = \{\}$
obtains $U\ V$ **where** *open* U *open* V $S \subseteq U$ $T \subseteq V$ $U \cap V = \{\}$
using *separation_closures* [*of* S T]
by (*metis assms closure_closed disjnt_def inf_commute*)

```
lemma separation_normal_local:
  fixes S :: 'a::euclidean_space set
  assumes US: closedin (top_of_set U) S
      and UT: closedin (top_of_set U) T
      and S ∩ T = {}
  obtains S' T' where openin (top_of_set U) S'
                      openin (top_of_set U) T'
                      S ⊆ S'  T ⊆ T'  S' ∩ T' = {}
proof (cases S = {} ∨ T = {})
  case True with that show ?thesis
    using UT US by (blast dest: closedin_subset)
next
  case False
  define f where f ≡ λx. setdist {x} T - setdist {x} S
  have contf: continuous_on U f
    unfolding f_def by (intro continuous_intros)
  show ?thesis
  proof (rule_tac S' = (U ∩ f -` {0<..}) and T' = (U ∩ f -` {..<0}) in that)
    show (U ∩ f -` {0<..}) ∩ (U ∩ f -` {..<0}) = {}
      by auto
    show openin (top_of_set U) (U ∩ f -` {0<..})
      by (rule continuous_openin_preimage [where T=UNIV]) (simp_all add:
contf)
  next
    show openin (top_of_set U) (U ∩ f -` {..<0})
      by (rule continuous_openin_preimage [where T=UNIV]) (simp_all add:
contf)
  next
    have S ⊆ U T ⊆ U
      using closedin_imp_subset assms by blast+
    then show S ⊆ U ∩ f -` {0<..} T ⊆ U ∩ f -` {..<0}
      using assms False by (force simp add: f_def setdist_sing_in_set intro!:
setdist_gt_0_closedin)+
  qed
qed

lemma separation_normal_compact:
  fixes S :: 'a::euclidean_space set
  assumes compact S closed T S ∩ T = {}
  obtains U V where open U compact(closure U) open V S ⊆ U T ⊆ V U ∩ V
= {}
proof -
  have closed S bounded S
    using assms by (auto simp: compact_eq_bounded_closed)
  then obtain r where r>0 and r: S ⊆ ball 0 r
    by (auto dest!: bounded_subset_ballD)
  have **: closed (T ∪ - ball 0 r) S ∩ (T ∪ - ball 0 r) = {}
    using assms r by blast+
```

then obtain U V **where** UV: *open* U *open* V $S \subseteq U$ $T \cup -$ *ball* 0 $r \subseteq V$ $U \cap V = \{\}$
by (*meson* ‹*closed* S› *separation_normal*)
then have *compact*(*closure* U)
by (*meson bounded_ball bounded_subset compact_closure compl_le_swap2 disjoint_eq_subset_Compl le_sup_iff*)
with UV **show** *thesis*
using *that* **by** *auto*
qed

6.0.11 Connectedness of the intersection of a chain

proposition *connected_chain*:
fixes $\mathcal{F}$ :: *'a* :: *euclidean_space set set*
assumes *cc*: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *compact* $S \wedge$ *connected* S
and *linear*: $\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
shows *connected*($\bigcap \mathcal{F}$)
proof (*cases* $\mathcal{F} = \{\}$)
case *True* **then show** *?thesis*
by *auto*
next
case *False*
then have *cf*: *compact*($\bigcap \mathcal{F}$)
by (*simp add*: *cc compact_Inter*)
have *False* **if** *AB*: *closed* A *closed* B $A \cap B = \{\}$
and *ABeq*: $A \cup B = \bigcap \mathcal{F}$ **and** $A \neq \{\}$ $B \neq \{\}$ **for** A B
proof −
obtain U V **where** *open* U *open* V $A \subseteq U$ $B \subseteq V$ $U \cap V = \{\}$
using *separation_normal* [*OF AB*] **by** *metis*
obtain K **where** $K \in \mathcal{F}$ *compact* K
using *cc False* **by** *blast*
then obtain N **where** *open* N **and** $K \subseteq N$
by *blast*
let *?C* = *insert* ($U \cup V$) ((λS. $N - S$) ‘ $\mathcal{F}$)
obtain $\mathcal{D}$ **where** $\mathcal{D} \subseteq$ *?C* *finite* $\mathcal{D}$ $K \subseteq \bigcup \mathcal{D}$
proof (*rule compactE* [*OF* ‹*compact* K›])
show $K \subseteq \bigcup$(*insert* ($U \cup V$) ((−) N ‘ $\mathcal{F}$))
using ‹$K \subseteq N$› *ABeq* ‹$A \subseteq U$› ‹$B \subseteq V$› **by** *auto*
show $\bigwedge B.\ B \in$ *insert* ($U \cup V$) ((−) N ‘ $\mathcal{F}$) $\Longrightarrow$ *open* B
by (*auto simp*: ‹*open* U› ‹*open* V› *open_Un* ‹*open* N› *cc compact_imp_closed open_Diff*)
qed
then have *finite*($\mathcal{D} - \{U \cup V\}$)
by *blast*
moreover have $\mathcal{D} - \{U \cup V\} \subseteq$ (λS. $N - S$) ‘ $\mathcal{F}$
using ‹$\mathcal{D} \subseteq$ *?C*› **by** *blast*
ultimately obtain $\mathcal{G}$ **where** $\mathcal{G} \subseteq \mathcal{F}$ *finite* $\mathcal{G}$ **and** *Deq*: $\mathcal{D} - \{U \cup V\} =$ (λS. $N - S$) ‘ $\mathcal{G}$
using *finite_subset_image* **by** *metis*

```
    obtain J where J ∈ F and J: (⋃S∈G. N − S) ⊆ N − J
    proof (cases G = {})
      case True
      with ‹F ≠ {}› that show ?thesis
        by auto
    next
      case False
      have ⋀S T. ⟦S ∈ G; T ∈ G⟧ ⟹ S ⊆ T ∨ T ⊆ S
        by (meson ‹G ⊆ F› in_mono local.linear)
      with ‹finite G› ‹G ≠ {}›
      have ∃J∈G. (⋃S∈G. N − S) ⊆ N − J
      proof induction
        case (insert X H)
        show ?case
        proof (cases H = {})
          case True then show ?thesis by auto
        next
          case False
          then have ⋀S T. ⟦S ∈ H; T ∈ H⟧ ⟹ S ⊆ T ∨ T ⊆ S
            by (simp add: insert.prems)
          with insert.IH False obtain J where J ∈ H and J: (⋃ Y∈H. N − Y)
⊆ N − J
            by metis
          have N − J ⊆ N − X ∨ N − X ⊆ N − J
            by (meson Diff_mono ‹J ∈ H› insert.prems(2) insert_iff order_refl)
          then show ?thesis
          proof
            assume N − J ⊆ N − X with J show ?thesis
              by auto
          next
            assume N − X ⊆ N − J
            with J have N − X ∪ ⋃ ((−) N ` H) ⊆ N − J
              by auto
            with ‹J ∈ H› show ?thesis
              by blast
          qed
        qed
      qed simp
      with ‹G ⊆ F› show ?thesis by (blast intro: that)
    qed
    have K ⊆ ⋃(insert (U ∪ V) (D − {U ∪ V}))
      using ‹K ⊆ ⋃D› by auto
    also have ... ⊆ (U ∪ V) ∪ (N − J)
     by (metis (no_types, opaque_lifting) Deq Un_subset_iff Un_upper2 J Union_insert
order_trans sup_ge1)
    finally have J ∩ K ⊆ U ∪ V
      by blast
    moreover have connected(J ∩ K)
      by (metis Int_absorb1 ‹J ∈ F› ‹K ∈ F› cc inf.orderE local.linear)
```

```
    moreover have U ∩ (J ∩ K) ≠ {}
      using ABeq ‹J ∈ ℱ› ‹K ∈ ℱ› ‹A ≠ {}› ‹A ⊆ U› by blast
    moreover have V ∩ (J ∩ K) ≠ {}
      using ABeq ‹J ∈ ℱ› ‹K ∈ ℱ› ‹B ≠ {}› ‹B ⊆ V› by blast
    ultimately show False
       using connectedD [of J ∩ K U V] ‹open U› ‹open V› ‹U ∩ V = {}›  by
auto
  qed
  with cf show ?thesis
    by (auto simp: connected_closed_set compact_imp_closed)
qed

lemma connected_chain_gen:
  fixes ℱ :: 'a :: euclidean_space set set
  assumes X: X ∈ ℱ compact X
      and cc: ⋀T. T ∈ ℱ ⟹ closed T ∧ connected T
      and linear: ⋀S T. S ∈ ℱ ∧ T ∈ ℱ ⟹ S ⊆ T ∨ T ⊆ S
  shows connected(⋂ℱ)
proof −
  have ⋂ℱ = (⋂T∈ℱ. X ∩ T)
    using X by blast
  moreover have connected (⋂T∈ℱ. X ∩ T)
  proof (rule connected_chain)
    show ⋀T. T ∈ (∩) X ` ℱ ⟹ compact T ∧ connected T
      using cc X by auto (metis inf.absorb2 inf.orderE local.linear)
    show ⋀S T. S ∈ (∩) X ` ℱ ∧ T ∈ (∩) X ` ℱ ⟹ S ⊆ T ∨ T ⊆ S
      using local.linear by blast
  qed
  ultimately show ?thesis
    by metis
qed

lemma connected_nest:
  fixes S :: 'a::linorder ⇒ 'b::euclidean_space set
  assumes S: ⋀n. compact(S n) ⋀n. connected(S n)
    and nest: ⋀m n. m ≤ n ⟹ S n ⊆ S m
  shows connected(⋂ (range S))
proof (rule connected_chain)
  show ⋀A T. A ∈ range S ∧ T ∈ range S ⟹ A ⊆ T ∨ T ⊆ A
  by (metis image_iff le_cases nest)
qed (use S in blast)

lemma connected_nest_gen:
  fixes S :: 'a::linorder ⇒ 'b::euclidean_space set
  assumes S: ⋀n. closed(S n) ⋀n. connected(S n) compact(S k)
    and nest: ⋀m n. m ≤ n ⟹ S n ⊆ S m
  shows connected(⋂ (range S))
proof (rule connected_chain_gen [of S k])
  show ⋀A T. A ∈ range S ∧ T ∈ range S ⟹ A ⊆ T ∨ T ⊆ A
```

by (*metis imageE le_cases nest*)
qed (*use S* **in** *auto*)

6.0.12 Proper maps, including projections out of compact sets

lemma *finite_indexed_bound*:
assumes *A*: *finite A* $\bigwedge x.\ x \in A \Longrightarrow \exists\, n::'a::linorder.\ P\ x\ n$
shows $\exists\, m.\ \forall\, x \in A.\ \exists\, k{\leq}m.\ P\ x\ k$
using *A*
proof (*induction A*)
case *empty* **then show** *?case* **by** *force*
next
case (*insert a A*)
then obtain *m n* **where** $\forall\, x \in A.\ \exists\, k{\leq}m.\ P\ x\ k\ P\ a\ n$
by *force*
then show *?case*
by (*metis dual_order.trans insert_iff le_cases*)
qed

proposition *proper_map*:
fixes *f* :: *'a::heine_borel* ⇒ *'b::heine_borel*
assumes *closedin* (*top_of_set S*) *K*
and *com*: $\bigwedge U.\ [\![U \subseteq T;\ compact\ U]\!] \Longrightarrow compact\ (S \cap f\ -`\ U)$
and $f\ `\ S \subseteq T$
shows *closedin* (*top_of_set T*) (*f ` K*)
proof −
have $K \subseteq S$
using *assms closedin_imp_subset* **by** *metis*
obtain *C* **where** *closed C* **and** *Keq*: $K = S \cap C$
using *assms* **by** (*auto simp*: *closedin_closed*)
have ∗: $y \in f\ `\ K$ **if** $y \in T$ **and** *y*: *y islimpt f ` K* **for** *y*
proof −
obtain *h* **where** $\forall\, n.\ (\exists\, x{\in}K.\ h\ n = f\ x) \wedge h\ n \neq y\ inj\ h$ **and** *hlim*: $(h \longrightarrow y)\ sequentially$
using ‹$y \in T$› *y* **by** (*force simp*: *limpt_sequential_inj*)
then obtain *X* **where** *X*: $\bigwedge n.\ X\ n \in K \wedge h\ n = f\ (X\ n) \wedge h\ n \neq y$
by *metis*
then have *fX*: $\bigwedge n.\ f\ (X\ n) = h\ n$
by *metis*
define Ψ **where** $\Psi \equiv \lambda n.\ \{a \in K.\ f\ a \in insert\ y\ (range\ (\lambda i.\ f\ (X\ (n + i))))\}$
have *compact* $(C \cap (S \cap f\ -`\ insert\ y\ (range\ (\lambda i.\ f(X(n + i))))))$ **for** *n*
proof (*intro closed_Int_compact* [*OF* ‹*closed C*› *com*] *compact_sequence_with_limit*)
show *insert y* (*range* $(\lambda i.\ f\ (X\ (n + i)))) \subseteq T$
using *X* ‹$K \subseteq S$› ‹$f\ `\ S \subseteq T$› ‹$y \in T$› **by** *blast*
show $(\lambda i.\ f\ (X\ (n + i))) \longrightarrow y$
by (*simp add*: *fX add.commute* [*of n*] *LIMSEQ_ignore_initial_segment* [*OF hlim*])
qed

```
    then have comf: compact (Ψ n) for n
      by (simp add: Keq Int_def Ψ_def conj_commute)
    have ne: ⋂ℱ ≠ {}
          if finite ℱ
            and ℱ: ⋀t. t ∈ ℱ ⟹ (∃ n. t = Ψ n)
          for ℱ
    proof −
      obtain m where m: ⋀t. t ∈ ℱ ⟹ ∃ k≤m. t = Ψ k
        by (rule exE [OF finite_indexed_bound [OF ‹finite ℱ› ℱ]], force+)
      have X m ∈ ⋂ℱ
        using X le_Suc_ex by (fastforce simp: Ψ_def dest: m)
      then show ?thesis by blast
    qed
    have (⋂ n. Ψ n) ≠ {}
    proof (rule compact_fip_Heine_Borel)
      show ⋀ℱ'. ⟦finite ℱ'; ℱ' ⊆ range Ψ⟧ ⟹ ⋂ ℱ' ≠ {}
        by (meson ne rangeE subset_eq)
    qed (use comf in blast)
    then obtain x where x ∈ K ⋀n. (f x = y ∨ (∃ u. f x = h (n + u)))
      by (force simp add: Ψ_def fX)
    then show ?thesis
        unfolding image_iff by (metis ‹inj h› le_add1 not_less_eq_eq rangeI
range_ex1_eq)
  qed
  with assms closedin_subset show ?thesis
    by (force simp: closedin_limpt)
qed
```

6.0.13 Closure of conic hulls

```
proposition closedin_conic_hull:
  fixes S :: 'a::euclidean_space set
  assumes compact T 0 ∉ T T ⊆ S
  shows   closedin (top_of_set (conic hull S)) (conic hull T)
proof −
  have **: compact ({0..} × T ∩ (λz. fst z *R snd z) −‘ K) (is compact ?L)
    if K ⊆ (λz. (fst z) *R snd z) ‘ ({0..} × S) compact K for K
  proof −
    obtain r where r > 0 and r: ⋀x. x ∈ K ⟹ norm x ≤ r
      by (metis ‹compact K› bounded_normE compact_imp_bounded)
    show ?thesis
      unfolding compact_eq_bounded_closed
    proof
      have bounded ({0..r / setdist{0}T} × T)
        by (simp add: assms(1) bounded_Times compact_imp_bounded)
      moreover have ?L ⊆ ({0..r / setdist{0}T} × T)
      proof clarsimp
        fix a b
        assume a *R b ∈ K and b ∈ T and 0 ≤ a
```

```
      have setdist {0} T ≠ 0
        using ‹b ∈ T› assms compact_imp_closed setdist_eq_0_closed by auto
      then have T0: setdist {0} T > 0
        using less_eq_real_def by fastforce
      then have a * setdist {0} T ≤ r
        by (smt (verit, ccfv_SIG) ‹0 ≤ a› ‹a *R b ∈ K› ‹b ∈ T› dist_0_norm
mult_mono' norm_scaleR r setdist_le_dist singletonI)
      with T0 ‹r>0› show a ≤ r / setdist {0} T
        by (simp add: divide_simps)
    qed
    ultimately show bounded ?L
      by (meson bounded_subset)
    show closed ?L
    proof (rule continuous_closed_preimage)
      show continuous_on ({0..} × T) (λz. fst z *R snd z)
        by (intro continuous_intros)
      show closed ({0::real..} × T)
        by (simp add: assms(1) closed_Times compact_imp_closed)
      show closed K
        by (simp add: compact_imp_closed that(2))
    qed
  qed
  show ?thesis
    unfolding conic_hull_as_image
  proof (rule proper_map)
    show  compact ({0..} × T ∩ (λz. fst z *R snd z) -‘ K) (is compact ?L)
    if K ⊆ (λz. (fst z) *R snd z) ‘ ({0..} × S) compact K for K
    proof −
      obtain r where r > 0 and r: ⋀x. x ∈ K ⟹ norm x ≤ r
        by (metis ‹compact K› bounded_normE compact_imp_bounded)
      show ?thesis
        unfolding compact_eq_bounded_closed
      proof
        have bounded ({0..r / setdist{0}T} × T)
          by (simp add: assms(1) bounded_Times compact_imp_bounded)
        moreover have ?L ⊆ ({0..r / setdist{0}T} × T)
        proof clarsimp
          fix a b
          assume a *R b ∈ K and b ∈ T and 0 ≤ a
          have setdist {0} T ≠ 0
            using ‹b ∈ T› assms compact_imp_closed setdist_eq_0_closed by auto
          then have T0: setdist {0} T > 0
            using less_eq_real_def by fastforce
          then have a * setdist {0} T ≤ r
            by (smt (verit, ccfv_SIG) ‹0 ≤ a› ‹a *R b ∈ K› ‹b ∈ T› dist_0_norm
mult_mono' norm_scaleR r setdist_le_dist singletonI)
          with T0 ‹r>0› show a ≤ r / setdist {0} T
            by (simp add: divide_simps)
```

```
      qed
      ultimately show bounded ?L
        by (meson bounded_subset)
      show closed ?L
      proof (rule continuous_closed_preimage)
        show continuous_on ({0..} × T) (λz. fst z *R snd z)
          by (intro continuous_intros)
        show closed ({0::real..} × T)
          by (simp add: assms(1) closed_Times compact_imp_closed)
        show closed K
          by (simp add: compact_imp_closed that(2))
      qed
    qed
  qed
  show (λz. fst z *R snd z) ` ({0::real..} × T) ⊆ (λz. fst z *R snd z) ` ({0..} ×
S)
    using ‹T ⊆ S› by force
 qed auto
qed

lemma closed_conic_hull:
 fixes S :: 'a::euclidean_space set
 assumes 0 ∈ rel_interior S ∨ compact S ∧ 0 ∉ S
 shows   closed(conic hull S)
 using assms
proof
 assume 0 ∈ rel_interior S
 then show closed (conic hull S)
   by (simp add: conic_hull_eq_span)
next
 assume compact S ∧ 0 ∉ S
 then have closedin (top_of_set UNIV) (conic hull S)
   using closedin_conic_hull by force
 then show closed (conic hull S)
   by simp
qed

lemma conic_closure:
 fixes S :: 'a::euclidean_space set
 shows conic S ⟹ conic(closure S)
 by (meson Convex.cone_def cone_closure conic_def)

lemma closure_conic_hull:
 fixes S :: 'a::euclidean_space set
 assumes 0 ∈ rel_interior S ∨ bounded S ∧ ~(0 ∈ closure S)
 shows   closure(conic hull S) = conic hull (closure S)
 using assms
proof
 assume 0 ∈ rel_interior S
```

then show *closure (conic hull S) = conic hull closure S*
by (*metis closed_affine_hull closure_closed closure_same_affine_hull closure_subset conic_hull_eq_affine_hull subsetD subset_rel_interior*)
next
have $\bigwedge x.\ x \in$ *conic hull closure S* $\Longrightarrow x \in$ *closure (conic hull S)*
by (*metis* (*no_types*, *opaque_lifting*) *closure_mono conic_closure conic_conic_hull subset_eq subset_hull*)
moreover
assume *bounded S* $\wedge$ *0* $\notin$ *closure S*
then have $\bigwedge x.\ x \in$ *closure (conic hull S)* $\Longrightarrow x \in$ *conic hull closure S*
by (*metis closed_conic_hull closure_Un_frontier closure_closed closure_mono compact_closure hull_Un_subset le_sup_iff subsetD*)
ultimately show *closure (conic hull S) = conic hull closure S*
by *blast*
qed

lemma *compact_continuous_image_eq*:
fixes *f* :: *'a*::*heine_borel* $\Rightarrow$ *'b*::*heine_borel*
assumes *f*: *inj_on f S*
shows *continuous_on S f* $\longleftrightarrow$ ($\forall$ *T*. *compact T* $\wedge$ *T* $\subseteq$ *S* $\longrightarrow$ *compact*(*f* ‘ *T*))
(**is** *?lhs = ?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis continuous_on_subset compact_continuous_image*)
next
assume *RHS*: *?rhs*
obtain *g* **where** *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
by (*metis inv_into_f_f f*)
then have $*$: (*S* $\cap$ *f* $-$‘ *U*) = *g* ‘ *U* **if** *U* $\subseteq$ *f* ‘ *S* **for** *U*
using *that* **by** *fastforce*
have *gfim*: *g* ‘ *f* ‘ *S* $\subseteq$ *S* **using** *gf* **by** *auto*
have $**$: *compact* (*f* ‘ *S* $\cap$ *g* $-$‘ *C*) **if** *C*: *C* $\subseteq$ *S compact C* **for** *C*
proof $-$
obtain *h* **where** *h C* $\in$ *C* $\wedge$ *h C* $\notin$ *S* $\vee$ *compact* (*f* ‘ *C*)
by (*force simp*: *C RHS*)
moreover have *f* ‘ *C* = (*f* ‘ *S* $\cap$ *g* $-$‘ *C*)
using *C gf* **by** *auto*
ultimately show *?thesis*
using *C* **by** *auto*
qed
show *?lhs*
using *proper_map* [*OF _ _ gfim*] $**$
by (*simp add*: *continuous_on_closed* $*$ *closedin_imp_subset*)
qed

6.0.14 Trivial fact: convexity equals connectedness for collinear sets

lemma *convex_connected_collinear*:

```
  fixes S :: 'a::euclidean_space set
  assumes collinear S
    shows convex S ⟷ connected S
proof
  assume convex S
  then show connected S
    using convex_connected by blast
next
  assume S: connected S
  show convex S
  proof (cases S = {})
    case True
    then show ?thesis by simp
  next
    case False
    then obtain a where a ∈ S by auto
    have collinear (affine hull S)
      by (simp add: assms collinear_affine_hull_collinear)
    then obtain z where z ≠ 0 ⋀x. x ∈ affine hull S ⟹ ∃c. x − a = c *R z
      by (meson ‹a ∈ S› collinear hull_inc)
    then obtain f where f: ⋀x. x ∈ affine hull S ⟹ x − a = f x *R z
      by metis
    then have inj_f: inj_on f (affine hull S)
      by (metis diff_add_cancel inj_onI)
    have diff: x − y = (f x − f y) *R z if x: x ∈ affine hull S and y: y ∈ affine
hull S for x y
    proof −
      have f x *R z = x − a
        by (simp add: f hull_inc x)
      moreover have f y *R z = y − a
        by (simp add: f hull_inc y)
      ultimately show ?thesis
        by (simp add: scaleR_left.diff)
    qed
    have cont_f: continuous_on (affine hull S) f
    proof (clarsimp simp: dist_norm continuous_on_iff diff)
      show ⋀x e. 0 < e ⟹ ∃d>0. ∀y ∈ affine hull S. |f y − f x| * norm z < d
⟶ |f y − f x| < e
      by (metis ‹z ≠ 0› mult_pos_pos mult_less_cancel_right_pos zero_less_norm_iff)
    qed
    then have conn_fS: connected (f ‘ S)
     by (meson S connected_continuous_image continuous_on_subset hull_subset)
    show ?thesis
    proof (clarsimp simp: convex_contains_segment)
      fix x y z
      assume x ∈ S y ∈ S z ∈ closed_segment x y
      have False if z ∉ S
      proof −
        have f ‘ (closed_segment x y) = closed_segment (f x) (f y)
```

```
      proof (rule continuous_injective_image_segment_1)
        show continuous_on (closed_segment x y) f
         by (meson ‹x ∈ S› ‹y ∈ S› convex_affine_hull convex_contains_segment
hull_inc continuous_on_subset [OF cont_f])
        show inj_on f (closed_segment x y)
         by (meson ‹x ∈ S› ‹y ∈ S› convex_affine_hull convex_contains_segment
hull_inc inj_on_subset [OF inj_f])
      qed
      then have fz: f z ∈ closed_segment (f x) (f y)
        using ‹z ∈ closed_segment x y› by blast
      have z ∈ affine hull S
        by (meson ‹x ∈ S› ‹y ∈ S› ‹z ∈ closed_segment x y› convex_affine_hull
convex_contains_segment hull_inc subset_eq)
      then have fz_notin: f z ∉ f ‘ S
        using hull_subset inj_f inj_onD that by fastforce
      moreover have {..<f z} ∩ f ‘ S ≠ {} {f z<..} ∩ f ‘ S ≠ {}
      proof −
        consider f x ≤ f z ∧ f z ≤ f y | f y ≤ f z ∧ f z ≤ f x
          using fz
          by (auto simp add: closed_segment_eq_real_ivl split: if_split_asm)
        then have {..<f z} ∩ f ‘ {x,y} ≠ {} ∧ {f z<..} ∩ f ‘ {x,y} ≠ {}
          by cases (use fz_notin ‹x ∈ S› ‹y ∈ S› in ‹auto simp: image_iff›)
        then show {..<f z} ∩ f ‘ S ≠ {} {f z<..} ∩ f ‘ S ≠ {}
          using ‹x ∈ S› ‹y ∈ S› by blast+
      qed
      ultimately show False
        using connectedD [OF conn_fS, of {..<f z} {f z<..}] by force
    qed
    then show z ∈ S by meson
  qed
 qed
qed

lemma compact_convex_collinear_segment_alt:
 fixes S :: 'a::euclidean_space set
 assumes S ≠ {} compact S connected S collinear S
 obtains a b where S = closed_segment a b
proof −
 obtain ξ where ξ ∈ S using ‹S ≠ {}› by auto
 have collinear (affine hull S)
   by (simp add: assms collinear_affine_hull_collinear)
 then obtain z where z ≠ 0 ⋀x. x ∈ affine hull S ⟹ ∃ c. x − ξ = c *R z
   by (meson ‹ξ ∈ S› collinear hull_inc)
 then obtain f where f: ⋀x. x ∈ affine hull S ⟹ x − ξ = f x *R z
   by metis
 let ?g = λr. r *R z + ξ
 have gf: ?g (f x) = x if x ∈ affine hull S for x
   by (metis diff_add_cancel f that)
 then have inj_f: inj_on f (affine hull S)
```

```
    by (metis inj_onI)
  have diff: x − y = (f x − f y) ∗R z if x: x ∈ affine hull S and y: y ∈ affine hull
S for x y
  proof −
    have f x ∗R z = x − ξ
      by (simp add: f hull_inc x)
    moreover have f y ∗R z = y − ξ
      by (simp add: f hull_inc y)
    ultimately show ?thesis
      by (simp add: scaleR_left.diff)
  qed
  have cont_f: continuous_on (affine hull S) f
  proof (clarsimp simp: dist_norm continuous_on_iff diff)
    show ⋀x e. 0 < e ⟹ ∃ d>0. ∀ y ∈ affine hull S. |f y  − f x| ∗ norm z < d
⟶ |f y  − f x| < e
    by (metis ‹z ≠ 0› mult_pos_pos mult_less_cancel_right_pos zero_less_norm_iff)
  qed
  then have connected (f ‘ S)
    by (meson ‹connected S› connected_continuous_image continuous_on_subset
hull_subset)
  moreover have compact (f ‘ S)
     by (meson ‹compact S› compact_continuous_image_eq cont_f hull_subset
inj_f)
  ultimately obtain x y where f ‘ S = {x..y}
    by (meson connected_compact_interval_1)
  then have fS_eq: f ‘ S = closed_segment x y
    using ‹S ≠ {}› closed_segment_eq_real_ivl by auto
  obtain a b where a ∈ S f a = x b ∈ S f b = y
    by (metis (full_types) ends_in_segment fS_eq imageE)
  have f ‘ (closed_segment a b) = closed_segment (f a) (f b)
  proof (rule continuous_injective_image_segment_1)
    show continuous_on (closed_segment a b) f
       by (meson ‹a ∈ S› ‹b ∈ S› convex_affine_hull convex_contains_segment
hull_inc continuous_on_subset [OF cont_f])
    show inj_on f (closed_segment a b)
       by (meson ‹a ∈ S› ‹b ∈ S› convex_affine_hull convex_contains_segment
hull_inc inj_on_subset [OF inj_f])
  qed
  then have f ‘ (closed_segment a b) = f ‘ S
    by (simp add: ‹f a = x› ‹f b = y› fS_eq)
  then have ?g ‘ f ‘ (closed_segment a b) = ?g ‘ f ‘ S
    by simp
  moreover have (λx. f x ∗R z + ξ) ‘ closed_segment a b = closed_segment a b
    unfolding image_def using ‹a ∈ S› ‹b ∈ S›
   by (safe; metis (mono_tags, lifting)  convex_affine_hull convex_contains_segment
gf hull_subset subsetCE)
  ultimately have closed_segment a b = S
    using gf by (simp add: image_comp o_def hull_inc cong: image_cong)
  then show ?thesis
```

using *that* **by** *blast*
qed

lemma *compact_convex_collinear_segment*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $S \neq \{\}$ *compact S convex S collinear S*
obtains *a b* **where** $S = closed\_segment\ a\ b$
using *assms convex_connected_collinear compact_convex_collinear_segment_alt*
by *blast*

lemma *proper_map_from_compact*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *contf*: *continuous_on S f* **and** *imf*: $f \in S \to T$ **and** *compact S*
closedin (*top_of_set T*) *K*
shows *compact* $(S \cap f -` K)$
by (*rule closedin_compact* [*OF* ‹*compact S*›] *continuous_closedin_preimage_gen assms*)+

lemma *proper_map_fst*:
assumes *compact T K* $\subseteq S$ *compact K*
shows *compact* $(S \times T \cap fst -` K)$
proof −
have $(S \times T \cap fst -` K) = K \times T$
using *assms* **by** *auto*
then show *?thesis* **by** (*simp add*: *assms compact_Times*)
qed

lemma *closed_map_fst*:
fixes $S :: 'a::euclidean\_space\ set$ **and** $T :: 'b::euclidean\_space\ set$
assumes *compact T closedin* (*top_of_set* $(S \times T)$) *c*
shows *closedin* (*top_of_set S*) (*fst* ` *c*)
proof −
have ∗: *fst* ` $(S \times T) \subseteq S$
by *auto*
show *?thesis*
using *proper_map* [*OF* _ _ ∗] **by** (*simp add*: *proper_map_fst assms*)
qed

lemma *proper_map_snd*:
assumes *compact S K* $\subseteq T$ *compact K*
shows *compact* $(S \times T \cap snd -` K)$
proof −
have $(S \times T \cap snd -` K) = S \times K$
using *assms* **by** *auto*
then show *?thesis* **by** (*simp add*: *assms compact_Times*)
qed

lemma *closed_map_snd*:

fixes S :: $'a$::*euclidean_space set* **and** T :: $'b$::*euclidean_space set*
assumes *compact S closedin* (*top_of_set* ($S \times T$)) *c*
shows *closedin* (*top_of_set T*) (*snd* ‘ *c*)
proof −
have ∗: *snd* ‘ ($S \times T$) ⊆ *T*
by *auto*
show *?thesis*
using *proper_map* [*OF* _ _ ∗] **by** (*simp add*: *proper_map_snd assms*)
qed

lemma *closedin_compact_projection*:
fixes S :: $'a$::*euclidean_space set* **and** T :: $'b$::*euclidean_space set*
assumes *compact S* **and** *clo*: *closedin* (*top_of_set* ($S \times T$)) *U*
shows *closedin* (*top_of_set T*) {*y*. ∃*x*. *x* ∈ *S* ∧ (*x*, *y*) ∈ *U*}
proof −
have *U* ⊆ *S* × *T*
by (*metis clo closedin_imp_subset*)
then have {*y*. ∃*x*. *x* ∈ *S* ∧ (*x*, *y*) ∈ *U*} = *snd* ‘ *U*
by *force*
moreover have *closedin* (*top_of_set T*) (*snd* ‘ *U*)
by (*rule closed_map_snd* [*OF assms*])
ultimately show *?thesis*
by *simp*
qed

lemma *closed_compact_projection*:
fixes S :: $'a$::*euclidean_space set*
and T :: ($'a$ ∗ $'b$::*euclidean_space*) *set*
assumes *compact S* **and** *clo*: *closed T*
shows *closed* {*y*. ∃*x*. *x* ∈ *S* ∧ (*x*, *y*) ∈ *T*}
proof −
have ∗: {*y*. ∃*x*. *x* ∈ *S* ∧ *Pair x y* ∈ *T*} = {*y*. ∃*x*. *x* ∈ *S* ∧ *Pair x y* ∈ ((*S* × *UNIV*) ∩ *T*)}
by *auto*
show *?thesis*
unfolding ∗
by (*intro clo closedin_closed_Int closedin_closed_trans* [*OF* _ *closed_UNIV*] *closedin_compact_projection* [*OF* ‹*compact S*›])
qed

Representing affine hull as a finite intersection of hyperplanes

proposition *affine_hull_convex_Int_nonempty_interior*:
fixes S :: $'a$::*real_normed_vector set*
assumes *convex S S* ∩ *interior T* ≠ {}
shows *affine hull* (*S* ∩ *T*) = *affine hull S*
proof
show *affine hull* (*S* ∩ *T*) ⊆ *affine hull S*

```
    by (simp add: hull_mono)
next
  obtain a where a ∈ S a ∈ T and at: a ∈ interior T
    using assms interior_subset by blast
  then obtain e where e > 0 and e: cball a e ⊆ T
    using mem_interior_cball by blast
  have ∗: x ∈ (+) a ‘ span ((λx. x − a) ‘ (S ∩ T)) if x ∈ S for x
  proof (cases x = a)
    case True with that span_0 eq_add_iff image_def mem_Collect_eq show
?thesis
      by blast
  next
    case False
    define k where k = min (1/2) (e / norm (x−a))
    have k: 0 < k k < 1
      using ‹e > 0› False by (auto simp: k_def)
    then have xa: (x−a) = inverse k ∗R k ∗R (x−a)
      by simp
    have e / norm (x − a) ≥ k
      using k_def by linarith
    then have a + k ∗R (x − a) ∈ cball a e
      using ‹0 < k› False
      by (simp add: dist_norm) (simp add: field_simps)
    then have T: a + k ∗R (x − a) ∈ T
      using e by blast
    have S: a + k ∗R (x − a) ∈ S
      using k ‹a ∈ S› convexD [OF ‹convex S› ‹a ∈ S› ‹x ∈ S›, of 1−k k]
      by (simp add: algebra_simps)
    have inverse k ∗R k ∗R (x−a) ∈ span ((λx. x − a) ‘ (S ∩ T))
      by (intro span_mul [OF span_base] image_eqI [where x = a + k ∗R (x −
a)]) (auto simp: S T)
    with xa image_iff show ?thesis  by fastforce
  qed
  have S ⊆ affine hull (S ∩ T)
    by (force simp: ∗ ‹a ∈ S› ‹a ∈ T› hull_inc affine_hull_span_gen [of a])
  then show affine hull S ⊆ affine hull (S ∩ T)
    by (simp add: subset_hull)
qed

corollary affine_hull_convex_Int_open:
  fixes S :: 'a::real_normed_vector set
  assumes convex S open T S ∩ T ≠ {}
  shows affine hull (S ∩ T) = affine hull S
  using affine_hull_convex_Int_nonempty_interior assms interior_eq by blast

corollary affine_hull_affine_Int_nonempty_interior:
  fixes S :: 'a::real_normed_vector set
  assumes affine S S ∩ interior T ≠ {}
  shows affine hull (S ∩ T) = affine hull S
```

by (*simp add*: *affine_hull_convex_Int_nonempty_interior affine_imp_convex assms*)

corollary *affine_hull_affine_Int_open*:
fixes S :: *'a*::*real_normed_vector set*
assumes *affine S open T* $S \cap T \neq \{\}$
shows *affine hull* $(S \cap T)$ = *affine hull S*
by (*simp add*: *affine_hull_convex_Int_open affine_imp_convex assms*)

corollary *affine_hull_convex_Int_openin*:
fixes S :: *'a*::*real_normed_vector set*
assumes *convex S openin* (*top_of_set* (*affine hull S*)) *T* $S \cap T \neq \{\}$
shows *affine hull* $(S \cap T)$ = *affine hull S*
using *assms* **unfolding** *openin_open*
by (*metis affine_hull_convex_Int_open hull_subset inf.orderE inf_assoc*)

corollary *affine_hull_openin*:
fixes S :: *'a*::*real_normed_vector set*
assumes *openin* (*top_of_set* (*affine hull T*)) *S* $S \neq \{\}$
shows *affine hull S* = *affine hull T*
using *assms* **unfolding** *openin_open*
by (*metis affine_affine_hull affine_hull_affine_Int_open hull_hull*)

corollary *affine_hull_open*:
fixes S :: *'a*::*real_normed_vector set*
assumes *open S* $S \neq \{\}$
shows *affine hull S* = *UNIV*
by (*metis affine_hull_convex_Int_nonempty_interior assms convex_UNIV hull_UNIV inf_top.left_neutral interior_open*)

lemma *aff_dim_convex_Int_nonempty_interior*:
fixes S :: *'a*::*euclidean_space set*
shows $[\![$*convex S*; $S \cap$ *interior T* $\neq \{\}]\!] \Longrightarrow$ *aff_dim*$(S \cap T)$ = *aff_dim S*
using *aff_dim_affine_hull2 affine_hull_convex_Int_nonempty_interior* **by** *blast*

lemma *aff_dim_convex_Int_open*:
fixes S :: *'a*::*euclidean_space set*
shows $[\![$*convex S*; *open T*; $S \cap T \neq \{\}]\!] \Longrightarrow$ *aff_dim*$(S \cap T)$ = *aff_dim S*
using *aff_dim_convex_Int_nonempty_interior interior_eq* **by** *blast*

lemma *affine_hull_Diff*:
fixes *S*:: *'a*::*real_normed_vector set*
assumes *ope*: *openin* (*top_of_set S*) *S* **and** *finite F F* $\subset S$
shows *affine hull* $(S - F)$ = *affine hull S*
proof −
have *clo*: *closedin* (*top_of_set S*) *F*
using *assms finite_imp_closedin* **by** *auto*
moreover have $S - F \neq \{\}$
using *assms* **by** *auto*

ultimately show *?thesis*
by (*metis ope closedin_def topspace_euclidean_subtopology affine_hull_openin openin_trans*)
qed

lemma *affine_hull_halfspace_lt*:
fixes *a* :: *'a*::*euclidean_space*
shows *affine hull* {*x*. *a* · *x* < *r*} = (*if a* = *0* ∧ *r* ≤ *0 then* {} *else UNIV*)
using *halfspace_eq_empty_lt* [*of a r*]
by (*simp add*: *open_halfspace_lt affine_hull_open*)

lemma *affine_hull_halfspace_le*:
fixes *a* :: *'a*::*euclidean_space*
shows *affine hull* {*x*. *a* · *x* ≤ *r*} = (*if a* = *0* ∧ *r* < *0 then* {} *else UNIV*)
proof (*cases a* = *0*)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
then have *affine hull closure* {*x*. *a* · *x* < *r*} = *UNIV*
using *affine_hull_halfspace_lt closure_same_affine_hull* **by** *fastforce*
moreover have {*x*. *a* · *x* < *r*} ⊆ {*x*. *a* · *x* ≤ *r*}
by (*simp add*: *Collect_mono*)
ultimately show *?thesis* **using** *False antisym_conv hull_mono top_greatest*
by (*metis affine_hull_halfspace_lt*)
qed

lemma *affine_hull_halfspace_gt*:
fixes *a* :: *'a*::*euclidean_space*
shows *affine hull* {*x*. *a* · *x* > *r*} = (*if a* = *0* ∧ *r* ≥ *0 then* {} *else UNIV*)
using *halfspace_eq_empty_gt* [*of r a*]
by (*simp add*: *open_halfspace_gt affine_hull_open*)

lemma *affine_hull_halfspace_ge*:
fixes *a* :: *'a*::*euclidean_space*
shows *affine hull* {*x*. *a* · *x* ≥ *r*} = (*if a* = *0* ∧ *r* > *0 then* {} *else UNIV*)
using *affine_hull_halfspace_le* [*of* −*a* −*r*] **by** *simp*

lemma *aff_dim_halfspace_lt*:
fixes *a* :: *'a*::*euclidean_space*
shows *aff_dim* {*x*. *a* · *x* < *r*} =
(*if a* = *0* ∧ *r* ≤ *0 then* −*1 else DIM*(*'a*))
by *simp* (*metis aff_dim_open halfspace_eq_empty_lt open_halfspace_lt*)

lemma *aff_dim_halfspace_le*:
fixes *a* :: *'a*::*euclidean_space*
shows *aff_dim* {*x*. *a* · *x* ≤ *r*} =
(*if a* = *0* ∧ *r* < *0 then* −*1 else DIM*(*'a*))
proof −
have *int* (*DIM*(*'a*)) = *aff_dim* (*UNIV*::*'a set*)

```
    by (simp)
  then have aff_dim (affine hull {x. a • x ≤ r}) = DIM('a) if (a = 0 ⟶ r ≥
0)
    using that by (simp add: affine_hull_halfspace_le not_less)
  then show ?thesis
    by (force)
qed

lemma aff_dim_halfspace_gt:
  fixes a :: 'a::euclidean_space
  shows aff_dim {x. a • x > r} =
        (if a = 0 ∧ r ≥ 0 then -1 else DIM('a))
by simp (metis aff_dim_open halfspace_eq_empty_gt open_halfspace_gt)

lemma aff_dim_halfspace_ge:
  fixes a :: 'a::euclidean_space
  shows aff_dim {x. a • x ≥ r} =
        (if a = 0 ∧ r > 0 then -1 else DIM('a))
using aff_dim_halfspace_le [of -a -r] by simp

proposition aff_dim_eq_hyperplane:
  fixes S :: 'a::euclidean_space set
  shows aff_dim S = DIM('a) - 1 ⟷ (∃ a b. a ≠ 0 ∧ affine hull S = {x. a • x
= b})
  (is ?lhs = ?rhs)
proof (cases S = {})
  case True then show ?thesis
    by (auto simp: dest: hyperplane_eq_Ex)
next
  case False
  then obtain c where c ∈ S by blast
  show ?thesis
  proof (cases c = 0)
    case True
    have ?lhs ⟷ (∃ a. a ≠ 0 ∧ span ((λx. x - c) ` S) = {x. a • x = 0})
      by (simp add: aff_dim_eq_dim [of c] ‹c ∈ S› hull_inc dim_eq_hyperplane
del: One_nat_def)
    also have ... ⟷ ?rhs
      using span_zero [of S] True ‹c ∈ S› affine_hull_span_0 hull_inc
      by (fastforce simp add: affine_hull_span_gen [of c] ‹c = 0›)
    finally show ?thesis .
  next
    case False
    have xc_im: x ∈ (+) c ` {y. a • y = 0} if a • x = a • c for a x
    proof -
      have ∃ y. a • y = 0 ∧ c + y = x
      by (metis that add.commute diff_add_cancel inner_commute inner_diff_left
right_minus_eq)
      then show x ∈ (+) c ` {y. a • y = 0}
```

by *blast*
qed
have *2*: *span* $((\lambda x.\ x - c)\ `\ S) = \{x.\ a \cdot x = 0\}$
if $(+)\ c\ `\ span\ ((\lambda x.\ x - c)\ `\ S) = \{x.\ a \cdot x = b\}$ **for** *a b*
proof −
have $b = a \cdot c$
using *span_0 that* **by** *fastforce*
with *that* **have** $(+)\ c\ `\ span\ ((\lambda x.\ x - c)\ `\ S) = \{x.\ a \cdot x = a \cdot c\}$
by *simp*
then have $span\ ((\lambda x.\ x - c)\ `\ S) = (\lambda x.\ x - c)\ `\ \{x.\ a \cdot x = a \cdot c\}$
by (*metis* (*no_types*) *image_cong translation_galois uminus_add_conv_diff*)
also have $... = \{x.\ a \cdot x = 0\}$
by (*force simp*: *inner_distrib inner_diff_right*
intro: *image_eqI* [**where** $x{=}x{+}c$ **for** *x*])
finally show *?thesis* **.**
qed
have *?lhs* $= (\exists a.\ a \neq 0 \wedge span\ ((\lambda x.\ x - c)\ `\ S) = \{x.\ a \cdot x = 0\})$
by (*simp add*: *aff_dim_eq_dim* [*of c*] ‹$c \in S$› *hull_inc dim_eq_hyperplane del*: *One_nat_def*)
also have $... =$ *?rhs*
by (*fastforce simp add*: *affine_hull_span_gen* [*of c*] ‹$c \in S$› *hull_inc inner_distrib intro*: *xc_im intro*!: *2*)
finally show *?thesis* **.**
qed
qed

corollary *aff_dim_hyperplane* [*simp*]:
fixes *a* :: *'a*::*euclidean_space*
shows $a \neq 0 \Longrightarrow aff\_dim\ \{x.\ a \cdot x = r\} = DIM('a) - 1$
by (*metis aff_dim_eq_hyperplane affine_hull_eq affine_hyperplane*)

6.0.15 Some stepping theorems

lemma *aff_dim_insert*:
fixes *a* :: *'a*::*euclidean_space*
shows *aff_dim* (*insert a S*) = (*if a* ∈ *affine hull S then aff_dim S else aff_dim S* + *1*)
proof (*cases S* = {})
case *True* **then show** *?thesis*
by *simp*
next
case *False*
then obtain *x s'* **where** *S*: $S = insert\ x\ s'\ x \notin s'$
by (*meson Set.set_insert all_not_in_conv*)
show *?thesis* **using** *S*
by (*force simp add*: *affine_hull_insert_span_gen span_zero insert_commute* [*of a*] *aff_dim_eq_dim* [*of x*] *dim_insert*)
qed

lemma *affine_dependent_choose*:
fixes $a :: 'a :: euclidean\_space$
assumes $\neg(affine\_dependent\ S)$
shows $affine\_dependent(insert\ a\ S) \longleftrightarrow a \notin S \wedge a \in affine\ hull\ S$
(**is** *?lhs* = *?rhs*)
proof *safe*
assume $affine\_dependent\ (insert\ a\ S)$ **and** $a \in S$
then show *False*
using ‹$a \in S$› *assms insert_absorb* **by** *fastforce*
next
assume *lhs*: $affine\_dependent\ (insert\ a\ S)$
then have $a \notin S$
by (*metis* (*no_types*) *assms insert_absorb*)
moreover have *finite S*
using *affine_independent_iff_card assms* **by** *blast*
moreover have $aff\_dim\ (insert\ a\ S) \neq int\ (card\ S)$
using ‹*finite S*› *affine_independent_iff_card* ‹$a \notin S$› *lhs* **by** *fastforce*
ultimately show $a \in affine\ hull\ S$
by (*metis aff_dim_affine_independent aff_dim_insert assms*)
next
assume $a \notin S$ **and** $a \in affine\ hull\ S$
show $affine\_dependent\ (insert\ a\ S)$
by (*simp add*: ‹$a \in affine\ hull\ S$› ‹$a \notin S$› *affine_dependent_def*)
qed

lemma *affine_independent_insert*:
fixes $a :: 'a :: euclidean\_space$
shows $\llbracket\neg\ affine\_dependent\ S;\ a \notin affine\ hull\ S\rrbracket \Longrightarrow \neg\ affine\_dependent(insert\ a\ S)$
by (*simp add*: *affine_dependent_choose*)

lemma *subspace_bounded_eq_trivial*:
fixes $S :: 'a{::}real\_normed\_vector\ set$
assumes *subspace S*
shows $bounded\ S \longleftrightarrow S = \{0\}$
proof −
have *False* **if** $bounded\ S\ x \in S\ x \neq 0$ **for** *x*
proof −
obtain *B* **where** *B*: $\bigwedge y.\ y \in S \Longrightarrow norm\ y < B\ B > 0$
using ‹*bounded S*› **by** (*force simp*: *bounded_pos_less*)
have $(B\ /\ norm\ x) *_R x \in S$
using *assms subspace_mul* ‹$x \in S$› **by** *auto*
moreover have $norm\ ((B\ /\ norm\ x) *_R x) = B$
using *that B* **by** (*simp add*: *algebra_simps*)
ultimately show *False* **using** *B* **by** *force*
qed
then have $bounded\ S \Longrightarrow S = \{0\}$
using *assms subspace_0* **by** *fastforce*
then show *?thesis*

by *blast*
qed

lemma *affine_bounded_eq_trivial*:
fixes S :: *'a*::*real_normed_vector set*
assumes *affine S*
shows *bounded S* $\longleftrightarrow$ $S = \{\} \vee (\exists\, a.\ S = \{a\})$
proof (*cases* $S = \{\}$)
case *True* **then show** *?thesis*
by *simp*
next
case *False*
then obtain *b* **where** $b \in S$ **by** *blast*
with *False assms*
have *bounded S* $\Longrightarrow$ $S = \{b\}$
using *affine_diffs_subspace* [*OF assms* ‹$b \in S$›]
by (*metis* (*no_types*, *lifting*) *ab_group_add_class.ab_left_minus bounded_translation image_empty image_insert subspace_bounded_eq_trivial translation_invert*)
then show *?thesis* **by** *auto*
qed

lemma *affine_bounded_eq_lowdim*:
fixes S :: *'a*::*euclidean_space set*
assumes *affine S*
shows *bounded S* $\longleftrightarrow$ *aff_dim S* $\leq$ *0*
proof
show *aff_dim S* $\leq$ *0* $\Longrightarrow$ *bounded S*
by (*metis aff_dim_sing aff_dim_subset affine_dim_equal affine_sing all_not_in_conv assms bounded_empty bounded_insert dual_order.antisym empty_subsetI insert_subset*)
qed (*use affine_bounded_eq_trivial assms* **in** *fastforce*)

lemma *bounded_hyperplane_eq_trivial_0*:
fixes *a* :: *'a*::*euclidean_space*
assumes $a \neq 0$
shows *bounded* $\{x.\ a \cdot x = 0\}$ $\longleftrightarrow$ *DIM*(*'a*) = *1*
proof
assume *bounded* $\{x.\ a \cdot x = 0\}$
then have *0*: *aff_dim* $\{x.\ a \cdot x = 0\} \leq 0$
by (*simp add*: *affine_bounded_eq_lowdim affine_hyperplane*)
with *assms 0*
have *int DIM*(*'a*) = *1*
using *order_neq_le_trans* **by** *fastforce*
then show *DIM*(*'a*) = *1* **by** *auto*
next
assume *DIM*(*'a*) = *1*
then show *bounded* $\{x.\ a \cdot x = 0\}$
by (*simp add*: *affine_bounded_eq_lowdim affine_hyperplane assms*)
qed

lemma *bounded_hyperplane_eq_trivial*:
fixes $a :: 'a::euclidean\_space$
shows $bounded\ \{x.\ a \cdot x = r\} \longleftrightarrow (if\ a = 0\ then\ r \neq 0\ else\ DIM('a) = 1)$
proof −
{ assume $r \neq 0\ a \neq 0$
have $aff\_dim\ \{x.\ y \cdot x = 0\} = aff\_dim\ \{x.\ a \cdot x = r\}$ **if** $y \neq 0$ **for** $y::'a$
by (*metis that* ‹$a \neq 0$› *aff_dim_hyperplane*)
then have $bounded\ \{x.\ a \cdot x = r\} = (DIM('a) = Suc\ 0)$
by (*metis One_nat_def* ‹$a \neq 0$› *affine_bounded_eq_lowdim affine_hyperplane bounded_hyperplane_eq_trivial_0*)
}
then show *?thesis*
by (*auto simp*: *bounded_hyperplane_eq_trivial_0*)
qed

6.0.16 General case without assuming closure and getting non-strict separation

proposition *separating_hyperplane_closed_point_inset*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *convex S closed S* $S \neq \{\}\ z \notin S$
obtains *a b* **where** $a \in S\ (a - z) \cdot z < b\ \bigwedge x.\ x \in S \Longrightarrow b < (a - z) \cdot x$
proof −
obtain *y* **where** $y \in S$ **and** y: $\bigwedge u.\ u \in S \Longrightarrow dist\ z\ y \leq dist\ z\ u$
using *distance_attains_inf* [*of S z*] *assms* **by** *auto*
then have $*$: $(y - z) \cdot z < (y - z) \cdot z + (norm\ (y - z))^2\ /\ 2$
using ‹$y \in S$› ‹$z \notin S$› **by** *auto*
show *?thesis*
proof (*rule that* [*OF* ‹$y \in S$› $*$])
fix *x*
assume $x \in S$
have *yz*: $0 < (y - z) \cdot (y - z)$
using ‹$y \in S$› ‹$z \notin S$› **by** *auto*
{ assume *0*: $0 < ((z - y) \cdot (x - y))$
with *any_closest_point_dot* [*OF* ‹*convex S*› ‹*closed S*›]
have *False*
using *y* ‹$x \in S$› ‹$y \in S$› *not_less* **by** *blast*
}
then have $0 \leq ((y - z) \cdot (x - y))$
by (*force simp*: *not_less inner_diff_left*)
with *yz* **have** $0 < 2 * ((y - z) \cdot (x - y)) + (y - z) \cdot (y - z)$
by (*simp add*: *algebra_simps*)
then show $(y - z) \cdot z + (norm\ (y - z))^2\ /\ 2 < (y - z) \cdot x$
by (*simp add*: *field_simps inner_diff_left inner_diff_right dot_square_norm* [*symmetric*])
qed
qed

lemma *separating_hyperplane_closed_0_inset*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *convex S closed S* $S \neq \{\}$ $0 \notin S$
obtains *a b* **where** $a \in S$ $a \neq 0$ $0 < b$ $\bigwedge x.\ x \in S \Longrightarrow a \cdot x > b$
using *separating_hyperplane_closed_point_inset* [*OF assms*] **by** *simp* (*metis* ‹*0* $\notin$ *S*›)

proposition *separating_hyperplane_set_0_inspan*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *convex S* $S \neq \{\}$ $0 \notin S$
obtains *a* **where** $a \in span\ S$ $a \neq 0$ $\bigwedge x.\ x \in S \Longrightarrow 0 \leq a \cdot x$
proof −
define *k* **where** [*abs_def*]: $k\ c = \{x.\ 0 \leq c \cdot x\}$ **for** $c :: 'a$
have $span\ S \cap frontier\ (cball\ 0\ 1) \cap \bigcap f' \neq \{\}$
if f': *finite* f' $f' \subseteq k\ `\ S$ **for** f'
proof −
obtain *C* **where** $C \subseteq S$ *finite C* **and** C: $f' = k\ `\ C$
using *finite_subset_image* [*OF f'*] **by** *blast*
obtain *a* **where** $a \in S$ $a \neq 0$
using ‹$S \neq \{\}$› ‹$0 \notin S$› *ex_in_conv* **by** *blast*
then have $norm\ (a\ /_R\ (norm\ a)) = 1$
by *simp*
moreover have $a\ /_R\ (norm\ a) \in span\ S$
by (*simp add*: ‹$a \in S$› *span_scale span_base*)
ultimately have *ass*: $a\ /_R\ (norm\ a) \in span\ S \cap sphere\ 0\ 1$
by *simp*
show *?thesis*
proof (*cases* $C = \{\}$)
case *True* **with** *C ass* **show** *?thesis*
by *auto*
next
case *False*
have *closed* (*convex hull C*)
using ‹*finite C*› *compact_eq_bounded_closed finite_imp_compact_convex_hull*
by *auto*
moreover have *convex hull C* $\neq \{\}$
by (*simp add*: *False*)
moreover have $0 \notin$ *convex hull C*
by (*metis* ‹$C \subseteq S$› ‹*convex S*› ‹$0 \notin S$› *convex_hull_subset hull_same*
insert_absorb insert_subset)
ultimately obtain *a b*
where $a \in$ *convex hull C* $a \neq 0$ $0 < b$
and *ab*: $\bigwedge x.\ x \in$ *convex hull C* $\Longrightarrow a \cdot x > b$
using *separating_hyperplane_closed_0_inset* **by** *blast*
then have $a \in S$
by (*metis* ‹$C \subseteq S$› *assms*(*1*) *subsetCE subset_hull*)
moreover have $norm\ (a\ /_R\ (norm\ a)) = 1$
using ‹$a \neq 0$› **by** *simp*

```
      moreover have a /R (norm a) ∈ span S
        by (simp add: ‹a ∈ S› span_scale span_base)
      ultimately have ass: a /R (norm a) ∈ span S ∩ sphere 0 1
        by simp
      have ⋀x. ⟦a ≠ 0; x ∈ C⟧ ⟹ 0 ≤ x · a
         using ab ‹0 < b› by (metis hull_inc inner_commute less_eq_real_def
less_trans)
      then have aa: a /R (norm a) ∈ (⋂c∈C. {x. 0 ≤ c · x})
        by (auto simp add: field_split_simps)
      show ?thesis
        unfolding C k_def
        using ass aa Int_iff empty_iff by force
    qed
  qed
  moreover have ⋀T. T ∈ k ` S ⟹ closed T
    using closed_halfspace_ge k_def by blast
  ultimately have (span S ∩ frontier(cball 0 1)) ∩ (⋂ (k ` S)) ≠ {}
     by (metis compact_imp_fip closed_Int_compact closed_span compact_cball
compact_frontier)
  then show ?thesis
    unfolding set_eq_iff k_def
    by simp (metis inner_commute norm_eq_zero that zero_neq_one)
qed


lemma separating_hyperplane_set_point_inaff:
  fixes S :: 'a::euclidean_space set
  assumes convex S S ≠ {} and zno: z ∉ S
  obtains a b where (z + a) ∈ affine hull (insert z S)
               and a ≠ 0 and a · z ≤ b
               and ⋀x. x ∈ S ⟹ a · x ≥ b
proof −
  from separating_hyperplane_set_0_inspan [of image (λx. −z + x) S]
  have convex ((+) (− z) ` S)
    using ‹convex S› by simp
  moreover have (+) (− z) ` S ≠ {}
    by (simp add: ‹S ≠ {}›)
  moreover have 0 ∉ (+) (− z) ` S
    using zno by auto
  ultimately obtain a where a ∈ span ((+) (− z) ` S) a ≠ 0
                  and a:  ⋀x. x ∈ ((+) (− z) ` S) ⟹ 0 ≤ a · x
    using separating_hyperplane_set_0_inspan [of image (λx. −z + x) S]
    by blast
  then have szx: ⋀x. x ∈ S ⟹ a · z ≤ a · x
     by (metis (no_types, lifting) imageI inner_minus_right inner_right_distrib
minus_add neg_le_0_iff_le neg_le_iff_le real_add_le_0_iff)
  moreover
  have z + a ∈ affine hull insert z S
    using ‹a ∈ span ((+) (− z) ` S)› affine_hull_insert_span_gen by blast
```

```
  ultimately show ?thesis
    using ‹a ≠ 0› szx that by auto
qed

proposition supporting_hyperplane_rel_boundary:
  fixes S :: 'a::euclidean_space set
  assumes convex S x ∈ S and xno: x ∉ rel_interior S
  obtains a where a ≠ 0
            and ⋀y. y ∈ S ⟹ a · x ≤ a · y
            and ⋀y. y ∈ rel_interior S ⟹ a · x < a · y
proof −
  obtain a b where aff: (x + a) ∈ affine hull (insert x (rel_interior S))
                and a ≠ 0 and a · x ≤ b
                and ageb: ⋀u. u ∈ (rel_interior S) ⟹ a · u ≥ b
    using separating_hyperplane_set_point_inaff [of rel_interior S x] assms
    by (auto simp: rel_interior_eq_empty convex_rel_interior)
  have le_ay: a · x ≤ a · y if y ∈ S for y
  proof −
    have con: continuous_on (closure (rel_interior S)) ((·) a)
      by (rule continuous_intros continuous_on_subset | blast)+
    have y: y ∈ closure (rel_interior S)
      using ‹convex S› closure_def convex_closure_rel_interior ‹y ∈ S›
      by fastforce
    show ?thesis
      using continuous_ge_on_closure [OF con y] ageb ‹a · x ≤ b›
      by fastforce
  qed
  have 3: a · x < a · y if y ∈ rel_interior S for y
  proof −
    obtain e where 0 < e y ∈ S and e: cball y e ∩ affine hull S ⊆ S
      using ‹y ∈ rel_interior S› by (force simp: rel_interior_cball)
    define y' where y' = y − (e / norm a) *R ((x + a) − x)
    have y' ∈ cball y e
      unfolding y'_def using ‹0 < e› by force
    moreover have y' ∈ affine hull S
      unfolding y'_def
      by (metis ‹x ∈ S› ‹y ∈ S› ‹convex S› aff affine_affine_hull hull_redundant
            rel_interior_same_affine_hull hull_inc mem_affine_3_minus2)
    ultimately have y' ∈ S
      using e by auto
    have a · x ≤ a · y
      using le_ay ‹a ≠ 0› ‹y ∈ S› by blast
    moreover have a · x ≠ a · y
      using le_ay [OF ‹y' ∈ S›] ‹a ≠ 0› ‹0 < e› not_le
    by (fastforce simp add: y'_def inner_diff dot_square_norm power2_eq_square)
    ultimately show ?thesis by force
  qed
  show ?thesis
    by (rule that [OF ‹a ≠ 0› le_ay 3])
```

qed

lemma *supporting_hyperplane_relative_frontier*:
fixes S :: *'a::euclidean_space set*
assumes *convex* S $x \in$ *closure* S $x \notin$ *rel_interior* S
obtains a **where** $a \neq 0$
and $\bigwedge y.\ y \in$ *closure* $S \Longrightarrow a \cdot x \leq a \cdot y$
and $\bigwedge y.\ y \in$ *rel_interior* $S \Longrightarrow a \cdot x < a \cdot y$
using *supporting_hyperplane_rel_boundary* [*of closure* S x]
by (*metis assms convex_closure convex_rel_interior_closure*)

6.0.17 Some results on decomposing convex hulls: intersections, simplicial subdivision

lemma
fixes S :: *'a::euclidean_space set*
assumes ¬ *affine_dependent*($S \cup T$)
shows *convex_hull_Int_subset*: *convex hull* $S \cap$ *convex hull* $T \subseteq$ *convex hull* ($S \cap T$) (**is** *?C*)
and *affine_hull_Int_subset*: *affine hull* $S \cap$ *affine hull* $T \subseteq$ *affine hull* ($S \cap T$) (**is** *?A*)
proof −
have [*simp*]: *finite* S *finite* T
using *aff_independent_finite assms* **by** *blast+*
have *sum* u ($S \cap T$) $= 1 \wedge$
$(\sum v{\in}S \cap T.\ u\ v *_R v) = (\sum v{\in}S.\ u\ v *_R v)$
if [*simp*]: *sum* u $S = 1$
sum v $T = 1$
and *eq*: $(\sum x{\in}T.\ v\ x *_R x) = (\sum x{\in}S.\ u\ x *_R x)$ **for** u v
proof −
define f **where** $f\ x = (\textit{if } x \in S \textit{ then } u\ x \textit{ else } 0) - (\textit{if } x \in T \textit{ then } v\ x \textit{ else } 0)$ **for** x
have *sum* f ($S \cup T$) $= 0$
by (*simp add*: *f_def sum_Un sum_subtractf flip*: *sum.inter_restrict*)
moreover have $(\sum x{\in}(S \cup T).\ f\ x *_R x) = 0$
by (*simp add*: *eq f_def sum_Un scaleR_left_diff_distrib sum_subtractf if_smult flip*: *sum.inter_restrict cong*: *if_cong*)
ultimately have $\bigwedge v.\ v \in S \cup T \Longrightarrow f\ v = 0$
using *aff_independent_finite assms* **unfolding** *affine_dependent_explicit*
by *blast*
then have u [*simp*]: $\bigwedge x.\ x \in S \Longrightarrow u\ x = (\textit{if } x \in T \textit{ then } v\ x \textit{ else } 0)$
by (*simp add*: *f_def*) *presburger*
have *sum* u ($S \cap T$) $=$ *sum* u S
by (*simp add*: *sum.inter_restrict*)
then have *sum* u ($S \cap T$) $= 1$
using *that* **by** *linarith*
moreover have $(\sum v{\in}S \cap T.\ u\ v *_R v) = (\sum v{\in}S.\ u\ v *_R v)$
by (*auto simp*: *if_smult sum.inter_restrict intro*: *sum.cong*)
ultimately show *?thesis*

```
    by force
  qed
  then show ?A ?C
    by (auto simp: convex_hull_finite affine_hull_finite)
qed


proposition affine_hull_Int:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent(S ∪ T)
    shows affine hull (S ∩ T) = affine hull S ∩ affine hull T
  by (simp add: affine_hull_Int_subset assms hull_mono subset_antisym)

proposition convex_hull_Int:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent(S ∪ T)
    shows convex hull (S ∩ T) = convex hull S ∩ convex hull T
  by (simp add: convex_hull_Int_subset assms hull_mono subset_antisym)

proposition
  fixes S :: 'a::euclidean_space set set
  assumes ¬ affine_dependent (⋃ S)
    shows affine_hull_Inter: affine hull (⋂ S) = (⋂ T∈S. affine hull T) (is ?A)
     and convex_hull_Inter: convex hull (⋂ S) = (⋂ T∈S. convex hull T) (is ?C)
proof −
  have finite S
    using aff_independent_finite assms finite_UnionD by blast
  then have ?A ∧ ?C using assms
  proof (induction S rule: finite_induct)
    case empty then show ?case by auto
  next
    case (insert T F)
    then show ?case
    proof (cases F={})
      case True then show ?thesis by simp
    next
      case False
      with insert.prems have [simp]: ¬ affine_dependent (T ∪ ⋂ F)
        by (auto intro: affine_dependent_subset)
      have [simp]: ¬ affine_dependent (⋃ F)
        using affine_independent_subset insert.prems by fastforce
      show ?thesis
        by (simp add: affine_hull_Int convex_hull_Int insert.IH)
    qed
  qed
  then show ?A ?C
    by auto
qed
```

proposition *in_convex_hull_exchange_unique*:
fixes S :: *'a::euclidean_space set*
assumes *naff*: ¬ *affine_dependent* S **and** a: $a \in$ *convex hull* S
and S: $T \subseteq S$ $T' \subseteq S$
and x: $x \in$ *convex hull* (*insert* a T)
and x': $x \in$ *convex hull* (*insert* a T')
shows $x \in$ *convex hull* (*insert* a ($T \cap T'$))
proof (*cases* $a \in S$)
case *True*
then have ¬ *affine_dependent* (*insert* a $T \cup$ *insert* a T')
using *affine_dependent_subset assms* **by** *auto*
then have $x \in$ *convex hull* (*insert* a $T \cap$ *insert* a T')
by (*metis IntI convex_hull_Int* x x')
then show *?thesis*
by *simp*
next
case *False*
then have *anot*: $a \notin T$ $a \notin T'$
using *assms* **by** *auto*
have [*simp*]: *finite* S
by (*simp add*: *aff_independent_finite assms*)
then obtain b **where** *b0*: $\bigwedge s.\ s \in S \Longrightarrow 0 \leq b\ s$
and *b1*: *sum* b $S = 1$ **and** *aeq*: $a = (\sum s{\in}S.\ b\ s *_R s)$
using a **by** (*auto simp*: *convex_hull_finite*)
have *fin* [*simp*]: *finite* T *finite* T'
using *assms infinite_super* ‹*finite* S› **by** *blast+*
then obtain c c' **where** *c0*: $\bigwedge t.\ t \in$ *insert* a $T \Longrightarrow 0 \leq c\ t$
and *c1*: *sum* c (*insert* a T) = 1
and *xeq*: $x = (\sum t \in$ *insert* a $T.\ c\ t *_R t)$
and *c'0*: $\bigwedge t.\ t \in$ *insert* a $T' \Longrightarrow 0 \leq c'\ t$
and *c'1*: *sum* c' (*insert* a T') = 1
and *x'eq*: $x = (\sum t \in$ *insert* a $T'.\ c'\ t *_R t)$
using x x' **by** (*auto simp*: *convex_hull_finite*)
with *fin anot*
have *sumTT'*: *sum* c $T = 1 - c\ a$ *sum* c' $T' = 1 - c'\ a$
and *wsumT*: $(\sum t \in T.\ c\ t *_R t) = x - c\ a *_R a$
by *simp_all*
have *wsumT'*: $(\sum t \in T'.\ c'\ t *_R t) = x - c'\ a *_R a$
using *x'eq fin anot* **by** *simp*
define *cc* **where** *cc* ≡ $\lambda x.$ *if* $x \in T$ *then* $c\ x$ *else* 0
define *cc'* **where** *cc'* ≡ $\lambda x.$ *if* $x \in T'$ *then* $c'\ x$ *else* 0
define *dd* **where** *dd* ≡ $\lambda x.$ *cc* $x -$ *cc'* $x + (c\ a - c'\ a) * b\ x$
have *sumSS'*: *sum cc* $S = 1 - c\ a$ *sum cc'* $S = 1 - c'\ a$
unfolding *cc_def cc'_def* **using** S
by (*simp_all add*: *Int_absorb1 Int_absorb2 sum_subtractf sum.inter_restrict* [*symmetric*] *sumTT'*)
have *wsumSS*: $(\sum t \in S.$ *cc* $t *_R t) = x - c\ a *_R a$ $(\sum t \in S.$ *cc'* $t *_R t) = x - c'\ a *_R a$
unfolding *cc_def cc'_def* **using** S

by (*simp_all add*: *Int_absorb1 Int_absorb2 if_smult sum.inter_restrict* [*symmetric*] *wsumT wsumT′ cong*: *if_cong*)
have *sum_dd0*: *sum dd S = 0*
unfolding *dd_def* **using** *S*
by (*simp add*: *sumSS′ comm_monoid_add_class.sum.distrib sum_subtractf algebra_simps sum_distrib_right* [*symmetric*] *b1*)
have $(\sum v{\in}S.\ (b\ v * x) *_R v) = x *_R (\sum v{\in}S.\ b\ v *_R v)$ **for** *x*
by (*simp add*: *pth_5 real_vector.scale_sum_right mult.commute*)
then have $*$: $(\sum v{\in}S.\ (b\ v * x) *_R v) = x *_R a$ **for** *x*
using *aeq* **by** *blast*
have $(\sum v \in S.\ dd\ v *_R v) = 0$
unfolding *dd_def* **using** *S*
by (*simp add*: $*$ *wsumSS sum.distrib sum_subtractf algebra_simps*)
then have *dd0*: *dd v = 0* **if** $v \in S$ **for** *v*
using *naff* [*unfolded affine_dependent_explicit not_ex*, *rule_format*, *of S dd*]
using *that sum_dd0* **by** *force*
consider $c'\ a \le c\ a \mid c\ a \le c'\ a$ **by** *linarith*
then show *?thesis*
proof *cases*
case *1*
then have *sum cc S* $\le$ *sum cc′ S*
by (*simp add*: *sumSS′*)
then have *le*: *cc x* $\le$ *cc′ x* **if** $x \in S$ **for** *x*
using *dd0* [*OF that*] *1 b0 mult_left_mono that*
by (*fastforce simp add*: *dd_def algebra_simps*)
have *cc0*: *cc x = 0* **if** $x \in S\ x \notin T \cap T'$ **for** *x*
using *le* [*OF* ‹$x \in S$›] *that c0*
by (*force simp*: *cc_def cc′_def split*: *if_split_asm*)
have *ge0*: $\forall x{\in}T \cap T'.\ 0 \le (cc(a := c\ a))\ x$
by (*simp add*: *c0 cc_def*)
have *sum* $(cc(a := c\ a))$ (*insert a* $(T \cap T')$) = *c a* + *sum* $(cc(a := c\ a))$ $(T \cap T')$
by (*simp add*: *anot*)
also have ... = *c a* + *sum* $(cc(a := c\ a))$ *S*
using ‹$T \subseteq S$› *False cc0 cc_def* ‹$a \notin S$› **by** (*fastforce intro*!: *sum.mono_neutral_left split*: *if_split_asm*)
also have ... = *c a* + (*1* − *c a*)
by (*metis* ‹$a \notin S$› *fun_upd_other sum.cong sumSS′*(*1*))
finally have *1*: *sum* $(cc(a := c\ a))$ (*insert a* $(T \cap T')$) = *1*
by *simp*
have $(\sum x{\in}$*insert a* $(T \cap T').\ (cc(a := c\ a))\ x *_R x) = c\ a *_R a + (\sum x \in T \cap T'.\ (cc(a := c\ a))\ x *_R x)$
by (*simp add*: *anot*)
also have ... = *c a* $*_R$ *a* + $(\sum x \in S.\ (cc(a := c\ a))\ x *_R x)$
using ‹$T \subseteq S$› *False cc0* **by** (*fastforce intro*!: *sum.mono_neutral_left split*: *if_split_asm*)
also have ... = *c a* $*_R$ *a* + *x* − *c a* $*_R$ *a*
by (*simp add*: *wsumSS* ‹$a \notin S$› *if_smult sum_delta_notmem*)
finally have *self*: $(\sum x{\in}$*insert a* $(T \cap T').\ (cc(a := c\ a))\ x *_R x) = x$

```
    by simp
  show ?thesis
    by (force simp: convex_hull_finite c0 intro!: ge0 1 self exI [where x = cc(a := c a)])
 next
  case 2
  then have sum cc' S ≤ sum cc S
    by (simp add: sumSS')
  then have le: cc' x ≤ cc x if x ∈ S for x
    using dd0 [OF that] 2 b0 mult_left_mono that
    by (fastforce simp add: dd_def algebra_simps)
  have cc0: cc' x = 0 if x ∈ S x ∉ T ∩ T' for x
    using le [OF ‹x ∈ S›] that c'0
    by (force simp: cc_def cc'_def split: if_split_asm)
  have ge0: ∀x∈T ∩ T'. 0 ≤ (cc'(a := c' a)) x
    by (simp add: c'0 cc'_def)
  have sum (cc'(a := c' a)) (insert a (T ∩ T')) = c' a + sum (cc'(a := c' a)) (T ∩ T')
    by (simp add: anot)
  also have ... = c' a + sum (cc'(a := c' a)) S
    using ‹T ⊆ S› False cc0 by (fastforce intro!: sum.mono_neutral_left split: if_split_asm)
  also have ... = c' a + (1 − c' a)
    by (metis ‹a ∉ S› fun_upd_other sum.cong sumSS')
  finally have 1: sum (cc'(a := c' a)) (insert a (T ∩ T')) = 1
    by simp
  have (∑x∈insert a (T ∩ T'). (cc'(a := c' a)) x *R x) = c' a *R a + (∑x ∈ T ∩ T'. (cc'(a := c' a)) x *R x)
    by (simp add: anot)
  also have ... = c' a *R a + (∑x ∈ S. (cc'(a := c' a)) x *R x)
    using ‹T ⊆ S› False cc0 by (fastforce intro!: sum.mono_neutral_left split: if_split_asm)
  also have ... = c a *R a + x − c a *R a
    by (simp add: wsumSS ‹a ∉ S› if_smult sum_delta_notmem)
  finally have self: (∑x∈insert a (T ∩ T'). (cc'(a := c' a)) x *R x) = x
    by simp
  show ?thesis
    by (force simp: convex_hull_finite c'0 intro!: ge0 1 self exI [where x = cc'(a := c' a)])
 qed
qed

corollary convex_hull_exchange_Int:
 fixes a :: 'a::euclidean_space
 assumes ¬ affine_dependent S a ∈ convex hull S T ⊆ S T' ⊆ S
 shows (convex hull (insert a T)) ∩ (convex hull (insert a T')) =
       convex hull (insert a (T ∩ T')) (is ?lhs = ?rhs)
proof (rule subset_antisym)
 show ?lhs ⊆ ?rhs
```

```
    using in_convex_hull_exchange_unique assms by blast
  show ?rhs ⊆ ?lhs
    by (metis hull_mono inf_le1 inf_le2 insert_inter_insert le_inf_iff)
qed

lemma Int_closed_segment:
  fixes b :: 'a::euclidean_space
  assumes b ∈ closed_segment a c ∨ ¬ collinear{a,b,c}
    shows closed_segment a b ∩ closed_segment b c = {b}
proof (cases c = a)
  case True
  then show ?thesis
    using assms collinear_3_eq_affine_dependent by fastforce
next
  case False
  from assms show ?thesis
  proof
    assume b ∈ closed_segment a c
    moreover have ¬ affine_dependent {a, c}
      by (simp)
    ultimately show ?thesis
      using False convex_hull_exchange_Int [of {a,c} b {a} {c}]
      by (simp add: segment_convex_hull insert_commute)
  next
    assume ncoll: ¬ collinear {a, b, c}
    have False if closed_segment a b ∩ closed_segment b c ≠ {b}
    proof −
      have b ∈ closed_segment a b and b ∈ closed_segment b c
        by auto
     with that obtain d where b ≠ d d ∈ closed_segment a b d ∈ closed_segment
b c
        by force
      then have d: collinear {a, d, b}  collinear {b, d, c}
        by (auto simp:  between_mem_segment between_imp_collinear)
      have collinear {a, b, c}
        by (metis (full_types) ‹b ≠ d› collinear_3_trans d insert_commute)
      with ncoll show False ..
    qed
    then show ?thesis
      by blast
  qed
qed

lemma affine_hull_finite_intersection_hyperplanes:
  fixes S :: 'a::euclidean_space set
  obtains F where
    finite F
    of_nat (card F) + aff_dim S = DIM('a)
    affine hull S = ⋂F
```

```
    ⋀h. h ∈ ℱ ⟹ ∃a b. a ≠ 0 ∧ h = {x. a · x = b}
proof −
  obtain b where b ⊆ S
            and indb: ¬ affine_dependent b
            and eq: affine hull S = affine hull b
    using affine_basis_exists by blast
  obtain c where indc: ¬ affine_dependent c and b ⊆ c
            and affc: affine hull c = UNIV
   by (metis extend_to_affine_basis affine_UNIV hull_same indb subset_UNIV)
  then have finite c
    by (simp add: aff_independent_finite)
  then have fbc: finite b card b ≤ card c
    using ‹b ⊆ c› infinite_super by (auto simp: card_mono)
  have imeq: (λx. affine hull x) ‘ ((λa. c − {a}) ‘ (c − b)) = ((λa. affine hull (c
− {a})) ‘ (c − b))
    by blast
  have card_cb: (card (c − b)) + aff_dim S = DIM('a)
  proof −
    have aff: aff_dim (UNIV::'a set) = aff_dim c
      by (metis aff_dim_affine_hull affc)
    have aff_dim b = aff_dim S
      by (metis (no_types) aff_dim_affine_hull eq)
    then have int (card b) = 1 + aff_dim S
      by (simp add: aff_dim_affine_independent indb)
    then show ?thesis
      using fbc aff
      by (simp add: ‹¬ affine_dependent c› ‹b ⊆ c› aff_dim_affine_independent
card_Diff_subset of_nat_diff)
  qed
  show ?thesis
  proof (cases c = b)
    case True show ?thesis
    proof
      show int (card {}) + aff_dim S = int DIM('a)
        using True card_cb by auto
      show affine hull S = ⋂ {}
        using True affc eq by blast
    qed auto
  next
    case False
    have ind: ¬ affine_dependent (⋃a∈c − b. c − {a})
      by (rule affine_independent_subset [OF indc]) auto
    have ∗: 1 + aff_dim (c − {t}) = int (DIM('a)) if t: t ∈ c for t
    proof −
      have insert t c = c
        using t by blast
      then show ?thesis
        by (metis (full_types) add.commute aff_dim_affine_hull aff_dim_insert
aff_dim_UNIV affc affine_dependent_def indc insert_Diff_single t)
```

```
    qed
    let ?F = (λx. affine hull x) ‘ ((λa. c − {a}) ‘ (c − b))
    show ?thesis
    proof
      have card ((λa. affine hull (c − {a})) ‘ (c − b)) = card (c − b)
      proof (rule card_image)
        show inj_on (λa. affine hull (c − {a})) (c − b)
          unfolding inj_on_def
       by (metis Diff_eq_empty_iff Diff_iff indc affine_dependent_def hull_subset
insert_iff)
      qed
      then show int (card ?F) + aff_dim S = int DIM('a)
        by (simp add: imeq card_cb)
      show affine hull S = ⋂ ?F
       by (metis Diff_eq_empty_iff INT_simps(4) UN_singleton order_refl ‹b ⊆
c› False eq double_diff affine_hull_Inter [OF ind])
      have ⋀a. ⟦a ∈ c; a ∉ b⟧ ⟹ aff_dim (c − {a}) = int (DIM('a) − Suc 0)
       by (metis ∗ DIM_ge_Suc0 One_nat_def add_diff_cancel_left′ int_ops(2)
of_nat_diff)
      then show ⋀h. h ∈ ?F ⟹ ∃ a b. a ≠ 0 ∧ h = {x. a · x = b}
        by (auto simp only: One_nat_def aff_dim_eq_hyperplane [symmetric])
      qed (use ‹finite c› in auto)
  qed
qed

lemma affine_hyperplane_sums_eq_UNIV_0:
  fixes S :: 'a :: euclidean_space set
  assumes affine S
     and 0 ∈ S and w ∈ S
     and a · w ≠ 0
   shows {x + y| x y. x ∈ S ∧ a · y = 0} = UNIV
proof −
  have subspace S
    by (simp add: assms subspace_affine)
  have span1: span {y. a · y = 0} ⊆ span {x + y |x y. x ∈ S ∧ a · y = 0}
    using ‹0 ∈ S› add.left_neutral by (intro span_mono) force
  have w ∉ span {y. a · y = 0}
    using ‹a · w ≠ 0› span_induct subspace_hyperplane by auto
  moreover have w ∈ span {x + y |x y. x ∈ S ∧ a · y = 0}
    using ‹w ∈ S›
   by (metis (mono_tags, lifting) inner_zero_right mem_Collect_eq pth_d span_base)
  ultimately have span2: span {y. a · y = 0} ≠ span {x + y |x y. x ∈ S ∧ a ·
y = 0}
    by blast
  have a ≠ 0 using assms inner_zero_left by blast
  then have DIM('a) − 1 = dim {y. a · y = 0}
    by (simp add: dim_hyperplane)
  also have ... < dim {x + y |x y. x ∈ S ∧ a · y = 0}
    using span1 span2 by (blast intro: dim_psubset)
```

finally have $DIM('a) - 1 < dim\ \{x + y\ |x\ y.\ x \in S \wedge a \cdot y = 0\}$ **.**
then have *DD*: $dim\ \{x + y\ |x\ y.\ x \in S \wedge a \cdot y = 0\} = DIM('a)$
using *antisym dim_subset_UNIV lowdim_subset_hyperplane not_le* **by** *fastforce*
have *subs*: *subspace* $\{x + y|\ x\ y.\ x \in S \wedge a \cdot y = 0\}$
using *subspace_sums* [*OF* ‹*subspace S*› *subspace_hyperplane*] **by** *simp*
moreover have *span* $\{x + y|\ x\ y.\ x \in S \wedge a \cdot y = 0\} = UNIV$
using *DD dim_eq_full* **by** *blast*
ultimately show *?thesis*
by (*simp add*: *subs*) (*metis* (*lifting*) *span_eq_iff subs*)
qed

proposition *affine_hyperplane_sums_eq_UNIV*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *affine S*
and $S \cap \{v.\ a \cdot v = b\} \neq \{\}$
and $S - \{v.\ a \cdot v = b\} \neq \{\}$
shows $\{x + y|\ x\ y.\ x \in S \wedge a \cdot y = b\} = UNIV$
proof (*cases* $a = 0$)
case *True* **with** *assms* **show** *?thesis*
by (*auto simp*: *if_splits*)
next
case *False*
obtain *c* **where** $c \in S$ **and** *c*: $a \cdot c = b$
using *assms* **by** *force*
with *affine_diffs_subspace* [*OF* ‹*affine S*›]
have *subspace* $((+)\ (-\ c)\ `\ S)$ **by** *blast*
then have *aff*: *affine* $((+)\ (-\ c)\ `\ S)$
by (*simp add*: *subspace_imp_affine*)
have *0*: $0 \in (+)\ (-\ c)\ `\ S$
by (*simp add*: ‹$c \in S$›)
obtain *d* **where** $d \in S$ **and** $a \cdot d \neq b$ **and** *dc*: $d - c \in (+)\ (-\ c)\ `\ S$
using *assms* **by** *auto*
then have *adc*: $a \cdot (d - c) \neq 0$
by (*simp add*: *c inner_diff_right*)
define *U* **where** $U \equiv \{x + y\ |x\ y.\ x \in (+)\ (-\ c)\ `\ S \wedge a \cdot y = 0\}$
have $u + v \in (+)\ (c+c)\ `\ U$
if $u \in S\ b = a \cdot v$ **for** *u v*
proof
show $u + v = c + c + (u + v - c - c)$
by (*simp add*: *algebra_simps*)
have $\exists x\ y.\ u + v - c - c = x + y \wedge (\exists xa \in S.\ x = xa - c) \wedge a \cdot y = 0$
proof (*intro exI conjI*)
show $u + v - c - c = (u - c) + (v - c)\ a \cdot (v - c) = 0$
by (*simp_all add*: *algebra_simps c that*)
qed (*use that* **in** *auto*)
then show $u + v - c - c \in U$
by (*auto simp*: *U_def image_def*)
qed

moreover have $[\![a \cdot v = 0; u \in S]\!]$
$\Longrightarrow \exists x\ ya.\ v + (u + c) = x + ya \wedge x \in S \wedge a \cdot ya = b$ **for** $v\ u$
by (*metis add.left_commute c inner_right_distrib pth_d*)
ultimately have $\{x + y\ |x\ y.\ x \in S \wedge a \cdot y = b\} = (+)\ (c+c)\ `\ U$
by (*fastforce simp*: *algebra_simps U_def*)
also have ... = *range* $((+)\ (c + c))$
by (*simp only*: *U_def affine_hyperplane_sums_eq_UNIV_0* [*OF aff 0 dc adc*])
also have ... = *UNIV*
by *simp*
finally show *?thesis* **.**
qed

lemma *aff_dim_sums_Int_0*:
assumes *affine S*
and *affine T*
and $0 \in S\ 0 \in T$
shows *aff_dim* $\{x + y|\ x\ y.\ x \in S \wedge y \in T\} = (aff\_dim\ S + aff\_dim\ T) - aff\_dim(S \cap T)$
proof −
have $0 \in \{x + y\ |x\ y.\ x \in S \wedge y \in T\}$
using *assms* **by** *force*
then have *0*: $0 \in affine\ hull\ \{x + y\ |x\ y.\ x \in S \wedge y \in T\}$
by (*metis* (*lifting*) *hull_inc*)
have *sub*: *subspace S subspace T*
using *assms* **by** (*auto simp*: *subspace_affine*)
show *?thesis*
using *dim_sums_Int* [*OF sub*] **by** (*simp add*: *aff_dim_zero assms 0 hull_inc*)
qed

proposition *aff_dim_sums_Int*:
assumes *affine S*
and *affine T*
and $S \cap T \neq \{\}$
shows *aff_dim* $\{x + y|\ x\ y.\ x \in S \wedge y \in T\} = (aff\_dim\ S + aff\_dim\ T) - aff\_dim(S \cap T)$
proof −
obtain *a* **where** *a*: $a \in S\ a \in T$ **using** *assms* **by** *force*
have *aff*: *affine* $((+)\ (-a)\ `\ S)$ *affine* $((+)\ (-a)\ `\ T)$
using *affine_translation* [*symmetric*, *of* − *a*] *assms* **by** (*simp_all cong*: *image_cong_simp*)
have *zero*: $0 \in ((+)\ (-a)\ `\ S)\ \ 0 \in ((+)\ (-a)\ `\ T)$
using *a assms* **by** *auto*
have $\{x + y\ |x\ y.\ x \in (+)\ (-\ a)\ `\ S \wedge y \in (+)\ (-\ a)\ `\ T\} =$
$(+)\ (-\ 2\ *_R\ a)\ `\ \{x + y|\ x\ y.\ x \in S \wedge y \in T\}$
by (*force simp*: *algebra_simps scaleR_2*)
moreover have $(+)\ (-\ a)\ `\ S \cap (+)\ (-\ a)\ `\ T = (+)\ (-\ a)\ `\ (S \cap T)$
by *auto*
ultimately show *?thesis*
using *aff_dim_sums_Int_0* [*OF aff zero*] *aff_dim_translation_eq*

```
    by (metis (lifting))
qed

lemma aff_dim_affine_Int_hyperplane:
  fixes a :: 'a::euclidean_space
  assumes affine S
    shows aff_dim(S ∩ {x. a · x = b}) =
            (if S ∩ {v. a · v = b} = {} then - 1
             else if S ⊆ {v. a · v = b} then aff_dim S
             else aff_dim S - 1)
proof (cases a = 0)
  case True with assms show ?thesis
    by auto
next
  case False
  then have aff_dim (S ∩ {x. a · x = b}) = aff_dim S - 1
          if x ∈ S a · x ≠ b and non: S ∩ {v. a · v = b} ≠ {} for x
  proof -
    have [simp]: {x + y| x y. x ∈ S ∧ a · y = b} = UNIV
      using affine_hyperplane_sums_eq_UNIV [OF assms non] that  by blast
    show ?thesis
      using aff_dim_sums_Int [OF assms affine_hyperplane non]
      by (simp add: of_nat_diff False)
  qed
  then show ?thesis
   by (metis (mono_tags, lifting) inf.orderE aff_dim_empty_eq mem_Collect_eq
subsetI)
qed

lemma aff_dim_lt_full:
  fixes S :: 'a::euclidean_space set
  shows aff_dim S < DIM('a) ⟷ (affine hull S ≠ UNIV)
by (metis (no_types) aff_dim_affine_hull aff_dim_le_DIM aff_dim_UNIV affine_hull_UNIV
less_le)

lemma aff_dim_openin:
  fixes S :: 'a::euclidean_space set
  assumes ope: openin (top_of_set T) S and affine T S ≠ {}
  shows aff_dim S = aff_dim T
proof -
  show ?thesis
  proof (rule order_antisym)
    show aff_dim S ≤ aff_dim T
      by (blast intro: aff_dim_subset [OF openin_imp_subset] ope)
  next
    obtain a where a ∈ S
      using ‹S ≠ {}› by blast
    have S ⊆ T
      using ope openin_imp_subset by auto
```

```
    then have a ∈ T
      using ‹a ∈ S› by auto
    then have subT': subspace ((λx. − a + x) ‘ T)
      using affine_diffs_subspace ‹affine T› by auto
     then obtain B where Bsub: B ⊆ ((λx. − a + x) ‘ T) and po: pairwise
orthogonal B
                    and eq1: ⋀x. x ∈ B ⟹ norm x = 1 and independent B
                    and cardB: card B = dim ((λx. − a + x) ‘ T)
                    and spanB: span B = ((λx. − a + x) ‘ T)
      by (rule orthonormal_basis_subspace) auto
    obtain e where 0 < e and e: cball a e ∩ T ⊆ S
      by (meson ‹a ∈ S› openin_contains_cball ope)
    have aff_dim T = aff_dim ((λx. − a + x) ‘ T)
      by (metis aff_dim_translation_eq)
    also have ... = dim ((λx. − a + x) ‘ T)
      using aff_dim_subspace subT' by blast
    also have ... = card B
      by (simp add: cardB)
    also have ... = card ((λx. e *R x) ‘ B)
      using ‹0 < e›  by (force simp: inj_on_def card_image)
    also have ... ≤ dim ((λx. − a + x) ‘ S)
    proof −
      have e': cball 0 e ∩ (λx. x − a) ‘ T ⊆ (λx. x − a) ‘ S
        using e by (auto simp: dist_norm norm_minus_commute subset_eq)
      have (λx. e *R x) ‘ B ⊆ cball 0 e ∩ (λx. x − a) ‘ T
        using Bsub ‹0 < e› eq1 subT' ‹a ∈ T› by (auto simp: subspace_def)
      then have (λx. e *R x) ‘ B ⊆ (λx. x − a) ‘ S
        using e' by blast
      moreover
      have inj_on ((*R) e) (span B)
        using ‹0 < e› inj_on_def by fastforce
      then have independent ((λx. e *R x) ‘ B)
       using linear_scale_self ‹independent B› linear_dependent_inj_imageD by
blast
      ultimately show ?thesis
        by (auto simp: intro!: independent_card_le_dim)
    qed
    also have ... = aff_dim S
     using ‹a ∈ S› aff_dim_eq_dim hull_inc by (force cong: image_cong_simp)
    finally show aff_dim T ≤ aff_dim S .
  qed
qed

lemma dim_openin:
  fixes S :: 'a::euclidean_space set
  assumes ope: openin (top_of_set T) S and subspace T S ≠ {}
  shows dim S = dim T
proof (rule order_antisym)
  show dim S ≤ dim T
```

```
    by (metis ope dim_subset openin_subset topspace_euclidean_subtopology)
next
  have dim T = aff_dim S
    using aff_dim_openin
    by (metis aff_dim_subspace ‹subspace T› ‹S ≠ {}› ope subspace_affine)
  also have ... ≤ dim S
    by (metis aff_dim_subset aff_dim_subspace dim_span span_superset
        subspace_span)
  finally show dim T ≤ dim S by simp
qed
```

6.0.18 Lower-dimensional affine subsets are nowhere dense

```
proposition dense_complement_subspace:
  fixes S :: 'a :: euclidean_space set
  assumes dim_less: dim T < dim S and subspace S shows closure(S - T) = S
proof -
  have closure(S - U) = S if dim U < dim S U ⊆ S for U
  proof -
    have span U ⊂ span S
      by (metis neq_iff psubsetI span_eq_dim span_mono that)
   then obtain a where a ≠ 0 a ∈ span S and a: ⋀y. y ∈ span U ⟹ orthogonal
a y
      using orthogonal_to_subspace_exists_gen by metis
    show ?thesis
    proof
      have closed S
        by (simp add: ‹subspace S› closed_subspace)
      then show closure (S - U) ⊆ S
        by (simp add: closure_minimal)
      show S ⊆ closure (S - U)
      proof (clarsimp simp: closure_approachable)
        fix x and e::real
        assume x ∈ S 0 < e
        show ∃y∈S - U. dist y x < e
        proof (cases x ∈ U)
          case True
          let ?y = x + (e/2 / norm a) *R a
          show ?thesis
          proof
            show dist ?y x < e
              using ‹0 < e› by (simp add: dist_norm)
          next
            have ?y ∈ S
              by (metis ‹a ∈ span S› ‹x ∈ S› assms(2) span_eq_iff subspace_add
subspace_scale)
            moreover have ?y ∉ U
            proof -
              have e/2 / norm a ≠ 0
```

```
            using ‹0 < e› ‹a ≠ 0› by auto
          then show ?thesis
               by (metis True ‹a ≠ 0› a orthogonal_scaleR orthogonal_self
real_vector.scale_eq_0_iff span_add_eq span_base)
        qed
        ultimately show ?y ∈ S − U by blast
      qed
    next
      case False
      with ‹0 < e› ‹x ∈ S› show ?thesis by force
    qed
  qed
 qed
 moreover have S − S ∩ T = S−T
  by blast
 moreover have dim (S ∩ T) < dim S
  by (metis dim_less dim_subset inf.cobounded2 inf.orderE inf.strict_boundedE
not_le)
 ultimately show ?thesis
  by force
qed

corollary dense_complement_affine:
 fixes S :: 'a :: euclidean_space set
 assumes less: aff_dim T < aff_dim S and affine S shows closure(S − T) = S
proof (cases S ∩ T = {})
 case True
 then show ?thesis
  by (metis Diff_triv affine_hull_eq ‹affine S› closure_same_affine_hull clo-
sure_subset hull_subset subset_antisym)
next
 case False
 then obtain z where z: z ∈ S ∩ T by blast
 then have subspace ((+) (− z) ‘ S)
  by (meson IntD1 affine_diffs_subspace ‹affine S›)
 moreover have int (dim ((+) (− z) ‘ T)) < int (dim ((+) (− z) ‘ S))
thm aff_dim_eq_dim
  using z less by (simp add: aff_dim_eq_dim_subtract [of z] hull_inc cong:
image_cong_simp)
 ultimately have closure(((+) (− z) ‘ S) − ((+) (− z) ‘ T)) = ((+) (− z) ‘ S)
  by (simp add: dense_complement_subspace)
 then show ?thesis
  by (metis closure_translation translation_diff translation_invert)
qed

corollary dense_complement_openin_affine_hull:
 fixes S :: 'a :: euclidean_space set
 assumes less: aff_dim T < aff_dim S
```

and *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
shows *closure*(*S* − *T*) = *closure S*
proof −
have *affine hull S* − *T* ⊆ *affine hull S*
by *blast*
then have *closure* (*S* ∩ *closure* (*affine hull S* − *T*)) = *closure* (*S* ∩ (*affine hull S* − *T*))
by (*rule closure_openin_Int_closure* [*OF ope*])
then show *?thesis*
by (*metis Int_Diff aff_dim_affine_hull affine_affine_hull dense_complement_affine hull_subset inf.orderE less*)
qed

corollary *dense_complement_convex*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *aff_dim T* < *aff_dim S convex S*
shows *closure*(*S* − *T*) = *closure S*
proof
show *closure* (*S* − *T*) ⊆ *closure S*
by (*simp add*: *closure_mono*)
have *closure* (*rel_interior S* − *T*) = *closure* (*rel_interior S*)
by (*simp add*: *assms dense_complement_openin_affine_hull openin_rel_interior rel_interior_aff_dim rel_interior_same_affine_hull*)
then show *closure S* ⊆ *closure* (*S* − *T*)
by (*metis Diff_mono* ‹*convex S*› *closure_mono convex_closure_rel_interior order_refl rel_interior_subset*)
qed

corollary *dense_complement_convex_closed*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *aff_dim T* < *aff_dim S convex S closed S*
shows *closure*(*S* − *T*) = *S*
by (*simp add*: *assms dense_complement_convex*)

6.0.19 Parallel slices, etc

If we take a slice out of a set, we can do it perpendicularly, with the normal vector to the slice parallel to the affine hull.

proposition *affine_parallel_slice*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *affine S*
and *S* ∩ {*x*. *a* · *x* ≤ *b*} ≠ {}
and ¬ (*S* ⊆ {*x*. *a* · *x* ≤ *b*})
obtains *a' b'* **where** *a'* ≠ *0*
S ∩ {*x*. *a'* · *x* ≤ *b'*} = *S* ∩ {*x*. *a* · *x* ≤ *b*}
S ∩ {*x*. *a'* · *x* = *b'*} = *S* ∩ {*x*. *a* · *x* = *b*}
⋀*w*. *w* ∈ *S* ⟹ (*w* + *a'*) ∈ *S*
proof (*cases S* ∩ {*x*. *a* · *x* = *b*} = {})
case *True*

```
  then obtain u v where u ∈ S v ∈ S a • u ≤ b a • v > b
    using assms by (auto simp: not_le)
  define η where η = u + ((b − a • u) / (a • v − a • u)) *R (v − u)
  have η ∈ S
    by (simp add: η_def ‹u ∈ S› ‹v ∈ S› ‹affine S› mem_affine_3_minus)
  moreover have a • η = b
    using ‹a • u ≤ b› ‹b < a • v›
    by (simp add: η_def algebra_simps) (simp add: field_simps)
  ultimately have False
    using True by force
  then show ?thesis ..
next
  case False
  then obtain z where z ∈ S and z: a • z = b
    using assms by auto
  with affine_diffs_subspace [OF ‹affine S›]
  have sub: subspace ((+) (− z) ` S) by blast
  then have aff: affine ((+) (− z) ` S) and span: span ((+) (− z) ` S) = ((+)
(− z) ` S)
    by (auto simp: subspace_imp_affine)
  obtain a' a'' where a': a' ∈ span ((+) (− z) ` S) and a: a = a' + a''
                  and ⋀w. w ∈ span ((+) (− z) ` S) ⟹ orthogonal a'' w
    using orthogonal_subspace_decomp_exists [of (+) (− z) ` S a] by metis
  then have ⋀w. w ∈ S ⟹ a'' • (w−z) = 0
    by (simp add: span_base orthogonal_def)
  then have a'': ⋀w. w ∈ S ⟹ a'' • w = (a − a') • z
    by (simp add: a inner_diff_right)
  then have ba'': ⋀w. w ∈ S ⟹ a'' • w = b − a' • z
    by (simp add: inner_diff_left z)
  show ?thesis
  proof (cases a' = 0)
    case True
    with a assms True a'' diff_zero less_irrefl show ?thesis
      by auto
  next
    case False
    show ?thesis
    proof
      show S ∩ {x. a' • x ≤ a' • z} = S ∩ {x. a • x ≤ b}
        S ∩ {x. a' • x = a' • z} = S ∩ {x. a • x = b}
        by (auto simp: a ba'' inner_left_distrib)
      have ⋀w. w ∈ (+) (− z) ` S ⟹ (w + a') ∈ (+) (− z) ` S
        by (metis subspace_add a' span_eq_iff sub)
      then show ⋀w. w ∈ S ⟹ (w + a') ∈ S
        by fastforce
    qed (use False in auto)
  qed
qed
```

lemma *diffs_affine_hull_span*:
assumes $a \in S$
shows $(\lambda x.\ x - a)\ `\ (\mathit{affine\ hull}\ S) = \mathit{span}\ ((\lambda x.\ x - a)\ `\ S)$
proof −
have $*$: $((\lambda x.\ x - a)\ `\ (S - \{a\})) = ((\lambda x.\ x - a)\ `\ S) - \{0\}$
by (*auto simp*: *algebra_simps*)
show *?thesis*
by (*auto simp add*: *algebra_simps affine_hull_span2* [*OF assms*] $*$)
qed

lemma *aff_dim_dim_affine_diffs*:
fixes $S :: 'a :: \mathit{euclidean\_space\ set}$
assumes *affine S a* $\in S$
shows *aff_dim S* = *dim* $((\lambda x.\ x - a)\ `\ S)$
proof −
obtain *B* **where** *aff*: *affine hull B* = *affine hull S*
and *ind*: ¬ *affine_dependent B*
and *card*: *of_nat* (*card B*) = *aff_dim S* + *1*
using *aff_dim_basis_exists* **by** *blast*
then have $B \neq \{\}$ **using** *assms*
by (*metis affine_hull_eq_empty ex_in_conv*)
then obtain *c* **where** $c \in B$ **by** *auto*
then have $c \in S$
by (*metis aff affine_hull_eq* ‹*affine S*› *hull_inc*)
have *xy*: $x - c = y - a \longleftrightarrow y = x + 1 *_R (a - c)$ **for** *x y c* **and** $a::'a$
by (*auto simp*: *algebra_simps*)
have $*$: $(\lambda x.\ x - c)\ `\ S = (\lambda x.\ x - a)\ `\ S$
using *assms* ‹$c \in S$›
by (*auto simp*: *image_iff xy*; *metis mem_affine_3_minus pth_1*)
have *affS*: *affine hull S* = *S*
by (*simp add*: ‹*affine S*›)
have *aff_dim S* = *of_nat* (*card B*) − *1*
using *card* **by** *simp*
also have ... = *dim* $((\lambda x.\ x - c)\ `\ B)$
using *affine_independent_card_dim_diffs* [*OF ind* ‹$c \in B$›]
by (*simp add*: *affine_independent_card_dim_diffs* [*OF ind* ‹$c \in B$›])
also have ... = *dim* $((\lambda x.\ x - c)\ `\ (\mathit{affine\ hull}\ B))$
by (*simp add*: *diffs_affine_hull_span* ‹$c \in B$›)
also have ... = *dim* $((\lambda x.\ x - a)\ `\ S)$
by (*simp add*: *affS aff* $*$)
finally show *?thesis* **.**
qed

lemma *aff_dim_linear_image_le*:
assumes *linear f*
shows *aff_dim*(*f* ` *S*) ≤ *aff_dim S*
proof −
have *aff_dim* (*f* ` *T*) ≤ *aff_dim T* **if** *affine T* **for** *T*
proof (*cases T* = {})

```
    case True then show ?thesis by (simp add: aff_dim_geq)
  next
    case False
    then obtain a where a ∈ T by auto
    have 1: ((λx. x − f a) ‘ f ‘ T) = {x − f a |x. x ∈ f ‘ T}
      by auto
    have 2: {x − f a| x. x ∈ f ‘ T} = f ‘ ((λx. x − a) ‘ T)
      by (force simp: linear_diff [OF assms])
    have aff_dim (f ‘ T) = int (dim {x − f a |x. x ∈ f ‘ T})
       by (simp add: ‹a ∈ T› hull_inc aff_dim_eq_dim [of f a] 1 cong: image_cong_simp)
    also have ... = int (dim (f ‘ ((λx. x − a) ‘ T)))
      by (force simp: linear_diff [OF assms] 2)
    also have ... ≤ int (dim ((λx. x − a) ‘ T))
      by (simp add: dim_image_le [OF assms])
    also have ... ≤ aff_dim T
      by (simp add: aff_dim_dim_affine_diffs [symmetric] ‹a ∈ T› ‹affine T›)
    finally show ?thesis .
  qed
  then
  have aff_dim (f ‘ (affine hull S)) ≤ aff_dim (affine hull S)
    using affine_affine_hull [of S] by blast
  then show ?thesis
    using affine_hull_linear_image assms linear_conv_bounded_linear by fastforce
qed

lemma aff_dim_injective_linear_image [simp]:
  assumes linear f inj f
    shows aff_dim (f ‘ S) = aff_dim S
proof (rule antisym)
  show aff_dim (f ‘ S) ≤ aff_dim S
    by (simp add: aff_dim_linear_image_le assms(1))
next
  obtain g where linear g g ∘ f = id
    using assms(1) assms(2) linear_injective_left_inverse by blast
  then have aff_dim S ≤ aff_dim(g ‘ f ‘ S)
    by (simp add: image_comp)
  also have ... ≤ aff_dim (f ‘ S)
    by (simp add: ‹linear g› aff_dim_linear_image_le)
  finally show aff_dim S ≤ aff_dim (f ‘ S) .
qed


lemma choose_affine_subset:
  assumes affine S −1 ≤ d and dle: d ≤ aff_dim S
  obtains T where affine T T ⊆ S aff_dim T = d
proof (cases d = −1 ∨ S={})
  case True with assms show ?thesis
```

```
    by (metis aff_dim_empty affine_empty bot.extremum that eq_iff)
next
  case False
  with assms obtain a where a ∈ S 0 ≤ d by auto
  with assms have ss: subspace ((+) (− a) ‘ S)
    by (simp add: affine_diffs_subspace_subtract cong: image_cong_simp)
  have nat d ≤ dim ((+) (− a) ‘ S)
     by (metis aff_dim_subspace aff_dim_translation_eq dle nat_int nat_mono
ss)
  then obtain T where subspace T and Tsb: T ⊆ span ((+) (− a) ‘ S)
                  and Tdim: dim T = nat d
    using choose_subspace_of_subspace [of nat d (+) (− a) ‘ S] by blast
  then have affine T
    using subspace_affine by blast
  then have affine ((+) a ‘ T)
    by (metis affine_hull_eq affine_hull_translation)
  moreover have (+) a ‘ T ⊆ S
  proof −
    have T ⊆ (+) (− a) ‘ S
      by (metis (no_types) span_eq_iff Tsb ss)
    then show (+) a ‘ T ⊆ S
      using add_ac by auto
  qed
  moreover have aff_dim ((+) a ‘ T) = d
   by (simp add: aff_dim_subspace Tdim ‹0 ≤ d› ‹subspace T› aff_dim_translation_eq)
  ultimately show ?thesis
    by (rule that)
qed
```

6.0.20 Paracompactness

```
proposition paracompact:
  fixes S :: 'a :: {metric_space,second_countable_topology} set
  assumes S ⊆ ⋃C and opC: ⋀T. T ∈ C ⟹ open T
  obtains C' where S ⊆ ⋃ C'
              and ⋀U. U ∈ C' ⟹ open U ∧ (∃ T. T ∈ C ∧ U ⊆ T)
              and ⋀x. x ∈ S
                   ⟹ ∃ V. open V ∧ x ∈ V ∧ finite {U. U ∈ C' ∧ (U ∩ V ≠ {})}
proof (cases S = {})
  case True with that show ?thesis by blast
next
  case False
  have ∃ T U. x ∈ U ∧ open U ∧ closure U ⊆ T ∧ T ∈ C if x ∈ S for x
  proof −
    obtain T where x ∈ T T ∈ C open T
      using assms ‹x ∈ S› by blast
    then obtain e where e > 0 cball x e ⊆ T
      by (force simp: open_contains_cball)
    then show ?thesis
```

by (*meson open_ball ‹T ∈ 𝒞› ball_subset_cball centre_in_ball closed_cball closure_minimal dual_order.trans*)
qed
then obtain *F G* **where** *Gin*: $x \in G\ x$ **and** *oG*: *open* $(G\ x)$
and *clos*: *closure* $(G\ x) \subseteq F\ x$ **and** *Fin*: $F\ x \in \mathcal{C}$
if $x \in S$ **for** x
by *metis*
then obtain $\mathcal{F}$ **where** $\mathcal{F} \subseteq G\ `\ S$ *countable* $\mathcal{F}\ \bigcup\mathcal{F} = \bigcup(G\ `\ S)$
using *Lindelof* [*of* $G\ `\ S$] **by** (*metis image_iff*)
then obtain *K* **where** *K*: $K \subseteq S$ *countable K* **and** *eq*: $\bigcup(G\ `\ K) = \bigcup(G\ `\ S)$
by (*metis countable_subset_image*)
with *False Gin* **have** $K \neq \{\}$ **by** *force*
then obtain $a :: nat \Rightarrow 'a$ **where** *range a* = *K*
by (*metis range_from_nat_into ‹countable K›*)
then have *odif*: $\bigwedge n.$ *open* $(F\ (a\ n) - \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < n\})$
using $‹K \subseteq S›$ *Fin opC* **by** (*fastforce simp add:*)
let *?C* = *range* $(\lambda n.\ F(a\ n) - \bigcup\{closure(G(a\ m))\ |m.\ m < n\})$
have *enum_S*: $\exists n.\ x \in F(a\ n) \wedge x \in G(a\ n)$ **if** $x \in S$ **for** x
proof −
have $\exists y \in K.\ x \in G\ y$ **using** *eq that Gin* **by** *fastforce*
then show *?thesis*
using *clos K ‹range a = K› closure_subset* **by** *blast*
qed
show *?thesis*
proof
show $S \subseteq$ *Union ?C*
proof
fix x **assume** $x \in S$
define n **where** $n \equiv LEAST\ n.\ x \in F(a\ n)$
have *n*: $x \in F(a\ n)$
using *enum_S* [*OF* $‹x \in S›$] **by** (*force simp*: *n_def intro*: *LeastI*)
have *notn*: $x \notin F(a\ m)$ **if** $m < n$ **for** m
using *that not_less_Least* **by** (*force simp*: *n_def*)
then have $x \notin \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < n\}$
using n $‹K \subseteq S›$ $‹range\ a = K›$ *clos notn* **by** *fastforce*
with n **show** $x \in$ *Union ?C*
by *blast*
qed
show $\bigwedge U.\ U \in ?C \Longrightarrow$ *open* $U \wedge (\exists T.\ T \in \mathcal{C} \wedge U \subseteq T)$
using *Fin* $‹K \subseteq S›$ $‹range\ a = K›$ **by** (*auto simp*: *odif*)
show $\exists V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U.\ U \in ?C \wedge (U \cap V \neq \{\})\}$ **if** $x \in S$ **for** x
proof −
obtain n **where** *n*: $x \in F(a\ n)\ x \in G(a\ n)$
using $‹x \in S›$ *enum_S* **by** *auto*
have $\{U \in ?C.\ U \cap G\ (a\ n) \neq \{\}\} \subseteq (\lambda n.\ F(a\ n) - \bigcup\{closure(G(a\ m))\ |m.\ m < n\})\ `\ atMost\ n$
proof *clarsimp*
fix k **assume** $(F\ (a\ k) - \bigcup\{closure\ (G\ (a\ m))\ |m.\ m < k\}) \cap G\ (a\ n) \neq$

```
{}
      then have k ≤ n
        by auto (metis closure_subset not_le subsetCE)
      then show F (a k) − ⋃{closure (G (a m)) |m. m < k}
            ∈ (λn. F (a n) − ⋃{closure (G (a m)) |m. m < n}) ‘ {..n}
        by force
    qed
     moreover have finite ((λn. F(a n) − ⋃{closure(G(a m)) |m. m < n}) ‘
atMost n)
      by force
    ultimately have ∗: finite {U ∈ ?C. U ∩ G (a n) ≠ {}}
      using finite_subset by blast
    have a n ∈ S
      using ‹K ⊆ S› ‹range a = K› by blast
    then show ?thesis
      by (blast intro: oG n ∗)
  qed
 qed
qed

corollary paracompact_closedin:
 fixes S :: 'a :: {metric_space,second_countable_topology} set
 assumes cin: closedin (top_of_set U) S
    and oin: ⋀T. T ∈ C ⟹ openin (top_of_set U) T
    and S ⊆ ⋃C
 obtains C' where S ⊆ ⋃ C'
           and ⋀V. V ∈ C' ⟹ openin (top_of_set U) V ∧ (∃ T. T ∈ C ∧ V
⊆ T)
           and ⋀x. x ∈ U
                 ⟹ ∃ V. openin (top_of_set U) V ∧ x ∈ V ∧
                       finite {X. X ∈ C' ∧ (X ∩ V ≠ {})}
proof −
 have ∃ Z. open Z ∧ (T = U ∩ Z) if T ∈ C for T
   using oin [OF that] by (auto simp: openin_open)
 then obtain F where opF: open (F T) and intF: U ∩ F T = T if T ∈ C for
T
   by metis
 obtain K where K: closed K U ∩ K = S
   using cin by (auto simp: closedin_closed)
 have 1: U ⊆ ⋃(insert (− K) (F ‘ C))
   by clarsimp (metis Int_iff Union_iff ‹U ∩ K = S› ‹S ⊆ ⋃C› subsetD intF)
 have 2: ⋀T. T ∈ insert (− K) (F ‘ C) ⟹ open T
   using ‹closed K› by (auto simp: opF)
 obtain D where U ⊆ ⋃D
          and D1: ⋀U. U ∈ D ⟹ open U ∧ (∃ T. T ∈ insert (− K) (F ‘ C) ∧
U ⊆ T)
          and D2: ⋀x. x ∈ U ⟹ ∃ V. open V ∧ x ∈ V ∧ finite {U ∈ D. U ∩
V ≠ {}}
   by (blast intro: paracompact [OF 1 2])
```

```
  let ?C = {U ∩ V |V. V ∈ 𝒟 ∧ (V ∩ K ≠ {})}
  show ?thesis
  proof (rule_tac 𝒞' = {U ∩ V |V. V ∈ 𝒟 ∧ (V ∩ K ≠ {})} in that)
    show S ⊆ ⋃ ?C
      using ‹U ∩ K = S› ‹U ⊆ ⋃𝒟› K by (blast dest!: subsetD)
    show ⋀V. V ∈ ?C ⟹ openin (top_of_set U) V ∧ (∃ T. T ∈ 𝒞 ∧ V ⊆ T)
      using D1 intF by fastforce
    have *: {X. (∃ V. X = U ∩ V ∧ V ∈ 𝒟 ∧ V ∩ K ≠ {}) ∧ X ∩ (U ∩ V) ≠
{}} ⊆
          (λx. U ∩ x) ` {U ∈ 𝒟. U ∩ V ≠ {}} for V
      by blast
    show ∃ V. openin (top_of_set U) V ∧ x ∈ V ∧ finite {X ∈ ?C. X ∩ V ≠
{}}
      if x ∈ U for x
    proof −
      from D2 [OF that] obtain V where open V x ∈ V finite {U ∈ 𝒟. U ∩ V
≠ {}}
        by auto
      with * show ?thesis
        by (rule_tac x=U ∩ V in exI) (auto intro: that finite_subset [OF *])
    qed
  qed
qed

corollary paracompact_closed:
  fixes S :: 'a :: {metric_space,second_countable_topology} set
  assumes closed S
      and opC: ⋀T. T ∈ 𝒞 ⟹ open T
      and S ⊆ ⋃𝒞
  obtains 𝒞' where S ⊆ ⋃𝒞'
              and ⋀U. U ∈ 𝒞' ⟹ open U ∧ (∃ T. T ∈ 𝒞 ∧ U ⊆ T)
              and ⋀x. ∃ V. open V ∧ x ∈ V ∧
                           finite {U. U ∈ 𝒞' ∧ (U ∩ V ≠ {})}
  by (rule paracompact_closedin [of UNIV S 𝒞]) (auto simp: assms)
```

6.0.21 Closed-graph characterization of continuity

```
lemma continuous_closed_graph_gen:
  fixes T :: 'b::real_normed_vector set
  assumes contf: continuous_on S f and fim: f ` S ⊆ T
    shows closedin (top_of_set (S × T)) ((λx. Pair x (f x)) ` S)
proof −
  have eq: ((λx. Pair x (f x)) ` S) = (S × T ∩ (λz. (f ∘ fst)z − snd z) −` {0})
    using fim by auto
  show ?thesis
    unfolding eq
   by (intro continuous_intros continuous_closedin_preimage continuous_on_subset
[OF contf]) auto
qed
```

lemma *continuous_closed_graph_eq*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *compact T* **and** *fim*: *f* ∈ *S* → *T*
shows *continuous_on S f* ⟷
closedin (*top_of_set* (*S* × *T*)) ((λ*x*. *Pair x* (*f x*)) ‘ *S*)
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* **if** *?rhs*
proof (*clarsimp simp add*: *continuous_on_closed_gen* [*OF fim*])
fix *U*
assume *U*: *closedin* (*top_of_set T*) *U*
have *eq*: (*S* ∩ *f* −‘ *U*) = *fst* ‘ (((λ*x*. *Pair x* (*f x*)) ‘ *S*) ∩ (*S* × *U*))
by (*force simp*: *image_iff*)
show *closedin* (*top_of_set S*) (*S* ∩ *f* −‘ *U*)
by (*simp add*: *U closedin_Int closedin_Times closed_map_fst* [*OF* ‹*compact T*›] *that eq*)
qed
with *continuous_closed_graph_gen assms* **show** *?thesis* **by** *blast*
qed

lemma *continuous_closed_graph*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*real_normed_vector*
assumes *closed S* **and** *contf*: *continuous_on S f*
shows *closed* ((λ*x*. *Pair x* (*f x*)) ‘ *S*)
proof (*rule closedin_closed_trans*)
show *closedin* (*top_of_set* (*S* × *UNIV*)) ((λ*x*. (*x*, *f x*)) ‘ *S*)
by (*rule continuous_closed_graph_gen* [*OF contf subset_UNIV*])
qed (*simp add*: ‹*closed S*› *closed_Times*)

lemma *continuous_from_closed_graph*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *compact T* **and** *fim*: *f* ∈ *S* → *T* **and** *clo*: *closed* ((λ*x*. *Pair x* (*f x*)) ‘ *S*)
shows *continuous_on S f*
using *fim clo*
by (*auto intro*: *closed_subset simp*: *continuous_closed_graph_eq* [*OF* ‹*compact T*› *fim*])

lemma *continuous_on_Un_local_open*:
assumes *opS*: *openin* (*top_of_set* (*S* ∪ *T*)) *S*
and *opT*: *openin* (*top_of_set* (*S* ∪ *T*)) *T*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T f*
shows *continuous_on* (*S* ∪ *T*) *f*
using *pasting_lemma* [*of* {*S*,*T*} *top_of_set* (*S* ∪ *T*) *id euclidean* λ*i*. *f f*] *contf contg opS opT*
by (*simp add*: *subtopology_subtopology*) (*metis inf.absorb2 openin_imp_subset*)

lemma *continuous_on_cases_local_open*:
assumes *opS*: *openin* (*top_of_set* (*S* ∪ *T*)) *S*
and *opT*: *openin* (*top_of_set* (*S* ∪ *T*)) *T*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on T g*
and *fg*: ⋀*x*. *x* ∈ *S* ∧ ¬*P x* ∨ *x* ∈ *T* ∧ *P x* ⟹ *f x* = *g x*
shows *continuous_on* (*S* ∪ *T*) (λ*x*. *if P x then f x else g x*)
proof −
have ⋀*x*. *x* ∈ *S* ⟹ (*if P x then f x else g x*) = *f x* ⋀*x*. *x* ∈ *T* ⟹ (*if P x then f x else g x*) = *g x*
by (*simp_all add*: *fg*)
then have *continuous_on S* (λ*x*. *if P x then f x else g x*) *continuous_on T* (λ*x*. *if P x then f x else g x*)
by (*simp_all add*: *contf contg cong*: *continuous_on_cong*)
then show *?thesis*
by (*rule continuous_on_Un_local_open* [*OF opS opT*])
qed

6.0.22 The union of two collinear segments is another segment

proposition *in_convex_hull_exchange*:
fixes *a* :: *'a*::*euclidean_space*
assumes *a*: *a* ∈ *convex hull S* **and** *xS*: *x* ∈ *convex hull S*
obtains *b* **where** *b* ∈ *S x* ∈ *convex hull* (*insert a* (*S* − {*b*}))
proof (*cases a* ∈ *S*)
case *True*
with *xS insert_Diff that* **show** *?thesis* **by** *fastforce*
next
case *False*
show *?thesis*
proof (*cases finite S* ∧ *card S* ≤ *Suc* (*DIM*(*'a*)))
case *True*
then obtain *u* **where** *u0*: ⋀*i*. *i* ∈ *S* ⟹ 0 ≤ *u i* **and** *u1*: *sum u S* = 1
and *ua*: (∑ *i*∈*S*. *u i* ∗*R i*) = *a*
using *a* **by** (*auto simp*: *convex_hull_finite*)
obtain *v* **where** *v0*: ⋀*i*. *i* ∈ *S* ⟹ 0 ≤ *v i* **and** *v1*: *sum v S* = 1
and *vx*: (∑ *i*∈*S*. *v i* ∗*R i*) = *x*
using *True xS* **by** (*auto simp*: *convex_hull_finite*)
show *?thesis*
proof (*cases* ∃ *b*. *b* ∈ *S* ∧ *v b* = 0)
case *True*
then obtain *b* **where** *b*: *b* ∈ *S v b* = 0
by *blast*
show *?thesis*
proof
have *fin*: *finite* (*insert a* (*S* − {*b*}))
using *sum.infinite v1* **by** *fastforce*
show *x* ∈ *convex hull insert a* (*S* − {*b*})
unfolding *convex_hull_finite* [*OF fin*] *mem_Collect_eq*

```
      proof (intro conjI exI ballI)
        have (∑ x ∈ insert a (S − {b}). if x = a then 0 else v x) =
              (∑ x ∈ S − {b}. if x = a then 0 else v x)
          using fin by (force intro: sum.mono_neutral_right)
        also have ... = (∑ x ∈ S − {b}. v x)
          using b False by (auto intro!: sum.cong split: if_split_asm)
        also have ... = (∑ x∈S. v x)
          by (metis ‹v b = 0› diff_zero sum.infinite sum_diff1 u1 zero_neq_one)
        finally show (∑ x∈insert a (S − {b}). if x = a then 0 else v x) = 1
          by (simp add: v1)
        show ⋀x. x ∈ insert a (S − {b}) ⟹ 0 ≤ (if x = a then 0 else v x)
          by (auto simp: v0)
        have (∑ x ∈ insert a (S − {b}). (if x = a then 0 else v x) *R x) =
              (∑ x ∈ S − {b}. (if x = a then 0 else v x) *R x)
          using fin by (force intro: sum.mono_neutral_right)
        also have ... = (∑ x ∈ S − {b}. v x *R x)
          using b False by (auto intro!: sum.cong split: if_split_asm)
        also have ... = (∑ x∈S. v x *R x)
          by (metis (no_types, lifting) b(2) diff_zero fin finite.emptyI finite_Diff2
finite_insert scale_eq_0_iff sum_diff1)
        finally show (∑ x∈insert a (S − {b}). (if x = a then 0 else v x) *R x) =
x
          by (simp add: vx)
      qed
    qed (rule ‹b ∈ S›)
  next
    case False
    have le_Max: u i / v i ≤ Max ((λi. u i / v i) ‘ S) if i ∈ S for i
      by (simp add: True that)
    have Max ((λi. u i / v i) ‘ S) ∈ (λi. u i / v i) ‘ S
      using True v1 by (auto intro: Max_in)
    then obtain b where b ∈ S and beq: Max ((λb. u b / v b) ‘ S) = u b / v b
      by blast
    then have 0 ≠ u b / v b
      using le_Max beq divide_le_0_iff le_numeral_extra(2) sum_nonpos u1
      by (metis False eq_iff v0)
    then have  0 < u b 0 < v b
      using False ‹b ∈ S› u0 v0 by force+
    have fin: finite (insert a (S − {b}))
      using sum.infinite v1 by fastforce
    show ?thesis
    proof
      show x ∈ convex hull insert a (S − {b})
        unfolding convex_hull_finite [OF fin] mem_Collect_eq
      proof (intro conjI exI ballI)
        have (∑ x ∈ insert a (S − {b}). if x=a then v b / u b else v x − (v b / u
b) * u x) =
              v b / u b + (∑ x ∈ S − {b}. v x − (v b / u b) * u x)
          using ‹a ∉ S› ‹b ∈ S› True
```

```
          by (auto intro!: sum.cong split: if_split_asm)
        also have ... = v b / u b + (∑x ∈ S - {b}. v x) - (v b / u b) * (∑x ∈
S - {b}. u x)
          by (simp add: Groups_Big.sum_subtractf sum_distrib_left)
        also have ... = (∑x∈S. v x)
            using ‹0 < u b› True  by (simp add: Groups_Big.sum_diff1 u1
field_simps)
        finally show sum (λx. if x=a then v b / u b else v x - (v b / u b) * u x)
(insert a (S - {b})) = 1
          by (simp add: v1)
        show 0 ≤ (if i = a then v b / u b else v i - v b / u b * u i)
          if i ∈ insert a (S - {b}) for i
          using ‹0 < u b› ‹0 < v b› v0 [of i] le_Max [of i] beq that False
          by (auto simp: field_simps split: if_split_asm)
        have (∑x∈insert a (S - {b}). (if x=a then v b / u b else v x - v b / u b
* u x) *R x) =
              (v b / u b) *R a + (∑x∈S - {b}. (v x - v b / u b * u x) *R x)
        using ‹a ∉ S› ‹b ∈ S› True by (auto intro!: sum.cong split: if_split_asm)
        also have ... = (v b / u b) *R a + (∑x ∈ S - {b}. v x *R x) - (v b / u
b) *R (∑x ∈ S - {b}. u x *R x)
          by (simp add: Groups_Big.sum_subtractf scaleR_left_diff_distrib
sum_distrib_left scale_sum_right)
        also have ... = (∑x∈S. v x *R x)
           using ‹0 < u b› True  by (simp add: ua vx Groups_Big.sum_diff1
algebra_simps)
        finally
        show (∑x∈insert a (S - {b}). (if x=a then v b / u b else v x - v b / u
b * u x) *R x) = x
          by (simp add: vx)
      qed
    qed (rule ‹b ∈ S›)
  qed
 next
  case False
  obtain T where finite T T ⊆ S and caT: card T ≤ Suc (DIM('a)) and xT:
x ∈ convex hull T
    using xS by (auto simp: caratheodory [of S])
  with False obtain b where b: b ∈ S b ∉ T
    by (metis antisym subsetI)
  show ?thesis
  proof
    show x ∈ convex hull insert a (S - {b})
      using  ‹T ⊆ S› b by (blast intro: subsetD [OF hull_mono xT])
  qed (rule ‹b ∈ S›)
 qed
qed

lemma convex_hull_exchange_Union:
 fixes a :: 'a::euclidean_space
```

assumes $a \in$ *convex hull* S
shows *convex hull* $S = (\bigcup b \in S.$ *convex hull* (*insert* a $(S - \{b\})))$ (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
by (*blast intro*: *in_convex_hull_exchange* [*OF assms*])
show *?rhs* $\subseteq$ *?lhs*
proof *clarify*
fix x b
assume$b \in S$ $x \in$ *convex hull insert* a $(S - \{b\})$
then show $x \in$ *convex hull* S **if** $b \in S$
by (*metis* (*no_types*) *that assms order_refl hull_mono hull_redundant insert_Diff_single insert_subset subsetCE*)
qed
qed

lemma *Un_closed_segment*:
fixes a :: $'a$::*euclidean_space*
assumes $b \in$ *closed_segment* a c
shows *closed_segment* a $b \cup$ *closed_segment* b $c =$ *closed_segment* a c
proof (*cases* $c = a$)
case *True*
with *assms* **show** *?thesis* **by** *simp*
next
case *False*
with *assms* **have** *convex hull* $\{a, b\} \cup$ *convex hull* $\{b, c\} = (\bigcup ba \in \{a, c\}.$ *convex hull insert* b $(\{a, c\} - \{ba\}))$
by (*auto simp*: *insert_Diff_if insert_commute*)
then show *?thesis*
using *convex_hull_exchange_Union*
by (*metis assms segment_convex_hull*)
qed

lemma *Un_open_segment*:
fixes a :: $'a$::*euclidean_space*
assumes $b \in$ *open_segment* a c
shows *open_segment* a $b \cup \{b\} \cup$ *open_segment* b $c =$ *open_segment* a c (**is** *?lhs* = *?rhs*)
proof −
have *b*: $b \in$ *closed_segment* a c
by (*simp add*: *assms open_closed_segment*)
have $*$: *?rhs* $\subseteq$ *insert* b (*open_segment* a $b \cup$ *open_segment* b c)
if $\{b,c,a\} \cup$ *open_segment* a $b \cup$ *open_segment* b $c = \{c,a\} \cup$ *?rhs*
proof −
have *insert* a (*insert* c (*insert* b (*open_segment* a $b \cup$ *open_segment* b c))) = *insert* a (*insert* c (*?rhs*))
using *that* **by** (*simp add*: *insert_commute*)
then show *?thesis*
by (*metis* (*no_types*) *Diff_cancel Diff_eq_empty_iff Diff_insert2 open_segment_def*)

qed
show *?thesis*
proof
show *?lhs* $\subseteq$ *?rhs*
by (*simp add*: *assms b subset_open_segment*)
show *?rhs* $\subseteq$ *?lhs*
using *Un_closed_segment* [*OF b*] $*$
by (*simp add*: *closed_segment_eq_open insert_commute*)
qed
qed

6.0.23 Covering an open set by a countable chain of compact sets

proposition *open_Union_compact_subsets*:
fixes S :: *'a*::*euclidean_space set*
assumes *open S*
obtains C **where** $\bigwedge n.$ *compact*$(C\ n)$ $\bigwedge n.$ $C\ n \subseteq S$
$\bigwedge n.$ $C\ n \subseteq$ *interior*$(C(Suc\ n))$
$\bigcup$(*range* C) $= S$
$\bigwedge K.$ $[\![$*compact* K; $K \subseteq S]\!] \Longrightarrow \exists N.\ \forall n{\geq}N.\ K \subseteq (C\ n)$
proof (*cases* $S = \{\}$)
case *True*
then show *?thesis*
by (*rule_tac* $C = \lambda n.\ \{\}$ **in** *that*) *auto*
next
case *False*
then obtain a **where** $a \in S$
by *auto*
let *?C* $= \lambda n.$ *cball a* (*real n*) $-$ ($\bigcup x \in -S.\ \bigcup e \in$ *ball 0* (*1 / real*(*Suc n*)). $\{x + e\}$)
have $\exists N.\ \forall n{\geq}N.\ K \subseteq (f\ n)$
if $\bigwedge n.$ *compact*$(f\ n)$ **and** *sub_int*: $\bigwedge n.$ $f\ n \subseteq$ *interior* $(f(Suc\ n))$
and *eq*: $\bigcup$(*range* f) $= S$ **and** *compact* K $K \subseteq S$ **for** f K
proof $-$
have $*$: $\forall n.\ f\ n \subseteq (\bigcup n.$ *interior* $(f\ n))$
by (*meson Sup_upper2 UNIV_I* ‹$\bigwedge n.$ $f\ n \subseteq$ *interior* $(f\ (Suc\ n))$› *image_iff*)
have *mono*: $\bigwedge m\ n.\ m \leq n \Longrightarrow f\ m \subseteq f\ n$
by (*meson dual_order.trans interior_subset lift_Suc_mono_le sub_int*)
obtain I **where** *finite* I **and** I: $K \subseteq (\bigcup i{\in}I.$ *interior* $(f\ i))$
proof (*rule compactE_image* [*OF* ‹*compact* K›])
show $K \subseteq (\bigcup n.$ *interior* $(f\ n))$
using ‹$K \subseteq S$› ‹$\bigcup(f\ `\ UNIV) = S$› $*$ **by** *blast*
qed *auto*
{ fix n
assume n: *Max* $I \leq n$
have $(\bigcup i{\in}I.$ *interior* $(f\ i)) \subseteq f\ n$
by (*rule UN_least*) (*meson dual_order.trans interior_subset mono I Max_ge* [*OF* ‹*finite I*›] n)

```
    then have K ⊆ f n
      using I by auto
  }
  then show ?thesis
    by blast
 qed
 moreover have ∃f. (∀n. compact(f n)) ∧ (∀n. (f n) ⊆ S) ∧ (∀n. (f n) ⊆
interior(f(Suc n))) ∧
                  ((⋃(range f) = S))
 proof (intro exI conjI allI)
  show ⋀n. compact (?C n)
    by (auto simp: compact_diff open_sums)
  show ⋀n. ?C n ⊆ S
    by auto
  show ?C n ⊆ interior (?C (Suc n)) for n
  proof (simp add: interior_diff, rule Diff_mono)
    show cball a (real n) ⊆ ball a (1 + real n)
      by (simp add: cball_subset_ball_iff)
    have cl: closed (⋃x∈- S. ⋃e∈cball 0 (1 / (2 + real n)). {x + e})
      using assms by (auto intro: closed_compact_sums)
    have closure (⋃x∈- S. ⋃y∈ball 0 (1 / (2 + real n)). {x + y})
        ⊆ (⋃x ∈ -S. ⋃e ∈ cball 0 (1 / (2 + real n)). {x + e})
      by (intro closure_minimal UN_mono ball_subset_cball order_refl cl)
    also have ... ⊆ (⋃x ∈ -S. ⋃y∈ball 0 (1 / (1 + real n)). {x + y})
      by (simp add: cball_subset_ball_iff field_split_simps UN_mono)
    finally show closure (⋃x∈- S. ⋃y∈ball 0 (1 / (2 + real n)). {x + y})
                 ⊆ (⋃x ∈ -S. ⋃y∈ball 0 (1 / (1 + real n)). {x + y}) .
  qed
  have S ⊆ ⋃ (range ?C)
  proof
    fix x
    assume x: x ∈ S
    then obtain e where e > 0 and e: ball x e ⊆ S
      using assms open_contains_ball by blast
    then obtain N1 where N1 > 0 and N1: real N1 > 1/e
      using reals_Archimedean2
      by (metis divide_less_0_iff less_eq_real_def neq0_conv not_le of_nat_0
of_nat_1 of_nat_less_0_iff)
    obtain N2 where N2: norm(x - a) ≤ real N2
      by (meson real_arch_simple)
    have N12: inverse((N1 + N2) + 1) ≤ inverse(N1)
      using ‹N1 > 0› by (auto simp: field_split_simps)
    have x ≠ y + z if y ∉ S norm z < 1 / (1 + (real N1 + real N2)) for y z
    proof -
      have e * real N1 < e * (1 + (real N1 + real N2))
        by (simp add: ‹0 < e›)
      then have 1 / (1 + (real N1 + real N2)) < e
        using N1 ‹e > 0›
      by (metis divide_less_eq less_trans mult.commute of_nat_add of_nat_less_0_iff
```

of_nat_Suc)
      then have $x - z \in ball\ x\ e$
        using *that* **by** *simp*
      then have $x - z \in S$
        using *e* **by** *blast*
      with *that* **show** *?thesis*
        by *auto*
    qed
    with *N2* **show** $x \in \bigcup\ (range\ ?C)$
    by (*rule_tac a = N1+N2* **in** *UN_I*) (*auto simp*: *dist_norm norm_minus_commute*)
  qed
  then show $\bigcup\ (range\ ?C) = S$ **by** *auto*
 qed
 ultimately show *?thesis*
  using *that* **by** *metis*
qed

6.0.24 Orthogonal complement

definition *orthogonal_comp* (‹(‹*open_block notation=‹postfix* $\perp$›› _$^\perp$)› [*80*] *80*)
 where *orthogonal_comp* $W \equiv \{x.\ \forall y \in W.\ orthogonal\ y\ x\}$

proposition *subspace_orthogonal_comp*: *subspace* ($W^\perp$)
 unfolding *subspace_def orthogonal_comp_def orthogonal_def*
 by (*auto simp*: *inner_right_distrib*)

lemma *orthogonal_comp_anti_mono*:
 assumes $A \subseteq B$
 shows $B^\perp \subseteq A^\perp$
proof
 fix *x* **assume** *x*: $x \in B^\perp$
 show $x \in orthogonal\_comp\ A$ **using** *x* **unfolding** *orthogonal_comp_def*
  by (*simp add*: *orthogonal_def*, *metis assms in_mono*)
qed

lemma *orthogonal_comp_null* [*simp*]: $\{0\}^\perp = UNIV$
 by (*auto simp*: *orthogonal_comp_def orthogonal_def*)

lemma *orthogonal_comp_UNIV* [*simp*]: $UNIV^\perp = \{0\}$
 unfolding *orthogonal_comp_def orthogonal_def*
 by *auto* (*use inner_eq_zero_iff* **in** *blast*)

lemma *orthogonal_comp_subset*: $U \subseteq U^{\perp\perp}$
 by (*auto simp*: *orthogonal_comp_def orthogonal_def inner_commute*)

lemma *subspace_sum_minimal*:
 assumes $S \subseteq U$ $T \subseteq U$ *subspace U*
 shows $S + T \subseteq U$
proof

```
  fix x
  assume x ∈ S + T
  then obtain xs xt where xs ∈ S xt ∈ T x = xs+xt
    by (meson set_plus_elim)
  then show x ∈ U
    by (meson assms subsetCE subspace_add)
qed

proposition subspace_sum_orthogonal_comp:
  fixes U :: 'a :: euclidean_space set
  assumes subspace U
  shows U + U^⊥ = UNIV
proof −
  obtain B where B ⊆ U
    and ortho: pairwise orthogonal B ⋀x. x ∈ B ⟹ norm x = 1
    and independent B card B = dim U span B = U
    using orthonormal_basis_subspace [OF assms] by metis
  then have finite B
    by (simp add: indep_card_eq_dim_span)
  have ∗: ∀ x∈B. ∀ y∈B. x • y = (if x=y then 1 else 0)
    using ortho norm_eq_1 by (auto simp: orthogonal_def pairwise_def)
  { fix v
    let ?u = ∑ b∈B. (v • b) ∗_R b
    have v = ?u + (v − ?u)
      by simp
    moreover have ?u ∈ U
      by (metis (no_types, lifting) ‹span B = U› assms subspace_sum span_base
span_mul)
    moreover have (v − ?u) ∈ U^⊥
    proof (clarsimp simp: orthogonal_comp_def orthogonal_def)
      fix y
      assume y ∈ U
      with ‹span B = U› span_finite [OF ‹finite B›]
      obtain u where u: y = (∑ b∈B. u b ∗_R b)
        by auto
      have b • (v − ?u) = 0 if b ∈ B for b
        using that ‹finite B›
        by (simp add: ∗ algebra_simps inner_sum_right if_distrib [of (∗)v for v]
inner_commute cong: if_cong)
      then show y • (v − ?u) = 0
        by (simp add: u inner_sum_left)
    qed
    ultimately have v ∈ U + U^⊥
      using set_plus_intro by fastforce
  } then show ?thesis
    by auto
qed

lemma orthogonal_Int_0:
```

assumes *subspace U*
shows $U \cap U^\perp = \{0\}$
using *orthogonal_comp_def orthogonal_self*
by (*force simp*: *assms subspace_0 subspace_orthogonal_comp*)

lemma *orthogonal_comp_self*:
fixes $U :: 'a :: euclidean\_space\ set$
assumes *subspace U*
shows $U^{\perp\perp} = U$
proof
have *ssU'*: *subspace* $(U^\perp)$
by (*simp add*: *subspace_orthogonal_comp*)
have $u \in U$ **if** $u \in U^{\perp\perp}$ **for** u
proof −
obtain $v\ w$ **where** $u = v+w\ v \in U\ w \in U^\perp$
using *subspace_sum_orthogonal_comp* [*OF assms*] *set_plus_elim* **by** *blast*
then have $u-v \in U^\perp$
by *simp*
moreover have $v \in U^{\perp\perp}$
using ‹$v \in U$› *orthogonal_comp_subset* **by** *blast*
then have $u-v \in U^{\perp\perp}$
by (*simp add*: *subspace_diff subspace_orthogonal_comp that*)
ultimately have $u-v = 0$
using *orthogonal_Int_0 ssU'* **by** *blast*
with ‹$v \in U$› **show** *?thesis*
by *auto*
qed
then show $U^{\perp\perp} \subseteq U$
by *auto*
qed (*use orthogonal_comp_subset* **in** *auto*)

lemma *ker_orthogonal_comp_adjoint*:
fixes $f :: 'm::euclidean\_space \Rightarrow 'n::euclidean\_space$
assumes *linear f*
shows $f -` \{0\} = (range\ (adjoint\ f))^\perp$
proof −
have $\bigwedge x.\ [\![\forall y.\ y \cdot f\ x = 0]\!] \Longrightarrow f\ x = 0$
using *assms inner_commute all_zero_iff* **by** *metis*
then show *?thesis*
using *assms*
by (*auto simp*: *orthogonal_comp_def orthogonal_def adjoint_works inner_commute*)
qed

6.0.25 A non-injective linear function maps into a hyperplane.

lemma *linear_surj_adj_imp_inj*:
fixes $f :: 'm::euclidean\_space \Rightarrow 'n::euclidean\_space$
assumes *linear f surj* (*adjoint f*)

```
  shows inj f
proof −
  have ∃ x. y = adjoint f x for y
    using assms by (simp add: surjD)
  then show inj f
    using assms unfolding inj_on_def image_def
    by (metis (no_types) adjoint_works euclidean_eqI)
qed

— https://mathonline.wikidot.com/injectivity-and-surjectivity-of-the-adjoint-of-a-linear-map
lemma surj_adjoint_iff_inj [simp]:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
  shows  surj (adjoint f) ⟷ inj f
proof
  assume surj (adjoint f)
  then show inj f
    by (simp add: assms linear_surj_adj_imp_inj)
next
  assume inj f
  have f −‘ {0} = {0}
    using assms ‹inj f› linear_0 linear_injective_0 by fastforce
  moreover have f −‘ {0} = range (adjoint f)⊥
    by (intro ker_orthogonal_comp_adjoint assms)
  ultimately have range (adjoint f)⊥⊥ = UNIV
    by (metis orthogonal_comp_null)
  then show surj (adjoint f)
    using adjoint_linear ‹linear f›
    by (subst (asm) orthogonal_comp_self)
      (simp add: adjoint_linear linear_subspace_image)
qed

lemma inj_adjoint_iff_surj [simp]:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes linear f
  shows  inj (adjoint f) ⟷ surj f
proof
  assume inj (adjoint f)
  have (adjoint f) −‘ {0} = {0}
   by (metis ‹inj (adjoint f)› adjoint_linear assms surj_adjoint_iff_inj ker_orthogonal_comp_adjoint
orthogonal_comp_UNIV)
  then have (range(f))⊥ = {0}
     by (metis (no_types, opaque_lifting) adjoint_adjoint adjoint_linear assms
ker_orthogonal_comp_adjoint set_zero)
  then show surj f
   by (metis ‹inj (adjoint f)› adjoint_adjoint adjoint_linear assms surj_adjoint_iff_inj)
next
  assume surj f
  then have range f = (adjoint f −‘ {0})⊥
```

by (*simp add*: *adjoint_adjoint adjoint_linear assms ker_orthogonal_comp_adjoint*)
then have $\{0\} = \mathit{adjoint}\ f\ -`\ \{0\}$
using ‹*surj f*› *adjoint_adjoint adjoint_linear assms ker_orthogonal_comp_adjoint*
by *force*
then show *inj* (*adjoint f*)
by (*simp add*: ‹*surj f*› *adjoint_adjoint adjoint_linear assms linear_surj_adj_imp_inj*)
qed

lemma *linear_singular_into_hyperplane*:
fixes $f :: 'n::\mathit{euclidean\_space} \Rightarrow 'n$
assumes *linear f*
shows $\neg\ \mathit{inj}\ f \longleftrightarrow (\exists a.\ a \neq 0 \wedge (\forall x.\ a \cdot f\ x = 0))$ (**is** _ = *?rhs*)
proof
assume $\neg \mathit{inj}\ f$
then show *?rhs*
using *all_zero_iff*
by (*metis* (*no_types*, *opaque_lifting*) *adjoint_clauses*(*2*) *adjoint_linear assms*
linear_injective_0 linear_injective_imp_surjective linear_surj_adj_imp_inj)
next
assume *?rhs*
then show $\neg \mathit{inj}\ f$
by (*metis assms linear_injective_isomorphism all_zero_iff*)
qed

lemma *linear_singular_image_hyperplane*:
fixes $f :: 'n::\mathit{euclidean\_space} \Rightarrow 'n$
assumes *linear f* $\neg \mathit{inj}\ f$
obtains *a* **where** $a \neq 0\ \bigwedge S.\ f\ `\ S \subseteq \{x.\ a \cdot x = 0\}$
using *assms* **by** (*fastforce simp add*: *linear_singular_into_hyperplane*)

end

6.1 Path-Connectedness

theory *Path_Connected*
imports
Starlike
T1_Spaces
begin

6.1.1 Paths and Arcs

definition $\mathit{path} :: (\mathit{real} \Rightarrow 'a::\mathit{topological\_space}) \Rightarrow \mathit{bool}$
where $\mathit{path}\ g \equiv \mathit{continuous\_on}\ \{0..1\}\ g$

definition $\mathit{pathstart} :: (\mathit{real} \Rightarrow 'a::\mathit{topological\_space}) \Rightarrow 'a$
where $\mathit{pathstart}\ g \equiv g\ 0$

definition $\mathit{pathfinish} :: (\mathit{real} \Rightarrow 'a::\mathit{topological\_space}) \Rightarrow 'a$

where *pathfinish g* ≡ *g 1*

definition *path_image* :: (*real* ⇒ *'a*::*topological_space*) ⇒ *'a set*
where *path_image g* ≡ *g* ‘ {*0* .. *1*}

definition *reversepath* :: (*real* ⇒ *'a*::*topological_space*) ⇒ *real* ⇒ *'a*
where *reversepath g* ≡ (λ*x*. *g*(*1* − *x*))

definition *joinpaths* :: (*real* ⇒ *'a*::*topological_space*) ⇒ (*real* ⇒ *'a*) ⇒ *real* ⇒ *'a*
(**infixr** ‹+++› *75*)
where *g1* +++ *g2* ≡ (λ*x*. *if x* ≤ *1*/*2 then g1* (*2* ∗ *x*) *else g2* (*2* ∗ *x* − *1*))

definition *loop_free* :: (*real* ⇒ *'a*::*topological_space*) ⇒ *bool*
where *loop_free g* ≡ ∀ *x*∈{*0*..*1*}. ∀ *y*∈{*0*..*1*}. *g x* = *g y* ⟶ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*

definition *simple_path* :: (*real* ⇒ *'a*::*topological_space*) ⇒ *bool*
where *simple_path g* ≡ *path g* ∧ *loop_free g*

definition *arc* :: (*real* ⇒ *'a* :: *topological_space*) ⇒ *bool*
where *arc g* ≡ *path g* ∧ *inj_on g* {*0*..*1*}

6.1.2 Invariance theorems

lemma *path_eq*: *path p* ⟹ (⋀*t*. *t* ∈ {*0*..*1*} ⟹ *p t* = *q t*) ⟹ *path q*
using *continuous_on_eq path_def* **by** *blast*

lemma *path_continuous_image*: *path g* ⟹ *continuous_on* (*path_image g*) *f* ⟹ *path*(*f* ∘ *g*)
unfolding *path_def path_image_def*
using *continuous_on_compose* **by** *blast*

lemma *path_translation_eq*:
fixes *g* :: *real* ⇒ *'a* :: *real_normed_vector*
shows *path*((λ*x*. *a* + *x*) ∘ *g*) = *path g*
using *continuous_on_translation_eq path_def* **by** *blast*

lemma *path_linear_image_eq*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f*
shows *path*(*f* ∘ *g*) = *path g*
proof −
from *linear_injective_left_inverse* [*OF assms*]
obtain *h* **where** *h*: *linear h h* ∘ *f* = *id*
by *blast*
with *assms* **show** *?thesis*
by (*metis comp_assoc id_comp linear_continuous_on linear_linear path_continuous_image*)
qed

lemma *pathstart_translation*: *pathstart*((λ*x*. *a* + *x*) ∘ *g*) = *a* + *pathstart g*
by (*simp add*: *pathstart_def*)

lemma *pathstart_linear_image_eq*: *linear f* ⟹ *pathstart*(*f* ∘ *g*) = *f*(*pathstart g*)
by (*simp add*: *pathstart_def*)

lemma *pathfinish_translation*: *pathfinish*((λ*x*. *a* + *x*) ∘ *g*) = *a* + *pathfinish g*
by (*simp add*: *pathfinish_def*)

lemma *pathfinish_linear_image*: *linear f* ⟹ *pathfinish*(*f* ∘ *g*) = *f*(*pathfinish g*)
by (*simp add*: *pathfinish_def*)

lemma *path_image_translation*: *path_image*((λ*x*. *a* + *x*) ∘ *g*) = (λ*x*. *a* + *x*) ‘ (*path_image g*)
by (*simp add*: *image_comp path_image_def*)

lemma *path_image_linear_image*: *linear f* ⟹ *path_image*(*f* ∘ *g*) = *f* ‘ (*path_image g*)
by (*simp add*: *image_comp path_image_def*)

lemma *reversepath_translation*: *reversepath*((λ*x*. *a* + *x*) ∘ *g*) = (λ*x*. *a* + *x*) ∘ *reversepath g*
by (*rule ext*) (*simp add*: *reversepath_def*)

lemma *reversepath_linear_image*: *linear f* ⟹ *reversepath*(*f* ∘ *g*) = *f* ∘ *reversepath g*
by (*rule ext*) (*simp add*: *reversepath_def*)

lemma *joinpaths_translation*:
((λ*x*. *a* + *x*) ∘ *g1*) +++ ((λ*x*. *a* + *x*) ∘ *g2*) = (λ*x*. *a* + *x*) ∘ (*g1* +++ *g2*)
by (*rule ext*) (*simp add*: *joinpaths_def*)

lemma *joinpaths_linear_image*: *linear f* ⟹ (*f* ∘ *g1*) +++ (*f* ∘ *g2*) = *f* ∘ (*g1* +++ *g2*)
by (*rule ext*) (*simp add*: *joinpaths_def*)

lemma *loop_free_translation_eq*:
fixes *g* :: *real* ⇒ *'a*::*euclidean_space*
shows *loop_free*((λ*x*. *a* + *x*) ∘ *g*) = *loop_free g*
by (*simp add*: *loop_free_def*)

lemma *simple_path_translation_eq*:
fixes *g* :: *real* ⇒ *'a*::*euclidean_space*
shows *simple_path*((λ*x*. *a* + *x*) ∘ *g*) = *simple_path g*
by (*simp add*: *simple_path_def loop_free_translation_eq path_translation_eq*)

lemma *loop_free_linear_image_eq*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f*

shows *loop_free*(*f* ∘ *g*) = *loop_free g*
using *assms inj_on_eq_iff* [*of f*] **by** (*auto simp*: *loop_free_def*)

lemma *simple_path_linear_image_eq*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f*
shows *simple_path*(*f* ∘ *g*) = *simple_path g*
using *assms*
by (*simp add*: *loop_free_linear_image_eq path_linear_image_eq simple_path_def*)

lemma *simple_pathI* [*intro?*]:
assumes *path p*
assumes ⋀*x y*. *0* ≤ *x* ⟹ *x* < *y* ⟹ *y* ≤ *1* ⟹ *p x* = *p y* ⟹ *x* = *0* ∧ *y* = *1*
shows *simple_path p*
unfolding *simple_path_def loop_free_def*
proof (*intro ballI conjI impI*)
fix *x y* **assume** *x* ∈ {*0*..*1*} *y* ∈ {*0*..*1*} *p x* = *p y*
thus *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*
by (*metis assms*(*2*) *atLeastAtMost_iff linorder_less_linear*)
qed *fact*+

lemma *arcD*: *arc p* ⟹ *p x* = *p y* ⟹ *x* ∈ {*0*..*1*} ⟹ *y* ∈ {*0*..*1*} ⟹ *x* = *y*
by (*auto simp*: *arc_def inj_on_def*)

lemma *arc_translation_eq*:
fixes *g* :: *real* ⇒ *'a*::*euclidean_space*
shows *arc*((λ*x*. *a* + *x*) ∘ *g*) ⟷ *arc g*
by (*auto simp*: *arc_def inj_on_def path_translation_eq*)

lemma *arc_linear_image_eq*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f*
shows *arc*(*f* ∘ *g*) = *arc g*
using *assms inj_on_eq_iff* [*of f*]
by (*auto simp*: *arc_def inj_on_def path_linear_image_eq*)

6.1.3 Basic lemmas about paths

lemma *path_of_real*: *path complex_of_real*
unfolding *path_def* **by** (*intro continuous_intros*)

lemma *path_const*: *path* (λ*t*. *a*) **for** *a*::*'a*::*real_normed_vector*
unfolding *path_def* **by** (*intro continuous_intros*)

lemma *path_minus*: *path g* ⟹ *path* (λ*t*. − *g t*) **for** *g*::*real*⇒*'a*::*real_normed_vector*
unfolding *path_def* **by** (*intro continuous_intros*)

lemma *path_add*: ⟦*path f*; *path g*⟧ ⟹ *path* (λ*t*. *f t* + *g t*) **for** *f*::*real*⇒*'a*::*real_normed_vector*
unfolding *path_def* **by** (*intro continuous_intros*)

lemma *path_diff*: $[\![path\ f; path\ g]\!] \Longrightarrow path\ (\lambda t.\ f\ t - g\ t)$ **for** f::*real*⇒'*a*::*real_normed_vector*
unfolding *path_def* **by** (*intro continuous_intros*)

lemma *path_mult*: $[\![path\ f; path\ g]\!] \Longrightarrow path\ (\lambda t.\ f\ t * g\ t)$ **for** f::*real*⇒'*a*::*real_normed_field*
unfolding *path_def* **by** (*intro continuous_intros*)

lemma *pathin_iff_path_real* [*simp*]: *pathin euclideanreal g* ⟷ *path g*
by (*simp add*: *pathin_def path_def*)

lemma *continuous_on_path*: *path f* ⟹ $t \subseteq \{0..1\}$ ⟹ *continuous_on t f*
using *continuous_on_subset path_def* **by** *blast*

lemma *inj_on_imp_loop_free*: *inj_on g* $\{0..1\}$ ⟹ *loop_free g*
by (*simp add*: *inj_onD loop_free_def*)

lemma *arc_imp_simple_path*: *arc g* ⟹ *simple_path g*
by (*simp add*: *arc_def inj_on_imp_loop_free simple_path_def*)

lemma *arc_imp_path*: *arc g* ⟹ *path g*
using *arc_def* **by** *blast*

lemma *arc_imp_inj_on*: *arc g* ⟹ *inj_on g* $\{0..1\}$
by (*auto simp*: *arc_def*)

lemma *simple_path_imp_path*: *simple_path g* ⟹ *path g*
using *simple_path_def* **by** *blast*

lemma *loop_free_cases*: *loop_free g* ⟹ *inj_on g* $\{0..1\}$ ∨ *pathfinish g* = *pathstart g*
by (*force simp*: *inj_on_def loop_free_def pathfinish_def pathstart_def*)

lemma *simple_path_cases*: *simple_path g* ⟹ *arc g* ∨ *pathfinish g* = *pathstart g*
using *arc_def loop_free_cases simple_path_def* **by** *blast*

lemma *simple_path_imp_arc*: *simple_path g* ⟹ *pathfinish g* ≠ *pathstart g* ⟹ *arc g*
using *simple_path_cases* **by** *auto*

lemma *arc_distinct_ends*: *arc g* ⟹ *pathfinish g* ≠ *pathstart g*
unfolding *arc_def inj_on_def pathfinish_def pathstart_def*
by *fastforce*

lemma *arc_simple_path*: *arc g* ⟷ *simple_path g* ∧ *pathfinish g* ≠ *pathstart g*
using *arc_distinct_ends arc_imp_simple_path simple_path_cases* **by** *blast*

lemma *simple_path_eq_arc*: *pathfinish g* ≠ *pathstart g* ⟹ (*simple_path g* = *arc g*)
by (*simp add*: *arc_simple_path*)

lemma *path_image_const* [*simp*]: *path_image* (λ*t*. *a*) = {*a*}
by (*force simp*: *path_image_def*)

lemma *path_image_nonempty* [*simp*]: *path_image g* ≠ {}
unfolding *path_image_def image_is_empty box_eq_empty*
by *auto*

lemma *pathstart_in_path_image*[*intro*]: *pathstart g* ∈ *path_image g*
unfolding *pathstart_def path_image_def*
by *auto*

lemma *pathfinish_in_path_image*[*intro*]: *pathfinish g* ∈ *path_image g*
unfolding *pathfinish_def path_image_def*
by *auto*

lemma *connected_path_image*[*intro*]: *path g* ⟹ *connected* (*path_image g*)
unfolding *path_def path_image_def*
using *connected_continuous_image connected_Icc* **by** *blast*

lemma *compact_path_image*[*intro*]: *path g* ⟹ *compact* (*path_image g*)
unfolding *path_def path_image_def*
using *compact_continuous_image connected_Icc* **by** *blast*

lemma *reversepath_reversepath*[*simp*]: *reversepath* (*reversepath g*) = *g*
unfolding *reversepath_def*
by *auto*

lemma *pathstart_reversepath*[*simp*]: *pathstart* (*reversepath g*) = *pathfinish g*
unfolding *pathstart_def reversepath_def pathfinish_def*
by *auto*

lemma *pathfinish_reversepath*[*simp*]: *pathfinish* (*reversepath g*) = *pathstart g*
unfolding *pathstart_def reversepath_def pathfinish_def*
by *auto*

lemma *reversepath_o*: *reversepath g* = *g* ∘ (−)*1*
by (*auto simp*: *reversepath_def*)

lemma *pathstart_join*[*simp*]: *pathstart* (*g1* +++ *g2*) = *pathstart g1*
unfolding *pathstart_def joinpaths_def pathfinish_def*
by *auto*

lemma *pathfinish_join*[*simp*]: *pathfinish* (*g1* +++ *g2*) = *pathfinish g2*
unfolding *pathstart_def joinpaths_def pathfinish_def*
by *auto*

lemma *path_image_reversepath*[*simp*]: *path_image* (*reversepath g*) = *path_image g*

by (*metis cancel_comm_monoid_add_class.diff_cancel diff_zero image_comp image_diff_atLeastAtMost path_image_def reversepath_o*)

lemma *path_reversepath* [*simp*]: *path* (*reversepath g*) ⟷ *path g*
by (*metis continuous_on_compose continuous_on_op_minus image_comp image_ident path_def path_image_def path_image_reversepath reversepath_o reversepath_reversepath*)

lemma *arc_reversepath*:
assumes *arc g* **shows** *arc*(*reversepath g*)
proof −
have *injg*: *inj_on g* {*0..1*}
using *assms*
by (*simp add*: *arc_def*)
have ∗∗: ⋀*x y*::*real*. *1*−*x* = *1*−*y* ⟹ *x* = *y*
by *simp*
show *?thesis*
using *assms* **by** (*clarsimp simp*: *arc_def intro*!: *inj_onI*) (*simp add*: *inj_onD reversepath_def* ∗∗)
qed

lemma *loop_free_reversepath*:
assumes *loop_free g* **shows** *loop_free*(*reversepath g*)
using *assms* **by** (*simp add*: *reversepath_def loop_free_def Ball_def*) (*smt* (*verit*))

lemma *simple_path_reversepath*: *simple_path g* ⟹ *simple_path* (*reversepath g*)
by (*simp add*: *loop_free_reversepath simple_path_def*)

lemmas *reversepath_simps* =
path_reversepath path_image_reversepath pathstart_reversepath pathfinish_reversepath

lemma *path_join*[*simp*]:
assumes *pathfinish g1* = *pathstart g2*
shows *path* (*g1* +++ *g2*) ⟷ *path g1* ∧ *path g2*
unfolding *path_def pathfinish_def pathstart_def*
proof *safe*
assume *cont*: *continuous_on* {*0..1*} (*g1* +++ *g2*)
have *g1*: *continuous_on* {*0..1*} *g1* ⟷ *continuous_on* {*0..1*} ((*g1* +++ *g2*) ∘ (λ*x*. *x* / *2*))
by (*intro continuous_on_cong refl*) (*auto simp*: *joinpaths_def*)
have *g2*: *continuous_on* {*0..1*} *g2* ⟷ *continuous_on* {*0..1*} ((*g1* +++ *g2*) ∘ (λ*x*. *x* / *2* + *1*/*2*))
using *assms*
by (*intro continuous_on_cong refl*) (*auto simp*: *joinpaths_def pathfinish_def pathstart_def*)
show *continuous_on* {*0..1*} *g1* **and** *continuous_on* {*0..1*} *g2*
unfolding *g1 g2*
by (*auto intro*!: *continuous_intros continuous_on_subset*[*OF cont*] *simp del*: *o_apply*)

next
assume *g1g2*: *continuous_on* {*0..1*} *g1 continuous_on* {*0..1*} *g2*
have *01*: {*0 .. 1*} = {*0..1/2*} ∪ {*1/2 .. 1*::*real*}
by *auto*
{
fix *x* :: *real*
assume *0* ≤ *x* **and** *x* ≤ *1*
then have *x* ∈ (λ*x*. *x* ∗ *2*) ‘ {*0..1 / 2*}
by (*intro image_eqI*[**where** *x*=*x/2*]) *auto*
}
note *1* = *this*
{
fix *x* :: *real*
assume *0* ≤ *x* **and** *x* ≤ *1*
then have *x* ∈ (λ*x*. *x* ∗ *2* − *1*) ‘ {*1 / 2..1*}
by (*intro image_eqI*[**where** *x*=*x/2* + *1/2*]) *auto*
}
note *2* = *this*
show *continuous_on* {*0..1*} (*g1* +++ *g2*)
using *assms*
unfolding *joinpaths_def 01*
apply (*intro continuous_on_cases closed_atLeastAtMost g1g2*[*THEN continuous_on_compose2*] *continuous_intros*)
apply (*auto simp*: *field_simps pathfinish_def pathstart_def intro*!: *1 2*)
done
qed

6.1.4 Path Images

lemma *bounded_path_image*: *path g* ⟹ *bounded*(*path_image g*)
by (*simp add*: *compact_imp_bounded compact_path_image*)

lemma *closed_path_image*:
fixes *g* :: *real* ⇒ *'a*::*t2_space*
shows *path g* ⟹ *closed*(*path_image g*)
by (*metis compact_path_image compact_imp_closed*)

lemma *connected_simple_path_image*: *simple_path g* ⟹ *connected*(*path_image g*)
by (*metis connected_path_image simple_path_imp_path*)

lemma *compact_simple_path_image*: *simple_path g* ⟹ *compact*(*path_image g*)
by (*metis compact_path_image simple_path_imp_path*)

lemma *bounded_simple_path_image*: *simple_path g* ⟹ *bounded*(*path_image g*)
by (*metis bounded_path_image simple_path_imp_path*)

lemma *closed_simple_path_image*:
fixes *g* :: *real* ⇒ *'a*::*t2_space*

shows *simple_path g* $\Longrightarrow$ *closed*(*path_image g*)
by (*metis closed_path_image simple_path_imp_path*)

lemma *connected_arc_image*: *arc g* $\Longrightarrow$ *connected*(*path_image g*)
by (*metis connected_path_image arc_imp_path*)

lemma *compact_arc_image*: *arc g* $\Longrightarrow$ *compact*(*path_image g*)
by (*metis compact_path_image arc_imp_path*)

lemma *bounded_arc_image*: *arc g* $\Longrightarrow$ *bounded*(*path_image g*)
by (*metis bounded_path_image arc_imp_path*)

lemma *closed_arc_image*:
fixes *g* :: *real* $\Rightarrow$ *'a*::*t2_space*
shows *arc g* $\Longrightarrow$ *closed*(*path_image g*)
by (*metis closed_path_image arc_imp_path*)

lemma *path_image_join_subset*: *path_image* (*g1* +++ *g2*) $\subseteq$ *path_image g1* $\cup$ *path_image g2*
unfolding *path_image_def joinpaths_def*
by *auto*

lemma *subset_path_image_join*:
assumes *path_image g1* $\subseteq$ *S* **and** *path_image g2* $\subseteq$ *S*
shows *path_image* (*g1* +++ *g2*) $\subseteq$ *S*
using *path_image_join_subset*[*of g1 g2*] **and** *assms*
by *auto*

lemma *path_image_join*:
assumes *pathfinish g1* = *pathstart g2*
shows *path_image*(*g1* +++ *g2*) = *path_image g1* $\cup$ *path_image g2*
proof −
have *path_image g1* $\subseteq$ *path_image* (*g1* +++ *g2*)
proof (*clarsimp simp*: *path_image_def joinpaths_def*)
fix *u*::*real*
assume *0* $\leq$ *u u* $\leq$ *1*
then show *g1 u* $\in$ (*λx. g1* (*2* $*$ *x*)) ' ({*0..1*} $\cap$ {*x. x* $*$ *2* $\leq$ *1*})
by (*rule_tac x*=*u*/*2* **in** *image_eqI*) *auto*
qed
moreover
have §: *g2 u* $\in$ (*λx. g2* (*2* $*$ *x* − *1*)) ' ({*0..1*} $\cap$ {*x.* ¬ *x* $*$ *2* $\leq$ *1*})
if *0* < *u u* $\leq$ *1* **for** *u*
using *that assms*
by (*rule_tac x*=(*u*+*1*)/*2* **in** *image_eqI*) (*auto simp*: *field_simps pathfinish_def pathstart_def*)
have *g2 0* $\in$ (*λx. g1* (*2* $*$ *x*)) ' ({*0..1*} $\cap$ {*x. x* $*$ *2* $\leq$ *1*})
using *assms*
by (*rule_tac x*=*1*/*2* **in** *image_eqI*) (*auto simp*: *pathfinish_def pathstart_def*)
then have *path_image g2* $\subseteq$ *path_image* (*g1* +++ *g2*)

by (*auto simp*: *path_image_def joinpaths_def intro*!: §)
ultimately show *?thesis*
using *path_image_join_subset* **by** *blast*
qed

lemma *not_in_path_image_join*:
assumes $x \notin$ *path_image g1* **and** $x \notin$ *path_image g2*
shows $x \notin$ *path_image* (*g1* +++ *g2*)
using *assms* **and** *path_image_join_subset*[*of g1 g2*]
by *auto*

lemma *pathstart_compose*: *pathstart*($f \circ p$) = f(*pathstart p*)
by (*simp add*: *pathstart_def*)

lemma *pathfinish_compose*: *pathfinish*($f \circ p$) = f(*pathfinish p*)
by (*simp add*: *pathfinish_def*)

lemma *path_image_compose*: *path_image* ($f \circ p$) = f ‘ (*path_image p*)
by (*simp add*: *image_comp path_image_def*)

lemma *path_compose_join*: $f \circ$ (*p* +++ *q*) = ($f \circ p$) +++ ($f \circ q$)
by (*rule ext*) (*simp add*: *joinpaths_def*)

lemma *path_compose_reversepath*: $f \circ$ *reversepath p* = *reversepath*($f \circ p$)
by (*rule ext*) (*simp add*: *reversepath_def*)

lemma *joinpaths_eq*:
($\bigwedge t.\ t \in \{0..1\} \Longrightarrow p\ t = p'\ t$) $\Longrightarrow$
($\bigwedge t.\ t \in \{0..1\} \Longrightarrow q\ t = q'\ t$)
$\Longrightarrow$ $t \in \{0..1\} \Longrightarrow$ (*p* +++ *q*) *t* = (*p'* +++ *q'*) *t*
by (*auto simp*: *joinpaths_def*)

lemma *loop_free_inj_on*: *loop_free g* $\Longrightarrow$ *inj_on g* {*0*<..<*1*}
by (*force simp*: *inj_on_def loop_free_def*)

lemma *simple_path_inj_on*: *simple_path g* $\Longrightarrow$ *inj_on g* {*0*<..<*1*}
using *loop_free_inj_on simple_path_def* **by** *auto*

6.1.5 Simple paths with the endpoints removed

lemma *simple_path_endless*:
assumes *simple_path c*
shows *path_image c* − {*pathstart c*,*pathfinish c*} = *c* ‘ {*0*<..<*1*} (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using *less_eq_real_def* **by** (*auto simp*: *path_image_def pathstart_def pathfinish_def*)
show *?rhs* ⊆ *?lhs*

using *assms*
apply (*simp add*: *image_subset_iff path_image_def pathstart_def pathfinish_def simple_path_def loop_free_def Ball_def*)
by (*smt* (*verit*))
qed

lemma *connected_simple_path_endless*:
assumes *simple_path c*
shows *connected*(*path_image c* − {*pathstart c*,*pathfinish c*})
proof −
have *continuous_on* {*0*<..<*1*} *c*
using *assms* **by** (*simp add*: *simple_path_def continuous_on_path path_def subset_iff*)
then have *connected* (*c* ‘ {*0*<..<*1*})
using *connected_Ioo connected_continuous_image* **by** *blast*
then show *?thesis*
using *assms* **by** (*simp add*: *simple_path_endless*)
qed

lemma *nonempty_simple_path_endless*:
simple_path c ⟹ *path_image c* − {*pathstart c*,*pathfinish c*} ≠ {}
by (*simp add*: *simple_path_endless*)

lemma *simple_path_continuous_image*:
assumes *simple_path f continuous_on* (*path_image f*) *g inj_on g* (*path_image f*)
shows *simple_path* (*g* ∘ *f*)
unfolding *simple_path_def*
proof
show *path* (*g* ∘ *f*)
using *assms* **unfolding** *simple_path_def* **by** (*intro path_continuous_image*) *auto*
from *assms* **have** [*simp*]: *g* (*f x*) = *g* (*f y*) ⟷ *f x* = *f y* **if** *x* ∈ {*0..1*} *y* ∈ {*0..1*}
for *x y*
unfolding *inj_on_def path_image_def* **using** *that* **by** *fastforce*
show *loop_free* (*g* ∘ *f*)
using *assms*(*1*) **by** (*auto simp*: *loop_free_def simple_path_def*)
qed

6.1.6 The operations on paths

lemma *path_image_subset_reversepath*: *path_image*(*reversepath g*) ≤ *path_image g*
by *simp*

lemma *path_imp_reversepath*: *path g* ⟹ *path*(*reversepath g*)
by *simp*

lemma *half_bounded_equal*: *1* ≤ *x* ∗ *2* ⟹ *x* ∗ *2* ≤ *1* ⟷ *x* = (*1*/*2*::*real*)

by *simp*

lemma *continuous_on_joinpaths*:
assumes *continuous_on* {*0..1*} *g1* *continuous_on* {*0..1*} *g2* *pathfinish g1* = *pathstart g2*
shows *continuous_on* {*0..1*} (*g1* +++ *g2*)
using *assms path_def path_join* **by** *blast*

lemma *path_join_imp*: ⟦*path g1*; *path g2*; *pathfinish g1* = *pathstart g2*⟧ ⟹ *path*(*g1* +++ *g2*)
by *simp*

lemma *arc_join*:
assumes *arc g1* *arc g2*
pathfinish g1 = *pathstart g2*
path_image g1 ∩ *path_image g2* ⊆ {*pathstart g2*}
shows *arc*(*g1* +++ *g2*)
proof −
have *injg1*: *inj_on g1* {*0..1*} **and** *injg2*: *inj_on g2* {*0..1*}
and *g11*: *g1 1* = *g2 0* **and** *sb*: *g1* ‘ {*0..1*} ∩ *g2* ‘ {*0..1*} ⊆ {*g2 0*}
using *assms*
by (*auto simp*: *arc_def pathfinish_def pathstart_def path_image_def*)
{ fix *x* **and** *y*::*real*
assume *xy*: *g2* (*2* ∗ *x* − *1*) = *g1* (*2* ∗ *y*) *x* ≤ *1* *0* ≤ *y* *y* ∗ *2* ≤ *1* ¬ *x* ∗ *2* ≤ *1*
then have *g1* (*2* ∗ *y*) = *g2 0*
using *sb* **by** *force*
then have *False*
using *xy inj_onD injg2* **by** *fastforce*
} note ∗ = *this*
have *inj_on* (*g1* +++ *g2*) {*0..1*}
using *inj_onD* [*OF injg1*] *inj_onD* [*OF injg2*] ∗
by (*simp add*: *inj_on_def joinpaths_def Ball_def*) (*smt* (*verit*))
then show *?thesis*
using *arc_def assms path_join_imp* **by** *blast*
qed

lemma *simple_path_join_loop*:
assumes *arc g1* *arc g2*
pathfinish g1 = *pathstart g2* *pathfinish g2* = *pathstart g1*
path_image g1 ∩ *path_image g2* ⊆ {*pathstart g1*, *pathstart g2*}
shows *simple_path*(*g1* +++ *g2*)
proof −
have *injg1*: *inj_on g1* {*0..1*} **and** *injg2*: *inj_on g2* {*0..1*}
using *assms* **by** (*auto simp add*: *arc_def*)
have *g12*: *g1 1* = *g2 0*
and *g21*: *g2 1* = *g1 0*
and *sb*: *g1* ‘ {*0..1*} ∩ *g2* ‘ {*0..1*} ⊆ {*g1 0*, *g2 0*}
using *assms*
by (*simp_all add*: *arc_def pathfinish_def pathstart_def path_image_def*)

```
  { fix x and y::real
    assume g2_eq: g2 (2 * x - 1) = g1 (2 * y)
      and xyI: x ≠ 1 ∨ y ≠ 0
      and xy: x ≤ 1 0 ≤ y  y * 2 ≤ 1 ¬ x * 2 ≤ 1
    then consider g1 (2 * y) = g1 0 | g1 (2 * y) = g2 0
      using sb by force
    then have False
    proof cases
      case 1
      then have y = 0
        using xy g2_eq by (auto dest!: inj_onD [OF injg1])
      then show ?thesis
        using xy g2_eq xyI by (auto dest: inj_onD [OF injg2] simp flip: g21)
    next
      case 2
      then have 2*x = 1
        using g2_eq g12 inj_onD [OF injg2] atLeastAtMost_iff xy(1) xy(4) by
fastforce
      with xy show False by auto
    qed
  } note * = this
  have loop_free(g1 +++ g2)
    using inj_onD [OF injg1] inj_onD [OF injg2] *
    by (simp add: loop_free_def joinpaths_def Ball_def) (smt (verit))
  then show ?thesis
    by (simp add: arc_imp_path assms simple_path_def)
qed

lemma reversepath_joinpaths:
    pathfinish g1 = pathstart g2 ⟹ reversepath(g1 +++ g2) = reversepath g2
+++ reversepath g1
  unfolding reversepath_def pathfinish_def pathstart_def joinpaths_def
  by (rule ext) (auto simp: mult.commute)
```

6.1.7 Some reversed and "if and only if" versions of joining theorems

```
lemma path_join_path_ends:
  fixes g1 :: real ⇒ 'a::metric_space
  assumes path(g1 +++ g2) path g2
    shows pathfinish g1 = pathstart g2
proof (rule ccontr)
  define e where e = dist (g1 1) (g2 0)
  assume Neg: pathfinish g1 ≠ pathstart g2
  then have 0 < dist (pathfinish g1) (pathstart g2)
    by auto
  then have e > 0
    by (metis e_def pathfinish_def pathstart_def)
  then have ∀ e>0. ∃ d>0. ∀ x'∈{0..1}. dist x' 0 < d ⟶ dist (g2 x') (g2 0) < e
```

using ‹*path g2*› *atLeastAtMost_iff zero_le_one* **unfolding** *path_def continuous_on_iff*
by *blast*
then obtain *d1* **where** *d1 > 0*
and *d1*: $\bigwedge x'$. $[\![x' \in \{0..1\}; \mathit{norm}\ x' < d1]\!] \Longrightarrow \mathit{dist}\ (g2\ x')\ (g2\ 0) < e/2$
by (*metis* ‹*0 < e*› *half_gt_zero_iff norm_conv_dist*)
obtain *d2* **where** *d2 > 0*
and *d2*: $\bigwedge x'$. $[\![x' \in \{0..1\}; \mathit{dist}\ x'\ (1/2) < d2]\!]$
$\Longrightarrow \mathit{dist}\ ((g1 +\!\!+\!\!+ g2)\ x')\ (g1\ 1) < e/2$
using *assms*(*1*) ‹*e > 0*› **unfolding** *path_def continuous_on_iff*
apply (*drule_tac x=1/2* **in** *bspec*, *simp*)
apply (*drule_tac x=e/2* **in** *spec*, *force simp*: *joinpaths_def*)
done
have *int01_1*: *min (1/2) (min d1 d2) / 2* $\in \{0..1\}$
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def*)
have *dist1*: *norm (min (1 / 2) (min d1 d2) / 2) < d1*
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def dist_norm*)
have *int01_2*: *1/2 + min (1/2) (min d1 d2) / 4* $\in \{0..1\}$
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def*)
have *dist2*: *dist (1 / 2 + min (1 / 2) (min d1 d2) / 4) (1 / 2) < d2*
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def dist_norm*)
have [*simp*]: $\neg$ *min (1 / 2) (min d1 d2)* ≤ 0
using ‹*d1 > 0*› ‹*d2 > 0*› **by** (*simp add*: *min_def*)
have *dist (g2 (min (1 / 2) (min d1 d2) / 2)) (g1 1) < e/2*
dist (g2 (min (1 / 2) (min d1 d2) / 2)) (g2 0) < e/2
using *d1* [*OF int01_1 dist1*] *d2* [*OF int01_2 dist2*] **by** (*simp_all add*: *joinpaths_def*)
then have *dist (g1 1) (g2 0) < e/2 + e/2*
using *dist_triangle_half_r e_def* **by** *blast*
then show *False*
by (*simp add*: *e_def* [*symmetric*])
qed

lemma *path_join_eq* [*simp*]:
fixes *g1* :: *real* $\Rightarrow$ *'a*::*metric_space*
assumes *path g1 path g2*
shows *path*$(g1 +\!\!+\!\!+ g2) \longleftrightarrow$ *pathfinish g1 = pathstart g2*
using *assms* **by** (*metis path_join_path_ends path_join_imp*)

lemma *simple_path_joinE*:
assumes *simple_path*$(g1 +\!\!+\!\!+ g2)$ **and** *pathfinish g1 = pathstart g2*
obtains *arc g1 arc g2*
path_image g1 $\cap$ *path_image g2* $\subseteq$ $\{$*pathstart g1*, *pathstart g2*$\}$
proof $-$
have $*$: $\bigwedge x\ y$. $[\![0 \leq x; x \leq 1; 0 \leq y; y \leq 1; (g1 +\!\!+\!\!+ g2)\ x = (g1 +\!\!+\!\!+ g2)\ y]\!]$
$\Longrightarrow x = y \vee x = 0 \wedge y = 1 \vee x = 1 \wedge y = 0$
using *assms* **by** (*simp add*: *simple_path_def loop_free_def*)
have *path g1*
using *assms path_join simple_path_imp_path* **by** *blast*

```
  moreover have inj_on g1 {0..1}
  proof (clarsimp simp: inj_on_def)
    fix x y
    assume g1 x = g1 y 0 ≤ x x ≤ 1 0 ≤ y y ≤ 1
    then show x = y
      using * [of x/2 y/2] by (simp add: joinpaths_def split_ifs)
  qed
  ultimately have arc g1
    using assms  by (simp add: arc_def)
  have [simp]: g2 0 = g1 1
    using assms by (metis pathfinish_def pathstart_def)
  have path g2
    using assms path_join simple_path_imp_path by blast
  moreover have inj_on g2 {0..1}
  proof (clarsimp simp: inj_on_def)
    fix x y
    assume g2 x = g2 y 0 ≤ x x ≤ 1 0 ≤ y y ≤ 1
    then show x = y
      using * [of (x+1) / 2 (y+1) / 2]
      by (force simp: joinpaths_def split_ifs field_split_simps)
  qed
  ultimately have arc g2
    using assms  by (simp add: arc_def)
  have g2 y = g1 0 ∨ g2 y = g1 1
       if g1 x = g2 y 0 ≤ x x ≤ 1 0 ≤ y y ≤ 1 for x y
      using * [of x / 2 (y + 1) / 2] that
      by (auto simp: joinpaths_def split_ifs field_split_simps)
  then have path_image g1 ∩ path_image g2 ⊆ {pathstart g1, pathstart g2}
    by (fastforce simp: pathstart_def pathfinish_def path_image_def)
  with ‹arc g1› ‹arc g2› show ?thesis using that by blast
qed

lemma simple_path_join_loop_eq:
  assumes pathfinish g2 = pathstart g1 pathfinish g1 = pathstart g2
  shows simple_path(g1 +++ g2) ⟷
            arc g1 ∧ arc g2 ∧ path_image g1 ∩ path_image g2 ⊆ {pathstart g1,
pathstart g2}
  by (metis assms simple_path_joinE simple_path_join_loop)

lemma arc_join_eq:
  assumes pathfinish g1 = pathstart g2
    shows arc(g1 +++ g2) ⟷
          arc g1 ∧ arc g2 ∧ path_image g1 ∩ path_image g2 ⊆ {pathstart g2}
          (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    using reversepath_simps assms
    by (smt (verit, best) Int_commute arc_reversepath arc_simple_path in_mono
insertE pathstart_join
```

reversepath_joinpaths simple_path_joinE subsetI)
next
assume *?rhs* **then show** *?lhs*
using *assms*
by (*fastforce simp*: *pathfinish_def pathstart_def intro*!: *arc_join*)
qed

lemma *arc_join_eq_alt*:
pathfinish g1 = pathstart g2
$\Longrightarrow$ *arc*(*g1 +++ g2*) $\longleftrightarrow$ *arc g1* $\wedge$ *arc g2* $\wedge$ *path_image g1* $\cap$ *path_image g2* = {*pathstart g2*}
using *pathfinish_in_path_image* **by** (*fastforce simp*: *arc_join_eq*)

Symmetry and loops

lemma *path_sym*:
⟦*pathfinish p = pathstart q*; *pathfinish q = pathstart p*⟧ $\Longrightarrow$ *path*(*p +++ q*) $\longleftrightarrow$ *path*(*q +++ p*)
by *auto*

lemma *simple_path_sym*:
⟦*pathfinish p = pathstart q*; *pathfinish q = pathstart p*⟧
$\Longrightarrow$ *simple_path*(*p +++ q*) $\longleftrightarrow$ *simple_path*(*q +++ p*)
by (*metis* (*full_types*) *inf_commute insert_commute simple_path_joinE simple_path_join_loop*)

lemma *path_image_sym*:
⟦*pathfinish p = pathstart q*; *pathfinish q = pathstart p*⟧
$\Longrightarrow$ *path_image*(*p +++ q*) = *path_image*(*q +++ p*)
by (*simp add*: *path_image_join sup_commute*)

lemma *simple_path_joinI*:
assumes *arc p1 arc p2 pathfinish p1 = pathstart p2*
assumes *path_image p1* $\cap$ *path_image p2*
$\subseteq$ *insert* (*pathstart p2*) (*if pathstart p1 = pathfinish p2 then* {*pathstart p1*} *else* {})
shows *simple_path* (*p1 +++ p2*)
by (*smt* (*verit*, *best*) *Int_commute arc_imp_simple_path arc_join assms insert_commute simple_path_join_loop simple_path_sym*)

lemma *simple_path_join3I*:
assumes *arc p1 arc p2 arc p3*
assumes *path_image p1* $\cap$ *path_image p2* $\subseteq$ {*pathstart p2*}
assumes *path_image p2* $\cap$ *path_image p3* $\subseteq$ {*pathstart p3*}
assumes *path_image p1* $\cap$ *path_image p3* $\subseteq$ {*pathstart p1*} $\cap$ {*pathfinish p3*}
assumes *pathfinish p1 = pathstart p2 pathfinish p2 = pathstart p3*
shows *simple_path* (*p1 +++ p2 +++ p3*)
using *assms* **by** (*intro simple_path_joinI arc_join*) (*auto simp*: *path_image_join*)

6.1.8 The joining of paths is associative

lemma *path_assoc*:
 ⟦*pathfinish p = pathstart q*; *pathfinish q = pathstart r*⟧
  ⟹ *path*(*p +++ (q +++ r*)) ⟷ *path*((*p +++ q*) *+++ r*)
 by *simp*

lemma *simple_path_assoc*:
 assumes *pathfinish p = pathstart q pathfinish q = pathstart r*
  shows *simple_path* (*p +++* (*q +++ r*)) ⟷ *simple_path* ((*p +++ q*) *+++ r*)
proof (*cases pathstart p = pathfinish r*)
 case *True* **show** *?thesis*
 proof
  assume *simple_path* (*p +++ q +++ r*)
  with *assms True* **show** *simple_path* ((*p +++ q*) *+++ r*)
  by (*fastforce simp add*: *simple_path_join_loop_eq arc_join_eq path_image_join*
      dest: *arc_distinct_ends* [*of r*])
 next
  assume *0*: *simple_path* ((*p +++ q*) *+++ r*)
  with *assms True* **have** *q*: *pathfinish r* ∉ *path_image q*
   using *arc_distinct_ends*
  by (*fastforce simp add*: *simple_path_join_loop_eq arc_join_eq path_image_join*)
  have *pathstart r* ∉ *path_image p*
   using *assms*
   by (*metis 0 IntI arc_distinct_ends arc_join_eq_alt empty_iff insert_iff*
      pathfinish_in_path_image pathfinish_join simple_path_joinE)
  with *assms 0 q True* **show** *simple_path* (*p +++ q +++ r*)
   by (*auto simp*: *simple_path_join_loop_eq arc_join_eq path_image_join*
      dest!: *subsetD* [*OF _ IntI*])
 qed
next
 case *False*
 { fix *x* :: *'a*
  assume *a*: *path_image p* ∩ *path_image q* ⊆ {*pathstart q*}
      (*path_image p* ∪ *path_image q*) ∩ *path_image r* ⊆ {*pathstart r*}
      x ∈ *path_image p x* ∈ *path_image r*
  have *pathstart r* ∈ *path_image q*
   by (*metis assms*(*2*) *pathfinish_in_path_image*)
  with *a* **have** *x = pathstart q*
   by *blast*
 }
 with *False assms* **show** *?thesis*
  by (*auto simp*: *simple_path_eq_arc simple_path_join_loop_eq arc_join_eq*
path_image_join)
qed

lemma *arc_assoc*:
  ⟦*pathfinish p = pathstart q*; *pathfinish q = pathstart r*⟧
   ⟹ *arc*(*p +++* (*q +++ r*)) ⟷ *arc*((*p +++ q*) *+++ r*)

by (*simp add*: *arc_simple_path simple_path_assoc*)

6.1.9 Subpath

definition *subpath* :: *real* ⇒ *real* ⇒ (*real* ⇒ *'a*) ⇒ *real* ⇒ *'a*::*real_normed_vector*
where *subpath a b g* ≡ λ*x*. *g*((*b* − *a*) ∗ *x* + *a*)

lemma *path_image_subpath_gen*:
fixes *g* :: _ ⇒ *'a*::*real_normed_vector*
shows *path_image*(*subpath u v g*) = *g* ‘ (*closed_segment u v*)
by (*auto simp add*: *closed_segment_real_eq path_image_def subpath_def*)

lemma *path_image_subpath*:
fixes *g* :: *real* ⇒ *'a*::*real_normed_vector*
shows *path_image*(*subpath u v g*) = (*if u* ≤ *v then g* ‘ {*u*..*v*} *else g* ‘ {*v*..*u*})
by (*simp add*: *path_image_subpath_gen closed_segment_eq_real_ivl*)

lemma *path_image_subpath_commute*:
fixes *g* :: *real* ⇒ *'a*::*real_normed_vector*
shows *path_image*(*subpath u v g*) = *path_image*(*subpath v u g*)
by (*simp add*: *path_image_subpath_gen closed_segment_eq_real_ivl*)

lemma *path_subpath* [*simp*]:
fixes *g* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *path g u* ∈ {*0*..*1*} *v* ∈ {*0*..*1*}
shows *path*(*subpath u v g*)
proof −
have *continuous_on* {*u*..*v*} *g continuous_on* {*v*..*u*} *g*
using *assms continuous_on_path* **by** *fastforce*+
then have *continuous_on* {*0*..*1*} (*g* ∘ (λ*x*. ((*v*−*u*) ∗ *x*+ *u*)))
by (*intro continuous_intros*; *simp add*: *image_affinity_atLeastAtMost* [**where** *c*=*u*])
then show *?thesis*
by (*simp add*: *path_def subpath_def*)
qed

lemma *pathstart_subpath* [*simp*]: *pathstart*(*subpath u v g*) = *g*(*u*)
by (*simp add*: *pathstart_def subpath_def*)

lemma *pathfinish_subpath* [*simp*]: *pathfinish*(*subpath u v g*) = *g*(*v*)
by (*simp add*: *pathfinish_def subpath_def*)

lemma *subpath_trivial* [*simp*]: *subpath 0 1 g* = *g*
by (*simp add*: *subpath_def*)

lemma *subpath_reversepath*: *subpath 1 0 g* = *reversepath g*
by (*simp add*: *reversepath_def subpath_def*)

lemma *reversepath_subpath*: *reversepath*(*subpath u v g*) = *subpath v u g*

by (*simp add*: *reversepath_def subpath_def algebra_simps*)

lemma *subpath_translation*: *subpath u v* ((λ*x*. *a* + *x*) ∘ *g*) = (λ*x*. *a* + *x*) ∘ *subpath u v g*
by (*rule ext*) (*simp add*: *subpath_def*)

lemma *subpath_image*: *subpath u v* (*f* ∘ *g*) = *f* ∘ *subpath u v g*
by (*rule ext*) (*simp add*: *subpath_def*)

lemma *affine_ineq*:
fixes *x* :: *'a*::*linordered_idom*
assumes *x* ≤ *1 v* ≤ *u*
shows *v* + *x* ∗ *u* ≤ *u* + *x* ∗ *v*
proof −
have (*1*−*x*)∗(*u*−*v*) ≥ *0*
using *assms* **by** *auto*
then show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *sum_le_prod1*:
fixes *a*::*real* **shows** ⟦*a* ≤ *1*; *b* ≤ *1*⟧ ⟹ *a* + *b* ≤ *1* + *a* ∗ *b*
by (*metis add.commute affine_ineq mult.right_neutral*)

lemma *simple_path_subpath_eq*:
simple_path(*subpath u v g*) ⟷
path(*subpath u v g*) ∧ *u*≠*v* ∧
(∀ *x y*. *x* ∈ *closed_segment u v* ∧ *y* ∈ *closed_segment u v* ∧ *g x* = *g y*
⟶ *x* = *y* ∨ *x* = *u* ∧ *y* = *v* ∨ *x* = *v* ∧ *y* = *u*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have *p*: *path* (λ*x*. *g* ((*v* − *u*) ∗ *x* + *u*))
and *sim*: (⋀*x y*. ⟦*x*∈{*0*..*1*}; *y*∈{*0*..*1*}; *g* ((*v* − *u*) ∗ *x* + *u*) = *g* ((*v* − *u*) ∗ *y* + *u*)⟧
⟹ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*)
by (*auto simp*: *simple_path_def loop_free_def subpath_def*)
{ fix *x y*
assume *x* ∈ *closed_segment u v y* ∈ *closed_segment u v g x* = *g y*
then have *x* = *y* ∨ *x* = *u* ∧ *y* = *v* ∨ *x* = *v* ∧ *y* = *u*
using *sim* [*of* (*x*−*u*)/(*v*−*u*) (*y*−*u*)/(*v*−*u*)] *p*
by (*auto split*: *if_split_asm simp add*: *closed_segment_real_eq image_affinity_atLeastAtMost*)
(*simp_all add*: *field_split_simps*)
} moreover
have *path*(*subpath u v g*) ∧ *u*≠*v*
using *sim* [*of 1/3 2/3*] *p*
by (*auto simp*: *subpath_def*)
ultimately show *?rhs*
by *metis*

next
assume *?rhs*
then
have *d1*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ u \le x;\ x \le v;\ u \le y;\ y \le v]\!] \Longrightarrow x = y \vee x = u \wedge y = v \vee x = v \wedge y = u$
and *d2*: $\bigwedge x\ y.\ [\![g\ x = g\ y;\ v \le x;\ x \le u;\ v \le y;\ y \le u]\!] \Longrightarrow x = y \vee x = u \wedge y = v \vee x = v \wedge y = u$
and *ne*: $u < v \vee v < u$
and *psp*: *path* (*subpath u v g*)
by (*auto simp*: *closed_segment_real_eq image_affinity_atLeastAtMost*)
have [*simp*]: $\bigwedge x.\ u + x * v = v + x * u \longleftrightarrow u{=}v \vee x{=}1$
by *algebra*
show *?lhs* **using** *psp ne*
unfolding *simple_path_def loop_free_def subpath_def*
by (*fastforce simp add*: *algebra_simps affine_ineq mult_left_mono crossproduct_eq dest*: *d1 d2*)
qed

lemma *arc_subpath_eq*:
$arc(subpath\ u\ v\ g) \longleftrightarrow path(subpath\ u\ v\ g) \wedge u{\neq}v \wedge inj\_on\ g\ (closed\_segment\ u\ v)$
by (*smt* (*verit*, *best*) *arc_simple_path closed_segment_commute ends_in_segment*(*2*) *inj_on_def pathfinish_subpath pathstart_subpath simple_path_subpath_eq*)

lemma *simple_path_subpath*:
assumes $simple\_path\ g\ u \in \{0..1\}\ v \in \{0..1\}\ u \neq v$
shows $simple\_path(subpath\ u\ v\ g)$
using *assms*
unfolding *simple_path_subpath_eq*
by (*force simp*: *simple_path_def loop_free_def closed_segment_real_eq image_affinity_atLeastAtMost*)

lemma *arc_simple_path_subpath*:
$[\![simple\_path\ g;\ u \in \{0..1\};\ v \in \{0..1\};\ g\ u \neq g\ v]\!] \Longrightarrow arc(subpath\ u\ v\ g)$
by (*force intro*: *simple_path_subpath simple_path_imp_arc*)

lemma *arc_subpath_arc*:
$[\![arc\ g;\ u \in \{0..1\};\ v \in \{0..1\};\ u \neq v]\!] \Longrightarrow arc(subpath\ u\ v\ g)$
by (*meson arc_def arc_imp_simple_path arc_simple_path_subpath inj_onD*)

lemma *arc_simple_path_subpath_interior*:
$[\![simple\_path\ g;\ u \in \{0..1\};\ v \in \{0..1\};\ u \neq v;\ |u{-}v| < 1]\!] \Longrightarrow arc(subpath\ u\ v\ g)$
by (*force simp*: *simple_path_def loop_free_def intro*: *arc_simple_path_subpath*)

lemma *path_image_subpath_subset*:
$[\![u \in \{0..1\};\ v \in \{0..1\}]\!] \Longrightarrow path\_image(subpath\ u\ v\ g) \subseteq path\_image\ g$
by (*metis atLeastAtMost_iff atLeastatMost_subset_iff path_image_def path_image_subpath*

subset_image_iff)

lemma *join_subpaths_middle*: *subpath* (*0*) ((*1* / *2*)) *p* +++ *subpath* ((*1* / *2*)) *1* *p* = *p*
by (*rule ext*) (*simp add*: *joinpaths_def subpath_def field_split_simps*)

6.1.10 There is a subpath to the frontier

lemma *subpath_to_frontier_explicit*:
fixes *S* :: *'a*::*metric_space set*
assumes *g*: *path g* **and** *pathfinish g* ∉ *S*
obtains *u* **where** *0* ≤ *u u* ≤ *1*
⋀*x*. *0* ≤ *x* ∧ *x* < *u* ⟹ *g x* ∈ *interior S*
(*g u* ∉ *interior S*) (*u* = *0* ∨ *g u* ∈ *closure S*)
proof −
have *gcon*: *continuous_on* {*0..1*} *g*
using *g* **by** (*simp add*: *path_def*)
moreover have *bounded* ({*u*. *g u* ∈ *closure* (− *S*)} ∩ {*0..1*})
using *compact_eq_bounded_closed* **by** *fastforce*
ultimately have *com*: *compact* ({*0..1*} ∩ {*u*. *g u* ∈ *closure* (− *S*)})
using *closed_vimage_Int*
by (*metis* (*full_types*) *Int_commute closed_atLeastAtMost closed_closure compact_eq_bounded_closed vimage_def*)
have *1* ∈ {*u*. *g u* ∈ *closure* (− *S*)}
using *assms* **by** (*simp add*: *pathfinish_def closure_def*)
then have *dis*: {*0..1*} ∩ {*u*. *g u* ∈ *closure* (− *S*)} ≠ {}
using *atLeastAtMost_iff zero_le_one* **by** *blast*
then obtain *u* **where** *0* ≤ *u u* ≤ *1* **and** *gu*: *g u* ∈ *closure* (− *S*)
and *umin*: ⋀*t*. ⟦*0* ≤ *t*; *t* ≤ *1*; *g t* ∈ *closure* (− *S*)⟧ ⟹ *u* ≤ *t*
using *compact_attains_inf* [*OF com dis*] **by** *fastforce*
then have *umin'*: ⋀*t*. ⟦*0* ≤ *t*; *t* ≤ *1*; *t* < *u*⟧ ⟹ *g t* ∈ *S*
using *closure_def* **by** *fastforce*
have §: *g u* ∈ *closure S* **if** *u* ≠ *0*
proof −
have *u* > *0* **using** *that* ‹*0* ≤ *u*› **by** *auto*
{ fix *e*::*real* **assume** *e* > *0*
obtain *d* **where** *d*>*0* **and** *d*: ⋀*x'*. ⟦*x'* ∈ {*0..1*}; *dist x' u* ≤ *d*⟧ ⟹ *dist* (*g x'*) (*g u*) < *e*
using *continuous_onE* [*OF gcon* _ ‹*e* > *0*›] ‹*0* ≤ _› ‹_ ≤ *1*› *atLeastAtMost_iff* **by** *auto*
have ∗: *dist* (*max 0* (*u* − *d* / *2*)) *u* ≤ *d*
using ‹*0* ≤ *u*› ‹*u* ≤ *1*› ‹*d* > *0*› **by** (*simp add*: *dist_real_def*)
have ∃ *y*∈*S*. *dist y* (*g u*) < *e*
using ‹*0* < *u*› ‹*u* ≤ *1*› ‹*d* > *0*›
by (*force intro*: *d* [*OF* _ ∗] *umin'*)
}
then show *?thesis*
by (*simp add*: *frontier_def closure_approachable*)
qed

```
  show ?thesis
  proof
    show ⋀x. 0 ≤ x ∧ x < u ⟹ g x ∈ interior S
      using ‹u ≤ 1› interior_closure umin by fastforce
    show g u ∉ interior S
      by (simp add: gu interior_closure)
  qed (use ‹0 ≤ u› ‹u ≤ 1› § in auto)
qed

lemma subpath_to_frontier_strong:
    assumes g: path g and pathfinish g ∉ S
    obtains u where 0 ≤ u u ≤ 1 g u ∉ interior S
                    u = 0 ∨ (∀x. 0 ≤ x ∧ x < 1 ⟶ subpath 0 u g x ∈ interior S)
∧  g u ∈ closure S
proof −
  obtain u where 0 ≤ u u ≤ 1
            and gxin: ⋀x. 0 ≤ x ∧ x < u ⟹ g x ∈ interior S
            and gunot: (g u ∉ interior S) and u0: (u = 0 ∨ g u ∈ closure S)
    using subpath_to_frontier_explicit [OF assms] by blast
  show ?thesis
  proof
    show g u ∉ interior S
      using gunot by blast
  qed (use ‹0 ≤ u› ‹u ≤ 1› u0 in ‹(force simp: subpath_def gxin)+›)
qed

lemma subpath_to_frontier:
    assumes g: path g and g0: pathstart g ∈ closure S and g1: pathfinish g ∉ S
    obtains u where 0 ≤ u u ≤ 1 g u ∈ frontier S path_image(subpath 0 u g) −
{g u} ⊆ interior S
proof −
  obtain u where 0 ≤ u u ≤ 1
            and notin: g u ∉ interior S
            and disj: u = 0 ∨
                      (∀x. 0 ≤ x ∧ x < 1 ⟶ subpath 0 u g x ∈ interior S) ∧ g u ∈
closure S
                      (is _ ∨ ?P)
    using subpath_to_frontier_strong [OF g g1] by blast
  show ?thesis
  proof
    show g u ∈ frontier S
      by (metis DiffI disj frontier_def g0 notin pathstart_def)
    show path_image (subpath 0 u g) − {g u} ⊆ interior S
      using disj
    proof
      assume u = 0
      then show ?thesis
        by (simp add: path_image_subpath)
    next
```

```
      assume P: ?P
      show ?thesis
      proof (clarsimp simp add: path_image_subpath_gen)
        fix y
        assume y: y ∈ closed_segment 0 u g y ∉ interior S
        with ‹0 ≤ u› have 0 ≤ y y ≤ u
          by (auto simp: closed_segment_eq_real_ivl split: if_split_asm)
        then have y=u ∨ subpath 0 u g (y/u) ∈ interior S
          using P less_eq_real_def by force
        then show g y = g u
          using y by (auto simp: subpath_def split: if_split_asm)
      qed
    qed
  qed (use ‹0 ≤ u› ‹u ≤ 1› in auto)
qed

lemma exists_path_subpath_to_frontier:
    fixes S :: 'a::real_normed_vector set
    assumes path g pathstart g ∈ closure S pathfinish g ∉ S
    obtains h where path h pathstart h = pathstart g path_image h ⊆ path_image
g
                   path_image h − {pathfinish h} ⊆ interior S
                   pathfinish h ∈ frontier S
proof −
  obtain u where u: 0 ≤ u u ≤ 1 g u ∈ frontier S (path_image(subpath 0 u g) −
{g u}) ⊆ interior S
    using subpath_to_frontier [OF assms] by blast
  show ?thesis
  proof
    show path_image (subpath 0 u g) ⊆ path_image g
      by (simp add: path_image_subpath_subset u)
    show pathstart (subpath 0 u g) = pathstart g
      by (metis pathstart_def pathstart_subpath)
  qed (use assms u in ‹auto simp: path_image_subpath›)
qed

lemma exists_path_subpath_to_frontier_closed:
    fixes S :: 'a::real_normed_vector set
    assumes S: closed S and g: path g and g0: pathstart g ∈ S and g1: pathfinish
g ∉ S
    obtains h where path h pathstart h = pathstart g path_image h ⊆ path_image
g ∩ S
                   pathfinish h ∈ frontier S
  by (smt (verit, del_insts) Diff_iff Int_iff S closure_closed exists_path_subpath_to_frontier

      frontier_def g g0 g1 interior_subset singletonD subset_eq)
```

6.1.11 Shift Path to Start at Some Given Point

definition *shiftpath* :: *real* ⇒ (*real* ⇒ 'a::*topological_space*) ⇒ *real* ⇒ 'a
where *shiftpath a f* = (λ*x*. *if* (*a* + *x*) ≤ *1 then f* (*a* + *x*) *else f* (*a* + *x* − *1*))

lemma *shiftpath_alt_def*: *shiftpath a f* = (λ*x*. *if x* ≤ *1*−*a then f* (*a* + *x*) *else f* (*a* + *x* − *1*))
by (*auto simp*: *shiftpath_def*)

lemma *pathstart_shiftpath*: *a* ≤ *1* ⟹ *pathstart* (*shiftpath a g*) = *g a*
unfolding *pathstart_def shiftpath_def* **by** *auto*

lemma *pathfinish_shiftpath*:
assumes *0* ≤ *a*
and *pathfinish g* = *pathstart g*
shows *pathfinish* (*shiftpath a g*) = *g a*
using *assms*
unfolding *pathstart_def pathfinish_def shiftpath_def*
by *auto*

lemma *endpoints_shiftpath*:
assumes *pathfinish g* = *pathstart g*
and *a* ∈ {*0* .. *1*}
shows *pathfinish* (*shiftpath a g*) = *g a*
and *pathstart* (*shiftpath a g*) = *g a*
using *assms*
by (*simp_all add*: *pathstart_shiftpath pathfinish_shiftpath*)

lemma *closed_shiftpath*:
assumes *pathfinish g* = *pathstart g*
and *a* ∈ {*0*..*1*}
shows *pathfinish* (*shiftpath a g*) = *pathstart* (*shiftpath a g*)
using *endpoints_shiftpath*[*OF assms*]
by *auto*

lemma *path_shiftpath*:
assumes *path g*
and *pathfinish g* = *pathstart g*
and *a* ∈ {*0*..*1*}
shows *path* (*shiftpath a g*)
proof −
have ∗: {*0* .. *1*} = {*0* .. *1*−*a*} ∪ {*1*−*a* .. *1*}
using *assms*(*3*) **by** *auto*
have ∗∗: ⋀*x*. *x* + *a* = *1* ⟹ *g* (*x* + *a* − *1*) = *g* (*x* + *a*)
by (*smt* (*verit*, *best*) *assms*(*2*) *pathfinish_def pathstart_def*)
show *?thesis*
unfolding *path_def shiftpath_def* ∗
proof (*rule continuous_on_closed_Un*)
have *contg*: *continuous_on* {*0*..*1*} *g*
using ‹*path g*› *path_def* **by** *blast*

```
    show continuous_on {0..1−a} (λx. if a + x ≤ 1 then g (a + x) else g (a + x
− 1))
    proof (rule continuous_on_eq)
      show continuous_on {0..1−a} (g ∘ (+) a)
        by (intro continuous_intros continuous_on_subset [OF contg]) (use ‹a ∈
{0..1}› in auto)
    qed auto
    show continuous_on {1−a..1} (λx. if a + x ≤ 1 then g (a + x) else g (a + x
− 1))
    proof (rule continuous_on_eq)
      show continuous_on {1−a..1} (g ∘ (+) (a − 1))
        by (intro continuous_intros continuous_on_subset [OF contg]) (use ‹a ∈
{0..1}› in auto)
    qed (auto simp: ∗∗ add.commute add_diff_eq)
  qed auto
qed

lemma shiftpath_shiftpath:
  assumes pathfinish g = pathstart g
    and a ∈ {0..1}
    and x ∈ {0..1}
  shows shiftpath (1 − a) (shiftpath a g) x = g x
  using assms
  unfolding pathfinish_def pathstart_def shiftpath_def
  by auto

lemma path_image_shiftpath:
  assumes a: a ∈ {0..1}
    and pathfinish g = pathstart g
  shows path_image (shiftpath a g) = path_image g
proof −
  { fix x
    assume g: g 1 = g 0 x ∈ {0..1::real} and gne: ⋀y. y∈{0..1} ∩ {x. ¬ a + x
≤ 1} ⟹ g x ≠ g (a + y − 1)
    then have ∃ y∈{0..1} ∩ {x. a + x ≤ 1}. g x = g (a + y)
    proof (cases a ≤ x)
      case False
      then show ?thesis
        apply (rule_tac x=1 + x − a in bexI)
        using g gne[of 1 + x − a] a by (force simp: field_simps)+
    next
      case True
      then show ?thesis
        using g a  by (rule_tac x=x − a in bexI) (auto simp: field_simps)
    qed
  }
  then show ?thesis
    using assms
    unfolding shiftpath_def path_image_def pathfinish_def pathstart_def
```

by (*auto simp*: *image_iff*)
qed

lemma *loop_free_shiftpath*:
assumes *loop_free g pathfinish g* = *pathstart g* **and** *a*: *0* ≤ *a a* ≤ *1*
shows *loop_free* (*shiftpath a g*)
unfolding *loop_free_def*
proof (*intro conjI impI ballI*)
show *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*
if *x* ∈ {*0*..*1*} *y* ∈ {*0*..*1*} *shiftpath a g x* = *shiftpath a g y* **for** *x y*
using *that a assms* **unfolding** *shiftpath_def loop_free_def*
by (*smt* (*verit*, *ccfv_threshold*) *atLeastAtMost_iff*)
qed

lemma *simple_path_shiftpath*:
assumes *simple_path g pathfinish g* = *pathstart g* **and** *a*: *0* ≤ *a a* ≤ *1*
shows *simple_path* (*shiftpath a g*)
using *assms loop_free_shiftpath path_shiftpath simple_path_def* **by** *fastforce*

6.1.12 Straight-Line Paths

definition *linepath* :: *'a*::*real_normed_vector* ⇒ *'a* ⇒ *real* ⇒ *'a*
where *linepath a b* = (λ*x*. (*1* − *x*) ∗$_R$ *a* + *x* ∗$_R$ *b*)

lemma *pathstart_linepath*[*simp*]: *pathstart* (*linepath a b*) = *a*
unfolding *pathstart_def linepath_def*
by *auto*

lemma *pathfinish_linepath*[*simp*]: *pathfinish* (*linepath a b*) = *b*
unfolding *pathfinish_def linepath_def*
by *auto*

lemma *linepath_inner*: *linepath a b x* • *v* = *linepath* (*a* • *v*) (*b* • *v*) *x*
by (*simp add*: *linepath_def algebra_simps*)

lemma *Re_linepath'*: *Re* (*linepath a b x*) = *linepath* (*Re a*) (*Re b*) *x*
by (*simp add*: *linepath_def*)

lemma *Im_linepath'*: *Im* (*linepath a b x*) = *linepath* (*Im a*) (*Im b*) *x*
by (*simp add*: *linepath_def*)

lemma *linepath_0'*: *linepath a b 0* = *a*
by (*simp add*: *linepath_def*)

lemma *linepath_1'*: *linepath a b 1* = *b*
by (*simp add*: *linepath_def*)

lemma *continuous_linepath_at*[*intro*]: *continuous* (*at x*) (*linepath a b*)
unfolding *linepath_def*

by (*intro continuous_intros*)

lemma *continuous_on_linepath* [*intro*,*continuous_intros*]: *continuous_on s* (*linepath a b*)
using *continuous_linepath_at*
by (*auto intro*!: *continuous_at_imp_continuous_on*)

lemma *path_linepath*[*iff*]: *path* (*linepath a b*)
unfolding *path_def*
by (*rule continuous_on_linepath*)

lemma *path_image_linepath*[*simp*]: *path_image* (*linepath a b*) = *closed_segment a b*
unfolding *path_image_def segment linepath_def*
by *auto*

lemma *reversepath_linepath*[*simp*]: *reversepath* (*linepath a b*) = *linepath b a*
unfolding *reversepath_def linepath_def*
by *auto*

lemma *linepath_0* [*simp*]: *linepath 0 b x* = *x* $*_R$ *b*
by (*simp add*: *linepath_def*)

lemma *linepath_cnj*: *cnj* (*linepath a b x*) = *linepath* (*cnj a*) (*cnj b*) *x*
by (*simp add*: *linepath_def*)

lemma *arc_linepath*:
assumes *a* ≠ *b* **shows** [*simp*]: *arc* (*linepath a b*)
proof −
{
fix *x y* :: *real*
assume *x* $*_R$ *b* + *y* $*_R$ *a* = *x* $*_R$ *a* + *y* $*_R$ *b*
then have (*x* − *y*) $*_R$ *a* = (*x* − *y*) $*_R$ *b*
by (*simp add*: *algebra_simps*)
with *assms* **have** *x* = *y*
by *simp*
}
then show *?thesis*
unfolding *arc_def inj_on_def*
by (*fastforce simp*: *algebra_simps linepath_def*)
qed

lemma *simple_path_linepath*[*intro*]: *a* ≠ *b* ⟹ *simple_path* (*linepath a b*)
by (*simp add*: *arc_imp_simple_path*)

lemma *linepath_trivial* [*simp*]: *linepath a a x* = *a*
by (*simp add*: *linepath_def real_vector.scale_left_diff_distrib*)

lemma *linepath_refl*: *linepath a a* = (λ*x*. *a*)

by *auto*

lemma *subpath_refl*: *subpath a a g = linepath (g a) (g a)*
by (*simp add*: *subpath_def linepath_def algebra_simps*)

lemma *linepath_of_real*: (*linepath* (*of_real a*) (*of_real b*) *x*) = *of_real* ((*1* − *x*)∗*a* + *x*∗*b*)
by (*simp add*: *scaleR_conv_of_real linepath_def*)

lemma *of_real_linepath*: *of_real* (*linepath a b x*) = *linepath* (*of_real a*) (*of_real b*) *x*
by (*metis linepath_of_real mult.right_neutral of_real_def real_scaleR_def*)

lemma *inj_on_linepath*:
assumes *a* ≠ *b* **shows** *inj_on* (*linepath a b*) {*0..1*}
using *arc_imp_inj_on arc_linepath assms* **by** *blast*

lemma *linepath_le_1*:
fixes *a*::′*a*::*linordered_idom* **shows** ⟦*a* ≤ *1*; *b* ≤ *1*; *0* ≤ *u*; *u* ≤ *1*⟧ ⟹ (*1* − *u*) ∗ *a* + *u* ∗ *b* ≤ *1*
using *mult_left_le* [*of a 1*−*u*] *mult_left_le* [*of b u*] **by** *auto*

lemma *linepath_in_path*:
shows *x* ∈ {*0..1*} ⟹ *linepath a b x* ∈ *closed_segment a b*
by (*auto simp*: *segment linepath_def*)

lemma *linepath_image_01*: *linepath a b* ‘ {*0..1*} = *closed_segment a b*
by (*auto simp*: *segment linepath_def*)

lemma *linepath_in_convex_hull*:
fixes *x*::*real*
assumes *a* ∈ *convex hull S*
and *b* ∈ *convex hull S*
and *0*≤*x x*≤*1*
shows *linepath a b x* ∈ *convex hull S*
by (*meson assms atLeastAtMost_iff convex_contains_segment convex_convex_hull linepath_in_path subset_eq*)

lemma *Re_linepath*: *Re*(*linepath* (*of_real a*) (*of_real b*) *x*) = (*1* − *x*)∗*a* + *x*∗*b*
by (*simp add*: *linepath_def*)

lemma *Im_linepath*: *Im*(*linepath* (*of_real a*) (*of_real b*) *x*) = *0*
by (*simp add*: *linepath_def*)

lemma *bounded_linear_linepath*:
assumes *bounded_linear f*
shows *f* (*linepath a b x*) = *linepath* (*f a*) (*f b*) *x*
proof −
interpret *f*: *bounded_linear f* **by** *fact*

show *?thesis* **by** (*simp add*: *linepath_def f.add f.scale*)
qed

lemma *bounded_linear_linepath'*:
assumes *bounded_linear f*
shows *f ∘ linepath a b = linepath (f a) (f b)*
using *bounded_linear_linepath*[*OF assms*] **by** (*simp add*: *fun_eq_iff*)

lemma *linepath_cnj'*: *cnj ∘ linepath a b = linepath (cnj a) (cnj b)*
by (*simp add*: *linepath_def fun_eq_iff*)

lemma *differentiable_linepath* [*intro*]: *linepath a b differentiable at x within A*
by (*auto simp*: *linepath_def*)

lemma *has_vector_derivative_linepath_within*:
(*linepath a b has_vector_derivative* (*b − a*)) (*at x within S*)
by (*force intro*: *derivative_eq_intros simp add*: *linepath_def has_vector_derivative_def algebra_simps*)

6.1.13 Segments via convex hulls

lemma *segments_subset_convex_hull*:
closed_segment a b ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment a c ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment b c ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment b a ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment c a ⊆ (*convex hull* {*a*,*b*,*c*})
closed_segment c b ⊆ (*convex hull* {*a*,*b*,*c*})
by (*auto simp*: *segment_convex_hull linepath_of_real elim*!: *rev_subsetD* [*OF _ hull_mono*])

lemma *midpoints_in_convex_hull*:
assumes *x ∈ convex hull s y ∈ convex hull s*
shows *midpoint x y ∈ convex hull s*
using *assms closed_segment_subset_convex_hull csegment_midpoint_subset* **by** *blast*

lemma *not_in_interior_convex_hull_3*:
fixes *a* :: *complex*
shows *a ∉ interior*(*convex hull* {*a*,*b*,*c*})
b ∉ interior(*convex hull* {*a*,*b*,*c*})
c ∉ interior(*convex hull* {*a*,*b*,*c*})
by (*auto simp*: *card_insert_le_m1 not_in_interior_convex_hull*)

lemma *midpoint_in_closed_segment* [*simp*]: *midpoint a b ∈ closed_segment a b*
using *midpoints_in_convex_hull segment_convex_hull* **by** *blast*

lemma *midpoint_in_open_segment* [*simp*]: *midpoint a b ∈ open_segment a b ⟷ a ≠ b*

by (*simp add*: *open_segment_def*)

lemma *continuous_IVT_local_extremum*:
fixes f :: $'a$::*euclidean_space* ⇒ *real*
assumes *contf*: *continuous_on* (*closed_segment a b*) f
and *ab*: $a \neq b$ $f\ a = f\ b$
obtains z **where** $z \in$ *open_segment a b*
$(\forall w \in$ *closed_segment a b*. $(f\ w) \leq (f\ z)) \vee$
$(\forall w \in$ *closed_segment a b*. $(f\ z) \leq (f\ w))$
proof −
obtain c **where** $c \in$ *closed_segment a b* **and** *c*: $\bigwedge y.\ y \in$ *closed_segment a b* ⟹ $f\ y \leq f\ c$
using *continuous_attains_sup* [*of closed_segment a b f*] *contf* **by** *auto*
moreover
obtain d **where** $d \in$ *closed_segment a b* **and** *d*: $\bigwedge y.\ y \in$ *closed_segment a b* ⟹ $f\ d \leq f\ y$
using *continuous_attains_inf* [*of closed_segment a b f*] *contf* **by** *auto*
ultimately show *?thesis*
by (*smt* (*verit*) *UnE ab closed_segment_eq_open empty_iff insert_iff midpoint_in_open_segment that*)
qed

An injective map into R is also an open map w.r.T. the universe, and conversely.

proposition *injective_eq_1d_open_map_UNIV*:
fixes f :: *real* ⇒ *real*
assumes *contf*: *continuous_on S f* **and** *S*: *is_interval S*
shows *inj_on f S* ⟷ $(\forall T.$ *open* $T \wedge T \subseteq S \longrightarrow$ *open*$(f\ `\ T))$
(**is** *?lhs* = *?rhs*)
proof *safe*
fix T
assume *injf*: *?lhs* **and** *open* T **and** $T \subseteq S$
have $\exists U.$ *open* $U \wedge f\ x \in U \wedge U \subseteq f\ `\ T$ **if** $x \in T$ **for** x
proof −
obtain δ **where** $\delta > 0$ **and** δ: *cball x* $\delta \subseteq T$
using ‹*open T*› ‹$x \in T$› *open_contains_cball_eq* **by** *blast*
show *?thesis*
proof (*intro exI conjI*)
have *closed_segment* $(x-\delta)$ $(x+\delta) = \{x-\delta..x+\delta\}$
using ‹$0 < \delta$› **by** (*auto simp*: *closed_segment_eq_real_ivl*)
also have ... $\subseteq S$
using δ ‹$T \subseteq S$› **by** (*auto simp*: *dist_norm subset_eq*)
finally have $f\ `$ (*open_segment* $(x-\delta)$ $(x+\delta)$) = *open_segment* $(f\ (x-\delta))$ $(f\ (x+\delta))$
using *continuous_injective_image_open_segment_1*
by (*metis continuous_on_subset* [*OF contf*] *inj_on_subset* [*OF injf*])
then show *open* $(f\ `\ \{x-\delta<..<x+\delta\})$
using ‹$0 < \delta$› **by** (*simp add*: *open_segment_eq_real_ivl*)
show $f\ x \in f\ `\ \{x - \delta<..<x + \delta\}$

```
      by (auto simp: ‹δ > 0›)
    show f ‘ {x − δ<..<x + δ} ⊆ f ‘ T
      using δ by (auto simp: dist_norm subset_iff)
  qed
 qed
 with open_subopen show open (f ‘ T)
  by blast
next
 assume R: ?rhs
 have False if xy: x ∈ S y ∈ S and f x = f y x ≠ y for x y
 proof −
  have open (f ‘ open_segment x y)
   using R
  by (metis S convex_contains_open_segment is_interval_convex open_greaterThanLessThan
open_segment_eq_real_ivl xy)
  moreover
  have continuous_on (closed_segment x y) f
  by (meson S closed_segment_subset contf continuous_on_subset is_interval_convex
that)
  then obtain ξ where ξ ∈ open_segment x y
                and ξ: (∀ w ∈ closed_segment x y. (f w) ≤ (f ξ)) ∨
                       (∀ w ∈ closed_segment x y. (f ξ) ≤ (f w))
   using continuous_IVT_local_extremum [of x y f] ‹f x = f y› ‹x ≠ y› by blast
  ultimately obtain e where e>0 and e: ⋀u. dist u (f ξ) < e ⟹ u ∈ f ‘
open_segment x y
   using open_dist by (metis image_eqI)
  have fin: f ξ + (e/2) ∈ f ‘ open_segment x y f ξ − (e/2) ∈ f ‘ open_segment
x y
   using e [of f ξ + (e/2)] e [of f ξ − (e/2)] ‹e > 0› by (auto simp: dist_norm)
  show ?thesis
   using ξ ‹0 < e› fin open_closed_segment by fastforce
 qed
 then show ?lhs
  by (force simp: inj_on_def)
qed
```

6.1.14 Bounding a point away from a path

```
lemma not_on_path_ball:
 fixes g :: real ⇒ 'a::heine_borel
 assumes path g
  and z: z ∉ path_image g
 shows ∃ e > 0. ball z e ∩ path_image g = {}
proof −
 have closed (path_image g)
  by (simp add: ‹path g› closed_path_image)
 then obtain a where a ∈ path_image g ∀ y ∈ path_image g. dist z a ≤ dist z y
  by (auto intro: distance_attains_inf[OF _ path_image_nonempty, of g z])
 then show ?thesis
```

by (*rule_tac x=dist z a* **in** *exI*) (*use dist_commute z* **in** *auto*)
qed

lemma *not_on_path_cball*:
fixes *g* :: *real* ⇒ *'a*::*heine_borel*
assumes *path g*
and *z* ∉ *path_image g*
shows ∃ *e*>*0*. *cball z e* ∩ (*path_image g*) = {}
by (*smt* (*verit*, *ccfv_threshold*) *open_ball assms centre_in_ball inf.orderE inf_assoc inf_bot_right not_on_path_ball open_contains_cball_eq*)

6.1.15 Path component

Original formalization by Tom Hales

definition *path_component S x y* ≡
(∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

abbreviation
path_component_set S x ≡ *Collect* (*path_component S x*)

lemmas *path_defs* = *path_def pathstart_def pathfinish_def path_image_def path_component_def*

lemma *path_component_mem*:
assumes *path_component S x y*
shows *x* ∈ *S* **and** *y* ∈ *S*
using *assms*
unfolding *path_defs*
by *auto*

lemma *path_component_refl*:
assumes *x* ∈ *S*
shows *path_component S x x*
using *assms*
unfolding *path_defs*
by (*metis* (*full_types*) *assms continuous_on_const image_subset_iff path_image_def*)

lemma *path_component_refl_eq*: *path_component S x x* ⟷ *x* ∈ *S*
by (*auto intro*!: *path_component_mem path_component_refl*)

lemma *path_component_sym*: *path_component S x y* ⟹ *path_component S y x*
unfolding *path_component_def*
by (*metis* (*no_types*) *path_image_reversepath path_reversepath pathfinish_reversepath pathstart_reversepath*)

lemma *path_component_trans*:
assumes *path_component S x y* **and** *path_component S y z*
shows *path_component S x z*
using *assms*
unfolding *path_component_def*

by (*metis path_join pathfinish_join pathstart_join subset_path_image_join*)

lemma *path_component_of_subset*: $S \subseteq T \Longrightarrow$ *path_component S x y* $\Longrightarrow$ *path_component T x y*
unfolding *path_component_def* **by** *auto*

lemma *path_component_linepath*:
fixes $S :: 'a$::*real_normed_vector set*
shows *closed_segment a b* $\subseteq S \Longrightarrow$ *path_component S a b*
unfolding *path_component_def* **by** *fastforce*

Path components as sets

lemma *path_component_set*:
path_component_set S x =
$\{y.$ $(\exists g.$ *path g* $\wedge$ *path_image g* $\subseteq S \wedge$ *pathstart g* $= x \wedge$ *pathfinish g* $= y)\}$
by (*auto simp*: *path_component_def*)

lemma *path_component_subset*: *path_component_set S x* $\subseteq S$
by (*auto simp*: *path_component_mem*(*2*))

lemma *path_component_eq_empty*: *path_component_set S x* $= \{\} \longleftrightarrow x \notin S$
using *path_component_mem path_component_refl_eq*
by *fastforce*

lemma *path_component_mono*:
$S \subseteq T \Longrightarrow$ (*path_component_set S x*) $\subseteq$ (*path_component_set T x*)
by (*simp add*: *Collect_mono path_component_of_subset*)

lemma *path_component_eq*:
y $\in$ *path_component_set S x* $\Longrightarrow$ *path_component_set S y* = *path_component_set S x*
by (*metis* (*no_types*, *lifting*) *Collect_cong mem_Collect_eq path_component_sym path_component_trans*)

6.1.16 Path connectedness of a space

definition *path_connected S* $\longleftrightarrow$
$(\forall x \in S.\ \forall y \in S.\ \exists g.$ *path g* $\wedge$ *path_image g* $\subseteq S \wedge$ *pathstart g* $= x \wedge$ *pathfinish g* $= y)$

lemma *path_connectedin_iff_path_connected_real* [*simp*]:
path_connectedin euclideanreal S $\longleftrightarrow$ *path_connected S*
by (*simp add*: *path_connectedin path_connected_def path_defs image_subset_iff_funcset*)

lemma *path_connected_component*: *path_connected S* $\longleftrightarrow (\forall x \in S.\ \forall y \in S.$ *path_component S x y*)
unfolding *path_connected_def path_component_def* **by** *auto*

lemma *path_connected_component_set*: *path_connected S* ⟷ (∀ *x*∈*S*. *path_component_set S x* = *S*)
unfolding *path_connected_component path_component_subset*
using *path_component_mem* **by** *blast*

lemma *path_component_maximal*:
⟦*x* ∈ *T*; *path_connected T*; *T* ⊆ *S*⟧ ⟹ *T* ⊆ (*path_component_set S x*)
by (*metis path_component_mono path_connected_component_set*)

lemma *convex_imp_path_connected*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *convex S*
shows *path_connected S*
unfolding *path_connected_def*
using *assms convex_contains_segment* **by** *fastforce*

lemma *path_connected_UNIV* [*iff*]: *path_connected* (*UNIV* :: *'a*::*real_normed_vector set*)
by (*simp add*: *convex_imp_path_connected*)

lemma *path_component_UNIV*: *path_component_set UNIV x* = (*UNIV* :: *'a*::*real_normed_vector set*)
using *path_connected_component_set* **by** *auto*

lemma *path_connected_imp_connected*:
assumes *path_connected S*
shows *connected S*
proof (*rule connectedI*)
fix *e1 e2*
assume *as*: *open e1 open e2 S* ⊆ *e1* ∪ *e2 e1* ∩ *e2* ∩ *S* = {} *e1* ∩ *S* ≠ {} *e2* ∩ *S* ≠ {}
then obtain *x1 x2* **where** *obt*:*x1* ∈ *e1* ∩ *S x2* ∈ *e2* ∩ *S*
by *auto*
then obtain *g* **where** *g*: *path g path_image g* ⊆ *S* **and** *pg*: *pathstart g* = *x1 pathfinish g* = *x2*
using *assms*[*unfolded path_connected_def*,*rule_format*,*of x1 x2*] **by** *auto*
have ∗: *connected* {*0..1*::*real*}
by (*auto intro*!: *convex_connected*)
have {*0..1*} ⊆ {*x* ∈ {*0..1*}. *g x* ∈ *e1*} ∪ {*x* ∈ {*0..1*}. *g x* ∈ *e2*}
using *as*(*3*) *g*(*2*)[*unfolded path_defs*] **by** *blast*
moreover have {*x* ∈ {*0..1*}. *g x* ∈ *e1*} ∩ {*x* ∈ {*0..1*}. *g x* ∈ *e2*} = {}
using *as*(*4*) *g*(*2*)[*unfolded path_defs*]
unfolding *subset_eq*
by *auto*
moreover have {*x* ∈ {*0..1*}. *g x* ∈ *e1*} ≠ {} ∧ {*x* ∈ {*0..1*}. *g x* ∈ *e2*} ≠ {}
by (*smt* (*verit*, *ccfv_threshold*) *IntE atLeastAtMost_iff empty_iff pg mem_Collect_eq obt pathfinish_def pathstart_def*)
ultimately show *False*
using ∗[*unfolded connected_local not_ex*, *rule_format*,

```
    of {0..1} ∩ g -` e1 {0..1} ∩ g -` e2]
  using continuous_openin_preimage_gen[OF g(1)[unfolded path_def] as(1)]
  using continuous_openin_preimage_gen[OF g(1)[unfolded path_def] as(2)]
  by auto
qed

lemma open_path_component:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open (path_component_set S x)
  unfolding open_contains_ball
 by (metis assms centre_in_ball convex_ball convex_imp_path_connected equals0D
openE
      path_component_eq path_component_eq_empty path_component_maximal)

lemma open_non_path_component:
  fixes S :: 'a::real_normed_vector set
  assumes open S
  shows open (S - path_component_set S x)
  unfolding open_contains_ball
proof
  fix y
  assume y: y ∈ S - path_component_set S x
  then obtain e where e: e > 0 ball y e ⊆ S
    using assms openE by auto
  show ∃e>0. ball y e ⊆ S - path_component_set S x
  proof (intro exI conjI subsetI DiffI notI)
    show ⋀x. x ∈ ball y e ⟹ x ∈ S
      using e by blast
    show False if z ∈ ball y e z ∈ path_component_set S x for z
    by (metis (no_types, lifting) Diff_iff centre_in_ball convex_ball convex_imp_path_connected

        path_component_eq path_component_maximal subsetD that y e)
  qed (use e in auto)
qed

lemma connected_open_path_connected:
  fixes S :: 'a::real_normed_vector set
  assumes open S
    and connected S
  shows path_connected S
  unfolding path_connected_component_set
proof (rule, rule, rule path_component_subset, rule)
  fix x y
  assume x ∈ S and y ∈ S
  show y ∈ path_component_set S x
  proof (rule ccontr)
    assume ¬ ?thesis
    moreover have path_component_set S x ∩ S ≠ {}
```

```
      using ‹x ∈ S› path_component_eq_empty path_component_subset[of S x]
      by auto
    ultimately
    show False
    using ‹y ∈ S› open_non_path_component[OF ‹open S›] open_path_component[OF
‹open S›]
      using ‹connected S›[unfolded connected_def not_ex, rule_format,
        of path_component_set S x S − path_component_set S x]
      by auto
  qed
qed

lemma path_connected_continuous_image:
  assumes contf: continuous_on S f
    and path_connected S
  shows path_connected (f ‘ S)
  unfolding path_connected_def
proof clarsimp
  fix x y
  assume x: x ∈ S and y: y ∈ S
  with ‹path_connected S›
  show ∃g. path g ∧ path_image g ⊆ f ‘ S ∧ pathstart g = f x ∧ pathfinish g = f y
    unfolding path_defs path_connected_def
    using continuous_on_subset[OF contf]
   by (smt (verit, ccfv_threshold) continuous_on_compose2 image_eqI image_subset_iff)
qed

lemma path_connected_translationI:
  fixes a :: 'a :: topological_group_add
  assumes path_connected S shows path_connected ((λx. a + x) ‘ S)
  by (intro path_connected_continuous_image assms continuous_intros)

lemma path_connected_translation:
  fixes a :: 'a :: topological_group_add
  shows path_connected ((λx. a + x) ‘ S) = path_connected S
proof −
  have ∀x y. (+) (x::'a) ‘ (+) (0 − x) ‘ y = y
    by (simp add: image_image)
  then show ?thesis
    by (metis (no_types) path_connected_translationI)
qed

lemma path_connected_segment [simp]:
    fixes a :: 'a::real_normed_vector
    shows path_connected (closed_segment a b)
  by (simp add: convex_imp_path_connected)

lemma path_connected_open_segment [simp]:
    fixes a :: 'a::real_normed_vector
```

shows *path_connected* (*open_segment a b*)
by (*simp add*: *convex_imp_path_connected*)

lemma *homeomorphic_path_connectedness*:
S homeomorphic T $\Longrightarrow$ *path_connected S* $\longleftrightarrow$ *path_connected T*
unfolding *homeomorphic_def homeomorphism_def* **by** (*metis path_connected_continuous_image*)

lemma *path_connected_empty* [*simp*]: *path_connected* {}
unfolding *path_connected_def* **by** *auto*

lemma *path_connected_singleton* [*simp*]: *path_connected* {*a*}
unfolding *path_connected_def pathstart_def pathfinish_def path_image_def*
using *path_def* **by** *fastforce*

lemma *path_connected_Un*:
assumes *path_connected S*
and *path_connected T*
and $S \cap T \neq \{\}$
shows *path_connected* ($S \cup T$)
unfolding *path_connected_component*
proof (*intro ballI*)
fix *x y*
assume *x*: $x \in S \cup T$ **and** *y*: $y \in S \cup T$
from *assms* **obtain** *z* **where** *z*: $z \in S$ $z \in T$
by *auto*
with *x y* **show** *path_component* ($S \cup T$) *x y*
by (*smt* (*verit*) *assms*(*1*,*2*) *in_mono mem_Collect_eq path_component_eq path_component_maximal*
sup.bounded_iff sup.cobounded2 sup_ge1)
qed

lemma *path_connected_UNION*:
assumes $\bigwedge i.\ i \in A \Longrightarrow$ *path_connected* (*S i*)
and $\bigwedge i.\ i \in A \Longrightarrow z \in S\ i$
shows *path_connected* ($\bigcup i{\in}A.\ S\ i$)
unfolding *path_connected_component*
proof *clarify*
fix *x i y j*
assume $*$: $i \in A$ $x \in S\ i$ $j \in A$ $y \in S\ j$
then have *path_component* (*S i*) *x z* **and** *path_component* (*S j*) *z y*
using *assms* **by** (*simp_all add*: *path_connected_component*)
then have *path_component* ($\bigcup i{\in}A.\ S\ i$) *x z* **and** *path_component* ($\bigcup i{\in}A.\ S\ i$) *z y*
using $*$(*1*,*3*) **by** (*meson SUP_upper path_component_of_subset*)+
then show *path_component* ($\bigcup i{\in}A.\ S\ i$) *x y*
by (*rule path_component_trans*)
qed

lemma *path_component_path_image_pathstart*:

assumes *p*: *path p* **and** *x*: $x \in$ *path_image p*
shows *path_component* (*path_image p*) (*pathstart p*) *x*
proof −
obtain *y* **where** *x*: $x = p\ y$ **and** *y*: $0 \le y\ y \le 1$
using *x* **by** (*auto simp*: *path_image_def*)
show *?thesis*
unfolding *path_component_def*
proof (*intro exI conjI*)
have *continuous_on* ((∗) *y* ‘ {*0..1*}) *p*
by (*simp add*: *continuous_on_path image_mult_atLeastAtMost_if p y*)
then have *continuous_on* {*0..1*} (*p* ∘ ((∗) *y*))
using *continuous_on_compose continuous_on_mult_const* **by** *blast*
then show *path* (λ*u*. *p* (*y* ∗ *u*))
by (*simp add*: *path_def*)
show *path_image* (λ*u*. *p* (*y* ∗ *u*)) ⊆ *path_image p*
using *y mult_le_one* **by** (*fastforce simp*: *path_image_def image_iff*)
qed (*auto simp*: *pathstart_def pathfinish_def x*)
qed

lemma *path_connected_path_image*: *path p* ⟹ *path_connected*(*path_image p*)
unfolding *path_connected_component*
by (*meson path_component_path_image_pathstart path_component_sym path_component_trans*)

lemma *path_connected_path_component* [*simp*]:
path_connected (*path_component_set S x*)
by (*smt* (*verit*) *mem_Collect_eq path_component_def path_component_eq path_component_maximal*

path_connected_component path_connected_path_image pathstart_in_path_image)

lemma *path_component*:
path_component S x y ⟷ (∃ *t*. *path_connected t* ∧ *t* ⊆ *S* ∧ *x* ∈ *t* ∧ *y* ∈ *t*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis path_component_def path_connected_path_image pathfinish_in_path_image*
pathstart_in_path_image)
next
assume *?rhs* **then show** *?lhs*
by (*meson path_component_of_subset path_connected_component*)
qed

lemma *path_component_path_component* [*simp*]:
path_component_set (*path_component_set S x*) *x* = *path_component_set S x*
by (*metis* (*full_types*) *mem_Collect_eq path_component_eq_empty path_component_refl*
path_connected_component_set path_connected_path_component)

lemma *path_component_subset_connected_component*:
(*path_component_set S x*) ⊆ (*connected_component_set S x*)
proof (*cases x* ∈ *S*)

case *True* **show** *?thesis*
by (*simp add*: *True connected_component_maximal path_component_refl path_component_subset path_connected_imp_connected*)
next
case *False* **then show** *?thesis*
using *path_component_eq_empty* **by** *auto*
qed

6.1.17 Lemmas about path-connectedness

lemma *path_connected_linear_image*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*
assumes *path_connected S bounded_linear f*
shows *path_connected*(*f* ‘ *S*)
by (*auto simp*: *linear_continuous_on assms path_connected_continuous_image*)

lemma *is_interval_path_connected*: *is_interval S* ⟹ *path_connected S*
by (*simp add*: *convex_imp_path_connected is_interval_convex*)

lemma *path_connected_Ioi*[*simp*]: *path_connected* {*a*<..} **for** *a* :: *real*
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connected_Ici*[*simp*]: *path_connected* {*a*..} **for** *a* :: *real*
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connected_Iio*[*simp*]: *path_connected* {..<*a*} **for** *a* :: *real*
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connected_Iic*[*simp*]: *path_connected* {..*a*} **for** *a* :: *real*
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connected_Ioo*[*simp*]: *path_connected* {*a*<..<*b*} **for** *a b* :: *real*
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connected_Ioc*[*simp*]: *path_connected* {*a*<..*b*} **for** *a b* :: *real*
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connected_Ico*[*simp*]: *path_connected* {*a*..<*b*} **for** *a b* :: *real*
by (*simp add*: *convex_imp_path_connected*)

lemma *path_connectedin_path_image*:
assumes *pathin X g* **shows** *path_connectedin X* (*g* ‘ ({*0*..*1*}))
unfolding *pathin_def*
proof (*rule path_connectedin_continuous_map_image*)
show *continuous_map* (*subtopology euclideanreal* {*0*..*1*}) *X g*
using *assms pathin_def* **by** *blast*
qed (*auto simp*: *is_interval_1 is_interval_path_connected*)

lemma *path_connected_space_subconnected*:

path_connected_space X ⟷
(∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. ∃ *S*. *path_connectedin X S* ∧ *x* ∈ *S* ∧ *y* ∈ *S*)
by (*metis path_connectedin path_connectedin_topspace path_connected_space_def*)

lemma *connectedin_path_image*: *pathin X g* ⟹ *connectedin X* (*g* ‘ ({*0..1*}))
by (*simp add*: *path_connectedin_imp_connectedin path_connectedin_path_image*)

lemma *compactin_path_image*: *pathin X g* ⟹ *compactin X* (*g* ‘ ({*0..1*}))
unfolding *pathin_def*
by (*rule image_compactin* [*of top_of_set* {*0..1*}]) *auto*

lemma *linear_homeomorphism_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f*
obtains *g* **where** *homeomorphism* (*f* ‘ *S*) *S g f*
proof −
obtain *g* **where** *linear g g* ∘ *f* = *id*
using *assms linear_injective_left_inverse* **by** *blast*
then have *homeomorphism* (*f* ‘ *S*) *S g f*
using *assms* **unfolding** *homeomorphism_def*
by (*auto simp*: *eq_id_iff* [*symmetric*] *image_comp linear_conv_bounded_linear linear_continuous_on*)
then show *thesis* **..**
qed

lemma *linear_homeomorphic_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f*
shows *S homeomorphic f* ‘ *S*
by (*meson homeomorphic_def homeomorphic_sym linear_homeomorphism_image* [*OF assms*])

lemma *path_connected_Times*:
assumes *path_connected s path_connected t*
shows *path_connected* (*s* × *t*)
proof (*simp add*: *path_connected_def Sigma_def*, *clarify*)
fix *x1 y1 x2 y2*
assume *x1* ∈ *s y1* ∈ *t x2* ∈ *s y2* ∈ *t*
obtain *g* **where** *path g* **and** *g*: *path_image g* ⊆ *s* **and** *gs*: *pathstart g* = *x1* **and** *gf*: *pathfinish g* = *x2*
using ‹*x1* ∈ *s*› ‹*x2* ∈ *s*› *assms* **by** (*force simp*: *path_connected_def*)
obtain *h* **where** *path h* **and** *h*: *path_image h* ⊆ *t* **and** *hs*: *pathstart h* = *y1* **and** *hf*: *pathfinish h* = *y2*
using ‹*y1* ∈ *t*› ‹*y2* ∈ *t*› *assms* **by** (*force simp*: *path_connected_def*)
have *path* (λ*z*. (*x1*, *h z*))
using ‹*path h*›
unfolding *path_def*

by (*intro continuous_intros continuous_on_compose2* [**where** *g* = *Pair* _]; *force*)
moreover have *path* (λ*z*. (*g z*, *y2*))
using ‹*path g*›
unfolding *path_def*
by (*intro continuous_intros continuous_on_compose2* [**where** *g* = *Pair* _]; *force*)
ultimately have *1*: *path* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*)))
by (*metis hf gs path_join_imp pathstart_def pathfinish_def*)
have *path_image* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*))) ⊆ *path_image* (λ*z*. (*x1*, *h z*)) ∪ *path_image* (λ*z*. (*g z*, *y2*))
by (*rule Path_Connected.path_image_join_subset*)
also have ... ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)})
using *g h* ‹*x1* ∈ *s*› ‹*y2* ∈ *t*› **by** (*force simp*: *path_image_def*)
finally have *2*: *path_image* ((λ*z*. (*x1*, *h z*)) +++ (λ*z*. (*g z*, *y2*))) ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)}) **.**
show ∃*g*. *path g* ∧ *path_image g* ⊆ (⋃*x*∈*s*. ⋃*x1*∈*t*. {(*x*, *x1*)}) ∧
pathstart g = (*x1*, *y1*) ∧ *pathfinish g* = (*x2*, *y2*)
using *1 2 gf hs*
by (*metis* (*no_types*, *lifting*) *pathfinish_def pathfinish_join pathstart_def pathstart_join*)
qed

lemma *is_interval_path_connected_1*:
fixes *s* :: *real set*
shows *is_interval s* ⟷ *path_connected s*
using *is_interval_connected_1 is_interval_path_connected path_connected_imp_connected*
by *blast*

6.1.18 Path components

lemma *Union_path_component* [*simp*]:
Union {*path_component_set S x* |*x*. *x* ∈ *S*} = *S*
using *path_component_subset path_component_refl* **by** *blast*

lemma *path_component_disjoint*:
disjnt (*path_component_set S a*) (*path_component_set S b*) ⟷
(*a* ∉ *path_component_set S b*)
unfolding *disjnt_iff*
using *path_component_sym path_component_trans* **by** *blast*

lemma *path_component_eq_eq*:
path_component S x = *path_component S y* ⟷
(*x* ∉ *S*) ∧ (*y* ∉ *S*) ∨ *x* ∈ *S* ∧ *y* ∈ *S* ∧ *path_component S x y*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis* (*no_types*) *path_component_mem*(*1*) *path_component_refl*)
next

assume *?rhs* **then show** *?lhs*
proof
assume $x \notin S \wedge y \notin S$ **then show** *?lhs*
by (*metis Collect_empty_eq_bot path_component_eq_empty*)
next
assume *S*: $x \in S \wedge y \in S \wedge$ *path_component S x y* **show** *?lhs*
by (*rule ext*) (*metis S path_component_trans path_component_sym*)
qed
qed

lemma *path_component_unique*:
assumes $x \in C$ $C \subseteq S$ *path_connected C*
$\bigwedge C'$. $[\![x \in C'; C' \subseteq S;$ *path_connected* $C']\!] \Longrightarrow C' \subseteq C$
shows *path_component_set S x* $= C$
by (*smt* (*verit*, *best*) *Collect_cong assms path_component path_component_of_subset path_connected_component_set*)

lemma *path_component_intermediate_subset*:
path_component_set U a $\subseteq T \wedge T \subseteq U$
$\Longrightarrow$ *path_component_set T a* $=$ *path_component_set U a*
by (*metis* (*no_types*) *path_component_mono path_component_path_component subset_antisym*)

lemma *complement_path_component_Union*:
fixes $x :: 'a ::$ *topological_space*
shows $S -$ *path_component_set S x* $=$
$\bigcup(\{$*path_component_set S y*$|\ y.\ y \in S\} - \{$*path_component_set S x*$\})$
proof $-$
have $*$: $(\bigwedge x.\ x \in S - \{a\} \Longrightarrow$ *disjnt a x*$) \Longrightarrow \bigcup S - a = \bigcup(S - \{a\})$
for *a*::*'a set* **and** *S*
by (*auto simp*: *disjnt_def*)
have $\bigwedge y.\ y \in \{$*path_component_set S x* $|x.\ x \in S\} - \{$*path_component_set S x*$\}$
$\Longrightarrow$ *disjnt* (*path_component_set S x*) *y*
using *path_component_disjoint path_component_eq* **by** *fastforce*
then have $\bigcup\{$*path_component_set S x* $|x.\ x \in S\} -$ *path_component_set S x* $=$
$\bigcup(\{$*path_component_set S y* $|y.\ y \in S\} - \{$*path_component_set S x*$\})$
by (*meson* $*$)
then show *?thesis* **by** *simp*
qed

6.1.19 Path components

definition *path_component_of*
where *path_component_of X x y* $\equiv \exists g.$ *pathin X g* $\wedge$ *g 0* $= x \wedge$ *g 1* $= y$

abbreviation *path_component_of_set*
where *path_component_of_set X x* $\equiv$ *Collect* (*path_component_of X x*)

definition *path_components_of* :: *'a topology* ⇒ *'a set set*
where *path_components_of X* ≡ *path_component_of_set X ' topspace X*

lemma *pathin_canon_iff*: *pathin* (*top_of_set T*) *g* ⟷ *path g* ∧ *g* ∈ {*0..1*} → *T*
by (*simp add*: *path_def pathin_def image_subset_iff_funcset*)

lemma *path_component_of_canon_iff* [*simp*]:
path_component_of (*top_of_set T*) *a b* ⟷ *path_component T a b*
by (*simp add*: *path_component_of_def pathin_canon_iff path_defs image_subset_iff_funcset*)

lemma *path_component_in_topspace*:
path_component_of X x y ⟹ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X*
by (*auto simp*: *path_component_of_def pathin_def continuous_map_def*)

lemma *path_component_of_refl*:
path_component_of X x x ⟷ *x* ∈ *topspace X*
by (*metis path_component_in_topspace path_component_of_def pathin_const*)

lemma *path_component_of_sym*:
assumes *path_component_of X x y*
shows *path_component_of X y x*
using *assms*
apply (*clarsimp simp*: *path_component_of_def pathin_def*)
apply (*rule_tac x=g* ∘ (λ*t. 1 − t*) **in** *exI*)
apply (*auto intro*!: *continuous_map_compose simp*: *continuous_map_in_subtopology continuous_on_op_minus*)
done

lemma *path_component_of_sym_iff*:
path_component_of X x y ⟷ *path_component_of X y x*
by (*metis path_component_of_sym*)

lemma *continuous_map_cases_le*:
assumes *contp*: *continuous_map X euclideanreal p*
and *contq*: *continuous_map X euclideanreal q*
and *contf*: *continuous_map* (*subtopology X* {*x. x* ∈ *topspace X* ∧ *p x* ≤ *q x*}) *Y f*
and *contg*: *continuous_map* (*subtopology X* {*x. x* ∈ *topspace X* ∧ *q x* ≤ *p x*}) *Y g*
and *fg*: ⋀*x*. ⟦*x* ∈ *topspace X*; *p x* = *q x*⟧ ⟹ *f x* = *g x*
shows *continuous_map X Y* (λ*x. if p x* ≤ *q x then f x else g x*)
proof −
have *continuous_map X Y* (λ*x. if q x* − *p x* ∈ {*0..*} *then f x else g x*)
proof (*rule continuous_map_cases_function*)
show *continuous_map X euclideanreal* (λ*x. q x* − *p x*)
by (*intro contp contq continuous_intros*)
show *continuous_map* (*subtopology X* {*x* ∈ *topspace X. q x* − *p x* ∈ *euclideanreal*

```
closure_of {0..}}) Y f
      by (simp add: contf)
   show continuous_map (subtopology X {x ∈ topspace X. q x − p x ∈ euclideanreal
closure_of (topspace euclideanreal − {0..})}) Y g
      by (simp add: contg flip: Compl_eq_Diff_UNIV)
  qed (auto simp: fg)
  then show ?thesis
    by simp
qed

lemma continuous_map_cases_lt:
  assumes contp: continuous_map X euclideanreal p
    and contq: continuous_map X euclideanreal q
    and contf: continuous_map (subtopology X {x. x ∈ topspace X ∧ p x ≤ q x})
Y f
    and contg: continuous_map (subtopology X {x. x ∈ topspace X ∧ q x ≤ p x})
Y g
    and fg: ⋀x. ⟦x ∈ topspace X; p x = q x⟧ ⟹ f x = g x
  shows continuous_map X Y (λx. if p x < q x then f x else g x)
proof −
  have continuous_map X Y (λx. if q x − p x ∈ {0<..} then f x else g x)
  proof (rule continuous_map_cases_function)
    show continuous_map X euclideanreal (λx. q x − p x)
      by (intro contp contq continuous_intros)
   show continuous_map (subtopology X {x ∈ topspace X. q x − p x ∈ euclideanreal
closure_of {0<..}}) Y f
      by (simp add: contf)
   show continuous_map (subtopology X {x ∈ topspace X. q x − p x ∈ euclideanreal
closure_of (topspace euclideanreal − {0<..})}) Y g
      by (simp add: contg flip: Compl_eq_Diff_UNIV)
  qed (auto simp: fg)
  then show ?thesis
    by simp
qed

lemma path_component_of_trans:
  assumes path_component_of X x y and path_component_of X y z
  shows path_component_of X x z
  unfolding path_component_of_def pathin_def
proof −
  let ?T01 = top_of_set {0..1::real}
  obtain g1 g2 where g1: continuous_map ?T01 X g1 x = g1 0 y = g1 1
    and g2: continuous_map ?T01 X g2 g2 0 = g1 1 z = g2 1
    using assms unfolding path_component_of_def pathin_def by blast
  let ?g = λx. if x ≤ 1/2 then (g1 ∘ (λt. 2 * t)) x else (g2 ∘ (λt. 2 * t −1)) x
  show ∃g. continuous_map ?T01 X g ∧ g 0 = x ∧ g 1 = z
  proof (intro exI conjI)
    show continuous_map (subtopology euclideanreal {0..1}) X ?g
   proof (intro continuous_map_cases_le continuous_map_compose, force, force)
```

show *continuous_map* (*subtopology ?T01* {$x \in$ *topspace ?T01*. $x \leq 1/2$}) *?T01* ((∗) *2*)
by (*auto simp*: *continuous_map_in_subtopology continuous_map_from_subtopology*)
have *continuous_map*
(*subtopology* (*top_of_set* {*0..1*}) {*x*. $0 \leq x \wedge x \leq 1 \wedge 1 \leq x * 2$})
euclideanreal (λt. $2 * t - 1$)
by (*intro continuous_intros*) (*force intro*: *continuous_map_from_subtopology*)
then show *continuous_map* (*subtopology ?T01* {$x \in$ *topspace ?T01*. $1/2 \leq x$}) *?T01* (λt. $2 * t - 1$)
by (*force simp*: *continuous_map_in_subtopology*)
show (*g1* ∘ (∗) *2*) *x* = (*g2* ∘ (λt. $2 * t - 1$)) *x* **if** $x \in$ *topspace ?T01 x* = *1/2* **for** *x*
using *that* **by** (*simp add*: *g2*(*2*) *mult.commute continuous_map_from_subtopology*)
qed (*auto simp*: *g1 g2*)
qed (*auto simp*: *g1 g2*)
qed

lemma *path_component_of_mono*:
⟦*path_component_of* (*subtopology X S*) *x y*; $S \subseteq T$⟧ ⟹ *path_component_of* (*subtopology X T*) *x y*
unfolding *path_component_of_def*
by (*metis subsetD pathin_subtopology*)

lemma *path_component_of*:
path_component_of X x y ⟷ (∃ *T*. *path_connectedin X T* ∧ $x \in T \wedge y \in T$)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis atLeastAtMost_iff image_eqI order_refl path_component_of_def path_connectedin_path_image zero_le_one*)
next
assume *?rhs* **then show** *?lhs*
by (*metis path_component_of_def path_connectedin*)
qed

lemma *path_component_of_set*:
path_component_of X x y ⟷ (∃ *g*. *pathin X g* ∧ *g 0* = *x* ∧ *g 1* = *y*)
by (*auto simp*: *path_component_of_def*)

lemma *path_component_of_subset_topspace*:
Collect(*path_component_of X x*) ⊆ *topspace X*
using *path_component_in_topspace* **by** *fastforce*

lemma *path_component_of_eq_empty*:
Collect(*path_component_of X x*) = {} ⟷ ($x \notin$ *topspace X*)
using *path_component_in_topspace path_component_of_refl* **by** *fastforce*

lemma *path_connected_space_iff_path_component*:
path_connected_space X ⟷ (∀ *x* ∈ *topspace X*. ∀ *y* ∈ *topspace X*. *path_component_of*

X x y)
by (*simp add*: *path_component_of path_connected_space_subconnected*)

lemma *path_connected_space_imp_path_component_of*:
⟦*path_connected_space X*; *a* ∈ *topspace X*; *b* ∈ *topspace X*⟧
⟹ *path_component_of X a b*
by (*simp add*: *path_connected_space_iff_path_component*)

lemma *path_connected_space_path_component_set*:
path_connected_space X ⟷ (∀ *x* ∈ *topspace X*. *Collect*(*path_component_of X*
x) = *topspace X*)
using *path_component_of_subset_topspace path_connected_space_iff_path_component*
by *fastforce*

lemma *path_component_of_maximal*:
⟦*path_connectedin X s*; *x* ∈ *s*⟧ ⟹ *s* ⊆ *Collect*(*path_component_of X x*)
using *path_component_of* **by** *fastforce*

lemma *path_component_of_equiv*:
path_component_of X x y ⟷ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *path_component_of*
X x = *path_component_of X y*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *fun_eq_iff path_component_in_topspace*
by (*metis path_component_in_topspace path_component_of_sym path_component_of_trans*)
qed (*simp add*: *path_component_of_refl*)

lemma *path_component_of_disjoint*:
disjnt (*Collect* (*path_component_of X x*)) (*Collect* (*path_component_of X y*))
⟷
~(*path_component_of X x y*)
by (*force simp*: *disjnt_def path_component_of_eq_empty path_component_of_equiv*)

lemma *path_component_of_eq*:
path_component_of X x = *path_component_of X y* ⟷
(*x* ∉ *topspace X*) ∧ (*y* ∉ *topspace X*) ∨
x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *path_component_of X x y*
by (*metis Collect_empty_eq_bot path_component_of_eq_empty path_component_of_equiv*)

lemma *path_component_of_aux*:
path_component_of X x y
⟹ *path_component_of* (*subtopology X* (*Collect* (*path_component_of X*
x))) *x y*
by (*meson path_component_of path_component_of_maximal path_connectedin_subtopology*)

lemma *path_connectedin_path_component_of*:
path_connectedin X (*Collect* (*path_component_of X x*))

proof −
have *topspace* (*subtopology X* (*path_component_of_set X x*)) = *path_component_of_set X x*
by (*meson path_component_of_subset_topspace topspace_subtopology_subset*)
then have *path_connected_space* (*subtopology X* (*path_component_of_set X x*))
by (*metis mem_Collect_eq path_component_of_aux path_component_of_equiv path_connected_space_iff_path_component*)
then show *?thesis*
by (*simp add*: *path_component_of_subset_topspace path_connectedin_def*)
qed

lemma *path_connectedin_euclidean* [*simp*]:
path_connectedin euclidean S ⟷ *path_connected S*
by (*auto simp*: *path_connectedin_def path_connected_space_iff_path_component path_connected_component*)

lemma *path_connected_space_euclidean_subtopology* [*simp*]:
path_connected_space(*subtopology euclidean S*) ⟷ *path_connected S*
using *path_connectedin_topspace* **by** *force*

lemma *Union_path_components_of*:
⋃(*path_components_of X*) = *topspace X*
by (*auto simp*: *path_components_of_def path_component_of_equiv*)

lemma *path_components_of_maximal*:
⟦*C* ∈ *path_components_of X*; *path_connectedin X S*; ~*disjnt C S*⟧ ⟹ *S* ⊆ *C*
by (*smt* (*verit*, *ccfv_SIG*) *disjnt_iff imageE mem_Collect_eq path_component_of_equiv path_component_of_maximal path_components_of_def*)

lemma *pairwise_disjoint_path_components_of*:
pairwise disjnt (*path_components_of X*)
by (*auto simp*: *path_components_of_def pairwise_def path_component_of_disjoint path_component_of_equiv*)

lemma *complement_path_components_of_Union*:
C ∈ *path_components_of X* ⟹ *topspace X* − *C* = ⋃(*path_components_of X* − {*C*})
by (*metis Union_path_components_of bot.extremum ccpo_Sup_singleton diff_Union_pairwise_disjoint insert_subsetI pairwise_disjoint_path_components_of*)

lemma *nonempty_path_components_of*:
assumes *C* ∈ *path_components_of X* **shows** *C* ≠ {}
by (*metis assms imageE path_component_of_eq_empty path_components_of_def*)

lemma *path_components_of_subset*: *C* ∈ *path_components_of X* ⟹ *C* ⊆ *topspace X*

by (*auto simp*: *path_components_of_def path_component_of_equiv*)

lemma *path_connectedin_path_components_of*:
C ∈ *path_components_of X* ⟹ *path_connectedin X C*
by (*auto simp*: *path_components_of_def path_connectedin_path_component_of*)

lemma *path_component_in_path_components_of*:
Collect (*path_component_of X a*) ∈ *path_components_of X* ⟷ *a* ∈ *topspace X*
by (*metis imageI nonempty_path_components_of path_component_of_eq_empty path_components_of_def*)

lemma *path_connectedin_Union*:
assumes *A*: ⋀*S*. *S* ∈ *A* ⟹ *path_connectedin X S* **and** ⋂ *A* ≠ {}
shows *path_connectedin X* (⋃ *A*)
proof −
obtain *a* **where** ⋀*S*. *S* ∈ *A* ⟹ *a* ∈ *S*
using *assms* **by** *blast*
then have ⋀*x*. *x* ∈ *topspace* (*subtopology X* (⋃ *A*)) ⟹ *path_component_of* (*subtopology X* (⋃ *A*)) *a x*
unfolding *topspace_subtopology path_component_of*
by (*metis* (*full_types*) *IntD2 Union_iff Union_upper A path_connectedin_subtopology*)
then show *?thesis*
using *A* **unfolding** *path_connectedin_def*
by (*metis Sup_le_iff path_component_of_equiv path_connected_space_iff_path_component*)
qed

lemma *path_connectedin_Un*:
⟦*path_connectedin X S*; *path_connectedin X T*; *S* ∩ *T* ≠ {}⟧
⟹ *path_connectedin X* (*S* ∪ *T*)
by (*blast intro*: *path_connectedin_Union* [*of* {*S*,*T*}, *simplified*])

lemma *path_connected_space_iff_components_eq*:
path_connected_space X ⟷
(∀ *C* ∈ *path_components_of X*. ∀ *C′* ∈ *path_components_of X*. *C* = *C′*)
unfolding *path_components_of_def*
proof (*intro iffI ballI*)
assume ∀ *C* ∈ *path_component_of_set X* ‘ *topspace X*.
∀ *C′* ∈ *path_component_of_set X* ‘ *topspace X*. *C* = *C′*
then show *path_connected_space X*
using *path_component_of_refl path_connected_space_iff_path_component* **by** *fastforce*
qed (*auto simp*: *path_connected_space_path_component_set*)

lemma *path_components_of_eq_empty*:
path_components_of X = {} ⟷ *X* = *trivial_topology*
by (*metis image_is_empty path_components_of_def subtopology_eq_discrete_topology_empty*)

lemma *path_components_of_empty_space*:

path_components_of trivial_topology = {}
by (*simp add*: *path_components_of_eq_empty*)

lemma *path_components_of_subset_singleton*:
path_components_of X ⊆ {S} ⟷
path_connected_space X ∧ (topspace X = {} ∨ topspace X = S)
proof (*cases topspace X = {}*)
case *True*
then show *?thesis*
by (*auto simp*: *path_components_of_empty_space path_connected_space_topspace_empty*)
next
case *False*
have (*path_components_of X = {S}*) ⟷ (*path_connected_space X ∧ topspace X = S*)
by (*metis False Set.set_insert ex_in_conv insert_iff path_component_in_path_components_of path_connected_space_iff_components_eq path_connected_space_path_component_set*)
with *False* **show** *?thesis*
by (*simp add*: *path_components_of_eq_empty subset_singleton_iff*)
qed

lemma *path_connected_space_iff_components_subset_singleton*:
path_connected_space X ⟷ (∃a. path_components_of X ⊆ {a})
by (*simp add*: *path_components_of_subset_singleton*)

lemma *path_components_of_eq_singleton*:
path_components_of X = {S} ⟷ path_connected_space X ∧ topspace X ≠ {} ∧ S = topspace X
by (*metis cSup_singleton insert_not_empty path_components_of_subset_singleton subset_singleton_iff*)

lemma *path_components_of_path_connected_space*:
path_connected_space X ⟹ path_components_of X = (if topspace X = {} then {} else {topspace X})
by (*simp add*: *path_components_of_eq_empty path_components_of_eq_singleton*)

lemma *path_component_subset_connected_component_of*:
path_component_of_set X x ⊆ connected_component_of_set X x
proof (*cases x ∈ topspace X*)
case *True*
then show *?thesis*
by (*simp add*: *connected_component_of_maximal path_component_of_refl path_connectedin_imp_connectedin path_connectedin_path_component_of*)
next
case *False*
then show *?thesis*
using *path_component_of_eq_empty* **by** *fastforce*
qed

lemma *exists_path_component_of_superset*:
assumes *S*: *path_connectedin X S* **and** *ne*: *topspace X ≠ {}*
obtains *C* **where** *C ∈ path_components_of X S ⊆ C*
by (*metis S ne ex_in_conv path_component_in_path_components_of path_component_of_maximal path_component_of_subset_topspace subset_eq that*)

lemma *path_component_of_eq_overlap*:
path_component_of X x = path_component_of X y ⟷
(x ∉ topspace X) ∧ (y ∉ topspace X) ∨
Collect (path_component_of X x) ∩ Collect (path_component_of X y) ≠ {}
by (*metis disjnt_def empty_iff inf_bot_right mem_Collect_eq path_component_of_disjoint path_component_of_eq path_component_of_eq_empty*)

lemma *path_component_of_nonoverlap*:
Collect (path_component_of X x) ∩ Collect (path_component_of X y) = {}
⟷
(x ∉ topspace X) ∨ (y ∉ topspace X) ∨
path_component_of X x ≠ path_component_of X y
by (*metis inf.idem path_component_of_eq_empty path_component_of_eq_overlap*)

lemma *path_component_of_overlap*:
Collect (path_component_of X x) ∩ Collect (path_component_of X y) ≠ {}
⟷
x ∈ topspace X ∧ y ∈ topspace X ∧ path_component_of X x = path_component_of X y
by (*meson path_component_of_nonoverlap*)

lemma *path_components_of_disjoint*:
⟦C ∈ path_components_of X; C′ ∈ path_components_of X⟧ ⟹ disjnt C C′
⟷ C ≠ C′
by (*auto simp*: *path_components_of_def path_component_of_disjoint path_component_of_equiv*)

lemma *path_components_of_overlap*:
⟦C ∈ path_components_of X; C′ ∈ path_components_of X⟧ ⟹ C ∩ C′ ≠ {}
⟷ C = C′
by (*auto simp*: *path_components_of_def path_component_of_equiv*)

lemma *path_component_of_unique*:
⟦x ∈ C; path_connectedin X C; ⋀C′. ⟦x ∈ C′; path_connectedin X C′⟧ ⟹ C′ ⊆ C⟧
⟹ Collect (path_component_of X x) = C
by (*meson subsetD eq_iff path_component_of_maximal path_connectedin_path_component_of*)

lemma *path_component_of_discrete_topology* [*simp*]:
Collect (path_component_of (discrete_topology U) x) = (if x ∈ U then {x} else {})
proof −
have *⋀C′. ⟦x ∈ C′; path_connectedin (discrete_topology U) C′⟧ ⟹ C′ ⊆ {x}*
by (*metis path_connectedin_discrete_topology subsetD singletonD*)

then have $x \in U \Longrightarrow$ *Collect* (*path_component_of* (*discrete_topology U*) *x*) = {*x*}
by (*simp add*: *path_component_of_unique*)
then show *?thesis*
using *path_component_in_topspace* **by** *fastforce*
qed

lemma *path_component_of_discrete_topology_iff* [*simp*]:
path_component_of (*discrete_topology U*) *x y* $\longleftrightarrow$ *x* $\in$ *U* $\wedge$ *y*=*x*
by (*metis empty_iff insertI1 mem_Collect_eq path_component_of_discrete_topology singletonD*)

lemma *path_components_of_discrete_topology* [*simp*]:
path_components_of (*discrete_topology U*) = (*λx*. {*x*}) ' *U*
by (*auto simp*: *path_components_of_def image_def fun_eq_iff*)

lemma *homeomorphic_map_path_component_of*:
assumes *f*: *homeomorphic_map X Y f* **and** *x*: *x* $\in$ *topspace X*
shows *Collect* (*path_component_of Y* (*f x*)) = *f* ' *Collect*(*path_component_of X x*)
proof −
obtain *g* **where** *g*: *homeomorphic_maps X Y f g*
using *f homeomorphic_map_maps* **by** *blast*
show *?thesis*
proof
have *Collect* (*path_component_of Y* (*f x*)) $\subseteq$ *topspace Y*
by (*simp add*: *path_component_of_subset_topspace*)
moreover have *g* ' *Collect*(*path_component_of Y* (*f x*)) $\subseteq$ *Collect* (*path_component_of X* (*g* (*f x*)))
using *f g x* **unfolding** *homeomorphic_maps_def*
by (*metis image_Collect_subsetI image_eqI mem_Collect_eq path_component_of_equiv path_component_of_maximal*
path_connectedin_continuous_map_image path_connectedin_path_component_of)
ultimately show *Collect* (*path_component_of Y* (*f x*)) $\subseteq$ *f* ' *Collect* (*path_component_of X x*)
using *g x* **unfolding** *homeomorphic_maps_def continuous_map_def image_iff subset_iff*
by *metis*
show *f* ' *Collect* (*path_component_of X x*) $\subseteq$ *Collect* (*path_component_of Y* (*f x*))
proof (*rule path_component_of_maximal*)
show *path_connectedin Y* (*f* ' *Collect* (*path_component_of X x*))
by (*meson f homeomorphic_map_path_connectedness_eq path_connectedin_path_component_of*)
qed (*simp add*: *path_component_of_refl x*)
qed
qed

lemma *homeomorphic_map_path_components_of*:
assumes *homeomorphic_map X Y f*

shows *path_components_of Y = (image f) ‘ (path_components_of X)*
unfolding *path_components_of_def homeomorphic_imp_surjective_map [OF assms, symmetric]*
using *assms homeomorphic_map_path_component_of* **by** *fastforce*

6.1.20 Paths and path-connectedness

lemma *path_connected_space_quotient_map_image*:
⟦quotient_map X Y q; path_connected_space X⟧ ⟹ path_connected_space Y
by (*metis path_connectedin_continuous_map_image path_connectedin_topspace quotient_imp_continuous_map quotient_imp_surjective_map*)

lemma *path_connected_space_retraction_map_image*:
⟦retraction_map X Y r; path_connected_space X⟧ ⟹ path_connected_space Y
using *path_connected_space_quotient_map_image retraction_imp_quotient_map*
by *blast*

lemma *path_connected_space_prod_topology*:
path_connected_space(prod_topology X Y) ⟷
topspace(prod_topology X Y) = {} ∨ path_connected_space X ∧ path_connected_space Y
proof (*cases topspace(prod_topology X Y) = {}*)
case *True*
then show *?thesis*
using *path_connected_space_topspace_empty* **by** *force*
next
case *False*
have *path_connected_space (prod_topology X Y)*
if *X*: *path_connected_space X* **and** *Y*: *path_connected_space Y*
proof (*clarsimp simp*: *path_connected_space_def*)
fix *x y x′ y′*
assume *x ∈ topspace X* **and** *y ∈ topspace Y* **and** *x′ ∈ topspace X* **and** *y′ ∈ topspace Y*
obtain *f* **where** *pathin X f f 0 = x f 1 = x′*
by (*meson X ‹x ∈ topspace X› ‹x′ ∈ topspace X› path_connected_space_def*)
obtain *g* **where** *pathin Y g g 0 = y g 1 = y′*
by (*meson Y ‹y ∈ topspace Y› ‹y′ ∈ topspace Y› path_connected_space_def*)
show *∃h. pathin (prod_topology X Y) h ∧ h 0 = (x,y) ∧ h 1 = (x′,y′)*
proof (*intro exI conjI*)
show *pathin (prod_topology X Y) (λt. (f t, g t))*
using *‹pathin X f› ‹pathin Y g›* **by** (*simp add*: *continuous_map_paired pathin_def*)
show *(λt. (f t, g t)) 0 = (x, y)*
using *‹f 0 = x› ‹g 0 = y›* **by** *blast*
show *(λt. (f t, g t)) 1 = (x′, y′)*
using *‹f 1 = x′› ‹g 1 = y′›* **by** *blast*
qed
qed
then show *?thesis*

by (*metis False path_connected_space_quotient_map_image prod_topology_trivial1 prod_topology_trivial2*
quotient_map_fst quotient_map_snd topspace_discrete_topology)
qed

lemma *path_connectedin_Times*:
path_connectedin (*prod_topology X Y*) (*S* × *T*) ⟷
S = {} ∨ *T* = {} ∨ *path_connectedin X S* ∧ *path_connectedin Y T*
by (*auto simp add*: *path_connectedin_def subtopology_Times path_connected_space_prod_topology*)

6.1.21 Path components

lemma *path_component_of_subtopology_eq*:
path_component_of (*subtopology X U*) *x* = *path_component_of X x* ⟷ *path_component_of_set X x* ⊆ *U*
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*metis path_connectedin_path_component_of path_connectedin_subtopology*)
next
show *?rhs* ⟹ *?lhs*
unfolding *fun_eq_iff*
by (*metis path_connectedin_subtopology path_component_of path_component_of_aux path_component_of_mono*)
qed

lemma *path_components_of_subtopology*:
assumes *C* ∈ *path_components_of X C* ⊆ *U*
shows *C* ∈ *path_components_of* (*subtopology X U*)
using *assms path_component_of_refl path_component_of_subtopology_eq topspace_subtopology*
by (*smt* (*verit*) *imageE path_component_in_path_components_of path_components_of_def*)

lemma *path_imp_connected_component_of*:
path_component_of X x y ⟹ *connected_component_of X x y*
by (*metis in_mono mem_Collect_eq path_component_subset_connected_component_of*)

lemma *finite_path_components_of_finite*:
finite(*topspace X*) ⟹ *finite*(*path_components_of X*)
by (*simp add*: *Union_path_components_of finite_UnionD*)

lemma *path_component_of_continuous_image*:
⟦*continuous_map X X′ f*; *path_component_of X x y*⟧ ⟹ *path_component_of X′* (*f x*) (*f y*)
by (*meson image_eqI path_component_of path_connectedin_continuous_map_image*)

lemma *path_component_of_pair* [*simp*]:
path_component_of_set (*prod_topology X Y*) (*x*,*y*) =
path_component_of_set X x × *path_component_of_set Y y* (**is** *?lhs* = *?rhs*)
proof (*cases ?lhs* = {})

case *True*
then show *?thesis*
by (*metis Sigma_empty1 Sigma_empty2 mem_Sigma_iff path_component_of_eq_empty topspace_prod_topology*)
next
case *False*
then have *path_component_of X x x path_component_of Y y y*
using *path_component_of_eq_empty path_component_of_refl* **by** *fastforce+*
moreover
have *path_connectedin* (*prod_topology X Y*) (*path_component_of_set X x* × *path_component_of_set Y y*)
by (*metis path_connectedin_Times path_connectedin_path_component_of*)
moreover have *path_component_of X x a path_component_of Y y b*
if (*x*, *y*) ∈ *C′* (*a*,*b*) ∈ *C′* **and** *path_connectedin* (*prod_topology X Y*) *C′* **for** *C′ a b*
by (*smt* (*verit*, *best*) *that continuous_map_fst continuous_map_snd fst_conv snd_conv path_component_of path_component_of_continuous_image*)+
ultimately show *?thesis*
by (*intro path_component_of_unique*) *auto*
qed

lemma *path_components_of_prod_topology*:
path_components_of (*prod_topology X Y*) =
(λ(*C*,*D*). *C* × *D*) ‘ (*path_components_of X* × *path_components_of Y*)
by (*force simp add*: *image_iff path_components_of_def*)

lemma *path_components_of_prod_topology′*:
path_components_of (*prod_topology X Y*) =
{*C* × *D* |*C D*. *C* ∈ *path_components_of X* ∧ *D* ∈ *path_components_of Y*}
by (*auto simp*: *path_components_of_prod_topology*)

lemma *path_component_of_product_topology*:
path_component_of_set (*product_topology X I*) *f* =
(*if f* ∈ *extensional I then PiE I* (λ*i*. *path_component_of_set* (*X i*) (*f i*)) *else* {})
(**is** *?lhs* = *?rhs*)
proof (*cases path_component_of_set* (*product_topology X I*) *f* = {})
case *True*
then show *?thesis*
by (*smt* (*verit*) *PiE_eq_empty_iff PiE_iff path_component_of_eq_empty topspace_product_topology*)
next
case *False*
then have [*simp*]: *f* ∈ *extensional I*
by (*auto simp*: *path_component_of_eq_empty PiE_iff path_component_of_equiv*)
show *?thesis*
proof (*intro path_component_of_unique*)
show *f* ∈ *?rhs*
using *False path_component_of_eq_empty path_component_of_refl* **by** *force*

show *path_connectedin* (*product_topology X I*) (*if f ∈ extensional I then* Π_E *i∈I. path_component_of_set* (*X i*) (*f i*) *else* {})
by (*simp add*: *path_connectedin_PiE path_connectedin_path_component_of*)
fix *C′*
assume *f ∈ C′* **and** *C′*: *path_connectedin* (*product_topology X I*) *C′*
show *C′ ⊆ ?rhs*
proof −
have *C′ ⊆ extensional I*
using *PiE_def C′ path_connectedin_subset_topspace* **by** *fastforce*
with ‹*f ∈ C′*› *C′* **show** *?thesis*
apply (*clarsimp simp*: *PiE_iff subset_iff*)
by (*smt* (*verit*, *ccfv_threshold*) *continuous_map_product_projection path_component_of path_component_of_continuous_image*)
qed
qed
qed

lemma *path_components_of_product_topology*:
path_components_of (*product_topology X I*) =
{*PiE I B* |*B*. ∀ *i* ∈ *I*. *B i* ∈ *path_components_of*(*X i*)} (**is** *?lhs*=*?rhs*)
proof
show *?lhs ⊆ ?rhs*
unfolding *path_components_of_def image_subset_iff*
by (*smt* (*verit*) *image_iff mem_Collect_eq path_component_of_product_topology topspace_product_topology_alt*)
next
show *?rhs ⊆ ?lhs*
proof
fix *F*
assume *F ∈ ?rhs*
then obtain *B* **where** *B*: *F* = Pi_E *I B*
and ∀ *i*∈*I*. ∃ *x*∈*topspace* (*X i*). *B i* = *path_component_of_set* (*X i*) *x*
by (*force simp add*: *path_components_of_def image_iff*)
then obtain *f* **where** *ftop*: ⋀*i*. *i ∈ I* ⟹ *f i ∈ topspace* (*X i*)
and *BF*: ⋀*i*. *i ∈ I* ⟹ *B i* = *path_component_of_set* (*X i*) (*f i*)
by *metis*
then have *F* = *path_component_of_set* (*product_topology X I*) (*restrict f I*)
by (*metis* (*mono_tags*, *lifting*) *B PiE_cong path_component_of_product_topology restrict_apply′ restrict_extensional*)
then show *F ∈ ?lhs*
by (*simp add*: *ftop path_component_in_path_components_of*)
qed
qed

6.1.22 Sphere is path-connected

lemma *path_connected_punctured_universe*:
assumes *2* ≤ *DIM*(*′a*::*euclidean_space*)
shows *path_connected* (− {*a*::*′a*})

proof −
let *?A* = {*x*::′*a*. ∃ *i*∈*Basis*. *x* · *i* < *a* · *i*}
let *?B* = {*x*::′*a*. ∃ *i*∈*Basis*. *a* · *i* < *x* · *i*}

have *A*: *path_connected ?A*
unfolding *Collect_bex_eq*
proof (*rule path_connected_UNION*)
fix *i* :: ′*a*
assume *i* ∈ *Basis*
then show (∑ *i*∈*Basis*. (*a* · *i* − *1*)∗$_R$ *i*) ∈ {*x*::′*a*. *x* · *i* < *a* · *i*}
by *simp*
show *path_connected* {*x*. *x* · *i* < *a* · *i*}
using *convex_imp_path_connected* [*OF convex_halfspace_lt*, *of i a* · *i*]
by (*simp add*: *inner_commute*)
qed
have *B*: *path_connected ?B*
unfolding *Collect_bex_eq*
proof (*rule path_connected_UNION*)
fix *i* :: ′*a*
assume *i* ∈ *Basis*
then show (∑ *i*∈*Basis*. (*a* · *i* + *1*) ∗$_R$ *i*) ∈ {*x*::′*a*. *a* · *i* < *x* · *i*}
by *simp*
show *path_connected* {*x*. *a* · *i* < *x* · *i*}
using *convex_imp_path_connected* [*OF convex_halfspace_gt*, *of a* · *i i*]
by (*simp add*: *inner_commute*)
qed
obtain *S* :: ′*a set* **where** *S* ⊆ *Basis* **and** *card S* = *Suc* (*Suc 0*)
using *obtain_subset_with_card_n*[*OF assms*] **by** (*force simp add*: *eval_nat_numeral*)
then obtain *b0 b1* :: ′*a* **where** *b0* ∈ *Basis* **and** *b1* ∈ *Basis* **and** *b0* ≠ *b1*
unfolding *card_Suc_eq* **by** *auto*
then have *a* + *b0* − *b1* ∈ *?A* ∩ *?B*
by (*auto simp*: *inner_simps inner_Basis*)
then have *?A* ∩ *?B* ≠ {}
by *fast*
with *A B* **have** *path_connected* (*?A* ∪ *?B*)
by (*rule path_connected_Un*)
also have *?A* ∪ *?B* = {*x*. ∃ *i*∈*Basis*. *x* · *i* ≠ *a* · *i*}
unfolding *neq_iff bex_disj_distrib Collect_disj_eq* **..**
also have ... = {*x*. *x* ≠ *a*}
unfolding *euclidean_eq_iff* [**where** ′*a*=′*a*]
by (*simp add*: *Bex_def*)
also have ... = − {*a*}
by *auto*
finally show *?thesis* **.**
qed

corollary *connected_punctured_universe*:
2 ≤ *DIM*(′*N*::*euclidean_space*) ⟹ *connected*(− {*a*::′*N*})
by (*simp add*: *path_connected_punctured_universe path_connected_imp_connected*)

proposition *path_connected_sphere*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *2* ≤ *DIM*(*'a*)
shows *path_connected*(*sphere a r*)
proof (*cases r 0*::*real rule*: *linorder_cases*)
case *greater*
then have *eq*: (*sphere* (*0*::*'a*) *r*) = (λ*x*. (*r* / *norm x*) ∗$_R$ *x*) ‘ (− {*0*::*'a*})
by (*force simp*: *image_iff split*: *if_split_asm*)
have *continuous_on* (− {*0*::*'a*}) (λ*x*. (*r* / *norm x*) ∗$_R$ *x*)
by (*intro continuous_intros*) *auto*
then have *path_connected* ((λ*x*. (*r* / *norm x*) ∗$_R$ *x*) ‘ (− {*0*::*'a*}))
by (*intro path_connected_continuous_image path_connected_punctured_universe assms*)
with *eq* **have** *path_connected*((+) *a* ‘ (*sphere* (*0*::*'a*) *r*))
by (*simp add*: *path_connected_translation*)
then show *?thesis*
by (*metis add.right_neutral sphere_translation*)
qed *auto*

lemma *connected_sphere*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *2* ≤ *DIM*(*'a*)
shows *connected*(*sphere a r*)
using *path_connected_sphere* [*OF assms*]
by (*simp add*: *path_connected_imp_connected*)

corollary *path_connected_complement_bounded_convex*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *bounded S convex S* **and** *2*: *2* ≤ *DIM*(*'a*)
shows *path_connected* (− *S*)
proof (*cases S* = {})
case *True* **then show** *?thesis*
using *convex_imp_path_connected* **by** *auto*
next
case *False*
then obtain *a* **where** *a* ∈ *S* **by** *auto*
have § [*rule_format*]: ∀ *y*∈*S*. ∀ *u*. *0* ≤ *u* ∧ *u* ≤ *1* ⟶ (*1* − *u*) ∗$_R$ *a* + *u* ∗$_R$ *y* ∈ *S*
using ‹*convex S*› ‹*a* ∈ *S*› **by** (*simp add*: *convex_alt*)
{ fix *x y* **assume** *x* ∉ *S y* ∉ *S*
then have *x* ≠ *a y* ≠ *a* **using** ‹*a* ∈ *S*› **by** *auto*
then have *bxy*: *bounded*(*insert x* (*insert y S*))
by (*simp add*: ‹*bounded S*›)
then obtain *B*::*real* **where** *B*: *0* < *B* **and** *Bx*: *norm* (*a* − *x*) < *B* **and** *By*: *norm* (*a* − *y*) < *B*
and *S* ⊆ *ball a B*
using *bounded_subset_ballD* [*OF bxy*, *of a*] **by** (*auto simp*: *dist_norm*)

define C **where** $C = B \mathbin{/} \mathit{norm}(x - a)$
let *?Cxa* $= a + C *_R (x - a)$
{ fix u
assume u: $(1 - u) *_R x + u *_R$ *?Cxa* $\in S$ **and** $0 \le u\ u \le 1$
have CC: $1 \le 1 + (C - 1) * u$
using ‹$x \neq a$› ‹$0 \le u$› Bx
by (*auto simp add*: C_*def norm_minus_commute*)
have $*$: $\bigwedge v.\ (1 - u) *_R x + u *_R (a + v *_R (x - a)) = a + (1 + (v - 1) * u) *_R (x - a)$
by (*simp add*: *algebra_simps*)
have $a + ((1 \mathbin{/} (1 + C * u - u)) *_R x + ((u \mathbin{/} (1 + C * u - u)) *_R a + (C * u \mathbin{/} (1 + C * u - u)) *_R x)) =$
$(1 + (u \mathbin{/} (1 + C * u - u))) *_R a + ((1 \mathbin{/} (1 + C * u - u)) + (C * u \mathbin{/} (1 + C * u - u))) *_R x$
by (*simp add*: *algebra_simps*)
also have $\ldots = (1 + (u \mathbin{/} (1 + C * u - u))) *_R a + (1 + (u \mathbin{/} (1 + C * u - u))) *_R x$
using CC **by** (*simp add*: *field_simps*)
also have $\ldots = x + (1 + (u \mathbin{/} (1 + C * u - u))) *_R a + (u \mathbin{/} (1 + C * u - u)) *_R x$
by (*simp add*: *algebra_simps*)
also have $\ldots = x + ((1 \mathbin{/} (1 + C * u - u)) *_R a +$
$((u \mathbin{/} (1 + C * u - u)) *_R x + (C * u \mathbin{/} (1 + C * u - u)) *_R a))$
using CC **by** (*simp add*: *field_simps*) (*simp add*: *add_divide_distrib scaleR_add_left*)
finally have *xeq*: $(1 - 1 \mathbin{/} (1 + (C - 1) * u)) *_R a + (1 \mathbin{/} (1 + (C - 1) * u)) *_R (a + (1 + (C - 1) * u) *_R (x - a)) = x$
by (*simp add*: *algebra_simps*)
have *False*
using § [*of* $a + (1 + (C - 1) * u) *_R (x - a)$ $1 \mathbin{/} (1 + (C - 1) * u)$]
using u ‹$x \neq a$› ‹$x \notin S$› ‹$0 \le u$› CC
by (*auto simp*: *xeq* $*$)
}
then have *pcx*: *path_component* $(- S)$ x *?Cxa*
by (*force simp*: *closed_segment_def intro*!: *path_component_linepath*)
define D **where** $D = B \mathbin{/} \mathit{norm}(y - a)$ — massive duplication with the proof above
let *?Dya* $= a + D *_R (y - a)$
{ fix u
assume u: $(1 - u) *_R y + u *_R$ *?Dya* $\in S$ **and** $0 \le u\ u \le 1$
have DD: $1 \le 1 + (D - 1) * u$
using ‹$y \neq a$› ‹$0 \le u$› By
by (*auto simp add*: D_*def norm_minus_commute*)
have $*$: $\bigwedge v.\ (1 - u) *_R y + u *_R (a + v *_R (y - a)) = a + (1 + (v - 1) * u) *_R (y - a)$
by (*simp add*: *algebra_simps*)
have $a + ((1 \mathbin{/} (1 + D * u - u)) *_R y + ((u \mathbin{/} (1 + D * u - u)) *_R a + (D * u \mathbin{/} (1 + D * u - u)) *_R y)) =$
$(1 + (u \mathbin{/} (1 + D * u - u))) *_R a + ((1 \mathbin{/} (1 + D * u - u)) + (D * u$

```
/ (1 + D * u - u))) *R y
      by (simp add: algebra_simps)
    also have ... = (1 + (u / (1 + D * u - u))) *R a + (1 + (u / (1 + D *
u - u))) *R y
      using DD by (simp add: field_simps)
    also have ... = y + (1 + (u / (1 + D * u - u))) *R a + (u / (1 + D * u
- u)) *R y
      by (simp add: algebra_simps)
    also have ... = y + ((1 / (1 + D * u - u)) *R a +
          ((u / (1 + D * u - u)) *R y + (D * u / (1 + D * u - u)) *R a))
        using DD by (simp add: field_simps) (simp add: add_divide_distrib
scaleR_add_left)
    finally have xeq: (1 - 1 / (1 + (D - 1) * u)) *R a + (1 / (1 + (D - 1)
* u)) *R (a + (1 + (D - 1) * u) *R (y - a)) = y
      by (simp add: algebra_simps)
    have False
      using § [of a + (1 + (D - 1) * u) *R (y - a) 1 / (1 + (D - 1) * u)]
      using u ‹y ≠ a› ‹y ∉ S› ‹0 ≤ u› DD
      by (auto simp: xeq *)
  }
  then have pdy: path_component (- S) y ?Dya
    by (force simp: closed_segment_def intro!: path_component_linepath)
  have pyx: path_component (- S) ?Dya ?Cxa
  proof (rule path_component_of_subset)
    show sphere a B ⊆ - S
    using ‹S ⊆ ball a B› by (force simp: ball_def dist_norm norm_minus_commute)
    have aB: ?Dya ∈ sphere a B ?Cxa ∈ sphere a B
      using ‹x ≠ a› using ‹y ≠ a› B by (auto simp: dist_norm C_def D_def)
    then show path_component (sphere a B) ?Dya ?Cxa
      using path_connected_sphere [OF 2] path_connected_component by blast
  qed
  have path_component (- S) x y
    by (metis path_component_trans path_component_sym pcx pdy pyx)
 }
 then show ?thesis
  by (auto simp: path_connected_component)
qed

lemma connected_complement_bounded_convex:
  fixes S :: 'a :: euclidean_space set
  assumes bounded S convex S 2 ≤ DIM('a)
    shows  connected (- S)
 using path_connected_complement_bounded_convex [OF assms] path_connected_imp_connected
by blast

lemma connected_diff_ball:
  fixes S :: 'a :: euclidean_space set
  assumes connected S cball a r ⊆ S 2 ≤ DIM('a)
    shows connected (S - ball a r)
```

proof (*rule connected_diff_open_from_closed* [*OF ball_subset_cball*])
show *connected* (*cball a r − ball a r*)
using *assms connected_sphere* **by** (*auto simp*: *cball_diff_eq_sphere*)
qed (*auto simp*: *assms dist_norm*)

proposition *connected_open_delete*:
assumes *open S connected S* **and** *2*: *2* ≤ *DIM*('*N*::*euclidean_space*)
shows *connected*(*S* − {*a*::'*N*})
proof (*cases a* ∈ *S*)
case *True*
with ‹*open S*› **obtain** ε **where** ε > *0* **and** ε: *cball a* ε ⊆ *S*
using *open_contains_cball_eq* **by** *blast*
define *b* **where** *b* ≡ *a* + ε *∗*$_R$ (*SOME i*. *i* ∈ *Basis*)
have *dist a b* = ε
by (*simp add*: *b_def dist_norm SOME_Basis* ‹*0* < ε› *less_imp_le*)
with ε **have** *b* ∈ ⋂{*S* − *ball a r* |*r*. *0* < *r* ∧ *r* < ε}
by *auto*
then have *nonemp*: (⋂{*S* − *ball a r* |*r*. *0* < *r* ∧ *r* < ε}) = {} ⟹ *False*
by *auto*
have *con*: ⋀*r*. *r* < ε ⟹ *connected* (*S* − *ball a r*)
using ε **by** (*force intro*: *connected_diff_ball* [*OF* ‹*connected S*› _ *2*])
have *x* ∈ ⋃{*S* − *ball a r* |*r*. *0* < *r* ∧ *r* < ε} **if** *x* ∈ *S* − {*a*} **for** *x*
using *that* ‹*0* < ε›
by (*intro UnionI* [*of S* − *ball a* (*min* ε (*dist a x*) / *2*)]) *auto*
then have *S* − {*a*} = ⋃{*S* − *ball a r* | *r*. *0* < *r* ∧ *r* < ε}
by *auto*
then show *?thesis*
by (*auto intro*: *connected_Union con dest*!: *nonemp*)
next
case *False* **then show** *?thesis*
by (*simp add*: ‹*connected S*›)
qed

corollary *path_connected_open_delete*:
assumes *open S connected S* **and** *2*: *2* ≤ *DIM*('*N*::*euclidean_space*)
shows *path_connected*(*S* − {*a*::'*N*})
by (*simp add*: *assms connected_open_delete connected_open_path_connected open_delete*)

corollary *path_connected_punctured_ball*:
2 ≤ *DIM*('*N*::*euclidean_space*) ⟹ *path_connected*(*ball a r* − {*a*::'*N*})
by (*simp add*: *path_connected_open_delete*)

corollary *connected_punctured_ball*:
2 ≤ *DIM*('*N*::*euclidean_space*) ⟹ *connected*(*ball a r* − {*a*::'*N*})
by (*simp add*: *connected_open_delete*)

corollary *connected_open_delete_finite*:
fixes *S T*::'*a*::*euclidean_space set*

```
  assumes S: open S connected S and 2: 2 ≤ DIM('a) and finite T
  shows connected(S − T)
  using ‹finite T› S
proof (induct T)
  case empty
  show ?case using ‹connected S› by simp
next
  case (insert x T)
  then have connected (S−T)
    by auto
  moreover have open (S − T)
    using finite_imp_closed[OF ‹finite T›] ‹open S› by auto
  ultimately have connected (S − T − {x})
    using connected_open_delete[OF _ _ 2] by auto
  thus ?case by (metis Diff_insert)
qed

lemma sphere_1D_doubleton_zero:
  assumes 1: DIM('a) = 1 and r > 0
  obtains x y::'a::euclidean_space
    where sphere 0 r = {x,y} ∧ dist x y = 2∗r
proof −
  obtain b::'a where b: Basis = {b}
    using 1 card_1_singletonE by blast
  show ?thesis
  proof (intro that conjI)
    have x = norm x ∗R b ∨ x = − norm x ∗R b if r = norm x for x
    proof −
      have xb: (x • b) ∗R b = x
        using euclidean_representation [of x, unfolded b] by force
      then have norm ((x • b) ∗R b) = norm x
        by simp
      with b have |x • b| = norm x
        using norm_Basis by (simp add: b)
      with xb show ?thesis
        by (metis (mono_tags, opaque_lifting) abs_eq_iff abs_norm_cancel)
    qed
    with ‹r > 0› b show sphere 0 r = {r ∗R b, − r ∗R b}
      by (force simp: sphere_def dist_norm)
    have dist (r ∗R b) (− r ∗R b) = norm (r ∗R b + r ∗R b)
      by (simp add: dist_norm)
    also have ... = norm ((2∗r) ∗R b)
      by (metis mult_2 scaleR_add_left)
    also have ... = 2∗r
      using ‹r > 0› b norm_Basis by fastforce
    finally show dist (r ∗R b) (− r ∗R b) = 2∗r .
  qed
qed
```

lemma *sphere_1D_doubleton*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *DIM('a)* = *1* **and** *r* > *0*
obtains *x y* **where** *sphere a r* = {*x*,*y*} ∧ *dist x y* = *2*∗*r*
using *sphere_1D_doubleton_zero* [*OF assms*] *dist_add_cancel image_empty image_insert*
by (*metis* (*no_types*, *opaque_lifting*) *add.right_neutral sphere_translation*)

lemma *psubset_sphere_Compl_connected*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S*: *S* ⊂ *sphere a r* **and** *0* < *r* **and** *2*: *2* ≤ *DIM('a)*
shows *connected*(− *S*)
proof −
have *S* ⊆ *sphere a r*
using *S* **by** *blast*
obtain *b* **where** *dist a b* = *r* **and** *b* ∉ *S*
using *S mem_sphere* **by** *blast*
have *CS*: − *S* = {*x*. *dist a x* ≤ *r* ∧ (*x* ∉ *S*)} ∪ {*x*. *r* ≤ *dist a x* ∧ (*x* ∉ *S*)}
by *auto*
have {*x*. *dist a x* ≤ *r* ∧ *x* ∉ *S*} ∩ {*x*. *r* ≤ *dist a x* ∧ *x* ∉ *S*} ≠ {}
using ‹*b* ∉ *S*› ‹*dist a b* = *r*› **by** *blast*
moreover have *connected* {*x*. *dist a x* ≤ *r* ∧ *x* ∉ *S*}
using *assms*
by (*force intro*: *connected_intermediate_closure* [*of ball a r*])
moreover have *connected* {*x*. *r* ≤ *dist a x* ∧ *x* ∉ *S*}
proof (*rule connected_intermediate_closure* [*of* − *cball a r*])
show {*x*. *r* ≤ *dist a x* ∧ *x* ∉ *S*} ⊆ *closure* (− *cball a r*)
using *interior_closure* **by** (*force intro*: *connected_complement_bounded_convex*)
qed (*use assms connected_complement_bounded_convex* **in** *auto*)
ultimately show *?thesis*
by (*simp add*: *CS connected_Un*)
qed

6.1.23 Every annulus is a connected set

lemma *path_connected_2DIM_I*:
fixes *a* :: *'N*::*euclidean_space*
assumes *2*: *2* ≤ *DIM('N)* **and** *pc*: *path_connected* {*r*. *0* ≤ *r* ∧ *P r*}
shows *path_connected* {*x*. *P*(*norm*(*x* − *a*))}
proof −
have {*x*. *P*(*norm*(*x* − *a*))} = (+) *a* ‘ {*x*. *P*(*norm x*)}
by *force*
moreover have *path_connected* {*x*::*'N*. *P*(*norm x*)}
proof −
let *?D* = {*x*. *0* ≤ *x* ∧ *P x*} × *sphere* (*0*::*'N*) *1*
have *x* ∈ (λ*z*. *fst z* ∗*R* *snd z*) ‘ *?D*
if *P* (*norm x*) **for** *x*::*'N*
proof (*cases x*=*0*)
case *True*

```
    with that show ?thesis
      apply (simp add: image_iff)
    by (metis (no_types) mem_sphere_0 order_refl vector_choose_size zero_le_one)
  next
    case False
    with that show ?thesis
      by (rule_tac x=(norm x, x /R norm x) in image_eqI) auto
  qed
  then have *: {x::'N. P(norm x)} = (λz. fst z *R snd z) ` ?D
    by auto
  have continuous_on ?D (λz:: real×'N. fst z *R snd z)
    by (intro continuous_intros)
  moreover have path_connected ?D
    by (metis path_connected_Times [OF pc] path_connected_sphere 2)
  ultimately show ?thesis
    by (simp add: * path_connected_continuous_image)
 qed
 ultimately show ?thesis
  using path_connected_translation by metis
qed

proposition path_connected_annulus:
 fixes a :: 'N::euclidean_space
 assumes 2 ≤ DIM('N)
 shows path_connected {x. r1 < norm(x − a) ∧ norm(x − a) < r2}
       path_connected {x. r1 < norm(x − a) ∧ norm(x − a) ≤ r2}
       path_connected {x. r1 ≤ norm(x − a) ∧ norm(x − a) < r2}
       path_connected {x. r1 ≤ norm(x − a) ∧ norm(x − a) ≤ r2}
 by (auto simp: is_interval_def intro!: is_interval_convex convex_imp_path_connected
path_connected_2DIM_I [OF assms])

proposition connected_annulus:
 fixes a :: 'N::euclidean_space
 assumes 2 ≤ DIM('N::euclidean_space)
 shows connected {x. r1 < norm(x − a) ∧ norm(x − a) < r2}
       connected {x. r1 < norm(x − a) ∧ norm(x − a) ≤ r2}
       connected {x. r1 ≤ norm(x − a) ∧ norm(x − a) < r2}
       connected {x. r1 ≤ norm(x − a) ∧ norm(x − a) ≤ r2}
 by (auto simp: path_connected_annulus [OF assms] path_connected_imp_connected)
```

6.1.24 Relations between components and path components

```
lemma open_connected_component:
 fixes S :: 'a::real_normed_vector set
 assumes open S
 shows open (connected_component_set S x)
proof (clarsimp simp: open_contains_ball)
 fix y
 assume xy: connected_component S x y
```

then obtain e **where** $e>0$ *ball* y $e \subseteq S$
using *assms connected_component_in openE* **by** *blast*
then show $\exists e>0.$ *ball* y $e \subseteq$ *connected_component_set* S x
by (*metis xy centre_in_ball connected_ball connected_component_eq_eq connected_component_in connected_component_maximal*)
qed

corollary *open_components*:
fixes S :: *'a::real_normed_vector set*
shows $[\![$*open* u; $S \in$ *components* $u]\!] \Longrightarrow$ *open* S
by (*simp add*: *components_iff*) (*metis open_connected_component*)

lemma *in_closure_connected_component*:
fixes S :: *'a::real_normed_vector set*
assumes x: $x \in S$ **and** S: *open* S
shows $x \in$ *closure* (*connected_component_set* S y) $\longleftrightarrow$ $x \in$ *connected_component_set* S y
proof −
have x *islimpt connected_component_set* S $y \Longrightarrow$ *connected_component* S y x
by (*metis* (*no_types*, *lifting*) S *connected_component_eq connected_component_refl islimptE mem_Collect_eq open_connected_component* x)
then show *?thesis*
by (*auto simp*: *closure_def*)
qed

lemma *connected_disjoint_Union_open_pick*:
assumes *pairwise disjnt* B
$\bigwedge S.\ S \in A \Longrightarrow$ *connected* $S \wedge S \neq \{\}$
$\bigwedge S.\ S \in B \Longrightarrow$ *open* S
$\bigcup A \subseteq \bigcup B$
$S \in A$
obtains T **where** $T \in B$ $S \subseteq T$ $S \cap \bigcup (B - \{T\}) = \{\}$
proof −
have $S \subseteq \bigcup B$ *connected* S $S \neq \{\}$
using *assms* ‹$S \in A$› **by** *blast+*
then obtain T **where** $T \in B$ $S \cap T \neq \{\}$
by (*metis Sup_inf_eq_bot_iff inf.absorb_iff2 inf_commute*)
have *1*: *open* T **by** (*simp add*: ‹$T \in B$› *assms*)
have *2*: *open* $(\bigcup (B-\{T\}))$ **using** *assms* **by** *blast*
have *3*: $S \subseteq T \cup \bigcup (B - \{T\})$ **using** ‹$S \subseteq \bigcup B$› **by** *blast*
have $T \cap \bigcup (B - \{T\}) = \{\}$ **using** ‹$T \in B$› ‹*pairwise disjnt B*›
by (*auto simp*: *pairwise_def disjnt_def*)
then have *4*: $T \cap \bigcup (B - \{T\}) \cap S = \{\}$ **by** *auto*
from *connectedD* [*OF* ‹*connected S*› *1 2 4 3*]
have $S \cap \bigcup (B-\{T\}) = \{\}$
by (*auto simp*: *Int_commute* ‹$S \cap T \neq \{\}$›)
with ‹$T \in B$› *3 that* **show** *?thesis*
by (*metis IntI UnE empty_iff subsetD subsetI*)
qed

lemma *connected_disjoint_Union_open_subset*:
  assumes *A*: *pairwise disjnt A* **and** *B*: *pairwise disjnt B*
    and *SA*: $\bigwedge S.\ S \in A \Longrightarrow \mathit{open}\ S \wedge \mathit{connected}\ S \wedge S \neq \{\}$
    and *SB*: $\bigwedge S.\ S \in B \Longrightarrow \mathit{open}\ S \wedge \mathit{connected}\ S \wedge S \neq \{\}$
    and *eq* [*simp*]: $\bigcup A = \bigcup B$
   shows $A \subseteq B$
proof
  fix *S*
  assume $S \in A$
  obtain *T* **where** $T \in B\ S \subseteq T\ S \cap \bigcup (B - \{T\}) = \{\}$
    using *SA SB* ‹$S \in A$› *connected_disjoint_Union_open_pick* [*OF B*, *of A*] *eq order_refl* **by** *blast*
  moreover obtain S' **where** $S' \in A\ T \subseteq S'\ T \cap \bigcup (A - \{S'\}) = \{\}$
    using *SA SB* ‹$T \in B$› *connected_disjoint_Union_open_pick* [*OF A*, *of B*] *eq order_refl* **by** *blast*
  ultimately have $S' = S$
    by (*metis A Int_subset_iff SA* ‹$S \in A$› *disjnt_def inf.orderE pairwise_def*)
  with ‹$T \subseteq S'$› **have** $T \subseteq S$ **by** *simp*
  with ‹$S \subseteq T$› **have** $S = T$ **by** *blast*
  with ‹$T \in B$› **show** $S \in B$ **by** *simp*
qed

lemma *connected_disjoint_Union_open_unique*:
  assumes *A*: *pairwise disjnt A* **and** *B*: *pairwise disjnt B*
    and *SA*: $\bigwedge S.\ S \in A \Longrightarrow \mathit{open}\ S \wedge \mathit{connected}\ S \wedge S \neq \{\}$
    and *SB*: $\bigwedge S.\ S \in B \Longrightarrow \mathit{open}\ S \wedge \mathit{connected}\ S \wedge S \neq \{\}$
    and *eq* [*simp*]: $\bigcup A = \bigcup B$
   shows $A = B$
by (*metis subset_antisym connected_disjoint_Union_open_subset assms*)

proposition *components_open_unique*:
 fixes $S :: 'a{::}\mathit{real\_normed\_vector}\ \mathit{set}$
  assumes *pairwise disjnt A* $\bigcup A = S$
        $\bigwedge X.\ X \in A \Longrightarrow \mathit{open}\ X \wedge \mathit{connected}\ X \wedge X \neq \{\}$
   shows *components S* $= A$
proof −
  have *open S* **using** *assms* **by** *blast*
  show *?thesis*
  proof (*rule connected_disjoint_Union_open_unique*)
    show *disjoint* (*components S*)
      by (*simp add*: *components_eq disjnt_def pairwise_def*)
 qed (*use* ‹*open S*› **in** ‹*simp_all add*: *assms open_components in_components_connected in_components_nonempty*›)
qed

6.1.25 Existence of unbounded components

lemma *cobounded_unbounded_component*:

```
    fixes S :: 'a :: euclidean_space set
    assumes bounded (-S)
      shows ∃x. x ∈ S ∧ ¬ bounded (connected_component_set S x)
proof -
  obtain i::'a where i: i ∈ Basis
    using nonempty_Basis by blast
  obtain B where B: B>0 -S ⊆ ball 0 B
    using bounded_subset_ballD [OF assms, of 0] by auto
  then have *: ⋀x. B ≤ norm x ⟹ x ∈ S
    by (force simp: ball_def dist_norm)
  have unbounded_inner: ¬ bounded {x. inner i x ≥ B}
  proof (clarsimp simp: bounded_def dist_norm)
    fix e x
    show ∃y. B ≤ i • y ∧ ¬ norm (x - y) ≤ e
      using i
      by (rule_tac x=x + (max B e + 1 + |i • x|) *R i in exI) (auto simp:
inner_right_distrib)
  qed
  have §: ⋀x. B ≤ i • x ⟹ x ∈ S
     using * Basis_le_norm [OF i] by (metis abs_ge_self inner_commute or-
der_trans)
  have {x. B ≤ i • x} ⊆ connected_component_set S (B *R i)
   by (intro connected_component_maximal) (auto simp: i intro: convex_connected
convex_halfspace_ge [of B] §)
  then have ¬ bounded (connected_component_set S (B *R i))
    using bounded_subset unbounded_inner by blast
  moreover have B *R i ∈ S
    by (rule *) (simp add: norm_Basis [OF i])
  ultimately show ?thesis
    by blast
qed

lemma cobounded_unique_unbounded_component:
    fixes S :: 'a :: euclidean_space set
    assumes bs: bounded (-S) and 2 ≤ DIM('a)
        and bo: ¬ bounded(connected_component_set S x)
                ¬ bounded(connected_component_set S y)
      shows connected_component_set S x = connected_component_set S y
proof -
  obtain i::'a where i: i ∈ Basis
    using nonempty_Basis by blast
  obtain B where B>0 and B: -S ⊆ ball 0 B
    using bounded_subset_ballD [OF bs, of 0] by auto
  then have *: ⋀x. B ≤ norm x ⟹ x ∈ S
    by (force simp: ball_def dist_norm)
  obtain x' y' where x': connected_component S x x' norm x' > B
  and  y': connected_component S y y' norm y' > B
    using ‹B>0› bo bounded_pos by (metis linorder_not_le mem_Collect_eq)
  have x'y': connected_component S x' y'
```

unfolding *connected_component_def*
proof (*intro exI conjI*)
show *connected* (− *ball 0 B* :: *'a set*)
using *assms* **by** (*auto intro*: *connected_complement_bounded_convex*)
qed (*use x' y' dist_norm* ∗ **in** *auto*)
show *?thesis*
using *x' y' x'y'*
by (*metis connected_component_eq mem_Collect_eq*)
qed

lemma *cobounded_unbounded_components*:
fixes *S* :: *'a* :: *euclidean_space set*
shows *bounded* (−*S*) ⟹ ∃ *c*. *c* ∈ *components S* ∧ ¬*bounded c*
by (*metis cobounded_unbounded_component components_def imageI*)

lemma *cobounded_unique_unbounded_components*:
fixes *S* :: *'a* :: *euclidean_space set*
shows ⟦*bounded* (− *S*); *c* ∈ *components S*; ¬ *bounded c*; *c'* ∈ *components S*;
¬ *bounded c'*; *2* ≤ *DIM*(*'a*)⟧ ⟹ *c'* = *c*
unfolding *components_iff*
by (*metis cobounded_unique_unbounded_component*)

lemma *cobounded_has_bounded_component*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *bounded* (− *S*) ¬ *connected S 2* ≤ *DIM*(*'a*)
obtains *C* **where** *C* ∈ *components S bounded C*
by (*meson cobounded_unique_unbounded_components connected_eq_connected_components_eq assms*)

6.1.26 The *inside* and *outside* of a Set

The inside comprises the points in a bounded connected component of the set's complement. The outside comprises the points in unbounded connected component of the complement.

definition *inside* **where**
inside S ≡ {*x*. (*x* ∉ *S*) ∧ *bounded*(*connected_component_set* (− *S*) *x*)}

definition *outside* **where**
outside S ≡ −*S* ∩ {*x*. ¬ *bounded*(*connected_component_set* (− *S*) *x*)}

lemma *outside*: *outside S* = {*x*. ¬ *bounded*(*connected_component_set* (− *S*) *x*)}
by (*auto simp*: *outside_def*) (*metis Compl_iff bounded_empty connected_component_eq_empty*)

lemma *inside_no_overlap* [*simp*]: *inside S* ∩ *S* = {}
by (*auto simp*: *inside_def*)

lemma *outside_no_overlap* [*simp*]:
outside S ∩ *S* = {}
by (*auto simp*: *outside_def*)

lemma *inside_Int_outside* [*simp*]: *inside S* ∩ *outside S* = {}
by (*auto simp*: *inside_def outside_def*)

lemma *inside_Un_outside* [*simp*]: *inside S* ∪ *outside S* = (− *S*)
by (*auto simp*: *inside_def outside_def*)

lemma *inside_eq_outside*:
inside S = *outside S* ⟷ *S* = *UNIV*
by (*auto simp*: *inside_def outside_def*)

lemma *inside_outside*: *inside S* = (− (*S* ∪ *outside S*))
by (*force simp*: *inside_def outside*)

lemma *outside_inside*: *outside S* = (− (*S* ∪ *inside S*))
by (*auto simp*: *inside_outside*) (*metis IntI equals0D outside_no_overlap*)

lemma *union_with_inside*: *S* ∪ *inside S* = − *outside S*
by (*auto simp*: *inside_outside*) (*simp add*: *outside_inside*)

lemma *union_with_outside*: *S* ∪ *outside S* = − *inside S*
by (*simp add*: *inside_outside*)

lemma *outside_mono*: *S* ⊆ *T* ⟹ *outside T* ⊆ *outside S*
by (*auto simp*: *outside bounded_subset connected_component_mono*)

lemma *inside_mono*: *S* ⊆ *T* ⟹ *inside S* − *T* ⊆ *inside T*
by (*auto simp*: *inside_def bounded_subset connected_component_mono*)

lemma *segment_bound_lemma*:
fixes *u*::*real*
assumes *x* ≥ *B y* ≥ *B* 0 ≤ *u u* ≤ 1
shows (1 − *u*) ∗ *x* + *u* ∗ *y* ≥ *B*
by (*smt* (*verit*) *assms convex_bound_le ge_iff_diff_ge_0 minus_add_distrib*
mult_minus_right neg_le_iff_le)

lemma *cobounded_outside*:
fixes *S* :: *'a* :: *real_normed_vector set*
assumes *bounded S* **shows** *bounded* (− *outside S*)
proof −
obtain *B* **where** *B*: *B*>0 *S* ⊆ *ball* 0 *B*
using *bounded_subset_ballD* [*OF assms*, *of* 0] **by** *auto*
{ fix *x*::*'a* **and** *C*::*real*
assume *Bno*: *B* ≤ *norm x* **and** *C*: 0 < *C*
have ∃ *y*. *connected_component* (− *S*) *x y* ∧ *norm y* > *C*
proof (*cases x* = 0)
case *True* **with** *B Bno* **show** *?thesis* **by** *force*
next
case *False*

```
      have closed_segment x (((B + C) / norm x) *R x) ⊆ - ball 0 B
      proof
        fix w
        assume w ∈ closed_segment x (((B + C) / norm x) *R x)
        then obtain u where
          w: w = (1 - u + u * (B + C) / norm x) *R x 0 ≤ u u ≤ 1
         by (auto simp add: closed_segment_def real_vector_class.scaleR_add_left
[symmetric])
        with False B C have B ≤ (1 - u) * norm x + u * (B + C)
          using segment_bound_lemma [of B norm x B + C u] Bno
          by simp
        with False B C show w ∈ - ball 0 B
          using distrib_right [of _ _ norm x]
          by (simp add: ball_def w not_less)
      qed
      also have ... ⊆ -S
        by (simp add: B)
      finally have ∃ T. connected T ∧ T ⊆ - S ∧ x ∈ T ∧ ((B + C) / norm x)
*R x ∈ T
        by (rule_tac x=closed_segment x (((B+C)/norm x) *R x) in exI) simp
      with False B
      show ?thesis
      by (rule_tac x=((B+C)/norm x) *R x in exI) (simp add: connected_component_def)
    qed
  }
  then show ?thesis
    apply (simp add: outside_def assms)
    apply (rule bounded_subset [OF bounded_ball [of 0 B]])
    apply (force simp: dist_norm not_less bounded_pos)
    done
qed

lemma unbounded_outside:
    fixes S :: 'a::{real_normed_vector, perfect_space} set
    shows bounded S ⟹ ¬ bounded(outside S)
  using cobounded_imp_unbounded cobounded_outside by blast

lemma bounded_inside:
    fixes S :: 'a::{real_normed_vector, perfect_space} set
    shows bounded S ⟹ bounded(inside S)
  by (simp add: bounded_Int cobounded_outside inside_outside)

lemma connected_outside:
    fixes S :: 'a::euclidean_space set
    assumes bounded S 2 ≤ DIM('a)
    shows connected(outside S)
  apply (clarsimp simp add: connected_iff_connected_component outside)
  apply (rule_tac S=connected_component_set (- S) x in connected_component_of_subset)
  apply (metis (no_types) assms cobounded_unbounded_component cobounded_unique_unbounded_com
```

connected_component_eq_eq connected_component_idemp double_complement mem_Collect_eq)
by (*simp add*: *Collect_mono connected_component_eq*)

lemma *outside_connected_component_lt*:
outside S = {*x*. ∀ *B*. ∃ *y*. *B* < *norm*(*y*) ∧ *connected_component* (− *S*) *x y*}
proof −
have ⋀*x B*. *x* ∈ *outside S* ⟹ ∃ *y*. *B* < *norm y* ∧ *connected_component* (− *S*) *x y*
by (*metis boundedI linorder_not_less mem_Collect_eq outside*)
moreover
have ⋀*x*. ∀ *B*. ∃ *y*. *B* < *norm y* ∧ *connected_component* (− *S*) *x y* ⟹ *x* ∈ *outside S*
by (*metis bounded_pos linorder_not_less mem_Collect_eq outside*)
ultimately show *?thesis* **by** *auto*
qed

lemma *outside_connected_component_le*:
outside S = {*x*. ∀ *B*. ∃ *y*. *B* ≤ *norm*(*y*) ∧ *connected_component* (− *S*) *x y*}
apply (*simp add*: *outside_connected_component_lt Set.set_eq_iff*)
by (*meson gt_ex leD le_less_linear less_imp_le order.trans*)

lemma *not_outside_connected_component_lt*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S*: *bounded S* **and** *2* ≤ *DIM*(*'a*)
shows − (*outside S*) = {*x*. ∀ *B*. ∃ *y*. *B* < *norm*(*y*) ∧ ¬ *connected_component* (− *S*) *x y*}
proof −
obtain *B*::*real* **where** *B*: *0* < *B* **and** *Bno*: ⋀*x*. *x* ∈ *S* ⟹ *norm x* ≤ *B*
using *S* [*simplified bounded_pos*] **by** *auto*
have *cyz*: *connected_component* (− *S*) *y z*
if *yz*: *B* < *norm z B* < *norm y* **for** *y*::*'a* **and** *z*::*'a*
proof −
have *connected_component* (− *cball 0 B*) *y z*
using *assms yz*
by (*force simp*: *dist_norm intro*: *connected_componentI* [*OF* _ *subset_refl*] *connected_complement_bounded_convex*)
then show *?thesis*
by (*metis connected_component_of_subset Bno Compl_anti_mono mem_cball_0 subset_iff*)
qed
show *?thesis*
apply (*auto simp*: *outside bounded_pos*)
apply (*metis Compl_iff bounded_iff cobounded_imp_unbounded mem_Collect_eq not_le*)
by (*metis B connected_component_trans cyz not_le*)
qed

lemma *not_outside_connected_component_le*:
fixes *S* :: *'a*::*euclidean_space set*

assumes S: *bounded* S $2 \leq DIM('a)$
shows $-$ (*outside* S) = $\{x.\ \forall B.\ \exists y.\ B \leq norm(y) \wedge \neg$ *connected_component* $(-$ $S)$ x $y\}$
unfolding *not_outside_connected_component_lt* [*OF assms*]
by (*metis* (*no_types*, *opaque_lifting*) *dual_order.strict_trans1 gt_ex pinf*(*8*))

lemma *inside_connected_component_lt*:
fixes S :: $'a$::*euclidean_space set*
assumes S: *bounded* S $2 \leq DIM('a)$
shows *inside* $S = \{x.\ (x \notin S) \wedge (\forall B.\ \exists y.\ B < norm(y) \wedge \neg$ *connected_component* $(-$ $S)$ x $y)\}$
by (*auto simp*: *inside_outside not_outside_connected_component_lt* [*OF assms*])

lemma *inside_connected_component_le*:
fixes S :: $'a$::*euclidean_space set*
assumes S: *bounded* S $2 \leq DIM('a)$
shows *inside* $S = \{x.\ (x \notin S) \wedge (\forall B.\ \exists y.\ B \leq norm(y) \wedge \neg$ *connected_component* $(-$ $S)$ x $y)\}$
by (*auto simp*: *inside_outside not_outside_connected_component_le* [*OF assms*])

lemma *inside_subset*:
assumes *connected* U **and** $\neg$ *bounded* U **and** $T \cup U = -$ S
shows *inside* $S \subseteq T$
using *bounded_subset* [*of connected_component_set* $(-$ $S)$ _ U] *assms*
by (*metis* (*no_types*, *lifting*) *ComplI Un_iff connected_component_maximal inside_def mem_Collect_eq subsetI*)

lemma *frontier_not_empty*:
fixes S :: $'a$ :: *real_normed_vector set*
shows $[\![S \neq \{\}; S \neq UNIV]\!] \Longrightarrow$ *frontier* $S \neq \{\}$
using *connected_Int_frontier* [*of UNIV S*] **by** *auto*

lemma *frontier_eq_empty*:
fixes S :: $'a$ :: *real_normed_vector set*
shows *frontier* $S = \{\} \longleftrightarrow S = \{\} \vee S = UNIV$
using *frontier_UNIV frontier_empty frontier_not_empty* **by** *blast*

lemma *frontier_of_connected_component_subset*:
fixes S :: $'a$::*real_normed_vector set*
shows *frontier*(*connected_component_set* S x) $\subseteq$ *frontier* S
proof $-$
{ fix y
assume *y1*: $y \in$ *closure* (*connected_component_set* S x)
and *y2*: $y \notin$ *interior* (*connected_component_set* S x)
have $y \in$ *closure* S
using *y1 closure_mono connected_component_subset* **by** *blast*
moreover have $z \in$ *interior* (*connected_component_set* S x)
if *0* $<$ e *ball* y $e \subseteq$ *interior* S *dist* y $z < e$ **for** e z
proof $-$

```
      have ball y e ⊆ connected_component_set S y
        using connected_component_maximal that interior_subset
        by (metis centre_in_ball connected_ball subset_trans)
      then show ?thesis
        using y1 apply (simp add: closure_approachable open_contains_ball_eq
[OF open_interior])
      by (metis connected_component_eq dist_commute mem_Collect_eq mem_ball
mem_interior subsetD ‹0 < e› y2)
    qed
    then have y ∉ interior S
      using y2 by (force simp: open_contains_ball_eq [OF open_interior])
    ultimately have y ∈ frontier S
      by (auto simp: frontier_def)
  }
  then show ?thesis by (auto simp: frontier_def)
qed

lemma frontier_Union_subset_closure:
  fixes F :: 'a::real_normed_vector set set
  shows frontier(⋃ F) ⊆ closure(⋃ t ∈ F. frontier t)
proof -
  have ∃ y∈F. ∃ y∈frontier y. dist y x < e
       if T ∈ F y ∈ T dist y x < e
          x ∉ interior (⋃ F) 0 < e for x y e T
  proof (cases x ∈ T)
    case True with that show ?thesis
      by (metis Diff_iff Sup_upper closure_subset contra_subsetD dist_self fron-
tier_def interior_mono)
  next
    case False
    have §: closed_segment x y ∩ T ≠ {} closed_segment x y - T ≠ {}
      using ‹y ∈ T› False by blast+
    obtain c where c ∈ closed_segment x y c ∈ frontier T
      using False connected_Int_frontier [OF connected_segment §] by auto
    with that show ?thesis
      by (smt (verit) dist_norm segment_bound1)
  qed
  then show ?thesis
    by (fastforce simp add: frontier_def closure_approachable)
qed

lemma frontier_Union_subset:
  fixes F :: 'a::real_normed_vector set set
  shows finite F ⟹ frontier(⋃ F) ⊆ (⋃ t ∈ F. frontier t)
 by (metis closed_UN closure_closed frontier_Union_subset_closure frontier_closed)

lemma frontier_of_components_subset:
  fixes S :: 'a::real_normed_vector set
  shows C ∈ components S ⟹ frontier C ⊆ frontier S
```

by (*metis Path_Connected.frontier_of_connected_component_subset components_iff*)

lemma *frontier_of_components_closed_complement*:
fixes $S :: 'a::real\_normed\_vector\ set$
shows $[\![closed\ S;\ C \in components\ (-\ S)]\!] \Longrightarrow frontier\ C \subseteq S$
using *frontier_complement frontier_of_components_subset frontier_subset_eq*
by *blast*

lemma *frontier_minimal_separating_closed*:
fixes $S :: 'a::real\_normed\_vector\ set$
assumes *closed S*
and *nconn*: $\neg\ connected(-\ S)$
and *C*: $C \in components\ (-\ S)$
and *conn*: $\bigwedge T.\ [\![closed\ T;\ T \subset S]\!] \Longrightarrow connected(-\ T)$
shows $frontier\ C = S$
proof (*rule ccontr*)
assume $frontier\ C \neq S$
then have $frontier\ C \subset S$
using *frontier_of_components_closed_complement* [*OF* ‹*closed S*› *C*] **by** *blast*
then have $connected(-\ (frontier\ C))$
by (*simp add*: *conn*)
have $\neg\ connected(-\ (frontier\ C))$
unfolding *connected_def not_not*
proof (*intro exI conjI*)
show *open C*
using *C* ‹*closed S*› *open_components* **by** *blast*
show *open* $(-\ closure\ C)$
by *blast*
show $C \cap -\ closure\ C \cap -\ frontier\ C = \{\}$
using *closure_subset* **by** *blast*
show $C \cap -\ frontier\ C \neq \{\}$
using *C* ‹*open C*› *components_eq frontier_disjoint_eq* **by** *fastforce*
show $-\ frontier\ C \subseteq C \cup -\ closure\ C$
by (*simp add*: ‹*open C*› *closed_Compl frontier_closures*)
then show $-\ closure\ C \cap -\ frontier\ C \neq \{\}$
by (*metis C Compl_Diff_eq Un_Int_eq*(*4*) *Un_commute* ‹*frontier C* ⊂ *S*›
‹*open C*› *compl_le_compl_iff frontier_def in_components_subset interior_eq leD*
sup_bot.right_neutral)
qed
then show *False*
using ‹*connected* $(-\ frontier\ C)$› **by** *blast*
qed

lemma *connected_component_UNIV* [*simp*]:
fixes $x :: 'a::real\_normed\_vector$
shows $connected\_component\_set\ UNIV\ x = UNIV$
using *connected_iff_eq_connected_component_set* [*of UNIV*::*'a set*] *connected_UNIV*
by *auto*

lemma *connected_component_eq_UNIV*:
  fixes $x :: 'a$::*real_normed_vector*
  shows *connected_component_set s x* $=$ *UNIV* $\longleftrightarrow$ $s =$ *UNIV*
 using *connected_component_in connected_component_UNIV* **by** *blast*

lemma *components_UNIV* [*simp*]: *components UNIV* $= \{$*UNIV* $:: 'a$::*real_normed_vector set*$\}$
 by (*auto simp*: *components_eq_sing_iff*)

lemma *interior_inside_frontier*:
  fixes $S :: 'a$::*real_normed_vector set*
  assumes *bounded S*
    shows *interior S* $\subseteq$ *inside* (*frontier S*)
proof −
 { fix *x y*
  assume *x*: $x \in$ *interior S* **and** *y*: $y \notin S$
    and *cc*: *connected_component* (− *frontier S*) *x y*
  have *connected_component_set* (− *frontier S*) $x \cap$ *frontier S* $\neq \{\}$
  proof (*rule connected_Int_frontier*; *simp add*: *set_eq_iff*)
    show $\exists u$. *connected_component* (− *frontier S*) *x u* $\wedge$ $u \in S$
      by (*meson cc connected_component_in connected_component_refl_eq interior_subset subsetD x*)
    show $\exists u$. *connected_component* (− *frontier S*) *x u* $\wedge$ $u \notin S$
      using *y cc* **by** *blast*
  qed
  then have *bounded* (*connected_component_set* (− *frontier S*) *x*)
    using *connected_component_in* **by** *auto*
 }
 then show *?thesis*
  using *bounded_subset* [*OF assms*]
  by (*metis* (*no_types*, *lifting*) *Diff_iff frontier_def inside_def mem_Collect_eq subsetI*)
qed

lemma *inside_empty* [*simp*]: *inside* $\{\}$ $= (\{\} :: 'a :: \{$*real_normed_vector*, *perfect_space*$\}$ *set*)
 by (*simp add*: *inside_def*)

lemma *outside_empty* [*simp*]: *outside* $\{\}$ $= (UNIV :: 'a :: \{$*real_normed_vector*, *perfect_space*$\}$ *set*)
 using *inside_empty inside_Un_outside* **by** *blast*

lemma *inside_same_component*:
  $[\![$*connected_component* (− *S*) *x y*; $x \in$ *inside S*$]\!] \Longrightarrow y \in$ *inside S*
 using *connected_component_eq connected_component_in*
 by (*fastforce simp add*: *inside_def*)

lemma *outside_same_component*:
  $[\![$*connected_component* (− *S*) *x y*; $x \in$ *outside S*$]\!] \Longrightarrow y \in$ *outside S*

```
  using connected_component_eq connected_component_in
  by (fastforce simp add: outside_def)

lemma convex_in_outside:
  fixes S :: 'a :: {real_normed_vector, perfect_space} set
  assumes S: convex S and z: z ∉ S
    shows z ∈ outside S
proof (cases S={})
  case True then show ?thesis by simp
next
  case False then obtain a where a ∈ S by blast
  with z have zna: z ≠ a by auto
  { assume bounded (connected_component_set (- S) z)
   with bounded_pos_less obtain B where B>0 and B: ⋀x. connected_component
(- S) z x ⟹ norm x < B
      by (metis mem_Collect_eq)
    define C where C = (B + 1 + norm z) / norm (z-a)
    have C > 0
    using ‹0 < B› zna by (simp add: C_def field_split_simps add_strict_increasing)
    have |norm (z + C *R (z-a)) - norm (C *R (z-a))| ≤ norm z
      by (metis add_diff_cancel norm_triangle_ineq3)
    moreover have norm (C *R (z-a)) > norm z + B
      using zna ‹B>0› by (simp add: C_def le_max_iff_disj)
    ultimately have C: norm (z + C *R (z-a)) > B by linarith
    { fix u::real
     assume u: 0≤u u≤1 and ins: (1 - u) *R z + u *R (z + C *R (z - a)) ∈ S
      then have Cpos: 1 + u * C > 0
        by (meson ‹0 < C› add_pos_nonneg less_eq_real_def zero_le_mult_iff
zero_less_one)
      then have *: (1 / (1 + u * C)) *R z + (u * C / (1 + u * C)) *R z = z
        by (simp add: scaleR_add_left [symmetric] field_split_simps)
      then have False
        using convexD_alt [OF S ‹a ∈ S› ins, of 1/(u*C + 1)] ‹C>0› ‹z ∉ S›
Cpos u
        by (simp add: * field_split_simps)
    } note contra = this
    have connected_component (- S) z (z + C *R (z-a))
    proof (rule connected_componentI [OF connected_segment])
      show closed_segment z (z + C *R (z - a)) ⊆ - S
        using contra by (force simp add: closed_segment_def)
    qed auto
    then have False
      using zna B [of z + C *R (z-a)] C
      by (auto simp: field_split_simps max_mult_distrib_right)
  }
  then show ?thesis
    by (auto simp: outside_def z)
qed
```

lemma *outside_convex*:
fixes $S :: 'a :: \{real\_normed\_vector, perfect\_space\}$ *set*
assumes *convex S*
shows *outside S* = − *S*
by (*metis ComplD assms convex_in_outside equalityI inside_Un_outside subsetI sup.cobounded2*)

lemma *outside_singleton* [*simp*]:
fixes $x :: 'a :: \{real\_normed\_vector, perfect\_space\}$
shows *outside* $\{x\} = -\{x\}$
by (*auto simp*: *outside_convex*)

lemma *inside_convex*:
fixes $S :: 'a :: \{real\_normed\_vector, perfect\_space\}$ *set*
shows *convex S* ⟹ *inside S* = {}
by (*simp add*: *inside_outside outside_convex*)

lemma *inside_singleton* [*simp*]:
fixes $x :: 'a :: \{real\_normed\_vector, perfect\_space\}$
shows *inside* $\{x\} = \{\}$
by (*auto simp*: *inside_convex*)

lemma *outside_subset_convex*:
fixes $S :: 'a :: \{real\_normed\_vector, perfect\_space\}$ *set*
shows ⟦*convex T*; *S* ⊆ *T*⟧ ⟹ − *T* ⊆ *outside S*
using *outside_convex outside_mono* **by** *blast*

lemma *outside_Un_outside_Un*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *S* ∩ *outside*(*T* ∪ *U*) = {}
shows *outside*(*T* ∪ *U*) ⊆ *outside*(*T* ∪ *S*)
proof
fix *x*
assume *x*: *x* ∈ *outside* (*T* ∪ *U*)
have *Y* ⊆ − *S* **if** *connected Y Y* ⊆ − *T Y* ⊆ − *U x* ∈ *Y u* ∈ *Y* **for** *u Y*
proof −
have *Y* ⊆ *connected_component_set* (− (*T* ∪ *U*)) *x*
by (*simp add*: *connected_component_maximal that*)
also have ... ⊆ *outside*(*T* ∪ *U*)
by (*metis* (*mono_tags*, *lifting*) *Collect_mono mem_Collect_eq outside outside_same_component x*)
finally have *Y* ⊆ *outside*(*T* ∪ *U*) **.**
with *assms* **show** *?thesis* **by** *auto*
qed
with *x* **show** *x* ∈ *outside* (*T* ∪ *S*)
by (*simp add*: *outside_connected_component_lt connected_component_def*) *meson*
qed

lemma *outside_frontier_misses_closure*:
  fixes S :: *'a::real_normed_vector set*
  assumes *bounded S*
  shows *outside(frontier S)* $\subseteq -$ *closure S*
using *assms frontier_def interior_inside_frontier outside_inside* **by** *fastforce*

lemma *outside_frontier_eq_complement_closure*:
fixes S :: *'a* :: {*real_normed_vector*, *perfect_space*} *set*
  assumes *bounded S convex S*
    shows *outside(frontier S)* $= -$ *closure S*
by (*metis Diff_subset assms convex_closure frontier_def outside_frontier_misses_closure*
      outside_subset_convex subset_antisym)

lemma *inside_frontier_eq_interior*:
   fixes S :: *'a* :: {*real_normed_vector*, *perfect_space*} *set*
   shows $[\![$*bounded S*; *convex S*$]\!] \Longrightarrow$ *inside(frontier S)* $=$ *interior S*
unfolding *inside_outside outside_frontier_eq_complement_closure*
using *closure_subset interior_subset* **by** (*auto simp*: *frontier_def*)

lemma *open_inside*:
  fixes S :: *'a::real_normed_vector set*
  assumes *closed S*
    shows *open* (*inside S*)
proof −
{ fix x **assume** x: $x \in$ *inside S*
  have *open* (*connected_component_set* (− *S*) *x*)
    using *assms open_connected_component* **by** *blast*
 then obtain *e* **where** *e*: *e>0* **and** *e*: $\bigwedge$*y*. *dist y x* $<$ *e* $\longrightarrow$ *connected_component*
(− *S*) *x y*
    using *dist_not_less_zero*
    apply (*simp add*: *open_dist*)
     by (*metis* (*no_types*, *lifting*) *Compl_iff connected_component_refl_eq inside_def mem_Collect_eq x*)
  then have $\exists$ *e>0*. *ball x e* $\subseteq$ *inside S*
    by (*metis e dist_commute inside_same_component mem_ball subsetI x*)
}
then show *?thesis*
  by (*simp add*: *open_contains_ball*)
qed

lemma *open_outside*:
  fixes S :: *'a::real_normed_vector set*
  assumes *closed S*
    shows *open* (*outside S*)
proof −
{ fix x **assume** x: $x \in$ *outside S*
  have *open* (*connected_component_set* (− *S*) *x*)
    using *assms open_connected_component* **by** *blast*
 then obtain *e* **where** *e*: *e>0* **and** *e*: $\bigwedge$*y*. *dist y x* $<$ *e* $\longrightarrow$ *connected_component*

```
(− S) x y
      using dist_not_less_zero x
      by (auto simp add: open_dist outside_def intro: connected_component_refl)
    then have ∃ e>0. ball x e ⊆ outside S
      by (metis e dist_commute outside_same_component mem_ball subsetI x)
  }
  then show ?thesis
    by (simp add: open_contains_ball)
qed

lemma closure_inside_subset:
  fixes S :: 'a::real_normed_vector set
  assumes closed S
  shows closure(inside S) ⊆ S ∪ inside S
 by (metis assms closure_minimal open_closed open_outside sup.cobounded2 union_with_inside)

lemma frontier_inside_subset:
    fixes S :: 'a::real_normed_vector set
    assumes closed S
      shows frontier(inside S) ⊆ S
 using assms closure_inside_subset frontier_closures frontier_disjoint_eq open_inside
by fastforce

lemma closure_outside_subset:
    fixes S :: 'a::real_normed_vector set
    assumes closed S
      shows closure(outside S) ⊆ S ∪ outside S
 by (metis assms closed_open closure_minimal inside_outside open_inside sup_ge2)

lemma closed_path_image_Un_inside:
  fixes g :: real ⇒ 'a :: real_normed_vector
  assumes path g
  shows   closed (path_image g ∪ inside (path_image g))
 by (simp add: assms closed_Compl closed_path_image open_outside union_with_inside)

lemma frontier_outside_subset:
  fixes S :: 'a::real_normed_vector set
  assumes closed S
  shows frontier(outside S) ⊆ S
  unfolding frontier_def
 by (metis Diff_subset_conv assms closure_outside_subset interior_eq open_outside
sup_aci(1))

lemma inside_complement_unbounded_connected_empty:
    ⟦connected (− S); ¬ bounded (− S)⟧ ⟹ inside S = {}
  using inside_subset by blast

lemma inside_bounded_complement_connected_empty:
    fixes S :: 'a::{real_normed_vector, perfect_space} set
```

shows ⟦*connected* (− *S*); *bounded S*⟧ ⟹ *inside S* = {}
by (*metis inside_complement_unbounded_connected_empty cobounded_imp_unbounded*)

lemma *inside_inside*:
assumes *S* ⊆ *inside T*
shows *inside S* − *T* ⊆ *inside T*
unfolding *inside_def*
proof *clarify*
fix *x*
assume *x*: *x* ∉ *T x* ∉ *S* **and** *bo*: *bounded* (*connected_component_set* (− *S*) *x*)
show *bounded* (*connected_component_set* (− *T*) *x*)
proof (*cases S* ∩ *connected_component_set* (− *T*) *x* = {})
case *True* **then show** *?thesis*
by (*metis bounded_subset* [*OF bo*] *compl_le_compl_iff connected_component_idemp connected_component_mono disjoint_eq_subset_Compl double_compl*)
next
case *False*
then obtain *y* **where** *y*: *y* ∈ *S y* ∈ *connected_component_set* (− *T*) *x*
by (*meson disjoint_iff*)
then have *bounded* (*connected_component_set* (− *T*) *y*)
using *assms* [*unfolded inside_def*] **by** *blast*
with *y* **show** *?thesis*
by (*metis connected_component_eq*)
qed
qed

lemma *inside_inside_subset*: *inside*(*inside S*) ⊆ *S*
using *inside_inside union_with_outside* **by** *fastforce*

lemma *inside_outside_intersect_connected*:
⟦*connected T*; *inside S* ∩ *T* ≠ {}; *outside S* ∩ *T* ≠ {}⟧ ⟹ *S* ∩ *T* ≠ {}
apply (*simp add*: *inside_def outside_def ex_in_conv* [*symmetric*] *disjoint_eq_subset_Compl*, *clarify*)
by (*metis compl_le_swap1 connected_componentI connected_component_eq mem_Collect_eq*)

lemma *outside_bounded_nonempty*:
fixes *S* :: *'a* :: {*real_normed_vector*, *perfect_space*} *set*
assumes *bounded S* **shows** *outside S* ≠ {}
using *assms unbounded_outside* **by** *force*

lemma *outside_compact_in_open*:
fixes *S* :: *'a* :: {*real_normed_vector*,*perfect_space*} *set*
assumes *S*: *compact S* **and** *T*: *open T* **and** *S* ⊆ *T T* ≠ {}
shows *outside S* ∩ *T* ≠ {}
proof −
have *outside S* ≠ {}
by (*simp add*: *compact_imp_bounded outside_bounded_nonempty S*)
with *assms* **obtain** *a b* **where** *a*: *a* ∈ *outside S* **and** *b*: *b* ∈ *T* **by** *auto*
show *?thesis*

```
  proof (cases a ∈ T)
    case True with a show ?thesis by blast
  next
    case False
      have front: frontier T ⊆ − S
        using ‹S ⊆ T› frontier_disjoint_eq T by auto
      { fix γ
        assume path γ and pimg_sbs: path_image γ − {pathfinish γ} ⊆ interior
(− T)
          and pf: pathfinish γ ∈ frontier T and ps: pathstart γ = a
        define c where c = pathfinish γ
        have c ∈ −S unfolding c_def using front pf by blast
        moreover have open (−S) using S compact_imp_closed by blast
        ultimately obtain ε::real where ε > 0 and ε: cball c ε ⊆ −S
          using open_contains_cball[of −S] S by blast
        then obtain d where d ∈ T and d: dist d c < ε
          using closure_approachable [of c T] pf unfolding c_def
          by (metis Diff_iff frontier_def)
        then have d ∈ −S using ε
              using dist_commute by (metis contra_subsetD mem_cball not_le
not_less_iff_gr_or_eq)
        have pimg_sbs_cos: path_image γ ⊆ −S
          using ‹c ∈ − S› ‹S ⊆ T› c_def interior_subset pimg_sbs by fastforce
        have closed_segment c d ≤ cball c ε
          by (metis ‹0 < ε› centre_in_cball closed_segment_subset convex_cball d
dist_commute less_eq_real_def mem_cball)
        with ε have closed_segment c d ⊆ −S by blast
        moreover have con_gcd: connected (path_image γ ∪ closed_segment c d)
        by (rule connected_Un) (auto simp: c_def ‹path γ› connected_path_image)
        ultimately have connected_component (− S) a d
          unfolding connected_component_def using pimg_sbs_cos ps by blast
        then have outside S ∩ T ≠ {}
            using outside_same_component [OF _ a]  by (metis IntI ‹d ∈ T›
empty_iff)
      } note ∗ = this
      have pal: pathstart (linepath a b) ∈ closure (− T)
        by (auto simp: False closure_def)
      show ?thesis
       by (rule exists_path_subpath_to_frontier [OF path_linepath pal _ ∗]) (auto
simp: b)
  qed
qed

lemma inside_inside_compact_connected:
    fixes S :: 'a :: euclidean_space set
    assumes S: closed S and T: compact T and connected T S ⊆ inside T
      shows inside S ⊆ inside T
proof (cases inside T = {})
  case True with assms show ?thesis by auto
```

```
next
  case False
  consider DIM('a) = 1 | DIM('a) ≥ 2
    using antisym not_less_eq_eq by fastforce
  then show ?thesis
  proof cases
    case 1 then show ?thesis
            using connected_convex_1_gen assms False inside_convex by blast
  next
    case 2
    have bounded S
    using assms by (meson bounded_inside bounded_subset compact_imp_bounded)
    then have coms: compact S
      by (simp add: S compact_eq_bounded_closed)
    then have bst: bounded (S ∪ T)
      by (simp add: compact_imp_bounded T)
    then obtain r where 0 < r and r: S ∪ T ⊆ ball 0 r
      using bounded_subset_ballD by blast
    have outst: outside S ∩ outside T ≠ {}
    by (metis bounded_Un bounded_subset bst cobounded_outside disjoint_eq_subset_Compl
unbounded_outside)
    have S ∩ T = {} using assms
      by (metis disjoint_iff_not_equal inside_no_overlap subsetCE)
    moreover have outside S ∩ inside T ≠ {}
      by (meson False assms(4) compact_eq_bounded_closed coms open_inside
outside_compact_in_open T)
    ultimately have inside S ∩ T = {}
      using inside_outside_intersect_connected [OF ‹connected T›, of S]
    by (metis 2 compact_eq_bounded_closed coms connected_outside inf.commute
inside_outside_intersect_connected outst)
    then show ?thesis
      using inside_inside [OF ‹S ⊆ inside T›] by blast
  qed
qed

lemma connected_with_inside:
    fixes S :: 'a :: real_normed_vector set
    assumes S: closed S and cons: connected S
      shows connected(S ∪ inside S)
proof (cases S ∪ inside S = UNIV)
  case True with assms show ?thesis by auto
next
  case False
  then obtain b where b: b ∉ S b ∉ inside S by blast
  have ∗: ∃ y T. y ∈ S ∧ connected T ∧ a ∈ T ∧ y ∈ T ∧ T ⊆ (S ∪ inside S)
    if a ∈ S ∪ inside S for a
    using that
  proof
    assume a ∈ S then show ?thesis
```

```
    using cons by blast
  next
    assume a: a ∈ inside S
    then have ain: a ∈ closure (inside S)
      by (simp add: closure_def)
    obtain h where h: path h pathstart h = a
                      path_image h − {pathfinish h} ⊆ interior (inside S)
                      pathfinish h ∈ frontier (inside S)
      using ain b
     by (metis exists_path_subpath_to_frontier path_linepath pathfinish_linepath
pathstart_linepath)
    moreover
    have h1S: pathfinish h ∈ S
      using S h frontier_inside_subset by blast
    moreover
    have path_image h ⊆ S ∪ inside S
      using IntD1 S h1S h interior_eq open_inside by fastforce
    ultimately show ?thesis by blast
  qed
  show ?thesis
    apply (simp add: connected_iff_connected_component)
    apply (clarsimp simp add: connected_component_def dest!: ∗)
    subgoal for x y u u' T t'
      by (rule_tac x = S ∪ T ∪ t' in exI) (auto intro!: connected_Un cons)
    done
qed
```

The proof is virtually the same as that above.

```
lemma connected_with_outside:
    fixes S :: 'a :: real_normed_vector set
    assumes S: closed S and cons: connected S
      shows connected(S ∪ outside S)
proof (cases S ∪ outside S = UNIV)
  case True with assms show ?thesis by auto
next
  case False
  then obtain b where b: b ∉ S b ∉ outside S by blast
  have ∗: ∃ y T. y ∈ S ∧ connected T ∧ a ∈ T ∧ y ∈ T ∧ T ⊆ (S ∪ outside S)
if a ∈ (S ∪ outside S) for a
  using that proof
    assume a ∈ S then show ?thesis
      by (rule_tac x=a in exI, rule_tac x={a} in exI, simp)
  next
    assume a: a ∈ outside S
    then have ain: a ∈ closure (outside S)
      by (simp add: closure_def)
    obtain h where h: path h pathstart h = a
                      path_image h − {pathfinish h} ⊆ interior (outside S)
                      pathfinish h ∈ frontier (outside S)
```

using *ain b*
by (*metis exists_path_subpath_to_frontier path_linepath pathfinish_linepath pathstart_linepath*)
moreover
have *h1S*: *pathfinish h* ∈ *S*
using *S frontier_outside_subset h*(*4*) **by** *blast*
moreover
have *path_image h* ⊆ *S* ∪ *outside S*
using *IntD1 S h1S h interior_eq open_outside* **by** *fastforce*
ultimately show *?thesis*
by *blast*
qed
show *?thesis*
apply (*simp add*: *connected_iff_connected_component*)
apply (*clarsimp simp add*: *connected_component_def dest*!: ∗)
subgoal for *x y u u′ T t′*
by (*rule_tac x*=(*S* ∪ *T* ∪ *t′*) **in** *exI*) (*auto intro*!: *connected_Un cons*)
done
qed

lemma *inside_inside_eq_empty* [*simp*]:
fixes *S* :: *′a* :: {*real_normed_vector*, *perfect_space*} *set*
assumes *S*: *closed S* **and** *cons*: *connected S*
shows *inside* (*inside S*) = {}
proof −
have *connected* (− *inside S*)
by (*metis S connected_with_outside cons union_with_outside*)
then show *?thesis*
by (*metis bounded_Un inside_complement_unbounded_connected_empty unbounded_outside union_with_outside*)
qed

lemma *inside_in_components*:
inside S ∈ *components* (− *S*) ⟷ *connected*(*inside S*) ∧ *inside S* ≠ {} (**is** *?lhs* = *?rhs*)
proof
assume *R*: *?rhs*
then have ⋀*x*. ⟦*x* ∈ *S*; *x* ∈ *inside S*⟧ ⟹ ¬ *connected* (*inside S*)
by (*simp add*: *inside_outside*)
with *R* **show** *?lhs*
unfolding *in_components_maximal*
by (*auto intro*: *inside_same_component connected_componentI*)
qed (*simp add*: *in_components_maximal*)

The proof is like that above.

lemma *outside_in_components*:
outside S ∈ *components* (− *S*) ⟷ *connected*(*outside S*) ∧ *outside S* ≠ {} (**is** *?lhs* = *?rhs*)
proof

assume *R*: *?rhs*
then have $\bigwedge x.\ [\![x \in S;\ x \in \mathit{outside}\ S]\!] \Longrightarrow \neg\ \mathit{connected}\ (\mathit{outside}\ S)$
by (*meson disjoint_iff outside_no_overlap*)
with *R* **show** *?lhs*
unfolding *in_components_maximal*
by (*auto intro*: *outside_same_component connected_componentI*)
qed (*simp add*: *in_components_maximal*)

lemma *bounded_unique_outside*:
fixes $S :: 'a :: \mathit{euclidean\_space}\ \mathit{set}$
assumes *bounded S DIM*$('a) \geq 2$
shows $(c \in \mathit{components}\ (-S) \wedge \neg\ \mathit{bounded}\ c) \longleftrightarrow c = \mathit{outside}\ S$
using *assms*
by (*metis cobounded_unique_unbounded_components connected_outside double_compl outside_bounded_nonempty*
outside_in_components unbounded_outside)

6.1.27 Condition for an open map's image to contain a ball

proposition *ball_subset_open_map_image*:
fixes $f :: 'a{::}\mathit{heine\_borel} \Rightarrow 'b :: \{\mathit{real\_normed\_vector}, \mathit{heine\_borel}\}$
assumes *contf*: *continuous_on* (*closure S*) *f*
and *oint*: *open* (*f ‘ interior S*)
and *le_no*: $\bigwedge z.\ z \in \mathit{frontier}\ S \Longrightarrow r \leq \mathit{norm}(f\ z - f\ a)$
and *bounded S a* $\in S$ $0 < r$
shows *ball* (*f a*) *r* $\subseteq$ *f ‘ S*
proof (*cases f ‘ S* $=$ *UNIV*)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
then have *closed* (*frontier* (*f ‘ S*)) *frontier* (*f ‘ S*) $\neq \{\}$
using ‹*a* $\in$ *S*› **by** (*auto simp*: *frontier_eq_empty*)
then obtain *w* **where** *w*: *w* $\in$ *frontier* (*f ‘ S*)
and *dw_le*: $\bigwedge y.\ y \in \mathit{frontier}\ (f\ `\ S) \Longrightarrow \mathit{norm}\ (f\ a - w) \leq \mathit{norm}\ (f\ a - y)$
by (*auto simp add*: *dist_norm intro*: *distance_attains_inf* [*of frontier*(*f ‘ S*) *f a*])
then obtain ξ **where** ξ: $\bigwedge n.\ \xi\ n \in f\ `\ S$ **and** *tendsw*: $\xi \longrightarrow w$
by (*metis Diff_iff frontier_def closure_sequential*)
then have $\bigwedge n.\ \exists x \in S.\ \xi\ n = f\ x$ **by** *force*
then obtain *z* **where** *zs*: $\bigwedge n.\ z\ n \in S$ **and** *fz*: $\bigwedge n.\ \xi\ n = f\ (z\ n)$
by *metis*
then obtain *y K* **where** *y*: *y* $\in$ *closure S* **and** *strict_mono* $(K :: \mathit{nat} \Rightarrow \mathit{nat})$
and *Klim*: $(z \circ K) \longrightarrow y$
using ‹*bounded S*›
unfolding *compact_closure* [*symmetric*] *compact_def* **by** (*meson closure_subset subset_iff*)
then have *ftendsw*: $((\lambda n.\ f\ (z\ n)) \circ K) \longrightarrow w$
by (*metis LIMSEQ_subseq_LIMSEQ fun.map_cong0 fz tendsw*)
have *zKs*: $\bigwedge n.\ (z \circ K)\ n \in S$ **by** (*simp add*: *zs*)

have *fz*: $f \circ z = \xi \ \ (\lambda n.\ f\ (z\ n)) = \xi$
using *fz* **by** *auto*
then have $(\xi \circ K) \longrightarrow f\ y$
by (*metis* (*no_types*) *Klim zKs y contf comp_assoc continuous_on_closure_sequentially*)
with *fz* **have** *wy*: $w = f\ y$ **using** *fz LIMSEQ_unique ftendsw* **by** *auto*
have $r \leq$ *norm* $(f\ y - f\ a)$
proof (*rule le_no*)
show $y \in$ *frontier* S
using *w wy oint* **by** (*force simp*: *imageI image_mono interiorI interior_subset frontier_def y*)
qed
then have $\bigwedge y.\ [\![$*norm* $(f\ a - y) < r$; $y \in$ *frontier* $(f\ `\ S)]\!] \Longrightarrow$ *False*
by (*metis dw_le norm_minus_commute not_less order_trans wy*)
then have *ball* $(f\ a)\ r \cap$ *frontier* $(f\ `\ S) = \{\}$
by (*metis disjoint_iff_not_equal dist_norm mem_ball*)
moreover
have *ball* $(f\ a)\ r \cap f\ `\ S \neq \{\}$
using ‹$a \in S$› ‹$0 < r$› *centre_in_ball* **by** *blast*
ultimately show *?thesis*
by (*meson connected_Int_frontier connected_ball diff_shunt_var*)
qed

Special characterizations of classes of functions into and out of R.

lemma *Hausdorff_space_euclidean* [*simp*]: *Hausdorff_space* (*euclidean* :: 'a::*metric_space topology*)
proof −
have $\exists\, U\ V.$ *open* $U \wedge$ *open* $V \wedge x \in U \wedge y \in V \wedge$ *disjnt* $U\ V$
if $x \neq y$ **for** $x\ y$ :: 'a
proof (*intro exI conjI*)
let *?r* = *dist x y* / *2*
have [*simp*]: *?r* > *0*
by (*simp add*: *that*)
show *open* (*ball x ?r*) *open* (*ball y ?r*) $x \in$ (*ball x ?r*) $y \in$ (*ball y ?r*)
by (*auto simp add*: *that*)
show *disjnt* (*ball x ?r*) (*ball y ?r*)
unfolding *disjnt_def* **by** (*simp add*: *disjoint_ballI*)
qed
then show *?thesis*
by (*simp add*: *Hausdorff_space_def*)
qed

proposition *embedding_map_into_euclideanreal*:
assumes *path_connected_space X*
shows *embedding_map X euclideanreal f* ⟷
continuous_map X euclideanreal f ∧ *inj_on f* (*topspace X*)
proof *safe*
show *continuous_map X euclideanreal f*
if *embedding_map X euclideanreal f*

```
    using continuous_map_in_subtopology homeomorphic_imp_continuous_map
that
    unfolding embedding_map_def by blast
  show inj_on f (topspace X)
    if embedding_map X euclideanreal f
    using that homeomorphic_imp_injective_map
    unfolding embedding_map_def by blast
  show embedding_map X euclideanreal f
    if cont: continuous_map X euclideanreal f and inj: inj_on f (topspace X)
  proof −
    obtain g where gf: ⋀x. x ∈ topspace X ⟹ g (f x) = x
      using inv_into_f_f [OF inj] by auto
    show ?thesis
     unfolding embedding_map_def homeomorphic_map_maps homeomorphic_maps_def
    proof (intro exI conjI)
      show continuous_map X (top_of_set (f ‘ topspace X)) f
        by (simp add: cont continuous_map_in_subtopology)
      let ?S = f ‘ topspace X
      have eq: {x ∈ ?S. g x ∈ U} = f ‘ U if openin X U for U
        using openin_subset [OF that] by (auto simp: gf)
      have 1: g ‘ ?S ⊆ topspace X
        using eq by blast
      have openin (top_of_set ?S) {x ∈ ?S. g x ∈ T}
        if openin X T for T
      proof −
        have T ⊆ topspace X
          by (simp add: openin_subset that)
        have RR: ∀x ∈ ?S ∩ g −‘ T. ∃d>0. ∀x′ ∈ ?S ∩ ball x d. g x′ ∈ T
        proof (clarsimp simp add: gf)
          have pcS: path_connectedin euclidean ?S
         using assms cont path_connectedin_continuous_map_image path_connectedin_topspace
by blast
          show ∃d>0. ∀x′∈f ‘ topspace X ∩ ball (f x) d. g x′ ∈ T
            if x ∈ T for x
          proof −
            have x: x ∈ topspace X
              using ‹T ⊆ topspace X› ‹x ∈ T› by blast
            obtain u v d where 0 < d u ∈ topspace X v ∈ topspace X
                      and sub_fuv: ?S ∩ {f x − d .. f x + d} ⊆ {f u..f v}
            proof (cases ∃u ∈ topspace X. f u < f x)
              case True
              then obtain u where u: u ∈ topspace X f u < f x ..
              show ?thesis
              proof (cases ∃v ∈ topspace X. f x < f v)
                case True
                then obtain v where v: v ∈ topspace X f x < f v ..
                show ?thesis
                proof
                  let ?d = min (f x − f u) (f v − f x)
```

```
      show 0 < ?d
        by (simp add: ‹f u < f x› ‹f x < f v›)
      show f ‘ topspace X ∩ {f x − ?d..f x + ?d} ⊆ {f u..f v}
        by fastforce
    qed (auto simp: u v)
  next
    case False
    show ?thesis
    proof
      let ?d = f x − f u
      show 0 < ?d
        by (simp add: u)
      show f ‘ topspace X ∩ {f x − ?d..f x + ?d} ⊆ {f u..f x}
        using x u False by auto
    qed (auto simp: x u)
  qed
next
  case False
  note no_u = False
  show ?thesis
  proof (cases ∃ v ∈ topspace X. f x < f v)
    case True
    then obtain v where v: v ∈ topspace X f x < f v ..
    show ?thesis
    proof
      let ?d = f v − f x
      show 0 < ?d
        by (simp add: v)
      show f ‘ topspace X ∩ {f x − ?d..f x + ?d} ⊆ {f x..f v}
        using False by auto
    qed (auto simp: x v)
  next
    case False
    show ?thesis
    proof
      show f ‘ topspace X ∩ {f x − 1..f x + 1} ⊆ {f x..f x}
        using False no_u by fastforce
    qed (auto simp: x)
  qed
qed
then obtain h where pathin X h h 0 = u h 1 = v
  using assms unfolding path_connected_space_def by blast
obtain C where compactin X C connectedin X C u ∈ C v ∈ C
proof
  show compactin X (h ‘ {0..1})
    using that by (simp add: ‹pathin X h› compactin_path_image)
  show connectedin X (h ‘ {0..1})
    using ‹pathin X h› connectedin_path_image by blast
qed (use ‹h 0 = u› ‹h 1 = v› in auto)
```

```
        have continuous_map (subtopology euclideanreal (?S ∩ {f x − d .. f x + d})) (subtopology X C) g
         proof (rule continuous_inverse_map)
          show compact_space (subtopology X C)
           using ‹compactin X C› compactin_subspace by blast
          show continuous_map (subtopology X C) euclideanreal f
           by (simp add: cont continuous_map_from_subtopology)
          have {f u .. f v} ⊆ f ‘ topspace (subtopology X C)
          proof (rule connected_contains_Icc)
           show connected (f ‘ topspace (subtopology X C))
            using connectedin_continuous_map_image [OF cont]
                  by (simp add: ‹compactin X C› ‹connectedin X C› compactin_subset_topspace inf_absorb2)
           show f u ∈ f ‘ topspace (subtopology X C)
            by (simp add: ‹u ∈ C› ‹u ∈ topspace X›)
           show f v ∈ f ‘ topspace (subtopology X C)
            by (simp add: ‹v ∈ C› ‹v ∈ topspace X›)
          qed
         then show f ‘ topspace X ∩ {f x − d..f x + d} ⊆ f ‘ topspace (subtopology X C)
           using sub_fuv by blast
         qed (auto simp: gf)
        then have contg: continuous_map (subtopology euclideanreal (?S ∩ {f x − d .. f x + d})) X g
          using continuous_map_in_subtopology by blast
        have ∃e>0. ∀x ∈ ?S ∩ {f x − d .. f x + d} ∩ ball (f x) e. g x ∈ T
         using openin_continuous_map_preimage [OF contg ‹openin X T›] x ‹x ∈ T› ‹0 < d›
          unfolding openin_euclidean_subtopology_iff
          by (force simp: gf dist_commute)
        then obtain e where e > 0 ∧ (∀x∈f ‘ topspace X ∩ {f x − d..f x + d} ∩ ball (f x) e. g x ∈ T)
          by metis
        with ‹0 < d› have min d e > 0 ∀u. u ∈ topspace X ⟶ |f x − f u| < min d e ⟶ u ∈ T
          using dist_real_def gf by force+
        then show ?thesis
          by (metis (full_types) Int_iff dist_real_def image_iff mem_ball gf)
       qed
      qed
      then obtain d where d: ⋀r. r ∈ ?S ∩ g −‘ T ⟹
            d r > 0 ∧ (∀x ∈ ?S ∩ ball r (d r). g x ∈ T)
       by metis
      show ?thesis
       unfolding openin_subtopology
      proof (intro exI conjI)
       show {x ∈ ?S. g x ∈ T} = (⋃r ∈ ?S ∩ g −‘ T. ball r (d r)) ∩ f ‘ topspace X
         using d by (auto simp: gf)
```

qed *auto*
qed
then show *continuous_map* (*top_of_set ?S*) *X g*
by (*simp add*: *1 continuous_map*)
qed (*auto simp*: *gf*)
qed
qed

An injective function into R is a homeomorphism and so an open map.

lemma *injective_into_1d_eq_homeomorphism*:
fixes *f* :: *'a*::*topological_space* ⇒ *real*
assumes *f*: *continuous_on S f* **and** *S*: *path_connected S*
shows *inj_on f S* ⟷ (∃ *g*. *homeomorphism S* (*f* ‘ *S*) *f g*)
proof
show ∃ *g*. *homeomorphism S* (*f* ‘ *S*) *f g*
if *inj_on f S*
proof −
have *embedding_map* (*top_of_set S*) *euclideanreal f*
using *that embedding_map_into_euclideanreal* [*of top_of_set S f*] *assms* **by** *auto*
then show *?thesis*
unfolding *embedding_map_def topspace_euclidean_subtopology*
by (*metis f homeomorphic_map_closedness_eq homeomorphism_injective_closed_map that*)
qed
qed (*metis homeomorphism_def inj_onI*)

lemma *injective_into_1d_imp_open_map*:
fixes *f* :: *'a*::*topological_space* ⇒ *real*
assumes *continuous_on S f path_connected S inj_on f S openin* (*subtopology euclidean S*) *T*
shows *openin* (*subtopology euclidean* (*f* ‘ *S*)) (*f* ‘ *T*)
using *assms homeomorphism_imp_open_map injective_into_1d_eq_homeomorphism* **by** *blast*

lemma *homeomorphism_into_1d*:
fixes *f* :: *'a*::*topological_space* ⇒ *real*
assumes *path_connected S continuous_on S f f* ‘ *S* = *T inj_on f S*
shows ∃ *g*. *homeomorphism S T f g*
using *assms injective_into_1d_eq_homeomorphism* **by** *blast*

6.1.28 Rectangular paths

definition *rectpath* **where**
rectpath a1 a3 = (*let a2* = *Complex* (*Re a3*) (*Im a1*); *a4* = *Complex* (*Re a1*) (*Im a3*)
in linepath a1 a2 +++ *linepath a2 a3* +++ *linepath a3 a4* +++ *linepath a4 a1*)

lemma *path_rectpath* [*simp*, *intro*]: *path* (*rectpath a b*)
by (*simp add*: *Let_def rectpath_def*)

lemma *pathstart_rectpath* [*simp*]: *pathstart* (*rectpath a1 a3*) = *a1*
by (*simp add*: *rectpath_def Let_def*)

lemma *pathfinish_rectpath* [*simp*]: *pathfinish* (*rectpath a1 a3*) = *a1*
by (*simp add*: *rectpath_def Let_def*)

lemma *simple_path_rectpath* [*simp*, *intro*]:
assumes *Re a1* ≠ *Re a3 Im a1* ≠ *Im a3*
shows *simple_path* (*rectpath a1 a3*)
unfolding *rectpath_def Let_def* **using** *assms*
by (*intro simple_path_join_loop arc_join arc_linepath*)
(*auto simp*: *complex_eq_iff path_image_join closed_segment_same_Re closed_segment_same_Im*)

lemma *path_image_rectpath*:
assumes *Re a1* ≤ *Re a3 Im a1* ≤ *Im a3*
shows *path_image* (*rectpath a1 a3*) =
{*z*. *Re z* ∈ {*Re a1*, *Re a3*} ∧ *Im z* ∈ {*Im a1*..*Im a3*}} ∪
{*z*. *Im z* ∈ {*Im a1*, *Im a3*} ∧ *Re z* ∈ {*Re a1*..*Re a3*}} (**is** *?lhs* = *?rhs*)
proof −
define *a2 a4* **where** *a2* = *Complex* (*Re a3*) (*Im a1*) **and** *a4* = *Complex* (*Re a1*) (*Im a3*)
have *?lhs* = *closed_segment a1 a2* ∪ *closed_segment a2 a3* ∪
closed_segment a4 a3 ∪ *closed_segment a1 a4*
by (*simp_all add*: *rectpath_def Let_def path_image_join closed_segment_commute a2_def a4_def Un_assoc*)
also have ... = *?rhs* **using** *assms*
by (*auto simp*: *rectpath_def Let_def path_image_join a2_def a4_def closed_segment_same_Re closed_segment_same_Im closed_segment_eq_real_ivl*)
finally show *?thesis* **.**
qed

lemma *path_image_rectpath_subset_cbox*:
assumes *Re a* ≤ *Re b Im a* ≤ *Im b*
shows *path_image* (*rectpath a b*) ⊆ *cbox a b*
using *assms* **by** (*auto simp*: *path_image_rectpath in_cbox_complex_iff*)

lemma *path_image_rectpath_inter_box*:
assumes *Re a* ≤ *Re b Im a* ≤ *Im b*
shows *path_image* (*rectpath a b*) ∩ *box a b* = {}
using *assms* **by** (*auto simp*: *path_image_rectpath in_box_complex_iff*)

lemma *path_image_rectpath_cbox_minus_box*:
assumes *Re a* ≤ *Re b Im a* ≤ *Im b*
shows *path_image* (*rectpath a b*) = *cbox a b* − *box a b*
using *assms* **by** (*auto simp*: *path_image_rectpath in_cbox_complex_iff in_box_complex_iff*)

end

6.2 Neighbourhood bases and Locally path-connected spaces

theory *Locally*
imports
Path_Connected Function_Topology Sum_Topology
begin

6.2.1 Neighbourhood Bases

Useful for "local" properties of various kinds

definition *neighbourhood_base_at* **where**
neighbourhood_base_at x P X ≡
∀ *W*. *openin X W* ∧ *x* ∈ *W*
⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*)

definition *neighbourhood_base_of* **where**
neighbourhood_base_of P X ≡
∀ *x* ∈ *topspace X*. *neighbourhood_base_at x P X*

lemma *neighbourhood_base_of*:
neighbourhood_base_of P X ⟷
(∀ *W x*. *openin X W* ∧ *x* ∈ *W*
⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*))
unfolding *neighbourhood_base_at_def neighbourhood_base_of_def*
using *openin_subset* **by** *blast*

lemma *neighbourhood_base_at_mono*:
⟦*neighbourhood_base_at x P X*; ⋀*S*. ⟦*P S*; *x* ∈ *S*⟧ ⟹ *Q S*⟧ ⟹ *neighbourhood_base_at x Q X*
unfolding *neighbourhood_base_at_def*
by (*meson subset_eq*)

lemma *neighbourhood_base_of_mono*:
⟦*neighbourhood_base_of P X*; ⋀*S*. *P S* ⟹ *Q S*⟧ ⟹ *neighbourhood_base_of Q X*
unfolding *neighbourhood_base_of_def*
using *neighbourhood_base_at_mono* **by** *force*

lemma *open_neighbourhood_base_at*:
(⋀*S*. ⟦*P S*; *x* ∈ *S*⟧ ⟹ *openin X S*)
⟹ *neighbourhood_base_at x P X* ⟷ (∀ *W*. *openin X W* ∧ *x* ∈ *W* ⟶ (∃ *U*. *P U* ∧ *x* ∈ *U* ∧ *U* ⊆ *W*))
unfolding *neighbourhood_base_at_def*
by (*meson subsetD*)

lemma *open_neighbourhood_base_of*:
(∀ *S*. *P S* ⟶ *openin X S*)
⟹ *neighbourhood_base_of P X* ⟷ (∀ *W x*. *openin X W* ∧ *x* ∈ *W* ⟶ (∃ *U*. *P U* ∧ *x* ∈ *U* ∧ *U* ⊆ *W*))
by (*smt* (*verit*) *neighbourhood_base_of subsetD*)

lemma *neighbourhood_base_of_open_subset*:
⟦*neighbourhood_base_of P X*; *openin X S*⟧
⟹ *neighbourhood_base_of P* (*subtopology X S*)
by (*smt* (*verit*, *ccfv_SIG*) *neighbourhood_base_of openin_open_subtopology subset_trans*)

lemma *neighbourhood_base_of_topology_base*:
openin X = *arbitrary union_of* (λ*W*. *W* ∈ ℬ)
⟹ *neighbourhood_base_of P X* ⟷
(∀ *W x*. *W* ∈ ℬ ∧ *x* ∈ *W* ⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*))
by (*smt* (*verit*, *del_insts*) *neighbourhood_base_of openin_topology_base_unique subset_trans*)

lemma *neighbourhood_base_at_unlocalized*:
assumes ⋀*S T*. ⟦*P S*; *openin X T*; *x* ∈ *T*; *T* ⊆ *S*⟧ ⟹ *P T*
shows *neighbourhood_base_at x P X*
⟷ (*x* ∈ *topspace X* ⟶ (∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *topspace X*))
(**is** *?lhs* = *?rhs*)
proof
assume *R*: *?rhs* **show** *?lhs*
unfolding *neighbourhood_base_at_def*
proof *clarify*
fix *W*
assume *openin X W x* ∈ *W*
then have *x* ∈ *topspace X*
using *openin_subset* **by** *blast*
with *R* **obtain** *U V* **where** *openin X U P V x* ∈ *U U* ⊆ *V V* ⊆ *topspace X*
by *blast*
then show ∃ *U V*. *openin X U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*
by (*metis IntI* ‹*openin X W*› ‹*x* ∈ *W*› *assms inf_le1 inf_le2 openin_Int*)
qed
qed (*simp add*: *neighbourhood_base_at_def*)

lemma *neighbourhood_base_at_with_subset*:
⟦*openin X U*; *x* ∈ *U*⟧
⟹ (*neighbourhood_base_at x P X* ⟷ *neighbourhood_base_at x* (λ*T*. *T* ⊆ *U* ∧ *P T*) *X*)
unfolding *neighbourhood_base_at_def* **by** (*metis IntI Int_subset_iff openin_Int*)

lemma *neighbourhood_base_of_with_subset*:

neighbourhood_base_of P X ⟷ neighbourhood_base_of (λt. t ⊆ topspace X ∧ P t) X
using *neighbourhood_base_at_with_subset*
by (*fastforce simp add*: *neighbourhood_base_of_def*)

6.2.2 Locally path-connected spaces

definition *weakly_locally_path_connected_at*
where *weakly_locally_path_connected_at x X ≡ neighbourhood_base_at x (path_connectedin X) X*

definition *locally_path_connected_at*
where *locally_path_connected_at x X ≡*
neighbourhood_base_at x (λU. openin X U ∧ path_connectedin X U) X

definition *locally_path_connected_space*
where *locally_path_connected_space X ≡ neighbourhood_base_of (path_connectedin X) X*

lemma *locally_path_connected_space_alt*:
locally_path_connected_space X ⟷ neighbourhood_base_of (λU. openin X U ∧ path_connectedin X U) X
(**is** *?P = ?Q*)
and *locally_path_connected_space_eq_open_path_component_of*:
locally_path_connected_space X ⟷
(∀ U x. openin X U ∧ x ∈ U ⟶ openin X (Collect (path_component_of (subtopology X U) x)))
(**is** *?P = ?R*)
proof −
have *?P* **if** *?Q*
using *locally_path_connected_space_def neighbourhood_base_of_mono that*
by *auto*
moreover have *?R* **if** *P*: *?P*
proof −
show *?thesis*
proof *clarify*
fix *U y*
assume *openin X U y ∈ U*
have *∃ T. openin X T ∧ x ∈ T ∧ T ⊆ Collect (path_component_of (subtopology X U) y)*
if *path_component_of (subtopology X U) y x* **for** *x*

proof −
have *x ∈ U*
using *path_component_of_equiv that topspace_subtopology* **by** *fastforce*
then have *∃ Ua. openin X Ua ∧ (∃ V. path_connectedin X V ∧ x ∈ Ua ∧ Ua ⊆ V ∧ V ⊆ U)*
using *P*
by (*simp add*: ‹*openin X U*› *locally_path_connected_space_def neighbour-*

hood_base_of)
then show *?thesis*
by (*metis dual_order.trans path_component_of_equiv path_component_of_maximal path_connectedin_subtopology subset_iff that*)
qed
then show *openin X* (*Collect* (*path_component_of* (*subtopology X U*) *y*))
using *openin_subopen* **by** *force*
qed
qed
moreover have *?Q* **if** *?R*
by (*smt* (*verit*) *mem_Collect_eq open_neighbourhood_base_of openin_subset path_component_of_refl*
path_connectedin_path_component_of path_connectedin_subtopology that topspace_subtopology_subset)
ultimately show *?P = ?Q ?P = ?R*
by *blast+*
qed

lemma *locally_path_connected_space*:
locally_path_connected_space X
$\longleftrightarrow$ ($\forall$ *V x*. *openin X V* $\wedge$ *x* $\in$ *V* $\longrightarrow$ ($\exists$ *U*. *openin X U* $\wedge$ *path_connectedin X U* $\wedge$ *x* $\in$ *U* $\wedge$ *U* $\subseteq$ *V*))
by (*simp add*: *locally_path_connected_space_alt open_neighbourhood_base_of*)

lemma *locally_path_connected_space_open_path_components*:
locally_path_connected_space X $\longleftrightarrow$
($\forall$ *U C*. *openin X U* $\wedge$ *C* $\in$ *path_components_of*(*subtopology X U*) $\longrightarrow$ *openin X C*)
apply (*simp add*: *locally_path_connected_space_eq_open_path_component_of path_components_of_def*)
by (*smt* (*verit*, *ccfv_threshold*) *Int_iff image_iff openin_subset subset_iff*)

lemma *openin_path_component_of_locally_path_connected_space*:
locally_path_connected_space X $\Longrightarrow$ *openin X* (*Collect* (*path_component_of X x*))
using *locally_path_connected_space_eq_open_path_component_of openin_subopen path_component_of_eq_empty*
by *fastforce*

lemma *openin_path_components_of_locally_path_connected_space*:
$[\![$*locally_path_connected_space X*; *C* $\in$ *path_components_of X*$]\!] \Longrightarrow$ *openin X C*
using *locally_path_connected_space_open_path_components* **by** *force*

lemma *closedin_path_components_of_locally_path_connected_space*:
$[\![$*locally_path_connected_space X*; *C* $\in$ *path_components_of X*$]\!] \Longrightarrow$ *closedin X C*
unfolding *closedin_def*
by (*metis Diff_iff complement_path_components_of_Union openin_clauses*(*3*)

```
openin_closedin_eq
      openin_path_components_of_locally_path_connected_space)

lemma closedin_path_component_of_locally_path_connected_space:
  assumes locally_path_connected_space X
  shows closedin X (Collect (path_component_of X x))
proof (cases x ∈ topspace X)
  case True
  then show ?thesis
   by (simp add: assms closedin_path_components_of_locally_path_connected_space
path_component_in_path_components_of)
next
  case False
  then show ?thesis
    by (metis closedin_empty path_component_of_eq_empty)
qed

lemma weakly_locally_path_connected_at:
   weakly_locally_path_connected_at x X ⟷
    (∀ V. openin X V ∧ x ∈ V
          ⟶ (∃ U. openin X U ∧ x ∈ U ∧ U ⊆ V ∧
                (∀ y ∈ U. ∃ C. path_connectedin X C ∧ C ⊆ V ∧ x ∈ C ∧ y ∈ C)))
         (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
   by (smt (verit) neighbourhood_base_at_def subset_iff weakly_locally_path_connected_at_def)
next
  have ∗: ∃ V. path_connectedin X V ∧ U ⊆ V ∧ V ⊆ W
    if (∀ y∈U. ∃ C. path_connectedin X C ∧ C ⊆ W ∧ x ∈ C ∧ y ∈ C)
    for W U
  proof (intro exI conjI)
    let ?V = (Collect (path_component_of (subtopology X W) x))
      show path_connectedin X (Collect (path_component_of (subtopology X W)
x))
      by (meson path_connectedin_path_component_of path_connectedin_subtopology)
      show U ⊆ ?V
        by (auto simp: path_component_of path_connectedin_subtopology that)
      show ?V ⊆ W
      by (meson path_connectedin_path_component_of path_connectedin_subtopology)
    qed
  assume ?rhs
  then show ?lhs
   unfolding weakly_locally_path_connected_at_def neighbourhood_base_at_def
by (metis ∗)
qed

lemma locally_path_connected_space_im_kleinen:
   locally_path_connected_space X ⟷
     (∀ V x. openin X V ∧ x ∈ V
```

```
          ⟶ (∃ U. openin X U ∧
                x ∈ U ∧ U ⊆ V ∧
                (∀ y ∈ U. ∃ C. path_connectedin X C ∧
                      C ⊆ V ∧ x ∈ C ∧ y ∈ C)))
      (is ?lhs = ?rhs)
proof
  show ?lhs ⟹ ?rhs
    by (metis locally_path_connected_space)
  assume ?rhs
  then show ?lhs
    unfolding locally_path_connected_space_def neighbourhood_base_of
   by (metis neighbourhood_base_at_def weakly_locally_path_connected_at weakly_locally_path_connected_at_de
qed

lemma locally_path_connected_space_open_subset:
  ⟦locally_path_connected_space X; openin X S⟧
      ⟹ locally_path_connected_space (subtopology X S)
  by (smt (verit, best) locally_path_connected_space openin_open_subtopology
path_connectedin_subtopology subset_trans)

lemma locally_path_connected_space_quotient_map_image:
  assumes f: quotient_map X Y f and X: locally_path_connected_space X
  shows locally_path_connected_space Y
  unfolding locally_path_connected_space_open_path_components
proof clarify
  fix V C
  assume V: openin Y V and c: C ∈ path_components_of (subtopology Y V)
  then have sub: C ⊆ topspace Y
   using path_connectedin_path_components_of path_connectedin_subset_topspace
path_connectedin_subtopology by blast
  have openin X {x ∈ topspace X. f x ∈ C}
  proof (subst openin_subopen, clarify)
    fix x
    assume x: x ∈ topspace X and f x ∈ C
    let ?T = Collect (path_component_of (subtopology X {z ∈ topspace X. f z ∈
V}) x)
    show ∃ T. openin X T ∧ x ∈ T ∧ T ⊆ {x ∈ topspace X. f x ∈ C}
    proof (intro exI conjI)
      have ∗: ∃ U. openin X U ∧ ?T ∈ path_components_of (subtopology X U)
      proof (intro exI conjI)
        show openin X ({z ∈ topspace X. f z ∈ V})
          using V f openin_subset quotient_map_def by fastforce
        have x ∈ topspace (subtopology X {z ∈ topspace X. f z ∈ V})
          using ‹f x ∈ C› c path_components_of_subset x by force
        then show Collect (path_component_of (subtopology X {z ∈ topspace X.
f z ∈ V}) x)
            ∈ path_components_of (subtopology X {z ∈ topspace X. f z ∈ V})
          by (meson path_component_in_path_components_of)
      qed
```

```
      with X show openin X ?T
        using locally_path_connected_space_open_path_components by blast
      show x: x ∈ ?T
      using ∗ nonempty_path_components_of path_component_of_eq path_component_of_eq_empty
by fastforce
      have f ‘ ?T ⊆ C
      proof (rule path_components_of_maximal)
        show C ∈ path_components_of (subtopology Y V)
          by (simp add: c)
        show path_connectedin (subtopology Y V) (f ‘ ?T)
        proof −
        have quotient_map (subtopology X {a ∈ topspace X. f a ∈ V}) (subtopology
Y V) f
            by (simp add: V f quotient_map_restriction)
          then show ?thesis
         by (meson path_connectedin_continuous_map_image path_connectedin_path_component_of
quotient_imp_continuous_map)
        qed
        show ¬ disjnt C (f ‘ ?T)
          by (metis (no_types, lifting) ‹f x ∈ C› x disjnt_iff image_eqI)
      qed
      then show ?T ⊆ {x ∈ topspace X. f x ∈ C}
        by (force simp: path_component_of_equiv)
    qed
  qed
  then show openin Y C
    using f sub by (simp add: quotient_map_def)
qed

lemma homeomorphic_locally_path_connected_space:
  assumes X homeomorphic_space Y
  shows locally_path_connected_space X ⟷ locally_path_connected_space Y
  using assms
   unfolding homeomorphic_space_def homeomorphic_map_def homeomorphic_maps_map
    by (metis locally_path_connected_space_quotient_map_image)

lemma locally_path_connected_space_retraction_map_image:
   ⟦retraction_map X Y r; locally_path_connected_space X⟧
        ⟹ locally_path_connected_space Y
 using Abstract_Topology.retraction_imp_quotient_map locally_path_connected_space_quotient_map
by blast

lemma locally_path_connected_space_euclideanreal: locally_path_connected_space
euclideanreal
  unfolding locally_path_connected_space_def neighbourhood_base_of
  proof clarsimp
  fix W and x :: real
  assume open W x ∈ W
  then obtain e where e > 0 and e: ⋀x'. |x' − x| < e ⟶ x' ∈ W
```

by (*auto simp*: *open_real*)
then show ∃ *U*. *open U* ∧ (∃ *V*. *path_connected V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *W*)
by (*force intro*!: *convex_imp_path_connected exI* [**where** *x* = {*x*−*e*<..<*x*+*e*}])
qed

lemma *locally_path_connected_space_discrete_topology*:
locally_path_connected_space (*discrete_topology U*)
using *locally_path_connected_space_open_path_components* **by** *fastforce*

lemma *path_component_eq_connected_component_of*:
assumes *locally_path_connected_space X*
shows *path_component_of_set X x* = *connected_component_of_set X x*
proof (*cases x* ∈ *topspace X*)
case *True*
have *path_component_of_set X x* ⊆ *connected_component_of_set X x*
by (*simp add*: *path_component_subset_connected_component_of*)
moreover have *closedin X* (*path_component_of_set X x*)
by (*simp add*: *assms closedin_path_component_of_locally_path_connected_space*)
moreover have *openin X* (*path_component_of_set X x*)
by (*simp add*: *assms openin_path_component_of_locally_path_connected_space*)
moreover have *path_component_of_set X x* ≠ {}
by (*meson True path_component_of_eq_empty*)
ultimately show *?thesis*
using *connectedin_connected_component_of* [*of X x*] **unfolding** *connectedin_def*
by (*metis closedin_subset_topspace connected_space_clopen_in*
subset_openin_subtopology topspace_subtopology_subset)
next
case *False*
then show *?thesis*
using *connected_component_of_eq_empty path_component_of_eq_empty* **by** *fastforce*
qed

lemma *path_components_eq_connected_components_of*:
locally_path_connected_space X ⟹ (*path_components_of X* = *connected_components_of X*)
by (*simp add*: *path_components_of_def connected_components_of_def image_def path_component_eq_connected_component_of*)

lemma *path_connected_eq_connected_space*:
locally_path_connected_space X
⟹ *path_connected_space X* ⟷ *connected_space X*
by (*metis connected_components_of_subset_sing path_components_eq_connected_components_of path_components_of_subset_singleton*)

lemma *locally_path_connected_space_product_topology*:
locally_path_connected_space(*product_topology X I*) ⟷

```
      (product_topology X I) = trivial_topology ∨
      finite {i. i ∈ I ∧ ~path_connected_space(X i)} ∧
      (∀ i ∈ I. locally_path_connected_space(X i))
  (is ?lhs ⟷ ?empty ∨ ?rhs)
proof (cases ?empty)
 case True
 then show ?thesis
  by (simp add: locally_path_connected_space_def neighbourhood_base_of openin_closedin_eq)
next
 case False
 then obtain z where z: z ∈ (Π_E i∈I. topspace (X i))
  using discrete_topology_unique_derived_set by (fastforce iff: null_topspace_iff_trivial)
 have ?rhs if L: ?lhs
 proof −
  obtain U C where U: openin (product_topology X I) U
    and V: path_connectedin (product_topology X I) C
    and z ∈ U U ⊆ C and Csub: C ⊆ (Π_E i∈I. topspace (X i))
   by (metis L locally_path_connected_space openin_topspace topspace_product_topology
z)
  then obtain V where finV: finite {i ∈ I. V i ≠ topspace (X i)}
    and XV: ⋀i. i∈I ⟹ openin (X i) (V i) and z ∈ Pi_E I V and subU: Pi_E
I V ⊆ U
    by (force simp: openin_product_topology_alt)
  show ?thesis
  proof (intro conjI ballI)
    have path_connected_space (X i) if i ∈ I V i = topspace (X i) for i
    proof −
      have pc: path_connectedin (X i) ((λx. x i) ‘ C)
     by (metis V continuous_map_product_projection path_connectedin_continuous_map_image
that(1))
      moreover have ((λx. x i) ‘ C) = topspace (X i)
      proof
        show (λx. x i) ‘ C ⊆ topspace (X i)
          by (simp add: pc path_connectedin_subset_topspace)
        have V i ⊆ (λx. x i) ‘ (Π_E i∈I. V i)
           by (metis ‹z ∈ Pi_E I V› empty_iff image_projection_PiE order_refl
that(1))
        also have ... ⊆ (λx. x i) ‘ U
          using subU by blast
        finally show topspace (X i) ⊆ (λx. x i) ‘ C
          using ‹U ⊆ C› that by blast
      qed
      ultimately show ?thesis
        by (simp add: path_connectedin_topspace)
    qed
    then have {i ∈ I. ¬ path_connected_space (X i)} ⊆ {i ∈ I. V i ≠ topspace
(X i)}
      by blast
    with finV show finite {i ∈ I. ¬ path_connected_space (X i)}
```

using *finite_subset* **by** *blast*
next
show *locally_path_connected_space* (*X i*) **if** $i \in I$ **for** *i*
by (*meson False L locally_path_connected_space_quotient_map_image quotient_map_product_projection that*)
qed
qed
moreover have *?lhs* **if** *R*: *?rhs*
proof (*clarsimp simp add*: *locally_path_connected_space_def neighbourhood_base_of*)
fix *F z*
assume *openin* (*product_topology X I*) *F* **and** $z \in F$
then obtain *W* **where** *finW*: *finite* $\{i \in I.\ W\ i \neq$ *topspace* (*X i*)$\}$
and *opeW*: $\bigwedge i.\ i \in I \Longrightarrow$ *openin* (*X i*) (*W i*) **and** $z \in Pi_E\ I\ W\ Pi_E\ I\ W \subseteq F$
by (*auto simp*: *openin_product_topology_alt*)
have $\forall i \in I.\ \exists U\ C.$ *openin* (*X i*) *U* $\wedge$ *path_connectedin* (*X i*) *C* $\wedge$ $z\ i \in U \wedge U \subseteq C \wedge C \subseteq W\ i \wedge$
(*W i* = *topspace* (*X i*) $\wedge$
path_connected_space (*X i*) $\longrightarrow$ *U* = *topspace* (*X i*) $\wedge$ *C* = *topspace* (*X i*))
(**is** $\forall i \in I.$ *?*Φ *i*)
proof
fix *i* **assume** $i \in I$
have *locally_path_connected_space* (*X i*)
by (*simp add*: *R* ‹$i \in I$›)
moreover have $*$:*openin* (*X i*) (*W i*) $z\ i \in W\ i$
using ‹$z \in Pi_E\ I\ W$› *opeW* ‹$i \in I$› **by** *auto*
ultimately obtain *U C* **where** *UC*: *openin* (*X i*) *U path_connectedin* (*X i*) *C z i* $\in$ *U U* $\subseteq$ *C C* $\subseteq$ *W i*
using ‹$i \in I$› **by** (*force simp*: *locally_path_connected_space_def neighbourhood_base_of*)
show *?*Φ *i*
by (*metis UC* $*$ *openin_subset path_connectedin_topspace*)
qed
then obtain *U C* **where**
$*$: $\bigwedge i.\ i \in I \Longrightarrow$ *openin* (*X i*) (*U i*) $\wedge$ *path_connectedin* (*X i*) (*C i*) $\wedge$ $z\ i \in (U\ i) \wedge (U\ i) \subseteq (C\ i) \wedge (C\ i) \subseteq W\ i \wedge$
(*W i* = *topspace* (*X i*) $\wedge$ *path_connected_space* (*X i*)
$\longrightarrow$ (*U i*) = *topspace* (*X i*) $\wedge$ (*C i*) = *topspace* (*X i*))
by *metis*
let *?A* = $\{i \in I.\ \neg$ *path_connected_space* (*X i*)$\} \cup \{i \in I.\ W\ i \neq$ *topspace* (*X i*)$\}$
have $\{i \in I.\ U\ i \neq$ *topspace* (*X i*)$\} \subseteq$ *?A*
by (*clarsimp simp add*: $*$)
moreover have *finite ?A*
by (*simp add*: *that finW*)
ultimately have *finite* $\{i \in I.\ U\ i \neq$ *topspace* (*X i*)$\}$
using *finite_subset* **by** *auto*
with $*$ **have** *openin* (*product_topology X I*) (Pi_E *I U*)

```
      by (simp add: openin_PiE_gen)
    then show ∃ U. openin (product_topology X I) U ∧
            (∃ V. path_connectedin (product_topology X I) V ∧ z ∈ U ∧ U ⊆ V
∧ V ⊆ F)
      by (metis (no_types, opaque_lifting) subsetI ‹z ∈ Pi_E I W› ‹Pi_E I W ⊆ F›
∗ path_connectedin_PiE
          PiE_iff PiE_mono order.trans)
  qed
  ultimately show ?thesis
    using False by blast
qed

lemma locally_path_connected_is_realinterval:
  assumes is_interval S
  shows locally_path_connected_space(subtopology euclideanreal S)
  unfolding locally_path_connected_space_def
proof (clarsimp simp add: neighbourhood_base_of openin_subtopology_alt)
  fix a U
  assume a ∈ S and a ∈ U and open U
  then obtain r where r > 0 and r: ball a r ⊆ U
    by (metis open_contains_ball_eq)
  show ∃ W. open W ∧ (∃ V. path_connectedin (top_of_set S) V ∧ a ∈ W ∧ S
∩ W ⊆ V ∧ V ⊆ S ∧ V ⊆ U)
  proof (intro exI conjI)
    show path_connectedin (top_of_set S) (S ∩ ball a r)
    by (simp add: assms is_interval_Int is_interval_ball_real is_interval_path_connected
path_connectedin_subtopology)
    show a ∈ ball a r
      by (simp add: ‹0 < r›)
  qed (use ‹0 < r› r in auto)
qed

lemma locally_path_connected_real_interval:
 locally_path_connected_space (subtopology euclideanreal{a..b})
  locally_path_connected_space (subtopology euclideanreal{a<..<b})
  using locally_path_connected_is_realinterval
  by (auto simp add: is_interval_1)

lemma locally_path_connected_space_prod_topology:
   locally_path_connected_space (prod_topology X Y) ⟷
     (prod_topology X Y) = trivial_topology ∨
    locally_path_connected_space X ∧ locally_path_connected_space Y (is ?lhs=?rhs)
proof (cases (prod_topology X Y) = trivial_topology)
  case True
  then show ?thesis
    using locally_path_connected_space_discrete_topology by force
next
  case False
  then have ne: X ≠ trivial_topology Y ≠ trivial_topology
```

```
    by simp_all
  show ?thesis
  proof
    assume ?lhs then show ?rhs
     by (meson locally_path_connected_space_quotient_map_image ne(1) ne(2)
quotient_map_fst quotient_map_snd)
  next
    assume ?rhs
   with False have X: locally_path_connected_space X and Y: locally_path_connected_space
Y
     by auto
    show ?lhs
     unfolding locally_path_connected_space_def neighbourhood_base_of
    proof clarify
     fix UV x y
     assume UV: openin (prod_topology X Y) UV and (x,y) ∈ UV
     obtain A B where W12: openin X A ∧ openin Y B ∧ x ∈ A ∧ y ∈ B ∧ A
× B ⊆ UV
       using X Y by (metis UV ‹(x,y) ∈ UV› openin_prod_topology_alt)
     then obtain C D K L
       where openin X C path_connectedin X K x ∈ C C ⊆ K K ⊆ A
         openin Y D path_connectedin Y L y ∈ D D ⊆ L L ⊆ B
       by (metis X Y locally_path_connected_space)
     with W12 ‹openin X C› ‹openin Y D›
    show ∃ U V. openin (prod_topology X Y) U ∧ path_connectedin (prod_topology
X Y) V ∧ (x, y) ∈ U ∧ U ⊆ V ∧ V ⊆ UV
       apply (rule_tac x=C × D in exI)
       apply (rule_tac x=K × L in exI)
       apply (fastforce simp: openin_prod_Times_iff path_connectedin_Times)
       done
    qed
  qed
qed

lemma locally_path_connected_space_sum_topology:
   locally_path_connected_space(sum_topology X I) ⟷
    (∀ i ∈ I. locally_path_connected_space (X i)) (is ?lhs=?rhs)
proof
  assume ?lhs then show ?rhs
   by (smt (verit) homeomorphic_locally_path_connected_space locally_path_connected_space_open_subset
topological_property_of_sum_component)
next
  assume R: ?rhs
  show ?lhs
  proof (clarsimp simp add: locally_path_connected_space_def neighbourhood_base_of
forall_openin_sum_topology imp_conjL)
    fix W i x
    assume ope: ∀ i∈I. openin (X i) (W i)
      and i ∈ I and x ∈ W i
```

then obtain U V **where** U: *openin* $(X\ i)$ U **and** V: *path_connectedin* $(X\ i)$ V
and $x \in U$ $U \subseteq V$ $V \subseteq W\ i$
by (*metis* R ‹$i \in I$› ‹$x \in W\ i$› *locally_path_connected_space*)
show $\exists U.$ *openin* (*sum_topology* $X\ I$) $U \wedge (\exists V.$ *path_connectedin* (*sum_topology* $X\ I$) $V \wedge (i, x) \in U \wedge U \subseteq V \wedge V \subseteq$ *Sigma* $I\ W$)
proof (*intro exI conjI*)
show *openin* (*sum_topology* $X\ I$) (*Pair* i ‘ U)
by (*meson* U ‹$i \in I$› *open_map_component_injection open_map_def*)
show *path_connectedin* (*sum_topology* $X\ I$) (*Pair* i ‘ V)
by (*meson* V ‹$i \in I$› *continuous_map_component_injection path_connectedin_continuous_map_i*
show *Pair* i ‘ $V \subseteq$ *Sigma* $I\ W$
using ‹$V \subseteq W\ i$› ‹$i \in I$› **by** *force*
qed (*use* ‹$x \in U$› ‹$U \subseteq V$› **in** *auto*)
qed
qed

6.2.3 Locally connected spaces

definition *weakly_locally_connected_at*
where *weakly_locally_connected_at* $x\ X \equiv$ *neighbourhood_base_at* x (*connectedin* X) X

definition *locally_connected_at*
where *locally_connected_at* $x\ X \equiv$
neighbourhood_base_at x ($\lambda U.$ *openin* $X\ U \wedge$ *connectedin* $X\ U$) X

definition *locally_connected_space*
where *locally_connected_space* $X \equiv$ *neighbourhood_base_of* (*connectedin* X) X

lemma *locally_connected_A*: $(\forall\ U\ x.$ *openin* $X\ U \wedge x \in U \longrightarrow$ *openin* X (*connected_component_of_set* (*subtopology* $X\ U$) x))
$\Longrightarrow$ *neighbourhood_base_of* ($\lambda U.$ *openin* $X\ U \wedge$ *connectedin* $X\ U$) X
unfolding *neighbourhood_base_of*
by (*metis* (*full_types*) *connected_component_of_refl connectedin_connected_component_of connectedin_subtopology mem_Collect_eq openin_subset topspace_subtopology_subset*)

lemma *locally_connected_B*: *locally_connected_space* $X \Longrightarrow$
$(\forall\ U\ x.$ *openin* $X\ U \wedge x \in U \longrightarrow$ *openin* X (*connected_component_of_set* (*subtopology* $X\ U$) x))
unfolding *locally_connected_space_def neighbourhood_base_of*
apply (*erule all_forward*)
apply *clarify*
apply (*subst openin_subopen*)
by (*smt* (*verit*, *ccfv_threshold*) *Ball_Collect connected_component_of_def connected_component_of_equiv connectedin_subtopology in_mono mem_Collect_eq*)

lemma *locally_connected_C*: *neighbourhood_base_of* ($\lambda U.$ *openin* $X\ U \wedge$ *con-*

nectedin X U) *X* ⟹ *locally_connected_space X*
using *locally_connected_space_def neighbourhood_base_of_mono* **by** *auto*

lemma *locally_connected_space_alt*:
locally_connected_space X ⟷ *neighbourhood_base_of* (λ*U*. *openin X U* ∧ *connectedin X U*) *X*
using *locally_connected_A locally_connected_B locally_connected_C* **by** *blast*

lemma *locally_connected_space_eq_open_connected_component_of*:
locally_connected_space X ⟷
(∀ *U x*. *openin X U* ∧ *x* ∈ *U*
⟶ *openin X* (*connected_component_of_set* (*subtopology X U*) *x*))
by (*meson locally_connected_A locally_connected_B locally_connected_C*)

lemma *locally_connected_space*:
locally_connected_space X ⟷
(∀ *V x*. *openin X V* ∧ *x* ∈ *V* ⟶ (∃ *U*. *openin X U* ∧ *connectedin X U* ∧ *x* ∈ *U* ∧ *U* ⊆ *V*))
by (*simp add*: *locally_connected_space_alt open_neighbourhood_base_of*)

lemma *locally_path_connected_imp_locally_connected_space*:
locally_path_connected_space X ⟹ *locally_connected_space X*
by (*simp add*: *locally_connected_space_def locally_path_connected_space_def neighbourhood_base_of_mono path_connectedin_imp_connectedin*)

lemma *locally_connected_space_open_connected_components*:
locally_connected_space X ⟷
(∀ *U C*. *openin X U* ∧ *C* ∈ *connected_components_of*(*subtopology X U*) ⟶ *openin X C*)
unfolding *connected_components_of_def locally_connected_space_eq_open_connected_component_of*
by (*smt* (*verit*, *best*) *image_iff openin_subset topspace_subtopology_subset*)

lemma *openin_connected_component_of_locally_connected_space*:
locally_connected_space X ⟹ *openin X* (*connected_component_of_set X x*)
by (*metis connected_component_of_eq_empty locally_connected_B openin_empty openin_topspace subtopology_topspace*)

lemma *openin_connected_components_of_locally_connected_space*:
⟦*locally_connected_space X*; *C* ∈ *connected_components_of X*⟧ ⟹ *openin X C*
by (*metis locally_connected_space_open_connected_components openin_topspace subtopology_topspace*)

lemma *weakly_locally_connected_at*:
weakly_locally_connected_at x X ⟷
(∀ *V*. *openin X V* ∧ *x* ∈ *V*
⟶ (∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧
(∀ *y* ∈ *U*. ∃ *C*. *connectedin X C* ∧ *C* ⊆ *V* ∧ *x* ∈ *C* ∧ *y* ∈ *C*))) (**is**

```
?lhs=?rhs)
proof
  assume ?lhs then show ?rhs
    unfolding neighbourhood_base_at_def weakly_locally_connected_at_def
    by (meson subsetD subset_trans)
next
  assume R: ?rhs
  show ?lhs
    unfolding neighbourhood_base_at_def weakly_locally_connected_at_def
  proof clarify
    fix V
    assume openin X V and x ∈ V
    then obtain U where openin X U x ∈ U U ⊆ V
                  and U: ∀ y∈U. ∃ C. connectedin X C ∧ C ⊆ V ∧ x ∈ C ∧ y ∈ C
      using R by force
    show ∃ A B. openin X A ∧ connectedin X B ∧ x ∈ A ∧ A ⊆ B ∧ B ⊆ V
    proof (intro conjI exI)
      show connectedin X (connected_component_of_set (subtopology X V) x)
        by (meson connectedin_connected_component_of connectedin_subtopology)
      show U ⊆ connected_component_of_set (subtopology X V) x
        using connected_component_of_maximal U
        by (simp add: connected_component_of_def connectedin_subtopology subsetI)
      show connected_component_of_set (subtopology X V) x ⊆ V
        using connected_component_of_subset_topspace by fastforce
    qed (auto simp: ‹x ∈ U› ‹openin X U›)
  qed
qed

lemma locally_connected_space_iff_weak:
  locally_connected_space X ⟷ (∀ x ∈ topspace X. weakly_locally_connected_at
x X)
  by (simp add: locally_connected_space_def neighbourhood_base_of_def weakly_locally_connected_at_

lemma locally_connected_space_im_kleinen:
    locally_connected_space X ⟷
    (∀ V x. openin X V ∧ x ∈ V
          ⟶ (∃ U. openin X U ∧ x ∈ U ∧ U ⊆ V ∧
                  (∀ y ∈ U. ∃ C. connectedin X C ∧ C ⊆ V ∧ x ∈ C ∧ y ∈ C)))
  unfolding locally_connected_space_iff_weak weakly_locally_connected_at
  using openin_subset subsetD by fastforce

lemma locally_connected_space_open_subset:
  ⟦locally_connected_space X; openin X S⟧ ⟹ locally_connected_space (subtopology
X S)
  unfolding locally_connected_space_def neighbourhood_base_of
  by (smt (verit) connectedin_subtopology openin_open_subtopology subset_trans)

lemma locally_connected_space_quotient_map_image:
```

```
  assumes X: locally_connected_space X and f: quotient_map X Y f
  shows locally_connected_space Y
  unfolding locally_connected_space_open_connected_components
proof clarify
  fix V C
  assume openin Y V and C: C ∈ connected_components_of (subtopology Y V)
  then have C ⊆ topspace Y
    using connected_components_of_subset by force
  have ope1: openin X {a ∈ topspace X. f a ∈ V}
   using ‹openin Y V› f openin_continuous_map_preimage quotient_imp_continuous_map
by blast
  define Vf where Vf ≡ {z ∈ topspace X. f z ∈ V}
  have openin X {x ∈ topspace X. f x ∈ C}
  proof (clarsimp simp: openin_subopen [where S = {x ∈ topspace X. f x ∈ C}])
    fix x
    assume x ∈ topspace X and f x ∈ C
    show ∃ T. openin X T ∧ x ∈ T ∧ T ⊆ {x ∈ topspace X. f x ∈ C}
    proof (intro exI conjI)
      show openin X (connected_component_of_set (subtopology X Vf) x)
      by (metis Vf_def X connected_component_of_eq_empty locally_connected_B
ope1 openin_empty
                openin_subset topspace_subtopology_subset)
      show x_in_conn: x ∈ connected_component_of_set (subtopology X Vf) x
        using C Vf_def ‹f x ∈ C› ‹x ∈ topspace X› connected_component_of_refl
connected_components_of_subset by fastforce
      have connected_component_of_set (subtopology X Vf) x ⊆ topspace X ∩ Vf
        using connected_component_of_subset_topspace by fastforce
      moreover
      have f ‘ connected_component_of_set (subtopology X Vf) x ⊆ C
      proof (rule connected_components_of_maximal [where X = subtopology Y
V])
        show C ∈ connected_components_of (subtopology Y V)
          by (simp add: C)
        have §: quotient_map (subtopology X Vf) (subtopology Y V) f
          by (simp add: Vf_def ‹openin Y V› f quotient_map_restriction)
       then show connectedin (subtopology Y V) (f ‘ connected_component_of_set
(subtopology X Vf) x)
        by (metis connectedin_connected_component_of connectedin_continuous_map_image
quotient_imp_continuous_map)
        show ¬ disjnt C (f ‘ connected_component_of_set (subtopology X Vf) x)
          using ‹f x ∈ C› x_in_conn by (auto simp: disjnt_iff)
      qed
      ultimately
      show connected_component_of_set (subtopology X Vf) x ⊆ {x ∈ topspace
X. f x ∈ C}
        by blast
    qed
  qed
  then show openin Y C
```

using ‹$C \subseteq$ *topspace* Y› f *quotient_map_def* **by** *fastforce*
qed

lemma *locally_connected_space_retraction_map_image*:
⟦*retraction_map* X Y r; *locally_connected_space* X⟧
$\Longrightarrow$ *locally_connected_space* Y
using *locally_connected_space_quotient_map_image retraction_imp_quotient_map*
by *blast*

lemma *homeomorphic_locally_connected_space*:
X *homeomorphic_space* $Y \Longrightarrow$ *locally_connected_space* $X \longleftrightarrow$ *locally_connected_space* Y
by (*meson homeomorphic_map_def homeomorphic_space homeomorphic_space_sym locally_connected_space_quotient_map_image*)

lemma *locally_connected_space_euclideanreal*: *locally_connected_space euclideanreal*
by (*simp add*: *locally_path_connected_imp_locally_connected_space locally_path_connected_space_e*

lemma *locally_connected_is_realinterval*:
is_interval $S \Longrightarrow$ *locally_connected_space*(*subtopology euclideanreal* S)
by (*simp add*: *locally_path_connected_imp_locally_connected_space locally_path_connected_is_realin*

lemma *locally_connected_real_interval*:
locally_connected_space (*subtopology euclideanreal*{$a..b$})
locally_connected_space (*subtopology euclideanreal*{$a<..<b$})
using *connected_Icc is_interval_connected_1 locally_connected_is_realinterval*
by *auto*

lemma *locally_connected_space_discrete_topology*:
locally_connected_space (*discrete_topology* U)
by (*simp add*: *locally_path_connected_imp_locally_connected_space locally_path_connected_space_d*

lemma *locally_path_connected_imp_locally_connected_at*:
locally_path_connected_at x $X \Longrightarrow$ *locally_connected_at* x X
by (*simp add*: *locally_connected_at_def locally_path_connected_at_def neighbourhood_base_at_mono path_connectedin_imp_connectedin*)

lemma *weakly_locally_path_connected_imp_weakly_locally_connected_at*:
weakly_locally_path_connected_at x $X \Longrightarrow$ *weakly_locally_connected_at* x X
by (*metis path_connectedin_imp_connectedin weakly_locally_connected_at weakly_locally_path_conn*

lemma *interior_of_locally_connected_subspace_component*:
assumes X: *locally_connected_space* X
and C: $C \in$ *connected_components_of* (*subtopology* X S)
shows X *interior_of* $C = C \cap X$ *interior_of* S
proof −

```
  obtain Csub: C ⊆ topspace X C ⊆ S
   by (meson C connectedin_connected_components_of connectedin_subset_topspace
connectedin_subtopology)
  show ?thesis
  proof
    show X interior_of C ⊆ C ∩ X interior_of S
      by (simp add: Csub interior_of_mono interior_of_subset)
    have eq: X interior_of S = ⋃ (connected_components_of (subtopology X (X
interior_of S)))
       by (metis Union_connected_components_of interior_of_subset_topspace
topspace_subtopology_subset)
    moreover have C ∩ D ⊆ X interior_of C
      if D ∈ connected_components_of (subtopology X (X interior_of S)) for D
    proof (cases C ∩ D = {})
      case False
      have D ⊆ X interior_of C
      proof (rule interior_of_maximal)
        have connectedin (subtopology X S) D
        by (meson connectedin_connected_components_of connectedin_subtopology
interior_of_subset subset_trans that)
        then show D ⊆ C
          by (meson C False connected_components_of_maximal disjnt_def)
        show openin X D
       using X locally_connected_space_open_connected_components openin_interior_of
that by blast
      qed
      then show ?thesis
        by blast
    qed auto
    ultimately show C ∩ X interior_of S ⊆ X interior_of C
      by blast
  qed
qed


lemma frontier_of_locally_connected_subspace_component:
  assumes X: locally_connected_space X and closedin X S
    and C: C ∈ connected_components_of (subtopology X S)
  shows X frontier_of C = C ∩ X frontier_of S
proof −
  obtain Csub: C ⊆ topspace X C ⊆ S
   by (meson C connectedin_connected_components_of connectedin_subset_topspace
connectedin_subtopology)
  then have X closure_of C − X interior_of C = C ∩ X closure_of S − C ∩ X
interior_of S
    using assms
   apply (simp add: closure_of_closedin flip: interior_of_locally_connected_subspace_component)
   by (metis closedin_connected_components_of closedin_trans_full closure_of_eq
inf.orderE)
```

then show *?thesis*
by (*simp add*: *Diff_Int_distrib frontier_of_def*)
qed

lemma *locally_connected_space_prod_topology*:
locally_connected_space (*prod_topology X Y*) ⟷
(*prod_topology X Y*) = *trivial_topology* ∨
locally_connected_space X ∧ *locally_connected_space Y* (**is** *?lhs=?rhs*)
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
case *True*
then show *?thesis*
using *locally_connected_space_iff_weak* **by** *force*
next
case *False*
then have *ne*: *X* ≠ *trivial_topology Y* ≠ *trivial_topology*
by *simp_all*
show *?thesis*
proof
assume *?lhs* **then show** *?rhs*
by (*metis locally_connected_space_quotient_map_image ne quotient_map_fst quotient_map_snd*)
next
assume *?rhs*
with *False* **have** *X*: *locally_connected_space X* **and** *Y*: *locally_connected_space Y*
by *auto*
show *?lhs*
unfolding *locally_connected_space_def neighbourhood_base_of*
proof *clarify*
fix *UV x y*
assume *UV*: *openin* (*prod_topology X Y*) *UV* **and** (*x*,*y*) ∈ *UV*

obtain *A B* **where** *W12*: *openin X A* ∧ *openin Y B* ∧ *x* ∈ *A* ∧ *y* ∈ *B* ∧ *A* × *B* ⊆ *UV*
using *X Y* **by** (*metis UV* ‹(*x*,*y*) ∈ *UV*› *openin_prod_topology_alt*)
then obtain *C D K L*
where *openin X C connectedin X K x* ∈ *C C* ⊆ *K K* ⊆ *A*
openin Y D connectedin Y L y ∈ *D D* ⊆ *L L* ⊆ *B*
by (*metis X Y locally_connected_space*)
with *W12* ‹*openin X C*› ‹*openin Y D*›
show ∃ *U V*. *openin* (*prod_topology X Y*) *U* ∧ *connectedin* (*prod_topology X Y*) *V* ∧ (*x*, *y*) ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *UV*
apply (*rule_tac x*=*C* × *D* **in** *exI*)
apply (*rule_tac x*=*K* × *L* **in** *exI*)
apply (*auto simp*: *openin_prod_Times_iff connectedin_Times*)
done
qed
qed

qed

lemma *locally_connected_space_product_topology*:
locally_connected_space(*product_topology X I*) ⟷
(*product_topology X I*) = *trivial_topology* ∨
finite {*i*. *i* ∈ *I* ∧ ~*connected_space*(*X i*)} ∧
(∀ *i* ∈ *I*. *locally_connected_space*(*X i*))
(**is** *?lhs* ⟷ *?empty* ∨ *?rhs*)
proof (*cases ?empty*)
case *True*
then show *?thesis*
by (*simp add*: *locally_connected_space_def neighbourhood_base_of openin_closedin_eq*)
next
case *False*
then obtain *z* **where** *z*: *z* ∈ (Π$_E$ *i*∈*I*. *topspace* (*X i*))
using *discrete_topology_unique_derived_set* **by** (*fastforce iff*: *null_topspace_iff_trivial*)
have *?rhs* **if** *L*: *?lhs*
proof −
obtain *U C* **where** *U*: *openin* (*product_topology X I*) *U*
and *V*: *connectedin* (*product_topology X I*) *C*
and *z* ∈ *U U* ⊆ *C* **and** *Csub*: *C* ⊆ (Π$_E$ *i*∈*I*. *topspace* (*X i*))
using *L* **apply** (*clarsimp simp add*: *locally_connected_space_def neighbourhood_base_of*)
by (*metis openin_topspace topspace_product_topology z*)
then obtain *V* **where** *finV*: *finite* {*i* ∈ *I*. *V i* ≠ *topspace* (*X i*)}
and *XV*: ⋀*i*. *i*∈*I* ⟹ *openin* (*X i*) (*V i*) **and** *z* ∈ *Pi*$_E$ *I V* **and** *subU*: *Pi*$_E$ *I V* ⊆ *U*
by (*force simp*: *openin_product_topology_alt*)
show *?thesis*
proof (*intro conjI ballI*)
have *connected_space* (*X i*) **if** *i* ∈ *I V i* = *topspace* (*X i*) **for** *i*
proof −
have *pc*: *connectedin* (*X i*) ((λ*x*. *x i*) ‘ *C*)
by (*metis V connectedin_continuous_map_image continuous_map_product_projection that*(*1*))
moreover have ((λ*x*. *x i*) ‘ *C*) = *topspace* (*X i*)
proof
show (λ*x*. *x i*) ‘ *C* ⊆ *topspace* (*X i*)
by (*simp add*: *pc connectedin_subset_topspace*)
have *V i* ⊆ (λ*x*. *x i*) ‘ (Π$_E$ *i*∈*I*. *V i*)
by (*metis* ‹*z* ∈ *Pi*$_E$ *I V*› *empty_iff image_projection_PiE order_refl that*(*1*))
also have ... ⊆ (λ*x*. *x i*) ‘ *U*
using *subU* **by** *blast*
finally show *topspace* (*X i*) ⊆ (λ*x*. *x i*) ‘ *C*
using ‹*U* ⊆ *C*› *that* **by** *blast*
qed
ultimately show *?thesis*

```
        by (simp add: connectedin_topspace)
    qed
    then have {i ∈ I. ¬ connected_space (X i)} ⊆ {i ∈ I. V i ≠ topspace (X i)}
      by blast
    with finV show finite {i ∈ I. ¬ connected_space (X i)}
      using finite_subset by blast
  next
    show locally_connected_space (X i) if i ∈ I for i
      by (meson False L locally_connected_space_quotient_map_image quotient_map_product_projection that)
  qed
 qed
 moreover have ?lhs if R: ?rhs
 proof (clarsimp simp add: locally_connected_space_def neighbourhood_base_of)
  fix F z
  assume openin (product_topology X I) F and z ∈ F
  then obtain W where finW: finite {i ∈ I. W i ≠ topspace (X i)}
      and openW: ⋀i. i ∈ I ⟹ openin (X i) (W i) and z ∈ Pi_E I W Pi_E I W ⊆ F
    by (auto simp: openin_product_topology_alt)
  have ∀ i ∈ I. ∃ U C. openin (X i) U ∧ connectedin (X i) C ∧ z i ∈ U ∧ U ⊆ C ∧ C ⊆ W i ∧
                       (W i = topspace (X i) ∧
                       connected_space (X i) ⟶ U = topspace (X i) ∧ C = topspace (X i))
      (is ∀ i ∈ I. ?Φ i)
  proof
    fix i assume i ∈ I
    have locally_connected_space (X i)
      by (simp add: R ‹i ∈ I›)
    moreover have ∗: openin (X i) (W i) z i ∈ W i
      using ‹z ∈ Pi_E I W› openW ‹i ∈ I› by auto
    ultimately obtain U C where openin (X i) U connectedin (X i) C z i ∈ U U ⊆ C C ⊆ W i
      using ‹i ∈ I› by (force simp: locally_connected_space_def neighbourhood_base_of)
    then show ?Φ i
      by (metis ∗ connectedin_topspace openin_subset)
  qed
  then obtain U C where
    ∗: ⋀i. i ∈ I ⟹ openin (X i) (U i) ∧ connectedin (X i) (C i) ∧ z i ∈ (U i) ∧ (U i) ⊆ (C i) ∧ (C i) ⊆ W i ∧
                   (W i = topspace (X i) ∧ connected_space (X i)
                    ⟶ (U i) = topspace (X i) ∧ (C i) = topspace (X i))
    by metis
  let ?A = {i ∈ I. ¬ connected_space (X i)} ∪ {i ∈ I. W i ≠ topspace (X i)}
  have {i ∈ I. U i ≠ topspace (X i)} ⊆ ?A
    by (clarsimp simp add: ∗)
  moreover have finite ?A
```

```
    by (simp add: that finW)
  ultimately have finite {i ∈ I. U i ≠ topspace (X i)}
    using finite_subset by auto
  then have openin (product_topology X I) (Pi_E I U)
    using ∗ by (simp add: openin_PiE_gen)
  then show ∃ U. openin (product_topology X I) U ∧
        (∃ V. connectedin (product_topology X I) V ∧ z ∈ U ∧ U ⊆ V ∧ V ⊆
F)
    using ‹z ∈ Pi_E I W› ‹Pi_E I W ⊆ F› ∗
     by (metis (no_types, opaque_lifting) PiE_iff PiE_mono connectedin_PiE
subset_iff)
 qed
 ultimately show ?thesis
   using False by blast
qed

lemma locally_connected_space_sum_topology:
  locally_connected_space(sum_topology X I) ⟷
   (∀ i ∈ I. locally_connected_space (X i)) (is ?lhs=?rhs)
proof
 assume ?lhs then show ?rhs
  by (smt (verit) homeomorphic_locally_connected_space locally_connected_space_open_subset
topological_property_of_sum_component)
next
 assume R: ?rhs
 show ?lhs
 proof (clarsimp simp add: locally_connected_space_def neighbourhood_base_of
forall_openin_sum_topology imp_conjL)
   fix W i x
   assume ope: ∀ i∈I. openin (X i) (W i)
     and i ∈ I and x ∈ W i
   then obtain U V where U: openin (X i) U and V: connectedin (X i) V
        and x ∈ U U ⊆ V V ⊆ W i
     by (metis R ‹i ∈ I› ‹x ∈ W i› locally_connected_space)
   show ∃ U. openin (sum_topology X I) U ∧ (∃ V. connectedin (sum_topology
X I) V ∧ (i,x) ∈ U ∧ U ⊆ V ∧ V ⊆ Sigma I W)
   proof (intro exI conjI)
     show openin (sum_topology X I) (Pair i ‘ U)
       by (meson U ‹i ∈ I› open_map_component_injection open_map_def)
     show connectedin (sum_topology X I) (Pair i ‘ V)
         by (meson V ‹i ∈ I› continuous_map_component_injection connecte-
din_continuous_map_image)
     show Pair i ‘ V ⊆ Sigma I W
       using ‹V ⊆ W i› ‹i ∈ I› by force
   qed (use ‹x ∈ U› ‹U ⊆ V› in auto)
 qed
qed
```

6.2.4 Dimension of a topological space

Basic definition of the small inductive dimension relation. Works in any topological space.

inductive *dimension_le* :: [*'a topology*, *int*] ⇒ *bool* (**infix** ‹*dim'_le*› *50*)
where ⟦*−1 ≤ n*;
⋀*V a*. ⟦*openin X V*; *a ∈ V*⟧ ⟹ ∃ *U*. *a ∈ U ∧ U ⊆ V ∧ openin X U ∧ (subtopology X (X frontier_of U)) dim_le (n−1)*⟧
⟹ *X dim_le* (*n*::*int*)

lemma *dimension_le_neighbourhood_base*:
X dim_le n ⟷
−1 ≤ n ∧ neighbourhood_base_of (λ*U*. *openin X U ∧ (subtopology X (X frontier_of U)) dim_le (n−1)*) *X*
by (*smt* (*verit*, *best*) *dimension_le.simps open_neighbourhood_base_of*)

lemma *dimension_le_bound*: *X dim_le n* ⟹*−1 ≤ n*
using *dimension_le.simps* **by** *blast*

lemma *dimension_le_mono* [*rule_format*]:
assumes *X dim_le m*
shows *m ≤ n* ⟶ *X dim_le n*
using *assms*
proof (*induction arbitrary*: *n rule*: *dimension_le.induct*)
qed (*smt* (*verit*) *dimension_le.simps*)

inductive_simps *dim_le_minus2* [*simp*]: *X dim_le −2*

lemma *dimension_le_eq_empty* [*simp*]:
X dim_le −1 ⟷ *X = trivial_topology*
proof
show *X dim_le* (*−1*) ⟹ *X = trivial_topology*
by (*force intro*: *dimension_le.cases*)
next
show *X = trivial_topology* ⟹ *X dim_le* (*−1*)
using *dimension_le.simps openin_subset* **by** *fastforce*
qed

lemma *dimension_le_0_neighbourhood_base_of_clopen*:
X dim_le 0 ⟷ *neighbourhood_base_of* (λ*U*. *closedin X U ∧ openin X U*) *X*
proof −
have (*subtopology X* (*X frontier_of U*) *dim_le −1*) = *closedin X U*
if *openin X U* **for** *U*
using *that clopenin_eq_frontier_of openin_subset*
by (*fastforce simp add*: *subtopology_trivial_iff frontier_of_subset_topspace Int_absorb1*)
then show *?thesis*
by (*smt* (*verit*, *del_insts*) *dimension_le.simps open_neighbourhood_base_of*)
qed

lemma *dimension_le_subtopology*:
X dim_le n $\Longrightarrow$ *subtopology X S dim_le n*
proof (*induction arbitrary*: *S rule*: *dimension_le.induct*)
case (*1 n X*)
show *?case*
proof (*intro dimension_le.intros*)
show $-1 \leq n$
by (*simp add*: *1.hyps*)
fix U' *a*
assume U': *openin* (*subtopology X S*) U' **and** $a \in U'$
then obtain *U* **where** *U*: *openin X U U'* = $U \cap S$
by (*meson openin_subtopology*)
then obtain *V* **where** $a \in V$ $V \subseteq U$ *openin X V*
and *subV*: *subtopology X* (*X frontier_of V*) *dim_le n−1*
and *dimV*: $\bigwedge T$. *subtopology X* (*X frontier_of V* $\cap$ *T*) *dim_le n−1*
by (*metis 1.IH Int_iff* ‹*a* $\in$ *U'*› *subtopology_subtopology*)
show $\exists W.\ a \in W \wedge W \subseteq U' \wedge$ *openin* (*subtopology X S*) *W* $\wedge$ *subtopology* (*subtopology X S*) (*subtopology X S frontier_of W*) *dim_le n−1*
proof (*intro exI conjI*)
show $a \in S \cap V$ $S \cap V \subseteq U'$
using ‹*U'* = *U* $\cap$ *S*› ‹*a* $\in$ *U'*› ‹*a* $\in$ *V*› ‹*V* $\subseteq$ *U*› **by** *blast+*
show *openin* (*subtopology X S*) (*S* $\cap$ *V*)
by (*simp add*: ‹*openin X V*› *openin_subtopology_Int2*)
have *S* $\cap$ *subtopology X S frontier_of V* $\subseteq$ *X frontier_of V*
by (*simp add*: *frontier_of_subtopology_subset*)
then show *subtopology* (*subtopology X S*) (*subtopology X S frontier_of* (*S* $\cap$ *V*)) *dim_le n−1*
by (*metis dimV frontier_of_restrict inf.absorb_iff2 inf_left_idem subtopology_subtopology topspace_subtopology*)
qed
qed
qed

lemma *dimension_le_subtopologies*:
$[\![$*subtopology X T dim_le n*; $S \subseteq T$$]\!] \Longrightarrow$ (*subtopology X S*) *dim_le n*
by (*metis dimension_le_subtopology inf.absorb_iff2 subtopology_subtopology*)

lemma *dimension_le_eq_subtopology*:
(*subtopology X S*) *dim_le n* $\longleftrightarrow$
$-1 \leq n \wedge$
$(\forall V\ a.$ *openin X V* $\wedge$ $a \in V \wedge a \in S$
$\longrightarrow (\exists U.\ a \in U \wedge U \subseteq V \wedge$ *openin X U* $\wedge$
subtopology X (*subtopology X S frontier_of* (*S* $\cap$ *U*)) *dim_le* $(n-1)))$
proof −
have $*$: $(\exists T.\ a \in T \wedge T \cap S \subseteq V \cap S \wedge$ *openin X T* $\wedge$ *subtopology X* (*S* $\cap$ (*subtopology X S frontier_of* (*T* $\cap$ *S*))) *dim_le n−1*)
$\longleftrightarrow (\exists U.\ a \in U \wedge U \subseteq V \wedge$ *openin X U* $\wedge$ *subtopology X* (*subtopology X*

S frontier_of (*S* ∩ *U*)) *dim_le n−1*)
if *a* ∈ *V a* ∈ *S openin X V* **for** *a V*
proof −
have ∃ *U*. *a* ∈ *U* ∧ *U* ⊆ *V* ∧ *openin X U* ∧ *subtopology X* (*subtopology X S frontier_of* (*S* ∩ *U*)) *dim_le n−1*
if *a* ∈ *T* **and** *sub*: *T* ∩ *S* ⊆ *V* ∩ *S* **and** *openin X T*
and *dim*: *subtopology X* (*S* ∩ *subtopology X S frontier_of* (*T* ∩ *S*)) *dim_le n−1*
for *T*
proof (*intro exI conjI*)
show *openin X* (*T* ∩ *V*)
using ‹*openin X V*› ‹*openin X T*› **by** *blast*
show *subtopology X* (*subtopology X S frontier_of* (*S* ∩ (*T* ∩ *V*))) *dim_le n−1*
by (*metis dim frontier_of_subset_subtopology inf.boundedE inf_absorb2 inf_assoc inf_commute sub*)
qed (*use* ‹*a* ∈ *V*› ‹*a* ∈ *T*› **in** *auto*)
moreover have ∃ *T*. *a* ∈ *T* ∧ *T* ∩ *S* ⊆ *V* ∩ *S* ∧ *openin X T* ∧ *subtopology X* (*S* ∩ *subtopology X S frontier_of* (*T* ∩ *S*)) *dim_le n−1*
if *a* ∈ *U* **and** *U* ⊆ *V* **and** *openin X U*
and *dim*: *subtopology X* (*subtopology X S frontier_of* (*S* ∩ *U*)) *dim_le n−1*
for *U*
by (*metis that frontier_of_subset_subtopology inf_absorb2 inf_commute inf_le1 le_inf_iff*)
ultimately show *?thesis*
by *safe*
qed
show *?thesis*
apply (*simp add*: *dimension_le.simps* [*of* _ *n*] *subtopology_subtopology openin_subtopology flip*: ∗)
by (*safe*; *metis Int_iff inf_le2 le_inf_iff*)
qed

lemma *homeomorphic_space_dimension_le_aux*:
assumes *X homeomorphic_space Y X dim_le of_nat n − 1*
shows *Y dim_le of_nat n − 1*
using *assms*
proof (*induction n arbitrary*: *X Y*)
case *0*
then show *?case*
by (*simp add*: *dimension_le_eq_empty homeomorphic_empty_space*)
next
case (*Suc n*)
then have *X_dim_n*: *X dim_le n*
by *simp*
show *?case*
proof (*clarsimp simp add*: *dimension_le.simps* [*of Y n*])
fix *V b*

```
    assume openin Y V and b ∈ V
    obtain f g where fg: homeomorphic_maps X Y f g
      using ‹X homeomorphic_space Y› homeomorphic_space_def by blast
    then have openin X (g ‘ V)
    using ‹openin Y V› homeomorphic_map_openness_eq homeomorphic_maps_map
by blast
    then obtain U where g b ∈ U openin X U and gim: U ⊆ g ‘ V and sub:
subtopology X (X frontier_of U) dim_le int n − int 1
      using X_dim_n unfolding dimension_le.simps [of X n] by (metis ‹b ∈ V›
imageI of_nat_eq_1_iff)
    show ∃ U. b ∈ U ∧ U ⊆ V ∧ openin Y U ∧ subtopology Y (Y frontier_of U)
dim_le int n − 1
    proof (intro conjI exI)
      show b ∈ f ‘ U
        by (metis (no_types, lifting) ‹b ∈ V› ‹g b ∈ U› ‹openin Y V› fg homeo-
morphic_maps_map image_iff openin_subset subsetD)
      show f ‘ U ⊆ V
      by (smt (verit, ccfv_threshold) ‹openin Y V› fg gim homeomorphic_maps_map
image_iff openin_subset subset_iff)
      show openin Y (f ‘ U)
          using ‹openin X U› fg homeomorphic_map_openness_eq homeomor-
phic_maps_map by blast
      show subtopology Y (Y frontier_of f ‘ U) dim_le int n−1
      proof (rule Suc.IH)
        have homeomorphic_maps (subtopology X (X frontier_of U)) (subtopology
Y (Y frontier_of f ‘ U)) f g
          using ‹openin X U› fg
          by (metis frontier_of_subset_topspace homeomorphic_map_frontier_of
homeomorphic_maps_map homeomorphic_maps_subtopologies openin_subset topspace_subtopology
topspace_subtopology_subset)
      then show subtopology X (X frontier_of U) homeomorphic_space subtopology
Y (Y frontier_of f ‘ U)
          using homeomorphic_space_def by blast
        show subtopology X (X frontier_of U) dim_le int n−1
          using sub by fastforce
      qed
    qed
  qed
qed

lemma homeomorphic_space_dimension_le:
  assumes X homeomorphic_space Y
  shows X dim_le n ⟷ Y dim_le n
proof (cases n ≥ −1)
  case True
  then show ?thesis
    using homeomorphic_space_dimension_le_aux [of _ _ nat(n+1)]
    by (smt (verit) assms homeomorphic_space_sym nat_eq_iff)
next
```

case *False*
then show *?thesis*
by (*metis dimension_le_bound*)
qed

lemma *dimension_le_retraction_map_image*:
⟦*retraction_map X Y r*; *X dim_le n*⟧ ⟹ *Y dim_le n*
by (*meson dimension_le_subtopology homeomorphic_space_dimension_le retraction_map_def retraction_maps_section_image2*)

lemma *dimension_le_discrete_topology* [*simp*]: (*discrete_topology U*) *dim_le 0*
using *dimension_le.simps dimension_le_eq_empty* **by** *fastforce*

end

6.3 Some Uncountable Sets

theory *Uncountable_Sets*
imports *Path_Connected Continuum_Not_Denumerable*
begin

lemma *uncountable_closed_segment*:
fixes *a* :: *'a*::*real_normed_vector*
assumes *a* ≠ *b* **shows** *uncountable* (*closed_segment a b*)
unfolding *path_image_linepath* [*symmetric*] *path_image_def*
using *inj_on_linepath* [*OF assms*] *uncountable_closed_interval* [*of 0 1*]
countable_image_inj_on **by** *auto*

lemma *uncountable_open_segment*:
fixes *a* :: *'a*::*real_normed_vector*
assumes *a* ≠ *b* **shows** *uncountable* (*open_segment a b*)
by (*simp add*: *assms open_segment_def uncountable_closed_segment uncountable_minus_countable*)

lemma *uncountable_convex*:
fixes *a* :: *'a*::*real_normed_vector*
assumes *convex S a* ∈ *S b* ∈ *S a* ≠ *b*
shows *uncountable S*
proof −
have *uncountable* (*closed_segment a b*)
by (*simp add*: *uncountable_closed_segment assms*)
then show *?thesis*
by (*meson assms convex_contains_segment countable_subset*)
qed

lemma *uncountable_ball*:
fixes *a* :: *'a*::*euclidean_space*
assumes *r* > *0*
shows *uncountable* (*ball a r*)

proof −
have *uncountable* (*open_segment a* (*a* + *r* ∗$_R$ (*SOME i*. *i* ∈ *Basis*)))
by (*metis Basis_zero SOME_Basis add_cancel_right_right assms less_le scale_eq_0_iff uncountable_open_segment*)
moreover have *open_segment a* (*a* + *r* ∗$_R$ (*SOME i*. *i* ∈ *Basis*)) ⊆ *ball a r*
using *assms* **by** (*auto simp*: *in_segment algebra_simps dist_norm SOME_Basis*)
ultimately show *?thesis*
by (*metis countable_subset*)
qed

lemma *ball_minus_countable_nonempty*:
assumes *countable* (*A* :: *'a* :: *euclidean_space set*) *r* > *0*
shows *ball z r* − *A* ≠ {}
proof
assume ∗: *ball z r* − *A* = {}
have *uncountable* (*ball z r* − *A*)
by (*intro uncountable_minus_countable assms uncountable_ball*)
thus *False* **by** (*subst* (*asm*) ∗) *auto*
qed

lemma *uncountable_cball*:
fixes *a* :: *'a*::*euclidean_space*
assumes *r* > *0*
shows *uncountable* (*cball a r*)
using *assms countable_subset uncountable_ball* **by** *auto*

lemma *pairwise_disjnt_countable*:
fixes $\mathcal{N}$ :: *nat set set*
assumes *pairwise disjnt* $\mathcal{N}$
shows *countable* $\mathcal{N}$
by (*simp add*: *assms countable_disjoint_open_subsets open_discrete*)

lemma *pairwise_disjnt_countable_Union*:
assumes *countable* ($\bigcup\mathcal{N}$) **and** *pwd*: *pairwise disjnt* $\mathcal{N}$
shows *countable* $\mathcal{N}$
proof −
obtain *f* :: _ ⇒ *nat* **where** *f*: *inj_on f* ($\bigcup\mathcal{N}$)
using *assms* **by** *blast*
then have *pairwise disjnt* ($\bigcup$ *X* ∈ $\mathcal{N}$. {*f* ‘ *X*})
using *assms* **by** (*force simp*: *pairwise_def disjnt_inj_on_iff* [*OF f*])
then have *countable* ($\bigcup$ *X* ∈ $\mathcal{N}$. {*f* ‘ *X*})
using *pairwise_disjnt_countable* **by** *blast*
then show *?thesis*
by (*meson pwd countable_image_inj_on disjoint_image f inj_on_image pairwise_disjnt_countable*)
qed

lemma *connected_uncountable*:
fixes *S* :: *'a*::*metric_space set*

assumes *connected S a* ∈ *S b* ∈ *S a* ≠ *b* **shows** *uncountable S*
proof −
have *continuous_on S* (*dist a*)
by (*intro continuous_intros*)
then have *connected* (*dist a* ‘ *S*)
by (*metis connected_continuous_image* ‹*connected S*›)
then have *closed_segment 0* (*dist a b*) ⊆ (*dist a* ‘ *S*)
by (*simp add*: *assms closed_segment_subset is_interval_connected_1 is_interval_convex*)
then have *uncountable* (*dist a* ‘ *S*)
by (*metis* ‹*a* ≠ *b*› *countable_subset dist_eq_0_iff uncountable_closed_segment*)
then show *?thesis*
by *blast*
qed

lemma *path_connected_uncountable*:
fixes *S* :: *'a*::*metric_space set*
assumes *path_connected S a* ∈ *S b* ∈ *S a* ≠ *b* **shows** *uncountable S*
using *path_connected_imp_connected assms connected_uncountable* **by** *metis*

lemma *simple_path_image_uncountable*:
fixes *g* :: *real* ⇒ *'a*::*metric_space*
assumes *simple_path g*
shows *uncountable* (*path_image g*)
proof −
have *g 0* ∈ *path_image g g* (*1/2*) ∈ *path_image g*
by (*simp_all add*: *path_defs*)
moreover have *g 0* ≠ *g* (*1/2*)
using *assms* **by** (*fastforce simp add*: *simple_path_def loop_free_def*)
ultimately have ∀ *a*. ¬ *path_image g* ⊆ {*a*}
by *blast*
then show *?thesis*
using *assms connected_simple_path_image connected_uncountable* **by** *blast*
qed

lemma *arc_image_uncountable*:
fixes *g* :: *real* ⇒ *'a*::*metric_space*
assumes *arc g*
shows *uncountable* (*path_image g*)
by (*simp add*: *arc_imp_simple_path assms simple_path_image_uncountable*)

end

6.4 Homotopy of Maps

theory *Homotopy*
imports *Path_Connected Product_Topology Uncountable_Sets*
begin

definition *homotopic_with*

where
homotopic_with P X Y f g ≡
(∃*h*. *continuous_map* (*prod_topology* (*top_of_set* {*0..1*::*real*}) *X*) *Y h* ∧
(∀*x*. *h*(*0*, *x*) = *f x*) ∧
(∀*x*. *h*(*1*, *x*) = *g x*) ∧
(∀*t* ∈ {*0..1*}. *P*(λ*x*. *h*(*t*,*x*))))

p, *q* are functions $X \to Y$, and the property P restricts all intermediate maps. We often just want to require that P fixes some subset, but to include the case of a loop homotopy, it is convenient to have a general property P.

abbreviation *homotopic_with_canon* ::
[('*a*::*topological_space* ⇒ '*b*::*topological_space*) ⇒ *bool*, '*a set*, '*b set*, '*a* ⇒ '*b*, '*a* ⇒ '*b*] ⇒ *bool*
where
homotopic_with_canon P S T p q ≡ *homotopic_with P* (*top_of_set S*) (*top_of_set T*) *p q*

lemma *split_01*: {*0..1*::*real*} = {*0..1/2*} ∪ {*1/2..1*}
by *force*

lemma *split_01_prod*: {*0..1*::*real*} × *X* = ({*0..1/2*} × *X*) ∪ ({*1/2..1*} × *X*)
by *force*

lemma *image_Pair_const*: (λ*x*. (*x*, *c*)) ‘ *A* = *A* × {*c*}
by *auto*

lemma *fst_o_paired* [*simp*]: *fst* ∘ (λ(*x*,*y*). (*f x y*, *g x y*)) = (λ(*x*,*y*). *f x y*)
by *auto*

lemma *snd_o_paired* [*simp*]: *snd* ∘ (λ(*x*,*y*). (*f x y*, *g x y*)) = (λ(*x*,*y*). *g x y*)
by *auto*

lemma *continuous_on_o_Pair*: ⟦*continuous_on* (*T* × *X*) *h*; *t* ∈ *T*⟧ ⟹ *continuous_on X* (*h* ∘ *Pair t*)
by (*fast intro*: *continuous_intros elim*!: *continuous_on_subset*)

lemma *continuous_map_o_Pair*:
assumes *h*: *continuous_map* (*prod_topology X Y*) *Z h* **and** *t*: *t* ∈ *topspace X*
shows *continuous_map Y Z* (*h* ∘ *Pair t*)
by (*intro continuous_map_compose* [*OF* _ *h*] *continuous_intros*; *simp add*: *t*)

6.4.1 Trivial properties

We often want to just localize the ending function equality or whatever.

proposition *homotopic_with*:
assumes ⋀*h k*. (⋀*x*. *x* ∈ *topspace X* ⟹ *h x* = *k x*) ⟹ (*P h* ⟷ *P k*)
shows *homotopic_with P X Y p q* ⟷

```
        (∃h. continuous_map (prod_topology (subtopology euclideanreal {0..1})
X) Y h ∧
            (∀x ∈ topspace X. h(0,x) = p x) ∧
            (∀x ∈ topspace X. h(1,x) = q x) ∧
            (∀t ∈ {0..1}. P(λx. h(t, x))))
  unfolding homotopic_with_def
  apply (rule iffI, blast, clarify)
  apply (rule_tac x=λ(u,v). if v ∈ topspace X then h(u,v) else if u = 0 then p v
else q v in exI)
  apply simp
 by (smt (verit, best) SigmaE assms case_prod_conv continuous_map_eq topspace_prod_topology)

lemma homotopic_with_mono:
  assumes hom: homotopic_with P X Y f g
    and Q: ⋀h. ⟦continuous_map X Y h; P h⟧ ⟹ Q h
  shows homotopic_with Q X Y f g
  using hom unfolding homotopic_with_def
  by (force simp: o_def dest: continuous_map_o_Pair intro: Q)

lemma homotopic_with_imp_continuous_maps:
    assumes homotopic_with P X Y f g
    shows continuous_map X Y f ∧ continuous_map X Y g
proof −
  obtain h :: real × 'a ⇒ 'b
    where conth: continuous_map (prod_topology (top_of_set {0..1}) X) Y h
      and h: ∀x. h (0, x) = f x ∀x. h (1, x) = g x
    using assms by (auto simp: homotopic_with_def)
  have ∗: t ∈ {0..1} ⟹ continuous_map X Y (h ∘ (λx. (t,x))) for t
    by (rule continuous_map_compose [OF _ conth]) (simp add: o_def continu-
ous_map_pairwise)
  show ?thesis
    using h ∗[of 0] ∗[of 1] by (simp add: continuous_map_eq)
qed

lemma homotopic_with_imp_continuous:
    assumes homotopic_with_canon P X Y f g
    shows continuous_on X f ∧ continuous_on X g
 by (meson assms continuous_map_subtopology_eu homotopic_with_imp_continuous_maps)

lemma homotopic_with_imp_property:
  assumes homotopic_with P X Y f g
  shows P f ∧ P g
proof
  obtain h where h: ⋀x. h(0, x) = f x ⋀x. h(1, x) = g x and P: ⋀t. t ∈
{0..1::real} ⟹ P(λx. h(t,x))
    using assms by (force simp: homotopic_with_def)
  show P f P g
    using P [of 0] P [of 1] by (force simp: h)+
qed
```

lemma *homotopic_with_equal*:
assumes *P f P g* **and** *contf*: *continuous_map X Y f* **and** *fg*: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *f x* = *g x*
shows *homotopic_with P X Y f g*
unfolding *homotopic_with_def*
proof (*intro exI conjI allI ballI*)
let *?h* = λ(*t*::*real*,*x*). *if t* = *1 then g x else f x*
show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) *X*) *Y ?h*
proof (*rule continuous_map_eq*)
show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) *X*) *Y* (*f* $\circ$ *snd*)
by (*simp add*: *contf continuous_map_of_snd*)
qed (*auto simp*: *fg*)
show *P* ($\lambda x.$ *?h* (*t*, *x*)) **if** *t* $\in$ {*0..1*} **for** *t*
by (*cases t* = *1*) (*simp_all add*: *assms*)
qed *auto*

lemma *homotopic_with_imp_subset1*:
homotopic_with_canon P X Y f g $\Longrightarrow$ *f* ' *X* $\subseteq$ *Y*
by (*meson continuous_map_subtopology_eu homotopic_with_imp_continuous_maps*)

lemma *homotopic_with_imp_subset2*:
homotopic_with_canon P X Y f g $\Longrightarrow$ *g* ' *X* $\subseteq$ *Y*
by (*meson continuous_map_subtopology_eu homotopic_with_imp_continuous_maps*)

lemma *homotopic_with_imp_funspace1*:
homotopic_with_canon P X Y f g $\Longrightarrow$ *f* $\in$ *X* $\to$ *Y*
using *homotopic_with_imp_subset1* **by** *blast*

lemma *homotopic_with_imp_funspace2*:
homotopic_with_canon P X Y f g $\Longrightarrow$ *g* $\in$ *X* $\to$ *Y*
using *homotopic_with_imp_subset2* **by** *blast*

lemma *homotopic_with_subset_left*:
$[\![$*homotopic_with_canon P X Y f g*; *Z* $\subseteq$ *X*$]\!]$ $\Longrightarrow$ *homotopic_with_canon P Z Y f g*
unfolding *homotopic_with_def* **by** (*auto elim*!: *continuous_on_subset ex_forward*)

lemma *homotopic_with_subset_right*:
$[\![$*homotopic_with_canon P X Y f g*; *Y* $\subseteq$ *Z*$]\!]$ $\Longrightarrow$ *homotopic_with_canon P X Z f g*
unfolding *homotopic_with_def* **by** (*auto elim*!: *continuous_on_subset ex_forward*)

6.4.2 Homotopy with P is an equivalence relation

(on continuous functions mapping X into Y that satisfy P, though this only affects reflexivity)

lemma *homotopic_with_refl* [*simp*]: *homotopic_with P X Y f f* $\longleftrightarrow$ *continuous_map X Y f* $\wedge$ *P f*

by (*metis homotopic_with_equal homotopic_with_imp_continuous_maps homotopic_with_imp_property*)

lemma *homotopic_with_symD*:
assumes *homotopic_with P X Y f g*
shows *homotopic_with P X Y g f*
proof −
let *?I01* = *subtopology euclideanreal* {*0..1*}
let *?j* = λ*y*. (*1* − *fst y*, *snd y*)
have *1*: *continuous_map* (*prod_topology ?I01 X*) (*prod_topology euclideanreal X*) *?j*
by (*intro continuous_intros*; *simp add*: *continuous_map_subtopology_fst prod_topology_subtopology*)
have ∗: *continuous_map* (*prod_topology ?I01 X*) (*prod_topology ?I01 X*) *?j*
proof −
have *continuous_map* (*prod_topology ?I01 X*) (*subtopology* (*prod_topology euclideanreal X*) ({*0..1*} × *topspace X*)) *?j*
by (*simp add*: *continuous_map_into_subtopology* [*OF 1*] *image_subset_iff flip*: *image_subset_iff_funcset*)
then show *?thesis*
by (*simp add*: *prod_topology_subtopology*(*1*))
qed
show *?thesis*
using *assms*
apply (*clarsimp simp*: *homotopic_with_def*)
subgoal for *h*
by (*rule_tac x*=*h* ∘ (λ*y*. (*1* − *fst y*, *snd y*)) **in** *exI*) (*simp add*: *continuous_map_compose* [*OF* ∗])
done
qed

lemma *homotopic_with_sym*:
homotopic_with P X Y f g ⟷ *homotopic_with P X Y g f*
by (*metis homotopic_with_symD*)

proposition *homotopic_with_trans*:
assumes *homotopic_with P X Y f g homotopic_with P X Y g h*
shows *homotopic_with P X Y f h*
proof −
let *?X01* = *prod_topology* (*subtopology euclideanreal* {*0..1*}) *X*
obtain *k1 k2*
where *contk1*: *continuous_map ?X01 Y k1* **and** *contk2*: *continuous_map ?X01 Y k2*
and *k12*: ∀ *x*. *k1* (*1*, *x*) = *g x* ∀ *x*. *k2* (*0*, *x*) = *g x*
∀ *x*. *k1* (*0*, *x*) = *f x* ∀ *x*. *k2* (*1*, *x*) = *h x*
and *P*: ∀ *t*∈{*0..1*}. *P* (λ*x*. *k1* (*t*, *x*)) ∀ *t*∈{*0..1*}. *P* (λ*x*. *k2* (*t*, *x*))
using *assms* **by** (*auto simp*: *homotopic_with_def*)
define *k* **where** *k* ≡ λ*y*. *if fst y* ≤ *1*/*2*
then (*k1* ∘ (λ*x*. (*2* ∗$_R$ *fst x*, *snd x*))) *y*
else (*k2* ∘ (λ*x*. (*2* ∗$_R$ *fst x* −*1*, *snd x*))) *y*

```
  have keq: k1 (2 * u, v) = k2 (2 * u -1, v) if u = 1/2 for u v
    by (simp add: k12 that)
  show ?thesis
    unfolding homotopic_with_def
  proof (intro exI conjI)
    show continuous_map ?X01 Y k
      unfolding k_def
    proof (rule continuous_map_cases_le)
      show fst: continuous_map ?X01 euclideanreal fst
        using continuous_map_fst continuous_map_in_subtopology by blast
      show continuous_map ?X01 euclideanreal (λx. 1/2)
        by simp
     show continuous_map (subtopology ?X01 {y ∈ topspace ?X01. fst y ≤ 1/2})
Y
              (k1 ∘ (λx. (2 *R fst x, snd x)))
      apply (intro fst continuous_map_compose [OF _ contk1] continuous_intros
continuous_map_into_subtopology continuous_map_from_subtopology | simp)+
        by (force simp: prod_topology_subtopology)
     show continuous_map (subtopology ?X01 {y ∈ topspace ?X01. 1/2 ≤ fst y})
Y
              (k2 ∘ (λx. (2 *R fst x -1, snd x)))
      apply (intro fst continuous_map_compose [OF _ contk2] continuous_intros
continuous_map_into_subtopology continuous_map_from_subtopology | simp)+
        by (force simp: prod_topology_subtopology)
      show (k1 ∘ (λx. (2 *R fst x, snd x))) y = (k2 ∘ (λx. (2 *R fst x -1, snd
x))) y
        if y ∈ topspace ?X01 and fst y = 1/2 for y
        using that by (simp add: keq)
    qed
    show ∀x. k (0, x) = f x
      by (simp add: k12 k_def)
    show ∀x. k (1, x) = h x
      by (simp add: k12 k_def)
    show ∀t∈{0..1}. P (λx. k (t, x))
    proof
      fix t show t∈{0..1} ⟹ P (λx. k (t, x))
        by (cases t ≤ 1/2) (auto simp: k_def P)
    qed
  qed
qed

lemma homotopic_with_id2:
  (⋀x. x ∈ topspace X ⟹ g (f x) = x) ⟹ homotopic_with (λx. True) X X (g ∘
f) id
  by (metis comp_apply continuous_map_id eq_id_iff homotopic_with_equal ho-
motopic_with_symD)
```

6.4.3 Continuity lemmas

lemma *homotopic_with_compose_continuous_map_left*:
 ⟦*homotopic_with p X1 X2 f g*; *continuous_map X2 X3 h*; ⋀*j*. *p j* ⟹ *q*(*h* ∘ *j*)⟧
 ⟹ *homotopic_with q X1 X3* (*h* ∘ *f*) (*h* ∘ *g*)
 unfolding *homotopic_with_def*
 apply *clarify*
 subgoal for *k*
 by (*rule_tac x*=*h* ∘ *k* **in** *exI*) (*rule conjI continuous_map_compose* | *simp add*: *o_def*)+
 done

lemma *homotopic_with_compose_continuous_map_right*:
 assumes *hom*: *homotopic_with p X2 X3 f g* **and** *conth*: *continuous_map X1 X2 h*
 and *q*: ⋀*j*. *p j* ⟹ *q*(*j* ∘ *h*)
 shows *homotopic_with q X1 X3* (*f* ∘ *h*) (*g* ∘ *h*)
proof −
 obtain *k*
 where *contk*: *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0..1*}) *X2*) *X3 k*
 and *k*: ∀ *x*. *k* (*0*, *x*) = *f x* ∀ *x*. *k* (*1*, *x*) = *g x* **and** *p*: ⋀*t*. *t*∈{*0..1*} ⟹ *p* (λ*x*. *k* (*t*, *x*))
 using *hom* **unfolding** *homotopic_with_def* **by** *blast*
 have *hsnd*: *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0..1*}) *X1*) *X2* (*h* ∘ *snd*)
 by (*rule continuous_map_compose* [*OF continuous_map_snd conth*])
 let *?h* = *k* ∘ (λ(*t*,*x*). (*t*,*h x*))
 show *?thesis*
 unfolding *homotopic_with_def*
 proof (*intro exI conjI allI ballI*)
 have *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) *X1*)
 (*prod_topology* (*top_of_set* {*0..1*::*real*}) *X2*) (λ(*t*, *x*). (*t*, *h x*))
 by (*metis* (*mono_tags*, *lifting*) *case_prod_beta*′ *comp_def continuous_map_eq continuous_map_fst continuous_map_pairedI hsnd*)
 then show *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0..1*}) *X1*) *X3 ?h*
 by (*intro conjI continuous_map_compose* [*OF _ contk*])
 show *q* (λ*x*. *?h* (*t*, *x*)) **if** *t* ∈ {*0..1*} **for** *t*
 using *q* [*OF p* [*OF that*]] **by** (*simp add*: *o_def*)
 qed (*auto simp*: *k*)
qed

corollary *homotopic_compose*:
 assumes *homotopic_with* (λ*x*. *True*) *X Y f f*′ *homotopic_with* (λ*x*. *True*) *Y Z g g*′
 shows *homotopic_with* (λ*x*. *True*) *X Z* (*g* ∘ *f*) (*g*′ ∘ *f*′)
 by (*metis assms homotopic_with_compose_continuous_map_left homotopic_with_compose_continuou*
homotopic_with_imp_continuous_maps homotopic_with_trans)

proposition *homotopic_with_compose_continuous_right*:
  ⟦*homotopic_with_canon* (λ*f*. *p* (*f* ∘ *h*)) *X Y f g*; *continuous_on W h*; *h* ∈ *W* → *X*⟧
  ⟹ *homotopic_with_canon p W Y* (*f* ∘ *h*) (*g* ∘ *h*)
 by (*simp add*: *homotopic_with_compose_continuous_map_right image_subset_iff_funcset*)

proposition *homotopic_with_compose_continuous_left*:
  ⟦*homotopic_with_canon* (λ*f*. *p* (*h* ∘ *f*)) *X Y f g*; *continuous_on Y h*; *h* ∈ *Y* → *Z*⟧
  ⟹ *homotopic_with_canon p X Z* (*h* ∘ *f*) (*h* ∘ *g*)
 by (*simp add*: *homotopic_with_compose_continuous_map_left image_subset_iff_funcset*)

lemma *homotopic_from_subtopology*:
  homotopic_with P X X′ f g ⟹ *homotopic_with P* (*subtopology X S*) *X′ f g*
 by (*metis continuous_map_id_subt homotopic_with_compose_continuous_map_right o_id*)

lemma *homotopic_on_emptyI*:
 assumes *P f P g*
 shows *homotopic_with P trivial_topology X f g*
 by (*metis assms continuous_map_on_empty empty_iff homotopic_with_equal topspace_discrete_topology*)

lemma *homotopic_on_empty*:
  (*homotopic_with P trivial_topology X f g* ⟷ *P f* ∧ *P g*)
 using *homotopic_on_emptyI homotopic_with_imp_property* **by** *metis*

lemma *homotopic_with_canon_on_empty*: *homotopic_with_canon* (λ*x*. *True*) {} *t f g*
 by (*auto intro*: *homotopic_with_equal*)

lemma *homotopic_constant_maps*:
  homotopic_with (λ*x*. *True*) *X X′* (λ*x*. *a*) (λ*x*. *b*) ⟷
   X = *trivial_topology* ∨ *path_component_of X′ a b* (**is** *?lhs* = *?rhs*)
proof (*cases X* = *trivial_topology*)
 case *False*
 then obtain *c* **where** *c*: *c* ∈ *topspace X*
   by *fastforce*
 have ∃ *g*. *continuous_map* (*top_of_set* {*0..1*::*real*}) *X′ g* ∧ *g 0* = *a* ∧ *g 1* = *b*
   if *x* ∈ *topspace X* **and** *hom*: *homotopic_with* (λ*x*. *True*) *X X′* (λ*x*. *a*) (λ*x*. *b*)
for *x*
 proof −
   obtain *h* :: *real* × *'a* ⇒ *'b*
     where *conth*: *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) *X*) *X′ h*
       and *h*: ⋀*x*. *h* (*0*, *x*) = *a* ⋀*x*. *h* (*1*, *x*) = *b*
     using *hom* **by** (*auto simp*: *homotopic_with_def*)
   have *cont*: *continuous_map* (*top_of_set* {*0..1*}) *X′* (*h* ∘ (λ*t*. (*t*, *c*)))
     by (*rule continuous_map_compose* [*OF* _ *conth*] *continuous_intros* | *simp add*: *c*)+

```
    then show ?thesis
      by (force simp: h)
  qed
  moreover have homotopic_with (λx. True) X X' (λx. g 0) (λx. g 1)
    if x ∈ topspace X a = g 0 b = g 1 continuous_map (top_of_set {0..1}) X' g
    for x and g :: real ⇒ 'b
    unfolding homotopic_with_def
    by (force intro!: continuous_map_compose continuous_intros c that)
  ultimately show ?thesis
    using False
    by (metis c path_component_of_set pathin_def)
qed (simp add: homotopic_on_empty)

proposition homotopic_with_eq:
   assumes h: homotopic_with P X Y f g
       and f': ⋀x. x ∈ topspace X ⟹ f' x = f x
       and g': ⋀x. x ∈ topspace X ⟹ g' x = g x
       and P:  (⋀h k. (⋀x. x ∈ topspace X ⟹ h x = k x) ⟹ P h ⟷ P k)
   shows homotopic_with P X Y f' g'
  by (smt (verit, ccfv_SIG) assms homotopic_with)

lemma homotopic_with_prod_topology:
  assumes homotopic_with p X1 Y1 f f' and homotopic_with q X2 Y2 g g'
    and r: ⋀i j. ⟦p i; q j⟧ ⟹ r(λ(x,y). (i x, j y))
  shows homotopic_with r (prod_topology X1 X2) (prod_topology Y1 Y2)
                         (λz. (f(fst z),g(snd z))) (λz. (f'(fst z), g'(snd z)))
proof −
  obtain h
    where h: continuous_map (prod_topology (subtopology euclideanreal {0..1})
X1) Y1 h
      and h0: ⋀x. h (0, x) = f x
      and h1: ⋀x. h (1, x) = f' x
      and p: ⋀t. ⟦0 ≤ t; t ≤ 1⟧ ⟹ p (λx. h (t,x))
    using assms unfolding homotopic_with_def by auto
  obtain k
    where k: continuous_map (prod_topology (subtopology euclideanreal {0..1})
X2) Y2 k
      and k0: ⋀x. k (0, x) = g x
      and k1: ⋀x. k (1, x) = g' x
      and q: ⋀t. ⟦0 ≤ t; t ≤ 1⟧ ⟹ q (λx. k (t,x))
    using assms unfolding homotopic_with_def by auto
  let ?hk = λ(t,x,y). (h(t,x), k(t,y))
  show ?thesis
    unfolding homotopic_with_def
  proof (intro conjI allI exI)
   show continuous_map (prod_topology (subtopology euclideanreal {0..1}) (prod_topology
X1 X2))
                       (prod_topology Y1 Y2) ?hk
      unfolding continuous_map_pairwise case_prod_unfold
```

by (*rule conjI continuous_map_pairedI continuous_intros continuous_map_id* [*unfolded id_def*]
continuous_map_fst_of [*unfolded o_def*] *continuous_map_snd_of* [*unfolded o_def*]
continuous_map_compose [*OF _ h*, *unfolded o_def*]
continuous_map_compose [*OF _ k*, *unfolded o_def*])+
next
fix *x*
show *?hk* (*0*, *x*) = (*f* (*fst x*), *g* (*snd x*)) *?hk* (*1*, *x*) = (*f'* (*fst x*), *g'* (*snd x*))
by (*auto simp*: *case_prod_beta h0 k0 h1 k1*)
qed (*auto simp*: *p q r*)
qed

lemma *homotopic_with_product_topology*:
assumes *ht*: ⋀*i*. *i* ∈ *I* ⟹ *homotopic_with* (*p i*) (*X i*) (*Y i*) (*f i*) (*g i*)
and *pq*: ⋀*h*. (⋀*i*. *i* ∈ *I* ⟹ *p i* (*h i*)) ⟹ *q*(λ*x*. (λ*i*∈*I*. *h i* (*x i*)))
shows *homotopic_with q* (*product_topology X I*) (*product_topology Y I*)
(λ*z*. (λ*i*∈*I*. (*f i*) (*z i*))) (λ*z*. (λ*i*∈*I*. (*g i*) (*z i*)))
proof −
obtain *h*
where *h*: ⋀*i*. *i* ∈ *I* ⟹ *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0..1*}) (*X i*)) (*Y i*) (*h i*)
and *h0*: ⋀*i x*. *i* ∈ *I* ⟹ *h i* (*0*, *x*) = *f i x*
and *h1*: ⋀*i x*. *i* ∈ *I* ⟹ *h i* (*1*, *x*) = *g i x*
and *p*: ⋀*i t*. ⟦*i* ∈ *I*; *t* ∈ {*0..1*}⟧ ⟹ *p i* (λ*x*. *h i* (*t*,*x*))
using *ht* **unfolding** *homotopic_with_def* **by** *metis*
show *?thesis*
unfolding *homotopic_with_def*
proof (*intro conjI allI exI*)
let *?h* = λ(*t*,*z*). λ*i*∈*I*. *h i* (*t*,*z i*)
have *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0..1*}) (*product_topology X I*))
(*Y i*) (λ*x*. *h i* (*fst x*, *snd x i*)) **if** *i* ∈ *I* **for** *i*
proof −
have §: *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*product_topology X I*)) (*X i*) (λ*x*. *snd x i*)
using *continuous_map_componentwise continuous_map_snd that* **by** *fastforce*
show *?thesis*
unfolding *continuous_map_pairwise case_prod_unfold*
by (*intro conjI that* § *continuous_intros continuous_map_compose* [*OF _ h*, *unfolded o_def*])
qed
then show *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0..1*}) (*product_topology X I*))
(*product_topology Y I*) *?h*
by (*auto simp*: *continuous_map_componentwise case_prod_beta*)
show *?h* (*0*, *x*) = (λ*i*∈*I*. *f i* (*x i*)) *?h* (*1*, *x*) = (λ*i*∈*I*. *g i* (*x i*)) **for** *x*

```
    by (auto simp: case_prod_beta h0 h1)
  show ∀ t∈{0..1}. q (λx. ?h (t, x))
    by (force intro: p pq)
 qed
qed
```

Homotopic triviality implicitly incorporates path-connectedness.

```
lemma homotopic_triviality:
 shows  (∀ f g. continuous_on S f ∧ f ∈ S → T ∧
                continuous_on S g ∧ g ∈ S → T
                ⟶ homotopic_with_canon (λx. True) S T f g) ⟷
        (S = {} ∨ path_connected T) ∧
         (∀ f. continuous_on S f ∧ f ∈ S → T ⟶ (∃ c. homotopic_with_canon
(λx. True) S T f (λx. c)))
        (is ?lhs = ?rhs)
proof (cases S = {} ∨ T = {})
 case True then show ?thesis
   by (auto simp: homotopic_on_emptyI simp flip: image_subset_iff_funcset)
next
 case False show ?thesis
 proof
   assume LHS [rule_format]: ?lhs
   have pab: path_component T a b if a ∈ T b ∈ T for a b
   proof −
     have homotopic_with_canon (λx. True) S T (λx. a) (λx. b)
       by (simp add: LHS image_subset_iff that)
     then show ?thesis
       using False homotopic_constant_maps [of top_of_set S top_of_set T a b]
      by (metis path_component_of_canon_iff topspace_discrete_topology topspace_euclidean_subtopolo
   qed
   moreover
   have ∃ c. homotopic_with_canon (λx. True) S T f (λx. c) if continuous_on S
f f ∈ S → T for f
     using False LHS continuous_on_const that by blast
   ultimately show ?rhs
     by (simp add: path_connected_component)
 next
   assume RHS: ?rhs
   with False have T: path_connected T
     by blast
   show ?lhs
   proof clarify
     fix f g
     assume continuous_on S f f ∈ S → T continuous_on S g g ∈ S → T
     obtain c d where c: homotopic_with_canon (λx. True) S T f (λx. c) and
d: homotopic_with_canon (λx. True) S T g (λx. d)
       using RHS ‹continuous_on S f› ‹continuous_on S g› ‹f ∈ S → T› ‹g ∈ S
→ T› by presburger
     with T have path_component T c d
```

by (*metis False ex_in_conv homotopic_with_imp_subset2 image_subset_iff path_connected_component*)
then have *homotopic_with_canon* (λ*x*. *True*) *S T* (λ*x*. *c*) (λ*x*. *d*)
by (*simp add*: *homotopic_constant_maps*)
with *c d* **show** *homotopic_with_canon* (λ*x*. *True*) *S T f g*
by (*meson homotopic_with_symD homotopic_with_trans*)
qed
qed
qed

6.4.4 Homotopy of paths, maintaining the same endpoints

definition *homotopic_paths* :: ['*a set*, *real* ⇒ '*a*, *real* ⇒ '*a*::*topological_space*] ⇒ *bool*
where
homotopic_paths S p q ≡
homotopic_with_canon (λ*r*. *pathstart r* = *pathstart p* ∧ *pathfinish r* = *pathfinish p*) {*0*..*1*} *S p q*

lemma *homotopic_paths*:
homotopic_paths S p q ⟷
(∃ *h*. *continuous_on* ({*0*..*1*} × {*0*..*1*}) *h* ∧
h ∈ ({*0*..*1*} × {*0*..*1*}) → *S* ∧
(∀ *x* ∈ {*0*..*1*}. *h*(*0*,*x*) = *p x*) ∧
(∀ *x* ∈ {*0*..*1*}. *h*(*1*,*x*) = *q x*) ∧
(∀ *t* ∈ {*0*..*1*::*real*}. *pathstart*(*h* ∘ *Pair t*) = *pathstart p* ∧
pathfinish(*h* ∘ *Pair t*) = *pathfinish p*))
by (*auto simp*: *homotopic_paths_def homotopic_with pathstart_def pathfinish_def*)

proposition *homotopic_paths_imp_pathstart*:
homotopic_paths S p q ⟹ *pathstart p* = *pathstart q*
by (*metis* (*mono_tags*, *lifting*) *homotopic_paths_def homotopic_with_imp_property*)

proposition *homotopic_paths_imp_pathfinish*:
homotopic_paths S p q ⟹ *pathfinish p* = *pathfinish q*
by (*metis* (*mono_tags*, *lifting*) *homotopic_paths_def homotopic_with_imp_property*)

lemma *homotopic_paths_imp_path*:
homotopic_paths S p q ⟹ *path p* ∧ *path q*
using *homotopic_paths_def homotopic_with_imp_continuous_maps path_def continuous_map_subtopology_eu* **by** *blast*

lemma *homotopic_paths_imp_subset*:
homotopic_paths S p q ⟹ *path_image p* ⊆ *S* ∧ *path_image q* ⊆ *S*
by (*metis* (*mono_tags*) *continuous_map_subtopology_eu homotopic_paths_def homotopic_with_imp_continuous_maps path_image_def*)

proposition *homotopic_paths_refl* [*simp*]: *homotopic_paths S p p* ⟷ *path p* ∧ *path_image p* ⊆ *S*

by (*simp add*: *homotopic_paths_def path_def path_image_def*)

proposition *homotopic_paths_sym*: *homotopic_paths S p q* ⟹ *homotopic_paths S q p*
by (*metis* (*mono_tags*) *homotopic_paths_def homotopic_paths_imp_pathfinish homotopic_paths_imp_pathstart homotopic_with_symD*)

proposition *homotopic_paths_sym_eq*: *homotopic_paths S p q* ⟷ *homotopic_paths S q p*
by (*metis homotopic_paths_sym*)

proposition *homotopic_paths_trans* [*trans*]:
assumes *homotopic_paths S p q homotopic_paths S q r*
shows *homotopic_paths S p r*
using *assms homotopic_paths_imp_pathfinish homotopic_paths_imp_pathstart*
unfolding *homotopic_paths_def*
by (*smt* (*verit*, *ccfv_SIG*) *homotopic_with_mono homotopic_with_trans*)

proposition *homotopic_paths_eq*:
⟦*path p*; *path_image p* ⊆ *S*; ⋀*t*. *t* ∈ {*0..1*} ⟹ *p t* = *q t*⟧ ⟹ *homotopic_paths S p q*
by (*smt* (*verit*, *best*) *homotopic_paths homotopic_paths_refl*)

proposition *homotopic_paths_reparametrize*:
assumes *path p*
and *pips*: *path_image p* ⊆ *S*
and *contf*: *continuous_on* {*0..1*} *f*
and *f01* :*f* ∈ {*0..1*} → {*0..1*}
and [*simp*]: *f*(*0*) = *0 f*(*1*) = *1*
and *q*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *q*(*t*) = *p*(*f t*)
shows *homotopic_paths S p q*
proof −
have *contp*: *continuous_on* {*0..1*} *p*
by (*metis* ‹*path p*› *path_def*)
then have *continuous_on* {*0..1*} (*p* ∘ *f*)
by (*meson assms*(*4*) *contf continuous_on_compose continuous_on_subset image_subset_iff_funcset*)
then have *path q*
by (*simp add*: *path_def*) (*metis q continuous_on_cong*)
have *piqs*: *path_image q* ⊆ *S*
by (*smt* (*verit*, *ccfv_threshold*) *Pi_iff assms*(*2*) *assms*(*4*) *assms*(*7*) *image_subset_iff path_defs*(*4*))
have *fb0*: ⋀*a b*. ⟦*0* ≤ *a*; *a* ≤ *1*; *0* ≤ *b*; *b* ≤ *1*⟧ ⟹ *0* ≤ (*1* − *a*) ∗ *f b* + *a* ∗ *b*
using *f01* **by** *force*
have *fb1*: ⟦*0* ≤ *a*; *a* ≤ *1*; *0* ≤ *b*; *b* ≤ *1*⟧ ⟹ (*1* − *a*) ∗ *f b* + *a* ∗ *b* ≤ *1* **for** *a b*
by (*intro convex_bound_le*) (*use f01* **in** *auto*)
have *homotopic_paths S q p*
proof (*rule homotopic_paths_trans*)
show *homotopic_paths S q* (*p* ∘ *f*)

using *q* **by** (*force intro*: *homotopic_paths_eq* [*OF* ‹*path q*› *piqs*])
next
show *homotopic_paths S* (*p* ∘ *f*) *p*
using *pips* [*unfolded path_image_def*]
apply (*simp add*: *homotopic_paths_def homotopic_with_def*)
apply (*rule_tac x*=*p* ∘ (λ*y*. (1 − (*fst y*)) *$*_R$ ((*f* ∘ *snd*) *y*) + (*fst y*) *$*_R$ *snd y*) **in** *exI*)
apply (*rule conjI contf continuous_intros continuous_on_subset* [*OF contp*] | *simp*)+
by (*auto simp*: *fb0 fb1 pathstart_def pathfinish_def*)
qed
then show *?thesis*
by (*simp add*: *homotopic_paths_sym*)
qed

lemma *homotopic_paths_subset*: ⟦*homotopic_paths S p q*; *S* ⊆ *t*⟧ ⟹ *homotopic_paths t p q*
unfolding *homotopic_paths* **by** *fast*

A slightly ad-hoc but useful lemma in constructing homotopies.

lemma *continuous_on_homotopic_join_lemma*:
fixes *q* :: [*real*,*real*] ⇒ *'a*::*topological_space*
assumes *p*: *continuous_on* ({*0..1*} × {*0..1*}) (λ*y*. *p* (*fst y*) (*snd y*)) (**is** *continuous_on ?A ?p*)
and *q*: *continuous_on* ({*0..1*} × {*0..1*}) (λ*y*. *q* (*fst y*) (*snd y*)) (**is** *continuous_on ?A ?q*)
and *pf*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *pathfinish*(*p t*) = *pathstart*(*q t*)
shows *continuous_on* ({*0..1*} × {*0..1*}) (λ*y*. (*p*(*fst y*) +++ *q*(*fst y*)) (*snd y*))
proof −
have §: (λ*t*. *p* (*fst t*) (*2* ∗ *snd t*)) = *?p* ∘ (λ*y*. (*fst y*, *2* ∗ *snd y*))
(λ*t*. *q* (*fst t*) (*2* ∗ *snd t* − *1*)) = *?q* ∘ (λ*y*. (*fst y*, *2* ∗ *snd y* − *1*))
by *force*+
show *?thesis*
unfolding *joinpaths_def*
proof (*rule continuous_on_cases_le*)
show *continuous_on* {*y* ∈ *?A*. *snd y* ≤ *1/2*} (λ*t*. *p* (*fst t*) (*2* ∗ *snd t*))
continuous_on {*y* ∈ *?A*. *1/2* ≤ *snd y*} (λ*t*. *q* (*fst t*) (*2* ∗ *snd t* − *1*))
continuous_on ?A snd
unfolding §
by (*rule continuous_intros continuous_on_subset* [*OF p*] *continuous_on_subset* [*OF q*] | *force*)+
qed (*use pf* **in** ‹*auto simp*: *mult.commute pathstart_def pathfinish_def*›)
qed

Congruence properties of homotopy w.r.t. path-combining operations.

lemma *homotopic_paths_reversepath_D*:
assumes *homotopic_paths S p q*
shows *homotopic_paths S* (*reversepath p*) (*reversepath q*)
using *assms*

apply (*simp add*: *homotopic_paths_def homotopic_with_def*, *clarify*)
apply (*rule_tac x*=*h* ∘ (λ*x*. (*fst x*, *1* − *snd x*)) **in** *exI*)
apply (*rule conjI continuous_intros*)+
apply (*auto simp*: *reversepath_def pathstart_def pathfinish_def elim*!: *continuous_on_subset*)
done

proposition *homotopic_paths_reversepath*:
homotopic_paths S (*reversepath p*) (*reversepath q*) ⟷ *homotopic_paths S p q*
using *homotopic_paths_reversepath_D* **by** *force*

proposition *homotopic_paths_join*:
⟦*homotopic_paths S p p*′; *homotopic_paths S q q*′; *pathfinish p* = *pathstart q*⟧ ⟹ *homotopic_paths S* (*p* +++ *q*) (*p*′ +++ *q*′)
apply (*clarsimp simp*: *homotopic_paths_def homotopic_with_def*)
apply (*rename_tac k1 k2*)
apply (*rule_tac x*=(λ*y*. ((*k1* ∘ *Pair* (*fst y*)) +++ (*k2* ∘ *Pair* (*fst y*))) (*snd y*)) **in** *exI*)
apply (*intro conjI continuous_intros continuous_on_homotopic_join_lemma*; *force simp*: *joinpaths_def pathstart_def pathfinish_def path_image_def*)
done

proposition *homotopic_paths_continuous_image*:
⟦*homotopic_paths S f g*; *continuous_on S h*; *h* ∈ *S* → *t*⟧ ⟹ *homotopic_paths t* (*h* ∘ *f*) (*h* ∘ *g*)
unfolding *homotopic_paths_def*
by (*simp add*: *homotopic_with_compose_continuous_map_left pathfinish_compose pathstart_compose image_subset_iff_funcset*)

6.4.5 Group properties for homotopy of paths

So taking equivalence classes under homotopy would give the fundamental group

proposition *homotopic_paths_rid*:
assumes *path p path_image p* ⊆ *S*
shows *homotopic_paths S* (*p* +++ *linepath* (*pathfinish p*) (*pathfinish p*)) *p*
proof −
have §: *continuous_on* {*0..1*} (λ*t*::*real*. *if t* ≤ *1/2 then 2* ∗*R t else 1*)
unfolding *split_01*
by (*rule continuous_on_cases continuous_intros* | *force simp*: *pathfinish_def joinpaths_def*)+
show *?thesis*
apply (*rule homotopic_paths_sym*)
using *assms* **unfolding** *pathfinish_def joinpaths_def*
by (*intro* § *continuous_on_cases continuous_intros homotopic_paths_reparametrize* [**where** *f* = λ*t*. *if t* ≤ *1/2 then 2* ∗*R t else 1*]; *force*)
qed

proposition *homotopic_paths_lid*:
⟦*path p*; *path_image p* ⊆ *S*⟧ ⟹ *homotopic_paths S* (*linepath* (*pathstart p*) (*pathstart p*) +++ *p*) *p*
using *homotopic_paths_rid* [*of reversepath p S*]
by (*metis homotopic_paths_reversepath path_image_reversepath path_reversepath pathfinish_linepath*
pathfinish_reversepath reversepath_joinpaths reversepath_linepath)

lemma *homotopic_paths_rid'*:
assumes *path p path_image p* ⊆ *s x* = *pathfinish p*
shows *homotopic_paths s* (*p* +++ *linepath x x*) *p*
using *homotopic_paths_rid*[*of p s*] *assms* **by** *simp*

lemma *homotopic_paths_lid'*:
⟦*path p*; *path_image p* ⊆ *s*; *x* = *pathstart p*⟧ ⟹ *homotopic_paths s* (*linepath x x* +++ *p*) *p*
using *homotopic_paths_lid*[*of p s*] **by** *simp*

proposition *homotopic_paths_assoc*:
⟦*path p*; *path_image p* ⊆ *S*; *path q*; *path_image q* ⊆ *S*; *path r*; *path_image r* ⊆ *S*; *pathfinish p* = *pathstart q*;
pathfinish q = *pathstart r*⟧
⟹ *homotopic_paths S* (*p* +++ (*q* +++ *r*)) ((*p* +++ *q*) +++ *r*)
apply (*subst homotopic_paths_sym*)
apply (*rule homotopic_paths_reparametrize*
[**where** *f* = λ*t*. *if* *t* ≤ *1/2 then inverse 2* ∗$_R$ *t*
else if *t* ≤ *3 / 4 then t* − (*1 / 4*)
else 2 ∗$_R$ *t* − *1*])
apply (*simp_all del*: *le_divide_eq_numeral1 add*: *subset_path_image_join*)
apply (*rule continuous_on_cases_1 continuous_intros* | *auto simp*: *joinpaths_def*)+
done

proposition *homotopic_paths_rinv*:
assumes *path p path_image p* ⊆ *S*
shows *homotopic_paths S* (*p* +++ *reversepath p*) (*linepath* (*pathstart p*) (*pathstart p*))
proof −
have *p*: *continuous_on* {*0..1*} *p*
using *assms* **by** (*auto simp*: *path_def*)
let *?A* = {*0..1*} × {*0..1*}
have *continuous_on ?A* (λ*x*. (*subpath 0* (*fst x*) *p* +++ *reversepath* (*subpath 0* (*fst x*) *p*)) (*snd x*))
unfolding *joinpaths_def subpath_def reversepath_def path_def add_0_right diff_0_right*
proof (*rule continuous_on_cases_le*)
show *continuous_on* {*x* ∈ *?A*. *snd x* ≤ *1/2*} (λ*t*. *p* (*fst t* ∗ (*2* ∗ *snd t*)))
continuous_on {*x* ∈ *?A*. *1/2* ≤ *snd x*} (λ*t*. *p* (*fst t* ∗ (*1* − (*2* ∗ *snd t* − *1*))))

continuous_on ?A snd
by (*intro continuous_on_compose2* [*OF p*] *continuous_intros*; *auto simp*: *mult_le_one*)+
qed (*auto simp*: *algebra_simps*)
then show *?thesis*
using *assms*
apply (*subst homotopic_paths_sym_eq*)
unfolding *homotopic_paths_def homotopic_with_def*
apply (*rule_tac x*=(λ*y*. (*subpath 0* (*fst y*) *p* +++ *reversepath*(*subpath 0* (*fst y*) *p*)) (*snd y*)) **in** *exI*)
apply (*force simp*: *mult_le_one path_defs joinpaths_def subpath_def reversepath_def*)
done
qed

proposition *homotopic_paths_linv*:
assumes *path p path_image p* ⊆ *S*
shows *homotopic_paths S* (*reversepath p* +++ *p*) (*linepath* (*pathfinish p*) (*pathfinish p*))
using *homotopic_paths_rinv* [*of reversepath p S*] *assms* **by** *simp*

6.4.6 Homotopy of loops without requiring preservation of endpoints

definition *homotopic_loops* :: *'a*::*topological_space set* ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *bool* **where**
homotopic_loops S p q ≡
homotopic_with_canon (λ*r*. *pathfinish r* = *pathstart r*) {*0..1*} *S p q*

lemma *homotopic_loops*:
homotopic_loops S p q ⟷
(∃*h*. *continuous_on* ({*0..1*::*real*} × {*0..1*}) *h* ∧
image h ({*0..1*} × {*0..1*}) ⊆ *S* ∧
(∀ *x* ∈ {*0..1*}. *h*(*0*,*x*) = *p x*) ∧
(∀ *x* ∈ {*0..1*}. *h*(*1*,*x*) = *q x*) ∧
(∀ *t* ∈ {*0..1*}. *pathfinish*(*h* ∘ *Pair t*) = *pathstart*(*h* ∘ *Pair t*)))
by (*simp add*: *homotopic_loops_def pathstart_def pathfinish_def homotopic_with*)

proposition *homotopic_loops_imp_loop*:
homotopic_loops S p q ⟹ *pathfinish p* = *pathstart p* ∧ *pathfinish q* = *pathstart q*
using *homotopic_with_imp_property homotopic_loops_def* **by** *blast*

proposition *homotopic_loops_imp_path*:
homotopic_loops S p q ⟹ *path p* ∧ *path q*
unfolding *homotopic_loops_def path_def*
using *homotopic_with_imp_continuous_maps continuous_map_subtopology_eu*
by *blast*

proposition *homotopic_loops_imp_subset*:

homotopic_loops S p q ⟹ *path_image p* ⊆ *S* ∧ *path_image q* ⊆ *S*
unfolding *homotopic_loops_def path_image_def*
by (*meson continuous_map_subtopology_eu homotopic_with_imp_continuous_maps*)

proposition *homotopic_loops_refl*:
homotopic_loops S p p ⟷
path p ∧ *path_image p* ⊆ *S* ∧ *pathfinish p* = *pathstart p*
by (*simp add*: *homotopic_loops_def path_image_def path_def*)

proposition *homotopic_loops_sym*: *homotopic_loops S p q* ⟹ *homotopic_loops S q p*
by (*simp add*: *homotopic_loops_def homotopic_with_sym*)

proposition *homotopic_loops_sym_eq*: *homotopic_loops S p q* ⟷ *homotopic_loops S q p*
by (*metis homotopic_loops_sym*)

proposition *homotopic_loops_trans*:
⟦*homotopic_loops S p q*; *homotopic_loops S q r*⟧ ⟹ *homotopic_loops S p r*
unfolding *homotopic_loops_def* **by** (*blast intro*: *homotopic_with_trans*)

proposition *homotopic_loops_subset*:
⟦*homotopic_loops S p q*; *S* ⊆ *t*⟧ ⟹ *homotopic_loops t p q*
by (*fastforce simp*: *homotopic_loops*)

proposition *homotopic_loops_eq*:
⟦*path p*; *path_image p* ⊆ *S*; *pathfinish p* = *pathstart p*; ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*t*) = *q*(*t*)⟧
⟹ *homotopic_loops S p q*
unfolding *homotopic_loops_def path_image_def path_def pathstart_def pathfinish_def*
by (*auto intro*: *homotopic_with_eq* [*OF homotopic_with_refl* [**where** *f* = *p*, *THEN iffD2*]])

proposition *homotopic_loops_continuous_image*:
⟦*homotopic_loops S f g*; *continuous_on S h*; *h* ∈ *S* → *t*⟧ ⟹ *homotopic_loops t* (*h* ∘ *f*) (*h* ∘ *g*)
unfolding *homotopic_loops_def*
by (*simp add*: *homotopic_with_compose_continuous_map_left pathfinish_def pathstart_def image_subset_iff_funcset*)

6.4.7 Relations between the two variants of homotopy

proposition *homotopic_paths_imp_homotopic_loops*:
⟦*homotopic_paths S p q*; *pathfinish p* = *pathstart p*; *pathfinish q* = *pathstart p*⟧
⟹ *homotopic_loops S p q*
by (*auto simp*: *homotopic_with_def homotopic_paths_def homotopic_loops_def*)

proposition *homotopic_loops_imp_homotopic_paths_null*:

assumes *homotopic_loops S p* (*linepath a a*)
shows *homotopic_paths S p* (*linepath* (*pathstart p*) (*pathstart p*))
proof −
have *path p* **by** (*metis assms homotopic_loops_imp_path*)
have *ploop*: *pathfinish p* = *pathstart p* **by** (*metis assms homotopic_loops_imp_loop*)
have *pip*: *path_image p* ⊆ *S* **by** (*metis assms homotopic_loops_imp_subset*)
let *?A* = {*0..1*::*real*} × {*0..1*::*real*}
obtain *h* **where** *conth*: *continuous_on ?A h*
and *hs*: *h* ∈ *?A* → *S*
and *h0*[*simp*]: ⋀*x*. *x* ∈ {*0..1*} ⟹ *h*(*0*,*x*) = *p x*
and *h1*[*simp*]: ⋀*x*. *x* ∈ {*0..1*} ⟹ *h*(*1*,*x*) = *a*
and *ends*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *pathfinish* (*h* ∘ *Pair t*) = *pathstart* (*h* ∘ *Pair t*)
using *assms* **by** (*auto simp*: *homotopic_loops homotopic_with image_subset_iff_funcset*)
have *conth0*: *path* (λ*u*. *h* (*u*, *0*))
unfolding *path_def*
proof (*rule continuous_on_compose* [*of* _ _ *h*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*x*, *0*)) ‘ {*0..1*}) *h*
by (*force intro*: *continuous_on_subset* [*OF conth*])
qed (*force intro*: *continuous_intros*)
have *pih0*: *path_image* (λ*u*. *h* (*u*, *0*)) ⊆ *S*
using *hs* **by** (*force simp*: *path_image_def*)
have *c1*: *continuous_on ?A* (λ*x*. *h* (*fst x* ∗ *snd x*, *0*))
proof (*rule continuous_on_compose* [*of* _ _ *h*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*fst x* ∗ *snd x*, *0*)) ‘ *?A*) *h*
by (*force simp*: *mult_le_one intro*: *continuous_on_subset* [*OF conth*])
qed (*force intro*: *continuous_intros*)+
have *c2*: *continuous_on ?A* (λ*x*. *h* (*fst x* − *fst x* ∗ *snd x*, *0*))
proof (*rule continuous_on_compose* [*of* _ _ *h*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*fst x* − *fst x* ∗ *snd x*, *0*)) ‘ *?A*) *h*
by (*auto simp*: *algebra_simps add_increasing2 mult_left_le intro*: *continuous_on_subset* [*OF conth*])
qed (*force intro*: *continuous_intros*)
have [*simp*]: ⋀*t*. ⟦*0* ≤ *t* ∧ *t* ≤ *1*⟧ ⟹ *h* (*t*, *1*) = *h* (*t*, *0*)
using *ends* **by** (*simp add*: *pathfinish_def pathstart_def*)
have *adhoc_le*: *c* ∗ *4* ≤ *1* + *c* ∗ (*d* ∗ *4*) **if** ¬ *d* ∗ *4* ≤ *3 0* ≤ *c c* ≤ *1* **for** *c d*::*real*
proof −
have *c* ∗ *3* ≤ *c* ∗ (*d* ∗ *4*) **using** *that less_eq_real_def* **by** *auto*
with ‹*c* ≤ *1*› **show** *?thesis* **by** *fastforce*
qed
have ∗: ⋀*p x*. ⟦*path p* ∧ *path*(*reversepath p*);
path_image p ⊆ *S* ∧ *path_image*(*reversepath p*) ⊆ *S*;
pathfinish p = *pathstart*(*linepath a a* +++ *reversepath p*) ∧
pathstart(*reversepath p*) = *a* ∧ *pathstart p* = *x*⟧
⟹ *homotopic_paths S* (*p* +++ *linepath a a* +++ *reversepath p*) (*linepath x x*)
by (*metis homotopic_paths_lid homotopic_paths_join*
homotopic_paths_trans homotopic_paths_sym homotopic_paths_rinv)

have *1*: *homotopic_paths S p* (*p +++ linepath* (*pathfinish p*) (*pathfinish p*))
using ‹*path p*› *homotopic_paths_rid homotopic_paths_sym pip* **by** *blast*
moreover have *homotopic_paths S* (*p +++ linepath* (*pathfinish p*) (*pathfinish p*))
(*linepath* (*pathstart p*) (*pathstart p*) *+++ p +++ linepath* (*pathfinish p*) (*pathfinish p*))
using *homotopic_paths_lid* [*of p +++ linepath* (*pathfinish p*) (*pathfinish p*) *S*]
by (*metis 1 homotopic_paths_imp_path homotopic_paths_imp_subset homotopic_paths_sym pathstart_join*)
moreover
have *homotopic_paths S* (*linepath* (*pathstart p*) (*pathstart p*) *+++ p +++ linepath* (*pathfinish p*) (*pathfinish p*))
((λ*u. h* (*u*, *0*)) *+++ linepath a a +++ reversepath* (λ*u. h* (*u*, *0*)))
unfolding *homotopic_paths_def homotopic_with_def*
proof (*intro exI strip conjI*)
let *?h* = λ*y.* (*subpath 0* (*fst y*) (λ*u. h* (*u*, *0*)) *+++* (λ*u. h* (*Pair* (*fst y*) *u*)) *+++ subpath* (*fst y*) *0* (λ*u. h* (*u*, *0*))) (*snd y*)
have *continuous_on ?A ?h*
by (*intro continuous_on_homotopic_join_lemma*; *simp add*: *path_defs joinpaths_def subpath_def conth c1 c2*)
moreover have *?h* ∈ *?A* → *S*
using *hs*
unfolding *joinpaths_def subpath_def*
by (*force simp*: *algebra_simps mult_le_one mult_left_le intro*: *adhoc_le*)
ultimately show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*top_of_set* {*0..1*})) (*top_of_set S*) *?h*
by (*simp add*: *subpath_reversepath image_subset_iff_funcset*)
qed (*use ploop* **in** ‹*simp_all add*: *reversepath_def path_defs joinpaths_def o_def subpath_def conth c1 c2*›)
moreover have *homotopic_paths S* ((λ*u. h* (*u*, *0*)) *+++ linepath a a +++ reversepath* (λ*u. h* (*u*, *0*))) (*linepath* (*pathstart p*) (*pathstart p*))
by (*rule* ∗; *simp add*: *pih0 pathstart_def pathfinish_def conth0*; *simp add*: *reversepath_def joinpaths_def*)
ultimately show *?thesis*
by (*blast intro*: *homotopic_paths_trans*)
qed

proposition *homotopic_loops_conjugate*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *path p path q* **and** *pip*: *path_image p* ⊆ *S* **and** *piq*: *path_image q* ⊆ *S*
and *pq*: *pathfinish p* = *pathstart q* **and** *qloop*: *pathfinish q* = *pathstart q*
shows *homotopic_loops S* (*p +++ q +++ reversepath p*) *q*
proof −
have *contp*: *continuous_on* {*0..1*} *p* **using** ‹*path p*› [*unfolded path_def*] **by** *blast*
have *contq*: *continuous_on* {*0..1*} *q* **using** ‹*path q*› [*unfolded path_def*] **by** *blast*
let *?A* = {*0..1*::*real*} × {*0..1*::*real*}

have *c1*: *continuous_on ?A* (λ*x*. *p* ((*1* − *fst x*) ∗ *snd x* + *fst x*))
proof (*rule continuous_on_compose* [*of* _ _ *p*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*1* − *fst x*) ∗ *snd x* + *fst x*) ‘ *?A*) *p*
by (*auto intro*: *continuous_on_subset* [*OF contp*] *simp*: *algebra_simps add_increasing2 mult_right_le_one_le sum_le_prod1*)
qed (*force intro*: *continuous_intros*)
have *c2*: *continuous_on ?A* (λ*x*. *p* ((*fst x* − *1*) ∗ *snd x* + *1*))
proof (*rule continuous_on_compose* [*of* _ _ *p*, *unfolded o_def*])
show *continuous_on* ((λ*x*. (*fst x* − *1*) ∗ *snd x* + *1*) ‘ *?A*) *p*
by (*auto intro*: *continuous_on_subset* [*OF contp*] *simp*: *algebra_simps add_increasing2 mult_left_le_one_le*)
qed (*force intro*: *continuous_intros*)

have *ps1*: ⋀*a b*. ⟦*b* ∗ *2* ≤ *1*; *0* ≤ *b*; *0* ≤ *a*; *a* ≤ *1*⟧ ⟹ *p* ((*1* − *a*) ∗ (*2* ∗ *b*) + *a*) ∈ *S*
using *sum_le_prod1*
by (*force simp*: *algebra_simps add_increasing2 mult_left_le intro*: *pip* [*unfolded path_image_def*, *THEN subsetD*])
have *ps2*: ⋀*a b*. ⟦¬ *4* ∗ *b* ≤ *3*; *b* ≤ *1*; *0* ≤ *a*; *a* ≤ *1*⟧ ⟹ *p* ((*a* − *1*) ∗ (*4* ∗ *b* − *3*) + *1*) ∈ *S*
apply (*rule pip* [*unfolded path_image_def*, *THEN subsetD*])
apply (*rule image_eqI*, *blast*)
apply (*simp add*: *algebra_simps*)
by (*metis add_mono affine_ineq linear mult.commute mult.left_neutral mult_right_mono add.commute zero_le_numeral*)
have *qs*: ⋀*a b*. ⟦*4* ∗ *b* ≤ *3*; ¬ *b* ∗ *2* ≤ *1*⟧ ⟹ *q* (*4* ∗ *b* − *2*) ∈ *S*
using *path_image_def piq* **by** *fastforce*
have *homotopic_loops S* (*p* +++ *q* +++ *reversepath p*)
(*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*))
unfolding *homotopic_loops_def homotopic_with_def*
proof (*intro exI strip conjI*)
let *?h* = (λ*y*. (*subpath* (*fst y*) *1 p* +++ *q* +++ *subpath 1* (*fst y*) *p*) (*snd y*))
have *continuous_on ?A* (λ*y*. *q* (*snd y*))
by (*force simp*: *contq intro*: *continuous_on_compose* [*of* _ _ *q*, *unfolded o_def*] *continuous_on_id continuous_on_snd*)
then have *continuous_on ?A ?h*
using *pq qloop*
by (*intro continuous_on_homotopic_join_lemma*) (*auto simp*: *path_defs joinpaths_def subpath_def c1 c2*)
then show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*top_of_set* {*0..1*})) (*top_of_set S*) *?h*
by (*auto simp*: *joinpaths_def subpath_def ps1 ps2 qs*)
show *?h* (*1*,*x*) = (*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*)) *x* **for** *x*
using *pq* **by** (*simp add*: *pathfinish_def subpath_refl*)
qed (*auto simp*: *subpath_reversepath*)
moreover have *homotopic_loops S* (*linepath* (*pathstart q*) (*pathstart q*) +++ *q* +++ *linepath* (*pathstart q*) (*pathstart q*)) *q*

proof −
have *homotopic_paths S* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q*) *q*
using ‹*path q*› *homotopic_paths_lid qloop piq* **by** *auto*
hence *1*: ⋀*f*. *homotopic_paths S f q* ∨ ¬ *homotopic_paths S f* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q*)
using *homotopic_paths_trans* **by** *blast*
hence *homotopic_paths S* (*linepath* (*pathfinish q*) (*pathfinish q*) +++ *q* +++ *linepath* (*pathfinish q*) (*pathfinish q*)) *q*
by (*smt* (*verit*, *best*) ‹*path q*› *homotopic_paths_imp_path homotopic_paths_imp_subset homotopic_paths_lid*
homotopic_paths_rid homotopic_paths_trans pathstart_join piq qloop)
thus *?thesis*
by (*metis* (*no_types*) *qloop homotopic_loops_sym homotopic_paths_imp_homotopic_loops homotopic_paths_imp_pathfinish homotopic_paths_sym*)
qed
ultimately show *?thesis*
by (*blast intro*: *homotopic_loops_trans*)
qed

lemma *homotopic_paths_loop_parts*:
assumes *loops*: *homotopic_loops S* (*p* +++ *reversepath q*) (*linepath a a*) **and** *path q*
shows *homotopic_paths S p q*
proof −
have *paths*: *homotopic_paths S* (*p* +++ *reversepath q*) (*linepath* (*pathstart p*) (*pathstart p*))
using *homotopic_loops_imp_homotopic_paths_null* [*OF loops*] **by** *simp*
then have *path p*
using ‹*path q*› *homotopic_loops_imp_path loops path_join path_join_path_ends path_reversepath* **by** *blast*
show *?thesis*
proof (*cases pathfinish p* = *pathfinish q*)
case *True*
obtain *pipq*: *path_image p* ⊆ *S path_image q* ⊆ *S*
by (*metis Un_subset_iff paths* ‹*path p*› ‹*path q*› *homotopic_loops_imp_subset homotopic_paths_imp_path loops*
path_image_join path_image_reversepath path_imp_reversepath path_join_eq)
have *homotopic_paths S p* (*p* +++ (*linepath* (*pathfinish p*) (*pathfinish p*)))
using ‹*path p*› ‹*path_image p* ⊆ *S*› *homotopic_paths_rid homotopic_paths_sym* **by** *blast*
moreover have *homotopic_paths S* (*p* +++ (*linepath* (*pathfinish p*) (*pathfinish p*))) (*p* +++ (*reversepath q* +++ *q*))
by (*simp add*: *True* ‹*path p*› ‹*path q*› *pipq homotopic_paths_join homotopic_paths_linv homotopic_paths_sym*)
moreover have *homotopic_paths S* (*p* +++ (*reversepath q* +++ *q*)) ((*p* +++ *reversepath q*) +++ *q*)
by (*simp add*: *True* ‹*path p*› ‹*path q*› *homotopic_paths_assoc pipq*)
moreover have *homotopic_paths S* ((*p* +++ *reversepath q*) +++ *q*) (*linepath* (*pathstart p*) (*pathstart p*) +++ *q*)

by (*simp add*: ‹*path q*› *homotopic_paths_join paths pipq*)
ultimately show *?thesis*
by (*metis* ‹*path q*› *homotopic_paths_imp_path homotopic_paths_lid homotopic_paths_trans path_join_path_ends pathfinish_linepath pipq*(*2*))
next
case *False*
then show *?thesis*
using ‹*path q*› *homotopic_loops_imp_path loops path_join_path_ends* **by** *fastforce*
qed
qed

6.4.8 Homotopy of "nearby" function, paths and loops

lemma *homotopic_with_linear*:
fixes *f g* :: _ ⇒ *'b*::*real_normed_vector*
assumes *contf*: *continuous_on S f*
and *contg*:*continuous_on S g*
and *sub*: ⋀*x*. *x* ∈ *S* ⟹ *closed_segment* (*f x*) (*g x*) ⊆ *t*
shows *homotopic_with_canon* (λ*z*. *True*) *S t f g*
unfolding *homotopic_with_def*
apply (*rule_tac x*=λ*y*. ((*1* − (*fst y*)) ∗*R* *f*(*snd y*) + (*fst y*) ∗*R* *g*(*snd y*)) **in** *exI*)
using *sub closed_segment_def*
by (*fastforce intro*: *continuous_intros continuous_on_subset* [*OF contf*] *continuous_on_compose2* [**where** *g*=*f*]
continuous_on_subset [*OF contg*] *continuous_on_compose2* [**where** *g*=*g*])

lemma *homotopic_paths_linear*:
fixes *g h* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *path g path h pathstart h* = *pathstart g pathfinish h* = *pathfinish g*
⋀*t*. *t* ∈ {*0*..*1*} ⟹ *closed_segment* (*g t*) (*h t*) ⊆ *S*
shows *homotopic_paths S g h*
using *assms*
unfolding *path_def*
apply (*simp add*: *closed_segment_def pathstart_def pathfinish_def homotopic_paths_def homotopic_with_def*)
apply (*rule_tac x*=λ*y*. ((*1* − (*fst y*)) ∗*R* (*g* ∘ *snd*) *y* + (*fst y*) ∗*R* (*h* ∘ *snd*) *y*) **in** *exI*)
apply (*intro conjI subsetI continuous_intros*; *force*)
done

lemma *homotopic_loops_linear*:
fixes *g h* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *path g path h pathfinish g* = *pathstart g pathfinish h* = *pathstart h*
⋀*t x*. *t* ∈ {*0*..*1*} ⟹ *closed_segment* (*g t*) (*h t*) ⊆ *S*
shows *homotopic_loops S g h*
using *assms*
unfolding *path_defs homotopic_loops_def homotopic_with_def*

apply (*rule_tac x*=λ*y*. ((1 − (*fst y*)) ∗$_R$ *g*(*snd y*) + (*fst y*) ∗$_R$ *h*(*snd y*)) **in** *exI*)
by (*force simp*: *closed_segment_def intro*!: *continuous_intros intro*: *continuous_on_compose2* [**where** *g*=*g*] *continuous_on_compose2* [**where** *g*=*h*])

lemma *homotopic_paths_nearby_explicit*:
assumes §: *path g path h pathstart h* = *pathstart g pathfinish h* = *pathfinish g*
and *no*: ⋀*t x*. ⟦*t* ∈ {0..1}; *x* ∉ *S*⟧ ⟹ *norm*(*h t* − *g t*) < *norm*(*g t* − *x*)
shows *homotopic_paths S g h*
using *homotopic_paths_linear* [*OF* §] **by** (*metis linorder_not_le no norm_minus_commute segment_bound1 subsetI*)

lemma *homotopic_loops_nearby_explicit*:
assumes §: *path g path h pathfinish g* = *pathstart g pathfinish h* = *pathstart h*
and *no*: ⋀*t x*. ⟦*t* ∈ {0..1}; *x* ∉ *S*⟧ ⟹ *norm*(*h t* − *g t*) < *norm*(*g t* − *x*)
shows *homotopic_loops S g h*
using *homotopic_loops_linear* [*OF* §] **by** (*metis linorder_not_le no norm_minus_commute segment_bound1 subsetI*)

lemma *homotopic_nearby_paths*:
fixes *g h* :: *real* ⇒ *'a*::*euclidean_space*
assumes *path g open S path_image g* ⊆ *S*
shows ∃*e*. 0 < *e* ∧
(∀*h*. *path h* ∧
pathstart h = *pathstart g* ∧ *pathfinish h* = *pathfinish g* ∧
(∀*t* ∈ {0..1}. *norm*(*h t* − *g t*) < *e*) ⟶ *homotopic_paths S g h*)
proof −
obtain *e* **where** *e* > 0 **and** *e*: ⋀*x y*. *x* ∈ *path_image g* ⟹ *y* ∈ − *S* ⟹ *e* ≤ *dist x y*
using *separate_compact_closed* [*of path_image g* −*S*] *assms* **by** *force*
show *?thesis*
using *e* [*unfolded dist_norm*] ‹*e* > 0›
by (*fastforce simp*: *path_image_def intro*!: *homotopic_paths_nearby_explicit assms exI*)
qed

lemma *homotopic_nearby_loops*:
fixes *g h* :: *real* ⇒ *'a*::*euclidean_space*
assumes *path g open S path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows ∃*e*. 0 < *e* ∧
(∀*h*. *path h* ∧ *pathfinish h* = *pathstart h* ∧
(∀*t* ∈ {0..1}. *norm*(*h t* − *g t*) < *e*) ⟶ *homotopic_loops S g h*)
proof −
obtain *e* **where** *e* > 0 **and** *e*: ⋀*x y*. *x* ∈ *path_image g* ⟹ *y* ∈ − *S* ⟹ *e* ≤ *dist x y*
using *separate_compact_closed* [*of path_image g* −*S*] *assms* **by** *force*
show *?thesis*
using *e* [*unfolded dist_norm*] ‹*e* > 0›
by (*fastforce simp*: *path_image_def intro*!: *homotopic_loops_nearby_explicit assms exI*)

qed

6.4.9 Homotopy and subpaths

lemma *homotopic_join_subpaths1*:
assumes *path g* **and** *pag*: *path_image g* ⊆ *S*
and *u*: *u* ∈ {*0..1*} **and** *v*: *v* ∈ {*0..1*} **and** *w*: *w* ∈ {*0..1*} *u* ≤ *v v* ≤ *w*
shows *homotopic_paths S* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
proof −
have *1*: *t* ∗ *2* ≤ *1* ⟹ *u* + *t* ∗ (*v* ∗ *2*) ≤ *v* + *t* ∗ (*u* ∗ *2*) **for** *t*
using *affine_ineq* ‹*u* ≤ *v*› **by** *fastforce*
have *2*: *t* ∗ *2* > *1* ⟹ *u* + (*2*∗*t* − *1*) ∗ *v* ≤ *v* + (*2*∗*t* − *1*) ∗ *w* **for** *t*
by (*metis add_mono_thms_linordered_semiring*(*1*) *diff_gt_0_iff_gt less_eq_real_def mult.commute mult_right_mono* ‹*u* ≤ *v*› ‹*v* ≤ *w*›)
have *t2*: ⋀*t*::*real*. *t*∗*2* = *1* ⟹ *t* = *1*/*2* **by** *auto*
have *homotopic_paths* (*path_image g*) (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
proof (*cases w* = *u*)
case *True*
then show *?thesis*
by (*metis* ‹*path g*› *homotopic_paths_rinv path_image_subpath_subset path_subpath pathstart_subpath reversepath_subpath subpath_refl u v*)
next
case *False*
let *?f* = λ*t*. *if* *t* ≤ *1*/*2* *then inverse*((*w* − *u*)) ∗*R* (*2* ∗ (*v* − *u*)) ∗*R* *t*
else inverse((*w* − *u*)) ∗*R* ((*v* − *u*) + (*w* − *v*) ∗*R* (*2* ∗*R* *t* − *1*))
show *?thesis*
proof (*rule homotopic_paths_sym* [*OF homotopic_paths_reparametrize* [**where** *f* = *?f*]])
show *path* (*subpath u w g*)
using *assms*(*1*) *path_subpath u w*(*1*) **by** *blast*
show *path_image* (*subpath u w g*) ⊆ *path_image g*
by (*meson path_image_subpath_subset u w*(*1*))
show *continuous_on* {*0..1*} *?f*
unfolding *split_01*
by (*rule continuous_on_cases continuous_intros* | *force simp*: *pathfinish_def joinpaths_def dest*!: *t2*)+
show *?f* ∈ {*0..1*} → {*0..1*}
using *False assms*
by (*force simp*: *field_simps not_le mult_left_mono affine_ineq dest*!: *1 2*)
show (*subpath u v g* +++ *subpath v w g*) *t* = *subpath u w g* (*?f t*) **if** *t* ∈ {*0..1*} **for** *t*
using *assms*
unfolding *joinpaths_def subpath_def* **by** (*auto simp*: *divide_simps add.commute mult.commute mult.left_commute*)
qed (*use False* **in** *auto*)
qed
then show *?thesis*

by (*rule homotopic_paths_subset* [*OF* _ *pag*])
qed

lemma *homotopic_join_subpaths2*:
assumes *homotopic_paths S* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
shows *homotopic_paths S* (*subpath w v g* +++ *subpath v u g*) (*subpath w u g*)
by (*metis assms homotopic_paths_reversepath_D pathfinish_subpath pathstart_subpath reversepath_joinpaths reversepath_subpath*)

lemma *homotopic_join_subpaths3*:
assumes *hom*: *homotopic_paths S* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)
and *path g* **and** *pag*: *path_image g* ⊆ *S*
and *u*: *u* ∈ {*0..1*} **and** *v*: *v* ∈ {*0..1*} **and** *w*: *w* ∈ {*0..1*}
shows *homotopic_paths S* (*subpath v w g* +++ *subpath w u g*) (*subpath v u g*)
proof −
obtain *wvg*: *path* (*subpath w v g*) *path_image* (*subpath w v g*) ⊆ *S*
and *wug*: *path* (*subpath w u g*) *path_image* (*subpath w u g*) ⊆ *S*
and *vug*: *path* (*subpath v u g*) *path_image* (*subpath v u g*) ⊆ *S*
by (*meson* ‹*path g*› *pag path_image_subpath_subset path_subpath subset_trans u v w*)
have *homotopic_paths S* (*subpath u w g* +++ *subpath w v g*)
((*subpath u v g* +++ *subpath v w g*) +++ *subpath w v g*)
by (*simp add*: *hom homotopic_paths_join homotopic_paths_sym wvg*)
also have *homotopic_paths S* ... (*subpath u v g* +++ *subpath v w g* +++ *subpath w v g*)
using *wvg vug* ‹*path g*›
by (*metis homotopic_paths_assoc homotopic_paths_sym path_image_subpath_commute path_subpath*
pathfinish_subpath pathstart_subpath u v w)
also have *homotopic_paths S* ... (*subpath u v g* +++ *linepath* (*pathfinish* (*subpath u v g*)) (*pathfinish* (*subpath u v g*)))
using *wvg vug* ‹*path g*›
by (*metis homotopic_paths_join homotopic_paths_linv homotopic_paths_refl path_image_subpath_commute*
path_subpath pathfinish_subpath pathstart_join pathstart_subpath reversepath_subpath u v)
also have *homotopic_paths S* ... (*subpath u v g*)
using *vug* ‹*path g*› **by** (*metis homotopic_paths_rid path_image_subpath_commute path_subpath u v*)
finally have *homotopic_paths S* (*subpath u w g* +++ *subpath w v g*) (*subpath u v g*) **.**
then show *?thesis*
using *homotopic_join_subpaths2* **by** *blast*
qed

proposition *homotopic_join_subpaths*:
⟦*path g*; *path_image g* ⊆ *S*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *w* ∈ {*0..1*}⟧
⟹ *homotopic_paths S* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)

by (*smt* (*verit*, *del_insts*) *homotopic_join_subpaths1 homotopic_join_subpaths2 homotopic_join_subpaths3*)

Relating homotopy of trivial loops to path-connectedness.

lemma *path_component_imp_homotopic_points*:
assumes *path_component S a b*
shows *homotopic_loops S* (*linepath a a*) (*linepath b b*)
proof −
obtain *g* :: *real* ⇒ *'a* **where** *g*: *continuous_on* {*0..1*} *g g* ∈ {*0..1*} → *S g 0* = *a g 1* = *b*
using *assms* **by** (*auto simp*: *path_defs*)
then have *continuous_on* ({*0..1*} × {*0..1*}) (*g* ∘ *fst*)
by (*fastforce intro*!: *continuous_intros*)+
with *g* **show** *?thesis*
by (*auto simp*: *homotopic_loops_def homotopic_with_def path_defs Pi_iff*)
qed

lemma *homotopic_loops_imp_path_component_value*:
⟦*homotopic_loops S p q*; *0* ≤ *t*; *t* ≤ *1*⟧ ⟹ *path_component S* (*p t*) (*q t*)
apply (*clarsimp simp*: *homotopic_loops_def homotopic_with_def path_defs*)
apply (*rule_tac x*=*h* ∘ (λ*u*. (*u*, *t*)) **in** *exI*)
apply (*fastforce elim*!: *continuous_on_subset intro*!: *continuous_intros*)
done

lemma *homotopic_points_eq_path_component*:
homotopic_loops S (*linepath a a*) (*linepath b b*) ⟷ *path_component S a b*
using *homotopic_loops_imp_path_component_value path_component_imp_homotopic_points*
by *fastforce*

lemma *path_connected_eq_homotopic_points*:
path_connected S ⟷
(∀ *a b*. *a* ∈ *S* ∧ *b* ∈ *S* ⟶ *homotopic_loops S* (*linepath a a*) (*linepath b b*))
by (*auto simp*: *path_connected_def path_component_def homotopic_points_eq_path_component*)

6.4.10 Simply connected sets

defined as "all loops are homotopic (as loops)

definition *simply_connected* **where**
simply_connected S ≡
∀ *p q*. *path p* ∧ *pathfinish p* = *pathstart p* ∧ *path_image p* ⊆ *S* ∧
path q ∧ *pathfinish q* = *pathstart q* ∧ *path_image q* ⊆ *S*
⟶ *homotopic_loops S p q*

lemma *simply_connected_empty* [*iff*]: *simply_connected* {}
by (*simp add*: *simply_connected_def*)

lemma *simply_connected_imp_path_connected*:
fixes *S* :: *_*::*real_normed_vector set*
shows *simply_connected S* ⟹ *path_connected S*

by (*simp add*: *simply_connected_def path_connected_eq_homotopic_points*)

lemma *simply_connected_imp_connected*:
fixes *S* :: _::*real_normed_vector set*
shows *simply_connected S* ⟹ *connected S*
by (*simp add*: *path_connected_imp_connected simply_connected_imp_path_connected*)

lemma *simply_connected_eq_contractible_loop_any*:
fixes *S* :: _::*real_normed_vector set*
shows *simply_connected S* ⟷
(∀ *p a*. *path p* ∧ *path_image p* ⊆ *S* ∧ *pathfinish p* = *pathstart p* ∧ *a* ∈ *S*
⟶ *homotopic_loops S p* (*linepath a a*))
(**is** *?lhs* = *?rhs*)
proof
assume *?rhs* **then show** *?lhs*
unfolding *simply_connected_def*
by (*metis pathfinish_in_path_image subsetD homotopic_loops_trans homotopic_loops_sym*)
qed (*force simp*: *simply_connected_def*)

lemma *simply_connected_eq_contractible_loop_some*:
fixes *S* :: _::*real_normed_vector set*
shows *simply_connected S* ⟷
path_connected S ∧
(∀ *p*. *path p* ∧ *path_image p* ⊆ *S* ∧ *pathfinish p* = *pathstart p*
⟶ (∃ *a*. *a* ∈ *S* ∧ *homotopic_loops S p* (*linepath a a*)))
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *simply_connected_eq_contractible_loop_any* **by** (*blast intro*: *simply_connected_imp_path_connected*)
next
assume *?rhs*
then show *?lhs*
by (*meson homotopic_loops_trans path_connected_eq_homotopic_points simply_connected_eq_contractible_loop_any*)
qed

lemma *simply_connected_eq_contractible_loop_all*:
fixes *S* :: _::*real_normed_vector set*
shows *simply_connected S* ⟷
S = {} ∨
(∃ *a* ∈ *S*. ∀ *p*. *path p* ∧ *path_image p* ⊆ *S* ∧ *pathfinish p* = *pathstart p*
⟶ *homotopic_loops S p* (*linepath a a*))
by (*meson ex_in_conv homotopic_loops_sym homotopic_loops_trans simply_connected_def simply_connected_eq_contractible_loop_any*)

lemma *simply_connected_eq_contractible_path*:
fixes *S* :: _::*real_normed_vector set*

```
  shows simply_connected S ⟷
          path_connected S ∧
          (∀ p. path p ∧ path_image p ⊆ S ∧ pathfinish p = pathstart p
           ⟶ homotopic_paths S p (linepath (pathstart p) (pathstart p)))
     (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    unfolding simply_connected_imp_path_connected
   by (metis simply_connected_eq_contractible_loop_some homotopic_loops_imp_homotopic_paths_n
next
  assume  ?rhs
  then show ?lhs
   using homotopic_paths_imp_homotopic_loops simply_connected_eq_contractible_loop_some
by fastforce
qed

lemma simply_connected_eq_homotopic_paths:
  fixes S :: _::real_normed_vector set
  shows simply_connected S ⟷
          path_connected S ∧
          (∀ p q. path p ∧ path_image p ⊆ S ∧
                  path q ∧ path_image q ⊆ S ∧
                  pathstart q = pathstart p ∧ pathfinish q = pathfinish p
                  ⟶ homotopic_paths S p q)
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have pc: path_connected S
        and ∗:  ⋀p. ⟦path p; path_image p ⊆ S;
                     pathfinish p = pathstart p⟧
                    ⟹ homotopic_paths S p (linepath (pathstart p) (pathstart p))
    by (auto simp: simply_connected_eq_contractible_path)
  have homotopic_paths S p q
        if path p path_image p ⊆ S path q
           path_image q ⊆ S pathstart q = pathstart p
           pathfinish q = pathfinish p for p q
  proof −
    have homotopic_paths S p (p +++ reversepath q +++ q)
      using that
       by (smt (verit, best) homotopic_paths_join homotopic_paths_linv homo-
topic_paths_rid homotopic_paths_sym
          homotopic_paths_trans pathstart_linepath)
    also have homotopic_paths S ... ((p +++ reversepath q) +++ q)
      by (simp add: that homotopic_paths_assoc)
    also have homotopic_paths S ... (linepath (pathstart q) (pathstart q) +++ q)
      using ∗ [of p +++ reversepath q] that
     by (simp add: homotopic_paths_assoc homotopic_paths_join path_image_join)
    also have homotopic_paths S ... q
```

```
    using that homotopic_paths_lid by blast
   finally show ?thesis .
  qed
  then show ?rhs
   by (blast intro: pc *)
next
  assume ?rhs
  then show ?lhs
   by (force simp: simply_connected_eq_contractible_path)
qed

proposition simply_connected_Times:
  fixes S :: 'a::real_normed_vector set and T :: 'b::real_normed_vector set
  assumes S: simply_connected S and T: simply_connected T
   shows simply_connected(S × T)
proof −
  have homotopic_loops (S × T) p (linepath (a, b) (a, b))
       if path p path_image p ⊆ S × T p 1 = p 0 a ∈ S b ∈ T
       for p a b
  proof −
   have path (fst ∘ p)
     by (simp add: continuous_on_fst Path_Connected.path_continuous_image
[OF ‹path p›])
   moreover have path_image (fst ∘ p) ⊆ S
     using that by (force simp: path_image_def)
   ultimately have p1: homotopic_loops S (fst ∘ p) (linepath a a)
     using S that
      by (simp add: simply_connected_eq_contractible_loop_any pathfinish_def
pathstart_def)
   have path (snd ∘ p)
     by (simp add: continuous_on_snd Path_Connected.path_continuous_image
[OF ‹path p›])
   moreover have path_image (snd ∘ p) ⊆ T
     using that by (force simp: path_image_def)
   ultimately have p2: homotopic_loops T (snd ∘ p) (linepath b b)
     using T that
      by (simp add: simply_connected_eq_contractible_loop_any pathfinish_def
pathstart_def)
   show ?thesis
     using p1 p2 unfolding homotopic_loops
     apply clarify
     subgoal for h k
       by (rule_tac x=λz. (h z, k z) in exI) (force intro: continuous_intros simp:
path_defs)
     done
  qed
  with assms show ?thesis
   by (simp add: simply_connected_eq_contractible_loop_any pathfinish_def path-
start_def)
```

qed

6.4.11 Contractible sets

definition *contractible* **where**
contractible S ≡ ∃ *a*. *homotopic_with_canon* (λ*x*. *True*) *S S id* (λ*x*. *a*)

proposition *contractible_imp_simply_connected*:
fixes *S* :: _::*real_normed_vector set*
assumes *contractible S* **shows** *simply_connected S*
proof (*cases S* = {})
case *True* **then show** *?thesis* **by** *force*
next
case *False*
obtain *a* **where** *a*: *homotopic_with_canon* (λ*x*. *True*) *S S id* (λ*x*. *a*)
using *assms* **by** (*force simp*: *contractible_def*)
then have *a* ∈ *S*
using *False homotopic_with_imp_funspace2* **by** *fastforce*
have ∀ *p*. *path p* ∧
path_image p ⊆ *S* ∧ *pathfinish p* = *pathstart p* ⟶
homotopic_loops S p (*linepath a a*)
using *a* **apply** (*clarsimp simp*: *homotopic_loops_def homotopic_with_def path_defs*)
apply (*rule_tac x*=(*h* ∘ (λ*y*. (*fst y*, (*p* ∘ *snd*) *y*))) **in** *exI*)
apply (*intro conjI continuous_on_compose continuous_intros*; *force elim*: *continuous_on_subset*)
done
with ‹*a* ∈ *S*› **show** *?thesis*
by (*auto simp*: *simply_connected_eq_contractible_loop_all False*)
qed

corollary *contractible_imp_connected*:
fixes *S* :: _::*real_normed_vector set*
shows *contractible S* ⟹ *connected S*
by (*simp add*: *contractible_imp_simply_connected simply_connected_imp_connected*)

lemma *contractible_imp_path_connected*:
fixes *S* :: _::*real_normed_vector set*
shows *contractible S* ⟹ *path_connected S*
by (*simp add*: *contractible_imp_simply_connected simply_connected_imp_path_connected*)

lemma *nullhomotopic_through_contractible*:
fixes *S* :: _::*topological_space set*
assumes *f*: *continuous_on S f f* ∈ *S* → *T*
and *g*: *continuous_on T g g* ∈ *T* → *U*
and *T*: *contractible T*
obtains *c* **where** *homotopic_with_canon* (λ*h*. *True*) *S U* (*g* ∘ *f*) (λ*x*. *c*)
proof −
obtain *b* **where** *b*: *homotopic_with_canon* (λ*x*. *True*) *T T id* (λ*x*. *b*)

using *assms* **by** (*force simp*: *contractible_def*)
have *homotopic_with_canon* (λ*f*. *True*) *T U* (*g* ∘ *id*) (*g* ∘ (λ*x*. *b*))
by (*metis b continuous_map_subtopology_eu g homotopic_with_compose_continuous_map_left image_subset_iff_funcset*)
then have *homotopic_with_canon* (λ*f*. *True*) *S U* (*g* ∘ *id* ∘ *f*) (*g* ∘ (λ*x*. *b*) ∘ *f*)
by (*simp add*: *f homotopic_with_compose_continuous_map_right image_subset_iff_funcset*)
then show *?thesis*
by (*simp add*: *comp_def that*)
qed

lemma *nullhomotopic_into_contractible*:
assumes *f*: *continuous_on S f f* ∈ *S* → *T*
and *T*: *contractible T*
obtains *c* **where** *homotopic_with_canon* (λ*h*. *True*) *S T f* (λ*x*. *c*)
by (*rule nullhomotopic_through_contractible* [*OF f*, *of id T*]) (*use assms* **in** *auto*)

lemma *nullhomotopic_from_contractible*:
assumes *f*: *continuous_on S f f* ∈ *S* → *T*
and *S*: *contractible S*
obtains *c* **where** *homotopic_with_canon* (λ*h*. *True*) *S T f* (λ*x*. *c*)
by (*auto simp*: *comp_def intro*: *nullhomotopic_through_contractible* [*OF continuous_on_id _ f S*])

lemma *homotopic_through_contractible*:
fixes *S* :: _::*real_normed_vector set*
assumes *continuous_on S f1 f1* ∈ *S* → *T*
continuous_on T g1 g1 ∈ *T* → *U*
continuous_on S f2 f2 ∈ *S* → *T*
continuous_on T g2 g2 ∈ *T* → *U*
contractible T path_connected U
shows *homotopic_with_canon* (λ*h*. *True*) *S U* (*g1* ∘ *f1*) (*g2* ∘ *f2*)
proof −
obtain *c1* **where** *c1*: *homotopic_with_canon* (λ*h*. *True*) *S U* (*g1* ∘ *f1*) (λ*x*. *c1*)
by (*rule nullhomotopic_through_contractible* [*of S f1 T g1 U*]) (*use assms* **in** *auto*)
obtain *c2* **where** *c2*: *homotopic_with_canon* (λ*h*. *True*) *S U* (*g2* ∘ *f2*) (λ*x*. *c2*)
by (*rule nullhomotopic_through_contractible* [*of S f2 T g2 U*]) (*use assms* **in** *auto*)
have *S* = {} ∨ (∃ *t*. *path_connected t* ∧ *t* ⊆ *U* ∧ *c2* ∈ *t* ∧ *c1* ∈ *t*)
proof (*cases S* = {})
case *True* **then show** *?thesis* **by** *force*
next
case *False*
with *c1 c2* **have** *c1* ∈ *U c2* ∈ *U*
using *homotopic_with_imp_continuous_maps*
by (*metis PiE equals0I homotopic_with_imp_funspace2*)+
with ‹*path_connected U*› **show** *?thesis* **by** *blast*
qed
then have *homotopic_with_canon* (λ*h*. *True*) *S U* (λ*x*. *c2*) (λ*x*. *c1*)

by (*auto simp*: *path_component homotopic_constant_maps*)
then show *?thesis*
using *c1 c2 homotopic_with_symD homotopic_with_trans* **by** *blast*
qed

lemma *homotopic_into_contractible*:
fixes *S* :: *'a*::*real_normed_vector set* **and** *T*:: *'b*::*real_normed_vector set*
assumes *f*: *continuous_on S f f* ∈ *S* → *T*
and *g*: *continuous_on S g g* ∈ *S* → *T*
and *T*: *contractible T*
shows *homotopic_with_canon* (λ*h*. *True*) *S T f g*
using *homotopic_through_contractible* [*of S f T id T g id*]
by (*simp add*: *assms contractible_imp_path_connected*)

lemma *homotopic_from_contractible*:
fixes *S* :: *'a*::*real_normed_vector set* **and** *T*:: *'b*::*real_normed_vector set*
assumes *f*: *continuous_on S f f* ∈ *S* → *T*
and *g*: *continuous_on S g g* ∈ *S* → *T*
and *contractible S path_connected T*
shows *homotopic_with_canon* (λ*h*. *True*) *S T f g*
using *homotopic_through_contractible* [*of S id S f T id g*]
by (*simp add*: *assms contractible_imp_path_connected*)

6.4.12 Starlike sets

definition *starlike S* ⟷ (∃ *a*∈*S*. ∀ *x*∈*S*. *closed_segment a x* ⊆ *S*)

lemma *starlike_UNIV* [*simp*]: *starlike UNIV*
by (*simp add*: *starlike_def*)

lemma *convex_imp_starlike*:
convex S ⟹ *S* ≠ {} ⟹ *starlike S*
unfolding *convex_contains_segment starlike_def* **by** *auto*

lemma *starlike_convex_tweak_boundary_points*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *convex S S* ≠ {} **and** *ST*: *rel_interior S* ⊆ *T* **and** *TS*: *T* ⊆ *closure S*
shows *starlike T*
proof −
have *rel_interior S* ≠ {}
by (*simp add*: *assms rel_interior_eq_empty*)
with *ST* **obtain** *a* **where** *a*: *a* ∈ *rel_interior S* **and** *a* ∈ *T* **by** *blast*
have ⋀*x*. *x* ∈ *T* ⟹ *open_segment a x* ⊆ *rel_interior S*
by (*rule rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *a*]) (*use assms* **in** *auto*)
then have ∀ *x*∈*T*. *a* ∈ *T* ∧ *open_segment a x* ⊆ *T*
using *ST* **by** (*blast intro*: *a* ‹*a* ∈ *T*› *rel_interior_closure_convex_segment* [*OF* ‹*convex S*› *a*])
then show *?thesis*

unfolding *starlike_def* **using** *bexI* [*OF* _ ‹$a \in T$›]
by (*simp add*: *closed_segment_eq_open*)
qed

lemma *starlike_imp_contractible_gen*:
fixes S :: '*a*::*real_normed_vector set*
assumes S: *starlike S*
and P: $\bigwedge a\ T.\ [\![a \in S;\ 0 \le T;\ T \le 1]\!] \Longrightarrow P(\lambda x.\ (1 - T) *_R x + T *_R a)$
obtains *a* **where** *homotopic_with_canon P S S* ($\lambda x.\ x$) ($\lambda x.\ a$)
proof −
obtain *a* **where** $a \in S$ **and** *a*: $\bigwedge x.\ x \in S \Longrightarrow$ *closed_segment a x* $\subseteq S$
using S **by** (*auto simp*: *starlike_def*)
have $\bigwedge t\ b.\ 0 \le t \wedge t \le 1 \Longrightarrow$
$\exists u.\ (1 - t) *_R b + t *_R a = (1 - u) *_R a + u *_R b \wedge 0 \le u \wedge u \le 1$
by (*metis add_diff_cancel_right' diff_ge_0_iff_ge le_add_diff_inverse pth_c*(1))
then have ($\lambda y.\ (1 - fst\ y) *_R snd\ y + fst\ y *_R a$) ‘ ($\{0..1\} \times S$) $\subseteq S$
using *a* [*unfolded closed_segment_def*] **by** *force*
then have *homotopic_with_canon P S S* ($\lambda x.\ x$) ($\lambda x.\ a$)
using ‹$a \in S$›
unfolding *homotopic_with_def*
apply (*rule_tac x*=$\lambda y.\ (1 - (fst\ y)) *_R snd\ y + (fst\ y) *_R a$ **in** *exI*)
apply (*force simp*: *P intro*: *continuous_intros*)
done
then show *?thesis*
using *that* **by** *blast*
qed

lemma *starlike_imp_contractible*:
fixes S :: '*a*::*real_normed_vector set*
shows *starlike S* $\Longrightarrow$ *contractible S*
using *starlike_imp_contractible_gen contractible_def* **by** (*fastforce simp*: *id_def*)

lemma *contractible_UNIV* [*simp*]: *contractible* (*UNIV* :: '*a*::*real_normed_vector set*)
by (*simp add*: *starlike_imp_contractible*)

lemma *starlike_imp_simply_connected*:
fixes S :: '*a*::*real_normed_vector set*
shows *starlike S* $\Longrightarrow$ *simply_connected S*
by (*simp add*: *contractible_imp_simply_connected starlike_imp_contractible*)

lemma *convex_imp_simply_connected*:
fixes S :: '*a*::*real_normed_vector set*
shows *convex S* $\Longrightarrow$ *simply_connected S*
using *convex_imp_starlike starlike_imp_simply_connected* **by** *blast*

lemma *starlike_imp_path_connected*:
fixes S :: '*a*::*real_normed_vector set*
shows *starlike S* $\Longrightarrow$ *path_connected S*

by (*simp add*: *simply_connected_imp_path_connected starlike_imp_simply_connected*)

lemma *starlike_imp_connected*:
fixes S :: *'a*::*real_normed_vector set*
shows *starlike* $S \Longrightarrow$ *connected* S
by (*simp add*: *path_connected_imp_connected starlike_imp_path_connected*)

lemma *is_interval_simply_connected_1*:
fixes S :: *real set*
shows *is_interval* $S \longleftrightarrow$ *simply_connected* S
by (*meson convex_imp_simply_connected is_interval_connected_1 is_interval_convex_1 simply_connected_imp_connected*)

lemma *contractible_empty* [*simp*]: *contractible* {}
by (*simp add*: *contractible_def homotopic_on_emptyI*)

lemma *contractible_convex_tweak_boundary_points*:
fixes S :: *'a*::*euclidean_space set*
assumes *convex* S **and** *TS*: *rel_interior* $S \subseteq T$ $T \subseteq$ *closure* S
shows *contractible* T
by (*metis assms closure_eq_empty contractible_empty empty_subsetI starlike_convex_tweak_boundary_points starlike_imp_contractible subset_antisym*)

lemma *convex_imp_contractible*:
fixes S :: *'a*::*real_normed_vector set*
shows *convex* $S \Longrightarrow$ *contractible* S
using *contractible_empty convex_imp_starlike starlike_imp_contractible* **by** *blast*

lemma *contractible_sing* [*simp*]:
fixes a :: *'a*::*real_normed_vector*
shows *contractible* $\{a\}$
by (*rule convex_imp_contractible* [*OF convex_singleton*])

lemma *is_interval_contractible_1*:
fixes S :: *real set*
shows *is_interval* $S \longleftrightarrow$ *contractible* S
using *contractible_imp_simply_connected convex_imp_contractible is_interval_convex_1 is_interval_simply_connected_1* **by** *auto*

lemma *contractible_Times*:
fixes S :: *'a*::*euclidean_space set* **and** T :: *'b*::*euclidean_space set*
assumes S: *contractible* S **and** T: *contractible* T
shows *contractible* $(S \times T)$
proof −
obtain *a h* **where** *conth*: *continuous_on* $(\{0..1\} \times S)$ h
and *hsub*: $h \in (\{0..1\} \times S) \to S$
and [*simp*]: $\bigwedge x.\ x \in S \Longrightarrow h\ (0,\ x) = x$
and [*simp*]: $\bigwedge x.\ x \in S \Longrightarrow\ h\ (1{::}real,\ x) = a$
using S **by** (*force simp*: *contractible_def homotopic_with*)

obtain *b k* **where** *contk*: *continuous_on* ({*0..1*} × *T*) *k*
and *ksub*: *k* ∈ ({*0..1*} × *T*) → *T*
and [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *k* (*0*, *x*) = *x*
and [*simp*]: ⋀*x*. *x* ∈ *T* ⟹ *k* (*1*::*real*, *x*) = *b*
using *T* **by** (*force simp*: *contractible_def homotopic_with*)
show *?thesis*
apply (*simp add*: *contractible_def homotopic_with*)
apply (*rule exI* [**where** *x*=*a*])
apply (*rule exI* [**where** *x*=*b*])
apply (*rule exI* [**where** *x* = λ*z*. (*h* (*fst z*, *fst*(*snd z*)), *k* (*fst z*, *snd*(*snd z*)))])
using *hsub ksub*
apply (*fastforce intro*!: *continuous_intros continuous_on_compose2* [*OF conth*] *continuous_on_compose2* [*OF contk*])
done
qed

6.4.13 Local versions of topological properties in general

definition *locally* :: ('*a*::*topological_space set* ⇒ *bool*) ⇒ '*a set* ⇒ *bool*
where
locally P S ≡
∀ *w x*. *openin* (*top_of_set S*) *w* ∧ *x* ∈ *w*
⟶ (∃ *U V*. *openin* (*top_of_set S*) *U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *w*)

lemma *locallyI*:
assumes ⋀*w x*. ⟦*openin* (*top_of_set S*) *w*; *x* ∈ *w*⟧
⟹ ∃ *U V*. *openin* (*top_of_set S*) *U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V* ⊆ *w*
shows *locally P S*
using *assms* **by** (*force simp*: *locally_def*)

lemma *locallyE*:
assumes *locally P S openin* (*top_of_set S*) *w x* ∈ *w*
obtains *U V* **where** *openin* (*top_of_set S*) *U P V x* ∈ *U U* ⊆ *V V* ⊆ *w*
using *assms* **unfolding** *locally_def* **by** *meson*

lemma *locally_mono*:
assumes *locally P S* ⋀*T*. *P T* ⟹ *Q T*
shows *locally Q S*
by (*metis assms locally_def*)

lemma *locally_open_subset*:
assumes *locally P S openin* (*top_of_set S*) *t*
shows *locally P t*
by (*smt* (*verit*, *ccfv_SIG*) *assms order.trans locally_def openin_imp_subset openin_subset_trans openin_trans*)

lemma *locally_diff_closed*:

⟦*locally P S*; *closedin* (*top_of_set S*) *t*⟧ ⟹ *locally P* (*S* − *t*)
using *locally_open_subset closedin_def* **by** *fastforce*

lemma *locally_empty* [*iff*]: *locally P* {}
by (*simp add*: *locally_def openin_subtopology*)

lemma *locally_singleton* [*iff*]:
fixes *a* :: 'a::*metric_space*
shows *locally P* {*a*} ⟷ *P* {*a*}
proof −
have ∀ *x*::*real*. ¬ 0 < *x* ⟹ *P* {*a*}
using *zero_less_one* **by** *blast*
then show *?thesis*
unfolding *locally_def*
by (*auto simp*: *openin_euclidean_subtopology_iff subset_singleton_iff conj_disj_distribR*)
qed

lemma *locally_iff*:
locally P S ⟷
(∀ *T x*. *open T* ∧ *x* ∈ *S* ∩ *T* ⟶ (∃ *U*. *open U* ∧ (∃ *V*. *P V* ∧ *x* ∈ *S* ∩ *U* ∧
S ∩ *U* ⊆ *V* ∧ *V* ⊆ *S* ∩ *T*)))
by (*smt* (*verit*) *locally_def openin_open*)

lemma *locally_Int*:
assumes *S*: *locally P S* **and** *T*: *locally P T*
and *P*: ⋀*S T*. *P S* ∧ *P T* ⟹ *P*(*S* ∩ *T*)
shows *locally P* (*S* ∩ *T*)
unfolding *locally_iff*
proof *clarify*
fix *A x*
assume *open A x* ∈ *A x* ∈ *S x* ∈ *T*
then obtain *U1 V1 U2 V2*
where *open U1 P V1 x* ∈ *S* ∩ *U1 S* ∩ *U1* ⊆ *V1* ∧ *V1* ⊆ *S* ∩ *A*
open U2 P V2 x ∈ *T* ∩ *U2 T* ∩ *U2* ⊆ *V2* ∧ *V2* ⊆ *T* ∩ *A*
using *S T* **unfolding** *locally_iff* **by** (*meson IntI*)
then have *S* ∩ *T* ∩ (*U1* ∩ *U2*) ⊆ *V1* ∩ *V2 V1* ∩ *V2* ⊆ *S* ∩ *T* ∩ *A x* ∈ *S* ∩
T ∩ (*U1* ∩ *U2*)
by *blast*+
moreover have *P* (*V1* ∩ *V2*)
by (*simp add*: *P* ‹*P V1*› ‹*P V2*›)
ultimately show ∃ *U*. *open U* ∧ (∃ *V*. *P V* ∧ *x* ∈ *S* ∩ *T* ∩ *U* ∧ *S* ∩ *T* ∩ *U* ⊆
V ∧ *V* ⊆ *S* ∩ *T* ∩ *A*)
using ‹*open U1*› ‹*open U2*› **by** *blast*
qed

lemma *locally_Times*:
fixes *S* :: ('a::*metric_space*) *set* **and** *T* :: ('b::*metric_space*) *set*
assumes *PS*: *locally P S* **and** *QT*: *locally Q T* **and** *R*: ⋀*S T*. *P S* ∧ *Q T* ⟹

```
R(S × T)
  shows locally R (S × T)
    unfolding locally_def
proof (clarify)
  fix W x y
  assume W: openin (top_of_set (S × T)) W and xy: (x, y) ∈ W
  then obtain U V where openin (top_of_set S) U x ∈ U
                        openin (top_of_set T) V y ∈ V U × V ⊆ W
    using Times_in_interior_subtopology by metis
  then obtain U1 U2 V1 V2
        where opeS: openin (top_of_set S) U1 ∧ P U2 ∧ x ∈ U1 ∧ U1 ⊆ U2 ∧
U2 ⊆ U
          and opeT: openin (top_of_set T) V1 ∧ Q V2 ∧ y ∈ V1 ∧ V1 ⊆ V2 ∧
V2 ⊆ V
    by (meson PS QT locallyE)
  then have openin (top_of_set (S × T)) (U1 × V1)
    by (simp add: openin_Times)
  moreover have R (U2 × V2)
    by (simp add: R opeS opeT)
  moreover have U1 × V1 ⊆ U2 × V2 ∧ U2 × V2 ⊆ W
    using opeS opeT ‹U × V ⊆ W› by auto
  ultimately show ∃ U V. openin (top_of_set (S × T)) U ∧ R V ∧ (x,y) ∈ U
∧ U ⊆ V ∧ V ⊆ W
    using opeS opeT by auto
qed


proposition homeomorphism_locally_imp:
  fixes S :: 'a::metric_space set and T :: 'b::t2_space set
  assumes S: locally P S and hom: homeomorphism S T f g
      and Q: ⋀S S'. ⟦P S; homeomorphism S S' f g⟧ ⟹ Q S'
    shows locally Q T
proof (clarsimp simp: locally_def)
  fix W y
  assume y ∈ W and openin (top_of_set T) W
  then obtain A where T: open A W = T ∩ A
    by (force simp: openin_open)
  then have W ⊆ T by auto
  have f: ⋀x. x ∈ S ⟹ g(f x) = x f ‘ S = T continuous_on S f
   and g: ⋀y. y ∈ T ⟹ f(g y) = y g ‘ T = S continuous_on T g
    using hom by (auto simp: homeomorphism_def)
  have gw: g ‘ W = S ∩ f −‘ W
    using ‹W ⊆ T› g by force
  have openin (top_of_set S) (g ‘ W)
    using ‹openin (top_of_set T) W› continuous_on_open f gw by auto
  then obtain U V
    where osu: openin (top_of_set S) U and uv: P V g y ∈ U U ⊆ V V ⊆ g ‘ W
    by (metis S ‹y ∈ W› image_eqI locallyE)
  have V ⊆ S using uv by (simp add: gw)
```

```
  have fv: f ‘ V = T ∩ {x. g x ∈ V}
    using ‹f ‘ S = T› f ‹V ⊆ S› by auto
  have contvf: continuous_on V f
    using ‹V ⊆ S› continuous_on_subset f(3) by blast
  have openin (top_of_set (g ‘ T)) U
    using ‹g ‘ T = S› by (simp add: osu)
  then have openin (top_of_set T) (T ∩ g −‘ U)
    using ‹continuous_on T g› continuous_on_open [THEN iffD1] by blast
  moreover have ∃ V. Q V ∧ y ∈ (T ∩ g −‘ U) ∧ (T ∩ g −‘ U) ⊆ V ∧ V ⊆ W
  proof (intro exI conjI)
    show f ‘ V ⊆ W
     using uv using Int_lower2 gw image_subsetI mem_Collect_eq subset_iff by
auto
    then have contvg: continuous_on (f ‘ V) g
      using ‹W ⊆ T› continuous_on_subset [OF g(3)] by blast
    have V ⊆ g ‘ f ‘ V
      by (metis ‹V ⊆ S› hom homeomorphism_def homeomorphism_of_subsets
order_refl)
    then have homv: homeomorphism V (f ‘ V) f g
      using ‹V ⊆ S› f by (auto simp: homeomorphism_def contvf contvg)
    show Q (f ‘ V)
      using Q homv ‹P V› by blast
    show y ∈ T ∩ g −‘ U
      using T(2) ‹y ∈ W› ‹g y ∈ U› by blast
    show T ∩ g −‘ U ⊆ f ‘ V
      using g(1) image_iff uv(3) by fastforce
  qed
  ultimately show ∃ U. openin (top_of_set T) U ∧ (∃ v. Q v ∧ y ∈ U ∧ U ⊆ v
∧ v ⊆ W)
    by meson
qed

lemma homeomorphism_locally:
  fixes f:: 'a::metric_space ⇒ 'b::metric_space
  assumes homeomorphism S T f g
      and ⋀S T. homeomorphism S T f g ⟹ (P S ⟷ Q T)
    shows locally P S ⟷ locally Q T
  by (smt (verit) assms homeomorphism_locally_imp homeomorphism_symD)

lemma homeomorphic_locally:
  fixes S:: 'a::metric_space set and T:: 'b::metric_space set
  assumes hom: S homeomorphic T
        and iff: ⋀X Y. X homeomorphic Y ⟹ (P X ⟷ Q Y)
    shows locally P S ⟷ locally Q T
 by (smt (verit, ccfv_SIG) hom homeomorphic_def homeomorphism_locally homeomorphism_locally_imp iff)

lemma homeomorphic_local_compactness:
  fixes S:: 'a::metric_space set and T:: 'b::metric_space set
```

shows *S homeomorphic T* $\Longrightarrow$ *locally compact S* $\longleftrightarrow$ *locally compact T*
by (*simp add*: *homeomorphic_compactness homeomorphic_locally*)

lemma *locally_translation*:
fixes *P* :: *'a* :: *real_normed_vector set* $\Rightarrow$ *bool*
shows ($\bigwedge$*S*. *P* ((+) *a* ' *S*) = *P S*) $\Longrightarrow$ *locally P* ((+) *a* ' *S*) = *locally P S*
using *homeomorphism_locally* [*OF homeomorphism_translation*]
by (*metis* (*full_types*) *homeomorphism_image2*)

lemma *locally_injective_linear_image*:
fixes *f* :: *'a*::*euclidean_space* $\Rightarrow$ *'b*::*euclidean_space*
assumes *f*: *linear f inj f* **and** *iff*: $\bigwedge$*S*. *P* (*f* ' *S*) $\longleftrightarrow$ *Q S*
shows *locally P* (*f* ' *S*) $\longleftrightarrow$ *locally Q S*
by (*smt* (*verit*) *f homeomorphism_image2 homeomorphism_locally iff linear_homeomorphism_image*)

lemma *locally_open_map_image*:
fixes *f* :: *'a*::*real_normed_vector* $\Rightarrow$ *'b*::*real_normed_vector*
assumes *P*: *locally P S*
and *f*: *continuous_on S f*
and *oo*: $\bigwedge$*T*. *openin* (*top_of_set S*) *T* $\Longrightarrow$ *openin* (*top_of_set* (*f* ' *S*)) (*f* ' *T*)
and *Q*: $\bigwedge$*T*. $[\![T \subseteq S; P\ T]\!] \Longrightarrow Q(f\ `\ T)$
shows *locally Q* (*f* ' *S*)
proof (*clarsimp simp*: *locally_def*)
fix *W y*
assume *oiw*: *openin* (*top_of_set* (*f* ' *S*)) *W* **and** $y \in W$
then have $W \subseteq f\ `\ S$ **by** (*simp add*: *openin_euclidean_subtopology_iff*)
have *oivf*: *openin* (*top_of_set S*) ($S \cap f -` W$)
by (*rule continuous_on_open* [*THEN iffD1*, *rule_format*, *OF f oiw*])
then obtain *x* **where** $x \in S$ *f x* = *y*
using ‹$W \subseteq f\ `\ S$› ‹$y \in W$› **by** *blast*
then obtain *U V*
where *openin* (*top_of_set S*) *U P V x* $\in$ *U U* $\subseteq$ *V V* $\subseteq S \cap f -` W$
by (*metis IntI P* ‹$y \in W$› *locallyE oivf vimageI*)
then have *openin* (*top_of_set* (*f* ' *S*)) (*f* ' *U*)
by (*simp add*: *oo*)
then show $\exists X$. *openin* (*top_of_set* (*f* ' *S*)) *X* $\wedge$ ($\exists$ *Y*. *Q Y* $\wedge$ $y \in X \wedge X \subseteq Y \wedge Y \subseteq W$)
using *Q* ‹*P V*› ‹*U* $\subseteq$ *V*› ‹*V* $\subseteq S \cap f -` W$› ‹*f x* = *y*› ‹*x* $\in$ *U*› **by** *blast*
qed

6.4.14 An induction principle for connected sets

proposition *connected_induction*:
assumes *connected S*
and *opD*: $\bigwedge$*T a*. $[\![$*openin* (*top_of_set S*) *T*; $a \in T]\!] \Longrightarrow \exists z.\ z \in T \wedge P\ z$
and *opI*: $\bigwedge$*a*. $a \in S$
$\Longrightarrow \exists T.$ *openin* (*top_of_set S*) $T \wedge a \in T \wedge$
($\forall x \in T.\ \forall y \in T.\ P\ x \wedge P\ y \wedge Q\ x \longrightarrow Q\ y$)

```
    and etc: a ∈ S b ∈ S P a P b Q a
  shows Q b
proof −
 let ?A = {b. ∃ T. openin (top_of_set S) T ∧ b ∈ T ∧ (∀ x∈T. P x ⟶ Q x)}
 let ?B = {b. ∃ T. openin (top_of_set S) T ∧ b ∈ T ∧ (∀ x∈T. P x ⟶ ¬ Q x)}
 have ?A ∩ ?B = {}
   by (clarsimp simp: set_eq_iff) (metis (no_types, opaque_lifting) Int_iff opD
openin_Int)
 moreover have S ⊆ ?A ∪ ?B
   by clarsimp (meson opI)
 moreover have openin (top_of_set S) ?A
   by (subst openin_subopen, blast)
 moreover have openin (top_of_set S) ?B
   by (subst openin_subopen, blast)
 ultimately have ?A = {} ∨ ?B = {}
   by (metis (no_types, lifting) ‹connected S› connected_openin)
 then show ?thesis
   by clarsimp (meson opI etc)
qed

lemma connected_equivalence_relation_gen:
 assumes connected S
    and etc: a ∈ S b ∈ S P a P b
    and trans: ⋀x y z. ⟦R x y; R y z⟧ ⟹ R x z
    and opD: ⋀T a. ⟦openin (top_of_set S) T; a ∈ T⟧ ⟹ ∃ z. z ∈ T ∧ P z
    and opI: ⋀a. a ∈ S
          ⟹ ∃ T. openin (top_of_set S) T ∧ a ∈ T ∧
                (∀ x ∈ T. ∀ y ∈ T. P x ∧ P y ⟶ R x y)
  shows R a b
proof −
 have ⋀a b c. ⟦a ∈ S; P a; b ∈ S; c ∈ S; P b; P c; R a b⟧ ⟹ R a c
   apply (rule connected_induction [OF ‹connected S› opD], simp_all)
   by (meson trans opI)
 then show ?thesis by (metis etc opI)
qed

lemma connected_induction_simple:
 assumes connected S
    and etc: a ∈ S b ∈ S P a
    and opI: ⋀a. a ∈ S
          ⟹ ∃ T. openin (top_of_set S) T ∧ a ∈ T ∧
                (∀ x ∈ T. ∀ y ∈ T. P x ⟶ P y)
  shows P b
 by (rule connected_induction [OF ‹connected S› _, where P = λx. True])
   (use opI etc in auto)

lemma connected_equivalence_relation:
 assumes connected S
    and etc: a ∈ S b ∈ S
```

and *sym*: $\bigwedge x\ y.\ [\![R\ x\ y;\ x \in S;\ y \in S]\!] \Longrightarrow R\ y\ x$
and *trans*: $\bigwedge x\ y\ z.\ [\![R\ x\ y;\ R\ y\ z;\ x \in S;\ y \in S;\ z \in S]\!] \Longrightarrow R\ x\ z$
and *opI*: $\bigwedge a.\ a \in S \Longrightarrow \exists\, T.\ openin\ (top\_of\_set\ S)\ T \wedge a \in T \wedge (\forall x \in T.\ R\ a\ x)$
shows $R\ a\ b$
proof −
have $\bigwedge a\ b\ c.\ [\![a \in S;\ b \in S;\ c \in S;\ R\ a\ b]\!] \Longrightarrow R\ a\ c$
by (*smt* (*verit*, *ccfv_threshold*) *connected_induction_simple* [*OF* ‹*connected S*›]
assms openin_imp_subset subset_eq)
then show *?thesis* **by** (*metis etc opI*)
qed

lemma *locally_constant_imp_constant*:
assumes *connected S*
and *opI*: $\bigwedge a.\ a \in S \Longrightarrow \exists\, T.\ openin\ (top\_of\_set\ S)\ T \wedge a \in T \wedge (\forall x \in T.\ f\ x = f\ a)$
shows *f constant_on S*
proof −
have $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow f\ x = f\ y$
apply (*rule connected_equivalence_relation* [*OF* ‹*connected S*›], *simp_all*)
by (*metis opI*)
then show *?thesis*
by (*metis constant_on_def*)
qed

lemma *locally_constant*:
assumes *connected S*
shows *locally* ($\lambda U.$ *f constant_on U*) *S* $\longleftrightarrow$ *f constant_on S* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*smt* (*verit*, *del_insts*) *assms constant_on_def locally_constant_imp_constant locally_def openin_subtopology_self subset_iff*)
next
assume *?rhs* **then show** *?lhs*
by (*metis constant_on_subset locallyI openin_imp_subset order_refl*)
qed

6.4.15 Basic properties of local compactness

proposition *locally_compact*:
fixes *S* :: *'a* :: *metric_space set*
shows
locally compact S $\longleftrightarrow$
$(\forall x \in S.\ \exists u\ v.\ x \in u \wedge u \subseteq v \wedge v \subseteq S \wedge$
openin (*top_of_set S*) *u* $\wedge$ *compact v*)
(**is** *?lhs* = *?rhs*)
proof

assume *?lhs*
then show *?rhs*
by (*meson locallyE openin_subtopology_self*)
next
assume *r* [*rule_format*]: *?rhs*
have ∗: ∃ *u v*.
openin (*top_of_set S*) *u* ∧
compact v ∧ *x* ∈ *u* ∧ *u* ⊆ *v* ∧ *v* ⊆ *S* ∩ *T*
if *open T x* ∈ *S x* ∈ *T* **for** *x T*
proof −
obtain *U V* **where** *uv*: *x* ∈ *U U* ⊆ *V V* ⊆ *S compact V openin* (*top_of_set S*) *U*
using *r* [*OF* ‹*x* ∈ *S*›] **by** *auto*
obtain *e* **where** *e*>*0* **and** *e*: *cball x e* ⊆ *T*
using *open_contains_cball* ‹*open T*› ‹*x* ∈ *T*› **by** *blast*
show *?thesis*
apply (*rule_tac x*=(*S* ∩ *ball x e*) ∩ *U* **in** *exI*)
apply (*rule_tac x*=*cball x e* ∩ *V* **in** *exI*)
using *that* ‹*e* > *0*› *e uv*
apply *auto*
done
qed
show *?lhs*
by (*rule locallyI*) (*metis* ∗ *Int_iff openin_open*)
qed

lemma *locally_compactE*:
fixes *S* :: *'a* :: *metric_space set*
assumes *locally compact S*
obtains *u v* **where** ⋀*x*. *x* ∈ *S* ⟹ *x* ∈ *u x* ∧ *u x* ⊆ *v x* ∧ *v x* ⊆ *S* ∧
openin (*top_of_set S*) (*u x*) ∧ *compact* (*v x*)
using *assms* **unfolding** *locally_compact* **by** *metis*

lemma *locally_compact_alt*:
fixes *S* :: *'a* :: *heine_borel set*
shows *locally compact S* ⟷
(∀ *x* ∈ *S*. ∃ *U*. *x* ∈ *U* ∧
openin (*top_of_set S*) *U* ∧ *compact*(*closure U*) ∧ *closure U* ⊆ *S*)
by (*smt* (*verit*, *ccfv_threshold*) *bounded_subset closure_closed closure_mono closure_subset*
compact_closure compact_imp_closed order.trans locally_compact)

lemma *locally_compact_Int_cball*:
fixes *S* :: *'a* :: *heine_borel set*
shows *locally compact S* ⟷ (∀ *x* ∈ *S*. ∃ *e*. *0* < *e* ∧ *closed*(*cball x e* ∩ *S*))
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have ⋀*x U V e*. ⟦*U* ⊆ *V*; *V* ⊆ *S*; *compact V*; *0* < *e*; *cball x e* ∩ *S* ⊆ *U*⟧

```
      ⟹ closed (cball x e ∩ S)
     by (metis compact_Int compact_cball compact_imp_closed inf.absorb_iff2
inf.assoc inf.orderE)
  with L show ?rhs
    by (meson locally_compactE openin_contains_cball)
next
  assume R: ?rhs
  show ?lhs unfolding locally_compact
  proof
    fix x
    assume x ∈ S
    then obtain e where e>0 and compact (cball x e ∩ S)
      by (metis Int_commute compact_Int_closed compact_cball inf.right_idem
R)
    moreover have ∀ y∈ball x e ∩ S. ∃ε>0. cball y ε ∩ S ⊆ ball x e
    by (meson Elementary_Metric_Spaces.open_ball IntD1 le_infI1 open_contains_cball_eq)
    moreover have openin (top_of_set S) (ball x e ∩ S)
      by (simp add: inf_commute openin_open_Int)
    ultimately show ∃ U V. x ∈ U ∧ U ⊆ V ∧ V ⊆ S ∧ openin (top_of_set S)
U ∧ compact V
    by (metis Int_iff ‹0 < e› ‹x ∈ S› ball_subset_cball centre_in_ball inf_commute
inf_le1 inf_mono order_refl)
  qed
qed

lemma locally_compact_compact:
  fixes S :: 'a :: heine_borel set
  shows locally compact S ⟷
        (∀ K. K ⊆ S ∧ compact K
            ⟶ (∃ U V. K ⊆ U ∧ U ⊆ V ∧ V ⊆ S ∧
                      openin (top_of_set S) U ∧ compact V))
       (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain u v where
    uv: ⋀x. x ∈ S ⟹ x ∈ u x ∧ u x ⊆ v x ∧ v x ⊆ S ∧
                          openin (top_of_set S) (u x) ∧ compact (v x)
    by (metis locally_compactE)
  have ∗: ∃ U V. K ⊆ U ∧ U ⊆ V ∧ V ⊆ S ∧ openin (top_of_set S) U ∧
compact V
        if K ⊆ S compact K for K
  proof −
    have ⋀C. (∀ c∈C. openin (top_of_set K) c) ∧ K ⊆ ⋃ C ⟹
                ∃ D⊆C. finite D ∧ K ⊆ ⋃ D
      using that by (simp add: compact_eq_openin_cover)
    moreover have ∀ c ∈ (λx. K ∩ u x) ‘ K. openin (top_of_set K) c
    using that by clarify (metis subsetD inf.absorb_iff2 openin_subset openin_subtopology_Int_subset
topspace_euclidean_subtopology uv)
    moreover have K ⊆ ⋃((λx. K ∩ u x) ‘ K)
```

```
    using that by clarsimp (meson subsetCE uv)
   ultimately obtain D where D ⊆ (λx. K ∩ u x) ‘ K finite D K ⊆ ⋃ D
    by metis
   then obtain T where T: T ⊆ K finite T K ⊆ ⋃((λx. K ∩ u x) ‘ T)
    by (metis finite_subset_image)
   have Tuv: ⋃(u ‘ T) ⊆ ⋃(v ‘ T)
    using T that by (force dest!: uv)
   moreover
   have openin (top_of_set S) (⋃ (u ‘ T))
    using T that uv by fastforce
   moreover
   obtain compact (⋃ (v ‘ T)) ⋃ (v ‘ T) ⊆ S
    by (metis T UN_subset_iff ‹K ⊆ S› compact_UN subset_iff uv)
   ultimately show ?thesis
    using T by auto
  qed
  show ?rhs
   by (blast intro: ∗)
next
  assume ?rhs
  then show ?lhs
   apply (clarsimp simp: locally_compact)
   apply (drule_tac x={x} in spec, simp)
   done
qed

lemma open_imp_locally_compact:
  fixes S :: 'a :: heine_borel set
  assumes open S
   shows locally compact S
proof −
  have ∗: ∃ U V. x ∈ U ∧ U ⊆ V ∧ V ⊆ S ∧ openin (top_of_set S) U ∧ compact
V
        if x ∈ S for x
  proof −
   obtain e where e>0 and e: cball x e ⊆ S
    using open_contains_cball assms ‹x ∈ S› by blast
   have ope: openin (top_of_set S) (ball x e)
    by (meson e open_ball ball_subset_cball dual_order.trans open_subset)
   show ?thesis
    by (meson ‹0 < e› ball_subset_cball centre_in_ball compact_cball e ope)
  qed
  show ?thesis
   unfolding locally_compact by (blast intro: ∗)
qed

lemma closed_imp_locally_compact:
  fixes S :: 'a :: heine_borel set
  assumes closed S
```

```
    shows locally compact S
proof −
  have ∗: ∃ U V. x ∈ U ∧ U ⊆ V ∧ V ⊆ S ∧ openin (top_of_set S) U ∧ compact
V
          if x ∈ S for x
    apply (rule_tac x = S ∩ ball x 1 in exI, rule_tac x = S ∩ cball x 1 in exI)
    using ‹x ∈ S› assms by auto
  show ?thesis
    unfolding locally_compact by (blast intro: ∗)
qed

lemma locally_compact_UNIV: locally compact (UNIV :: 'a :: heine_borel set)
  by (simp add: closed_imp_locally_compact)

lemma locally_compact_Int:
  fixes S :: 'a :: t2_space set
  shows ⟦locally compact S; locally compact T⟧ ⟹ locally compact (S ∩ T)
  by (simp add: compact_Int locally_Int)

lemma locally_compact_closedin:
  fixes S :: 'a :: heine_borel set
  shows ⟦closedin (top_of_set S) T; locally compact S⟧
        ⟹ locally compact T
  unfolding closedin_closed
  using closed_imp_locally_compact locally_compact_Int by blast

lemma locally_compact_delete:
    fixes S :: 'a :: t1_space set
    shows locally compact S ⟹ locally compact (S − {a})
  by (auto simp: openin_delete locally_open_subset)

lemma locally_closed:
  fixes S :: 'a :: heine_borel set
  shows locally closed S ⟷ locally compact S
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    unfolding locally_def
    apply (elim all_forward imp_forward asm_rl exE)
    apply (rename_tac U V)
    apply (rule_tac x = U ∩ ball x 1 in exI)
    apply (rule_tac x = V ∩ cball x 1 in exI)
    apply (force intro: openin_trans)
    done
next
  assume ?rhs then show ?lhs
    using compact_eq_bounded_closed locally_mono by blast
qed
```

lemma *locally_compact_openin_Un*:
fixes S :: *'a*::*euclidean_space set*
assumes *LCS*: *locally compact S* **and** *LCT*: *locally compact T*
and *opS*: *openin* (*top_of_set* ($S \cup T$)) S
and *opT*: *openin* (*top_of_set* ($S \cup T$)) T
shows *locally compact* ($S \cup T$)
proof −
have $\exists e{>}0.$ *closed* (*cball x e* ∩ ($S \cup T$)) **if** $x \in S$ **for** x
proof −
obtain *e1* **where** $e1 > 0$ **and** *e1*: *closed* (*cball x e1* ∩ S)
using *LCS* ‹$x \in S$› **unfolding** *locally_compact_Int_cball* **by** *blast*
moreover obtain *e2* **where** $e2 > 0$ **and** *e2*: *cball x e2* ∩ ($S \cup T$) ⊆ S
by (*meson* ‹$x \in S$› *opS openin_contains_cball*)
then have *cball x e2* ∩ ($S \cup T$) = *cball x e2* ∩ S
by *force*
ultimately have *closed* (*cball x* (*min e1 e2*) ∩ ($S \cup T$))
by (*metis* (*no_types*, *lifting*) *cball_min_Int closed_Int closed_cball inf_assoc inf_commute*)
then show *?thesis*
by (*metis* ‹$0 < e1$› ‹$0 < e2$› *min_def*)
qed
moreover have $\exists e{>}0.$ *closed* (*cball x e* ∩ ($S \cup T$)) **if** $x \in T$ **for** x
proof −
obtain *e1* **where** $e1 > 0$ **and** *e1*: *closed* (*cball x e1* ∩ T)
using *LCT* ‹$x \in T$› **unfolding** *locally_compact_Int_cball* **by** *blast*
moreover obtain *e2* **where** $e2 > 0$ **and** *e2*: *cball x e2* ∩ ($S \cup T$) ⊆ T
by (*meson* ‹$x \in T$› *opT openin_contains_cball*)
then have *cball x e2* ∩ ($S \cup T$) = *cball x e2* ∩ T
by *force*
moreover have *closed* (*cball x e1* ∩ (*cball x e2* ∩ T))
by (*metis closed_Int closed_cball e1 inf_left_commute*)
ultimately show *?thesis*
by (*rule_tac x*=*min e1 e2* **in** *exI*) (*simp add*: ‹$0 < e2$› *cball_min_Int inf_assoc*)
qed
ultimately show *?thesis*
by (*force simp*: *locally_compact_Int_cball*)
qed

lemma *locally_compact_closedin_Un*:
fixes S :: *'a*::*euclidean_space set*
assumes *LCS*: *locally compact S* **and** *LCT*:*locally compact T*
and *clS*: *closedin* (*top_of_set* ($S \cup T$)) S
and *clT*: *closedin* (*top_of_set* ($S \cup T$)) T
shows *locally compact* ($S \cup T$)
proof −
have $\exists e{>}0.$ *closed* (*cball x e* ∩ ($S \cup T$)) **if** $x \in S$ $x \in T$ **for** x
proof −

```
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ S)
      using LCS ‹x ∈ S› unfolding locally_compact_Int_cball by blast
    moreover
    obtain e2 where e2 > 0 and e2: closed (cball x e2 ∩ T)
      using LCT ‹x ∈ T› unfolding locally_compact_Int_cball by blast
    moreover have closed (cball x (min e1 e2) ∩ (S ∪ T))
      by (smt (verit) Int_Un_distrib2 Int_commute cball_min_Int closed_Int
closed_Un closed_cball e1 e2 inf_left_commute)
    ultimately show ?thesis
      by (rule_tac x=min e1 e2 in exI) linarith
  qed
  moreover
  have ∃e>0. closed (cball x e ∩ (S ∪ T)) if x: x ∈ S x ∉ T for x
  proof −
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ S)
      using LCS ‹x ∈ S› unfolding locally_compact_Int_cball by blast
    moreover
    obtain e2 where e2>0 and cball x e2 ∩ (S ∪ T) ⊆ S − T
      using clT x by (fastforce simp: openin_contains_cball closedin_def)
    then have closed (cball x e2 ∩ T)
    proof −
      have {} = T − (T − cball x e2)
        using Diff_subset Int_Diff ‹cball x e2 ∩ (S ∪ T) ⊆ S − T› by auto
      then show ?thesis
        by (simp add: Diff_Diff_Int inf_commute)
    qed
    with e1 have closed ((cball x e1 ∩ cball x e2) ∩ (S ∪ T))
      apply (simp add: inf_commute inf_sup_distrib2)
      by (metis closed_Int closed_Un closed_cball inf_assoc inf_left_commute)
    then have closed (cball x (min e1 e2) ∩ (S ∪ T))
      by (simp add: cball_min_Int inf_commute)
    ultimately show ?thesis
      using ‹0 < e2› by (rule_tac x=min e1 e2 in exI) linarith
  qed
  moreover
  have ∃e>0. closed (cball x e ∩ (S ∪ T)) if x: x ∉ S x ∈ T for x
  proof −
    obtain e1 where e1 > 0 and e1: closed (cball x e1 ∩ T)
      using LCT ‹x ∈ T› unfolding locally_compact_Int_cball by blast
    moreover
    obtain e2 where e2>0 and cball x e2 ∩ (S ∪ T) ⊆ S ∪ T − S
      using clS x by (fastforce simp: openin_contains_cball closedin_def)
    then have closed (cball x e2 ∩ S)
      by (metis Diff_disjoint Int_empty_right closed_empty inf.left_commute
inf.orderE inf_sup_absorb)
    with e1 have closed ((cball x e1 ∩ cball x e2) ∩ (S ∪ T))
      apply (simp add: inf_commute inf_sup_distrib2)
      by (metis closed_Int closed_Un closed_cball inf_assoc inf_left_commute)
    then have closed (cball x (min e1 e2) ∩ (S ∪ T))
```

```
    by (auto simp: cball_min_Int)
   ultimately show ?thesis
    using ‹0 < e2› by (rule_tac x=min e1 e2 in exI) linarith
  qed
  ultimately show ?thesis
   by (auto simp: locally_compact_Int_cball)
qed

lemma locally_compact_Times:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  shows ⟦locally compact S; locally compact T⟧ ⟹ locally compact (S × T)
  by (auto simp: compact_Times locally_Times)

lemma locally_compact_compact_subopen:
  fixes S :: 'a :: heine_borel set
  shows
   locally compact S ⟷
    (∀ K T. K ⊆ S ∧ compact K ∧ open T ∧ K ⊆ T
         ⟶ (∃ U V. K ⊆ U ∧ U ⊆ V ∧ U ⊆ T ∧ V ⊆ S ∧
                   openin (top_of_set S) U ∧ compact V))
   (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  show ?rhs
  proof clarify
    fix K :: 'a set and T :: 'a set
    assume K ⊆ S and compact K and open T and K ⊆ T
    obtain U V where K ⊆ U U ⊆ V V ⊆ S compact V
                and ope: openin (top_of_set S) U
      using L unfolding locally_compact_compact by (meson ‹K ⊆ S› ‹compact
K›)
    show ∃ U V. K ⊆ U ∧ U ⊆ V ∧ U ⊆ T ∧ V ⊆ S ∧
              openin (top_of_set S) U ∧ compact V
    proof (intro exI conjI)
      show K ⊆ U ∩ T
        by (simp add: ‹K ⊆ T› ‹K ⊆ U›)
      show U ∩ T ⊆ closure(U ∩ T)
        by (rule closure_subset)
      show closure (U ∩ T) ⊆ S
         by (metis ‹U ⊆ V› ‹V ⊆ S› ‹compact V› closure_closed closure_mono
compact_imp_closed inf.cobounded1 subset_trans)
      show openin (top_of_set S) (U ∩ T)
        by (simp add: ‹open T› ope openin_Int_open)
      show compact (closure (U ∩ T))
           by (meson Int_lower1 ‹U ⊆ V› ‹compact V› bounded_subset com-
pact_closure compact_eq_bounded_closed)
    qed auto
  qed
next
```

assume *?rhs* **then show** *?lhs*
unfolding *locally_compact_compact*
by (*metis open_openin openin_topspace subtopology_superset top.extremum topspace_euclidean_subtopology*)
qed

6.4.16 Sura-Bura's results about compact components of sets

proposition *Sura_Bura_compact*:
fixes S :: *'a::euclidean_space set*
assumes *compact S* **and** C: $C \in$ *components S*
shows $C = \bigcap \{T.\ C \subseteq T \wedge$ *openin* (*top_of_set S*) $T \wedge$
closedin (*top_of_set S*) $T\}$
(**is** $C = \bigcap$ *?T*)
proof
obtain x **where** x: C = *connected_component_set S x* **and** $x \in S$
using C **by** (*auto simp*: *components_def*)
have $C \subseteq S$
by (*simp add*: C *in_components_subset*)
have $\bigcap$ *?T* $\subseteq$ *connected_component_set S x*
proof (*rule connected_component_maximal*)
have $x \in C$
by (*simp add*: ‹$x \in S$› x)
then show $x \in \bigcap$ *?T*
by *blast*
have *clo*: *closed* ($\bigcap$ *?T*)
by (*simp add*: ‹*compact S*› *closed_Inter closedin_compact_eq compact_imp_closed*)
have *False*
if *K1*: *closedin* (*top_of_set* ($\bigcap$ *?T*)) *K1* **and**
K2: *closedin* (*top_of_set* ($\bigcap$ *?T*)) *K2* **and**
K12_Int: *K1* $\cap$ *K2* = {} **and** *K12_Un*: *K1* $\cup$ *K2* = $\bigcap$ *?T* **and** *K1* $\neq$ {} *K2* $\neq$ {}
for *K1 K2*
proof −
have *closed K1 closed K2*
using *closedin_closed_trans clo K1 K2* **by** *blast+*
then obtain *V1 V2* **where** *open V1 open V2 K1* $\subseteq$ *V1 K2* $\subseteq$ *V2* **and** *V12*: *V1* $\cap$ *V2* = {}
using *separation_normal* ‹*K1* $\cap$ *K2* = {}› **by** *metis*
have *SV12_ne*: (*S* − (*V1* $\cup$ *V2*)) $\cap$ ($\bigcap$ *?T*) $\neq$ {}
proof (*rule compact_imp_fip*)
show *compact* (*S* − (*V1* $\cup$ *V2*))
by (*simp add*: ‹*open V1*› ‹*open V2*› ‹*compact S*› *compact_diff open_Un*)
show *cloT*: *closed T* **if** $T \in$ *?T* **for** T
using *that* ‹*compact S*›
by (*force intro*: *closedin_closed_trans simp add*: *compact_imp_closed*)
show (*S* − (*V1* $\cup$ *V2*)) $\cap \bigcap \mathcal{F} \neq$ {} **if** *finite* $\mathcal{F}$ **and** $\mathcal{F}$: $\mathcal{F} \subseteq$ *?T* **for** $\mathcal{F}$
proof
assume *djo*: (*S* − (*V1* $\cup$ *V2*)) $\cap \bigcap \mathcal{F}$ = {}

```
        obtain D where openD: openin (top_of_set S) D
                and cloD: closedin (top_of_set S) D
                and C ⊆ D and DV12: D ⊆ V1 ∪ V2
        proof (cases ℱ = {})
          case True
          with ‹C ⊆ S› djo that show ?thesis
            by force
        next
          case False show ?thesis
          proof
            show ope: openin (top_of_set S) (⋂ℱ)
              using openin_Inter ‹finite ℱ› False ℱ by blast
            then show closedin (top_of_set S) (⋂ℱ)
              by (meson cloT ℱ closed_Inter closed_subset openin_imp_subset
subset_eq)
            show C ⊆ ⋂ℱ
              using ℱ by auto
            show ⋂ℱ ⊆ V1 ∪ V2
              using ope djo openin_imp_subset by fastforce
          qed
        qed
        have connected C
          by (simp add: x)
        have closed D
          using ‹compact S› cloD closedin_closed_trans compact_imp_closed by
blast
        have cloV1: closedin (top_of_set D) (D ∩ closure V1)
          and cloV2: closedin (top_of_set D) (D ∩ closure V2)
          by (simp_all add: closedin_closed_Int)
        moreover have D ∩ closure V1 = D ∩ V1 D ∩ closure V2 = D ∩ V2
          using ‹D ⊆ V1 ∪ V2› ‹open V1› ‹open V2› V12
       by (auto simp: closure_subset [THEN subsetD] closure_iff_nhds_not_empty,
blast+)
        ultimately have cloDV1: closedin (top_of_set D) (D ∩ V1)
                  and cloDV2:  closedin (top_of_set D) (D ∩ V2)
          by metis+
        then obtain U1 U2 where closed U1 closed U2
            and D1: D ∩ V1 = D ∩ U1 and D2: D ∩ V2 = D ∩ U2
          by (auto simp: closedin_closed)
        have D ∩ U1 ∩ C ≠ {}
        proof
          assume D ∩ U1 ∩ C = {}
          then have ∗: C ⊆ D ∩ V2
            using D1 DV12 ‹C ⊆ D› by auto
          have 1: openin (top_of_set S) (D ∩ V2)
            by (simp add: ‹open V2› opeD openin_Int_open)
          have 2: closedin (top_of_set S) (D ∩ V2)
            using cloD cloDV2 closedin_trans by blast
          have ⋂ ?T ⊆ D ∩ V2
```

```
            by (rule Inter_lower) (use * 1 2 in simp)
          then show False
            using K1 V12 ‹K1 ≠ {}› ‹K1 ⊆ V1› closedin_imp_subset by blast
        qed
        moreover have D ∩ U2 ∩ C ≠ {}
        proof
          assume D ∩ U2 ∩ C = {}
          then have *: C ⊆ D ∩ V1
            using D2 DV12 ‹C ⊆ D› by auto
          have 1: openin (top_of_set S) (D ∩ V1)
            by (simp add: ‹open V1› opeD openin_Int_open)
          have 2: closedin (top_of_set S) (D ∩ V1)
            using cloD cloDV1 closedin_trans by blast
          have ⋂ ?T ⊆ D ∩ V1
            by (rule Inter_lower) (use * 1 2 in simp)
          then show False
            using K2 V12 ‹K2 ≠ {}› ‹K2 ⊆ V2› closedin_imp_subset by blast
        qed
        ultimately show False
         using ‹connected C› [unfolded connected_closed, simplified, rule_format,
of concl: D ∩ U1 D ∩ U2]
          using ‹C ⊆ D› D1 D2 V12 DV12 ‹closed U1› ‹closed U2› ‹closed D›
          by blast
      qed
    qed
    show False
      by (metis (full_types) DiffE UnE Un_upper2 SV12_ne ‹K1 ⊆ V1› ‹K2 ⊆
V2› disjoint_iff_not_equal subsetCE sup_ge1 K12_Un)
  qed
  then show connected (⋂ ?T)
    by (auto simp: connected_closedin_eq)
  show ⋂ ?T ⊆ S
    by (fastforce simp: C in_components_subset)
 qed
 with x show ⋂ ?T ⊆ C by simp
qed auto


corollary Sura_Bura_clopen_subset:
 fixes S :: 'a::euclidean_space set
 assumes S: locally compact S and C: C ∈ components S and compact C
    and U: open U C ⊆ U
 obtains K where openin (top_of_set S) K compact K C ⊆ K K ⊆ U
proof (rule ccontr)
 assume ¬ thesis
 with that have neg: ∄K. openin (top_of_set S) K ∧ compact K ∧ C ⊆ K ∧ K
⊆ U
   by metis
 obtain V K where C ⊆ V V ⊆ U V ⊆ K K ⊆ S compact K
```

```
            and opeSV: openin (top_of_set S) V
  using S U ‹compact C› by (meson C in_components_subset locally_compact_compact_subopen)
 let ?T = {T. C ⊆ T ∧ openin (top_of_set K) T ∧ compact T ∧ T ⊆ K}
 have CK: C ∈ components K
   by (meson C ‹C ⊆ V› ‹K ⊆ S› ‹V ⊆ K› components_intermediate_subset
subset_trans)
 with ‹compact K›
 have C = ⋂{T. C ⊆ T ∧ openin (top_of_set K) T ∧ closedin (top_of_set K)
T}
   by (simp add: Sura_Bura_compact)
 then have Ceq: C = ⋂ ?T
   by (simp add: closedin_compact_eq ‹compact K›)
 obtain W where open W and W: V = S ∩ W
   using opeSV by (auto simp: openin_open)
 have −(U ∩ W) ∩ ⋂ ?T ≠ {}
 proof (rule closed_imp_fip_compact)
   show − (U ∩ W) ∩ ⋂ℱ ≠ {}
     if finite ℱ and ℱ: ℱ ⊆ ?T for ℱ
   proof (cases ℱ = {})
     case True
     have False if U = UNIV W = UNIV
     proof −
       have V = S
         by (simp add: W ‹W = UNIV›)
       with neg show False
         using ‹C ⊆ V› ‹K ⊆ S› ‹V ⊆ K› ‹V ⊆ U› ‹compact K› by auto
     qed
     with True show ?thesis
       by auto
   next
     case False
     show ?thesis
     proof
       assume − (U ∩ W) ∩ ⋂ℱ = {}
       then have FUW: ⋂ℱ ⊆ U ∩ W
         by blast
       have C ⊆ ⋂ℱ
         using ℱ by auto
       moreover have compact (⋂ℱ)
         by (metis (no_types, lifting) compact_Inter False mem_Collect_eq sub-
setCE ℱ)
       moreover have ⋂ℱ ⊆ K
         using False that(2) by fastforce
       moreover have opeKF: openin (top_of_set K) (⋂ℱ)
         using False ℱ ‹finite ℱ› by blast
       then have opeVF: openin (top_of_set V) (⋂ℱ)
        using W ‹K ⊆ S› ‹V ⊆ K› opeKF ‹⋂ℱ ⊆ K› FUW openin_subset_trans
by fastforce
       then have openin (top_of_set S) (⋂ℱ)
```

```
        by (metis opeSV openin_trans)
      moreover have ⋂ℱ ⊆ U
        by (meson ‹V ⊆ U› opeVF dual_order.trans openin_imp_subset)
      ultimately show False
        using neg by blast
    qed
  qed
  qed (use ‹open W› ‹open U› in auto)
  with W Ceq ‹C ⊆ V› ‹C ⊆ U› show False
    by auto
qed


corollary Sura_Bura_clopen_subset_alt:
  fixes S :: 'a::euclidean_space set
  assumes S: locally compact S and C: C ∈ components S and compact C
     and opeSU: openin (top_of_set S) U and C ⊆ U
  obtains K where openin (top_of_set S) K compact K C ⊆ K K ⊆ U
proof −
  obtain V where open V U = S ∩ V
    using opeSU by (auto simp: openin_open)
  with ‹C ⊆ U› have C ⊆ V
    by auto
  then show ?thesis
    using Sura_Bura_clopen_subset [OF S C ‹compact C› ‹open V›]
    by (metis ‹U = S ∩ V› inf.bounded_iff openin_imp_subset that)
qed

corollary Sura_Bura:
  fixes S :: 'a::euclidean_space set
  assumes locally compact S C ∈ components S compact C
  shows C = ⋂ {K. C ⊆ K ∧ compact K ∧ openin (top_of_set S) K}
        (is C = ?rhs)
proof
  show ?rhs ⊆ C
  proof (clarsimp, rule ccontr)
    fix x
    assume ∗: ∀X. C ⊆ X ∧ compact X ∧ openin (top_of_set S) X ⟶ x ∈ X
      and x ∉ C
    obtain U V where open U open V {x} ⊆ U C ⊆ V U ∩ V = {}
      using separation_normal [of {x} C]
      by (metis Int_empty_left ‹x ∉ C› ‹compact C› closed_empty closed_insert
compact_imp_closed insert_disjoint(1))
    have x ∉ V
      using ‹U ∩ V = {}› ‹{x} ⊆ U› by blast
    then show False
     by (meson ∗ Sura_Bura_clopen_subset ‹C ⊆ V› ‹open V› assms(1) assms(2)
assms(3) subsetCE)
  qed
```

qed *blast*

6.4.17 Special cases of local connectedness and path connectedness

lemma *locally_connected_1*:
assumes
$\bigwedge V\ x.\ [\![openin\ (top\_of\_set\ S)\ V;\ x \in V]\!] \Longrightarrow \exists U.\ openin\ (top\_of\_set\ S)\ U \wedge connected\ U \wedge x \in U \wedge U \subseteq V$
shows *locally connected S*
by (*metis assms locally_def*)

lemma *locally_connected_2*:
assumes *locally connected S*
openin (*top_of_set S*) *t*
$x \in t$
shows *openin* (*top_of_set S*) (*connected_component_set t x*)
proof −
{ fix $y :: 'a$
let *?SS* = *top_of_set S*
assume *1*: *openin ?SS t*
$\forall w\ x.\ openin\ ?SS\ w \wedge x \in w \longrightarrow (\exists u.\ openin\ ?SS\ u \wedge (\exists v.\ connected\ v \wedge x \in u \wedge u \subseteq v \wedge v \subseteq w))$
and *connected_component t x y*
then have $y \in t$ **and** *y*: $y \in connected\_component\_set\ t\ x$
using *connected_component_subset* **by** *blast+*
obtain *F* **where**
$\forall x\ y.\ (\exists w.\ openin\ ?SS\ w \wedge (\exists u.\ connected\ u \wedge x \in w \wedge w \subseteq u \wedge u \subseteq y)) = (openin\ ?SS\ (F\ x\ y) \wedge (\exists u.\ connected\ u \wedge x \in F\ x\ y \wedge F\ x\ y \subseteq u \wedge u \subseteq y))$
by *moura*
then obtain *G* **where**
$\forall a\ A.\ (\exists U.\ openin\ ?SS\ U \wedge (\exists V.\ connected\ V \wedge a \in U \wedge U \subseteq V \wedge V \subseteq A)) = (openin\ ?SS\ (F\ a\ A) \wedge connected\ (G\ a\ A) \wedge a \in F\ a\ A \wedge F\ a\ A \subseteq G\ a\ A \wedge G\ a\ A \subseteq A)$
by *moura*
then have ∗: *openin ?SS* $(F\ y\ t) \wedge connected\ (G\ y\ t) \wedge y \in F\ y\ t \wedge F\ y\ t \subseteq G\ y\ t \wedge G\ y\ t \subseteq t$
using *1* ‹$y \in t$› **by** *presburger*
have $G\ y\ t \subseteq connected\_component\_set\ t\ y$
by (*metis* (*no_types*) ∗ *connected_component_eq_self connected_component_mono contra_subsetD*)
then have $\exists A.\ openin\ ?SS\ A \wedge y \in A \wedge A \subseteq connected\_component\_set\ t\ x$
by (*metis* (*no_types*) ∗ *connected_component_eq dual_order.trans y*)
}
then show *?thesis*
using *assms openin_subopen* **by** (*force simp*: *locally_def*)
qed

lemma *locally_connected_3*:

assumes $\bigwedge t\ x.\ [\![openin\ (top\_of\_set\ S)\ t;\ x \in t]\!]$
$\Longrightarrow openin\ (top\_of\_set\ S)$
$(connected\_component\_set\ t\ x)$
$openin\ (top\_of\_set\ S)\ v\ x \in v$
shows $\exists u.\ openin\ (top\_of\_set\ S)\ u \wedge connected\ u \wedge x \in u \wedge u \subseteq v$
using *assms connected_component_subset* **by** *fastforce*

lemma *locally_connected*:
locally connected S $\longleftrightarrow$
$(\forall v\ x.\ openin\ (top\_of\_set\ S)\ v \wedge x \in v$
$\longrightarrow (\exists u.\ openin\ (top\_of\_set\ S)\ u \wedge connected\ u \wedge x \in u \wedge u \subseteq v))$
by (*metis locally_connected_1 locally_connected_2 locally_connected_3*)

lemma *locally_connected_open_connected_component*:
locally connected S $\longleftrightarrow$
$(\forall t\ x.\ openin\ (top\_of\_set\ S)\ t \wedge x \in t$
$\longrightarrow openin\ (top\_of\_set\ S)\ (connected\_component\_set\ t\ x))$
by (*metis locally_connected_1 locally_connected_2 locally_connected_3*)

lemma *locally_path_connected_1*:
assumes
$\bigwedge v\ x.\ [\![openin\ (top\_of\_set\ S)\ v;\ x \in v]\!]$
$\Longrightarrow \exists u.\ openin\ (top\_of\_set\ S)\ u \wedge path\_connected\ u \wedge x \in u \wedge u \subseteq v$
shows *locally path_connected S*
by (*force simp*: *locally_def dest*: *assms*)

lemma *locally_path_connected_2*:
assumes *locally path_connected S*
$openin\ (top\_of\_set\ S)\ t$
$x \in t$
shows $openin\ (top\_of\_set\ S)\ (path\_component\_set\ t\ x)$
proof −
{ fix $y :: 'a$
let *?SS* = *top_of_set S*
assume *1*: *openin ?SS t*
$\forall w\ x.\ openin\ ?SS\ w \wedge x \in w \longrightarrow (\exists u.\ openin\ ?SS\ u \wedge (\exists v.\ path\_connected\ v \wedge x \in u \wedge u \subseteq v \wedge v \subseteq w))$
and *path_component t x y*
then have $y \in t$ **and** *y*: $y \in path\_component\_set\ t\ x$
using *path_component_mem*(*2*) **by** *blast*+
obtain *F* **where**
$\forall x\ y.\ (\exists w.\ openin\ ?SS\ w \wedge (\exists u.\ path\_connected\ u \wedge x \in w \wedge w \subseteq u \wedge u \subseteq y)) = (openin\ ?SS\ (F\ x\ y) \wedge (\exists u.\ path\_connected\ u \wedge x \in F\ x\ y \wedge F\ x\ y \subseteq u \wedge u \subseteq y))$
by *moura*
then obtain *G* **where**
$\forall a\ A.\ (\exists U.\ openin\ ?SS\ U \wedge (\exists V.\ path\_connected\ V \wedge a \in U \wedge U \subseteq V \wedge V \subseteq A)) = (openin\ ?SS\ (F\ a\ A) \wedge path\_connected\ (G\ a\ A) \wedge a \in F\ a\ A \wedge F\ a\ A \subseteq G\ a\ A \wedge G\ a\ A \subseteq A)$

```
    by moura
  then have *: openin ?SS (F y t) ∧ path_connected (G y t) ∧ y ∈ F y t ∧ F y
t ⊆ G y t ∧ G y t ⊆ t
    using 1 ‹y ∈ t› by presburger
  have G y t ⊆ path_component_set t y
    using * path_component_maximal rev_subsetD by blast
  then have ∃A. openin ?SS A ∧ y ∈ A ∧ A ⊆ path_component_set t x
  by (metis * ‹G y t ⊆ path_component_set t y› dual_order.trans path_component_eq
y)
 }
 then show ?thesis
  using assms openin_subopen by (force simp: locally_def)
qed

lemma locally_path_connected_3:
 assumes ⋀t x. ⟦openin (top_of_set S) t; x ∈ t⟧
           ⟹ openin (top_of_set S) (path_component_set t x)
         openin (top_of_set S) v x ∈ v
  shows ∃u. openin (top_of_set S) u ∧ path_connected u ∧ x ∈ u ∧ u ⊆ v
proof −
 have path_component v x x
  by (meson assms(3) path_component_refl)
 then show ?thesis
  by (metis assms mem_Collect_eq path_component_subset path_connected_path_component)
qed

proposition locally_path_connected:
 locally path_connected S ⟷
  (∀V x. openin (top_of_set S) V ∧ x ∈ V
       ⟶ (∃U. openin (top_of_set S) U ∧ path_connected U ∧ x ∈ U ∧ U ⊆
V))
 by (metis locally_path_connected_1 locally_path_connected_2 locally_path_connected_3)

proposition locally_path_connected_open_path_component:
 locally path_connected S ⟷
  (∀t x. openin (top_of_set S) t ∧ x ∈ t
       ⟶ openin (top_of_set S) (path_component_set t x))
 by (metis locally_path_connected_1 locally_path_connected_2 locally_path_connected_3)

lemma locally_connected_open_component:
 locally connected S ⟷
  (∀t c. openin (top_of_set S) t ∧ c ∈ components t
       ⟶ openin (top_of_set S) c)
by (metis components_iff locally_connected_open_connected_component)

proposition locally_connected_im_kleinen:
 locally connected S ⟷
  (∀v x. openin (top_of_set S) v ∧ x ∈ v
      ⟶ (∃u. openin (top_of_set S) u ∧
```

```
            x ∈ u ∧ u ⊆ v ∧
            (∀ y. y ∈ u ⟶ (∃ c. connected c ∧ c ⊆ v ∧ x ∈ c ∧ y ∈ c))))
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    by (fastforce simp: locally_connected)
next
  assume ?rhs
  have ∗: ∃ T. openin (top_of_set S) T ∧ x ∈ T ∧ T ⊆ c
      if openin (top_of_set S) t and c: c ∈ components t and x ∈ c for t c x
  proof −
    from that ‹?rhs› [rule_format, of t x]
    obtain u where u:
      openin (top_of_set S) u ∧ x ∈ u ∧ u ⊆ t ∧
      (∀ y. y ∈ u ⟶ (∃ c. connected c ∧ c ⊆ t ∧ x ∈ c ∧ y ∈ c))
      using in_components_subset by auto
    obtain F :: 'a set ⇒ 'a set ⇒ 'a where
      ∀ x y. (∃ z. z ∈ x ∧ y = connected_component_set x z) = (F x y ∈ x ∧ y =
connected_component_set x (F x y))
      by moura
    then have F: F t c ∈ t ∧ c = connected_component_set t (F t c)
      by (meson components_iff c)
    obtain G :: 'a set ⇒ 'a set ⇒ 'a where
       G: ∀ x y. (∃ z. z ∈ y ∧ z ∉ x) = (G x y ∈ y ∧ G x y ∉ x)
      by moura
     have G c u ∉ u ∨ G c u ∈ c
    using F by (metis (full_types) u connected_componentI connected_component_eq
mem_Collect_eq that(3))
    then show ?thesis
      using G u by auto
  qed
  show ?lhs
    unfolding locally_connected_open_component by (meson ∗ openin_subopen)
qed

proposition locally_path_connected_im_kleinen:
  locally path_connected S ⟷
   (∀ v x. openin (top_of_set S) v ∧ x ∈ v
       ⟶ (∃ u. openin (top_of_set S) u ∧
            x ∈ u ∧ u ⊆ v ∧
            (∀ y. y ∈ u ⟶ (∃ p. path p ∧ path_image p ⊆ v ∧
                                pathstart p = x ∧ pathfinish p = y))))
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    apply (simp add: locally_path_connected path_connected_def)
    apply (erule all_forward ex_forward imp_forward conjE | simp)+
```

```
    by (meson dual_order.trans)
next
  assume ?rhs
  have ∗: ∃ T. openin (top_of_set S) T ∧
              x ∈ T ∧ T ⊆ path_component_set u z
      if openin (top_of_set S) u and z ∈ u and c: path_component u z x for u z
x
  proof −
    have x ∈ u
      by (meson c path_component_mem(2))
    with that ‹?rhs› [rule_format, of u x]
    obtain U where U:
      openin (top_of_set S) U ∧ x ∈ U ∧ U ⊆ u ∧
        (∀ y. y ∈ U ⟶ (∃ p. path p ∧ path_image p ⊆ u ∧ pathstart p = x ∧
pathfinish p = y))
       by blast
    show ?thesis
       by (metis U c mem_Collect_eq path_component_def path_component_eq
subsetI)
  qed
  show ?lhs
    unfolding locally_path_connected_open_path_component
    using ∗ openin_subopen by fastforce
qed

lemma locally_path_connected_imp_locally_connected:
  locally path_connected S ⟹ locally connected S
using locally_mono path_connected_imp_connected by blast

lemma locally_connected_components:
  ⟦locally connected S; c ∈ components S⟧ ⟹ locally connected c
by (meson locally_connected_open_component locally_open_subset openin_subtopology_self)

lemma locally_path_connected_components:
  ⟦locally path_connected S; c ∈ components S⟧ ⟹ locally path_connected c
by (meson locally_connected_open_component locally_open_subset locally_path_connected_imp_locall
openin_subtopology_self)

lemma locally_path_connected_connected_component:
  locally path_connected S ⟹ locally path_connected (connected_component_set
S x)
by (metis components_iff connected_component_eq_empty locally_empty locally_path_connected_comp

lemma open_imp_locally_path_connected:
  fixes S :: 'a :: real_normed_vector set
  assumes open S
  shows locally path_connected S
proof (rule locally_mono)
  show locally convex S
```

using *assms* **unfolding** *locally_def*
by (*meson open_ball centre_in_ball convex_ball openE open_subset openin_imp_subset openin_open_trans subset_trans*)
show $\bigwedge T{::}'a\ set.\ convex\ T \Longrightarrow path\_connected\ T$
using *convex_imp_path_connected* **by** *blast*
qed

lemma *open_imp_locally_connected*:
fixes $S :: 'a :: real\_normed\_vector\ set$
shows $open\ S \Longrightarrow locally\ connected\ S$
by (*simp add*: *locally_path_connected_imp_locally_connected open_imp_locally_path_connected*)

lemma *locally_path_connected_UNIV*: *locally path_connected* (*UNIV*::$'a$ :: *real_normed_vector set*)
by (*simp add*: *open_imp_locally_path_connected*)

lemma *locally_connected_UNIV*: *locally connected* (*UNIV*::$'a$ :: *real_normed_vector set*)
by (*simp add*: *open_imp_locally_connected*)

lemma *openin_connected_component_locally_connected*:
locally connected S
$\Longrightarrow$ *openin* (*top_of_set S*) (*connected_component_set S x*)
by (*metis connected_component_eq_empty locally_connected_2 openin_empty openin_subtopology_self*)

lemma *openin_components_locally_connected*:
$[\![locally\ connected\ S;\ c \in components\ S]\!] \Longrightarrow openin\ (top\_of\_set\ S)\ c$
using *locally_connected_open_component openin_subtopology_self* **by** *blast*

lemma *openin_path_component_locally_path_connected*:
locally path_connected S
$\Longrightarrow$ *openin* (*top_of_set S*) (*path_component_set S x*)
by (*metis* (*no_types*) *empty_iff locally_path_connected_2 openin_subopen openin_subtopology_self path_component_eq_empty*)

lemma *closedin_path_component_locally_path_connected*:
assumes *locally path_connected S*
shows *closedin* (*top_of_set S*) (*path_component_set S x*)
proof −
have *openin* (*top_of_set S*) ($\bigcup$ ({*path_component_set S y* |*y*. $y \in S$} − {*path_component_set S x*}))
using *locally_path_connected_2 assms* **by** *fastforce*
then show *?thesis*
by (*simp add*: *closedin_def path_component_subset complement_path_component_Union*)
qed

lemma *convex_imp_locally_path_connected*:
fixes $S :: 'a:: real\_normed\_vector\ set$

assumes *convex S*
shows *locally path_connected S*
proof (*clarsimp simp*: *locally_path_connected*)
fix *V x*
assume *openin* (*top_of_set S*) *V* **and** $x \in V$
then obtain *T e* **where** $V = S \cap T$ $x \in S$ $0 < e$ *ball x e* $\subseteq T$
by (*metis Int_iff openE openin_open*)
then have *openin* (*top_of_set S*) ($S \cap$ *ball x e*) *path_connected* ($S \cap$ *ball x e*)
by (*simp_all add*: *assms convex_Int convex_imp_path_connected openin_open_Int*)
then show $\exists U$. *openin* (*top_of_set S*) $U \wedge$ *path_connected* $U \wedge x \in U \wedge U \subseteq V$
using ‹*0 < e*› ‹$V = S \cap T$› ‹*ball x e* $\subseteq T$› ‹$x \in S$› **by** *auto*
qed

lemma *convex_imp_locally_connected*:
fixes *S* :: *'a*:: *real_normed_vector set*
shows *convex S* $\Longrightarrow$ *locally connected S*
by (*simp add*: *locally_path_connected_imp_locally_connected convex_imp_locally_path_connected*)

6.4.18 Relations between components and path components

lemma *path_component_eq_connected_component*:
assumes *locally path_connected S*
shows (*path_component S x* = *connected_component S x*)
proof (*cases* $x \in S$)
case *True*
have *openin* (*top_of_set* (*connected_component_set S x*)) (*path_component_set S x*)
proof (*rule openin_subset_trans*)
show *openin* (*top_of_set S*) (*path_component_set S x*)
by (*simp add*: *True assms locally_path_connected_2*)
show *connected_component_set S x* $\subseteq S$
by (*simp add*: *connected_component_subset*)
qed (*simp add*: *path_component_subset_connected_component*)
moreover have *closedin* (*top_of_set* (*connected_component_set S x*)) (*path_component_set S x*)
proof (*rule closedin_subset_trans* [*of S*])
show *closedin* (*top_of_set S*) (*path_component_set S x*)
by (*simp add*: *assms closedin_path_component_locally_path_connected*)
show *connected_component_set S x* $\subseteq S$
by (*simp add*: *connected_component_subset*)
qed (*simp add*: *path_component_subset_connected_component*)
ultimately have $*$: *path_component_set S x* = *connected_component_set S x*
by (*metis connected_connected_component connected_clopen True path_component_eq_empty*)
then show *?thesis*
by *blast*
next
case *False* **then show** *?thesis*
by (*metis Collect_empty_eq_bot connected_component_eq_empty path_component_eq_empty*)

qed

lemma *path_component_eq_connected_component_set*:
locally path_connected S $\Longrightarrow$ (*path_component_set S x* = *connected_component_set S x*)
by (*simp add*: *path_component_eq_connected_component*)

lemma *locally_path_connected_path_component*:
locally path_connected S $\Longrightarrow$ *locally path_connected* (*path_component_set S x*)
using *locally_path_connected_connected_component path_component_eq_connected_component*
by *fastforce*

lemma *open_path_connected_component*:
fixes *S* :: *'a* :: *real_normed_vector set*
shows *open S* $\Longrightarrow$ *path_component S x* = *connected_component S x*
by (*simp add*: *path_component_eq_connected_component open_imp_locally_path_connected*)

lemma *open_path_connected_component_set*:
fixes *S* :: *'a* :: *real_normed_vector set*
shows *open S* $\Longrightarrow$ *path_component_set S x* = *connected_component_set S x*
by (*simp add*: *open_path_connected_component*)

proposition *locally_connected_quotient_image*:
assumes *lcS*: *locally connected S*
and *oo*: $\bigwedge T.\ T \subseteq f\ `\ S$
$\Longrightarrow$ *openin* (*top_of_set S*) (*S* $\cap$ *f* $-`$ *T*) $\longleftrightarrow$
openin (*top_of_set* (*f* ` *S*)) *T*
shows *locally connected* (*f* ` *S*)
proof (*clarsimp simp*: *locally_connected_open_component*)
fix *U C*
assume *opefSU*: *openin* (*top_of_set* (*f* ` *S*)) *U* **and** *C* $\in$ *components U*
then have *C* $\subseteq$ *U U* $\subseteq$ *f* ` *S*
by (*meson in_components_subset openin_imp_subset*)+
then have *openin* (*top_of_set* (*f* ` *S*)) *C* $\longleftrightarrow$
openin (*top_of_set S*) (*S* $\cap$ *f* $-`$ *C*)
by (*auto simp*: *oo*)
moreover have *openin* (*top_of_set S*) (*S* $\cap$ *f* $-`$ *C*)
proof (*subst openin_subopen*, *clarify*)
fix *x*
assume *x* $\in$ *S f x* $\in$ *C*
show $\exists T.$ *openin* (*top_of_set S*) *T* $\wedge$ *x* $\in$ *T* $\wedge$ *T* $\subseteq$ (*S* $\cap$ *f* $-`$ *C*)
proof (*intro conjI exI*)
show *openin* (*top_of_set S*) (*connected_component_set* (*S* $\cap$ *f* $-`$ *U*) *x*)
proof (*rule ccontr*)
assume $**$: $\neg$ *openin* (*top_of_set S*) (*connected_component_set* (*S* $\cap$ *f* $-`$ *U*) *x*)
then have *x* $\notin$ (*S* $\cap$ *f* $-`$ *U*)
using ‹*U* $\subseteq$ *f* ` *S*› *opefSU lcS locally_connected_2 oo* **by** *blast*

```
      with ** show False
     by (metis (no_types) connected_component_eq_empty empty_iff openin_subopen)
    qed
  next
    show x ∈ connected_component_set (S ∩ f -` U) x
     using ‹C ⊆ U› ‹f x ∈ C› ‹x ∈ S› by auto
  next
    have contf: continuous_on S f
     by (simp add: continuous_on_open oo openin_imp_subset)
    then have continuous_on (connected_component_set (S ∩ f -` U) x) f
    by (meson connected_component_subset continuous_on_subset inf.boundedE)
    then have connected (f ` connected_component_set (S ∩ f -` U) x)
    by (rule connected_continuous_image [OF _ connected_connected_component])
    moreover have f ` connected_component_set (S ∩ f -` U) x ⊆ U
     using connected_component_in by blast
    moreover have C ∩ f ` connected_component_set (S ∩ f -` U) x ≠ {}
     using ‹C ⊆ U› ‹f x ∈ C› ‹x ∈ S› by fastforce
    ultimately have fC: f ` (connected_component_set (S ∩ f -` U) x) ⊆ C
     by (rule components_maximal [OF ‹C ∈ components U›])
    have cUC: connected_component_set (S ∩ f -` U) x ⊆ (S ∩ f -` C)
     using connected_component_subset fC by blast
   have connected_component_set (S ∩ f -` U) x ⊆ connected_component_set
(S ∩ f -` C) x
    proof -
     { assume x ∈ connected_component_set (S ∩ f -` U) x
       then have ?thesis
         using cUC connected_component_idemp connected_component_mono
by blast }
      then show ?thesis
       using connected_component_eq_empty by auto
    qed
    also have ... ⊆ (S ∩ f -` C)
     by (rule connected_component_subset)
    finally show connected_component_set (S ∩ f -` U) x ⊆ (S ∩ f -` C) .
  qed
 qed
 ultimately show openin (top_of_set (f ` S)) C
  by metis
qed
```

The proof resembles that above but is not identical!

```
proposition locally_path_connected_quotient_image:
 assumes lcS: locally path_connected S
    and oo: ⋀T. T ⊆ f ` S
            ⟹ openin (top_of_set S) (S ∩ f -` T) ⟷ openin (top_of_set (f
` S)) T
  shows locally path_connected (f ` S)
proof (clarsimp simp: locally_path_connected_open_path_component)
 fix U y
```

```
  assume opefSU: openin (top_of_set (f ‘ S)) U and y ∈ U
  then have path_component_set U y ⊆ U U ⊆ f ‘ S
    by (meson path_component_subset openin_imp_subset)+
  then have openin (top_of_set (f ‘ S)) (path_component_set U y) ⟷
            openin (top_of_set S) (S ∩ f −‘ path_component_set U y)
  proof −
    have path_component_set U y ⊆ f ‘ S
      using ‹U ⊆ f ‘ S› ‹path_component_set U y ⊆ U› by blast
    then show ?thesis
      using oo by blast
  qed
  moreover have openin (top_of_set S) (S ∩ f −‘ path_component_set U y)
  proof (subst openin_subopen, clarify)
    fix x
    assume x ∈ S and Uyfx: path_component U y (f x)
    then have f x ∈ U
      using path_component_mem by blast
   show ∃ T. openin (top_of_set S) T ∧ x ∈ T ∧ T ⊆ (S ∩ f −‘ path_component_set
U y)
    proof (intro conjI exI)
      show openin (top_of_set S) (path_component_set (S ∩ f −‘ U) x)
      proof (rule ccontr)
        assume **: ¬ openin (top_of_set S) (path_component_set (S ∩ f −‘ U)
x)
        then have x ∉ (S ∩ f −‘ U)
       by (metis (no_types, lifting) ‹U ⊆ f ‘ S› opefSU lcS oo locally_path_connected_open_path_component)
        then show False
          using ** ‹path_component_set U y ⊆ U› ‹x ∈ S› ‹path_component U y
(f x)› by blast
      qed
    next
      show x ∈ path_component_set (S ∩ f −‘ U) x
        by (simp add: ‹f x ∈ U› ‹x ∈ S› path_component_refl)
    next
      have contf: continuous_on S f
        by (simp add: continuous_on_open oo openin_imp_subset)
      then have continuous_on (path_component_set (S ∩ f −‘ U) x) f
        by (meson Int_lower1 continuous_on_subset path_component_subset)
      then have path_connected (f ‘ path_component_set (S ∩ f −‘ U) x)
        by (simp add: path_connected_continuous_image)
      moreover have f ‘ path_component_set (S ∩ f −‘ U) x ⊆ U
        using path_component_mem by fastforce
      moreover have f x ∈ f ‘ path_component_set (S ∩ f −‘ U) x
        by (force simp: ‹x ∈ S› ‹f x ∈ U› path_component_refl_eq)
     ultimately have f ‘ (path_component_set (S ∩ f −‘ U) x) ⊆ path_component_set
U (f x)
        by (meson path_component_maximal)
       also have ... ⊆ path_component_set U y
          by (simp add: Uyfx path_component_maximal path_component_subset
```

path_component_sym)
finally have *fC*: *f* ‘ (*path_component_set* (*S* ∩ *f* −‘ *U*) *x*) ⊆ *path_component_set* *U* *y* **.**
have *cUC*: *path_component_set* (*S* ∩ *f* −‘ *U*) *x* ⊆ (*S* ∩ *f* −‘ *path_component_set* *U* *y*)
using *path_component_subset fC* **by** *blast*
have *path_component_set* (*S* ∩ *f* −‘ *U*) *x* ⊆ *path_component_set* (*S* ∩ *f* −‘ *path_component_set* *U* *y*) *x*
proof −
have ⋀*a*. *path_component_set* (*path_component_set* (*S* ∩ *f* −‘ *U*) *x*) *a* ⊆ *path_component_set* (*S* ∩ *f* −‘ *path_component_set* *U* *y*) *a*
using *cUC path_component_mono* **by** *blast*
then show *?thesis*
using *path_component_path_component* **by** *blast*
qed
also have ... ⊆ (*S* ∩ *f* −‘ *path_component_set* *U* *y*)
by (*rule path_component_subset*)
finally show *path_component_set* (*S* ∩ *f* −‘ *U*) *x* ⊆ (*S* ∩ *f* −‘ *path_component_set* *U* *y*) **.**
qed
qed
ultimately show *openin* (*top_of_set* (*f* ‘ *S*)) (*path_component_set* *U* *y*)
by *metis*
qed

6.4.19 Components, continuity, openin, closedin

lemma *continuous_on_components_gen*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*topological_space*
assumes ⋀*C*. *C* ∈ *components* *S* ⟹ *openin* (*top_of_set* *S*) *C* ∧ *continuous_on* *C* *f*
shows *continuous_on* *S* *f*
proof (*clarsimp simp*: *continuous_openin_preimage_eq*)
fix *t* :: *'b* *set*
assume *open* *t*
have ∗: *S* ∩ *f* −‘ *t* = (⋃ *c* ∈ *components* *S*. *c* ∩ *f* −‘ *t*)
by *auto*
show *openin* (*top_of_set* *S*) (*S* ∩ *f* −‘ *t*)
unfolding ∗ **using** ‹*open* *t*› *assms continuous_openin_preimage_gen openin_trans* *openin_Union* **by** *blast*
qed

lemma *continuous_on_components*:
fixes *f* :: *'a*::*topological_space* ⇒ *'b*::*topological_space*
assumes *locally connected* *S* ⋀*C*. *C* ∈ *components* *S* ⟹ *continuous_on* *C* *f*
shows *continuous_on* *S* *f*
proof (*rule continuous_on_components_gen*)
fix *C*
assume *C* ∈ *components* *S*

then show *openin* (*top_of_set S*) *C* ∧ *continuous_on C f*
by (*simp add*: *assms openin_components_locally_connected*)
qed

lemma *continuous_on_components_eq*:
locally connected S
⟹ (*continuous_on S f* ⟷ (∀ *c* ∈ *components S*. *continuous_on c f*))
by (*meson continuous_on_components continuous_on_subset in_components_subset*)

lemma *continuous_on_components_open*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *open S*
⋀*c*. *c* ∈ *components S* ⟹ *continuous_on c f*
shows *continuous_on S f*
using *continuous_on_components open_imp_locally_connected assms* **by** *blast*

lemma *continuous_on_components_open_eq*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *open S* ⟹ (*continuous_on S f* ⟷ (∀ *c* ∈ *components S*. *continuous_on c f*))
using *continuous_on_subset in_components_subset*
by (*blast intro*: *continuous_on_components_open*)

lemma *closedin_union_complement_components*:
assumes *U*: *locally connected U*
and *S*: *closedin* (*top_of_set U*) *S*
and *cuS*: *c* ⊆ *components*(*U* − *S*)
shows *closedin* (*top_of_set U*) (*S* ∪ ⋃ *c*)
proof −
have *di*: (⋀*S T*. *S* ∈ *c* ∧ *T* ∈ *c'* ⟹ *disjnt S T*) ⟹ *disjnt* (⋃ *c*) (⋃ *c'*) **for** *c'*
by (*simp add*: *disjnt_def*) *blast*
have *S* ⊆ *U*
using *S closedin_imp_subset* **by** *blast*
moreover have *U* − *S* = ⋃ *c* ∪ ⋃(*components* (*U* − *S*) − *c*)
by (*metis Diff_partition Union_components Union_Un_distrib assms*(*3*))
moreover have *disjnt* (⋃ *c*) (⋃(*components* (*U* − *S*) − *c*))
apply (*rule di*)
by (*metis di DiffD1 DiffD2 assms*(*3*) *components_nonoverlap disjnt_def subsetCE*)
ultimately have *eq*: *S* ∪ ⋃ *c* = *U* − (⋃(*components*(*U* − *S*) − *c*))
by (*auto simp*: *disjnt_def*)
have ∗: *openin* (*top_of_set U*) (⋃(*components* (*U* − *S*) − *c*))
proof (*rule openin_Union* [*OF openin_trans* [*of U* − *S*]])
show *openin* (*top_of_set* (*U* − *S*)) *T* **if** *T* ∈ *components* (*U* − *S*) − *c* **for** *T*
using *that* **by** (*simp add*: *U S locally_diff_closed openin_components_locally_connected*)
show *openin* (*top_of_set U*) (*U* − *S*) **if** *T* ∈ *components* (*U* − *S*) − *c* **for** *T*
using *that* **by** (*simp add*: *openin_diff S*)
qed
have *closedin* (*top_of_set U*) (*U* − ⋃ (*components* (*U* − *S*) − *c*))

by (*metis closedin_diff closedin_topspace topspace_euclidean_subtopology* ∗)
then have *openin* (*top_of_set U*) (*U* − (*U* − ⋃(*components* (*U* − *S*) − *c*)))
by (*simp add*: *openin_diff*)
then show *?thesis*
by (*force simp*: *eq closedin_def*)
qed

lemma *closed_union_complement_components*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *S*: *closed S* **and** *c*: *c* ⊆ *components*(− *S*)
shows *closed*(*S* ∪ ⋃ *c*)
proof −
have *closedin* (*top_of_set UNIV*) (*S* ∪ ⋃ *c*)
by (*metis Compl_eq_Diff_UNIV S c closed_closedin closedin_union_complement_components locally_connected_UNIV subtopology_UNIV*)
then show *?thesis* **by** *simp*
qed

lemma *closedin_Un_complement_component*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *u*: *locally connected u*
and *S*: *closedin* (*top_of_set u*) *S*
and *c*: *c* ∈ *components*(*u* − *S*)
shows *closedin* (*top_of_set u*) (*S* ∪ *c*)
proof −
have *closedin* (*top_of_set u*) (*S* ∪ ⋃{*c*})
using *c* **by** (*blast intro*: *closedin_union_complement_components* [*OF u S*])
then show *?thesis*
by *simp*
qed

lemma *closed_Un_complement_component*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *S*: *closed S* **and** *c*: *c* ∈ *components*(−*S*)
shows *closed* (*S* ∪ *c*)
by (*metis Compl_eq_Diff_UNIV S c closed_closedin closedin_Un_complement_component locally_connected_UNIV subtopology_UNIV*)

6.4.20 Existence of isometry between subspaces of same dimension

lemma *isometry_subset_subspace*:
fixes *S* :: *'a*::*euclidean_space set*
and *T* :: *'b*::*euclidean_space set*
assumes *S*: *subspace S*
and *T*: *subspace T*
and *d*: *dim S* ≤ *dim T*
obtains *f* **where** *linear f f* ∈ *S* → *T* ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) = *norm x*
proof −

```
  obtain B where B ⊆ S and Borth: pairwise orthogonal B
            and B1: ⋀x. x ∈ B ⟹ norm x = 1
            and independent B finite B card B = dim S span B = S
    by (metis orthonormal_basis_subspace [OF S] independent_imp_finite)
  obtain C where C ⊆ T and Corth: pairwise orthogonal C
            and C1:⋀x. x ∈ C ⟹ norm x = 1
            and independent C finite C card C = dim T span C = T
    by (metis orthonormal_basis_subspace [OF T] independent_imp_finite)
  obtain fb where fb ` B ⊆ C inj_on fb B
   by (metis ‹card B = dim S› ‹card C = dim T› ‹finite B› ‹finite C› card_le_inj
d)
  then have pairwise_orth_fb: pairwise (λv j. orthogonal (fb v) (fb j)) B
    using Corth unfolding pairwise_def inj_on_def
    by (blast intro: orthogonal_clauses)
  obtain f where linear f and ffb: ⋀x. x ∈ B ⟹ f x = fb x
    using linear_independent_extend ‹independent B› by fastforce
  have span (f ` B) ⊆ span C
    by (metis ‹fb ` B ⊆ C› ffb image_cong span_mono)
  then have f ` S ⊆ T
    unfolding ‹span B = S› ‹span C = T› span_linear_image[OF ‹linear f›] .
  have [simp]: ⋀x. x ∈ B ⟹ norm (fb x) = norm x
    using B1 C1 ‹fb ` B ⊆ C› by auto
  have norm (f x) = norm x if x ∈ S for x
  proof −
    interpret linear f by fact
    obtain a where x: x = (∑ v ∈ B. a v *R v)
      using ‹finite B› ‹span B = S› ‹x ∈ S› span_finite by fastforce
    have norm (f x)^2 = norm (∑ v∈B. a v *R fb v)^2 by (simp add: sum scale
ffb x)
    also have ... = (∑ v∈B. norm ((a v *R fb v))^2)
    proof (rule norm_sum_Pythagorean [OF ‹finite B›])
      show pairwise (λv j. orthogonal (a v *R fb v) (a j *R fb j)) B
        by (rule pairwise_ortho_scaleR [OF pairwise_orth_fb])
    qed
    also have ... = norm x ^2
      by (simp add: x pairwise_ortho_scaleR Borth norm_sum_Pythagorean [OF
‹finite B›])
    finally show ?thesis
      by (simp add: norm_eq_sqrt_inner)
  qed
  then show ?thesis
    by (meson ‹f ` S ⊆ T› ‹linear f› image_subset_iff_funcset that)
qed

proposition isometries_subspaces:
  fixes S :: 'a::euclidean_space set
    and T :: 'b::euclidean_space set
  assumes S: subspace S
      and T: subspace T
```

```
    and d: dim S = dim T
  obtains f g where linear f linear g f ‘ S = T g ‘ T = S
                    ⋀x. x ∈ S ⟹ norm(f x) = norm x
                    ⋀x. x ∈ T ⟹ norm(g x) = norm x
                    ⋀x. x ∈ S ⟹ g(f x) = x
                    ⋀x. x ∈ T ⟹ f(g x) = x
proof −
  obtain B where B ⊆ S and Borth: pairwise orthogonal B
            and B1: ⋀x. x ∈ B ⟹ norm x = 1
            and independent B finite B card B = dim S span B = S
    by (metis orthonormal_basis_subspace [OF S] independent_imp_finite)
  obtain C where C ⊆ T and Corth: pairwise orthogonal C
            and C1:⋀x. x ∈ C ⟹ norm x = 1
            and independent C finite C card C = dim T span C = T
    by (metis orthonormal_basis_subspace [OF T] independent_imp_finite)
  obtain fb where bij_betw fb B C
    by (metis ‹finite B› ‹finite C› bij_betw_iff_card ‹card B = dim S› ‹card C =
dim T› d)
  then have pairwise_orth_fb: pairwise (λv j. orthogonal (fb v) (fb j)) B
    using Corth unfolding pairwise_def inj_on_def bij_betw_def
    by (blast intro: orthogonal_clauses)
  obtain f where linear f and ffb: ⋀x. x ∈ B ⟹ f x = fb x
    using linear_independent_extend ‹independent B› by fastforce
  interpret f: linear f by fact
  define gb where gb ≡ inv_into B fb
  then have pairwise_orth_gb: pairwise (λv j. orthogonal (gb v) (gb j)) C
    using Borth ‹bij_betw fb B C› unfolding pairwise_def bij_betw_def by force
  obtain g where linear g and ggb: ⋀x. x ∈ C ⟹ g x = gb x
    using linear_independent_extend ‹independent C› by fastforce
  interpret g: linear g by fact
  have span (f ‘ B) ⊆ span C
    by (metis ‹bij_betw fb B C› bij_betw_imp_surj_on eq_iff ffb image_cong)
  then have f ‘ S ⊆ T
    unfolding ‹span B = S› ‹span C = T› span_linear_image[OF ‹linear f›] .
  have [simp]: ⋀x. x ∈ B ⟹ norm (fb x) = norm x
    using B1 C1 ‹bij_betw fb B C› bij_betw_imp_surj_on by fastforce
  have f [simp]: norm (f x) = norm x g (f x) = x if x ∈ S for x
  proof −
    obtain a where x: x = (∑ v ∈ B. a v *R v)
      using ‹finite B› ‹span B = S› ‹x ∈ S› span_finite by fastforce
    have f x = (∑ v ∈ B. f (a v *R v))
      using linear_sum [OF ‹linear f›] x by auto
    also have ... = (∑ v ∈ B. a v *R f v)
      by (simp add: f.sum f.scale)
    also have ... = (∑ v ∈ B. a v *R fb v)
      by (simp add: ffb cong: sum.cong)
    finally have *: f x = (∑ v∈B. a v *R fb v) .
    then have (norm (f x))^2 = (norm (∑ v∈B. a v *R fb v))^2 by simp
    also have ... = (∑ v∈B. norm ((a v *R fb v))^2)
```

```
    proof (rule norm_sum_Pythagorean [OF ‹finite B›])
      show pairwise (λv j. orthogonal (a v *R fb v) (a j *R fb j)) B
        by (rule pairwise_ortho_scaleR [OF pairwise_orth_fb])
    qed
    also have ... = (norm x)^2
      by (simp add: x pairwise_ortho_scaleR Borth norm_sum_Pythagorean [OF
‹finite B›])
    finally show norm (f x) = norm x
      by (simp add: norm_eq_sqrt_inner)
    have g (f x) = g (∑ v∈B. a v *R fb v) by (simp add: *)
    also have ... = (∑ v∈B. g (a v *R fb v))
      by (simp add: g.sum g.scale)
    also have ... = (∑ v∈B. a v *R g (fb v))
      by (simp add: g.scale)
    also have ... = (∑ v∈B. a v *R v)
    proof (rule sum.cong [OF refl])
      show a x *R g (fb x) = a x *R x if x ∈ B for x
        using that ‹bij_betw fb B C› bij_betwE bij_betw_inv_into_left gb_def ggb
by fastforce
    qed
    also have ... = x
      using x by blast
    finally show g (f x) = x .
  qed
  have [simp]: ⋀x. x ∈ C ⟹ norm (gb x) = norm x
   by (metis B1 C1 ‹bij_betw fb B C› bij_betw_imp_surj_on gb_def inv_into_into)
  have g [simp]: f (g x) = x if x ∈ T for x
  proof −
    obtain a where x: x = (∑ v ∈ C. a v *R v)
      using ‹finite C› ‹span C = T› ‹x ∈ T› span_finite by fastforce
    have g x = (∑ v ∈ C. g (a v *R v))
      by (simp add: x g.sum)
    also have ... = (∑ v ∈ C. a v *R g v)
      by (simp add: g.scale)
    also have ... = (∑ v ∈ C. a v *R gb v)
      by (simp add: ggb cong: sum.cong)
    finally have f (g x) = f (∑ v∈C. a v *R gb v) by simp
    also have ... = (∑ v∈C. f (a v *R gb v))
      by (simp add: f.scale f.sum)
    also have ... = (∑ v∈C. a v *R f (gb v))
      by (simp add: f.scale f.sum)
    also have ... = (∑ v∈C. a v *R v)
      using ‹bij_betw fb B C›
     by (simp add: bij_betw_def gb_def bij_betw_inv_into_right ffb inv_into_into)
    also have ... = x
      using x by blast
    finally show f (g x) = x .
  qed
  have gim: g ‘ T = S
```

```
    by (metis (full_types) S T ‹f ‘ S ⊆ T› d dim_eq_span dim_image_le f(2)
g.linear_axioms
        image_iff linear_subspace_image span_eq_iff subset_iff)
  have fim: f ‘ S = T
    using ‹g ‘ T = S› image_iff by fastforce
  have [simp]: norm (g x) = norm x if x ∈ T for x
    using fim that by auto
  show ?thesis
    by (rule that [OF ‹linear f› ‹linear g›]) (simp_all add: fim gim)
qed

corollary isometry_subspaces:
  fixes S :: 'a::euclidean_space set
    and T :: 'b::euclidean_space set
  assumes S: subspace S
      and T: subspace T
      and d: dim S = dim T
  obtains f where linear f f ‘ S = T ⋀x. x ∈ S ⟹ norm(f x) = norm x
using isometries_subspaces [OF assms]
by metis

corollary isomorphisms_UNIV_UNIV:
  assumes DIM('M) = DIM('N)
  obtains f::'M::euclidean_space ⇒'N::euclidean_space and g
  where linear f linear g
                ⋀x. norm(f x) = norm x ⋀y. norm(g y) = norm y
                ⋀x. g (f x) = x ⋀y. f(g y) = y
  using assms by (auto intro: isometries_subspaces [of UNIV::'M set UNIV::'N
set])

lemma homeomorphic_subspaces:
  fixes S :: 'a::euclidean_space set
    and T :: 'b::euclidean_space set
  assumes S: subspace S
      and T: subspace T
      and d: dim S = dim T
    shows S homeomorphic T
proof −
  obtain f g where linear f linear g f ‘ S = T g ‘ T = S
                ⋀x. x ∈ S ⟹ g(f x) = x ⋀x. x ∈ T ⟹ f(g x) = x
    by (blast intro: isometries_subspaces [OF assms])
  then show ?thesis
    unfolding homeomorphic_def homeomorphism_def
    apply (rule_tac x=f in exI, rule_tac x=g in exI)
    apply (auto simp: linear_continuous_on linear_conv_bounded_linear)
    done
qed

lemma homeomorphic_affine_sets:
```

assumes *affine S affine T aff_dim S = aff_dim T*
shows *S homeomorphic T*
proof (*cases S = {} ∨ T = {}*)
case *True* **with** *assms aff_dim_empty homeomorphic_empty* **show** *?thesis*
by *metis*
next
case *False*
then obtain *a b* **where** *ab*: *a ∈ S b ∈ T* **by** *auto*
then have *ss*: *subspace ((+) (− a) ‘ S) subspace ((+) (− b) ‘ T)*
using *affine_diffs_subspace assms* **by** *blast+*
have *dd*: *dim ((+) (− a) ‘ S) = dim ((+) (− b) ‘ T)*
using *assms ab* **by** (*simp add*: *aff_dim_eq_dim* [*OF hull_inc*] *image_def*)
have *S homeomorphic ((+) (− a) ‘ S)*
by (*fact homeomorphic_translation*)
also have ... *homeomorphic ((+) (− b) ‘ T)*
by (*rule homeomorphic_subspaces* [*OF ss dd*])
also have ... *homeomorphic T*
using *homeomorphic_translation* [*of T − b*] **by** (*simp add*: *homeomorphic_sym* [*of T*])
finally show *?thesis* **.**
qed

6.4.21 Retracts, in a general sense, preserve (co)homotopic triviality)

locale *Retracts* =
fixes *S h t k*
assumes *conth*: *continuous_on S h*
and *imh*: *h ‘ S = t*
and *contk*: *continuous_on t k*
and *imk*: *k ∈ t → S*
and *idhk*: ⋀*y*. *y ∈ t ⟹ h*(*k y*) = *y*

begin

lemma *homotopically_trivial_retraction_gen*:
assumes *P*: ⋀*f*. ⟦*continuous_on U f*; *f ∈ U → t*; *Q f*⟧ ⟹ *P*(*k ∘ f*)
and *Q*: ⋀*f*. ⟦*continuous_on U f*; *f ∈ U → S*; *P f*⟧ ⟹ *Q*(*h ∘ f*)
and *Qeq*: ⋀*h k*. (⋀*x*. *x ∈ U ⟹ h x = k x*) ⟹ *Q h = Q k*
and *hom*: ⋀*f g*. ⟦*continuous_on U f*; *f ∈ U → S*; *P f*;
continuous_on U g; *g ∈ U → S*; *P g*⟧
⟹ *homotopic_with_canon P U S f g*
and *contf*: *continuous_on U f* **and** *imf*: *f ∈ U → t* **and** *Qf*: *Q f*
and *contg*: *continuous_on U g* **and** *img*: *g ∈ U → t* **and** *Qg*: *Q g*
shows *homotopic_with_canon Q U t f g*
proof −
have *continuous_on U* (*k ∘ f*)
by (*meson contf continuous_on_compose continuous_on_subset contk funcset_image imf*)

```
  moreover have (k ∘ f) ‘ U ⊆ S
    using imf imk by fastforce
  moreover have P (k ∘ f)
    by (simp add: P Qf contf imf)
  moreover have continuous_on U (k ∘ g)
     by (meson contg continuous_on_compose continuous_on_subset contk funcset_image img)
  moreover have (k ∘ g) ‘ U ⊆ S
    using img imk by fastforce
  moreover have P (k ∘ g)
    by (simp add: P Qg contg img)
  ultimately have homotopic_with_canon P U S (k ∘ f) (k ∘ g)
    by (simp add: hom image_subset_iff)
  then have homotopic_with_canon Q U t (h ∘ (k ∘ f)) (h ∘ (k ∘ g))
   apply (rule homotopic_with_compose_continuous_left [OF homotopic_with_mono])
    using Q conth imh by force+
  then show ?thesis
  proof (rule homotopic_with_eq; simp)
    show ⋀h k. (⋀x. x ∈ U ⟹ h x = k x) ⟹ Q h = Q k
      using Qeq topspace_euclidean_subtopology by blast
    show ⋀x. x ∈ U ⟹ f x = h (k (f x)) ⋀x. x ∈ U ⟹ g x = h (k (g x))
      using idhk imf img by fastforce+
  qed
qed

lemma homotopically_trivial_retraction_null_gen:
  assumes P: ⋀f. ⟦continuous_on U f; f ∈ U → t; Q f⟧ ⟹ P(k ∘ f)
      and Q: ⋀f. ⟦continuous_on U f; f ∈ U → S; P f⟧ ⟹ Q(h ∘ f)
      and Qeq: ⋀h k. (⋀x. x ∈ U ⟹ h x = k x) ⟹ Q h = Q k
      and hom: ⋀f. ⟦continuous_on U f; f ∈ U → S; P f⟧
                   ⟹ ∃c. homotopic_with_canon P U S f (λx. c)
      and contf: continuous_on U f and imf:f ∈ U → t and Qf: Q f
  obtains c where homotopic_with_canon Q U t f (λx. c)
proof −
  have feq: ⋀x. x ∈ U ⟹ (h ∘ (k ∘ f)) x = f x using idhk imf by auto
  have continuous_on U (k ∘ f)
     by (meson contf continuous_on_compose continuous_on_subset contk funcset_image imf)
  moreover have (k ∘ f) ∈ U → S
    using imf imk by fastforce
  moreover have P (k ∘ f)
    by (simp add: P Qf contf imf)
  ultimately obtain c where homotopic_with_canon P U S (k ∘ f) (λx. c)
    by (metis hom)
  then have homotopic_with_canon Q U t (h ∘ (k ∘ f)) (h ∘ (λx. c))
   apply (rule homotopic_with_compose_continuous_left [OF homotopic_with_mono])
    using Q conth imh by force+
  then have homotopic_with_canon Q U t f (λx. h c)
  proof (rule homotopic_with_eq)
```

```
    show ⋀x. x ∈ topspace (top_of_set U) ⟹ f x = (h ∘ (k ∘ f)) x
      using feq by auto
    show ⋀h k. (⋀x. x ∈ topspace (top_of_set U) ⟹ h x = k x) ⟹ Q h = Q k
      using Qeq topspace_euclidean_subtopology by blast
  qed auto
  then show ?thesis
    using that by blast
qed

lemma cohomotopically_trivial_retraction_gen:
  assumes P: ⋀f. ⟦continuous_on t f; f ∈ t → U; Q f⟧ ⟹ P(f ∘ h)
      and Q: ⋀f. ⟦continuous_on S f; f ∈ S → U; P f⟧ ⟹ Q(f ∘ k)
      and Qeq: ⋀h k. (⋀x. x ∈ t ⟹ h x = k x) ⟹ Q h = Q k
      and hom: ⋀f g. ⟦continuous_on S f; f ∈ S → U; P f;
                      continuous_on S g; g ∈ S → U; P g⟧
                      ⟹ homotopic_with_canon P S U f g
      and contf: continuous_on t f and imf: f ∈ t → U and Qf: Q f
      and contg: continuous_on t g and img: g ∈ t → U and Qg: Q g
    shows homotopic_with_canon Q t U f g
proof −
  have feq: ⋀x. x ∈ t ⟹ (f ∘ h ∘ k) x = f x using idhk imf by auto
  have geq: ⋀x. x ∈ t ⟹ (g ∘ h ∘ k) x = g x using idhk img by auto
  have continuous_on S (f ∘ h)
    using contf conth continuous_on_compose imh by blast
  moreover have (f ∘ h) ∈ S → U
    using imf imh by fastforce
  moreover have P (f ∘ h)
    by (simp add: P Qf contf imf)
  moreover have continuous_on S (g ∘ h)
   using contg continuous_on_compose continuous_on_subset conth imh by blast
  moreover have (g ∘ h) ∈ S → U
    using img imh by fastforce
  moreover have P (g ∘ h)
    by (simp add: P Qg contg img)
  ultimately have homotopic_with_canon P S U (f ∘ h) (g ∘ h)
    by (simp add: hom)
  then have homotopic_with_canon Q t U (f ∘ h ∘ k) (g ∘ h ∘ k)
   apply (rule homotopic_with_compose_continuous_right [OF homotopic_with_mono])
    using Q contk imk by force+
  then show ?thesis
  proof (rule homotopic_with_eq)
    show f x = (f ∘ h ∘ k) x g x = (g ∘ h ∘ k) x
      if x ∈ topspace (top_of_set t) for x
      using feq geq that by force+
  qed (use Qeq topspace_euclidean_subtopology in blast)
qed

lemma cohomotopically_trivial_retraction_null_gen:
  assumes P: ⋀f. ⟦continuous_on t f; f ∈ t → U; Q f⟧ ⟹ P(f ∘ h)
```

```
      and Q: ⋀f. ⟦continuous_on S f; f ∈ S → U; P f⟧ ⟹ Q(f ∘ k)
      and Qeq: ⋀h k. (⋀x. x ∈ t ⟹ h x = k x) ⟹ Q h = Q k
      and hom: ⋀f g. ⟦continuous_on S f; f ∈ S → U; P f⟧
                  ⟹ ∃c. homotopic_with_canon P S U f (λx. c)
      and contf: continuous_on t f and imf: f ∈ t → U and Qf: Q f
  obtains c where homotopic_with_canon Q t U f (λx. c)
proof −
  have feq: ⋀x. x ∈ t ⟹ (f ∘ h ∘ k) x = f x using idhk imf by auto
  have continuous_on S (f ∘ h)
    using contf conth continuous_on_compose imh by blast
  moreover have (f ∘ h) ∈ S → U
    using imf imh by fastforce
  moreover have P (f ∘ h)
    by (simp add: P Qf contf imf)
  ultimately obtain c where homotopic_with_canon P S U (f ∘ h) (λx. c)
    by (metis hom)
  then have §: homotopic_with_canon Q t U (f ∘ h ∘ k) ((λx. c) ∘ k)
  proof (rule homotopic_with_compose_continuous_right [OF homotopic_with_mono])
    show ⋀h. ⟦continuous_map (top_of_set S) (top_of_set U) h; P h⟧ ⟹ Q (h
∘ k)
      using Q by auto
  qed (use contk imk in force)+
  moreover have homotopic_with_canon Q t U f (λx. c)
    using homotopic_with_eq [OF §] feq Qeq by fastforce
  ultimately show ?thesis
    using that by blast
qed

end

lemma simply_connected_retraction_gen:
  shows ⟦simply_connected S; continuous_on S h; h ‘ S = T;
         continuous_on T k; k ∈ T → S; ⋀y. y ∈ T ⟹ h(k y) = y⟧
        ⟹ simply_connected T
apply (simp add: simply_connected_def path_def path_image_def homotopic_loops_def,
clarify)
apply (rule Retracts.homotopically_trivial_retraction_gen
       [of S h _ k _ λp. pathfinish p = pathstart p  λp. pathfinish p = pathstart p])
apply (simp_all add: Retracts_def pathfinish_def pathstart_def image_subset_iff_funcset)
done

lemma homeomorphic_simply_connected:
    ⟦S homeomorphic T; simply_connected S⟧ ⟹ simply_connected T
  by (auto simp: homeomorphic_def homeomorphism_def intro: simply_connected_retraction_gen)

lemma homeomorphic_simply_connected_eq:
    S homeomorphic T ⟹ (simply_connected S ⟷ simply_connected T)
  by (metis homeomorphic_simply_connected homeomorphic_sym)
```

6.4.22 Homotopy equivalence

6.4.23 Homotopy equivalence of topological spaces.

definition *homotopy_equivalent_space*
(**infix** ‹*homotopy'_equivalent'_space*› *50*)
where *X homotopy_equivalent_space Y* ≡
(∃*f g*. *continuous_map X Y f* ∧
continuous_map Y X g ∧
homotopic_with (λ*x*. *True*) *X X* (*g* ∘ *f*) *id* ∧
homotopic_with (λ*x*. *True*) *Y Y* (*f* ∘ *g*) *id*)

lemma *homeomorphic_imp_homotopy_equivalent_space*:
X homeomorphic_space Y ⟹ *X homotopy_equivalent_space Y*
unfolding *homeomorphic_space_def homotopy_equivalent_space_def*
apply (*erule ex_forward*)+
by (*simp add*: *homotopic_with_equal homotopic_with_sym homeomorphic_maps_def*)

lemma *homotopy_equivalent_space_refl*:
X homotopy_equivalent_space X
by (*simp add*: *homeomorphic_imp_homotopy_equivalent_space homeomorphic_space_refl*)

lemma *homotopy_equivalent_space_sym*:
X homotopy_equivalent_space Y ⟷ *Y homotopy_equivalent_space X*
by (*meson homotopy_equivalent_space_def*)

lemma *homotopy_eqv_trans* [*trans*]:
assumes *1*: *X homotopy_equivalent_space Y* **and** *2*: *Y homotopy_equivalent_space U*
shows *X homotopy_equivalent_space U*
proof −
obtain *f1 g1* **where** *f1*: *continuous_map X Y f1*
and *g1*: *continuous_map Y X g1*
and *hom1*: *homotopic_with* (λ*x*. *True*) *X X* (*g1* ∘ *f1*) *id*
homotopic_with (λ*x*. *True*) *Y Y* (*f1* ∘ *g1*) *id*
using *1* **by** (*auto simp*: *homotopy_equivalent_space_def*)
obtain *f2 g2* **where** *f2*: *continuous_map Y U f2*
and *g2*: *continuous_map U Y g2*
and *hom2*: *homotopic_with* (λ*x*. *True*) *Y Y* (*g2* ∘ *f2*) *id*
homotopic_with (λ*x*. *True*) *U U* (*f2* ∘ *g2*) *id*
using *2* **by** (*auto simp*: *homotopy_equivalent_space_def*)
have *homotopic_with* (λ*f*. *True*) *X Y* (*g2* ∘ *f2* ∘ *f1*) (*id* ∘ *f1*)
using *f1 hom2*(*1*) *homotopic_with_compose_continuous_map_right* **by** *metis*
then have *homotopic_with* (λ*f*. *True*) *X Y* (*g2* ∘ (*f2* ∘ *f1*)) (*id* ∘ *f1*)
by (*simp add*: *o_assoc*)
then have *homotopic_with* (λ*x*. *True*) *X X*
(*g1* ∘ (*g2* ∘ (*f2* ∘ *f1*))) (*g1* ∘ (*id* ∘ *f1*))
by (*simp add*: *g1 homotopic_with_compose_continuous_map_left*)
moreover have *homotopic_with* (λ*x*. *True*) *X X* (*g1* ∘ *id* ∘ *f1*) *id*
using *hom1* **by** *simp*

ultimately have *SS*: *homotopic_with* (λ*x*. *True*) *X X* (*g1* ∘ *g2* ∘ (*f2* ∘ *f1*)) *id*
by (*metis comp_assoc homotopic_with_trans id_comp*)
have *homotopic_with* (λ*f*. *True*) *U Y* (*f1* ∘ *g1* ∘ *g2*) (*id* ∘ *g2*)
using *g2 hom1*(*2*) *homotopic_with_compose_continuous_map_right* **by** *fastforce*
then have *homotopic_with* (λ*f*. *True*) *U Y* (*f1* ∘ (*g1* ∘ *g2*)) (*id* ∘ *g2*)
by (*simp add*: *o_assoc*)
then have *homotopic_with* (λ*x*. *True*) *U U*
(*f2* ∘ (*f1* ∘ (*g1* ∘ *g2*))) (*f2* ∘ (*id* ∘ *g2*))
by (*simp add*: *f2 homotopic_with_compose_continuous_map_left*)
moreover have *homotopic_with* (λ*x*. *True*) *U U* (*f2* ∘ *id* ∘ *g2*) *id*
using *hom2* **by** *simp*
ultimately have *UU*: *homotopic_with* (λ*x*. *True*) *U U* (*f2* ∘ *f1* ∘ (*g1* ∘ *g2*)) *id*
by (*simp add*: *fun.map_comp hom2*(*2*) *homotopic_with_trans*)
show *?thesis*
unfolding *homotopy_equivalent_space_def*
by (*blast intro*: *f1 f2 g1 g2 continuous_map_compose SS UU*)
qed

lemma *deformation_retraction_imp_homotopy_equivalent_space*:
⟦*homotopic_with* (λ*x*. *True*) *X X* (*S* ∘ *r*) *id*; *retraction_maps X Y r S*⟧
⟹ *X homotopy_equivalent_space Y*
unfolding *homotopy_equivalent_space_def retraction_maps_def*
using *homotopic_with_id2* **by** *fastforce*

lemma *deformation_retract_imp_homotopy_equivalent_space*:
⟦*homotopic_with* (λ*x*. *True*) *X X r id*; *retraction_maps X Y r id*⟧
⟹ *X homotopy_equivalent_space Y*
using *deformation_retraction_imp_homotopy_equivalent_space* **by** *force*

lemma *deformation_retract_of_space*:
S ⊆ *topspace X* ∧
(∃ *r*. *homotopic_with* (λ*x*. *True*) *X X id r* ∧ *retraction_maps X* (*subtopology X S*) *r id*) ⟷
S retract_of_space X ∧ (∃ *f*. *homotopic_with* (λ*x*. *True*) *X X id f* ∧ *f* ‘ (*topspace X*) ⊆ *S*)
proof (*cases S* ⊆ *topspace X*)
case *True*
moreover have (∃ *r*. *homotopic_with* (λ*x*. *True*) *X X id r* ∧ *retraction_maps X* (*subtopology X S*) *r id*)
⟷ (*S retract_of_space X* ∧ (∃ *f*. *homotopic_with* (λ*x*. *True*) *X X id f* ∧ *f* ‘ *topspace X* ⊆ *S*))
unfolding *retract_of_space_def*
proof *safe*
fix *f r*
assume *f*: *homotopic_with* (λ*x*. *True*) *X X id f*
and *fS*: *f* ‘ *topspace X* ⊆ *S*
and *r*: *continuous_map X* (*subtopology X S*) *r*
and *req*: ∀ *x*∈*S*. *r x* = *x*

show $\exists r.$ *homotopic_with* $(\lambda x.\ True)$ *X X id r* $\wedge$ *retraction_maps X* (*subtopology X S*) *r id*
proof (*intro exI conjI*)
have *homotopic_with* $(\lambda x.\ True)$ *X X f r*
proof (*rule homotopic_with_eq*)
show *homotopic_with* $(\lambda x.\ True)$ *X X* $(r \circ f)$ $(r \circ id)$
by (*metis continuous_map_into_fulltopology f homotopic_with_compose_continuous_map_left homotopic_with_symD r*)
show $f\ x = (r \circ f)\ x$ **if** $x \in$ *topspace X* **for** *x*
using *that fS req* **by** *auto*
qed *auto*
then show *homotopic_with* $(\lambda x.\ True)$ *X X id r*
by (*rule homotopic_with_trans* [*OF f*])
next
show *retraction_maps X* (*subtopology X S*) *r id*
by (*simp add*: *r req retraction_maps_def*)
qed
qed (*use True* **in** ‹*auto simp*: *retraction_maps_def topspace_subtopology_subset continuous_map_in_subtopology*›)
ultimately show *?thesis* **by** *simp*
qed (*auto simp*: *retract_of_space_def retraction_maps_def*)

6.4.24 Contractible spaces

The definition (which agrees with "contractible" on subsets of Euclidean space) is a little cryptic because we don't in fact assume that the constant "a" is in the space. This forces the convention that the empty space / set is contractible, avoiding some special cases.

definition *contractible_space* **where**
contractible_space X $\equiv \exists a.$ *homotopic_with* $(\lambda x.\ True)$ *X X id* $(\lambda x.\ a)$

lemma *contractible_space_top_of_set* [*simp*]:*contractible_space* (*top_of_set S*) $\longleftrightarrow$ *contractible S*
by (*auto simp*: *contractible_space_def contractible_def*)

lemma *contractible_space_empty* [*simp*]:
contractible_space trivial_topology
unfolding *contractible_space_def homotopic_with_def*
apply (*rule_tac x*=*undefined* **in** *exI*)
apply (*rule_tac x*=λ(*t*,*x*). *if t = 0 then x else undefined* **in** *exI*)
apply (*auto simp*: *continuous_map_on_empty*)
done

lemma *contractible_space_singleton* [*simp*]:
contractible_space (*discrete_topology*{*a*})
unfolding *contractible_space_def homotopic_with_def*
apply (*rule_tac x*=*a* **in** *exI*)
apply (*rule_tac x*=λ(*t*,*x*). *if t = 0 then x else a* **in** *exI*)

apply (*auto intro*: *continuous_map_eq* [**where** $f = \lambda z.\ a$])
done

lemma *contractible_space_subset_singleton*:
$topspace\ X \subseteq \{a\} \Longrightarrow contractible\_space\ X$
by (*metis contractible_space_empty contractible_space_singleton null_topspace_iff_trivial subset_singletonD subtopology_eq_discrete_topology_sing*)

lemma *contractible_space_subtopology_singleton* [*simp*]:
$contractible\_space\ (subtopology\ X\ \{a\})$
by (*meson contractible_space_subset_singleton insert_subset path_connectedin_singleton path_connectedin_subtopology subsetI*)

lemma *contractible_space*:
$contractible\_space\ X \longleftrightarrow$
$X = trivial\_topology \vee$
$(\exists\, a \in topspace\ X.\ homotopic\_with\ (\lambda x.\ True)\ X\ X\ id\ (\lambda x.\ a))$
proof (*cases* $X = trivial\_topology$)
case *False*
then show *?thesis*
using *homotopic_with_imp_continuous_maps* **by** (*fastforce simp*: *contractible_space_def*)
qed (*simp add*: *contractible_space_empty*)

lemma *contractible_imp_path_connected_space*:
assumes *contractible_space X* **shows** *path_connected_space X*
proof (*cases* $X = trivial\_topology$)
case *False*
have $*$: *path_connected_space X*
if $a \in topspace\ X$ **and** *conth*: *continuous_map* (*prod_topology* (*top_of_set* $\{0..1\}$) *X*) *X h*
and *h*: $\forall x.\ h\ (0,\ x) = x\ \forall x.\ h\ (1,\ x) = a$
for *a* **and** *h* :: $real \times 'a \Rightarrow 'a$
proof −
have *path_component_of X b a* **if** $b \in topspace\ X$ **for** *b*
unfolding *path_component_of_def*
proof (*intro exI conjI*)
let *?g* = $h \circ (\lambda x.\ (x,b))$
show *pathin X ?g*
unfolding *pathin_def*
proof (*rule continuous_map_compose* [*OF* _ *conth*])
show *continuous_map* (*top_of_set* $\{0..1\}$) (*prod_topology* (*top_of_set* $\{0..1\}$) *X*) $(\lambda x.\ (x,\ b))$
using *that* **by** (*auto intro*!: *continuous_intros*)
qed
qed (*use h* **in** *auto*)
then show *?thesis*
by (*metis path_component_of_equiv path_connected_space_iff_path_component*)
qed
show *?thesis*

using *assms False* **by** (*auto simp*: *contractible_space homotopic_with_def* ∗)
qed (*simp add*: *path_connected_space_topspace_empty*)

lemma *contractible_imp_connected_space*:
contractible_space X ⟹ *connected_space X*
by (*simp add*: *contractible_imp_path_connected_space path_connected_imp_connected_space*)

lemma *contractible_space_alt*:
contractible_space X ⟷ (∀ *a* ∈ *topspace X*. *homotopic_with* (λ*x*. *True*) *X X*
id (λ*x*. *a*)) (**is** *?lhs* = *?rhs*)
proof
assume *X*: *?lhs*
then obtain *a* **where** *a*: *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *a*)
by (*auto simp*: *contractible_space_def*)
show *?rhs*
proof
show *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *b*) **if** *b* ∈ *topspace X* **for** *b*
proof (*rule homotopic_with_trans* [*OF a*])
show *homotopic_with* (λ*x*. *True*) *X X* (λ*x*. *a*) (λ*x*. *b*)
using *homotopic_constant_maps path_connected_space_imp_path_component_of*
by (*metis X a contractible_imp_path_connected_space homotopic_with_sym*
homotopic_with_trans path_component_of_equiv that)
qed
qed
next
assume *R*: *?rhs*
then show *?lhs*
using *contractible_space_def* **by** *fastforce*
qed

lemma *compose_const* [*simp*]: *f* ∘ (λ*x*. *a*) = (λ*x*. *f a*) (λ*x*. *a*) ∘ *g* = (λ*x*. *a*)
by (*simp_all add*: *o_def*)

lemma *nullhomotopic_through_contractible_space*:
assumes *f*: *continuous_map X Y f* **and** *g*: *continuous_map Y Z g* **and** *Y*:
contractible_space Y
obtains *c* **where** *homotopic_with* (λ*h*. *True*) *X Z* (*g* ∘ *f*) (λ*x*. *c*)
proof −
obtain *b* **where** *b*: *homotopic_with* (λ*x*. *True*) *Y Y id* (λ*x*. *b*)
using *Y* **by** (*auto simp*: *contractible_space_def*)
show *thesis*
using *homotopic_with_compose_continuous_map_right*
[*OF homotopic_with_compose_continuous_map_left* [*OF b g*] *f*]
by (*force simp*: *that*)
qed

lemma *nullhomotopic_into_contractible_space*:
assumes *f*: *continuous_map X Y f* **and** *Y*: *contractible_space Y*

obtains c **where** *homotopic_with* $(\lambda h.\ True)\ X\ Y\ f\ (\lambda x.\ c)$
using *nullhomotopic_through_contractible_space* $[OF\ f\ \_\ Y]$
by (*metis continuous_map_id id_comp*)

lemma *nullhomotopic_from_contractible_space*:
assumes f: *continuous_map* $X\ Y\ f$ **and** X: *contractible_space* X
obtains c **where** *homotopic_with* $(\lambda h.\ True)\ X\ Y\ f\ (\lambda x.\ c)$
using *nullhomotopic_through_contractible_space* $[OF\ \_\ f\ X]$
by (*metis comp_id continuous_map_id*)

lemma *homotopy_dominated_contractibility*:
assumes f: *continuous_map* $X\ Y\ f$ **and** g: *continuous_map* $Y\ X\ g$
and *hom*: *homotopic_with* $(\lambda x.\ True)\ Y\ Y\ (f \circ g)\ id$ **and** X: *contractible_space* X
shows *contractible_space* Y
proof −
obtain c **where** c: *homotopic_with* $(\lambda h.\ True)\ X\ Y\ f\ (\lambda x.\ c)$
using *nullhomotopic_from_contractible_space* $[OF\ f\ X]$ **.**
have *homotopic_with* $(\lambda x.\ True)\ Y\ Y\ (f \circ g)\ (\lambda x.\ c)$
using *homotopic_with_compose_continuous_map_right* $[OF\ c\ g]$ **by** *fastforce*
then have *homotopic_with* $(\lambda x.\ True)\ Y\ Y\ id\ (\lambda x.\ c)$
using *homotopic_with_trans* $[OF\ \_\ hom]$ *homotopic_with_symD* **by** *blast*
then show *?thesis*
unfolding *contractible_space_def* **..**
qed

lemma *homotopy_equivalent_space_contractibility*:
X *homotopy_equivalent_space* $Y \Longrightarrow$ (*contractible_space* $X \longleftrightarrow$ *contractible_space* Y)
unfolding *homotopy_equivalent_space_def*
by (*blast intro*: *homotopy_dominated_contractibility*)

lemma *homeomorphic_space_contractibility*:
X *homeomorphic_space* Y
$\Longrightarrow$ (*contractible_space* $X \longleftrightarrow$ *contractible_space* Y)
by (*simp add*: *homeomorphic_imp_homotopy_equivalent_space homotopy_equivalent_space_contracti*

lemma *homotopic_through_contractible_space*:
continuous_map $X\ Y\ f \wedge$
continuous_map $X\ Y\ f' \wedge$
continuous_map $Y\ Z\ g \wedge$
continuous_map $Y\ Z\ g' \wedge$
contractible_space $Y \wedge$ *path_connected_space* Z
$\Longrightarrow$ *homotopic_with* $(\lambda h.\ True)\ X\ Z\ (g \circ f)\ (g' \circ f')$
using *nullhomotopic_through_contractible_space* $[of\ X\ Y\ f\ Z\ g]$
using *nullhomotopic_through_contractible_space* $[of\ X\ Y\ f'\ Z\ g']$
by (*smt* (*verit*) *continuous_map_const homotopic_constant_maps homotopic_with_imp_continuous_*
homotopic_with_symD homotopic_with_trans path_connected_space_imp_path_component_of)

lemma *homotopic_from_contractible_space*:
continuous_map X Y f ∧ *continuous_map X Y g* ∧
contractible_space X ∧ *path_connected_space Y*
⟹ *homotopic_with* (λ*x*. *True*) *X Y f g*
by (*metis comp_id continuous_map_id homotopic_through_contractible_space*)

lemma *homotopic_into_contractible_space*:
continuous_map X Y f ∧ *continuous_map X Y g* ∧
contractible_space Y
⟹ *homotopic_with* (λ*x*. *True*) *X Y f g*
by (*metis continuous_map_id contractible_imp_path_connected_space homotopic_through_contractible_space id_comp*)

lemma *contractible_eq_homotopy_equivalent_singleton_subtopology*:
contractible_space X ⟷
X = *trivial_topology* ∨ (∃ *a* ∈ *topspace X*. *X homotopy_equivalent_space* (*subtopology X* {*a*}))(**is** *?lhs* = *?rhs*)
proof (*cases X* = *trivial_topology*)
case *False*
show *?thesis*
proof
assume *?lhs*
then obtain *a* **where** *a*: *homotopic_with* (λ*x*. *True*) *X X id* (λ*x*. *a*)
by (*auto simp*: *contractible_space_def*)
then have *a* ∈ *topspace X*
by (*metis False continuous_map_const homotopic_with_imp_continuous_maps*)
then have *homotopic_with* (λ*x*. *True*) (*subtopology X* {*a*}) (*subtopology X* {*a*}) *id* (λ*x*. *a*)
using *connectedin_absolute connectedin_sing contractible_space_alt contractible_space_subtopology_singleton* **by** *fastforce*
then have *X homotopy_equivalent_space subtopology X* {*a*}
unfolding *homotopy_equivalent_space_def* **using** ‹*a* ∈ *topspace X*›
by (*metis* (*full_types*) *a comp_id continuous_map_const continuous_map_id_subt empty_subsetI homotopic_with_symD*
id_comp insertI1 insert_subset topspace_subtopology_subset)
with ‹*a* ∈ *topspace X*› **show** *?rhs*
by *blast*
next
assume *?rhs*
then show *?lhs*
by (*meson False contractible_space_subtopology_singleton homotopy_equivalent_space_contractibility*)
qed
qed (*simp add*: *contractible_space_empty*)

lemma *contractible_space_retraction_map_image*:
assumes *retraction_map X Y f* **and** *X*: *contractible_space X*
shows *contractible_space Y*
proof −
obtain *g* **where** *f*: *continuous_map X Y f* **and** *g*: *continuous_map Y X g* **and**

```
fg: ∀ y ∈ topspace Y. f(g y) = y
    using assms by (auto simp: retraction_map_def retraction_maps_def)
  obtain a where a: homotopic_with (λx. True) X X id (λx. a)
    using X by (auto simp: contractible_space_def)
  have homotopic_with (λx. True) Y Y id (λx. f a)
  proof (rule homotopic_with_eq)
    show homotopic_with (λx. True) Y Y (f ∘ id ∘ g) (f ∘ (λx. a) ∘ g)
    using f g a homotopic_with_compose_continuous_map_left homotopic_with_compose_continuous_
by metis
  qed (use fg in auto)
  then show ?thesis
    unfolding contractible_space_def by blast
qed

lemma contractible_space_prod_topology:
   contractible_space(prod_topology X Y) ⟷
    X = trivial_topology ∨ Y = trivial_topology ∨ contractible_space X ∧ con-
tractible_space Y
proof (cases X = trivial_topology ∨ Y = trivial_topology)
  case True
  then have (prod_topology X Y) = trivial_topology
    by simp
  then show ?thesis
    by (auto simp: contractible_space_empty)
next
  case False
  have contractible_space(prod_topology X Y) ⟷ contractible_space X ∧ con-
tractible_space Y
  proof safe
    assume XY: contractible_space (prod_topology X Y)
    with False have retraction_map (prod_topology X Y) X fst
      by (auto simp: contractible_space False retraction_map_fst)
    then show contractible_space X
      by (rule contractible_space_retraction_map_image [OF _ XY])
    have retraction_map (prod_topology X Y) Y snd
     using False XY by (auto simp: contractible_space False retraction_map_snd)
    then show contractible_space Y
      by (rule contractible_space_retraction_map_image [OF _ XY])
  next
    assume contractible_space X and contractible_space Y
    with False obtain a b
      where a ∈ topspace X and a: homotopic_with (λx. True) X X id (λx. a)
        and b ∈ topspace Y and b: homotopic_with (λx. True) Y Y id (λx. b)
      by (auto simp: contractible_space)
    with False show contractible_space (prod_topology X Y)
      apply (simp add: contractible_space)
      apply (rule_tac x=a in bexI)
       apply (rule_tac x=b in bexI)
      using homotopic_with_prod_topology [OF a b]
```

```
      apply (metis (no_types, lifting) case_prod_Pair case_prod_beta' eq_id_iff)
       apply auto
      done
  qed
  with False show ?thesis
    by auto
qed


lemma contractible_space_product_topology:
  contractible_space(product_topology X I) ⟷
    (product_topology X I) = trivial_topology ∨ (∀ i ∈ I. contractible_space(X i))
proof (cases (product_topology X I) = trivial_topology)
  case False
  have 1: contractible_space (X i)
    if XI: contractible_space (product_topology X I) and i ∈ I
    for i
  proof (rule contractible_space_retraction_map_image [OF _ XI])
    show retraction_map (product_topology X I) (X i) (λx. x i)
      using False by (simp add: retraction_map_product_projection ‹i ∈ I›)
  qed
  have 2: contractible_space (product_topology X I)
    if x ∈ topspace (product_topology X I) and cs: ∀ i∈I. contractible_space (X i)
    for x :: 'a ⇒ 'b
  proof −
    obtain f where f: ⋀i. i∈I ⟹ homotopic_with (λx. True) (X i) (X i) id (λx.
f i)
      using cs unfolding contractible_space_def by metis
    have homotopic_with (λx. True)
                        (product_topology X I) (product_topology X I) id (λx. restrict
f I)
      by (rule homotopic_with_eq [OF homotopic_with_product_topology [OF f]])
(auto)
    then show ?thesis
      by (auto simp: contractible_space_def)
  qed
  show ?thesis
    using False 1 2 by (meson equals0I subtopology_eq_discrete_topology_empty)
qed auto


lemma contractible_space_subtopology_euclideanreal [simp]:
  contractible_space(subtopology euclideanreal S) ⟷ is_interval S
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have path_connectedin (subtopology euclideanreal S) S
   using contractible_imp_path_connected_space path_connectedin_topspace path_connectedin_absolute
    by (simp add: contractible_imp_path_connected)
```

```
  then show ?rhs
    by (simp add: is_interval_path_connected_1)
next
  assume ?rhs
  then have convex S
    by (simp add: is_interval_convex_1)
  show ?lhs
  proof (cases S = {})
    case False
    then obtain z where z ∈ S
      by blast
    show ?thesis
      unfolding contractible_space_def homotopic_with_def
    proof (intro exI conjI allI)
      note § = convexD [OF ‹convex S›, simplified]
      show continuous_map (prod_topology (top_of_set {0..1}) (top_of_set S))
(top_of_set S)
                          (λ(t,x). (1 − t) ∗ x + t ∗ z)
        using ‹z ∈ S›
        by (auto simp: case_prod_unfold intro!: continuous_intros §)
    qed auto
  qed (simp add: contractible_space_empty)
qed


corollary contractible_space_euclideanreal: contractible_space euclideanreal
proof −
  have contractible_space (subtopology euclideanreal UNIV)
    using contractible_space_subtopology_euclideanreal by blast
  then show ?thesis
    by simp
qed


abbreviation homotopy_eqv :: 'a::topological_space set ⇒ 'b::topological_space
set ⇒ bool
            (infix ‹homotopy'_eqv› 50)
 where S homotopy_eqv T ≡ top_of_set S homotopy_equivalent_space top_of_set
T

lemma homeomorphic_imp_homotopy_eqv: S homeomorphic T ⟹ S homotopy_eqv
T
 unfolding homeomorphic_def homeomorphism_def homotopy_equivalent_space_def
 by (metis continuous_map_subtopology_eu homotopic_with_id2 openin_imp_subset
openin_subtopology_self topspace_euclidean_subtopology)

lemma homotopy_eqv_inj_linear_image:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f inj f
```

shows (*f ' S*) *homotopy_eqv S*
by (*metis assms homeomorphic_sym homeomorphic_imp_homotopy_eqv linear_homeomorphic_image*)

lemma *homotopy_eqv_translation*:
fixes *S* :: *'a*::*real_normed_vector set*
shows (+) *a ' S homotopy_eqv S*
using *homeomorphic_imp_homotopy_eqv homeomorphic_translation homeomorphic_sym* **by** *blast*

lemma *homotopy_eqv_homotopic_triviality_imp*:
fixes *S* :: *'a*::*real_normed_vector set*
and *T* :: *'b*::*real_normed_vector set*
and *U* :: *'c*::*real_normed_vector set*
assumes *S homotopy_eqv T*
and *f*: *continuous_on U f f* ∈ *U* → *T*
and *g*: *continuous_on U g g* ∈ *U* → *T*
and *homUS*: ⋀*f g*. ⟦*continuous_on U f*; *f* ∈ *U* → *S*;
continuous_on U g; *g* ∈ *U* → *S*⟧
⟹ *homotopic_with_canon* (λ*x*. *True*) *U S f g*
shows *homotopic_with_canon* (λ*x*. *True*) *U T f g*
proof −
obtain *h k* **where** *h*: *continuous_on S h h* ∈ *S* → *T*
and *k*: *continuous_on T k k* ∈ *T* → *S*
and *hom*: *homotopic_with_canon* (λ*x*. *True*) *S S* (*k* ∘ *h*) *id*
homotopic_with_canon (λ*x*. *True*) *T T* (*h* ∘ *k*) *id*
using *assms* **by** (*force simp*: *homotopy_equivalent_space_def image_subset_iff_funcset*)
have *homotopic_with_canon* (λ*f*. *True*) *U S* (*k* ∘ *f*) (*k* ∘ *g*)
proof (*rule homUS*)
show *continuous_on U* (*k* ∘ *f*) *continuous_on U* (*k* ∘ *g*)
using *continuous_on_compose continuous_on_subset f g k* **by** (*metis funcset_image*)+
qed (*use f g k* **in** ‹(*force simp*: *o_def*)+›)
then have *homotopic_with_canon* (λ*x*. *True*) *U T* (*h* ∘ (*k* ∘ *f*)) (*h* ∘ (*k* ∘ *g*))
by (*simp add*: *h homotopic_with_compose_continuous_map_left image_subset_iff_funcset*)
moreover have *homotopic_with_canon* (λ*x*. *True*) *U T* (*h* ∘ *k* ∘ *f*) (*id* ∘ *f*)
by (*rule homotopic_with_compose_continuous_right* [**where** *X*=*T* **and** *Y*=*T*]; *simp add*: *hom f*)
moreover have *homotopic_with_canon* (λ*x*. *True*) *U T* (*h* ∘ *k* ∘ *g*) (*id* ∘ *g*)
by (*rule homotopic_with_compose_continuous_right* [**where** *X*=*T* **and** *Y*=*T*]; *simp add*: *hom g*)
ultimately show *homotopic_with_canon* (λ*x*. *True*) *U T f g*
unfolding *o_assoc*
by (*metis homotopic_with_trans homotopic_with_sym id_comp*)
qed

lemma *homotopy_eqv_homotopic_triviality*:
fixes *S* :: *'a*::*real_normed_vector set*
and *T* :: *'b*::*real_normed_vector set*

```
    and U :: 'c::real_normed_vector set
  assumes S homotopy_eqv T
    shows (∀f g. continuous_on U f ∧ f ∈ U → S ∧
                 continuous_on U g ∧ g ∈ U → S
                 ⟶ homotopic_with_canon (λx. True) U S f g) ⟷
           (∀f g. continuous_on U f ∧ f ∈ U → T ∧
                  continuous_on U g ∧ g ∈ U → T
                  ⟶ homotopic_with_canon (λx. True) U T f g)
      (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
    by (metis assms homotopy_eqv_homotopic_triviality_imp)
next
  assume ?rhs
  moreover
  have T homotopy_eqv S
    using assms homotopy_equivalent_space_sym by blast
  ultimately show ?lhs
    by (blast intro: homotopy_eqv_homotopic_triviality_imp)
qed


lemma homotopy_eqv_cohomotopic_triviality_null_imp:
  fixes S :: 'a::real_normed_vector set
    and T :: 'b::real_normed_vector set
    and U :: 'c::real_normed_vector set
  assumes S homotopy_eqv T
      and f: continuous_on T f f ∈ T → U
      and homSU: ⋀f. ⟦continuous_on S f; f ∈ S → U⟧
                     ⟹ ∃c. homotopic_with_canon (λx. True) S U f (λx. c)
  obtains c where homotopic_with_canon (λx. True) T U f (λx. c)
proof −
  obtain h k where h: continuous_on S h h ∈ S → T
               and k: continuous_on T k k ∈ T → S
               and hom: homotopic_with_canon (λx. True) S S (k ∘ h) id
                        homotopic_with_canon (λx. True) T T (h ∘ k) id
   using assms by (force simp: homotopy_equivalent_space_def image_subset_iff_funcset)
  obtain c where homotopic_with_canon (λx. True) S U (f ∘ h) (λx. c)
  proof (rule exE [OF homSU])
    show continuous_on S (f ∘ h)
      by (metis continuous_on_compose continuous_on_subset f h funcset_image)
  qed (use f h in force)
  then have homotopic_with_canon (λx. True) T U ((f ∘ h) ∘ k) ((λx. c) ∘ k)
    by (rule homotopic_with_compose_continuous_right [where X=S]) (use k in
auto)
  moreover have homotopic_with_canon (λx. True) T U (f ∘ id) (f ∘ (h ∘ k))
    by (rule homotopic_with_compose_continuous_left [where Y=T])
       (use f in ‹auto simp: hom homotopic_with_symD›)
```

ultimately show *?thesis*
using *that homotopic_with_trans* **by** (*fastforce simp*: *o_def*)
qed

lemma *homotopy_eqv_cohomotopic_triviality_null*:
fixes S :: *'a*::*real_normed_vector set*
and T :: *'b*::*real_normed_vector set*
and U :: *'c*::*real_normed_vector set*
assumes S *homotopy_eqv* T
shows $(\forall f.\ continuous\_on\ S\ f \wedge f \in S \to U$
$\longrightarrow (\exists c.\ homotopic\_with\_canon\ (\lambda x.\ True)\ S\ U\ f\ (\lambda x.\ c))) \longleftrightarrow$
$(\forall f.\ continuous\_on\ T\ f \wedge f \in T \to U$
$\longrightarrow (\exists c.\ homotopic\_with\_canon\ (\lambda x.\ True)\ T\ U\ f\ (\lambda x.\ c)))$
by (*rule iffI*; *metis assms homotopy_eqv_cohomotopic_triviality_null_imp homotopy_equivalent_space_sym*)

Similar to the proof above

lemma *homotopy_eqv_homotopic_triviality_null_imp*:
fixes S :: *'a*::*real_normed_vector set*
and T :: *'b*::*real_normed_vector set*
and U :: *'c*::*real_normed_vector set*
assumes S *homotopy_eqv* T
and f: *continuous_on U f* $f \in U \to T$
and *homSU*: $\bigwedge f.\ [\![continuous\_on\ U\ f; f \in U \to S]\!]$
$\Longrightarrow \exists c.\ homotopic\_with\_canon\ (\lambda x.\ True)\ U\ S\ f\ (\lambda x.\ c)$
shows $\exists c.\ homotopic\_with\_canon\ (\lambda x.\ True)\ U\ T\ f\ (\lambda x.\ c)$
proof −
obtain $h\ k$ **where** h: *continuous_on S h* $h \in S \to T$
and k: *continuous_on T k* $k \in T \to S$
and *hom*: *homotopic_with_canon* $(\lambda x.\ True)\ S\ S\ (k \circ h)\ id$
homotopic_with_canon $(\lambda x.\ True)\ T\ T\ (h \circ k)\ id$
using *assms* **by** (*force simp*: *homotopy_equivalent_space_def image_subset_iff_funcset*)
obtain *c*::*'a* **where** *homotopic_with_canon* $(\lambda x.\ True)\ U\ S\ (k \circ f)\ (\lambda x.\ c)$
proof (*rule exE* [*OF homSU* [*of* $k \circ f$]])
show *continuous_on U* $(k \circ f)$
using *continuous_on_compose continuous_on_subset f k* **by** (*metis funcset_image*)
qed (*use f k* **in** *force*)
then have *homotopic_with_canon* $(\lambda x.\ True)\ U\ T\ (h \circ (k \circ f))\ (h \circ (\lambda x.\ c))$
by (*rule homotopic_with_compose_continuous_left* [**where** $Y{=}S$]) (*use h* **in** *auto*)
moreover have *homotopic_with_canon* $(\lambda x.\ True)\ U\ T\ (id \circ f)\ ((h \circ k) \circ f)$
by (*rule homotopic_with_compose_continuous_right* [**where** $X{=}T$])
(*use f* **in** ‹*auto simp*: *hom homotopic_with_symD*›)
ultimately show *?thesis*
using *homotopic_with_trans* **by** (*fastforce simp*: *o_def*)
qed

lemma *homotopy_eqv_homotopic_triviality_null*:

fixes S :: $'a$::*real_normed_vector set*
and T :: $'b$::*real_normed_vector set*
and U :: $'c$::*real_normed_vector set*
assumes S *homotopy_eqv* T
shows $(\forall f.$ *continuous_on* U $f \wedge f \in U \to S$
$\longrightarrow (\exists c.$ *homotopic_with_canon* $(\lambda x.$ *True*$)$ U S f $(\lambda x.$ $c))) \longleftrightarrow$
$(\forall f.$ *continuous_on* U $f \wedge f \in U \to T$
$\longrightarrow (\exists c.$ *homotopic_with_canon* $(\lambda x.$ *True*$)$ U T f $(\lambda x.$ $c)))$
by (*rule iffI*; *metis assms homotopy_eqv_homotopic_triviality_null_imp homotopy_equivalent_space_sym*)

lemma *homotopy_eqv_contractible_sets*:
fixes S :: $'a$::*real_normed_vector set*
and T :: $'b$::*real_normed_vector set*
assumes *contractible* S *contractible* T $S = \{\} \longleftrightarrow T = \{\}$
shows S *homotopy_eqv* T
proof (*cases* $S = \{\}$)
case *True* **with** *assms* **show** *?thesis*
using *homeomorphic_imp_homotopy_eqv* **by** *fastforce*
next
case *False*
with *assms* **obtain** a b **where** $a \in S$ $b \in T$
by *auto*
then show *?thesis*
unfolding *homotopy_equivalent_space_def*
apply (*rule_tac* x=$\lambda x.$ b **in** *exI*, *rule_tac* x=$\lambda x.$ a **in** *exI*)
apply (*intro assms conjI continuous_on_id'* *homotopic_into_contractible*; *force*)
done
qed

lemma *homotopy_eqv_empty1* [*simp*]:
fixes S :: $'a$::*real_normed_vector set*
shows S *homotopy_eqv* $(\{\}::'b::$*real_normed_vector set*$) \longleftrightarrow S = \{\}$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis continuous_map_subtopology_eu empty_iff equalityI homotopy_equivalent_space_def image_subset_iff subsetI*)
qed (*use homeomorphic_imp_homotopy_eqv* **in** *force*)

lemma *homotopy_eqv_empty2* [*simp*]:
fixes S :: $'a$::*real_normed_vector set*
shows $(\{\}::'b::$*real_normed_vector set*$)$ *homotopy_eqv* $S \longleftrightarrow S = \{\}$
using *homotopy_equivalent_space_sym homotopy_eqv_empty1* **by** *blast*

lemma *homotopy_eqv_contractibility*:
fixes S :: $'a$::*real_normed_vector set* **and** T :: $'b$::*real_normed_vector set*
shows S *homotopy_eqv* $T \Longrightarrow$ (*contractible* $S \longleftrightarrow$ *contractible* T)

by (*meson contractible_space_top_of_set homotopy_equivalent_space_contractibility*)

lemma *homotopy_eqv_sing*:
fixes S :: *'a::real_normed_vector set* **and** a :: *'b::real_normed_vector*
shows S *homotopy_eqv* $\{a\} \longleftrightarrow S \neq \{\} \wedge$ *contractible* S
by (*metis contractible_sing empty_not_insert homotopy_eqv_contractibility homotopy_eqv_contractible_sets homotopy_eqv_empty2*)

lemma *homeomorphic_contractible_eq*:
fixes S :: *'a::real_normed_vector set* **and** T :: *'b::real_normed_vector set*
shows S *homeomorphic* $T \Longrightarrow$ (*contractible* $S \longleftrightarrow$ *contractible* T)
by (*simp add*: *homeomorphic_imp_homotopy_eqv homotopy_eqv_contractibility*)

lemma *homeomorphic_contractible*:
fixes S :: *'a::real_normed_vector set* **and** T :: *'b::real_normed_vector set*
shows $[\![$*contractible* S; S *homeomorphic* $T]\!] \Longrightarrow$ *contractible* T
by (*metis homeomorphic_contractible_eq*)

6.4.25 Misc other results

lemma *bounded_connected_Compl_real*:
fixes S :: *real set*
assumes *bounded* S **and** *conn*: *connected*$(-S)$
shows $S = \{\}$
proof −
obtain a b **where** $S \subseteq$ *box* a b
by (*meson assms bounded_subset_box_symmetric*)
then have $a \notin S$ $b \notin S$
by *auto*
then have $\forall x.\ a \leq x \wedge x \leq b \longrightarrow x \in -S$
by (*meson Compl_iff conn connected_iff_interval*)
then show *?thesis*
using ‹$S \subseteq$ *box* a b› **by** *auto*
qed

corollary *bounded_path_connected_Compl_real*:
fixes S :: *real set*
assumes *bounded* S *path_connected*$(-S)$ **shows** $S = \{\}$
by (*simp add*: *assms bounded_connected_Compl_real path_connected_imp_connected*)

lemma *bounded_connected_Compl_1*:
fixes S :: *'a::*{*euclidean_space*} *set*
assumes *bounded* S **and** *conn*: *connected*$(-S)$ **and** *1*: *DIM*$('a) = 1$
shows $S = \{\}$
proof −
have *DIM*$('a) =$ *DIM*$(real)$
by (*simp add*: *1*)
then obtain f::*'a* ⇒ *real* **and** g
where *linear* f $\bigwedge x.$ *norm*$(f\ x) =$ *norm* x **and** *fg*: $\bigwedge x.\ g(f\ x) = x$ $\bigwedge y.\ f(g\ y) = y$

by (*rule isomorphisms_UNIV_UNIV*) *blast*
with ‹*bounded S*› **have** *bounded* (*f* ‘ *S*)
using *bounded_linear_image linear_linear* **by** *blast*
have *bij f* **by** (*metis fg bijI'*)
have *connected* (*f* ‘ (−*S*))
using *connected_linear_image assms* ‹*linear f*› **by** *blast*
moreover have *f* ‘ (−*S*) = − (*f* ‘ *S*)
by (*simp add*: ‹*bij f*› *bij_image_Compl_eq*)
finally have *connected* (− (*f* ‘ *S*))
by *simp*
then have *f* ‘ *S* = {}
using ‹*bounded* (*f* ‘ *S*)› *bounded_connected_Compl_real* **by** *blast*
then show *?thesis*
by *blast*
qed

lemma *connected_card_eq_iff_nontrivial*:
fixes *S* :: *'a*::*metric_space set*
shows *connected S* ⟹ *uncountable S* ⟷ ¬(∃ *a*. *S* ⊆ {*a*})
by (*metis connected_uncountable finite.emptyI finite.insertI rev_finite_subset singleton_iff subsetI uncountable_infinite*)

lemma *connected_finite_iff_sing*:
fixes *S* :: *'a*::*metric_space set*
assumes *connected S*
shows *finite S* ⟷ *S* = {} ∨ (∃ *a*. *S* = {*a*})
using *assms connected_uncountable countable_finite* **by** *blast*

6.4.26 Some simple positive connection theorems

proposition *path_connected_convex_diff_countable*:
fixes *U* :: *'a*::*euclidean_space set*
assumes *convex U* ¬ *collinear U countable S*
shows *path_connected*(*U* − *S*)
proof (*clarsimp simp*: *path_connected_def*)
fix *a b*
assume *a* ∈ *U a* ∉ *S b* ∈ *U b* ∉ *S*
let *?m* = *midpoint a b*
show ∃ *g*. *path g* ∧ *path_image g* ⊆ *U* − *S* ∧ *pathstart g* = *a* ∧ *pathfinish g* = *b*
proof (*cases a* = *b*)
case *True*
then show *?thesis*
by (*metis DiffI* ‹*a* ∈ *U*› ‹*a* ∉ *S*› *path_component_def path_component_refl*)
next
case *False*
then have *a* ≠ *?m b* ≠ *?m*
using *midpoint_eq_endpoint* **by** *fastforce*+
have *?m* ∈ *U*
using ‹*a* ∈ *U*› ‹*b* ∈ *U*› ‹*convex U*› *convex_contains_segment* **by** *force*

obtain c **where** $c \in U$ **and** *nc_abc*: $\neg$ *collinear* $\{a,b,c\}$
by (*metis False* ‹$a \in U$› ‹$b \in U$› ‹$\neg$ *collinear U*› *collinear_triples insert_absorb*)
have *ncoll_mca*: $\neg$ *collinear* $\{?m,c,a\}$
by (*metis* (*full_types*) ‹$a \neq ?m$› *collinear_3_trans collinear_midpoint insert_commute nc_abc*)
have *ncoll_mcb*: $\neg$ *collinear* $\{?m,c,b\}$
by (*metis* (*full_types*) ‹$b \neq ?m$› *collinear_3_trans collinear_midpoint insert_commute nc_abc*)
have $c \neq ?m$
by (*metis collinear_midpoint insert_commute nc_abc*)
then have *closed_segment ?m c* $\subseteq$ U
by (*simp add*: ‹$c \in U$› ‹$?m \in U$› ‹*convex U*› *closed_segment_subset*)
then obtain z **where** z: $z \in$ *closed_segment ?m c*
and *disjS*: (*closed_segment a z* $\cup$ *closed_segment z b*) $\cap$ $S = \{\}$
proof $-$
have *False* **if** *closed_segment ?m c* $\subseteq \{z.$ (*closed_segment a z* $\cup$ *closed_segment z b*) $\cap$ $S \neq \{\}\}$
proof $-$
have *closb*: *closed_segment ?m c* $\subseteq$
$\{z \in$ *closed_segment ?m c. closed_segment a z* $\cap$ $S \neq \{\}\}$ $\cup$ $\{z \in$ *closed_segment ?m c. closed_segment z b* $\cap$ $S \neq \{\}\}$
using *that* **by** *blast*
have $*$: *countable* $\{z \in$ *closed_segment ?m c. closed_segment z u* $\cap$ $S \neq \{\}\}$
if $u \in U$ $u \notin S$ **and** *ncoll*: $\neg$ *collinear* $\{?m,\ c,\ u\}$ **for** u
proof $-$
have $**$: *False* **if** *x1*: *x1* $\in$ *closed_segment ?m c* **and** *x2*: *x2* $\in$ *closed_segment ?m c*
and *x1* $\neq$ *x2 x1* $\neq$ u
and w: $w \in$ *closed_segment x1 u* $w \in$ *closed_segment x2 u*
and $w \in S$ **for** *x1 x2 w*
proof $-$
have *x1* $\in$ *affine hull* $\{?m,c\}$ *x2* $\in$ *affine hull* $\{?m,c\}$
using *segment_as_ball x1 x2* **by** *auto*
then have *coll_x1*: *collinear* $\{x1,\ ?m,\ c\}$ **and** *coll_x2*: *collinear* $\{?m,\ c,\ x2\}$
by (*simp_all add*: *affine_hull_3_imp_collinear*) (*metis affine_hull_3_imp_collinear insert_commute*)
have $\neg$ *collinear* $\{x1,\ u,\ x2\}$
proof
assume *collinear* $\{x1,\ u,\ x2\}$
then have *collinear* $\{?m,\ c,\ u\}$
by (*metis* (*full_types*) ‹$c \neq ?m$› *coll_x1 coll_x2 collinear_3_trans insert_commute ncoll* ‹*x1* $\neq$ *x2*›)
with *ncoll* **show** *False* **..**
qed
then have *closed_segment x1 u* $\cap$ *closed_segment u x2* $= \{u\}$
by (*blast intro*!: *Int_closed_segment*)
then have $w = u$

```
            using closed_segment_commute w by auto
           show ?thesis
            using ‹u ∉ S› ‹w = u› that(7) by auto
          qed
         then have disj: disjoint ((⋃z∈closed_segment ?m c. {closed_segment z
u ∩ S}))
           by (fastforce simp: pairwise_def disjnt_def)
         have cou: countable ((⋃z ∈ closed_segment ?m c. {closed_segment z u ∩
S}) − {{}})
           apply (rule pairwise_disjnt_countable_Union [OF _ pairwise_subset
[OF disj]])
           apply (rule countable_subset [OF _ ‹countable S›], auto)
           done
           define f where f ≡ λX. (THE z. z ∈ closed_segment ?m c ∧ X =
closed_segment z u ∩ S)
         show ?thesis
          proof (rule countable_subset [OF _ countable_image [OF cou, where
f=f]], clarify)
           fix x
           assume x: x ∈ closed_segment ?m c closed_segment x u ∩ S ≠ {}
           show x ∈ f ‘ ((⋃z∈closed_segment ?m c. {closed_segment z u ∩ S}) −
{{}})
           proof (rule_tac x=closed_segment x u ∩ S in image_eqI)
            show x = f (closed_segment x u ∩ S)
             unfolding f_def
             by (rule the_equality [symmetric]) (use x in ‹auto dest: ∗∗›)
           qed (use x in auto)
         qed
        qed
        have uncountable (closed_segment ?m c)
         by (metis ‹c ≠ ?m› uncountable_closed_segment)
        then show False
          using closb ∗ [OF ‹a ∈ U› ‹a ∉ S› ncoll_mca] ∗ [OF ‹b ∈ U› ‹b ∉ S›
ncoll_mcb]
         by (simp add: closed_segment_commute countable_subset)
      qed
      then show ?thesis
       by (force intro: that)
    qed
    show ?thesis
    proof (intro exI conjI)
      have path_image (linepath a z +++ linepath z b) ⊆ U
        by (metis ‹a ∈ U› ‹b ∈ U› ‹closed_segment ?m c ⊆ U› z ‹convex U›
closed_segment_subset contra_subsetD path_image_linepath subset_path_image_join)
      with disjS show path_image (linepath a z +++ linepath z b) ⊆ U − S
       by (force simp: path_image_join)
    qed auto
  qed
qed
```

corollary *connected_convex_diff_countable*:
fixes U :: *'a*::*euclidean_space set*
assumes *convex* U ¬ *collinear* U *countable* S
shows *connected*$(U - S)$
by (*simp add*: *assms path_connected_convex_diff_countable path_connected_imp_connected*)

lemma *path_connected_punctured_convex*:
assumes *convex* S **and** *aff*: *aff_dim* $S \neq 1$
shows *path_connected*$(S - \{a\})$
proof −
consider *aff_dim* $S = -1$ | *aff_dim* $S = 0$ | *aff_dim* $S \geq 2$
using *assms aff_dim_geq* [*of* S] **by** *linarith*
then show *?thesis*
proof *cases*
assume *aff_dim* $S = -1$
then show *?thesis*
by (*metis aff_dim_empty empty_Diff path_connected_empty*)
next
assume *aff_dim* $S = 0$
then show *?thesis*
by (*metis aff_dim_eq_0 Diff_cancel Diff_empty Diff_insert0 convex_empty convex_imp_path_connected path_connected_singleton singletonD*)
next
assume *ge2*: *aff_dim* $S \geq 2$
then have ¬ *collinear* S
proof (*clarsimp simp*: *collinear_affine_hull*)
fix u v
assume $S \subseteq$ *affine hull* $\{u, v\}$
then have *aff_dim* $S \leq$ *aff_dim* $\{u, v\}$
by (*metis* (*no_types*) *aff_dim_affine_hull aff_dim_subset*)
with *ge2* **show** *False*
by (*metis* (*no_types*) *aff_dim_2 antisym aff not_numeral_le_zero one_le_numeral order_trans*)
qed
moreover have *countable* $\{a\}$
by *simp*
ultimately show *?thesis*
by (*metis path_connected_convex_diff_countable* [*OF* ‹*convex* S›])
qed
qed

lemma *connected_punctured_convex*:
shows ⟦*convex* S; *aff_dim* $S \neq 1$⟧ ⟹ *connected*$(S - \{a\})$
using *path_connected_imp_connected path_connected_punctured_convex* **by** *blast*

lemma *path_connected_complement_countable*:
fixes S :: *'a*::*euclidean_space set*

```
  assumes 2 ≤ DIM('a) countable S
  shows path_connected(- S)
proof -
  have ¬ collinear (UNIV::'a set)
    using assms by (auto simp: collinear_aff_dim [of UNIV :: 'a set])
  then have path_connected(UNIV - S)
    by (simp add: ‹countable S› path_connected_convex_diff_countable)
  then show ?thesis
    by (simp add: Compl_eq_Diff_UNIV)
qed

proposition path_connected_openin_diff_countable:
  fixes S :: 'a::euclidean_space set
  assumes connected S and ope: openin (top_of_set (affine hull S)) S
      and ¬ collinear S countable T
    shows path_connected(S - T)
proof (clarsimp simp: path_connected_component)
  fix x y
  assume xy: x ∈ S x ∉ T y ∈ S y ∉ T
  show path_component (S - T) x y
  proof (rule connected_equivalence_relation_gen [OF ‹connected S›, where P
= λx. x ∉ T])
    show ∃z. z ∈ U ∧ z ∉ T if opeU: openin (top_of_set S) U and x ∈ U for
U x
    proof -
      have openin (top_of_set (affine hull S)) U
        using opeU ope openin_trans by blast
      with ‹x ∈ U› obtain r where Usub: U ⊆ affine hull S and r > 0
                                  and subU: ball x r ∩ affine hull S ⊆ U
        by (auto simp: openin_contains_ball)
      with ‹x ∈ U› have x: x ∈ ball x r ∩ affine hull S
        by auto
      have ¬ S ⊆ {x}
        using ‹¬ collinear S›  collinear_subset by blast
      then obtain x' where x' ≠ x x' ∈ S
        by blast
      obtain y where y: y ≠ x y ∈ ball x r ∩ affine hull S
      proof
        show x + (r / 2 / norm(x' - x)) *R (x' - x) ≠ x
          using ‹x' ≠ x› ‹r > 0› by auto
        show x + (r / 2 / norm (x' - x)) *R (x' - x) ∈ ball x r ∩ affine hull S
          using ‹x' ≠ x› ‹r > 0› ‹x' ∈ S› x
          by (simp add: dist_norm mem_affine_3_minus hull_inc)
      qed
      have convex (ball x r ∩ affine hull S)
        by (simp add: affine_imp_convex convex_Int)
      with x y subU have uncountable U
        by (meson countable_subset uncountable_convex)
      then have ¬ U ⊆ T
```

```
      using ‹countable T› countable_subset by blast
    then show ?thesis by blast
  qed
  show ∃U. openin (top_of_set S) U ∧ x ∈ U ∧
        (∀x∈U. ∀y∈U. x ∉ T ∧ y ∉ T ⟶ path_component (S − T) x y)
      if x ∈ S for x
  proof −
    obtain r where Ssub: S ⊆ affine hull S and r > 0
          and subS: ball x r ∩ affine hull S ⊆ S
      using ope ‹x ∈ S› by (auto simp: openin_contains_ball)
    then have conv: convex (ball x r ∩ affine hull S)
      by (simp add: affine_imp_convex convex_Int)
    have ¬ aff_dim (affine hull S) ≤ 1
      using ‹¬ collinear S› collinear_aff_dim by auto
    then have ¬ aff_dim (ball x r ∩ affine hull S) ≤ 1
    by (metis (no_types, opaque_lifting) aff_dim_convex_Int_open IntI open_ball
‹0 < r› aff_dim_affine_hull affine_affine_hull affine_imp_convex centre_in_ball
empty_iff hull_subset inf_commute subsetCE that)
    then have ¬ collinear (ball x r ∩ affine hull S)
      by (simp add: collinear_aff_dim)
    then have ∗: path_connected ((ball x r ∩ affine hull S) − T)
     by (rule path_connected_convex_diff_countable [OF conv _ ‹countable T›])
    have ST: ball x r ∩ affine hull S − T ⊆ S − T
      using subS by auto
    show ?thesis
    proof (intro exI conjI)
      show x ∈ ball x r ∩ affine hull S
        using ‹x ∈ S› ‹r > 0› by (simp add: hull_inc)
      have openin (top_of_set (affine hull S)) (ball x r ∩ affine hull S)
        by (subst inf.commute) (simp add: openin_Int_open)
      then show openin (top_of_set S) (ball x r ∩ affine hull S)
        by (rule openin_subset_trans [OF _ subS Ssub])
    qed (use ∗ path_component_trans in ‹auto simp: path_connected_component
path_component_of_subset [OF ST]›)
  qed
 qed (use xy path_component_trans in auto)
qed

corollary connected_openin_diff_countable:
 fixes S :: 'a::euclidean_space set
 assumes connected S and ope: openin (top_of_set (affine hull S)) S
     and ¬ collinear S countable T
   shows connected(S − T)
 by (metis path_connected_imp_connected path_connected_openin_diff_countable
[OF assms])

corollary path_connected_open_diff_countable:
 fixes S :: 'a::euclidean_space set
 assumes 2 ≤ DIM('a) open S connected S countable T
```

shows *path_connected*$(S - T)$
proof (*cases* $S = \{\}$)
case *True*
then show *?thesis*
by (*simp*)
next
case *False*
show *?thesis*
proof (*rule path_connected_openin_diff_countable*)
show *openin* (*top_of_set* (*affine hull S*)) *S*
by (*simp add*: *assms hull_subset open_subset*)
show ¬ *collinear S*
using *assms False* **by** (*simp add*: *collinear_aff_dim aff_dim_open*)
qed (*simp_all add*: *assms*)
qed

corollary *connected_open_diff_countable*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $2 \leq DIM('a)$ *open S connected S countable T*
shows *connected*$(S - T)$
by (*simp add*: *assms path_connected_imp_connected path_connected_open_diff_countable*)

6.4.27 Self-homeomorphisms shuffling points about

The theorem *homeomorphism_moving_points_exists*

lemma *homeomorphism_moving_point_1*:
fixes $a :: 'a::euclidean\_space$
assumes *affine T a* $\in T$ **and** u: $u \in ball\ a\ r \cap T$
obtains *f g* **where** *homeomorphism* (*cball a r* $\cap$ *T*) (*cball a r* $\cap$ *T*) *f g*
$f\ a = u\ \bigwedge x.\ x \in sphere\ a\ r \Longrightarrow f\ x = x$
proof −
have *nou*: *norm* $(u - a) < r$ **and** $u \in T$
using *u* **by** (*auto simp*: *dist_norm norm_minus_commute*)
then have $0 < r$
by (*metis DiffD1 Diff_Diff_Int ball_eq_empty centre_in_ball not_le u*)
define *f* **where** $f \equiv \lambda x.\ (1 - norm(x - a) / r) *_R (u - a) + x$
have $*$: *False* **if** *eq*: $x + (norm\ y / r) *_R u = y + (norm\ x / r) *_R u$
and *nou*: *norm u* $< r$ **and** *yx*: *norm y* $<$ *norm x* **for** *x y* **and** $u::'a$
proof −
have $x = y + (norm\ x / r - (norm\ y / r)) *_R u$
using *eq* **by** (*simp add*: *algebra_simps*)
then have *norm x* = *norm* $(y + ((norm\ x - norm\ y) / r) *_R u)$
by (*metis diff_divide_distrib*)
also have $\ldots \leq norm\ y + norm(((norm\ x - norm\ y) / r) *_R u)$
using *norm_triangle_ineq* **by** *blast*
also have $\ldots = norm\ y + (norm\ x - norm\ y) * (norm\ u / r)$
using *yx* ‹$r > 0$›
by (*simp add*: *field_split_simps*)
also have $\ldots < norm\ y + (norm\ x - norm\ y) * 1$

```
    proof (subst add_less_cancel_left)
      show (norm x − norm y) ∗ (norm u / r) < (norm x − norm y) ∗ 1
      proof (rule mult_strict_left_mono)
        show norm u / r < 1
          using ‹0 < r› divide_less_eq_1_pos nou by blast
      qed (simp add: yx)
    qed
    also have ... = norm x
      by simp
    finally show False by simp
  qed
  have inj f
    unfolding f_def
  proof (clarsimp simp: inj_on_def)
    fix x y
    assume (1 − norm (x − a) / r) ∗R (u − a) + x =
           (1 − norm (y − a) / r) ∗R (u − a) + y
    then have eq: (x − a) + (norm (y − a) / r) ∗R (u − a) = (y − a) + (norm
(x − a) / r) ∗R (u − a)
      by (auto simp: algebra_simps)
    show x=y
    proof (cases norm (x − a) = norm (y − a))
      case True
      then show ?thesis
        using eq by auto
    next
      case False
      then consider norm (x − a) < norm (y − a) | norm (x − a) > norm (y −
a)
        by linarith
      then have False
      proof cases
        case 1 show False
          using ∗ [OF _ nou 1] eq by simp
      next
        case 2 with ∗ [OF eq nou] show False
          by auto
      qed
      then show x=y ..
    qed
  qed
  then have inj_onf: inj_on f (cball a r ∩ T)
    using inj_on_Int by fastforce
  have contf: continuous_on (cball a r ∩ T) f
    unfolding f_def using ‹0 < r› by (intro continuous_intros) blast
  have fim: f ‘ (cball a r ∩ T) = cball a r ∩ T
  proof
    have ∗: norm (y + (1 − norm y / r) ∗R u) ≤ r if norm y ≤ r norm u < r
for y u::'a
```

```
    proof −
      have norm (y + (1 − norm y / r) ∗R u) ≤ norm y + norm((1 − norm y /
r) ∗R u)
        using norm_triangle_ineq by blast
      also have ... = norm y + abs(1 − norm y / r) ∗ norm u
        by simp
      also have ... ≤ r
      proof −
        have (r − norm u) ∗ (r − norm y) ≥ 0
          using that by auto
        then have r ∗ norm u + r ∗ norm y ≤ r ∗ r + norm u ∗ norm y
          by (simp add: algebra_simps)
        then show ?thesis
        using that ‹0 < r› by (simp add: abs_if field_simps)
      qed
      finally show ?thesis .
    qed
    have f ‘ (cball a r) ⊆ cball a r
      using ∗ nou
      apply (clarsimp simp: dist_norm norm_minus_commute f_def)
      by (metis diff_add_eq diff_diff_add diff_diff_eq2 norm_minus_commute)
    moreover have f ‘ T ⊆ T
      unfolding f_def using ‹affine T› ‹a ∈ T› ‹u ∈ T›
      by (force simp: add.commute mem_affine_3_minus)
    ultimately show f ‘ (cball a r ∩ T) ⊆ cball a r ∩ T
      by blast
  next
    show cball a r ∩ T ⊆ f ‘ (cball a r ∩ T)
    proof (clarsimp simp: dist_norm norm_minus_commute)
      fix x
      assume x: norm (x − a) ≤ r and x ∈ T
      have ∃ v ∈ {0..1}. ((1 − v) ∗ r − norm ((x − a) − v ∗R (u − a))) · 1 = 0
       by (rule ivt_decreasing_component_on_1) (auto simp: x continuous_intros)
      then obtain v where 0 ≤ v v ≤ 1
        and v: (1 − v) ∗ r = norm ((x − a) − v ∗R (u − a))
        by auto
      then have n: norm (a − (x − v ∗R (u − a))) = r − r ∗ v
        by (simp add: field_simps norm_minus_commute)
      show x ∈ f ‘ (cball a r ∩ T)
      proof (rule image_eqI)
        show x = f (x − v ∗R (u − a))
          using ‹r > 0› v by (simp add: f_def) (simp add: field_simps)
        have x − v ∗R (u − a) ∈ cball a r
          using ‹r > 0›‹0 ≤ v›
          by (simp add: dist_norm n)
        moreover have x − v ∗R (u − a) ∈ T
          by (simp add: f_def ‹u ∈ T› ‹x ∈ T› assms mem_affine_3_minus2)
        ultimately show x − v ∗R (u − a) ∈ cball a r ∩ T
          by blast
```

```
    qed
   qed
  qed
  have compact (cball a r ∩ T)
   by (simp add: affine_closed compact_Int_closed ‹affine T›)
  then obtain g where homeomorphism (cball a r ∩ T) (cball a r ∩ T) f g
   by (metis homeomorphism_compact [OF _ contf fim inj_onf])
  then show thesis
   apply (rule_tac f=f in that)
   using ‹r > 0› by (simp_all add: f_def dist_norm norm_minus_commute)
qed

corollary homeomorphism_moving_point_2:
  fixes a :: 'a::euclidean_space
  assumes affine T a ∈ T and u: u ∈ ball a r ∩ T and v: v ∈ ball a r ∩ T
  obtains f g where homeomorphism (cball a r ∩ T) (cball a r ∩ T) f g
                    f u = v ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ f x = x
proof −
  have 0 < r
   by (metis DiffD1 Diff_Diff_Int ball_eq_empty centre_in_ball not_le u)
  obtain f1 g1 where hom1: homeomorphism (cball a r ∩ T) (cball a r ∩ T) f1
g1
                and f1 a = u and f1: ⋀x. x ∈ sphere a r ⟹ f1 x = x
   using homeomorphism_moving_point_1 [OF ‹affine T› ‹a ∈ T› u] by blast
  obtain f2 g2 where hom2: homeomorphism (cball a r ∩ T) (cball a r ∩ T) f2
g2
                and f2 a = v and f2: ⋀x. x ∈ sphere a r ⟹ f2 x = x
   using homeomorphism_moving_point_1 [OF ‹affine T› ‹a ∈ T› v] by blast
  show ?thesis
  proof
   show homeomorphism (cball a r ∩ T) (cball a r ∩ T) (f2 ∘ g1) (f1 ∘ g2)
    by (metis homeomorphism_compose homeomorphism_symD hom1 hom2)
   have g1 u = a
    using ‹0 < r› ‹f1 a = u› assms hom1 homeomorphism_apply1 by fastforce
   then show (f2 ∘ g1) u = v
    by (simp add: ‹f2 a = v›)
   show ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ (f2 ∘ g1) x = x
    using f1 f2 hom1 homeomorphism_apply1 by fastforce
  qed
qed


corollary homeomorphism_moving_point_3:
  fixes a :: 'a::euclidean_space
  assumes affine T a ∈ T and ST: ball a r ∩ T ⊆ S S ⊆ T
     and u: u ∈ ball a r ∩ T and v: v ∈ ball a r ∩ T
  obtains f g where homeomorphism S S f g
                    f u = v {x. ¬ (f x = x ∧ g x = x)} ⊆ ball a r ∩ T
proof −
```

```
  obtain f g where hom: homeomorphism (cball a r ∩ T) (cball a r ∩ T) f g
                and f u = v and fid: ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ f x = x
    using homeomorphism_moving_point_2 [OF ‹affine T› ‹a ∈ T› u v] by blast
  have gid: ⋀x. ⟦x ∈ sphere a r; x ∈ T⟧ ⟹ g x = x
    using fid hom homeomorphism_apply1 by fastforce
  define ff where ff ≡ λx. if x ∈ ball a r ∩ T then f x else x
  define gg where gg ≡ λx. if x ∈ ball a r ∩ T then g x else x
  show ?thesis
  proof
    show homeomorphism S S ff gg
    proof (rule homeomorphismI)
      have continuous_on ((cball a r ∩ T) ∪ (T - ball a r)) ff
        unfolding ff_def
        using homeomorphism_cont1 [OF hom]
        by (intro continuous_on_cases) (auto simp: affine_closed ‹affine T› fid)
      then show continuous_on S ff
        by (rule continuous_on_subset) (use ST in auto)
      have continuous_on ((cball a r ∩ T) ∪ (T - ball a r)) gg
        unfolding gg_def
        using homeomorphism_cont2 [OF hom]
        by (intro continuous_on_cases) (auto simp: affine_closed ‹affine T› gid)
      then show continuous_on S gg
        by (rule continuous_on_subset) (use ST in auto)
      show ff ‘ S ⊆ S
      proof (clarsimp simp: ff_def)
        fix x
        assume x ∈ S and x: dist a x < r and x ∈ T
        then have f x ∈ cball a r ∩ T
          using homeomorphism_image1 [OF hom] by force
        then show f x ∈ S
          using ST(1) ‹x ∈ T› gid hom homeomorphism_def x by fastforce
      qed
      show gg ‘ S ⊆ S
      proof (clarsimp simp: gg_def)
        fix x
        assume x ∈ S and x: dist a x < r and x ∈ T
        then have g x ∈ cball a r ∩ T
          using homeomorphism_image2 [OF hom] by force
        then have g x ∈ ball a r
          using homeomorphism_apply2 [OF hom]
                by (metis Diff_Diff_Int Diff_iff ‹x ∈ T› cball_def fid le_less
mem_Collect_eq mem_ball mem_sphere x)
        then show g x ∈ S
          using ST(1) ‹g x ∈ cball a r ∩ T› by force
        qed
      show ⋀x. x ∈ S ⟹ gg (ff x) = x
        unfolding ff_def gg_def
          using homeomorphism_apply1 [OF hom] homeomorphism_image1 [OF
hom]
```

```
        by simp (metis Int_iff homeomorphism_apply1 [OF hom] fid image_eqI
less_eq_real_def mem_cball mem_sphere)
      show ⋀x. x ∈ S ⟹ ff (gg x) = x
        unfolding ff_def gg_def
          using homeomorphism_apply2 [OF hom] homeomorphism_image2 [OF
hom]
        by simp (metis Int_iff fid image_eqI less_eq_real_def mem_cball mem_sphere)
    qed
    show ff u = v
      using u by (auto simp: ff_def ‹f u = v›)
    show {x. ¬ (ff x = x ∧ gg x = x)} ⊆ ball a r ∩ T
      by (auto simp: ff_def gg_def)
  qed
qed


proposition homeomorphism_moving_point:
  fixes a :: 'a::euclidean_space
  assumes ope: openin (top_of_set (affine hull S)) S
      and S ⊆ T
      and TS: T ⊆ affine hull S
      and S: connected S a ∈ S b ∈ S
  obtains f g where homeomorphism T T f g f a = b
                    {x. ¬ (f x = x ∧ g x = x)} ⊆ S
                    bounded {x. ¬ (f x = x ∧ g x = x)}
proof −
  have 1: ∃h k. homeomorphism T T h k ∧ h (f d) = d ∧
            {x. ¬ (h x = x ∧ k x = x)} ⊆ S ∧ bounded {x. ¬ (h x = x ∧ k x = x)}
       if d ∈ S f d ∈ S and homfg: homeomorphism T T f g
       and S: {x. ¬ (f x = x ∧ g x = x)} ⊆ S
       and bo: bounded {x. ¬ (f x = x ∧ g x = x)} for d f g
  proof (intro exI conjI)
    show homgf: homeomorphism T T g f
      by (metis homeomorphism_symD homfg)
    then show g (f d) = d
      by (meson ‹S ⊆ T› homeomorphism_def subsetD ‹d ∈ S›)
    show {x. ¬ (g x = x ∧ f x = x)} ⊆ S
      using S by blast
    show bounded {x. ¬ (g x = x ∧ f x = x)}
      using bo by (simp add: conj_commute)
  qed
  have 2: ∃f g. homeomorphism T T f g ∧ f x = f2 (f1 x) ∧
              {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧ bounded {x. ¬ (f x = x ∧ g x =
x)}
          if x ∈ S f1 x ∈ S f2 (f1 x) ∈ S
             and hom: homeomorphism T T f1 g1 homeomorphism T T f2 g2
             and sub: {x. ¬ (f1 x = x ∧ g1 x = x)} ⊆ S  {x. ¬ (f2 x = x ∧ g2 x
= x)} ⊆ S
             and bo: bounded {x. ¬ (f1 x = x ∧ g1 x = x)}  bounded {x. ¬ (f2 x
```

```
= x ∧ g2 x = x)}
            for x f1 f2 g1 g2
  proof (intro exI conjI)
    show homgf: homeomorphism T T (f2 ∘ f1) (g1 ∘ g2)
      by (metis homeomorphism_compose hom)
    then show (f2 ∘ f1) x = f2 (f1 x)
      by force
    show {x. ¬ ((f2 ∘ f1) x = x ∧ (g1 ∘ g2) x = x)} ⊆ S
      using sub by force
    have bounded ({x. ¬(f1 x = x ∧ g1 x = x)} ∪ {x. ¬(f2 x = x ∧ g2 x = x)})
      using bo by simp
    then show bounded {x. ¬ ((f2 ∘ f1) x = x ∧ (g1 ∘ g2) x = x)}
      by (rule bounded_subset) auto
  qed
  have 3: ∃ U. openin (top_of_set S) U ∧
              d ∈ U ∧
              (∀ x∈U.
                  ∃f g. homeomorphism T T f g ∧ f d = x ∧
                        {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧
                        bounded {x. ¬ (f x = x ∧ g x = x)})
           if d ∈ S for d
  proof −
    obtain r where r > 0 and r: ball d r ∩ affine hull S ⊆ S
      by (metis ‹d ∈ S› ope openin_contains_ball)
    have ∗: ∃f g. homeomorphism T T f g ∧ f d = e ∧
                  {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧
                  bounded {x. ¬ (f x = x ∧ g x = x)} if e ∈ S e ∈ ball d r for e
      apply (rule homeomorphism_moving_point_3 [of affine hull S d r T d e])
      using r ‹S ⊆ T› TS that
          apply (auto simp: ‹d ∈ S› ‹0 < r› hull_inc)
      using bounded_subset by blast
    show ?thesis
      by (rule_tac x=S ∩ ball d r in exI) (fastforce simp: openin_open_Int ‹0 <
r› that intro: ∗)
  qed
  have ∃f g. homeomorphism T T f g ∧ f a = b ∧
             {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧ bounded {x. ¬ (f x = x ∧ g x = x)}
    by (rule connected_equivalence_relation [OF S]; blast intro: 1 2 3)
  then show ?thesis
    using that by auto
qed


lemma homeomorphism_moving_points_exists_gen:
  assumes K: finite K ⋀i. i ∈ K ⟹ x i ∈ S ∧ y i ∈ S
           pairwise (λi j. (x i ≠ x j) ∧ (y i ≠ y j)) K
      and 2 ≤ aff_dim S
      and ope: openin (top_of_set (affine hull S)) S
      and S ⊆ T T ⊆ affine hull S connected S
```

```
  shows ∃f g. homeomorphism T T f g ∧ (∀i∈K. f(x i) = y i) ∧
              {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧ bounded {x. ¬ (f x = x ∧ g x = x)}
  using assms
proof (induction K)
  case empty
  then show ?case
    by (force simp: homeomorphism_ident)
next
  case (insert i K)
  then have xney: ⋀j. ⟦j ∈ K; j ≠ i⟧ ⟹ x i ≠ x j ∧ y i ≠ y j
      and pw: pairwise (λi j. x i ≠ x j ∧ y i ≠ y j) K
      and x i ∈ S y i ∈ S
      and xyS: ⋀i. i ∈ K ⟹ x i ∈ S ∧ y i ∈ S
    by (simp_all add: pairwise_insert)
  obtain f g where homfg: homeomorphism T T f g and feq: ⋀i. i ∈ K ⟹ f(x
i) = y i
              and fg_sub: {x. ¬ (f x = x ∧ g x = x)} ⊆ S
              and bo_fg: bounded {x. ¬ (f x = x ∧ g x = x)}
    using insert.IH [OF xyS pw] insert.prems by (blast intro: that)
  then have ∃f g. homeomorphism T T f g ∧ (∀i∈K. f(x i) = y i) ∧
                {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧ bounded {x. ¬ (f x = x ∧ g x =
x)}
    using insert by blast
  have aff_eq: affine hull (S - y ` K) = affine hull S
  proof (rule affine_hull_Diff [OF ope])
    show finite (y ` K)
      by (simp add: insert.hyps(1))
    show y ` K ⊂ S
      using ‹y i ∈ S› insert.hyps(2) xney xyS by fastforce
  qed
  have f_in_S: f x ∈ S if x ∈ S for x
    using homfg fg_sub homeomorphism_apply1 ‹S ⊆ T›
  proof −
    have (f (f x) ≠ f x ∨ g (f x) ≠ f x) ∨ f x ∈ S
      by (metis ‹S ⊆ T› homfg subsetD homeomorphism_apply1 that)
    then show ?thesis
      using fg_sub by force
  qed
  obtain h k where homhk: homeomorphism T T h k and heq: h (f (x i)) = y i
              and hk_sub: {x. ¬ (h x = x ∧ k x = x)} ⊆ S - y ` K
              and bo_hk:  bounded {x. ¬ (h x = x ∧ k x = x)}
  proof (rule homeomorphism_moving_point [of S - y`K T f(x i) y i])
    show openin (top_of_set (affine hull (S - y ` K))) (S - y ` K)
        by (simp add: aff_eq openin_diff finite_imp_closedin image_subset_iff
hull_inc insert xyS)
    show S - y ` K ⊆ T
      using ‹S ⊆ T› by auto
    show T ⊆ affine hull (S - y ` K)
      using insert by (simp add: aff_eq)
```

```
    show connected (S − y ‘ K)
    proof (rule connected_openin_diff_countable [OF ‹connected S› ope])
      show ¬ collinear S
        using collinear_aff_dim ‹2 ≤ aff_dim S› by force
      show countable (y ‘ K)
        using countable_finite insert.hyps(1) by blast
    qed
    have ⋀k. ⟦f (x i) = y k; k ∈ K⟧ ⟹ False
          by (metis feq homfg ‹x i ∈ S› homeomorphism_def ‹S ⊆ T› ‹i ∉ K›
subsetCE xney xyS)
    then show f (x i) ∈ S − y ‘ K
      by (auto simp: f_in_S ‹x i ∈ S›)
    show y i ∈ S − y ‘ K
      using insert.hyps xney by (auto simp: ‹y i ∈ S›)
  qed blast
  show ?case
  proof (intro exI conjI)
    show homeomorphism T T (h ∘ f) (g ∘ k)
      using homfg homhk homeomorphism_compose by blast
    show ∀ i ∈ insert i K. (h ∘ f) (x i) = y i
      using feq hk_sub by (auto simp: heq)
    show {x. ¬ ((h ∘ f) x = x ∧ (g ∘ k) x = x)} ⊆ S
      using fg_sub hk_sub by force
    have bounded ({x. ¬(f x = x ∧ g x = x)} ∪ {x. ¬(h x = x ∧ k x = x)})
      using bo_fg bo_hk bounded_Un by blast
    then show bounded {x. ¬ ((h ∘ f) x = x ∧ (g ∘ k) x = x)}
      by (rule bounded_subset) auto
  qed
qed

proposition homeomorphism_moving_points_exists:
  fixes S :: 'a::euclidean_space set
  assumes 2: 2 ≤ DIM('a) open S connected S S ⊆ T finite K
      and KS: ⋀i. i ∈ K ⟹ x i ∈ S ∧ y i ∈ S
      and pw: pairwise (λi j. (x i ≠ x j) ∧ (y i ≠ y j)) K
      and S: S ⊆ T T ⊆ affine hull S connected S
  obtains f g where homeomorphism T T f g ⋀i. i ∈ K ⟹ f(x i) = y i
                    {x. ¬ (f x = x ∧ g x = x)} ⊆ S bounded {x. (¬ (f x = x ∧ g x =
x))}
proof (cases S = {})
  case True
  then show ?thesis
    using KS homeomorphism_ident that by fastforce
next
  case False
  then have affS: affine hull S = UNIV
    by (simp add: affine_hull_open ‹open S›)
  then have ope: openin (top_of_set (affine hull S)) S
    using ‹open S› open_openin by auto
```

have $2 \leq DIM('a)$ **by** (*rule 2*)
also have $\ldots = aff\_dim\ (UNIV :: 'a\ set)$
by *simp*
also have $\ldots \leq aff\_dim\ S$
by (*metis aff_dim_UNIV aff_dim_affine_hull aff_dim_le_DIM affS*)
finally have $2 \leq aff\_dim\ S$
by *linarith*
then show *?thesis*
using *homeomorphism_moving_points_exists_gen* [*OF* ‹*finite K*› *KS pw* _ *ope S*] *that* **by** *fastforce*
qed

The theorem *homeomorphism_grouping_points_exists*

lemma *homeomorphism_grouping_point_1*:
fixes *a*::*real* **and** *c*::*real*
assumes $a < b\ c < d$
obtains *f g* **where** *homeomorphism* (*cbox a b*) (*cbox c d*) *f g f a* $= c$ *f b* $= d$
proof −
define *f* **where** $f \equiv \lambda x.\ ((d - c)\ /\ (b - a)) * x + (c - a * ((d - c)\ /\ (b - a)))$
have $\exists g.$ *homeomorphism* (*cbox a b*) (*cbox c d*) *f g*
proof (*rule homeomorphism_compact*)
show *continuous_on* (*cbox a b*) *f*
unfolding *f_def* **by** (*intro continuous_intros*)
have $f\ `\ \{a..b\} = \{c..d\}$
unfolding *f_def image_affinity_atLeastAtMost*
using *assms sum_sqs_eq* **by** (*auto simp*: *field_split_simps*)
then show *f* ‘ *cbox a b* $=$ *cbox c d*
by *auto*
show *inj_on f* (*cbox a b*)
unfolding *f_def inj_on_def* **using** *assms* **by** *auto*
qed *auto*
then obtain *g* **where** *homeomorphism* (*cbox a b*) (*cbox c d*) *f g* **..**
then show *?thesis*
proof
show $f\ a = c$
by (*simp add*: *f_def*)
show $f\ b = d$
using *assms sum_sqs_eq* [*of a b*] **by** (*auto simp*: *f_def field_split_simps*)
qed
qed

lemma *homeomorphism_grouping_point_2*:
fixes *a*::*real* **and** *w*::*real*
assumes *hom_ab*: *homeomorphism* (*cbox a b*) (*cbox u v*) *f1 g1*
and *hom_bc*: *homeomorphism* (*cbox b c*) (*cbox v w*) *f2 g2*
and $b \in cbox\ a\ c\ v \in cbox\ u\ w$
and *eq*: *f1 a* $= u$ *f1 b* $= v$ *f2 b* $= v$ *f2 c* $= w$

obtains *f g* **where** *homeomorphism* (*cbox a c*) (*cbox u w*) *f g f a* = *u f c* = *w*
⋀*x*. *x* ∈ *cbox a b* ⟹ *f x* = *f1 x* ⋀*x*. *x* ∈ *cbox b c* ⟹ *f x* = *f2 x*
proof −
have *le*: *a* ≤ *b b* ≤ *c u* ≤ *v v* ≤ *w*
using *assms* **by** *simp_all*
then have *ac*: *cbox a c* = *cbox a b* ∪ *cbox b c* **and** *uw*: *cbox u w* = *cbox u v* ∪ *cbox v w*
by *auto*
define *f* **where** *f* ≡ λ*x*. *if x* ≤ *b then f1 x else f2 x*
have ∃ *g*. *homeomorphism* (*cbox a c*) (*cbox u w*) *f g*
proof (*rule homeomorphism_compact*)
have *cf1*: *continuous_on* (*cbox a b*) *f1*
using *hom_ab homeomorphism_cont1* **by** *blast*
have *cf2*: *continuous_on* (*cbox b c*) *f2*
using *hom_bc homeomorphism_cont1* **by** *blast*
show *continuous_on* (*cbox a c*) *f*
unfolding *f_def* **using** *le eq*
by (*force intro*: *continuous_on_cases_le* [*OF continuous_on_subset* [*OF cf1*] *continuous_on_subset* [*OF cf2*]])
have *f* ‘ *cbox a b* = *f1* ‘ *cbox a b f* ‘ *cbox b c* = *f2* ‘ *cbox b c*
unfolding *f_def* **using** *eq* **by** *force*+
then show *f* ‘ *cbox a c* = *cbox u w*
unfolding *ac uw image_Un* **by** (*metis hom_ab hom_bc homeomorphism_def*)
have *neq12*: *f1 x* ≠ *f2 y* **if** *x*: *a* ≤ *x x* ≤ *b* **and** *y*: *b* < *y y* ≤ *c* **for** *x y*
proof −
have *f1 x* ∈ *cbox u v*
by (*metis hom_ab homeomorphism_def image_eqI mem_box_real*(2) *x*)
moreover have *f2 y* ∈ *cbox v w*
by (*metis* (*full_types*) *hom_bc homeomorphism_def image_subset_iff mem_box_real*(2) *not_le not_less_iff_gr_or_eq order_refl y*)
moreover have *f2 y* ≠ *f2 b*
by (*metis cancel_comm_monoid_add_class.diff_cancel diff_gt_0_iff_gt hom_bc homeomorphism_def le*(2) *less_imp_le less_numeral_extra*(3) *mem_box_real*(2) *order_refl y*)
ultimately show *?thesis*
using *le eq* **by** *simp*
qed
have *inj_on f1* (*cbox a b*)
by (*metis* (*full_types*) *hom_ab homeomorphism_def inj_onI*)
moreover have *inj_on f2* (*cbox b c*)
by (*metis* (*full_types*) *hom_bc homeomorphism_def inj_onI*)
ultimately show *inj_on f* (*cbox a c*)
apply (*simp* (*no_asm*) *add*: *inj_on_def*)
apply (*simp add*: *f_def inj_on_eq_iff*)
using *neq12* **by** *force*
qed *auto*
then obtain *g* **where** *homeomorphism* (*cbox a c*) (*cbox u w*) *f g* **..**
then show *?thesis*
using *eq f_def le that* **by** *force*

qed

lemma *homeomorphism_grouping_point_3*:
fixes *a*::*real*
assumes *cbox_sub*: *cbox c d* ⊆ *box a b cbox u v* ⊆ *box a b*
and *box_ne*: *box c d* ≠ {} *box u v* ≠ {}
obtains *f g* **where** *homeomorphism* (*cbox a b*) (*cbox a b*) *f g f a* = *a f b* = *b*
⋀*x*. *x* ∈ *cbox c d* ⟹ *f x* ∈ *cbox u v*
proof −
have *less*: *a* < *c a* < *u d* < *b v* < *b c* < *d u* < *v cbox c d* ≠ {}
using *assms*
by (*simp_all add*: *cbox_sub subset_eq*)
obtain *f1 g1* **where** *1*: *homeomorphism* (*cbox a c*) (*cbox a u*) *f1 g1*
and *f1_eq*: *f1 a* = *a f1 c* = *u*
using *homeomorphism_grouping_point_1* [*OF* ‹*a* < *c*› ‹*a* < *u*›] **.**
obtain *f2 g2* **where** *2*: *homeomorphism* (*cbox c d*) (*cbox u v*) *f2 g2*
and *f2_eq*: *f2 c* = *u f2 d* = *v*
using *homeomorphism_grouping_point_1* [*OF* ‹*c* < *d*› ‹*u* < *v*›] **.**
obtain *f3 g3* **where** *3*: *homeomorphism* (*cbox d b*) (*cbox v b*) *f3 g3*
and *f3_eq*: *f3 d* = *v f3 b* = *b*
using *homeomorphism_grouping_point_1* [*OF* ‹*d* < *b*› ‹*v* < *b*›] **.**
obtain *f4 g4* **where** *4*: *homeomorphism* (*cbox a d*) (*cbox a v*) *f4 g4* **and** *f4 a* = *a f4 d* = *v*
and *f4_eq*: ⋀*x*. *x* ∈ *cbox a c* ⟹ *f4 x* = *f1 x* ⋀*x*. *x* ∈ *cbox c d* ⟹ *f4 x* = *f2 x*
using *homeomorphism_grouping_point_2* [*OF 1 2*] *less* **by** (*auto simp*: *f1_eq f2_eq*)
obtain *f g* **where** *fg*: *homeomorphism* (*cbox a b*) (*cbox a b*) *f g f a* = *a f b* = *b*
and *f_eq*: ⋀*x*. *x* ∈ *cbox a d* ⟹ *f x* = *f4 x* ⋀*x*. *x* ∈ *cbox d b* ⟹ *f x* = *f3 x*
using *homeomorphism_grouping_point_2* [*OF 4 3*] *less* **by** (*auto simp*: *f4_eq f3_eq f2_eq f1_eq*)
show *?thesis*
proof (*rule that* [*OF fg*])
show *f x* ∈ *cbox u v* **if** *x* ∈ *cbox c d* **for** *x*
using *that f4_eq f_eq homeomorphism_image1* [*OF 2*]
by (*metis atLeastAtMost_iff box_real*(2) *image_eqI less*(1) *less_eq_real_def order_trans*)
qed
qed

lemma *homeomorphism_grouping_point_4*:
fixes *T* :: *real set*
assumes *open U open S connected S U* ≠ {} *finite K K* ⊆ *S U* ⊆ *S S* ⊆ *T*
obtains *f g* **where** *homeomorphism T T f g*
⋀*x*. *x* ∈ *K* ⟹ *f x* ∈ *U* {*x*. (¬ (*f x* = *x* ∧ *g x* = *x*))} ⊆ *S*
bounded {*x*. (¬ (*f x* = *x* ∧ *g x* = *x*))}
proof −

```
  obtain c d where box c d ≠ {} cbox c d ⊆ U
  proof −
    obtain u where u ∈ U
      using ‹U ≠ {}› by blast
    then obtain e where e > 0 cball u e ⊆ U
      using ‹open U› open_contains_cball by blast
    then show ?thesis
      by (rule_tac c=u and d=u+e in that) (auto simp: dist_norm subset_iff)
  qed
  have compact K
    by (simp add: ‹finite K› finite_imp_compact)
  obtain a b where box a b ≠ {} K ⊆ cbox a b cbox a b ⊆ S
  proof (cases K = {})
    case True then show ?thesis
      using ‹box c d ≠ {}› ‹cbox c d ⊆ U› ‹U ⊆ S› that by blast
  next
    case False
    then obtain a b where a ∈ K b ∈ K
          and a: ⋀x. x ∈ K ⟹ a ≤ x and b: ⋀x. x ∈ K ⟹ x ≤ b
      using compact_attains_inf compact_attains_sup by (metis ‹compact K›)+
    obtain e where e > 0 cball b e ⊆ S
      using ‹open S› open_contains_cball
      by (metis ‹b ∈ K› ‹K ⊆ S› subsetD)
    show ?thesis
    proof
      show box a (b + e) ≠ {}
        using ‹0 < e› ‹b ∈ K› a by force
      show K ⊆ cbox a (b + e)
        using ‹0 < e› a b by fastforce
      have a ∈ S
        using ‹a ∈ K› assms(6) by blast
      have b + e ∈ S
        using ‹0 < e› ‹cball b e ⊆ S›  by (force simp: dist_norm)
      show cbox a (b + e) ⊆ S
        using ‹a ∈ S› ‹b + e ∈ S› ‹connected S› connected_contains_Icc by auto
    qed
  qed
  obtain w z where cbox w z ⊆ S and sub_wz: cbox a b ∪ cbox c d ⊆ box w z
  proof −
    have a ∈ S b ∈ S
      using ‹box a b ≠ {}› ‹cbox a b ⊆ S› by auto
    moreover have c ∈ S d ∈ S
      using ‹box c d ≠ {}› ‹cbox c d ⊆ U› ‹U ⊆ S› by force+
    ultimately have min a c ∈ S max b d ∈ S
      by linarith+
    then obtain e1 e2 where e1 > 0 cball (min a c) e1 ⊆ S e2 > 0 cball (max
b d) e2 ⊆ S
      using ‹open S› open_contains_cball by metis
    then have ∗: min a c − e1 ∈ S max b d + e2 ∈ S
```

```
    by (auto simp: dist_norm)
  show ?thesis
  proof
    show cbox (min a c − e1) (max b d+ e2) ⊆ S
      using ∗ ‹connected S› connected_contains_Icc by auto
    show cbox a b ∪ cbox c d ⊆ box (min a c − e1) (max b d + e2)
      using ‹0 < e1› ‹0 < e2› by auto
  qed
 qed
 then
 obtain f g where hom: homeomorphism (cbox w z) (cbox w z) f g
             and f w = w f z = z
             and fin: ⋀x. x ∈ cbox a b ⟹ f x ∈ cbox c d
  using homeomorphism_grouping_point_3 [of a b w z c d]
  using ‹box a b ≠ {}› ‹box c d ≠ {}› by blast
 have contfg: continuous_on (cbox w z) f continuous_on (cbox w z) g
  using hom homeomorphism_def by blast+
 define f′ where f′ ≡ λx. if x ∈ cbox w z then f x else x
 define g′ where g′ ≡ λx. if x ∈ cbox w z then g x else x
 show ?thesis
 proof
  have T: cbox w z ∪ (T − box w z) = T
   using ‹cbox w z ⊆ S› ‹S ⊆ T› by auto
  show homeomorphism T T f′ g′
  proof
   have clo: closedin (top_of_set (cbox w z ∪ (T − box w z))) (T − box w z)
    by (metis Diff_Diff_Int Diff_subset T closedin_def open_box openin_open_Int
topspace_euclidean_subtopology)
   have ⋀x. ⟦w ≤ x ∧ x ≤ z; w < x ⟶ ¬ x < z⟧ ⟹ f x = x
    using ‹f w = w› ‹f z = z› by auto
   moreover have ⋀x. ⟦w ≤ x ∧ x ≤ z; w < x ⟶ ¬ x < z⟧ ⟹ g x = x
    using ‹f w = w› ‹f z = z› hom homeomorphism_apply1 by fastforce
   ultimately
   have continuous_on (cbox w z ∪ (T − box w z)) f′ continuous_on (cbox w z
∪ (T − box w z)) g′
    unfolding f′_def g′_def
     by (intro continuous_on_cases_local contfg continuous_on_id clo; auto
simp: closed_subset)+
   then show continuous_on T f′ continuous_on T g′
    by (simp_all only: T)
   show f′ ‘ T ⊆ T
    unfolding f′_def
   by clarsimp (metis ‹cbox w z ⊆ S› ‹S ⊆ T› subsetD hom homeomorphism_def
imageI mem_box_real(2))
   show g′ ‘ T ⊆ T
    unfolding g′_def
   by clarsimp (metis ‹cbox w z ⊆ S› ‹S ⊆ T› subsetD hom homeomorphism_def
imageI mem_box_real(2))
   show ⋀x. x ∈ T ⟹ g′ (f′ x) = x
```

```
      unfolding f'_def g'_def
       using homeomorphism_apply1 [OF hom] homeomorphism_image1 [OF
hom] by fastforce
     show ⋀y. y ∈ T ⟹ f' (g' y) = y
      unfolding f'_def g'_def
       using homeomorphism_apply2 [OF hom] homeomorphism_image2 [OF
hom] by fastforce
   qed
   show ⋀x. x ∈ K ⟹ f' x ∈ U
    using fin sub_wz ‹K ⊆ cbox a b› ‹cbox c d ⊆ U› by (force simp: f'_def)
   show {x. ¬ (f' x = x ∧ g' x = x)} ⊆ S
    using ‹cbox w z ⊆ S› by (auto simp: f'_def g'_def)
   show bounded {x. ¬ (f' x = x ∧ g' x = x)}
   proof (rule bounded_subset [of cbox w z])
    show bounded (cbox w z)
     using bounded_cbox by blast
    show {x. ¬ (f' x = x ∧ g' x = x)} ⊆ cbox w z
     by (auto simp: f'_def g'_def)
   qed
  qed
qed

proposition homeomorphism_grouping_points_exists:
  fixes S :: 'a::euclidean_space set
  assumes open U open S connected S U ≠ {} finite K K ⊆ S U ⊆ S S ⊆ T
  obtains f g where homeomorphism T T f g {x. (¬ (f x = x ∧ g x = x))} ⊆ S
                    bounded {x. (¬ (f x = x ∧ g x = x))} ⋀x. x ∈ K ⟹ f x ∈ U
proof (cases 2 ≤ DIM('a))
  case True
  have TS: T ⊆ affine hull S
   using affine_hull_open assms by blast
  have infinite U
   using ‹open U› ‹U ≠ {}› finite_imp_not_open by blast
  then obtain P where P ⊆ U finite P card K = card P
   using infinite_arbitrarily_large by metis
  then obtain γ where γ: bij_betw γ K P
   using ‹finite K› finite_same_card_bij by blast
  obtain f g where homeomorphism T T f g ⋀i. i ∈ K ⟹ f (id i) = γ i {x. ¬
(f x = x ∧ g x = x)} ⊆ S bounded {x. ¬ (f x = x ∧ g x = x)}
   proof (rule homeomorphism_moving_points_exists [OF True ‹open S› ‹con-
nected S› ‹S ⊆ T› ‹finite K›])
   show ⋀i. i ∈ K ⟹ id i ∈ S ∧ γ i ∈ S
    using ‹P ⊆ U› ‹bij_betw γ K P› ‹K ⊆ S› ‹U ⊆ S› bij_betwE by blast
   show pairwise (λi j. id i ≠ id j ∧ γ i ≠ γ j) K
    using γ by (auto simp: pairwise_def bij_betw_def inj_on_def)
  qed (use affine_hull_open assms that in auto)
  then show ?thesis
   using γ ‹P ⊆ U› bij_betwE by (fastforce simp: intro!: that)
next
```

case *False*
with *DIM_positive* **have** *DIM*('*a*) = *1*
by (*simp add*: *dual_order.antisym*)
then obtain *h*::'*a* ⇒*real* **and** *j*
where *linear h linear j*
and *noh*: ⋀*x*. *norm*(*h x*) = *norm x* **and** *noj*: ⋀*y*. *norm*(*j y*) = *norm y*
and *hj*: ⋀*x*. *j*(*h x*) = *x* ⋀*y*. *h*(*j y*) = *y*
and *ranh*: *surj h*
using *isomorphisms_UNIV_UNIV*
by (*metis* (*mono_tags*, *opaque_lifting*) *DIM_real UNIV_eq_I range_eqI*)
obtain *f g* **where** *hom*: *homeomorphism* (*h* ' *T*) (*h* ' *T*) *f g*
and *f*: ⋀*x*. *x* ∈ *h* ' *K* ⟹ *f x* ∈ *h* ' *U*
and *sub*: {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)} ⊆ *h* ' *S*
and *bou*: *bounded* {*x*. ¬ (*f x* = *x* ∧ *g x* = *x*)}
apply (*rule homeomorphism_grouping_point_4* [*of h* ' *U h* ' *S h* ' *K h* ' *T*])
by (*simp_all add*: *assms image_mono* ‹*linear h*› *open_surjective_linear_image connected_linear_image ranh*)
have *jf*: *j* (*f* (*h x*)) = *x* ⟷ *f* (*h x*) = *h x* **for** *x*
by (*metis hj*)
have *jg*: *j* (*g* (*h x*)) = *x* ⟷ *g* (*h x*) = *h x* **for** *x*
by (*metis hj*)
have *cont_hj*: *continuous_on X h* *continuous_on Y j* **for** *X Y*
by (*simp_all add*: ‹*linear h*› ‹*linear j*› *linear_linear linear_continuous_on*)
show *?thesis*
proof
show *homeomorphism T T* (*j* ∘ *f* ∘ *h*) (*j* ∘ *g* ∘ *h*)
proof
show *continuous_on T* (*j* ∘ *f* ∘ *h*) *continuous_on T* (*j* ∘ *g* ∘ *h*)
using *hom homeomorphism_def*
by (*blast intro*: *continuous_on_compose cont_hj*)+
show (*j* ∘ *f* ∘ *h*) ' *T* ⊆ *T* (*j* ∘ *g* ∘ *h*) ' *T* ⊆ *T*
by *auto* (*metis* (*mono_tags*, *opaque_lifting*) *hj*(*1*) *hom homeomorphism_def imageE imageI*)+
show ⋀*x*. *x* ∈ *T* ⟹ (*j* ∘ *g* ∘ *h*) ((*j* ∘ *f* ∘ *h*) *x*) = *x*
using *hj hom homeomorphism_apply1* **by** *fastforce*
show ⋀*y*. *y* ∈ *T* ⟹ (*j* ∘ *f* ∘ *h*) ((*j* ∘ *g* ∘ *h*) *y*) = *y*
using *hj hom homeomorphism_apply2* **by** *fastforce*
qed
show {*x*. ¬ ((*j* ∘ *f* ∘ *h*) *x* = *x* ∧ (*j* ∘ *g* ∘ *h*) *x* = *x*)} ⊆ *S*
proof (*clarsimp simp*: *jf jg hj*)
show *f* (*h x*) = *h x* ⟶ *g* (*h x*) ≠ *h x* ⟹ *x* ∈ *S* **for** *x*
using *sub* [*THEN subsetD*, *of h x*] *hj* **by** *simp* (*metis imageE*)
qed
have *bounded* (*j* ' {*x*. (¬ (*f x* = *x* ∧ *g x* = *x*))})
by (*rule bounded_linear_image* [*OF bou*]) (*use* ‹*linear j*› *linear_conv_bounded_linear* **in** *auto*)
moreover
have ∗: {*x*. ¬((*j* ∘ *f* ∘ *h*) *x* = *x* ∧ (*j* ∘ *g* ∘ *h*) *x* = *x*)} = *j* ' {*x*. (¬ (*f x* = *x* ∧ *g x* = *x*))}

```
    using hj by (auto simp: jf jg image_iff, metis+)
   ultimately show bounded {x. ¬ ((j ∘ f ∘ h) x = x ∧ (j ∘ g ∘ h) x = x)}
    by metis
   show ⋀x. x ∈ K ⟹ (j ∘ f ∘ h) x ∈ U
    using f hj by fastforce
 qed
qed


proposition homeomorphism_grouping_points_exists_gen:
 fixes S :: 'a::euclidean_space set
 assumes opeU: openin (top_of_set S) U
    and opeS: openin (top_of_set (affine hull S)) S
    and U ≠ {} finite K K ⊆ S and S: S ⊆ T T ⊆ affine hull S connected S
 obtains f g where homeomorphism T T f g {x. (¬ (f x = x ∧ g x = x))} ⊆ S
                   bounded {x. (¬ (f x = x ∧ g x = x))} ⋀x. x ∈ K ⟹ f x ∈ U
proof (cases 2 ≤ aff_dim S)
 case True
 have opeU': openin (top_of_set (affine hull S)) U
  using opeS opeU openin_trans by blast
 obtain u where u ∈ U u ∈ S
  using ‹U ≠ {}› opeU openin_imp_subset by fastforce+
 have infinite U
 proof (rule infinite_openin [OF opeU ‹u ∈ U›])
  show u islimpt S
   using True ‹u ∈ S› assms(8) connected_imp_perfect_aff_dim by fastforce
 qed
 then obtain P where P ⊆ U finite P card K = card P
  using infinite_arbitrarily_large by metis
 then obtain γ where γ: bij_betw γ K P
  using ‹finite K› finite_same_card_bij by blast
 have ∃f g. homeomorphism T T f g ∧ (∀ i ∈ K. f(id i) = γ i) ∧
           {x. ¬ (f x = x ∧ g x = x)} ⊆ S ∧ bounded {x. ¬ (f x = x ∧ g x = x)}
  proof (rule homeomorphism_moving_points_exists_gen [OF ‹finite K› _ _
True opeS S])
   show ⋀i. i ∈ K ⟹ id i ∈ S ∧ γ i ∈ S
    by (metis id_apply opeU openin_contains_cball subsetCE ‹P ⊆ U› ‹bij_betw
γ K P› ‹K ⊆ S› bij_betwE)
   show pairwise (λi j. id i ≠ id j ∧ γ i ≠ γ j) K
    using γ by (auto simp: pairwise_def bij_betw_def inj_on_def)
 qed
 then show ?thesis
  using γ ‹P ⊆ U› bij_betwE by (fastforce simp: intro!: that)
next
 case False
 with aff_dim_geq [of S] consider aff_dim S = -1 | aff_dim S = 0 | aff_dim
S = 1 by linarith
 then show ?thesis
 proof cases
```

```
    assume aff_dim S = -1
    then have S = {}
      using aff_dim_empty by blast
    then have False
      using ‹U ≠ {}› ‹K ⊆ S› openin_imp_subset [OF opeU] by blast
    then show ?thesis ..
  next
    assume aff_dim S = 0
    then obtain a where S = {a}
      using aff_dim_eq_0 by blast
    then have K ⊆ U
      using ‹U ≠ {}› ‹K ⊆ S› openin_imp_subset [OF opeU] by blast
    show ?thesis
      using ‹K ⊆ U› by (intro that [of id id]) (auto intro: homeomorphismI)
  next
    assume aff_dim S = 1
    then have affine hull S homeomorphic (UNIV :: real set)
      by (auto simp: homeomorphic_affine_sets)
    then obtain h::'a⇒real and j where homhj: homeomorphism (affine hull S)
UNIV h j
      using homeomorphic_def by blast
    then have h: ⋀x. x ∈ affine hull S ⟹ j(h(x)) = x and j: ⋀y. j y ∈ affine
hull S ∧ h(j y) = y
      by (auto simp: homeomorphism_def)
    have connh: connected (h ‘ S)
      by (meson Topological_Spaces.connected_continuous_image ‹connected S›
homeomorphism_cont1 homeomorphism_of_subsets homhj hull_subset top_greatest)
    have hUS: h ‘ U ⊆ h ‘ S
      by (meson homeomorphism_imp_open_map homeomorphism_of_subsets
homhj hull_subset opeS opeU open_UNIV openin_open_eq)
    have opn: openin (top_of_set (affine hull S)) U ⟹ open (h ‘ U) for U
      using homeomorphism_imp_open_map [OF homhj] by simp
    have open (h ‘ U) open (h ‘ S)
      by (auto intro: opeS opeU openin_trans opn)
    then obtain f g where hom: homeomorphism (h ‘ T) (h ‘ T) f g
                and f: ⋀x. x ∈ h ‘ K ⟹ f x ∈ h ‘ U
                and sub: {x. ¬ (f x = x ∧ g x = x)} ⊆ h ‘ S
                and bou: bounded {x. ¬ (f x = x ∧ g x = x)}
      apply (rule homeomorphism_grouping_points_exists [of h ‘ U h ‘ S h ‘ K h
‘ T])
      using assms by (auto simp: connh hUS)
    have jf: ⋀x. x ∈ affine hull S ⟹ j (f (h x)) = x ⟷ f (h x) = h x
      by (metis h j)
    have jg: ⋀x. x ∈ affine hull S ⟹ j (g (h x)) = x ⟷ g (h x) = h x
      by (metis h j)
    have cont_hj: continuous_on T h  continuous_on Y j for Y
    proof (rule continuous_on_subset [OF _ ‹T ⊆ affine hull S›])
      show continuous_on (affine hull S) h
        using homeomorphism_def homhj by blast
```

qed (*meson continuous_on_subset homeomorphism_def homhj top_greatest*)
define f' **where** $f' \equiv \lambda x.$ *if* $x \in$ *affine hull* S *then* $(j \circ f \circ h)\ x$ *else* x
define g' **where** $g' \equiv \lambda x.$ *if* $x \in$ *affine hull* S *then* $(j \circ g \circ h)\ x$ *else* x
show *?thesis*
proof
show *homeomorphism* $T\ T\ f'\ g'$
proof
have *continuous_on* $T\ (j \circ f \circ h)$
using *hom homeomorphism_def* **by** (*intro continuous_on_compose cont_hj*) *blast*
then show *continuous_on* $T\ f'$
apply (*rule continuous_on_eq*)
using ‹$T \subseteq$ *affine hull* S› f'*_def* **by** *auto*
have *continuous_on* $T\ (j \circ g \circ h)$
using *hom homeomorphism_def* **by** (*intro continuous_on_compose cont_hj*) *blast*
then show *continuous_on* $T\ g'$
apply (*rule continuous_on_eq*)
using ‹$T \subseteq$ *affine hull* S› g'*_def* **by** *auto*
show f' ‘ $T \subseteq T$
proof (*clarsimp simp*: f'*_def*)
fix x **assume** $x \in T$
then have $f\ (h\ x) \in h$ ‘ T
by (*metis* (*no_types*) *hom homeomorphism_def image_subset_iff subset_refl*)
then show $j\ (f\ (h\ x)) \in T$
using ‹$T \subseteq$ *affine hull* S› h **by** *auto*
qed
show g' ‘ $T \subseteq T$
proof (*clarsimp simp*: g'*_def*)
fix x **assume** $x \in T$
then have $g\ (h\ x) \in h$ ‘ T
by (*metis* (*no_types*) *hom homeomorphism_def image_subset_iff subset_refl*)
then show $j\ (g\ (h\ x)) \in T$
using ‹$T \subseteq$ *affine hull* S› h **by** *auto*
qed
show $\bigwedge x.\ x \in T \Longrightarrow g'\ (f'\ x) = x$
using $h\ j$ *hom homeomorphism_apply1* **by** (*fastforce simp*: f'*_def* g'*_def*)
show $\bigwedge y.\ y \in T \Longrightarrow f'\ (g'\ y) = y$
using $h\ j$ *hom homeomorphism_apply2* **by** (*fastforce simp*: f'*_def* g'*_def*)
qed
next
have §: $\bigwedge x\ y.\ [\![x \in$ *affine hull* S; $h\ x = h\ y$; $y \in S]\!] \Longrightarrow x \in S$
by (*metis h hull_inc*)
show $\{x.\ \neg\ (f'\ x = x \wedge g'\ x = x)\} \subseteq S$
using *sub* **by** (*simp add*: f'*_def* g'*_def jf jg*) (*force elim*: §)
next
have *compact* (j ‘ *closure* $\{x.\ \neg\ (f\ x = x \wedge g\ x = x)\}$)

```
      using bou by (auto simp: compact_continuous_image cont_hj)
    then have bounded (j ` {x. ¬ (f x = x ∧ g x = x)})
      by (rule bounded_closure_image [OF compact_imp_bounded])
    moreover
    have *: {x ∈ affine hull S. j (f (h x)) ≠ x ∨ j (g (h x)) ≠ x} = j ` {x. (¬ (f
x = x ∧ g x = x))}
      using h j by (auto simp: image_iff; metis)
    ultimately have bounded {x ∈ affine hull S. j (f (h x)) ≠ x ∨ j (g (h x)) ≠
x}
      by metis
    then show bounded {x. ¬ (f' x = x ∧ g' x = x)}
      by (simp add: f'_def g'_def Collect_mono bounded_subset)
  next
    show f' x ∈ U if x ∈ K for x
    proof −
      have U ⊆ S
        using opeU openin_imp_subset by blast
      then have j (f (h x)) ∈ U
        using f h hull_subset that by fastforce
      then show f' x ∈ U
        using ‹K ⊆ S› S f'_def that by auto
    qed
  qed
 qed
qed
```

6.4.28 Nullhomotopic mappings

A mapping out of a sphere is nullhomotopic iff it extends to the ball. This even works out in the degenerate cases when the radius is ≤ 0, and we also don't need to explicitly assume continuity since it's already implicit in both sides of the equivalence.

```
lemma nullhomotopic_from_lemma:
  assumes contg: continuous_on (cball a r − {a}) g
      and fa: ⋀e. 0 < e
              ⟹ ∃ d. 0 < d ∧ (∀ x. x ≠ a ∧ norm(x − a) < d ⟶ norm(g x − f
a) < e)
      and r: ⋀x. x ∈ cball a r ∧ x ≠ a ⟹ f x = g x
    shows continuous_on (cball a r) f
proof (clarsimp simp: continuous_on_eq_continuous_within Ball_def)
  fix x
  assume x: dist a x ≤ r
  show continuous (at x within cball a r) f
  proof (cases x=a)
    case True
    then show ?thesis
      by (metis continuous_within_eps_delta fa dist_norm dist_self r)
  next
```

```
    case False
    show ?thesis
    proof (rule continuous_transform_within [where f=g and d = norm(x−a)])
      have ∃ d>0. ∀ x′∈cball a r.
                    dist x′ x < d ⟶ dist (g x′) (g x) < e if e>0 for e
      proof −
        obtain d where d > 0
          and d: ⋀x′. ⟦dist x′ a ≤ r; x′ ≠ a; dist x′ x < d⟧ ⟹
                                dist (g x′) (g x) < e
          using contg False x ‹e>0›
           unfolding continuous_on_iff by (fastforce simp: dist_commute intro:
that)
        show ?thesis
          using ‹d > 0› ‹x ≠ a›
          by (rule_tac x=min d (norm(x − a)) in exI)
             (auto simp: dist_commute dist_norm [symmetric]  intro!: d)
      qed
      then show continuous (at x within cball a r) g
        using contg False by (auto simp: continuous_within_eps_delta)
      show 0 < norm (x − a)
        using False by force
      show x ∈ cball a r
        by (simp add: x)
      show ⋀x′. ⟦x′ ∈ cball a r; dist x′ x < norm (x − a)⟧
        ⟹ g x′ = f x′
        by (metis dist_commute dist_norm less_le r)
    qed
  qed
qed

proposition nullhomotopic_from_sphere_extension:
  fixes f :: 'M::euclidean_space ⇒ 'a::real_normed_vector
  shows (∃ c. homotopic_with_canon (λx. True) (sphere a r) S f (λx. c)) ⟷
         (∃ g. continuous_on (cball a r) g ∧ g ‘ (cball a r) ⊆ S ∧
               (∀ x ∈ sphere a r. g x = f x))
         (is ?lhs = ?rhs)
proof (cases r 0::real rule: linorder_cases)
  case less
  then show ?thesis
    by (simp add: homotopic_on_emptyI)
next
  case equal
  show ?thesis
  proof
    assume L: ?lhs
    with equal have [simp]: f a ∈ S
      using homotopic_with_imp_subset1 by fastforce
    obtain h:: real × 'M ⇒ 'a
      where h: continuous_on ({0..1} × {a}) h h ‘ ({0..1} × {a}) ⊆ S h (0, a)
```

```
= f a
      using L equal by (auto simp: homotopic_with)
    then have continuous_on (cball a r) (λx. h (0, a)) (λx. h (0, a)) ‘ cball a r
⊆ S
      by (auto simp: equal)
    then show ?rhs
      using h(3) local.equal by force
  next
    assume ?rhs
    then show ?lhs
      using equal continuous_on_const by (force simp: homotopic_with)
  qed
next
  case greater
  let ?P = continuous_on {x. norm(x − a) = r} f ∧ f ‘ {x. norm(x − a) = r}
⊆ S
  have ?P if ?lhs using that
  proof
    fix c
    assume c: homotopic_with_canon (λx. True) (sphere a r) S f (λx. c)
    then have contf: continuous_on (sphere a r) f
      by (metis homotopic_with_imp_continuous)
    moreover have fim: f ‘ sphere a r ⊆ S
     by (meson continuous_map_subtopology_eu c homotopic_with_imp_continuous_maps)
    show ?P
     using contf fim by (auto simp: sphere_def dist_norm norm_minus_commute)
  qed
  moreover have ?P if ?rhs using that
  proof
    fix g
    assume g: continuous_on (cball a r) g ∧ g ‘ cball a r ⊆ S ∧ (∀ xa∈sphere a r.
g xa = f xa)
    then have f ‘ {x. norm (x − a) = r} ⊆ S
      using sphere_cball [of a r] unfolding image_subset_iff sphere_def
      by (metis dist_commute dist_norm mem_Collect_eq subset_eq)
    with g show ?P
      by (auto simp: dist_norm norm_minus_commute elim!: continuous_on_eq
[OF continuous_on_subset])
  qed
  moreover have ?thesis if ?P
  proof
    assume ?lhs
    then obtain c where homotopic_with_canon (λx. True) (sphere a r) S (λx.
c) f
      using homotopic_with_sym by blast
    then obtain h where conth: continuous_on ({0..1::real} × sphere a r) h
                    and him: h ‘ ({0..1} × sphere a r) ⊆ S
                    and h: ⋀x. h(0, x) = c ⋀x. h(1, x) = f x
      by (auto simp: homotopic_with_def)
```

```
    obtain b1::'M where b1 ∈ Basis
      using SOME_Basis by auto
    have c ∈ h ` ({0..1} × sphere a r)
    proof
      show c = h (0, a + r *R b1)
        by (simp add: h)
      show (0, a + r *R b1) ∈ {0..1::real} × sphere a r
        using greater ‹b1 ∈ Basis› by (auto simp: dist_norm)
    qed
    then have c ∈ S
      using him by blast
    have uconth: uniformly_continuous_on ({0..1::real} × (sphere a r)) h
      by (force intro: compact_Times conth compact_uniformly_continuous)
    let ?g = λx. h (norm (x − a)/r,
                    a + (if x = a then r *R b1 else (r / norm(x − a)) *R (x − a)))
    let ?g' = λx. h (norm (x − a)/r, a + (r / norm(x − a)) *R (x − a))
    show ?rhs
    proof (intro exI conjI)
      have continuous_on (cball a r − {a}) ?g'
        using greater
      by (force simp: dist_norm norm_minus_commute intro: continuous_on_compose2
[OF conth] continuous_intros)
      then show continuous_on (cball a r) ?g
      proof (rule nullhomotopic_from_lemma)
       show ∃ d>0. ∀ x. x ≠ a ∧ norm (x − a) < d ⟶ norm (?g' x − ?g a) < e
if 0 < e for e
        proof −
          obtain d where 0 < d
            and d: ⋀x x'. ⟦x ∈ {0..1} × sphere a r; x' ∈ {0..1} × sphere a r; norm
( x' − x) < d⟧
                    ⟹ norm (h x' − h x) < e
            using uniformly_continuous_onE [OF uconth ‹0 < e›] by (auto simp:
dist_norm)
          have *: norm (h (norm (x − a) / r,
                      a + (r / norm (x − a)) *R (x − a)) − h (0, a + r *R b1)) <
e (is  norm (?ha − ?hb) < e)
                  if x ≠ a norm (x − a) < r norm (x − a) < d * r for x
          proof −
            have norm (?ha − ?hb) = norm (?ha − h (0, a + (r / norm (x − a))
*R (x − a)))
              by (simp add: h)
            also have ... < e
              using greater ‹0 < d› ‹b1 ∈ Basis› that
              by (intro d) (simp_all add: dist_norm, simp add: field_simps)
            finally show ?thesis .
          qed
          show ?thesis
            using greater ‹0 < d›
            by (rule_tac x = min r (d * r) in exI) (auto simp: *)
```

```
      qed
      show ⋀x. x ∈ cball a r ∧ x ≠ a ⟹ ?g x = ?g' x
        by auto
    qed
  next
    show ?g ‘ cball a r ⊆ S
      using greater him ‹c ∈ S›
      by (force simp: h dist_norm norm_minus_commute)
  next
    show ∀x∈sphere a r. ?g x = f x
      using greater by (auto simp: h dist_norm norm_minus_commute)
  qed
next
  assume ?rhs
  then obtain g where contg: continuous_on (cball a r) g
                  and gim: g ‘ cball a r ⊆ S
                  and gf: ∀x ∈ sphere a r. g x = f x
    by auto
  let ?h = λy. g (a + (fst y) *R (snd y − a))
  have continuous_on ({0..1} × sphere a r) ?h
  proof (rule continuous_on_compose2 [OF contg])
    show continuous_on ({0..1} × sphere a r) (λx. a + fst x *R (snd x − a))
      by (intro continuous_intros)
    qed (auto simp: dist_norm norm_minus_commute mult_left_le_one_le)
  moreover
  have ?h ‘ ({0..1} × sphere a r) ⊆ S
    by (auto simp: dist_norm norm_minus_commute mult_left_le_one_le gim
[THEN subsetD])
  moreover
  have ∀x∈sphere a r. ?h (0, x) = g a ∀x∈sphere a r. ?h (1, x) = f x
    by (auto simp: dist_norm norm_minus_commute mult_left_le_one_le gf)
  ultimately have homotopic_with_canon (λx. True) (sphere a r) S (λx. g a) f
    by (auto simp: homotopic_with)
  then show ?lhs
    using homotopic_with_symD by blast
qed
ultimately
show ?thesis by meson
qed

end
```

6.5 Euclidean space and n-spheres, as subtopologies of n-dimensional space

```
theory Abstract_Euclidean_Space
imports Homotopy Locally
begin
```

6.5.1 Euclidean spaces as abstract topologies

definition *Euclidean_space* :: *nat* ⇒ (*nat* ⇒ *real*) *topology*
where *Euclidean_space n* ≡ *subtopology* (*powertop_real UNIV*) {*x*. ∀ *i*≥*n*. *x i* = *0*}

lemma *topspace_Euclidean_space*:
topspace(*Euclidean_space n*) = {*x*. ∀ *i*≥*n*. *x i* = *0*}
by (*simp add*: *Euclidean_space_def*)

lemma *nontrivial_Euclidean_space*: *Euclidean_space n* ≠ *trivial_topology*
using *topspace_Euclidean_space* [*of n*] **by** *force*

lemma *subset_Euclidean_space* [*simp*]:
topspace(*Euclidean_space m*) ⊆ *topspace*(*Euclidean_space n*) ⟷ *m* ≤ *n*
apply (*simp add*: *topspace_Euclidean_space subset_iff*, *safe*)
apply (*drule_tac x*=(λ*i*. *if i* < *m then 1 else 0*) **in** *spec*)
apply *auto*
using *not_less* **by** *fastforce*

lemma *topspace_Euclidean_space_alt*:
topspace(*Euclidean_space n*) = (⋂ *i* ∈ {*n*..}. {*x*. *x* ∈ *topspace*(*powertop_real UNIV*) ∧ *x i* ∈ {*0*}})
by (*auto simp*: *topspace_Euclidean_space*)

lemma *closedin_Euclidean_space*:
closedin (*powertop_real UNIV*) (*topspace*(*Euclidean_space n*))
proof −
have *closedin* (*powertop_real UNIV*) {*x*. *x i* = *0*} **if** *n* ≤ *i* **for** *i*
proof −
have *closedin* (*powertop_real UNIV*) {*x* ∈ *topspace* (*powertop_real UNIV*). *x i* ∈ {*0*}}
proof (*rule closedin_continuous_map_preimage*)
show *continuous_map* (*powertop_real UNIV*) *euclideanreal* (λ*x*. *x i*)
by (*metis UNIV_I continuous_map_product_coordinates*)
show *closedin euclideanreal* {*0*}
by *simp*
qed
then show *?thesis*
by *auto*
qed
then show *?thesis*
unfolding *topspace_Euclidean_space_alt*
by *force*
qed

lemma *closedin_Euclidean_imp_closed*: *closedin* (*Euclidean_space m*) *S* ⟹ *closed S*
by (*metis Euclidean_space_def closed_closedin closedin_Euclidean_space closedin_closed_subtopology euclidean_product_topology topspace_Euclidean_space*)

lemma *closedin_Euclidean_space_iff*:
closedin (*Euclidean_space m*) *S* ⟷ *closed S* ∧ *S* ⊆ *topspace* (*Euclidean_space m*)
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
using *closedin_closed_subtopology topspace_Euclidean_space*
by (*fastforce simp*: *topspace_Euclidean_space_alt closedin_Euclidean_imp_closed*)
show *?rhs* ⟹ *?lhs*
apply (*simp add*: *closedin_subtopology Euclidean_space_def*)
by (*metis* (*no_types*) *closed_closedin euclidean_product_topology inf.orderE*)
qed

lemma *continuous_map_componentwise_Euclidean_space*:
continuous_map X (*Euclidean_space n*) (λ*x i*. *if i* < *n then f x i else 0*) ⟷
(∀ *i* < *n*. *continuous_map X euclideanreal* (λ*x*. *f x i*))
proof −
have ∗: *continuous_map X euclideanreal* (λ*x*. *if k* < *n then f x k else 0*)
if ⋀*i*. *i*<*n* ⟹ *continuous_map X euclideanreal* (λ*x*. *f x i*) **for** *k*
by (*intro continuous_intros that*)
show *?thesis*
unfolding *Euclidean_space_def continuous_map_in_subtopology*
by (*fastforce simp*: *continuous_map_componentwise_UNIV* ∗ *elim*: *continuous_map_eq*)
qed

lemma *continuous_map_Euclidean_space_add* [*continuous_intros*]:
⟦*continuous_map X* (*Euclidean_space n*) *f*; *continuous_map X* (*Euclidean_space n*) *g*⟧
⟹ *continuous_map X* (*Euclidean_space n*) (λ*x i*. *f x i* + *g x i*)
unfolding *Euclidean_space_def continuous_map_in_subtopology*
by (*fastforce simp add*: *continuous_map_componentwise_UNIV continuous_map_add*)

lemma *continuous_map_Euclidean_space_diff* [*continuous_intros*]:
⟦*continuous_map X* (*Euclidean_space n*) *f*; *continuous_map X* (*Euclidean_space n*) *g*⟧
⟹ *continuous_map X* (*Euclidean_space n*) (λ*x i*. *f x i* − *g x i*)
unfolding *Euclidean_space_def continuous_map_in_subtopology*
by (*fastforce simp add*: *continuous_map_componentwise_UNIV continuous_map_diff*)

lemma *continuous_map_Euclidean_space_iff*:
continuous_map (*Euclidean_space m*) *euclidean g*
= *continuous_on* (*topspace* (*Euclidean_space m*)) *g*
proof
assume *continuous_map* (*Euclidean_space m*) *euclidean g*
then have *continuous_map* (*top_of_set* {*f*. ∀ *n*≥*m*. *f n* = *0*}) *euclidean g*
by (*simp add*: *Euclidean_space_def euclidean_product_topology*)
then show *continuous_on* (*topspace* (*Euclidean_space m*)) *g*

by (*metis continuous_map_subtopology_eu subtopology_topspace topspace_Euclidean_space*)
next
assume *continuous_on* (*topspace* (*Euclidean_space m*)) *g*
then have *continuous_map* (*top_of_set* {*f*. ∀ *n*≥*m*. *f n* = *0*}) *euclidean g*
by (*metis* (*lifting*) *continuous_map_into_fulltopology continuous_map_subtopology_eu order_refl topspace_Euclidean_space*)
then show *continuous_map* (*Euclidean_space m*) *euclidean g*
by (*simp add*: *Euclidean_space_def euclidean_product_topology*)
qed

lemma *cm_Euclidean_space_iff_continuous_on*:
continuous_map (*subtopology* (*Euclidean_space m*) *S*) (*Euclidean_space n*) *f*
⟷ *continuous_on* (*topspace* (*subtopology* (*Euclidean_space m*) *S*)) *f* ∧
f ∈ (*topspace* (*subtopology* (*Euclidean_space m*) *S*)) → *topspace* (*Euclidean_space n*)
(**is** *?P* ⟷ *?Q* ∧ *?R*)
proof −
have *?Q* **if** *?P*
proof −
have ⋀*n*. *Euclidean_space n* = *top_of_set* {*f*. ∀ *m*≥*n*. *f m* = *0*}
by (*simp add*: *Euclidean_space_def euclidean_product_topology*)
with *that* **show** *?thesis*
by (*simp add*: *subtopology_subtopology*)
qed
moreover
have *?R* **if** *?P*
using *that* **by** (*simp add*: *image_subset_iff continuous_map_def*)
moreover
have *?P* **if** *?Q ?R*
proof −
have *continuous_map* (*top_of_set* (*topspace* (*subtopology* (*subtopology* (*powertop_real UNIV*) {*f*. ∀ *n*≥*m*. *f n* = *0*}) *S*))) (*top_of_set* (*topspace* (*subtopology* (*powertop_real UNIV*) {*f*. ∀ *na*≥*n*. *f na* = *0*}))) *f*
using *Euclidean_space_def that* **by** *auto*
then show *?thesis*
by (*simp add*: *Euclidean_space_def euclidean_product_topology subtopology_subtopology*)
qed
ultimately show *?thesis*
by *auto*
qed

lemma *homeomorphic_Euclidean_space_product_topology*:
Euclidean_space n homeomorphic_space product_topology (λ*i*. *euclideanreal*) {..<*n*}
proof −
have *cm*: *continuous_map* (*product_topology* (λ*i*. *euclideanreal*) {..<*n*})
euclideanreal (λ*x*. *if k* < *n then x k else 0*) **for** *k*
by (*auto intro*: *continuous_map_if continuous_map_product_projection*)

show *?thesis*
unfolding *homeomorphic_space_def homeomorphic_maps_def*
apply (*rule_tac x=λf. restrict f {..<n} in exI*)
apply (*rule_tac x=λf i. if i < n then f i else 0 in exI*)
apply (*simp add*: *Euclidean_space_def continuous_map_in_subtopology*)
apply (*intro conjI continuous_map_from_subtopology*)
apply (*force simp*: *continuous_map_componentwise cm intro*: *continuous_map_product_projection*)+
done
qed

lemma *contractible_Euclidean_space* [*simp*]: *contractible_space* (*Euclidean_space n*)
using *homeomorphic_Euclidean_space_product_topology contractible_space_euclideanreal contractible_space_product_topology homeomorphic_space_contractibility* **by** *blast*

lemma *path_connected_Euclidean_space*: *path_connected_space* (*Euclidean_space n*)
by (*simp add*: *contractible_imp_path_connected_space*)

lemma *connected_Euclidean_space*: *connected_space* (*Euclidean_space n*)
by (*simp add*: *contractible_imp_connected_space*)

lemma *locally_path_connected_Euclidean_space*:
locally_path_connected_space (*Euclidean_space n*)
apply (*simp add*: *homeomorphic_locally_path_connected_space* [*OF homeomorphic_Euclidean_space_product_topology* [*of n*]]
locally_path_connected_space_product_topology)
using *locally_path_connected_space_euclideanreal* **by** *auto*

lemma *compact_Euclidean_space* [*simp*]:
compact_space (*Euclidean_space n*) ⟷ *n = 0*
using *homeomorphic_compact_space* [*OF homeomorphic_Euclidean_space_product_topology*]

by (*auto simp*: *product_topology_trivial_iff compact_space_product_topology*)

6.5.2 n-dimensional spheres

definition *nsphere* **where**
nsphere n ≡ *subtopology* (*Euclidean_space* (*Suc n*)) { *x*. (∑ *i*≤*n*. *x i* ^ *2*) = *1* }

lemma *nsphere*:
nsphere n = subtopology (*powertop_real UNIV*)
{*x*. (∑ *i*≤*n*. *x i* ^ *2*) = *1* ∧ (∀ *i*>*n*. *x i* = *0*)}
by (*simp add*: *nsphere_def Euclidean_space_def subtopology_subtopology Suc_le_eq Collect_conj_eq Int_commute*)

lemma *continuous_map_nsphere_projection*: *continuous_map* (*nsphere n*) *eu-*

clideanreal (λ*x*. *x k*)
unfolding *nsphere*
by (*blast intro*: *continuous_map_from_subtopology* [*OF continuous_map_product_projection*])

lemma *in_topspace_nsphere*: (λ*n*. *if n = 0 then 1 else 0*) ∈ *topspace* (*nsphere n*)
by (*simp add*: *nsphere_def topspace_Euclidean_space power2_eq_square if_distrib* [**where** *f* = λ*x*. *x* ∗ _] *cong*: *if_cong*)

lemma *nonempty_nsphere* [*simp*]: (*nsphere n*) ≠ *trivial_topology*
by (*metis discrete_topology_unique empty_iff in_topspace_nsphere*)

lemma *subtopology_nsphere_equator*:
subtopology (*nsphere* (*Suc n*)) {*x*. *x*(*Suc n*) = *0*} = *nsphere n*
proof −
have ({*x*. (∑ *i*≤*n*. (*x i*)²) + (*x* (*Suc n*))² = *1* ∧ (∀ *i*>*Suc n*. *x i* = *0*)} ∩ {*x*. *x* (*Suc n*) = *0*})
= {*x*. (∑ *i*≤*n*. (*x i*)²) = *1* ∧ (∀ *i*>*n*. *x i* = (*0*::*real*))}
using *Suc_lessI* [*of n*] **by** (*fastforce simp*: *set_eq_iff*)
then show *?thesis*
by (*simp add*: *nsphere subtopology_subtopology*)
qed

lemma *topspace_nsphere_minus1*:
assumes *x*: *x* ∈ *topspace* (*nsphere n*) **and** *x n* = *0*
shows *x* ∈ *topspace* (*nsphere* (*n* − *Suc 0*))
proof (*cases n* = *0*)
case *True*
then show *?thesis*
using *x* **by** *auto*
next
case *False*
have *subt_eq*: *nsphere* (*n* − *Suc 0*) = *subtopology* (*nsphere n*) {*x*. *x n* = *0*}
by (*metis False Suc_pred le_zero_eq not_le subtopology_nsphere_equator*)
with *x* **show** *?thesis*
by (*simp add*: *assms*)
qed

lemma *continuous_map_nsphere_reflection*:
continuous_map (*nsphere n*) (*nsphere n*) (λ*x i*. *if i* = *k then* −*x i else x i*)
proof −
have *cm*: *continuous_map* (*powertop_real UNIV*) *euclideanreal* (λ*x*. *if j* = *k then* − *x j else x j*) **for** *j*
proof (*cases j*=*k*)
case *True*
then show *?thesis*
by *simp* (*metis UNIV_I continuous_map_product_projection*)
next
case *False*
then show *?thesis*

```
    by (auto intro: continuous_map_product_projection)
  qed
  have eq: (if i = k then x k * x k else x i * x i) = x i * x i for i and x :: nat ⇒
real
    by simp
  show ?thesis
   apply (simp add: nsphere continuous_map_in_subtopology continuous_map_componentwise_UNIV
                    continuous_map_from_subtopology cm)
   apply (intro conjI allI impI continuous_intros continuous_map_from_subtopology
continuous_map_product_projection)
      apply (auto simp: power2_eq_square if_distrib [where f = λx. x * _] eq
cong: if_cong)
    done
qed


proposition contractible_space_upper_hemisphere:
  assumes k ≤ n
  shows contractible_space(subtopology (nsphere n) {x. x k ≥ 0})
proof −
  define p:: nat ⇒ real where p ≡ λi. if i = k then 1 else 0
  have p ∈ topspace(nsphere n)
    using assms
    by (simp add: nsphere p_def power2_eq_square if_distrib [where f = λx. x *
_] cong: if_cong)
  let ?g = λx i. x i / sqrt(∑ j≤n. x j ^ 2)
  let ?h = λ(t,q) i. (1 − t) * q i + t * p i
  let ?Y = subtopology (Euclidean_space (Suc n)) {x. 0 ≤ x k ∧ (∃ i≤n. x i ≠
0)}
  have continuous_map (prod_topology (top_of_set {0..1}) (subtopology (nsphere
n) {x. 0 ≤ x k}))
                      (subtopology (nsphere n) {x. 0 ≤ x k}) (?g ∘ ?h)
  proof (rule continuous_map_compose)
    have ∗: ⟦0 ≤ b k; (∑ i≤n. (b i)^2) = 1; ∀ i>n. b i = 0; 0 ≤ a; a ≤ 1⟧
          ⟹ ∃ i. (i = k ⟶ (1 − a) * b k + a ≠ 0) ∧
                (i ≠ k ⟶ i ≤ n ∧ a ≠ 1 ∧ b i ≠ 0) for a::real and b
      apply (cases a ≠ 1 ∧ b k = 0; simp)
       apply (metis (no_types, lifting) atMost_iff sum.neutral zero_power2)
       by (metis add.commute add_le_same_cancel2 diff_ge_0_iff_ge diff_zero
less_eq_real_def mult_eq_0_iff mult_nonneg_nonneg not_le numeral_One zero_neq_numeral)
   show continuous_map (prod_topology (top_of_set {0..1}) (subtopology (nsphere
n) {x. 0 ≤ x k})) ?Y ?h
      using assms
      apply (auto simp: ∗ nsphere continuous_map_componentwise_UNIV
             prod_topology_subtopology subtopology_subtopology case_prod_unfold
            continuous_map_in_subtopology Euclidean_space_def p_def if_distrib
[where f = λx. _ * x] cong: if_cong)
     apply (intro continuous_map_prod_snd continuous_intros continuous_map_from_subtopology)
       apply auto
```

```
    done
  next
   have 1: ⋀x i. ⟦ i ≤ n; x i ≠ 0⟧ ⟹ (∑ i≤n. (x i / sqrt (∑ j≤n. (x j)²))²) = 1
    by (force simp: sum_nonneg sum_nonneg_eq_0_iff field_split_simps simp
flip: sum_divide_distrib)
   have cm: continuous_map ?Y (nsphere n) (λx i. x i / sqrt (∑ j≤n. (x j)²))
     unfolding Euclidean_space_def nsphere subtopology_subtopology continu-
ous_map_in_subtopology
   proof (intro continuous_intros conjI)
    show continuous_map
          (subtopology (powertop_real UNIV) ({x. ∀ i≥Suc n. x i = 0} ∩ {x. 0
≤ x k ∧ (∃ i≤n. x i ≠ 0)}))
          (powertop_real UNIV) (λx i. x i / sqrt (∑ j≤n. (x j)²))
     unfolding continuous_map_componentwise
     by (intro continuous_intros conjI ballI) (auto simp: sum_nonneg_eq_0_iff)
   qed (auto simp: 1)
   show continuous_map ?Y (subtopology (nsphere n) {x. 0 ≤ x k}) (λx i. x i /
sqrt (∑ j≤n. (x j)²))
    by (force simp: cm sum_nonneg continuous_map_in_subtopology if_distrib
[where f = λx. _ * x] cong: if_cong)
  qed
  moreover have (?g ∘ ?h) (0, x) = x
   if x ∈ topspace (subtopology (nsphere n) {x. 0 ≤ x k}) for x
   using that
   by (simp add: assms nsphere)
  moreover
  have (?g ∘ ?h) (1, x) = p
   if x ∈ topspace (subtopology (nsphere n) {x. 0 ≤ x k}) for x
   by (force simp: assms p_def power2_eq_square if_distrib [where f = λx. x *
_] cong: if_cong)
  ultimately
  show ?thesis
   apply (simp add: contractible_space_def homotopic_with)
   apply (rule_tac x=p in exI)
   apply (rule_tac x=?g ∘ ?h in exI, force)
   done
qed


corollary contractible_space_lower_hemisphere:
  assumes k ≤ n
  shows contractible_space(subtopology (nsphere n) {x. x k ≤ 0})
proof −
  have contractible_space (subtopology (nsphere n) {x. 0 ≤ x k}) = ?thesis
  proof (rule homeomorphic_space_contractibility)
   show subtopology (nsphere n) {x. 0 ≤ x k} homeomorphic_space subtopology
(nsphere n) {x. x k ≤ 0}
    unfolding homeomorphic_space_def homeomorphic_maps_def
    apply (rule_tac x=λx i. if i = k then −(x i) else x i in exI)+
```

```
    apply (auto simp: continuous_map_in_subtopology continuous_map_from_subtopology
                continuous_map_nsphere_reflection)
     done
  qed
  then show ?thesis
    using contractible_space_upper_hemisphere [OF assms] by metis
qed


proposition nullhomotopic_nonsurjective_sphere_map:
  assumes f: continuous_map (nsphere p) (nsphere p) f
    and fim: f ' (topspace(nsphere p)) ≠ topspace(nsphere p)
  obtains a where homotopic_with (λx. True) (nsphere p) (nsphere p) f (λx. a)
proof −
  obtain a where a: a ∈ topspace(nsphere p) a ∉ f ' (topspace(nsphere p))
    using fim continuous_map_image_subset_topspace f by blast
  then have a1: (∑ i≤p. (a i)^2) = 1 and a0: ⋀i. i > p ⟹ a i = 0
    by (simp_all add: nsphere)
  have f1: (∑ j≤p. (f x j)^2) = 1 if x ∈ topspace (nsphere p) for x
  proof −
    have f x ∈ topspace (nsphere p)
      using continuous_map_image_subset_topspace f that by blast
    then show ?thesis
      by (simp add: nsphere)
  qed
  show thesis
  proof
    let ?g = λx i. x i / sqrt(∑ j≤p. x j ^ 2)
    let ?h = λ(t,x) i. (1 − t) * f x i − t * a i
    let ?Y = subtopology (Euclidean_space(Suc p)) (− {λi. 0})
    let ?T01 = top_of_set {0..1::real}
    have 1: continuous_map (prod_topology ?T01 (nsphere p)) (nsphere p) (?g ∘
?h)
    proof (rule continuous_map_compose)
      have continuous_map (prod_topology ?T01 (nsphere p)) euclideanreal ((λx.
f x k) ∘ snd) for k
        unfolding nsphere
        apply (simp add: continuous_map_of_snd flip: null_topspace_iff_trivial)
        apply (rule continuous_map_compose [of _ nsphere p f, unfolded o_def])
        using f apply (simp add: nsphere)
        by (simp add: continuous_map_nsphere_projection)
      then have continuous_map (prod_topology ?T01 (nsphere p)) euclideanreal
(λr. ?h r k)
        for k
        unfolding case_prod_unfold o_def
          by (intro continuous_map_into_fulltopology [OF continuous_map_fst]
continuous_intros) auto
      moreover have ?h ' ({0..1} × topspace (nsphere p)) ⊆ {x. ∀ i≥Suc p. x i =
0}
```

using *continuous_map_image_subset_topspace* [*OF f*]
by (*auto simp*: *nsphere image_subset_iff a0*)
moreover have (λ*i*. *0*) ∉ *?h* ‘ ({*0..1*} × *topspace* (*nsphere p*))
proof *clarify*
fix *t b*
assume *eq*: (λ*i*. *0*) = (λ*i*. (*1* − *t*) ∗ *f b i* − *t* ∗ *a i*) **and** *t* ∈ {*0..1*} **and** *b*: *b* ∈ *topspace* (*nsphere p*)
have (*1* − *t*)^2 = (∑ *i*≤*p*. ((*1* − *t*) ∗ *f b i*) ^ *2*)
using *f1* [*OF b*] **by** (*simp add*: *power_mult_distrib flip*: *sum_distrib_left*)
also have ... = (∑ *i*≤*p*. (*t* ∗ *a i*) ^ *2*)
using *eq* **by** (*simp add*: *fun_eq_iff*)
also have ... = *t*^2
using *a1* **by** (*simp add*: *power_mult_distrib flip*: *sum_distrib_left*)
finally have *1* − *t* = *t*
by (*simp add*: *power2_eq_iff*)
then have ∗: *t* = *1*/*2*
by *simp*
have *fba*: *f b* ≠ *a*
using *a*(*2*) *b* **by** *blast*
then show *False*
using *eq* **unfolding** ∗ **by** (*simp add*: *fun_eq_iff*)
qed
ultimately show *continuous_map* (*prod_topology ?T01* (*nsphere p*)) *?Y ?h*
by (*simp add*: *Euclidean_space_def continuous_map_in_subtopology continuous_map_componentwise_UNIV*)
next
have ∗: ⟦∀ *i*≥*Suc p*. *x i* = *0*; *x* ≠ (λ*i*. *0*)⟧ ⟹ (∑ *j*≤*p*. (*x j*)^2) ≠ *0* **for** *x* :: *nat* ⇒ *real*
by (*force simp*: *fun_eq_iff not_less_eq_eq sum_nonneg_eq_0_iff*)
show *continuous_map ?Y* (*nsphere p*) *?g*
apply (*simp add*: *Euclidean_space_def continuous_map_in_subtopology continuous_map_componentwise_UNIV*
nsphere continuous_map_componentwise subtopology_subtopology)
apply (*intro conjI allI continuous_intros continuous_map_from_subtopology* [*OF continuous_map_product_projection*])
apply (*simp_all add*: ∗)
apply (*force simp*: *sum_nonneg fun_eq_iff not_less_eq_eq sum_nonneg_eq_0_iff power_divide simp flip*: *sum_divide_distrib*)
done
qed
have *2*: (*?g* ∘ *?h*) (*0*, *x*) = *f x* **if** *x* ∈ *topspace* (*nsphere p*) **for** *x*
using *that f1* **by** *simp*
have *3*: (*?g* ∘ *?h*) (*1*, *x*) = (λ*i*. − *a i*) **for** *x*
using *a* **by** (*force simp*: *field_split_simps nsphere*)
then show *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) *f* (λ*x*. (λ*i*. − *a i*))
by (*force simp*: *homotopic_with intro*: *1 2 3*)
qed
qed

lemma *Hausdorff_Euclidean_space*:
Hausdorff_space (*Euclidean_space n*)
unfolding *Euclidean_space_def*
by (*rule Hausdorff_space_subtopology*) (*metis Hausdorff_space_euclidean Hausdorff_space_product_topology*)

end

6.6 Various Forms of Topological Spaces

theory *Abstract_Topological_Spaces*
imports *Lindelof_Spaces Locally Abstract_Euclidean_Space Sum_Topology FSigma*
begin

6.6.1 Connected topological spaces

lemma *connected_space_eq_frontier_eq_empty*:
connected_space X $\longleftrightarrow$ ($\forall S.\ S \subseteq$ *topspace X* $\wedge$ *X frontier_of S* $= \{\} \longrightarrow S = \{\} \vee S =$ *topspace X*)
by (*meson clopenin_eq_frontier_of connected_space_clopen_in*)

lemma *connected_space_frontier_eq_empty*:
connected_space X $\wedge$ $S \subseteq$ *topspace X*
$\Longrightarrow$ (*X frontier_of S* $= \{\} \longleftrightarrow S = \{\} \vee S =$ *topspace X*)
by (*meson connected_space_eq_frontier_eq_empty frontier_of_empty frontier_of_topspace*)

lemma *connectedin_eq_subset_separated_union*:
connectedin X C $\longleftrightarrow$
$C \subseteq$ *topspace X* $\wedge$ ($\forall S\ T.$ *separatedin X S T* $\wedge$ $C \subseteq S \cup T \longrightarrow C \subseteq S \vee C \subseteq T$) (**is** *?lhs=?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *connectedin_subset_topspace connectedin_subset_separated_union* **by** *blast*
next
assume *?rhs*
then show *?lhs*
by (*metis closure_of_subset connectedin_separation dual_order.eq_iff inf.orderE separatedin_def sup.boundedE*)
qed

lemma *connectedin_clopen_cases*:
⟦*connectedin X C*; *closedin X T*; *openin X T*⟧ $\Longrightarrow$ $C \subseteq T \vee$ *disjnt C T*
by (*metis Diff_eq_empty_iff Int_empty_right clopenin_eq_frontier_of connectedin_Int_frontier_of disjnt_def*)

lemma *connected_space_retraction_map_image*:
⟦*retraction_map X X' r*; *connected_space X*⟧ $\Longrightarrow$ *connected_space X'*

using *connected_space_quotient_map_image retraction_imp_quotient_map* **by** *blast*

lemma *connectedin_imp_perfect_gen*:
assumes *X*: *t1_space X* **and** *S*: *connectedin X S* **and** *nontriv*: $\nexists a.\ S = \{a\}$
shows *S* ⊆ *X derived_set_of S*
unfolding *derived_set_of_def*
proof (*intro subsetI CollectI conjI strip*)
show *XS*: *x* ∈ *topspace X* **if** *x* ∈ *S* **for** *x*
using *that S connectedin* **by** *fastforce*
show ∃ *y*. *y* ≠ *x* ∧ *y* ∈ *S* ∧ *y* ∈ *T*
if *x* ∈ *S* **and** *x* ∈ *T* ∧ *openin X T* **for** *x T*
proof −
have *opeXx*: *openin X* (*topspace X* − {*x*})
by (*meson X openin_topspace t1_space_openin_delete_alt*)
moreover
have *S* ⊆ *T* ∪ (*topspace X* − {*x*})
using *XS that*(*2*) **by** *auto*
moreover have (*topspace X* − {*x*}) ∩ *S* ≠ {}
by (*metis Diff_triv S connectedin double_diff empty_subsetI inf_commute insert_subsetI nontriv that*(*1*))
ultimately show *?thesis*
using *that connectedinD* [*OF S*, *of T topspace X* − {*x*}]
by *blast*
qed
qed

lemma *connectedin_imp_perfect*:
⟦*Hausdorff_space X*; *connectedin X S*; $\nexists a.\ S = \{a\}$⟧ ⟹ *S* ⊆ *X derived_set_of S*
by (*simp add*: *Hausdorff_imp_t1_space connectedin_imp_perfect_gen*)

6.6.2 The notion of "separated between" (complement of "connected between)"

definition *separated_between*
where *separated_between X S T* ≡
∃ *U V*. *openin X U* ∧ *openin X V* ∧ *U* ∪ *V* = *topspace X* ∧ *disjnt U V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V*

lemma *separated_between_alt*:
separated_between X S T ⟷
(∃ *U V*. *closedin X U* ∧ *closedin X V* ∧ *U* ∪ *V* = *topspace X* ∧ *disjnt U V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V*)
unfolding *separated_between_def*
by (*metis separatedin_open_sets separation_closedin_Un_gen subtopology_topspace*
separatedin_closed_sets separation_openin_Un_gen)

lemma *separated_between*:
separated_between X S T ⟷
(∃ *U*. *closedin X U* ∧ *openin X U* ∧ *S* ⊆ *U* ∧ *T* ⊆ *topspace X* − *U*)
unfolding *separated_between_def closedin_def disjnt_def*
by (*smt* (*verit*, *del_insts*) *Diff_cancel Diff_disjoint Diff_partition Un_Diff Un_Diff_Int openin_subset*)

lemma *separated_between_mono*:
⟦*separated_between X S T*; *S'* ⊆ *S*; *T'* ⊆ *T*⟧ ⟹ *separated_between X S' T'*
by (*meson order.trans separated_between*)

lemma *separated_between_refl*:
separated_between X S S ⟷ *S* = {}
unfolding *separated_between_def*
by (*metis Un_empty_right disjnt_def disjnt_empty2 disjnt_subset2 disjnt_sym le_iff_inf openin_empty openin_topspace*)

lemma *separated_between_sym*:
separated_between X S T ⟷ *separated_between X T S*
by (*metis disjnt_sym separated_between_alt sup_commute*)

lemma *separated_between_imp_subset*:
separated_between X S T ⟹ *S* ⊆ *topspace X* ∧ *T* ⊆ *topspace X*
by (*metis le_supI1 le_supI2 separated_between_def*)

lemma *separated_between_empty*:
(*separated_between X* {} *S* ⟷ *S* ⊆ *topspace X*) ∧ (*separated_between X S* {} ⟷ *S* ⊆ *topspace X*)
by (*metis Diff_empty bot.extremum closedin_empty openin_empty separated_between separated_between_imp_subset separated_between_sym*)

lemma *separated_between_Un*:
separated_between X S (*T* ∪ *U*) ⟷ *separated_between X S T* ∧ *separated_between X S U*
by (*auto simp*: *separated_between*)

lemma *separated_between_Un'*:
separated_between X (*S* ∪ *T*) *U* ⟷ *separated_between X S U* ∧ *separated_between X T U*
by (*simp add*: *separated_between_Un separated_between_sym*)

lemma *separated_between_imp_disjoint*:
separated_between X S T ⟹ *disjnt S T*
by (*meson disjnt_iff separated_between_def subsetD*)

lemma *separated_between_imp_separatedin*:
separated_between X S T ⟹ *separatedin X S T*
by (*meson separated_between_def separatedin_mono separatedin_open_sets*)

lemma *separated_between_full*:
assumes $S \cup T = \text{topspace } X$
shows *separated_between X S T* $\longleftrightarrow$ *disjnt S T* $\wedge$ *closedin X S* $\wedge$ *openin X S* $\wedge$ *closedin X T* $\wedge$ *openin X T*
proof −
have *separated_between X S T* $\longrightarrow$ *separatedin X S T*
by (*simp add*: *separated_between_imp_separatedin*)
then show *?thesis*
unfolding *separated_between_def*
by (*metis assms separation_closedin_Un_gen separation_openin_Un_gen subset_refl subtopology_topspace*)
qed

lemma *separated_between_eq_separatedin*:
$S \cup T = \text{topspace } X \Longrightarrow$ (*separated_between X S T* $\longleftrightarrow$ *separatedin X S T*)
by (*simp add*: *separated_between_full separatedin_full*)

lemma *separated_between_pointwise_left*:
assumes *compactin X S*
shows *separated_between X S T* $\longleftrightarrow$
$(S = \{\} \longrightarrow T \subseteq \text{topspace } X) \wedge (\forall x \in S.\ \text{separated\_between } X\ \{x\}\ T)$
(**is** *?lhs=?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *separated_between_imp_subset separated_between_mono* **by** *fastforce*
next
assume *R*: *?rhs*
then have $T \subseteq \text{topspace } X$
by (*meson equals0I separated_between_imp_subset*)
show *?lhs*
proof −
obtain *U* **where** *U*: $\forall x \in S.\ \text{openin } X\ (U\ x)$
$\forall x \in S.\ \exists V.\ \text{openin } X\ V \wedge U\ x \cup V = \text{topspace } X \wedge \text{disjnt } (U\ x)\ V \wedge \{x\} \subseteq U\ x \wedge T \subseteq V$
using *R* **unfolding** *separated_between_def* **by** *metis*
then have $S \subseteq \bigcup (U\ `\ S)$
by *blast*
then obtain *K* **where** *finite K* $K \subseteq S$ **and** *K*: $S \subseteq (\bigcup i{\in}K.\ U\ i)$
using *assms U* **unfolding** *compactin_def* **by** (*smt* (*verit*) *finite_subset_image imageE*)
show *?thesis*
unfolding *separated_between*
proof (*intro conjI exI*)
have $\bigwedge x.\ x \in K \Longrightarrow \text{closedin } X\ (U\ x)$
by (*smt* (*verit*) ‹$K \subseteq S$› *Diff_cancel U*(*2*) *Un_Diff Un_Diff_Int disjnt_def openin_closedin_eq subsetD*)
then show *closedin X* $(\bigcup\ (U\ `\ K))$
by (*metis* (*mono_tags*, *lifting*) ‹*finite K*› *closedin_Union finite_imageI*

image_iff)
show *openin X* (⋃ (*U* ‘ *K*))
using *U*(*1*) ‹*K* ⊆ *S*› **by** *blast*
show *S* ⊆ ⋃ (*U* ‘ *K*)
by (*simp add*: *K*)
have ⋀*x i*. ⟦*x* ∈ *T*; *i* ∈ *K*; *x* ∈ *U i*⟧ ⟹ *False*
by (*meson U*(*2*) ‹*K* ⊆ *S*› *disjnt_iff subsetD*)
then show *T* ⊆ *topspace X* − ⋃ (*U* ‘ *K*)
using ‹*T* ⊆ *topspace X*› **by** *auto*
qed
qed
qed

lemma *separated_between_pointwise_right*:
compactin X T
⟹ *separated_between X S T* ⟷ (*T* = {} ⟶ *S* ⊆ *topspace X*) ∧ (∀ *y* ∈ *T*. *separated_between X S* {*y*})
by (*meson separated_between_pointwise_left separated_between_sym*)

lemma *separated_between_closure_of*:
S ⊆ *topspace X* ⟹ *separated_between X* (*X closure_of S*) *T* ⟷ *separated_between X S T*
by (*meson closure_of_minimal_eq separated_between_alt*)

lemma *separated_between_closure_of′*:
T ⊆ *topspace X* ⟹ *separated_between X S* (*X closure_of T*) ⟷ *separated_between X S T*
by (*meson separated_between_closure_of separated_between_sym*)

lemma *separated_between_closure_of_eq*:
separated_between X S T ⟷ *S* ⊆ *topspace X* ∧ *separated_between X* (*X closure_of S*) *T*
by (*metis separated_between_closure_of separated_between_imp_subset*)

lemma *separated_between_closure_of_eq′*:
separated_between X S T ⟷ *T* ⊆ *topspace X* ∧ *separated_between X S* (*X closure_of T*)
by (*metis separated_between_closure_of′ separated_between_imp_subset*)

lemma *separated_between_frontier_of_eq′*:
separated_between X S T ⟷
T ⊆ *topspace X* ∧ *disjnt S T* ∧ *separated_between X S* (*X frontier_of T*) (**is** *?lhs*=*?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis interior_of_union_frontier_of separated_between_Un separated_between_closure_of_eq′*
separated_between_imp_disjoint)

next
assume *R*: *?rhs*
then obtain *U* **where** *U*: *closedin X U openin X U S ⊆ U X closure_of T − X interior_of T ⊆ topspace X − U*
by (*metis frontier_of_def separated_between*)
show *?lhs*
proof (*rule separated_between_mono* [*of _ S X closure_of T*])
have *separated_between X S T*
unfolding *separated_between*
proof (*intro conjI exI*)
show *S ⊆ U − T T ⊆ topspace X − (U − T)*
using *R U*(*3*) **by** (*force simp*: *disjnt_iff*)+
have *T ⊆ X closure_of T*
by (*simp add*: *R closure_of_subset*)
then have ∗: *U − T = U − X interior_of T*
using *U*(*4*) *interior_of_subset* **by** *fastforce*
then show *closedin X (U − T)*
by (*simp add*: *U*(*1*) *closedin_diff*)
have *U ∩ X frontier_of T = {}*
using *U*(*4*) *frontier_of_def* **by** *fastforce*
then show *openin X (U − T)*
by (*metis ∗ Diff_Un U*(*2*) *Un_Diff_Int Un_Int_eq*(*1*) *closedin_closure_of interior_of_union_frontier_of openin_diff sup_bot_right*)
qed
then show *separated_between X S* (*X closure_of T*)
by (*simp add*: *R separated_between_closure_of′*)
qed (*auto simp*: *R closure_of_subset*)
qed

lemma *separated_between_frontier_of_eq*:
separated_between X S T ⟷ S ⊆ topspace X ∧ disjnt S T ∧ separated_between X (*X frontier_of S*) *T*
by (*metis disjnt_sym separated_between_frontier_of_eq′ separated_between_sym*)

lemma *separated_between_frontier_of*:
⟦*S ⊆ topspace X*; *disjnt S T*⟧
⟹ (*separated_between X* (*X frontier_of S*) *T ⟷ separated_between X S T*)
using *separated_between_frontier_of_eq* **by** *blast*

lemma *separated_between_frontier_of′*:
⟦*T ⊆ topspace X*; *disjnt S T*⟧
⟹ (*separated_between X S* (*X frontier_of T*) *⟷ separated_between X S T*)
using *separated_between_frontier_of_eq′* **by** *auto*

lemma *connected_space_separated_between*:
connected_space X ⟷ (∀ *S T*. *separated_between X S T ⟶ S = {} ∨ T = {}*) (**is** *?lhs=?rhs*)
proof
assume *?lhs* **then show** *?rhs*

by (*metis Diff_cancel connected_space_clopen_in separated_between subset_empty*)
next
assume *?rhs* **then show** *?lhs*
by (*meson connected_space_eq_not_separated separated_between_eq_separatedin*)
qed

lemma *connected_space_imp_separated_between_trivial*:
connected_space X
$\Longrightarrow$ (*separated_between X S T* $\longleftrightarrow$ *S* = {} $\wedge$ *T* $\subseteq$ *topspace X* $\vee$ *S* $\subseteq$ *topspace X* $\wedge$ *T* = {})
by (*metis connected_space_separated_between separated_between_empty*)

6.6.3 Connected components

lemma *connected_component_of_subtopology_eq*:
connected_component_of (*subtopology X U*) *a* = *connected_component_of X a*
$\longleftrightarrow$
connected_component_of_set X a $\subseteq$ *U*
by (*force simp*: *connected_component_of_set connectedin_subtopology connected_component_of_def fun_eq_iff subset_iff*)

lemma *connected_components_of_subtopology*:
assumes *C* $\in$ *connected_components_of X C* $\subseteq$ *U*
shows *C* $\in$ *connected_components_of* (*subtopology X U*)
proof −
obtain *a* **where** *a*: *connected_component_of_set X a* $\subseteq$ *U* **and** *a* $\in$ *topspace X*
and *Ceq*: *C* = *connected_component_of_set X a*
using *assms* **by** (*force simp*: *connected_components_of_def*)
then have *a* $\in$ *U*
by (*simp add*: *connected_component_of_refl in_mono*)
then have *connected_component_of_set X a* = *connected_component_of_set* (*subtopology X U*) *a*
by (*metis a connected_component_of_subtopology_eq*)
then show *?thesis*
by (*simp add*: *Ceq* ‹*a* $\in$ *U*› ‹*a* $\in$ *topspace X*› *connected_component_in_connected_components_of*)
qed

lemma *open_in_finite_connected_components*:
assumes *finite*(*connected_components_of X*) *C* $\in$ *connected_components_of X*
shows *openin X C*
proof −
have *closedin X* (*topspace X* − *C*)
by (*metis DiffD1 assms closedin_Union closedin_connected_components_of complement_connected_components_of_Union finite_Diff*)
then show *?thesis*
by (*simp add*: *assms connected_components_of_subset openin_closedin*)
qed
thm *connected_component_of_eq_overlap*

lemma *connected_components_of_disjoint*:
assumes $C \in$ *connected_components_of* X $C' \in$ *connected_components_of* X
shows (*disjnt* C $C' \longleftrightarrow (C \neq C')$)
using *assms nonempty_connected_components_of pairwiseD pairwise_disjoint_connected_components*
by *fastforce*

lemma *connected_components_of_overlap*:
$[\![C \in$ *connected_components_of* X; $C' \in$ *connected_components_of* $X]\!] \Longrightarrow C \cap C' \neq \{\} \longleftrightarrow C = C'$
by (*meson connected_components_of_disjoint disjnt_def*)

lemma *pairwise_separated_connected_components_of*:
pairwise (*separatedin* X) (*connected_components_of* X)
by (*simp add*: *closedin_connected_components_of connected_components_of_disjoint pairwiseI separatedin_closed_sets*)

lemma *finite_connected_components_of_finite*:
finite(*topspace* X) $\Longrightarrow$ *finite*(*connected_components_of* X)
by (*simp add*: *Union_connected_components_of finite_UnionD*)

lemma *connected_component_of_unique*:
$[\![x \in C$; *connectedin* X C; $\bigwedge C'.\ x \in C' \wedge$ *connectedin* X $C' \Longrightarrow C' \subseteq C]\!]$
$\Longrightarrow$ *connected_component_of_set* X $x = C$
by (*meson connected_component_of_maximal connectedin_connected_component_of subsetD subset_antisym*)

lemma *closedin_connected_component_of_subtopology*:
$[\![C \in$ *connected_components_of* (*subtopology* X s); X *closure_of* $C \subseteq s]\!] \Longrightarrow$ *closedin* X C
by (*metis closedin_Int_closure_of closedin_connected_components_of closure_of_eq inf.absorb_iff2*)

lemma *connected_component_of_discrete_topology*:
connected_component_of_set (*discrete_topology* U) $x =$ (*if* $x \in U$ *then* $\{x\}$ *else* $\{\}$)
by (*simp add*: *locally_path_connected_space_discrete_topology flip*: *path_component_eq_connected_c*

lemma *connected_components_of_discrete_topology*:
connected_components_of (*discrete_topology* U) $= (\lambda x.\ \{x\})$ ‘ U
by (*simp add*: *connected_component_of_discrete_topology connected_components_of_def*)

lemma *connected_component_of_continuous_image*:
$[\![$*continuous_map* X Y f; *connected_component_of* X x $y]\!]$
$\Longrightarrow$ *connected_component_of* Y (f x) (f y)
by (*meson connected_component_of_def connectedin_continuous_map_image image_eqI*)

lemma *homeomorphic_map_connected_component_of*:
assumes *homeomorphic_map* X Y f **and** x: $x \in$ *topspace* X

shows *connected_component_of_set Y (f x) = f ‘ (connected_component_of_set X x)*
proof −
obtain *g* **where** *g*: *continuous_map X Y f*
continuous_map Y X g $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ *g (f x) = x*
$\bigwedge y.\ y \in$ *topspace Y* $\Longrightarrow$ *f (g y) = y*
using *assms(1) homeomorphic_map_maps homeomorphic_maps_def* **by** *fastforce*
show *?thesis*
using *connected_component_in_topspace [of Y] x g*
connected_component_of_continuous_image [of X Y f]
connected_component_of_continuous_image [of Y X g]
by *force*
qed

lemma *homeomorphic_map_connected_components_of*:
assumes *homeomorphic_map X Y f*
shows *connected_components_of Y = (image f) ‘ (connected_components_of X)*
proof −
have *topspace Y = f ‘ topspace X*
by (*metis assms homeomorphic_imp_surjective_map*)
with *homeomorphic_map_connected_component_of [OF assms]* **show** *?thesis*
by (*auto simp*: *connected_components_of_def image_iff*)
qed

lemma *connected_component_of_pair*:
connected_component_of_set (prod_topology X Y) (x,y) =
connected_component_of_set X x × connected_component_of_set Y y
proof (*cases x ∈ topspace X ∧ y ∈ topspace Y*)
case *True*
show *?thesis*
proof (*rule connected_component_of_unique*)
show (*x*, *y*) ∈ *connected_component_of_set X x × connected_component_of_set Y y*
using *True* **by** (*simp add*: *connected_component_of_refl*)
show *connectedin (prod_topology X Y) (connected_component_of_set X x × connected_component_of_set Y y)*
by (*metis connectedin_Times connectedin_connected_component_of*)
show *C* ⊆ *connected_component_of_set X x × connected_component_of_set Y y*
if (*x*, *y*) ∈ *C* ∧ *connectedin (prod_topology X Y) C* **for** *C*
using *that* **unfolding** *connected_component_of_def*
apply *clarsimp*
by (*metis* (*no_types*) *connectedin_continuous_map_image continuous_map_fst continuous_map_snd fst_conv imageI snd_conv*)
qed
next
case *False* **then show** *?thesis*

by (*metis Sigma_empty1 Sigma_empty2 connected_component_of_eq_empty mem_Sigma_iff topspace_prod_topology*)
qed

lemma *connected_components_of_prod_topology*:
connected_components_of (*prod_topology X Y*) =
{*C* × *D* | *C D*. *C* ∈ *connected_components_of X* ∧ *D* ∈ *connected_components_of Y*} (**is** *?lhs*=*?rhs*)
proof
show *?lhs* ⊆ *?rhs*
apply (*clarsimp simp*: *connected_components_of_def*)
by (*metis* (*no_types*) *connected_component_of_pair imageI*)
next
show *?rhs* ⊆ *?lhs*
using *connected_component_of_pair*
by (*fastforce simp*: *connected_components_of_def*)
qed

lemma *connected_component_of_product_topology*:
connected_component_of_set (*product_topology X I*) *x* =
(*if x* ∈ *extensional I then PiE I* (λ*i*. *connected_component_of_set* (*X i*) (*x i*)) *else* {})
(**is** *?lhs* = *If* _ *?R* _)
proof (*cases x* ∈ *topspace*(*product_topology X I*))
case *True*
have *?lhs* = (Π$_E$ *i*∈*I*. *connected_component_of_set* (*X i*) (*x i*))
if *xX*: ⋀*i*. *i*∈*I* ⟹ *x i* ∈ *topspace* (*X i*) **and** *ext*: *x* ∈ *extensional I*
proof (*rule connected_component_of_unique*)
show *x* ∈ *?R*
by (*simp add*: *PiE_iff connected_component_of_refl local.ext xX*)
show *connectedin* (*product_topology X I*) *?R*
by (*simp add*: *connectedin_PiE connectedin_connected_component_of*)
show *C* ⊆ *?R*
if *x* ∈ *C* ∧ *connectedin* (*product_topology X I*) *C* **for** *C*
proof −
have *C* ⊆ *extensional I*
using *PiE_def connectedin_subset_topspace that* **by** *fastforce*
have ⋀*y*. *y* ∈ *C* ⟹ *y* ∈ (Π *i*∈*I*. *connected_component_of_set* (*X i*) (*x i*))
apply (*simp add*: *connected_component_of_def Pi_def*)
by (*metis connectedin_continuous_map_image continuous_map_product_projection imageI that*)
then show *?thesis*
using *PiE_def* ‹*C* ⊆ *extensional I*› **by** *fastforce*
qed
qed
with *True* **show** *?thesis*
by (*simp add*: *PiE_iff*)
next

case *False*
then show *?thesis*
by (*smt* (*verit*, *best*) *PiE_eq_empty_iff PiE_iff connected_component_of_eq_empty topspace_product_topology*)
qed

lemma *connected_components_of_product_topology*:
connected_components_of (*product_topology X I*) =
{*PiE I B* |*B*. ∀ *i* ∈ *I*. *B i* ∈ *connected_components_of*(*X i*)} (**is** *?lhs*=*?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*auto simp*: *connected_components_of_def connected_component_of_product_topology PiE_iff*)
show *?rhs* ⊆ *?lhs*
proof
fix *F*
assume *F* ∈ *?rhs*
then obtain *B* **where** *Feq*: *F* = *Pi*$_E$ *I B* **and**
∀ *i*∈*I*. ∃ *x*∈*topspace* (*X i*). *B i* = *connected_component_of_set* (*X i*) *x*
by (*force simp*: *connected_components_of_def connected_component_of_product_topology image_iff*)
then obtain *f* **where**
f: ⋀*i*. *i* ∈ *I* ⟹ *f i* ∈ *topspace* (*X i*) ∧ *B i* = *connected_component_of_set* (*X i*) (*f i*)
by *metis*
then have (λ*i*∈*I*. *f i*) ∈ ((Π$_E$ *i*∈*I*. *topspace* (*X i*)) ∩ *extensional I*)
by *simp*
with *f* **show** *F* ∈ *?lhs*
unfolding *Feq connected_components_of_def connected_component_of_product_topology image_iff*
by (*smt* (*verit*, *del_insts*) *PiE_cong restrict_PiE_iff restrict_apply' restrict_extensional topspace_product_topology*)
qed
qed

6.6.4 Monotone maps (in the general topological sense)

definition *monotone_map*
where *monotone_map X Y f* ==
f ‘ (*topspace X*) ⊆ *topspace Y* ∧
(∀ *y* ∈ *topspace Y*. *connectedin X* {*x* ∈ *topspace X*. *f x* = *y*})

lemma *monotone_map*:
monotone_map X Y f ⟷
f ‘ (*topspace X*) ⊆ *topspace Y* ∧ (∀ *y*. *connectedin X* {*x* ∈ *topspace X*. *f x* = *y*})
apply (*simp add*: *monotone_map_def*)
by (*metis* (*mono_tags*, *lifting*) *connectedin_empty* [*of X*] *Collect_empty_eq image_subset_iff*)

lemma *monotone_map_in_subtopology*:
monotone_map X (subtopology Y S) f ⟷ monotone_map X Y f ∧ f ‘ (topspace X) ⊆ S
by (*smt* (*verit*, *del_insts*) *le_inf_iff monotone_map topspace_subtopology*)

lemma *monotone_map_from_subtopology*:
assumes *monotone_map X Y f*
⋀*x y*. ⟦*x ∈ topspace X*; *y ∈ topspace X*; *x ∈ S*; *f x = f y*⟧ ⟹ *y ∈ S*
shows *monotone_map (subtopology X S) Y f*
proof −
have ⋀*y*. *y ∈ topspace Y* ⟹ *connectedin X* {*x ∈ topspace X*. *x ∈ S ∧ f x = y*}
by (*smt* (*verit*) *Collect_cong assms connectedin_empty empty_def monotone_map_def*)
then show *?thesis*
using *assms* **by** (*auto simp*: *monotone_map_def connectedin_subtopology*)
qed

lemma *monotone_map_restriction*:
monotone_map X Y f ∧ {*x ∈ topspace X*. *f x ∈ v*} = *u*
⟹ *monotone_map* (*subtopology X u*) (*subtopology Y v*) *f*
by (*smt* (*verit*, *best*) *IntI Int_Collect image_subset_iff mem_Collect_eq monotone_map monotone_map_from_subtopology topspace_subtopology*)

lemma *injective_imp_monotone_map*:
assumes *f ‘ topspace X ⊆ topspace Y inj_on f* (*topspace X*)
shows *monotone_map X Y f*
unfolding *monotone_map_def*
proof (*intro conjI assms strip*)
fix *y*
assume *y ∈ topspace Y*
then have {*x ∈ topspace X*. *f x = y*} = {} ∨ (∃ *a ∈ topspace X*. {*x ∈ topspace X*. *f x = y*} = {*a*})
using *assms*(*2*) **unfolding** *inj_on_def* **by** *blast*
then show *connectedin X* {*x ∈ topspace X*. *f x = y*}
by (*metis* (*no_types*, *lifting*) *connectedin_empty connectedin_sing*)
qed

lemma *embedding_imp_monotone_map*:
embedding_map X Y f ⟹ *monotone_map X Y f*
by (*metis* (*no_types*) *embedding_map_def homeomorphic_eq_everything_map inf.absorb_iff2 injective_imp_monotone_map topspace_subtopology*)

lemma *section_imp_monotone_map*:
section_map X Y f ⟹ *monotone_map X Y f*
by (*simp add*: *embedding_imp_monotone_map section_imp_embedding_map*)

lemma *homeomorphic_imp_monotone_map*:

homeomorphic_map X Y f ⟹ *monotone_map X Y f*
by (*meson section_and_retraction_eq_homeomorphic_map section_imp_monotone_map*)

proposition *connected_space_monotone_quotient_map_preimage*:
assumes *f*: *monotone_map X Y f quotient_map X Y f* **and** *connected_space Y*
shows *connected_space X*
proof (*rule ccontr*)
assume ¬ *connected_space X*
then obtain *U V* **where** *openin X U openin X V U* ∩ *V* = {}
U ≠ {} *V* ≠ {} **and** *topUV*: *topspace X* ⊆ *U* ∪ *V*
by (*auto simp*: *connected_space_def*)
then have *UVsub*: *U* ⊆ *topspace X V* ⊆ *topspace X*
by (*auto simp*: *openin_subset*)
have ¬ *connected_space Y*
unfolding *connected_space_def not_not*
proof (*intro exI conjI*)
show *topspace Y* ⊆ *f*‘*U* ∪ *f*‘*V*
by (*metis f*(*2*) *image_Un quotient_imp_surjective_map subset_Un_eq topUV*)
show *f*‘*U* ≠ {}
by (*simp add*: ‹*U* ≠ {}›)
show (*f*‘*V*) ≠ {}
by (*simp add*: ‹*V* ≠ {}›)
have ∗: *y* ∉ *f* ‘ *V* **if** *y* ∈ *f* ‘ *U* **for** *y*
proof −
have §: *connectedin X* {*x* ∈ *topspace X*. *f x* = *y*}
using *f*(*1*) *monotone_map* **by** *fastforce*
show *?thesis*
using *connectedinD* [*OF* § ‹*openin X U*› ‹*openin X V*›] *UVsub topUV* ‹*U* ∩ *V* = {}› *that*
by (*force simp*: *disjoint_iff*)
qed
then show *f*‘*U* ∩ *f*‘*V* = {}
by *blast*
show *openin Y* (*f*‘*U*)
using *f* ‹*openin X U*› *topUV* ∗ **unfolding** *quotient_map_saturated_open*
by *force*
show *openin Y* (*f*‘*V*)
using *f* ‹*openin X V*› *topUV* ∗ **unfolding** *quotient_map_saturated_open*
by *force*
qed
then show *False*
by (*simp add*: *assms*)
qed

lemma *connectedin_monotone_quotient_map_preimage*:
assumes *monotone_map X Y f quotient_map X Y f connectedin Y C openin Y C* ∨ *closedin Y C*
shows *connectedin X* {*x* ∈ *topspace X*. *f x* ∈ *C*}

proof −
have *connected_space* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*})
proof −
have *connected_space* (*subtopology Y C*)
using ‹*connectedin Y C*› *connectedin_def* **by** *blast*
moreover have *quotient_map* (*subtopology X* {*a* ∈ *topspace X*. *f a* ∈ *C*}) (*subtopology Y C*) *f*
by (*simp add*: *assms quotient_map_restriction*)
ultimately show *?thesis*
using ‹*monotone_map X Y f*› *connected_space_monotone_quotient_map_preimage monotone_map_restriction* **by** *blast*
qed
then show *?thesis*
by (*simp add*: *connectedin_def*)
qed

lemma *monotone_open_map*:
assumes *continuous_map X Y f open_map X Y f* **and** *fim*: *f* ‘ (*topspace X*) = *topspace Y*
shows *monotone_map X Y f* ⟷ (∀ *C*. *connectedin Y C* ⟶ *connectedin X* {*x* ∈ *topspace X*. *f x* ∈ *C*})
(**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
unfolding *connectedin_def*
proof (*intro strip conjI*)
fix *C*
assume *C*: *C* ⊆ *topspace Y* ∧ *connected_space* (*subtopology Y C*)
show *connected_space* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*})
proof (*rule connected_space_monotone_quotient_map_preimage*)
show *monotone_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*}) (*subtopology Y C*) *f*
by (*simp add*: *L monotone_map_restriction*)
show *quotient_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*}) (*subtopology Y C*) *f*
proof (*rule continuous_open_imp_quotient_map*)
show *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*}) (*subtopology Y C*) *f*
using *assms continuous_map_from_subtopology continuous_map_in_subtopology* **by** *fastforce*
qed (*use open_map_restriction assms* **in** *fastforce*)+
qed (*simp add*: *C*)
qed *auto*
next
assume *?rhs*
then have ∀ *y*. *connectedin Y* {*y*} ⟶ *connectedin X* {*x* ∈ *topspace X*. *f x* = *y*}
by (*smt* (*verit*) *Collect_cong singletonD singletonI*)
then show *?lhs*

by (*simp add*: *fim monotone_map_def*)
qed

lemma *monotone_closed_map*:
assumes *continuous_map X Y f closed_map X Y f* **and** *fim*: *f* ' (*topspace X*) = *topspace Y*
shows *monotone_map X Y f* ⟷ (∀ *C*. *connectedin Y C* ⟶ *connectedin X* {*x* ∈ *topspace X*. *f x* ∈ *C*})
(**is** *?lhs=?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
unfolding *connectedin_def*
proof (*intro strip conjI*)
fix *C*
assume *C*: *C* ⊆ *topspace Y* ∧ *connected_space* (*subtopology Y C*)
show *connected_space* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*})
proof (*rule connected_space_monotone_quotient_map_preimage*)
show *monotone_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*}) (*subtopology Y C*) *f*
by (*simp add*: *L monotone_map_restriction*)
show *quotient_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*}) (*subtopology Y C*) *f*
proof (*rule continuous_closed_imp_quotient_map*)
show *continuous_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *C*}) (*subtopology Y C*) *f*
using *assms continuous_map_from_subtopology continuous_map_in_subtopology*
by *fastforce*
qed (*use closed_map_restriction assms* **in** *fastforce*)+
qed (*simp add*: *C*)
qed *auto*
next
assume *?rhs*
then have ∀ *y*. *connectedin Y* {*y*} ⟶ *connectedin X* {*x* ∈ *topspace X*. *f x* = *y*}
by (*smt* (*verit*) *Collect_cong singletonD singletonI*)
then show *?lhs*
by (*simp add*: *fim monotone_map_def*)
qed

6.6.5 Other countability properties

definition *second_countable*
where *second_countable X* ≡
∃ℬ. *countable* ℬ ∧ (∀ *V* ∈ ℬ. *openin X V*) ∧
(∀ *U x*. *openin X U* ∧ *x* ∈ *U* ⟶ (∃ *V* ∈ ℬ. *x* ∈ *V* ∧ *V* ⊆ *U*))

definition *first_countable*
where *first_countable X* ≡
∀ *x* ∈ *topspace X*.

$\exists \mathcal{B}.\ countable\ \mathcal{B} \wedge (\forall\, V \in \mathcal{B}.\ openin\ X\ V) \wedge$
$(\forall\, U.\ openin\ X\ U \wedge x \in U \longrightarrow (\exists\, V \in \mathcal{B}.\ x \in V \wedge V \subseteq U))$

definition *separable_space*
where *separable_space X* $\equiv$
$\exists\, C.\ countable\ C \wedge C \subseteq topspace\ X \wedge X\ closure\_of\ C = topspace\ X$

lemma *second_countable*:
second_countable X $\longleftrightarrow$
$(\exists \mathcal{B}.\ countable\ \mathcal{B} \wedge openin\ X = arbitrary\ union\_of\ (\lambda x.\ x \in \mathcal{B}))$
by (*smt* (*verit*) *openin_topology_base_unique second_countable_def*)

lemma *second_countable_subtopology*:
assumes *second_countable X*
shows *second_countable* (*subtopology X S*)
proof −
obtain $\mathcal{B}$ **where** $\mathcal{B}$: *countable* $\mathcal{B}$ $\bigwedge V.\ V \in \mathcal{B} \Longrightarrow openin\ X\ V$
$\bigwedge U\ x.\ openin\ X\ U \wedge x \in U \longrightarrow (\exists\, V \in \mathcal{B}.\ x \in V \wedge V \subseteq U)$
using *assms* **by** (*auto simp*: *second_countable_def*)
show *?thesis*
unfolding *second_countable_def*
proof (*intro exI conjI*)
show $\forall\, V{\in}((\cap)S)\ `\ \mathcal{B}.\ openin$ (*subtopology X S*) *V*
using *openin_subtopology_Int2* $\mathcal{B}$ **by** *blast*
show $\forall\, U\ x.\ openin$ (*subtopology X S*) $U \wedge x \in U \longrightarrow (\exists\, V{\in}((\cap)S)\ `\ \mathcal{B}.\ x \in V \wedge V \subseteq U)$
using $\mathcal{B}$ **unfolding** *openin_subtopology*
by (*smt* (*verit*, *del_insts*) *IntI image_iff inf_commute inf_le1 subset_iff*)
qed (*use* $\mathcal{B}$ **in** *auto*)
qed

lemma *second_countable_discrete_topology*:
second_countable(*discrete_topology U*) $\longleftrightarrow$ *countable U* (**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
then
obtain $\mathcal{B}$ **where** $\mathcal{B}$: *countable* $\mathcal{B}$ $\bigwedge V.\ V \in \mathcal{B} \Longrightarrow V \subseteq U$
$\bigwedge W\ x.\ W \subseteq U \wedge x \in W \longrightarrow (\exists\, V \in \mathcal{B}.\ x \in V \wedge V \subseteq W)$
by (*auto simp*: *second_countable_def*)
then have $\{x\} \in \mathcal{B}$ **if** $x \in U$ **for** *x*
by (*metis empty_subsetI insertCI insert_subset subset_antisym that*)
then show *?rhs*
by (*smt* (*verit*) *countable_subset image_subsetI* ‹*countable* $\mathcal{B}$› *countable_image_inj_on* [*OF* _ *inj_singleton*])
next
assume *?rhs*
then show *?lhs*
unfolding *second_countable_def*

by (*rule_tac x*=($\lambda x.\ \{x\}$) ‘ *U* **in** *exI*) *auto*
qed

lemma *second_countable_open_map_image*:
assumes *continuous_map X Y f open_map X Y f*
and *fim*: *f* ‘ (*topspace X*) = *topspace Y* **and** *second_countable X*
shows *second_countable Y*
proof −
have *openXYf*: $\bigwedge U.$ *openin X U* ⟶ *openin Y* (*f* ‘ *U*)
using *assms* **by** (*auto simp*: *open_map_def*)
obtain $\mathcal{B}$ **where** $\mathcal{B}$: *countable* $\mathcal{B}$ $\bigwedge V.\ V \in \mathcal{B} \Longrightarrow$ *openin X V*
and ∗: $\bigwedge U\ x.$ *openin X U* ∧ *x* ∈ *U* ⟶ (∃ *V* ∈ $\mathcal{B}$. *x* ∈ *V* ∧ *V* ⊆ *U*)
using *assms* **by** (*auto simp*: *second_countable_def*)
show *?thesis*
unfolding *second_countable_def*
proof (*intro exI conjI strip*)
fix *V y*
assume *V*: *openin Y V* ∧ *y* ∈ *V*
then obtain *x* **where** *x* ∈ *topspace X* **and** *x*: *f x* = *y*
by (*metis fim image_iff openin_subset subsetD*)

then obtain *W* **where** *W*∈$\mathcal{B}$ *x* ∈ *W W* ⊆ {*x* ∈ *topspace X*. *f x* ∈ *V*}
using ∗ [*of* {*x* ∈ *topspace X*. *f x* ∈ *V*} *x*] *V assms openin_continuous_map_preimage*

by *force*
then show ∃ *W* ∈ (*image f*) ‘ $\mathcal{B}$. *y* ∈ *W* ∧ *W* ⊆ *V*
using *x* **by** *auto*
qed (*use* $\mathcal{B}$ *openXYf* **in** *auto*)
qed

lemma *homeomorphic_space_second_countability*:
X homeomorphic_space Y ⟹ (*second_countable X* ⟷ *second_countable Y*)
by (*meson homeomorphic_eq_everything_map homeomorphic_space homeomorphic_space_sym second_countable_open_map_image*)

lemma *second_countable_retraction_map_image*:
⟦*retraction_map X Y r*; *second_countable X*⟧ ⟹ *second_countable Y*
using *hereditary_imp_retractive_property homeomorphic_space_second_countability second_countable_subtopology* **by** *blast*

lemma *second_countable_imp_first_countable*:
second_countable X ⟹ *first_countable X*
by (*metis first_countable_def second_countable_def*)

lemma *first_countable_subtopology*:
assumes *first_countable X*
shows *first_countable* (*subtopology X S*)
unfolding *first_countable_def*
proof

```
  fix x
  assume x ∈ topspace (subtopology X S)
  then obtain ℬ where countable ℬ and ℬ: ⋀V. V ∈ ℬ ⟹ openin X V
    ⋀U. openin X U ∧ x ∈ U ⟶ (∃ V ∈ ℬ. x ∈ V ∧ V ⊆ U)
    using assms first_countable_def by force
  show ∃ℬ. countable ℬ ∧ (∀ V∈ℬ. openin (subtopology X S) V) ∧ (∀ U. openin
(subtopology X S) U ∧ x ∈ U ⟶ (∃ V∈ℬ. x ∈ V ∧ V ⊆ U))
  proof (intro exI conjI strip)
    show countable (((∩)S) ` ℬ)
      using ‹countable ℬ› by blast
    show openin (subtopology X S) V if V ∈ (((∩)S) ` ℬ for V
      using ℬ openin_subtopology_Int2 that by fastforce
    show ∃ V∈(((∩)S) ` ℬ. x ∈ V ∧ V ⊆ U
      if openin (subtopology X S) U ∧ x ∈ U for U
      using that ℬ(2) by (clarsimp simp: openin_subtopology) (meson le_infI2)
  qed
qed

lemma first_countable_discrete_topology:
   first_countable (discrete_topology U)
 unfolding first_countable_def topspace_discrete_topology openin_discrete_topology
proof
  fix x assume x ∈ U
  show ∃ℬ. countable ℬ ∧ (∀ V∈ℬ. V ⊆ U) ∧ (∀ Ua. Ua ⊆ U ∧ x ∈ Ua ⟶
(∃ V∈ℬ. x ∈ V ∧ V ⊆ Ua))
    using ‹x ∈ U› by (rule_tac x={{x}} in exI) auto
qed

lemma first_countable_open_map_image:
  assumes continuous_map X Y f open_map X Y f
   and fim: f ` (topspace X) = topspace Y and first_countable X
 shows first_countable Y
  unfolding first_countable_def
proof
  fix y
  assume y ∈ topspace Y
  have openXYf: ⋀U. openin X U ⟶ openin Y (f ` U)
    using assms by (auto simp: open_map_def)
  then obtain x where x: x ∈ topspace X f x = y
    by (metis ‹y ∈ topspace Y› fim imageE)
  obtain ℬ where ℬ: countable ℬ ⋀V. V ∈ ℬ ⟹ openin X V
    and ∗: ⋀U. openin X U ∧ x ∈ U ⟶ (∃ V ∈ ℬ. x ∈ V ∧ V ⊆ U)
    using assms x first_countable_def by force
  show ∃ℬ. countable ℬ ∧
            (∀ V∈ℬ. openin Y V) ∧ (∀ U. openin Y U ∧ y ∈ U ⟶ (∃ V∈ℬ. y ∈
V ∧ V ⊆ U))
  proof (intro exI conjI strip)
    fix V assume openin Y V ∧ y ∈ V
    then have ∃ W∈ℬ. x ∈ W ∧ W ⊆ {x ∈ topspace X. f x ∈ V}
```

using ∗ [*of* {*x* ∈ *topspace X*. *f x* ∈ *V*}] *assms openin_continuous_map_preimage x*
by *fastforce*
then show ∃ *V'* ∈ (*image f*) ‘ *ℬ*. *y* ∈ *V'* ∧ *V'* ⊆ *V*
using *image_mono x* **by** *auto*
qed (*use ℬ openXYf* **in** *force*)+
qed

lemma *homeomorphic_space_first_countability*:
X homeomorphic_space Y ⟹ *first_countable X* ⟷ *first_countable Y*
by (*meson first_countable_open_map_image homeomorphic_eq_everything_map homeomorphic_space homeomorphic_space_sym*)

lemma *first_countable_retraction_map_image*:
⟦*retraction_map X Y r*; *first_countable X*⟧ ⟹ *first_countable Y*
using *first_countable_subtopology hereditary_imp_retractive_property homeomorphic_space_first_countability* **by** *blast*

lemma *separable_space_open_subset*:
assumes *separable_space X openin X S*
shows *separable_space* (*subtopology X S*)
proof −
obtain *C* **where** *C*: *countable C C* ⊆ *topspace X X closure_of C* = *topspace X*
by (*meson assms separable_space_def*)
then have ⋀*x T*. ⟦*x* ∈ *topspace X*; *x* ∈ *T*; *openin* (*subtopology X S*) *T*⟧
⟹ ∃ *y*. *y* ∈ *S* ∧ *y* ∈ *C* ∧ *y* ∈ *T*
by (*smt* (*verit*) ‹*openin X S*› *in_closure_of openin_open_subtopology subsetD*)
with *C* ‹*openin X S*› **show** *?thesis*
unfolding *separable_space_def*
by (*rule_tac x*=*S* ∩ *C* **in** *exI*) (*force simp*: *in_closure_of*)
qed

lemma *separable_space_continuous_map_image*:
assumes *separable_space X continuous_map X Y f*
and *fim*: *f* ‘ (*topspace X*) = *topspace Y*
shows *separable_space Y*
proof −
have *cont*: ⋀*S*. *f* ‘ (*X closure_of S*) ⊆ *Y closure_of f* ‘ *S*
by (*simp add*: *assms continuous_map_image_closure_subset*)
obtain *C* **where** *C*: *countable C C* ⊆ *topspace X X closure_of C* = *topspace X*
by (*meson assms separable_space_def*)
then show *?thesis*
unfolding *separable_space_def*
by (*metis cont fim closure_of_subset_topspace countable_image image_mono subset_antisym*)
qed

lemma *separable_space_quotient_map_image*:
⟦*quotient_map X Y q*; *separable_space X*⟧ ⟹ *separable_space Y*

by (*meson quotient_imp_continuous_map quotient_imp_surjective_map separable_space_continuous_map_image*)

lemma *separable_space_retraction_map_image*:
 ⟦*retraction_map X Y r*; *separable_space X*⟧ ⟹ *separable_space Y*
 using *retraction_imp_quotient_map separable_space_quotient_map_image* **by** *blast*

lemma *homeomorphic_separable_space*:
 X homeomorphic_space Y ⟹ (*separable_space X* ⟷ *separable_space Y*)
 by (*meson homeomorphic_eq_everything_map homeomorphic_maps_map homeomorphic_space_def separable_space_continuous_map_image*)

lemma *separable_space_discrete_topology*:
 separable_space(*discrete_topology U*) ⟷ *countable U*
 by (*metis countable_Int2 discrete_topology_closure_of dual_order.refl inf.orderE separable_space_def topspace_discrete_topology*)

lemma *second_countable_imp_separable_space*:
 assumes *second_countable X*
 shows *separable_space X*
proof −
 obtain ℬ **where** ℬ: *countable* ℬ ⋀*V*. *V* ∈ ℬ ⟹ *openin X V*
  and ∗: ⋀*U x*. *openin X U* ∧ *x* ∈ *U* ⟶ (∃ *V* ∈ ℬ. *x* ∈ *V* ∧ *V* ⊆ *U*)
  using *assms* **by** (*auto simp*: *second_countable_def*)
 obtain *c* **where** *c*: ⋀*V*. ⟦*V* ∈ ℬ; *V* ≠ {}⟧ ⟹ *c V* ∈ *V*
  by (*metis all_not_in_conv*)
 then have ∗∗: ⋀*x*. *x* ∈ *topspace X* ⟹ *x* ∈ *X closure_of c* ‘ (ℬ − {{}})
  using ∗ **by** (*force simp*: *closure_of_def*)
 show *?thesis*
  unfolding *separable_space_def*
 proof (*intro exI conjI*)
  show *countable* (*c* ‘ (ℬ−{{}}))
   using ℬ(*1*) **by** *blast*
  show (*c* ‘ (ℬ−{{}})) ⊆ *topspace X*
   using ℬ(*2*) *c openin_subset* **by** *fastforce*
  show *X closure_of* (*c* ‘ (ℬ−{{}})) = *topspace X*
   by (*meson* ∗∗ *closure_of_subset_topspace subsetI subset_antisym*)
 qed
qed

lemma *second_countable_imp_Lindelof_space*:
 assumes *second_countable X*
 shows *Lindelof_space X*
unfolding *Lindelof_space_def*
proof *clarify*
 fix *U*
 assume ∀ *U* ∈ *U*. *openin X U* **and** *UU*: ⋃*U* = *topspace X*
 obtain ℬ **where** ℬ: *countable* ℬ ⋀*V*. *V* ∈ ℬ ⟹ *openin X V*

and $*$: $\bigwedge U\ x.\ openin\ X\ U \wedge x \in U \longrightarrow (\exists\, V \in \mathcal{B}.\ x \in V \wedge V \subseteq U)$
using *assms* **by** (*auto simp*: *second_countable_def*)
define $\mathcal{B}'$ **where** $\mathcal{B}' = \{B \in \mathcal{B}.\ \exists\, U.\ U \in \mathcal{U} \wedge B \subseteq U\}$
have $\mathcal{B}'$: *countable* $\mathcal{B}'$ $\bigcup \mathcal{B}' = \bigcup \mathcal{U}$
using $\mathcal{B}$ **using** $*$ ‹$\forall\, U{\in}\mathcal{U}.\ openin\ X\ U$› **by** (*fastforce simp*: $\mathcal{B}'$*_def*)+
have $\bigwedge b.\ \exists\, U.\ b \in \mathcal{B}' \longrightarrow U \in \mathcal{U} \wedge b \subseteq U$
by (*simp add*: $\mathcal{B}'$*_def*)
then obtain G **where** G: $\bigwedge b.\ b \in \mathcal{B}' \longrightarrow G\ b \in \mathcal{U} \wedge b \subseteq G\ b$
by *metis*
with $\mathcal{B}'$ *UU* **show** $\exists \mathcal{V}.$ *countable* $\mathcal{V} \wedge \mathcal{V} \subseteq \mathcal{U} \wedge \bigcup \mathcal{V} =$ *topspace* X
by (*rule_tac x*=G ‘ $\mathcal{B}'$ **in** *exI*) *fastforce*
qed

6.6.6 Neigbourhood bases EXTRAS

Neigbourhood bases: useful for "local" properties of various kinds

lemma *openin_topology_neighbourhood_base_unique*:
openin X = *arbitrary union_of* $P \longleftrightarrow$
$(\forall\, u.\ P\ u \longrightarrow openin\ X\ u) \wedge neighbourhood\_base\_of\ P\ X$
by (*smt* (*verit*, *best*) *open_neighbourhood_base_of openin_topology_base_unique*)

lemma *neighbourhood_base_at_topology_base*:
openin X = *arbitrary union_of* b
$\Longrightarrow$ (*neighbourhood_base_at* x P $X \longleftrightarrow$
$(\forall\, w.\ b\ w \wedge x \in w \longrightarrow (\exists\, u\ v.\ openin\ X\ u \wedge P\ v \wedge x \in u \wedge u \subseteq v \wedge v \subseteq w)))$
apply (*simp add*: *neighbourhood_base_at_def*)
by (*smt* (*verit*, *del_insts*) *openin_topology_base_unique subset_trans*)

lemma *neighbourhood_base_of_unlocalized*:
assumes $\bigwedge S\ t.\ P\ S \wedge openin\ X\ t \wedge (t \neq \{\}) \wedge t \subseteq S \Longrightarrow P\ t$
shows *neighbourhood_base_of* P $X \longleftrightarrow$
$(\forall\, x \in topspace\ X.\ \exists\, u\ v.\ openin\ X\ u \wedge P\ v \wedge x \in u \wedge u \subseteq v \wedge v \subseteq topspace\ X)$
apply (*simp add*: *neighbourhood_base_of_def*)
by (*smt* (*verit*, *ccfv_SIG*) *assms empty_iff neighbourhood_base_at_unlocalized*)

lemma *neighbourhood_base_at_discrete_topology*:
neighbourhood_base_at x P (*discrete_topology* u) $\longleftrightarrow x \in u \Longrightarrow P\ \{x\}$
apply (*simp add*: *neighbourhood_base_at_def*)
by (*smt* (*verit*) *empty_iff empty_subsetI insert_subset singletonI subsetD subset_singletonD*)

lemma *neighbourhood_base_of_discrete_topology*:
neighbourhood_base_of P (*discrete_topology* u) $\longleftrightarrow (\forall\, x \in u.\ P\ \{x\})$
apply (*simp add*: *neighbourhood_base_of_def*)
using *neighbourhood_base_at_discrete_topology*[*of* _ P u]
by (*metis empty_subsetI insert_subset neighbourhood_base_at_def openin_discrete_topology singletonI*)

lemma *second_countable_neighbourhood_base_alt*:
second_countable X $\longleftrightarrow$
($\exists \mathcal{B}$. *countable* $\mathcal{B} \wedge (\forall V \in \mathcal{B}$. *openin X V*$) \wedge$ *neighbourhood_base_of* (λA. $A \in \mathcal{B}$) *X*)
by (*metis* (*full_types*) *openin_topology_neighbourhood_base_unique second_countable*)

lemma *first_countable_neighbourhood_base_alt*:
first_countable X $\longleftrightarrow$
($\forall x \in$ *topspace X*. $\exists \mathcal{B}$. *countable* $\mathcal{B} \wedge (\forall V \in \mathcal{B}$. *openin X V*$) \wedge$ *neighbourhood_base_at x* (λV. $V \in \mathcal{B}$) *X*)
unfolding *first_countable_def*
apply (*intro ball_cong refl ex_cong conj_cong*)
by (*metis* (*mono_tags*, *lifting*) *open_neighbourhood_base_at*)

lemma *second_countable_neighbourhood_base*:
second_countable X $\longleftrightarrow$
($\exists \mathcal{B}$. *countable* $\mathcal{B} \wedge$ *neighbourhood_base_of* (λV. $V \in \mathcal{B}$) *X*) (**is** *?lhs=?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *second_countable_neighbourhood_base_alt* **by** *blast*
next
assume *?rhs*
then obtain $\mathcal{B}$ **where** *countable* $\mathcal{B}$
and $\mathcal{B}$: $\bigwedge W\ x$. *openin X W* $\wedge$ $x \in W \longrightarrow (\exists U$. *openin X U* $\wedge$ ($\exists V$. $V \in \mathcal{B} \wedge x \in U \wedge U \subseteq V \wedge V \subseteq W$))
by (*metis neighbourhood_base_of*)
then show *?lhs*
unfolding *second_countable_neighbourhood_base_alt neighbourhood_base_of*
apply (*rule_tac x*=(λu. *X interior_of u*) ' $\mathcal{B}$ **in** *exI*)
by (*smt* (*verit*, *best*) *interior_of_eq interior_of_mono countable_image image_iff openin_interior_of*)
qed

lemma *first_countable_neighbourhood_base*:
first_countable X $\longleftrightarrow$
($\forall x \in$ *topspace X*. $\exists \mathcal{B}$. *countable* $\mathcal{B} \wedge$ *neighbourhood_base_at x* (λV. $V \in \mathcal{B}$) *X*) (**is** *?lhs=?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*metis first_countable_neighbourhood_base_alt*)
next
assume *R*: *?rhs*
show *?lhs*
unfolding *first_countable_neighbourhood_base_alt*
proof
fix *x*

assume $x \in$ *topspace* X
with R **obtain** $\mathcal{B}$ **where** *countable* $\mathcal{B}$ **and** $\mathcal{B}$: *neighbourhood_base_at* x $(\lambda V.$ $V \in \mathcal{B})$ X
by *blast*
then
show $\exists \mathcal{B}.$ *countable* $\mathcal{B} \wedge$ *Ball* $\mathcal{B}$ (*openin* X) $\wedge$ *neighbourhood_base_at* x $(\lambda V.$ $V \in \mathcal{B})$ X
unfolding *neighbourhood_base_at_def*
apply (*rule_tac* x=$(\lambda u.$ X *interior_of* $u)$ ‘ $\mathcal{B}$ **in** *exI*)
by (*smt* (*verit*, *best*) *countable_image image_iff interior_of_eq interior_of_mono openin_interior_of*)
qed
qed

6.6.7 T_0 spaces and the Kolmogorov quotient

definition *t0_space* **where**
t0_space $X \equiv$
$\forall x \in$ *topspace* $X.$ $\forall y \in$ *topspace* $X.$ $x \neq y \longrightarrow (\exists U.$ *openin* X $U \wedge (x \notin U \longleftrightarrow y \in U))$

lemma *t0_space_expansive*:
$[\![$*topspace* $Y =$ *topspace* X; $\bigwedge U.$ *openin* X $U \Longrightarrow$ *openin* Y $U]\!] \Longrightarrow$ *t0_space* $X \Longrightarrow$ *t0_space* Y
by (*metis t0_space_def*)

lemma *t1_imp_t0_space*: *t1_space* $X \Longrightarrow$ *t0_space* X
by (*metis t0_space_def t1_space_def*)

lemma *t1_eq_symmetric_t0_space_alt*:
t1_space $X \longleftrightarrow$
t0_space $X \wedge$
$(\forall x \in$ *topspace* $X.$ $\forall y \in$ *topspace* $X.$ $x \in X$ *closure_of* $\{y\} \longleftrightarrow y \in X$ *closure_of* $\{x\})$
apply (*simp add*: *t0_space_def t1_space_def closure_of_def*)
by (*smt* (*verit*, *best*) *openin_topspace*)

lemma *t1_eq_symmetric_t0_space*:
t1_space $X \longleftrightarrow$ *t0_space* $X \wedge (\forall x\ y.$ $x \in X$ *closure_of* $\{y\} \longleftrightarrow y \in X$ *closure_of* $\{x\})$
by (*auto simp*: *t1_eq_symmetric_t0_space_alt in_closure_of*)

lemma *Hausdorff_imp_t0_space*:
Hausdorff_space $X \Longrightarrow$ *t0_space* X
by (*simp add*: *Hausdorff_imp_t1_space t1_imp_t0_space*)

lemma *t0_space*:
t0_space $X \longleftrightarrow$
$(\forall x \in$ *topspace* $X.$ $\forall y \in$ *topspace* $X.$ $x \neq y \longrightarrow (\exists C.$ *closedin* X $C \wedge (x \notin C$

$\longleftrightarrow$ $y \in C$)))
unfolding *t0_space_def* **by** (*metis Diff_iff closedin_def openin_closedin_eq*)

lemma *homeomorphic_t0_space*:
assumes *X homeomorphic_space Y*
shows *t0_space X* $\longleftrightarrow$ *t0_space Y*
proof −
obtain *f* **where** *f*: *homeomorphic_map X Y f* **and** *F*: *inj_on f* (*topspace X*)
and *topspace Y* = *f* ‘ *topspace X*
by (*metis assms homeomorphic_imp_injective_map homeomorphic_imp_surjective_map homeomorphic_space*)
with *inj_on_image_mem_iff* [*OF F*]
show *?thesis*
apply (*simp add*: *t0_space_def homeomorphic_eq_everything_map continuous_map_def open_map_def inj_on_def*)
by (*smt* (*verit*) *mem_Collect_eq openin_subset*)
qed

lemma *t0_space_closure_of_sing*:
t0_space X $\longleftrightarrow$
($\forall x \in$ *topspace X*. $\forall y \in$ *topspace X*. *X closure_of* {*x*} = *X closure_of* {*y*} $\longrightarrow$ *x* = *y*)
by (*simp add*: *t0_space_def closure_of_def set_eq_iff*) (*smt* (*verit*))

lemma *t0_space_discrete_topology*: *t0_space* (*discrete_topology S*)
by (*simp add*: *Hausdorff_imp_t0_space*)

lemma *t0_space_subtopology*: *t0_space X* $\Longrightarrow$ *t0_space* (*subtopology X U*)
by (*simp add*: *t0_space_def openin_subtopology*) (*metis Int_iff*)

lemma *t0_space_retraction_map_image*:
⟦*retraction_map X Y r*; *t0_space X*⟧ $\Longrightarrow$ *t0_space Y*
using *hereditary_imp_retractive_property homeomorphic_t0_space t0_space_subtopology*
by *blast*

lemma *t0_space_prod_topologyI*: ⟦*t0_space X*; *t0_space Y*⟧ $\Longrightarrow$ *t0_space* (*prod_topology X Y*)
by (*simp add*: *t0_space_closure_of_sing closure_of_Times closure_of_eq_empty_gen times_eq_iff flip*: *sing_Times_sing insert_Times_insert*)

lemma *t0_space_prod_topology_iff*:
t0_space (*prod_topology X Y*) $\longleftrightarrow$ *prod_topology X Y* = *trivial_topology* ∨ *t0_space X* ∧ *t0_space Y* (**is** *?lhs*=*?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*metis prod_topology_trivial_iff retraction_map_fst retraction_map_snd t0_space_retraction_map_image*)

qed (*metis t0_space_discrete_topology t0_space_prod_topologyI*)

proposition *t0_space_product_topology*:
  t0_space (*product_topology X I*) ⟷ *product_topology X I = trivial_topology*
∨ (∀ *i* ∈ *I*. *t0_space* (*X i*))
  (**is** *?lhs=?rhs*)
proof
 assume *?lhs*
 then show *?rhs*
  by (*meson retraction_map_product_projection t0_space_retraction_map_image*)
next
 assume *R*: *?rhs*
 show *?lhs*
 proof (*cases product_topology X I = trivial_topology*)
  case *True*
  then show *?thesis*
   by (*simp add*: *t0_space_def*)
 next
  case *False*
  show *?thesis*
   unfolding *t0_space*
  proof (*intro strip*)
   fix *x y*
   assume *x*: *x* ∈ *topspace* (*product_topology X I*)
    and *y*: *y* ∈ *topspace* (*product_topology X I*)
    and *x* ≠ *y*
   then obtain *i* **where** *i* ∈ *I x i* ≠ *y i*
    by (*metis PiE_ext topspace_product_topology*)
   then have *t0_space* (*X i*)
    using *False R* **by** *blast*
   then obtain *U* **where** *closedin* (*X i*) *U* (*x i* ∉ *U* ⟷ *y i* ∈ *U*)
    by (*metis t0_space PiE_mem* ‹*i* ∈ *I*› ‹*x i* ≠ *y i*› *topspace_product_topology*
x y)
   with ‹*i* ∈ *I*› *x y* **show** ∃ *U*. *closedin* (*product_topology X I*) *U* ∧ (*x* ∉ *U*) =
(*y* ∈ *U*)
    by (*rule_tac x=PiE I* (λ*j*. *if j = i then U else topspace*(*X j*)) **in** *exI*)
     (*simp add*: *closedin_product_topology PiE_iff*)
  qed
 qed
qed

6.6.8 Kolmogorov quotients

definition *Kolmogorov_quotient*
 where *Kolmogorov_quotient X* ≡ λ*x*. @*y*. ∀ *U*. *openin X U* ⟶ (*y* ∈ *U* ⟷ *x*
∈ *U*)

lemma *Kolmogorov_quotient_in_open*:
  openin X U ⟹ (*Kolmogorov_quotient X x* ∈ *U* ⟷ *x* ∈ *U*)

by (*smt* (*verit*, *ccfv_SIG*) *Kolmogorov_quotient_def someI_ex*)

lemma *Kolmogorov_quotient_in_topspace*:
Kolmogorov_quotient X x ∈ *topspace X* ⟷ *x* ∈ *topspace X*
by (*simp add*: *Kolmogorov_quotient_in_open*)

lemma *Kolmogorov_quotient_in_closed*:
closedin X C ⟹ (*Kolmogorov_quotient X x* ∈ *C* ⟷ *x* ∈ *C*)
unfolding *closedin_def*
by (*meson DiffD2 DiffI Kolmogorov_quotient_in_open Kolmogorov_quotient_in_topspace in_mono*)

lemma *continuous_map_Kolmogorov_quotient*:
continuous_map X X (*Kolmogorov_quotient X*)
using *Kolmogorov_quotient_in_open openin_subopen openin_subset*
by (*fastforce simp*: *continuous_map_def Kolmogorov_quotient_in_topspace*)

lemma *open_map_Kolmogorov_quotient_explicit*:
openin X U ⟹ *Kolmogorov_quotient X ' U* = *Kolmogorov_quotient X ' topspace X* ∩ *U*
using *Kolmogorov_quotient_in_open openin_subset* **by** *fastforce*

lemma *open_map_Kolmogorov_quotient_gen*:
open_map (*subtopology X S*) (*subtopology X* (*Kolmogorov_quotient X ' S*)) (*Kolmogorov_quotient X*)
proof (*clarsimp simp*: *open_map_def openin_subtopology_alt image_iff*)
fix *U*
assume *openin X U*
then have *Kolmogorov_quotient X ' (S* ∩ *U)* = *Kolmogorov_quotient X ' S* ∩ *U*
using *Kolmogorov_quotient_in_open* [*of X U*] **by** *auto*
then show ∃ *V*. *openin X V* ∧ *Kolmogorov_quotient X ' (S* ∩ *U)* = *Kolmogorov_quotient X ' S* ∩ *V*
using ‹*openin X U*› **by** *blast*
qed

lemma *open_map_Kolmogorov_quotient*:
open_map X (*subtopology X* (*Kolmogorov_quotient X ' topspace X*)) (*Kolmogorov_quotient X*)
by (*metis open_map_Kolmogorov_quotient_gen subtopology_topspace*)

lemma *closed_map_Kolmogorov_quotient_explicit*:
closedin X U ⟹ *Kolmogorov_quotient X ' U* = *Kolmogorov_quotient X ' topspace X* ∩ *U*
using *closedin_subset* **by** (*fastforce simp*: *Kolmogorov_quotient_in_closed*)

lemma *closed_map_Kolmogorov_quotient_gen*:
closed_map (*subtopology X S*) (*subtopology X* (*Kolmogorov_quotient X ' S*))

(*Kolmogorov_quotient X*)
using *Kolmogorov_quotient_in_closed* **by** (*force simp*: *closed_map_def closedin_subtopology_alt image_iff*)

lemma *closed_map_Kolmogorov_quotient*:
closed_map X (*subtopology X* (*Kolmogorov_quotient X ‘ topspace X*))
(*Kolmogorov_quotient X*)
by (*metis closed_map_Kolmogorov_quotient_gen subtopology_topspace*)

lemma *quotient_map_Kolmogorov_quotient_gen*:
quotient_map (*subtopology X S*) (*subtopology X* (*Kolmogorov_quotient X ‘ S*)) (*Kolmogorov_quotient X*)
proof (*intro continuous_open_imp_quotient_map*)
show *continuous_map* (*subtopology X S*) (*subtopology X* (*Kolmogorov_quotient X ‘ S*)) (*Kolmogorov_quotient X*)
by (*simp add*: *continuous_map_Kolmogorov_quotient continuous_map_from_subtopology continuous_map_in_subtopology image_mono*)
show *open_map* (*subtopology X S*) (*subtopology X* (*Kolmogorov_quotient X ‘ S*)) (*Kolmogorov_quotient X*)
using *open_map_Kolmogorov_quotient_gen* **by** *blast*
show *Kolmogorov_quotient X ‘ topspace* (*subtopology X S*) = *topspace* (*subtopology X* (*Kolmogorov_quotient X ‘ S*))
by (*force simp*: *Kolmogorov_quotient_in_open*)
qed

lemma *quotient_map_Kolmogorov_quotient*:
quotient_map X (*subtopology X* (*Kolmogorov_quotient X ‘ topspace X*)) (*Kolmogorov_quotient X*)
by (*metis quotient_map_Kolmogorov_quotient_gen subtopology_topspace*)

lemma *Kolmogorov_quotient_eq*:
Kolmogorov_quotient X x = *Kolmogorov_quotient X y* ⟷
(∀ *U*. *openin X U* ⟶ (*x* ∈ *U* ⟷ *y* ∈ *U*)) (**is** *?lhs*=*?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis Kolmogorov_quotient_in_open*)
next
assume *?rhs* **then show** *?lhs*
by (*simp add*: *Kolmogorov_quotient_def*)
qed

lemma *Kolmogorov_quotient_eq_alt*:
Kolmogorov_quotient X x = *Kolmogorov_quotient X y* ⟷
(∀ *U*. *closedin X U* ⟶ (*x* ∈ *U* ⟷ *y* ∈ *U*)) (**is** *?lhs*=*?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis Kolmogorov_quotient_in_closed*)
next
assume *?rhs* **then show** *?lhs*

by (*smt* (*verit*) *Diff_iff Kolmogorov_quotient_eq closedin_topspace in_mono openin_closedin_eq*)
qed

lemma *Kolmogorov_quotient_continuous_map*:
assumes *continuous_map X Y f t0_space Y* **and** *x*: $x \in$ *topspace X*
shows f (*Kolmogorov_quotient X x*) = f *x*
using *assms* **unfolding** *continuous_map_def t0_space_def*
by (*smt* (*verit*, *ccfv_threshold*) *Kolmogorov_quotient_in_open Kolmogorov_quotient_in_topspace PiE mem_Collect_eq*)

lemma *t0_space_Kolmogorov_quotient*:
t0_space (*subtopology X* (*Kolmogorov_quotient X ‘ topspace X*))
apply (*clarsimp simp*: *t0_space_def*)
by (*smt* (*verit*, *best*) *Kolmogorov_quotient_eq imageE image_eqI open_map_Kolmogorov_quotient open_map_def*)

lemma *Kolmogorov_quotient_id*:
t0_space X $\Longrightarrow$ *x* $\in$ *topspace X* $\Longrightarrow$ *Kolmogorov_quotient X x* = *x*
by (*metis Kolmogorov_quotient_in_open Kolmogorov_quotient_in_topspace t0_space_def*)

lemma *Kolmogorov_quotient_idemp*:
Kolmogorov_quotient X (*Kolmogorov_quotient X x*) = *Kolmogorov_quotient X x*
by (*simp add*: *Kolmogorov_quotient_eq Kolmogorov_quotient_in_open*)

lemma *retraction_maps_Kolmogorov_quotient*:
retraction_maps X
(*subtopology X* (*Kolmogorov_quotient X ‘ topspace X*))
(*Kolmogorov_quotient X*) *id*
unfolding *retraction_maps_def continuous_map_in_subtopology*
using *Kolmogorov_quotient_idemp continuous_map_Kolmogorov_quotient* **by** *force*

lemma *retraction_map_Kolmogorov_quotient*:
retraction_map X
(*subtopology X* (*Kolmogorov_quotient X ‘ topspace X*))
(*Kolmogorov_quotient X*)
using *retraction_map_def retraction_maps_Kolmogorov_quotient* **by** *blast*

lemma *retract_of_space_Kolmogorov_quotient_image*:
Kolmogorov_quotient X ‘ topspace X retract_of_space X
proof −
have *continuous_map X X* (*Kolmogorov_quotient X*)
by (*simp add*: *continuous_map_Kolmogorov_quotient*)
then have *Kolmogorov_quotient X ‘ topspace X* $\subseteq$ *topspace X*
by (*simp add*: *continuous_map_image_subset_topspace*)
then show *?thesis*
by (*meson retract_of_space_retraction_maps retraction_maps_Kolmogorov_quotient*)

qed

lemma *Kolmogorov_quotient_lift_exists*:
assumes $S \subseteq$ *topspace X t0_space Y* **and** f: *continuous_map* (*subtopology X S*) *Y f*
obtains g **where** *continuous_map* (*subtopology X* (*Kolmogorov_quotient X ' S*)) *Y g*
$\bigwedge x.\ x \in S \Longrightarrow g(\text{Kolmogorov\_quotient } X\ x) = f\ x$
proof −
have $\bigwedge x\ y.\ [\![x \in S;\ y \in S;$ *Kolmogorov_quotient X x = Kolmogorov_quotient X y*$]\!] \Longrightarrow f\ x = f\ y$
using *assms*
apply (*simp add*: *Kolmogorov_quotient_eq t0_space_def continuous_map_def Int_absorb1 openin_subtopology*)
by (*smt* (*verit*, *del_insts*) *Int_iff mem_Collect_eq Pi_iff*)
then obtain g **where** g: *continuous_map* (*subtopology X* (*Kolmogorov_quotient X ' S*)) *Y g*
g ' (topspace X ∩ *Kolmogorov_quotient X ' S) = f ' S*
$\bigwedge x.\ x \in S \Longrightarrow g$ (*Kolmogorov_quotient X x*) $= f\ x$
using *quotient_map_lift_exists* [*OF quotient_map_Kolmogorov_quotient_gen* [*of X S*] *f*]
by (*metis assms*(*1*) *topspace_subtopology topspace_subtopology_subset*)
show *?thesis*
proof qed (*use g* **in** *auto*)
qed

6.6.9 Closed diagonals and graphs

lemma *Hausdorff_space_closedin_diagonal*:
Hausdorff_space X ⟷ *closedin* (*prod_topology X X*) ((λ*x*. (*x*,*x*)) ' *topspace X*)
proof −
have §: ((λ*x*. (*x*, *x*)) ' *topspace X*) ⊆ *topspace X* × *topspace X*
by *auto*
show *?thesis*
apply (*simp add*: *closedin_def openin_prod_topology_alt Hausdorff_space_def disjnt_iff* §)
apply (*intro all_cong1 imp_cong ex_cong1 conj_cong refl*)
by (*force dest*!: *openin_subset*)+
qed

lemma *closed_map_diag_eq*:
closed_map X (*prod_topology X X*) (λ*x*. (*x*,*x*)) ⟷ *Hausdorff_space X*
proof −
have *section_map X* (*prod_topology X X*) (λ*x*. (*x*, *x*))
unfolding *section_map_def retraction_maps_def*
by (*smt* (*verit*) *continuous_map_fst continuous_map_of_fst continuous_map_on_empty continuous_map_pairwise fst_conv fst_diag_fst snd_diag_fst*)
then have *embedding_map X* (*prod_topology X X*) (λ*x*. (*x*, *x*))
by (*rule section_imp_embedding_map*)

```
  then show ?thesis
   using Hausdorff_space_closedin_diagonal embedding_imp_closed_map_eq by
blast
qed

lemma proper_map_diag_eq [simp]:
   proper_map X (prod_topology X X) (λx. (x,x)) ⟷ Hausdorff_space X
 by (simp add: closed_map_diag_eq inj_on_convol_ident injective_imp_proper_eq_closed_map)

lemma closedin_continuous_maps_eq:
  assumes Hausdorff_space Y and f: continuous_map X Y f and g: continu-
ous_map X Y g
  shows closedin X {x ∈ topspace X. f x = g x}
proof −
  have §:{x ∈ topspace X. f x = g x} = {x ∈ topspace X. (f x,g x) ∈ ((λy.(y,y)) ‘
topspace Y)}
    using f continuous_map_image_subset_topspace by fastforce
  show ?thesis
    unfolding §
  proof (intro closedin_continuous_map_preimage)
    show continuous_map X (prod_topology Y Y) (λx. (f x, g x))
      by (simp add: continuous_map_pairedI f g)
    show closedin (prod_topology Y Y) ((λy. (y, y)) ‘ topspace Y)
      using Hausdorff_space_closedin_diagonal assms by blast
  qed
qed

lemma forall_in_closure_of:
  assumes x ∈ X closure_of S ⋀x. x ∈ S ⟹ P x
    and closedin X {x ∈ topspace X. P x}
  shows P x
 by (smt (verit, ccfv_threshold) Diff_iff assms closedin_def in_closure_of mem_Collect_eq)

lemma forall_in_closure_of_eq:
  assumes x: x ∈ X closure_of S
    and Y: Hausdorff_space Y
    and f: continuous_map X Y f and g: continuous_map X Y g
    and fg: ⋀x. x ∈ S ⟹ f x = g x
  shows f x = g x
proof −
  have closedin X {x ∈ topspace X. f x = g x}
    using Y closedin_continuous_maps_eq f g by blast
  then show ?thesis
    using forall_in_closure_of [OF x fg]
    by fastforce
qed

lemma retract_of_space_imp_closedin:
  assumes Hausdorff_space X and S: S retract_of_space X
```

shows *closedin X S*
proof −
obtain *r* **where** *r*: *continuous_map X* (*subtopology X S*) *r* ∀ *x*∈*S*. *r x* = *x*
using *assms* **by** (*meson retract_of_space_def*)
then have §: *S* = {*x* ∈ *topspace X*. *r x* = *x*}
using *S retract_of_space_imp_subset* **by** (*force simp*: *continuous_map_def Pi_iff*)
show *?thesis*
unfolding §
using *r continuous_map_into_fulltopology assms*
by (*force intro*: *closedin_continuous_maps_eq*)
qed

lemma *homeomorphic_maps_graph*:
homeomorphic_maps X (*subtopology* (*prod_topology X Y*) ((λ*x*. (*x*, *f x*)) ‘ (*topspace X*)))
(λ*x*. (*x*, *f x*)) *fst* ⟷ *continuous_map X Y f*
(**is** *?lhs*=*?rhs*)
proof
assume *?lhs*
then
have *h*: *homeomorphic_map X* (*subtopology* (*prod_topology X Y*) ((λ*x*. (*x*, *f x*)) ‘ *topspace X*)) (λ*x*. (*x*, *f x*))
by (*auto simp*: *homeomorphic_maps_map*)
have *f* = *snd* ∘ (λ*x*. (*x*, *f x*))
by *force*
then show *?rhs*
by (*metis* (*no_types*, *lifting*) *h continuous_map_in_subtopology continuous_map_snd_of homeomorphic_eq_everything_map*)
next
assume *?rhs*
then show *?lhs*
unfolding *homeomorphic_maps_def*
by (*smt* (*verit*, *del_insts*) *continuous_map_eq continuous_map_subtopology_fst embedding_map_def*
embedding_map_graph homeomorphic_eq_everything_map image_cong image_iff prod.sel(*1*))
qed

6.6.10 KC spaces, those where all compact sets are closed.

definition *kc_space*
where *kc_space X* ≡ ∀ *S*. *compactin X S* ⟶ *closedin X S*

lemma *kc_space_euclidean*: *kc_space* (*euclidean* :: *'a*::*metric_space topology*)
by (*simp add*: *compact_imp_closed kc_space_def*)

lemma *kc_space_expansive*:
⟦*kc_space X*; *topspace Y* = *topspace X*; ⋀*U*. *openin X U* ⟹ *openin Y U*⟧

```
    ⟹ kc_space Y
  by (meson compactin_contractive kc_space_def topology_finer_closedin)

lemma compactin_imp_closedin_gen:
   ⟦kc_space X; compactin X S⟧ ⟹ closedin X S
  using kc_space_def by blast

lemma Hausdorff_imp_kc_space: Hausdorff_space X ⟹ kc_space X
  by (simp add: compactin_imp_closedin kc_space_def)

lemma kc_imp_t1_space: kc_space X ⟹ t1_space X
  by (simp add: finite_imp_compactin kc_space_def t1_space_closedin_finite)

lemma kc_space_subtopology:
   kc_space X ⟹ kc_space(subtopology X S)
  by (metis closedin_Int_closure_of closure_of_eq compactin_subtopology inf.absorb2
kc_space_def)

lemma kc_space_discrete_topology:
   kc_space(discrete_topology U)
  using Hausdorff_space_discrete_topology compactin_imp_closedin kc_space_def
by blast

lemma kc_space_continuous_injective_map_preimage:
  assumes kc_space Y continuous_map X Y f and injf: inj_on f (topspace X)
  shows kc_space X
  unfolding kc_space_def
proof (intro strip)
  fix S
  assume S: compactin X S
  have S = {x ∈ topspace X. f x ∈ f ‘ S}
    using S compactin_subset_topspace inj_onD [OF injf] by blast
  with assms S show closedin X S
   by (metis (no_types, lifting) Collect_cong closedin_continuous_map_preimage
compactin_imp_closedin_gen image_compactin)
qed

lemma kc_space_retraction_map_image:
  assumes retraction_map X Y r kc_space X
  shows kc_space Y
proof −
  obtain s where s: continuous_map X Y r continuous_map Y X s ⋀x. x ∈
topspace Y ⟹ r (s x) = x
    using assms by (force simp: retraction_map_def retraction_maps_def)
  then have inj: inj_on s (topspace Y)
    by (metis inj_on_def)
  show ?thesis
    unfolding kc_space_def
  proof (intro strip)
```

fix S
assume S: *compactin* Y S
have $S = \{x \in$ *topspace* Y. s $x \in s$ ‘ $S\}$
using S *compactin_subset_topspace inj_onD* [*OF inj*] **by** *blast*
with *assms* S **show** *closedin* Y S
by (*meson compactin_imp_closedin_gen inj kc_space_continuous_injective_map_preimage s*(*2*))
qed
qed

lemma *homeomorphic_kc_space*:
X *homeomorphic_space* $Y \Longrightarrow$ *kc_space* $X \longleftrightarrow$ *kc_space* Y
by (*meson homeomorphic_eq_everything_map homeomorphic_space homeomorphic_space_sym kc_space_continuous_injective_map_preimage*)

lemma *compact_kc_eq_maximal_compact_space*:
assumes *compact_space* X
shows *kc_space* $X \longleftrightarrow$
$(\forall Y$. *topspace* Y = *topspace* $X \wedge (\forall S$. *openin* X $S \longrightarrow$ *openin* Y $S) \wedge$ *compact_space* $Y \longrightarrow Y = X)$ (**is** *?lhs=?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*metis closedin_compact_space compactin_contractive kc_space_def topology_eq topology_finer_closedin*)
next
assume R: *?rhs*
show *?lhs*
unfolding *kc_space_def*
proof (*intro strip*)
fix S
assume S: *compactin* X S
define Y **where**
$Y \equiv$ *topology* (*arbitrary union_of* (*finite intersection_of* (λA. A = *topspace* $X - S \vee$ *openin* X A)
relative_to (*topspace* X)))
have *topspace* Y = *topspace* X
by (*auto simp*: Y*_def*)
have *openin* X $T \longrightarrow$ *openin* Y T **for** T
by (*simp add*: Y*_def arbitrary_union_of_inc finite_intersection_of_inc openin_subbase openin_subset relative_to_subset_inc*)
have *compact_space* Y
proof (*rule Alexander_subbase_alt*)
show $\exists \mathcal{F}'$. *finite* $\mathcal{F}' \wedge \mathcal{F}' \subseteq \mathcal{C} \wedge$ *topspace* $X \subseteq \bigcup \mathcal{F}'$
if $\mathcal{C}$: $\mathcal{C} \subseteq$ *insert* (*topspace* $X - S$) (*Collect* (*openin* X)) **and** *sub*: *topspace* $X \subseteq \bigcup \mathcal{C}$ **for** $\mathcal{C}$
proof −
consider $\mathcal{C} \subseteq$ *Collect* (*openin* X) | $\mathcal{V}$ **where** $\mathcal{C}$ = *insert* (*topspace* $X - S$) $\mathcal{V}$ $\mathcal{V} \subseteq$ *Collect* (*openin* X)

```
        using C by (metis insert_Diff subset_insert_iff)
       then show ?thesis
       proof cases
        case 1
        then show ?thesis
          by (metis assms compact_space_alt mem_Collect_eq subsetD that(2))
       next
        case 2
        then have S ⊆ ⋃V
          using S sub compactin_subset_topspace by blast
        with 2 obtain F where finite F ∧ F ⊆ V ∧ S ⊆ ⋃F
          using S unfolding compactin_def by (metis Ball_Collect)
        with 2 show ?thesis
          by (rule_tac x=insert (topspace X − S) F in exI) blast
       qed
      qed
    qed (auto simp: Y_def)
    have Y = X
      using R ‹⋀S. openin X S ⟶ openin Y S› ‹compact_space Y› ‹topspace Y
= topspace X› by blast
    moreover have openin Y (topspace X − S)
      by (simp add: Y_def arbitrary_union_of_inc finite_intersection_of_inc
openin_subbase relative_to_subset_inc)
    ultimately show closedin X S
      unfolding closedin_def using S compactin_subset_topspace by blast
  qed
qed

lemma continuous_imp_closed_map_gen:
  ⟦compact_space X; kc_space Y; continuous_map X Y f⟧ ⟹ closed_map X Y
f
  by (meson closed_map_def closedin_compact_space compactin_imp_closedin_gen
image_compactin)

lemma kc_space_compact_subtopologies:
  kc_space X ⟷ (∀K. compactin X K ⟶ kc_space(subtopology X K)) (is
?lhs=?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (auto simp: kc_space_def closedin_closed_subtopology compactin_subtopology)
next
  assume R: ?rhs
  show ?lhs
    unfolding kc_space_def
  proof (intro strip)
    fix K
    assume K: compactin X K
    then have K ⊆ topspace X
```

```
    by (simp add: compactin_subset_topspace)
  moreover have X closure_of K ⊆ K
  proof
    fix x
    assume x: x ∈ X closure_of K
    have kc_space (subtopology X K)
      by (simp add: R ‹compactin X K›)
    have compactin X (insert x K)
     by (metis K x compactin_Un compactin_sing in_closure_of insert_is_Un)
    then show x ∈ K
    by (metis R x K Int_insert_left_if1 closedin_Int_closure_of compact_imp_compactin_subtopology
          insertCI kc_space_def subset_insertI)
  qed
  ultimately show closedin X K
    using closure_of_subset_eq by blast
 qed
qed

lemma kc_space_compact_prod_topology:
 assumes compact_space X
 shows kc_space(prod_topology X X) ⟷ Hausdorff_space X (is ?lhs=?rhs)
proof
 assume L: ?lhs
 show ?rhs
  unfolding closed_map_diag_eq [symmetric]
 proof (intro continuous_imp_closed_map_gen)
  show continuous_map X (prod_topology X X) (λx. (x, x))
   by (intro continuous_intros)
 qed (use L assms in auto)
next
 assume ?rhs then show ?lhs
  by (simp add: Hausdorff_imp_kc_space Hausdorff_space_prod_topology)
qed

lemma kc_space_prod_topology:
 kc_space(prod_topology X X) ⟷ (∀ K. compactin X K ⟶ Hausdorff_space(subtopology
X K)) (is ?lhs=?rhs)
proof
 assume ?lhs
 then show ?rhs
  by (metis compactin_subspace kc_space_compact_prod_topology kc_space_subtopology
subtopology_Times)
next
 assume R: ?rhs
 have kc_space (subtopology (prod_topology X X) L) if compactin (prod_topology
X X) L for L
 proof −
  define K where K ≡ fst ‘ L ∪ snd ‘ L
```

```
    have L ⊆ K × K
      by (force simp: K_def)
    have compactin X K
      by (metis K_def compactin_Un continuous_map_fst continuous_map_snd
image_compactin that)
    then have Hausdorff_space (subtopology X K)
      by (simp add: R)
    then have kc_space (prod_topology (subtopology X K) (subtopology X K))
    by (simp add: ‹compactin X K› compact_space_subtopology kc_space_compact_prod_topology)
   then have kc_space (subtopology (prod_topology (subtopology X K) (subtopology
X K)) L)
      using kc_space_subtopology by blast
    then show ?thesis
      using ‹L ⊆ K × K› subtopology_Times subtopology_subtopology
      by (metis (no_types, lifting) Sigma_cong inf.absorb_iff2)
  qed
  then show ?lhs
    using kc_space_compact_subtopologies by blast
qed

lemma kc_space_prod_topology_alt:
   kc_space(prod_topology X X) ⟷
        kc_space X ∧
        (∀ K. compactin X K ⟶ Hausdorff_space(subtopology X K))
 using Hausdorff_imp_kc_space kc_space_compact_subtopologies kc_space_prod_topology
by blast

proposition kc_space_prod_topology_left:
  assumes X: kc_space X and Y: Hausdorff_space Y
  shows kc_space (prod_topology X Y)
  unfolding kc_space_def
proof (intro strip)
  fix K
  assume K: compactin (prod_topology X Y) K
  then have K ⊆ topspace X × topspace Y
    using compactin_subset_topspace topspace_prod_topology by blast
 moreover have ∃ T. openin (prod_topology X Y) T ∧ (a,b) ∈ T ∧ T ⊆ (topspace
X × topspace Y) − K
    if ab: (a,b) ∈ (topspace X × topspace Y) − K for a b
  proof −
    have compactin Y {b}
      using that by force
    moreover
    have compactin Y {y ∈ topspace Y. (a,y) ∈ K}
    proof −
      have compactin (prod_topology X Y) (K ∩ {a} × topspace Y)
        using that compact_Int_closedin [OF K]
        by (simp add: X closedin_prod_Times_iff compactin_imp_closedin_gen)
      moreover have subtopology (prod_topology X Y) (K ∩ {a} × topspace Y)
```

homeomorphic_space
subtopology Y {y ∈ topspace Y. (a, y) ∈ K}
unfolding *homeomorphic_space_def homeomorphic_maps_def*
using *that*
apply (*rule_tac x=snd* **in** *exI*)
apply (*rule_tac x=Pair a* **in** *exI*)
by (*force simp*: *continuous_map_in_subtopology continuous_map_from_subtopology continuous_map_subtopology_snd continuous_map_paired*)
ultimately show *?thesis*
by (*simp add*: *compactin_subspace homeomorphic_compact_space*)
qed
moreover have *disjnt* {*b*} {*y ∈ topspace Y. (a,y) ∈ K*}
using *ab* **by** *force*
ultimately obtain *V U* **where** *VU*: *openin Y V openin Y U* {*b*} ⊆ *V* {*y ∈ topspace Y. (a,y) ∈ K*} ⊆ *U disjnt V U*
using *Hausdorff_space_compact_separation* [*OF Y*] **by** *blast*
define *V′* **where** *V′ ≡ topspace Y − U*
have *W*: *closedin Y V′* {*y ∈ topspace Y. (a,y) ∈ K*} ⊆ *topspace Y − V′ disjnt V* (*topspace Y − V′*)
using *VU* **by** (*auto simp*: *V′_def disjnt_iff*)
with *VU* **obtain** *V* ⊆ *topspace Y V′* ⊆ *topspace Y*
by (*meson closedin_subset openin_closedin_eq*)
then obtain *b ∈ V disjnt* {*y ∈ topspace Y. (a,y) ∈ K*} *V′ V* ⊆ *V′*
using *VU* **unfolding** *disjnt_iff V′_def* **by** *force*
define *C* **where** *C ≡ image fst* (*K* ∩ {*z ∈ topspace*(*prod_topology X Y*). *snd z ∈ V′*})
have *closedin* (*prod_topology X Y*) {*z ∈ topspace* (*prod_topology X Y*). *snd z ∈ V′*}
using *closedin_continuous_map_preimage* ‹*closedin Y V′*› *continuous_map_snd* **by** *blast*
then have *compactin X C*
unfolding *C_def* **by** (*meson K compact_Int_closedin continuous_map_fst image_compactin*)
then have *closedin X C*
using *assms* **by** (*auto simp*: *kc_space_def*)
show *?thesis*
proof (*intro exI conjI*)
show *openin* (*prod_topology X Y*) ((*topspace X − C*) × *V*)
by (*simp add*: *VU* ‹*closedin X C*› *openin_diff openin_prod_Times_iff*)
have *a ∉ C*
using *VU* **by** (*auto simp*: *C_def V′_def*)
then show (*a*, *b*) ∈ (*topspace X − C*) × *V*
using ‹*a ∉ C*› ‹*b ∈ V*› *that* **by** *blast*
show (*topspace X − C*) × *V* ⊆ *topspace X* × *topspace Y − K*
using ‹*V* ⊆ *V′*› ‹*V* ⊆ *topspace Y*›
apply (*simp add*: *C_def*)
by (*smt* (*verit*, *ccfv_threshold*) *DiffE DiffI IntI SigmaE SigmaI image_eqI mem_Collect_eq prod.sel*(*1*) *snd_conv subset_iff*)
qed

qed
ultimately show *closedin* (*prod_topology X Y*) *K*
by (*metis surj_pair closedin_def openin_subopen topspace_prod_topology*)
qed

lemma *kc_space_prod_topology_right*:
⟦*Hausdorff_space X*; *kc_space Y*⟧ ⟹ *kc_space* (*prod_topology X Y*)
using *kc_space_prod_topology_left homeomorphic_kc_space homeomorphic_space_prod_topology_su*
by *blast*

6.6.11 Technical results about proper maps, perfect maps, etc

lemma *compact_imp_proper_map_gen*:
assumes *Y*: ⋀*S*. ⟦*S* ⊆ *topspace Y*; ⋀*K*. *compactin Y K* ⟹ *compactin Y* (*S* ∩ *K*)⟧
⟹ *closedin Y S*
and *fim*: *f* ‘ (*topspace X*) ⊆ *topspace Y*
and *f*: *continuous_map X Y f* ∨ *kc_space X*
and *YX*: ⋀*K*. *compactin Y K* ⟹ *compactin X* {*x* ∈ *topspace X*. *f x* ∈ *K*}
shows *proper_map X Y f*
unfolding *proper_map_alt closed_map_def*
proof (*intro conjI strip*)
fix *C*
assume *C*: *closedin X C*
show *closedin Y* (*f* ‘ *C*)
proof (*intro Y*)
show *f* ‘ *C* ⊆ *topspace Y*
using *C closedin_subset fim* **by** *blast*
fix *K*
assume *K*: *compactin Y K*
define *A* **where** *A* ≡ {*x* ∈ *topspace X*. *f x* ∈ *K*}
have *eq*: *f* ‘ *C* ∩ *K* = *f* ‘ ({*x* ∈ *topspace X*. *f x* ∈ *K*} ∩ *C*)
using *C closedin_subset* **by** *auto*
show *compactin Y* (*f* ‘ *C* ∩ *K*)
unfolding *eq*
proof (*rule image_compactin*)
show *compactin* (*subtopology X A*) ({*x* ∈ *topspace X*. *f x* ∈ *K*} ∩ *C*)
proof (*rule closedin_compact_space*)
show *compact_space* (*subtopology X A*)
by (*simp add*: *A_def K YX compact_space_subtopology*)
show *closedin* (*subtopology X A*) ({*x* ∈ *topspace X*. *f x* ∈ *K*} ∩ *C*)
using *A_def C closedin_subtopology* **by** *blast*
qed
have *continuous_map* (*subtopology X A*) (*subtopology Y K*) *f* **if** *kc_space X*
unfolding *continuous_map_closedin*
proof (*intro conjI strip*)
show *f* ∈ *topspace* (*subtopology X A*) → *topspace* (*subtopology Y K*)
using *A_def K compactin_subset_topspace* **by** *fastforce*

```
      next
        fix C
        assume C: closedin (subtopology Y K) C
       show closedin (subtopology X A) {x ∈ topspace (subtopology X A). f x ∈ C}
        proof (rule compactin_imp_closedin_gen)
          show kc_space (subtopology X A)
            by (simp add: kc_space_subtopology that)
         have [simp]: {x ∈ topspace X. f x ∈ K ∧ f x ∈ C} = {x ∈ topspace X. f
x ∈ C}
            using C closedin_imp_subset by auto
          have compactin (subtopology Y K) C
           by (simp add: C K closedin_compact_space compact_space_subtopology)
          then have compactin X {x ∈ topspace X. x ∈ A ∧ f x ∈ C}
            by (auto simp: A_def compactin_subtopology dest: YX)
         then show compactin (subtopology X A) {x ∈ topspace (subtopology X A).
f x ∈ C}
            by (auto simp add: compactin_subtopology)
        qed
      qed
      with f show continuous_map (subtopology X A) Y f
       using continuous_map_from_subtopology continuous_map_in_subtopology
by blast
    qed
  qed
qed (simp add: YX)

lemma tube_lemma_left:
  assumes W: openin (prod_topology X Y) W and C: compactin X C
    and y: y ∈ topspace Y and subW: C × {y} ⊆ W
  shows ∃ U V. openin X U ∧ openin Y V ∧ C ⊆ U ∧ y ∈ V ∧ U × V ⊆ W
proof (cases C = {})
  case True
  with y show ?thesis by auto
next
  case False
  have ∃ U V. openin X U ∧ openin Y V ∧ x ∈ U ∧ y ∈ V ∧ U × V ⊆ W
    if x ∈ C for x
    using W openin_prod_topology_alt subW subsetD that by fastforce
  then obtain U V where UV: ⋀x. x ∈ C ⟹ openin X (U x) ∧ openin Y (V
x) ∧ x ∈ U x ∧ y ∈ V x ∧ U x × V x ⊆ W
    by metis
  then obtain D where D: finite D D ⊆ C C ⊆ ⋃ (U ‘ D)
    using compactinD [OF C, of U‘C]
    by (smt (verit) UN_I finite_subset_image imageE subsetI)
  show ?thesis
  proof (intro exI conjI)
    show openin X (⋃ (U ‘ D)) openin Y (⋂ (V ‘ D))
      using D False UV by blast+
    show y ∈ ⋂ (V ‘ D) C ⊆ ⋃ (U ‘ D) ⋃(U ‘ D) × ⋂(V ‘ D) ⊆ W
```

```
    using D UV by force+
  qed
qed

lemma Wallace_theorem_prod_topology:
  assumes compactin X K compactin Y L
    and W: openin (prod_topology X Y) W and subW: K × L ⊆ W
  obtains U V where openin X U openin Y V K ⊆ U L ⊆ V U × V ⊆ W
proof −
  have ⋀y. y ∈ L ⟹ ∃ U V. openin X U ∧ openin Y V ∧ K ⊆ U ∧ y ∈ V ∧ U
× V ⊆ W
  proof (intro tube_lemma_left assms)
    fix y assume y ∈ L
    show y ∈ topspace Y
      using assms ‹y ∈ L› compactin_subset_topspace by blast
    show K × {y} ⊆ W
      using ‹y ∈ L› subW by force
  qed
  then obtain U V where UV:
        ⋀y. y ∈ L ⟹ openin X (U y) ∧ openin Y (V y) ∧ K ⊆ U y ∧ y ∈ V y
∧ U y × V y ⊆ W
    by metis
  then obtain M where finite M M ⊆ L and M: L ⊆ ⋃ (V ‘ M)
    using ‹compactin Y L› unfolding compactin_def
    by (smt (verit) UN_iff finite_subset_image imageE subset_iff)
  show thesis
  proof (cases M={})
    case True
    with M have L={}
      by blast
    then show ?thesis
      using ‹compactin X K› compactin_subset_topspace that by fastforce
  next
    case False
    show ?thesis
    proof
      show openin X (⋂ (U‘M))
        using False UV ‹M ⊆ L› ‹finite M› by blast
      show openin Y (⋃ (V‘M))
        using UV ‹M ⊆ L› by blast
      show K ⊆ ⋂ (U‘M)
        by (meson INF_greatest UV ‹M ⊆ L› subsetD)
      show L ⊆ ⋃ (V‘M)
        by (simp add: M)
      show ⋂ (U‘M) × ⋃ (V‘M) ⊆ W
        using UV ‹M ⊆ L› by fastforce
    qed
  qed
qed
```

lemma *proper_map_prod*:
proper_map (*prod_topology X Y*) (*prod_topology X′ Y′*) (λ(*x*,*y*). (*f x*, *g y*)) ⟷
(*prod_topology X Y*) = *trivial_topology* ∨ *proper_map X X′ f* ∧ *proper_map Y Y′ g*
(**is** *?lhs* ⟷ _ ∨ *?rhs*)
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
case *True* **then show** *?thesis* **by** *auto*
next
case *False*
then have *ne*: *topspace X* ≠ {} *topspace Y* ≠ {}
by *auto*
define *h* **where** *h* ≡ λ(*x*,*y*). (*f x*, *g y*)
have *proper_map X X′ f proper_map Y Y′ g* **if** *?lhs*
proof −
have *cm*: *closed_map X X′ f closed_map Y Y′ g*
using *that False closed_map_prod proper_imp_closed_map* **by** *blast+*
show *proper_map X X′ f*
proof (*clarsimp simp add*: *proper_map_def cm*)
fix *y*
assume *y*: *y* ∈ *topspace X′*
obtain *z* **where** *z*: *z* ∈ *topspace Y*
using *ne* **by** *blast*
then have *eq*: {*x* ∈ *topspace X*. *f x* = *y*} =
fst ‘ {*u* ∈ *topspace X* × *topspace Y*. *h u* = (*y*,*g z*)}
by (*force simp*: *h_def*)
show *compactin X* {*x* ∈ *topspace X*. *f x* = *y*}
unfolding *eq*
proof (*intro image_compactin*)
have *g z* ∈ *topspace Y′*
by (*meson closed_map_def closedin_subset closedin_topspace cm image_subset_iff z*)
with *y* **show** *compactin* (*prod_topology X Y*) {*u* ∈ *topspace X* × *topspace Y*. (*h u*) = (*y*, *g z*)}
using *that* **by** (*simp add*: *h_def proper_map_def*)
show *continuous_map* (*prod_topology X Y*) *X fst*
by (*simp add*: *continuous_map_fst*)
qed
qed
show *proper_map Y Y′ g*
proof (*clarsimp simp add*: *proper_map_def cm*)
fix *y*
assume *y*: *y* ∈ *topspace Y′*
obtain *z* **where** *z*: *z* ∈ *topspace X*
using *ne* **by** *blast*
then have *eq*: {*x* ∈ *topspace Y*. *g x* = *y*} =
snd ‘ {*u* ∈ *topspace X* × *topspace Y*. *h u* = (*f z*,*y*)}
by (*force simp*: *h_def*)
show *compactin Y* {*x* ∈ *topspace Y*. *g x* = *y*}

```
      unfolding eq
     proof (intro image_compactin)
      have f z ∈ topspace X'
          by (meson closed_map_def closedin_subset closedin_topspace cm im-
age_subset_iff z)
       with y show compactin (prod_topology X Y) {u ∈ topspace X × topspace
Y. (h u) = (f z, y)}
        using that by (simp add: proper_map_def h_def)
       show continuous_map (prod_topology X Y) Y snd
        by (simp add: continuous_map_snd)
     qed
    qed
  qed
  moreover
  { assume R: ?rhs
    then have fgim: f ∈ topspace X → topspace X' g ∈ topspace Y → topspace
Y'
         and cm: closed_map X X' f closed_map Y Y' g
     by (auto simp: proper_map_def closed_map_imp_subset_topspace)
    have closed_map (prod_topology X Y) (prod_topology X' Y') h
     unfolding closed_map_fibre_neighbourhood imp_conjL
    proof (intro conjI strip)
     show h ∈ topspace (prod_topology X Y) → topspace (prod_topology X' Y')
      unfolding h_def using fgim by auto
     fix W w
     assume W: openin (prod_topology X Y) W
      and w: w ∈ topspace (prod_topology X' Y')
      and subW: {x ∈ topspace (prod_topology X Y). h x = w} ⊆ W
    then obtain x' y' where weq: w = (x',y') x' ∈ topspace X' y' ∈ topspace Y'
      by auto
     have eq: {u ∈ topspace X × topspace Y. h u = (x',y')} = {x ∈ topspace X.
f x = x'} × {y ∈ topspace Y. g y = y'}
      by (auto simp: h_def)
     obtain U V where openin X U openin Y V U × V ⊆ W
      and U: {x ∈ topspace X. f x = x'} ⊆ U
      and V: {x ∈ topspace Y. g x = y'} ⊆ V
     proof (rule Wallace_theorem_prod_topology)
      show compactin X {x ∈ topspace X. f x = x'} compactin Y {x ∈ topspace
Y. g x = y'}
       using R weq unfolding proper_map_def closed_map_fibre_neighbourhood
by fastforce+
      show {x ∈ topspace X. f x = x'} × {x ∈ topspace Y. g x = y'} ⊆ W
       using weq subW by (auto simp: h_def)
     qed (use W in auto)
    obtain U' where openin X' U' x' ∈ U' and U': {x ∈ topspace X. f x ∈ U'}
⊆ U
     using cm U ‹openin X U› weq unfolding closed_map_fibre_neighbourhood
by meson
     obtain V' where openin Y' V' y' ∈ V' and V': {x ∈ topspace Y. g x ∈
```

```
V'} ⊆ V
      using cm V ‹openin Y V› weq unfolding closed_map_fibre_neighbourhood
by meson
      show ∃V. openin (prod_topology X' Y') V ∧ w ∈ V ∧ {x ∈ topspace
(prod_topology X Y). h x ∈ V} ⊆ W
     proof (intro conjI exI)
       show openin (prod_topology X' Y') (U' × V')
         by (simp add: ‹openin X' U'› ‹openin Y' V'› openin_prod_Times_iff)
       show w ∈ U' × V'
         using ‹x' ∈ U'› ‹y' ∈ V'› weq by blast
       show {x ∈ topspace (prod_topology X Y). h x ∈ U' × V'} ⊆ W
         using ‹U × V ⊆ W› U' V' h_def by auto
     qed
   qed
   moreover
    have compactin (prod_topology X Y) {u ∈ topspace X × topspace Y. h u =
(w, z)}
     if w ∈ topspace X' and z ∈ topspace Y' for w z
   proof −
     have eq: {u ∈ topspace X × topspace Y. h u = (w,z)} =
              {u ∈ topspace X. f u = w} × {y. y ∈ topspace Y ∧ g y = z}
       by (auto simp: h_def)
     show ?thesis
       using R that by (simp add: eq compactin_Times proper_map_def)
   qed
   ultimately have ?lhs
     by (auto simp: h_def proper_map_def)
 }
 ultimately show ?thesis using False by metis
qed

lemma proper_map_paired:
 assumes Hausdorff_space X ∧ proper_map X Y f ∧ proper_map X Z g ∨
       Hausdorff_space Y ∧ continuous_map X Y f ∧ proper_map X Z g ∨
       Hausdorff_space Z ∧ proper_map X Y f ∧ continuous_map X Z g
 shows proper_map X (prod_topology Y Z) (λx. (f x,g x))
 using assms
proof (elim disjE conjE)
 assume §: Hausdorff_space X proper_map X Y f proper_map X Z g
 have eq: (λx. (f x, g x)) = (λ(x, y). (f x, g y)) ∘ (λx. (x, x))
   by auto
 show proper_map X (prod_topology Y Z) (λx. (f x, g x))
   unfolding eq
 proof (rule proper_map_compose)
   show proper_map X (prod_topology X X) (λx. (x,x))
     by (simp add: §)
    show proper_map (prod_topology X X) (prod_topology Y Z) (λ(x,y). (f x, g
y))
     by (simp add: § proper_map_prod)
```

qed
next
assume §: *Hausdorff_space Y continuous_map X Y f proper_map X Z g*
have *eq*: $(\lambda x.\ (f\ x,\ g\ x)) = (\lambda(x,y).\ (x,g\ y)) \circ (\lambda x.\ (f\ x,x))$
by *auto*
show *proper_map X (prod_topology Y Z) (λx. (f x, g x))*
unfolding *eq*
proof (*rule proper_map_compose*)
show *proper_map X (prod_topology Y X) (λx. (f x,x))*
by (*simp add*: § *proper_map_paired_continuous_map_left*)
show *proper_map (prod_topology Y X) (prod_topology Y Z) (λ(x,y). (x,g y))*
by (*simp add*: § *proper_map_prod proper_map_id* [*unfolded id_def*])
qed
next
assume §: *Hausdorff_space Z proper_map X Y f continuous_map X Z g*
have *eq*: $(\lambda x.\ (f\ x,\ g\ x)) = (\lambda(x,y).\ (f\ x,y)) \circ (\lambda x.\ (x,g\ x))$
by *auto*
show *proper_map X (prod_topology Y Z) (λx. (f x, g x))*
unfolding *eq*
proof (*rule proper_map_compose*)
show *proper_map X (prod_topology X Z) (λx. (x, g x))*
using § *proper_map_paired_continuous_map_right* **by** *auto*
show *proper_map (prod_topology X Z) (prod_topology Y Z) (λ(x,y). (f x,y))*
by (*simp add*: § *proper_map_prod proper_map_id* [*unfolded id_def*])
qed
qed

lemma *proper_map_pairwise*:
assumes
Hausdorff_space X ∧ *proper_map X Y* (*fst* ∘ *f*) ∧ *proper_map X Z* (*snd* ∘ *f*)
∨
Hausdorff_space Y ∧ *continuous_map X Y* (*fst* ∘ *f*) ∧ *proper_map X Z* (*snd*
∘ *f*) ∨
Hausdorff_space Z ∧ *proper_map X Y* (*fst* ∘ *f*) ∧ *continuous_map X Z* (*snd*
∘ *f*)
shows *proper_map X* (*prod_topology Y Z*) *f*
using *proper_map_paired* [*OF assms*] **by** (*simp add*: *o_def*)

lemma *proper_map_from_composition_right*:
assumes *Hausdorff_space Y proper_map X Z* (*g* ∘ *f*) **and** *contf*: *continuous_map X Y f*
and *contg*: *continuous_map Y Z g*
shows *proper_map X Y f*
proof −
define *YZ* **where** *YZ* ≡ *subtopology* (*prod_topology Y Z*) ((λx. (x, g x)) ‘
topspace Y)
have *proper_map X Y* (*fst* ∘ (λx. (*f x*, (*g* ∘ *f*) *x*)))
proof (*rule proper_map_compose*)
have [*simp*]: *x* ∈ *topspace X* ⟹ *f x* ∈ *topspace Y* **for** *x*

using *contf continuous_map_preimage_topspace* **by** *auto*
show *proper_map X YZ* (λ*x*. (*f x*, (*g* ∘ *f*) *x*))
unfolding *YZ_def*
using *assms*
by (*force intro*!: *proper_map_into_subtopology proper_map_paired simp*: *o_def image_iff*)+
show *proper_map YZ Y fst*
using *contg*
by (*simp flip*: *homeomorphic_maps_graph add*: *YZ_def homeomorphic_maps_map homeomorphic_imp_proper_map*)
qed
moreover have *fst* ∘ (λ*x*. (*f x*, (*g* ∘ *f*) *x*)) = *f*
by *auto*
ultimately show *?thesis*
by *auto*
qed

lemma *perfect_map_from_composition_right*:
⟦*Hausdorff_space Y*; *perfect_map X Z* (*g* ∘ *f*);
continuous_map X Y f; *continuous_map Y Z g*; *f* ‘ *topspace X* = *topspace Y*⟧
⟹ *perfect_map X Y f*
by (*meson perfect_map_def proper_map_from_composition_right*)

lemma *perfect_map_from_composition_right_inj*:
⟦*perfect_map X Z* (*g* ∘ *f*); *f* ‘ *topspace X* = *topspace Y*;
continuous_map X Y f; *continuous_map Y Z g*; *inj_on g* (*topspace Y*)⟧
⟹ *perfect_map X Y f*
by (*meson continuous_map_openin_preimage_eq perfect_map_def proper_map_from_composition_right_inj*)

6.6.12 Regular spaces

Regular spaces are *not* a priori assumed to be Hausdorff or T_1

definition *regular_space*
where *regular_space X* ≡
∀ *C a*. *closedin X C* ∧ *a* ∈ *topspace X* − *C*
⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *a* ∈ *U* ∧ *C* ⊆ *V* ∧ *disjnt U V*)

lemma *homeomorphic_regular_space_aux*:
assumes *hom*: *X homeomorphic_space Y* **and** *X*: *regular_space X*
shows *regular_space Y*
proof −
obtain *f g* **where** *hmf*: *homeomorphic_map X Y f* **and** *hmg*: *homeomorphic_map Y X g*
and *fg*: (∀ *x* ∈ *topspace X*. *g*(*f x*) = *x*) ∧ (∀ *y* ∈ *topspace Y*. *f*(*g y*) = *y*)
using *assms X homeomorphic_maps_map homeomorphic_space_def* **by** *fastforce*
show *?thesis*
unfolding *regular_space_def*

proof *clarify*
fix *C a*
assume *Y*: *closedin Y C a* ∈ *topspace Y* **and** *a* ∉ *C*
then obtain *closedin X* (*g* ‘ *C*) *g a* ∈ *topspace X g a* ∉ *g* ‘ *C*
using ‹*closedin Y C*› *hmg homeomorphic_map_closedness_eq*
by (*smt* (*verit*, *ccfv_SIG*) *fg homeomorphic_imp_surjective_map image_iff in_mono*)
then obtain *S T* **where** *ST*: *openin X S openin X T g a* ∈ *S g*‘*C* ⊆ *T disjnt S T*
using *X* **unfolding** *regular_space_def* **by** (*metis DiffI*)
then have *openin Y* (*f*‘*S*) *openin Y* (*f*‘*T*)
by (*meson hmf homeomorphic_map_openness_eq*)+
moreover have *a* ∈ *f*‘*S* ∧ *C* ⊆ *f*‘*T*
using *ST* **by** (*smt* (*verit*, *best*) *Y closedin_subset fg image_eqI subset_iff*)
moreover have *disjnt* (*f*‘*S*) (*f*‘*T*)
using *ST* **by** (*smt* (*verit*, *ccfv_SIG*) *disjnt_iff fg image_iff openin_subset subsetD*)
ultimately show ∃ *U V*. *openin Y U* ∧ *openin Y V* ∧ *a* ∈ *U* ∧ *C* ⊆ *V* ∧ *disjnt U V*
by *metis*
qed
qed

lemma *homeomorphic_regular_space*:
X homeomorphic_space Y
⟹ (*regular_space X* ⟷ *regular_space Y*)
by (*meson homeomorphic_regular_space_aux homeomorphic_space_sym*)

lemma *regular_space*:
regular_space X ⟷
(∀ *C a*. *closedin X C* ∧ *a* ∈ *topspace X* − *C*
⟶ (∃ *U*. *openin X U* ∧ *a* ∈ *U* ∧ *disjnt C* (*X closure_of U*)))
unfolding *regular_space_def*
proof (*intro all_cong1 imp_cong refl ex_cong1*)
fix *C a U*
assume *C*: *closedin X C* ∧ *a* ∈ *topspace X* − *C*
show (∃ *V*. *openin X U* ∧ *openin X V* ∧ *a* ∈ *U* ∧ *C* ⊆ *V* ∧ *disjnt U V*)
⟷ (*openin X U* ∧ *a* ∈ *U* ∧ *disjnt C* (*X closure_of U*)) (**is** *?lhs*=*?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*smt* (*verit*, *best*) *disjnt_iff in_closure_of subsetD*)
next
assume *R*: *?rhs*
then have *disjnt U* (*topspace X* − *X closure_of U*)
by (*meson DiffD2 closure_of_subset disjnt_iff openin_subset subsetD*)
moreover have *C* ⊆ *topspace X* − *X closure_of U*
by (*meson C DiffI R closedin_subset disjnt_iff subset_eq*)
ultimately show *?lhs*

```
    using R by (rule_tac x=topspace X - X closure_of U in exI) auto
   qed
qed

lemma neighbourhood_base_of_closedin:
  neighbourhood_base_of (closedin X) X ⟷ regular_space X (is ?lhs=?rhs)
proof -
  have ?lhs ⟷ (∀ W x. openin X W ∧ x ∈ W ⟶
                (∃ U. openin X U ∧ (∃ V. closedin X V ∧ x ∈ U ∧ U ⊆ V ∧ V ⊆
W)))
    by (simp add: neighbourhood_base_of)
  also have ... ⟷ (∀ W x. closedin X W ∧ x ∈ topspace X - W ⟶
                  (∃ U. openin X U ∧ (∃ V. closedin X V ∧ x ∈ U ∧ U ⊆ V ∧ V
⊆ topspace X - W)))
   by (smt (verit) Diff_Diff_Int closedin_def inf.absorb_iff2 openin_closedin_eq)
  also have ... ⟷ ?rhs
  proof -
    have §: (∃ V. closedin X V ∧ x ∈ U ∧ U ⊆ V ∧ V ⊆ topspace X - W)
        ⟷ (∃ V. openin X V ∧ x ∈ U ∧ W ⊆ V ∧ disjnt U V) (is ?lhs=?rhs)
      if openin X U  closedin X W x ∈ topspace X x ∉ W for W U x
    proof
      assume ?lhs with ‹closedin X W› show ?rhs
       unfolding closedin_def by (smt (verit) Diff_mono disjnt_Diff1 double_diff
subset_eq)
    next
      assume ?rhs with ‹openin X U› show ?lhs
        unfolding openin_closedin_eq disjnt_def
         by (smt (verit) Diff_Diff_Int Diff_disjoint Diff_eq_empty_iff Int_Diff
inf.orderE)
    qed
    show ?thesis
      unfolding regular_space_def
      by (intro all_cong1 ex_cong1 imp_cong refl) (metis § DiffE)
  qed
  finally show ?thesis .
qed

lemma regular_space_discrete_topology [simp]:
   regular_space (discrete_topology S)
 using neighbourhood_base_of_closedin neighbourhood_base_of_discrete_topology
by fastforce

lemma regular_space_subtopology:
 regular_space X ⟹ regular_space (subtopology X S)
  unfolding regular_space_def openin_subtopology_alt closedin_subtopology_alt
disjnt_iff
  by clarsimp (smt (verit, best) inf.orderE inf_le1 le_inf_iff)
```

lemma *regular_space_retraction_map_image*:
  ⟦*retraction_map X Y r*; *regular_space X*⟧ ⟹ *regular_space Y*
  using *hereditary_imp_retractive_property homeomorphic_regular_space regular_space_subtopology* **by** *blast*

lemma *regular_t0_imp_Hausdorff_space*:
  ⟦*regular_space X*; *t0_space X*⟧ ⟹ *Hausdorff_space X*
 apply (*clarsimp simp*: *regular_space_def t0_space Hausdorff_space_def*)
 by (*metis disjnt_sym subsetD*)

lemma *regular_t0_eq_Hausdorff_space*:
  regular_space X ⟹ (*t0_space X* ⟷ *Hausdorff_space X*)
 using *Hausdorff_imp_t0_space regular_t0_imp_Hausdorff_space* **by** *blast*

lemma *regular_t1_imp_Hausdorff_space*:
  ⟦*regular_space X*; *t1_space X*⟧ ⟹ *Hausdorff_space X*
 by (*simp add*: *regular_t0_imp_Hausdorff_space t1_imp_t0_space*)

lemma *regular_t1_eq_Hausdorff_space*:
  regular_space X ⟹ *t1_space X* ⟷ *Hausdorff_space X*
 using *regular_t0_imp_Hausdorff_space t1_imp_t0_space t1_or_Hausdorff_space*
by *blast*

lemma *compact_Hausdorff_imp_regular_space*:
 assumes *compact_space X Hausdorff_space X*
 shows *regular_space X*
 unfolding *regular_space_def*
proof *clarify*
 fix *S a*
 assume *closedin X S* **and** *a* ∈ *topspace X* **and** *a* ∉ *S*
 then show ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *a* ∈ *U* ∧ *S* ⊆ *V* ∧ *disjnt U V*
  using *assms* **unfolding** *Hausdorff_space_compact_sets*
  by (*metis closedin_compact_space compactin_sing disjnt_empty1 insert_subset disjnt_insert1*)
qed

lemma *neighbourhood_base_of_closed_Hausdorff_space*:
  regular_space X ∧ *Hausdorff_space X* ⟷
   neighbourhood_base_of (λ*C*. *closedin X C* ∧ *Hausdorff_space*(*subtopology X C*)) *X* (**is** *?lhs*=*?rhs*)
proof
 assume *?lhs* **then show** *?rhs*
  by (*simp add*: *Hausdorff_space_subtopology neighbourhood_base_of_closedin*)
next
 assume *?rhs* **then show** *?lhs*
  by (*metis* (*mono_tags*, *lifting*) *Hausdorff_space_closed_neighbourhood neighbourhood_base_of neighbourhood_base_of_closedin openin_topspace*)
qed

lemma *locally_compact_imp_kc_eq_Hausdorff_space*:
neighbourhood_base_of (*compactin X*) *X* ⟹ *kc_space X* ⟷ *Hausdorff_space X*
by (*metis Hausdorff_imp_kc_space kc_imp_t1_space kc_space_def neighbourhood_base_of_closedin neighbourhood_base_of_mono regular_t1_imp_Hausdorff_space*)

lemma *regular_space_compact_closed_separation*:
assumes *X*: *regular_space X*
and *S*: *compactin X S*
and *T*: *closedin X T*
and *disjnt S T*
shows ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*
proof (*cases S*={})
case *True*
then show *?thesis*
by (*meson T closedin_def disjnt_empty1 empty_subsetI openin_empty openin_topspace*)
next
case *False*
then have ∃ *U V*. *x* ∈ *S* ⟶ *openin X U* ∧ *openin X V* ∧ *x* ∈ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V* **for** *x*
using *assms* **unfolding** *regular_space_def*
by (*smt* (*verit*) *Diff_iff compactin_subset_topspace disjnt_iff subsetD*)
then obtain *U V* **where** *UV*: ⋀*x*. *x* ∈ *S* ⟹ *openin X* (*U x*) ∧ *openin X* (*V x*) ∧ *x* ∈ (*U x*) ∧ *T* ⊆ (*V x*) ∧ *disjnt* (*U x*) (*V x*)
by *metis*
then obtain *ℱ* **where** *finite ℱ ℱ* ⊆ *U* ‘ *S S* ⊆ ⋃ *ℱ*
using *S* **unfolding** *compactin_def* **by** (*smt* (*verit*) *UN_iff image_iff subsetI*)
then obtain *K* **where** *finite K K* ⊆ *S* **and** *K*: *S* ⊆ ⋃(*U* ‘ *K*)
by (*metis finite_subset_image*)
show *?thesis*
proof (*intro exI conjI*)
show *openin X* (⋃(*U* ‘ *K*))
using ‹*K* ⊆ *S*› *UV* **by** *blast*
show *openin X* (⋂(*V* ‘ *K*))
using *False K UV* ‹*K* ⊆ *S*› ‹*finite K*› **by** *blast*
show *S* ⊆ ⋃(*U* ‘ *K*)
by (*simp add*: *K*)
show *T* ⊆ ⋂(*V* ‘ *K*)
using *UV* ‹*K* ⊆ *S*› **by** *blast*
show *disjnt* (⋃(*U* ‘ *K*)) (⋂(*V* ‘ *K*))
by (*smt* (*verit*) *Inter_iff UN_E UV* ‹*K* ⊆ *S*› *disjnt_iff image_eqI subset_iff*)
qed
qed

lemma *regular_space_compact_closed_sets*:
regular_space X ⟷
(∀ *S T*. *compactin X S* ∧ *closedin X T* ∧ *disjnt S T*
⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*)) (**is** *?lhs*=*?rhs*)

proof
assume *?lhs* **then show** *?rhs*
using *regular_space_compact_closed_separation* **by** *fastforce*
next
assume *R*: *?rhs*
show *?lhs*
unfolding *regular_space*
proof *clarify*
fix *S x*
assume *closedin X S* **and** $x \in$ *topspace X* **and** $x \notin S$
then obtain *U V* **where** *openin X U* $\wedge$ *openin X V* $\wedge$ $\{x\} \subseteq U \wedge S \subseteq V \wedge$ *disjnt U V*
by (*metis R compactin_sing disjnt_empty1 disjnt_insert1*)
then show $\exists U$. *openin X U* $\wedge$ $x \in U \wedge$ *disjnt S* (*X closure_of U*)
by (*smt* (*verit*, *best*) *disjnt_iff in_closure_of insert_subset subsetD*)
qed
qed

lemma *regular_space_prod_topology*:
regular_space (*prod_topology X Y*) $\longleftrightarrow$
$X =$ *trivial_topology* $\vee$ $Y =$ *trivial_topology* $\vee$ *regular_space X* $\wedge$ *regular_space Y* (**is** *?lhs=?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*metis regular_space_retraction_map_image retraction_map_fst retraction_map_snd*)
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases X* $=$ *trivial_topology* $\vee$ *Y* $=$ *trivial_topology*)
case *True* **then show** *?thesis* **by** *auto*
next
case *False*
then have *regular_space X regular_space Y*
using *R* **by** *auto*
show *?thesis*
unfolding *neighbourhood_base_of_closedin* [*symmetric*] *neighbourhood_base_of*
proof *clarify*
fix *W x y*
assume *W*: *openin* (*prod_topology X Y*) *W* **and** $(x,y) \in W$
then obtain *U V* **where** *U*: *openin X U* $x \in U$ **and** *V*: *openin Y V* $y \in V$
and $U \times V \subseteq W$
by (*metis openin_prod_topology_alt*)
obtain *D1 C1* **where** *1*: *openin X D1 closedin X C1* $x \in$ *D1 D1* $\subseteq$ *C1 C1* $\subseteq$ *U*
by (*metis* ‹*regular_space X*› *U neighbourhood_base_of neighbourhood_base_of_closedin*)
obtain *D2 C2* **where** *2*: *openin Y D2 closedin Y C2 y* $\in$ *D2 D2* $\subseteq$ *C2 C2* $\subseteq$

```
V
      by (metis ‹regular_space Y› V neighbourhood_base_of neighbourhood_base_of_closedin)
      show ∃ U V. openin (prod_topology X Y) U ∧ closedin (prod_topology X Y)
V ∧
                  (x,y) ∈ U ∧ U ⊆ V ∧ V ⊆ W
      proof (intro conjI exI)
        show openin (prod_topology X Y) (D1 × D2)
          by (simp add: 1 2 openin_prod_Times_iff)
        show closedin (prod_topology X Y) (C1 × C2)
          by (simp add: 1 2 closedin_prod_Times_iff)
      qed (use 1 2 ‹U × V ⊆ W› in auto)
    qed
  qed
qed

lemma regular_space_product_topology:
   regular_space (product_topology X I) ⟷
    (product_topology X I) = trivial_topology ∨ (∀ i ∈ I. regular_space (X i)) (is
?lhs=?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (meson regular_space_retraction_map_image retraction_map_product_projection)
next
  assume R: ?rhs
  show ?lhs
  proof (cases product_topology X I = trivial_topology)
    case True
    then show ?thesis
      by auto
  next
    case False
    then obtain x where x: x ∈ topspace (product_topology X I)
      by (meson ex_in_conv null_topspace_iff_trivial)
    define ℱ where ℱ ≡ {Pi_E I U |U. finite {i ∈ I. U i ≠ topspace (X i)}
                        ∧ (∀ i∈I. openin (X i) (U i))}
    have oo: openin (product_topology X I) = arbitrary union_of (λW. W ∈ ℱ)
      by (simp add: ℱ_def openin_product_topology product_topology_base_alt)
    have ∃ U V. openin (product_topology X I) U ∧ closedin (product_topology X
I) V ∧ x ∈ U ∧ U ⊆ V ∧ V ⊆ Pi_E I W
      if fin: finite {i ∈ I. W i ≠ topspace (X i)}
        and opeW: ⋀k. k ∈ I ⟹ openin (X k) (W k) and x: x ∈ PiE I W for W
x
    proof −
      have ⋀i. i ∈ I ⟹ ∃ U V. openin (X i) U ∧ closedin (X i) V ∧ x i ∈ U ∧
U ⊆ V ∧ V ⊆ W i
       by (metis False PiE_iff R neighbourhood_base_of neighbourhood_base_of_closedin
opeW x)
      then obtain U C where UC:
```

⋀i. i ∈ I ⟹ openin (X i) (U i) ∧ closedin (X i) (C i) ∧ x i ∈ U i ∧ U i ⊆ C i ∧ C i ⊆ W i
by *metis*
define *PI* **where** *PI* ≡ λV. *PiE I* (λi. *if W i* = *topspace*(*X i*) *then topspace*(*X i*) *else V i*)
show *?thesis*
proof (*intro conjI exI*)
have ∀ i∈I. *W i* ≠ *topspace* (*X i*) ⟶ *openin* (*X i*) (*U i*)
using *UC* **by** *force*
with *fin* **show** *openin* (*product_topology X I*) (*PI U*)
by (*simp add*: *Collect_mono_iff PI_def openin_PiE_gen rev_finite_subset*)
show *closedin* (*product_topology X I*) (*PI C*)
by (*simp add*: *UC closedin_product_topology PI_def*)
show *x* ∈ *PI U*
using *UC x* **by** (*fastforce simp*: *PI_def*)
show *PI U* ⊆ *PI C*
by (*smt* (*verit*) *UC Orderings.order_eq_iff PiE_mono PI_def*)
show *PI C* ⊆ Pi_E *I W*
by (*simp add*: *UC PI_def subset_PiE*)
qed
qed
then have *neighbourhood_base_of* (*closedin* (*product_topology X I*)) (*product_topology X I*)
unfolding *neighbourhood_base_of_topology_base* [*OF oo*] **by** (*force simp*: *ℱ_def*)
then show *?thesis*
by (*simp add*: *neighbourhood_base_of_closedin*)
qed
qed

lemma *closed_map_paired_gen_aux*:
assumes *regular_space Y* **and** *f*: *closed_map Z X f* **and** *g*: *closed_map Z Y g*
and *clo*: ⋀y. *y* ∈ *topspace X* ⟹ *closedin Z* {*x* ∈ *topspace Z*. *f x* = *y*}
and *contg*: *continuous_map Z Y g*
shows *closed_map Z* (*prod_topology X Y*) (λx. (*f x*, *g x*))
unfolding *closed_map_def*
proof (*intro strip*)
fix *C* **assume** *closedin Z C*
then have *C* ⊆ *topspace Z*
by (*simp add*: *closedin_subset*)
have *f* ∈ *topspace Z* → *topspace X g* ∈ *topspace Z* → *topspace Y*
by (*simp_all add*: *assms closed_map_imp_subset_topspace*)
show *closedin* (*prod_topology X Y*) ((λx. (*f x*, *g x*)) ‘ *C*)
unfolding *closedin_def topspace_prod_topology*
proof (*intro conjI*)
have *closedin Y* (*g* ‘ *C*)
using ‹*closedin Z C*› *assms*(*3*) *closed_map_def* **by** *blast*
with *assms* **show** (λx. (*f x*, *g x*)) ‘ *C* ⊆ *topspace X* × *topspace Y*
by (*smt* (*verit*) *SigmaI* ‹*closedin Z C*› *closed_map_def closedin_subset im-*

```
age_subset_iff)
  have ∗: ∃ T. openin (prod_topology X Y) T ∧ (y1,y2) ∈ T ∧ T ⊆ topspace X
× topspace Y − (λx. (f x, g x)) ‘ C
    if (y1,y2) ∉ (λx. (f x, g x)) ‘ C and y1: y1 ∈ topspace X and y2: y2 ∈
topspace Y
    for y1 y2
  proof −
    define A where A ≡ topspace Z − (C ∩ {x ∈ topspace Z. f x = y1})
    have A: openin Z A {x ∈ topspace Z. g x = y2} ⊆ A
      using that ‹closedin Z C› clo that(2) by (auto simp: A_def)
    obtain V0 where openin Y V0 ∧ y2 ∈ V0 and UA: {x ∈ topspace Z. g x
∈ V0} ⊆ A
      using g A y2 unfolding closed_map_fibre_neighbourhood by blast
    then obtain V V′ where VV: openin Y V ∧ closedin Y V′ ∧ y2 ∈ V ∧ V
⊆ V′ and V′ ⊆ V0
      by (metis (no_types, lifting) ‹regular_space Y› neighbourhood_base_of
neighbourhood_base_of_closedin)
    with UA have subA: {x ∈ topspace Z. g x ∈ V′} ⊆ A
      by blast
    show ?thesis
    proof −
      define B where B ≡ topspace Z − (C ∩ {x ∈ topspace Z. g x ∈ V′})
      have openin Z B
        using VV ‹closedin Z C› contg by (fastforce simp: B_def continu-
ous_map_closedin)
      have {x ∈ topspace Z. f x = y1} ⊆ B
        using A_def subA by (auto simp: A_def B_def)
      then obtain U where openin X U y1 ∈ U and subB: {x ∈ topspace Z. f
x ∈ U} ⊆ B
        using ‹openin Z B› y1 f unfolding closed_map_fibre_neighbourhood by
meson
      show ?thesis
      proof (intro conjI exI)
        show openin (prod_topology X Y) (U × V)
          by (metis VV ‹openin X U› openin_prod_Times_iff)
        show (y1, y2) ∈ U × V
          by (simp add: VV ‹y1 ∈ U›)
        show U × V ⊆ topspace X × topspace Y − (λx. (f x, g x)) ‘ C
          using VV ‹C ⊆ topspace Z› ‹openin X U› subB
          by (force simp: image_iff B_def subset_iff dest: openin_subset)
      qed
    qed
  qed
  then show openin (prod_topology X Y) (topspace X × topspace Y − (λx. (f
x, g x)) ‘ C)
    by (smt (verit, ccfv_threshold) Diff_iff SigmaE openin_subopen)
 qed
qed
```

lemma *closed_map_paired_gen*:
assumes *f*: *closed_map Z X f* **and** *g*: *closed_map Z Y g*
and *D*: (*regular_space X* ∧ *continuous_map Z X f* ∧ (∀ *z* ∈ *topspace Y*. *closedin Z* {*x* ∈ *topspace Z*. *g x* = *z*})
∨ *regular_space Y* ∧ *continuous_map Z Y g* ∧ (∀ *y* ∈ *topspace X*. *closedin Z* {*x* ∈ *topspace Z*. *f x* = *y*}))
(**is** *?RSX* ∨ *?RSY*)
shows *closed_map Z* (*prod_topology X Y*) (λ*x*. (*f x*, *g x*))
using *D*
proof
assume *RSX*: *?RSX*
have *eq*: (λ*x*. (*f x*, *g x*)) = (λ(*x*,*y*). (*y*,*x*)) ∘ (λ*x*. (*g x*, *f x*))
by *auto*
show *?thesis*
unfolding *eq*
proof (*rule closed_map_compose*)
show *closed_map Z* (*prod_topology Y X*) (λ*x*. (*g x*, *f x*))
using *RSX closed_map_paired_gen_aux f g* **by** *fastforce*
show *closed_map* (*prod_topology Y X*) (*prod_topology X Y*) (λ(*x*, *y*). (*y*, *x*))
using *homeomorphic_imp_closed_map homeomorphic_map_swap* **by** *blast*
qed
qed (*blast intro*: *assms closed_map_paired_gen_aux*)

lemma *closed_map_paired*:
assumes *closed_map Z X f* **and** *contf*: *continuous_map Z X f*
closed_map Z Y g **and** *contg*: *continuous_map Z Y g*
and *D*: *t1_space X* ∧ *regular_space Y* ∨ *regular_space X* ∧ *t1_space Y*
shows *closed_map Z* (*prod_topology X Y*) (λ*x*. (*f x*, *g x*))
proof (*rule closed_map_paired_gen*)
show *regular_space X* ∧ *continuous_map Z X f* ∧ (∀ *z*∈*topspace Y*. *closedin Z* {*x* ∈ *topspace Z*. *g x* = *z*}) ∨ *regular_space Y* ∧ *continuous_map Z Y g* ∧ (∀ *y*∈*topspace X*. *closedin Z* {*x* ∈ *topspace Z*. *f x* = *y*})
using *D contf contg*
by (*smt* (*verit*, *del_insts*) *Collect_cong closedin_continuous_map_preimage t1_space_closedin_singleton singleton_iff*)
qed (*use assms* **in** *auto*)

lemma *closed_map_pairwise*:
assumes *closed_map Z X* (*fst* ∘ *f*) *continuous_map Z X* (*fst* ∘ *f*)
closed_map Z Y (*snd* ∘ *f*) *continuous_map Z Y* (*snd* ∘ *f*)
t1_space X ∧ *regular_space Y* ∨ *regular_space X* ∧ *t1_space Y*
shows *closed_map Z* (*prod_topology X Y*) *f*
proof −
have *closed_map Z* (*prod_topology X Y*) (λ*a*. ((*fst* ∘ *f*) *a*, (*snd* ∘ *f*) *a*))
using *assms closed_map_paired* **by** *blast*
then show *?thesis*
by *auto*
qed

lemma *continuous_imp_proper_map*:
⟦*compact_space X*; *kc_space Y*; *continuous_map X Y f*⟧ ⟹ *proper_map X Y f*
by (*simp add*: *continuous_closed_imp_proper_map continuous_imp_closed_map_gen kc_imp_t1_space*)

lemma *tube_lemma_right*:
assumes *W*: *openin* (*prod_topology X Y*) *W* **and** *C*: *compactin Y C*
and *x*: *x* ∈ *topspace X* **and** *subW*: {*x*} × *C* ⊆ *W*
shows ∃ *U V*. *openin X U* ∧ *openin Y V* ∧ *x* ∈ *U* ∧ *C* ⊆ *V* ∧ *U* × *V* ⊆ *W*
proof (*cases C* = {})
case *True*
with *x* **show** *?thesis* **by** *auto*
next
case *False*
have ∃ *U V*. *openin X U* ∧ *openin Y V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *U* × *V* ⊆ *W*
if *y* ∈ *C* **for** *y*
using *W openin_prod_topology_alt subW subsetD that* **by** *fastforce*
then obtain *U V* **where** *UV*: ⋀*y*. *y* ∈ *C* ⟹ *openin X* (*U y*) ∧ *openin Y* (*V y*) ∧ *x* ∈ *U y* ∧ *y* ∈ *V y* ∧ *U y* × *V y* ⊆ *W*
by *metis*
then obtain *D* **where** *D*: *finite D D* ⊆ *C C* ⊆ ⋃ (*V* ‘ *D*)
using *compactinD* [*OF C*, *of V*‘*C*]
by (*smt* (*verit*) *UN_I finite_subset_image imageE subsetI*)
show *?thesis*
proof (*intro exI conjI*)
show *openin X* (⋂ (*U* ‘ *D*)) *openin Y* (⋃ (*V* ‘ *D*))
using *D False UV* **by** *blast*+
show *x* ∈ ⋂ (*U* ‘ *D*) *C* ⊆ ⋃ (*V* ‘ *D*) ⋂ (*U* ‘ *D*) × ⋃ (*V* ‘ *D*) ⊆ *W*
using *D UV* **by** *force*+
qed
qed

lemma *closed_map_fst*:
assumes *compact_space Y*
shows *closed_map* (*prod_topology X Y*) *X fst*
proof −
have ∗: {*x* ∈ *topspace X* × *topspace Y*. *fst x* ∈ *U*} = *U* × *topspace Y*
if *U* ⊆ *topspace X* **for** *U*
using *that* **by** *force*
have ∗∗: ⋀*U y*. ⟦*openin* (*prod_topology X Y*) *U*; *y* ∈ *topspace X*;
{*x* ∈ *topspace X* × *topspace Y*. *fst x* = *y*} ⊆ *U*⟧
⟹ ∃ *V*. *openin X V* ∧ *y* ∈ *V* ∧ *V* × *topspace Y* ⊆ *U*
using *tube_lemma_right*[*of X Y _ topspace Y*] *assms* **by** (*fastforce simp*: *compact_space_def*)
show *?thesis*

unfolding *closed_map_fibre_neighbourhood*
by (*force simp*: ∗ *openin_subset cong*: *conj_cong intro*: ∗∗)
qed

lemma *closed_map_snd*:
assumes *compact_space X*
shows *closed_map* (*prod_topology X Y*) *Y snd*
proof −
have *snd* = *fst o prod.swap*
by *force*
moreover have *closed_map* (*prod_topology X Y*) *Y* (*fst o prod.swap*)
proof (*rule closed_map_compose*)
show *closed_map* (*prod_topology X Y*) (*prod_topology Y X*) *prod.swap*
by (*metis* (*no_types*, *lifting*) *homeomorphic_imp_closed_map homeomorphic_map_eq homeomorphic_map_swap prod.swap_def split_beta*)
show *closed_map* (*prod_topology Y X*) *Y fst*
by (*simp add*: *closed_map_fst assms*)
qed
ultimately show *?thesis*
by *metis*
qed

lemma *closed_map_paired_closed_map_right*:
⟦*closed_map X Y f*; *regular_space X*;
⋀*y*. *y* ∈ *topspace Y* ⟹ *closedin X* {*x* ∈ *topspace X*. *f x* = *y*}⟧
⟹ *closed_map X* (*prod_topology X Y*) (λ*x*. (*x*, *f x*))
by (*rule closed_map_paired_gen* [*OF closed_map_id*, *unfolded id_def*]) *auto*

lemma *closed_map_paired_closed_map_left*:
assumes *closed_map X Y f regular_space X*
⋀*y*. *y* ∈ *topspace Y* ⟹ *closedin X* {*x* ∈ *topspace X*. *f x* = *y*}
shows *closed_map X* (*prod_topology Y X*) (λ*x*. (*f x*, *x*))
proof −
have *eq*: (λ*x*. (*f x*, *x*)) = (λ(*x*,*y*). (*y*,*x*)) ∘ (λ*x*. (*x*, *f x*))
by *auto*
show *?thesis*
unfolding *eq*
proof (*rule closed_map_compose*)
show *closed_map X* (*prod_topology X Y*) (λ*x*. (*x*, *f x*))
by (*simp add*: *assms closed_map_paired_closed_map_right*)
show *closed_map* (*prod_topology X Y*) (*prod_topology Y X*) (λ(*x*, *y*). (*y*, *x*))
using *homeomorphic_imp_closed_map homeomorphic_map_swap* **by** *blast*
qed
qed

lemma *closed_map_imp_closed_graph*:
assumes *closed_map X Y f regular_space X*
⋀*y*. *y* ∈ *topspace Y* ⟹ *closedin X* {*x* ∈ *topspace X*. *f x* = *y*}
shows *closedin* (*prod_topology X Y*) ((λ*x*. (*x*, *f x*)) ‘ *topspace X*)

using *assms closed_map_def closed_map_paired_closed_map_right* **by** *blast*

lemma *proper_map_paired_closed_map_right*:
assumes *closed_map X Y f regular_space X*
⋀*y*. *y* ∈ *topspace Y* ⟹ *closedin X* {*x* ∈ *topspace X*. *f x* = *y*}
shows *proper_map X* (*prod_topology X Y*) (λ*x*. (*x*, *f x*))
by (*simp add*: *assms closed_injective_imp_proper_map inj_on_def closed_map_paired_closed_map_right*)

lemma *proper_map_paired_closed_map_left*:
assumes *closed_map X Y f regular_space X*
⋀*y*. *y* ∈ *topspace Y* ⟹ *closedin X* {*x* ∈ *topspace X*. *f x* = *y*}
shows *proper_map X* (*prod_topology Y X*) (λ*x*. (*f x*, *x*))
by (*simp add*: *assms closed_injective_imp_proper_map inj_on_def closed_map_paired_closed_map_left*)

proposition *regular_space_continuous_proper_map_image*:
assumes *regular_space X* **and** *contf*: *continuous_map X Y f* **and** *pmapf*: *proper_map X Y f*
and *fim*: *f* ‘ (*topspace X*) = *topspace Y*
shows *regular_space Y*
unfolding *regular_space_def*
proof *clarify*
fix *C y*
assume *closedin Y C* **and** *y* ∈ *topspace Y* **and** *y* ∉ *C*
have *closed_map X Y f* (∀ *y* ∈ *topspace Y*. *compactin X* {*x* ∈ *topspace X*. *f x* = *y*})
using *pmapf proper_map_def* **by** *force+*
moreover have *closedin X* {*z* ∈ *topspace X*. *f z* ∈ *C*}
using ‹*closedin Y C*› *contf continuous_map_closedin* **by** *fastforce*
moreover have *disjnt* {*z* ∈ *topspace X*. *f z* = *y*} {*z* ∈ *topspace X*. *f z* ∈ *C*}
using ‹*y* ∉ *C*› *disjnt_iff* **by** *blast*
ultimately
obtain *U V* **where** *UV*: *openin X U openin X V* {*z* ∈ *topspace X*. *f z* = *y*} ⊆ *U* {*z* ∈ *topspace X*. *f z* ∈ *C*} ⊆ *V*
and *dUV*: *disjnt U V*
using ‹*y* ∈ *topspace Y*› ‹*regular_space X*› **unfolding** *regular_space_compact_closed_sets*
by *meson*

have ∗: ⋀*U T*. *openin X U* ∧ *T* ⊆ *topspace Y* ∧ {*x* ∈ *topspace X*. *f x* ∈ *T*} ⊆ *U* ⟶
(∃ *V*. *openin Y V* ∧ *T* ⊆ *V* ∧ {*x* ∈ *topspace X*. *f x* ∈ *V*} ⊆ *U*)
using ‹*closed_map X Y f*› **unfolding** *closed_map_preimage_neighbourhood*
by *blast*
obtain *V1* **where** *V1*: *openin Y V1* ∧ *y* ∈ *V1* **and** *sub1*: {*x* ∈ *topspace X*. *f x* ∈ *V1*} ⊆ *U*
using ∗ [*of U* {*y*}] *UV* ‹*y* ∈ *topspace Y*› **by** *auto*
moreover
obtain *V2* **where** *openin Y V2* ∧ *C* ⊆ *V2* **and** *sub2*: {*x* ∈ *topspace X*. *f x* ∈ *V2*} ⊆ *V*
by (*smt* (*verit*, *ccfv_SIG*) ∗ *UV* ‹*closedin Y C*› *closedin_subset mem_Collect_eq*

subset_iff)
moreover have *disjnt V1 V2*
proof −
have $\bigwedge x.\ [\![\forall x.\ x \in U \longrightarrow x \notin V;\ x \in V1;\ x \in V2]\!] \Longrightarrow False$
by (*smt* (*verit*) *V1 fim image_iff mem_Collect_eq openin_subset sub1 sub2 subsetD*)
with *dUV* **show** *?thesis* **by** (*auto simp*: *disjnt_iff*)
qed
ultimately show $\exists U\ V.$ *openin Y U* $\wedge$ *openin Y V* $\wedge$ $y \in U \wedge C \subseteq V \wedge$ *disjnt U V*
by *meson*
qed

lemma *regular_space_perfect_map_image*:
$[\![$*regular_space X*; *perfect_map X Y f*$]\!] \Longrightarrow$ *regular_space Y*
by (*meson perfect_map_def regular_space_continuous_proper_map_image*)

proposition *regular_space_perfect_map_image_eq*:
assumes *Hausdorff_space X* **and** *perf*: *perfect_map X Y f*
shows *regular_space X* $\longleftrightarrow$ *regular_space Y* (**is** *?lhs=?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *perf regular_space_perfect_map_image* **by** *blast*
next
assume *R*: *?rhs*
have *continuous_map X Y f proper_map X Y f* **and** *fim*: *f ‘ (topspace X) = topspace Y*
using *perf* **by** (*auto simp*: *perfect_map_def*)
then have *closed_map X Y f* **and** *preYf*: $(\forall y \in$ *topspace Y*. *compactin X* $\{x \in$ *topspace X*. $f\ x = y\})$
by (*simp_all add*: *proper_map_def*)
show *?lhs*
unfolding *regular_space_def*
proof *clarify*
fix *C x*
assume *closedin X C* **and** $x \in$ *topspace X* **and** $x \notin C$
obtain *U1 U2* **where** *openin X U1 openin X U2* $\{x\} \subseteq$ *U1* **and** *disjnt U1 U2*
and *subV*: $C \cap \{z \in$ *topspace X*. $f\ z = f\ x\} \subseteq$ *U2*
proof (*rule Hausdorff_space_compact_separation* [*of X* $\{x\}$ $C \cap \{z \in$ *topspace X*. $f\ z = f\ x\}$, *OF* ‹*Hausdorff_space X*›])
show *compactin X* $\{x\}$
by (*simp add*: ‹$x \in$ *topspace X*›)
show *compactin X* $(C \cap \{z \in$ *topspace X*. $f\ z = f\ x\})$
using ‹*closedin X C*› *fim* ‹$x \in$ *topspace X*› *closed_Int_compactin preYf* **by** *fastforce*
show *disjnt* $\{x\}$ $(C \cap \{z \in$ *topspace X*. $f\ z = f\ x\})$
using ‹$x \notin C$› **by** *force*
qed

```
    have closedin Y (f ‘ (C − U2))
      using ‹closed_map X Y f› ‹closedin X C› ‹openin X U2› closed_map_def
by blast
    moreover
    have f x ∈ topspace Y − f ‘ (C − U2)
    using ‹closedin X C› ‹continuous_map X Y f› ‹x ∈ topspace X› closedin_subset
continuous_map_def subV
      by (fastforce simp: Pi_iff)
    ultimately
    obtain V1 V2 where VV: openin Y V1 openin Y V2 f x ∈ V1
              and subV2: f ‘ (C − U2) ⊆ V2 and disjnt V1 V2
      by (meson R regular_space_def)
    show ∃ U U'. openin X U ∧ openin X U' ∧ x ∈ U ∧ C ⊆ U' ∧ disjnt U U'
    proof (intro exI conjI)
      show openin X (U1 ∩ {x ∈ topspace X. f x ∈ V1})
       using VV(1) ‹continuous_map X Y f› ‹openin X U1› continuous_map by
fastforce
      show openin X (U2 ∪ {x ∈ topspace X. f x ∈ V2})
       using VV(2) ‹continuous_map X Y f› ‹openin X U2› continuous_map by
fastforce
      show x ∈ U1 ∩ {x ∈ topspace X. f x ∈ V1}
       using VV(3) ‹x ∈ topspace X› ‹{x} ⊆ U1› by auto
      show C ⊆ U2 ∪ {x ∈ topspace X. f x ∈ V2}
       using ‹closedin X C› closedin_subset subV2 by auto
      show disjnt (U1 ∩ {x ∈ topspace X. f x ∈ V1}) (U2 ∪ {x ∈ topspace X. f x
∈ V2})
       using ‹disjnt U1 U2› ‹disjnt V1 V2› by (auto simp: disjnt_iff)
    qed
  qed
qed
```

6.6.13 Locally compact spaces

```
definition locally_compact_space
  where locally_compact_space X ≡
    ∀ x ∈ topspace X. ∃ U K. openin X U ∧ compactin X K ∧ x ∈ U ∧ U ⊆ K

lemma homeomorphic_locally_compact_spaceD:
  assumes X: locally_compact_space X and X homeomorphic_space Y
  shows locally_compact_space Y
proof −
  obtain f where hmf: homeomorphic_map X Y f
    using assms homeomorphic_space by blast
  then have eq: topspace Y = f ‘ (topspace X)
    by (simp add: homeomorphic_imp_surjective_map)
  have ∃ V K. openin Y V ∧ compactin Y K ∧ f x ∈ V ∧ V ⊆ K
    if x ∈ topspace X openin X U compactin X K x ∈ U U ⊆ K for x U K
    using that
   by (meson hmf homeomorphic_map_compactness_eq homeomorphic_map_openness_eq
```

image_mono image_eqI)
with *X* **show** *?thesis*
by (*smt* (*verit*) *eq image_iff locally_compact_space_def*)
qed

lemma *homeomorphic_locally_compact_space*:
assumes *X homeomorphic_space Y*
shows *locally_compact_space X* ⟷ *locally_compact_space Y*
by (*meson assms homeomorphic_locally_compact_spaceD homeomorphic_space_sym*)

lemma *locally_compact_space_retraction_map_image*:
assumes *retraction_map X Y r* **and** *X*: *locally_compact_space X*
shows *locally_compact_space Y*
proof −
obtain *s* **where** *s*: *retraction_maps X Y r s*
using *assms retraction_map_def* **by** *blast*
obtain *T* **where** *T retract_of_space X* **and** *Teq*: *T* = *s* ‘ *topspace Y*
using *retraction_maps_section_image1 s* **by** *blast*
then obtain *r* **where** *r*: *continuous_map X* (*subtopology X T*) *r* ∀ *x*∈*T*. *r x* = *x*
by (*meson retract_of_space_def*)
have *locally_compact_space* (*subtopology X T*)
unfolding *locally_compact_space_def openin_subtopology_alt*
proof *clarsimp*
fix *x*
assume *x* ∈ *topspace X x* ∈ *T*
obtain *U K* **where** *UK*: *openin X U* ∧ *compactin X K* ∧ *x* ∈ *U* ∧ *U* ⊆ *K*
by (*meson X* ‹*x* ∈ *topspace X*› *locally_compact_space_def*)
then have *compactin* (*subtopology X T*) (*r* ‘ *K*) ∧ *T* ∩ *U* ⊆ *r* ‘ *K*
by (*smt* (*verit*) *IntD1 image_compactin image_iff inf_le2 r subset_iff*)
then show ∃ *U*. *openin X U* ∧ (∃ *K*. *compactin* (*subtopology X T*) *K* ∧ *x* ∈ *U* ∧ *T* ∩ *U* ⊆ *K*)
using *UK* **by** *auto*
qed
with *Teq* **show** *?thesis*
using *homeomorphic_locally_compact_space retraction_maps_section_image2 s* **by** *blast*
qed

lemma *compact_imp_locally_compact_space*:
compact_space X ⟹ *locally_compact_space X*
using *compact_space_def locally_compact_space_def* **by** *blast*

lemma *neighbourhood_base_imp_locally_compact_space*:
neighbourhood_base_of (*compactin X*) *X* ⟹ *locally_compact_space X*
by (*metis locally_compact_space_def neighbourhood_base_of openin_topspace*)

lemma *locally_compact_imp_neighbourhood_base*:
assumes *loc*: *locally_compact_space X* **and** *reg*: *regular_space X*

shows *neighbourhood_base_of* (*compactin X*) *X*
unfolding *neighbourhood_base_of*
proof *clarify*
fix *W x*
assume *openin X W* **and** $x \in W$
then obtain *U K* **where** *openin X U compactin X K* $x \in U$ $U \subseteq K$
by (*metis loc locally_compact_space_def openin_subset subsetD*)
moreover have *openin X* $(U \cap W) \wedge x \in U \cap W$
using ‹*openin X W*› ‹$x \in W$› ‹*openin X U*› ‹$x \in U$› **by** *blast*
then have $\exists u'\ v.$ *openin X u'* $\wedge$ *closedin X v* $\wedge$ $x \in u' \wedge u' \subseteq v \wedge v \subseteq U \wedge v \subseteq W$
using *reg*
by (*metis le_infE neighbourhood_base_of neighbourhood_base_of_closedin*)
then show $\exists U\ V.$ *openin X U* $\wedge$ *compactin X V* $\wedge$ $x \in U \wedge U \subseteq V \wedge V \subseteq W$
by (*meson* ‹$U \subseteq K$› ‹*compactin X K*› *closed_compactin subset_trans*)
qed

lemma *Hausdorff_regular*: $[\![$*Hausdorff_space X*; *neighbourhood_base_of* (*compactin X*) *X*$]\!] \Longrightarrow$ *regular_space X*
by (*metis compactin_imp_closedin neighbourhood_base_of_closedin neighbourhood_base_of_mono*)

lemma *locally_compact_Hausdorff_imp_regular_space*:
assumes *loc*: *locally_compact_space X* **and** *Hausdorff_space X*
shows *regular_space X*
unfolding *neighbourhood_base_of_closedin* [*symmetric*] *neighbourhood_base_of*
proof *clarify*
fix *W x*
assume *openin X W* **and** $x \in W$
then have $x \in$ *topspace X*
using *openin_subset* **by** *blast*
then obtain *U K* **where** *openin X U compactin X K* **and** *UK*: $x \in U$ $U \subseteq K$
by (*meson loc locally_compact_space_def*)
with ‹*Hausdorff_space X*› **have** *regular_space* (*subtopology X K*)
using *Hausdorff_space_subtopology compact_Hausdorff_imp_regular_space compact_space_subtopology* **by** *blast*
then have $\exists U'\ V'.$ *openin* (*subtopology X K*) *U'* $\wedge$ *closedin* (*subtopology X K*) *V'* $\wedge$ $x \in U' \wedge U' \subseteq V' \wedge V' \subseteq K \cap W$
unfolding *neighbourhood_base_of_closedin* [*symmetric*] *neighbourhood_base_of*
by (*meson IntI* ‹$U \subseteq K$› ‹*openin X W*› ‹$x \in U$› ‹$x \in W$› *openin_subtopology_Int2 subsetD*)
then obtain *V C* **where** *openin X V closedin X C* **and** *VC*: $x \in K \cap V$ $K \cap V \subseteq K \cap C$ $K \cap C \subseteq K \cap W$
by (*metis Int_commute closedin_subtopology openin_subtopology*)
show $\exists U\ V.$ *openin X U* $\wedge$ *closedin X V* $\wedge$ $x \in U \wedge U \subseteq V \wedge V \subseteq W$
proof (*intro conjI exI*)
show *openin X* $(U \cap V)$
using ‹*openin X U*› ‹*openin X V*› **by** *blast*
show *closedin X* $(K \cap C)$

using ‹*closedin X C*› ‹*compactin X K*› *compactin_imp_closedin* ‹*Hausdorff_space X*› **by** *blast*
qed (*use UK VC* **in** *auto*)
qed

lemma *locally_compact_space_neighbourhood_base*:
Hausdorff_space X ∨ *regular_space X*
⟹ *locally_compact_space X* ⟷ *neighbourhood_base_of* (*compactin X*) *X*
by (*metis locally_compact_imp_neighbourhood_base locally_compact_Hausdorff_imp_regular_space neighbourhood_base_imp_locally_compact_space*)

lemma *locally_compact_Hausdorff_or_regular*:
locally_compact_space X ∧ (*Hausdorff_space X* ∨ *regular_space X*) ⟷ *locally_compact_space X* ∧ *regular_space X*
using *locally_compact_Hausdorff_imp_regular_space* **by** *blast*

lemma *locally_compact_space_compact_closedin*:
assumes *Hausdorff_space X* ∨ *regular_space X*
shows *locally_compact_space X* ⟷
(∀ *x* ∈ *topspace X*. ∃ *U K*. *openin X U* ∧ *compactin X K* ∧ *closedin X K* ∧ *x* ∈ *U* ∧ *U* ⊆ *K*)
using *locally_compact_Hausdorff_or_regular* **unfolding** *locally_compact_space_def*
by (*metis assms closed_compactin inf.absorb_iff2 le_infE neighbourhood_base_of neighbourhood_base_of_closedin*)

lemma *locally_compact_space_compact_closure_of*:
assumes *Hausdorff_space X* ∨ *regular_space X*
shows *locally_compact_space X* ⟷
(∀ *x* ∈ *topspace X*. ∃ *U*. *openin X U* ∧ *compactin X* (*X closure_of U*) ∧ *x* ∈ *U*) (**is** *?lhs=?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*metis assms closed_compactin closedin_closure_of closure_of_eq closure_of_mono locally_compact_space_compact_closedin*)
next
assume *?rhs* **then show** *?lhs*
by (*meson closure_of_subset locally_compact_space_def openin_subset*)
qed

lemma *locally_compact_space_neighbourhood_base_closedin*:
assumes *Hausdorff_space X* ∨ *regular_space X*
shows *locally_compact_space X* ⟷ *neighbourhood_base_of* (λ*C*. *compactin X C* ∧ *closedin X C*) *X* (**is** *?lhs=?rhs*)
proof
assume *L*: *?lhs*
then have *regular_space X*
using *assms locally_compact_Hausdorff_imp_regular_space* **by** *blast*

with *L* **have** *neighbourhood_base_of* (*compactin X*) *X*
by (*simp add*: *locally_compact_imp_neighbourhood_base*)
with ‹*regular_space X*› **show** *?rhs*
by (*smt* (*verit*, *ccfv_threshold*) *closed_compactin neighbourhood_base_of subset_trans neighbourhood_base_of_closedin*)
next
assume *?rhs* **then show** *?lhs*
using *neighbourhood_base_imp_locally_compact_space neighbourhood_base_of_mono*
by *blast*
qed

lemma *locally_compact_space_neighbourhood_base_closure_of*:
assumes *Hausdorff_space X* ∨ *regular_space X*
shows *locally_compact_space X* ⟷ *neighbourhood_base_of* (λ*T*. *compactin X* (*X closure_of T*)) *X*
(**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
then have *regular_space X*
using *assms locally_compact_Hausdorff_imp_regular_space* **by** *blast*
with *L* **have** *neighbourhood_base_of* (λ*A*. *compactin X A* ∧ *closedin X A*) *X*
using *locally_compact_space_neighbourhood_base_closedin* **by** *blast*
then show *?rhs*
by (*simp add*: *closure_of_closedin neighbourhood_base_of_mono*)
next
assume *?rhs* **then show** *?lhs*
unfolding *locally_compact_space_def neighbourhood_base_of*
by (*meson closure_of_subset openin_topspace subset_trans*)
qed

lemma *locally_compact_space_neighbourhood_base_open_closure_of*:
assumes *Hausdorff_space X* ∨ *regular_space X*
shows *locally_compact_space X* ⟷
neighbourhood_base_of (λ*U*. *openin X U* ∧ *compactin X* (*X closure_of U*)) *X*
(**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
then have *regular_space X*
using *assms locally_compact_Hausdorff_imp_regular_space* **by** *blast*
then have *neighbourhood_base_of* (λ*T*. *compactin X* (*X closure_of T*)) *X*
using *L locally_compact_space_neighbourhood_base_closure_of* **by** *auto*
with *L* **show** *?rhs*
unfolding *neighbourhood_base_of*
by (*meson closed_compactin closure_of_closure_of closure_of_eq closure_of_mono subset_trans*)
next
assume *?rhs* **then show** *?lhs*
unfolding *locally_compact_space_def neighbourhood_base_of*

by (*meson closure_of_subset openin_topspace subset_trans*)
qed

lemma *locally_compact_space_compact_closed_compact*:
assumes *Hausdorff_space X* ∨ *regular_space X*
shows *locally_compact_space X* ⟷
(∀ *K*. *compactin X K*
⟶ (∃ *U L*. *openin X U* ∧ *compactin X L* ∧ *closedin X L* ∧ *K* ⊆ *U* ∧ *U* ⊆ *L*))
(**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
then obtain *U L* **where** *UL*: ∀ *x* ∈ *topspace X*. *openin X* (*U x*) ∧ *compactin X* (*L x*) ∧ *closedin X* (*L x*) ∧ *x* ∈ *U x* ∧ *U x* ⊆ *L x*
unfolding *locally_compact_space_compact_closedin* [*OF assms*]
by *metis*
show *?rhs*
proof *clarify*
fix *K*
assume *compactin X K*
then have *K* ⊆ *topspace X*
by (*simp add*: *compactin_subset_topspace*)
then have ∗: (∀ *U*∈*U* ‘ *K*. *openin X U*) ∧ *K* ⊆ ⋃ (*U* ‘ *K*)
using *UL* **by** *blast*
with ‹*compactin X K*› **obtain** *KF* **where** *KF*: *finite KF KF* ⊆ *K K* ⊆ ⋃(*U* ‘ *KF*)
by (*metis compactinD finite_subset_image*)
show ∃ *U L*. *openin X U* ∧ *compactin X L* ∧ *closedin X L* ∧ *K* ⊆ *U* ∧ *U* ⊆ *L*
proof (*intro conjI exI*)
show *openin X* (⋃ (*U* ‘ *KF*))
using ∗ ‹*KF* ⊆ *K*› **by** *fastforce*
show *compactin X* (⋃ (*L* ‘ *KF*))
by (*smt* (*verit*) *UL* ‹*K* ⊆ *topspace X*› *KF compactin_Union finite_imageI imageE subset_iff*)
show *closedin X* (⋃ (*L* ‘ *KF*))
by (*smt* (*verit*) *UL* ‹*K* ⊆ *topspace X*› *KF closedin_Union finite_imageI imageE subsetD*)
qed (*use UL* ‹*K* ⊆ *topspace X*› *KF* **in** *auto*)
qed
next
assume *?rhs* **then show** *?lhs*
by (*metis compactin_sing insert_subset locally_compact_space_def*)
qed

lemma *locally_compact_regular_space_neighbourhood_base*:
locally_compact_space X ∧ *regular_space X* ⟷
neighbourhood_base_of (λ*C*. *compactin X C* ∧ *closedin X C*) *X*
using *locally_compact_space_neighbourhood_base_closedin neighbourhood_base_of_closedin neighbourhood_base_of_mono* **by** *blast*

lemma *locally_compact_kc_space*:
neighbourhood_base_of (*compactin X*) *X* ∧ *kc_space X* ⟷
locally_compact_space X ∧ *Hausdorff_space X*
using *Hausdorff_imp_kc_space locally_compact_imp_kc_eq_Hausdorff_space locally_compact_space_neighbourhood_base* **by** *blast*

lemma *locally_compact_kc_space_alt*:
neighbourhood_base_of (*compactin X*) *X* ∧ *kc_space X* ⟷
locally_compact_space X ∧ *Hausdorff_space X* ∧ *regular_space X*
using *Hausdorff_regular locally_compact_kc_space* **by** *blast*

lemma *locally_compact_kc_imp_regular_space*:
⟦*neighbourhood_base_of* (*compactin X*) *X*; *kc_space X*⟧ ⟹ *regular_space X*
using *Hausdorff_regular locally_compact_imp_kc_eq_Hausdorff_space* **by** *blast*

lemma *kc_locally_compact_space*:
kc_space X
⟹ *neighbourhood_base_of* (*compactin X*) *X* ⟷ *locally_compact_space X* ∧ *Hausdorff_space X* ∧ *regular_space X*
using *Hausdorff_regular locally_compact_kc_space* **by** *blast*

lemma *locally_compact_space_closed_subset*:
assumes *loc*: *locally_compact_space X* **and** *closedin X S*
shows *locally_compact_space* (*subtopology X S*)
proof (*clarsimp simp*: *locally_compact_space_def*)
fix *x* **assume** *x*: *x* ∈ *topspace X x* ∈ *S*
then obtain *U K* **where** *UK*: *openin X U* ∧ *compactin X K* ∧ *x* ∈ *U* ∧ *U* ⊆ *K*
by (*meson loc locally_compact_space_def*)
show ∃ *U*. *openin* (*subtopology X S*) *U* ∧
(∃ *K*. *compactin* (*subtopology X S*) *K* ∧ *x* ∈ *U* ∧ *U* ⊆ *K*)
proof (*intro conjI exI*)
show *openin* (*subtopology X S*) (*S* ∩ *U*)
by (*simp add*: *UK openin_subtopology_Int2*)
show *compactin* (*subtopology X S*) (*S* ∩ *K*)
by (*simp add*: *UK assms*(*2*) *closed_Int_compactin compactin_subtopology*)
qed (*use UK x* **in** *auto*)
qed

lemma *locally_compact_space_open_subset*:
assumes *X*: *Hausdorff_space X* ∨ *regular_space X* **and** *loc*: *locally_compact_space X* **and** *openin X S*
shows *locally_compact_space* (*subtopology X S*)
proof (*clarsimp simp*: *locally_compact_space_def*)
fix *x* **assume** *x*: *x* ∈ *topspace X x* ∈ *S*
then obtain *U K* **where** *UK*: *openin X U compactin X K x* ∈ *U U* ⊆ *K*
by (*meson loc locally_compact_space_def*)
moreover have *reg*: *regular_space X*

```
    using X loc locally_compact_Hausdorff_imp_regular_space by blast
  moreover have openin X (U ∩ S)
    by (simp add: UK ‹openin X S› openin_Int)
  ultimately obtain V C
      where VC: openin X V closedin X C x ∈ V V ⊆ C C ⊆ U C ⊆ S
   by (metis ‹x ∈ S› IntI le_inf_iff neighbourhood_base_of neighbourhood_base_of_closedin)
  show ∃U. openin (subtopology X S) U ∧
           (∃K. compactin (subtopology X S) K ∧ x ∈ U ∧ U ⊆ K)
  proof (intro conjI exI)
    show openin (subtopology X S) V
      using VC by (meson ‹openin X S› openin_open_subtopology order_trans)
    show compactin (subtopology X S) (C ∩ K)
      using UK VC closed_Int_compactin compactin_subtopology by fastforce
  qed (use UK VC x in auto)
qed

lemma locally_compact_space_discrete_topology:
   locally_compact_space (discrete_topology U)
 by (simp add: neighbourhood_base_imp_locally_compact_space neighbourhood_base_of_discrete_topo

lemma locally_compact_space_continuous_open_map_image:
  ⟦continuous_map X X' f; open_map X X' f;
   f ‘ topspace X = topspace X'; locally_compact_space X⟧ ⟹ locally_compact_space
X'
unfolding locally_compact_space_def open_map_def
  by (smt (verit, ccfv_SIG) image_compactin image_iff image_mono)

lemma locally_compact_subspace_openin_closure_of:
  assumes Hausdorff_space X and S: S ⊆ topspace X
    and loc: locally_compact_space (subtopology X S)
  shows openin (subtopology X (X closure_of S)) S
  unfolding openin_subopen [where S=S]
proof clarify
  fix a assume a ∈ S
  then obtain T K where ∗: openin X T compactin X K K ⊆ S a ∈ S a ∈ T S
∩ T ⊆ K
    using loc unfolding locally_compact_space_def
 by (metis IntE S compactin_subtopology inf_commute openin_subtopology topspace_subtopology_subset
  have T ∩ X closure_of S ⊆ X closure_of (T ∩ S)
    by (simp add: ∗(1) openin_Int_closure_of_subset)
  also have ... ⊆ S
   using ∗ ‹Hausdorff_space X› by (metis closure_of_minimal compactin_imp_closedin
order.trans inf_commute)
  finally have T ∩ X closure_of S ⊆ T ∩ S by simp
  then have openin (subtopology X (X closure_of S)) (T ∩ S)
    unfolding openin_subtopology using ‹openin X T› S closure_of_subset by
fastforce
  with ∗ show ∃T. openin (subtopology X (X closure_of S)) T ∧ a ∈ T ∧ T ⊆
S
```

by *blast*
qed

lemma *locally_compact_subspace_closed_Int_openin*:
⟦*Hausdorff_space X* ∧ *S* ⊆ *topspace X* ∧ *locally_compact_space*(*subtopology X S*)⟧
⟹ ∃ *C U*. *closedin X C* ∧ *openin X U* ∧ *C* ∩ *U* = *S*
by (*metis closedin_closure_of inf_commute locally_compact_subspace_openin_closure_of openin_subtopology*)

lemma *locally_compact_subspace_open_in_closure_of_eq*:
assumes *Hausdorff_space X* **and** *loc*: *locally_compact_space X*
shows *openin* (*subtopology X* (*X closure_of S*)) *S* ⟷ *S* ⊆ *topspace X* ∧ *locally_compact_space*(*subtopology X S*) (**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
then obtain *S* ⊆ *topspace X regular_space X*
using *assms locally_compact_Hausdorff_imp_regular_space openin_subset* **by** *fastforce*
then have *locally_compact_space* (*subtopology* (*subtopology X* (*X closure_of S*)) *S*)
by (*simp add*: *L loc locally_compact_space_closed_subset locally_compact_space_open_subset regular_space_subtopology*)
then show *?rhs*
by (*metis L inf.orderE inf_commute le_inf_iff openin_subset subtopology_subtopology topspace_subtopology*)
next
assume *?rhs* **then show** *?lhs*
using *assms locally_compact_subspace_openin_closure_of* **by** *blast*
qed

lemma *locally_compact_subspace_closed_Int_openin_eq*:
assumes *Hausdorff_space X* **and** *loc*: *locally_compact_space X*
shows (∃ *C U*. *closedin X C* ∧ *openin X U* ∧ *C* ∩ *U* = *S*) ⟷ *S* ⊆ *topspace X* ∧ *locally_compact_space*(*subtopology X S*) (**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
then obtain *C U* **where** *closedin X C openin X U* **and** *Seq*: *S* = *C* ∩ *U*
by *blast*
then have *C* ⊆ *topspace X*
by (*simp add*: *closedin_subset*)
have *locally_compact_space* (*subtopology* (*subtopology X C*) (*topspace* (*subtopology X C*) ∩ *U*))
proof (*rule locally_compact_space_open_subset*)
show *locally_compact_space* (*subtopology X C*)
by (*simp add*: ‹*closedin X C*› *loc locally_compact_space_closed_subset*)
show *openin* (*subtopology X C*) (*topspace* (*subtopology X C*) ∩ *U*)
by (*simp add*: ‹*openin X U*› *Int_left_commute inf_commute openin_Int openin_subtopology_Int2*)

```
  qed (simp add: Hausdorff_space_subtopology ‹Hausdorff_space X›)
  then show ?rhs
    by (metis Seq ‹C ⊆ topspace X› inf.coboundedI1 subtopology_subtopology
subtopology_topspace)
next
  assume ?rhs then show ?lhs
    using assms locally_compact_subspace_closed_Int_openin by blast
qed

lemma dense_locally_compact_openin_Hausdorff_space:
  ⟦Hausdorff_space X; S ⊆ topspace X; X closure_of S = topspace X;
    locally_compact_space (subtopology X S)⟧ ⟹ openin X S
  by (metis locally_compact_subspace_openin_closure_of subtopology_topspace)

lemma locally_compact_space_prod_topology:
  locally_compact_space (prod_topology X Y) ⟷
        (prod_topology X Y) = trivial_topology ∨
        locally_compact_space X ∧ locally_compact_space Y (is ?lhs=?rhs)
proof (cases (prod_topology X Y) = trivial_topology)
  case True
  then show ?thesis
    using locally_compact_space_discrete_topology by force
next
  case False
  then obtain w z where wz: w ∈ topspace X z ∈ topspace Y
    by fastforce
  show ?thesis
  proof
    assume L: ?lhs then show ?rhs
    by (metis locally_compact_space_retraction_map_image prod_topology_trivial_iff
retraction_map_fst retraction_map_snd)
  next
    assume R: ?rhs
    show ?lhs
      unfolding locally_compact_space_def
    proof clarsimp
      fix x y
      assume x ∈ topspace X and y ∈ topspace Y
      obtain U C where openin X U compactin X C x ∈ U U ⊆ C
        by (meson False R ‹x ∈ topspace X› locally_compact_space_def)
      obtain V D where openin Y V compactin Y D y ∈ V V ⊆ D
        by (meson False R ‹y ∈ topspace Y› locally_compact_space_def)
      show ∃U. openin (prod_topology X Y) U ∧ (∃K. compactin (prod_topology
X Y) K ∧ (x, y) ∈ U ∧ U ⊆ K)
      proof (intro exI conjI)
        show openin (prod_topology X Y) (U × V)
          by (simp add: ‹openin X U› ‹openin Y V› openin_prod_Times_iff)
        show compactin (prod_topology X Y) (C × D)
          by (simp add: ‹compactin X C› ‹compactin Y D› compactin_Times)
```

```
      show (x, y) ∈ U × V
        by (simp add: ‹x ∈ U› ‹y ∈ V›)
      show U × V ⊆ C × D
        by (simp add: Sigma_mono ‹U ⊆ C› ‹V ⊆ D›)
    qed
  qed
 qed
qed

lemma locally_compact_space_product_topology:
  locally_compact_space(product_topology X I) ⟷
       product_topology X I = trivial_topology ∨
       finite {i ∈ I. ¬ compact_space(X i)} ∧ (∀ i ∈ I. locally_compact_space(X
i)) (is ?lhs=?rhs)
proof (cases (product_topology X I) = trivial_topology)
 case True
 then show ?thesis
   by (simp add: locally_compact_space_def)
next
 case False
 show ?thesis
 proof
   assume L: ?lhs
   obtain z where z: z ∈ topspace (product_topology X I)
     using False
     by (meson ex_in_conv null_topspace_iff_trivial)
    with L z obtain U C where openin (product_topology X I) U compactin
(product_topology X I) C z ∈ U U ⊆ C
     by (meson locally_compact_space_def)
   then obtain V where finV: finite {i ∈ I. V i ≠ topspace (X i)} and ∀ i ∈ I.
openin (X i) (V i)
                  and z ∈ PiE I V PiE I V ⊆ U
     by (auto simp: openin_product_topology_alt)
   have compact_space (X i) if i ∈ I V i = topspace (X i) for i
   proof −
     have compactin (X i) ((λx. x i) ‘ C)
       using ‹compactin (product_topology X I) C› image_compactin
       by (metis continuous_map_product_projection ‹i ∈ I›)
     moreover have V i ⊆ (λx. x i) ‘ C
     proof −
       have V i ⊆ (λx. x i) ‘ Pi_E I V
         by (metis ‹z ∈ Pi_E I V› empty_iff image_projection_PiE order_refl ‹i
∈ I›)
       also have ... ⊆ (λx. x i) ‘ C
         using ‹U ⊆ C› ‹Pi_E I V ⊆ U› by blast
       finally show ?thesis .
     qed
     ultimately show ?thesis
       by (metis closed_compactin closedin_topspace compact_space_def that(2))
```

```
    qed
    with finV have finite {i ∈ I. ¬ compact_space (X i)}
      by (metis (mono_tags, lifting) mem_Collect_eq finite_subset subsetI)
    moreover have locally_compact_space (X i) if i ∈ I for i
      by (meson False L locally_compact_space_retraction_map_image retrac-
tion_map_product_projection that)
    ultimately show ?rhs by metis
  next
    assume R: ?rhs
    show ?lhs
      unfolding locally_compact_space_def
    proof clarsimp
      fix z
      assume z: z ∈ (Π_E i∈I. topspace (X i))
      have ∃ U C. openin (X i) U ∧ compactin (X i) C ∧ z i ∈ U ∧ U ⊆ C ∧
                  (compact_space(X i) ⟶ U = topspace(X i) ∧ C = topspace(X
i))
        if i ∈ I for i
        using that R z unfolding locally_compact_space_def compact_space_def
      by (metis (no_types, lifting) False PiE_mem openin_topspace set_eq_subset)
      then obtain U C where UC: ⋀i. i ∈ I ⟹
            openin (X i) (U i) ∧ compactin (X i) (C i) ∧ z i ∈ U i ∧ U i ⊆ C i ∧
              (compact_space(X i) ⟶ U i = topspace(X i) ∧ C i = topspace(X
i))
        by metis
    show ∃ U. openin (product_topology X I) U ∧ (∃ K. compactin (product_topology
X I) K ∧ z ∈ U ∧ U ⊆ K)
      proof (intro exI conjI)
        show openin (product_topology X I) (Pi_E I U)
      by (smt (verit) Collect_cong False R UC compactin_subspace openin_PiE_gen
subset_antisym subtopology_topspace)
        show compactin (product_topology X I) (Pi_E I C)
          by (simp add: UC compactin_PiE)
      qed (use UC z in blast)+
    qed
  qed
qed

lemma locally_compact_space_sum_topology:
  locally_compact_space (sum_topology X I) ⟷ (∀ i ∈ I. locally_compact_space
(X i)) (is ?lhs=?rhs)
proof
  assume ?lhs then show ?rhs
   by (metis closed_map_component_injection embedding_map_imp_homeomorphic_space
embedding_map_component_injection
       embedding_imp_closed_map_eq homeomorphic_locally_compact_space lo-
cally_compact_space_closed_subset)
next
  assume R: ?rhs
```

show *?lhs*
unfolding *locally_compact_space_def*
proof *clarsimp*
fix *i y*
assume $i \in I$ **and** *y*: $y \in$ *topspace* $(X\ i)$
then obtain *U K* **where** *UK*: *openin* $(X\ i)$ *U compactin* $(X\ i)$ *K* $y \in U\ U \subseteq K$
using *R* **by** (*fastforce simp*: *locally_compact_space_def*)
then show $\exists U.$ *openin* (*sum_topology X I*) *U* $\wedge$ $(\exists K.$ *compactin* (*sum_topology X I*) *K* $\wedge$ $(i, y) \in U \wedge U \subseteq K)$
by (*metis* ‹$i \in I$› *continuous_map_component_injection image_compactin image_mono imageI open_map_component_injection open_map_def*)
qed
qed

lemma *locally_compact_space_euclidean*:
locally_compact_space (*euclidean*::*'a*::*heine_borel topology*)
unfolding *locally_compact_space_def*
proof (*intro strip*)
fix *x*::*'a*
assume $x \in$ *topspace euclidean*
have *ball x 1* $\subseteq$ *cball x 1*
by *auto*
then show $\exists U\ K.$ *openin euclidean U* $\wedge$ *compactin euclidean K* $\wedge$ $x \in U \wedge U \subseteq K$
by (*metis Elementary_Metric_Spaces.open_ball centre_in_ball compact_cball compactin_euclidean_iff open_openin zero_less_one*)
qed

lemma *locally_compact_Euclidean_space*:
locally_compact_space(*Euclidean_space n*)
using *homeomorphic_locally_compact_space* [*OF homeomorphic_Euclidean_space_product_topology*]

using *locally_compact_space_product_topology locally_compact_space_euclidean*
by *fastforce*

proposition *quotient_map_prod_right*:
assumes *loc*: *locally_compact_space Z*
and *reg*: *Hausdorff_space Z* $\vee$ *regular_space Z*
and *f*: *quotient_map X Y f*
shows *quotient_map* (*prod_topology Z X*) (*prod_topology Z Y*) $(\lambda(x,y).\ (x,f\ y))$
proof −
define *h* **where** $h \equiv (\lambda(x{::}'a,y).\ (x,f\ y))$
have *continuous_map* (*prod_topology Z X*) *Y* (*f o snd*)
by (*simp add*: *continuous_map_of_snd f quotient_imp_continuous_map*)
then have *cmh*: *continuous_map* (*prod_topology Z X*) (*prod_topology Z Y*) *h*
by (*simp add*: *h_def continuous_map_paired split_def continuous_map_fst o_def*)

```
  have fim: f ‘ topspace X = topspace Y
    by (simp add: f quotient_imp_surjective_map)
  moreover
  have openin (prod_topology Z X) {u ∈ topspace Z × topspace X. h u ∈ W}
   ⟷ openin (prod_topology Z Y) W   (is ?lhs=?rhs)
    if W: W ⊆ topspace Z × topspace Y for W
  proof
    define S where S ≡ {u ∈ topspace Z × topspace X. h u ∈ W}
    assume ?lhs
    then have L: openin (prod_topology Z X) S
      using S_def by blast
    have ∃ T. openin (prod_topology Z Y) T ∧ (x0, z0) ∈ T ∧ T ⊆ W
      if §: (x0,z0) ∈ W for x0 z0
    proof −
      have x0: x0 ∈ topspace Z
        using W that by blast
      obtain y0 where y0: y0 ∈ topspace X f y0 = z0
        by (metis W fim imageE insert_absorb insert_subset mem_Sigma_iff §)
      then have (x0, y0) ∈ S
        by (simp add: S_def h_def that x0)
      have continuous_map Z (prod_topology Z X) (λx. (x, y0))
        by (simp add: continuous_map_paired y0)
      with openin_continuous_map_preimage [OF _ L]
      have ope_ZS: openin Z {x ∈ topspace Z. (x,y0) ∈ S}
        by blast
      obtain U U' where openin Z U compactin Z U' closedin Z U'
        x0 ∈ U  U ⊆ U' U' ⊆ {x ∈ topspace Z. (x,y0) ∈ S}
        using loc ope_ZS x0 ‹(x0, y0) ∈ S›
        by (force simp: locally_compact_space_neighbourhood_base_closedin [OF
reg]
            neighbourhood_base_of)
      then have D: U' × {y0} ⊆ S
        by (auto simp: )
      define V where V ≡ {z ∈ topspace Y. U' × {y ∈ topspace X. f y = z} ⊆
S}
      have z0 ∈ V
        using D y0 Int_Collect fim by (fastforce simp: h_def V_def S_def)
      have openin X {x ∈ topspace X. f x ∈ V} ⟹ openin Y V
        using f unfolding V_def quotient_map_def subset_iff
        by (smt (verit, del_insts) Collect_cong mem_Collect_eq)
      moreover have openin X {x ∈ topspace X. f x ∈ V}
      proof −
        let ?Z = subtopology Z U'
        have ∗: {x ∈ topspace X. f x ∈ V} = topspace X − snd ‘ (U' × topspace
X − S)
          by (force simp: V_def S_def h_def simp flip: fim)
        have compact_space ?Z
          using ‹compactin Z U'› compactin_subspace by auto
        moreover have closedin (prod_topology ?Z X) (U' × topspace X − S)
```

```
        by (simp add: L ‹closedin Z U'› closedin_closed_subtopology closedin_diff
closedin_prod_Times_iff
            prod_topology_subtopology(1))
       ultimately show ?thesis
        using ∗ closed_map_snd closed_map_def by fastforce
     qed
     ultimately have openin Y V
      by metis
     show ?thesis
     proof (intro conjI exI)
      show openin (prod_topology Z Y) (U × V)
       by (simp add: openin_prod_Times_iff ‹openin Z U› ‹openin Y V›)
      show (x0, z0) ∈ U × V
       by (simp add: ‹x0 ∈ U› ‹z0 ∈ V›)
      show U × V ⊆ W
       using ‹U ⊆ U'› by (force simp: V_def S_def h_def simp flip: fim)
     qed
    qed
    with openin_subopen show ?rhs by force
  next
    assume ?rhs then show ?lhs
     using openin_continuous_map_preimage cmh by fastforce
  qed
  ultimately show ?thesis
    by (fastforce simp: image_iff quotient_map_def h_def)
qed

lemma quotient_map_prod_left:
  assumes loc: locally_compact_space Z
    and reg: Hausdorff_space Z ∨ regular_space Z
    and f: quotient_map X Y f
  shows quotient_map (prod_topology X Z) (prod_topology Y Z) (λ(x,y). (f x,y))
proof −
  have (λ(x,y). (f x,y)) = prod.swap ∘ (λ(x,y). (x,f y)) ∘ prod.swap
    by force
  then
  show ?thesis
    apply (rule ssubst)
  proof (intro quotient_map_compose)
    show quotient_map (prod_topology X Z) (prod_topology Z X) prod.swap
      quotient_map (prod_topology Z Y) (prod_topology Y Z) prod.swap
     using homeomorphic_map_def homeomorphic_map_swap quotient_map_eq
by fastforce+
    show quotient_map (prod_topology Z X) (prod_topology Z Y) (λ(x, y). (x, f
y))
     by (simp add: f loc quotient_map_prod_right reg)
  qed
qed
```

lemma *locally_compact_space_perfect_map_preimage*:
assumes *locally_compact_space X'* **and** *f*: *perfect_map X X' f*
shows *locally_compact_space X*
unfolding *locally_compact_space_def*
proof (*intro strip*)
fix *x*
assume *x*: *x* ∈ *topspace X*
then obtain *U K* **where** *openin X' U compactin X' K f x* ∈ *U U* ⊆ *K*
using *assms* **unfolding** *locally_compact_space_def perfect_map_def*
by (*metis* (*no_types*, *lifting*) *continuous_map_closedin Pi_iff*)
show ∃ *U K*. *openin X U* ∧ *compactin X K* ∧ *x* ∈ *U* ∧ *U* ⊆ *K*
proof (*intro exI conjI*)
have *continuous_map X X' f*
using *f perfect_map_def* **by** *blast*
then show *openin X* {*x* ∈ *topspace X*. *f x* ∈ *U*}
by (*simp add*: ‹*openin X' U*› *continuous_map*)
show *compactin X* {*x* ∈ *topspace X*. *f x* ∈ *K*}
using ‹*compactin X' K*› *f perfect_imp_proper_map proper_map_alt* **by** *blast*
qed (*use x* ‹*f x* ∈ *U*› ‹*U* ⊆ *K*› **in** *auto*)
qed

6.6.14 Special characterizations of classes of functions into and out of R

lemma *monotone_map_into_euclideanreal_alt*:
assumes *continuous_map X euclideanreal f*
shows (∀ *k*. *is_interval k* ⟶ *connectedin X* {*x* ∈ *topspace X*. *f x* ∈ *k*}) ⟷
connected_space X ∧ *monotone_map X euclideanreal f* (**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof
show *connected_space X*
using *L connected_space_subconnected* **by** *blast*
have *connectedin X* {*x* ∈ *topspace X*. *f x* ∈ {*y*}} **for** *y*
by (*metis L is_interval_1 nle_le singletonD*)
then show *monotone_map X euclideanreal f*
by (*simp add*: *monotone_map*)
qed
next
assume *R*: *?rhs*
then
have ∗: *False*
if *a* < *b closedin X U closedin X V U* ≠ {} *V* ≠ {} *disjnt U V*
and *UV*: {*x* ∈ *topspace X*. *f x* ∈ {*a*..*b*}} = *U* ∪ *V*
and *dis*: *disjnt U* {*x* ∈ *topspace X*. *f x* = *b*} *disjnt V* {*x* ∈ *topspace X*. *f x* = *a*}
for *a b U V*
proof −

```
    define E1 where E1 ≡ U ∪ {x ∈ topspace X. f x ∈ {c. c ≤ a}}
    define E2 where E2 ≡ V ∪ {x ∈ topspace X. f x ∈ {c. b ≤ c}}
    have closedin X {x ∈ topspace X. f x ≤ a} closedin X {x ∈ topspace X. b ≤ f
x}
      using assms continuous_map_upper_lower_semicontinuous_le by blast+
    then have closedin X E1 closedin X E2
      unfolding E1_def E2_def using that by auto
    moreover
    have E1 ∩ E2 = {}
      unfolding E1_def E2_def using ‹a<b› ‹disjnt U V› dis UV
      by (simp add: disjnt_def set_eq_iff) (smt (verit))
    have topspace X ⊆ E1 ∪ E2
      unfolding E1_def E2_def using UV by fastforce
    have E1 = {} ∨ E2 = {}
      using R connected_space_closedin
      using ‹E1 ∩ E2 = {}› ‹closedin X E1› ‹closedin X E2› ‹topspace X ⊆ E1 ∪
E2› by blast
    then show False
      using E1_def E2_def ‹U ≠ {}› ‹V ≠ {}› by fastforce
  qed
  show ?lhs
  proof (intro strip)
    fix K :: real set
    assume is_interval K
    have False
      if a ∈ K b ∈ K and clo: closedin X U closedin X V
        and UV: {x. x ∈ topspace X ∧ f x ∈ K} ⊆ U ∪ V
                U ∩ V ∩ {x. x ∈ topspace X ∧ f x ∈ K} = {}
        and nondis: ¬ disjnt U {x. x ∈ topspace X ∧ f x = a}
                    ¬ disjnt V {x. x ∈ topspace X ∧ f x = b}
     for a b U V
    proof −
      have ∀ y. connectedin X {x. x ∈ topspace X ∧ f x = y}
        using R monotone_map by fastforce
      then have **: False if p ∈ U ∧ q ∈ V ∧ f p = f q ∧ f q ∈ K for p q
        unfolding connectedin_closedin
        using ‹a ∈ K› ‹b ∈ K› UV clo that
       by (smt (verit, ccfv_threshold) closedin_subset disjoint_iff mem_Collect_eq
subset_iff)
      consider a < b | a = b | b < a
        by linarith
      then show ?thesis
      proof cases
        case 1
        define W where W ≡ {x ∈ topspace X. f x ∈ {a..b}}
        have closedin X W
          unfolding W_def
          by (metis (no_types) assms closed_real_atLeastAtMost closed_closedin
continuous_map_closedin)
```

```
      show ?thesis
      proof (rule * [OF 1 , of U ∩ W V ∩ W])
        show closedin X (U ∩ W) closedin X (V ∩ W)
          using ‹closedin X W› clo by auto
        show U ∩ W ≠ {} V ∩ W ≠ {}
          using nondis 1 by (auto simp: disjnt_iff W_def)
        show disjnt (U ∩ W) (V ∩ W)
          using ‹is_interval K› unfolding is_interval_1 disjnt_iff W_def
              by (metis (mono_tags, lifting) ‹a ∈ K› ‹b ∈ K› ** Int_Collect
atLeastAtMost_iff)
        have ⋀x. ⟦x ∈ topspace X; a ≤ f x; f x ≤ b⟧ ⟹ x ∈ U ∨ x ∈ V
          using ‹a ∈ K› ‹b ∈ K› ‹is_interval K› UV unfolding is_interval_1
disjnt_iff
          by blast
        then show {x ∈ topspace X. f x ∈ {a..b}} = U ∩ W ∪ V ∩ W
          by (auto simp: W_def)
        show disjnt (U ∩ W) {x ∈ topspace X. f x = b} disjnt (V ∩ W) {x ∈
topspace X. f x = a}
          using ** ‹a ∈ K› ‹b ∈ K› nondis by (force simp: disjnt_iff)+
      qed
    next
      case 2
      then show ?thesis
        using ** nondis ‹b ∈ K› by (force simp add: disjnt_iff)
    next
      case 3
      define W where W ≡ {x ∈ topspace X. f x ∈ {b..a}}
      have closedin X W
        unfolding W_def
        by (metis (no_types) assms closed_real_atLeastAtMost closed_closedin
continuous_map_closedin)
      show ?thesis
      proof (rule * [OF 3, of V ∩ W U ∩ W])
        show closedin X (U ∩ W) closedin X (V ∩ W)
          using ‹closedin X W› clo by auto
        show U ∩ W ≠ {} V ∩ W ≠ {}
          using nondis 3 by (auto simp: disjnt_iff W_def)
        show disjnt (V ∩ W) (U ∩ W)
          using ‹is_interval K› unfolding is_interval_1 disjnt_iff W_def
              by (metis (mono_tags, lifting) ‹a ∈ K› ‹b ∈ K› ** Int_Collect
atLeastAtMost_iff)
        have ⋀x. ⟦x ∈ topspace X; b ≤ f x; f x ≤ a⟧ ⟹ x ∈ U ∨ x ∈ V
          using ‹a ∈ K› ‹b ∈ K› ‹is_interval K› UV unfolding is_interval_1
disjnt_iff
          by blast
        then show {x ∈ topspace X. f x ∈ {b..a}} = V ∩ W ∪ U ∩ W
          by (auto simp: W_def)
        show disjnt (V ∩ W) {x ∈ topspace X. f x = a} disjnt (U ∩ W) {x ∈
topspace X. f x = b}
```

```
          using ** ‹a ∈ K› ‹b ∈ K› nondis by (force simp: disjnt_iff)+
        qed
      qed
    qed
    then show connectedin X {x ∈ topspace X. f x ∈ K}
      unfolding connectedin_closedin disjnt_iff by blast
  qed
qed

lemma monotone_map_into_euclideanreal:
  ⟦connected_space X; continuous_map X euclideanreal f⟧
   ⟹ monotone_map X euclideanreal f ⟷
      (∀k. is_interval k ⟶ connectedin X {x ∈ topspace X. f x ∈ k})
  by (simp add: monotone_map_into_euclideanreal_alt)

lemma monotone_map_euclideanreal_alt:
  (∀I::real set. is_interval I ⟶ is_interval {x::real. x ∈ S ∧ f x ∈ I}) ⟷
   is_interval S ∧ (mono_on S f ∨ antimono_on S f) (is ?lhs=?rhs)
proof
  assume L [rule_format]: ?lhs
  show ?rhs
  proof
    show is_interval S
      using L is_interval_1 by auto
    have False if a ∈ S b ∈ S c ∈ S a<b b<c and d: f a < f b ∧ f c < f b ∨ f a
> f b ∧ f c > f b for a b c
      using d
    proof
      assume f a < f b ∧ f c < f b
      then show False
        using L [of {y.  y < f b}] unfolding is_interval_1
        by (smt (verit, best) mem_Collect_eq that)
    next
      assume f b < f a ∧ f b < f c
      then show False
        using L [of {y.  y > f b}] unfolding is_interval_1
        by (smt (verit, best) mem_Collect_eq that)
    qed
    then show mono_on S f ∨ monotone_on S (≤) (≥) f
      unfolding monotone_on_def by (smt (verit))
  qed
next
  assume ?rhs then show ?lhs
    unfolding is_interval_1 monotone_on_def by simp meson
qed


lemma monotone_map_euclideanreal:
  fixes S :: real set
```

shows
⟦*is_interval S*; *continuous_on S f*⟧ ⟹
monotone_map (*top_of_set S*) *euclideanreal f* ⟷ (*mono_on S f* ∨ *monotone_on S* (≤) (≥) *f*)
using *monotone_map_euclideanreal_alt*
by (*simp add*: *monotone_map_into_euclideanreal connectedin_subtopology is_interval_connected_1*)

lemma *injective_eq_monotone_map*:
fixes *f* :: *real* ⇒ *real*
assumes *is_interval S continuous_on S f*
shows *inj_on f S* ⟷ *strict_mono_on S f* ∨ *strict_antimono_on S f*
by (*metis assms injective_imp_monotone_map monotone_map_euclideanreal strict_antimono_iff_antimono*
strict_mono_iff_mono top_greatest topspace_euclidean topspace_euclidean_subtopology)

6.6.15 Normal spaces

definition *normal_space*
where *normal_space X* ≡
∀ *S T*. *closedin X S* ∧ *closedin X T* ∧ *disjnt S T*
⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*)

lemma *normal_space_retraction_map_image*:
assumes *r*: *retraction_map X Y r* **and** *X*: *normal_space X*
shows *normal_space Y*
unfolding *normal_space_def*
proof *clarify*
fix *S T*
assume *closedin Y S* **and** *closedin Y T* **and** *disjnt S T*
obtain *r′* **where** *r′*: *retraction_maps X Y r r′*
using *r retraction_map_def* **by** *blast*
have *closedin X* {*x* ∈ *topspace X*. *r x* ∈ *S*} *closedin X* {*x* ∈ *topspace X*. *r x* ∈ *T*}
using *closedin_continuous_map_preimage* ‹*closedin Y S*› ‹*closedin Y T*› *r′*
by (*auto simp*: *retraction_maps_def*)
moreover
have *disjnt* {*x* ∈ *topspace X*. *r x* ∈ *S*} {*x* ∈ *topspace X*. *r x* ∈ *T*}
using ‹*disjnt S T*› **by** (*auto simp*: *disjnt_def*)
ultimately
obtain *U V* **where** *UV*: *openin X U* ∧ *openin X V* ∧ {*x* ∈ *topspace X*. *r x* ∈ *S*} ⊆ *U* ∧ {*x* ∈ *topspace X*. *r x* ∈ *T*} ⊆ *V disjnt U V*
by (*meson X normal_space_def*)
show ∃ *U V*. *openin Y U* ∧ *openin Y V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*
proof (*intro exI conjI*)
show *openin Y* {*x* ∈ *topspace Y*. *r′ x* ∈ *U*} *openin Y* {*x* ∈ *topspace Y*. *r′ x* ∈ *V*}
using *openin_continuous_map_preimage UV r′*
by (*auto simp*: *retraction_maps_def*)

show $S \subseteq \{x \in \text{topspace } Y.\ r'\ x \in U\}$ $T \subseteq \{x \in \text{topspace } Y.\ r'\ x \in V\}$
using *openin_continuous_map_preimage UV r'* ‹*closedin Y S*› ‹*closedin Y T*›
by (*auto simp add*: *closedin_def continuous_map_closedin retraction_maps_def subset_iff Pi_iff*)
show *disjnt* $\{x \in \text{topspace } Y.\ r'\ x \in U\}$ $\{x \in \text{topspace } Y.\ r'\ x \in V\}$
using ‹*disjnt U V*› **by** (*auto simp*: *disjnt_def*)
qed
qed

lemma *homeomorphic_normal_space*:
X homeomorphic_space Y $\Longrightarrow$ *normal_space X* $\longleftrightarrow$ *normal_space Y*
unfolding *homeomorphic_space_def*
by (*meson homeomorphic_imp_retraction_maps homeomorphic_maps_sym normal_space_retraction_map_image retraction_map_def*)

lemma *normal_space*:
normal_space X $\longleftrightarrow$
$(\forall S\ T.$ *closedin X S* $\wedge$ *closedin X T* $\wedge$ *disjnt S T*
$\longrightarrow (\exists U.$ *openin X U* $\wedge$ $S \subseteq U$ $\wedge$ *disjnt T* (*X closure_of U*)))
proof −
have $(\exists V.$ *openin X U* $\wedge$ *openin X V* $\wedge$ $S \subseteq U$ $\wedge$ $T \subseteq V$ $\wedge$ *disjnt U V*) $\longleftrightarrow$ *openin X U* $\wedge$ $S \subseteq U$ $\wedge$ *disjnt T* (*X closure_of U*)
(**is** *?lhs=?rhs*)
if *closedin X S closedin X T disjnt S T* **for** *S T U*
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*smt* (*verit*, *best*) *disjnt_iff in_closure_of subsetD*)
assume *R*: *?rhs*
then have $(U \cup S) \cap (\text{topspace } X - X \text{ closure\_of } U) = \{\}$
by (*metis Diff_eq_empty_iff Int_Diff Int_Un_eq*(*4*) *closure_of_subset inf.orderE openin_subset*)
moreover have $T \subseteq \text{topspace } X - X \text{ closure\_of } U$
by (*meson DiffI R closedin_subset disjnt_iff subsetD subsetI that*(*2*))
ultimately show *?lhs*
by (*metis R closedin_closure_of closedin_def disjnt_def sup.orderE*)
qed
then show *?thesis*
unfolding *normal_space_def* **by** *meson*
qed

lemma *normal_space_alt*:
normal_space X $\longleftrightarrow$
$(\forall S\ U.$ *closedin X S* $\wedge$ *openin X U* $\wedge$ $S \subseteq U \longrightarrow (\exists V.$ *openin X V* $\wedge$ $S \subseteq V$ $\wedge$ *X closure_of V* $\subseteq U$))
proof −
have $\exists V.$ *openin X V* $\wedge$ $S \subseteq V$ $\wedge$ *X closure_of V* $\subseteq U$
if $\bigwedge T.$ *closedin X T* $\longrightarrow$ *disjnt S T* $\longrightarrow (\exists U.$ *openin X U* $\wedge$ $S \subseteq U$ $\wedge$ *disjnt T* (*X closure_of U*))

closedin X S openin X U S ⊆ U
for *S U*
using *that*
by (*smt* (*verit*) *Diff_eq_empty_iff Int_Diff closure_of_subset_topspace disjnt_def inf.orderE inf_commute openin_closedin_eq*)
moreover have ∃ *U*. *openin X U ∧ S ⊆ U ∧ disjnt T* (*X closure_of U*)
if ⋀*U*. *openin X U ∧ S ⊆ U ⟶* (∃ *V*. *openin X V ∧ S ⊆ V ∧ X closure_of V ⊆ U*)
and *closedin X S closedin X T disjnt S T*
for *S T*
using *that*
by (*smt* (*verit*) *Diff_Diff_Int Diff_eq_empty_iff Int_Diff closedin_def disjnt_def inf.absorb_iff2 inf.orderE*)
ultimately show *?thesis*
by (*fastforce simp*: *normal_space*)
qed

lemma *normal_space_closures*:
normal_space X ⟷
(∀ *S T*. *S ⊆ topspace X ∧ T ⊆ topspace X ∧*
disjnt (*X closure_of S*) (*X closure_of T*)
⟶ (∃ *U V*. *openin X U ∧ openin X V ∧ S ⊆ U ∧ T ⊆ V ∧ disjnt U V*))
(**is** *?lhs=?rhs*)
proof
show *?lhs ⟹ ?rhs*
by (*meson closedin_closure_of closure_of_subset normal_space_def order.trans*)
show *?rhs ⟹ ?lhs*
by (*metis closedin_subset closure_of_eq normal_space_def*)
qed

lemma *normal_space_disjoint_closures*:
normal_space X ⟷
(∀ *S T*. *closedin X S ∧ closedin X T ∧ disjnt S T*
⟶ (∃ *U V*. *openin X U ∧ openin X V ∧ S ⊆ U ∧ T ⊆ V ∧*
disjnt (*X closure_of U*) (*X closure_of V*)))
(**is** *?lhs=?rhs*)
proof
show *?lhs ⟹ ?rhs*
by (*metis closedin_closure_of normal_space*)
show *?rhs ⟹ ?lhs*
by (*smt* (*verit*) *closure_of_subset disjnt_iff normal_space openin_subset subset_eq*)
qed

lemma *normal_space_dual*:
normal_space X ⟷
(∀ *U V*. *openin X U ⟶ openin X V ∧ U ∪ V = topspace X*
⟶ (∃ *S T*. *closedin X S ∧ closedin X T ∧ S ⊆ U ∧ T ⊆ V ∧ S ∪ T =*

```
topspace X))
  (is _ = ?rhs)
proof −
  have normal_space X ⟷
       (∀ U V. closedin X U ⟶ closedin X V ⟶ disjnt U V ⟶
            (∃ S T. ¬ (openin X S ∧ openin X T ⟶
                       ¬ (U ⊆ S ∧ V ⊆ T ∧ disjnt S T))))
    unfolding normal_space_def by meson
  also have ... ⟷ (∀ U V. openin X U ⟶ openin X V ∧ disjnt (topspace X −
U) (topspace X − V) ⟶
            (∃ S T. ¬ (openin X S ∧ openin X T ⟶
                       ¬ (topspace X − U ⊆ S ∧ topspace X − V ⊆ T ∧ disjnt S
T))))
    by (auto simp: all_closedin)
  also have ... ⟷ ?rhs
  proof −
    have ∗: disjnt (topspace X − U) (topspace X − V) ⟷ U ∪ V = topspace X
      if U ⊆ topspace X V ⊆ topspace X for U V
      using that by (auto simp: disjnt_iff)
    show ?thesis
      using ex_closedin ∗
       apply (simp add: ex_closedin ∗ [OF openin_subset openin_subset] cong:
conj_cong)
      apply (intro all_cong1 ex_cong1 imp_cong refl)
     by (smt (verit, best) ∗ Diff_Diff_Int Diff_subset Diff_subset_conv inf.orderE
inf_commute openin_subset sup_commute)
  qed
  finally show ?thesis .
qed


lemma normal_t1_imp_Hausdorff_space:
  assumes normal_space X t1_space X
  shows Hausdorff_space X
  unfolding Hausdorff_space_def
proof clarify
  fix x y
  assume xy: x ∈ topspace X y ∈ topspace X x ≠ y
  then have disjnt {x} {y}
    by (auto simp: disjnt_iff)
  then show ∃ U V. openin X U ∧ openin X V ∧ x ∈ U ∧ y ∈ V ∧ disjnt U V
    using assms xy closedin_t1_singleton normal_space_def
    by (metis singletonI subsetD)
qed

lemma normal_t1_eq_Hausdorff_space:
   normal_space X ⟹ t1_space X ⟷ Hausdorff_space X
  using normal_t1_imp_Hausdorff_space t1_or_Hausdorff_space by blast
```

lemma *normal_t1_imp_regular_space*:
  ⟦*normal_space X*; *t1_space X*⟧ ⟹ *regular_space X*
 by (*metis compactin_imp_closedin normal_space_def normal_t1_eq_Hausdorff_space regular_space_compact_closed_sets*)

lemma *compact_Hausdorff_or_regular_imp_normal_space*:
  ⟦*compact_space X*; *Hausdorff_space X* ∨ *regular_space X*⟧
      ⟹ *normal_space X*
 by (*metis Hausdorff_space_compact_sets closedin_compact_space normal_space_def regular_space_compact_closed_sets*)

lemma *normal_space_discrete_topology*:
  normal_space(*discrete_topology U*)
 by (*metis discrete_topology_closure_of inf_le2 normal_space_alt*)

lemma *normal_space_fsigmas*:
 normal_space X ⟷
   (∀ *S T*. *fsigma_in X S* ∧ *fsigma_in X T* ∧ *separatedin X S T*
         ⟶ (∃ *U B*. *openin X U* ∧ *openin X B* ∧ *S* ⊆ *U* ∧ *T* ⊆ *B* ∧ *disjnt U B*)) (**is** *?lhs=?rhs*)
proof
 assume *L*: *?lhs*
 show *?rhs*
 proof *clarify*
  fix *S T*
  assume *fsigma_in X S*
   then obtain *C* **where** *C*: ⋀*n*. *closedin X* (*C n*) ⋀*n*. *C n* ⊆ *C* (*Suc n*) ⋃ (*range C*) = *S*
    by (*meson fsigma_in_ascending*)
  assume *fsigma_in X T*
   then obtain *D* **where** *D*: ⋀*n*. *closedin X* (*D n*) ⋀*n*. *D n* ⊆ *D* (*Suc n*) ⋃ (*range D*) = *T*
    by (*meson fsigma_in_ascending*)
  assume *separatedin X S T*
  have ⋀*n*. *disjnt* (*D n*) (*X closure_of S*)
   by (*metis D(3)* ‹*separatedin X S T*› *disjnt_Union1 disjnt_def rangeI separatedin_def*)
  then have ⋀*n*. ∃ *V V'*. *openin X V* ∧ *openin X V'* ∧ *D n* ⊆ *V* ∧ *X closure_of S* ⊆ *V'* ∧ *disjnt V V'*
    by (*metis D(1) L closedin_closure_of normal_space_def*)
  then obtain *V V'* **where** *V*: ⋀*n*. *openin X* (*V n*) **and** ⋀*n*. *openin X* (*V' n*) ⋀*n*. *disjnt* (*V n*) (*V' n*)
       and *DV*: ⋀*n*. *D n* ⊆ *V n*
       and *subV'*: ⋀*n*. *X closure_of S* ⊆ *V' n*
    by *metis*
  then have *VV*: *V' n* ∩ *X closure_of V n* = {} **for** *n*
    using *openin_Int_closure_of_eq_empty* [*of X V' n V n*] **by** (*simp add*: *Int_commute disjnt_def*)
  have ⋀*n*. *disjnt* (*C n*) (*X closure_of T*)

by (*metis C(3) ‹separatedin X S T› disjnt_Union1 disjnt_def rangeI separatedin_def*)
then have $\bigwedge n.\ \exists U\ U'.$ *openin X U* $\wedge$ *openin X U'* $\wedge$ *C n* $\subseteq$ *U* $\wedge$ *X closure_of T* $\subseteq$ *U'* $\wedge$ *disjnt U U'*
by (*metis C(1) L closedin_closure_of normal_space_def*)
then obtain *U U'* **where** *U*: $\bigwedge n.$ *openin X* (*U n*) **and** $\bigwedge n.$ *openin X* (*U' n*) $\bigwedge n.$ *disjnt* (*U n*) (*U' n*)
and *CU*: $\bigwedge n.$ *C n* $\subseteq$ *U n*
and *subU'*: $\bigwedge n.$ *X closure_of T* $\subseteq$ *U' n*
by *metis*
then have *UU*: *U' n* $\cap$ *X closure_of U n* = {} **for** *n*
using *openin_Int_closure_of_eq_empty* [*of X U' n U n*] **by** (*simp add: Int_commute disjnt_def*)
show $\exists\, U\ B.$ *openin X U* $\wedge$ *openin X B* $\wedge$ *S* $\subseteq$ *U* $\wedge$ *T* $\subseteq$ *B* $\wedge$ *disjnt U B*
proof (*intro conjI exI*)
have $\bigwedge S\ n.$ *closedin X* ($\bigcup m{\le}n.$ *X closure_of V m*)
by (*force intro*: *closedin_Union*)
then show *openin X* ($\bigcup n.$ *U n* − ($\bigcup m{\le}n.$ *X closure_of V m*))
using *U* **by** *blast*
have $\bigwedge S\ n.$ *closedin X* ($\bigcup m{\le}n.$ *X closure_of U m*)
by (*force intro*: *closedin_Union*)
then show *openin X* ($\bigcup n.$ *V n* − ($\bigcup m{\le}n.$ *X closure_of U m*))
using *V* **by** *blast*
have *S* $\subseteq$ *topspace X*
by (*simp add*: *‹fsigma_in X S› fsigma_in_subset*)
then show *S* $\subseteq$ ($\bigcup n.$ *U n* − ($\bigcup m{\le}n.$ *X closure_of V m*))
apply (*clarsimp simp*: *Ball_def*)
by (*metis VV C(3) CU IntI UN_E closure_of_subset empty_iff subV' subsetD*)
have *T* $\subseteq$ *topspace X*
by (*simp add*: *‹fsigma_in X T› fsigma_in_subset*)
then show *T* $\subseteq$ ($\bigcup n.$ *V n* − ($\bigcup m{\le}n.$ *X closure_of U m*))
apply (*clarsimp simp*: *Ball_def*)
by (*metis UU D(3) DV IntI UN_E closure_of_subset empty_iff subU' subsetD*)
have $\bigwedge x\ m\ n.\ [\![x \in U\ n;\ x \in V\ m;\ \forall k{\le}m.\ x \notin X\ closure\_of\ U\ k]\!] \Longrightarrow \exists k{\le}n.$ *x* $\in$ *X closure_of V k*
by (*meson U V closure_of_subset nat_le_linear openin_subset subsetD*)
then show *disjnt* ($\bigcup n.$ *U n* − ($\bigcup m{\le}n.$ *X closure_of V m*)) ($\bigcup n.$ *V n* − ($\bigcup m{\le}n.$ *X closure_of U m*))
by (*force simp*: *disjnt_iff*)
qed
qed
next
show *?rhs* $\Longrightarrow$ *?lhs*
by (*simp add*: *closed_imp_fsigma_in normal_space_def separatedin_closed_sets*)
qed

lemma *normal_space_fsigma_subtopology*:

assumes *normal_space X fsigma_in X S*
shows *normal_space* (*subtopology X S*)
unfolding *normal_space_fsigmas*
proof *clarify*
fix *T U*
assume *fsigma_in* (*subtopology X S*) *T*
and *fsigma_in* (*subtopology X S*) *U*
and *TU*: *separatedin* (*subtopology X S*) *T U*
then obtain *A B* **where** *openin X A* ∧ *openin X B* ∧ *T* ⊆ *A* ∧ *U* ⊆ *B* ∧ *disjnt A B*
by (*metis assms fsigma_in_fsigma_subtopology normal_space_fsigmas separatedin_subtopology*)
then
show ∃ *A B*. *openin* (*subtopology X S*) *A* ∧ *openin* (*subtopology X S*) *B* ∧ *T* ⊆ *A* ∧
U ⊆ *B* ∧ *disjnt A B*
using *TU*
by (*force simp add*: *separatedin_subtopology openin_subtopology_alt disjnt_iff*)
qed

lemma *normal_space_closed_subtopology*:
assumes *normal_space X closedin X S*
shows *normal_space* (*subtopology X S*)
by (*simp add*: *assms closed_imp_fsigma_in normal_space_fsigma_subtopology*)

lemma *normal_space_continuous_closed_map_image*:
assumes *normal_space X* **and** *contf*: *continuous_map X Y f*
and *clof*: *closed_map X Y f* **and** *fim*: *f ‘ topspace X* = *topspace Y*
shows *normal_space Y*
unfolding *normal_space_def*
proof *clarify*
fix *S T*
assume *closedin Y S* **and** *closedin Y T* **and** *disjnt S T*
have *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *S*} *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *T*}
using ‹*closedin Y S*› ‹*closedin Y T*› *closedin_continuous_map_preimage contf*
by *auto*
moreover
have *disjnt* {*x* ∈ *topspace X*. *f x* ∈ *S*} {*x* ∈ *topspace X*. *f x* ∈ *T*}
using ‹*disjnt S T*› **by** (*auto simp*: *disjnt_iff*)
ultimately
obtain *U V* **where** *closedin X U closedin X V*
and *subXU*: {*x* ∈ *topspace X*. *f x* ∈ *S*} ⊆ *topspace X* − *U*
and *subXV*: {*x* ∈ *topspace X*. *f x* ∈ *T*} ⊆ *topspace X* − *V*
and *dis*: *disjnt* (*topspace X* − *U*) (*topspace X* − *V*)
using ‹*normal_space X*› **by** (*force simp add*: *normal_space_def ex_openin*)
have *closedin Y* (*f ‘ U*) *closedin Y* (*f ‘ V*)
using ‹*closedin X U*› ‹*closedin X V*› *clof closed_map_def* **by** *blast*+
moreover have *S* ⊆ *topspace Y* − *f ‘ U*

using ‹*closedin Y S*› ‹*closedin X U*› *subXU* **by** (*force dest*: *closedin_subset*)
moreover have $T \subseteq$ *topspace Y* $-$ *f* ‘ *V*
using ‹*closedin Y T*› ‹*closedin X V*› *subXV* **by** (*force dest*: *closedin_subset*)
moreover have *disjnt* (*topspace Y* $-$ *f* ‘ *U*) (*topspace Y* $-$ *f* ‘ *V*)
using *fim dis* **by** (*force simp add*: *disjnt_iff*)
ultimately show $\exists$ *U V*. *openin Y U* $\wedge$ *openin Y V* $\wedge$ *S* $\subseteq$ *U* $\wedge$ *T* $\subseteq$ *V* $\wedge$ *disjnt U V*
by (*force simp add*: *ex_openin*)
qed

6.6.16 Hereditary topological properties

definition *hereditarily*
where *hereditarily P X* $\equiv$
$\forall$ *S*. *S* $\subseteq$ *topspace X* $\longrightarrow$ *P*(*subtopology X S*)

lemma *hereditarily*:
hereditarily P X $\longleftrightarrow$ ($\forall$ *S*. *P*(*subtopology X S*))
by (*metis Int_lower1 hereditarily_def subtopology_restrict*)

lemma *hereditarily_mono*:
⟦*hereditarily P X*; $\bigwedge$*x*. *P x* $\Longrightarrow$ *Q x*⟧ $\Longrightarrow$ *hereditarily Q X*
by (*simp add*: *hereditarily*)

lemma *hereditarily_inc*:
hereditarily P X $\Longrightarrow$ *P X*
by (*metis hereditarily subtopology_topspace*)

lemma *hereditarily_subtopology*:
hereditarily P X $\Longrightarrow$ *hereditarily P* (*subtopology X S*)
by (*simp add*: *hereditarily subtopology_subtopology*)

lemma *hereditarily_normal_space_continuous_closed_map_image*:
assumes *X*: *hereditarily normal_space X* **and** *contf*: *continuous_map X Y f*
and *clof*: *closed_map X Y f* **and** *fim*: *f* ‘ (*topspace X*) = *topspace Y*
shows *hereditarily normal_space Y*
unfolding *hereditarily_def*
proof (*intro strip*)
fix *T*
assume *T* $\subseteq$ *topspace Y*
then have *nx*: *normal_space* (*subtopology X* {*x* $\in$ *topspace X*. *f x* $\in$ *T*})
by (*meson X hereditarily*)
moreover have *continuous_map* (*subtopology X* {*x* $\in$ *topspace X*. *f x* $\in$ *T*}) (*subtopology Y T*) *f*
by (*simp add*: *contf continuous_map_from_subtopology continuous_map_in_subtopology image_subset_iff*)
moreover have *closed_map* (*subtopology X* {*x* $\in$ *topspace X*. *f x* $\in$ *T*}) (*subtopology Y T*) *f*
by (*simp add*: *clof closed_map_restriction*)

ultimately show *normal_space* (*subtopology Y T*)
using *fim normal_space_continuous_closed_map_image* **by** *fastforce*
qed

lemma *homeomorphic_hereditarily_normal_space*:
X homeomorphic_space Y
⟹ (*hereditarily normal_space X* ⟷ *hereditarily normal_space Y*)
by (*meson hereditarily_normal_space_continuous_closed_map_image homeomorphic_eq_everything_map*
homeomorphic_space homeomorphic_space_sym)

lemma *hereditarily_normal_space_retraction_map_image*:
⟦*retraction_map X Y r*; *hereditarily normal_space X*⟧ ⟹ *hereditarily normal_space Y*
by (*smt* (*verit*) *hereditarily_subtopology hereditary_imp_retractive_property homeomorphic_hereditarily_normal_space*)

6.6.17 Limits in a topological space

lemma *limitin_const_iff*:
assumes *t1_space X* ¬ *trivial_limit F*
shows *limitin X* (λ*k*. *a*) *l F* ⟷ *l* ∈ *topspace X* ∧ *a* = *l* (**is** *?lhs*=*?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *assms* **unfolding** *limitin_def t1_space_def* **by** (*metis eventually_const openin_topspace*)
next
assume *?rhs* **then show** *?lhs*
using *assms* **by** (*auto simp*: *limitin_def t1_space_def*)
qed

lemma *compactin_sequence_with_limit*:
assumes *lim*: *limitin X σ l sequentially* **and** *S* ⊆ *range σ* **and** *SX*: *S* ⊆ *topspace X*
shows *compactin X* (*insert l S*)
unfolding *compactin_def*
proof (*intro conjI strip*)
show *insert l S* ⊆ *topspace X*
by (*meson SX insert_subset lim limitin_topspace*)
fix *U*
assume §: *Ball U* (*openin X*) ∧ *insert l S* ⊆ ⋃ *U*
have ∃ *V*. *finite V* ∧ *V* ⊆ *U* ∧ (∃ *t* ∈ *V*. *l* ∈ *t*) ∧ *S* ⊆ ⋃ *V*
if ∗: ∀ *x* ∈ *S*. ∃ *T* ∈ *U*. *x* ∈ *T* **and** *T* ∈ *U l* ∈ *T* **for** *T*
proof −
obtain *V* **where** *V*: ⋀*x*. *x* ∈ *S* ⟹ *V x* ∈ *U* ∧ *x* ∈ *V x*
using ∗ **by** *metis*
obtain *N* **where** *N*: ⋀*n*. *N* ≤ *n* ⟹ *σ n* ∈ *T*
by (*meson* § ‹*T* ∈ *U*› ‹*l* ∈ *T*› *lim limitin_sequentially*)
show *?thesis*

```
    proof (intro conjI exI)
      have x ∈ T
        if x ∈ S and ∀A. (∀x∈S. (∀n≤N. x ≠ σ n) ∨ A ≠ V x) ∨ x ∉ A for x
        by (metis (no_types) N V that assms(2) imageE nle_le subsetD)
      then show S ⊆ ⋃ (insert T (V ` (S ∩ σ ` {0..N})))
        by force
    qed (use V that in auto)
  qed
  then show ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ insert l S ⊆ ⋃ ℱ
    by (smt (verit, best) Union_iff § insert_subset subsetD)
qed

lemma limitin_Hausdorff_unique:
  assumes limitin X f l1 F limitin X f l2 F ¬ trivial_limit F Hausdorff_space X
  shows l1 = l2
proof (rule ccontr)
  assume l1 ≠ l2
  with assms obtain U V where openin X U openin X V l1 ∈ U l2 ∈ V disjnt
U V
    by (metis Hausdorff_space_def limitin_topspace)
  then have eventually (λx. f x ∈ U) F eventually (λx. f x ∈ V) F
    using assms by (fastforce simp: limitin_def)+
  then have ∃x. f x ∈ U ∧ f x ∈ V
    using assms eventually_elim2 filter_eq_iff by fastforce
  with assms ‹disjnt U V› show False
    by (meson disjnt_iff)
qed

lemma limitin_kc_unique:
  assumes kc_space X and lim1: limitin X f l1 sequentially and lim2: limitin X
f l2 sequentially
  shows l1 = l2
proof (rule ccontr)
  assume l1 ≠ l2
  define A where A ≡ insert l1 (range f − {l2})
  have l1 ∈ topspace X
    using lim1 limitin_def by fastforce
  moreover have compactin X (insert l1 (topspace X ∩ (range f − {l2})))
    by (meson Diff_subset compactin_sequence_with_limit inf_le1 inf_le2 lim1
subset_trans)
  ultimately have compactin X (topspace X ∩ A)
    by (simp add: A_def)
  then have OXA: openin X (topspace X − A)
   by (metis Diff_Diff_Int Diff_subset ‹kc_space X› kc_space_def openin_closedin_eq)
  have l2 ∈ topspace X − A
    using ‹l1 ≠ l2› A_def lim2 limitin_topspace by fastforce
  then have ∀ F x in sequentially. f x = l2
    using limitinD [OF lim2 OXA] by (auto simp: A_def eventually_conj_iff)
  then show False
```

using *limitin_transform_eventually* [*OF _ lim1*]
limitin_const_iff [*OF kc_imp_t1_space trivial_limit_sequentially*]
using ‹*l1* ≠ *l2*› ‹*kc_space X*› **by** *fastforce*
qed

lemma *limitin_closedin*:
assumes *lim*: *limitin X f l F*
and *closedin X S* **and** *ev*: *eventually* (λ*x*. *f x* ∈ *S*) *F* ¬ *trivial_limit F*
shows *l* ∈ *S*
proof (*rule ccontr*)
assume *l* ∉ *S*
have ∀ *F x in F*. *f x* ∈ *topspace X* − *S*
by (*metis Diff_iff* ‹*l* ∉ *S*› ‹*closedin X S*› *closedin_def lim limitin_def*)
with *ev eventually_elim2 trivial_limit_def* **show** *False*
by *force*
qed

6.6.18 Quasi-components

definition *quasi_component_of* :: *'a topology* ⇒ *'a* ⇒ *'a* ⇒ *bool*
where
quasi_component_of X x y ≡
x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧
(∀ *T*. *closedin X T* ∧ *openin X T* ⟶ (*x* ∈ *T* ⟷ *y* ∈ *T*))

abbreviation *quasi_component_of_set S x* ≡ *Collect* (*quasi_component_of S x*)

definition *quasi_components_of* :: *'a topology* ⇒ (*'a set*) *set*
where
quasi_components_of X = *quasi_component_of_set X* ‘ *topspace X*

lemma *quasi_component_in_topspace*:
quasi_component_of X x y ⟹ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X*
by (*simp add*: *quasi_component_of_def*)

lemma *quasi_component_of_refl* [*simp*]:
quasi_component_of X x x ⟷ *x* ∈ *topspace X*
by (*simp add*: *quasi_component_of_def*)

lemma *quasi_component_of_sym*:
quasi_component_of X x y ⟷ *quasi_component_of X y x*
by (*meson quasi_component_of_def*)

lemma *quasi_component_of_trans*:
⟦*quasi_component_of X x y*; *quasi_component_of X y z*⟧ ⟹ *quasi_component_of X x z*
by (*simp add*: *quasi_component_of_def*)

lemma *quasi_component_of_subset_topspace*:

quasi_component_of_set X x ⊆ topspace X
using *quasi_component_of_def* **by** *fastforce*

lemma *quasi_component_of_eq_empty*:
quasi_component_of_set X x = {} ⟷ (x ∉ topspace X)
using *quasi_component_of_def* **by** *fastforce*

lemma *quasi_component_of*:
quasi_component_of X x y ⟷
x ∈ topspace X ∧ y ∈ topspace X ∧ (∀ T. x ∈ T ∧ closedin X T ∧ openin X T ⟶ y ∈ T)
unfolding *quasi_component_of_def* **by** (*metis Diff_iff closedin_def openin_closedin_eq*)

lemma *quasi_component_of_alt*:
quasi_component_of X x y ⟷
x ∈ topspace X ∧ y ∈ topspace X ∧
¬ (∃ U V. openin X U ∧ openin X V ∧ U ∪ V = topspace X ∧ disjnt U V ∧ x ∈ U ∧ y ∈ V)
(**is** *?lhs = ?rhs*)
proof
show *?lhs ⟹ ?rhs*
unfolding *quasi_component_of_def*
by (*metis disjnt_iff separatedin_full separatedin_open_sets*)
show *?rhs ⟹ ?lhs*
unfolding *quasi_component_of_def*
by (*metis Diff_disjoint Diff_iff Un_Diff_cancel closedin_def disjnt_def inf_commute sup.orderE sup_commute*)
qed

lemma *quasi_components_lepoll_topspace*: *quasi_components_of X ≲ topspace X*
by (*simp add*: *image_lepoll quasi_components_of_def*)

lemma *quasi_component_of_separated*:
quasi_component_of X x y ⟷
x ∈ topspace X ∧ y ∈ topspace X ∧
¬ (∃ U V. separatedin X U V ∧ U ∪ V = topspace X ∧ x ∈ U ∧ y ∈ V)
by (*meson quasi_component_of_alt separatedin_full separatedin_open_sets*)

lemma *quasi_component_of_subtopology*:
quasi_component_of (subtopology X s) x y ⟹ quasi_component_of X x y
unfolding *quasi_component_of_def*
by (*simp add*: *closedin_subtopology*) (*metis Int_iff inf_commute openin_subtopology_Int2*)

lemma *quasi_component_of_mono*:
quasi_component_of (subtopology X S) x y ∧ S ⊆ T
⟹ quasi_component_of (subtopology X T) x y
by (*metis inf.absorb_iff2 quasi_component_of_subtopology subtopology_subtopology*)

lemma *quasi_component_of_equiv*:
quasi_component_of X x y ⟷
x ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *quasi_component_of X x* = *quasi_component_of X y*
using *quasi_component_of_def* **by** *fastforce*

lemma *quasi_component_of_disjoint* [*simp*]:
disjnt (*quasi_component_of_set X x*) (*quasi_component_of_set X y*) ⟷ ¬ (*quasi_component_of X x y*)
by (*metis disjnt_iff quasi_component_of_equiv mem_Collect_eq*)

lemma *quasi_component_of_eq*:
quasi_component_of X x = *quasi_component_of X y* ⟷
(*x* ∉ *topspace X* ∧ *y* ∉ *topspace X*)
∨ *x* ∈ *topspace X* ∧ *y* ∈ *topspace X* ∧ *quasi_component_of X x y*
by (*metis Collect_empty_eq_bot quasi_component_of_eq_empty quasi_component_of_equiv*)

lemma *topspace_imp_quasi_components_of*:
assumes *x* ∈ *topspace X*
obtains *C* **where** *C* ∈ *quasi_components_of X x* ∈ *C*
by (*metis assms imageI mem_Collect_eq quasi_component_of_refl quasi_components_of_def*)

lemma *Union_quasi_components_of*: ⋃ (*quasi_components_of X*) = *topspace X*
by (*auto simp*: *quasi_components_of_def quasi_component_of_def*)

lemma *pairwise_disjoint_quasi_components_of*:
pairwise disjnt (*quasi_components_of X*)
by (*auto simp*: *quasi_components_of_def quasi_component_of_def disjoint_def*)

lemma *complement_quasi_components_of_Union*:
assumes *C* ∈ *quasi_components_of X*
shows *topspace X* − *C* = ⋃ (*quasi_components_of X* − {*C*}) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using *Union_quasi_components_of* **by** *fastforce*
show *?rhs* ⊆ *?lhs*
using *assms*
using *quasi_component_of_equiv* **by** (*fastforce simp add*: *quasi_components_of_def image_iff subset_iff*)
qed

lemma *nonempty_quasi_components_of*:
C ∈ *quasi_components_of X* ⟹ *C* ≠ {}
by (*metis imageE quasi_component_of_eq_empty quasi_components_of_def*)

lemma *quasi_components_of_subset*:
C ∈ *quasi_components_of X* ⟹ *C* ⊆ *topspace X*

using *Union_quasi_components_of* **by** *force*

lemma *quasi_component_in_quasi_components_of*:
quasi_component_of_set X a ∈ quasi_components_of X ⟷ a ∈ topspace X
by (*metis* (*no_types*, *lifting*) *image_iff quasi_component_of_eq_empty quasi_components_of_def*)

lemma *quasi_components_of_eq_empty* [*simp*]:
quasi_components_of X = {} ⟷ X = trivial_topology
by (*simp add*: *quasi_components_of_def*)

lemma *quasi_components_of_empty_space* [*simp*]:
quasi_components_of trivial_topology = {}
by *simp*

lemma *quasi_component_of_set*:
quasi_component_of_set X x =
(if x ∈ topspace X
then ⋂ {t. closedin X t ∧ openin X t ∧ x ∈ t}
else {})
by (*auto simp*: *quasi_component_of*)

lemma *closedin_quasi_component_of*: *closedin X* (*quasi_component_of_set X x*)
by (*auto simp*: *quasi_component_of_set*)

lemma *closedin_quasi_components_of*:
C ∈ quasi_components_of X ⟹ closedin X C
by (*auto simp*: *quasi_components_of_def closedin_quasi_component_of*)

lemma *openin_finite_quasi_components*:
⟦*finite*(*quasi_components_of X*); *C ∈ quasi_components_of X*⟧ *⟹ openin X C*
apply (*simp add*:*openin_closedin_eq quasi_components_of_subset complement_quasi_components_of_Union*)
by (*meson DiffD1 closedin_Union closedin_quasi_components_of finite_Diff*)

lemma *quasi_component_of_eq_overlap*:
quasi_component_of X x = quasi_component_of X y ⟷
(x ∉ topspace X ∧ y ∉ topspace X) ∨
¬ (quasi_component_of_set X x ∩ quasi_component_of_set X y = {})
using *quasi_component_of_equiv* **by** *fastforce*

lemma *quasi_component_of_nonoverlap*:
quasi_component_of_set X x ∩ quasi_component_of_set X y = {} ⟷
(x ∉ topspace X) ∨ (y ∉ topspace X) ∨
¬ (quasi_component_of X x = quasi_component_of X y)
by (*metis inf.idem quasi_component_of_eq_empty quasi_component_of_eq_overlap*)

lemma *quasi_component_of_overlap*:
¬ (quasi_component_of_set X x ∩ quasi_component_of_set X y = {}) ⟷
x ∈ topspace X ∧ y ∈ topspace X ∧ quasi_component_of X x = quasi_component_of

X y
by (*meson quasi_component_of_nonoverlap*)

lemma *quasi_components_of_disjoint*:
⟦*C* ∈ *quasi_components_of X*; *D* ∈ *quasi_components_of X*⟧ ⟹ *disjnt C D* ⟷ *C* ≠ *D*
by (*metis disjnt_self_iff_empty nonempty_quasi_components_of pairwiseD pairwise_disjoint_quasi_components_of*)

lemma *quasi_components_of_overlap*:
⟦*C* ∈ *quasi_components_of X*; *D* ∈ *quasi_components_of X*⟧ ⟹ ¬ (*C* ∩ *D* = {}) ⟷ *C* = *D*
by (*metis disjnt_def quasi_components_of_disjoint*)

lemma *pairwise_separated_quasi_components_of*:
pairwise (*separatedin X*) (*quasi_components_of X*)
by (*metis closedin_quasi_components_of pairwise_def pairwise_disjoint_quasi_components_of separatedin_closed_sets*)

lemma *finite_quasi_components_of_finite*:
finite(*topspace X*) ⟹ *finite*(*quasi_components_of X*)
by (*simp add*: *Union_quasi_components_of finite_UnionD*)

lemma *connected_imp_quasi_component_of*:
assumes *connected_component_of X x y*
shows *quasi_component_of X x y*
proof −
have *x* ∈ *topspace X y* ∈ *topspace X*
by (*meson assms connected_component_of_equiv*)+
with *assms* **show** *?thesis*
apply (*clarsimp simp add*: *quasi_component_of connected_component_of_def*)
by (*meson connectedin_clopen_cases disjnt_iff subsetD*)
qed

lemma *connected_component_subset_quasi_component_of*:
connected_component_of_set X x ⊆ *quasi_component_of_set X x*
using *connected_imp_quasi_component_of* **by** *force*

lemma *quasi_component_as_connected_component_Union*:
quasi_component_of_set X x =
⋃ (*connected_component_of_set X* ‘ *quasi_component_of_set X x*)
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using *connected_component_of_refl quasi_component_of* **by** *fastforce*
show *?rhs* ⊆ *?lhs*
apply (*rule SUP_least*)
by (*simp add*: *connected_component_subset_quasi_component_of quasi_component_of_equiv*)
qed

lemma *quasi_components_as_connected_components_Union*:
assumes $C \in$ *quasi_components_of* X
obtains $\mathcal{T}$ **where** $\mathcal{T} \subseteq$ *connected_components_of* X $\bigcup \mathcal{T} = C$
proof −
obtain x **where** $x \in$ *topspace* X **and** *Ceq*: $C =$ *quasi_component_of_set* X x
by (*metis assms imageE quasi_components_of_def*)
define $\mathcal{T}$ **where** $\mathcal{T} \equiv$ *connected_component_of_set* X ‘ *quasi_component_of_set* X x
show *thesis*
proof
show $\mathcal{T} \subseteq$ *connected_components_of* X
by (*simp add*: $\mathcal{T}$*_def connected_components_of_def image_mono quasi_component_of_subset_topspace*)
show $\bigcup \mathcal{T} = C$
by (*metis* $\mathcal{T}$*_def Ceq quasi_component_as_connected_component_Union*)
qed
qed

lemma *path_imp_quasi_component_of*:
path_component_of X x $y \Longrightarrow$ *quasi_component_of* X x y
by (*simp add*: *connected_imp_quasi_component_of path_imp_connected_component_of*)

lemma *path_component_subset_quasi_component_of*:
path_component_of_set X $x \subseteq$ *quasi_component_of_set* X x
by (*simp add*: *Collect_mono path_imp_quasi_component_of*)

lemma *connected_space_iff_quasi_component*:
connected_space $X \longleftrightarrow (\forall x \in$ *topspace* $X. \forall y \in$ *topspace* $X.$ *quasi_component_of* X x $y)$
unfolding *connected_space_clopen_in closedin_def quasi_component_of*
by *blast*

lemma *connected_space_imp_quasi_component_of*:
$[\![$*connected_space* X; $a \in$ *topspace* X; $b \in$ *topspace* $X]\!] \Longrightarrow$ *quasi_component_of* X a b
by (*simp add*: *connected_space_iff_quasi_component*)

lemma *connected_space_quasi_component_set*:
connected_space $X \longleftrightarrow (\forall x \in$ *topspace* $X.$ *quasi_component_of_set* X $x =$ *topspace* $X)$
by (*metis Ball_Collect connected_space_iff_quasi_component quasi_component_of_subset_topspace subset_antisym*)

lemma *connected_space_iff_quasi_components_eq*:
connected_space $X \longleftrightarrow$
$(\forall C \in$ *quasi_components_of* $X. \forall D \in$ *quasi_components_of* $X.$ $C = D)$
apply (*simp add*: *quasi_components_of_def*)
by (*metis connected_space_iff_quasi_component mem_Collect_eq quasi_component_of_equiv*)

lemma *quasi_components_of_subset_sing*:
quasi_components_of X ⊆ {S} ⟷ connected_space X ∧ (X = trivial_topology ∨ topspace X = S)
proof (*cases quasi_components_of X = {}*)
case *True*
then show *?thesis*
by (*simp add*: *subset_singleton_iff*)
next
case *False*
then show *?thesis*
apply (*simp add*: *connected_space_iff_quasi_components_eq subset_iff Ball_def*)
by (*metis False Union_quasi_components_of ccpo_Sup_singleton insert_iff is_singletonE is_singletonI'*)
qed

lemma *connected_space_iff_quasi_components_subset_sing*:
connected_space X ⟷ (∃ a. quasi_components_of X ⊆ {a})
by (*simp add*: *quasi_components_of_subset_sing*)

lemma *quasi_components_of_eq_singleton*:
quasi_components_of X = {S} ⟷
connected_space X ∧ ¬ (X = trivial_topology) ∧ S = topspace X
by (*metis empty_not_insert quasi_components_of_eq_empty quasi_components_of_subset_sing subset_singleton_iff*)

lemma *quasi_components_of_connected_space*:
connected_space X
⟹ quasi_components_of X = (if X = trivial_topology then {} else {topspace X})
by (*simp add*: *quasi_components_of_eq_singleton*)

lemma *separated_between_singletons*:
separated_between X {x} {y} ⟷
x ∈ topspace X ∧ y ∈ topspace X ∧ ¬ (quasi_component_of X x y)
proof (*cases x ∈ topspace X ∧ y ∈ topspace X*)
case *True*
then show *?thesis*
by (*auto simp add*: *separated_between_def quasi_component_of_alt*)
qed (*use separated_between_imp_subset* **in** *blast*)

lemma *quasi_component_nonseparated*:
quasi_component_of X x y ⟷ x ∈ topspace X ∧ y ∈ topspace X ∧ ¬ (separated_between X {x} {y})
by (*metis quasi_component_of_equiv separated_between_singletons*)

lemma *separated_between_quasi_component_pointwise_left*:
assumes *C ∈ quasi_components_of X*
shows *separated_between X C S ⟷ (∃ x ∈ C. separated_between X {x} S)* (**is** *?lhs = ?rhs*)

proof
show *?lhs* ⟹ *?rhs*
using *assms quasi_components_of_disjoint separated_between_mono* **by** *fastforce*
next
assume *?rhs*
then obtain *y* **where** *separated_between X {y} S* **and** *y* ∈ *C*
by *metis*
with *assms* **show** *?lhs*
by (*force simp add*: *separated_between quasi_components_of_def quasi_component_of_def*)
qed

lemma *separated_between_quasi_component_pointwise_right*:
C ∈ *quasi_components_of X* ⟹ *separated_between X S C* ⟷ (∃ *x* ∈ *C*. *separated_between X S {x}*)
by (*simp add*: *separated_between_quasi_component_pointwise_left separated_between_sym*)

lemma *separated_between_quasi_component_point*:
assumes *C* ∈ *quasi_components_of X*
shows *separated_between X C {x}* ⟷ *x* ∈ *topspace X* − *C* (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*meson DiffI disjnt_insert2 insert_subset separated_between_imp_disjoint separated_between_imp_subset*)
next
assume *?rhs*
with *assms* **show** *?lhs*
unfolding *quasi_components_of_def image_iff Diff_iff separated_between_quasi_component_pointwise_left* [*OF assms*]
by (*metis mem_Collect_eq quasi_component_of_refl separated_between_singletons*)
qed

lemma *separated_between_point_quasi_component*:
C ∈ *quasi_components_of X* ⟹ *separated_between X {x} C* ⟷ *x* ∈ *topspace X* − *C*
by (*simp add*: *separated_between_quasi_component_point separated_between_sym*)

lemma *separated_between_quasi_component_compact*:
⟦*C* ∈ *quasi_components_of X*; *compactin X K*⟧ ⟹ (*separated_between X C K* ⟷ *disjnt C K*)
unfolding *disjnt_iff*
using *compactin_subset_topspace quasi_components_of_subset separated_between_pointwise_right separated_between_quasi_component_point* **by** *fastforce*

lemma *separated_between_compact_quasi_component*:
⟦*compactin X K*; *C* ∈ *quasi_components_of X*⟧ ⟹ *separated_between X K C* ⟷ *disjnt K C*
using *disjnt_sym separated_between_quasi_component_compact separated_between_sym*
by *blast*

lemma *separated_between_quasi_components*:
assumes *C*: $C \in$ *quasi_components_of* X **and** *D*: $D \in$ *quasi_components_of* X
shows *separated_between* X C $D \longleftrightarrow$ *disjnt* C D (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*simp add*: *separated_between_imp_disjoint*)
next
assume *?rhs*
obtain x y **where** x: C = *quasi_component_of_set* X x **and** $x \in C$
and y: D = *quasi_component_of_set* X y **and** $y \in D$
using *assms* **by** (*auto simp*: *quasi_components_of_def*)
then have *separated_between* X $\{x\}$ $\{y\}$
using ‹*disjnt* C D› *separated_between_singletons* **by** *fastforce*
with ‹$x \in C$› ‹$y \in D$› **show** *?lhs*
by (*auto simp*: *assms separated_between_quasi_component_pointwise_left separated_between_quasi_component_pointwise_right*)
qed

lemma *quasi_eq_connected_component_of_eq*:
quasi_component_of X x = *connected_component_of* X $x \longleftrightarrow$
connectedin X (*quasi_component_of_set* X x) (**is** *?lhs* = *?rhs*)
proof (*cases* $x \in$ *topspace* X)
case *True*
show *?thesis*
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*simp add*: *connectedin_connected_component_of*)
next
assume *?rhs*
then have $\bigwedge y$. *quasi_component_of* X x y = *connected_component_of* X x y
by (*metis connected_component_of_def connected_imp_quasi_component_of mem_Collect_eq quasi_component_of_equiv*)
then show *?lhs*
by *force*
qed
next
case *False*
then show *?thesis*
by (*metis Collect_empty_eq_bot connected_component_of_eq_empty connectedin_empty quasi_component_of_eq_empty*)
qed

lemma *connected_quasi_component_of*:
assumes $C \in$ *quasi_components_of* X
shows $C \in$ *connected_components_of* $X \longleftrightarrow$ *connectedin* X C (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
using *assms*

by (*simp add*: *connectedin_connected_components_of*)
next
assume *?rhs*
with *assms* **show** *?lhs*
unfolding *quasi_components_of_def connected_components_of_def image_iff*
by (*metis quasi_eq_connected_component_of_eq*)
qed

lemma *quasi_component_of_clopen_cases*:
⟦$C \in$ *quasi_components_of* X; *closedin* X T; *openin* X T⟧ $\Longrightarrow$ $C \subseteq T \vee$ *disjnt* C T
by (*smt* (*verit*) *disjnt_iff image_iff mem_Collect_eq quasi_component_of_def quasi_components_of_def subset_iff*)

lemma *quasi_components_of_set*:
assumes $C \in$ *quasi_components_of* X
shows $\bigcap \{T.$ *closedin* X $T \wedge$ *openin* X $T \wedge$ $C \subseteq T\} = C$ (**is** *?lhs* = *?rhs*)
proof
have $x \in C$ **if** $x \in \bigcap \{T.$ *closedin* X $T \wedge$ *openin* X $T \wedge$ $C \subseteq T\}$ **for** x
proof (*rule ccontr*)
assume $x \notin C$
have $x \in$ *topspace* X
using *assms quasi_components_of_subset that* **by** *force*
then have *separated_between* X C $\{x\}$
by (*simp add*: ‹$x \notin C$› *assms separated_between_quasi_component_point*)
with *that* **show** *False*
by (*auto simp*: *separated_between*)
qed
then show *?lhs* $\subseteq$ *?rhs*
by *auto*
qed *blast*

lemma *open_quasi_eq_connected_components_of*:
assumes *openin* X C
shows $C \in$ *quasi_components_of* $X \longleftrightarrow C \in$ *connected_components_of* X (**is** *?lhs* = *?rhs*)
proof (*cases closedin* X C)
case *True*
show *?thesis*
proof
assume *L*: *?lhs*
have $T = \{\} \vee T =$ *topspace* $X \cap C$
if *openin* (*subtopology* X C) T *closedin* (*subtopology* X C) T **for** T
proof −
have $C \subseteq T \vee$ *disjnt* C T
by (*meson L True assms closedin_trans_full openin_trans_full quasi_component_of_clopen_cases that*)
with *that* **show** *?thesis*
by (*metis Int_absorb2 True closedin_imp_subset closure_of_subset_eq*

```
disjnt_def inf_absorb2)
    qed
    with L assms show ?rhs
      by (simp add: connected_quasi_component_of connected_space_clopen_in
connectedin_def openin_subset)
  next
    assume ?rhs
   then obtain x where x ∈ topspace X and x: C = connected_component_of_set
X x
      by (metis connected_components_of_def imageE)
    have C = quasi_component_of_set X x
     using True assms connected_component_of_refl connected_imp_quasi_component_of
quasi_component_of_def x by fastforce
    then show ?lhs
      using ‹x ∈ topspace X› quasi_components_of_def by fastforce
  qed
next
  case False
  then show ?thesis
    using closedin_connected_components_of closedin_quasi_components_of by
blast
qed

lemma quasi_component_of_continuous_image:
  assumes f: continuous_map X Y f and qc: quasi_component_of X x y
  shows quasi_component_of Y (f x) (f y)
  unfolding quasi_component_of_def
proof (intro strip conjI)
  show f x ∈ topspace Y f y ∈ topspace Y
   using assms by (simp_all add: continuous_map_def quasi_component_of_def
Pi_iff)
  fix T
  assume closedin Y T ∧ openin Y T
  with assms show (f x ∈ T) = (f y ∈ T)
   by (smt (verit) continuous_map_closedin continuous_map_def mem_Collect_eq
quasi_component_of_def)
qed

lemma quasi_component_of_discrete_topology:
   quasi_component_of_set (discrete_topology U) x = (if x ∈ U then {x} else {})
proof −
  have quasi_component_of_set (discrete_topology U) y = {y} if y ∈ U for y
    using that
    apply (simp add: set_eq_iff quasi_component_of_def)
    by (metis Set.set_insert insertE subset_insertI)
  then show ?thesis
    by (simp add: quasi_component_of)
qed
```

lemma *quasi_components_of_discrete_topology*:
quasi_components_of (*discrete_topology U*) = (λ*x*. {*x*}) ' *U*
by (*auto simp add*: *quasi_components_of_def quasi_component_of_discrete_topology*)

lemma *homeomorphic_map_quasi_component_of*:
assumes *hmf*: *homeomorphic_map X Y f* **and** *x* ∈ *topspace X*
shows *quasi_component_of_set Y* (*f x*) = *f* ' (*quasi_component_of_set X x*)
proof −
obtain *g* **where** *hmg*: *homeomorphic_map Y X g*
and *contf*: *continuous_map X Y f* **and** *contg*: *continuous_map Y X g*
and *fg*: (∀ *x* ∈ *topspace X*. *g*(*f x*) = *x*) ∧ (∀ *y* ∈ *topspace Y*. *f*(*g y*) = *y*)
by (*smt* (*verit*, *best*) *hmf homeomorphic_map_maps homeomorphic_maps_def*)
show *?thesis*
proof
show *quasi_component_of_set Y* (*f x*) ⊆ *f* ' *quasi_component_of_set X x*
using *quasi_component_of_continuous_image* [*OF contg*]
‹*x* ∈ *topspace X*› *fg image_iff quasi_component_of_subset_topspace* **by**
fastforce
show *f* ' *quasi_component_of_set X x* ⊆ *quasi_component_of_set Y* (*f x*)
using *quasi_component_of_continuous_image* [*OF contf*] **by** *blast*
qed
qed

lemma *homeomorphic_map_quasi_components_of*:
assumes *homeomorphic_map X Y f*
shows *quasi_components_of Y* = *image* (*image f*) (*quasi_components_of X*)
using *assms*
proof −
have ∃ *x*∈*topspace X*. *quasi_component_of_set Y y* = *f* ' *quasi_component_of_set*
X x
if *y* ∈ *topspace Y* **for** *y*
by (*metis that assms homeomorphic_imp_surjective_map homeomorphic_map_quasi_component_of*
image_iff)
moreover have ∃ *x*∈*topspace Y*. *f* ' *quasi_component_of_set X u* = *quasi_component_of_set*
Y x
if *u* ∈ *topspace X* **for** *u*
by (*metis that assms homeomorphic_imp_surjective_map homeomorphic_map_quasi_component_of*
imageI)
ultimately show *?thesis*
by (*auto simp*: *quasi_components_of_def image_iff*)
qed

lemma *openin_quasi_component_of_locally_connected_space*:
assumes *locally_connected_space X*
shows *openin X* (*quasi_component_of_set X x*)
proof −
have ∗: *openin X* (*connected_component_of_set X x*)
by (*simp add*: *assms openin_connected_component_of_locally_connected_space*)

moreover have *connected_component_of_set X x = quasi_component_of_set X x*
using ∗ *closedin_connected_component_of connected_component_of_refl connected_imp_quasi_component_of quasi_component_of_def* **by** *fastforce*
ultimately show *?thesis*
by *simp*
qed

lemma *openin_quasi_components_of_locally_connected_space*:
locally_connected_space X ∧ c ∈ quasi_components_of X ⟹ openin X c
by (*smt* (*verit*, *best*) *image_iff openin_quasi_component_of_locally_connected_space quasi_components_of_def*)

lemma *quasi_eq_connected_components_of_alt*:
quasi_components_of X = connected_components_of X ⟷ (∀ C ∈ quasi_components_of X. connectedin X C)
(**is** *?lhs = ?rhs*)
proof
assume *R*: *?rhs*
moreover have *connected_components_of X ⊆ quasi_components_of X*
using *R* **unfolding** *quasi_components_of_def connected_components_of_def*
by (*force simp flip*: *quasi_eq_connected_component_of_eq*)
ultimately show *?lhs*
using *connected_quasi_component_of* **by** *blast*
qed (*use connected_quasi_component_of* **in** *blast*)

lemma *connected_subset_quasi_components_of_pointwise*:
connected_components_of X ⊆ quasi_components_of X ⟷
(∀ x ∈ topspace X. quasi_component_of X x = connected_component_of X x)
(**is** *?lhs = ?rhs*)
proof
assume *L*: *?lhs*
have *connectedin X* (*quasi_component_of_set X x*) **if** *x ∈ topspace X* **for** *x*
proof −
have ∃ *y*∈*topspace X. connected_component_of_set X x = quasi_component_of_set X y*
using *L that* **by** (*force simp*: *quasi_components_of_def connected_components_of_def image_subset_iff*)
then show *?thesis*
by (*metis connected_component_of_equiv connectedin_connected_component_of mem_Collect_eq quasi_component_of_eq*)
qed
then show *?rhs*
by (*simp add*: *quasi_eq_connected_component_of_eq*)
qed (*simp add*: *connected_components_of_def quasi_components_of_def*)

lemma *quasi_subset_connected_components_of_pointwise*:

quasi_components_of X ⊆ connected_components_of X ⟷
(∀ x ∈ topspace X. quasi_component_of X x = connected_component_of X x)
by (*simp add*: *connected_quasi_component_of image_subset_iff quasi_components_of_def quasi_eq_connected_component_of_eq*)

lemma *quasi_eq_connected_components_of_pointwise*:
quasi_components_of X = connected_components_of X ⟷
(∀ x ∈ topspace X. quasi_component_of X x = connected_component_of X x)
using *connected_subset_quasi_components_of_pointwise quasi_subset_connected_components_of_pointwise*
by *fastforce*

lemma *quasi_eq_connected_components_of_pointwise_alt*:
quasi_components_of X = connected_components_of X ⟷
(∀ x. quasi_component_of X x = connected_component_of X x)
unfolding *quasi_eq_connected_components_of_pointwise*
by (*metis connectedin_empty quasi_component_of_eq_empty quasi_eq_connected_component_of_eq*)

lemma *quasi_eq_connected_components_of_inclusion*:
quasi_components_of X = connected_components_of X ⟷
connected_components_of X ⊆ quasi_components_of X ∨
quasi_components_of X ⊆ connected_components_of X
by (*simp add*: *connected_subset_quasi_components_of_pointwise dual_order.eq_iff quasi_subset_connected_components_of_pointwise*)

lemma *quasi_eq_connected_components_of*:
finite(*connected_components_of X*) ∨
finite(*quasi_components_of X*) ∨
locally_connected_space X ∨
compact_space X ∧ (*Hausdorff_space X* ∨ *regular_space X* ∨ *normal_space X*)
⟹ *quasi_components_of X = connected_components_of X*
proof (*elim disjE*)
show *quasi_components_of X = connected_components_of X*
if *finite* (*connected_components_of X*)
unfolding *quasi_eq_connected_components_of_inclusion*
using *that open_in_finite_connected_components open_quasi_eq_connected_components_of*
by *blast*
show *quasi_components_of X = connected_components_of X*
if *finite* (*quasi_components_of X*)
unfolding *quasi_eq_connected_components_of_inclusion*
using *that open_quasi_eq_connected_components_of openin_finite_quasi_components*
by *blast*
show *quasi_components_of X = connected_components_of X*
if *locally_connected_space X*
unfolding *quasi_eq_connected_components_of_inclusion*
using *that open_quasi_eq_connected_components_of openin_quasi_components_of_locally_connected_space*
by *auto*
show *quasi_components_of X = connected_components_of X*

if *compact_space X* ∧ (*Hausdorff_space X* ∨ *regular_space X* ∨ *normal_space X*)
proof −
show *?thesis*
unfolding *quasi_eq_connected_components_of_alt*
proof (*intro strip*)
fix *C*
assume *C*: *C* ∈ *quasi_components_of X*
then have *cloC*: *closedin X C*
by (*simp add*: *closedin_quasi_components_of*)
have *normal_space X*
using *that compact_Hausdorff_or_regular_imp_normal_space* **by** *blast*
show *connectedin X C*
proof (*clarsimp simp add*: *connectedin_def connected_space_closedin_eq closedin_closed_subtopology cloC closedin_subset* [*OF cloC*])
fix *S T*
assume *S* ⊆ *C* **and** *closedin X S* **and** *S* ∩ *T* = {} **and** *SUT*: *S* ∪ *T* = *topspace X* ∩ *C*
and *T*: *T* ⊆ *C T* ≠ {} **and** *closedin X T*
with ‹*normal_space X*› **obtain** *U V* **where** *UV*: *openin X U openin X V S* ⊆ *U T* ⊆ *V disjnt U V*
by (*meson disjnt_def normal_space_def*)
moreover have *compactin X* (*topspace X* − (*U* ∪ *V*))
using *UV that* **by** (*intro closedin_compact_space closedin_diff openin_Un*) *auto*
ultimately have *separated_between X C* (*topspace X* − (*U* ∪ *V*)) ⟷ *disjnt C* (*topspace X* − (*U* ∪ *V*))
by (*simp add*: ‹*C* ∈ *quasi_components_of X*› *separated_between_quasi_component_compact*)
moreover have *disjnt C* (*topspace X* − (*U* ∪ *V*))
using *UV SUT disjnt_def* **by** *fastforce*
ultimately have *separated_between X C* (*topspace X* − (*U* ∪ *V*))
by *simp*
then obtain *A B* **where** *openin X A openin X B A* ∪ *B* = *topspace X disjnt A B C* ⊆ *A*
and *subB*: *topspace X* − (*U* ∪ *V*) ⊆ *B*
by (*meson separated_between_def*)
have *B* ∪ *U* = *topspace X* − (*A* ∩ *V*)
proof
show *B* ∪ *U* ⊆ *topspace X* − *A* ∩ *V*
using ‹*openin X U*› ‹*disjnt U V*› ‹*disjnt A B*› ‹*openin X B*› *disjnt_iff openin_closedin_eq* **by** *fastforce*
show *topspace X* − *A* ∩ *V* ⊆ *B* ∪ *U*
using ‹*A* ∪ *B* = *topspace X*› *subB* **by** *fastforce*
qed
then have *closedin X* (*B* ∪ *U*)
using ‹*openin X V*› ‹*openin X A*› **by** *auto*
then have *C* ⊆ *B* ∪ *U* ∨ *disjnt C* (*B* ∪ *U*)
using *quasi_component_of_clopen_cases* [*OF C*] ‹*openin X U*› ‹*openin X B*› **by** *blast*

```
      with UV show S = {}
        by (metis UnE ‹C ⊆ A› ‹S ⊆ C› T ‹disjnt A B› all_not_in_conv
disjnt_Un2 disjnt_iff subset_eq)
    qed
  qed
 qed
qed


lemma quasi_eq_connected_component_of:
  finite(connected_components_of X) ∨
    finite(quasi_components_of X) ∨
    locally_connected_space X ∨
    compact_space X ∧ (Hausdorff_space X ∨ regular_space X ∨ normal_space
X)
    ⟹ quasi_component_of X x = connected_component_of X x
 by (metis quasi_eq_connected_components_of quasi_eq_connected_components_of_pointwise_alt)
```

6.6.19 Additional quasicomponent and continuum properties like Boundary Bumping

```
lemma cut_wire_fence_theorem_gen:
  assumes compact_space X and X: Hausdorff_space X ∨ regular_space X ∨
normal_space X
    and S: compactin X S and T: closedin X T
    and dis: ⋀C. connectedin X C ⟹ disjnt C S ∨ disjnt C T
  shows separated_between X S T
  proof −
  have x ∈ topspace X if x ∈ S and T = {} for x
    using that S compactin_subset_topspace by auto
  moreover have separated_between X {x} {y} if x ∈ S and y ∈ T for x y
  proof (cases x ∈ topspace X ∧ y ∈ topspace X)
    case True
    then have ¬ connected_component_of X x y
      by (meson dis connected_component_of_def disjnt_iff that)
    with True X ‹compact_space X› show ?thesis
    by (metis quasi_component_nonseparated quasi_eq_connected_component_of)
  next
    case False
    then show ?thesis
      using S T compactin_subset_topspace closedin_subset that by blast
  qed
  ultimately show ?thesis
    using assms
   by (simp add: separated_between_pointwise_left separated_between_pointwise_right

            closedin_compact_space closedin_subset)
qed
```

lemma *cut_wire_fence_theorem*:
  ⟦*compact_space X*; *Hausdorff_space X*; *closedin X S*; *closedin X T*;
    ⋀*C*. *connectedin X C* ⟹ *disjnt C S* ∨ *disjnt C T*⟧
      ⟹ *separated_between X S T*
 by (*simp add*: *closedin_compact_space cut_wire_fence_theorem_gen*)

lemma *separated_between_from_closed_subtopology*:
 assumes *XC*: *separated_between* (*subtopology X C*) *S* (*X frontier_of C*)
   and *ST*: *separated_between* (*subtopology X C*) *S T*
 shows *separated_between X S T*
proof −
 obtain *U* **where** *clo*: *closedin* (*subtopology X C*) *U* **and** *ope*: *openin* (*subtopology X C*) *U*
          and *S* ⊆ *U* **and** *sub*: *X frontier_of C* ∪ *T* ⊆ *topspace* (*subtopology X C*) − *U*
   by (*meson assms separated_between separated_between_Un*)
 then have *X frontier_of C* ∪ *T* ⊆ *topspace X* ∩ *C* − *U*
   by *auto*
 have *closedin X* (*topspace X* ∩ *C*)
  by (*metis XC frontier_of_restrict frontier_of_subset_eq inf_le1 separated_between_imp_subset topspace_subtopology*)
 then have *closedin X U*
   by (*metis clo closedin_closed_subtopology subtopology_restrict*)
 moreover have *openin* (*subtopology X C*) *U* ⟷ *openin X U* ∧ *U* ⊆ *C*
   using *disjnt_iff sub* **by** (*force intro*!: *openin_subset_topspace_eq*)
 with *ope* **have** *openin X U*
   by *blast*
 moreover have *T* ⊆ *topspace X* − *U*
   using *ope openin_closedin_eq sub* **by** *auto*
 ultimately show *?thesis*
   using ‹*S* ⊆ *U*› *separated_between* **by** *blast*
qed

lemma *separated_between_from_closed_subtopology_frontier*:
  separated_between (*subtopology X T*) *S* (*X frontier_of T*)
      ⟹ *separated_between X S* (*X frontier_of T*)
 using *separated_between_from_closed_subtopology* **by** *blast*

lemma *separated_between_from_frontier_of_closed_subtopology*:
 assumes *separated_between* (*subtopology X T*) *S* (*X frontier_of T*)
 shows *separated_between X S* (*topspace X* − *T*)
proof −
 have *disjnt S* (*topspace X* − *T*)
   using *assms disjnt_iff separated_between_imp_subset* **by** *fastforce*
 then show *?thesis*
  by (*metis Diff_subset assms frontier_of_complement separated_between_from_closed_subtopology separated_between_frontier_of_eq'*)
qed

lemma *separated_between_compact_connected_component*:
assumes *locally_compact_space X Hausdorff_space X*
and *C*: *C* ∈ *connected_components_of X*
and *compactin X C closedin X T disjnt C T*
shows *separated_between X C T*
proof −
have *Csub*: *C* ⊆ *topspace X*
by (*simp add*: *assms*(*4*) *compactin_subset_topspace*)
have *Hausdorff_space* (*subtopology X* (*topspace X* − *T*))
using *Hausdorff_space_subtopology assms*(*2*) **by** *blast*
moreover have *compactin* (*subtopology X* (*topspace X* − *T*)) *C*
using *assms Csub* **by** (*metis Diff_Int_distrib Diff_empty compact_imp_compactin_subtopology disjnt_def le_iff_inf*)
moreover have *locally_compact_space* (*subtopology X* (*topspace X* − *T*))
by (*meson assms closedin_def locally_compact_Hausdorff_imp_regular_space locally_compact_space_open_subset*)
ultimately
obtain *N L* **where** *openin X N compactin X L closedin X L C* ⊆ *N N* ⊆ *L*
and *Lsub*: *L* ⊆ *topspace X* − *T*
using ‹*Hausdorff_space X*› ‹*closedin X T*›
apply (*simp add*: *locally_compact_space_compact_closed_compact compactin_subtopology*)
by (*meson closedin_def compactin_imp_closedin openin_trans_full*)
then have *disC*: *disjnt C* (*topspace X* − *L*)
by (*meson DiffD2 disjnt_iff subset_iff*)
have *separated_between* (*subtopology X L*) *C* (*X frontier_of L*)
proof (*rule cut_wire_fence_theorem*)
show *compact_space* (*subtopology X L*)
by (*simp add*: ‹*compactin X L*› *compact_space_subtopology*)
show *Hausdorff_space* (*subtopology X L*)
by (*simp add*: *Hausdorff_space_subtopology* ‹*Hausdorff_space X*›)
show *closedin* (*subtopology X L*) *C*
by (*meson* ‹*C* ⊆ *N*› ‹*N* ⊆ *L*› ‹*Hausdorff_space X*› ‹*compactin X C*› *closedin_subset_topspace compactin_imp_closedin subset_trans*)
show *closedin* (*subtopology X L*) (*X frontier_of L*)
by (*simp add*: ‹*closedin X L*› *closedin_frontier_of closedin_subset_topspace frontier_of_subset_closedin*)
show *disjnt D C* ∨ *disjnt D* (*X frontier_of L*)
if *connectedin* (*subtopology X L*) *D* **for** *D*
proof (*rule ccontr*)
assume ¬ (*disjnt D C* ∨ *disjnt D* (*X frontier_of L*))
moreover have *connectedin X D*
using *connectedin_subtopology that* **by** *blast*
ultimately show *False*
using *that connected_components_of_maximal* [*of C X D*] *C*
apply (*simp add*: *disjnt_iff*)
by (*metis Diff_eq_empty_iff* ‹*C* ⊆ *N*› ‹*N* ⊆ *L*› ‹*openin X N*› *disjoint_iff frontier_of_openin_straddle_Int*(*2*) *subsetD*)
qed
qed

then have *separated_between X* (*X frontier_of C*) (*topspace X − L*)
using *separated_between_from_frontier_of_closed_subtopology separated_between_frontier_of_eq*
by *blast*
with ‹*closedin X T*›
separated_between_frontier_of [*OF Csub disC*]
show *?thesis*
unfolding *separated_between* **by** (*smt* (*verit*) *Diff_iff Lsub closedin_subset subset_iff*)
qed

lemma *wilder_locally_compact_component_thm*:
assumes *locally_compact_space X Hausdorff_space X*
and *C ∈ connected_components_of X compactin X C openin X W C ⊆ W*
obtains *U V* **where** *openin X U openin X V disjnt U V U ∪ V = topspace X C ⊆ U U ⊆ W*
proof −
have *closedin X* (*topspace X − W*)
using ‹*openin X W*› **by** *blast*
moreover have *disjnt C* (*topspace X − W*)
using ‹*C ⊆ W*› *disjnt_def* **by** *fastforce*
ultimately have *separated_between X C* (*topspace X − W*)
using *separated_between_compact_connected_component assms* **by** *blast*
then show *thesis*
by (*smt* (*verit*, *del_insts*) *DiffI disjnt_iff openin_subset separated_between_def subset_iff that*)
qed

lemma *compact_quasi_eq_connected_components_of*:
assumes *locally_compact_space X Hausdorff_space X compactin X C*
shows *C ∈ quasi_components_of X ⟷ C ∈ connected_components_of X*
proof −
have *compactin X* (*connected_component_of_set X x*)
if *x ∈ topspace X compactin X* (*quasi_component_of_set X x*) **for** *x*
proof (*rule closed_compactin*)
show *compactin X* (*quasi_component_of_set X x*)
by (*simp add*: *that*)
show *connected_component_of_set X x ⊆ quasi_component_of_set X x*
by (*simp add*: *connected_component_subset_quasi_component_of*)
show *closedin X* (*connected_component_of_set X x*)
by (*simp add*: *closedin_connected_component_of*)
qed
moreover have *connected_component_of X x = quasi_component_of X x*
if §: *x ∈ topspace X compactin X* (*connected_component_of_set X x*) **for** *x*
proof −
have ⋀*y*. *connected_component_of X x y ⟹ quasi_component_of X x y*
by (*simp add*: *connected_imp_quasi_component_of*)
moreover have *False* **if** *non*: ¬ *connected_component_of X x y* **and** *quasi*: *quasi_component_of X x y* **for** *y*
proof −

```
      have y ∈ topspace X
        by (meson quasi_component_of_equiv that)
      then have closedin X {y}
        by (simp add: ‹Hausdorff_space X› compactin_imp_closedin)
      moreover have disjnt (connected_component_of_set X x) {y}
        by (simp add: non)
      moreover have ¬ separated_between X (connected_component_of_set X x)
{y}
        using § quasi separated_between_pointwise_left
      by (fastforce simp: quasi_component_nonseparated connected_component_of_refl)
      ultimately show False
      using assms by (metis § connected_component_in_connected_components_of
separated_between_compact_connected_component)
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately show ?thesis
    using ‹compactin X C› unfolding connected_components_of_def image_iff
quasi_components_of_def by metis
qed


lemma boundary_bumping_theorem_closed_gen:
  assumes connected_space X locally_compact_space X Hausdorff_space X closedin
X S
    S ≠ topspace X and C: compactin X C C ∈ connected_components_of (subtopology
X S)
  shows C ∩ X frontier_of S ≠ {}
proof
  assume §: C ∩ X frontier_of S = {}
  consider C ≠ {} X frontier_of S ⊆ topspace X | C ⊆ topspace X S = {}
   using C by (metis frontier_of_subset_topspace nonempty_connected_components_of)
  then show False
  proof cases
    case 1
    have separated_between (subtopology X S) C (X frontier_of S)
    proof (rule separated_between_compact_connected_component)
      show compactin (subtopology X S) C
      using C compact_imp_compactin_subtopology connected_components_of_subset
by fastforce
      show closedin (subtopology X S) (X frontier_of S)
       by (simp add: ‹closedin X S› closedin_frontier_of closedin_subset_topspace
frontier_of_subset_closedin)
      show disjnt C (X frontier_of S)
        using § by (simp add: disjnt_def)
    qed (use assms Hausdorff_space_subtopology locally_compact_space_closed_subset
in auto)
    then have separated_between X C (X frontier_of S)
```

```
      using separated_between_from_closed_subtopology by auto
    then have X frontier_of S = {}
     using ‹C ≠ {}› ‹connected_space X› connected_space_separated_between by
blast
    moreover have C ⊆ S
      using C connected_components_of_subset by fastforce
    ultimately show False
    using 1 assms by (metis closedin_subset connected_space_eq_frontier_eq_empty
subset_empty)
  next
    case 2
    then show False
      using C connected_components_of_eq_empty by fastforce
  qed
qed

lemma boundary_bumping_theorem_closed:
  assumes connected_space X compact_space X Hausdorff_space X closedin X S
          S ≠ topspace X C ∈ connected_components_of(subtopology X S)
  shows C ∩ X frontier_of S ≠ {}
 by (meson assms boundary_bumping_theorem_closed_gen closedin_compact_space
closedin_connected_components_of
            closedin_trans_full compact_imp_locally_compact_space)


lemma intermediate_continuum_exists:
  assumes connected_space X locally_compact_space X Hausdorff_space X
    and C: compactin X C connectedin X C C ≠ {} C ≠ topspace X
    and U: openin X U C ⊆ U
  obtains D where compactin X D connectedin X D C ⊂ D D ⊂ U
proof −
  have C ⊆ topspace X
    by (simp add: C compactin_subset_topspace)
  with C obtain a where a: a ∈ topspace X a ∉ C
    by blast
  moreover have compactin (subtopology X (U − {a})) C
   by (simp add: C U a compact_imp_compactin_subtopology subset_Diff_insert)
  moreover have Hausdorff_space (subtopology X (U − {a}))
    using Hausdorff_space_subtopology assms(3) by blast
  moreover
  have locally_compact_space (subtopology X (U − {a}))
    by (rule locally_compact_space_open_subset)
     (auto simp: locally_compact_Hausdorff_imp_regular_space open_in_Hausdorff_delete
assms)
  ultimately obtain V K where V: openin X V a ∉ V V ⊆ U and K: compactin
X K a ∉ K K ⊆ U
    and cloK: closedin (subtopology X (U − {a})) K and C ⊆ V V ⊆ K
     using locally_compact_space_compact_closed_compact [of subtopology X (U
− {a})] assms
```

by (*smt* (*verit*, *del_insts*) *Diff_empty compactin_subtopology open_in_Hausdorff_delete openin_open_subtopology subset_Diff_insert*)
then obtain *D* **where** *D*: *D* ∈ *connected_components_of* (*subtopology X K*) **and** *C* ⊆ *D*
using *C*
by (*metis compactin_subset_topspace connected_component_in_connected_components_of connected_component_of_maximal connectedin_subtopology subset_empty subset_eq topspace_subtopology_subset*)
show *thesis*
proof
have *cloD*: *closedin* (*subtopology X K*) *D*
by (*simp add*: *D closedin_connected_components_of*)
then have *XKD*: *compactin* (*subtopology X K*) *D*
by (*simp add*: *K closedin_compact_space compact_space_subtopology*)
then show *compactin X D*
using *compactin_subtopology_imp_compact* **by** *blast*
show *connectedin X D*
using *D connectedin_connected_components_of connectedin_subtopology* **by** *blast*
have *K* ≠ *topspace X*
using *K a* **by** *blast*
moreover have *V* ⊆ *X interior_of K*
by (*simp add*: ‹*openin X V*› ‹*V* ⊆ *K*› *interior_of_maximal*)
ultimately have *C* ≠ *D*
using *boundary_bumping_theorem_closed_gen* [*of X K C*] *D* ‹*C* ⊆ *V*›
by (*auto simp add*: *assms K compactin_imp_closedin frontier_of_def*)
then show *C* ⊂ *D*
using ‹*C* ⊆ *D*› **by** *blast*
have *D* ⊆ *U*
using *K*(*3*) ‹*closedin* (*subtopology X K*) *D*› *closedin_imp_subset* **by** *blast*
moreover have *D* ≠ *U*
using *K XKD* ‹*C* ⊂ *D*› *assms*
by (*metis* ‹*K* ≠ *topspace X*› *cloD closedin_imp_subset compactin_imp_closedin connected_space_clopen_in inf_bot_left inf_le2 subset_antisym*)
ultimately
show *D* ⊂ *U* **by** *blast*
qed
qed

lemma *boundary_bumping_theorem_gen*:
assumes *X*: *connected_space X locally_compact_space X Hausdorff_space X*
and *S* ⊂ *topspace X* **and** *C*: *C* ∈ *connected_components_of*(*subtopology X S*)
and *compC*: *compactin X* (*X closure_of C*)
shows *X frontier_of C* ∩ *X frontier_of S* ≠ {}
proof −
have *Csub*: *C* ⊆ *topspace X C* ⊆ *S* **and** *connectedin X C*
using *C connectedin_connected_components_of connectedin_subset_topspace*

connectedin_subtopology
by *fastforce+*
have $C \neq \{\}$
using *C nonempty_connected_components_of* **by** *blast*
obtain *X interior_of C* $\subseteq$ *X interior_of S X closure_of C* $\subseteq$ *X closure_of S*
by (*simp add*: *Csub closure_of_mono interior_of_mono*)
moreover have *False* **if** *X closure_of C* $\subseteq$ *X interior_of S*
proof −
have *X closure_of C* = *C*
by (*meson C closedin_connected_component_of_subtopology closure_of_eq interior_of_subset order_trans that*)
with *that* **have** *C* $\subseteq$ *X interior_of S*
by *simp*
then obtain *D* **where** *compactin X D* **and** *connectedin X D* **and** *C* $\subset$ *D* **and** *D* $\subset$ *X interior_of S*
using *intermediate_continuum_exists assms* ‹*X closure_of C* = *C*› *compC Csub*
by (*metis* ‹*C* $\neq$ {}› ‹*connectedin X C*› *openin_interior_of psubsetE*)
then have *D* $\subseteq$ *C*
by (*metis C* ‹*C* $\neq$ {}› *connected_components_of_maximal connectedin_subtopology disjnt_def inf.orderE interior_of_subset order_trans psubsetE*)
then show *False*
using ‹*C* $\subset$ *D*› **by** *blast*
qed
ultimately show *?thesis*
by (*smt* (*verit*, *ccfv_SIG*) *DiffI disjoint_iff_not_equal frontier_of_def subset_eq*)
qed

lemma *boundary_bumping_theorem*:
⟦*connected_space X*; *compact_space X*; *Hausdorff_space X*; *S* $\subset$ *topspace X*;
C $\in$ *connected_components_of*(*subtopology X S*)⟧
$\Longrightarrow$ *X frontier_of C* $\cap$ *X frontier_of S* $\neq$ {}
by (*simp add*: *boundary_bumping_theorem_gen closedin_compact_space compact_imp_locally_compact_space*)

6.6.20 Compactly generated spaces (k-spaces)

These don't have to be Hausdorff

definition *k_space* **where**
k_space X ≡
∀ *S*. *S* $\subseteq$ *topspace X* $\longrightarrow$
(*closedin X S* $\longleftrightarrow$ (∀ *K*. *compactin X K* $\longrightarrow$ *closedin* (*subtopology X K*) (*K* $\cap$ *S*)))

lemma *k_space*:
k_space X $\longleftrightarrow$
(∀ *S*. *S* $\subseteq$ *topspace X* ∧
(∀ *K*. *compactin X K* $\longrightarrow$ *closedin* (*subtopology X K*) (*K* $\cap$ *S*)) $\longrightarrow$ *closedin*

```
X S)
  by (metis closedin_subtopology inf_commute k_space_def)

lemma k_space_open:
   k_space X ⟷
    (∀S. S ⊆ topspace X ∧
         (∀K. compactin X K ⟶ openin (subtopology X K) (K ∩ S)) ⟶ openin
X S)
proof −
  have openin X S
    if k_space X S ⊆ topspace X
      and ∀K. compactin X K ⟶ openin (subtopology X K) (K ∩ S) for S
    using that unfolding k_space openin_closedin_eq
    by (metis Diff_Int_distrib2 Diff_subset inf_commute topspace_subtopology)
  moreover have k_space X
    if ∀S. S ⊆ topspace X ∧ (∀K. compactin X K ⟶ openin (subtopology X K)
(K ∩ S)) ⟶ openin X S
    unfolding k_space openin_closedin_eq
    by (simp add: Diff_Int_distrib closedin_def inf_commute that)
  ultimately show ?thesis
    by blast
qed

lemma k_space_alt:
   k_space X ⟷
    (∀S. S ⊆ topspace X
        ⟶ (openin X S ⟷ (∀K. compactin X K ⟶ openin (subtopology X K)
(K ∩ S))))
  by (meson k_space_open openin_subtopology_Int2)

lemma k_space_quotient_map_image:
  assumes q: quotient_map X Y q and X: k_space X
  shows k_space Y
  unfolding k_space
proof clarify
  fix S
  assume S ⊆ topspace Y and S: ∀K. compactin Y K ⟶ closedin (subtopology
Y K) (K ∩ S)
  then have iff: closedin X {x ∈ topspace X. q x ∈ S} ⟷ closedin Y S
    using q quotient_map_closedin by fastforce
  have closedin (subtopology X K) (K ∩ {x ∈ topspace X. q x ∈ S}) if compactin
X K for K
  proof −
    have {x ∈ topspace X. q x ∈ q ‘ K} ∩ K = K
      using compactin_subset_topspace that by blast
    then have ∗: subtopology X K = subtopology (subtopology X {x ∈ topspace X.
q x ∈ q ‘ K}) K
      by (simp add: subtopology_subtopology)
    have ∗∗: K ∩ {x ∈ topspace X. q x ∈ S} =
```

```
          K ∩ {x ∈ topspace (subtopology X {x ∈ topspace X. q x ∈ q ‘ K}). q x
∈ q ‘ K ∩ S}
      by auto
    have K ⊆ topspace X
      by (simp add: compactin_subset_topspace that)
    show ?thesis
      unfolding * **
  proof (intro closedin_continuous_map_preimage closedin_subtopology_Int_closed)
       show continuous_map (subtopology X {x ∈ topspace X. q x ∈ q ‘ K})
(subtopology Y (q ‘ K)) q
     by (auto simp add: continuous_map_in_subtopology continuous_map_from_subtopology
q quotient_imp_continuous_map)
      show closedin (subtopology Y (q ‘ K)) (q ‘ K ∩ S)
       by (meson S image_compactin q quotient_imp_continuous_map that)
    qed
  qed
  then have closedin X {x ∈ topspace X. q x ∈ S}
    by (metis (no_types, lifting) X k_space mem_Collect_eq subsetI)
  with iff show closedin Y S by simp
qed

lemma k_space_retraction_map_image:
   ⟦retraction_map X Y r; k_space X⟧ ⟹ k_space Y
  using k_space_quotient_map_image retraction_imp_quotient_map by blast

lemma homeomorphic_k_space:
   X homeomorphic_space Y ⟹ k_space X ⟷ k_space Y
  by (meson homeomorphic_map_def homeomorphic_space homeomorphic_space_sym
k_space_quotient_map_image)

lemma k_space_perfect_map_image:
   ⟦k_space X; perfect_map X Y f⟧ ⟹ k_space Y
  using k_space_quotient_map_image perfect_imp_quotient_map by blast

lemma locally_compact_imp_k_space:
  assumes locally_compact_space X
  shows k_space X
  unfolding k_space
proof clarify
  fix S
  assume S ⊆ topspace X and S: ∀ K. compactin X K ⟶ closedin (subtopology
X K) (K ∩ S)
  have False if non: ¬ (X closure_of S ⊆ S)
  proof −
    obtain x where x ∈ X closure_of S x ∉ S
      using non by blast
    then have x ∈ topspace X
      by (simp add: in_closure_of)
    then obtain K U where openin X U compactin X K x ∈ U U ⊆ K
```

```
    by (meson assms locally_compact_space_def)
  then show False
    using ‹x ∈ X closure_of S› openin_Int_closure_of_eq [OF ‹openin X U›]
     by (smt (verit, ccfv_threshold) Int_iff S ‹x ∉ S› closedin_Int_closure_of
inf.orderE inf_assoc)
 qed
 then show closedin X S
   using S ‹S ⊆ topspace X› closure_of_subset_eq by blast
qed

lemma compact_imp_k_space:
  compact_space X ⟹ k_space X
 by (simp add: compact_imp_locally_compact_space locally_compact_imp_k_space)

lemma k_space_discrete_topology: k_space(discrete_topology U)
 by (simp add: k_space_open)

lemma k_space_closed_subtopology:
 assumes k_space X closedin X C
 shows k_space (subtopology X C)
unfolding k_space compactin_subtopology
proof clarsimp
 fix S
 assume Ssub: S ⊆ topspace X S ⊆ C
    and S: ∀K. compactin X K ∧ K ⊆ C ⟶ closedin (subtopology (subtopology
X C) K) (K ∩ S)
 have closedin (subtopology X K) (K ∩ S) if compactin X K for K
 proof −
   have closedin (subtopology (subtopology X C) (K ∩ C)) ((K ∩ C) ∩ S)
     by (simp add: S ‹closedin X C› compact_Int_closedin that)
   then show ?thesis
     using ‹closedin X C› Ssub by (auto simp add: closedin_subtopology)
 qed
 then show closedin (subtopology X C) S
   by (metis Ssub ‹k_space X› closedin_subset_topspace k_space_def)
qed

lemma k_space_subtopology:
 assumes 1: ⋀T. ⟦T ⊆ topspace X; T ⊆ S;
                ⋀K. compactin X K ⟹ closedin (subtopology X (K ∩ S)) (K ∩
T)⟧ ⟹ closedin (subtopology X S) T
 assumes 2: ⋀K. compactin X K ⟹ k_space(subtopology X (K ∩ S))
 shows k_space (subtopology X S)
 unfolding k_space
proof (intro conjI strip)
 fix U
 assume §: U ⊆ topspace (subtopology X S) ∧ (∀K. compactin (subtopology X S)
K ⟶ closedin (subtopology (subtopology X S) K) (K ∩ U))
 have closedin (subtopology X (K ∩ S)) (K ∩ U) if compactin X K for K
```

proof −
have $K \cap U \subseteq$ *topspace* (*subtopology X* ($K \cap S$))
using § **by** *auto*
moreover
have $\bigwedge K'$. *compactin* (*subtopology X* ($K \cap S$)) $K' \Longrightarrow$ *closedin* (*subtopology* (*subtopology X* ($K \cap S$)) K') ($K' \cap K \cap U$)
by (*metis* § *compactin_subtopology inf.orderE inf_commute subtopology_subtopology*)
ultimately show *?thesis*
by (*metis* (*no_types*, *opaque_lifting*) *2 inf.assoc k_space_def that*)
qed
then show *closedin* (*subtopology X S*) *U*
using *1* § **by** *auto*
qed

lemma *k_space_subtopology_open*:
assumes *1*: $\bigwedge T$. $[\![T \subseteq$ *topspace X*; $T \subseteq S$;
$\bigwedge K$. *compactin X K* $\Longrightarrow$ *openin* (*subtopology X* ($K \cap S$)) ($K \cap T$)$]\!] \Longrightarrow$ *openin* (*subtopology X S*) *T*
assumes *2*: $\bigwedge K$. *compactin X K* $\Longrightarrow$ *k_space*(*subtopology X* ($K \cap S$))
shows *k_space* (*subtopology X S*)
unfolding *k_space_open*
proof (*intro conjI strip*)
fix *U*
assume §: $U \subseteq$ *topspace* (*subtopology X S*) $\wedge$ ($\forall K$. *compactin* (*subtopology X S*) $K \longrightarrow$ *openin* (*subtopology* (*subtopology X S*) *K*) ($K \cap U$))
have *openin* (*subtopology X* ($K \cap S$)) ($K \cap U$) **if** *compactin X K* **for** *K*
proof −
have $K \cap U \subseteq$ *topspace* (*subtopology X* ($K \cap S$))
using § **by** *auto*
moreover
have $\bigwedge K'$. *compactin* (*subtopology X* ($K \cap S$)) $K' \Longrightarrow$ *openin* (*subtopology* (*subtopology X* ($K \cap S$)) K') ($K' \cap K \cap U$)
by (*metis* § *compactin_subtopology inf.orderE inf_commute subtopology_subtopology*)
ultimately show *?thesis*
by (*metis* (*no_types*, *opaque_lifting*) *2 inf.assoc k_space_open that*)
qed
then show *openin* (*subtopology X S*) *U*
using *1* § **by** *auto*
qed

lemma *k_space_open_subtopology_aux*:
assumes *kc_space X compact_space X openin X V*
shows *k_space* (*subtopology X V*)
proof (*clarsimp simp*: *k_space subtopology_subtopology compactin_subtopology Int_absorb1*)
fix *S*
assume $S \subseteq$ *topspace X*
and $S \subseteq V$
and *S*: $\forall K$. *compactin X K* $\wedge$ $K \subseteq V \longrightarrow$ *closedin* (*subtopology X K*) ($K \cap$

```
S)
  then have V ⊆ topspace X
    using assms openin_subset by blast
  have S = V ∩ ((topspace X - V) ∪ S)
    using ‹S ⊆ V› by auto
  moreover have closedin (subtopology X V) (V ∩ ((topspace X - V) ∪ S))
  proof (intro closedin_subtopology_Int_closed compactin_imp_closedin_gen ‹kc_space
X›)
    show compactin X (topspace X - V ∪ S)
      unfolding compactin_def
    proof (intro conjI strip)
      show topspace X - V ∪ S ⊆ topspace X
        by (simp add: ‹S ⊆ topspace X›)
      fix 𝒰
      assume 𝒰: Ball 𝒰 (openin X) ∧ topspace X - V ∪ S ⊆ ⋃𝒰
      moreover
      have compactin X (topspace X - V)
        using assms closedin_compact_space by blast
      ultimately obtain 𝒢 where finite 𝒢 𝒢 ⊆ 𝒰 and 𝒢: topspace X - V ⊆ ⋃𝒢
        unfolding compactin_def using ‹V ⊆ topspace X› by (metis le_sup_iff)
      then have topspace X - ⋃𝒢 ⊆ V
        by blast
     then have closedin (subtopology X (topspace X - ⋃𝒢)) ((topspace X - ⋃𝒢)
∩ S)
          by (meson S 𝒰 ‹𝒢 ⊆ 𝒰› ‹compact_space X› closedin_compact_space
openin_Union openin_closedin_eq subset_iff)
      then have compactin X ((topspace X - ⋃𝒢) ∩ S)
      by (meson 𝒰 ‹𝒢 ⊆ 𝒰›‹compact_space X› closedin_compact_space closedin_trans_full
openin_Union openin_closedin_eq subset_iff)
      then obtain ℋ where finite ℋ ℋ ⊆ 𝒰 (topspace X - ⋃𝒢) ∩ S ⊆ ⋃ℋ
          unfolding compactin_def by (smt (verit, best) 𝒰 inf_le2 subset_trans
sup.boundedE)
      with 𝒢 have topspace X - V ∪ S ⊆ ⋃(𝒢 ∪ ℋ)
        using ‹S ⊆ topspace X› by auto
      then show ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ topspace X - V ∪ S ⊆ ⋃ℱ
        by (metis ‹𝒢 ⊆ 𝒰› ‹ℋ ⊆ 𝒰› ‹finite 𝒢› ‹finite ℋ› finite_Un le_sup_iff)
    qed
  qed
  ultimately show closedin (subtopology X V) S
    by metis
qed


lemma k_space_open_subtopology:
  assumes X: kc_space X ∨ Hausdorff_space X ∨ regular_space X and k_space
X openin X S
  shows k_space(subtopology X S)
proof (rule k_space_subtopology_open)
  fix T
```

```
  assume T ⊆ topspace X
    and T ⊆ S
    and T: ⋀K. compactin X K ⟹ openin (subtopology X (K ∩ S)) (K ∩ T)
  have openin (subtopology X K) (K ∩ T) if compactin X K for K
    by (smt (verit, ccfv_threshold) T assms(3) inf_assoc inf_commute openin_Int
openin_subtopology that)
  then show openin (subtopology X S) T
   by (metis ‹T ⊆ S› ‹T ⊆ topspace X› assms(2) k_space_alt subset_openin_subtopology)
next
  fix K
  assume compactin X K
  then have KS: openin (subtopology X K) (K ∩ S)
    by (simp add: ‹openin X S› openin_subtopology_Int2)
  have XK: compact_space (subtopology X K)
    by (simp add: ‹compactin X K› compact_space_subtopology)
  show k_space (subtopology X (K ∩ S))
    using X
  proof (rule disjE)
    assume kc_space X
    then show k_space (subtopology X (K ∩ S))
      using k_space_open_subtopology_aux [of subtopology X K K ∩ S]
      by (simp add: KS XK kc_space_subtopology subtopology_subtopology)
  next
    assume Hausdorff_space X ∨ regular_space X
    then have locally_compact_space (subtopology (subtopology X K) (K ∩ S))
      using locally_compact_space_open_subset Hausdorff_space_subtopology KS
XK
        compact_imp_locally_compact_space regular_space_subtopology by blast
    then show k_space (subtopology X (K ∩ S))
      by (simp add: locally_compact_imp_k_space subtopology_subtopology)
  qed
qed

lemma k_kc_space_subtopology:
  ⟦k_space X; kc_space X; openin X S ∨ closedin X S⟧ ⟹ k_space(subtopology
X S) ∧ kc_space(subtopology X S)
 by (metis k_space_closed_subtopology k_space_open_subtopology kc_space_subtopology)


lemma k_space_as_quotient_explicit:
  k_space X ⟷ quotient_map (sum_topology (subtopology X) {K. compactin X
K}) X snd
proof −
  have [simp]: {x ∈ topspace X. x ∈ K ∧ x ∈ U} = K ∩ U if U ⊆ topspace X
for K U
    using that by blast
  show ?thesis
    apply (simp add: quotient_map_def openin_sum_topology snd_image_Sigma
k_space_alt)
```

by (*smt* (*verit*, *del_insts*) *Union_iff compactin_sing inf.orderE mem_Collect_eq singletonI subsetI*)
qed

lemma *k_space_as_quotient*:
fixes $X :: 'a$ *topology*
shows *k_space* $X \longleftrightarrow (\exists\, q.\ \exists\, Y :: ('a\ set * 'a)\ topology.\ locally\_compact\_space\ Y \wedge quotient\_map\ Y\ X\ q)$
(**is** *?lhs=?rhs*)
proof
show *k_space* X **if** *?rhs*
using *that k_space_quotient_map_image locally_compact_imp_k_space* **by** *blast*
next
assume *k_space* X
show *?rhs*
proof (*intro exI conjI*)
show *locally_compact_space* (*sum_topology* (*subtopology* X) $\{K.\ compactin\ X\ K\}$)
by (*simp add*: *compact_imp_locally_compact_space compact_space_subtopology locally_compact_space_sum_topology*)
show *quotient_map* (*sum_topology* (*subtopology* X) $\{K.\ compactin\ X\ K\}$) X *snd*
using ‹*k_space* X› *k_space_as_quotient_explicit* **by** *blast*
qed
qed

lemma *k_space_prod_topology_left*:
assumes X: *locally_compact_space* X *Hausdorff_space* $X \vee$ *regular_space* X **and** *k_space* Y
shows *k_space* (*prod_topology* X Y)
proof −
obtain q **and** $Z :: ('b\ set * 'b)\ topology$ **where** *locally_compact_space* Z **and** q: *quotient_map* Z Y q
using ‹*k_space* Y› *k_space_as_quotient* **by** *blast*
then show *?thesis*
using *quotient_map_prod_right* [*OF* X q] X *k_space_quotient_map_image locally_compact_imp_k_space*
locally_compact_space_prod_topology **by** *blast*
qed

lemma *k_space_prod_topology_right*:
assumes *k_space* X **and** Y: *locally_compact_space* Y *Hausdorff_space* $Y \vee$ *regular_space* Y
shows *k_space* (*prod_topology* X Y)
using *assms homeomorphic_k_space homeomorphic_space_prod_topology_swap k_space_prod_topology_left* **by** *blast*

lemma *continuous_map_from_k_space*:
assumes *k_space X* **and** *f*: $\bigwedge K$. *compactin X K* $\Longrightarrow$ *continuous_map*(*subtopology X K*) *Y f*
shows *continuous_map X Y f*
proof −
have $\bigwedge x$. *x* ∈ *topspace X* $\Longrightarrow$ *f x* ∈ *topspace Y*
by (*metis compactin_absolute compactin_sing f image_compactin image_empty image_insert*)
moreover have *closedin X* {*x* ∈ *topspace X*. *f x* ∈ *C*} **if** *closedin Y C* **for** *C*
proof −
have {*x* ∈ *topspace X*. *f x* ∈ *C*} ⊆ *topspace X*
by *fastforce*
moreover
have *eq*: *K* ∩ {*x* ∈ *topspace X*. *f x* ∈ *C*} = {*x*. *x* ∈ *topspace*(*subtopology X K*) ∧ *f x* ∈ (*f* ‘ *K* ∩ *C*)} **for** *K*
by *auto*
have *closedin* (*subtopology X K*) (*K* ∩ {*x* ∈ *topspace X*. *f x* ∈ *C*}) **if** *compactin X K* **for** *K*
unfolding *eq*
proof (*rule closedin_continuous_map_preimage*)
show *continuous_map* (*subtopology X K*) (*subtopology Y* (*f*‘*K*)) *f*
by (*simp add*: *continuous_map_in_subtopology f image_mono that*)
show *closedin* (*subtopology Y* (*f*‘*K*)) (*f* ‘ *K* ∩ *C*)
using ‹*closedin Y C*› *closedin_subtopology* **by** *blast*
qed
ultimately show *?thesis*
using ‹*k_space X*› *k_space* **by** *blast*
qed
ultimately show *?thesis*
by (*simp add*: *continuous_map_closedin*)
qed

lemma *closed_map_into_k_space*:
assumes *k_space Y* **and** *fim*: *f* ∈ (*topspace X*) → *topspace Y*
and *f*: $\bigwedge K$. *compactin Y K*
$\Longrightarrow$ *closed_map*(*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *K*}) (*subtopology Y K*) *f*
shows *closed_map X Y f*
unfolding *closed_map_def*
proof (*intro strip*)
fix *C*
assume *closedin X C*
have *closedin* (*subtopology Y K*) (*K* ∩ *f* ‘ *C*)
if *compactin Y K* **for** *K*
proof −
have *eq*: *K* ∩ *f* ‘ *C* = *f* ‘ ({*x* ∈ *topspace X*. *f x* ∈ *K*} ∩ *C*)
using ‹*closedin X C*› *closedin_subset* **by** *auto*
show *?thesis*
unfolding *eq*

by (*metis* (*no_types*, *lifting*) ‹*closedin X C*› *closed_map_def closedin_subtopology f inf_commute that*)
qed
then show *closedin Y* (*f ‘ C*)
using ‹*k_space Y*› **unfolding** *k_space*
by (*meson* ‹*closedin X C*› *closedin_subset order.trans fim funcset_image subset_image_iff*)
qed

Essentially the same proof

lemma *open_map_into_k_space*:
assumes *k_space Y* **and** *fim*: $f \in$ (*topspace X*) $\to$ *topspace Y*
and *f*: $\bigwedge K$. *compactin Y K*
$\Longrightarrow$ *open_map* (*subtopology X* {$x \in$ *topspace X*. *f x* $\in K$}) (*subtopology Y K*) *f*
shows *open_map X Y f*
unfolding *open_map_def*
proof (*intro strip*)
fix *C*
assume *openin X C*
have *openin* (*subtopology Y K*) (*K* $\cap$ *f ‘ C*)
if *compactin Y K* **for** *K*
proof −
have *eq*: *K* $\cap$ *f ‘ C* = *f ‘* ({$x \in$ *topspace X*. *f x* $\in K$} $\cap$ *C*)
using ‹*openin X C*› *openin_subset* **by** *auto*
show *?thesis*
unfolding *eq*
by (*metis* (*no_types*, *lifting*) ‹*openin X C*› *open_map_def openin_subtopology f inf_commute that*)
qed
then show *openin Y* (*f ‘ C*)
using ‹*k_space Y*› **unfolding** *k_space_open*
by (*meson* ‹*openin X C*› *openin_subset order.trans fim funcset_image subset_image_iff*)
qed

lemma *quotient_map_into_k_space*:
fixes *f* :: *'a* $\Rightarrow$ *'b*
assumes *k_space Y* **and** *cmf*: *continuous_map X Y f*
and *fim*: *f ‘* (*topspace X*) = *topspace Y*
and *f*: $\bigwedge k$. *compactin Y k*
$\Longrightarrow$ *quotient_map* (*subtopology X* {$x \in$ *topspace X*. *f x* $\in k$})
(*subtopology Y k*) *f*
shows *quotient_map X Y f*
proof −
have *closedin Y C*
if *C* $\subseteq$ *topspace Y* **and** *K*: *closedin X* {$x \in$ *topspace X*. *f x* $\in C$} **for** *C*
proof −
have *closedin* (*subtopology Y K*) (*K* $\cap$ *C*) **if** *compactin Y K* **for** *K*

```
    proof −
      define Kf where Kf ≡ {x ∈ topspace X. f x ∈ K}
      have ∗: K ∩ C ⊆ topspace Y ∧ K ∩ C ⊆ K
        using ‹C ⊆ topspace Y› by blast
     then have eq: closedin (subtopology X Kf) (Kf ∩ {x ∈ topspace X. f x ∈ C})
=
                  closedin (subtopology Y K) (K ∩ C)
        using f [OF that] ∗ unfolding quotient_map_closedin Kf_def
       by (smt (verit, ccfv_SIG) Collect_cong Int_def compactin_subset_topspace
mem_Collect_eq that topspace_subtopology topspace_subtopology_subset)
      have dd: {x ∈ topspace X ∩ Kf. f x ∈ K ∩ C} = Kf ∩ {x ∈ topspace X. f x
∈ C}
        by (auto simp add: Kf_def)
      have closedin (subtopology X Kf) {x ∈ topspace X. x ∈ Kf ∧ f x ∈ K ∧ f x
∈ C}
        using K closedin_subtopology by (fastforce simp add: Kf_def)
      with K closedin_subtopology_Int_closed eq show ?thesis
        by blast
    qed
    then show ?thesis
      using ‹k_space Y› that unfolding k_space by blast
  qed
  moreover have closedin X {x ∈ topspace X. f x ∈ K}
    if K ⊆ topspace Y closedin Y K for K
    using that cmf continuous_map_closedin by fastforce
  ultimately show ?thesis
    unfolding quotient_map_closedin using fim by blast
qed

lemma quotient_map_into_k_space_eq:
  assumes k_space Y kc_space Y
  shows quotient_map X Y f ⟷
        continuous_map X Y f ∧ f ‘ (topspace X) = topspace Y ∧
        (∀ K. compactin Y K
          ⟶ quotient_map (subtopology X {x ∈ topspace X. f x ∈ K}) (subtopology
Y K) f)
  using assms
 by (auto simp: kc_space_def intro: quotient_map_into_k_space quotient_map_restriction
      dest: quotient_imp_continuous_map quotient_imp_surjective_map)

lemma open_map_into_k_space_eq:
  assumes k_space Y
  shows open_map X Y f ⟷
        f ∈ (topspace X) → topspace Y ∧
        (∀ k. compactin Y k
            ⟶ open_map (subtopology X {x ∈ topspace X. f x ∈ k}) (subtopology
Y k) f)
 using assms open_map_imp_subset_topspace open_map_into_k_space open_map_restriction
by fastforce
```

lemma *closed_map_into_k_space_eq*:
assumes *k_space Y*
shows *closed_map X Y f* ⟷
f ∈ (*topspace X*) → *topspace Y* ∧
(∀ *k*. *compactin Y k*
⟶ *closed_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *k*}) (*subtopology Y k*) *f*)
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*simp add*: *closed_map_imp_subset_topspace closed_map_restriction*)
show *?rhs* ⟹ *?lhs*
by (*simp add*: *assms closed_map_into_k_space*)
qed

lemma *proper_map_into_k_space*:
assumes *k_space Y* **and** *fim*: *f* ∈ (*topspace X*) → *topspace Y*
and *f*: ⋀*K*. *compactin Y K*
⟹ *proper_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *K*})
(*subtopology Y K*) *f*
shows *proper_map X Y f*
proof −
have *closed_map X Y f*
by (*meson assms closed_map_into_k_space fim proper_map_def*)
with *f topspace_subtopology_subset* **show** *?thesis*
apply (*simp add*: *proper_map_alt*)
by (*smt* (*verit*, *best*) *Collect_cong compactin_absolute*)
qed

lemma *proper_map_into_k_space_eq*:
assumes *k_space Y*
shows *proper_map X Y f* ⟷
f ∈ (*topspace X*) → *topspace Y* ∧
(∀ *K*. *compactin Y K*
⟶ *proper_map* (*subtopology X* {*x* ∈ *topspace X*. *f x* ∈ *K*}) (*subtopology Y K*) *f*)
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*simp add*: *proper_map_imp_subset_topspace proper_map_restriction*)
show *?rhs* ⟹ *?lhs*
by (*simp add*: *assms funcset_image proper_map_into_k_space*)
qed

lemma *compact_imp_proper_map*:
assumes *k_space Y kc_space Y* **and** *fim*: *f* ∈ (*topspace X*) → *topspace Y*
and *f*: *continuous_map X Y f* ∨ *kc_space X*
and *comp*: ⋀*K*. *compactin Y K* ⟹ *compactin X* {*x* ∈ *topspace X*. *f x* ∈ *K*}

shows *proper_map X Y f*
proof (*rule compact_imp_proper_map_gen*)
fix *S*
assume $S \subseteq$ *topspace Y*
and $\bigwedge K$. *compactin Y K* $\Longrightarrow$ *compactin Y* $(S \cap K)$
with *assms* **show** *closedin Y S*
by (*simp add*: *closedin_subset_topspace inf_commute k_space kc_space_def*)
qed (*use assms* **in** *auto*)

lemma *proper_eq_compact_map*:
assumes *k_space Y kc_space Y*
and *f*: *continuous_map X Y f* $\vee$ *kc_space X*
shows *proper_map X Y f* $\longleftrightarrow$
$f \in$ (*topspace X*) $\to$ *topspace Y* $\wedge$
$(\forall K$. *compactin Y K* $\longrightarrow$ *compactin X* $\{x \in$ *topspace X*. $f\, x \in K\}$)
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
using ‹*k_space Y*› *compactin_proper_map_preimage proper_map_into_k_space_eq*
by *blast*
qed (*use assms compact_imp_proper_map* **in** *auto*)

lemma *compact_imp_perfect_map*:
assumes *k_space Y kc_space Y* **and** *f* ‘ (*topspace X*) = *topspace Y*
and *continuous_map X Y f*
and $\bigwedge K$. *compactin Y K* $\Longrightarrow$ *compactin X* $\{x \in$ *topspace X*. $f\, x \in K\}$
shows *perfect_map X Y f*
by (*simp add*: *assms compact_imp_proper_map perfect_map_def flip*: *image_subset_iff_funcset*)

end

6.7 Abstract Metric Spaces

theory *Abstract_Metric_Spaces*
imports *Elementary_Metric_Spaces Abstract_Limits Abstract_Topological_Spaces*
begin

locale *Metric_space* =
fixes *M* :: *'a set* **and** *d* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *real*
assumes *nonneg* [*simp*]: $\bigwedge x\ y$. $0 \le d\ x\ y$
assumes *commute*: $\bigwedge x\ y$. $d\ x\ y = d\ y\ x$
assumes *zero* [*simp*]: $\bigwedge x\ y$. $[\![x \in M;\ y \in M]\!] \Longrightarrow d\ x\ y = 0 \longleftrightarrow x{=}y$
assumes *triangle*: $\bigwedge x\ y\ z$. $[\![x \in M;\ y \in M;\ z \in M]\!] \Longrightarrow d\ x\ z \le d\ x\ y + d\ y\ z$

Link with the type class version

interpretation *Met_TC*: *Metric_space UNIV dist*
by (*simp add*: *dist_commute dist_triangle Metric_space.intro*)

context *Metric_space*
begin

lemma *subspace*: $M' \subseteq M \Longrightarrow$ *Metric_space* M' *d*
by (*simp add*: *commute in_mono Metric_space.intro triangle*)

lemma *abs_mdist* [*simp*] : $|d\ x\ y| = d\ x\ y$
by *simp*

lemma *mdist_pos_less*: $[\![x \neq y;\ x \in M;\ y \in M]\!] \Longrightarrow 0 < d\ x\ y$
by (*metis less_eq_real_def nonneg zero*)

lemma *mdist_zero* [*simp*]: $x \in M \Longrightarrow d\ x\ x = 0$
by *simp*

lemma *mdist_pos_eq* [*simp*]: $[\![x \in M;\ y \in M]\!] \Longrightarrow 0 < d\ x\ y \longleftrightarrow x \neq y$
using *mdist_pos_less zero* **by** *fastforce*

lemma *triangle'*: $[\![x \in M;\ y \in M;\ z \in M]\!] \Longrightarrow d\ x\ z \leq d\ x\ y + d\ z\ y$
by (*simp add*: *commute triangle*)

lemma *triangle''*: $[\![x \in M;\ y \in M;\ z \in M]\!] \Longrightarrow d\ x\ z \leq d\ y\ x + d\ y\ z$
by (*simp add*: *commute triangle*)

lemma *mdist_reverse_triangle*: $[\![x \in M;\ y \in M;\ z \in M]\!] \Longrightarrow |d\ x\ y - d\ y\ z| \leq d\ x\ z$
by (*smt* (*verit*) *commute triangle*)

Open and closed balls

definition *mball* **where** $mball\ x\ r \equiv \{y.\ x \in M \wedge y \in M \wedge d\ x\ y < r\}$
definition *mcball* **where** $mcball\ x\ r \equiv \{y.\ x \in M \wedge y \in M \wedge d\ x\ y \leq r\}$

lemma *in_mball* [*simp*]: $y \in mball\ x\ r \longleftrightarrow x \in M \wedge y \in M \wedge d\ x\ y < r$
by (*simp add*: *mball_def*)

lemma *centre_in_mball_iff* [*iff*]: $x \in mball\ x\ r \longleftrightarrow x \in M \wedge 0 < r$
using *in_mball mdist_zero* **by** *force*

lemma *mball_subset_mspace*: $mball\ x\ r \subseteq M$
by *auto*

lemma *mball_eq_empty*: $mball\ x\ r = \{\} \longleftrightarrow (x \notin M) \vee r \leq 0$
by (*smt* (*verit*, *best*) *Collect_empty_eq centre_in_mball_iff mball_def nonneg*)

lemma *mball_subset*: $[\![d\ x\ y + a \leq b;\ y \in M]\!] \Longrightarrow mball\ x\ a \subseteq mball\ y\ b$
by (*smt* (*verit*) *commute in_mball subsetI triangle*)

lemma *disjoint_mball*: $r + r' \leq d\ x\ x' \Longrightarrow disjnt\ (mball\ x\ r)\ (mball\ x'\ r')$
by (*smt* (*verit*) *commute disjnt_iff in_mball triangle*)

lemma *mball_subset_concentric*: $r \leq s \Longrightarrow mball\ x\ r \subseteq mball\ x\ s$
by *auto*

lemma *in_mcball* [*simp*]: $y \in mcball\ x\ r \longleftrightarrow x \in M \wedge y \in M \wedge d\ x\ y \leq r$
by (*simp add*: *mcball_def*)

lemma *centre_in_mcball_iff* [*iff*]: $x \in mcball\ x\ r \longleftrightarrow x \in M \wedge 0 \leq r$
using *mdist_zero* **by** *force*

lemma *mcball_eq_empty*: $mcball\ x\ r = \{\} \longleftrightarrow (x \notin M) \vee r < 0$
by (*smt* (*verit*, *best*) *Collect_empty_eq centre_in_mcball_iff empty_iff mcball_def nonneg*)

lemma *mcball_subset_mspace*: $mcball\ x\ r \subseteq M$
by *auto*

lemma *mball_subset_mcball*: $mball\ x\ r \subseteq mcball\ x\ r$
by *auto*

lemma *mcball_subset*: $[\![d\ x\ y + a \leq b;\ y \in M]\!] \Longrightarrow mcball\ x\ a \subseteq mcball\ y\ b$
by (*smt* (*verit*) *in_mcball mdist_reverse_triangle subsetI*)

lemma *mcball_subset_concentric*: $r \leq s \Longrightarrow mcball\ x\ r \subseteq mcball\ x\ s$
by *force*

lemma *mcball_subset_mball*: $[\![d\ x\ y + a < b;\ y \in M]\!] \Longrightarrow mcball\ x\ a \subseteq mball\ y\ b$
by (*smt* (*verit*) *commute in_mball in_mcball subsetI triangle*)

lemma *mcball_subset_mball_concentric*: $a < b \Longrightarrow mcball\ x\ a \subseteq mball\ x\ b$
by *force*

end

6.7.1 Metric topology

context *Metric_space*
begin

definition *mopen* **where**
$mopen\ U \equiv U \subseteq M \wedge (\forall x.\ x \in U \longrightarrow (\exists r{>}0.\ mball\ x\ r \subseteq U))$

definition *mtopology* :: *'a topology* **where**
$mtopology \equiv topology\ mopen$

lemma *is_topology_metric_topology* [*iff*]: *istopology mopen*
proof −
have $\bigwedge S\ T.\ [\![mopen\ S;\ mopen\ T]\!] \Longrightarrow mopen\ (S \cap T)$
by (*smt* (*verit*, *del_insts*) *Int_iff in_mball mopen_def subset_eq*)

moreover have $\bigwedge\mathcal{K}.\ (\forall K{\in}\mathcal{K}.\ mopen\ K) \longrightarrow mopen\ (\bigcup\mathcal{K})$
using *mopen_def* **by** *fastforce*
ultimately show *?thesis*
by (*simp add*: *istopology_def*)
qed

lemma *openin_mtopology*: *openin mtopology U* $\longleftrightarrow$ $U \subseteq M \wedge (\forall x.\ x \in U \longrightarrow (\exists r{>}0.\ mball\ x\ r \subseteq U))$
by (*simp add*: *mopen_def mtopology_def*)

lemma *topspace_mtopology* [*simp*]: *topspace mtopology* $= M$
by (*meson order.refl mball_subset_mspace openin_mtopology openin_subset openin_topspace subset_antisym zero_less_one*)

lemma *subtopology_mspace* [*simp*]: *subtopology mtopology M* $=$ *mtopology*
by (*metis subtopology_topspace topspace_mtopology*)

lemma *open_in_mspace* [*iff*]: *openin mtopology M*
by (*metis openin_topspace topspace_mtopology*)

lemma *closedin_mspace* [*iff*]: *closedin mtopology M*
by (*metis closedin_topspace topspace_mtopology*)

lemma *openin_mball* [*iff*]: *openin mtopology* (*mball x r*)
proof −
have $\bigwedge y.\ [\![x \in M;\ d\ x\ y < r]\!] \Longrightarrow \exists s{>}0.\ mball\ y\ s \subseteq mball\ x\ r$
by (*metis add_diff_cancel_left' add_diff_eq commute less_add_same_cancel1 mball_subset order_refl*)
then show *?thesis*
by (*auto simp*: *openin_mtopology*)
qed

lemma *mtopology_base*:
mtopology $=$ *topology*(*arbitrary union_of* $(\lambda U.\ \exists x \in M.\ \exists r{>}0.\ U = mball\ x\ r)$)
proof −
have $\bigwedge S.\ \exists x\ r.\ x \in M \wedge 0 < r \wedge S = mball\ x\ r \Longrightarrow$ *openin mtopology S*
using *openin_mball* **by** *blast*
moreover have $\bigwedge U\ x.\ [\![$*openin mtopology U*; $x \in U]\!] \Longrightarrow \exists B.\ (\exists x\ r.\ x \in M \wedge 0 < r \wedge B = mball\ x\ r) \wedge x \in B \wedge B \subseteq U$
by (*metis centre_in_mball_iff in_mono openin_mtopology*)
ultimately show *?thesis*
by (*smt* (*verit*) *topology_base_unique*)
qed

lemma *closedin_metric*:
closedin mtopology C $\longleftrightarrow$ $C \subseteq M \wedge (\forall x.\ x \in M - C \longrightarrow (\exists r{>}0.\ disjnt\ C\ (mball\ x\ r)))$ (**is** *?lhs* $=$ *?rhs*)
proof

show *?lhs* $\Longrightarrow$ *?rhs*
unfolding *closedin_def openin_mtopology*
by (*metis Diff_disjoint disjnt_def disjnt_subset2 topspace_mtopology*)
show *?rhs* $\Longrightarrow$ *?lhs*
unfolding *closedin_def openin_mtopology disjnt_def*
by (*metis Diff_subset Diff_triv Int_Diff Int_commute inf.absorb_iff2 mball_subset_mspace topspace_mtopology*)
qed

lemma *closedin_mcball* [*iff*]: *closedin mtopology* (*mcball x r*)
proof −
have $\exists ra{>}0.$ *disjnt* (*mcball x r*) (*mball y ra*) **if** $x \notin M$ **for** *y*
by (*metis disjnt_empty1 gt_ex mcball_eq_empty that*)
moreover have *disjnt* (*mcball x r*) (*mball y* (*d x y* − *r*)) **if** $y \in M$ *d x y* $> r$ **for** *y*
using *that disjnt_iff in_mball in_mcball mdist_reverse_triangle* **by** *force*
ultimately show *?thesis*
using *closedin_metric mcball_subset_mspace* **by** *fastforce*
qed

lemma *mball_iff_mcball*: $(\exists r{>}0.$ *mball x r* $\subseteq U) = (\exists r{>}0.$ *mcball x r* $\subseteq U)$
by (*meson dense mball_subset_mcball mcball_subset_mball_concentric order_trans*)

lemma *openin_mtopology_mcball*:
openin mtopology U $\longleftrightarrow U \subseteq M \wedge (\forall x.\ x \in U \longrightarrow (\exists r.\ 0 < r \wedge$ *mcball x r* $\subseteq U))$
by (*simp add*: *mball_iff_mcball openin_mtopology*)

lemma *metric_derived_set_of*:
mtopology derived_set_of S $= \{x \in M.\ \forall r{>}0.\ \exists y{\in}S.\ y{\neq}x \wedge y \in$ *mball x r*$\}$ (**is** *?lhs=?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs*
unfolding *openin_mtopology derived_set_of_def*
by *clarsimp* (*metis in_mball openin_mball openin_mtopology zero*)
show *?rhs* $\subseteq$ *?lhs*
unfolding *openin_mtopology derived_set_of_def*
by *clarify* (*metis subsetD topspace_mtopology*)
qed

lemma *metric_closure_of*:
mtopology closure_of S $= \{x \in M.\ \forall r{>}0.\ \exists y \in S.\ y \in$ *mball x r*$\}$
proof −
have $\bigwedge x\ r.\ [\![0 < r;\ x \in$ *mtopology closure_of S*$]\!] \Longrightarrow \exists y{\in}S.\ y \in$ *mball x r*
by (*metis centre_in_mball_iff in_closure_of openin_mball topspace_mtopology*)
moreover have $\bigwedge x\ T.\ [\![x \in M;\ \forall r{>}0.\ \exists y{\in}S.\ y \in$ *mball x r*$]\!] \Longrightarrow x \in$ *mtopology closure_of S*
by (*smt* (*verit*) *in_closure_of in_mball openin_mtopology subsetD topspace_mtopology*)
ultimately show *?thesis*

by (*auto simp*: *in_closure_of*)
qed

lemma *metric_closure_of_alt*:
mtopology closure_of S = {*x* ∈ *M*. ∀ *r*>0. ∃ *y* ∈ *S*. *y* ∈ *mcball x r*}
proof −
have ⋀*x r*. ⟦∀ *r*>0. *x* ∈ *M* ∧ (∃ *y*∈*S*. *y* ∈ *mcball x r*); 0 < *r*⟧ ⟹ ∃ *y*∈*S*. *y* ∈ *M* ∧ *d x y* < *r*
by (*meson dense in_mcball le_less_trans*)
then show *?thesis*
by (*fastforce simp*: *metric_closure_of in_closure_of*)
qed

lemma *metric_interior_of*:
mtopology interior_of S = {*x* ∈ *M*. ∃ *ε*>0. *mball x ε* ⊆ *S*} (**is** *?lhs*=*?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using *interior_of_maximal_eq openin_mtopology* **by** *fastforce*
show *?rhs* ⊆ *?lhs*
using *interior_of_def openin_mball* **by** *fastforce*
qed

lemma *metric_interior_of_alt*:
mtopology interior_of S = {*x* ∈ *M*. ∃ *ε*>0. *mcball x ε* ⊆ *S*}
by (*fastforce simp*: *mball_iff_mcball metric_interior_of*)

lemma *in_interior_of_mball*:
x ∈ *mtopology interior_of S* ⟷ *x* ∈ *M* ∧ (∃ *ε*>0. *mball x ε* ⊆ *S*)
using *metric_interior_of* **by** *force*

lemma *in_interior_of_mcball*:
x ∈ *mtopology interior_of S* ⟷ *x* ∈ *M* ∧ (∃ *ε*>0. *mcball x ε* ⊆ *S*)
using *metric_interior_of_alt* **by** *force*

lemma *Hausdorff_space_mtopology*: *Hausdorff_space mtopology*
unfolding *Hausdorff_space_def*
proof *clarify*
fix *x y*
assume *x*: *x* ∈ *topspace mtopology* **and** *y*: *y* ∈ *topspace mtopology* **and** *x* ≠ *y*
then have *gt0*: *d x y* / 2 > 0
by *auto*
have *disjnt* (*mball x* (*d x y* / 2)) (*mball y* (*d x y* / 2))
by (*simp add*: *disjoint_mball*)
then show ∃ *U V*. *openin mtopology U* ∧ *openin mtopology V* ∧ *x* ∈ *U* ∧ *y* ∈ *V* ∧ *disjnt U V*
by (*metis centre_in_mball_iff gt0 openin_mball topspace_mtopology x y*)
qed

6.7.2 Bounded sets

definition *mbounded* **where** $mbounded\ S \longleftrightarrow (\exists\, x\ B.\ S \subseteq mcball\ x\ B)$

lemma *mbounded_pos*: $mbounded\ S \longleftrightarrow (\exists\, x\ B.\ 0 < B \wedge S \subseteq mcball\ x\ B)$
proof −
  have $\exists\, x'\ r'.\ 0 < r' \wedge S \subseteq mcball\ x'\ r'$ **if** $S \subseteq mcball\ x\ r$ **for** $x\ r$
    by (*metis gt_ex less_eq_real_def linorder_not_le mcball_subset_concentric order_trans that*)
  then show *?thesis*
    by (*auto simp*: *mbounded_def*)
qed

lemma *mbounded_alt*:
  $mbounded\ S \longleftrightarrow S \subseteq M \wedge (\exists\, B.\ \forall\, x \in S.\ \forall\, y \in S.\ d\ x\ y \leq B)$
proof −
  have $\bigwedge x\ B.\ S \subseteq mcball\ x\ B \Longrightarrow \forall\, x{\in}S.\ \forall\, y{\in}S.\ d\ x\ y \leq 2 * B$
    by (*smt* (*verit*, *best*) *commute in_mcball subsetD triangle*)
  then show *?thesis*
    unfolding *mbounded_def* **by** (*metis in_mcball in_mono subsetI*)
qed

lemma *mbounded_alt_pos*:
  $mbounded\ S \longleftrightarrow S \subseteq M \wedge (\exists\, B{>}0.\ \forall\, x \in S.\ \forall\, y \in S.\ d\ x\ y \leq B)$
  by (*smt* (*verit*, *del_insts*) *gt_ex mbounded_alt*)

lemma *mbounded_subset*: $[\![mbounded\ T;\ S \subseteq T]\!] \Longrightarrow mbounded\ S$
  by (*meson mbounded_def order_trans*)

lemma *mbounded_subset_mspace*: $mbounded\ S \Longrightarrow S \subseteq M$
  by (*simp add*: *mbounded_alt*)

lemma *mbounded*:
  $mbounded\ S \longleftrightarrow S = \{\} \vee (\forall\, x \in S.\ x \in M) \wedge (\exists\, y\ B.\ y \in M \wedge (\forall\, x \in S.\ d\ y\ x \leq B))$
  by (*meson all_not_in_conv in_mcball mbounded_def subset_iff*)

lemma *mbounded_empty* [*iff*]: $mbounded\ \{\}$
  by (*simp add*: *mbounded*)

lemma *mbounded_mcball*: $mbounded\ (mcball\ x\ r)$
  using *mbounded_def* **by** *auto*

lemma *mbounded_mball* [*iff*]: $mbounded\ (mball\ x\ r)$
  by (*meson mball_subset_mcball mbounded_def*)

lemma *mbounded_insert*: $mbounded\ (insert\ a\ S) \longleftrightarrow a \in M \wedge mbounded\ S$
proof −
  have $\bigwedge y\ B.\ [\![y \in M;\ \forall\, x{\in}S.\ d\ y\ x \leq B]\!]$

$\Longrightarrow \exists y.\ y \in M \wedge (\exists B \geq d\ y\ a.\ \forall x{\in}S.\ d\ y\ x \leq B)$
by (*metis order.trans nle_le*)
then show *?thesis*
by (*auto simp*: *mbounded*)
qed

lemma *mbounded_Int*: *mbounded S* $\Longrightarrow$ *mbounded* $(S \cap T)$
by (*meson inf_le1 mbounded_subset*)

lemma *mbounded_Un*: *mbounded* $(S \cup T) \longleftrightarrow$ *mbounded S* $\wedge$ *mbounded T* (**is** *?lhs=?rhs*)
proof
assume *R*: *?rhs*
show *?lhs*
proof (*cases* $S{=}\{\} \vee T{=}\{\}$)
case *True* **then show** *?thesis*
using *R* **by** *auto*
next
case *False*
obtain *x y B C* **where** $S \subseteq$ *mcball x B* $T \subseteq$ *mcball y C* $B > 0$ $C > 0$ $x \in M$ $y \in M$
using *R mbounded_pos*
by (*metis False mcball_eq_empty subset_empty*)
then have $S \cup T \subseteq$ *mcball x* $(B + C + d\ x\ y)$
by (*smt* (*verit*) *commute dual_order.trans le_supI mcball_subset mdist_pos_eq*)
then show *?thesis*
using *mbounded_def* **by** *blast*
qed
next
show *?lhs* $\Longrightarrow$ *?rhs*
using *mbounded_def* **by** *auto*
qed

lemma *mbounded_Union*:
$[\![$*finite* $\mathcal{F}$; $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *mbounded X*$]\!] \Longrightarrow$ *mbounded* $(\bigcup \mathcal{F})$
by (*induction* $\mathcal{F}$ *rule*: *finite_induct*) (*auto simp*: *mbounded_Un*)

lemma *mbounded_closure_of*:
mbounded S $\Longrightarrow$ *mbounded* (*mtopology closure_of S*)
by (*meson closedin_mcball closure_of_minimal mbounded_def*)

lemma *mbounded_closure_of_eq*:
$S \subseteq M \Longrightarrow$ (*mbounded* (*mtopology closure_of S*) $\longleftrightarrow$ *mbounded S*)
by (*metis closure_of_subset mbounded_closure_of mbounded_subset topspace_mtopology*)

lemma *maxdist_thm*:
assumes *mbounded S*
and $x \in S$

```
    and y ∈ S
  shows  d x y = (SUP z∈S. |d x z − d z y|)
proof −
  have |d x z − d z y| ≤ d x y if z ∈ S for z
    by (metis all_not_in_conv assms mbounded mdist_reverse_triangle that)
  moreover have d x y ≤ r
    if ⋀z. z ∈ S ⟹ |d x z − d z y| ≤ r for r :: real
    using that assms mbounded_subset_mspace mdist_zero by fastforce
  ultimately show ?thesis
    by (intro cSup_eq [symmetric]) auto
qed


lemma metric_eq_thm: ⟦S ⊆ M; x ∈ S; y ∈ S⟧ ⟹ (x = y) = (∀ z∈S. d x z =
d y z)
  by (metis commute  subset_iff zero)

lemma compactin_imp_mbounded:
  assumes compactin mtopology S
  shows mbounded S
proof −
  have S ⊆ M
    and com: ⋀𝒰. ⟦∀ U∈𝒰. openin mtopology U; S ⊆ ⋃𝒰⟧ ⟹ ∃ℱ. finite ℱ ∧ ℱ
⊆ 𝒰 ∧ S ⊆ ⋃ℱ
    using assms by (auto simp: compactin_def mbounded_def)
  show ?thesis
  proof (cases S = {})
    case False
    with ‹S ⊆ M› obtain a where a ∈ S a ∈ M
      by blast
    with ‹S ⊆ M› gt_ex have S ⊆ ⋃(range (mball a))
      by force
    then obtain ℱ where finite ℱ ℱ ⊆ range (mball a) S ⊆ ⋃ℱ
      by (metis (no_types, opaque_lifting) com imageE openin_mball)
  then show ?thesis
      using mbounded_Union mbounded_subset by fastforce
  qed auto
qed


end

lemma mcball_eq_cball [simp]: Met_TC.mcball = cball
  by force

lemma mball_eq_ball [simp]: Met_TC.mball = ball
  by force

lemma mopen_eq_open [simp]: Met_TC.mopen = open
```

by (*force simp*: *open_contains_ball Met_TC.mopen_def*)

lemma *limitin_iff_tendsto* [*iff*]: *limitin Met_TC.mtopology σ x F = tendsto σ x F*
by (*simp add*: *Met_TC.mtopology_def*)

lemma *mtopology_is_euclidean* [*simp*]: *Met_TC.mtopology = euclidean*
by (*simp add*: *Met_TC.mtopology_def*)

lemma *mbounded_iff_bounded* [*iff*]: *Met_TC.mbounded A ⟷ bounded A*
by (*metis Met_TC.mbounded UNIV_I all_not_in_conv bounded_def*)

6.7.3 Subspace of a metric space

locale *Submetric = Metric_space +*
fixes *A*
assumes *subset*: *A ⊆ M*

sublocale *Submetric ⊆ sub*: *Metric_space A d*
by (*simp add*: *subset subspace*)

context *Submetric*
begin

lemma *mball_submetric_eq*: *sub.mball a r = (if a ∈ A then A ∩ mball a r else {})*
and *mcball_submetric_eq*: *sub.mcball a r = (if a ∈ A then A ∩ mcball a r else {})*
using *subset* **by** *force+*

lemma *mtopology_submetric*: *sub.mtopology = subtopology mtopology A*
unfolding *topology_eq*
proof (*intro allI iffI*)
fix *S*
assume *openin sub.mtopology S*
then have ∃ *T*. *openin* (*subtopology mtopology A*) *T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S* **if** *x* ∈ *S* **for** *x*
by (*metis mball_submetric_eq openin_mball openin_subtopology_Int2 sub.centre_in_mball_iff sub.openin_mtopology subsetD that*)
then show *openin* (*subtopology mtopology A*) *S*
by (*meson openin_subopen*)
next
fix *S*
assume *openin* (*subtopology mtopology A*) *S*
then obtain *T* **where** *openin mtopology T S = T ∩ A*
by (*meson openin_subtopology*)
then have *mopen T*
by (*simp add*: *mopen_def openin_mtopology*)
then have *sub.mopen* (*T ∩ A*)

unfolding *sub.mopen_def mopen_def*
by (*metis inf.coboundedI2 mball_submetric_eq Int_iff* ‹$S = T \cap A$› *inf.bounded_iff subsetI*)
then show *openin sub.mtopology S*
using ‹$S = T \cap A$› *sub.mopen_def sub.openin_mtopology* **by** *force*
qed

lemma *mbounded_submetric*: *sub.mbounded* $T \longleftrightarrow$ *mbounded* $T \wedge T \subseteq A$
by (*meson mbounded_alt sub.mbounded_alt subset subset_trans*)

end

lemma (**in** *Metric_space*) *submetric_empty* [*iff*]: *Submetric M d* {}
proof qed *auto*

6.7.4 Abstract type of metric spaces

typedef *'a metric* = {(*M*::*'a set*,*d*). *Metric_space M d*}
morphisms *dest_metric metric*
proof −
have *Metric_space* {} (λ*x y*. *0*)
by (*auto simp*: *Metric_space_def*)
then show *?thesis*
by *blast*
qed

definition *mspace* **where** *mspace m* ≡ *fst* (*dest_metric m*)

definition *mdist* **where** *mdist m* ≡ *snd* (*dest_metric m*)

lemma *Metric_space_mspace_mdist* [*iff*]: *Metric_space* (*mspace m*) (*mdist m*)
by (*metis Product_Type.Collect_case_prodD dest_metric mdist_def mspace_def*)

lemma *mdist_nonneg* [*simp*]: ⋀*x y*. *0* ≤ *mdist m x y*
by (*metis Metric_space_def Metric_space_mspace_mdist*)

lemma *mdist_commute*: ⋀*x y*. *mdist m x y* = *mdist m y x*
by (*metis Metric_space_def Metric_space_mspace_mdist*)

lemma *mdist_zero* [*simp*]: ⋀*x y*. ⟦*x* ∈ *mspace m*; *y* ∈ *mspace m*⟧ ⟹ *mdist m x y* = *0* ⟷ *x*=*y*
by (*meson Metric_space.zero Metric_space_mspace_mdist*)

lemma *mdist_triangle*: ⋀*x y z*. ⟦*x* ∈ *mspace m*; *y* ∈ *mspace m*; *z* ∈ *mspace m*⟧ ⟹ *mdist m x z* ≤ *mdist m x y* + *mdist m y z*
by (*meson Metric_space.triangle Metric_space_mspace_mdist*)

lemma (**in** *Metric_space*) *mspace_metric*[*simp*]:
mspace (*metric* (*M*,*d*)) = *M*

by (*simp add*: *metric_inverse mspace_def subspace*)

lemma (**in** *Metric_space*) *mdist_metric*[*simp*]:
mdist (*metric* (*M*,*d*)) = *d*
by (*simp add*: *mdist_def metric_inverse subspace*)

lemma *metric_collapse* [*simp*]: *metric* (*mspace m*, *mdist m*) = *m*
by (*simp add*: *dest_metric_inverse mdist_def mspace_def*)

definition *mtopology_of* :: *'a metric* ⇒ *'a topology*
where *mtopology_of* ≡ λ*m*. *Metric_space.mtopology* (*mspace m*) (*mdist m*)

lemma *topspace_mtopology_of* [*simp*]: *topspace* (*mtopology_of m*) = *mspace m*
by (*simp add*: *Metric_space.topspace_mtopology Metric_space_mspace_mdist mtopology_of_def*)

lemma (**in** *Metric_space*) *mtopology_of* [*simp*]:
mtopology_of (*metric* (*M*,*d*)) = *mtopology*
by (*simp add*: *mtopology_of_def*)

definition *mball_of* ≡ λ*m*. *Metric_space.mball* (*mspace m*) (*mdist m*)

lemma *in_mball_of* [*simp*]: *y* ∈ *mball_of m x r* ⟷ *x* ∈ *mspace m* ∧ *y* ∈ *mspace m* ∧ *mdist m x y* < *r*
by (*simp add*: *Metric_space.in_mball mball_of_def*)

lemma (**in** *Metric_space*) *mball_of* [*simp*]:
mball_of (*metric* (*M*,*d*)) = *mball*
by (*simp add*: *mball_of_def*)

definition *mcball_of* ≡ λ*m*. *Metric_space.mcball* (*mspace m*) (*mdist m*)

lemma *in_mcball_of* [*simp*]: *y* ∈ *mcball_of m x r* ⟷ *x* ∈ *mspace m* ∧ *y* ∈ *mspace m* ∧ *mdist m x y* ≤ *r*
by (*simp add*: *Metric_space.in_mcball mcball_of_def*)

lemma (**in** *Metric_space*) *mcball_of* [*simp*]:
mcball_of (*metric* (*M*,*d*)) = *mcball*
by (*simp add*: *mcball_of_def*)

definition *euclidean_metric* ≡ *metric* (*UNIV*,*dist*)

lemma *mspace_euclidean_metric* [*simp*]: *mspace euclidean_metric* = *UNIV*
by (*simp add*: *euclidean_metric_def*)

lemma *mdist_euclidean_metric* [*simp*]: *mdist euclidean_metric* = *dist*
by (*simp add*: *euclidean_metric_def*)

lemma *mtopology_of_euclidean* [*simp*]: *mtopology_of euclidean_metric* = *euclidean*

by (*simp add*: *Met_TC.mtopology_def mtopology_of_def*)

Allows reference to the current metric space within the locale as a value

definition (**in** *Metric_space*) *Self* ≡ *metric* (*M*,*d*)

lemma (**in** *Metric_space*) *mspace_Self* [*simp*]: *mspace Self* = *M*
by (*simp add*: *Self_def*)

lemma (**in** *Metric_space*) *mdist_Self* [*simp*]: *mdist Self* = *d*
by (*simp add*: *Self_def*)

Subspace of a metric space

definition *submetric* **where**
submetric ≡ λ*m S*. *metric* (*S* ∩ *mspace m*, *mdist m*)

lemma *mspace_submetric* [*simp*]: *mspace* (*submetric m S*) = *S* ∩ *mspace m*
unfolding *submetric_def*
by (*meson Metric_space.subspace inf_le2 Metric_space_mspace_mdist Metric_space.mspace_metric*)

lemma *mdist_submetric* [*simp*]: *mdist* (*submetric m S*) = *mdist m*
unfolding *submetric_def*
by (*meson Metric_space.subspace inf_le2 Metric_space.mdist_metric Metric_space_mspace_mdist*)

lemma *submetric_UNIV* [*simp*]: *submetric m UNIV* = *m*
by (*simp add*: *submetric_def dest_metric_inverse mdist_def mspace_def*)

lemma *submetric_submetric* [*simp*]:
submetric (*submetric m S*) *T* = *submetric m* (*S* ∩ *T*)
by (*metis submetric_def Int_assoc inf_commute mdist_submetric mspace_submetric*)

lemma *submetric_mspace* [*simp*]:
submetric m (*mspace m*) = *m*
by (*simp add*: *submetric_def dest_metric_inverse mdist_def mspace_def*)

lemma *submetric_restrict*:
submetric m S = *submetric m* (*mspace m* ∩ *S*)
by (*metis submetric_mspace submetric_submetric*)

lemma *mtopology_of_submetric*: *mtopology_of* (*submetric m A*) = *subtopology* (*mtopology_of m*) *A*
proof −
interpret *Submetric mspace m mdist m A* ∩ *mspace m*
using *Metric_space_mspace_mdist Submetric.intro Submetric_axioms.intro inf_le2* **by** *blast*
have *sub.mtopology* = *subtopology* (*mtopology_of m*) *A*
by (*metis inf_commute mtopology_of_def mtopology_submetric subtopology_mspace subtopology_subtopology*)
then show *?thesis*

by (*simp add*: *submetric_def*)
qed

6.7.5 The discrete metric

locale *discrete_metric* =
fixes *M* :: *'a set*

definition (**in** *discrete_metric*) *dd* :: *'a* ⇒ *'a* ⇒ *real*
where *dd* ≡ λ*x y*::*'a*. *if x*=*y then 0 else 1*

lemma *metric_M_dd*: *Metric_space M discrete_metric.dd*
by (*simp add*: *discrete_metric.dd_def Metric_space.intro*)

sublocale *discrete_metric* ⊆ *disc*: *Metric_space M dd*
by (*simp add*: *metric_M_dd*)

lemma (**in** *discrete_metric*) *mopen_singleton*:
assumes *x* ∈ *M* **shows** *disc.mopen* {*x*}
proof −
have *disc.mball x* (*1*/*2*) ⊆ {*x*}
by (*smt* (*verit*) *dd_def disc.in_mball less_divide_eq_1_pos singleton_iff subsetI*)
with *assms* **show** *?thesis*
using *disc.mopen_def half_gt_zero_iff zero_less_one* **by** *blast*
qed

lemma (**in** *discrete_metric*) *mtopology_discrete_metric*:
disc.mtopology = *discrete_topology M*
proof −
have ⋀*x*. *x* ∈ *M* ⟹ *openin disc.mtopology* {*x*}
by (*simp add*: *disc.mtopology_def mopen_singleton*)
then show *?thesis*
by (*metis disc.topspace_mtopology discrete_topology_unique*)
qed

lemma (**in** *discrete_metric*) *discrete_ultrametric*:
dd x z ≤ *max* (*dd x y*) (*dd y z*)
by (*simp add*: *dd_def*)

lemma (**in** *discrete_metric*) *dd_le1*: *dd x y* ≤ *1*
by (*simp add*: *dd_def*)

lemma (**in** *discrete_metric*) *mbounded_discrete_metric*: *disc.mbounded S* ⟷ *S* ⊆ *M*
by (*meson dd_le1 disc.mbounded_alt*)

6.7.6 Metrizable spaces

definition *metrizable_space* **where**
metrizable_space X ≡ ∃ *M d*. *Metric_space M d* ∧ *X* = *Metric_space.mtopology M d*

lemma (**in** *Metric_space*) *metrizable_space_mtopology*: *metrizable_space mtopology*
using *local.Metric_space_axioms metrizable_space_def* **by** *blast*

lemma (**in** *Metric_space*) *first_countable_mtopology*: *first_countable mtopology*
proof (*clarsimp simp add*: *first_countable_def*)
fix *x*
assume *x* ∈ *M*
define ℬ **where** ℬ ≡ *mball x* ' {*r* ∈ ℚ. *0* < *r*}
show ∃ℬ. *countable* ℬ ∧ (∀ *V*∈ℬ. *openin mtopology V*) ∧ (∀ *U*. *openin mtopology U* ∧ *x* ∈ *U* ⟶ (∃ *V*∈ℬ. *x* ∈ *V* ∧ *V* ⊆ *U*))
proof (*intro exI conjI ballI*)
show *countable* ℬ
by (*simp add*: ℬ_*def countable_rat*)
show ∀ *U*. *openin mtopology U* ∧ *x* ∈ *U* ⟶ (∃ *V*∈ℬ. *x* ∈ *V* ∧ *V* ⊆ *U*)
proof *clarify*
fix *U*
assume *openin mtopology U* **and** *x* ∈ *U*
then obtain *r* **where** *r*>*0* **and** *r*: *mball x r* ⊆ *U*
by (*meson openin_mtopology*)
then obtain *q* **where** *q* ∈ *Rats 0* < *q q* < *r*
using *Rats_dense_in_real* **by** *blast*
then show ∃ *V*∈ℬ. *x* ∈ *V* ∧ *V* ⊆ *U*
unfolding ℬ_*def* **using** ‹*x* ∈ *M*› *r* **by** *fastforce*
qed
qed (*auto simp*: ℬ_*def*)
qed

lemma *metrizable_imp_first_countable*:
metrizable_space X ⟹ *first_countable X*
by (*force simp*: *metrizable_space_def Metric_space.first_countable_mtopology*)

lemma *openin_mtopology_eq_open* [*simp*]: *openin Met_TC.mtopology* = *open*
by (*simp add*: *Met_TC.mtopology_def*)

lemma *closedin_mtopology_eq_closed* [*simp*]: *closedin Met_TC.mtopology* = *closed*
proof −
have (*euclidean*::'*a topology*) = *Met_TC.mtopology*
by (*simp add*: *Met_TC.mtopology_def*)
then show *?thesis*
using *closed_closedin* **by** *fastforce*
qed

lemma *compactin_mtopology_eq_compact* [*simp*]: *compactin Met_TC.mtopology*

= *compact*
by (*simp add*: *compactin_def compact_eq_Heine_Borel fun_eq_iff*) *meson*

lemma *metrizable_space_discrete_topology* [*simp*]:
metrizable_space(*discrete_topology U*)
by (*metis discrete_metric.mtopology_discrete_metric metric_M_dd metrizable_space_def*)

lemma *empty_metrizable_space*: *metrizable_space trivial_topology*
by *simp*

lemma *metrizable_space_subtopology*:
assumes *metrizable_space X*
shows *metrizable_space*(*subtopology X S*)
proof −
obtain *M d* **where** *Metric_space M d* **and** *X*: *X* = *Metric_space.mtopology M d*
using *assms metrizable_space_def* **by** *blast*
then interpret *Submetric M d M* ∩ *S*
by (*simp add*: *Submetric.intro Submetric_axioms_def*)
show *?thesis*
unfolding *metrizable_space_def*
by (*metis X mtopology_submetric sub.Metric_space_axioms subtopology_restrict topspace_mtopology*)
qed

lemma *homeomorphic_metrizable_space_aux*:
assumes *X homeomorphic_space Y metrizable_space X*
shows *metrizable_space Y*
proof −
obtain *M d* **where** *Metric_space M d* **and** *X*: *X* = *Metric_space.mtopology M d*
using *assms* **by** (*auto simp*: *metrizable_space_def*)
then interpret *m*: *Metric_space M d*
by *simp*
obtain *f g* **where** *hmf*: *homeomorphic_map X Y f* **and** *hmg*: *homeomorphic_map Y X g*
and *fg*: (∀ *x* ∈ *M*. *g*(*f x*) = *x*) ∧ (∀ *y* ∈ *topspace Y*. *f*(*g y*) = *y*)
using *assms X homeomorphic_maps_map homeomorphic_space_def* **by** *fastforce*
define *d′* **where** *d′ x y* ≡ *d* (*g x*) (*g y*) **for** *x y*
interpret *m′*: *Metric_space topspace Y d′*
unfolding *d′_def*
proof
show (*d* (*g x*) (*g y*) = *0*) = (*x* = *y*) **if** *x* ∈ *topspace Y y* ∈ *topspace Y* **for** *x y*
by (*metis fg X hmg homeomorphic_imp_surjective_map imageI m.topspace_mtopology m.zero that*)
show *d* (*g x*) (*g z*) ≤ *d* (*g x*) (*g y*) + *d* (*g y*) (*g z*)
if *x* ∈ *topspace Y* **and** *y* ∈ *topspace Y* **and** *z* ∈ *topspace Y* **for** *x y z*
by (*metis X that hmg homeomorphic_eq_everything_map imageI m.topspace_mtopology*

```
m.triangle)
  qed (auto simp: m.nonneg m.commute)
  have Y = Metric_space.mtopology (topspace Y) d'
    unfolding topology_eq
  proof (intro allI)
    fix S
    have openin m'.mtopology S if S: S ⊆ topspace Y and openin X (g ‘ S)
      unfolding m'.openin_mtopology
    proof (intro conjI that strip)
      fix y
      assume y ∈ S
      then obtain r where r>0 and r: m.mball (g y) r ⊆ g ‘ S
        using X ‹openin X (g ‘ S)› m.openin_mtopology using ‹y ∈ S› by auto
      then have g ‘ m'.mball y r ⊆ m.mball (g y) r
        using X d'_def hmg homeomorphic_imp_surjective_map by fastforce
      with S fg have m'.mball y r ⊆ S
        by (smt (verit, del_insts) image_iff m'.in_mball r subset_iff)
      then show ∃ r>0. m'.mball y r ⊆ S
        using ‹0 < r› by blast
    qed
    moreover have openin X (g ‘ S) if ope': openin m'.mtopology S
    proof −
      have ∃ r>0. m.mball (g y) r ⊆ g ‘ S if y ∈ S for y
      proof −
        have y: y ∈ topspace Y
          using m'.openin_mtopology ope' that by blast
        obtain r where r > 0 and r: m'.mball y r ⊆ S
          using ope' by (meson ‹y ∈ S› m'.openin_mtopology)
        moreover have ⋀x. ⟦x ∈ M; d (g y) x < r⟧ ⟹ ∃ u. u ∈ topspace Y ∧ d'
y u < r ∧ x = g u
          using fg X d'_def hmf homeomorphic_imp_surjective_map by fastforce
        ultimately have m.mball (g y) r ⊆ g ‘ m'.mball y r
          using y by (force simp: m'.openin_mtopology)
        then show ?thesis
          using ‹0 < r› r by blast
      qed
      then show ?thesis
       using X hmg homeomorphic_imp_surjective_map m.openin_mtopology ope'
openin_subset by fastforce
    qed
    ultimately have (S ⊆ topspace Y ∧ openin X (g ‘ S)) = openin m'.mtopology
S
      using m'.topspace_mtopology openin_subset by blast
    then show openin Y S = openin m'.mtopology S
      by (simp add: m'.mopen_def homeomorphic_map_openness_eq [OF hmg])
  qed
  then show ?thesis
    using m'.metrizable_space_mtopology by force
qed
```

lemma *homeomorphic_metrizable_space*:
assumes *X homeomorphic_space Y*
shows *metrizable_space X ⟷ metrizable_space Y*
using *assms homeomorphic_metrizable_space_aux homeomorphic_space_sym*
by *metis*

lemma *metrizable_space_retraction_map_image*:
retraction_map X Y r ∧ metrizable_space X
⟹ metrizable_space Y
using *hereditary_imp_retractive_property metrizable_space_subtopology homeomorphic_metrizable_space*
by *blast*

lemma *metrizable_imp_Hausdorff_space*:
metrizable_space X ⟹ Hausdorff_space X
by (*metis Metric_space.Hausdorff_space_mtopology metrizable_space_def*)

lemma *metrizable_imp_t1_space*:
metrizable_space X ⟹ t1_space X
by (*simp add*: *Hausdorff_imp_t1_space metrizable_imp_Hausdorff_space*)

lemma *closed_imp_gdelta_in*:
assumes *X*: *metrizable_space X* **and** *S*: *closedin X S*
shows *gdelta_in X S*
proof −
obtain *M d* **where** *Metric_space M d* **and** *Xeq*: *X* = *Metric_space.mtopology M d*
using *X metrizable_space_def* **by** *blast*
then interpret *M*: *Metric_space M d*
by *blast*
have *S* ⊆ *M*
using *M.closedin_metric* ‹*X* = *M.mtopology*› *S* **by** *blast*
show *?thesis*
proof (*cases S* = {})
case *True*
then show *?thesis*
by *simp*
next
case *False*
have ∃ *y*∈*S*. *d x y* < *inverse* (*1* + *real n*) **if** *x* ∈ *S* **for** *x n*
using ‹*S* ⊆ *M*› *M.mdist_zero* [*of x*] *that* **by** *force*
moreover
have *x* ∈ *S* **if** *x* ∈ *M* **and** §: ⋀*n*. ∃ *y*∈*S*. *d x y* < *inverse*(*Suc n*) **for** *x*
proof −
have ∗: ∃ *y*∈*S*. *d x y* < *ε* **if** *ε* > *0* **for** *ε*

```
        by (metis § that not0_implies_Suc order_less_le order_less_le_trans
real_arch_inverse)
      have closedin M.mtopology S
        using S by (simp add: Xeq)
      with ∗ ‹x ∈ M› show ?thesis
        by (force simp: M.closedin_metric disjnt_iff)
    qed
    ultimately have Seq: S = ⋂(range (λn. {x∈M. ∃y∈S. d x y < inverse(Suc
n)}))
      using ‹S ⊆ M› by force
    have openin M.mtopology {xa ∈ M. ∃y∈S. d xa y < inverse (1 + real n)} for
n
    proof (clarsimp simp: M.openin_mtopology)
      fix x y
      assume x ∈ M y ∈ S and dxy: d x y < inverse (1 + real n)
      then have ⋀z. ⟦z ∈ M; d x z < inverse (1 + real n) − d x y⟧ ⟹ ∃y∈S. d
z y < inverse (1 + real n)
        by (smt (verit) M.commute M.triangle ‹S ⊆ M› in_mono)
      with dxy show ∃r>0. M.mball x r ⊆ {z ∈ M. ∃y∈S. d z y < inverse (1 +
real n)}
        by (rule_tac x=inverse(Suc n) − d x y in exI) auto
    qed
    then have gdelta_in X (⋂(range (λn. {x∈M. ∃y∈S. d x y < inverse(Suc
n)})))
      by (force simp: Xeq intro: gdelta_in_Inter open_imp_gdelta_in)
    with Seq show ?thesis
      by presburger
  qed
qed

lemma open_imp_fsigma_in:
  ⟦metrizable_space X; openin X S⟧ ⟹ fsigma_in X S
 by (meson closed_imp_gdelta_in fsigma_in_gdelta_in openin_closedin openin_subset)

lemma metrizable_space_euclidean:
  metrizable_space (euclidean :: 'a::metric_space topology)
  using Met_TC.metrizable_space_mtopology by auto

lemma (in Metric_space) regular_space_mtopology:
   regular_space mtopology
unfolding regular_space_def
proof clarify
  fix C a
  assume C: closedin mtopology C and a: a ∈ topspace mtopology and a ∉ C
  have openin mtopology (topspace mtopology − C)
    by (simp add: C openin_diff)
  then obtain r where r>0 and r: mball a r ⊆ topspace mtopology − C
    unfolding openin_mtopology using ‹a ∉ C› a by auto
  show ∃U V. openin mtopology U ∧ openin mtopology V ∧ a ∈ U ∧ C ⊆ V ∧
```

disjnt U V
proof (*intro exI conjI*)
show *a* ∈ *mball a* (*r*/2)
using ‹0 < *r*› *a* **by** *force*
show *C* ⊆ *topspace mtopology* − *mcball a* (*r*/2)
using *C* ‹0 < *r*› *r* **by** (*fastforce simp*: *closedin_metric*)
qed (*auto simp*: *openin_mball closedin_mcball openin_diff disjnt_iff*)
qed

lemma *metrizable_imp_regular_space*:
metrizable_space X ⟹ *regular_space X*
by (*metis Metric_space.regular_space_mtopology metrizable_space_def*)

lemma *regular_space_euclidean*:
regular_space (*euclidean* :: 'a::*metric_space topology*)
by (*simp add*: *metrizable_imp_regular_space metrizable_space_euclidean*)

6.7.7 Limits at a point in a topological space

lemma (**in** *Metric_space*) *eventually_atin_metric*:
eventually P (*atin mtopology a*) ⟷
(*a* ∈ *M* ⟶ (∃δ>0. ∀ *x*. *x* ∈ *M* ∧ 0 < *d x a* ∧ *d x a* < δ ⟶ *P x*)) (**is** *?lhs=?rhs*)
proof (*cases a* ∈ *M*)
case *True*
show *?thesis*
proof
assume *L*: *?lhs*
with *True* **obtain** *U* **where** *openin mtopology U a* ∈ *U* **and** *U*: ∀ *x*∈*U* − {*a*}. *P x*
by (*auto simp*: *eventually_atin*)
then obtain *r* **where** *r*>0 **and** *mball a r* ⊆ *U*
by (*meson openin_mtopology*)
with *U* **show** *?rhs*
by (*smt* (*verit*, *ccfv_SIG*) *commute in_mball insert_Diff_single insert_iff subset_iff*)
next
assume *?rhs*
then obtain δ **where** δ>0 **and** δ: ∀ *x*. *x* ∈ *M* ∧ 0 < *d x a* ∧ *d x a* < δ ⟶ *P x*
using *True* **by** *blast*
then have ∀ *x* ∈ *mball a* δ − {*a*}. *P x*
by (*simp add*: *commute*)
then show *?lhs*
unfolding *eventually_atin openin_mtopology*
by (*metis True* ‹0 < δ› *centre_in_mball_iff openin_mball openin_mtopology*)

qed
qed *auto*

6.7.8 Normal spaces and metric spaces

lemma (**in** *Metric_space*) *normal_space_mtopology*:
normal_space mtopology
unfolding *normal_space_def*
proof *clarify*
fix *S T*
assume *closedin mtopology S*
then have $\bigwedge x.\ x \in M - S \Longrightarrow (\exists\, r{>}0.\ mball\ x\ r \subseteq M - S)$
by (*simp add*: *closedin_def openin_mtopology*)
then obtain δ **where** *d0*: $\bigwedge x.\ x \in M - S \Longrightarrow \delta\ x > 0 \wedge mball\ x\ (\delta\ x) \subseteq M - S$
by *metis*
assume *closedin mtopology T*
then have $\bigwedge x.\ x \in M - T \Longrightarrow (\exists\, r{>}0.\ mball\ x\ r \subseteq M - T)$
by (*simp add*: *closedin_def openin_mtopology*)
then obtain ε **where** *e*: $\bigwedge x.\ x \in M - T \Longrightarrow \varepsilon\ x > 0 \wedge mball\ x\ (\varepsilon\ x) \subseteq M - T$
by *metis*
assume *disjnt S T*
have $S \subseteq M\ T \subseteq M$
using ‹*closedin mtopology S*› ‹*closedin mtopology T*› *closedin_metric* **by** *blast+*
have δ: $\bigwedge x.\ x \in T \Longrightarrow \delta\ x > 0 \wedge mball\ x\ (\delta\ x) \subseteq M - S$
by (*meson DiffI* ‹$T \subseteq M$› ‹*disjnt S T*› *d0 disjnt_iff subsetD*)
have ε: $\bigwedge x.\ x \in S \Longrightarrow \varepsilon\ x > 0 \wedge mball\ x\ (\varepsilon\ x) \subseteq M - T$
by (*meson Diff_iff* ‹$S \subseteq M$› ‹*disjnt S T*› *disjnt_iff e subsetD*)
show $\exists\, U\ V.$ *openin mtopology U* $\wedge$ *openin mtopology V* $\wedge\ S \subseteq U \wedge\ T \subseteq V \wedge$ *disjnt U V*
proof (*intro exI conjI*)
show *openin mtopology* $(\bigcup x{\in}S.\ mball\ x\ (\varepsilon\ x\ /\ 2))$ *openin mtopology* $(\bigcup x{\in}T.\ mball\ x\ (\delta\ x\ /\ 2))$
by *force+*
show $S \subseteq (\bigcup x{\in}S.\ mball\ x\ (\varepsilon\ x\ /\ 2))$
using ε ‹$S \subseteq M$› **by** *force*
show $T \subseteq (\bigcup x{\in}T.\ mball\ x\ (\delta\ x\ /\ 2))$
using δ ‹$T \subseteq M$› **by** *force*
show *disjnt* $(\bigcup x{\in}S.\ mball\ x\ (\varepsilon\ x\ /\ 2))\ (\bigcup x{\in}T.\ mball\ x\ (\delta\ x\ /\ 2))$
using $\varepsilon\ \delta$
apply (*clarsimp simp*: *disjnt_iff subset_iff*)
by (*smt* (*verit*, *ccfv_SIG*) *field_sum_of_halves triangle'*)
qed
qed

lemma *metrizable_imp_normal_space*:
metrizable_space X $\Longrightarrow$ *normal_space X*
by (*metis Metric_space.normal_space_mtopology metrizable_space_def*)

6.7.9 Topological limitin in metric spaces

lemma (**in** *Metric_space*) *limitin_mspace*:

limitin mtopology f l F $\Longrightarrow$ *l* $\in$ *M*
using *limitin_topspace* **by** *fastforce*

lemma (**in** *Metric_space*) *limitin_metric_unique*:
⟦*limitin mtopology f l1 F*; *limitin mtopology f l2 F*; *F* $\neq$ *bot*⟧ $\Longrightarrow$ *l1* = *l2*
by (*meson Hausdorff_space_mtopology limitin_Hausdorff_unique*)

lemma (**in** *Metric_space*) *limitin_metric*:
limitin mtopology f l F $\longleftrightarrow$ *l* $\in$ *M* $\wedge$ ($\forall \varepsilon{>}0$. *eventually* (λx. *f x* $\in$ *M* $\wedge$ *d* (*f x*) *l* $<$ ε) *F*)
(**is** *?lhs=?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
unfolding *limitin_def*
proof (*intro conjI strip*)
show *l* $\in$ *M*
using *L limitin_mspace* **by** *blast*
fix ε::*real*
assume $\varepsilon{>}0$
then have $\forall_F$ *x in F*. *f x* $\in$ *mball l* ε
using *L openin_mball* **by** (*fastforce simp*: *limitin_def*)
then show $\forall_F$ *x in F*. *f x* $\in$ *M* $\wedge$ *d* (*f x*) *l* $<$ ε
using *commute eventually_mono* **by** *fastforce*
qed
next
assume *R*: *?rhs*
then show *?lhs*
by (*force simp*: *limitin_def commute openin_mtopology subset_eq elim*: *eventually_mono*)
qed

lemma (**in** *Metric_space*) *limit_metric_sequentially*:
limitin mtopology f l sequentially $\longleftrightarrow$
l $\in$ *M* $\wedge$ ($\forall \varepsilon{>}0$. $\exists N$. $\forall n{\geq}N$. *f n* $\in$ *M* $\wedge$ *d* (*f n*) *l* $<$ ε)
by (*auto simp*: *limitin_metric eventually_sequentially*)

lemma (**in** *Submetric*) *limitin_submetric_iff*:
limitin sub.mtopology f l F $\longleftrightarrow$
l $\in$ *A* $\wedge$ *eventually* (λx. *f x* $\in$ *A*) *F* $\wedge$ *limitin mtopology f l F* (**is** *?lhs=?rhs*)
by (*simp add*: *limitin_subtopology mtopology_submetric*)

lemma (**in** *Metric_space*) *metric_closedin_iff_sequentially_closed*:
closedin mtopology S $\longleftrightarrow$
S $\subseteq$ *M* $\wedge$ ($\forall \sigma$ *l*. *range* σ $\subseteq$ *S* $\wedge$ *limitin mtopology* σ *l sequentially* $\longrightarrow$ *l* $\in$ *S*)
(**is** *?lhs=?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*force simp*: *closedin_metric limitin_closedin range_subsetD*)

```
next
  assume R: ?rhs
  show ?lhs
    unfolding closedin_metric
  proof (intro conjI strip)
    show S ⊆ M
      using R by blast
    fix x
    assume x ∈ M − S
    have False if ∀ r>0. ∃ y. y ∈ M ∧ y ∈ S ∧ d x y < r
    proof −
      have ∀ n. ∃ y. y ∈ M ∧ y ∈ S ∧ d x y < inverse(Suc n)
        using that by auto
     then obtain σ where σ: ⋀n. σ n ∈ M ∧ σ n ∈ S ∧ d x (σ n) < inverse(Suc
n)
        by metis
      then have range σ ⊆ M
        by blast
      have ∃ N. ∀ n≥N. d x (σ n) < ε if ε>0 for ε
      proof −
        have real (Suc (nat ⌈inverse ε⌉)) ≥ inverse ε
          by linarith
        then have ∀ n ≥ nat ⌈inverse ε⌉. d x (σ n) < ε
            by (metis σ inverse_inverse_eq inverse_le_imp_le nat_ceiling_le_eq
nle_le not_less_eq_eq order.strict_trans2 that)
        then show ?thesis ..
      qed
      with σ have limitin mtopology σ x sequentially
        using ‹x ∈ M − S› commute limit_metric_sequentially by auto
      then show ?thesis
        by (metis R DiffD2 σ image_subset_iff ‹x ∈ M − S›)
    qed
    then show ∃ r>0. disjnt S (mball x r)
      by (meson disjnt_iff in_mball)
  qed
qed

lemma (in Metric_space) limit_atin_metric:
   limitin X f y (atin mtopology x) ⟷
      y ∈ topspace X ∧
      (x ∈ M
       ⟶ (∀ V. openin X V ∧ y ∈ V
             ⟶ (∃ δ>0.  ∀ x'. x' ∈ M ∧ 0 < d x' x ∧ d x' x < δ ⟶ f x' ∈ V)))
  by (force simp: limitin_def eventually_atin_metric)

lemma (in Metric_space) limitin_metric_dist_null:
   limitin mtopology f l F ⟷ l ∈ M ∧ eventually (λx. f x ∈ M) F ∧ ((λx. d (f
x) l) ⟶ 0) F
  by (simp add: limitin_metric tendsto_iff eventually_conj_iff all_conj_distrib
```

imp_conjR gt_ex)

6.7.10 Cauchy sequences and complete metric spaces

context *Metric_space*
begin

definition *MCauchy* :: (*nat* ⇒ *'a*) ⇒ *bool*
where *MCauchy σ* ≡ *range σ* ⊆ *M* ∧ (∀ *ε*>*0*. ∃ *N*. ∀ *n n'*. *N* ≤ *n* ⟶ *N* ≤ *n'* ⟶ *d* (*σ n*) (*σ n'*) < *ε*)

definition *mcomplete*
where *mcomplete* ≡ (∀ *σ*. *MCauchy σ* ⟶ (∃ *x*. *limitin mtopology σ x sequentially*))

lemma *mcomplete_empty* [*iff*]: *Metric_space.mcomplete* {} *d*
by (*simp add*: *Metric_space.MCauchy_def Metric_space.mcomplete_def subspace*)

lemma *MCauchy_imp_MCauchy_suffix*: *MCauchy σ* ⟹ *MCauchy* (*σ* ∘ (+)*n*)
unfolding *MCauchy_def image_subset_iff comp_apply*
by (*metis UNIV_I add.commute trans_le_add1*)

lemma *mcomplete*:
mcomplete ⟷
(∀ *σ*. (∀ *F n in sequentially*. *σ n* ∈ *M*) ∧
(∀ *ε*>*0*. ∃ *N*. ∀ *n n'*. *N* ≤ *n* ⟶ *N* ≤ *n'* ⟶ *d* (*σ n*) (*σ n'*) < *ε*) ⟶
(∃ *x*. *limitin mtopology σ x sequentially*)) (**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof *clarify*
fix *σ*
assume ∀ *F n in sequentially*. *σ n* ∈ *M*
and *σ*: ∀ *ε*>*0*. ∃ *N*. ∀ *n n'*. *N* ≤ *n* ⟶ *N* ≤ *n'* ⟶ *d* (*σ n*) (*σ n'*) < *ε*
then obtain *N* **where** ⋀*n*. *n*≥*N* ⟹ *σ n* ∈ *M*
by (*auto simp*: *eventually_sequentially*)
with *σ* **have** *MCauchy* (*σ* ∘ (+)*N*)
unfolding *MCauchy_def image_subset_iff comp_apply* **by** (*meson le_add1 trans_le_add2*)
then obtain *x* **where** *limitin mtopology* (*σ* ∘ (+)*N*) *x sequentially*
using *L MCauchy_imp_MCauchy_suffix mcomplete_def* **by** *blast*
then have *limitin mtopology σ x sequentially*
unfolding *o_def* **by** (*auto simp*: *add.commute limitin_sequentially_offset_rev*)
then show ∃ *x*. *limitin mtopology σ x sequentially* **..**
qed
qed (*simp add*: *mcomplete_def MCauchy_def image_subset_iff*)

lemma *mcomplete_empty_mspace*: *M* = {} ⟹ *mcomplete*

using *MCauchy_def mcomplete_def* **by** *blast*

lemma *MCauchy_const* [*simp*]: *MCauchy* ($\lambda n.\ a$) $\longleftrightarrow$ $a \in M$
using *MCauchy_def mdist_zero* **by** *auto*

lemma *convergent_imp_MCauchy*:
assumes *range* $\sigma \subseteq M$ **and** *lim*: *limitin mtopology* σ *l sequentially*
shows *MCauchy* σ
unfolding *MCauchy_def image_subset_iff*
proof (*intro conjI strip*)
fix ε::*real*
assume $\varepsilon > 0$
then have $\forall_F\ n$ *in sequentially*. $\sigma\ n \in M \wedge d\ (\sigma\ n)\ l < \varepsilon/2$
using *half_gt_zero lim limitin_metric* **by** *blast*
then obtain N **where** $\bigwedge n.\ n \geq N \Longrightarrow \sigma\ n \in M \wedge d\ (\sigma\ n)\ l < \varepsilon/2$
by (*force simp*: *eventually_sequentially*)
then show $\exists N.\ \forall n\ n'.\ N \leq n \longrightarrow N \leq n' \longrightarrow d\ (\sigma\ n)\ (\sigma\ n') < \varepsilon$
by (*smt* (*verit*) *limitin_mspace mdist_reverse_triangle field_sum_of_halves lim*)
qed (*use assms* **in** *blast*)

lemma *mcomplete_alt*:
mcomplete $\longleftrightarrow$ ($\forall \sigma$. *MCauchy* $\sigma \longleftrightarrow$ *range* $\sigma \subseteq M \wedge$ ($\exists x$. *limitin mtopology* σ *x sequentially*))
using *MCauchy_def convergent_imp_MCauchy mcomplete_def* **by** *blast*

lemma *MCauchy_subsequence*:
assumes *strict_mono r MCauchy* σ
shows *MCauchy* ($\sigma \circ r$)
proof −
have $d\ (\sigma\ (r\ n))\ (\sigma\ (r\ n')) < \varepsilon$
if $N \leq n\ N \leq n'$ *strict_mono r* $\forall n\ n'.\ N \leq n \longrightarrow N \leq n' \longrightarrow d\ (\sigma\ n)\ (\sigma\ n') < \varepsilon$
for $\varepsilon\ N\ n\ n'$
using *that* **by** (*meson le_trans strict_mono_imp_increasing*)
moreover have *range* ($\lambda x.\ \sigma\ (r\ x)$) $\subseteq M$
using *MCauchy_def assms* **by** *blast*
ultimately show *?thesis*
using *assms* **by** (*simp add*: *MCauchy_def*) *metis*
qed

lemma *MCauchy_offset*:
assumes *cau*: *MCauchy* ($\sigma \circ (+)k$) **and** σ: $\bigwedge n.\ n < k \Longrightarrow \sigma\ n \in M$
shows *MCauchy* σ
unfolding *MCauchy_def image_subset_iff*
proof (*intro conjI strip*)
fix n
show $\sigma\ n \in M$

using *assms*
unfolding *MCauchy_def image_subset_iff*
by (*metis UNIV_I comp_apply le_iff_add linorder_not_le*)
next
fix ε :: *real*
assume $\varepsilon > 0$
obtain N **where** $\forall n\ n'.\ N \leq n \longrightarrow N \leq n' \longrightarrow d\ ((\sigma \circ (+)k)\ n)\ ((\sigma \circ (+)k)\ n') < \varepsilon$
using *cau* ‹$\varepsilon > 0$› **by** (*fastforce simp*: *MCauchy_def*)
then show $\exists N.\ \forall n\ n'.\ N \leq n \longrightarrow N \leq n' \longrightarrow d\ (\sigma\ n)\ (\sigma\ n') < \varepsilon$
unfolding *o_def*
by (*intro exI* [**where** *x*=*k*+*N*]) (*smt* (*verit*, *del_insts*) *add.assoc le_add1 less_eqE*)
qed

lemma *MCauchy_convergent_subsequence*:
assumes *cau*: *MCauchy* σ **and** *strict_mono r*
and *lim*: *limitin mtopology* $(\sigma \circ r)$ *a sequentially*
shows *limitin mtopology* σ *a sequentially*
unfolding *limitin_metric*
proof (*intro conjI strip*)
show $a \in M$
by (*meson assms limitin_mspace*)
fix ε :: *real*
assume $\varepsilon > 0$
then obtain *N1* **where** *N1*: $\bigwedge n\ n'.\ [\![n \geq N1;\ n' \geq N1]\!] \Longrightarrow d\ (\sigma\ n)\ (\sigma\ n') < \varepsilon/2$
using *cau* **unfolding** *MCauchy_def* **by** (*meson half_gt_zero*)
obtain *N2* **where** *N2*: $\bigwedge n.\ n \geq N2 \Longrightarrow (\sigma \circ r)\ n \in M \wedge d\ ((\sigma \circ r)\ n)\ a < \varepsilon/2$
by (*metis* (*no_types*, *lifting*) *lim* ‹$\varepsilon > 0$› *half_gt_zero limit_metric_sequentially*)
have $\sigma\ n \in M \wedge d\ (\sigma\ n)\ a < \varepsilon$ **if** $n \geq max\ N1\ N2$ **for** n
proof (*intro conjI*)
show $\sigma\ n \in M$
using *MCauchy_def cau* **by** *blast*
have $N1 \leq r\ n$
by (*meson* ‹*strict_mono r*› *le_trans max.cobounded1 strict_mono_imp_increasing that*)
then show $d\ (\sigma\ n)\ a < \varepsilon$
using *N1*[*of n r n*] *N2*[*of n*] ‹$\sigma\ n \in M$› ‹$a \in M$› *triangle that* **by** *fastforce*
qed
then show $\forall_F\ n$ *in sequentially*. $\sigma\ n \in M \wedge d\ (\sigma\ n)\ a < \varepsilon$
using *eventually_sequentially* **by** *blast*
qed

lemma *MCauchy_interleaving_gen*:
MCauchy $(\lambda n.$ *if even n then* $x(n$ *div* $2)$ *else* $y(n$ *div* $2))$ $\longleftrightarrow$
$(MCauchy\ x \wedge MCauchy\ y \wedge (\lambda n.\ d\ (x\ n)\ (y\ n)) \longrightarrow 0)$ (**is** *?lhs*=*?rhs*)
proof
assume *L*: *?lhs*
have *evens*: *strict_mono* $(\lambda n{::}nat.\ 2 * n)$ **and** *odds*: *strict_mono* $(\lambda n{::}nat.$ *Suc*

```
(2 * n))
    by (auto simp: strict_mono_def)
  show ?rhs
  proof (intro conjI)
    show MCauchy x MCauchy y
      using MCauchy_subsequence [OF evens L] MCauchy_subsequence [OF odds
L] by (auto simp: o_def)
    show (λn. d (x n) (y n)) ⟶ 0
      unfolding LIMSEQ_iff
    proof (intro strip)
      fix ε :: real
      assume ε > 0
      then obtain N where N:
        ⋀n n'. ⟦n≥N; n'≥N⟧ ⟹ d (if even n then x (n div 2) else y (n div 2))
                                  (if even n' then x (n' div 2) else y (n' div 2)) < ε
        using L MCauchy_def by fastforce
      have d (x n) (y n) < ε if n≥N for n
        using N [of 2*n Suc(2*n)] that by auto
      then show ∃N. ∀n≥N. norm (d (x n) (y n) − 0) < ε
        by auto
    qed
  qed
next
  assume R: ?rhs
  show ?lhs
    unfolding MCauchy_def
  proof (intro conjI strip)
    show range (λn. if even n then x (n div 2) else y (n div 2)) ⊆ M
      using R by (auto simp: MCauchy_def)
    fix ε :: real
    assume ε > 0
    obtain Nx where Nx: ⋀n n'. ⟦n≥Nx; n'≥Nx⟧ ⟹ d (x n) (x n') < ε/2
      by (meson half_gt_zero MCauchy_def R ‹ε > 0›)
    obtain Ny where Ny: ⋀n n'. ⟦n≥Ny; n'≥Ny⟧ ⟹ d (y n) (y n') < ε/2
      by (meson half_gt_zero MCauchy_def R ‹ε > 0›)
    obtain Nxy where Nxy: ⋀n. n≥Nxy ⟹ d (x n) (y n) < ε/2
      using R ‹ε > 0› half_gt_zero unfolding LIMSEQ_iff
      by (metis abs_mdist diff_zero real_norm_def)
    define N where N ≡ 2 * Max{Nx,Ny,Nxy}
    show ∃N. ∀n n'. N ≤ n ⟶ N ≤ n' ⟶ d (if even n then x (n div 2) else y
(n div 2)) (if even n' then x (n' div 2) else y (n' div 2)) < ε
    proof (intro exI strip)
      fix n n'
      assume N ≤ n and N ≤ n'
      then have n div 2 ≥ Nx n div 2 ≥ Ny n div 2 ≥ Nxy n' div 2 ≥ Nx n' div
2 ≥ Ny
        by (auto simp: N_def)
      then have dxyn: d (x (n div 2)) (y (n div 2)) < ε/2
            and dxnn': d (x (n div 2)) (x (n' div 2)) < ε/2
```

```
          and dynn': d (y (n div 2)) (y (n' div 2)) < ε/2
        using Nx Ny Nxy by blast+
      have inM: x (n div 2) ∈ M x (n' div 2) ∈ My (n div 2) ∈ M y (n' div 2) ∈
M
        using MCauchy_def R by blast+
      show d (if even n then x (n div 2) else y (n div 2)) (if even n' then x (n' div
2) else y (n' div 2)) < ε
      proof (cases even n)
        case nt: True
        show ?thesis
        proof (cases even n')
          case True
          with ‹ε > 0› nt dxnn' show ?thesis by auto
        next
          case False
          with nt dxyn dynn' inM triangle show ?thesis
            by fastforce
        qed
      next
        case nf: False
        show ?thesis
        proof (cases even n')
          case True
          then show ?thesis
       by (smt (verit) ‹ε > 0› dxyn dxnn' triangle commute inM field_sum_of_halves)
        next
          case False
          with ‹ε > 0› nf dynn' show ?thesis by auto
        qed
      qed
    qed
  qed
qed

lemma MCauchy_interleaving:
   MCauchy (λn. if even n then σ(n div 2) else a) ⟷
    range σ ⊆ M ∧ limitin mtopology σ a sequentially  (is ?lhs=?rhs)
proof −
  have ?lhs ⟷ (MCauchy σ ∧ a ∈ M ∧ (λn. d (σ n) a) ⟶ 0)
    by (simp add: MCauchy_interleaving_gen [where y = λn. a])
  also have ... = ?rhs
     by (metis MCauchy_def always_eventually convergent_imp_MCauchy lim-
itin_metric_dist_null range_subsetD)
  finally show ?thesis .
qed

lemma mcomplete_nest:
   mcomplete ⟷
      (∀ C::nat ⇒'a set. (∀ n. closedin mtopology (C n)) ∧
```

```
        (∀ n. C n ≠ {}) ∧ decseq C ∧ (∀ ε>0. ∃ n a. C n ⊆ mcball a ε)
        ⟶ ⋂ (range C) ≠ {}) (is ?lhs=?rhs)
proof
  assume L: ?lhs
  show ?rhs
    unfolding imp_conjL
  proof (intro strip)
    fix C :: nat ⇒ 'a set
    assume clo: ∀ n. closedin mtopology (C n)
      and ne: ∀ n. C n ≠ ({}::'a set)
      and dec: decseq C
      and cover [rule_format]: ∀ ε>0. ∃ n a. C n ⊆ mcball a ε
    obtain σ where σ: ⋀n. σ n ∈ C n
      by (meson ne empty_iff set_eq_iff)
    have MCauchy σ
      unfolding MCauchy_def
    proof (intro conjI strip)
      show range σ ⊆ M
        using σ clo metric_closedin_iff_sequentially_closed by auto
      fix ε :: real
      assume ε > 0
      then obtain N a where N: C N ⊆ mcball a (ε/3)
        using cover by fastforce
      have d (σ m) (σ n) < ε if N ≤ m N ≤ n for m n
      proof −
        have d a (σ m) ≤ ε/3 d a (σ n) ≤ ε/3
          using dec N σ that by (fastforce simp: decseq_def)+
        then have d (σ m) (σ n) ≤ ε/3 + ε/3
          using triangle σ commute dec decseq_def subsetD that N
          by (smt (verit, ccfv_threshold) in_mcball)
        also have ... < ε
          using ‹ε > 0› by auto
        finally show ?thesis .
      qed
      then show ∃ N. ∀ m n. N ≤ m ⟶ N ≤ n ⟶ d (σ m) (σ n) < ε
        by blast
    qed
    then obtain x where x: limitin mtopology σ x sequentially
      using L mcomplete_def by blast
    have x ∈ C n for n
    proof (rule limitin_closedin [OF x])
      show closedin mtopology (C n)
        by (simp add: clo)
      show ∀ F x in sequentially. σ x ∈ C n
        by (metis σ dec decseq_def eventually_sequentiallyI subsetD)
    qed auto
    then show ⋂ (range C) ≠ {}
      by blast
qed
```

next
assume *R*: *?rhs*
show *?lhs*
unfolding *mcomplete_def*
proof (*intro strip*)
fix σ
assume *MCauchy* σ
then have *range* $\sigma \subseteq M$
using *MCauchy_def* **by** *blast*
define *C* **where** $C \equiv \lambda n.$ *mtopology closure_of* (σ ‘ {*n*..})
have $\forall n.$ *closedin mtopology* (*C n*)
by (*auto simp*: *C_def*)
moreover
have *ne*: $\bigwedge n.$ *C n* $\neq$ {}
using ‹*MCauchy* σ› **by** (*auto simp*: *C_def MCauchy_def disjnt_iff closure_of_eq_empty_gen*)
moreover
have *dec*: *decseq C*
unfolding *monotone_on_def*
proof (*intro strip*)
fix *m n*::*nat*
assume $m \leq n$
then have {*n*..} $\subseteq$ {*m*..}
by *auto*
then show *C n* $\subseteq$ *C m*
unfolding *C_def* **by** (*meson closure_of_mono image_mono*)
qed
moreover
have *C*: $\exists N\ u.$ *C N* $\subseteq$ *mcball u* ε **if** ε>*0* **for** ε
proof −
obtain *N* **where** $\bigwedge m\ n.\ N \leq m \wedge N \leq n \Longrightarrow d\ (\sigma\ m)\ (\sigma\ n) < \varepsilon$
by (*meson MCauchy_def* ‹*0* < ε› ‹*MCauchy* σ›)
then have σ ‘ {*N*..} $\subseteq$ *mcball* (σ *N*) ε
using *MCauchy_def* ‹*MCauchy* σ› **by** (*force simp*: *less_eq_real_def*)
then have *C N* $\subseteq$ *mcball* (σ *N*) ε
by (*simp add*: *C_def closure_of_minimal*)
then show *?thesis*
by *blast*
qed
ultimately obtain *l* **where** *x*: $l \in \bigcap$ (*range C*)
by (*metis R ex_in_conv*)
then have $*$: $\bigwedge\varepsilon\ N.\ 0 < \varepsilon \Longrightarrow \exists n'.\ N \leq n' \wedge l \in M \wedge \sigma\ n' \in M \wedge d\ l\ (\sigma\ n') < \varepsilon$
by (*force simp*: *C_def metric_closure_of*)
then have $l \in M$
using *gt_ex* **by** *blast*
show $\exists l.$ *limitin mtopology* σ *l sequentially*
unfolding *limitin_metric*
proof (*intro conjI strip exI*)

```
    show l ∈ M
      using ‹∀n. closedin mtopology (C n)› closedin_subset x by fastforce
    fix ε::real
    assume ε > 0
    obtain N where N: ⋀m n. N ≤ m ∧ N ≤ n ⟹ d (σ m) (σ n) < ε/2
      by (meson MCauchy_def ‹0 < ε› ‹MCauchy σ› half_gt_zero)
    with ∗ [of ε/2 N]
    have ∀n≥N. σ n ∈ M ∧ d (σ n) l < ε
    by (smt (verit) ‹range σ ⊆ M› commute field_sum_of_halves range_subsetD
triangle)
    then show ∀F n in sequentially. σ n ∈ M ∧ d (σ n) l < ε
      using eventually_sequentially by blast
   qed
  qed
qed


lemma mcomplete_nest_sing:
   mcomplete ⟷
    (∀C. (∀n. closedin mtopology (C n)) ∧
          (∀n. C n ≠ {}) ∧ decseq C ∧ (∀e>0. ∃n a. C n ⊆ mcball a e)
         ⟶ (∃l. l ∈ M ∧ ⋂ (range C) = {l}))
proof −
  have ∗: False
    if clo: ∀n. closedin mtopology (C n)
      and cover: ∀ε>0. ∃n a. C n ⊆ mcball a ε
      and no_sing: ⋀y. y ∈ M ⟹ ⋂ (range C) ≠ {y}
      and l: ∀n. l ∈ C n
    for C :: nat ⇒ 'a set and l
  proof −
    have inM: ⋀x. x ∈ ⋂ (range C) ⟹ x ∈ M
      using closedin_metric clo by fastforce
    then have l ∈ M
      by (simp add: l)
    have False if l': l' ∈ ⋂ (range C) and l' ≠ l for l'
    proof −
      have l' ∈ M
        using inM l' by blast
      obtain n a where na: C n ⊆ mcball a (d l l' / 3)
        using inM ‹l ∈ M› l' ‹l' ≠ l› cover by force
      then have d a l ≤ (d l l' / 3) d a l' ≤ (d l l' / 3) a ∈ M
        using l l' na in_mcball by auto
      then have d l l' ≤ (d l l' / 3) + (d l l' / 3)
        using ‹l ∈ M› ‹l' ∈ M› mdist_reverse_triangle by fastforce
      then show False
        using nonneg [of l l'] ‹l' ≠ l› ‹l ∈ M› ‹l' ∈ M› zero by force
    qed
    then show False
        by (metis l ‹l ∈ M› no_sing INT_I empty_iff insertI1 is_singletonE
```

```
is_singletonI')
  qed
  show ?thesis
    unfolding mcomplete_nest imp_conjL
    apply (intro all_cong1 imp_cong refl)
    using ∗
    by (smt (verit) Inter_iff ex_in_conv range_constant range_eqI)
qed

lemma mcomplete_fip:
   mcomplete ⟷
    (∀𝒞. (∀ C ∈ 𝒞. closedin mtopology C) ∧
        (∀ e>0. ∃ C a. C ∈ 𝒞 ∧ C ⊆ mcball a e) ∧ (∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒞 ⟶ ⋂
ℱ ≠ {})
        ⟶ ⋂ 𝒞 ≠ {})
   (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  show ?rhs
    unfolding mcomplete_nest_sing imp_conjL
  proof (intro strip)
    fix 𝒞 :: 'a set set
    assume clo: ∀ C∈𝒞. closedin mtopology C
      and cover: ∀ e>0. ∃ C a. C ∈ 𝒞 ∧ C ⊆ mcball a e
      and fip: ∀ℱ. finite ℱ ⟶ ℱ ⊆ 𝒞 ⟶ ⋂ ℱ ≠ {}
    then have ∀ n. ∃ C. C ∈ 𝒞 ∧ (∃ a. C ⊆ mcball a (inverse (Suc n)))
      by simp
    then obtain C where C: ⋀n. C n ∈ 𝒞
        and coverC: ⋀n. ∃ a. C n ⊆ mcball a (inverse (Suc n))
      by metis
    define D where D ≡ λn. ⋂ (C ‘ {..n})
    have cloD: closedin mtopology (D n) for n
      unfolding D_def using clo C by blast
    have neD: D n ≠ {} for n
      using fip C by (simp add: D_def image_subset_iff)
    have decD: decseq D
      by (force simp: D_def decseq_def)
    have coverD: ∃ n a. D n ⊆ mcball a ε if ε >0 for ε
    proof −
      obtain n where inverse (Suc n) < ε
        using ‹0 < ε› reals_Archimedean by blast
      then obtain a where C n ⊆ mcball a ε
       by (meson coverC less_eq_real_def mcball_subset_concentric order_trans)
      then show ?thesis
       unfolding D_def by blast
    qed
    have ∗: a ∈ ⋂𝒞 if a: ⋂ (range D) = {a} and a ∈ M for a
    proof −
      have aC: a ∈ C n for n
```

```
      using that by (auto simp: D_def)
    have eqa: ⋀u. (∀ n. u ∈ C n) ⟹ a = u
      using that by (auto simp: D_def)
    have a ∈ T if T ∈ 𝒞 for T
    proof −
      have cloT: closedin mtopology (T ∩ D n) for n
        using clo cloD that by blast
      have ⋂ (insert T (C ` {..n})) ≠ {} for n
        using that C by (intro fip [rule_format]) auto
      then have neT: T ∩ D n ≠ {} for n
        by (simp add: D_def)
      have decT: decseq (λn. T ∩ D n)
        by (force simp: D_def decseq_def)
      have coverT: ∃ n a. T ∩ D n ⊆ mcball a ε if ε >0 for ε
        by (meson coverD le_infI2 that)
      show ?thesis
        using L [unfolded mcomplete_nest_sing, rule_format, of λn. T ∩ D n] a
        by (force simp: cloT neT decT coverT)
    qed
    then show ?thesis by auto
  qed
  show ⋂ 𝒞 ≠ {}
    by (metis L cloD neD decD coverD ∗ empty_iff mcomplete_nest_sing)
 qed
next
 assume R [rule_format]: ?rhs
 show ?lhs
  unfolding mcomplete_nest imp_conjL
 proof (intro strip)
  fix C :: nat ⇒ 'a set
  assume clo: ∀ n. closedin mtopology (C n)
    and ne: ∀ n. C n ≠ {}
    and dec: decseq C
    and cover: ∀ε>0. ∃ n a. C n ⊆ mcball a ε
  have ⋂(C ` N) ≠ {} if finite N for N
  proof −
    obtain k where N ⊆ {..k}
      using ‹finite N› finite_nat_iff_bounded_le by auto
    with dec have C k ⊆ ⋂(C ` N) by (auto simp: decseq_def)
    then show ?thesis
      using ne by force
  qed
  with clo cover R [of range C] show ⋂ (range C) ≠ {}
   by (metis (no_types, opaque_lifting) finite_subset_image image_iff UNIV_I)
 qed
qed


lemma mcomplete_fip_sing:
```

```
   mcomplete ⟷
    (∀𝒞. (∀ C∈𝒞. closedin mtopology C) ∧
     (∀ e>0. ∃ c a. c ∈ 𝒞 ∧ c ⊆ mcball a e) ∧
     (∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒞 ⟶ ⋂ ℱ ≠ {}) ⟶
     (∃ l. l ∈ M ∧ ⋂ 𝒞 = {l}))
   (is ?lhs = ?rhs)
proof
  have ∗: l ∈ M ⋂ 𝒞 = {l}
    if clo: Ball 𝒞 (closedin mtopology)
      and cover: ∀ e>0. ∃ C a. C ∈ 𝒞 ∧ C ⊆ mcball a e
      and fin: ∀ℱ. finite ℱ ⟶ ℱ ⊆ 𝒞 ⟶ ⋂ ℱ ≠ {}
      and l: l ∈ ⋂ 𝒞
    for 𝒞 :: 'a set set and l
  proof −
    show l ∈ M
     by (meson Inf_lower2 clo cover gt_ex metric_closedin_iff_sequentially_closed
subsetD that(4))
    show ⋂ 𝒞 = {l}
    proof (cases 𝒞 = {})
      case True
      then show ?thesis
        using cover mbounded_pos by auto
    next
      case False
      have CM: ⋀a. a ∈ ⋂𝒞 ⟹ a ∈ M
        using False clo closedin_subset by fastforce
      have l' ∉ ⋂ 𝒞 if l' ≠ l for l'
      proof
        assume l': l' ∈ ⋂ 𝒞
        with CM have l' ∈ M by blast
        with that ‹l ∈ M› have gt0: 0 < d l l'
          by simp
        then obtain C a where C ∈ 𝒞 and C: C ⊆ mcball a (d l l' / 3)
          using cover [rule_format, of d l l' / 3] by auto
        then have d a l ≤ (d l l' / 3) d a l' ≤ (d l l' / 3) a ∈ M
          using l l' in_mcball by auto
        then have d l l' ≤ (d l l' / 3) + (d l l' / 3)
          using ‹l ∈ M› ‹l' ∈ M› mdist_reverse_triangle by fastforce
        with gt0 show False by auto
      qed
      then show ?thesis
        using l by fastforce
    qed
  qed
  assume L: ?lhs
  with ∗ show ?rhs
    unfolding mcomplete_fip imp_conjL ex_in_conv [symmetric]
    by (elim all_forward imp_forward2 asm_rl) (blast intro: elim: )
next
```

assume *?rhs* **then show** *?lhs*
unfolding *mcomplete_fip* **by** (*force elim*!: *all_forward*)
qed

end

definition *mcomplete_of* :: *'a metric* ⇒ *bool*
where *mcomplete_of* ≡ λ*m*. *Metric_space.mcomplete* (*mspace m*) (*mdist m*)

lemma (**in** *Metric_space*) *mcomplete_of* [*simp*]: *mcomplete_of* (*metric* (*M*,*d*)) = *mcomplete*
by (*simp add*: *mcomplete_of_def*)

lemma *mcomplete_trivial*: *Metric_space.mcomplete* {} (λ*x y*. *0*)
using *Metric_space.intro Metric_space.mcomplete_empty_mspace* **by** *force*

lemma *mcomplete_trivial_singleton*: *Metric_space.mcomplete* {λ*x*. *a*} (λ*x y*. *0*)
proof −
interpret *Metric_space* {λ*x*. *a*} λ*x y*. *0*
by *unfold_locales auto*
show *?thesis*
unfolding *mcomplete_def MCauchy_def image_subset_iff* **by** (*metis UNIV_I limit_metric_sequentially*)
qed

lemma *MCauchy_iff_Cauchy* [*iff*]: *Met_TC.MCauchy* = *Cauchy*
by (*force simp*: *Cauchy_def Met_TC.MCauchy_def*)

lemma *mcomplete_iff_complete* [*iff*]:
Met_TC.mcomplete TYPE('*a*::*metric_space*) ⟷ *complete* (*UNIV*::'*a set*)
by (*auto simp*: *Met_TC.mcomplete_def complete_def*)

context *Submetric*
begin

lemma *MCauchy_submetric*:
sub.MCauchy σ ⟷ *range* σ ⊆ *A* ∧ *MCauchy* σ
using *MCauchy_def sub.MCauchy_def subset* **by** *force*

lemma *closedin_mcomplete_imp_mcomplete*:
assumes *clo*: *closedin mtopology A* **and** *mcomplete*
shows *sub.mcomplete*
unfolding *sub.mcomplete_def*
proof (*intro strip*)
fix σ
assume *sub.MCauchy* σ
then have σ: *MCauchy* σ *range* σ ⊆ *A*
using *MCauchy_submetric* **by** *blast*+
then obtain *x* **where** *x*: *limitin mtopology* σ *x sequentially*

using ‹*mcomplete*› **unfolding** *mcomplete_def* **by** *blast*
then have $x \in A$
using σ *clo metric_closedin_iff_sequentially_closed* **by** *force*
with σ x **show** $\exists x$. *limitin sub.mtopology* σ x *sequentially*
using *limitin_submetric_iff range_subsetD* **by** *fastforce*
qed

lemma *sequentially_closedin_mcomplete_imp_mcomplete*:
assumes *mcomplete* **and** $\bigwedge\sigma$ *l. range* $\sigma \subseteq A \wedge$ *limitin mtopology* σ *l sequentially*
$\Longrightarrow l \in A$
shows *sub.mcomplete*
using *assms closedin_mcomplete_imp_mcomplete metric_closedin_iff_sequentially_closed subset* **by** *blast*

end

context *Metric_space*
begin

lemma *mcomplete_Un*:
assumes *A*: *Submetric M d A Metric_space.mcomplete A d*
and *B*: *Submetric M d B Metric_space.mcomplete B d*
shows *Submetric M d* $(A \cup B)$ *Metric_space.mcomplete* $(A \cup B)$ *d*
proof −
show *Submetric M d* $(A \cup B)$
by (*meson assms le_sup_iff Submetric_axioms_def Submetric_def*)
then interpret *MAB*: *Metric_space* $A \cup B$ *d*
by (*meson Submetric.subset subspace*)
interpret *MA*: *Metric_space A d*
by (*meson A Submetric.subset subspace*)
interpret *MB*: *Metric_space B d*
by (*meson B Submetric.subset subspace*)
show *Metric_space.mcomplete* $(A \cup B)$ *d*
unfolding *MAB.mcomplete_def*
proof (*intro strip*)
fix σ
assume *MAB.MCauchy* σ
then have *range* $\sigma \subseteq A \cup B$
using *MAB.MCauchy_def* **by** *blast*
then have *UNIV* $\subseteq \sigma -` A \cup \sigma -` B$
by *blast*
then consider *infinite* $(\sigma -` A) \mid$ *infinite* $(\sigma -` B)$
using *finite_subset* **by** *auto*
then show $\exists x$. *limitin MAB.mtopology* σ x *sequentially*
proof *cases*
case *1*
then obtain r **where** *strict_mono r* **and** r: $\bigwedge n$::*nat*. $r\ n \in \sigma -` A$
using *infinite_enumerate* **by** *blast*

```
      then have MA.MCauchy (σ ∘ r)
        using MA.MCauchy_def MAB.MCauchy_def MAB.MCauchy_subsequence
‹MAB.MCauchy σ› by auto
      with A obtain x where limitin MA.mtopology (σ ∘ r) x sequentially
        using MA.mcomplete_def by blast
      then have limitin MAB.mtopology (σ ∘ r) x sequentially
         by (metis MA.limit_metric_sequentially MAB.limit_metric_sequentially
UnCI)
      then show ?thesis
      using MAB.MCauchy_convergent_subsequence ‹MAB.MCauchy σ› ‹strict_mono
r› by blast
    next
      case 2
      then obtain r where strict_mono r and r: ⋀n::nat. r n ∈ σ −‘ B
        using infinite_enumerate by blast
      then have MB.MCauchy (σ ∘ r)
        using MB.MCauchy_def MAB.MCauchy_def MAB.MCauchy_subsequence
‹MAB.MCauchy σ› by auto
      with B obtain x where limitin MB.mtopology (σ ∘ r) x sequentially
        using MB.mcomplete_def by blast
      then have limitin MAB.mtopology (σ ∘ r) x sequentially
         by (metis MB.limit_metric_sequentially MAB.limit_metric_sequentially
UnCI)
      then show ?thesis
      using MAB.MCauchy_convergent_subsequence ‹MAB.MCauchy σ› ‹strict_mono
r› by blast
    qed
  qed
qed

lemma mcomplete_Union:
  assumes finite S
    and ⋀A. A ∈ S ⟹ Submetric M d A ⋀A. A ∈ S ⟹ Metric_space.mcomplete
A d
  shows   Submetric M d (⋃S) Metric_space.mcomplete (⋃S) d
  using assms
  by (induction rule: finite_induct) (auto simp: mcomplete_Un)

lemma mcomplete_Inter:
  assumes finite S S ≠ {}
    and sub: ⋀A. A ∈ S ⟹ Submetric M d A
    and comp: ⋀A. A ∈ S ⟹ Metric_space.mcomplete A d
  shows Submetric M d (⋂S) Metric_space.mcomplete (⋂S) d
proof −
  show Submetric M d (⋂S)
    using assms unfolding Submetric_def Submetric_axioms_def
    by (metis Inter_lower equals0I inf.orderE le_inf_iff)
  then interpret MS: Submetric M d ⋂S
    by (meson Submetric.subset subspace)
```

show *Metric_space.mcomplete* ($\bigcap \mathcal{S}$) *d*
unfolding *MS.sub.mcomplete_def*
proof (*intro strip*)
fix σ
assume *MS.sub.MCauchy* σ
then have *range* $\sigma \subseteq \bigcap \mathcal{S}$
using *MS.MCauchy_submetric* **by** *blast*
obtain *A* **where** $A \in \mathcal{S}$ **and** *A*: *Metric_space.mcomplete A d*
using *assms* **by** *blast*
then have *range* $\sigma \subseteq A$
using ‹*range* $\sigma \subseteq \bigcap \mathcal{S}$› **by** *blast*
interpret *SA*: *Submetric M d A*
by (*meson* ‹$A \in \mathcal{S}$› *sub Submetric.subset subspace*)
have *MCauchy* σ
using *MS.MCauchy_submetric* ‹*MS.sub.MCauchy* σ› **by** *blast*
then obtain *x* **where** *x*: *limitin SA.sub.mtopology* σ *x sequentially*
by (*metis A SA.sub.MCauchy_def SA.sub.mcomplete_alt MCauchy_def* ‹*range* $\sigma \subseteq A$›)
show $\exists x$. *limitin MS.sub.mtopology* σ *x sequentially*
unfolding *MS.limitin_submetric_iff*
proof (*intro exI conjI*)
show $x \in \bigcap\ \mathcal{S}$
proof *clarsimp*
fix *U*
assume $U \in \mathcal{S}$
interpret *SU*: *Submetric M d U*
by (*meson* ‹$U \in \mathcal{S}$› *sub Submetric.subset subspace*)
have *range* $\sigma \subseteq U$
using ‹$U \in \mathcal{S}$› ‹*range* $\sigma \subseteq \bigcap\ \mathcal{S}$› **by** *blast*
moreover have *Metric_space.mcomplete U d*
by (*simp add*: ‹$U \in \mathcal{S}$› *comp*)
ultimately obtain x' **where** x': *limitin SU.sub.mtopology* σ x' *sequentially*
using *MCauchy_def SU.sub.MCauchy_def SU.sub.mcomplete_alt* ‹*MCauchy* σ› **by** *meson*
have $x' = x$
proof (*intro limitin_metric_unique*)
show *limitin mtopology* σ x' *sequentially*
by (*meson SU.Submetric_axioms Submetric.limitin_submetric_iff* x')
show *limitin mtopology* σ *x sequentially*
by (*meson SA.Submetric_axioms Submetric.limitin_submetric_iff x*)
qed *auto*
then show $x \in U$
using *SU.sub.limitin_mspace* x' **by** *blast*
qed
show $\forall_F\ n$ *in sequentially*. $\sigma\ n \in \bigcap \mathcal{S}$
by (*meson* ‹*range* $\sigma \subseteq \bigcap\ \mathcal{S}$› *always_eventually range_subsetD*)
show *limitin mtopology* σ *x sequentially*
by (*meson SA.Submetric_axioms Submetric.limitin_submetric_iff x*)
qed

qed
qed

lemma *mcomplete_Int*:
assumes *A*: *Submetric M d A Metric_space.mcomplete A d*
and *B*: *Submetric M d B Metric_space.mcomplete B d*
shows *Submetric M d* $(A \cap B)$ *Metric_space.mcomplete* $(A \cap B)$ *d*
using *mcomplete_Inter* [*of* $\{A,B\}$] *assms* **by** *force+*

6.7.11 Totally bounded subsets of metric spaces

definition *mtotally_bounded*
where *mtotally_bounded S* $\equiv \forall \varepsilon{>}0.\ \exists K.$ *finite K* $\wedge$ *K* $\subseteq$ *S* $\wedge$ *S* $\subseteq (\bigcup x{\in}K.$ *mball x* $\varepsilon)$

lemma *mtotally_bounded_empty* [*iff*]: *mtotally_bounded* $\{\}$
by (*simp add*: *mtotally_bounded_def*)

lemma *finite_imp_mtotally_bounded*:
$[\![$*finite S*; *S* $\subseteq$ *M*$]\!] \Longrightarrow$ *mtotally_bounded S*
by (*auto simp*: *mtotally_bounded_def*)

lemma *mtotally_bounded_imp_subset*: *mtotally_bounded S* $\Longrightarrow$ *S* $\subseteq$ *M*
by (*force simp*: *mtotally_bounded_def intro*!: *zero_less_one*)

lemma *mtotally_bounded_sing* [*simp*]:
mtotally_bounded $\{x\} \longleftrightarrow x \in M$
by (*meson empty_subsetI finite.simps finite_imp_mtotally_bounded insert_subset mtotally_bounded_imp_subset*)

lemma *mtotally_bounded_Un*:
assumes *mtotally_bounded S mtotally_bounded T*
shows *mtotally_bounded* $(S \cup T)$
proof −
have $\exists K.$ *finite K* $\wedge$ *K* $\subseteq$ *S* $\cup$ *T* $\wedge$ *S* $\cup$ *T* $\subseteq (\bigcup x{\in}K.$ *mball x e*$)$
if *e*$>$*0* **and** *K*: *finite K* $\wedge$ *K* $\subseteq$ *S* $\wedge$ *S* $\subseteq (\bigcup x{\in}K.$ *mball x e*$)$
and *L*: *finite L* $\wedge$ *L* $\subseteq$ *T* $\wedge$ *T* $\subseteq (\bigcup x{\in}L.$ *mball x e*$)$ **for** *K L e*
using *that* **by** (*rule_tac x*=*K* $\cup$ *L* **in** *exI*) *auto*
with *assms* **show** *?thesis*
unfolding *mtotally_bounded_def* **by** *presburger*
qed

lemma *mtotally_bounded_Union*:
assumes *finite f* $\bigwedge S.$ *S* $\in$ *f* $\Longrightarrow$ *mtotally_bounded S*
shows *mtotally_bounded* $(\bigcup f)$
using *assms* **by** (*induction f*) (*auto simp*: *mtotally_bounded_Un*)

lemma *mtotally_bounded_imp_mbounded*:

assumes *mtotally_bounded S*
shows *mbounded S*
proof −
obtain *K* **where** *finite K* ∧ *K* ⊆ *S* ∧ *S* ⊆ (⋃*x*∈*K*. *mball x 1*)
using *assms* **by** (*force simp*: *mtotally_bounded_def*)
then show *?thesis*
by (*smt* (*verit*) *finite_imageI image_iff mbounded_Union mbounded_mball mbounded_subset*)
qed

lemma *mtotally_bounded_sequentially*:
mtotally_bounded S ⟷
S ⊆ *M* ∧ (∀ *σ*::*nat* ⇒ *'a*. *range σ* ⊆ *S* ⟶ (∃ *r*. *strict_mono r* ∧ *MCauchy* (*σ* ∘ *r*)))
(**is** _ ⟷ _ ∧ *?rhs*)
proof (*cases S* ⊆ *M*)
case *True*
show *?thesis*
proof −
{ fix *σ* :: *nat* ⇒ *'a*
assume *L*: *mtotally_bounded S* **and** *σ*: *range σ* ⊆ *S*
have ∃ *j* > *i*. *d* (*σ i*) (*σ j*) < *3*∗*ε*/*2* ∧ *infinite* (*σ* −‘ *mball* (*σ j*) (*ε*/*2*))
if *inf*: *infinite* (*σ* −‘ *mball* (*σ i*) *ε*) **and** *ε* > *0* **for** *i ε*
proof −
obtain *K* **where** *finite K K* ⊆ *S* **and** *K*: *S* ⊆ (⋃*x*∈*K*. *mball x* (*ε*/*4*))
by (*metis L mtotally_bounded_def* ‹*ε* > *0*› *zero_less_divide_iff zero_less_numeral*)
then have *K_imp_ex*: ⋀*y*. *y* ∈ *S* ⟹ ∃ *x*∈*K*. *d x y* < *ε*/*4*
by *fastforce*
have *False* **if** ∀ *x*∈*K*. *d x* (*σ i*) < *ε* + *ε*/*4* ⟶ *finite* (*σ* −‘ *mball x* (*ε*/*4*))
proof −
have ∃ *w*. *w* ∈ *K* ∧ *d w* (*σ i*) < *5* ∗ *ε*/*4* ∧ *d w* (*σ j*) < *ε*/*4*
if *d* (*σ i*) (*σ j*) < *ε* **for** *j*
proof −
obtain *w* **where** *w*: *d w* (*σ j*) < *ε*/*4 w* ∈ *K*
using *K_imp_ex σ* **by** *blast*
then have *d w* (*σ i*) < *ε* + *ε*/*4*
by (*smt* (*verit*, *ccfv_SIG*) *True* ‹*K* ⊆ *S*› *σ rangeI subset_eq that triangle'*)
with *w* **show** *?thesis*
using *in_mball* **by** *auto*
qed
then have (*σ* −‘ *mball* (*σ i*) *ε*) ⊆ (⋃*x*∈*K*. *if d x* (*σ i*) < *ε* + *ε*/*4 then σ* −‘ *mball x* (*ε*/*4*) *else* {})
using *True* ‹*K* ⊆ *S*› **by** *force*
then show *False*
using *finite_subset inf* ‹*finite K*› *that* **by** *fastforce*
qed
then obtain *x* **where** *x* ∈ *K* **and** *dxi*: *d x* (*σ i*) < *ε* + *ε*/*4* **and** *infx*:

infinite (σ $-$‘ *mball x* (ε/*4*))
by *blast*
then obtain *j* **where** $j \in (\sigma$ $-$‘ *mball x* (ε/*4*)) $-$ {..*i*}
using *bounded_nat_set_is_finite* **by** (*meson Diff_infinite_finite finite_atMost*)
then have $j > i$ **and** *dxj*: *d x* (σ *j*) $< \varepsilon/4$
by *auto*
have (σ $-$‘ *mball x* (ε/*4*)) $\subseteq$ (σ $-$‘ *mball y* (ε/*2*)) **if** *d x y* $< \varepsilon/4$ $y \in M$ **for** *y*
using *that* **by** (*simp add*: *mball_subset vimage_mono*)
then have *infj*: *infinite* (σ $-$‘ *mball* (σ *j*) (ε/*2*))
by (*meson True* ‹*d x* (σ *j*) $< \varepsilon/4$› σ *in_mono infx rangeI finite_subset*)
have σ *i* $\in$ *M* σ *j* $\in$ *M x* $\in$ *M*
using *True* ‹*K* $\subseteq$ *S*› ‹*x* $\in$ *K*› σ **by** *force+*
then have *d* (σ *i*) (σ *j*) $\leq$ *d x* (σ *i*) + *d x* (σ *j*)
using *triangle''* **by** *blast*
also have ... $<$ *3*$*$$\varepsilon$/*2*
using *dxi dxj* **by** *auto*
finally have *d* (σ *i*) (σ *j*) $<$ *3*$*$$\varepsilon$/*2* **.**
with ‹*i* $<$ *j*› *infj* **show** *?thesis* **by** *blast*
qed
then obtain *nxt* **where** *nxt*: $\bigwedge i\ \varepsilon.\ [\![\varepsilon > 0;$ *infinite* (σ $-$‘ *mball* (σ *i*) ε)$]\!] \Longrightarrow$
nxt i ε $>$ *i* $\wedge$ *d* (σ *i*) (σ (*nxt i* ε)) $<$ *3*$*$$\varepsilon$/*2* $\wedge$ *infinite* (σ $-$‘ *mball* (σ (*nxt i* ε)) (ε/*2*))
by *metis*
have *mbounded S*
using *L* **by** (*simp add*: *mtotally_bounded_imp_mbounded*)
then obtain *B* **where** *B*: $\forall$ *y* $\in$ *S*. *d* (σ *0*) *y* $\leq$ *B* **and** *B* $>$ *0*
by (*meson* σ *mbounded_alt_pos range_subsetD*)
define *eps* **where** *eps* $\equiv$ λ*n*. (*B*+*1*) / *2*^*n*
have [*simp*]: *eps* (*Suc n*) = *eps n* / *2* *eps n* $>$ *0* **for** *n*
using ‹*B* $>$ *0*› **by** (*auto simp*: *eps_def*)
have *UNIV* $\subseteq$ σ $-$‘ *mball* (σ *0*) (*B*+*1*)
using *B True* σ **unfolding** *image_iff subset_iff*
by (*smt* (*verit*, *best*) *UNIV_I in_mball vimageI*)
then have *inf0*: *infinite* (σ $-$‘ *mball* (σ *0*) (*eps 0*))
using *finite_subset* **by** (*auto simp*: *eps_def*)
define *r* **where** *r* $\equiv$ *rec_nat 0* (λ*n rec*. *nxt rec* (*eps n*))
have [*simp*]: *r 0* = *0 r* (*Suc n*) = *nxt* (*r n*) (*eps n*) **for** *n*
by (*auto simp*: *r_def*)
have σ*rM*[*simp*]: σ (*r n*) $\in$ *M* **for** *n*
using *True* σ **by** *blast*
have *inf*: *infinite* (σ $-$‘ *mball* (σ (*r n*)) (*eps n*)) **for** *n*
proof (*induction n*)
case *0* **then show** *?case*
by (*simp add*: *inf0*)
next
case (*Suc n*) **then show** *?case*

```
      using nxt [of eps n r n] by simp
    qed
    then have r (Suc n) > r n for n
      by (simp add: nxt)
    then have strict_mono r
      by (simp add: strict_mono_Suc_iff)
    have d_less: d (σ (r n)) (σ (r (Suc n))) < 3 * eps n / 2 for n
      using nxt [OF _ inf] by simp
    have eps_plus: eps (k + n) = eps n * (1/2)^k for k n
      by (simp add: eps_def power_add field_simps)
    have *: d (σ (r n)) (σ (r (k + n))) < 3 * eps n for n k
    proof −
      have d (σ (r n)) (σ (r (k+n))) ≤ 3/2 * eps n * (∑ i<k. (1/2)^i)
      proof (induction k)
        case 0 then show ?case
          by simp
      next
        case (Suc k)
        have d (σ (r n)) (σ (r (Suc k + n))) ≤ d (σ (r n)) (σ (r (k + n))) + d
(σ (r (k + n))) (σ (r (Suc (k + n))))
          by (metis σrM add.commute add_Suc_right triangle)
        with d_less[of k+n] Suc show ?case
          by (simp add: algebra_simps eps_plus)
      qed
      also have ... < 3/2 * eps n * 2
        using geometric_sum [of 1/2::real k] by simp
      finally show ?thesis by simp
    qed
    have ∃ N. ∀ n≥N. ∀ n'≥N. d (σ (r n)) (σ (r n')) < ε if ε > 0 for ε
    proof −
      define N where N ≡ nat ⌈(log 2 (6*(B+1) / ε))⌉
      have §: b ≤ 2 ^ nat ⌈log 2 b⌉ for b
        by (smt (verit) less_log_of_power real_nat_ceiling_ge)
      have N: 6 * eps N ≤ ε
      using § [of (6*(B+1) / ε)] that by (auto simp: N_def eps_def field_simps)
      have d (σ (r N)) (σ (r n)) < 3 * eps N if n ≥ N for n
        by (metis * add.commute nat_le_iff_add that)
     then have ∀ n≥N. ∀ n'≥N. d (σ (r n)) (σ (r n')) < 3 * eps N + 3 * eps N
        by (smt (verit, best) σrM triangle'')
      with N show ?thesis
        by fastforce
    qed
    then have MCauchy (σ ∘ r)
      unfolding MCauchy_def using True σ by auto
    then have ∃ r. strict_mono r ∧ MCauchy (σ ∘ r)
      using ‹strict_mono r› by blast
  }
  moreover
  { assume R: ?rhs
```

```
      have mtotally_bounded S
        unfolding mtotally_bounded_def
      proof (intro strip)
        fix ε :: real
        assume ε > 0
       have False if §: ⋀K. ⟦finite K; K ⊆ S⟧ ⟹ ∃s∈S. s ∉ (⋃x∈K. mball x ε)
        proof −
         obtain f where f: ⋀K. ⟦finite K; K ⊆ S⟧ ⟹ f K ∈ S ∧ f K ∉ (⋃x∈K.
mball x ε)
           using § by metis
         define σ where σ ≡ wfrec less_than (λseq n. f (seq ‘ {..<n}))
         have σ_eq: σ n = f (σ ‘ {..<n}) for n
           by (simp add: cut_apply def_wfrec [OF σ_def])
         have [simp]: σ n ∈ S for n
           using wf_less_than
         proof (induction n rule: wf_induct_rule)
           case (less n) with f show ?case
             by (auto simp: σ_eq [of n])
         qed
         then have range σ ⊆ S by blast
         have σ: p < n ⟹ ε ≤ d (σ p) (σ n) for n p
           using f[of σ ‘ {..<n}] True by (fastforce simp: σ_eq [of n] Ball_def)
         then obtain r where strict_mono r MCauchy (σ ∘ r)
           by (meson R ‹range σ ⊆ S›)
         with ‹0 < ε› obtain N
           where N: ⋀n n′. ⟦n≥N; n′≥N⟧ ⟹ d (σ (r n)) (σ (r n′)) < ε
           by (force simp: MCauchy_def)
         show ?thesis
           using N [of N Suc (r N)] ‹strict_mono r›
        by (smt (verit) Suc_le_eq σ le_SucI order_refl strict_mono_imp_increasing)
        qed
        then show ∃K. finite K ∧ K ⊆ S ∧ S ⊆ (⋃x∈K. mball x ε)
          by blast
      qed
    }
    ultimately show ?thesis
      using True by blast
  qed
qed (use mtotally_bounded_imp_subset in auto)


lemma mtotally_bounded_subset:
  ⟦mtotally_bounded S; T ⊆ S⟧ ⟹ mtotally_bounded T
  by (meson mtotally_bounded_sequentially order_trans)

lemma mtotally_bounded_submetric:
  assumes mtotally_bounded S S ⊆ T T ⊆ M
  shows Metric_space.mtotally_bounded T d S
proof −
```

```
  interpret Submetric M d T
    using ‹T ⊆ M› by unfold_locales
  show ?thesis
    using assms
    unfolding sub.mtotally_bounded_def mtotally_bounded_def
    by (force simp: subset_iff elim!: all_forward ex_forward)
qed

lemma mtotally_bounded_absolute:
   mtotally_bounded S ⟷ S ⊆ M ∧ Metric_space.mtotally_bounded S d S
proof −
  have mtotally_bounded S if S ⊆ M Metric_space.mtotally_bounded S d S
  proof −
    interpret Submetric M d S
      using ‹S ⊆ M› by unfold_locales
    show ?thesis
      using that
    by (meson MCauchy_submetric mtotally_bounded_sequentially sub.mtotally_bounded_sequentially)
  qed
  moreover have mtotally_bounded S ⟹ Metric_space.mtotally_bounded S d S
    by (simp add: mtotally_bounded_imp_subset mtotally_bounded_submetric)
  ultimately show ?thesis
    using mtotally_bounded_imp_subset by blast
qed

lemma mtotally_bounded_closure_of:
  assumes mtotally_bounded S
  shows mtotally_bounded (mtopology closure_of S)
proof −
  have S ⊆ M
    by (simp add: assms mtotally_bounded_imp_subset)
  have mtotally_bounded(mtopology closure_of S)
    unfolding mtotally_bounded_def
  proof (intro strip)
    fix ε::real
    assume ε > 0
    then obtain K where finite K K ⊆ S and K: S ⊆ (⋃x∈K. mball x (ε/2))
      by (metis assms mtotally_bounded_def half_gt_zero)
    have mtopology closure_of S ⊆ (⋃x∈K. mball x ε)
      unfolding metric_closure_of
    proof clarsimp
      fix x
      assume x ∈ M and x: ∀r>0. ∃y∈S. y ∈ M ∧ d x y < r
      then obtain y where y ∈ S and y: d x y < ε/2
        using ‹0 < ε› half_gt_zero by blast
      then obtain x' where x' ∈ K y ∈ mball x' (ε/2)
        using K by auto
      then have d x' x < ε/2 + ε/2
        using triangle y ‹x ∈ M› commute by fastforce
```

then show $\exists x'{\in}K.\ x' \in M \wedge d\ x'\ x < \varepsilon$
using ‹$K \subseteq S$› ‹$S \subseteq M$› ‹$x' \in K$› **by** *force*
qed
then show $\exists K.$ *finite* $K \wedge K \subseteq$ *mtopology closure_of* $S \wedge$ *mtopology closure_of* $S \subseteq (\bigcup x{\in}K.$ *mball* $x\ \varepsilon)$
using *closure_of_subset_Int* ‹$K \subseteq S$› ‹*finite* K› K **by** *fastforce*
qed
then show *?thesis*
by (*simp add*: *assms inf.absorb2 mtotally_bounded_imp_subset*)
qed

lemma *mtotally_bounded_closure_of_eq*:
$S \subseteq M \Longrightarrow$ *mtotally_bounded* (*mtopology closure_of* S) $\longleftrightarrow$ *mtotally_bounded* S
by (*metis closure_of_subset mtotally_bounded_closure_of mtotally_bounded_subset topspace_mtopology*)

lemma *mtotally_bounded_cauchy_sequence*:
assumes *MCauchy* σ
shows *mtotally_bounded* (*range* σ)
unfolding *MCauchy_def mtotally_bounded_def*
proof (*intro strip*)
fix ε::*real*
assume $\varepsilon > 0$
then obtain N **where** $\bigwedge n.\ N \leq n \Longrightarrow d\ (\sigma\ N)\ (\sigma\ n) < \varepsilon$
using *assms* **by** (*force simp*: *MCauchy_def*)
then have $\bigwedge m.\ \exists n{\leq}N.\ \sigma\ n \in M \wedge \sigma\ m \in M \wedge d\ (\sigma\ n)\ (\sigma\ m) < \varepsilon$
by (*metis MCauchy_def assms mdist_zero nle_le range_subsetD*)
then
show $\exists K.$ *finite* $K \wedge K \subseteq$ *range* $\sigma \wedge$ *range* $\sigma \subseteq (\bigcup x{\in}K.$ *mball* $x\ \varepsilon)$
by (*rule_tac* $x{=}\sigma$ ‘ $\{0..N\}$ **in** *exI*) *force*
qed

lemma *MCauchy_imp_mbounded*:
MCauchy $\sigma \Longrightarrow$ *mbounded* (*range* σ)
by (*simp add*: *mtotally_bounded_cauchy_sequence mtotally_bounded_imp_mbounded*)

6.7.12 Compactness in metric spaces

lemma *Bolzano_Weierstrass_property*:
assumes $S \subseteq U\ S \subseteq M$
shows
(∀ σ::*nat*⇒'*a*. *range* $\sigma \subseteq S$
$\longrightarrow$ ($\exists\, l\ r.\ l \in U \wedge$ *strict_mono* $r \wedge$ *limitin mtopology* ($\sigma \circ r$) l *sequentially*))
$\longleftrightarrow$
(∀ $T.\ T \subseteq S \wedge$ *infinite* $T \longrightarrow U \cap$ *mtopology derived_set_of* $T \neq \{\}$) (**is** *?lhs=?rhs*)
proof
assume L: *?lhs*

```
  show ?rhs
  proof clarify
    fix T
    assume T ⊆ S and infinite T
      and T: U ∩ mtopology derived_set_of T = {}
    then obtain σ :: nat⇒'a where inj σ range σ ⊆ T
      by (meson infinite_countable_subset)
    with L obtain l r where l ∈ U strict_mono r
          and lr: limitin mtopology (σ ∘ r) l sequentially
      by (meson ‹T ⊆ S› subset_trans)
    then obtain ε where ε > 0 and ε: ⋀y. y ∈ T ⟹ y = l ∨ ¬ d l y < ε
      using T ‹T ⊆ S› ‹S ⊆ M›
      by (force simp: metric_derived_set_of limitin_metric disjoint_iff)
    with lr have ∀F n in sequentially. σ (r n) ∈ M ∧ d (σ (r n)) l < ε
      by (auto simp: limitin_metric)
    then obtain N where N: d (σ (r N)) l < ε d (σ (r (Suc N))) l < ε
      using less_le_not_le by (auto simp: eventually_sequentially)
    moreover have σ (r N) ≠ l ∨ σ (r (Suc N)) ≠ l
      by (meson ‹inj σ› ‹strict_mono r› injD n_not_Suc_n strict_mono_eq)
    ultimately
    show False
      using ε ‹range σ ⊆ T› commute by fastforce
  qed
next
  assume R: ?rhs
  show ?lhs
  proof (intro strip)
    fix σ :: nat ⇒ 'a
    assume range σ ⊆ S
    show ∃l r. l ∈ U ∧ strict_mono r ∧ limitin mtopology (σ ∘ r) l sequentially
    proof (cases finite (range σ))
      case True
      then obtain m where infinite (σ −‘ {σ m})
        by (metis image_iff inf_img_fin_dom nat_not_finite)
      then obtain r where [iff]: strict_mono r and r: ⋀n::nat. r n ∈ σ −‘ {σ
m}
        using infinite_enumerate by blast
      have [iff]: σ m ∈ U σ m ∈ M
        using ‹range σ ⊆ S› assms by blast+
      show ?thesis
      proof (intro conjI exI)
        show limitin mtopology (σ ∘ r) (σ m) sequentially
          using r by (simp add: limitin_metric)
      qed auto
    next
      case False
      then obtain l where l ∈ U and l: l ∈ mtopology derived_set_of (range σ)
        by (meson R ‹range σ ⊆ S› disjoint_iff)
      then obtain g where g: ⋀ε. ε>0 ⟹ σ (g ε) ≠ l ∧ d l (σ (g ε)) < ε
```

```
    by (simp add: metric_derived_set_of) metis
  have range σ ⊆ M
    using ‹range σ ⊆ S› assms by auto
  have l ∈ M
    using l metric_derived_set_of by auto
  define E where — a construction to ensure monotonicity
    E ≡ λrec n. insert (inverse (Suc n)) ((λi. d l (σ i)) ‘ (⋃k<n. {0..rec k})) − {0}
  define r where r ≡ wfrec less_than (λrec n. g (Min (E rec n)))
  have (⋃k<n. {0..cut r less_than n k}) = (⋃k<n. {0..r k}) for n
    by (auto simp: cut_apply)
  then have r_eq: r n = g (Min (E r n)) for n
    by (metis E_def def_wfrec [OF r_def] wf_less_than)
  have dl_pos[simp]: d l (σ (r n)) > 0 for n
    using wf_less_than
  proof (induction n rule: wf_induct_rule)
    case (less n)
    then have ∗: Min (E r n) > 0
      using ‹l ∈ M› ‹range σ ⊆ M› by (auto simp: E_def image_subset_iff)
    show ?case
      using g [OF ∗] r_eq [of n]
      by (metis ‹l ∈ M› ‹range σ ⊆ M› mdist_pos_less range_subsetD)
  qed
  then have non_l: σ (r n) ≠ l for n
    using ‹range σ ⊆ M› mdist_pos_eq by blast
  have Min_pos: Min (E r n) > 0 for n
  using dl_pos ‹l ∈ M› ‹range σ ⊆ M› by (auto simp: E_def image_subset_iff)
  have d_small: d (σ(r n)) l < inverse(Suc n) for n
  proof −
    have d (σ(r n)) l < Min (E r n)
      by (simp add: ‹0 < Min (E r n)› commute g r_eq)
    also have ... ≤ inverse(Suc n)
      by (simp add: E_def)
    finally show ?thesis .
  qed
  have d_lt_d: d l (σ (r n)) < d l (σ i) if §: p < n i ≤ r p σ i ≠ l for i p n
  proof −
    have 1: d l (σ i) ∈ E r n
      using § ‹l ∈ M› ‹range σ ⊆ M›
      by (force simp: E_def image_subset_iff image_iff)
    have d l (σ (g (Min (E r n)))) < Min (E r n)
      by (rule conjunct2 [OF g [OF Min_pos]])
    also have Min (E r n) ≤ d l (σ i)
      using 1 unfolding E_def by (force intro!: Min.coboundedI)
    finally show ?thesis
      by (simp add: r_eq)
  qed
  have r: r p < r n if p < n for p n
  using d_lt_d [OF that] non_l by (meson linorder_not_le order_less_irrefl)
```

```
      show ?thesis
      proof (intro exI conjI)
        show strict_mono r
          by (simp add: r strict_monoI)
        show limitin mtopology (σ ∘ r) l sequentially
          unfolding limitin_metric
        proof (intro conjI strip ‹l ∈ M›)
          fix ε :: real
          assume ε > 0
          then have ∀F n in sequentially. inverse(Suc n) < ε
            using Archimedean_eventually_inverse by auto
          then show ∀F n in sequentially. (σ ∘ r) n ∈ M ∧ d ((σ ∘ r) n) l < ε
            by (smt (verit) ‹range σ ⊆ M› commute comp_apply d_small eventu-
ally_mono range_subsetD)
        qed
      qed (use ‹l ∈ U› in auto)
    qed
  qed
qed
```

More on Bolzano Weierstrass

```
lemma Bolzano_Weierstrass_A:
  assumes compactin mtopology S T ⊆ S infinite T
  shows S ∩ mtopology derived_set_of T ≠ {}
  by (simp add: assms compactin_imp_Bolzano_Weierstrass)

lemma Bolzano_Weierstrass_B:
  fixes σ :: nat ⇒ 'a
  assumes S ⊆ M range σ ⊆ S
    and ⋀T. ⟦T ⊆ S ∧ infinite T⟧ ⟹ S ∩ mtopology derived_set_of T ≠ {}
  shows ∃l r. l ∈ S ∧ strict_mono r ∧ limitin mtopology (σ ∘ r) l sequentially
  using Bolzano_Weierstrass_property assms by blast

lemma Bolzano_Weierstrass_C:
  assumes S ⊆ M
  assumes ⋀σ:: nat ⇒ 'a. range σ ⊆ S ⟹
            (∃l r. l ∈ S ∧ strict_mono r ∧ limitin mtopology (σ ∘ r) l sequentially)
  shows mtotally_bounded S
  unfolding mtotally_bounded_sequentially
  by (metis convergent_imp_MCauchy assms image_comp image_mono subset_UNIV
subset_trans)

lemma Bolzano_Weierstrass_D:
  assumes S ⊆ M S ⊆ ⋃C and openU: ⋀U. U ∈ C ⟹ openin mtopology U
  assumes §: (∀σ::nat⇒'a. range σ ⊆ S
        ⟶ (∃l r. l ∈ S ∧ strict_mono r ∧ limitin mtopology (σ ∘ r) l sequentially))
  shows ∃ε>0. ∀x ∈ S. ∃U ∈ C. mball x ε ⊆ U
```

proof (*rule ccontr*)
assume ¬ (∃ε>0. ∀ x ∈ S. ∃ U ∈ 𝒞. *mball* x ε ⊆ U)
then have ∀ n. ∃ x∈S. ∀ U∈𝒞. ¬ *mball* x (*inverse* (*Suc* n)) ⊆ U
by *simp*
then obtain σ **where** ⋀n. σ n ∈ S
and σ: ⋀n U. U ∈ 𝒞 ⟹ ¬ *mball* (σ n) (*inverse* (*Suc* n)) ⊆ U
by *metis*
then obtain l r **where** l ∈ S *strict_mono* r
and *lr*: *limitin mtopology* (σ ∘ r) l *sequentially*
by (*meson* § *image_subsetI*)
with ‹S ⊆ ⋃𝒞› **obtain** B **where** l ∈ B B ∈ 𝒞
by *auto*
then obtain ε **where** ε > 0 **and** ε: ⋀z. ⟦z ∈ M; d z l < ε⟧ ⟹ z ∈ B
by (*metis opeU* [*OF* ‹B ∈ 𝒞›] *commute in_mball openin_mtopology subset_iff*)
then have ∀ F n *in sequentially*. σ (r n) ∈ M ∧ d (σ (r n)) l < ε/2
using *lr half_gt_zero* **unfolding** *limitin_metric o_def* **by** *blast*
moreover have ∀ F n *in sequentially*. *inverse* (*real* (*Suc* n)) < ε/2
using *Archimedean_eventually_inverse* ‹0 < ε› *half_gt_zero* **by** *blast*
ultimately obtain n **where** n: d (σ (r n)) l < ε/2 *inverse* (*real* (*Suc* n)) < ε/2
by (*smt* (*verit*, *del_insts*) *eventually_sequentially le_add1 le_add2*)
have x ∈ B **if** d (σ (r n)) x < *inverse* (*Suc*(r n)) x ∈ M **for** x
proof −
have *rle*: *inverse* (*real* (*Suc* (r n))) ≤ *inverse* (*real* (*Suc* n))
using ‹*strict_mono* r› *strict_mono_imp_increasing* **by** *auto*
have d x l ≤ d (σ (r n)) x + d (σ (r n)) l
using *that* **by** (*metis triangle* ‹⋀n. σ n ∈ S› ‹l ∈ S› ‹S ⊆ M› *commute subsetD*)
also have ... < ε
using *that n rle* **by** *linarith*
finally show *?thesis*
by (*simp add*: ε *that*)
qed
then show *False*
using σ [*of* B r n] **by** (*simp add*: ‹B ∈ 𝒞› *subset_iff*)
qed

lemma *Bolzano_Weierstrass_E*:
assumes *mtotally_bounded* S S ⊆ M
and S: ⋀𝒞. ⟦⋀U. U ∈ 𝒞 ⟹ *openin mtopology* U; S ⊆ ⋃𝒞⟧ ⟹ ∃ε>0. ∀ x ∈ S. ∃ U ∈ 𝒞. *mball* x ε ⊆ U
shows *compactin mtopology* S
proof (*clarsimp simp*: *compactin_def assms*)
fix 𝒰 :: 'a *set set*
assume 𝒰: ∀ x∈𝒰. *openin mtopology* x **and** S ⊆ ⋃𝒰
then obtain ε **where** ε>0 **and** ε: ⋀x. x ∈ S ⟹ ∃ U ∈ 𝒰. *mball* x ε ⊆ U
by (*metis* S)
then obtain f **where** f: ⋀x. x ∈ S ⟹ f x ∈ 𝒰 ∧ *mball* x ε ⊆ f x
by *metis*

then obtain K **where** *finite* K $K \subseteq S$ **and** K: $S \subseteq (\bigcup x{\in}K.\ \mathit{mball}\ x\ \varepsilon)$
by (*metis* ‹$0 < \varepsilon$› ‹*mtotally_bounded* S› *mtotally_bounded_def*)
show $\exists \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge S \subseteq \bigcup \mathcal{F}$
proof (*intro conjI exI*)
show *finite* (f ‘ K)
by (*simp add*: ‹*finite* K›)
show f ‘ $K \subseteq \mathcal{U}$
using ‹$K \subseteq S$› f **by** *blast*
show $S \subseteq \bigcup (f$ ‘ $K)$
using K ‹$K \subseteq S$› **by** (*force dest*: f)
qed
qed

lemma *compactin_eq_Bolzano_Weierstrass*:
compactin mtopology $S \longleftrightarrow$
$S \subseteq M \wedge (\forall T.\ T \subseteq S \wedge \mathit{infinite}\ T \longrightarrow S \cap \mathit{mtopology\ derived\_set\_of}\ T \neq \{\})$
using *Bolzano_Weierstrass_C Bolzano_Weierstrass_D Bolzano_Weierstrass_E*
by (*smt* (*verit*, *del_insts*) *Bolzano_Weierstrass_property compactin_imp_Bolzano_Weierstrass compactin_subspace subset_refl topspace_mtopology*)

lemma *compactin_sequentially*:
shows *compactin mtopology* $S \longleftrightarrow$
$S \subseteq M \wedge$
$((\forall \sigma{::}\mathit{nat}{\Rightarrow}'a.\ \mathit{range}\ \sigma \subseteq S$
$\longrightarrow (\exists l\ r.\ l \in S \wedge \mathit{strict\_mono}\ r \wedge \mathit{limitin\ mtopology}\ (\sigma \circ r)\ l\ \mathit{sequentially})))$
by (*metis Bolzano_Weierstrass_property compactin_eq_Bolzano_Weierstrass subset_refl*)

lemma *compactin_imp_mtotally_bounded*:
compactin mtopology $S \Longrightarrow$ *mtotally_bounded* S
by (*simp add*: *Bolzano_Weierstrass_C compactin_sequentially*)

lemma *lebesgue_number*:
$[\![\mathit{compactin\ mtopology}\ S;\ S \subseteq \bigcup \mathcal{C};\ \bigwedge U.\ U \in \mathcal{C} \Longrightarrow \mathit{openin\ mtopology}\ U]\!]$
$\Longrightarrow \exists \varepsilon{>}0.\ \forall x \in S.\ \exists U \in \mathcal{C}.\ \mathit{mball}\ x\ \varepsilon \subseteq U$
by (*simp add*: *Bolzano_Weierstrass_D compactin_sequentially*)

lemma *compact_space_sequentially*:
compact_space mtopology $\longleftrightarrow$
$(\forall \sigma{::}\mathit{nat}{\Rightarrow}'a.\ \mathit{range}\ \sigma \subseteq M$
$\longrightarrow (\exists l\ r.\ l \in M \wedge \mathit{strict\_mono}\ r \wedge \mathit{limitin\ mtopology}\ (\sigma \circ r)\ l\ \mathit{sequentially}))$
by (*simp add*: *compact_space_def compactin_sequentially*)

lemma *compact_space_eq_Bolzano_Weierstrass*:
compact_space mtopology $\longleftrightarrow$
$(\forall S.\ S \subseteq M \wedge \mathit{infinite}\ S \longrightarrow \mathit{mtopology\ derived\_set\_of}\ S \neq \{\})$
using *Int_absorb1* [*OF derived_set_of_subset_topspace* [*of mtopology*]]
by (*force simp*: *compact_space_def compactin_eq_Bolzano_Weierstrass*)

lemma *compact_space_nest*:
  compact_space mtopology ⟷
   (∀ *C*. (∀ *n*::*nat*. *closedin mtopology* (*C n*)) ∧ (∀ *n*. *C n* ≠ {}) ∧ *decseq C* ⟶ ⋂(*range C*) ≠ {})
  (**is** *?lhs*=*?rhs*)
proof
 assume *L*: *?lhs*
 show *?rhs*
 proof *clarify*
  fix *C* :: *nat* ⇒ *'a set*
  assume ∀ *n*. *closedin mtopology* (*C n*)
    and ∀ *n*. *C n* ≠ {}
    and *decseq C*
    and ⋂ (*range C*) = {}
  then obtain *K* **where** *K*: *finite K* ⋂(*C* ‘ *K*) = {}
    by (*metis L compact_space_imp_nest*)
  then obtain *k* **where** *K* ⊆ {..*k*}
    using *finite_nat_iff_bounded_le* **by** *auto*
  then have *C k* ⊆ ⋂(*C* ‘ *K*)
    using ‹*decseq C*› **by** (*auto simp*:*decseq_def*)
  then show *False*
    by (*simp add*: *K* ‹∀ *n*. *C n* ≠ {}›)
 qed
next
 assume *R* [*rule_format*]: *?rhs*
 show *?lhs*
  unfolding *compact_space_sequentially*
 proof (*intro strip*)
  fix *σ* :: *nat* ⇒ *'a*
  assume *σ*: *range σ* ⊆ *M*
  have *mtopology closure_of σ* ‘ {*n*..} ≠ {} **for** *n*
   using ‹*range σ* ⊆ *M*› **by** (*auto simp*: *closure_of_eq_empty image_subset_iff*)
  moreover have *decseq* (λ*n*. *mtopology closure_of σ* ‘ {*n*..})
    using *closure_of_mono image_mono* **by** (*smt* (*verit*) *atLeast_subset_iff decseq_def*)
  ultimately obtain *l* **where** *l*: ⋀*n*. *l* ∈ *mtopology closure_of σ* ‘ {*n*..}
    using *R* [*of* λ*n*. *mtopology closure_of* (*σ* ‘ {*n*..})] **by** *auto*
  then have *l* ∈ *M* **and** ⋀*n*. ∀ *r*>*0*. ∃ *k*≥*n*. *σ k* ∈ *M* ∧ *d l* (*σ k*) < *r*
    using *metric_closure_of* **by** *fastforce*+
  then obtain *f* **where** *f*: ⋀*n r*. *r*>*0* ⟹ *f n r* ≥ *n* ∧ *σ* (*f n r*) ∈ *M* ∧ *d l* (*σ* (*f n r*)) < *r*
    by *metis*
  define *r* **where** *r* = *rec_nat* (*f 0 1*) (λ*n rec*. (*f* (*Suc rec*) (*inverse* (*Suc* (*Suc n*)))))
  have *r*: *d l* (*σ*(*r n*)) < *inverse*(*Suc n*) **for** *n*
    by (*induction n*) (*auto simp*: *rec_nat_0_imp* [*OF r_def*] *rec_nat_Suc_imp* [*OF r_def*] *f*)
  have *r n* < *r*(*Suc n*) **for** *n*

```
    by (simp add: Suc_le_lessD f r_def)
  then have strict_mono r
    by (simp add: strict_mono_Suc_iff)
  moreover have limitin mtopology (σ ∘ r) l sequentially
    proof (clarsimp simp: limitin_metric ‹l ∈ M›)
      fix ε :: real
      assume ε > 0
      then have (∀F n in sequentially. inverse (real (Suc n)) < ε)
        using Archimedean_eventually_inverse by blast
      then show ∀F n in sequentially. σ (r n) ∈ M ∧ d (σ (r n)) l < ε
        by eventually_elim (metis commute ‹range σ ⊆ M›  order_less_trans r
range_subsetD)
    qed
  ultimately show ∃l r. l ∈ M ∧ strict_mono r ∧ limitin mtopology (σ ∘ r) l
sequentially
    using ‹l ∈ M› by blast
 qed
qed


lemma (in discrete_metric) mcomplete_discrete_metric: disc.mcomplete
proof (clarsimp simp: disc.mcomplete_def)
 fix σ :: nat ⇒ 'a
 assume disc.MCauchy σ
 then obtain N where ⋀n. N ≤ n ⟹ σ N = σ n
  unfolding disc.MCauchy_def by (metis dd_def dual_order.refl order_less_irrefl
zero_less_one)
 moreover have range σ ⊆ M
   using ‹disc.MCauchy σ› disc.MCauchy_def by blast
 ultimately have limitin disc.mtopology σ (σ N) sequentially
   by (metis disc.limit_metric_sequentially disc.zero range_subsetD)
 then show ∃x. limitin disc.mtopology σ x sequentially ..
qed

lemma compact_space_imp_mcomplete: compact_space mtopology ⟹ mcomplete
 by (simp add: compact_space_nest mcomplete_nest)

lemma (in Submetric) compactin_imp_mcomplete:
   compactin mtopology A ⟹ sub.mcomplete
 by (simp add: compactin_subspace mtopology_submetric sub.compact_space_imp_mcomplete)

lemma (in Submetric) mcomplete_imp_closedin:
 assumes sub.mcomplete
 shows closedin mtopology A
proof −
 have l ∈ A
   if range σ ⊆ A and l: limitin mtopology σ l sequentially
   for σ :: nat ⇒ 'a and l
 proof −
```

```
    have sub.MCauchy σ
     using convergent_imp_MCauchy subset that by (force simp: MCauchy_submetric)
    then have limitin sub.mtopology σ l sequentially
      using assms unfolding sub.mcomplete_def
     using l limitin_metric_unique limitin_submetric_iff trivial_limit_sequentially
by blast
    then show ?thesis
      using limitin_submetric_iff by blast
  qed
  then show ?thesis
    using metric_closedin_iff_sequentially_closed subset by auto
qed

lemma (in Submetric) closedin_eq_mcomplete:
   mcomplete ⟹ (closedin mtopology A ⟷ sub.mcomplete)
  using closedin_mcomplete_imp_mcomplete mcomplete_imp_closedin by blast

lemma compact_space_eq_mcomplete_mtotally_bounded:
   compact_space mtopology ⟷ mcomplete ∧ mtotally_bounded M
 by (meson Bolzano_Weierstrass_C compact_space_imp_mcomplete compact_space_sequentially
limitin_mspace
            mcomplete_alt mtotally_bounded_sequentially subset_refl)


lemma compact_closure_of_imp_mtotally_bounded:
   ⟦compactin mtopology (mtopology closure_of S); S ⊆ M⟧
     ⟹ mtotally_bounded S
  using compactin_imp_mtotally_bounded mtotally_bounded_closure_of_eq by
blast

lemma mtotally_bounded_eq_compact_closure_of:
  assumes mcomplete
  shows mtotally_bounded S ⟷ S ⊆ M ∧ compactin mtopology (mtopology clo-
sure_of S)
  (is ?lhs=?rhs)
proof
  assume L: ?lhs
  show ?rhs
    unfolding compactin_subspace
  proof (intro conjI)
    show S ⊆ M
      using L by (simp add: mtotally_bounded_imp_subset)
    show mtopology closure_of S ⊆ topspace mtopology
      by (simp add: ‹S ⊆ M› closure_of_minimal)
    then have MSM: mtopology closure_of S ⊆ M
      by auto
    interpret S: Submetric M d mtopology closure_of S
    proof qed (use MSM in auto)
    have S.sub.mtotally_bounded (mtopology closure_of S)
```

```
    using L mtotally_bounded_absolute mtotally_bounded_closure_of by blast
   then
   show compact_space (subtopology mtopology (mtopology closure_of S))
   using S.closedin_mcomplete_imp_mcomplete S.mtopology_submetric S.sub.compact_space_eq_mcomplete_mt
assms by force
 qed
qed (auto simp: compact_closure_of_imp_mtotally_bounded)


lemma compact_closure_of_eq_Bolzano_Weierstrass:
  compactin mtopology (mtopology closure_of S) ⟷
   (∀ T. infinite T ∧ T ⊆ S ∧ T ⊆ M ⟶ mtopology derived_set_of T ≠ {})
(is ?lhs=?rhs)
proof
 assume L: ?lhs
 show ?rhs
 proof (intro strip)
  fix T
  assume T: infinite T ∧ T ⊆ S ∧ T ⊆ M
  show mtopology derived_set_of T ≠ {}
  proof (intro compact_closure_of_imp_Bolzano_Weierstrass)
   show compactin mtopology (mtopology closure_of S)
    by (simp add: L)
  qed (use T in auto)
 qed
next
 have compactin mtopology (mtopology closure_of S)
  if §: ⋀T. ⟦infinite T; T ⊆ S⟧ ⟹ mtopology derived_set_of T ≠ {} and S ⊆
M for S
  unfolding compactin_sequentially
 proof (intro conjI strip)
  show MSM: mtopology closure_of S ⊆ M
   using closure_of_subset_topspace by fastforce
  fix σ :: nat ⇒ 'a
  assume σ: range σ ⊆ mtopology closure_of S
  then have ∃ y ∈ S. d (σ n) y < inverse(Suc n) for n
    by (simp add: metric_closure_of image_subset_iff) (metis inverse_Suc
of_nat_Suc)
  then obtain τ where τ: ⋀n. τ n ∈ S ∧ d (σ n) (τ n) < inverse(Suc n)
   by metis
  then have range τ ⊆ S
   by blast
  moreover
   have ∗: ∀ T. T ⊆ S ∧ infinite T ⟶ mtopology closure_of S ∩ mtopology
derived_set_of T ≠ {}
    using §(1) derived_set_of_mono derived_set_of_subset_closure_of by
fastforce
  moreover have S ⊆ mtopology closure_of S
```

by (*simp add*: ‹$S \subseteq M$› *closure_of_subset*)
ultimately obtain $l\ r$ **where** *lr*:
$l \in$ *mtopology closure_of* S *strict_mono* r *limitin mtopology* $(\tau \circ r)\ l$ *sequentially*
using *Bolzano_Weierstrass_property* ‹$S \subseteq M$› **by** *metis*
then have $l \in M$
using *limitin_mspace* **by** *blast*
have *dr_less*: $d\ ((\sigma \circ r)\ n)\ ((\tau \circ r)\ n) < inverse(Suc\ n)$ **for** n
proof −
have $d\ ((\sigma \circ r)\ n)\ ((\tau \circ r)\ n) < inverse(Suc\ (r\ n))$
using τ **by** *auto*
also have ... $\leq inverse(Suc\ n)$
using *lr strict_mono_imp_increasing* **by** *auto*
finally show *?thesis* **.**
qed
have *limitin mtopology* $(\sigma \circ r)\ l$ *sequentially*
unfolding *limitin_metric*
proof (*intro conjI strip*)
show $l \in M$
using *limitin_mspace lr* **by** *blast*
fix ε :: *real*
assume $\varepsilon > 0$
then have $\forall_F\ n$ *in sequentially*. $(\tau \circ r)\ n \in M \wedge d\ ((\tau \circ r)\ n)\ l < \varepsilon/2$
using *lr half_gt_zero limitin_metric* **by** *blast*
moreover have $\forall_F\ n$ *in sequentially*. *inverse* (*real* (*Suc n*)) $< \varepsilon/2$
using *Archimedean_eventually_inverse* ‹$0 < \varepsilon$› *half_gt_zero* **by** *blast*
then have $\forall_F\ n$ *in sequentially*. $d\ ((\sigma \circ r)\ n)\ ((\tau \circ r)\ n) < \varepsilon/2$
by *eventually_elim* (*smt* (*verit*, *del_insts*) *dr_less*)
ultimately have $\forall_F\ n$ *in sequentially*. $d\ ((\sigma \circ r)\ n)\ l < \varepsilon/2 + \varepsilon/2$
by *eventually_elim* (*smt* (*verit*) *triangle* ‹$l \in M$› *MSM* σ *comp_apply order_trans range_subsetD*)
then show $\forall_F\ n$ *in sequentially*. $(\sigma \circ r)\ n \in M \wedge d\ ((\sigma \circ r)\ n)\ l < \varepsilon$
apply *eventually_elim*
using ‹*mtopology closure_of* $S \subseteq M$› σ **by** *auto*
qed
with *lr* **show** $\exists l\ r.\ l \in$ *mtopology closure_of* $S \wedge$ *strict_mono* $r \wedge$ *limitin mtopology* $(\sigma \circ r)\ l$ *sequentially*
by *blast*
qed
then show *?rhs* $\Longrightarrow$ *?lhs*
by (*metis Int_subset_iff closure_of_restrict inf_le1 topspace_mtopology*)
qed

end

lemma (**in** *discrete_metric*) *mtotally_bounded_discrete_metric*:
disc.mtotally_bounded $S \longleftrightarrow$ *finite* $S \wedge S \subseteq M$ (**is** *?lhs=?rhs*)
proof
assume *L*: *?lhs*

show *?rhs*
proof
show *finite S*
by (*metis* (*no_types*) *L closure_of_subset_Int compactin_discrete_topology disc.mtotally_bounded_eq_compact_closure_of*
disc.topspace_mtopology discrete_metric.mcomplete_discrete_metric inf.absorb_iff2 mtopology_discrete_metric finite_subset)
show $S \subseteq M$
by (*simp add*: *L disc.mtotally_bounded_imp_subset*)
qed
qed (*simp add*: *disc.finite_imp_mtotally_bounded*)

context *Metric_space*
begin

lemma *derived_set_of_infinite_openin_metric*:
mtopology derived_set_of S =
$\{x \in M.\ \forall U.\ x \in U \wedge$ *openin mtopology U* $\longrightarrow$ *infinite*$(S \cap U)\}$
by (*simp add*: *derived_set_of_infinite_openin Hausdorff_space_mtopology*)

lemma *derived_set_of_infinite_1*:
assumes *infinite* $(S \cap$ *mball x* $\varepsilon)$
shows *infinite* $(S \cap$ *mcball x* $\varepsilon)$
by (*meson Int_mono assms finite_subset mball_subset_mcball subset_refl*)

lemma *derived_set_of_infinite_2*:
assumes *openin mtopology U* $\bigwedge\varepsilon.\ 0 < \varepsilon \Longrightarrow$ *infinite* $(S \cap$ *mcball x* $\varepsilon)$ **and** $x \in U$
shows *infinite* $(S \cap U)$
by (*metis assms openin_mtopology_mcball finite_Int inf.absorb_iff2 inf_assoc*)

lemma *derived_set_of_infinite_mball*:
mtopology derived_set_of S $= \{x \in M.\ \forall e{>}0.$ *infinite*$(S \cap$ *mball x e*$)\}$
unfolding *derived_set_of_infinite_openin_metric*
by (*metis* (*no_types*, *opaque_lifting*) *centre_in_mball_iff openin_mball derived_set_of_infinite_1 derived_set_of_infinite_2*)

lemma *derived_set_of_infinite_mcball*:
mtopology derived_set_of S $= \{x \in M.\ \forall e{>}0.$ *infinite*$(S \cap$ *mcball x e*$)\}$
unfolding *derived_set_of_infinite_openin_metric*
by (*metis* (*no_types*, *opaque_lifting*) *centre_in_mball_iff openin_mball derived_set_of_infinite_1 derived_set_of_infinite_2*)

end

6.7.13 Continuous functions on metric spaces

context *Metric_space*

begin

lemma *continuous_map_to_metric*:
continuous_map X mtopology f $\longleftrightarrow$
($\forall x \in$ *topspace X*. $\forall \varepsilon{>}0$. $\exists U$. *openin X U* $\wedge$ $x \in U \wedge$ ($\forall y{\in}U$. *f y* $\in$ *mball* (*f x*) ε))
(**is** *?lhs=?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
unfolding *continuous_map_eq_topcontinuous_at topcontinuous_at_def*
by (*metis PiE centre_in_mball_iff openin_mball topspace_mtopology*)
next
assume *R*: *?rhs*
then have $\forall x{\in}$*topspace X*. *f x* $\in$ *M*
by (*meson gt_ex in_mball*)
moreover
have $\bigwedge x\ V$. $[\![x \in$ *topspace X*; *openin mtopology V*; *f x* $\in V]\!] \Longrightarrow \exists U$. *openin X U* $\wedge$ *x* $\in$ *U* $\wedge$ ($\forall y{\in}U$. *f y* $\in$ *V*)
unfolding *openin_mtopology* **by** (*metis Int_iff R inf.orderE*)
ultimately
show *?lhs*
by (*simp add*: *continuous_map_eq_topcontinuous_at topcontinuous_at_def*)
qed

lemma *continuous_map_from_metric*:
continuous_map mtopology X f $\longleftrightarrow$
f $\in$ *M* $\to$ *topspace X* $\wedge$
($\forall a \in M$. $\forall U$. *openin X U* $\wedge$ *f a* $\in$ *U* $\longrightarrow$ ($\exists r{>}0$. $\forall x$. *x* $\in$ *M* $\wedge$ *d a x* $<$ *r* $\longrightarrow$ *f x* $\in$ *U*))
proof (*cases f ' M* $\subseteq$ *topspace X*)
case *True*
then show *?thesis*
by (*fastforce simp*: *continuous_map openin_mtopology subset_eq*)
next
case *False*
then show *?thesis*
by (*simp add*: *continuous_map_def image_subset_iff_funcset*)
qed

An abstract formulation, since the limits do not have to be sequential

lemma *continuous_map_uniform_limit*:
assumes *contf*: $\forall_F\ \xi$ *in F*. *continuous_map X mtopology* (*f* ξ)
and *dfg*: $\bigwedge\varepsilon$. $0 < \varepsilon \Longrightarrow \forall_F\ \xi$ *in F*. $\forall x \in$ *topspace X*. *g x* $\in$ *M* $\wedge$ *d* (*f* ξ *x*) (*g x*) $< \varepsilon$
and *nontriv*: $\neg$ *trivial_limit F*
shows *continuous_map X mtopology g*
unfolding *continuous_map_to_metric*
proof (*intro strip*)
fix *x* **and** ε::*real*

assume $x \in$ *topspace* X **and** $\varepsilon > 0$
then obtain ξ **where** k: *continuous_map* X *mtopology* $(f\ \xi)$
and gM: $\forall x \in$ *topspace* $X.\ g\ x \in M$
and *third*: $\forall x \in$ *topspace* $X.\ d\ (f\ \xi\ x)\ (g\ x) < \varepsilon/3$
using *eventually_conj* [*OF contf*] *contf dfg* [*of* $\varepsilon/3$] *eventually_happens'* [*OF nontriv*]
by (*smt* (*verit*, *ccfv_SIG*) *zero_less_divide_iff*)
then obtain U **where** U: *openin* $X\ U\ x \in U$ **and** *Uthird*: $\forall y{\in}U.\ d\ (f\ \xi\ y)\ (f\ \xi\ x) < \varepsilon/3$
unfolding *continuous_map_to_metric*
by (*metis* ‹$0 < \varepsilon$› ‹$x \in$ *topspace* X› *commute divide_pos_pos in_mball zero_less_numeral*)
have *f_inM*: $f\ \xi\ y \in M$ **if** $y{\in}U$ **for** y
using U k *openin_subset that* **by** (*fastforce simp*: *continuous_map_def*)
have $d\ (g\ y)\ (g\ x) < \varepsilon$ **if** $y{\in}U$ **for** y
proof −
have $g\ y \in M$
using U gM *openin_subset that* **by** *blast*
have $d\ (g\ y)\ (g\ x) \le d\ (g\ y)\ (f\ \xi\ x) + d\ (f\ \xi\ x)\ (g\ x)$
by (*simp add*: U ‹$g\ y \in M$› ‹$x \in$ *topspace* X› *f_inM gM triangle*)
also have $\ldots \le d\ (g\ y)\ (f\ \xi\ y) + d\ (f\ \xi\ y)\ (f\ \xi\ x) + d\ (f\ \xi\ x)\ (g\ x)$
by (*simp add*: U ‹$g\ y \in M$› *commute f_inM that triangle'*)
also have $\ldots < \varepsilon/3 + \varepsilon/3 + \varepsilon/3$
by (*smt* (*verit*) $U(1)$ *Uthird* ‹$x \in$ *topspace* X› *commute openin_subset subsetD that third*)
finally show *?thesis* **by** *simp*
qed
with U gM **show** $\exists\, U.$ *openin* $X\ U \wedge x \in U \wedge (\forall y{\in}U.\ g\ y \in$ *mball* $(g\ x)\ \varepsilon)$
by (*metis commute in_mball in_mono openin_subset*)
qed

lemma *continuous_map_uniform_limit_alt*:
assumes *contf*: $\forall_F\ \xi$ *in* $F.$ *continuous_map* X *mtopology* $(f\ \xi)$
and *gim*: $g \in$ *topspace* $X \to M$
and *dfg*: $\bigwedge\varepsilon.\ 0 < \varepsilon \Longrightarrow \forall_F\ \xi$ *in* $F.\ \forall x \in$ *topspace* $X.\ d\ (f\ \xi\ x)\ (g\ x) < \varepsilon$
and *nontriv*: $\neg$ *trivial_limit* F
shows *continuous_map* X *mtopology* g
proof (*rule continuous_map_uniform_limit* [*OF contf*])
fix ε :: *real*
assume $\varepsilon > 0$
with *gim dfg* **show** $\forall_F\ \xi$ *in* $F.\ \forall x{\in}$*topspace* $X.\ g\ x \in M \wedge d\ (f\ \xi\ x)\ (g\ x) < \varepsilon$
by (*simp add*: *Pi_iff*)
qed (*use nontriv* **in** *auto*)

lemma *continuous_map_uniformly_Cauchy_limit*:
assumes *mcomplete*
assumes *contf*: $\forall_F\ n$ *in sequentially. continuous_map* X *mtopology* $(f\ n)$
and *Cauchy'*: $\bigwedge\varepsilon.\ \varepsilon > 0 \Longrightarrow \exists N.\ \forall m\ n\ x.\ N \le m \longrightarrow N \le n \longrightarrow x \in$ *topspace*

```
X ⟶ d (f m x) (f n x) < ε
  obtains g where
    continuous_map X mtopology g
    ⋀ε. 0 < ε ⟹ ∀F n in sequentially. ∀x∈topspace X. d (f n x) (g x) < ε
proof −
  have ⋀x. x ∈ topspace X ⟹ ∃l. limitin mtopology (λn. f n x) l sequentially
    using ‹mcomplete› [unfolded mcomplete, rule_format] assms
    unfolding continuous_map_def Pi_iff topspace_mtopology
    by (smt (verit, del_insts) eventually_mono)
  then obtain g where g: ⋀x. x ∈ topspace X ⟹ limitin mtopology (λn. f n x)
(g x) sequentially
    by metis
  show thesis
  proof
    show ∀F n in sequentially. ∀x∈topspace X. d (f n x) (g x) < ε
      if ε > 0 for ε :: real
    proof −
      obtain N where N: ⋀m n x. ⟦N ≤ m; N ≤ n; x ∈ topspace X⟧ ⟹ d (f m
x) (f n x) < ε/2
        by (meson Cauchy' ‹0 < ε› half_gt_zero)
      obtain P where P: ⋀n x. ⟦n ≥ P; x ∈ topspace X⟧ ⟹ f n x ∈ M
        using contf by (auto simp: eventually_sequentially continuous_map_def)
      show ?thesis
      proof (intro eventually_sequentiallyI strip)
        fix n x
        assume max N P ≤ n and x: x ∈ topspace X
       obtain L where g x ∈ M and L: ∀n≥L. f n x ∈ M ∧ d (f n x) (g x) < ε/2
          using g [OF x] ‹ε > 0› unfolding limitin_metric
          by (metis (no_types, lifting) eventually_sequentially half_gt_zero)
        define n' where n' ≡ Max{L,N,P}
        have L': ∀m ≥ n'. f m x ∈ M ∧ d (f m x) (g x) < ε/2
          using L by (simp add: n'_def)
        moreover
        have d (f n x) (f n' x) < ε/2
          using N [of n n' x] ‹max N P ≤ n› n'_def x by fastforce
        ultimately have d (f n x) (g x) < ε/2 + ε/2
            by (smt (verit, ccfv_SIG) P ‹g x ∈ M› ‹max N P ≤ n› le_refl
max.bounded_iff mdist_zero triangle' x)
        then show d (f n x) (g x) < ε by simp
      qed
    qed
    then show continuous_map X mtopology g
    by (smt (verit, del_insts) eventually_mono g limitin_mspace trivial_limit_sequentially
continuous_map_uniform_limit [OF contf])
  qed
qed

lemma metric_continuous_map:
  assumes Metric_space M' d'
```

```
  shows
   continuous_map mtopology (Metric_space.mtopology M' d') f ⟷
    f ‘ M ⊆ M' ∧ (∀ a ∈ M. ∀ε>0. ∃δ>0.  (∀ x. x ∈ M ∧ d a x < δ ⟶ d' (f a)
(f x) < ε))
   (is ?lhs = ?rhs)
proof −
  interpret M': Metric_space M' d'
    by (simp add: assms)
  show ?thesis
  proof
    assume L: ?lhs
    show ?rhs
    proof (intro conjI strip)
      show f ‘ M ⊆ M'
        using L by (auto simp: continuous_map_def)
      fix a and ε :: real
      assume a ∈ M and ε > 0
      then have openin mtopology {x ∈ M. f x ∈ M'.mball (f a) ε} f a ∈ M'
        using L unfolding continuous_map_def by fastforce+
      then obtain δ where δ > 0 mball a δ ⊆ {x ∈ M. f x ∈ M' ∧ d' (f a) (f x)
< ε}
        using ‹0 < ε› ‹a ∈ M› openin_mtopology by auto
      then show ∃δ>0. ∀ x. x ∈ M ∧ d a x < δ ⟶ d' (f a) (f x) < ε
        using ‹a ∈ M› in_mball by blast
    qed
  next
    assume R: ?rhs
    show ?lhs
      unfolding continuous_map_def
    proof (intro conjI strip)
      fix U
      assume openin M'.mtopology U
      then show openin mtopology {x ∈ topspace mtopology. f x ∈ U}
        using R
      by (force simp: continuous_map_def openin_mtopology M'.openin_mtopology
subset_iff)
    qed (use R in auto)
  qed
qed

end
```

6.7.14 Completely metrizable spaces

These spaces are topologically complete

```
definition completely_metrizable_space where
 completely_metrizable_space X ≡
      ∃ M d. Metric_space M d ∧ Metric_space.mcomplete M d ∧ X = Met-
ric_space.mtopology M d
```

lemma *empty_completely_metrizable_space*:
completely_metrizable_space trivial_topology
unfolding *completely_metrizable_space_def subtopology_eq_discrete_topology_empty* [*symmetric*]
by (*metis Metric_space.mcomplete_empty_mspace discrete_metric.mtopology_discrete_metric metric_M_dd*)

lemma *completely_metrizable_imp_metrizable_space*:
completely_metrizable_space X ⟹ *metrizable_space X*
using *completely_metrizable_space_def metrizable_space_def* **by** *auto*

lemma (**in** *Metric_space*) *completely_metrizable_space_mtopology*:
mcomplete ⟹ *completely_metrizable_space mtopology*
using *Metric_space_axioms completely_metrizable_space_def* **by** *blast*

lemma *completely_metrizable_space_discrete_topology*:
completely_metrizable_space (*discrete_topology U*)
unfolding *completely_metrizable_space_def*
by (*metis discrete_metric.mcomplete_discrete_metric discrete_metric.mtopology_discrete_metric metric_M_dd*)

lemma *completely_metrizable_space_euclidean*:
completely_metrizable_space (*euclidean*:: *'a*::*complete_space topology*)
using *Met_TC.completely_metrizable_space_mtopology complete_UNIV* **by** *auto*

lemma *completely_metrizable_space_closedin*:
assumes *X*: *completely_metrizable_space X* **and** *S*: *closedin X S*
shows *completely_metrizable_space*(*subtopology X S*)
proof −
obtain *M d* **where** *Metric_space M d* **and** *comp*: *Metric_space.mcomplete M d*

and *Xeq*: *X* = *Metric_space.mtopology M d*
using *assms completely_metrizable_space_def* **by** *blast*
then interpret *Metric_space M d*
by *blast*
show *?thesis*
unfolding *completely_metrizable_space_def*
proof (*intro conjI exI*)
show *Metric_space S d*
using *S Xeq closedin_subset subspace* **by** *force*
have *sub*: *Submetric_axioms M S*
by (*metis S Xeq closedin_metric Submetric_axioms_def*)
then show *Metric_space.mcomplete S d*
using *S Submetric.closedin_mcomplete_imp_mcomplete Submetric_def Xeq comp* **by** *blast*
show *subtopology X S* = *Metric_space.mtopology S d*
by (*metis Metric_space_axioms Xeq sub Submetric.intro Submetric.mtopology_submetric*)
qed

qed

lemma *completely_metrizable_space_cbox*: *completely_metrizable_space* (*top_of_set* (*cbox a b*))
using *closed_closedin completely_metrizable_space_closedin completely_metrizable_space_euclidean*
by *blast*

lemma *homeomorphic_completely_metrizable_space_aux*:
assumes *homXY*: *X homeomorphic_space Y* **and** *X*: *completely_metrizable_space X*
shows *completely_metrizable_space Y*
proof −
obtain *f g* **where** *hmf*: *homeomorphic_map X Y f* **and** *hmg*: *homeomorphic_map Y X g*
and *fg*: $\bigwedge x.\ x \in topspace\ X \Longrightarrow g(f\ x) = x$ $\bigwedge y.\ y \in topspace\ Y \Longrightarrow f(g\ y) = y$
and *fim*: $f \in topspace\ X \to topspace\ Y$ **and** *gim*: $g \in topspace\ Y \to topspace\ X$
using *homXY*
using *homeomorphic_space_unfold* **by** *blast*
obtain *M d* **where** *Md*: *Metric_space M d Metric_space.mcomplete M d* **and** *Xeq*: *X* = *Metric_space.mtopology M d*
using *X* **by** (*auto simp*: *completely_metrizable_space_def*)
then interpret *MX*: *Metric_space M d* **by** *metis*
define *D* **where** $D \equiv \lambda x\ y.\ d\ (g\ x)\ (g\ y)$
have *Metric_space* (*topspace Y*) *D*
proof
show $(D\ x\ y = 0) \longleftrightarrow (x = y)$ **if** $x \in topspace\ Y\ y \in topspace\ Y$ **for** *x y*
unfolding *D_def*
by (*metis that MX.topspace_mtopology MX.zero Xeq fg gim Pi_iff*)
show $D\ x\ z \le D\ x\ y + D\ y\ z$
if $x \in topspace\ Y\ y \in topspace\ Y\ z \in topspace\ Y$ **for** *x y z*
using *that MX.triangle Xeq gim* **by** (*auto simp*: *D_def*)
qed (*auto simp*: *D_def MX.commute*)
then interpret *MY*: *Metric_space topspace Y* $\lambda x\ y.\ D\ x\ y$ **by** *metis*
show *?thesis*
unfolding *completely_metrizable_space_def*
proof (*intro exI conjI*)
show *Metric_space* (*topspace Y*) *D*
using *MY.Metric_space_axioms* **by** *blast*
have *gball*: *g* ‘ *MY.mball y r* = *MX.mball* (*g y*) *r* **if** $y \in topspace\ Y$ **for** *y r*
using *that MX.topspace_mtopology Xeq gim hmg homeomorphic_imp_surjective_map*
unfolding *MX.mball_def MY.mball_def* **by** (*fastforce simp*: *D_def*)
have $\exists r{>}0.\ MY.mball\ y\ r \subseteq S$ **if** *openin Y S* **and** $y \in S$ **for** *S y*
proof −
have *openin X* (*g‘S*)
using *hmg homeomorphic_map_openness_eq that* **by** *auto*
then obtain *r* **where** *r*>*0 MX.mball* (*g y*) *r* ⊆ *g‘S*
using *MX.openin_mtopology Xeq* ‹*y* ∈ *S*› **by** *auto*

then show *?thesis*
by (*smt* (*verit*, *ccfv_SIG*) *MY.in_mball gball fg image_iff in_mono openin_subset subsetI that*(*1*))
qed
moreover have *openin Y S*
if $S \subseteq$ *topspace Y* **and** $\bigwedge y.\ y \in S \Longrightarrow \exists\, r{>}0.\ MY.mball\ y\ r \subseteq S$ **for** *S*
proof −
have $\bigwedge x.\ x \in g`S \Longrightarrow \exists\, r{>}0.\ MX.mball\ x\ r \subseteq g`S$
by (*smt* (*verit*) *gball imageE image_mono subset_iff that*)
then have *openin X* (*g`S*)
using *MX.openin_mtopology Xeq gim that*(*1*) **by** *auto*
then show *?thesis*
using *hmg homeomorphic_map_openness_eq that*(*1*) **by** *blast*
qed
ultimately show *Yeq*: $Y = MY.mtopology$
unfolding *topology_eq MY.openin_mtopology* **by** (*metis openin_subset*)

show *MY.mcomplete*
unfolding *MY.mcomplete_def*
proof (*intro strip*)
fix σ
assume σ: *MY.MCauchy* σ
have *MX.MCauchy* $(g \circ \sigma)$
unfolding *MX.MCauchy_def*
proof (*intro conjI strip*)
show *range* $(g \circ \sigma) \subseteq M$
using *MY.MCauchy_def Xeq* σ *gim* **by** *auto*
fix ε :: *real*
assume $\varepsilon > 0$
then obtain *N* **where** $\forall\, n\ n'.\ N \le n \longrightarrow N \le n' \longrightarrow D\ (\sigma\ n)\ (\sigma\ n') < \varepsilon$
using *MY.MCauchy_def* σ **by** *presburger*
then show $\exists\, N.\ \forall\, n\ n'.\ N \le n \longrightarrow N \le n' \longrightarrow d\ ((g \circ \sigma)\ n)\ ((g \circ \sigma)\ n') < \varepsilon$
by (*auto simp*: *o_def D_def*)
qed
then obtain *x* **where** *x*: *limitin MX.mtopology* $(g \circ \sigma)$ *x sequentially* $x \in$ *topspace X*
using *MX.limitin_mspace MX.topspace_mtopology Md Xeq* **unfolding** *MX.mcomplete_def*
by *blast*
with *x* **have** *limitin MY.mtopology* $(f \circ (g \circ \sigma))$ $(f\ x)$ *sequentially*
by (*metis Xeq Yeq continuous_map_limit hmf homeomorphic_imp_continuous_map*)
moreover have $f \circ (g \circ \sigma) = \sigma$
using ‹*MY.MCauchy* σ› **by** (*force simp*: *fg MY.MCauchy_def subset_iff*)
ultimately have *limitin MY.mtopology* σ $(f\ x)$ *sequentially* **by** *simp*
then show $\exists\, y.$ *limitin MY.mtopology* σ *y sequentially*
by *blast*
qed
qed

qed

lemma *homeomorphic_completely_metrizable_space*:
  X homeomorphic_space Y
    $\Longrightarrow$ *completely_metrizable_space X* $\longleftrightarrow$ *completely_metrizable_space Y*
 by (*meson homeomorphic_completely_metrizable_space_aux homeomorphic_space_sym*)

lemma *completely_metrizable_space_retraction_map_image*:
 assumes *r*: *retraction_map X Y r* **and** *X*: *completely_metrizable_space X*
 shows *completely_metrizable_space Y*
proof −
 obtain *s* **where** *s*: *retraction_maps X Y r s*
  using *r retraction_map_def* **by** *blast*
 then have *subtopology X* (*s* ‘ *topspace Y*) *homeomorphic_space Y*
  using *retraction_maps_section_image2* **by** *blast*
 then show *?thesis*
  by (*metis X retract_of_space_imp_closedin retraction_maps_section_image1*

    homeomorphic_completely_metrizable_space completely_metrizable_space_closedin

    completely_metrizable_imp_metrizable_space metrizable_imp_Hausdorff_space
s)
qed

6.7.15 Product metric

For the nicest fit with the main Euclidean theories, we choose the Euclidean product, though other definitions of the product work.

definition *prod_dist* ≡ λ*d1 d2* (*x*,*y*) (*x*′,*y*′). *sqrt*(*d1 x x*′ ^ *2* + *d2 y y*′ ^ *2*)

locale *Metric_space12* = *M1*: *Metric_space M1 d1* + *M2*: *Metric_space M2 d2*
for *M1 d1 M2 d2*

lemma (**in** *Metric_space12*) *prod_metric*: *Metric_space* (*M1* × *M2*) (*prod_dist d1 d2*)
proof
 fix *x y z*
 assume *xyz*: *x* ∈ *M1* × *M2 y* ∈ *M1* × *M2 z* ∈ *M1* × *M2*
 have *sqrt* ((*d1 x1 z1*)^2 + (*d2 x2 z2*)^2) ≤ *sqrt* ((*d1 x1 y1*)^2 + (*d2 x2 y2*)^2) + *sqrt* ((*d1 y1 z1*)^2 + (*d2 y2 z2*)^2)
   (**is** *sqrt ?L* ≤ *?R*)
  if *x* = (*x1*, *x2*) *y* = (*y1*, *y2*) *z* = (*z1*, *z2*)
  for *x1 x2 y1 y2 z1 z2*
 proof −
  have *tri*: *d1 x1 z1* ≤ *d1 x1 y1* + *d1 y1 z1 d2 x2 z2* ≤ *d2 x2 y2* + *d2 y2 z2*
   using *that xyz M1.triangle* [*of x1 y1 z1*] *M2.triangle* [*of x2 y2 z2*] **by** *auto*
  show *?thesis*
  proof (*rule real_le_lsqrt*)
   have *?L* ≤ (*d1 x1 y1* + *d1 y1 z1*)^2 + (*d2 x2 y2* + *d2 y2 z2*)^2

using *tri* **by** (*smt* (*verit*) *M1.nonneg M2.nonneg power_mono*)
also have ... $\leq$ *?R*^{2}
by (*metis real_sqrt_sum_squares_triangle_ineq sqrt_le_D*)
finally show *?L* $\leq$ *?R*^{2} **.**
qed *auto*
qed
then show *prod_dist d1 d2 x z* $\leq$ *prod_dist d1 d2 x y + prod_dist d1 d2 y z*
by (*simp add*: *prod_dist_def case_prod_unfold*)
qed (*auto simp*: *M1.commute M2.commute case_prod_unfold prod_dist_def*)

sublocale *Metric_space12* $\subseteq$ *Prod_metric*: *Metric_space M1*$\times$*M2 prod_dist d1 d2*
by (*simp add*: *prod_metric*)

For easy reference to theorems outside of the locale

lemma *Metric_space12_mspace_mdist*:
Metric_space12 (*mspace m1*) (*mdist m1*) (*mspace m2*) (*mdist m2*)
by (*simp add*: *Metric_space12_def*)

definition *prod_metric* **where**
prod_metric $\equiv$ λ*m1 m2*. *metric* (*mspace m1* $\times$ *mspace m2*, *prod_dist* (*mdist m1*) (*mdist m2*))

lemma *submetric_prod_metric*:
submetric (*prod_metric m1 m2*) (*S* $\times$ *T*) = *prod_metric* (*submetric m1 S*) (*submetric m2 T*)
apply (*simp add*: *prod_metric_def*)
by (*simp add*: *submetric_def Metric_space.mspace_metric Metric_space.mdist_metric Metric_space12.prod_metric Metric_space12_def Times_Int_Times*)

lemma *mspace_prod_metric* [*simp*]:
mspace (*prod_metric m1 m2*) = *mspace m1* $\times$ *mspace m2*
by (*simp add*: *prod_metric_def Metric_space.mspace_metric Metric_space12.prod_metric Metric_space12_mspace_mdist*)

lemma *mdist_prod_metric* [*simp*]:
mdist (*prod_metric m1 m2*) = *prod_dist* (*mdist m1*) (*mdist m2*)
by (*metis Metric_space.mdist_metric Metric_space12.prod_metric Metric_space12_mspace_mdist prod_metric_def*)

lemma *prod_dist_dist* [*simp*]: *prod_dist dist dist* = *dist*
by (*simp add*: *prod_dist_def dist_prod_def fun_eq_iff*)

lemma *prod_metric_euclidean* [*simp*]:
prod_metric euclidean_metric euclidean_metric = *euclidean_metric*
by (*simp add*: *prod_metric_def euclidean_metric_def*)

context *Metric_space12*
begin

lemma *component_le_prod_metric*:
d1 x1 x2 ≤ *prod_dist d1 d2* (*x1*,*y1*) (*x2*,*y2*) *d2 y1 y2* ≤ *prod_dist d1 d2* (*x1*,*y1*) (*x2*,*y2*)
by (*auto simp*: *prod_dist_def*)

lemma *prod_metric_le_components*:
⟦*x1* ∈ *M1*; *y1* ∈ *M1*; *x2* ∈ *M2*; *y2* ∈ *M2*⟧
⟹ *prod_dist d1 d2* (*x1*,*x2*) (*y1*,*y2*) ≤ *d1 x1 y1* + *d2 x2 y2*
by (*auto simp*: *prod_dist_def sqrt_sum_squares_le_sum*)

lemma *mball_prod_metric_subset*:
Prod_metric.mball (*x*,*y*) *r* ⊆ *M1.mball x r* × *M2.mball y r*
by *clarsimp* (*smt* (*verit*, *best*) *component_le_prod_metric*)

lemma *mcball_prod_metric_subset*:
Prod_metric.mcball (*x*,*y*) *r* ⊆ *M1.mcball x r* × *M2.mcball y r*
by *clarsimp* (*smt* (*verit*, *best*) *component_le_prod_metric*)

lemma *mball_subset_prod_metric*:
M1.mball x1 r1 × *M2.mball x2 r2* ⊆ *Prod_metric.mball* (*x1*,*x2*) (*r1* + *r2*)
using *prod_metric_le_components* **by** *force*

lemma *mcball_subset_prod_metric*:
M1.mcball x1 r1 × *M2.mcball x2 r2* ⊆ *Prod_metric.mcball* (*x1*,*x2*) (*r1* + *r2*)
using *prod_metric_le_components* **by** *force*

lemma *mtopology_prod_metric*:
Prod_metric.mtopology = *prod_topology M1.mtopology M2.mtopology*
unfolding *prod_topology_def*
proof (*rule topology_base_unique* [*symmetric*])
fix *U*
assume *U* ∈ {*S* × *T* |*S T*. *openin M1.mtopology S* ∧ *openin M2.mtopology T*}
then obtain *S T* **where** *Ueq*: *U* = *S* × *T*
and *S*: *openin M1.mtopology S* **and** *T*: *openin M2.mtopology T*
by *auto*
have *S* ⊆ *M1*
using *M1.openin_mtopology S* **by** *auto*
have *T* ⊆ *M2*
using *M2.openin_mtopology T* **by** *auto*
show *openin Prod_metric.mtopology U*
unfolding *Prod_metric.openin_mtopology*
proof (*intro conjI strip*)
show *U* ⊆ *M1* × *M2*
using *Ueq* **by** (*simp add*: *Sigma_mono* ‹*S* ⊆ *M1*› ‹*T* ⊆ *M2*›)
fix *z*
assume *z* ∈ *U*
then obtain *x1 x2* **where** *x1* ∈ *S x2* ∈ *T* **and** *zeq*: *z* = (*x1*,*x2*)
using *Ueq* **by** *blast*

```
    obtain r1 where r1>0 and r1: M1.mball x1 r1 ⊆ S
      by (meson M1.openin_mtopology ‹openin M1.mtopology S› ‹x1 ∈ S›)
    obtain r2 where r2>0 and r2: M2.mball x2 r2 ⊆ T
      by (meson M2.openin_mtopology ‹openin M2.mtopology T› ‹x2 ∈ T›)
    have Prod_metric.mball (x1,x2) (min r1 r2) ⊆ U
    proof (rule order_trans [OF mball_prod_metric_subset])
      show M1.mball x1 (min r1 r2) × M2.mball x2 (min r1 r2) ⊆ U
        using Ueq r1 r2 by force
    qed
    then show ∃r>0. Prod_metric.mball z r ⊆ U
      by (smt (verit, del_insts) zeq ‹0 < r1› ‹0 < r2›)
  qed
next
  fix U z
  assume openin Prod_metric.mtopology U and z ∈ U
  then have U ⊆ M1 × M2
    by (simp add: Prod_metric.openin_mtopology)
  then obtain x y where x ∈ M1 y ∈ M2 and zeq: z = (x,y)
    using ‹z ∈ U› by blast
  obtain r where r>0 and r: Prod_metric.mball (x,y) r ⊆ U
    by (metis Prod_metric.openin_mtopology ‹openin Prod_metric.mtopology U›
‹z ∈ U› zeq)
  define B1 where B1 ≡ M1.mball x (r/2)
  define B2 where B2 ≡ M2.mball y (r/2)
  have openin M1.mtopology B1 openin M2.mtopology B2
    by (simp_all add: B1_def B2_def)
  moreover have (x,y) ∈ B1 × B2
    using ‹r > 0› by (simp add: ‹x ∈ M1› ‹y ∈ M2› B1_def B2_def)
  moreover have B1 × B2 ⊆ U
    using r prod_metric_le_components by (force simp: B1_def B2_def)
  ultimately show ∃B. B ∈ {S × T |S T. openin M1.mtopology S ∧ openin
M2.mtopology T} ∧ z ∈ B ∧ B ⊆ U
    by (auto simp: zeq)
qed

lemma MCauchy_prod_metric:
  Prod_metric.MCauchy σ ⟷ M1.MCauchy (fst ∘ σ) ∧ M2.MCauchy (snd ∘ σ)
  (is ?lhs ⟷ ?rhs)
proof safe
  assume L: ?lhs
  then have range σ ⊆ M1 × M2
    using Prod_metric.MCauchy_def by blast
  then have 1: range (fst ∘ σ) ⊆ M1 and 2: range (snd ∘ σ) ⊆ M2
    by auto
  have N1: ∃N. ∀n≥N. ∀n'≥N. d1 (fst (σ n)) (fst (σ n')) < ε
    and N2: ∃N. ∀n≥N. ∀n'≥N. d2 (snd (σ n)) (snd (σ n')) < ε if ε>0 for ε
:: real
    using that L unfolding Prod_metric.MCauchy_def
  by (smt (verit, del_insts) add.commute add_less_imp_less_left add_right_mono
```

```
      component_le_prod_metric prod.collapse)+
  show M1.MCauchy (fst ∘ σ)
    using 1 N1 M1.MCauchy_def by auto
  have ∃N. ∀n≥N. ∀n'≥N. d2 (snd (σ n)) (snd (σ n')) < ε if ε>0 for ε :: real
    using that L unfolding Prod_metric.MCauchy_def
   by (smt (verit, del_insts) add.commute add_less_imp_less_left add_right_mono

      component_le_prod_metric prod.collapse)
  show M2.MCauchy (snd ∘ σ)
    using 2 N2 M2.MCauchy_def by auto
next
  assume M1: M1.MCauchy (fst ∘ σ) and M2: M2.MCauchy (snd ∘ σ)
  then have subM12: range (fst ∘ σ) ⊆ M1 range (snd ∘ σ) ⊆ M2
    using M1.MCauchy_def M2.MCauchy_def by blast+
  show ?lhs
    unfolding Prod_metric.MCauchy_def
  proof (intro conjI strip)
    show range σ ⊆ M1 × M2
    using subM12 by (smt (verit, best) SigmaI image_subset_iff o_apply prod.collapse)

    fix ε :: real
    assume ε > 0
   obtain N1 where N1: ⋀n n'. N1 ≤ n ⟹ N1 ≤ n' ⟹ d1 ((fst ∘ σ) n) ((fst
∘ σ) n') < ε/2
    by (meson M1.MCauchy_def ‹0 < ε› M1 zero_less_divide_iff zero_less_numeral)
    obtain N2 where N2: ⋀n n'. N2 ≤ n ⟹ N2 ≤ n' ⟹ d2 ((snd ∘ σ) n)
((snd ∘ σ) n') < ε/2
    by (meson M2.MCauchy_def ‹0 < ε› M2 zero_less_divide_iff zero_less_numeral)
    have prod_dist d1 d2 (σ n) (σ n') < ε
      if N1 ≤ n and N2 ≤ n and N1 ≤ n' and N2 ≤ n' for n n'
    proof −
      obtain a b a' b' where σ: σ n = (a,b) σ n' = (a',b')
        by fastforce+
      have prod_dist d1 d2 (a,b) (a',b') ≤ d1 a a' + d2 b b'
     by (metis ‹range σ ⊆ M1 × M2› σ mem_Sigma_iff prod_metric_le_components
range_subsetD)
      also have ... < ε/2 + ε/2
        using N1 N2 σ that by fastforce
      finally show ?thesis
        by (simp add: σ)
    qed
    then show ∃N. ∀n n'. N ≤ n ⟶ N ≤ n' ⟶ prod_dist d1 d2 (σ n) (σ n')
< ε
      by (metis order.trans linorder_le_cases)
  qed
qed
```

lemma *mcomplete_prod_metric*:
Prod_metric.mcomplete $\longleftrightarrow$ *M1* $= \{\} \vee$ *M2* $= \{\} \vee$ *M1.mcomplete* $\wedge$ *M2.mcomplete*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof (*cases M1* $= \{\} \vee$ *M2* $= \{\}$)
case *False*
then obtain *x y* **where** $x \in$ *M1* $y \in$ *M2*
by *blast*
have *M1.mcomplete* $\wedge$ *M2.mcomplete* $\Longrightarrow$ *Prod_metric.mcomplete*
by (*simp add*: *Prod_metric.mcomplete_def M1.mcomplete_def M2.mcomplete_def*
mtopology_prod_metric MCauchy_prod_metric limitin_pairwise)
moreover
{ assume *L*: *Prod_metric.mcomplete*
have *M1.mcomplete*
unfolding *M1.mcomplete_def*
proof (*intro strip*)
fix σ
assume *M1.MCauchy* σ
then have *Prod_metric.MCauchy* ($\lambda n.$ (σ *n*, *y*))
using ‹$y \in$ *M2*› **by** (*simp add*: *M1.MCauchy_def M2.MCauchy_def MCauchy_prod_metric*)
then obtain *z* **where** *limitin Prod_metric.mtopology* ($\lambda n.$ (σ *n*, *y*)) *z sequentially*
using *L Prod_metric.mcomplete_def* **by** *blast*
then show $\exists x.$ *limitin M1.mtopology* σ *x sequentially*
by (*auto simp*: *Prod_metric.mcomplete_def M1.mcomplete_def*
mtopology_prod_metric limitin_pairwise o_def)
qed
}
moreover
{ assume *L*: *Prod_metric.mcomplete*
have *M2.mcomplete*
unfolding *M2.mcomplete_def*
proof (*intro strip*)
fix σ
assume *M2.MCauchy* σ
then have *Prod_metric.MCauchy* ($\lambda n.$ (*x*, σ *n*))
using ‹$x \in$ *M1*› **by** (*simp add*: *M2.MCauchy_def M1.MCauchy_def MCauchy_prod_metric*)
then obtain *z* **where** *limitin Prod_metric.mtopology* ($\lambda n.$ (*x*, σ *n*)) *z sequentially*
using *L Prod_metric.mcomplete_def* **by** *blast*
then show $\exists x.$ *limitin M2.mtopology* σ *x sequentially*
by (*auto simp*: *Prod_metric.mcomplete_def M2.mcomplete_def*
mtopology_prod_metric limitin_pairwise o_def)
qed
}
ultimately show *?thesis*
using *False* **by** *blast*

qed *auto*

lemma *mbounded_prod_metric*:
Prod_metric.mbounded U $\longleftrightarrow$ *M1.mbounded* (*fst* ‘ *U*) ∧ *M2.mbounded* (*snd* ‘ *U*)
proof −
have (∃ *B*. *U* ⊆ *Prod_metric.mcball* (*x*,*y*) *B*)
$\longleftrightarrow$ ((∃ *B*. (*fst* ‘ *U*) ⊆ *M1.mcball x B*) ∧ (∃ *B*. (*snd* ‘ *U*) ⊆ *M2.mcball y B*))
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
for *x y*
proof *safe*
fix *B*
assume *U* ⊆ *Prod_metric.mcball* (*x*, *y*) *B*
then have (*fst* ‘ *U*) ⊆ *M1.mcball x B* (*snd* ‘ *U*) ⊆ *M2.mcball y B*
using *mcball_prod_metric_subset* **by** *fastforce+*
then show ∃ *B*. (*fst* ‘ *U*) ⊆ *M1.mcball x B* ∃ *B*. (*snd* ‘ *U*) ⊆ *M2.mcball y B*
by *auto*
next
fix *B1 B2*
assume (*fst* ‘ *U*) ⊆ *M1.mcball x B1* (*snd* ‘ *U*) ⊆ *M2.mcball y B2*
then have *fst* ‘ *U* × *snd* ‘ *U* ⊆ *M1.mcball x B1* × *M2.mcball y B2*
by *blast*
also have ... ⊆ *Prod_metric.mcball* (*x*, *y*) (*B1*+*B2*)
by (*intro mcball_subset_prod_metric*)
finally show ∃ *B*. *U* ⊆ *Prod_metric.mcball* (*x*, *y*) *B*
by (*metis subsetD subsetI subset_fst_snd*)
qed
then show *?thesis*
by (*simp add*: *M1.mbounded_def M2.mbounded_def Prod_metric.mbounded_def*)
qed

lemma *mbounded_Times*:
Prod_metric.mbounded (*S* × *T*) $\longleftrightarrow$ *S* = {} ∨ *T* = {} ∨ *M1.mbounded S* ∧ *M2.mbounded T*
by (*auto simp*: *mbounded_prod_metric*)

lemma *mtotally_bounded_Times*:
Prod_metric.mtotally_bounded (*S* × *T*) $\longleftrightarrow$
S = {} ∨ *T* = {} ∨ *M1.mtotally_bounded S* ∧ *M2.mtotally_bounded T*
(**is** *?lhs* $\longleftrightarrow$ _)
proof (*cases S* = {} ∨ *T* = {})
case *False*
then obtain *x y* **where** *x* ∈ *S y* ∈ *T*
by *auto*
have *M1.mtotally_bounded S* **if** *L*: *?lhs*
unfolding *M1.mtotally_bounded_sequentially*
proof (*intro conjI strip*)
show *S* ⊆ *M1*

```
    using Prod_metric.mtotally_bounded_imp_subset ‹y ∈ T› that by blast
   fix σ :: nat ⇒ 'a
   assume range σ ⊆ S
   with L obtain r where strict_mono r Prod_metric.MCauchy ((λn. (σ n,y))
∘ r)
    unfolding Prod_metric.mtotally_bounded_sequentially
    by (smt (verit) SigmaI ‹y ∈ T› image_subset_iff)
   then have M1.MCauchy (fst ∘ (λn. (σ n,y)) ∘ r)
    by (simp add: MCauchy_prod_metric o_def)
   with ‹strict_mono r› show ∃r. strict_mono r ∧ M1.MCauchy (σ ∘ r)
    by (auto simp: o_def)
 qed
 moreover
 have M2.mtotally_bounded T if L: ?lhs
  unfolding M2.mtotally_bounded_sequentially
 proof (intro conjI strip)
  show T ⊆ M2
   using Prod_metric.mtotally_bounded_imp_subset ‹x ∈ S› that by blast
  fix σ :: nat ⇒ 'b
  assume range σ ⊆ T
  with L obtain r where strict_mono r Prod_metric.MCauchy ((λn. (x,σ n))
∘ r)
   unfolding Prod_metric.mtotally_bounded_sequentially
   by (smt (verit) SigmaI ‹x ∈ S› image_subset_iff)
  then have M2.MCauchy (snd ∘ (λn. (x,σ n)) ∘ r)
   by (simp add: MCauchy_prod_metric o_def)
  with ‹strict_mono r› show ∃r. strict_mono r ∧ M2.MCauchy (σ ∘ r)
   by (auto simp: o_def)
 qed
 moreover have ?lhs if 1: M1.mtotally_bounded S and 2: M2.mtotally_bounded
T
  unfolding Prod_metric.mtotally_bounded_sequentially
 proof (intro conjI strip)
  show S × T ⊆ M1 × M2
   using that
   by (auto simp: M1.mtotally_bounded_sequentially M2.mtotally_bounded_sequentially)
  fix σ :: nat ⇒ 'a × 'b
  assume σ: range σ ⊆ S × T
  with 1 obtain r1 where r1: strict_mono r1 M1.MCauchy (fst ∘ σ ∘ r1)
    by (metis M1.mtotally_bounded_sequentially comp_apply image_subset_iff
mem_Sigma_iff prod.collapse)
  from σ 2 obtain r2 where r2: strict_mono r2 M2.MCauchy (snd ∘ σ ∘ r1 ∘
r2)
    apply (clarsimp simp: M2.mtotally_bounded_sequentially image_subset_iff)
    by (smt (verit, best) comp_apply mem_Sigma_iff prod.collapse)
  then have M1.MCauchy (fst ∘ σ ∘ r1 ∘ r2)
   by (simp add: M1.MCauchy_subsequence r1)
  with r2 have Prod_metric.MCauchy (σ ∘ (r1 ∘ r2))
   by (simp add: MCauchy_prod_metric o_def)
```

```
    then show ∃r. strict_mono r ∧ Prod_metric.MCauchy (σ ∘ r)
      using r1 r2 strict_mono_o by blast
  qed
  ultimately show ?thesis
    using False by blast
qed auto

lemma mtotally_bounded_prod_metric:
   Prod_metric.mtotally_bounded U ⟷
   M1.mtotally_bounded (fst ‘ U) ∧ M2.mtotally_bounded (snd ‘ U) (is ?lhs ⟷
?rhs)
proof
  assume L: ?lhs
  then have U ⊆ M1 × M2
   and ∗: ⋀σ. range σ ⊆ U ⟹ ∃r::nat⇒nat. strict_mono r ∧ Prod_metric.MCauchy
(σ∘r)
    by (simp_all add: Prod_metric.mtotally_bounded_sequentially)
  show ?rhs
   unfolding M1.mtotally_bounded_sequentially M2.mtotally_bounded_sequentially
  proof (intro conjI strip)
    show fst ‘ U ⊆ M1 snd ‘ U ⊆ M2
      using ‹U ⊆ M1 × M2› by auto
  next
    fix σ :: nat ⇒ 'a
    assume range σ ⊆ fst ‘ U
    then obtain ζ where ζ: ⋀n. σ n = fst (ζ n) ∧ ζ n ∈ U
      unfolding image_subset_iff image_iff by (meson UNIV_I)
    then obtain r where strict_mono r ∧ Prod_metric.MCauchy (ζ∘r)
      by (metis ∗ image_subset_iff)
    with ζ show ∃r. strict_mono r ∧ M1.MCauchy (σ ∘ r)
      by (auto simp: MCauchy_prod_metric o_def)
  next
    fix σ:: nat ⇒ 'b
    assume range σ ⊆ snd ‘ U
    then obtain ζ where ζ: ⋀n. σ n = snd (ζ n) ∧ ζ n ∈ U
      unfolding image_subset_iff image_iff by (meson UNIV_I)
    then obtain r where strict_mono r ∧ Prod_metric.MCauchy (ζ∘r)
      by (metis ∗ image_subset_iff)
    with ζ show ∃r. strict_mono r ∧ M2.MCauchy (σ ∘ r)
      by (auto simp: MCauchy_prod_metric o_def)
  qed
next
  assume ?rhs
  then have Prod_metric.mtotally_bounded ((fst ‘ U) × (snd ‘ U))
    by (simp add: mtotally_bounded_Times)
  then show ?lhs
    by (metis Prod_metric.mtotally_bounded_subset subset_fst_snd)
qed
```

end

lemma *metrizable_space_prod_topology*:
metrizable_space (*prod_topology X Y*) ⟷
(*prod_topology X Y*) = *trivial_topology* ∨ *metrizable_space X* ∧ *metrizable_space Y*
(**is** *?lhs* ⟷ *?rhs*)
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
case *False*
then obtain *x y* **where** *x* ∈ *topspace X y* ∈ *topspace Y*
by *fastforce*
show *?thesis*
proof
show *?rhs* ⟹ *?lhs*
unfolding *metrizable_space_def*
using *Metric_space12.mtopology_prod_metric*
by (*metis False Metric_space12.prod_metric Metric_space12_def*)
next
assume *L*: *?lhs*
have *metrizable_space* (*subtopology* (*prod_topology X Y*) (*topspace X* × {*y*}))
metrizable_space (*subtopology* (*prod_topology X Y*) ({*x*} × *topspace Y*))
using *L metrizable_space_subtopology* **by** *auto*
moreover
have (*subtopology* (*prod_topology X Y*) (*topspace X* × {*y*})) *homeomorphic_space X*
by (*metis* ‹*y* ∈ *topspace Y*› *homeomorphic_space_prod_topology_sing1 homeomorphic_space_sym prod_topology_subtopology*(*2*))
moreover
have (*subtopology* (*prod_topology X Y*) ({*x*} × *topspace Y*)) *homeomorphic_space Y*
by (*metis* ‹*x* ∈ *topspace X*› *homeomorphic_space_prod_topology_sing2 homeomorphic_space_sym prod_topology_subtopology*(*1*))
ultimately show *?rhs*
by (*simp add*: *homeomorphic_metrizable_space*)
qed
qed *auto*

lemma *completely_metrizable_space_prod_topology*:
completely_metrizable_space (*prod_topology X Y*) ⟷
(*prod_topology X Y*) = *trivial_topology* ∨
completely_metrizable_space X ∧ *completely_metrizable_space Y*
(**is** *?lhs* ⟷ *?rhs*)
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
case *False*
then obtain *x y* **where** *x* ∈ *topspace X y* ∈ *topspace Y*
by *fastforce*
show *?thesis*

proof
show *?rhs* $\Longrightarrow$ *?lhs*
unfolding *completely_metrizable_space_def*
by (*metis False Metric_space12.mtopology_prod_metric Metric_space12.mcomplete_prod_metric Metric_space12.prod_metric Metric_space12_def*)
next
assume *L*: *?lhs*
then have *Hausdorff_space* (*prod_topology X Y*)
by (*simp add*: *completely_metrizable_imp_metrizable_space metrizable_imp_Hausdorff_space*)
then have *H*: *Hausdorff_space X* $\wedge$ *Hausdorff_space Y*
using *False Hausdorff_space_prod_topology* **by** *blast*
then have *closedin* (*prod_topology X Y*) (*topspace X* $\times$ {*y*}) $\wedge$ *closedin* (*prod_topology X Y*) ({*x*} $\times$ *topspace Y*)
using ‹*x* $\in$ *topspace X*› ‹*y* $\in$ *topspace Y*›
by (*auto simp*: *closedin_Hausdorff_sing_eq closedin_prod_Times_iff*)
with *L* **have** *completely_metrizable_space*(*subtopology* (*prod_topology X Y*) (*topspace X* $\times$ {*y*}))
$\wedge$ *completely_metrizable_space*(*subtopology* (*prod_topology X Y*) ({*x*} $\times$ *topspace Y*))
by (*simp add*: *completely_metrizable_space_closedin*)
moreover
have (*subtopology* (*prod_topology X Y*) (*topspace X* $\times$ {*y*})) *homeomorphic_space X*
by (*metis* ‹*y* $\in$ *topspace Y*› *homeomorphic_space_prod_topology_sing1 homeomorphic_space_sym prod_topology_subtopology*(*2*))
moreover
have (*subtopology* (*prod_topology X Y*) ({*x*} $\times$ *topspace Y*)) *homeomorphic_space Y*
by (*metis* ‹*x* $\in$ *topspace X*› *homeomorphic_space_prod_topology_sing2 homeomorphic_space_sym prod_topology_subtopology*(*1*))
ultimately show *?rhs*
by (*simp add*: *homeomorphic_completely_metrizable_space*)
qed
next
case *True* **then show** *?thesis*
using *empty_completely_metrizable_space* **by** *auto*
qed

6.7.16 More sequential characterizations in a metric space

context *Metric_space*
begin

definition *decreasing_dist* :: (*nat* $\Rightarrow$ *'a*) $\Rightarrow$ *'a* $\Rightarrow$ *bool*
where *decreasing_dist* σ *x* $\equiv$ ($\forall$ *m n*. *m* < *n* $\longrightarrow$ *d* (σ *n*) *x* < *d* (σ *m*) *x*)

lemma *decreasing_dist_imp_inj*: *decreasing_dist* σ *a* $\Longrightarrow$ *inj* σ
by (*metis decreasing_dist_def dual_order.irrefl linorder_inj_onI'*)

lemma *eventually_atin_within_metric*:
eventually P (atin_within mtopology a S) $\longleftrightarrow$
$(a \in M \longrightarrow (\exists \delta{>}0.\ \forall x.\ x \in M \wedge x \in S \wedge 0 < d\ x\ a \wedge d\ x\ a < \delta \longrightarrow P\ x))$
(**is** *?lhs=?rhs*)
proof
assume *?lhs* **then show** *?rhs*
unfolding *eventually_atin_within openin_mtopology subset_iff*
by (*metis commute in_mball mdist_zero order_less_irrefl topspace_mtopology*)
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases a* $\in$ *M*)
case *True*
then obtain δ **where** $\delta > 0$ **and** δ: $\bigwedge x.\ [\![x \in M;\ x \in S;\ 0 < d\ x\ a;\ d\ x\ a < \delta]\!] \Longrightarrow P\ x$
using *R* **by** *blast*
then have *openin mtopology* (*mball a* δ) $\wedge$ ($\forall x \in$ *mball a* δ. $x \in S \wedge x \neq a \longrightarrow P\ x$)
by (*simp add*: *commute openin_mball*)
then show *?thesis*
by (*metis True* ‹0 < δ› *centre_in_mball_iff eventually_atin_within*)
next
case *False*
with *R* **show** *?thesis*
by (*simp add*: *eventually_atin_within*)
qed
qed

lemma *eventually_atin_within_A*:
assumes
($\bigwedge\sigma$. $[\![$*range* $\sigma \subseteq (S \cap M) - \{a\}$; *decreasing_dist* σ *a*;
inj σ; *limitin mtopology* σ *a sequentially*$]\!]$
$\Longrightarrow$ *eventually* ($\lambda n.$ *P* (σ *n*)) *sequentially*)
shows *eventually P* (*atin_within mtopology a S*)
proof −
have *False* **if** *SP*: $\bigwedge\delta.\ \delta{>}0 \Longrightarrow \exists x \in M{-}\{a\}.\ d\ x\ a < \delta \wedge x \in S \wedge \neg\ P\ x$ **and** $a \in M$
proof −
define Φ **where** $\Phi \equiv \lambda n\ x.\ x \in M{-}\{a\} \wedge d\ x\ a < inverse\ (Suc\ n) \wedge x \in S \wedge \neg\ P\ x$
obtain σ **where** σ: $\bigwedge n.\ \Phi\ n\ (\sigma\ n)$ **and** *dless*: $\bigwedge n.\ d\ (\sigma(Suc\ n))\ a < d\ (\sigma\ n)\ a$
proof −
obtain *x0* **where** *x0*: Φ *0 x0*
using *SP* [*OF zero_less_one*] **by** (*force simp*: Φ*_def*)
have $\exists y.\ \Phi\ (Suc\ n)\ y \wedge d\ y\ a < d\ x\ a$ **if** $\Phi\ n\ x$ **for** *n x*
using *SP* [*of min* (*inverse* (*Suc* (*Suc n*))) (*d x a*)] ‹*a* $\in$ *M*› *that*
by (*auto simp*: Φ*_def*)
then obtain *f* **where** *f*: $\bigwedge n\ x.\ \Phi\ n\ x \Longrightarrow \Phi\ (Suc\ n)\ (f\ n\ x) \wedge d\ (f\ n\ x)\ a <$

```
d x a
      by metis
    show thesis
      proof
        show Φ n (rec_nat x0 f n) for n
          by (induction n) (auto simp: x0 dest: f)
        with f show d (rec_nat x0 f (Suc n)) a < d (rec_nat x0 f n) a for n
          by auto
      qed
  qed
  have 1: range σ ⊆ (S ∩ M) − {a}
    using σ by (auto simp: Φ_def)
  have d (σ(Suc (m+n))) a < d (σ n) a for m n
    by (induction m) (auto intro: order_less_trans dless)
  then have 2: decreasing_dist σ a
    unfolding decreasing_dist_def by (metis add.commute less_imp_Suc_add)
  have ∀F xa in sequentially. d (σ xa) a < ε if ε > 0 for ε
  proof −
    obtain N where inverse (Suc N) < ε
      using ‹ε > 0› reals_Archimedean by blast
    with σ 2 show ?thesis
    unfolding decreasing_dist_def by (smt (verit, best) Φ_def eventually_at_top_dense)
  qed
  then have 4: limitin mtopology σ a sequentially
    using σ ‹a ∈ M› by (simp add: Φ_def limitin_metric)
  show False
   using 2 assms [OF 1 _ decreasing_dist_imp_inj 4] σ by (force simp: Φ_def)
 qed
 then show ?thesis
  by (fastforce simp: eventually_atin_within_metric)
qed


lemma eventually_atin_within_B:
 assumes ev: eventually P (atin_within mtopology a S)
   and ran: range σ ⊆ (S ∩ M) − {a}
   and lim: limitin mtopology σ a sequentially
 shows eventually (λn. P (σ n)) sequentially
proof −
 have a ∈ M
   using lim limitin_mspace by auto
 with ev obtain δ where 0 < δ
   and δ: ⋀σ. ⟦σ ∈ M; σ ∈ S; 0 < d σ a; d σ a < δ⟧ ⟹ P σ
   by (auto simp: eventually_atin_within_metric)
 then have ∗: ⋀n. σ n ∈ M ∧ d (σ n) a < δ ⟹ P (σ n)
   using ‹a ∈ M› ran by auto
 have ∀F n in sequentially. σ n ∈ M ∧ d (σ n) a < δ
   using lim ‹0 < δ› by (auto simp: limitin_metric)
 then show ?thesis
```

by (*simp add*: ∗ *eventually_mono*)
qed

lemma *eventually_atin_within_sequentially*:
eventually P (*atin_within mtopology a S*) ⟷
(∀ σ. *range* σ ⊆ (*S* ∩ *M*) − {*a*} ∧
limitin mtopology σ *a sequentially*
⟶ *eventually* (λ*n*. *P*(σ *n*)) *sequentially*)
by (*metis eventually_atin_within_A eventually_atin_within_B*)

lemma *eventually_atin_within_sequentially_inj*:
eventually P (*atin_within mtopology a S*) ⟷
(∀ σ. *range* σ ⊆ (*S* ∩ *M*) − {*a*} ∧ *inj* σ ∧
limitin mtopology σ *a sequentially*
⟶ *eventually* (λ*n*. *P*(σ *n*)) *sequentially*)
by (*metis eventually_atin_within_A eventually_atin_within_B*)

lemma *eventually_atin_within_sequentially_decreasing*:
eventually P (*atin_within mtopology a S*) ⟷
(∀ σ. *range* σ ⊆ (*S* ∩ *M*) − {*a*} ∧ *decreasing_dist* σ *a* ∧
limitin mtopology σ *a sequentially*
⟶ *eventually* (λ*n*. *P*(σ *n*)) *sequentially*)
by (*metis eventually_atin_within_A eventually_atin_within_B*)

lemma *eventually_atin_sequentially*:
eventually P (*atin mtopology a*) ⟷
(∀ σ. *range* σ ⊆ *M* − {*a*} ∧ *limitin mtopology* σ *a sequentially*
⟶ *eventually* (λ*n*. *P*(σ *n*)) *sequentially*)
using *eventually_atin_within_sequentially* [**where** *S*=*UNIV*] **by** *simp*

lemma *eventually_atin_sequentially_inj*:
eventually P (*atin mtopology a*) ⟷
(∀ σ. *range* σ ⊆ *M* − {*a*} ∧ *inj* σ ∧
limitin mtopology σ *a sequentially*
⟶ *eventually* (λ*n*. *P*(σ *n*)) *sequentially*)
using *eventually_atin_within_sequentially_inj* [**where** *S*=*UNIV*] **by** *simp*

lemma *eventually_atin_sequentially_decreasing*:
eventually P (*atin mtopology a*) ⟷
(∀ σ. *range* σ ⊆ *M* − {*a*} ∧ *decreasing_dist* σ *a* ∧
limitin mtopology σ *a sequentially*
⟶ *eventually* (λ*n*. *P*(σ *n*)) *sequentially*)
using *eventually_atin_within_sequentially_decreasing* [**where** *S*=*UNIV*] **by** *simp*

end

context *Metric_space12*

begin

lemma *limit_atin_sequentially_within*:
limitin M2.mtopology f l (*atin_within M1.mtopology a S*) ⟷
l ∈ *M2* ∧
(∀ σ. *range* σ ⊆ *S* ∩ *M1* − {*a*} ∧
limitin M1.mtopology σ *a sequentially*
⟶ *limitin M2.mtopology* (*f* ∘ σ) *l sequentially*)
by (*auto simp*: *M1.eventually_atin_within_sequentially limitin_def*)

lemma *limit_atin_sequentially_within_inj*:
limitin M2.mtopology f l (*atin_within M1.mtopology a S*) ⟷
l ∈ *M2* ∧
(∀ σ. *range* σ ⊆ *S* ∩ *M1* − {*a*} ∧ *inj* σ ∧
limitin M1.mtopology σ *a sequentially*
⟶ *limitin M2.mtopology* (*f* ∘ σ) *l sequentially*)
by (*auto simp*: *M1.eventually_atin_within_sequentially_inj limitin_def*)

lemma *limit_atin_sequentially_within_decreasing*:
limitin M2.mtopology f l (*atin_within M1.mtopology a S*) ⟷
l ∈ *M2* ∧
(∀ σ. *range* σ ⊆ *S* ∩ *M1* − {*a*} ∧ *M1.decreasing_dist* σ *a* ∧
limitin M1.mtopology σ *a sequentially*
⟶ *limitin M2.mtopology* (*f* ∘ σ) *l sequentially*)
by (*auto simp*: *M1.eventually_atin_within_sequentially_decreasing limitin_def*)

lemma *limit_atin_sequentially*:
limitin M2.mtopology f l (*atin M1.mtopology a*) ⟷
l ∈ *M2* ∧
(∀ σ. *range* σ ⊆ *M1* − {*a*} ∧
limitin M1.mtopology σ *a sequentially*
⟶ *limitin M2.mtopology* (*f* ∘ σ) *l sequentially*)
using *limit_atin_sequentially_within* [**where** *S*=*UNIV*] **by** *simp*

lemma *limit_atin_sequentially_inj*:
limitin M2.mtopology f l (*atin M1.mtopology a*) ⟷
l ∈ *M2* ∧
(∀ σ. *range* σ ⊆ *M1* − {*a*} ∧ *inj* σ ∧
limitin M1.mtopology σ *a sequentially*
⟶ *limitin M2.mtopology* (*f* ∘ σ) *l sequentially*)
using *limit_atin_sequentially_within_inj* [**where** *S*=*UNIV*] **by** *simp*

lemma *limit_atin_sequentially_decreasing*:
limitin M2.mtopology f l (*atin M1.mtopology a*) ⟷
l ∈ *M2* ∧
(∀ σ. *range* σ ⊆ *M1* − {*a*} ∧ *M1.decreasing_dist* σ *a* ∧
limitin M1.mtopology σ *a sequentially*
⟶ *limitin M2.mtopology* (*f* ∘ σ) *l sequentially*)
using *limit_atin_sequentially_within_decreasing* [**where** *S*=*UNIV*] **by** *simp*

end

An experiment: same result as within the locale, but using metric space variables

lemma *limit_atin_sequentially_within*:
limitin (*mtopology_of m2*) *f l* (*atin_within* (*mtopology_of m1*) *a S*) ⟷
l ∈ *mspace m2* ∧
(∀ σ. *range* σ ⊆ *S* ∩ *mspace m1* − {*a*} ∧
limitin (*mtopology_of m1*) σ *a sequentially*
⟶ *limitin* (*mtopology_of m2*) (*f* ∘ σ) *l sequentially*)
using *Metric_space12.limit_atin_sequentially_within* [*OF Metric_space12_mspace_mdist*]
by (*metis mtopology_of_def*)

context *Metric_space*
begin

lemma *atin_within_imp_M*:
atin_within mtopology x S ≠ *bot* ⟹ *x* ∈ *M*
by (*metis derived_set_of_trivial_limit in_derived_set_of topspace_mtopology*)

lemma *atin_within_sequentially_sequence*:
assumes *atin_within mtopology x S* ≠ *bot*
obtains σ **where** *range* σ ⊆ *S* ∩ *M* − {*x*}
decreasing_dist σ *x inj* σ *limitin mtopology* σ *x sequentially*
by (*metis eventually_atin_within_A eventually_False assms*)

lemma *derived_set_of_sequentially*:
mtopology derived_set_of S =
{*x* ∈ *M*. ∃ σ. *range* σ ⊆ *S* ∩ *M* − {*x*} ∧ *limitin mtopology* σ *x sequentially*}
proof −
have *False*
if *range* σ ⊆ *S* ∩ *M* − {*x*}
and *limitin mtopology* σ *x sequentially*
and *atin_within mtopology x S* = *bot*
for *x* σ
proof −
have ∀ *F n in sequentially*. *P* (σ *n*) **for** *P*
using *that* **by** (*metis eventually_atin_within_B eventually_bot*)
then show *False*
by (*meson eventually_False_sequentially eventually_mono*)
qed
then show *?thesis*
using *derived_set_of_trivial_limit*
by (*fastforce elim*!: *atin_within_sequentially_sequence intro*: *atin_within_imp_M*)
qed

lemma *derived_set_of_sequentially_alt*:

mtopology derived_set_of S =
{*x*. ∃*σ*. *range σ* ⊆ *S* − {*x*} ∧ *limitin mtopology σ x sequentially*}
proof −
have ∗: ∃*σ*. *range σ* ⊆ *S* ∩ *M* − {*x*} ∧ *limitin mtopology σ x sequentially*
if *σ*: *range σ* ⊆ *S* − {*x*} **and** *lim*: *limitin mtopology σ x sequentially* **for** *x σ*
proof −
obtain *N* **where** ∀ *n*≥*N*. *σ n* ∈ *M* ∧ *d* (*σ n*) *x* < *1*
using *lim limit_metric_sequentially* **by** *fastforce*
with *σ* **obtain** *a* **where** *a*:*a* ∈ *S* ∩ *M* − {*x*} **by** *auto*
show *?thesis*
proof (*intro conjI exI*)
show *range* (*λn*. *if σ n* ∈ *M then σ n else a*) ⊆ *S* ∩ *M* − {*x*}
using *a σ* **by** *fastforce*
show *limitin mtopology* (*λn*. *if σ n* ∈ *M then σ n else a*) *x sequentially*
using *lim limit_metric_sequentially* **by** *fastforce*
qed
qed
show *?thesis*
by (*auto simp*: *limitin_mspace derived_set_of_sequentially intro*!: ∗)
qed

lemma *derived_set_of_sequentially_inj*:
mtopology derived_set_of S =
{*x* ∈ *M*. ∃*σ*. *range σ* ⊆ *S* ∩ *M* − {*x*} ∧ *inj σ* ∧ *limitin mtopology σ x sequentially*}
proof −
have *False*
if *x* ∈ *M* **and** *range σ* ⊆ *S* ∩ *M* − {*x*}
and *limitin mtopology σ x sequentially*
and *atin_within mtopology x S* = *bot*
for *x σ*
proof −
have ∀ *F n in sequentially*. *P* (*σ n*) **for** *P*
using *that derived_set_of_sequentially_alt derived_set_of_trivial_limit* **by** *fastforce*
then show *False*
by (*meson eventually_False_sequentially eventually_mono*)
qed
then show *?thesis*
using *derived_set_of_trivial_limit*
by (*fastforce elim*!: *atin_within_sequentially_sequence intro*: *atin_within_imp_M*)
qed

lemma *derived_set_of_sequentially_inj_alt*:
mtopology derived_set_of S =
{*x*. ∃*σ*. *range σ* ⊆ *S* − {*x*} ∧ *inj σ* ∧ *limitin mtopology σ x sequentially*}
proof −
have ∃*σ*. *range σ* ⊆ *S* − {*x*} ∧ *inj σ* ∧ *limitin mtopology σ x sequentially*

if *atin_within mtopology x S ≠ bot* **for** *x*
by (*metis Diff_empty Int_subset_iff atin_within_sequentially_sequence subset_Diff_insert that*)
moreover have *False*
if *range* (λ*x*. *σ* (*x*::*nat*)) ⊆ *S* − {*x*}
and *limitin mtopology σ x sequentially*
and *atin_within mtopology x S* = *bot*
for *x σ*
proof −
have ∀ *F n in sequentially*. *P* (*σ n*) **for** *P*
using *that derived_set_of_sequentially_alt derived_set_of_trivial_limit* **by** *fastforce*
then show *False*
by (*meson eventually_False_sequentially eventually_mono*)
qed
ultimately show *?thesis*
using *derived_set_of_trivial_limit* **by** (*fastforce intro*: *atin_within_imp_M*)
qed

lemma *derived_set_of_sequentially_decreasing*:
mtopology derived_set_of S =
{*x* ∈ *M*. ∃*σ*. *range σ* ⊆ *S* − {*x*} ∧ *decreasing_dist σ x* ∧ *limitin mtopology σ x sequentially*}
proof −
have ∃*σ*. *range σ* ⊆ *S* − {*x*} ∧ *decreasing_dist σ x* ∧ *limitin mtopology σ x sequentially*
if *atin_within mtopology x S* ≠ *bot* **for** *x*
by (*metis Diff_empty atin_within_sequentially_sequence le_infE subset_Diff_insert that*)
moreover have *False*
if *x* ∈ *M* **and** *range σ* ⊆ *S* − {*x*}
and *limitin mtopology σ x sequentially*
and *atin_within mtopology x S* = *bot*
for *x σ*
proof −
have ∀ *F n in sequentially*. *P* (*σ n*) **for** *P*
using *that derived_set_of_sequentially_alt derived_set_of_trivial_limit* **by** *fastforce*
then show *False*
by (*meson eventually_False_sequentially eventually_mono*)
qed
ultimately show *?thesis*
using *derived_set_of_trivial_limit* **by** (*fastforce intro*: *atin_within_imp_M*)
qed

lemma *derived_set_of_sequentially_decreasing_alt*:
mtopology derived_set_of S =
{*x*. ∃*σ*. *range σ* ⊆ *S* − {*x*} ∧ *decreasing_dist σ x* ∧ *limitin mtopology σ x sequentially*}

using *derived_set_of_sequentially_alt derived_set_of_sequentially_decreasing*
by *auto*

lemma *closure_of_sequentially*:
mtopology closure_of S =
{*x* ∈ *M*. ∃*σ*. *range σ* ⊆ *S* ∩ *M* ∧ *limitin mtopology σ x sequentially*}
by (*auto simp*: *closure_of derived_set_of_sequentially*)

end

6.7.17 Three strong notions of continuity for metric spaces

Lipschitz continuity

definition *Lipschitz_continuous_map*
where *Lipschitz_continuous_map* ≡
λ*m1 m2 f*. *f* ∈ *mspace m1* → *mspace m2* ∧
(∃ *B*. ∀ *x* ∈ *mspace m1*. ∀ *y* ∈ *mspace m1*. *mdist m2* (*f x*) (*f y*) ≤ *B* ∗ *mdist m1 x y*)

lemma *Lipschitz_continuous_map_image*:
Lipschitz_continuous_map m1 m2 f ⟹ *f* ∈ *mspace m1* → *mspace m2*
by (*simp add*: *Lipschitz_continuous_map_def*)

lemma *Lipschitz_continuous_map_pos*:
Lipschitz_continuous_map m1 m2 f ⟷
f ∈ *mspace m1* → *mspace m2* ∧
(∃ *B*>*0*. ∀ *x* ∈ *mspace m1*. ∀ *y* ∈ *mspace m1*. *mdist m2* (*f x*) (*f y*) ≤ *B* ∗ *mdist m1 x y*)
proof −
have *B* ∗ *mdist m1 x y* ≤ (|*B*| + *1*) ∗ *mdist m1 x y* |*B*| + *1* > *0* **for** *x y B*
by (*auto simp*: *mult_right_mono*)
then show *?thesis*
unfolding *Lipschitz_continuous_map_def* **by** (*meson dual_order.trans*)
qed

lemma *Lipschitz_continuous_map_eq*:
assumes *Lipschitz_continuous_map m1 m2 f* ⋀*x*. *x* ∈ *mspace m1* ⟹ *f x* = *g x*
shows *Lipschitz_continuous_map m1 m2 g*
using *Lipschitz_continuous_map_def assms* **by** (*simp add*: *Lipschitz_continuous_map_pos Pi_iff*)

lemma *Lipschitz_continuous_map_from_submetric*:
assumes *Lipschitz_continuous_map m1 m2 f*
shows *Lipschitz_continuous_map* (*submetric m1 S*) *m2 f*
unfolding *Lipschitz_continuous_map_def*
proof
show *f* ∈ *mspace* (*submetric m1 S*) → *mspace m2*

using *Lipschitz_continuous_map_pos assms* **by** *fastforce*
qed (*use assms* **in** ‹*fastforce simp*: *Lipschitz_continuous_map_def*›)

lemma *Lipschitz_continuous_map_from_submetric_mono*:
⟦*Lipschitz_continuous_map* (*submetric m1 T*) *m2 f*; *S* ⊆ *T*⟧
⟹ *Lipschitz_continuous_map* (*submetric m1 S*) *m2 f*
by (*metis Lipschitz_continuous_map_from_submetric inf.absorb_iff2 submetric_submetric*)

lemma *Lipschitz_continuous_map_into_submetric*:
Lipschitz_continuous_map m1 (*submetric m2 S*) *f* ⟷
f ∈ *mspace m1* → *S* ∧ *Lipschitz_continuous_map m1 m2 f*
by (*auto simp*: *Lipschitz_continuous_map_def*)

lemma *Lipschitz_continuous_map_const*:
Lipschitz_continuous_map m1 m2 (λ*x*. *c*) ⟷
mspace m1 = {} ∨ *c* ∈ *mspace m2*
unfolding *Lipschitz_continuous_map_def Pi_iff*
by (*metis all_not_in_conv mdist_nonneg mdist_zero mult_1*)

lemma *Lipschitz_continuous_map_id*:
Lipschitz_continuous_map m1 m1 (λ*x*. *x*)
unfolding *Lipschitz_continuous_map_def* **by** (*metis funcset_id mult_1 order_refl*)

lemma *Lipschitz_continuous_map_compose*:
assumes *f*: *Lipschitz_continuous_map m1 m2 f* **and** *g*: *Lipschitz_continuous_map m2 m3 g*
shows *Lipschitz_continuous_map m1 m3* (*g* ∘ *f*)
unfolding *Lipschitz_continuous_map_def*
proof
show *g* ∘ *f* ∈ *mspace m1* → *mspace m3*
by (*smt* (*verit*, *best*) *Lipschitz_continuous_map_image Pi_iff comp_apply f g*)
obtain *B* **where** *B*: ∀ *x*∈*mspace m1*. ∀ *y*∈*mspace m1*. *mdist m2* (*f x*) (*f y*) ≤ *B* ∗ *mdist m1 x y*
using *assms* **unfolding** *Lipschitz_continuous_map_def* **by** *presburger*
obtain *C* **where** *C*>*0* **and** *C*: ∀ *x*∈*mspace m2*. ∀ *y*∈*mspace m2*. *mdist m3* (*g x*) (*g y*) ≤ *C* ∗ *mdist m2 x y*
using *assms* **unfolding** *Lipschitz_continuous_map_pos* **by** *metis*
show ∃ *B*. ∀ *x*∈*mspace m1*. ∀ *y*∈*mspace m1*. *mdist m3* ((*g* ∘ *f*) *x*) ((*g* ∘ *f*) *y*) ≤ *B* ∗ *mdist m1 x y*
proof (*intro strip exI*)
fix *x y*
assume §: *x* ∈ *mspace m1 y* ∈ *mspace m1*
then have *mdist m3* ((*g* ∘ *f*) *x*) ((*g* ∘ *f*) *y*) ≤ *C* ∗ *mdist m2* (*f x*) (*f y*)
using *C Lipschitz_continuous_map_image f* **by** *fastforce*
also have ... ≤ *C* ∗ *B* ∗ *mdist m1 x y*
by (*simp add*: § *B* ‹*0* < *C*›)

finally show *mdist m3* $((g \circ f)\ x)\ ((g \circ f)\ y) \leq C * B * mdist\ m1\ x\ y$ **.**
qed
qed

Uniform continuity

definition *uniformly_continuous_map*
where *uniformly_continuous_map* $\equiv$
$\lambda m1\ m2\ f.\ f \in mspace\ m1 \to mspace\ m2 \wedge$
$(\forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x \in mspace\ m1.\ \forall y \in mspace\ m1.$
$mdist\ m1\ y\ x < \delta \longrightarrow mdist\ m2\ (f\ y)\ (f\ x) < \varepsilon)$

lemma *uniformly_continuous_map_funspace*:
uniformly_continuous_map m1 m2 f $\Longrightarrow f \in mspace\ m1 \to mspace\ m2$
by (*simp add*: *uniformly_continuous_map_def*)

lemma *ucmap_A*:
assumes *uniformly_continuous_map m1 m2 f*
and $(\lambda n.\ mdist\ m1\ (\varrho\ n)\ (\sigma\ n)) \longrightarrow 0$
and *range* $\varrho \subseteq mspace\ m1$ *range* $\sigma \subseteq mspace\ m1$
shows $(\lambda n.\ mdist\ m2\ (f\ (\varrho\ n))\ (f\ (\sigma\ n))) \longrightarrow 0$
using *assms*
unfolding *uniformly_continuous_map_def image_subset_iff tendsto_iff*
apply *clarsimp*
by (*metis* (*mono_tags*, *lifting*) *eventually_sequentially*)

lemma *ucmap_B*:
assumes §: $\bigwedge \varrho\ \sigma.\ [\![range\ \varrho \subseteq mspace\ m1;\ range\ \sigma \subseteq mspace\ m1;$
$(\lambda n.\ mdist\ m1\ (\varrho\ n)\ (\sigma\ n)) \longrightarrow 0]\!]$
$\Longrightarrow (\lambda n.\ mdist\ m2\ (f\ (\varrho\ n))\ (f\ (\sigma\ n))) \longrightarrow 0$
and $0 < \varepsilon$
and ϱ: *range* $\varrho \subseteq mspace\ m1$
and σ: *range* $\sigma \subseteq mspace\ m1$
and $(\lambda n.\ mdist\ m1\ (\varrho\ n)\ (\sigma\ n)) \longrightarrow 0$
shows $\exists n.\ mdist\ m2\ (f\ (\varrho\ (n::nat)))\ (f\ (\sigma\ n)) < \varepsilon$
using § [*OF* $\varrho\ \sigma$] *assms* **by** (*meson LIMSEQ_le_const linorder_not_less*)

lemma *ucmap_C*:
assumes §: $\bigwedge \varrho\ \sigma\ \varepsilon.\ [\![\varepsilon > 0;\ range\ \varrho \subseteq mspace\ m1;\ range\ \sigma \subseteq mspace\ m1;$
$((\lambda n.\ mdist\ m1\ (\varrho\ n)\ (\sigma\ n)) \longrightarrow 0)]\!]$
$\Longrightarrow \exists n.\ mdist\ m2\ (f\ (\varrho\ n))\ (f\ (\sigma\ n)) < \varepsilon$
and *fim*: $f \in mspace\ m1 \to mspace\ m2$
shows *uniformly_continuous_map m1 m2 f*
proof −
{assume $\neg\ (\forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x{\in}mspace\ m1.\ \forall y{\in}mspace\ m1.\ mdist\ m1\ y\ x < \delta$
$\longrightarrow mdist\ m2\ (f\ y)\ (f\ x) < \varepsilon)$
then obtain ε **where** $\varepsilon > 0$
and $\bigwedge n.\ \exists x{\in}mspace\ m1.\ \exists y{\in}mspace\ m1.\ mdist\ m1\ y\ x < inverse(Suc\ n) \wedge$
$mdist\ m2\ (f\ y)\ (f\ x) \geq \varepsilon$

by (*meson inverse_Suc linorder_not_le*)
then obtain $\varrho\ \sigma$ **where** *space*: *range* $\varrho \subseteq$ *mspace m1 range* $\sigma \subseteq$ *mspace m1*
and *dist*: $\bigwedge n$. *mdist m1* $(\sigma\ n)\ (\varrho\ n) <$ *inverse*(*Suc n*) $\wedge$ *mdist m2* $(f(\sigma\ n))\ (f(\varrho\ n)) \geq \varepsilon$
by (*metis image_subset_iff*)
have *False*
using § [*OF* ‹$\varepsilon > 0$› *space*] *dist Lim_null_comparison*
by (*smt* (*verit*) *LIMSEQ_norm_0 inverse_eq_divide mdist_commute mdist_nonneg real_norm_def*)
}
moreover
have $t \in$ *mspace m2* **if** $t \in f$ ‘ *mspace m1* **for** t
using *fim that* **by** *blast*
ultimately show *?thesis*
by (*fastforce simp*: *uniformly_continuous_map_def*)
qed

lemma *uniformly_continuous_map_sequentially*:
uniformly_continuous_map m1 m2 f $\longleftrightarrow$
$f \in$ *mspace m1* $\to$ *mspace m2* $\wedge$
$(\forall\ \varrho\ \sigma$. *range* $\varrho \subseteq$ *mspace m1* $\wedge$ *range* $\sigma \subseteq$ *mspace m1* $\wedge$ $(\lambda n$. *mdist m1* $(\varrho\ n)\ (\sigma\ n)) \longrightarrow 0$
$\longrightarrow (\lambda n$. *mdist m2* $(f\ (\varrho\ n))\ (f\ (\sigma\ n))) \longrightarrow 0)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*simp add*: *ucmap_A uniformly_continuous_map_funspace*)
show *?rhs* $\Longrightarrow$ *?lhs*
by (*intro ucmap_B ucmap_C*) *auto*
qed

lemma *uniformly_continuous_map_sequentially_alt*:
uniformly_continuous_map m1 m2 f $\longleftrightarrow$
$f \in$ *mspace m1* $\to$ *mspace m2* $\wedge$
$(\forall \varepsilon{>}0.\ \forall\ \varrho\ \sigma$. *range* $\varrho \subseteq$ *mspace m1* $\wedge$ *range* $\sigma \subseteq$ *mspace m1* $\wedge$
$((\lambda n$. *mdist m1* $(\varrho\ n)\ (\sigma\ n)) \longrightarrow 0)$
$\longrightarrow (\exists\ n$. *mdist m2* $(f\ (\varrho\ n))\ (f\ (\sigma\ n)) < \varepsilon))$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
using *uniformly_continuous_map_funspace* **by** (*intro conjI strip ucmap_B* | *fastforce simp*: *ucmap_A*)+
show *?rhs* $\Longrightarrow$ *?lhs*
by (*intro ucmap_C*) *auto*
qed

lemma *uniformly_continuous_map_eq*:

⟦⋀*x*. *x* ∈ *mspace m1* ⟹ *f x* = *g x*; *uniformly_continuous_map m1 m2 f*⟧
⟹ *uniformly_continuous_map m1 m2 g*
by (*simp add*: *uniformly_continuous_map_def Pi_iff*)

lemma *uniformly_continuous_map_from_submetric*:
assumes *uniformly_continuous_map m1 m2 f*
shows *uniformly_continuous_map* (*submetric m1 S*) *m2 f*
unfolding *uniformly_continuous_map_def*
proof
show *f* ∈ *mspace* (*submetric m1 S*) → *mspace m2*
using *assms* **by** (*auto simp*: *uniformly_continuous_map_def*)
qed (*use assms* **in** ‹*force simp*: *uniformly_continuous_map_def*›)

lemma *uniformly_continuous_map_from_submetric_mono*:
⟦*uniformly_continuous_map* (*submetric m1 T*) *m2 f*; *S* ⊆ *T*⟧
⟹ *uniformly_continuous_map* (*submetric m1 S*) *m2 f*
by (*metis uniformly_continuous_map_from_submetric inf.absorb_iff2 submetric_submetric*)

lemma *uniformly_continuous_map_into_submetric*:
uniformly_continuous_map m1 (*submetric m2 S*) *f* ⟷
f ∈ *mspace m1* → *S* ∧ *uniformly_continuous_map m1 m2 f*
by (*auto simp*: *uniformly_continuous_map_def*)

lemma *uniformly_continuous_map_const*:
uniformly_continuous_map m1 m2 (λ*x*. *c*) ⟷
mspace m1 = {} ∨ *c* ∈ *mspace m2*
unfolding *uniformly_continuous_map_def Pi_iff*
by (*metis empty_iff equals0I mdist_zero*)

lemma *uniformly_continuous_map_id* [*simp*]:
uniformly_continuous_map m1 m1 (λ*x*. *x*)
by (*metis funcset_id uniformly_continuous_map_def*)

lemma *uniformly_continuous_map_compose*:
assumes *f*: *uniformly_continuous_map m1 m2 f* **and** *g*: *uniformly_continuous_map m2 m3 g*
shows *uniformly_continuous_map m1 m3* (*g* ∘ *f*)
using *f g* **unfolding** *uniformly_continuous_map_def comp_apply Pi_iff*
by *metis*

lemma *uniformly_continuous_map_real_const* [*simp*]:
uniformly_continuous_map m euclidean_metric (λ*x*. *c*)
by (*simp add*: *euclidean_metric_def uniformly_continuous_map_const*)

Equivalence between "abstract" and "type class" notions

lemma *uniformly_continuous_map_euclidean* [*simp*]:
uniformly_continuous_map (*submetric euclidean_metric S*) *euclidean_metric f*
= *uniformly_continuous_on S f*

by (*auto simp*: *uniformly_continuous_map_def uniformly_continuous_on_def*)

Cauchy continuity

definition *Cauchy_continuous_map* **where**
Cauchy_continuous_map ≡
λ*m1 m2 f*. ∀ σ. *Metric_space.MCauchy* (*mspace m1*) (*mdist m1*) σ
⟶ *Metric_space.MCauchy* (*mspace m2*) (*mdist m2*) (*f* ∘ σ)

lemma *Cauchy_continuous_map_euclidean* [*simp*]:
Cauchy_continuous_map (*submetric euclidean_metric S*) *euclidean_metric f*
= *Cauchy_continuous_on S f*
by (*auto simp*: *Cauchy_continuous_map_def Cauchy_continuous_on_def Cauchy_def Met_TC.subspace Metric_space.MCauchy_def*)

lemma *Cauchy_continuous_map_funspace*:
assumes *Cauchy_continuous_map m1 m2 f*
shows *f* ∈ *mspace m1* → *mspace m2*
proof *clarsimp*
fix *x*
assume *x* ∈ *mspace m1*
then have *Metric_space.MCauchy* (*mspace m1*) (*mdist m1*) (λ*n*. *x*)
by (*simp add*: *Metric_space.MCauchy_const Metric_space_mspace_mdist*)
then have *Metric_space.MCauchy* (*mspace m2*) (*mdist m2*) (*f* ∘ (λ*n*. *x*))
by (*meson Cauchy_continuous_map_def assms*)
then show *f x* ∈ *mspace m2*
by (*simp add*: *Metric_space.MCauchy_def* [*OF Metric_space_mspace_mdist*])
qed

lemma *Cauchy_continuous_map_eq*:
⟦⋀*x*. *x* ∈ *mspace m1* ⟹ *f x* = *g x*; *Cauchy_continuous_map m1 m2 f*⟧
⟹ *Cauchy_continuous_map m1 m2 g*
by (*simp add*: *image_subset_iff Cauchy_continuous_map_def Metric_space.MCauchy_def* [*OF Metric_space_mspace_mdist*])

lemma *Cauchy_continuous_map_from_submetric*:
assumes *Cauchy_continuous_map m1 m2 f*
shows *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
using *assms*
by (*simp add*: *image_subset_iff Cauchy_continuous_map_def Metric_space.MCauchy_def* [*OF Metric_space_mspace_mdist*])

lemma *Cauchy_continuous_map_from_submetric_mono*:
⟦*Cauchy_continuous_map* (*submetric m1 T*) *m2 f*; *S* ⊆ *T*⟧
⟹ *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
by (*metis Cauchy_continuous_map_from_submetric inf.absorb_iff2 submetric_submetric*)

lemma *Cauchy_continuous_map_into_submetric*:
Cauchy_continuous_map m1 (submetric m2 S) f ⟷
f ∈ *mspace m1* → *S* ∧ *Cauchy_continuous_map m1 m2 f* (**is** *?lhs* ⟷ *?rhs*)
proof −
have *?lhs* ⟹ *Cauchy_continuous_map m1 m2 f*
by (*simp add*: *Cauchy_continuous_map_def Metric_space.MCauchy_def* [*OF Metric_space_mspace_mdist*])
moreover have *?rhs* ⟹ *?lhs*
by (*auto simp*: *Cauchy_continuous_map_def Metric_space.MCauchy_def* [*OF Metric_space_mspace_mdist*])
ultimately show *?thesis*
by (*metis Cauchy_continuous_map_funspace Int_iff funcsetI funcset_mem mspace_submetric*)
qed

lemma *Cauchy_continuous_map_const* [*simp*]:
Cauchy_continuous_map m1 m2 (λ*x*. *c*) ⟷ *mspace m1* = {} ∨ *c* ∈ *mspace m2*
proof −
have *mspace m1* = {} ⟹ *Cauchy_continuous_map m1 m2* (λ*x*. *c*)
by (*simp add*: *Cauchy_continuous_map_def Metric_space.MCauchy_def Metric_space_mspace_mdist*)
moreover have *c* ∈ *mspace m2* ⟹ *Cauchy_continuous_map m1 m2* (λ*x*. *c*)
by (*simp add*: *Cauchy_continuous_map_def o_def Metric_space.MCauchy_const* [*OF Metric_space_mspace_mdist*])
ultimately show *?thesis*
using *Cauchy_continuous_map_funspace* **by** *blast*
qed

lemma *Cauchy_continuous_map_id* [*simp*]:
Cauchy_continuous_map m1 m1 (λ*x*. *x*)
by (*simp add*: *Cauchy_continuous_map_def o_def*)

lemma *Cauchy_continuous_map_compose*:
assumes *f*: *Cauchy_continuous_map m1 m2 f* **and** *g*: *Cauchy_continuous_map m2 m3 g*
shows *Cauchy_continuous_map m1 m3* (*g* ∘ *f*)
by (*metis* (*no_types*, *lifting*) *Cauchy_continuous_map_def f fun.map_comp g*)

lemma *Lipschitz_imp_uniformly_continuous_map*:
assumes *Lipschitz_continuous_map m1 m2 f*
shows *uniformly_continuous_map m1 m2 f*
proof −
have *f* ∈ *mspace m1* → *mspace m2*
by (*simp add*: *Lipschitz_continuous_map_image assms*)
moreover have ∃ *δ*>*0*. ∀ *x*∈*mspace m1*. ∀ *y*∈*mspace m1*. *mdist m1 y x* < *δ* ⟶ *mdist m2* (*f y*) (*f x*) < *ε*
if *ε* > *0* **for** *ε*
proof −

obtain B **where** $\forall x{\in}mspace\ m1.\ \forall y{\in}mspace\ m1.\ mdist\ m2\ (f\ x)\ (f\ y) \leq B * mdist\ m1\ x\ y$
and $B{>}0$
using *that assms* **by** (*force simp*: *Lipschitz_continuous_map_pos*)
then have $\forall x{\in}mspace\ m1.\ \forall y{\in}mspace\ m1.\ mdist\ m1\ y\ x < \varepsilon/B \longrightarrow mdist\ m2\ (f\ y)\ (f\ x) < \varepsilon$
by (*smt* (*verit*, *ccfv_SIG*) *less_divide_eq mdist_nonneg mult.commute that zero_less_divide_iff*)
with ‹$B{>}0$› **show** *?thesis*
by (*metis divide_pos_pos that*)
qed
ultimately show *?thesis*
by (*auto simp*: *uniformly_continuous_map_def*)
qed

lemma *uniformly_imp_Cauchy_continuous_map*:
$uniformly\_continuous\_map\ m1\ m2\ f \Longrightarrow Cauchy\_continuous\_map\ m1\ m2\ f$
unfolding *uniformly_continuous_map_def Cauchy_continuous_map_def*
apply (*simp add*: *image_subset_iff o_def Metric_space.MCauchy_def* [*OF Metric_space_mspace_mdist*])
by (*metis funcset_mem*)

lemma *locally_Cauchy_continuous_map*:
assumes $\varepsilon > 0$
and §: $\bigwedge x.\ x \in mspace\ m1 \Longrightarrow Cauchy\_continuous\_map\ (submetric\ m1\ (mball\_of\ m1\ x\ \varepsilon))\ m2\ f$
shows *Cauchy_continuous_map m1 m2 f*
unfolding *Cauchy_continuous_map_def*
proof (*intro strip*)
interpret *M1*: *Metric_space mspace m1 mdist m1*
by (*simp add*: *Metric_space_mspace_mdist*)
interpret *M2*: *Metric_space mspace m2 mdist m2*
by (*simp add*: *Metric_space_mspace_mdist*)
fix σ
assume σ: *M1.MCauchy* σ
with ‹$\varepsilon > 0$› **obtain** N **where** N: $\bigwedge n\ n'.\ [\![n{\geq}N;\ n'{\geq}N]\!] \Longrightarrow mdist\ m1\ (\sigma\ n)\ (\sigma\ n') < \varepsilon$
using *M1.MCauchy_def* **by** *fastforce*
then have *M1.mball* $(\sigma\ N)\ \varepsilon \subseteq mspace\ m1$
by (*auto simp*: *image_subset_iff M1.mball_def*)
then interpret *MS1*: *Metric_space mball_of m1* $(\sigma\ N)\ \varepsilon \cap mspace\ m1\ mdist\ m1$
by (*simp add*: *M1.subspace*)
show *M2.MCauchy* $(f \circ \sigma)$
proof (*rule M2.MCauchy_offset*)
have *M1.MCauchy* $(\sigma \circ (+)\ N)$
by (*simp add*: *M1.MCauchy_imp_MCauchy_suffix* σ)
moreover have *range* $(\sigma \circ (+)\ N) \subseteq M1.mball\ (\sigma\ N)\ \varepsilon$
using N [*OF order_refl*] *M1.MCauchy_def* σ **by** *fastforce*

```
    ultimately have MS1.MCauchy (σ ∘ (+) N)
    unfolding M1.MCauchy_def MS1.MCauchy_def by (simp add: mball_of_def)
    moreover have σ N ∈ mspace m1
      using M1.MCauchy_def σ by auto
    ultimately show M2.MCauchy (f ∘ σ ∘ (+) N)
      unfolding comp_assoc
    by (metis § Cauchy_continuous_map_def mdist_submetric mspace_submetric)
  next
    fix n
    have σ n ∈ mspace m1
    by (meson Metric_space.MCauchy_def Metric_space_mspace_mdist σ range_subsetD)
    then have σ n ∈ mball_of m1 (σ n) ε
    by (simp add: Metric_space.centre_in_mball_iff Metric_space_mspace_mdist
assms(1) mball_of_def)
    then show (f ∘ σ) n ∈ mspace m2
     using Cauchy_continuous_map_funspace [OF § [of σ n]] ‹σ n ∈ mspace m1›
by auto
  qed
qed

context Metric_space12
begin

lemma Cauchy_continuous_imp_continuous_map:
  assumes Cauchy_continuous_map (metric (M1,d1)) (metric (M2,d2)) f
  shows continuous_map M1.mtopology M2.mtopology f
proof (clarsimp simp: continuous_map_atin)
  fix x
  assume x ∈ M1
  show limitin M2.mtopology f (f x) (atin M1.mtopology x)
    unfolding limit_atin_sequentially
  proof (intro conjI strip)
    show f x ∈ M2
      using Cauchy_continuous_map_funspace ‹x ∈ M1› assms by fastforce
    fix σ
    assume range σ ⊆ M1 - {x} ∧ limitin M1.mtopology σ x sequentially
    then have M1.MCauchy (λn. if even n then σ (n div 2) else x)
      by (force simp: M1.MCauchy_interleaving)
    then have M2.MCauchy (f ∘ (λn. if even n then σ (n div 2) else x))
      using assms by (simp add: Cauchy_continuous_map_def)
    then show limitin M2.mtopology (f ∘ σ) (f x) sequentially
      using M2.MCauchy_interleaving [of f ∘ σ f x]
      by (simp add: o_def if_distrib cong: if_cong)
  qed
qed

lemma continuous_imp_Cauchy_continuous_map:
  assumes M1.mcomplete
    and f: continuous_map M1.mtopology M2.mtopology f
```

shows *Cauchy_continuous_map* (*metric* (*M1*,*d1*)) (*metric* (*M2*,*d2*)) *f*
unfolding *Cauchy_continuous_map_def*
proof *clarsimp*
fix σ
assume σ: *M1.MCauchy* σ
then obtain *y* **where** *y*: *limitin M1.mtopology* σ *y sequentially*
using *M1.mcomplete_def assms* **by** *blast*
have *range* (*f* $\circ$ σ) $\subseteq$ *M2*
using σ *f* **by** (*simp add*: *M2.subspace M1.MCauchy_def M1.metric_continuous_map image_subset_iff*)
then show *M2.MCauchy* (*f* $\circ$ σ)
using *continuous_map_limit* [*OF f y*] *M2.convergent_imp_MCauchy*
by *blast*
qed

end

The same outside the locale

lemma *Cauchy_continuous_imp_continuous_map*:
assumes *Cauchy_continuous_map m1 m2 f*
shows *continuous_map* (*mtopology_of m1*) (*mtopology_of m2*) *f*
using *assms Metric_space12.Cauchy_continuous_imp_continuous_map* [*OF Metric_space12_mspace_mdist*]
by (*auto simp*: *mtopology_of_def*)

lemma *continuous_imp_Cauchy_continuous_map*:
assumes *Metric_space.mcomplete* (*mspace m1*) (*mdist m1*)
and *continuous_map* (*mtopology_of m1*) (*mtopology_of m2*) *f*
shows *Cauchy_continuous_map m1 m2 f*
using *assms Metric_space12.continuous_imp_Cauchy_continuous_map* [*OF Metric_space12_mspace_mdist*]
by (*auto simp*: *mtopology_of_def*)

lemma *uniformly_continuous_imp_continuous_map*:
uniformly_continuous_map m1 m2 f
$\Longrightarrow$ *continuous_map* (*mtopology_of m1*) (*mtopology_of m2*) *f*
by (*simp add*: *Cauchy_continuous_imp_continuous_map uniformly_imp_Cauchy_continuous_map*)

lemma *Lipschitz_continuous_imp_continuous_map*:
Lipschitz_continuous_map m1 m2 f
$\Longrightarrow$ *continuous_map* (*mtopology_of m1*) (*mtopology_of m2*) *f*
by (*simp add*: *Lipschitz_imp_uniformly_continuous_map uniformly_continuous_imp_continuous_ma*

lemma *Lipschitz_imp_Cauchy_continuous_map*:
Lipschitz_continuous_map m1 m2 f
$\Longrightarrow$ *Cauchy_continuous_map m1 m2 f*
by (*simp add*: *Lipschitz_imp_uniformly_continuous_map uniformly_imp_Cauchy_continuous_map*)

lemma *Cauchy_imp_uniformly_continuous_map*:

assumes *f*: *Cauchy_continuous_map m1 m2 f*
and *tbo*: *Metric_space.mtotally_bounded* (*mspace m1*) (*mdist m1*) (*mspace m1*)
shows *uniformly_continuous_map m1 m2 f*
unfolding *uniformly_continuous_map_sequentially_alt imp_conjL*
proof (*intro conjI strip*)
show $f \in$ *mspace m1* $\to$ *mspace m2*
by (*simp add*: *Cauchy_continuous_map_funspace f*)
interpret *M1*: *Metric_space mspace m1 mdist m1*
by (*simp add*: *Metric_space_mspace_mdist*)
interpret *M2*: *Metric_space mspace m2 mdist m2*
by (*simp add*: *Metric_space_mspace_mdist*)
fix ε :: *real* **and** ϱ σ
assume $\varepsilon > 0$
and ϱ: *range* $\varrho \subseteq$ *mspace m1*
and σ: *range* $\sigma \subseteq$ *mspace m1*
and *0*: $(\lambda n.$ *mdist m1* $(\varrho\ n)\ (\sigma\ n)) \longrightarrow 0$
then obtain *r1* **where** *strict_mono r1* **and** *r1*: *M1.MCauchy* $(\varrho \circ r1)$
using *M1.mtotally_bounded_sequentially tbo* **by** *meson*
then obtain *r2* **where** *strict_mono r2* **and** *r2*: *M1.MCauchy* $(\sigma \circ r1 \circ r2)$
by (*metis M1.mtotally_bounded_sequentially tbo σ image_comp image_subset_iff range_subsetD*)
define *r* **where** $r \equiv r1 \circ r2$
have *r*: *strict_mono r*
by (*simp add*: *r_def* ‹*strict_mono r1*› ‹*strict_mono r2*› *strict_mono_o*)
define η **where** $\eta \equiv \lambda n.$ *if even n then* $(\varrho \circ r)$ $(n$ *div* $2)$ *else* $(\sigma \circ r)$ $(n$ *div* $2)$
have *M1.MCauchy* η
unfolding η*_def M1.MCauchy_interleaving_gen*
proof (*intro conjI*)
show *M1.MCauchy* $(\varrho \circ r)$
by (*simp add*: *M1.MCauchy_subsequence* ‹*strict_mono r2*› *fun.map_comp r1 r_def*)
show *M1.MCauchy* $(\sigma \circ r)$
by (*simp add*: *fun.map_comp r2 r_def*)
show $(\lambda n.$ *mdist m1* $((\varrho \circ r)\ n)\ ((\sigma \circ r)\ n)) \longrightarrow 0$
using *LIMSEQ_subseq_LIMSEQ* [*OF 0 r*] **by** (*simp add*: *o_def*)
qed
then have *Metric_space.MCauchy* (*mspace m2*) (*mdist m2*) $(f \circ \eta)$
by (*meson Cauchy_continuous_map_def f*)
then have $(\lambda n.$ *mdist m2* $(f\ (\varrho\ (r\ n)))\ (f\ (\sigma\ (r\ n)))) \longrightarrow 0$
using *M2.MCauchy_interleaving_gen* [*of f* $\circ\ \varrho \circ r\ f \circ \sigma \circ r$]
by (*simp add*: η*_def o_def if_distrib cong*: *if_cong*)
then show $\exists n.$ *mdist m2* $(f\ (\varrho\ n))\ (f\ (\sigma\ n)) < \varepsilon$
by (*meson LIMSEQ_le_const* ‹$0 < \varepsilon$› *linorder_not_le*)
qed

lemma *continuous_imp_uniformly_continuous_map*:
compact_space (*mtopology_of m1*) $\wedge$
continuous_map (*mtopology_of m1*) (*mtopology_of m2*) *f*

```
      ⟹ uniformly_continuous_map m1 m2 f
 by (simp add: Cauchy_imp_uniformly_continuous_map continuous_imp_Cauchy_continuous_map
              Metric_space.compact_space_eq_mcomplete_mtotally_bounded
              Metric_space_mspace_mdist mtopology_of_def)

lemma continuous_eq_Cauchy_continuous_map:
   Metric_space.mcomplete (mspace m1) (mdist m1) ⟹
   continuous_map (mtopology_of m1) (mtopology_of m2) f ⟷ Cauchy_continuous_map
m1 m2 f
 using Cauchy_continuous_imp_continuous_map continuous_imp_Cauchy_continuous_map
by blast

lemma continuous_eq_uniformly_continuous_map:
   compact_space (mtopology_of m1)
    ⟹ continuous_map (mtopology_of m1) (mtopology_of m2) f ⟷
       uniformly_continuous_map m1 m2 f
 using continuous_imp_uniformly_continuous_map uniformly_continuous_imp_continuous_map
by blast

lemma Cauchy_eq_uniformly_continuous_map:
   Metric_space.mtotally_bounded (mspace m1) (mdist m1) (mspace m1)
    ⟹ Cauchy_continuous_map m1 m2 f ⟷ uniformly_continuous_map m1
m2 f
 using Cauchy_imp_uniformly_continuous_map uniformly_imp_Cauchy_continuous_map
by blast

lemma Lipschitz_continuous_map_projections:
  Lipschitz_continuous_map (prod_metric m1 m2) m1 fst
  Lipschitz_continuous_map (prod_metric m1 m2) m2 snd
  by (simp add: Lipschitz_continuous_map_def prod_dist_def fst_Pi snd_Pi;
    metis mult_numeral_1 real_sqrt_sum_squares_ge1 real_sqrt_sum_squares_ge2)+

lemma Lipschitz_continuous_map_pairwise:
   Lipschitz_continuous_map m (prod_metric m1 m2) f ⟷
    Lipschitz_continuous_map m m1 (fst ∘ f) ∧ Lipschitz_continuous_map m m2
(snd ∘ f)
   (is ?lhs ⟷ ?rhs)
proof
  show ?lhs ⟹ ?rhs
   by (simp add: Lipschitz_continuous_map_compose Lipschitz_continuous_map_projections)
  have Lipschitz_continuous_map m (prod_metric m1 m2) (λx. (f1 x, f2 x))
    if f1: Lipschitz_continuous_map m m1 f1 and f2: Lipschitz_continuous_map
m m2 f2 for f1 f2
  proof −
    obtain B1 where B1 > 0
      and B1: ⋀x y. ⟦x ∈ mspace m; y ∈ mspace m⟧ ⟹ mdist m1 (f1 x) (f1 y) ≤
B1 ∗ mdist m x y
      by (meson Lipschitz_continuous_map_pos f1)
    obtain B2 where B2 > 0
```

```
    and B2: ⋀x y. ⟦x ∈ mspace m; y ∈ mspace m⟧ ⟹ mdist m2 (f2 x) (f2 y) ≤
B2 * mdist m x y
     by (meson Lipschitz_continuous_map_pos f2)
   show ?thesis
     unfolding Lipschitz_continuous_map_pos
   proof (intro exI conjI strip)
     have f1im: f1 ∈ mspace m → mspace m1
       by (simp add: Lipschitz_continuous_map_image f1)
     moreover have f2im: f2 ∈ mspace m → mspace m2
       by (simp add: Lipschitz_continuous_map_image f2)
     ultimately show (λx. (f1 x, f2 x)) ∈ mspace m → mspace (prod_metric m1
m2)
       by auto
     show B1+B2 > 0
       using ‹0 < B1› ‹0 < B2› by linarith
     fix x y
     assume xy: x ∈ mspace m y ∈ mspace m
     with f1im f2im have mdist (prod_metric m1 m2) (f1 x, f2 x) (f1 y, f2 y) ≤
mdist m1 (f1 x) (f1 y) + mdist m2 (f2 x) (f2 y)
       unfolding mdist_prod_metric
     by (intro Metric_space12.prod_metric_le_components [OF Metric_space12_mspace_mdist])
auto
     also have ... ≤ (B1+B2) * mdist m x y
     using B1 [OF xy] B2 [OF xy] by (simp add: vector_space_over_itself.scale_left_distrib)

    finally show mdist (prod_metric m1 m2) (f1 x, f2 x) (f1 y, f2 y) ≤ (B1+B2)
* mdist m x y .
   qed
 qed
 then show ?rhs ⟹ ?lhs
   by force
qed

lemma uniformly_continuous_map_pairwise:
  uniformly_continuous_map m (prod_metric m1 m2) f ⟷
   uniformly_continuous_map m m1 (fst ∘ f) ∧ uniformly_continuous_map m
m2 (snd ∘ f)
  (is ?lhs ⟷ ?rhs)
proof
 show ?lhs ⟹ ?rhs
  by (simp add: Lipschitz_continuous_map_projections Lipschitz_imp_uniformly_continuous_map
uniformly_continuous_map_compose)
 have uniformly_continuous_map m (prod_metric m1 m2) (λx. (f1 x, f2 x))
  if f1: uniformly_continuous_map m m1 f1 and f2: uniformly_continuous_map
m m2 f2 for f1 f2
 proof −
   show ?thesis
     unfolding uniformly_continuous_map_def
   proof (intro conjI strip)
```

have *f1im*: *f1* ∈ *mspace m* → *mspace m1*
by (*simp add*: *uniformly_continuous_map_funspace f1*)
moreover have *f2im*: *f2* ∈ *mspace m* → *mspace m2*
by (*simp add*: *uniformly_continuous_map_funspace f2*)
ultimately show (λ*x*. (*f1 x*, *f2 x*)) ∈ *mspace m* → *mspace* (*prod_metric m1 m2*)
by *auto*
fix ε:: *real*
assume ε > *0*
obtain *δ1* **where** *δ1>0*
and *δ1*: ⋀*x y*. ⟦*x* ∈ *mspace m*; *y* ∈ *mspace m*; *mdist m y x* < *δ1*⟧ ⟹ *mdist m1* (*f1 y*) (*f1 x*) < ε/*2*
by (*metis* ‹*0* < ε› *f1 half_gt_zero uniformly_continuous_map_def*)
obtain *δ2* **where** *δ2>0*
and *δ2*: ⋀*x y*. ⟦*x* ∈ *mspace m*; *y* ∈ *mspace m*; *mdist m y x* < *δ2*⟧ ⟹ *mdist m2* (*f2 y*) (*f2 x*) < ε/*2*
by (*metis* ‹*0* < ε› *f2 half_gt_zero uniformly_continuous_map_def*)
show ∃*δ*>*0*. ∀*x*∈*mspace m*. ∀*y*∈*mspace m*. *mdist m y x* < *δ* ⟶ *mdist* (*prod_metric m1 m2*) (*f1 y*, *f2 y*) (*f1 x*, *f2 x*) < ε
proof (*intro exI conjI strip*)
show *min δ1 δ2>0*
using ‹*0* < *δ1*› ‹*0* < *δ2*› **by** *auto*
fix *x y*
assume *xy*: *x* ∈ *mspace m y* ∈ *mspace m* **and** *d*: *mdist m y x* < *min δ1 δ2*
have ∗: *mdist m1* (*f1 y*) (*f1 x*) < ε/*2 mdist m2* (*f2 y*) (*f2 x*) < ε/*2*
using *δ1 δ2 d xy* **by** *auto*
have *mdist* (*prod_metric m1 m2*) (*f1 y*, *f2 y*) (*f1 x*, *f2 x*) ≤ *mdist m1* (*f1 y*) (*f1 x*) + *mdist m2* (*f2 y*) (*f2 x*)
unfolding *mdist_prod_metric* **using** *f1im f2im xy*
by (*intro Metric_space12.prod_metric_le_components* [*OF Metric_space12_mspace_mdist*]) *auto*
also have ... < ε/*2* + ε/*2*
using ∗ **by** *simp*
finally show *mdist* (*prod_metric m1 m2*) (*f1 y*, *f2 y*) (*f1 x*, *f2 x*) < ε
by *simp*
qed
qed
qed
then show *?rhs* ⟹ *?lhs*
by *force*
qed

lemma *Cauchy_continuous_map_pairwise*:
Cauchy_continuous_map m (*prod_metric m1 m2*) *f* ⟷ *Cauchy_continuous_map m m1* (*fst* ∘ *f*) ∧ *Cauchy_continuous_map m m2* (*snd* ∘ *f*)
by (*auto simp*: *Cauchy_continuous_map_def Metric_space12.MCauchy_prod_metric*[*OF Metric_space12_mspace_mdist*] *comp_assoc*)

lemma *Lipschitz_continuous_map_paired*:

Lipschitz_continuous_map m (prod_metric m1 m2) (λx. (f x, g x)) ⟷
Lipschitz_continuous_map m m1 f ∧ Lipschitz_continuous_map m m2 g
by (*simp add*: *Lipschitz_continuous_map_pairwise o_def*)

lemma *uniformly_continuous_map_paired*:
uniformly_continuous_map m (prod_metric m1 m2) (λx. (f x, g x)) ⟷
uniformly_continuous_map m m1 f ∧ uniformly_continuous_map m m2 g
by (*simp add*: *uniformly_continuous_map_pairwise o_def*)

lemma *Cauchy_continuous_map_paired*:
Cauchy_continuous_map m (prod_metric m1 m2) (λx. (f x, g x)) ⟷
Cauchy_continuous_map m m1 f ∧ Cauchy_continuous_map m m2 g
by (*simp add*: *Cauchy_continuous_map_pairwise o_def*)

lemma *mbounded_Lipschitz_continuous_image*:
assumes *f*: *Lipschitz_continuous_map m1 m2 f* **and** *S*: *Metric_space.mbounded*
(*mspace m1*) (*mdist m1*) *S*
shows *Metric_space.mbounded* (*mspace m2*) (*mdist m2*) (*f‘S*)
proof −
obtain *B* **where** *fim*: *f* ∈ *mspace m1* → *mspace m2*
and *B*>*0* **and** *B*: ⋀*x y*. ⟦*x* ∈ *mspace m1*; *y* ∈ *mspace m1*⟧ ⟹ *mdist m2* (*f x*)
(*f y*) ≤ *B* ∗ *mdist m1 x y*
by (*metis Lipschitz_continuous_map_pos f*)
show *?thesis*
unfolding *Metric_space.mbounded_alt_pos* [*OF Metric_space_mspace_mdist*]
proof
obtain *C* **where** *S* ⊆ *mspace m1* **and** *C*>*0* **and** *C*: ⋀*x y*. ⟦*x* ∈ *S*; *y* ∈ *S*⟧ ⟹
mdist m1 x y ≤ *C*
using *S* **by** (*auto simp*: *Metric_space.mbounded_alt_pos* [*OF Metric_space_mspace_mdist*])
show *f* ‘ *S* ⊆ *mspace m2*
using *fim* ‹*S* ⊆ *mspace m1*› **by** *blast*
have ⋀*x y*. ⟦*x* ∈ *S*; *y* ∈ *S*⟧ ⟹ *mdist m2* (*f x*) (*f y*) ≤ *B* ∗ *C*
by (*smt* (*verit*) *B C* ‹*0* < *B*› ‹*S* ⊆ *mspace m1*› *mdist_nonneg mult_mono*
subsetD)
moreover have *B*∗*C* > *0*
by (*simp add*: ‹*0* < *B*› ‹*0* < *C*›)
ultimately show ∃ *B*>*0*. ∀ *x*∈*f* ‘ *S*. ∀ *y*∈*f* ‘ *S*. *mdist m2 x y* ≤ *B*
by *auto*
qed
qed

lemma *mtotally_bounded_Cauchy_continuous_image*:
assumes *f*: *Cauchy_continuous_map m1 m2 f* **and** *S*: *Metric_space.mtotally_bounded*
(*mspace m1*) (*mdist m1*) *S*
shows *Metric_space.mtotally_bounded* (*mspace m2*) (*mdist m2*) (*f* ‘ *S*)
unfolding *Metric_space.mtotally_bounded_sequentially*[*OF Metric_space_mspace_mdist*]
proof (*intro conjI strip*)
have *S* ⊆ *mspace m1*
using *S* **by** (*simp add*: *Metric_space.mtotally_bounded_sequentially*[*OF Met-*

ric_space_mspace_mdist])
then show f ‘ $S \subseteq$ *mspace m2*
using *Cauchy_continuous_map_funspace f* **by** *blast*
fix σ :: *nat* $\Rightarrow$ $'b$
assume *range* $\sigma \subseteq f$ ‘ S
then have $\forall n.\ \exists x.\ \sigma\ n = f\ x \wedge x \in S$
by (*meson imageE range_subsetD*)
then obtain ϱ **where** ϱ: $\bigwedge n.\ \sigma\ n = f\ (\varrho\ n)$ *range* $\varrho \subseteq S$
by (*metis image_subset_iff*)
then have $\sigma = f \circ \varrho$
by *fastforce*
obtain r **where** *strict_mono r Metric_space.MCauchy* (*mspace m1*) (*mdist m1*)
($\varrho \circ r$)
by (*meson* ϱ *S Metric_space.mtotally_bounded_sequentially*[*OF Metric_space_mspace_mdist*])
then have *Metric_space.MCauchy* (*mspace m2*) (*mdist m2*) ($f \circ \varrho \circ r$)
using f **unfolding** *Cauchy_continuous_map_def* **by** (*metis fun.map_comp*)
then show $\exists r.$ *strict_mono* $r \wedge$ *Metric_space.MCauchy* (*mspace m2*) (*mdist*
m2) ($\sigma \circ r$)
using ‹$\sigma = f \circ \varrho$› ‹*strict_mono r*› **by** *blast*
qed

lemma *Lipschitz_coefficient_pos*:
assumes $x \in$ *mspace m* $y \in$ *mspace m* $f\ x \neq f\ y$
and $f \in$ *mspace m* $\to$ *mspace m2*
and $\bigwedge x\ y.\ [\![x \in$ *mspace m*; $y \in$ *mspace m*$]\!]$
$\Longrightarrow$ *mdist m2* ($f\ x$) ($f\ y$) $\leq k *$ *mdist m x y*
shows $0 < k$
using *assms* **by** (*smt* (*verit*, *best*) *Pi_iff mdist_nonneg mdist_zero mult_nonpos_nonneg*)

lemma *Lipschitz_continuous_map_metric*:
Lipschitz_continuous_map (*prod_metric m m*) *euclidean_metric* (λ(*x*,*y*). *mdist*
m x y)
proof −
have $\bigwedge x\ y\ x'\ y'.\ [\![x \in$ *mspace m*; $y \in$ *mspace m*; $x' \in$ *mspace m*; $y' \in$ *mspace m*$]\!]$
$\Longrightarrow |$*mdist m x y* − *mdist m x' y'*$| \leq 2 *$ *sqrt* ((*mdist m x x'*)^2 + (*mdist m*
y y')^2)
by (*smt* (*verit*, *del_insts*) *mdist_commute mdist_triangle real_sqrt_sum_squares_ge2*)
then show *?thesis*
by (*fastforce simp*: *Lipschitz_continuous_map_def prod_dist_def dist_real_def*)
qed

lemma *Lipschitz_continuous_map_mdist*:
assumes f: *Lipschitz_continuous_map m m' f*
and g: *Lipschitz_continuous_map m m' g*
shows *Lipschitz_continuous_map m euclidean_metric* ($\lambda x.$ *mdist m'* ($f\ x$) ($g\ x$))
(**is** *Lipschitz_continuous_map m _ ?h*)
proof −
have *eq*: *?h* = ((λ(*x*,*y*). *mdist m' x y*) $\circ$ ($\lambda x.$ ($f\ x$,$g\ x$)))
by *force*

```
  show ?thesis
    unfolding eq
  proof (rule Lipschitz_continuous_map_compose)
    show Lipschitz_continuous_map m (prod_metric m' m') (λx. (f x, g x))
      by (simp add: Lipschitz_continuous_map_paired f g)
   show Lipschitz_continuous_map (prod_metric m' m') euclidean_metric (λ(x,y).
mdist m' x y)
      by (simp add: Lipschitz_continuous_map_metric)
  qed
qed

lemma uniformly_continuous_map_mdist:
  assumes f: uniformly_continuous_map m m' f
    and g: uniformly_continuous_map m m' g
  shows uniformly_continuous_map m euclidean_metric (λx. mdist m' (f x) (g
x))
    (is uniformly_continuous_map m _ ?h)
proof −
  have eq: ?h = ((λ(x,y). mdist m' x y) ∘ (λx. (f x,g x)))
    by force
  show ?thesis
    unfolding eq
  proof (rule uniformly_continuous_map_compose)
    show uniformly_continuous_map m (prod_metric m' m') (λx. (f x, g x))
      by (simp add: uniformly_continuous_map_paired f g)
   show uniformly_continuous_map (prod_metric m' m') euclidean_metric (λ(x,y).
mdist m' x y)
    by (simp add: Lipschitz_continuous_map_metric Lipschitz_imp_uniformly_continuous_map)
  qed
qed

lemma Cauchy_continuous_map_mdist:
  assumes f: Cauchy_continuous_map m m' f
    and g: Cauchy_continuous_map m m' g
  shows Cauchy_continuous_map m euclidean_metric (λx. mdist m' (f x) (g x))
    (is Cauchy_continuous_map m _ ?h)
proof −
  have eq: ?h = ((λ(x,y). mdist m' x y) ∘ (λx. (f x,g x)))
    by force
  show ?thesis
    unfolding eq
  proof (rule Cauchy_continuous_map_compose)
    show Cauchy_continuous_map m (prod_metric m' m') (λx. (f x, g x))
      by (simp add: Cauchy_continuous_map_paired f g)
   show Cauchy_continuous_map (prod_metric m' m') euclidean_metric (λ(x,y).
mdist m' x y)
    by (simp add: Lipschitz_continuous_map_metric Lipschitz_imp_Cauchy_continuous_map)
  qed
qed
```

lemma *mtopology_of_prod_metric* [*simp*]:
mtopology_of (*prod_metric m1 m2*) = *prod_topology* (*mtopology_of m1*) (*mtopology_of m2*)
by (*simp add*: *mtopology_of_def Metric_space12.mtopology_prod_metric*[*OF Metric_space12_mspace_mdist*])

lemma *continuous_map_metric*:
continuous_map (*prod_topology* (*mtopology_of m*) (*mtopology_of m*)) *euclidean* (λ(*x*,*y*). *mdist m x y*)
using *Lipschitz_continuous_imp_continuous_map* [*OF Lipschitz_continuous_map_metric*]
by *auto*

lemma *continuous_map_mdist_alt*:
assumes *continuous_map X* (*prod_topology* (*mtopology_of m*) (*mtopology_of m*)) *f*
shows *continuous_map X euclidean* (λ*x*. *case_prod* (*mdist m*) (*f x*))
(**is** *continuous_map _ _ ?h*)
proof −
have *eq*: *?h* = *case_prod* (*mdist m*) ∘ *f*
by *force*
show *?thesis*
unfolding *eq*
using *assms continuous_map_compose continuous_map_metric* **by** *blast*
qed

lemma *continuous_map_mdist* [*continuous_intros*]:
assumes *f*: *continuous_map X* (*mtopology_of m*) *f*
and *g*: *continuous_map X* (*mtopology_of m*) *g*
shows *continuous_map X euclidean* (λ*x*. *mdist m* (*f x*) (*g x*))
(**is** *continuous_map X _ ?h*)
proof −
have *eq*: *?h* = ((λ(*x*,*y*). *mdist m x y*) ∘ (λ*x*. (*f x*,*g x*)))
by *force*
show *?thesis*
unfolding *eq*
proof (*rule continuous_map_compose*)
show *continuous_map X* (*prod_topology* (*mtopology_of m*) (*mtopology_of m*)) (λ*x*. (*f x*, *g x*))
by (*simp add*: *continuous_map_paired f g*)
qed (*simp add*: *continuous_map_metric*)
qed

lemma *continuous_on_mdist*:
a ∈ *mspace m* ⟹ *continuous_map* (*mtopology_of m*) *euclidean* (*mdist m a*)
by (*simp add*: *continuous_map_mdist*)

6.7.18 Isometries

lemma (**in** *Metric_space12*) *isometry_imp_embedding_map*:
assumes *fim*: $f \in M1 \to M2$ **and** *d*: $\bigwedge x\ y.\ [\![x \in M1;\ y \in M1]\!] \Longrightarrow d2\ (f\ x)\ (f\ y) = d1\ x\ y$
shows *embedding_map M1.mtopology M2.mtopology f*
proof −
have *inj_on f M1*
by (*metis M1.zero d inj_onI*)
then obtain *g* **where** *g*: $\bigwedge x.\ x \in M1 \Longrightarrow g\ (f\ x) = x$
by (*metis inv_into_f_f*)
have *homeomorphic_maps M1.mtopology* (*subtopology M2.mtopology* (*f ‘ topspace M1.mtopology*)) *f g*
unfolding *homeomorphic_maps_def*
proof (*intro conjI*; *clarsimp*)
show *continuous_map M1.mtopology* (*subtopology M2.mtopology* (*f ‘ M1*)) *f*
proof (*rule continuous_map_into_subtopology*)
show *continuous_map M1.mtopology M2.mtopology f*
by (*metis M1.metric_continuous_map M2.Metric_space_axioms d fim image_subset_iff_funcset*)
qed *simp*
have *Lipschitz_continuous_map* (*submetric* (*metric*(*M2*,*d2*)) (*f ‘ M1*)) (*metric*(*M1*,*d1*)) *g*
unfolding *Lipschitz_continuous_map_def*
proof (*intro conjI exI strip*; *simp*)
show $d1\ (g\ x)\ (g\ y) \le 1 * d2\ x\ y$ **if** $x \in f\ ‘\ M1 \wedge x \in M2$ **and** $y \in f\ ‘\ M1 \wedge y \in M2$ **for** *x y*
using *that d g* **by** *force*
qed (*use g* **in** *auto*)
then have *continuous_map* (*mtopology_of* (*submetric* (*metric*(*M2*,*d2*)) (*f ‘ M1*))) *M1.mtopology g*
using *Lipschitz_continuous_imp_continuous_map* **by** *force*
moreover have *mtopology_of* (*submetric* (*metric*(*M2*,*d2*)) (*f ‘ M1*)) = *subtopology M2.mtopology* (*f ‘ M1*)
by (*simp add*: *mtopology_of_submetric*)
ultimately show *continuous_map* (*subtopology M2.mtopology* (*f ‘ M1*)) *M1.mtopology g*
by *simp*
qed (*use g* **in** *auto*)
then show *?thesis*
by (*auto simp*: *embedding_map_def homeomorphic_map_maps*)
qed

lemma (**in** *Metric_space12*) *isometry_imp_homeomorphic_map*:
assumes *fim*: $f\ ‘\ M1 = M2$ **and** *d*: $\bigwedge x\ y.\ [\![x \in M1;\ y \in M1]\!] \Longrightarrow d2\ (f\ x)\ (f\ y) = d1\ x\ y$
shows *homeomorphic_map M1.mtopology M2.mtopology f*
by (*metis image_eqI M1.topspace_mtopology M2.subtopology_mspace d embedding_map_def fim isometry_imp_embedding_map Pi_iff*)

6.7.19 "Capped" equivalent bounded metrics and general product metrics

definition (**in** *Metric_space*) *capped_dist* **where**
capped_dist ≡ λδ *x y*. *if* δ > *0 then min* δ (*d x y*) *else d x y*

lemma (**in** *Metric_space*) *capped_dist*: *Metric_space M* (*capped_dist* δ)
proof
fix *x y*
assume *x* ∈ *M y* ∈ *M*
then show (*capped_dist* δ *x y* = *0*) = (*x* = *y*)
by (*smt* (*verit*, *best*) *capped_dist_def zero*)
fix *z*
assume *z* ∈ *M*
show *capped_dist* δ *x z* ≤ *capped_dist* δ *x y* + *capped_dist* δ *y z*
unfolding *capped_dist_def* **using** ‹*x* ∈ *M*› ‹*y* ∈ *M*› ‹*z* ∈ *M*›
by (*smt* (*verit*, *del_insts*) *Metric_space.mdist_pos_eq Metric_space_axioms mdist_reverse_triangle*)
qed (*use capped_dist_def commute* **in** *auto*)

definition *capped_metric* **where**
capped_metric δ *m* ≡ *metric*(*mspace m*, *Metric_space.capped_dist* (*mdist m*) δ)

lemma *capped_metric*:
capped_metric δ *m* = (*if* δ ≤ *0 then m else metric*(*mspace m*, λ*x y*. *min* δ (*mdist m x y*)))
proof −
interpret *Metric_space mspace m mdist m*
by (*simp add*: *Metric_space_mspace_mdist*)
show *?thesis*
by (*auto simp*: *capped_metric_def capped_dist_def*)
qed

lemma *capped_metric_mspace* [*simp*]:
mspace (*capped_metric* δ *m*) = *mspace m*
by (*simp add*: *Metric_space.capped_dist Metric_space.mspace_metric capped_metric_def*)

lemma *capped_metric_mdist*:
mdist (*capped_metric* δ *m*) = (λ*x y*. *if* δ ≤ *0 then mdist m x y else min* δ (*mdist m x y*))
by (*metis Metric_space.capped_dist Metric_space.capped_dist_def Metric_space.mdist_metric Metric_space_mspace_mdist capped_metric capped_metric_def leI*)

lemma *mdist_capped_le*: *mdist* (*capped_metric* δ *m*) *x y* ≤ *mdist m x y*
by (*simp add*: *capped_metric_mdist*)

lemma *mdist_capped*: δ > *0* ⟹ *mdist* (*capped_metric* δ *m*) *x y* ≤ δ
by (*simp add*: *capped_metric_mdist*)

lemma *mball_of_capped_metric* [*simp*]:
assumes $x \in mspace\ m\ r > \delta\ \delta > 0$
shows *mball_of* (*capped_metric* δ *m*) *x r* = *mspace m*
proof −
interpret *Metric_space mspace m mdist m*
by *auto*
have *Metric_space.mball* (*mspace m*) (*mdist* (*capped_metric* δ *m*)) *x r* ⊆ *mspace m*
by (*metis Metric_space.mball_subset_mspace Metric_space_mspace_mdist capped_metric_mspace*)
moreover have *mspace m* ⊆ *Metric_space.mball* (*mspace m*) (*mdist* (*capped_metric* δ *m*)) *x r*
by (*smt* (*verit*) *Metric_space.in_mball Metric_space_mspace_mdist assms capped_metric_mspace mdist_capped subset_eq*)
ultimately show *?thesis*
by (*simp add*: *mball_of_def*)
qed

lemma *Metric_space_capped_dist*[*simp*]:
Metric_space (*mspace m*) (*Metric_space.capped_dist* (*mdist m*) δ)
using *Metric_space.capped_dist Metric_space_mspace_mdist* **by** *blast*

lemma *mtopology_capped_metric*:
mtopology_of(*capped_metric* δ *m*) = *mtopology_of m*
proof (*cases* $\delta > 0$)
case *True*
interpret *Metric_space mspace m mdist m*
by (*simp add*: *Metric_space_mspace_mdist*)
interpret *Cap*: *Metric_space mspace m mdist* (*capped_metric* δ *m*)
by (*metis Metric_space_mspace_mdist capped_metric_mspace*)
show *?thesis*
unfolding *topology_eq*
proof
fix *S*
show *openin* (*mtopology_of* (*capped_metric* δ *m*)) *S* = *openin* (*mtopology_of m*) *S*
proof (*cases S* ⊆ *mspace m*)
case *True*
have *mball x r* ⊆ *Cap.mball x r* **for** *x r*
by (*smt* (*verit*, *ccfv_SIG*) *Cap.in_mball in_mball mdist_capped_le subsetI*)
moreover have ∃ *r*>0. *Cap.mball x r* ⊆ *S* **if** *r*>0 *x* ∈ *S* **and** *r*: *mball x r* ⊆ *S* **for** *r x*
proof (*intro exI conjI*)
show *min* (δ/2) *r* > 0
using ‹*r*>0› ‹δ>0› **by** *force*
show *Cap.mball x* (*min* (δ/2) *r*) ⊆ *S*
using *that*
by *clarsimp* (*smt* (*verit*) *capped_metric_mdist field_sum_of_halves*

```
in_mball subsetD)
      qed
      ultimately have (∃ r>0. Cap.mball x r ⊆ S) = (∃ r>0. mball x r ⊆ S) if x
∈ S for x
        by (meson subset_trans that)
      then show ?thesis
       by (simp add: mtopology_of_def openin_mtopology Cap.openin_mtopology)
    qed (simp add: openin_closedin_eq)
  qed
qed (simp add: capped_metric)
```

Might have been easier to prove this within the locale to start with (using Self)

```
lemma (in Metric_space) mtopology_capped_metric:
  Metric_space.mtopology M (capped_dist δ) = mtopology
  using mtopology_capped_metric [of δ metric(M,d)]
  by (simp add: Metric_space.mtopology_of capped_dist capped_metric_def)

lemma (in Metric_space) MCauchy_capped_metric:
  Metric_space.MCauchy M (capped_dist δ) σ ⟷ MCauchy σ
proof (cases δ > 0)
  case True
  interpret Cap: Metric_space M capped_dist δ
    by (simp add: capped_dist)
  show ?thesis
  proof
    assume σ: Cap.MCauchy σ
    show MCauchy σ
      unfolding MCauchy_def
    proof (intro conjI strip)
      show range σ ⊆ M
        using Cap.MCauchy_def σ by presburger
      fix ε :: real
      assume ε > 0
      with True σ
      obtain N where ∀ n n'. N ≤ n ⟶ N ≤ n' ⟶ capped_dist δ (σ n) (σ n')
< min δ ε
        unfolding Cap.MCauchy_def by (metis min_less_iff_conj)
      with True show ∃ N. ∀ n n'. N ≤ n ⟶ N ≤ n' ⟶ d (σ n) (σ n') < ε
        by (force simp: capped_dist_def)
    qed
  next
    assume MCauchy σ
    then show Cap.MCauchy σ
     unfolding MCauchy_def Cap.MCauchy_def by (force simp: capped_dist_def)
  qed
qed (simp add: capped_dist_def)
```

lemma (**in** *Metric_space*) *mcomplete_capped_metric*:
Metric_space.mcomplete M (*capped_dist δ*) ⟷ *mcomplete*
by (*simp add*: *MCauchy_capped_metric Metric_space.mcomplete_def capped_dist mtopology_capped_metric mcomplete_def*)

lemma *bounded_equivalent_metric*:
assumes $\delta > 0$
obtains m' **where** *mspace* m' = *mspace m mtopology_of* m' = *mtopology_of m* $\bigwedge x\ y.$ *mdist* m' x $y < \delta$
proof
let *?m* = *capped_metric* $(\delta/2)$ *m*
fix x y
show *mdist ?m x y* $< \delta$
by (*smt* (*verit*, *best*) *assms field_sum_of_halves mdist_capped*)
qed (*auto simp*: *mtopology_capped_metric*)

A technical lemma needed below

lemma *Sup_metric_cartesian_product*:
fixes *I m*
defines *S* ≡ *PiE I* (*mspace* ∘ *m*)
defines *D* ≡ λ*x y*. *if x* ∈ *S* ∧ *y* ∈ *S then SUP i*∈*I*. *mdist* (*m i*) (*x i*) (*y i*) *else 0*
defines m' ≡ *metric*(*S*,*D*)
assumes $I \neq \{\}$
and *c*: $\bigwedge i\ x\ y.$ ⟦*i* ∈ *I*; *x* ∈ *mspace*(*m i*); *y* ∈ *mspace*(*m i*)⟧ ⟹ *mdist* (*m i*) *x y* ≤ *c*
shows *Metric_space S D*
and ∀ *x* ∈ *S*. ∀ *y* ∈ *S*. ∀ *b*. *D x y* ≤ *b* ⟷ (∀ *i* ∈ *I*. *mdist* (*m i*) (*x i*) (*y i*) ≤ *b*) (**is** *?the2*)
proof −
have *bdd*: *bdd_above* ((λ*i*. *mdist* (*m i*) (*x i*) (*y i*)) ‘ *I*)
if *x* ∈ *S y* ∈ *S* **for** *x y*
using *c that* **by** (*force simp*: *S_def bdd_above_def*)
have *D_iff*: *D x y* ≤ *b* ⟷ (∀ *i* ∈ *I*. *mdist* (*m i*) (*x i*) (*y i*) ≤ *b*)
if *x* ∈ *S y* ∈ *S* **for** *x y b*
using *that* ‹*I* ≠ {}› **by** (*simp add*: *D_def PiE_iff cSup_le_iff bdd*)
show *Metric_space S D*
proof
fix *x y*
show *D0*: 0 ≤ *D x y*
using *bdd* ‹*I* ≠ {}›
by (*metis D_def D_iff Orderings.order_eq_iff dual_order.trans ex_in_conv mdist_nonneg*)
show *D x y* = *D y x*
by (*simp add*: *D_def mdist_commute*)
assume *x* ∈ *S* **and** *y* ∈ *S*
then
have *D x y* = 0 ⟷ (∀ *i*∈*I*. *mdist* (*m i*) (*x i*) (*y i*) = 0)
using *D0 D_iff* [*of x y 0*] *nle_le* **by** *fastforce*
also have ... ⟷ *x* = *y*

```
    using ‹x ∈ S› ‹y ∈ S› by (fastforce simp: S_def PiE_iff extensional_def)
   finally show (D x y = 0) ⟷ (x = y) .
   fix z
   assume z ∈ S
   have mdist (m i) (x i) (z i) ≤ D x y + D y z if i ∈ I for i
   proof −
    have mdist (m i) (x i) (z i) ≤ mdist (m i) (x i) (y i) + mdist (m i) (y i) (z
i)
     by (metis PiE_E S_def ‹x ∈ S› ‹y ∈ S› ‹z ∈ S› comp_apply mdist_triangle
that)
    also have ... ≤ D x y + D y z
     using ‹x ∈ S› ‹y ∈ S› ‹z ∈ S› by (meson D_iff add_mono order_refl that)
    finally show ?thesis .
   qed
   then show D x z ≤ D x y + D y z
    by (simp add: D_iff ‹x ∈ S› ‹z ∈ S›)
  qed
  then interpret Metric_space S D .
  show ?the2
  proof (intro strip)
   show (D x y ≤ b) = (∀ i∈I. mdist (m i) (x i) (y i) ≤ b)
    if x ∈ S and y ∈ S for x y b
    using that by (simp add: D_iff m'_def)
  qed
qed

lemma metrizable_topology_A:
  assumes metrizable_space (product_topology X I)
  shows (product_topology X I) = trivial_topology ∨ (∀ i ∈ I. metrizable_space
(X i))
   by (meson assms metrizable_space_retraction_map_image retraction_map_product_projection)

lemma metrizable_topology_C:
  assumes completely_metrizable_space (product_topology X I)
 shows (product_topology X I) = trivial_topology ∨ (∀ i ∈ I. completely_metrizable_space
(X i))
    by (meson assms completely_metrizable_space_retraction_map_image retrac-
tion_map_product_projection)

lemma metrizable_topology_B:
  fixes a X I
  defines L ≡ {i ∈ I. ∄ a. topspace (X i) ⊆ {a}}
  assumes topspace (product_topology X I) ≠ {}
   and met: metrizable_space (product_topology X I)
   and ⋀i. i ∈ I ⟹ metrizable_space (X i)
  shows  countable L
proof −
  have ⋀i. ∃ p q. i ∈ L ⟶ p ∈ topspace(X i) ∧ q ∈ topspace(X i) ∧ p ≠ q
   unfolding L_def by blast
```

then obtain $\varphi\ \psi$ **where** φ: $\bigwedge i.\ i \in L \Longrightarrow \varphi\ i \in topspace(X\ i) \wedge \psi\ i \in topspace(X\ i) \wedge \varphi\ i \neq \psi\ i$
by *metis*
obtain z **where** z: $z \in (\Pi_E\ i{\in}I.\ topspace\ (X\ i))$
using *assms*(2) **by** *fastforce*
define p **where** $p \equiv \lambda i.\ if\ i \in L\ then\ \varphi\ i\ else\ z\ i$
define q **where** $q \equiv \lambda i\ j.\ if\ j = i\ then\ \psi\ i\ else\ p\ j$
have p: $p \in topspace(product\_topology\ X\ I)$
using $z\ \varphi$ **by** (*auto simp*: *p_def L_def*)
then have q: $\bigwedge i.\ i \in L \Longrightarrow q\ i \in topspace\ (product\_topology\ X\ I)$
by (*auto simp*: *L_def q_def* φ)
have *fin*: *finite* $\{i \in L.\ q\ i \notin U\}$ **if** U: *openin* (*product_topology X I*) $U\ p \in U$ **for** U
proof −
obtain V **where** V: *finite* $\{i \in I.\ V\ i \neq topspace\ (X\ i)\}$ $(\forall\, i{\in}I.\ openin\ (X\ i)\ (V\ i))$ $p \in Pi_E\ I\ V$ $Pi_E\ I\ V \subseteq U$
using U **by** (*force simp*: *openin_product_topology_alt*)
moreover
have $V\ x \neq topspace\ (X\ x)$ **if** $x \in L$ **and** $q\ x \notin U$ **for** x
using *that V q*
by (*smt* (*verit*, *del_insts*) *PiE_iff q_def subset_eq topspace_product_topology*)
then have $\{i \in L.\ q\ i \notin U\} \subseteq \{i \in I.\ V\ i \neq topspace\ (X\ i)\}$
by (*force simp*: *L_def*)
ultimately show *?thesis*
by (*meson finite_subset*)
qed
obtain $M\ d$ **where** *Metric_space M d* **and** XI: *product_topology X I* = *Metric_space.mtopology M d*
using *met metrizable_space_def* **by** *blast*
then interpret *Metric_space M d*
by *blast*
define C **where** $C \equiv \bigcup n{::}nat.\ \{i \in L.\ q\ i \notin mball\ p\ (inverse\ (Suc\ n))\}$
have *finite* $\{i \in L.\ q\ i \notin mball\ p\ (inverse\ (real\ (Suc\ n)))\}$ **for** n
using *XI p* **by** (*intro fin*; *force*)
then have *countable C*
unfolding *C_def*
by (*meson countableI_type countable_UN countable_finite*)
moreover have $L \subseteq C$
proof (*clarsimp simp*: *C_def*)
fix i
assume $i \in L$ **and** $q\ i \in M$ **and** $p \in M$
then show $\exists n.\ \neg\ d\ p\ (q\ i) < inverse\ (1 + real\ n)$
using *reals_Archimedean* [*of d p* (*q i*)]
by (*metis* φ *mdist_pos_eq not_less_iff_gr_or_eq of_nat_Suc p_def q_def*)
qed
ultimately show *?thesis*
using *countable_subset* **by** *blast*
qed

lemma *metrizable_topology_DD*:
assumes *topspace* (*product_topology X I*) ≠ {}
and *co*: *countable* {*i* ∈ *I*. ∄*a*. *topspace* (*X i*) ⊆ {*a*}}
and *m*: ⋀*i*. *i* ∈ *I* ⟹ *X i* = *mtopology_of* (*m i*)
obtains *M d* **where** *Metric_space M d product_topology X I* = *Metric_space.mtopology M d*
(⋀*i*. *i* ∈ *I* ⟹ *mcomplete_of* (*m i*)) ⟹ *Metric_space.mcomplete M d*
proof (*cases I* = {})
case *True*
then show *?thesis*
by (*metis discrete_metric.mcomplete_discrete_metric discrete_metric.mtopology_discrete_metric metric_M_dd product_topology_empty_discrete that*)
next
case *False*
obtain *nk* **and** *C*:: *nat set* **where** *nk*: {*i* ∈ *I*. ∄*a*. *topspace* (*X i*) ⊆ {*a*}} = *nk* ‘ *C* **and** *inj_on nk C*
using *co* **by** (*force simp*: *countable_as_injective_image_subset*)
then obtain *kn* **where** *kn*: ⋀*w*. *w* ∈ *C* ⟹ *kn* (*nk w*) = *w*
by (*metis inv_into_f_f*)
define *cm* **where** *cm* ≡ λ*i*. *capped_metric* (*inverse*(*Suc*(*kn i*))) (*m i*)
have *mspace_cm*: *mspace* (*cm i*) = *mspace* (*m i*) **for** *i*
by (*simp add*: *cm_def*)
have *c1*: ⋀*i x y*. *mdist* (*cm i*) *x y* ≤ *1*
by (*simp add*: *cm_def capped_metric_mdist min_le_iff_disj divide_simps*)
then have *bdd*: *bdd_above* ((λ*i*. *mdist* (*cm i*) (*x i*) (*y i*)) ‘ *I*) **for** *x y*
by (*meson bdd_above.I2*)
define *M* **where** *M* ≡ *Pi*$_E$ *I* (*mspace* ∘ *cm*)
define *d* **where** *d* ≡ λ*x y*. *if x* ∈ *M* ∧ *y* ∈ *M then SUP i*∈*I*. *mdist* (*cm i*) (*x i*) (*y i*) *else 0*

have *d_le1*: *d x y* ≤ *1* **for** *x y*
using ‹*I* ≠ {}› *c1* **by** (*simp add*: *d_def bdd cSup_le_iff*)
with ‹*I* ≠ {}› *Sup_metric_cartesian_product* [*of I cm*]
have *Metric_space M d*
and ∗: ∀ *x*∈*M*. ∀ *y*∈*M*. ∀ *b*. (*d x y* ≤ *b*) ⟷ (∀ *i*∈*I*. *mdist* (*cm i*) (*x i*) (*y i*) ≤ *b*)
by (*auto simp*: *False bdd M_def d_def cSUP_le_iff intro*: *c1*)
then interpret *Metric_space M d*
by *metis*
have *le_d*: *mdist* (*cm i*) (*x i*) (*y i*) ≤ *d x y* **if** *i* ∈ *I x* ∈ *M y* ∈ *M* **for** *i x y*
using ∗ *that* **by** *blast*
have *product_m*: *PiE I* (λ*i*. *mspace* (*m i*)) = *topspace*(*product_topology X I*)
using *m* **by** *force*

define *m*′ **where** *m*′ = *metric* (*M*,*d*)
define *J* **where** *J* ≡ λ*U*. {*i* ∈ *I*. *U i* ≠ *topspace* (*X i*)}
have *1*: ∃ *U*. *finite* (*J U*) ∧ (∀ *i*∈*I*. *openin* (*X i*) (*U i*)) ∧ *x* ∈ *Pi*$_E$ *I U* ∧ *Pi*$_E$ *I U* ⊆ *mball z r*

```
    if x ∈ M z ∈ M and r: 0 < r d z x < r for x z r
  proof −
    have x: ⋀i. i ∈ I ⟹ x i ∈ topspace(X i)
      using M_def m mspace_cm that(1) by auto
    have z: ⋀i. i ∈ I ⟹ z i ∈ topspace(X i)
      using M_def m mspace_cm that(2) by auto
    obtain R where 0 < R d z x < R R < r
      using r dense by (smt (verit, ccfv_threshold))
    define U where U ≡ λi. if R ≤ inverse(Suc(kn i)) then mball_of (m i) (z i)
R else topspace(X i)
    show ?thesis
    proof (intro exI conjI)
      obtain n where n: real n * R > 1
        using ‹0 < R› ex_less_of_nat_mult by blast
      have finite (nk ‘ (C ∩ {..n}))
        by force
      moreover
      have ∃m. m ∈ C ∧ m ≤ n ∧ i = nk m
        if R: R ≤ inverse (1 + real (kn i)) and i ∈ I
          and neq: mball_of (m i) (z i) R ≠ topspace (X i) for i
      proof −
        interpret MI: Metric_space mspace (m i) mdist (m i)
          by auto
        have MI.mball (z i) R ≠ topspace (X i)
          by (metis mball_of_def neq)
        then have ∄a. topspace (X i) ⊆ {a}
          using ‹0 < R› m subset_antisym ‹i ∈ I› z by fastforce
        then have i ∈ nk ‘ C
          using nk ‹i ∈ I› by auto
        then show ?thesis
        by (smt (verit, ccfv_SIG) R ‹0 < R› image_iff kn lift_Suc_mono_less_iff
mult_mono n not_le_imp_less of_nat_0_le_iff of_nat_Suc right_inverse)
      qed
      then have J U ⊆ nk ‘ (C ∩ {..n})
        by (auto simp: image_iff Bex_def J_def U_def split: if_split_asm)
      ultimately show finite (J U)
        using finite_subset by blast
      show ∀i∈I. openin (X i) (U i)
          by (simp add: Metric_space.openin_mball U_def mball_of_def mtopol-
ogy_of_def m)
      have xin: x ∈ Pi_E I (topspace ∘ X)
        using M_def ‹x ∈ M› x by auto
      moreover
      have ⋀i. ⟦i ∈ I; R ≤ inverse (1 + real (kn i))⟧ ⟹ mdist (m i) (z i) (x i)
< R
        by (smt (verit, ccfv_SIG) ‹d z x < R› capped_metric_mdist cm_def le_d
of_nat_Suc that)
      ultimately show x ∈ Pi_E I U
        using m z by (auto simp: U_def PiE_iff)
```

```
      show Pi_E I U ⊆ mball z r
      proof
        fix y
        assume y: y ∈ Pi_E I U
        then have y ∈ M
          by (force simp: PiE_iff M_def U_def m mspace_cm split: if_split_asm)
        moreover
        have ∀ i∈I. mdist (cm i) (z i) (y i) ≤ R
            by (smt (verit) PiE_mem U_def cm_def in_mball_of inverse_Suc
mdist_capped mdist_capped_le y)
        then have d z y ≤ R
          by (simp add: ‹y ∈ M› ‹z ∈ M› *)
        ultimately show y ∈ mball z r
          using ‹R < r› ‹z ∈ M› by force
      qed
    qed
  qed
  have 2: ∃ r>0. mball x r ⊆ S
    if finite (J U) and x: x ∈ Pi_E I U and S: Pi_E I U ⊆ S
      and U: ⋀i. i∈I ⟹ openin (X i) (U i)
      and x ∈ S for U S x
  proof −
    { fix i
      assume i ∈ J U
      then have i ∈ I
        by (auto simp: J_def)
      then have openin (mtopology_of (m i)) (U i)
        using U m by force
      then have openin (mtopology_of (cm i)) (U i)
        by (simp add: Abstract_Metric_Spaces.mtopology_capped_metric cm_def)
      then have ∃ r>0. mball_of (cm i) (x i) r ⊆ U i
        using x
      by (simp add: Metric_space.openin_mtopology PiE_mem ‹i ∈ I› mball_of_def
mtopology_of_def)
    }
    then obtain rf where rf: ⋀j. j ∈ J U ⟹ rf j >0 ∧ mball_of (cm j) (x j)
(rf j) ⊆ U j
      by metis
    define r where r ≡ Min (insert 1 (rf ‘ J U))
    show ?thesis
    proof (intro exI conjI)
      show r > 0
        by (simp add: ‹finite (J U)› r_def rf)
      have r [simp]: ⋀j. j ∈ J U ⟹ r ≤ rf j r ≤ 1
        by (auto simp: r_def that(1))
      have *: mball_of (cm i) (x i) r ⊆ U i if i ∈ I for i
      proof (cases i ∈ J U)
        case True
        with r show ?thesis
```

by (*smt* (*verit*) *Metric_space.in_mball Metric_space_mspace_mdist mball_of_def rf subset_eq*)
next
case *False*
then show *?thesis*
by (*simp add*: *J_def cm_def m subset_eq that*)
qed
show *mball x r* $\subseteq$ *S*
by (*smt* (*verit*) *x* $*$ *in_mball_of M_def Metric_space.in_mball Metric_space_axioms PiE_iff le_d o_apply subset_eq S*)
qed
qed
have *3*: *x* $\in$ *M*
if §: $\bigwedge x.\ x{\in}S \Longrightarrow \exists\, U.\ \mathit{finite}\ (J\ U) \wedge (\forall\, i{\in}I.\ \mathit{openin}\ (X\ i)\ (U\ i)) \wedge x \in \mathit{Pi}_E\ I\ U \wedge \mathit{Pi}_E\ I\ U \subseteq S$
and *x* $\in$ *S* **for** *S x*
using § [*OF* ‹*x* $\in$ *S*›] *m openin_subset* **by** (*fastforce simp*: *M_def PiE_iff cm_def*)
show *thesis*
proof
show *Metric_space M d*
using *Metric_space_axioms* **by** *blast*
show *eq*: *product_topology X I* = *Metric_space.mtopology M d*
unfolding *topology_eq openin_mtopology openin_product_topology_alt*
using *J_def 1 2 3 subset_iff zero* **by** (*smt* (*verit*, *ccfv_threshold*))
show *mcomplete* **if** $\bigwedge i.\ i \in I \Longrightarrow \mathit{mcomplete\_of}\ (m\ i)$
unfolding *mcomplete_def*
proof (*intro strip*)
fix σ
assume σ: *MCauchy* σ
have $\exists\, y.\ i \in I \longrightarrow \mathit{limitin}\ (X\ i)\ (\lambda n.\ \sigma\ n\ i)\ y\ \mathit{sequentially}$ **for** *i*
proof (*cases i* $\in$ *I*)
case *True*
interpret *MI*: *Metric_space mspace* (*m i*) *mdist* (*m i*)
by *auto*
have $\bigwedge\sigma.$ *MI.MCauchy* $\sigma \longrightarrow (\exists\, x.\ \mathit{limitin}\ \mathit{MI.mtopology}\ \sigma\ x\ \mathit{sequentially})$
by (*meson MI.mcomplete_def True mcomplete_of_def that*)
moreover have *MI.MCauchy* ($\lambda n.\ \sigma\ n\ i$)
unfolding *MI.MCauchy_def*
proof (*intro conjI strip*)
show *range* ($\lambda n.\ \sigma\ n\ i$) $\subseteq$ *mspace* (*m i*)
by (*smt* (*verit*, *ccfv_threshold*) *MCauchy_def PiE_iff True* σ *eq image_subset_iff m topspace_mtopology topspace_mtopology_of topspace_product_topology*)
fix ε::*real*
define *r* **where** *r* $\equiv$ *min* ε (*inverse*(*Suc* (*kn i*)))
assume $\varepsilon > 0$
then have *r* $>$ *0*
by (*simp add*: *r_def*)
then obtain *N* **where** *N*: $\bigwedge n\ n'.\ N \le n \wedge N \le n' \Longrightarrow d\ (\sigma\ n)\ (\sigma\ n') <$

```
r
          using σ unfolding MCauchy_def by meson
        show ∃N. ∀n n'. N ≤ n ⟶ N ≤ n' ⟶ mdist (m i) (σ n i) (σ n' i) < ε
         proof (intro strip exI)
          fix n n'
          assume N ≤ n and N ≤ n'
          then have mdist (cm i) (σ n i) (σ n' i) < r
           using ∗
            by (smt (verit) Metric_space.MCauchy_def Metric_space_axioms N
True σ rangeI subsetD)
          then
          show mdist (m i) (σ n i) (σ n' i) < ε
           unfolding cm_def r_def
           by (smt (verit, ccfv_SIG) capped_metric_mdist)
         qed
       qed
       ultimately show ?thesis
        by (simp add: m mtopology_of_def)
     qed auto
    then obtain y where ⋀i. i ∈ I ⟹ limitin (X i) (λn. σ n i) (y i) sequentially
      by metis
     with σ show ∃x. limitin mtopology σ x sequentially
      apply (rule_tac x=λi∈I. y i in exI)
      apply (simp add: MCauchy_def limitin_componentwise flip: eq)
     by (metis eq eventually_at_top_linorder range_subsetD topspace_mtopology
topspace_product_topology)
   qed
  qed
qed


lemma metrizable_topology_D:
  assumes topspace (product_topology X I) ≠ {}
    and co: countable {i ∈ I. ∄a. topspace (X i) ⊆ {a}}
    and met: ⋀i. i ∈ I ⟹ metrizable_space (X i)
  shows metrizable_space (product_topology X I)
proof −
  have ⋀i. i ∈ I ⟹ ∃m. X i = mtopology_of m
    by (metis Metric_space.mtopology_of met metrizable_space_def)
  then obtain m where m: ⋀i. i ∈ I ⟹ X i = mtopology_of (m i)
    by metis
  then show ?thesis
     using metrizable_topology_DD [of X I m] assms by (force simp: metriz-
able_space_def)
qed

lemma metrizable_topology_E:
  assumes topspace (product_topology X I) ≠ {}
    and countable {i ∈ I. ∄a. topspace (X i) ⊆ {a}}
```

and *met*: $\bigwedge i.\ i \in I \Longrightarrow$ *completely_metrizable_space* $(X\ i)$
shows *completely_metrizable_space* (*product_topology* $X\ I$)
proof −
have $\bigwedge i.\ i \in I \Longrightarrow \exists\, m.$ *mcomplete_of* $m \wedge X\ i =$ *mtopology_of* m
using *met Metric_space.mtopology_of Metric_space.mcomplete_of* **unfolding** *completely_metrizable_space_def*
by *metis*
then obtain m **where** $\bigwedge i.\ i \in I \Longrightarrow$ *mcomplete_of* $(m\ i) \wedge X\ i =$ *mtopology_of* $(m\ i)$
by *metis*
then show *?thesis*
using *metrizable_topology_DD* [*of X I m*] *assms* **unfolding** *metrizable_space_def*
by (*metis* (*full_types*) *completely_metrizable_space_def*)
qed

proposition *metrizable_space_product_topology*:
metrizable_space (*product_topology* $X\ I$) $\longleftrightarrow$
(*product_topology* $X\ I$) = *trivial_topology* $\vee$
countable $\{i \in I.\ \neg\ (\exists\, a.$ *topspace*$(X\ i) \subseteq \{a\})\} \wedge$
$(\forall\, i \in I.$ *metrizable_space* $(X\ i))$
by (*metis* (*mono_tags*, *lifting*) *empty_metrizable_space metrizable_topology_A metrizable_topology_B metrizable_topology_D subtopology_eq_discrete_topology_empty*)

proposition *completely_metrizable_space_product_topology*:
completely_metrizable_space (*product_topology* $X\ I$) $\longleftrightarrow$
(*product_topology* $X\ I$) = *trivial_topology* $\vee$
countable $\{i \in I.\ \neg\ (\exists\, a.$ *topspace*$(X\ i) \subseteq \{a\})\} \wedge$
$(\forall\, i \in I.$ *completely_metrizable_space* $(X\ i))$
by (*smt* (*verit*, *del_insts*) *Collect_cong completely_metrizable_imp_metrizable_space empty_completely_metrizable_space metrizable_topology_B metrizable_topology_C metrizable_topology_E subtopology_eq_discrete_topology_empty*)

lemma *completely_metrizable_Euclidean_space*:
completely_metrizable_space(*Euclidean_space* n)
unfolding *Euclidean_space_def*
proof (*rule completely_metrizable_space_closedin*)
show *completely_metrizable_space* (*powertop_real* (*UNIV*::*nat set*))
by (*simp add*: *completely_metrizable_space_product_topology completely_metrizable_space_euclidean*)
show *closedin* (*powertop_real UNIV*) $\{x.\ \forall\, i{\geq}n.\ x\ i = 0\}$
using *closedin_Euclidean_space topspace_Euclidean_space* **by** *auto*
qed

lemma *metrizable_Euclidean_space*:
metrizable_space(*Euclidean_space* n)
by (*simp add*: *completely_metrizable_Euclidean_space completely_metrizable_imp_metrizable_space*)

lemma *locally_connected_Euclidean_space*:
locally_connected_space(*Euclidean_space* n)

by (*simp add*: *locally_path_connected_Euclidean_space locally_path_connected_imp_locally_connecte*

end

6.8 Infinite sums

In this theory, we introduce the definition of infinite sums, i.e., sums ranging over an infinite, potentially uncountable index set with no particular ordering. (This is different from series. Those are sums indexed by natural numbers, and the order of the index set matters.)

Our definition is quite standard: $s := \sum_{x\in A} f(x)$ is the limit of finite sums $s_F := \sum_{x\in F} f(x)$ for increasing F. That is, s is the limit of the net s_F where F are finite subsets of A ordered by inclusion. We believe that this is the standard definition for such sums. See, e.g., Definition 4.11 in [2]. This definition is quite general: it is well-defined whenever f takes values in some commutative monoid endowed with a Hausdorff topology. (Examples are reals, complex numbers, normed vector spaces, and more.)

theory *Infinite_Sum*
imports
Elementary_Topology
HOL−Library.Extended_Nonnegative_Real
HOL−Library.Complex_Order
begin

6.8.1 Definition and syntax

definition *HAS_SUM* :: ‹('*a* ⇒ '*b* :: {*comm_monoid_add*, *topological_space*}) ⇒ '*a set* ⇒ '*b* ⇒ *bool*›
where *has_sum_def*: ‹*HAS_SUM f A x* ≡ (*sum f* ⟶ *x*) (*finite_subsets_at_top A*)›

abbreviation *has_sum* (**infixr** ‹*has'_sum*› *46*) **where**
(*f has_sum S*) *A* ≡ *HAS_SUM f A S*

definition *summable_on* :: ('*a* ⇒ '*b*::{*comm_monoid_add*, *topological_space*}) ⇒ '*a set* ⇒ *bool* (**infixr** ‹*summable'_on*› *46*) **where**
f summable_on A ≡ (∃ *x*. (*f has_sum x*) *A*)

definition *infsum* :: ('*a* ⇒ '*b*::{*comm_monoid_add*,*t2_space*}) ⇒ '*a set* ⇒ '*b* **where**
infsum f A = (*if f summable_on A then Lim* (*finite_subsets_at_top A*) (*sum f*) *else 0*)

abbreviation *abs_summable_on* :: ('*a* ⇒ '*b*::*real_normed_vector*) ⇒ '*a set* ⇒ *bool* (**infixr** ‹*abs'_summable'_on*› *46*) **where**
f abs_summable_on A ≡ (λ*x*. *norm* (*f x*)) *summable_on A*

syntax (*ASCII*)
_infsum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b*::*topological_comm_monoid_add*
(‹(‹*indent=3 notation=‹binder INFSUM››INFSUM* (*_*/:*_*)./ *_*)› [*0*, *51*, *10*] *10*)
syntax
_infsum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b*::*topological_comm_monoid_add*
(‹(‹*indent=2 notation=‹binder* ∑∞››∑∞(*_*/∈*_*)./ *_*)› [*0*, *51*, *10*] *10*)
syntax_consts
_infsum ⇌ *infsum*
translations — Beware of argument permutation!
∑∞*i*∈*A*. *b* ⇌ *CONST infsum* (λ*i*. *b*) *A*

syntax (*ASCII*)
_univinfsum :: *pttrn* ⇒ *'a* ⇒ *'a* (‹(‹*indent=3 notation=‹binder INFSUM››INFSUM* *_*./ *_*)› [*0*, *10*] *10*)
syntax
_univinfsum :: *pttrn* ⇒ *'a* ⇒ *'a* (‹(‹*indent=2 notation=‹binder* ∑∞››∑∞*_*./ *_*)› [*0*, *10*] *10*)
syntax_consts
_univinfsum ⇌ *infsum*
translations
∑∞*x*. *t* ⇌ *CONST infsum* (λ*x*. *t*) (*CONST UNIV*)

syntax (*ASCII*)
_qinfsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* (‹(‹*indent=3 notation=‹binder INFSUM››INFSUM* *_* |/ *_*./ *_*)› [*0*, *0*, *10*] *10*)
syntax
_qinfsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* (‹(‹*indent=2 notation=‹binder* ∑∞››∑∞*_* | (*_*)./ *_*)› [*0*, *0*, *10*] *10*)
syntax_consts
_qinfsum ⇌ *infsum*
translations
∑∞*x*|*P*. *t* => *CONST infsum* (λ*x*. *t*) {*x*. *P*}
print_translation ‹
[(***const_syntax***‹*infsum*›, *K* (*Collect_binder_tr'* ***syntax_const***‹*_qinfsum*›))]
›

6.8.2 General properties

lemma *infsumI*:
fixes *f g* :: ‹*'a* ⇒ *'b*::{*comm_monoid_add*, *t2_space*}›
assumes ‹(*f has_sum x*) *A*›
shows ‹*infsum f A* = *x*›
by (*metis assms finite_subsets_at_top_neq_bot infsum_def summable_on_def has_sum_def tendsto_Lim*)

lemma *infsum_eqI*:
fixes *f g* :: ‹*'a* ⇒ *'b*::{*comm_monoid_add*, *t2_space*}›
assumes ‹*x* = *y*›

assumes ‹(*f has_sum x*) *A*›
assumes ‹(*g has_sum y*) *B*›
shows ‹*infsum f A* = *infsum g B*›
using *assms infsumI* **by** *blast*

lemma *infsum_eqI'*:
fixes *f g* :: ‹'*a* ⇒ '*b*::{*comm_monoid_add*, *t2_space*}›
assumes ‹⋀*x*. (*f has_sum x*) *A* ⟷ (*g has_sum x*) *B*›
shows ‹*infsum f A* = *infsum g B*›
by (*metis assms infsum_def infsum_eqI summable_on_def*)

lemma *infsum_not_exists*:
fixes *f* :: ‹'*a* ⇒ '*b*::{*comm_monoid_add*, *t2_space*}›
assumes ‹¬ *f summable_on A*›
shows ‹*infsum f A* = *0*›
by (*simp add*: *assms infsum_def*)

lemma *summable_iff_has_sum_infsum*: *f summable_on A* ⟷ (*f has_sum* (*infsum f A*)) *A*
using *infsumI summable_on_def* **by** *blast*

lemma *has_sum_infsum*[*simp*]:
assumes ‹*f summable_on S*›
shows ‹(*f has_sum* (*infsum f S*)) *S*›
using *assms summable_iff_has_sum_infsum* **by** *blast*

lemma *has_sum_cong_neutral*:
fixes *f g* :: ‹'*a* ⇒ '*b*::{*comm_monoid_add*, *topological_space*}›
assumes ‹⋀*x*. *x*∈*T*−*S* ⟹ *g x* = *0*›
assumes ‹⋀*x*. *x*∈*S*−*T* ⟹ *f x* = *0*›
assumes ‹⋀*x*. *x*∈*S*∩*T* ⟹ *f x* = *g x*›
shows (*f has_sum x*) *S* ⟷ (*g has_sum x*) *T*
proof −
have ‹*eventually P* (*filtermap* (*sum f*) (*finite_subsets_at_top S*))
= *eventually P* (*filtermap* (*sum g*) (*finite_subsets_at_top T*))› **for** *P*
proof
assume ‹*eventually P* (*filtermap* (*sum f*) (*finite_subsets_at_top S*))›
then obtain *F0* **where** ‹*finite F0*› **and** ‹*F0* ⊆ *S*› **and** *F0_P*: ‹⋀*F*. *finite F* ⟹ *F* ⊆ *S* ⟹ *F* ⊇ *F0* ⟹ *P* (*sum f F*)›
by (*metis* (*no_types*, *lifting*) *eventually_filtermap eventually_finite_subsets_at_top*)
define *F0'* **where** ‹*F0'* = *F0* ∩ *T*›
have [*simp*]: ‹*finite F0'*› ‹*F0'* ⊆ *T*›
by (*simp_all add*: *F0'_def* ‹*finite F0*›)
have ‹*P* (*sum g F*)› **if** ‹*finite F*› ‹*F* ⊆ *T*› ‹*F* ⊇ *F0'*› **for** *F*
proof −
have ‹*P* (*sum f* ((*F*∩*S*) ∪ (*F0*∩*S*)))›
by (*intro F0_P*) (*use* ‹*F0* ⊆ *S*› ‹*finite F0*› *that* **in** *auto*)
also have ‹*sum f* ((*F*∩*S*) ∪ (*F0*∩*S*)) = *sum g F*›
by (*intro sum.mono_neutral_cong*) (*use that* ‹*finite F0*› *F0'_def assms* **in**

auto)
    finally show *?thesis* **.**
   qed
  with ‹*F0'* ⊆ *T*› ‹*finite F0'*› **show** ‹*eventually P* (*filtermap* (*sum g*) (*finite_subsets_at_top T*))›
   by (*metis* (*no_types*, *lifting*) *eventually_filtermap eventually_finite_subsets_at_top*)
 next
   assume ‹*eventually P* (*filtermap* (*sum g*) (*finite_subsets_at_top T*))›
   then obtain *F0* **where** ‹*finite F0*› **and** ‹*F0* ⊆ *T*› **and** *F0_P*: ‹⋀*F*. *finite F* ⟹ *F* ⊆ *T* ⟹ *F* ⊇ *F0* ⟹ *P* (*sum g F*)›
   by (*metis* (*no_types*, *lifting*) *eventually_filtermap eventually_finite_subsets_at_top*)
   define *F0'* **where** ‹*F0'* = *F0* ∩ *S*›
   have [*simp*]: ‹*finite F0'*› ‹*F0'* ⊆ *S*›
    by (*simp_all add*: *F0'_def* ‹*finite F0*›)
   have ‹*P* (*sum f F*)› **if** ‹*finite F*› ‹*F* ⊆ *S*› ‹*F* ⊇ *F0'*› **for** *F*
   proof −
    have ‹*P* (*sum g* ((*F*∩*T*) ∪ (*F0*∩*T*)))›
     by (*intro F0_P*) (*use* ‹*F0* ⊆ *T*› ‹*finite F0*› *that* **in** *auto*)
    also have ‹*sum g* ((*F*∩*T*) ∪ (*F0*∩*T*)) = *sum f F*›
     by (*intro sum.mono_neutral_cong*) (*use that* ‹*finite F0*› *F0'_def assms* **in** *auto*)
    finally show *?thesis* **.**
   qed
  with ‹*F0'* ⊆ *S*› ‹*finite F0'*› **show** ‹*eventually P* (*filtermap* (*sum f*) (*finite_subsets_at_top S*))›
   by (*metis* (*no_types*, *lifting*) *eventually_filtermap eventually_finite_subsets_at_top*)
 qed

 then have *tendsto_x*: (*sum f* ⟶ *x*) (*finite_subsets_at_top S*) ⟷ (*sum g* ⟶ *x*) (*finite_subsets_at_top T*) **for** *x*
  by (*simp add*: *le_filter_def filterlim_def*)

 then show *?thesis*
  by (*simp add*: *has_sum_def*)
qed

lemma *summable_on_cong_neutral*:
 fixes *f g* :: ‹*'a* ⇒ *'b*::{*comm_monoid_add*, *topological_space*}›
 assumes ‹⋀*x*. *x*∈*T*−*S* ⟹ *g x* = *0*›
 assumes ‹⋀*x*. *x*∈*S*−*T* ⟹ *f x* = *0*›
 assumes ‹⋀*x*. *x*∈*S*∩*T* ⟹ *f x* = *g x*›
 shows *f summable_on S* ⟷ *g summable_on T*
 using *has_sum_cong_neutral*[*of T S g f*, *OF assms*]
 by (*simp add*: *summable_on_def*)

lemma *infsum_cong_neutral*:
 fixes *f g* :: ‹*'a* ⇒ *'b*::{*comm_monoid_add*, *t2_space*}›
 assumes ‹⋀*x*. *x*∈*T*−*S* ⟹ *g x* = *0*›
 assumes ‹⋀*x*. *x*∈*S*−*T* ⟹ *f x* = *0*›

assumes ‹$\bigwedge x.\ x{\in}S{\cap}T \Longrightarrow f\ x = g\ x$›
shows ‹*infsum f S* = *infsum g T*›
by (*smt* (*verit*, *best*) *assms has_sum_cong_neutral infsum_eqI'*)

lemma *has_sum_cong*:
assumes $\bigwedge x.\ x{\in}A \Longrightarrow f\ x = g\ x$
shows (*f has_sum x*) $A \longleftrightarrow$ (*g has_sum x*) A
using *assms* **by** (*intro has_sum_cong_neutral*) *auto*

lemma *summable_on_cong*:
assumes $\bigwedge x.\ x{\in}A \Longrightarrow f\ x = g\ x$
shows *f summable_on A* $\longleftrightarrow$ *g summable_on A*
by (*metis assms summable_on_def has_sum_cong*)

lemma *infsum_cong*:
assumes $\bigwedge x.\ x{\in}A \Longrightarrow f\ x = g\ x$
shows *infsum f A* = *infsum g A*
using *assms infsum_eqI' has_sum_cong* **by** *blast*

lemma *summable_on_cofin_subset*:
fixes *f* :: *'a* ⇒ *'b*::*topological_ab_group_add*
assumes *f summable_on A* **and** [*simp*]: *finite F*
shows *f summable_on* (*A* − *F*)
proof −
from *assms*(*1*) **obtain** *x* **where** *lim_f*: (*sum f* ⟶ *x*) (*finite_subsets_at_top A*)
unfolding *summable_on_def has_sum_def* **by** *auto*
define *F'* **where** *F'* = *F*∩*A*
with *assms* **have** *finite F'* **and** *A*−*F* = *A*−*F'*
by *auto*
have *filtermap* ((∪)*F'*) (*finite_subsets_at_top* (*A*−*F*)) ≤ *finite_subsets_at_top A*
proof (*rule filter_leI*)
fix *P* **assume** *eventually P* (*finite_subsets_at_top A*)
then obtain *X* **where** [*simp*]: *finite X* **and** *XA*: *X* ⊆ *A*
and *P*: ∀ *Y*. *finite Y* ∧ *X* ⊆ *Y* ∧ *Y* ⊆ *A* ⟶ *P Y*
unfolding *eventually_finite_subsets_at_top* **by** *auto*
define *X'* **where** *X'* = *X*−*F*
hence [*simp*]: *finite X'* **and** [*simp*]: *X'* ⊆ *A*−*F*
using *XA* **by** *auto*
hence *finite Y* ∧ *X'* ⊆ *Y* ∧ *Y* ⊆ *A* − *F* ⟶ *P* (*F'* ∪ *Y*) **for** *Y*
using *P XA* **unfolding** *X'_def* **using** *F'_def* ‹*finite F'*› **by** *blast*
thus *eventually P* (*filtermap* ((∪) *F'*) (*finite_subsets_at_top* (*A* − *F*)))
unfolding *eventually_filtermap eventually_finite_subsets_at_top*
by (*rule_tac x*=*X'* **in** *exI*, *simp*)
qed
with *lim_f* **have** (*sum f* ⟶ *x*) (*filtermap* ((∪)*F'*) (*finite_subsets_at_top* (*A*−*F*)))
using *tendsto_mono* **by** *blast*

```
  have ((λG. sum f (F' ∪ G)) ⟶ x) (finite_subsets_at_top (A − F))
    if ((sum f ∘ (∪) F') ⟶ x) (finite_subsets_at_top (A − F))
    using that unfolding o_def by auto
  hence ((λG. sum f (F' ∪ G)) ⟶ x) (finite_subsets_at_top (A−F))
    using tendsto_compose_filtermap [symmetric]
    by (simp add: ‹(sum f ⟶ x) (filtermap ((∪) F') (finite_subsets_at_top (A
− F)))›
        tendsto_compose_filtermap)
  have ∀ Y. finite Y ∧ Y ⊆ A − F ⟶ sum f (F' ∪ Y) = sum f F' + sum f Y
     by (metis Diff_disjoint Int_Diff ‹A − F = A − F'› ‹finite F'› inf.orderE
sum.union_disjoint)
  hence ∀F x in finite_subsets_at_top (A − F). sum f (F' ∪ x) = sum f F' +
sum f x
    unfolding eventually_finite_subsets_at_top
    using exI [where x = {}]
    by (simp add: ‹⋀P. P {} ⟹ ∃x. P x›)
  hence ((λG. sum f F' + sum f G) ⟶ x) (finite_subsets_at_top (A−F))
    using tendsto_cong [THEN iffD1 , rotated]
        ‹((λG. sum f (F' ∪ G)) ⟶ x) (finite_subsets_at_top (A − F))› by
fastforce
  hence ((λG. sum f F' + sum f G) ⟶ sum f F' + (x−sum f F')) (finite_subsets_at_top
(A−F))
    by simp
  hence (sum f ⟶ x − sum f F') (finite_subsets_at_top (A−F))
    using tendsto_add_const_iff by blast
  thus f summable_on (A − F)
    unfolding summable_on_def has_sum_def by auto
qed

lemma
  fixes f :: 'a ⇒ 'b::{topological_ab_group_add}
  assumes ‹(f has_sum b) B› and ‹(f has_sum a) A› and AB: A ⊆ B
  shows has_sum_Diff: (f has_sum (b − a)) (B − A)
proof −
  have finite_subsets1:
   finite_subsets_at_top (B − A) ≤ filtermap (λF. F − A) (finite_subsets_at_top
B)
  proof (rule filter_leI)
   fix P assume eventually P (filtermap (λF. F − A) (finite_subsets_at_top B))
   then obtain X where finite X and X ⊆ B
     and P: finite Y ∧ X ⊆ Y ∧ Y ⊆ B ⟶ P (Y − A) for Y
     unfolding eventually_filtermap eventually_finite_subsets_at_top by auto

   hence finite (X−A) and X−A ⊆ B − A
     by auto
   moreover have finite Y ∧ X−A ⊆ Y ∧ Y ⊆ B − A ⟶ P Y for Y
     using P[where Y=Y∪X] ‹finite X› ‹X ⊆ B›
       by (metis Diff_subset Int_Diff Un_Diff finite_Un inf.orderE le_sup_iff
sup.orderE sup_ge2)
```

```
  ultimately show eventually P (finite_subsets_at_top (B − A))
    unfolding eventually_finite_subsets_at_top by meson
 qed
 have finite_subsets2:
  filtermap (λF. F ∩ A) (finite_subsets_at_top B) ≤ finite_subsets_at_top A
  apply (rule filter_leI)
  using assms unfolding eventually_filtermap eventually_finite_subsets_at_top
    by (metis Int_subset_iff finite_Int inf_le2 subset_trans)

 from assms(1) have limB: (sum f ⟶ b) (finite_subsets_at_top B)
  using has_sum_def by auto
 from assms(2) have limA: (sum f ⟶ a) (finite_subsets_at_top A)
  using has_sum_def by blast
 have ((λF. sum f (F∩A)) ⟶ a) (finite_subsets_at_top B)
 proof (subst asm_rl [of (λF. sum f (F∩A)) = sum f ∘ (λF. F∩A)])
  show (λF. sum f (F ∩ A)) = sum f ∘ (λF. F ∩ A)
    unfolding o_def by auto
  show ((sum f ∘ (λF. F ∩ A)) ⟶ a) (finite_subsets_at_top B)
    unfolding o_def
    using tendsto_compose_filtermap finite_subsets2 limA tendsto_mono
      ‹(λF. sum f (F ∩ A)) = sum f ∘ (λF. F ∩ A)› by fastforce
 qed

 with limB have ((λF. sum f F − sum f (F∩A)) ⟶ b − a) (finite_subsets_at_top
B)
  using tendsto_diff by blast
 have sum f X − sum f (X ∩ A) = sum f (X − A) if finite X and X ⊆ B for
X :: 'a set
  using that by (metis add_diff_cancel_left' sum.Int_Diff)
 hence ∀F x in finite_subsets_at_top B. sum f x − sum f (x ∩ A) = sum f (x
− A)
  by (rule eventually_finite_subsets_at_top_weakI)
 hence ((λF. sum f (F−A)) ⟶ b − a) (finite_subsets_at_top B)
  using tendsto_cong [THEN iffD1 , rotated]
    ‹((λF. sum f F − sum f (F ∩ A)) ⟶ b − a) (finite_subsets_at_top B)›
by fastforce
 hence (sum f ⟶ b − a) (filtermap (λF. F−A) (finite_subsets_at_top B))
  by (subst tendsto_compose_filtermap[symmetric], simp add: o_def)
 thus ?thesis
  using finite_subsets1 has_sum_def tendsto_mono by blast
qed


lemma
 fixes f :: 'a ⇒ 'b::{topological_ab_group_add}
 assumes f summable_on B and f summable_on A and A ⊆ B
 shows summable_on_Diff: f summable_on (B−A)
 by (meson assms summable_on_def has_sum_Diff)
```

lemma
fixes $f :: 'a \Rightarrow 'b::\{topological\_ab\_group\_add,t2\_space\}$
assumes f *summable_on* B **and** f *summable_on* A **and** AB: $A \subseteq B$
shows *infsum_Diff*: *infsum* f $(B - A)$ = *infsum* f B − *infsum* f A
by (*metis AB assms has_sum_Diff infsumI summable_on_def*)

lemma *has_sum_mono_neutral*:
fixes $f :: 'a \Rightarrow 'b::\{ordered\_comm\_monoid\_add,linorder\_topology\}$

assumes ‹$(f$ *has_sum* $a)$ A› **and** $(g$ *has_sum* $b)$ B
assumes ‹$\bigwedge x.\ x \in A \cap B \Longrightarrow f\ x \leq g\ x$›
assumes ‹$\bigwedge x.\ x \in A - B \Longrightarrow f\ x \leq 0$›
assumes ‹$\bigwedge x.\ x \in B - A \Longrightarrow g\ x \geq 0$›
shows $a \leq b$
proof −
define f' g' **where** ‹$f'\ x = (if\ x \in A\ then\ f\ x\ else\ 0)$› **and** ‹$g'\ x = (if\ x \in B$ *then* $g\ x$ *else* $0)$› **for** x
have [*simp*]: ‹f *summable_on* A› ‹g *summable_on* B›
using *assms*(*1*,*2*) *summable_on_def* **by** *auto*
have ‹$(f'$ *has_sum* $a)$ $(A \cup B)$›
by (*smt* (*verit*, *best*) *DiffE IntE Un_iff f'_def assms*(*1*) *has_sum_cong_neutral*)
then have *f'_lim*: ‹$(sum\ f' \longrightarrow a)$ $(finite\_subsets\_at\_top\ (A \cup B))$›
by (*meson has_sum_def*)
have ‹$(g'$ *has_sum* $b)$ $(A \cup B)$›
by (*smt* (*verit*, *best*) *DiffD1 DiffD2 IntE UnCI g'_def assms*(*2*) *has_sum_cong_neutral*)
then have *g'_lim*: ‹$(sum\ g' \longrightarrow b)$ $(finite\_subsets\_at\_top\ (A \cup B))$›
using *has_sum_def* **by** *blast*

have $\bigwedge X\ i.\ [\![X \subseteq A \cup B;\ i \in X]\!] \Longrightarrow f'\ i \leq g'\ i$
using *assms* **by** (*auto simp*: *f'_def g'_def*)
then have ‹$\forall_F\ x$ *in finite_subsets_at_top* $(A \cup B).\ sum\ f'\ x \leq sum\ g'\ x$›
by (*intro eventually_finite_subsets_at_top_weakI sum_mono*)
then show *?thesis*
using *f'_lim finite_subsets_at_top_neq_bot g'_lim tendsto_le* **by** *blast*
qed

lemma *infsum_mono_neutral*:
fixes $f :: 'a \Rightarrow 'b::\{ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes f *summable_on* A **and** g *summable_on* B
assumes ‹$\bigwedge x.\ x \in A \cap B \Longrightarrow f\ x \leq g\ x$›
assumes ‹$\bigwedge x.\ x \in A - B \Longrightarrow f\ x \leq 0$›
assumes ‹$\bigwedge x.\ x \in B - A \Longrightarrow g\ x \geq 0$›
shows *infsum* f $A \leq$ *infsum* g B
by (*smt* (*verit*, *best*) *assms has_sum_infsum has_sum_mono_neutral*)

lemma *has_sum_mono*:
fixes $f :: 'a \Rightarrow 'b::\{ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes $(f$ *has_sum* $x)$ A **and** $(g$ *has_sum* $y)$ A
assumes ‹$\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$›

shows $x \leq y$
using *assms has_sum_mono_neutral* **by** *force*

lemma *infsum_mono*:
fixes $f :: 'a \Rightarrow 'b::\{ordered\_comm\_monoid\_add, linorder\_topology\}$
assumes f *summable_on* A **and** g *summable_on* A
assumes $\langle \bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x \rangle$
shows *infsum* $f\ A \leq$ *infsum* $g\ A$
by (*meson assms has_sum_infsum has_sum_mono*)

lemma *has_sum_finite*[*simp*]:
assumes *finite* F
shows (f *has_sum* (*sum* $f\ F$)) F
using *assms*
by (*auto intro*: *tendsto_Lim simp*: *finite_subsets_at_top_finite infsum_def has_sum_def principal_eq_bot_iff*)

lemma *summable_on_finite*[*simp*]:
fixes $f :: \langle 'a \Rightarrow 'b::\{comm\_monoid\_add, topological\_space\} \rangle$
assumes *finite* F
shows f *summable_on* F
using *assms summable_on_def has_sum_finite* **by** *blast*

lemma *infsum_finite*[*simp*]:
assumes *finite* F
shows *infsum* $f\ F =$ *sum* $f\ F$
by (*simp add*: *assms infsumI*)

lemma *has_sum_finite_approximation*:
fixes $f :: 'a \Rightarrow 'b::\{comm\_monoid\_add, metric\_space\}$
assumes (f *has_sum* x) A **and** $\varepsilon > 0$
shows $\exists F.$ *finite* $F \wedge F \subseteq A \wedge$ *dist* (*sum* $f\ F$) $x \leq \varepsilon$
proof −
have (*sum* $f \longrightarrow x$) (*finite_subsets_at_top* A)
by (*meson assms*(*1*) *has_sum_def*)
hence $*$: $\forall_F\ F$ *in* (*finite_subsets_at_top* A). *dist* (*sum* $f\ F$) $x < \varepsilon$
using *assms*(*2*) **by** (*rule tendstoD*)
thus *?thesis*
unfolding *eventually_finite_subsets_at_top* **by** *fastforce*
qed

lemma *infsum_finite_approximation*:
fixes $f :: 'a \Rightarrow 'b::\{comm\_monoid\_add, metric\_space\}$
assumes f *summable_on* A **and** $\varepsilon > 0$
shows $\exists F.$ *finite* $F \wedge F \subseteq A \wedge$ *dist* (*sum* $f\ F$) (*infsum* $f\ A$) $\leq \varepsilon$
proof −
from *assms* **have** (f *has_sum* (*infsum* $f\ A$)) A
by (*simp add*: *summable_iff_has_sum_infsum*)
from *this* **and** $\langle \varepsilon > 0 \rangle$ **show** *?thesis*

```
    by (rule has_sum_finite_approximation)
qed

lemma abs_summable_summable:
  fixes f :: ‹'a ⇒ 'b :: banach›
  assumes ‹f abs_summable_on A›
  shows ‹f summable_on A›
proof −
  from assms obtain L where lim: ‹(sum (λx. norm (f x)) ⟶ L) (finite_subsets_at_top
A)›
    unfolding has_sum_def summable_on_def by blast
  then have ∗: ‹cauchy_filter (filtermap (sum (λx. norm (f x))) (finite_subsets_at_top
A))›
    by (auto intro!: nhds_imp_cauchy_filter simp: filterlim_def)
  have ‹∃ P. eventually P (finite_subsets_at_top A) ∧
          (∀ F F'. P F ∧ P F' ⟶ dist (sum f F) (sum f F') < e)› if ‹e>0› for e
  proof −
    define d P where ‹d = e/4› and ‹P F ⟷ finite F ∧ F ⊆ A ∧ dist (sum
(λx. norm (f x)) F) L < d› for F
    then have ‹d > 0›
      by (simp add: d_def that)
    have ev_P: ‹eventually P (finite_subsets_at_top A)›
      using lim
      by (auto simp add: P_def[abs_def] ‹0 < d› eventually_conj_iff eventu-
ally_finite_subsets_at_top_weakI tendsto_iff)

    moreover have ‹dist (sum f F1) (sum f F2) < e› if ‹P F1› and ‹P F2› for
F1 F2
    proof −
      from ev_P
      obtain F' where ‹finite F'› and ‹F' ⊆ A› and P_sup_F': ‹finite F ∧ F ⊇
F' ∧ F ⊆ A ⟹ P F› for F
        by atomize_elim (simp add: eventually_finite_subsets_at_top)
      define F where ‹F = F' ∪ F1 ∪ F2›
      have ‹finite F› and ‹F ⊆ A›
        using F_def P_def[abs_def] that ‹finite F'› ‹F' ⊆ A› by auto
      have dist_F: ‹dist (sum (λx. norm (f x)) F) L < d›
        by (metis F_def ‹F ⊆ A› P_def P_sup_F' ‹finite F› le_supE order_refl)

      have dist_F_subset: ‹dist (sum f F) (sum f F') < 2∗d› if F': ‹F' ⊆ F› ‹P
F'› for F'
      proof −
        have ‹dist (sum f F) (sum f F') = norm (sum f (F−F'))›
          unfolding dist_norm using ‹finite F› F' by (subst sum_diff) auto
        also have ‹... ≤ norm (∑ x∈F−F'. norm (f x))›
          by (rule order.trans[OF sum_norm_le[OF order.refl]]) auto
        also have ‹... = dist (∑ x∈F. norm (f x)) (∑ x∈F'. norm (f x))›
          unfolding dist_norm using ‹finite F› F' by (subst sum_diff) auto
        also have ‹... < 2 ∗ d›
```

```
      using dist_F F' unfolding P_def dist_norm real_norm_def by linarith
    finally show ‹dist (sum f F) (sum f F') < 2*d› .
  qed

  have ‹dist (sum f F1) (sum f F2) ≤ dist (sum f F) (sum f F1) + dist (sum
f F) (sum f F2)›
    by (rule dist_triangle3)
  also have ‹... < 2 * d + 2 * d›
    by (intro add_strict_mono dist_F_subset that) (auto simp: F_def)
  also have ‹... ≤ e›
    by (auto simp: d_def)
  finally show ‹dist (sum f F1) (sum f F2) < e› .
 qed
 then show ?thesis
  using ev_P by blast
qed
then have ‹cauchy_filter (filtermap (sum f) (finite_subsets_at_top A))›
 by (simp add: cauchy_filter_metric_filtermap)
moreover have complete (UNIV::'b set)
 by (meson Cauchy_convergent UNIV_I complete_def convergent_def)
ultimately obtain L' where ‹(sum f ⟶ L') (finite_subsets_at_top A)›
 using complete_uniform[where S=UNIV] by (force simp add: filterlim_def)
then show ?thesis
 using summable_on_def has_sum_def by blast
qed
```

The converse of *abs_summable_summable* does not hold: Consider the Hilbert space of square-summable sequences. Let e_i denote the sequence with 1 in the ith position and 0 elsewhere. Let $f(i) := e_i/i$ for $i \geq 1$. We have $\neg$ *f abs_summable_on UNIV* because $\|f(i)\| = 1/i$ and thus the sum over $\|f(i)\|$ diverges. On the other hand, we have *f summable_on UNIV*; the limit is the sequence with $1/i$ in the ith position.

(We have not formalized this separating example here because to the best of our knowledge, this Hilbert space has not been formalized in Isabelle/HOL yet.)

```
lemma norm_has_sum_bound:
 fixes f :: 'b ⇒ 'a::real_normed_vector
  and A :: 'b set
 assumes ((λx. norm (f x)) has_sum n) A
 assumes (f has_sum a) A
 shows norm a ≤ n
proof −
 have norm a ≤ n + ε if ε>0 for ε
 proof−
  have ∃F. norm (a − sum f F) ≤ ε ∧ finite F ∧ F ⊆ A
   using has_sum_finite_approximation[where A=A and f=f and ε=ε] assms
‹0 < ε›
   by (metis dist_commute dist_norm)
```

```
    then obtain F where norm (a − sum f F) ≤ ε
      and finite F and F ⊆ A
      by (simp add: atomize_elim)
    hence norm a ≤ norm (sum f F) + ε
    by (metis add.commute diff_add_cancel dual_order.refl norm_triangle_mono)
    also have ... ≤ sum (λx. norm (f x)) F + ε
      using norm_sum by auto
    also have ... ≤ n + ε
    proof (intro add_right_mono [OF has_sum_mono_neutral])
      show ((λx. norm (f x)) has_sum (∑ x∈F. norm (f x))) F
        by (simp add: ‹finite F›)
    qed (use ‹F ⊆ A› assms in auto)
    finally show ?thesis
      by assumption
  qed
  thus ?thesis
    using linordered_field_class.field_le_epsilon by blast
qed

lemma norm_infsum_bound:
  fixes f :: 'b ⇒ 'a::real_normed_vector
    and A :: 'b set
  assumes f abs_summable_on A
  shows norm (infsum f A) ≤ infsum (λx. norm (f x)) A
proof (cases f summable_on A)
  case True
  have ((λx. norm (f x)) has_sum (∑ ∞x∈A. norm (f x))) A
    by (simp add: assms)
  then show ?thesis
    by (metis True has_sum_infsum norm_has_sum_bound)
next
  case False
  obtain t where t_def: (sum (λx. norm (f x)) ⟶ t) (finite_subsets_at_top
A)
    using assms unfolding summable_on_def has_sum_def by blast
  have sumpos: sum (λx. norm (f x)) X ≥ 0
    for X
    by (simp add: sum_nonneg)
  have tgeq0:t ≥ 0
  proof(rule ccontr)
    define S::real set where S = {s. s < 0}
    assume ¬ 0 ≤ t
    hence t < 0 by simp
    hence t ∈ S
      unfolding S_def by blast
    moreover have open S
      by (metis S_def lessThan_def open_real_lessThan)
    ultimately have ∀ F X in finite_subsets_at_top A. (∑ x∈X. norm (f x)) ∈ S
      using t_def unfolding tendsto_def by blast
```

hence $\exists X.\ (\sum x{\in}X.\ norm\ (f\ x)) \in S$
by (*metis* (*no_types*, *lifting*) *eventually_mono filterlim_iff finite_subsets_at_top_neq_bot tendsto_Lim*)
then obtain X **where** $(\sum x{\in}X.\ norm\ (f\ x)) \in S$
by *blast*
hence $(\sum x{\in}X.\ norm\ (f\ x)) < 0$
unfolding *S_def* **by** *auto*
thus *False* **by** (*simp add*: *leD sumpos*)
qed
have $\exists!h.\ (sum\ (\lambda x.\ norm\ (f\ x)) \longrightarrow h)\ (finite\_subsets\_at\_top\ A)$
using *t_def finite_subsets_at_top_neq_bot tendsto_unique* **by** *blast*
hence $t = (Topological\_Spaces.Lim\ (finite\_subsets\_at\_top\ A)\ (sum\ (\lambda x.\ norm\ (f\ x))))$
using *t_def* **unfolding** *Topological_Spaces.Lim_def*
by (*metis the_equality*)
hence $Lim\ (finite\_subsets\_at\_top\ A)\ (sum\ (\lambda x.\ norm\ (f\ x))) \geq 0$
using *tgeq0* **by** *blast*
thus *?thesis* **unfolding** *infsum_def*
using *False* **by** *auto*
qed

lemma *infsum_tendsto*:
assumes ‹*f summable_on S*›
shows ‹$((\lambda F.\ sum\ f\ F) \longrightarrow infsum\ f\ S)\ (finite\_subsets\_at\_top\ S)$›
using *assms has_sum_def has_sum_infsum* **by** *blast*

lemma *has_sum_0*:
assumes ‹$\bigwedge x.\ x{\in}M \Longrightarrow f\ x = 0$›
shows ‹$(f\ has\_sum\ 0)\ M$›
by (*metis assms finite.intros*(*1*) *has_sum_cong has_sum_cong_neutral has_sum_finite sum.neutral_const*)

lemma *summable_on_0*:
assumes ‹$\bigwedge x.\ x{\in}M \Longrightarrow f\ x = 0$›
shows ‹*f summable_on M*›
using *assms summable_on_def has_sum_0* **by** *blast*

lemma *infsum_0*:
assumes ‹$\bigwedge x.\ x{\in}M \Longrightarrow f\ x = 0$›
shows ‹$infsum\ f\ M = 0$›
by (*metis assms finite_subsets_at_top_neq_bot infsum_def has_sum_0 has_sum_def tendsto_Lim*)

Variants of *infsum_0* etc. suitable as simp-rules

lemma *infsum_0_simp*[*simp*]: ‹$infsum\ (\lambda\_.\ 0)\ M = 0$›
by (*simp_all add*: *infsum_0*)

lemma *summable_on_0_simp*[*simp*]: ‹$(\lambda\_.\ 0)$ *summable_on M*›
by (*simp_all add*: *summable_on_0*)

lemma *has_sum_0_simp*[*simp*]: ‹((λ_. 0) *has_sum* 0) *M›*
by (*simp_all add*: *has_sum_0*)

lemma *has_sum_add*:
fixes *f g* :: *'a* ⇒ *'b*::{*topological_comm_monoid_add*}
assumes ‹(*f has_sum a*) *A›*
assumes ‹(*g has_sum b*) *A›*
shows ‹((λ*x*. *f x* + *g x*) *has_sum* (*a* + *b*)) *A›*
proof −
from *assms* **have** *lim_f*: ‹(*sum f* ⟶ *a*) (*finite_subsets_at_top A*)›
and *lim_g*: ‹(*sum g* ⟶ *b*) (*finite_subsets_at_top A*)›
by (*simp_all add*: *has_sum_def*)
then have *lim*: ‹(*sum* (λ*x*. *f x* + *g x*) ⟶ *a* + *b*) (*finite_subsets_at_top A*)›
unfolding *sum.distrib* **by** (*rule tendsto_add*)
then show *?thesis*
by (*simp_all add*: *has_sum_def*)
qed

lemma *summable_on_add*:
fixes *f g* :: *'a* ⇒ *'b*::{*topological_comm_monoid_add*}
assumes ‹*f summable_on A›*
assumes ‹*g summable_on A›*
shows ‹(λ*x*. *f x* + *g x*) *summable_on A›*
by (*metis* (*full_types*) *assms summable_on_def has_sum_add*)

lemma *infsum_add*:
fixes *f g* :: *'a* ⇒ *'b*::{*topological_comm_monoid_add*, *t2_space*}
assumes ‹*f summable_on A›*
assumes ‹*g summable_on A›*
shows ‹*infsum* (λ*x*. *f x* + *g x*) *A* = *infsum f A* + *infsum g A›*
proof −
have ‹((λ*x*. *f x* + *g x*) *has_sum* (*infsum f A* + *infsum g A*)) *A›*
by (*simp add*: *assms has_sum_add*)
then show *?thesis*
using *infsumI* **by** *blast*
qed

lemma *has_sum_Un_disjoint*:
fixes *f* :: *'a* ⇒ *'b*::*topological_comm_monoid_add*
assumes (*f has_sum a*) *A*
assumes (*f has_sum b*) *B*
assumes *disj*: *A* ∩ *B* = {}
shows ‹(*f has_sum* (*a* + *b*)) (*A* ∪ *B*)›
proof −
define *fA fB* **where** ‹*fA x* = (*if x* ∈ *A then f x else* 0)›
and ‹*fB x* = (*if x* ∉ *A then f x else* 0)› **for** *x*

```
  have fA: ‹(fA has_sum a) (A ∪ B)›
   by (smt (verit, ccfv_SIG) DiffD1 DiffD2 UnCI fA_def assms(1) has_sum_cong_neutral
inf_sup_absorb)
  have fB: ‹(fB has_sum b) (A ∪ B)›
     by (smt (verit, best) DiffD1 DiffD2 IntE Un_iff fB_def assms(2) disj dis-
joint_iff has_sum_cong_neutral)
  have fAB: ‹f x = fA x + fB x› for x
    unfolding fA_def fB_def by simp
  show ?thesis
    unfolding fAB
    using fA fB by (rule has_sum_add)
qed

lemma summable_on_Un_disjoint:
  fixes f :: 'a ⇒ 'b::topological_comm_monoid_add
  assumes f summable_on A
  assumes f summable_on B
  assumes disj: A ∩ B = {}
  shows ‹f summable_on (A ∪ B)›
  by (meson assms disj summable_on_def has_sum_Un_disjoint)

lemma infsum_Un_disjoint:
  fixes f :: 'a ⇒ 'b::{topological_comm_monoid_add, t2_space}
  assumes f summable_on A
  assumes f summable_on B
  assumes disj: A ∩ B = {}
  shows ‹infsum f (A ∪ B) = infsum f A + infsum f B›
  by (intro infsumI has_sum_Un_disjoint has_sum_infsum assms)

lemma norm_summable_imp_has_sum:
  fixes f :: nat ⇒ 'a :: banach
  assumes summable (λn. norm (f n)) and f sums S
  shows   (f has_sum S) (UNIV :: nat set)
  unfolding has_sum_def tendsto_iff eventually_finite_subsets_at_top
proof clarsimp
  fix ε::real
  assume ε > 0
  from assms obtain S' where S': (λn. norm (f n)) sums S'
    by (auto simp: summable_def)
  with ‹ε > 0› obtain N where N: ⋀n. n ≥ N ⟹ |S' - (∑ i<n. norm (f i))|
< ε
     by (auto simp: tendsto_iff eventually_at_top_linorder sums_def dist_norm
abs_minus_commute)
  have dist (sum f Y) S < ε if finite Y {..<N} ⊆ Y for Y
  proof -
    from that have (λn. if n ∈ Y then 0 else f n) sums (S - sum f Y)
      by (intro sums_If_finite_set'[OF ‹f sums S›]) (auto simp: sum_negf)
    hence S - sum f Y = (∑ n. if n ∈ Y then 0 else f n)
      by (simp add: sums_iff)
```

also have *norm* ... $\leq$ ($\sum$ *n*. *norm* (*if n* $\in$ *Y then 0 else f n*))
by (*rule summable_norm*[*OF summable_comparison_test'*[*OF assms*(*1*)]])
auto
also have ... $\leq$ ($\sum$ *n*. *if n* < *N then 0 else norm* (*f n*))
using *that* **by** (*intro suminf_le summable_comparison_test'*[*OF assms*(*1*)])
auto
also have (*λn*. *if n* $\in$ {..<*N*} *then 0 else norm* (*f n*)) *sums* (*S'* − ($\sum$ *i*<*N*.
norm (*f i*)))
by (*intro sums_If_finite_set'*[*OF S'*]) (*auto simp*: *sum_negf*)
hence ($\sum$ *n*. *if n* < *N then 0 else norm* (*f n*)) = *S'* − ($\sum$ *i*<*N*. *norm* (*f i*))
by (*simp add*: *sums_iff*)
also have *S'* − ($\sum$ *i*<*N*. *norm* (*f i*)) $\leq$ |*S'* − ($\sum$ *i*<*N*. *norm* (*f i*))| **by** *simp*
also have ... < *ε* **by** (*rule N*) *auto*
finally show *?thesis* **by** (*simp add*: *dist_norm norm_minus_commute*)
qed
then show ∃ *X*. *finite X* ∧ (∀ *Y*. *finite Y* ∧ *X* ⊆ *Y* ⟶ *dist* (*sum f Y*) *S* < *ε*)
by (*meson finite_lessThan subset_UNIV*)
qed

lemma *norm_summable_imp_summable_on*:
fixes *f* :: *nat* ⇒ *'a* :: *banach*
assumes *summable* (*λn*. *norm* (*f n*))
shows *f summable_on UNIV*
using *norm_summable_imp_has_sum*[*OF assms*, *of suminf f*] *assms*
by (*auto simp*: *sums_iff summable_on_def dest*: *summable_norm_cancel*)

The following lemma indeed needs a complete space (as formalized by the premise *complete UNIV*). The following two counterexamples show this:

- Consider the real vector space V of sequences with finite support, and with the ℓ_2-norm (sum of squares). Let e_i denote the sequence with a 1 at position i. Let $f : \mathbb{Z} \to V$ be defined as $f(n) := e_{|n|}/n$ (with $f(0) := 0$). We have that $\sum_{n\in\mathbb{Z}} f(n) = 0$ (it even converges absolutely). But $\sum_{n\in\mathbb{N}} f(n)$ does not exist (it would converge against a sequence with infinite support).

- Let f be a positive rational valued function such that $\sum_{x\in B} f(x)$ is $\sqrt{2}$ and $\sum_{x\in A} f(x)$ is 1 (over the reals, with $A \subseteq B$). Then $\sum_{x\in B} f(x)$ does not exist over the rationals. But $\sum_{x\in A} f(x)$ exists.

The lemma also requires uniform continuity of the addition. And example of a topological group with continuous but not uniformly continuous addition would be the positive reals with the usual multiplication as the addition. We do not know whether the lemma would also hold for such topological groups.

lemma *summable_on_subset_aux*:
fixes *A B* **and** *f* :: ‹*'a* ⇒ *'b*::{*ab_group_add*, *uniform_space*}›
assumes ‹*complete* (*UNIV* :: *'b set*)›

assumes *plus_cont*: ‹*uniformly_continuous_on UNIV* (λ(*x*::′*b*,*y*). *x*+*y*)›
assumes ‹*f summable_on A*›
assumes ‹*B* ⊆ *A*›
shows ‹*f summable_on B*›
proof −
let *?filter_fB* = ‹*filtermap* (*sum f*) (*finite_subsets_at_top B*)›
from ‹*f summable_on A*›
obtain *S* **where** ‹(*sum f* ⟶ *S*) (*finite_subsets_at_top A*)› (**is** ‹(*sum f* ⟶ *S*) *?filter_A*›)
using *summable_on_def has_sum_def* **by** *blast*
then have *cauchy_fA*: ‹*cauchy_filter* (*filtermap* (*sum f*) (*finite_subsets_at_top A*))› (**is** ‹*cauchy_filter ?filter_fA*›)
by (*auto intro*!: *nhds_imp_cauchy_filter simp*: *filterlim_def*)

have ‹*cauchy_filter* (*filtermap* (*sum f*) (*finite_subsets_at_top B*))›
proof (*unfold cauchy_filter_def*, *rule filter_leI*)
fix *E* :: ‹(′*b*×′*b*) ⇒ *bool*› **assume** ‹*eventually E uniformity*›
then obtain *E*′ **where** ‹*eventually E*′ *uniformity*› **and** *E*′*E*′*E*: ‹*E*′ (*x*, *y*) ⟶ *E*′ (*y*, *z*) ⟶ *E* (*x*, *z*)› **for** *x y z*
using *uniformity_trans* **by** *blast*
obtain *D* **where** ‹*eventually D uniformity*› **and** *DE*: ‹*D* (*x*, *y*) ⟹ *E*′ (*x*+*c*, *y*+*c*)› **for** *x y c*
using *plus_cont* ‹*eventually E*′ *uniformity*›
unfolding *uniformly_continuous_on_uniformity filterlim_def le_filter_def uniformity_prod_def*
by (*auto simp*: *case_prod_beta eventually_filtermap eventually_prod_same uniformity_refl*)
have *DE*′: *E*′ (*x*, *y*) **if** *D* (*x* + *c*, *y* + *c*) **for** *x y c*
using *DE*[*of x* + *c y* + *c* −*c*] *that* **by** *simp*

from ‹*eventually D uniformity*› **and** *cauchy_fA* **have** ‹*eventually D* (*?filter_fA* ×$_F$ *?filter_fA*)›
unfolding *cauchy_filter_def le_filter_def* **by** *simp*
then obtain *P1 P2*
where *ev_P1*: ‹*eventually* (λ*F*. *P1* (*sum f F*)) *?filter_A*›
and *ev_P2*: ‹*eventually* (λ*F*. *P2* (*sum f F*)) *?filter_A*›
and *P1P2E*: ‹*P1 x* ⟹ *P2 y* ⟹ *D* (*x*, *y*)› **for** *x y*
unfolding *eventually_prod_filter eventually_filtermap*
by *auto*
from *ev_P1* **obtain** *F1* **where** *F1*: ‹*finite F1*› ‹*F1* ⊆ *A*› ‹⋀*F*. *F*⊇*F1* ⟹ *finite F* ⟹ *F*⊆*A* ⟹ *P1* (*sum f F*)›
by (*metis eventually_finite_subsets_at_top*)
from *ev_P2* **obtain** *F2* **where** *F2*: ‹*finite F2*› ‹*F2* ⊆ *A*› ‹⋀*F*. *F*⊇*F2* ⟹ *finite F* ⟹ *F*⊆*A* ⟹ *P2* (*sum f F*)›
by (*metis eventually_finite_subsets_at_top*)
define *F0 F0A F0B* **where** ‹*F0* ≡ *F1* ∪ *F2*› **and** ‹*F0A* ≡ *F0* − *B*› **and** ‹*F0B* ≡ *F0* ∩ *B*›
have [*simp*]: ‹*finite F0*› ‹*F0* ⊆ *A*›
using ‹*F1* ⊆ *A*› ‹*F2* ⊆ *A*› ‹*finite F1*› ‹*finite F2*› **unfolding** *F0_def* **by**

blast+

have ∗: *E′* (*sum f F1′*, *sum f F2′*)
if *F1′*⊇*F0B F2′*⊇*F0B finite F1′ finite F2′ F1′*⊆*B F2′*⊆*B* **for** *F1′ F2′*
proof (*intro DE′*[**where** *c* = *sum f F0A*] *P1P2E*)
have *P1* (*sum f* (*F1′* ∪ *F0A*))
using *that assms F1*(*1*,*2*) *F2*(*1*,*2*) **by** (*intro F1*) (*auto simp*: *F0A_def F0B_def F0_def*)
thus *P1* (*sum f F1′* + *sum f F0A*)
by (*subst* (*asm*) *sum.union_disjoint*) (*use that* **in** ‹*auto simp*: *F0A_def*›)
next
have *P2* (*sum f* (*F2′* ∪ *F0A*))
using *that assms F1*(*1*,*2*) *F2*(*1*,*2*) **by** (*intro F2*) (*auto simp*: *F0A_def F0B_def F0_def*)
thus *P2* (*sum f F2′* + *sum f F0A*)
by (*subst* (*asm*) *sum.union_disjoint*) (*use that* **in** ‹*auto simp*: *F0A_def*›)

qed

have *eventually* (λ*x*. *E′* (*x*, *sum f F0B*)) (*filtermap* (*sum f*) (*finite_subsets_at_top B*))
and *eventually* (λ*x*. *E′* (*sum f F0B*, *x*)) (*filtermap* (*sum f*) (*finite_subsets_at_top B*))
unfolding *eventually_filtermap eventually_finite_subsets_at_top*
by (*rule exI*[*of* _ *F0B*]; *use* ∗ **in** ‹*force simp*: *F0B_def*›)+
then
show ‹*eventually E* (*?filter_fB* ×$_F$ *?filter_fB*)›
unfolding *eventually_prod_filter*
using *E′E′E* **by** *blast*
qed

then obtain *x* **where** ‹*?filter_fB* ≤ *nhds x*›
using *cauchy_filter_complete_converges*[*of ?filter_fB UNIV*] ‹*complete* (*UNIV* :: _)›
by (*auto simp*: *filtermap_bot_iff*)
then have ‹(*sum f* ⟶ *x*) (*finite_subsets_at_top B*)›
by (*auto simp*: *filterlim_def*)
then show *?thesis*
by (*auto simp*: *summable_on_def has_sum_def*)
qed

A special case of *summable_on_subset_aux* for Banach spaces with fewer premises.

lemma *summable_on_subset_banach*:
fixes *A B* **and** *f* :: ‹*'a* ⇒ *'b*::*banach*›
assumes ‹*f summable_on A*›
assumes ‹*B* ⊆ *A*›
shows ‹*f summable_on B*›
by (*meson Cauchy_convergent UNIV_I assms complete_def convergent_def isU-*

Cont_plus summable_on_subset_aux)

lemma *has_sum_empty*[*simp*]: ‹(*f has_sum 0*) {}›
by (*meson ex_in_conv has_sum_0*)

lemma *summable_on_empty*[*simp*]: ‹*f summable_on* {}›
by *auto*

lemma *infsum_empty*[*simp*]: ‹*infsum f* {} = *0*›
by *simp*

lemma *sum_has_sum*:
fixes *f* :: *'a* ⇒ *'b*::*topological_comm_monoid_add*
assumes ‹*finite A*›
assumes ‹⋀*a*. *a* ∈ *A* ⟹ (*f has_sum* (*s a*)) (*B a*)›
assumes ‹⋀*a a'*. *a*∈*A* ⟹ *a'*∈*A* ⟹ *a*≠*a'* ⟹ *B a* ∩ *B a'* = {}›
shows ‹(*f has_sum* (*sum s A*)) (⋃*a*∈*A*. *B a*)›
using *assms*
proof (*induction*)
case *empty*
then show *?case*
by *simp*
next
case (*insert x A*)
have ‹(*f has_sum* (*s x*)) (*B x*)›
by (*simp add*: *insert.prems*)
moreover have *IH*: ‹(*f has_sum* (*sum s A*)) (⋃*a*∈*A*. *B a*)›
using *insert* **by** *simp*
ultimately have ‹(*f has_sum* (*s x* + *sum s A*)) (*B x* ∪ (⋃*a*∈*A*. *B a*))›
using *insert* **by** (*intro has_sum_Un_disjoint*) *auto*
then show *?case*
using *insert.hyps* **by** *auto*
qed

lemma *summable_on_finite_union_disjoint*:
fixes *f* :: *'a* ⇒ *'b*::*topological_comm_monoid_add*
assumes *finite*: ‹*finite A*›
assumes *conv*: ‹⋀*a*. *a* ∈ *A* ⟹ *f summable_on* (*B a*)›
assumes *disj*: ‹⋀*a a'*. *a*∈*A* ⟹ *a'*∈*A* ⟹ *a*≠*a'* ⟹ *B a* ∩ *B a'* = {}›
shows ‹*f summable_on* (⋃*a*∈*A*. *B a*)›
using *sum_has_sum* [*of A f B*] *assms* **unfolding** *summable_on_def* **by** *metis*

lemma *sum_infsum*:
fixes *f* :: *'a* ⇒ *'b*::{*topological_comm_monoid_add*, *t2_space*}
assumes *finite*: ‹*finite A*›
assumes *conv*: ‹⋀*a*. *a* ∈ *A* ⟹ *f summable_on* (*B a*)›
assumes *disj*: ‹⋀*a a'*. *a*∈*A* ⟹ *a'*∈*A* ⟹ *a*≠*a'* ⟹ *B a* ∩ *B a'* = {}›
shows ‹*sum* (λ*a*. *infsum f* (*B a*)) *A* = *infsum f* (⋃*a*∈*A*. *B a*)›

by (*metis* (*no_types*, *lifting*) *assms has_sum_infsum infsumI sum_has_sum*)

The lemmas *infsum_comm_additive_general* and *infsum_comm_additive* (and variants) below both state that the infinite sum commutes with a continuous additive function. *infsum_comm_additive_general* is stated more for more general type classes at the expense of a somewhat less compact formulation of the premises. E.g., by avoiding the constant *additive* which introduces an additional sort constraint (group instead of monoid). For example, extended reals (*ereal*, *ennreal*) are not covered by *infsum_comm_additive*.

lemma *has_sum_comm_additive_general*:
fixes $f :: \langle 'b :: \{comm\_monoid\_add, topological\_space\} \Rightarrow 'c :: \{comm\_monoid\_add, topological\_space\}\rangle$
assumes *f_sum*: $\langle \bigwedge F.\ finite\ F \Longrightarrow F \subseteq S \Longrightarrow sum\ (f \circ g)\ F = f\ (sum\ g\ F)\rangle$
— Not using *additive* because it would add sort constraint *ab_group_add*
assumes *cont*: $\langle f \ -x\rightarrow\ f\ x\rangle$
— For *t2_space*, this is equivalent to *isCont f x* by *isCont_def*.
assumes *infsum*: $\langle (g\ has\_sum\ x)\ S\rangle$
shows $\langle ((f \circ g)\ has\_sum\ (f\ x))\ S\rangle$
proof −
have $\langle (sum\ g \longrightarrow x)\ (finite\_subsets\_at\_top\ S)\rangle$
using *infsum has_sum_def* **by** *blast*
then have $\langle ((f \circ sum\ g) \longrightarrow f\ x)\ (finite\_subsets\_at\_top\ S)\rangle$
by (*meson cont filterlim_def tendsto_at_iff_tendsto_nhds tendsto_compose_filtermap tendsto_mono*)
then have $\langle (sum\ (f \circ g) \longrightarrow f\ x)\ (finite\_subsets\_at\_top\ S)\rangle$
using *tendsto_cong f_sum*
by (*simp add*: *Lim_transform_eventually eventually_finite_subsets_at_top_weakI*)
then show $\langle ((f \circ g)\ has\_sum\ (f\ x))\ S\rangle$
using *has_sum_def* **by** *blast*
qed

lemma *summable_on_comm_additive_general*:
fixes $f :: \langle 'b :: \{comm\_monoid\_add, topological\_space\} \Rightarrow 'c :: \{comm\_monoid\_add, topological\_space\}\rangle$
assumes $\langle \bigwedge F.\ finite\ F \Longrightarrow F \subseteq S \Longrightarrow sum\ (f \circ g)\ F = f\ (sum\ g\ F)\rangle$
— Not using *additive* because it would add sort constraint *ab_group_add*
assumes $\langle \bigwedge x.\ (g\ has\_sum\ x)\ S \Longrightarrow f\ -x\rightarrow\ f\ x\rangle$
— For *t2_space*, this is equivalent to *isCont f x* by *isCont_def*.
assumes $\langle g\ summable\_on\ S\rangle$
shows $\langle (f \circ g)\ summable\_on\ S\rangle$
by (*meson assms summable_on_def has_sum_comm_additive_general has_sum_def infsum_tendsto*)

lemma *infsum_comm_additive_general*:
fixes $f :: \langle 'b :: \{comm\_monoid\_add, t2\_space\} \Rightarrow 'c :: \{comm\_monoid\_add, t2\_space\}\rangle$
assumes *f_sum*: $\langle \bigwedge F.\ finite\ F \Longrightarrow F \subseteq S \Longrightarrow sum\ (f \circ g)\ F = f\ (sum\ g\ F)\rangle$
— Not using *additive* because it would add sort constraint *ab_group_add*
assumes $\langle isCont\ f\ (infsum\ g\ S)\rangle$
assumes $\langle g\ summable\_on\ S\rangle$
shows $\langle infsum\ (f \circ g)\ S = f\ (infsum\ g\ S)\rangle$

using *assms*
by (*intro infsumI has_sum_comm_additive_general has_sum_infsum*) (*auto simp*: *isCont_def*)

lemma *has_sum_comm_additive*:
fixes $f :: \langle 'b :: \{ab\_group\_add, topological\_space\} \Rightarrow 'c :: \{ab\_group\_add, topological\_space\}\rangle$
assumes ‹*additive f*›
assumes ‹$f \ -x\rightarrow f\ x$›
— For *t2_space*, this is equivalent to *isCont f x* by *isCont_def*.
assumes *infsum*: ‹(*g has_sum x*) *S*›
shows ‹((*f* ∘ *g*) *has_sum* (*f x*)) *S*›
using *assms*
by (*intro has_sum_comm_additive_general has_sum_infsum*) (*auto simp*: *isCont_def additive.sum*)

lemma *summable_on_comm_additive*:
fixes $f :: \langle 'b :: \{ab\_group\_add, t2\_space\} \Rightarrow 'c :: \{ab\_group\_add, topological\_space\}\rangle$
assumes ‹*additive f*›
assumes ‹*isCont f* (*infsum g S*)›
assumes ‹*g summable_on S*›
shows ‹(*f* ∘ *g*) *summable_on S*›
by (*meson assms summable_on_def has_sum_comm_additive has_sum_infsum isContD*)

lemma *infsum_comm_additive*:
fixes $f :: \langle 'b :: \{ab\_group\_add, t2\_space\} \Rightarrow 'c :: \{ab\_group\_add, t2\_space\}\rangle$
assumes ‹*additive f*›
assumes ‹*isCont f* (*infsum g S*)›
assumes ‹*g summable_on S*›
shows ‹*infsum* (*f* ∘ *g*) *S* = *f* (*infsum g S*)›
by (*rule infsum_comm_additive_general*; *auto simp*: *assms additive.sum*)

lemma *nonneg_bdd_above_has_sum*:
fixes $f :: \langle 'a \Rightarrow 'b :: \{conditionally\_complete\_linorder, ordered\_comm\_monoid\_add, linorder\_topology\}\rangle$
assumes ‹$\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$›
assumes ‹*bdd_above* (*sum f* ‘ {*F*. *F*⊆*A* ∧ *finite F*})›
shows ‹(*f has_sum* (*SUP F*∈{*F*. *finite F* ∧ *F*⊆*A*}. *sum f F*)) *A*›
proof −
have ‹(*sum f* ⟶ (*SUP F*∈{*F*. *finite F* ∧ *F*⊆*A*}. *sum f F*)) (*finite_subsets_at_top A*)›
proof (*rule order_tendstoI*)
fix *a* **assume** ‹*a* < (*SUP F*∈{*F*. *finite F* ∧ *F*⊆*A*}. *sum f F*)›
then obtain *F* **where** ‹*a* < *sum f F*› **and** ‹*finite F*› **and** ‹*F* ⊆ *A*›
by (*metis* (*mono_tags*, *lifting*) *Collect_cong Collect_empty_eq assms*(2) *empty_subsetI finite.emptyI less_cSUP_iff mem_Collect_eq*)
have $\bigwedge Y.\ [\![finite\ Y;\ F \subseteq Y;\ Y \subseteq A]\!] \Longrightarrow a < sum\ f\ Y$
by (*meson DiffE* ‹*a* < *sum f F*› *assms*(1) *less_le_trans subset_iff sum_mono2*)
then show ‹$\forall_F\ x$ *in finite_subsets_at_top A*. *a* < *sum f x*›

by (*metis* ‹$F \subseteq A$› ‹*finite* F› *eventually_finite_subsets_at_top*)
next
fix a **assume** $*$: ‹$(SUP\ F{\in}\{F.\ finite\ F \wedge F{\subseteq}A\}.\ sum\ f\ F) < a$›
have $sum\ f\ F \leq (SUP\ F{\in}\{F.\ finite\ F \wedge F{\subseteq}A\}.\ sum\ f\ F)$ **if** ‹$F{\subseteq}A$› **and** ‹*finite* F› **for** F
by (*rule cSUP_upper*) (*use that assms*(2) **in** ‹*auto simp*: *conj_commute*›)
then show ‹$\forall_F\ x$ *in finite_subsets_at_top* $A.\ sum\ f\ x < a$›
by (*metis* (*no_types*, *lifting*) $*$ *eventually_finite_subsets_at_top_weakI order_le_less_trans*)
qed
then show *?thesis*
using *has_sum_def* **by** *blast*
qed

lemma *nonneg_bdd_above_summable_on*:
fixes f :: ‹$'a \Rightarrow 'b$ :: {*conditionally_complete_linorder*, *ordered_comm_monoid_add*, *linorder_topology*}›
assumes ‹$\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$›
assumes ‹*bdd_above* (*sum* f ‘ $\{F.\ F{\subseteq}A \wedge finite\ F\}$)›
shows ‹f *summable_on* A›
using *assms summable_on_def nonneg_bdd_above_has_sum* **by** *blast*

lemma *nonneg_bdd_above_infsum*:
fixes f :: ‹$'a \Rightarrow 'b$ :: {*conditionally_complete_linorder*, *ordered_comm_monoid_add*, *linorder_topology*}›
assumes ‹$\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$›
assumes ‹*bdd_above* (*sum* f ‘ $\{F.\ F{\subseteq}A \wedge finite\ F\}$)›
shows ‹$infsum\ f\ A = (SUP\ F{\in}\{F.\ finite\ F \wedge F{\subseteq}A\}.\ sum\ f\ F)$›
using *assms* **by** (*auto intro*!: *infsumI nonneg_bdd_above_has_sum*)

lemma *nonneg_has_sum_complete*:
fixes f :: ‹$'a \Rightarrow 'b$ :: {*complete_linorder*, *ordered_comm_monoid_add*, *linorder_topology*}›
assumes ‹$\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$›
shows ‹$(f\ has\_sum\ (SUP\ F{\in}\{F.\ finite\ F \wedge F{\subseteq}A\}.\ sum\ f\ F))\ A$›
using *assms nonneg_bdd_above_has_sum* **by** *blast*

lemma *nonneg_summable_on_complete*:
fixes f :: ‹$'a \Rightarrow 'b$ :: {*complete_linorder*, *ordered_comm_monoid_add*, *linorder_topology*}›
assumes ‹$\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$›
shows ‹f *summable_on* A›
using *assms nonneg_bdd_above_summable_on* **by** *blast*

lemma *nonneg_infsum_complete*:
fixes f :: ‹$'a \Rightarrow 'b$ :: {*complete_linorder*, *ordered_comm_monoid_add*, *linorder_topology*}›
assumes ‹$\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$›
shows ‹$infsum\ f\ A = (SUP\ F{\in}\{F.\ finite\ F \wedge F{\subseteq}A\}.\ sum\ f\ F)$›
using *assms nonneg_bdd_above_infsum* **by** *blast*

lemma *has_sum_nonneg*:

fixes $f :: 'a \Rightarrow 'b::\{ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes $(f\ has\_sum\ a)\ M$
and $\bigwedge x.\ x \in M \Longrightarrow 0 \leq f\ x$
shows $a \geq 0$
by (*metis* (*no_types*, *lifting*) *DiffD1 assms empty_iff has_sum_0 has_sum_mono_neutral order_refl*)

lemma *infsum_nonneg*:
fixes $f :: 'a \Rightarrow 'b::\{ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes $\bigwedge x.\ x \in M \Longrightarrow 0 \leq f\ x$
shows $infsum\ f\ M \geq 0$ (**is** $?lhs \geq \_$)
by (*metis assms has_sum_infsum has_sum_nonneg infsum_not_exists linorder_linear*)

lemma *has_sum_mono2*:
fixes $f :: 'a \Rightarrow 'b::\{topological\_ab\_group\_add, ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes $(f\ has\_sum\ S)\ A\ (f\ has\_sum\ S')\ B\ A \subseteq B$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow f\ x \geq 0$
shows $S \leq S'$
by (*metis add_0 add_right_mono assms diff_add_cancel has_sum_Diff has_sum_nonneg*)

lemma *infsum_mono2*:
fixes $f :: 'a \Rightarrow 'b::\{topological\_ab\_group\_add, ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes $f\ summable\_on\ A\ f\ summable\_on\ B\ A \subseteq B$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow f\ x \geq 0$
shows $infsum\ f\ A \leq infsum\ f\ B$
by (*rule has_sum_mono2*[*OF has_sum_infsum has_sum_infsum*]) (*use assms*
in *auto*)

lemma *finite_sum_le_has_sum*:
fixes $f :: 'a \Rightarrow 'b::\{topological\_ab\_group\_add, ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes $(f\ has\_sum\ S)\ A\ finite\ B\ B \subseteq A$
assumes $\bigwedge x.\ x \in A - B \Longrightarrow f\ x \geq 0$
shows $sum\ f\ B \leq S$
by (*meson assms has_sum_finite has_sum_mono2*)

lemma *finite_sum_le_infsum*:
fixes $f :: 'a \Rightarrow 'b::\{topological\_ab\_group\_add, ordered\_comm\_monoid\_add,linorder\_topology\}$
assumes $f\ summable\_on\ A\ finite\ B\ B \subseteq A$
assumes $\bigwedge x.\ x \in A - B \Longrightarrow f\ x \geq 0$
shows $sum\ f\ B \leq infsum\ f\ A$
by (*rule finite_sum_le_has_sum*[*OF has_sum_infsum*]) (*use assms* **in** *auto*)

lemma *has_sum_reindex*:
assumes $\langle inj\_on\ h\ A\rangle$
shows $\langle(g\ has\_sum\ x)\ (h\ `\ A) \longleftrightarrow ((g \circ h)\ has\_sum\ x)\ A\rangle$
proof −
have $\langle(g\ has\_sum\ x)\ (h\ `\ A) \longleftrightarrow (sum\ g \longrightarrow x)\ (finite\_subsets\_at\_top\ (h\ `\ A))\rangle$
by (*simp add*: *has_sum_def*)

```
  also have ‹... ⟷ ((λF. sum g (h ‘ F)) ⟶ x) (finite_subsets_at_top A)›
    by (metis assms filterlim_filtermap filtermap_image_finite_subsets_at_top)
  also have ‹... ⟷ (sum (g ∘ h) ⟶ x) (finite_subsets_at_top A)›
 proof (intro tendsto_cong eventually_finite_subsets_at_top_weakI sum.reindex)
    show ⋀X. ⟦finite X; X ⊆ A⟧ ⟹ inj_on h X
      using assms subset_inj_on by blast
  qed
  also have ‹... ⟷ ((g ∘ h) has_sum x) A›
    by (simp add: has_sum_def)
  finally show ?thesis .
qed

lemma summable_on_reindex:
  assumes ‹inj_on h A›
  shows ‹g summable_on (h ‘ A) ⟷ (g ∘ h) summable_on A›
  by (simp add: assms summable_on_def has_sum_reindex)

lemma infsum_reindex:
  assumes ‹inj_on h A›
  shows ‹infsum g (h ‘ A) = infsum (g ∘ h) A›
  by (metis assms has_sum_infsum has_sum_reindex infsumI infsum_def)

lemma summable_on_reindex_bij_betw:
  assumes bij_betw g A B
  shows   (λx. f (g x)) summable_on A ⟷ f summable_on B
 by (smt (verit) assms bij_betw_def o_apply summable_on_cong summable_on_reindex)


lemma infsum_reindex_bij_betw:
  assumes bij_betw g A B
  shows   infsum (λx. f (g x)) A = infsum f B
  by (metis (mono_tags, lifting) assms bij_betw_def infsum_cong infsum_reindex
o_def)

lemma sum_uniformity:
 assumes plus_cont: ‹uniformly_continuous_on UNIV (λ(x::'b::{uniform_space,comm_monoid_add},y).
x+y)›
  assumes EE: ‹eventually E uniformity›
  obtains D where ‹eventually D uniformity›
    and ‹⋀M::'a set. ⋀f f' :: 'a ⇒ 'b. card M ≤ n ∧ (∀ m∈M. D (f m, f' m)) ⟹
E (sum f M, sum f' M)›
proof (atomize_elim, insert EE, induction n arbitrary: E rule:nat_induct)
  case 0
  then show ?case
   by (metis card_eq_0_iff equals0D le_zero_eq sum.infinite sum.not_neutral_contains_not_neutral
uniformity_refl)
next
  case (Suc n)
 from plus_cont[unfolded uniformly_continuous_on_uniformity filterlim_def le_filter_def,
```

rule_format, *OF Suc.prems*]
obtain *D1 D2* **where** ‹*eventually D1 uniformity*› **and** ‹*eventually D2 uniformity*›

and *D1D2E*: ‹*D1* (*x*, *y*) ⟹ *D2* (*x'*, *y'*) ⟹ *E* (*x* + *x'*, *y* + *y'*)› **for** *x y x' y'*
apply *atomize_elim*
by (*auto simp*: *eventually_prod_filter case_prod_beta uniformity_prod_def eventually_filtermap*)

from *Suc.IH*[*OF* ‹*eventually D2 uniformity*›]
obtain *D3* **where** ‹*eventually D3 uniformity*› **and** *D3*: ‹*card M* ≤ *n* ⟹ (∀ *m*∈*M*. *D3* (*f m*, *f' m*)) ⟹ *D2* (*sum f M*, *sum f' M*)›
for *M* :: ‹'*a set*› **and** *f f'*
by *metis*

define *D* **where** ‹*D x* ≡ *D1 x* ∧ *D3 x*› **for** *x*
have ‹*eventually D uniformity*›
using *D_def* ‹*eventually D1 uniformity*› ‹*eventually D3 uniformity*› *eventually_elim2* **by** *blast*

have ‹*E* (*sum f M*, *sum f' M*)›
if ‹*card M* ≤ *Suc n*› **and** *DM*: ‹∀ *m*∈*M*. *D* (*f m*, *f' m*)›
for *M* :: ‹'*a set*› **and** *f f'*
proof (*cases* ‹*card M* = *0*›)
case *True*
then show *?thesis*
by (*metis Suc.prems card_eq_0_iff sum.empty sum.infinite uniformity_refl*)
next
case *False*
with ‹*card M* ≤ *Suc n*› **obtain** *N x* **where** ‹*card N* ≤ *n*› **and** ‹*x* ∉ *N*› **and** ‹*M* = *insert x N*›
by (*metis card_Suc_eq less_Suc_eq_0_disj less_Suc_eq_le*)

from *DM* **have** ‹⋀*m*. *m*∈*N* ⟹ *D* (*f m*, *f' m*)›
using ‹*M* = *insert x N*› **by** *blast*
with *D3*[*OF* ‹*card N* ≤ *n*›]
have *D2_N*: ‹*D2* (*sum f N*, *sum f' N*)›
using *D_def* **by** *blast*

from *DM*
have ‹*D* (*f x*, *f' x*)›
using ‹*M* = *insert x N*› **by** *blast*
then have ‹*D1* (*f x*, *f' x*)›
by (*simp add*: *D_def*)

with *D2_N*
have ‹*E* (*f x* + *sum f N*, *f' x* + *sum f' N*)›
using *D1D2E* **by** *presburger*

then show ‹*E* (*sum f M*, *sum f' M*)›

```
      by (metis False ‹M = insert x N› ‹x ∉ N› card.infinite finite_insert
sum.insert)
  qed
  with ‹eventually D uniformity› show ?case
    by auto
qed

lemma has_sum_Sigma:
  fixes A :: 'a set and B :: 'a ⇒ 'b set
    and f :: ‹'a × 'b ⇒ 'c::{comm_monoid_add,uniform_space}›
  assumes plus_cont: ‹uniformly_continuous_on UNIV (λ(x::'c,y). x+y)›
  assumes summableAB: (f has_sum a) (Sigma A B)
  assumes summableB: ‹⋀x. x∈A ⟹ ((λy. f (x, y)) has_sum b x) (B x)›
  shows (b has_sum a) A
proof −
  define F FB FA where ‹F = finite_subsets_at_top (Sigma A B)› and ‹FB x
= finite_subsets_at_top (B x)›
    and ‹FA = finite_subsets_at_top A› for x

  from summableB
  have sum_b: ‹(sum (λy. f (x, y)) ⟶⟶ b x) (FB x)› if ‹x ∈ A› for x
    using FB_def[abs_def] has_sum_def that by auto
  from summableAB
  have sum_S: ‹(sum f ⟶⟶ a) F›
    using F_def has_sum_def by blast

  have finite_proj: ‹finite {b| b. (a,b) ∈ H}› if ‹finite H› for H :: ‹('a×'b) set›
and a
   by (metis (no_types, lifting) finite_imageI finite_subset image_eqI mem_Collect_eq
snd_conv subsetI that)

  have ‹(sum b ⟶⟶ a) FA›
  proof (rule tendsto_iff_uniformity[THEN iffD2, rule_format])
    fix E :: ‹('c × 'c) ⇒ bool›
    assume ‹eventually E uniformity›
    then obtain D where D_uni: ‹eventually D uniformity› and DDE': ‹⋀x y z.
D (x, y) ⟹ D (y, z) ⟹ E (x, z)›
      by (metis (no_types, lifting) ‹eventually E uniformity› uniformity_transE)
    from sum_S obtain G where ‹finite G› and ‹G ⊆ Sigma A B›
      and G_sum: ‹G ⊆ H ⟹ H ⊆ Sigma A B ⟹ finite H ⟹ D (sum f H,
a)› for H
      unfolding tendsto_iff_uniformity
    by (metis (mono_tags, lifting) D_uni F_def eventually_finite_subsets_at_top)
    have ‹finite (fst ‘ G)› and ‹fst ‘ G ⊆ A›
      using ‹finite G› ‹G ⊆ Sigma A B› by auto
    thm uniformity_prod_def
    define Ga where ‹Ga a = {b. (a,b) ∈ G}› for a
    have Ga_fin: ‹finite (Ga a)› and Ga_B: ‹Ga a ⊆ B a› for a
        using ‹finite G› ‹G ⊆ Sigma A B› finite_proj by (auto simp: Ga_def
```

finite_proj)

have ‹*E* (*sum b M*, *a*)› **if** ‹*M* ⊇ *fst* ‘ *G*› **and** ‹*finite M*› **and** ‹*M* ⊆ *A*› **for** *M*
proof −
define *FMB* **where** ‹*FMB* = *finite_subsets_at_top* (*Sigma M B*)›
have ‹*eventually* (λ*H*. *D* (∑ *a*∈*M*. *b a*, ∑ (*a*,*b*)∈*H*. *f* (*a*,*b*))) *FMB*›
proof −
obtain *D*′ **where** *D*′*_uni*: ‹*eventually D*′ *uniformity*›
and ‹*card M*′ ≤ *card M* ∧ (∀ *m*∈*M*′. *D*′ (*g m*, *g*′ *m*)) ⟹ *D* (*sum g M*′, *sum g*′ *M*′)›
for *M*′ :: ‹′*a set*› **and** *g g*′
using *sum_uniformity*[*OF plus_cont* ‹*eventually D uniformity*›] **by** *blast*
then have *D*′*_sum_D*: ‹(∀ *m*∈*M*. *D*′ (*g m*, *g*′ *m*)) ⟹ *D* (*sum g M*, *sum g*′ *M*)› **for** *g g*′
by *auto*

obtain *Ha* **where** ‹*Ha a* ⊇ *Ga a*› **and** *Ha_fin*: ‹*finite* (*Ha a*)› **and** *Ha_B*: ‹*Ha a* ⊆ *B a*›
and *D*′*_sum_Ha*: ‹*Ha a* ⊆ *L* ⟹ *L* ⊆ *B a* ⟹ *finite L* ⟹ *D*′ (*b a*, *sum* (λ*b*. *f* (*a*,*b*)) *L*)› **if** ‹*a* ∈ *A*› **for** *a L*
proof −
from *sum_b*[*unfolded tendsto_iff_uniformity*, *rule_format*, *OF _ D*′*_uni*[*THEN uniformity_sym*]]
obtain *Ha0* **where** ‹*finite* (*Ha0 a*)› **and** ‹*Ha0 a* ⊆ *B a*›
and ‹*Ha0 a* ⊆ *L* ⟹ *L* ⊆ *B a* ⟹ *finite L* ⟹ *D*′ (*b a*, *sum* (λ*b*. *f* (*a*,*b*)) *L*)› **if** ‹*a* ∈ *A*› **for** *a L*
unfolding *FB_def eventually_finite_subsets_at_top* **unfolding** *prod.case* **by** *metis*
moreover define *Ha* **where** ‹*Ha a* = *Ha0 a* ∪ *Ga a*› **for** *a*
ultimately show *?thesis*
using *that*[**where** *Ha*=*Ha*]
using *Ga_fin Ga_B* **by** *auto*
qed

have ‹*D* (∑ *a*∈*M*. *b a*, ∑ (*a*,*b*)∈*H*. *f* (*a*,*b*))› **if** ‹*finite H*› **and** ‹*H* ⊆ *Sigma M B*› **and** ‹*H* ⊇ *Sigma M Ha*› **for** *H*
proof −
define *Ha*′ **where** ‹*Ha*′ *a* = {*b*| *b*. (*a*,*b*) ∈ *H*}› **for** *a*
have [*simp*]: ‹*finite* (*Ha*′ *a*)› **and** [*simp*]: ‹*Ha*′ *a* ⊇ *Ha a*› **and** [*simp*]: ‹*Ha*′ *a* ⊆ *B a*› **if** ‹*a* ∈ *M*› **for** *a*
unfolding *Ha*′*_def* **using** ‹*finite H*› ‹*H* ⊆ *Sigma M B*› ‹*Sigma M Ha* ⊆ *H*› *that finite_proj* **by** *auto*
have ‹*Sigma M Ha*′ = *H*›
using *that* **by** (*auto simp*: *Ha*′*_def*)
then have ∗: ‹(∑ (*a*,*b*)∈*H*. *f* (*a*,*b*)) = (∑ *a*∈*M*. ∑ *b*∈*Ha*′ *a*. *f* (*a*,*b*))›
by (*simp add*: ‹*finite M*› *sum.Sigma*)
have ‹*D*′ (*b a*, *sum* (λ*b*. *f* (*a*,*b*)) (*Ha*′ *a*))› **if** ‹*a* ∈ *M*› **for** *a*
using *D*′*_sum_Ha* ‹*M* ⊆ *A*› *that* **by** *auto*
then have ‹*D* (∑ *a*∈*M*. *b a*, ∑ *a*∈*M*. *sum* (λ*b*. *f* (*a*,*b*)) (*Ha*′ *a*))›

```
        by (rule_tac D'_sum_D, auto)
      with ∗ show ?thesis
        by auto
    qed
    moreover have ‹Sigma M Ha ⊆ Sigma M B›
      using Ha_B ‹M ⊆ A› by auto
    ultimately show ?thesis
      unfolding FMB_def eventually_finite_subsets_at_top
    by (metis (no_types, lifting) Ha_fin finite_SigmaI subsetD that(2) that(3))
  qed
  moreover have ‹eventually (λH. D (∑(a,b)∈H. f (a,b), a)) FMB›
    unfolding FMB_def eventually_finite_subsets_at_top
  proof (rule exI[of _ G], safe)
    fix Y assume Y: finite Y G ⊆ Y Y ⊆ Sigma M B
    thus D (∑(a,b)∈Y. f (a, b), a)
      using G_sum[of Y] Y using that(3) by fastforce
  qed (use ‹finite G› ‹G ⊆ Sigma A B› that in auto)
  ultimately have ‹∀F x in FMB. E (sum b M, a)›
    by eventually_elim (use DDE' in auto)
  then show ‹E (sum b M, a)›
    using FMB_def by force
 qed
 then show ‹∀F x in FA. E (sum b x, a)›
  using ‹finite (fst ‘ G)› and ‹fst ‘ G ⊆ A›
  by (metis (mono_tags, lifting) FA_def eventually_finite_subsets_at_top)
qed
then show ?thesis
 by (simp add: FA_def has_sum_def)
qed

lemma summable_on_Sigma:
 fixes A :: 'a set and B :: 'a ⇒ 'b set
  and f :: ‹'a ⇒ 'b ⇒ 'c::{comm_monoid_add, t2_space, uniform_space}›
 assumes plus_cont: ‹uniformly_continuous_on UNIV (λ(x::'c,y). x+y)›
 assumes summableAB: (λ(x,y). f x y) summable_on (Sigma A B)
 assumes summableB: ‹⋀x. x∈A ⟹ (f x) summable_on (B x)›
 shows ‹(λx. infsum (f x) (B x)) summable_on A›
proof −
 from summableAB obtain a where a: ‹((λ(x,y). f x y) has_sum a) (Sigma A
B)›
  using has_sum_infsum by blast
 from summableB have b: ‹⋀x. x∈A ⟹ (f x has_sum infsum (f x) (B x)) (B
x)›
  by (auto intro!: has_sum_infsum)
 show ?thesis
  using plus_cont a b
   by (smt (verit) has_sum_Sigma[where f=‹λ(x,y). f x y›] has_sum_cong
old.prod.case summable_on_def)
qed
```

lemma *infsum_Sigma*:
fixes $A :: 'a\ set$ **and** $B :: 'a \Rightarrow 'b\ set$
and $f :: \langle 'a \times 'b \Rightarrow 'c::\{comm\_monoid\_add, t2\_space, uniform\_space\}\rangle$
assumes *plus_cont*: ‹*uniformly_continuous_on UNIV* $(\lambda(x::'c,y).\ x+y)$›
assumes *summableAB*: f *summable_on* (*Sigma A B*)
assumes *summableB*: $\langle\bigwedge x.\ x{\in}A \Longrightarrow (\lambda y.\ f\ (x,\ y))$ *summable_on* $(B\ x)\rangle$
shows *infsum* f (*Sigma A B*) = *infsum* $(\lambda x.$ *infsum* $(\lambda y.\ f\ (x,\ y))\ (B\ x))\ A$
proof −
from *summableAB* **have** *a*: ‹(f *has_sum infsum* f (*Sigma A B*)) (*Sigma A B*)›
using *has_sum_infsum* **by** *blast*
from *summableB* **have** *b*: $\langle\bigwedge x.\ x{\in}A \Longrightarrow ((\lambda y.\ f\ (x,\ y))$ *has_sum infsum* $(\lambda y.\ f\ (x,\ y))\ (B\ x))\ (B\ x)\rangle$
by (*auto intro*!: *has_sum_infsum*)
show *?thesis*
using *plus_cont a b* **by** (*auto intro*: *infsumI*[*symmetric*] *has_sum_Sigma simp*: *summable_on_def*)
qed

lemma *infsum_Sigma'*:
fixes $A :: 'a\ set$ **and** $B :: 'a \Rightarrow 'b\ set$
and $f :: \langle 'a \Rightarrow 'b \Rightarrow 'c::\{comm\_monoid\_add, t2\_space, uniform\_space\}\rangle$
assumes *plus_cont*: ‹*uniformly_continuous_on UNIV* $(\lambda(x::'c,y).\ x+y)$›
assumes *summableAB*: $(\lambda(x,y).\ f\ x\ y)$ *summable_on* (*Sigma A B*)
assumes *summableB*: $\langle\bigwedge x.\ x{\in}A \Longrightarrow (f\ x)$ *summable_on* $(B\ x)\rangle$
shows ‹*infsum* $(\lambda x.$ *infsum* $(f\ x)\ (B\ x))\ A =$ *infsum* $(\lambda(x,y).\ f\ x\ y)$ (*Sigma A B*)›
using *infsum_Sigma*[*of* ‹$\lambda(x,y).\ f\ x\ y$› $A\ B$]
using *assms* **by** *auto*

A special case of *infsum_Sigma* etc. for Banach spaces. It has less premises.

lemma
fixes $A :: 'a\ set$ **and** $B :: 'a \Rightarrow 'b\ set$
and $f :: \langle 'a \Rightarrow 'b \Rightarrow 'c::banach\rangle$
assumes [*simp*]: $(\lambda(x,y).\ f\ x\ y)$ *summable_on* (*Sigma A B*)
shows *infsum_Sigma'_banach*: ‹*infsum* $(\lambda x.$ *infsum* $(f\ x)\ (B\ x))\ A =$ *infsum* $(\lambda(x,y).\ f\ x\ y)$ (*Sigma A B*)› (**is** *?thesis1*)
and *summable_on_Sigma_banach*: ‹$(\lambda x.$ *infsum* $(f\ x)\ (B\ x))$ *summable_on* A› (**is** *?thesis2*)
proof −
have *fsum*: ‹$(f\ x)$ *summable_on* $(B\ x)$› **if** ‹$x \in A$› **for** x
proof −
from *assms*
have ‹$(\lambda(x,y).\ f\ x\ y)$ *summable_on* (*Pair* x ‘ $B\ x$)›
by (*meson image_subset_iff summable_on_subset_banach mem_Sigma_iff that*)
then have ‹$((\lambda(x,y).\ f\ x\ y) \circ$ *Pair* $x)$ *summable_on* $(B\ x)\rangle$
by (*metis summable_on_reindex inj_on_def prod.inject*)
then show *?thesis*

```
    by (auto simp: o_def)
  qed
  show ?thesis1
    using fsum assms infsum_Sigma' isUCont_plus by blast
  show ?thesis2
    using fsum assms isUCont_plus summable_on_Sigma by blast
qed

lemma infsum_Sigma_banach:
  fixes A :: 'a set and B :: 'a ⇒ 'b set
    and f :: ‹'a × 'b ⇒ 'c::banach›
  assumes [simp]: f summable_on (Sigma A B)
  shows ‹infsum (λx. infsum (λy. f (x,y)) (B x)) A = infsum f (Sigma A B)›
  using assms by (simp add: infsum_Sigma'_banach)

lemma infsum_swap:
  fixes A :: 'a set and B :: 'b set
  fixes f :: 'a ⇒ 'b ⇒ 'c::{comm_monoid_add,t2_space,uniform_space}
  assumes plus_cont: ‹uniformly_continuous_on UNIV (λ(x::'c,y). x+y)›
  assumes ‹(λ(x, y). f x y) summable_on (A × B)›
  assumes ‹⋀a. a∈A ⟹ (f a) summable_on B›
  assumes ‹⋀b. b∈B ⟹ (λa. f a b) summable_on A›
  shows ‹infsum (λx. infsum (λy. f x y) B) A = infsum (λy. infsum (λx. f x y)
A) B›
proof −
  have (λ(x, y). f y x) ∘ prod.swap summable_on A × B
    by (simp add: assms(2) summable_on_cong)
  then have fyx: ‹(λ(x, y). f y x) summable_on (B × A)›
   by (metis has_sum_reindex infsum_reindex inj_swap product_swap summable_iff_has_sum_infsum)
  have ‹infsum (λx. infsum (λy. f x y) B) A = infsum (λ(x,y). f x y) (A × B)›
    using assms infsum_Sigma' by blast
  also have ‹... = infsum (λ(x,y). f y x) (B × A)›
    apply (subst product_swap[symmetric])
    apply (subst infsum_reindex)
    using assms by (auto simp: o_def)
  also have ‹... = infsum (λy. infsum (λx. f x y) A) B›
    by (smt (verit) fyx assms(1) assms(4) infsum_Sigma' infsum_cong)
  finally show ?thesis .
qed

lemma infsum_swap_banach:
  fixes A :: 'a set and B :: 'b set
  fixes f :: 'a ⇒ 'b ⇒ 'c::banach
  assumes ‹(λ(x, y). f x y) summable_on (A × B)›
  shows infsum (λx. infsum (λy. f x y) B) A = infsum (λy. infsum (λx. f x y) A)
B
proof −
  have §: ‹(λ(x, y). f y x) summable_on (B × A)›
    by (metis (mono_tags, lifting) assms case_swap inj_swap o_apply prod-
```

uct_swap summable_on_cong summable_on_reindex)
have ‹*infsum* (λ*x*. *infsum* (λ*y*. *f x y*) *B*) *A* = *infsum* (λ(*x*,*y*). *f x y*) (*A* × *B*)›
using *assms infsum_Sigma'_banach* **by** *blast*
also have ‹... = *infsum* (λ(*x*,*y*). *f y x*) (*B* × *A*)›
apply (*subst product_swap*[*symmetric*])
apply (*subst infsum_reindex*)
using *assms* **by** (*auto simp*: *o_def*)
also have ‹... = *infsum* (λ*y*. *infsum* (λ*x*. *f x y*) *A*) *B*›
by (*metis* (*mono_tags*, *lifting*) § *infsum_Sigma'_banach infsum_cong*)
finally show *?thesis* **.**
qed

lemma *nonneg_infsum_le_0D*:
fixes *f* :: *'a* ⇒ *'b*::{*topological_ab_group_add*,*ordered_ab_group_add*,*linorder_topology*}
assumes *infsum f A* ≤ *0*
and *abs_sum*: *f summable_on A*
and *nneg*: ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ *0*
and *x* ∈ *A*
shows *f x* = *0*
proof (*rule ccontr*)
assume ‹*f x* ≠ *0*›
have *ex*: ‹*f summable_on* (*A*−{*x*})›
by (*rule summable_on_cofin_subset*) (*use assms* **in** *auto*)
have *pos*: ‹*infsum f* (*A* − {*x*}) ≥ *0*›
by (*rule infsum_nonneg*) (*use nneg* **in** *auto*)

have [*trans*]: ‹*x* ≥ *y* ⟹ *y* > *z* ⟹ *x* > *z*› **for** *x y z* :: *'b* **by** *auto*

have ‹*infsum f A* = *infsum f* (*A*−{*x*}) + *infsum f* {*x*}›
by (*subst infsum_Un_disjoint*[*symmetric*]) (*use assms ex* **in** ‹*auto simp*: *insert_absorb*›)
also have ‹... ≥ *infsum f* {*x*}› (**is** ‹_ ≥ ...›)
using *pos* **by** (*rule add_increasing*) *simp*
also have ‹... = *f x*› (**is** ‹_ = ...›)
by (*subst infsum_finite*) *auto*
also have ‹... > *0*›
using ‹*f x* ≠ *0*› *assms*(*4*) *nneg* **by** *fastforce*
finally show *False*
using *assms* **by** *auto*
qed

lemma *nonneg_has_sum_le_0D*:
fixes *f* :: *'a* ⇒ *'b*::{*topological_ab_group_add*,*ordered_ab_group_add*,*linorder_topology*}
assumes (*f has_sum a*) *A* ‹*a* ≤ *0*›
and ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ *0*
and *x* ∈ *A*
shows *f x* = *0*
by (*metis assms infsumI nonneg_infsum_le_0D summable_on_def*)

lemma *has_sum_cmult_left*:
fixes $f :: 'a \Rightarrow 'b :: \{topological\_semigroup\_mult, semiring\_0\}$
assumes ‹$(f\ has\_sum\ a)\ A$›
shows $((\lambda x.\ f\ x * c)\ has\_sum\ (a * c))\ A$
using *assms tendsto_mult_right*
by (*force simp add*: *has_sum_def sum_distrib_right*)

lemma *infsum_cmult_left*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, semiring\_0\}$
assumes ‹$c \neq 0 \Longrightarrow f\ summable\_on\ A$›
shows $infsum\ (\lambda x.\ f\ x * c)\ A = infsum\ f\ A * c$
using *assms has_sum_cmult_left infsumI summable_iff_has_sum_infsum* **by** *fastforce*

lemma *summable_on_cmult_left*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, semiring\_0\}$
assumes ‹$f\ summable\_on\ A$›
shows $(\lambda x.\ f\ x * c)\ summable\_on\ A$
using *assms summable_on_def has_sum_cmult_left* **by** *blast*

lemma *has_sum_cmult_right*:
fixes $f :: 'a \Rightarrow 'b :: \{topological\_semigroup\_mult, semiring\_0\}$
assumes ‹$(f\ has\_sum\ a)\ A$›
shows $((\lambda x.\ c * f\ x)\ has\_sum\ (c * a))\ A$
using *assms tendsto_mult_left*
by (*force simp add*: *has_sum_def sum_distrib_left*)

lemma *infsum_cmult_right*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, semiring\_0\}$
assumes ‹$c \neq 0 \Longrightarrow f\ summable\_on\ A$›
shows ‹$infsum\ (\lambda x.\ c * f\ x)\ A = c * infsum\ f\ A$›
using *assms has_sum_cmult_right infsumI summable_iff_has_sum_infsum* **by** *fastforce*

lemma *summable_on_cmult_right*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, semiring\_0\}$
assumes ‹$f\ summable\_on\ A$›
shows $(\lambda x.\ c * f\ x)\ summable\_on\ A$
using *assms summable_on_def has_sum_cmult_right* **by** *blast*

lemma *summable_on_cmult_left'*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, division\_ring\}$
assumes ‹$c \neq 0$›
shows $(\lambda x.\ f\ x * c)\ summable\_on\ A \longleftrightarrow f\ summable\_on\ A$
proof
assume ‹$f\ summable\_on\ A$›
then show ‹$(\lambda x.\ f\ x * c)\ summable\_on\ A$›
by (*rule summable_on_cmult_left*)
next

assume ‹$(\lambda x.\ f\ x * c)$ *summable_on* A›
then have ‹$(\lambda x.\ f\ x * c * inverse\ c)$ *summable_on* A›
by (*rule summable_on_cmult_left*)
then show ‹f *summable_on* A›
by (*smt* (*verit*, *del_insts*) *assms divide_inverse nonzero_divide_eq_eq summable_on_cong*)
qed

lemma *summable_on_cmult_right'*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, division\_ring\}$
assumes ‹$c \neq 0$›
shows $(\lambda x.\ c * f\ x)$ *summable_on* $A \longleftrightarrow f$ *summable_on* A
by (*metis* (*no_types*, *lifting*) *assms left_inverse mult.assoc mult_1 summable_on_cmult_right summable_on_cong*)

lemma *infsum_cmult_left'*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, division\_ring\}$
shows *infsum* $(\lambda x.\ f\ x * c)\ A =$ *infsum* $f\ A * c$
by (*metis* (*full_types*) *infsum_cmult_left infsum_not_exists mult_eq_0_iff summable_on_cmult_left*)

lemma *infsum_cmult_right'*:
fixes $f :: 'a \Rightarrow 'b :: \{t2\_space, topological\_semigroup\_mult, division\_ring\}$
shows *infsum* $(\lambda x.\ c * f\ x)\ A = c *$ *infsum* $f\ A$
by (*metis* (*full_types*) *infsum_cmult_right infsum_not_exists mult_eq_0_iff summable_on_cmult_right'*)

lemma *has_sum_constant*[*simp*]:
assumes ‹*finite* F›
shows ‹$((\lambda\_.\ c)$ *has_sum of_nat* $(card\ F) * c)\ F$›
by (*metis assms has_sum_finite sum_constant*)

lemma *infsum_constant*[*simp*]:
assumes ‹*finite* F›
shows ‹*infsum* $(\lambda\_.\ c)\ F =$ *of_nat* $(card\ F) * c$›
by (*simp add*: *assms*)

lemma *infsum_diverge_constant*:
— This probably does not really need all of *archimedean_field* but Isabelle/HOL has no type class such as, e.g., "archimedean ring".
fixes $c ::$ ‹$'a$::{*archimedean_field*, *comm_monoid_add*, *linorder_topology*, *topological_semigroup_mult*}›
assumes ‹*infinite* A› **and** ‹$c \neq 0$›
shows ‹$\neg\ (\lambda\_.\ c)$ *summable_on* A›
proof (*rule notI*)
assume ‹$(\lambda\_.\ c)$ *summable_on* A›
then have ‹$(\lambda\_.$ *inverse* $c * c)$ *summable_on* A›
by (*rule summable_on_cmult_right*)
then have [*simp*]: ‹$(\lambda\_.\ 1::'a)$ *summable_on* A›
using *assms* **by** *auto*
have ‹*infsum* $(\lambda\_.\ 1)\ A \geq d$› **for** $d :: 'a$

proof −
obtain *n* :: *nat* **where** ‹*of_nat n* ≥ *d*›
by (*meson real_arch_simple*)
from *assms*
obtain *F* **where** ‹*F* ⊆ *A*› **and** ‹*finite F*› **and** ‹*card F* = *n*›
by (*meson infinite_arbitrarily_large*)
note ‹*d* ≤ *of_nat n*›
also have ‹*of_nat n* = *infsum* (λ_. *1*::'*a*) *F*›
by (*simp add*: ‹*card F* = *n*› ‹*finite F*›)
also have ‹... ≤ *infsum* (λ_. *1*::'*a*) *A*›
apply (*rule infsum_mono_neutral*)
using ‹*finite F*› ‹*F* ⊆ *A*› **by** *auto*
finally show *?thesis* **.**
qed
then show *False*
by (*meson linordered_field_no_ub not_less*)
qed

lemma *has_sum_constant_archimedean*[*simp*]:
— This probably does not really need all of *archimedean_field* but Isabelle/HOL has no type class such as, e.g., "archimedean ring".
fixes *c* :: ‹'*a*::{*archimedean_field*, *comm_monoid_add*, *linorder_topology*, *topological_semigroup_mult*}›
shows ‹*infsum* (λ_. *c*) *A* = *of_nat* (*card A*) ∗ *c*›
by (*metis infsum_0 infsum_constant infsum_diverge_constant infsum_not_exists sum.infinite sum_constant*)

lemma *has_sum_uminus*:
fixes *f* :: ‹'*a* ⇒ '*b*::*topological_ab_group_add*›
shows ‹((λ*x*. − *f x*) *has_sum a*) *A* ⟷ (*f has_sum* (− *a*)) *A*›
by (*auto simp add*: *sum_negf*[*abs_def*] *tendsto_minus_cancel_left has_sum_def*)

lemma *summable_on_uminus*:
fixes *f* :: ‹'*a* ⇒ '*b*::*topological_ab_group_add*›
shows‹(λ*x*. − *f x*) *summable_on A* ⟷ *f summable_on A*›
by (*metis summable_on_def has_sum_uminus verit_minus_simplify*(*4*))

lemma *infsum_uminus*:
fixes *f* :: ‹'*a* ⇒ '*b*::{*topological_ab_group_add*, *t2_space*}›
shows ‹*infsum* (λ*x*. − *f x*) *A* = − *infsum f A*›
by (*metis* (*full_types*) *add.inverse_inverse add.inverse_neutral infsumI infsum_def has_sum_infsum has_sum_uminus*)

lemma *has_sum_le_finite_sums*:
fixes *a* :: ‹'*a*::{*comm_monoid_add*,*topological_space*,*linorder_topology*}›
assumes ‹(*f has_sum a*) *A*›
assumes ‹⋀*F*. *finite F* ⟹ *F* ⊆ *A* ⟹ *sum f F* ≤ *b*›
shows ‹*a* ≤ *b*›
by (*metis assms eventually_finite_subsets_at_top_weakI finite_subsets_at_top_neq_bot*

has_sum_def tendsto_upperbound)

lemma *infsum_le_finite_sums*:
fixes $b :: \langle 'a::\{comm\_monoid\_add,topological\_space,linorder\_topology\}\rangle$
assumes $\langle f\ summable\_on\ A\rangle$
assumes $\langle \bigwedge F.\ finite\ F \Longrightarrow F \subseteq A \Longrightarrow sum\ f\ F \leq b\rangle$
shows $\langle infsum\ f\ A \leq b\rangle$
by (*meson assms has_sum_infsum has_sum_le_finite_sums*)

lemma *summable_on_scaleR_left* [*intro*]:
fixes $c :: \langle 'a :: real\_normed\_vector\rangle$
assumes $c \neq 0 \Longrightarrow f\ summable\_on\ A$
shows $(\lambda x.\ f\ x *_R c)\ summable\_on\ A$
proof (*cases* $\langle c = 0\rangle$)
case *False*
then have $(\lambda y.\ y *_R c) \circ f\ summable\_on\ A$
using *assms* **by** (*auto simp add*: *scaleR_left.additive_axioms summable_on_comm_additive*)
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *comp_apply summable_on_cong*)
qed *auto*

lemma *summable_on_scaleR_right* [*intro*]:
fixes $f :: \langle 'a \Rightarrow 'b :: real\_normed\_vector\rangle$
assumes $c \neq 0 \Longrightarrow f\ summable\_on\ A$
shows $(\lambda x.\ c *_R f\ x)\ summable\_on\ A$
proof (*cases* $\langle c = 0\rangle$)
case *False*
then have $(*_R)\ c \circ f\ summable\_on\ A$
using *assms* **by** (*auto simp add*: *scaleR_right.additive_axioms summable_on_comm_additive*)
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *comp_apply summable_on_cong*)
qed *auto*

lemma *infsum_scaleR_left*:
fixes $c :: \langle 'a :: real\_normed\_vector\rangle$
assumes $c \neq 0 \Longrightarrow f\ summable\_on\ A$
shows $infsum\ (\lambda x.\ f\ x *_R c)\ A = infsum\ f\ A *_R c$
proof (*cases* $\langle c = 0\rangle$)
case *False*
then have $infsum\ ((\lambda y.\ y *_R c) \circ f)\ A = infsum\ f\ A *_R c$
using *assms* **by** (*auto simp add*: *scaleR_left.additive_axioms infsum_comm_additive*)
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *comp_apply infsum_cong*)
qed *auto*

lemma *infsum_scaleR_right*:
fixes $f :: \langle 'a \Rightarrow 'b :: real\_normed\_vector\rangle$

```
  shows   infsum (λx. c *R f x) A = c *R infsum f A
proof −
  consider (summable) ‹f summable_on A› | (c0) ‹c = 0› | (not_summable) ‹¬ f
summable_on A› ‹c ≠ 0›
    by auto
  then show ?thesis
  proof cases
    case summable
    then have infsum ((*R) c ∘ f) A = c *R infsum f A
      by (auto simp add: scaleR_right.additive_axioms infsum_comm_additive)
    then show ?thesis
      by (metis (mono_tags, lifting) comp_apply infsum_cong)
  next
    case c0
    then show ?thesis by auto
  next
    case not_summable
    have ‹¬ (λx. c *R f x) summable_on A›
    proof (rule notI)
      assume ‹(λx. c *R f x) summable_on A›
      then have ‹(λx. inverse c *R c *R f x) summable_on A›
        using summable_on_scaleR_right by blast
      with not_summable show False
        by simp
    qed
    then show ?thesis
      by (simp add: infsum_not_exists not_summable(1))
  qed
qed


lemma infsum_Un_Int:
  fixes f :: 'a ⇒ 'b::{topological_ab_group_add, t2_space}
  assumes f summable_on A − B f summable_on B − A ‹f summable_on A ∩
B›
  shows   infsum f (A ∪ B) = infsum f A + infsum f B − infsum f (A ∩ B)
proof −
  obtain ‹f summable_on A› ‹f summable_on B›
   using assms by (metis Int_Diff_Un Int_Diff_disjoint inf_commute summable_on_Un_disjoint)
  then have ‹infsum f (A ∪ B) = infsum f A + infsum f (B − A)›
    using assms(2) infsum_Un_disjoint by fastforce
  moreover have ‹infsum f (B − A) = infsum f B − infsum f (A ∩ B)›
    using assms by (metis Diff_Int2 Un_Int_eq(2) ‹f summable_on B› inf_le2
infsum_Diff)
  ultimately show ?thesis
    by auto
qed

lemma inj_combinator':
```

assumes $x \notin F$
shows ‹*inj_on* (λ(*g*, *y*). *g*(*x* := *y*)) (*Pi*$_E$ *F B* × *B x*)›
proof −
have *inj_on* ((λ(*y*, *g*). *g*(*x* := *y*)) ∘ *prod.swap*) (*Pi*$_E$ *F B* × *B x*)
using *inj_combinator*[*of x F B*] *assms* **by** (*intro comp_inj_on*) (*auto simp*: *product_swap*)
thus *?thesis*
by (*simp add*: *o_def*)
qed

lemma *infsum_prod_PiE*:
— See also *infsum_prod_PiE_abs* below with incomparable premises.
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c* :: {*comm_monoid_mult*, *topological_semigroup_mult*, *division_ring*, *banach*}
assumes *finite*: *finite A*
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x summable_on B x*
assumes (λ*g*. ∏ *x*∈*A*. *f x* (*g x*)) *summable_on* (*PiE A B*)
shows *infsum* (λ*g*. ∏ *x*∈*A*. *f x* (*g x*)) (*PiE A B*) = (∏ *x*∈*A*. *infsum* (*f x*) (*B x*))
proof (*use finite assms*(2−) **in** *induction*)
case *empty*
then show *?case*
by *auto*
next
case (*insert x F*)
have *pi*: ‹*Pi*$_E$ (*insert x F*) *B* = (λ(*g*,*y*). *g*(*x*:=*y*)) ‘ (*Pi*$_E$ *F B* × *B x*)›
unfolding *PiE_insert_eq*
by (*subst swap_product* [*symmetric*]) (*simp add*: *image_image case_prod_unfold*)
have *prod*: ‹(∏ *x'*∈*F*. *f x'* ((*p*(*x*:=*y*)) *x'*)) = (∏ *x'*∈*F*. *f x'* (*p x'*))› **for** *p y*
by (*rule prod.cong*) (*use insert.hyps* **in** *auto*)
have *inj*: ‹*inj_on* (λ(*g*, *y*). *g*(*x* := *y*)) (*Pi*$_E$ *F B* × *B x*)›
using ‹*x* ∉ *F*› **by** (*rule inj_combinator'*)

have *summable1*: ‹(λ*g*. ∏ *x*∈*insert x F*. *f x* (*g x*)) *summable_on Pi*$_E$ (*insert x F*) *B*›
using *insert.prems*(2) **.**
also have ‹*Pi*$_E$ (*insert x F*) *B* = (λ(*g*,*y*). *g*(*x*:=*y*)) ‘ (*Pi*$_E$ *F B* × *B x*)›
by (*simp only*: *pi*)
also have (λ*g*. ∏ *x*∈*insert x F*. *f x* (*g x*)) *summable_on* ... ⟷
((λ*g*. ∏ *x*∈*insert x F*. *f x* (*g x*)) ∘ (λ(*g*,*y*). *g*(*x*:=*y*))) *summable_on* (*Pi*$_E$ *F B* × *B x*)
using *inj* **by** (*rule summable_on_reindex*)
also have (∏ *z*∈*F*. *f z* ((*g*(*x* := *y*)) *z*)) = (∏ *z*∈*F*. *f z* (*g z*)) **for** *g y*
using *insert.hyps* **by** (*intro prod.cong*) *auto*
hence ((λ*g*. ∏ *x*∈*insert x F*. *f x* (*g x*)) ∘ (λ(*g*,*y*). *g*(*x*:=*y*))) =
(λ(*p*, *y*). *f x y* ∗ (∏ *x'*∈*F*. *f x'* (*p x'*)))
using *insert.hyps* **by** (*auto simp*: *fun_eq_iff cong*: *prod.cong_simp*)
finally have *summable2*: ‹(λ(*p*, *y*). *f x y* ∗ (∏ *x'*∈*F*. *f x'* (*p x'*))) *summable_on Pi*$_E$ *F B* × *B x*› **.**

then have ‹$(\lambda p.\ \sum_\infty y{\in}B\ x.\ f\ x\ y * (\prod x'{\in}F.\ f\ x'\ (p\ x'))) \ summable\_on\ Pi_E\ F\ B$›
by (*rule summable_on_Sigma_banach*)
then have ‹$(\lambda p.\ (\sum_\infty y{\in}B\ x.\ f\ x\ y) * (\prod x'{\in}F.\ f\ x'\ (p\ x'))) \ summable\_on\ Pi_E\ F\ B$›
by (*metis* (*mono_tags*, *lifting*) *infsum_cmult_left' infsum_cong summable_on_cong*)
then have *summable3*: ‹$(\lambda p.\ (\prod x'{\in}F.\ f\ x'\ (p\ x'))) \ summable\_on\ Pi_E\ F\ B$› **if** ‹$(\sum_\infty y{\in}B\ x.\ f\ x\ y) \neq 0$›
using *summable_on_cmult_right' that* **by** *blast*

have ‹$(\sum_\infty g{\in}Pi_E\ (insert\ x\ F)\ B.\ \prod x{\in}insert\ x\ F.\ f\ x\ (g\ x))$
$= (\sum_\infty (p,y){\in}Pi_E\ F\ B \times B\ x.\ \prod x'{\in}insert\ x\ F.\ f\ x'\ ((p(x:=y))\ x'))$›
by (*smt* (*verit*, *ccfv_SIG*) *comp_apply infsum_cong infsum_reindex inj pi prod.cong split_def*)
also have ‹$\ldots = (\sum_\infty (p,\ y){\in}Pi_E\ F\ B \times B\ x.\ f\ x\ y * (\prod x'{\in}F.\ f\ x'\ ((p(x:=y))\ x')))$›
using *insert.hyps* **by** *auto*
also have ‹$\ldots = (\sum_\infty (p,\ y){\in}Pi_E\ F\ B \times B\ x.\ f\ x\ y * (\prod x'{\in}F.\ f\ x'\ (p\ x')))$›
using *prod* **by** *presburger*
also have ‹$\ldots = (\sum_\infty p{\in}Pi_E\ F\ B.\ \sum_\infty y{\in}B\ x.\ f\ x\ y * (\prod x'{\in}F.\ f\ x'\ (p\ x')))$›
using *infsum_Sigma'_banach summable2* **by** *force*
also have ‹$\ldots = (\sum_\infty y{\in}B\ x.\ f\ x\ y) * (\sum_\infty p{\in}Pi_E\ F\ B.\ \prod x'{\in}F.\ f\ x'\ (p\ x'))$›
by (*smt* (*verit*) *infsum_cmult_left' infsum_cmult_right' infsum_cong*)
also have ‹$\ldots = (\prod x{\in}insert\ x\ F.\ infsum\ (f\ x)\ (B\ x))$›
using *insert summable3* **by** *auto*
finally show *?case*
by *simp*
qed

lemma *infsum_prod_PiE_abs*:
— See also *infsum_prod_PiE* above with incomparable premises.
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \{banach, real\_normed\_div\_algebra, comm\_semiring\_1\}$
assumes *finite*: *finite A*
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x\ abs\_summable\_on\ B\ x$
shows $infsum\ (\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x))\ (PiE\ A\ B) = (\prod x{\in}A.\ infsum\ (f\ x)\ (B\ x))$
proof (*use finite assms*(*2*) **in** *induction*)
case *empty*
then show *?case*
by *auto*
next
case (*insert x A*)

have *pi*: ‹$Pi_E\ (insert\ x\ F)\ B = (\lambda(g,y).\ g(x:=y))\ `\ (Pi_E\ F\ B \times B\ x)$› **for** *x F* **and** $B :: 'a \Rightarrow 'b\ set$
unfolding *PiE_insert_eq*
by (*subst swap_product* [*symmetric*]) (*simp add*: *image_image case_prod_unfold*)
have *prod*: ‹$(\prod x'{\in}A.\ f\ x'\ ((p(x:=y))\ x')) = (\prod x'{\in}A.\ f\ x'\ (p\ x'))$› **for** *p y*

```
    by (rule prod.cong) (use insert.hyps in auto)
  have inj: ‹inj_on (λ(g, y). g(x := y)) (Pi_E A B × B x)›
    using ‹x ∉ A› by (rule inj_combinator')

  define s where ‹s x = infsum (λy. norm (f x y)) (B x)› for x

  have ‹(∑ p∈P. norm (∏ x∈F. f x (p x))) ≤ prod s F›
    if P: ‹P ⊆ Pi_E F B› and [simp]: ‹finite P› ‹finite F›
      and sum: ‹⋀x. x ∈ F ⟹ f x abs_summable_on B x› for P F
  proof −
    define B' where ‹B' x = {p x| p. p∈P}› for x
    have fin_B'[simp]: ‹finite (B' x)› for x
      using that by (auto simp: B'_def)
    have [simp]: ‹finite (Pi_E F B')›
      by (simp add: finite_PiE)
    have [simp]: ‹P ⊆ Pi_E F B'›
      using that by (auto simp: B'_def)
    have B'B: ‹B' x ⊆ B x› if ‹x ∈ F› for x
      unfolding B'_def using P that
      by auto
    have s_bound: ‹(∑ y∈B' x. norm (f x y)) ≤ s x› if ‹x ∈ F› for x
    by (metis B'B fin_B' finite_sum_le_has_sum has_sum_infsum norm_ge_zero
s_def sum that)
    have ‹(∑ p∈P. norm (∏ x∈F. f x (p x))) ≤ (∑ p∈Pi_E F B'. norm (∏ x∈F. f
x (p x)))›
      by (simp add: sum_mono2)
    also have ‹... = (∑ p∈Pi_E F B'. ∏ x∈F. norm (f x (p x)))›
      by (simp add: prod_norm)
    also have ‹... = (∏ x∈F. ∑ y∈B' x. norm (f x y))›
    proof (use ‹finite F› in induction)
      case empty
      then show ?case by simp
    next
      case (insert x F)
      have inj: ‹inj_on (λ(g, y). g(x := y)) (Pi_E F B' × B' x)›
        by (simp add: inj_combinator' insert.hyps)
      then have ‹(∑ p∈Pi_E (insert x F) B'. ∏ x∈insert x F. norm (f x (p x)))
          = (∑ (p,y)∈Pi_E F B' × B' x. ∏ x'∈insert x F. norm (f x' ((p(x := y))
x')))›
        by (simp add: pi sum.reindex case_prod_unfold)
      also have ‹... = (∑ (p, y)∈Pi_E F B' × B' x. norm (f x y) * (∏ x'∈F. norm
(f x' (p x'))))›
        by (smt (verit, del_insts) fun_upd_apply insert.hyps prod.cong prod.insert
split_def sum.cong)
      also have ‹... = (∑ y∈B' x. norm (f x y)) * (∑ p∈Pi_E F B'. ∏ x'∈F. norm
(f x' (p x')))›
      by (simp add: sum_product sum.swap [of_ Pi_E F B'] sum.cartesian_product)
      also have ‹... = (∏ x∈insert x F. ∑ y∈B' x. norm (f x y))›
        using insert by force
```

```
    finally show ?case .
  qed
  also have ‹... ≤ (∏x∈F. s x)›
    using s_bound by (simp add: prod_mono sum_nonneg)
  finally show ?thesis .
 qed
 then have bdd_above
   (sum (λg. norm (∏x∈insert x A. f x (g x))) ` {F. F ⊆ Pi_E (insert x A) B ∧
finite F})
  using insert.hyps insert.prems by (intro bdd_aboveI) blast
 then have ‹(λg. ∏x∈insert x A. f x (g x)) abs_summable_on Pi_E (insert x A)
B›
  using nonneg_bdd_above_summable_on
  by (metis (mono_tags, lifting) Collect_cong norm_ge_zero)
 also have ‹Pi_E (insert x A) B = (λ(g,y). g(x:=y)) ` (Pi_E A B × B x)›
  by (simp only: pi)
 also have (λg. ∏x∈insert x A. f x (g x)) abs_summable_on ... ⟷
          ((λg. ∏x∈insert x A. f x (g x)) ∘ (λ(g,y). g(x:=y))) abs_summable_on
(Pi_E A B × B x)
  using inj by (subst summable_on_reindex) (auto simp: o_def)
 also have (∏z∈A. f z ((g(x := y)) z)) = (∏z∈A. f z (g z)) for g y
  using insert.hyps by (intro prod.cong) auto
 hence ((λg. ∏x∈insert x A. f x (g x)) ∘ (λ(g,y). g(x:=y))) =
          (λ(p, y). f x y * (∏x'∈A. f x' (p x')))
  using insert.hyps by (auto simp: fun_eq_iff cong: prod.cong_simp)
finally have summable2: ‹(λ(p, y). f x y * (∏x'∈A. f x' (p x'))) abs_summable_on
Pi_E A B × B x› .

 have ‹(∑∞g∈Pi_E (insert x A) B. ∏x∈insert x A. f x (g x))
   = (∑∞(p,y)∈Pi_E A B × B x. ∏x'∈insert x A. f x' ((p(x:=y)) x'))›
  using inj by (simp add: pi infsum_reindex o_def case_prod_unfold)
 also have ‹... = (∑∞(p,y) ∈ Pi_E A B × B x. f x y * (∏x'∈A. f x' (p x')))›
  using prod insert.hyps by auto
 also have ‹... = (∑∞p∈Pi_E A B. ∑∞y∈B x. f x y * (∏x'∈A. f x' (p x')))›
   using abs_summable_summable infsum_Sigma'_banach summable2 by fast-
force
 also have ‹... = (∑∞y∈B x. f x y) * (∑∞p∈Pi_E A B. ∏x'∈A. f x' (p x'))›
  by (smt (verit, best) infsum_cmult_left' infsum_cmult_right' infsum_cong)
 finally show ?case
  by (simp add: insert)
qed
```

6.8.3 Absolute convergence

```
lemma abs_summable_countable:
 assumes ‹f abs_summable_on A›
 shows ‹countable {x∈A. f x ≠ 0}›
proof −
 have fin: ‹finite {x∈A. norm (f x) ≥ t}› if ‹t > 0› for t
```

```
  proof (rule ccontr)
    assume *: ‹infinite {x ∈ A. t ≤ norm (f x)}›
    have ‹infsum (λx. norm (f x)) A ≥ b› for b
    proof −
      obtain b' where b': ‹of_nat b' ≥ b / t›
        by (meson real_arch_simple)
      from *
      obtain F where cardF: ‹card F ≥ b'› and ‹finite F› and F: ‹F ⊆ {x ∈ A.
t ≤ norm (f x)}›
        by (meson finite_if_finite_subsets_card_bdd nle_le)
      have ‹b ≤ of_nat b' * t›
        using b' ‹t > 0› by (simp add: field_simps split: if_splits)
      also have ‹... ≤ of_nat (card F) * t›
        by (simp add: cardF that)
      also have ‹... = sum (λx. t) F›
        by simp
      also have ‹... ≤ sum (λx. norm (f x)) F›
        by (metis (mono_tags, lifting) F in_mono mem_Collect_eq sum_mono)
      also have ‹... = infsum (λx. norm (f x)) F›
        using ‹finite F› by (rule infsum_finite[symmetric])
      also have ‹... ≤ infsum (λx. norm (f x)) A›
        by (rule infsum_mono_neutral) (use ‹finite F› assms F in auto)
      finally show ?thesis .
    qed
    then show False
      by (meson gt_ex linorder_not_less)
  qed
  have ‹countable (⋃i∈{1..}. {x∈A. norm (f x) ≥ 1/of_nat i})›
    by (rule countable_UN) (use fin in ‹auto intro!: countable_finite›)
  also have ‹... = {x∈A. f x ≠ 0}›
  proof safe
    fix x assume x: x ∈ A f x ≠ 0
    define i where i = max 1 (nat (ceiling (1 / norm (f x))))
    have i ≥ 1
      by (simp add: i_def)
    moreover have real i ≥ 1 / norm (f x)
      unfolding i_def by linarith
    hence 1 / real i ≤ norm (f x) using ‹f x ≠ 0›
      by (auto simp: divide_simps mult_ac)
    ultimately show x ∈ (⋃i∈{1..}. {x ∈ A. 1 / real i ≤ norm (f x)})
      using ‹x ∈ A› by auto
  qed auto
  finally show ?thesis .
qed


lemma summable_on_iff_abs_summable_on_real:
  fixes f :: ‹'a ⇒ real›
  shows ‹f summable_on A ⟷ f abs_summable_on A›
```

proof (*rule iffI*)
assume ‹*f summable_on A*›
define *n* A_p A_n
where ‹$n \equiv \lambda x.\ norm\ (f\ x)$› **and** ‹$A_p = \{x{\in}A.\ f\ x \geq 0\}$› **and** ‹$A_n = \{x{\in}A.\ f\ x < 0\}$› **for** *x*
have *A*: ‹$A_p \cup A_n = A$› ‹$A_p \cap A_n = \{\}$›
by (*auto simp*: A_p_*def* A_n_*def*)
from ‹*f summable_on A*› **have** ‹*f summable_on* A_p› ‹*f summable_on* A_n›
using A_p_*def* A_n_*def summable_on_subset_banach* **by** *fastforce+*
then have ‹*n summable_on* A_p›
by (*smt* (*verit*) A_p_*def n_def mem_Collect_eq real_norm_def summable_on_cong*)
moreover have ‹*n summable_on* A_n›
by (*smt* (*verit*, *best*) ‹*f summable_on* A_n› *summable_on_uminus* A_n_*def n_def summable_on_cong mem_Collect_eq real_norm_def*)
ultimately show ‹*n summable_on A*›
using *A summable_on_Un_disjoint* **by** *blast*
next
show ‹*f abs_summable_on A* ⟹ *f summable_on A*›
using *abs_summable_summable* **by** *blast*
qed

lemma *abs_summable_on_Sigma_iff*:
shows *f abs_summable_on Sigma A B* ⟷
(∀ *x*∈*A*. (λ*y*. *f* (*x*, *y*)) *abs_summable_on B x*) ∧
((λ*x*. *infsum* (λ*y*. *norm* (*f* (*x*, *y*))) (*B x*)) *abs_summable_on A*)
proof (*intro iffI conjI ballI*)
assume *asm*: ‹*f abs_summable_on Sigma A B*›
then have ‹(λ*x*. *infsum* (λ*y*. *norm* (*f* (*x*,*y*))) (*B x*)) *summable_on A*›
by (*simp add*: *cond_case_prod_eta summable_on_Sigma_banach*)
then show ‹(λ*x*. $\sum_\infty$*y*∈*B x*. *norm* (*f* (*x*, *y*))) *abs_summable_on A*›
using *summable_on_iff_abs_summable_on_real* **by** *force*

show ‹(λ*y*. *f* (*x*, *y*)) *abs_summable_on B x*› **if** ‹*x* ∈ *A*› **for** *x*
proof −
from *asm* **have** ‹*f abs_summable_on Pair x* ‘ *B x*›
by (*simp add*: *image_subset_iff summable_on_subset_banach that*)
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *o_def inj_on_def summable_on_reindex prod.inject summable_on_cong*)
qed
next
assume *asm*: ‹(∀ *x*∈*A*. (λ*xa*. *f* (*x*, *xa*)) *abs_summable_on B x*) ∧
(λ*x*. $\sum_\infty$*y*∈*B x*. *norm* (*f* (*x*, *y*))) *abs_summable_on A*›
have ‹($\sum$ *xy*∈*F*. *norm* (*f xy*)) ≤ ($\sum_\infty$*x*∈*A*. $\sum_\infty$*y*∈*B x*. *norm* (*f* (*x*, *y*)))›
if ‹*F* ⊆ *Sigma A B*› **and** [*simp*]: ‹*finite F*› **for** *F*
proof −
have [*simp*]: ‹(*SIGMA x*:*fst* ‘ *F*. {*y*. (*x*, *y*) ∈ *F*}) = *F*›
by (*auto intro*!: *set_eqI simp add*: *Domain.DomainI fst_eq_Domain*)
have [*simp*]: ‹*finite* {*y*. (*x*, *y*) ∈ *F*}› **for** *x*

by (*metis* ‹*finite F*› *Range.intros finite_Range finite_subset mem_Collect_eq subsetI*)
have ‹($\sum$ *xy*∈*F*. *norm* (*f xy*)) = ($\sum$ *x*∈*fst* ‘ *F*. $\sum$ *y*∈{*y*. (*x*,*y*)∈*F*}. *norm* (*f* (*x*,*y*)))›
by (*simp add*: *sum.Sigma*)
also have ‹... = ($\sum_\infty$ *x*∈*fst* ‘ *F*. $\sum_\infty$ *y*∈{*y*. (*x*,*y*)∈*F*}. *norm* (*f* (*x*,*y*)))›
by *auto*
also have ‹... ≤ ($\sum_\infty$ *x*∈*fst* ‘ *F*. $\sum_\infty$ *y*∈*B x*. *norm* (*f* (*x*,*y*)))›
using *asm that*(*1*) **by** (*intro infsum_mono infsum_mono_neutral*) *auto*
also have ‹... ≤ ($\sum_\infty$ *x*∈*A*. $\sum_\infty$ *y*∈*B x*. *norm* (*f* (*x*,*y*)))›
by (*rule infsum_mono_neutral*) (*use asm that*(*1*) **in** ‹*auto simp add*: *infsum_nonneg*›)
finally show *?thesis* **.**
qed
then show ‹*f abs_summable_on Sigma A B*›
by (*intro nonneg_bdd_above_summable_on*) (*auto simp*: *bdd_above_def*)
qed

lemma *abs_summable_on_comparison_test*:
assumes *g abs_summable_on A*
assumes $\bigwedge$*x*. *x* ∈ *A* ⟹ *norm* (*f x*) ≤ *norm* (*g x*)
shows *f abs_summable_on A*
proof (*rule nonneg_bdd_above_summable_on*)
show *bdd_above* (*sum* (λ*x*. *norm* (*f x*)) ‘ {*F*. *F* ⊆ *A* ∧ *finite F*})
proof (*rule bdd_aboveI2*)
fix *F* **assume** *F*: *F* ∈ {*F*. *F* ⊆ *A* ∧ *finite F*}
have ‹*sum* (λ*x*. *norm* (*f x*)) *F* ≤ *sum* (λ*x*. *norm* (*g x*)) *F*›
using *assms F* **by** (*intro sum_mono*) *auto*
also have ‹... = *infsum* (λ*x*. *norm* (*g x*)) *F*›
using *F* **by** *simp*
also have ‹... ≤ *infsum* (λ*x*. *norm* (*g x*)) *A*›
by (*smt* (*verit*) *F assms*(*1*) *infsum_mono2 mem_Collect_eq norm_ge_zero summable_on_subset_banach*)
finally show ($\sum$ *x*∈*F*. *norm* (*f x*)) ≤ ($\sum_\infty$ *x*∈*A*. *norm* (*g x*)) **.**
qed
qed *auto*

lemma *abs_summable_iff_bdd_above*:
fixes *f* :: ‹*'a* ⇒ *'b*::*real_normed_vector*›
shows ‹*f abs_summable_on A* ⟷ *bdd_above* (*sum* (λ*x*. *norm* (*f x*)) ‘ {*F*. *F*⊆*A* ∧ *finite F*})›
proof (*rule iffI*)
assume ‹*f abs_summable_on A*›
show ‹*bdd_above* (*sum* (λ*x*. *norm* (*f x*)) ‘ {*F*. *F* ⊆ *A* ∧ *finite F*})›
proof (*rule bdd_aboveI2*)
fix *F* **assume** *F*: *F* ∈ {*F*. *F* ⊆ *A* ∧ *finite F*}
show ($\sum$ *x*∈*F*. *norm* (*f x*)) ≤ ($\sum_\infty$ *x*∈*A*. *norm* (*f x*))
by (*rule finite_sum_le_infsum*) (*use* ‹*f abs_summable_on A*› *F* **in** *auto*)
qed

next
assume ‹*bdd_above* (*sum* (λ*x*. *norm* (*f x*)) ‘ {*F*. *F*⊆*A* ∧ *finite F*})›
then show ‹*f abs_summable_on A*›
by (*simp add*: *nonneg_bdd_above_summable_on*)
qed

lemma *abs_summable_product*:
fixes *x* :: *'a* ⇒ *'b*::{*real_normed_div_algebra*,*banach*,*second_countable_topology*}
assumes *x2_sum*: (λ*i*. (*x i*) ∗ (*x i*)) *abs_summable_on A*
and *y2_sum*: (λ*i*. (*y i*) ∗ (*y i*)) *abs_summable_on A*
shows (λ*i*. *x i* ∗ *y i*) *abs_summable_on A*
proof (*rule nonneg_bdd_above_summable_on*)
show *bdd_above* (*sum* (λ*xa*. *norm* (*x xa* ∗ *y xa*)) ‘ {*F*. *F* ⊆ *A* ∧ *finite F*})
proof (*rule bdd_aboveI2*)
fix *F* **assume** *F*: ‹*F* ∈ {*F*. *F* ⊆ *A* ∧ *finite F*}›
then have *r1*: *finite F* **and** *b4*: *F* ⊆ *A*
by *auto*

have *a1*: (∑∞*i*∈*F*. *norm* (*x i* ∗ *x i*)) ≤ (∑∞*i*∈*A*. *norm* (*x i* ∗ *x i*))
by (*metis* (*no_types*, *lifting*) *b4 infsum_mono2 norm_ge_zero summable_on_subset_banach x2_sum*)

have *norm* (*x i* ∗ *y i*) ≤ *norm* (*x i* ∗ *x i*) + *norm* (*y i* ∗ *y i*) **for** *i*
unfolding *norm_mult* **by** (*smt* (*verit*, *best*) *abs_norm_cancel mult_mono not_sum_squares_lt_zero*)
hence (∑*i*∈*F*. *norm* (*x i* ∗ *y i*)) ≤ (∑*i*∈*F*. *norm* (*x i* ∗ *x i*) + *norm* (*y i* ∗ *y i*))
by (*simp add*: *sum_mono*)
also have ... = (∑*i*∈*F*. *norm* (*x i* ∗ *x i*)) + (∑*i*∈*F*. *norm* (*y i* ∗ *y i*))
by (*simp add*: *sum.distrib*)
also have ... = (∑∞*i*∈*F*. *norm* (*x i* ∗ *x i*)) + (∑∞*i*∈*F*. *norm* (*y i* ∗ *y i*))
by (*simp add*: ‹*finite F*›)
also have ... ≤ (∑∞*i*∈*A*. *norm* (*x i* ∗ *x i*)) + (∑∞*i*∈*A*. *norm* (*y i* ∗ *y i*))
using *F assms*
by (*intro add_mono infsum_mono2*) *auto*
finally show ‹(∑*xa*∈*F*. *norm* (*x xa* ∗ *y xa*)) ≤ (∑∞*i*∈*A*. *norm* (*x i* ∗ *x i*)) + (∑∞*i*∈*A*. *norm* (*y i* ∗ *y i*))›
by *simp*
qed
qed *auto*

6.8.4 Extended reals and nats

lemma *summable_on_ennreal*[*simp*]: ‹(*f*::_ ⇒ *ennreal*) *summable_on S*› **and** *summable_on_enat*[*simp*]: ‹(*f*::_ ⇒ *enat*) *summable_on S*›
by (*simp_all add*: *nonneg_summable_on_complete*)

lemma *has_sum_superconst_infinite_ennreal*:
fixes *f* :: ‹*'a* ⇒ *ennreal*›

```
  assumes geqb: ‹⋀x. x ∈ S ⟹ f x ≥ b›
  assumes b: ‹b > 0›
  assumes ‹infinite S›
  shows (f has_sum ∞) S
proof −
  have ‹(sum f ⟶ ∞) (finite_subsets_at_top S)›
  proof (rule order_tendstoI)
    fix y :: ennreal assume ‹y < ∞›
    then have ‹y / b < ∞› ‹y < top›
      using b ennreal_divide_eq_top_iff top.not_eq_extremum by force+
    then obtain F where ‹finite F› and ‹F ⊆ S› and cardF: ‹card F > y / b›
      using ‹infinite S›
     by (metis ennreal_Ex_less_of_nat infinite_arbitrarily_large infinity_ennreal_def)
    moreover have ‹sum f Y > y› if ‹finite Y› and ‹F ⊆ Y› and ‹Y ⊆ S› for
Y
    proof −
      have ‹y < b * card F›
        by (metis b ‹y < top› cardF divide_less_ennreal ennreal_mult_eq_top_iff
gr_implies_not_zero mult.commute top.not_eq_extremum)
      also have ‹... ≤ b * card Y›
        by (meson b card_mono less_imp_le mult_left_mono of_nat_le_iff that)
      also have ‹... = sum (λ_. b) Y›
        by (simp add: mult.commute)
      also have ‹... ≤ sum f Y›
        using geqb by (meson subset_eq sum_mono that(3))
      finally show ?thesis .
    qed
    ultimately show ‹∀F x in finite_subsets_at_top S. y < sum f x›
      unfolding eventually_finite_subsets_at_top by auto
  qed auto
  then show ?thesis
    by (simp add: has_sum_def)
qed

lemma infsum_superconst_infinite_ennreal:
  fixes f :: ‹'a ⇒ ennreal›
  assumes ‹⋀x. x ∈ S ⟹ f x ≥ b›
  assumes ‹b > 0›
  assumes ‹infinite S›
  shows infsum f S = ∞
  using assms infsumI has_sum_superconst_infinite_ennreal by blast

lemma infsum_superconst_infinite_ereal:
  fixes f :: ‹'a ⇒ ereal›
  assumes geqb: ‹⋀x. x ∈ S ⟹ f x ≥ b›
  assumes b: ‹b > 0›
  assumes ‹infinite S›
  shows infsum f S = ∞
proof −
```

```
  obtain b' where b': ‹e2ennreal b' = b› and ‹b' > 0›
    using b by blast
  have 0 < e2ennreal b
    using b'
   by (metis dual_order.refl enn2ereal_e2ennreal gr_zeroI order_less_le zero_ennreal.abs_eq)
  hence ∗: ‹infsum (e2ennreal ∘ f) S = ∞›
    using assms b'
   by (intro infsum_superconst_infinite_ennreal[where b=b']) (auto intro!: e2en-
nreal_mono)
  have ‹infsum f S = infsum (enn2ereal ∘ (e2ennreal ∘ f)) S›
    using geqb b by (intro infsum_cong) (fastforce simp: enn2ereal_e2ennreal)
  also have ‹... = enn2ereal ∞›
   using ∗ by (simp add: infsum_comm_additive_general continuous_at_enn2ereal
nonneg_summable_on_complete)
  also have ‹... = ∞›
    by simp
  finally show ?thesis .
qed

lemma has_sum_superconst_infinite_ereal:
  fixes f :: ‹'a ⇒ ereal›
  assumes ‹⋀x. x ∈ S ⟹ f x ≥ b›
  assumes ‹b > 0›
  assumes ‹infinite S›
  shows (f has_sum ∞) S
 by (metis Infty_neq_0(1) assms infsum_def has_sum_infsum infsum_superconst_infinite_ereal)

lemma infsum_superconst_infinite_enat:
  fixes f :: ‹'a ⇒ enat›
  assumes geqb: ‹⋀x. x ∈ S ⟹ f x ≥ b›
  assumes b: ‹b > 0›
  assumes ‹infinite S›
  shows infsum f S = ∞
proof −
  have ‹ennreal_of_enat (infsum f S) = infsum (ennreal_of_enat ∘ f) S›
    by (simp flip: infsum_comm_additive_general)
  also have ‹... = ∞›
     by (metis assms(3) b comp_def ennreal_of_enat_0 ennreal_of_enat_le_iff
geqb infsum_superconst_infinite_ennreal leD leI)
  also have ‹... = ennreal_of_enat ∞›
    by simp
  finally show ?thesis
    by (rule ennreal_of_enat_inj[THEN iffD1])
qed

lemma has_sum_superconst_infinite_enat:
  fixes f :: ‹'a ⇒ enat›
  assumes ‹⋀x. x ∈ S ⟹ f x ≥ b›
  assumes ‹b > 0›
```

assumes ‹*infinite S*›
shows (*f has_sum* ∞) *S*
by (*metis assms i0_lb has_sum_infsum infsum_superconst_infinite_enat nonneg_summable_on_complete*)

This lemma helps to relate a real-valued infsum to a supremum over extended nonnegative reals.

lemma *infsum_nonneg_is_SUPREMUM_ennreal*:
fixes *f* :: *'a* ⇒ *real*
assumes *summable*: *f summable_on A*
and *fnn*: ⋀*x*. *x*∈*A* ⟹ *f x* ≥ *0*
shows *ennreal* (*infsum f A*) = (*SUP F*∈{*F*. *finite F* ∧ *F* ⊆ *A*}. (*ennreal* (*sum f F*)))
proof −
have §: ⋀*F*. ⟦*finite F*; *F* ⊆ *A*⟧ ⟹ *sum* (*ennreal* ∘ *f*) *F* = *ennreal* (*sum f F*)
by (*metis* (*mono_tags*, *lifting*) *comp_def fnn subsetD sum.cong sum_ennreal*)
then have ‹*ennreal* (*infsum f A*) = *infsum* (*ennreal* ∘ *f*) *A*›
by (*simp add*: *infsum_comm_additive_general local.summable*)
also have ‹... = (*SUP F*∈{*F*. *finite F* ∧ *F* ⊆ *A*}. (*ennreal* (*sum f F*)))›
by (*metis* (*mono_tags*, *lifting*) § *image_cong mem_Collect_eq nonneg_infsum_complete zero_le*)
finally show *?thesis* **.**
qed

This lemma helps to related a real-valued infsum to a supremum over extended reals.

lemma *infsum_nonneg_is_SUPREMUM_ereal*:
fixes *f* :: *'a* ⇒ *real*
assumes *summable*: *f summable_on A*
and *fnn*: ⋀*x*. *x*∈*A* ⟹ *f x* ≥ *0*
shows *ereal* (*infsum f A*) = (*SUP F*∈{*F*. *finite F* ∧ *F* ⊆ *A*}. (*ereal* (*sum f F*)))
proof −
have ⋀*F*. ⟦*finite F*; *F* ⊆ *A*⟧ ⟹ *sum* (*ereal* ∘ *f*) *F* = *ereal* (*sum f F*)
by *auto*
then have ‹*ereal* (*infsum f A*) = *infsum* (*ereal* ∘ *f*) *A*›
by (*simp add*: *infsum_comm_additive_general local.summable*)
also have ‹... = (*SUP F*∈{*F*. *finite F* ∧ *F* ⊆ *A*}. (*ereal* (*sum f F*)))›
by (*subst nonneg_infsum_complete*) (*simp_all add*: *assms*)
finally show *?thesis* **.**
qed

6.8.5 Real numbers

Most lemmas in the general property section already apply to real numbers. A few ones that are specific to reals are given here.

lemma *infsum_nonneg_is_SUPREMUM_real*:
fixes *f* :: *'a* ⇒ *real*
assumes *summable*: *f summable_on A*

and *fnn*: $\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$
shows *infsum f A* = (*SUP* $F{\in}\{F.\ finite\ F \wedge F \subseteq A\}$. (*sum f F*))
proof −
have ∗: *ereal* (*infsum f A*) = (*SUP* $F{\in}\{F.\ finite\ F \wedge F \subseteq A\}$. (*ereal* (*sum f F*)))
using *assms* **by** (*rule infsum_nonneg_is_SUPREMUM_ereal*)
also have ... = *ereal* (*SUP* $F{\in}\{F.\ finite\ F \wedge F \subseteq A\}$. (*sum f F*))
by (*metis* (*no_types*, *lifting*) ∗ *MInfty_neq_ereal*(2) *PInfty_neq_ereal*(2) *SUP_cong abs_eq_infinity_cases ereal_SUP*)
finally show *?thesis* **by** *simp*
qed

lemma *has_sum_nonneg_SUPREMUM_real*:
fixes *f* :: *'a* ⇒ *real*
assumes *f summable_on A* **and** $\bigwedge x.\ x{\in}A \Longrightarrow f\ x \geq 0$
shows (*f has_sum* (*SUP* $F{\in}\{F.\ finite\ F \wedge F \subseteq A\}$. (*sum f F*))) *A*
by (*metis* (*mono_tags*, *lifting*) *assms has_sum_infsum infsum_nonneg_is_SUPREMUM_real*)

lemma *summable_countable_real*:
fixes *f* :: ‹*'a* ⇒ *real*›
assumes ‹*f summable_on A*›
shows ‹*countable* $\{x{\in}A.\ f\ x \neq 0\}$›
using *abs_summable_countable assms summable_on_iff_abs_summable_on_real*
by *blast*

6.8.6 Complex numbers

lemma *has_sum_cnj_iff*[*simp*]:
fixes *f* :: ‹*'a* ⇒ *complex*›
shows ‹((λ*x*. *cnj* (*f x*)) *has_sum cnj a*) *M* ⟷ (*f has_sum a*) *M*›
by (*simp add*: *has_sum_def lim_cnj del*: *cnj_sum add*: *cnj_sum*[*symmetric*, *abs_def*, *of f*])

lemma *summable_on_cnj_iff*[*simp*]:
(λ*i*. *cnj* (*f i*)) *summable_on A* ⟷ *f summable_on A*
by (*metis complex_cnj_cnj summable_on_def has_sum_cnj_iff*)

lemma *infsum_cnj*[*simp*]: ‹*infsum* (λ*x*. *cnj* (*f x*)) *M* = *cnj* (*infsum f M*)›
by (*metis complex_cnj_zero infsumI has_sum_cnj_iff infsum_def summable_on_cnj_iff has_sum_infsum*)

lemma *has_sum_Re*:
assumes (*f has_sum a*) *M*
shows ((λ*x*. *Re* (*f x*)) *has_sum Re a*) *M*
using *has_sum_comm_additive*[**where** *f*=*Re*]
using *assms tendsto_Re* **by** (*fastforce simp add*: *o_def additive_def*)

lemma *infsum_Re*:
assumes *f summable_on M*

shows *infsum* (λ*x*. *Re* (*f x*)) *M* = *Re* (*infsum f M*)
by (*simp add*: *assms has_sum_Re infsumI*)

lemma *summable_on_Re*:
assumes *f summable_on M*
shows (λ*x*. *Re* (*f x*)) *summable_on M*
by (*metis assms has_sum_Re summable_on_def*)

lemma *has_sum_Im*:
assumes (*f has_sum a*) *M*
shows ((λ*x*. *Im* (*f x*)) *has_sum Im a*) *M*
using *has_sum_comm_additive*[**where** *f*=*Im*]
using *assms tendsto_Im* **by** (*fastforce simp add*: *o_def additive_def*)

lemma *infsum_Im*:
assumes *f summable_on M*
shows *infsum* (λ*x*. *Im* (*f x*)) *M* = *Im* (*infsum f M*)
by (*simp add*: *assms has_sum_Im infsumI*)

lemma *summable_on_Im*:
assumes *f summable_on M*
shows (λ*x*. *Im* (*f x*)) *summable_on M*
by (*metis assms has_sum_Im summable_on_def*)

lemma *nonneg_infsum_le_0D_complex*:
fixes *f* :: *'a* ⇒ *complex*
assumes *infsum f A* ≤ *0*
and *abs_sum*: *f summable_on A*
and *nneg*: ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ *0*
and *x* ∈ *A*
shows *f x* = *0*
proof −
have ‹*Im* (*f x*) = *0*›
using *assms*(*4*) *less_eq_complex_def nneg* **by** *auto*
moreover have ‹*Re* (*f x*) = *0*›
using *assms* **by** (*auto simp add*: *summable_on_Re infsum_Re less_eq_complex_def intro*: *nonneg_infsum_le_0D*[**where** *A*=*A*])
ultimately show *?thesis*
by (*simp add*: *complex_eqI*)
qed

lemma *nonneg_has_sum_le_0D_complex*:
fixes *f* :: *'a* ⇒ *complex*
assumes (*f has_sum a*) *A* **and** ‹*a* ≤ *0*›
and ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ *0* **and** *x* ∈ *A*
shows *f x* = *0*
by (*metis assms infsumI nonneg_infsum_le_0D_complex summable_on_def*)

The lemma *infsum_mono_neutral* above applies to various linear ordered

monoids such as the reals but not to the complex numbers. Thus we have a separate corollary for those:

lemma *infsum_mono_neutral_complex*:
fixes $f :: 'a \Rightarrow complex$
assumes [*simp*]: *f summable_on A*
and [*simp*]: *g summable_on B*
assumes ‹$\bigwedge x.\ x \in A \cap B \Longrightarrow f\ x \leq g\ x$›
assumes ‹$\bigwedge x.\ x \in A - B \Longrightarrow f\ x \leq 0$›
assumes ‹$\bigwedge x.\ x \in B - A \Longrightarrow g\ x \geq 0$›
shows ‹*infsum f A* ≤ *infsum g B*›
proof −
have ‹*infsum* (λ*x*. *Re* (*f x*)) *A* ≤ *infsum* (λ*x*. *Re* (*g x*)) *B*›
by (*smt* (*verit*) *assms infsum_cong infsum_mono_neutral less_eq_complex_def summable_on_Re zero_complex.simps*(*1*))
then have *Re*: ‹*Re* (*infsum f A*) ≤ *Re* (*infsum g B*)›
by (*metis assms*(*1*−*2*) *infsum_Re*)
have ‹*infsum* (λ*x*. *Im* (*f x*)) *A* = *infsum* (λ*x*. *Im* (*g x*)) *B*›
by (*smt* (*verit*, *best*) *assms*(*3*−*5*) *infsum_cong_neutral less_eq_complex_def zero_complex.simps*(*2*))
then have *Im*: ‹*Im* (*infsum f A*) = *Im* (*infsum g B*)›
by (*metis assms*(*1*−*2*) *infsum_Im*)
from *Re Im* **show** *?thesis*
by (*auto simp*: *less_eq_complex_def*)
qed

lemma *infsum_mono_complex*:
— For *real*, *infsum_mono* can be used. But *complex* does not have the right typeclass.
fixes *f g* :: $'a \Rightarrow complex$
assumes *f_sum*: *f summable_on A* **and** *g_sum*: *g summable_on A*
assumes *leq*: $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$
shows *infsum f A* ≤ *infsum g A*
by (*metis DiffE IntD1 f_sum g_sum infsum_mono_neutral_complex leq*)

lemma *infsum_nonneg_complex*:
fixes $f :: 'a \Rightarrow complex$
assumes *f summable_on M*
and $\bigwedge x.\ x \in M \Longrightarrow 0 \leq f\ x$
shows *infsum f M* ≥ *0* (**is** *?lhs* ≥ _)
by (*metis assms infsum_0_simp summable_on_0_simp infsum_mono_complex*)

lemma *infsum_cmod*:
assumes *f summable_on M*
and *fnn*: $\bigwedge x.\ x \in M \Longrightarrow 0 \leq f\ x$
shows *infsum* (λ*x*. *cmod* (*f x*)) *M* = *cmod* (*infsum f M*)
proof −
have ‹*complex_of_real* (*infsum* (λ*x*. *cmod* (*f x*)) *M*) = *infsum* (λ*x*. *complex_of_real* (*cmod* (*f x*))) *M*›

```
  proof (rule infsum_comm_additive[symmetric, unfolded o_def])
    have (λz. Re (f z)) summable_on M
      using assms summable_on_Re by blast
    also have ?this ⟷ f abs_summable_on M
      using fnn by (intro summable_on_cong) (auto simp: less_eq_complex_def
cmod_def)
    finally show ... .
  qed (auto simp: additive_def)
  also have ‹... = infsum f M›
    using fnn cmod_eq_Re complex_is_Real_iff less_eq_complex_def by (force
cong: infsum_cong)
  finally show ?thesis
   by (metis abs_of_nonneg infsum_def le_less_trans norm_ge_zero norm_infsum_bound
norm_of_real not_le order_refl)
qed


lemma summable_on_iff_abs_summable_on_complex:
  fixes f :: ‹'a ⇒ complex›
  shows ‹f summable_on A ⟷ f abs_summable_on A›
proof (rule iffI)
  assume ‹f summable_on A›
  define i r ni nr n where ‹i x = Im (f x)› and ‹r x = Re (f x)›
    and ‹ni x = norm (i x)› and ‹nr x = norm (r x)› and ‹n x = norm (f x)› for
x
  from ‹f summable_on A› have ‹i summable_on A›
    by (simp add: i_def[abs_def] summable_on_Im)
  then have [simp]: ‹ni summable_on A›
    using ni_def[abs_def] summable_on_iff_abs_summable_on_real by force

  from ‹f summable_on A› have ‹r summable_on A›
    by (simp add: r_def[abs_def] summable_on_Re)
  then have [simp]: ‹nr summable_on A›
   by (metis nr_def summable_on_cong summable_on_iff_abs_summable_on_real)

  have n_sum: ‹n x ≤ nr x + ni x› for x
    by (simp add: n_def nr_def ni_def r_def i_def cmod_le)

  have ∗: ‹(λx. nr x + ni x) summable_on A›
    by (simp add: summable_on_add)
  have bdd_above (sum n ‘ {F. F ⊆ A ∧ finite F})
    apply (rule bdd_aboveI[where M=‹infsum (λx. nr x + ni x) A›])
    using ∗ n_sum by (auto simp flip: infsum_finite simp: ni_def nr_def intro!:
infsum_mono_neutral)
  then show ‹n summable_on A›
    by (simp add: n_def nonneg_bdd_above_summable_on)
next
  show ‹f abs_summable_on A ⟹ f summable_on A›
    using abs_summable_summable by blast
```

qed

lemma *summable_countable_complex*:
fixes $f :: \langle 'a \Rightarrow complex\rangle$
assumes $\langle f\ summable\_on\ A\rangle$
shows $\langle countable\ \{x{\in}A.\ f\ x \neq 0\}\rangle$
using *abs_summable_countable assms summable_on_iff_abs_summable_on_complex*
by *blast*

inductive (**in** *topological_space*) *convergent_filter* :: $'a$ *filter* $\Rightarrow$ *bool* **where**
$F \leq nhds\ x \Longrightarrow convergent\_filter\ F$

lemma (**in** *topological_space*) *convergent_filter_iff*: *convergent_filter* $F \longleftrightarrow (\exists x.$
$F \leq nhds\ x)$
by (*auto simp*: *convergent_filter.simps*)

lemma (**in** *uniform_space*) *cauchy_filter_mono*:
cauchy_filter $F \Longrightarrow F' \leq F \Longrightarrow$ *cauchy_filter* F'
unfolding *cauchy_filter_def* **by** (*meson dual_order.trans prod_filter_mono*)

lemma (**in** *uniform_space*) *convergent_filter_cauchy*:
assumes *convergent_filter F*
shows *cauchy_filter F*
using *assms cauchy_filter_mono nhds_imp_cauchy_filter*[*OF order.refl*]
by (*auto simp*: *convergent_filter_iff*)

lemma (**in** *topological_space*) *convergent_filter_bot* [*simp*, *intro*]: *convergent_filter*
bot
by (*simp add*: *convergent_filter_iff*)

class *complete_uniform_space* = *uniform_space* +
assumes *cauchy_filter_convergent'*: *cauchy_filter* (F :: $'a$ *filter*) $\Longrightarrow F \neq bot$
$\Longrightarrow$ *convergent_filter F*

lemma (**in** *complete_uniform_space*)
cauchy_filter_convergent: *cauchy_filter* (F :: $'a$ *filter*) $\Longrightarrow$ *convergent_filter F*
using *cauchy_filter_convergent'*[*of F*] **by** (*cases* $F = bot$) *auto*

lemma (**in** *complete_uniform_space*) *convergent_filter_iff_cauchy*:
convergent_filter $F \longleftrightarrow$ *cauchy_filter F*
using *convergent_filter_cauchy cauchy_filter_convergent* **by** *blast*

definition *countably_generated_filter* :: $'a$ *filter* $\Rightarrow$ *bool* **where**
countably_generated_filter $F \longleftrightarrow (\exists U :: nat \Rightarrow 'a\ set.\ F = (INF\ (n{::}nat).$
principal $(U\ n)))$

lemma *countably_generated_filter_has_antimono_basis*:
assumes *countably_generated_filter F*
obtains $B :: nat \Rightarrow 'a\ set$
where *antimono B* **and** $F = (INF\ n.\ principal\ (B\ n))$ **and**
$\bigwedge P.\ eventually\ P\ F \longleftrightarrow (\exists i.\ \forall x{\in}B\ i.\ P\ x)$
proof −
from *assms* **obtain** *B* **where** *B*: $F = (INF\ (n{::}nat).\ principal\ (B\ n))$
unfolding *countably_generated_filter_def* **by** *blast*

define B' **where** $B' = (\lambda n.\ \bigcap k{\leq}n.\ B\ k)$
have *antimono B'*
unfolding *decseq_def B'_def* **by** *force*

have $(INF\ n.\ principal\ (B'\ n)) = (INF\ n.\ INF\ k{\in}\{..n\}.\ principal\ (B\ k))$
unfolding *B'_def* **by** (*intro INF_cong refl INF_principal_finite* [*symmetric*])
auto
also have $\ldots = (INF\ (n{::}nat).\ principal\ (B\ n))$
apply (*intro antisym*)
apply (*meson INF_lower INF_mono atMost_iff order_refl*)
apply (*meson INF_greatest INF_lower UNIV_I*)
done
also have $\ldots = F$
by (*simp add*: *B*)
finally have *F*: $F = (INF\ n.\ principal\ (B'\ n))$ **..**

moreover have $eventually\ P\ F \longleftrightarrow (\exists i.\ eventually\ P\ (principal\ (B'\ i)))$ **for** *P*
unfolding *F* **using** ‹*antimono B'*›
apply (*subst eventually_INF_base*)
apply (*auto simp*: *decseq_def*)
by (*meson nat_le_linear*)
ultimately show *?thesis*
using ‹*antimono B'*› *that*[*of B'*] **unfolding** *eventually_principal* **by** *blast*
qed

lemma (**in** *uniform_space*) *cauchy_filter_iff*:
cauchy_filter F $\longleftrightarrow$ $(\forall P.\ eventually\ P\ uniformity \longrightarrow (\exists X.\ eventually\ (\lambda x.\ x \in X)\ F \wedge (\forall z{\in}X{\times}X.\ P\ z)))$
unfolding *cauchy_filter_def le_filter_def*
apply (*auto simp*: *eventually_prod_same*)
apply (*metis* (*full_types*) *eventually_mono mem_Collect_eq*)
apply *blast*
done

lemma (**in** *uniform_space*) *controlled_sequences_convergent_imp_complete_aux_sequence*:
fixes $U :: nat \Rightarrow ('a \times 'a)\ set$
fixes $F :: 'a\ filter$
assumes *cauchy_filter F* $F \neq bot$
assumes $\bigwedge n.\ eventually\ (\lambda z.\ z \in U\ n)\ uniformity$

```
  obtains g G where
    antimono G ⋀n. g n ∈ G n
    ⋀n. eventually (λx. x ∈ G n) F ⋀n. G n × G n ⊆ U n
proof −
  have ∃ C. eventually (λx. x ∈ C) F ∧ C × C ⊆ U n for n
    using assms(1) assms(3)[of n] unfolding cauchy_filter_iff by blast
  then obtain G where G: ⋀n. eventually (λx. x ∈ G n) F ⋀n. G n × G n ⊆
U n
    by metis
  define G' where G' = (λn. ⋂k≤n. G k)
  have 1: eventually (λx. x ∈ G' n) F for n
    using G by (auto simp: G'_def intro: eventually_ball_finite)
  have 2: G' n × G' n ⊆ U n for n
    using G unfolding G'_def by fast
  have 3: antimono G'
    unfolding G'_def decseq_def by force

  have ∃ g. g ∈ G' n for n
    using 1 assms(2) eventually_happens' by auto
  then obtain g where g: ⋀n. g n ∈ G' n
    by metis
  from g 1 2 3 that[of G' g] show ?thesis
    by metis
qed

definition lift_filter :: ('a set ⇒ 'b filter) ⇒ 'a filter ⇒ 'b filter where
  lift_filter f F = (INF X∈{X. eventually (λx. x ∈ X) F}. f X)

lemma lift_filter_top [simp]: lift_filter g top = g UNIV
proof −
  have {X. ∀ x::'b. x ∈ X} = {UNIV}
    by auto
  thus ?thesis
    by (simp add: lift_filter_def)
qed

lemma eventually_lift_filter_iff:
  assumes mono g
  shows   eventually P (lift_filter g F) ⟷ (∃ X. eventually (λx. x ∈ X) F ∧
eventually P (g X))
  unfolding lift_filter_def
proof (subst eventually_INF_base, goal_cases)
  case 1
  thus ?case by (auto intro: exI[of _ UNIV])
next
  case (2 X Y)
  thus ?case
    by (auto intro!: exI[of _ X ∩ Y] eventually_conj monoD[OF assms])
qed auto
```

lemma *lift_filter_le*:
assumes *eventually* $(\lambda x.\ x \in X)\ F\ g\ X \leq F'$
shows *lift_filter g F* $\leq F'$
unfolding *lift_filter_def*
by (*metis INF_lower2 assms mem_Collect_eq*)

definition *lift_filter'* :: $('a\ set \Rightarrow 'b\ set) \Rightarrow 'a\ filter \Rightarrow 'b\ filter$ **where**
lift_filter' f F = *lift_filter* (*principal* $\circ$ *f*) *F*

lemma *lift_filter'_top* [*simp*]: *lift_filter' g top* = *principal* (*g UNIV*)
by (*simp add*: *lift_filter'_def*)

lemma *eventually_lift_filter'_iff*:
assumes *mono g*
shows *eventually P* (*lift_filter' g F*) $\longleftrightarrow$ $(\exists X.\ eventually\ (\lambda x.\ x \in X)\ F \wedge (\forall x{\in}g\ X.\ P\ x))$
unfolding *lift_filter'_def* **using** *assms*
by (*subst eventually_lift_filter_iff*) (*auto simp*: *mono_def eventually_principal*)

lemma *lift_filter'_le*:
assumes *eventually* $(\lambda x.\ x \in X)\ F\ principal\ (g\ X) \leq F'$
shows *lift_filter' g F* $\leq F'$
unfolding *lift_filter'_def* **using** *assms*
by (*intro lift_filter_le*[**where** $X = X$]) *auto*

lemma (**in** *uniform_space*) *comp_uniformity_le_uniformity*:
lift_filter' $(\lambda X.\ X\ O\ X)$ *uniformity* $\leq$ *uniformity*
unfolding *le_filter_def*
proof *safe*
fix *P* **assume** *P*: *eventually P uniformity*
have [*simp*]: *mono* $(\lambda X{::}('a \times 'a)\ set.\ X\ O\ X)$
by (*intro monoI*) *auto*
from *P* **obtain** *P'* **where** *P'*: *eventually P' uniformity* $(\bigwedge x\ y\ z.\ P'\ (x, y) \Longrightarrow P'\ (y, z) \Longrightarrow P\ (x, z))$
using *uniformity_transE* **by** *blast*
show *eventually P* (*lift_filter'* $(\lambda X.\ X\ O\ X)$ *uniformity*)
by (*auto simp*: *eventually_lift_filter'_iff intro*!: *exI*[*of* _ $\{x.\ P'\ x\}$] *P'*)
qed

lemma (**in** *uniform_space*) *comp_mem_uniformity_sets*:
assumes *eventually* $(\lambda z.\ z \in X)$ *uniformity*
obtains *Y* **where** *eventually* $(\lambda z.\ z \in Y)$ *uniformity Y O Y* $\subseteq X$
proof −
have [*simp*]: *mono* $(\lambda X{::}('a \times 'a)\ set.\ X\ O\ X)$
by (*intro monoI*) *auto*
have *eventually* $(\lambda z.\ z \in X)$ (*lift_filter'* $(\lambda X.\ X\ O\ X)$ *uniformity*)
using *assms comp_uniformity_le_uniformity* **using** *filter_leD* **by** *blast*

thus *?thesis* **using** *that*
by (*auto simp*: *eventually_lift_filter'_iff*)
qed

lemma (**in** *uniform_space*) *le_nhds_of_cauchy_adhp_aux*:
assumes $\bigwedge P.$ *eventually* P *uniformity* $\Longrightarrow$ $(\exists X.$ *eventually* $(\lambda y.\ y \in X)\ F \wedge (\forall z{\in}X \times X.\ P\ z) \wedge (\exists y.\ P\ (x,\ y) \wedge y \in X))$
shows $F \leq$ *nhds* x
unfolding *le_filter_def*
proof *safe*
fix P **assume** *eventually* P (*nhds* x)
hence $\forall_F\ z$ *in uniformity*. $z \in \{z.\ fst\ z = x \longrightarrow P\ (snd\ z)\}$
by (*simp add*: *eventually_nhds_uniformity case_prod_unfold*)
then obtain Y **where** Y: $\forall_F\ z$ *in uniformity*. $z \in Y\ Y\ O\ Y \subseteq \{z.\ fst\ z = x \longrightarrow P\ (snd\ z)\}$
using *comp_mem_uniformity_sets* **by** *blast*
obtain $X\ y$ **where** Xy: *eventually* $(\lambda y.\ y \in X)\ F\ X{\times}X \subseteq Y\ (x,\ y) \in Y\ y \in X$
using *assms*[*OF* $Y(1)$] **by** *blast*
have $*$: $P\ x$ **if** $x \in X$ **for** x
using $Y(2)$ $Xy(2{-}4)$ *that* **unfolding** *relcomp_unfold* **by** *force*
show *eventually* $P\ F$
by (*rule eventually_mono*[*OF* $Xy(1)$]) (*use* $*$ **in** *auto*)
qed

lemma (**in** *uniform_space*) *eventually_uniformity_imp_nhds*:
assumes *eventually* P *uniformity*
shows *eventually* $(\lambda y.\ P\ (x,\ y))$ (*nhds* x)
using *assms* **unfolding** *eventually_nhds_uniformity* **by** (*elim eventually_mono*) *auto*

lemma (**in** *uniform_space*) *controlled_sequences_convergent_imp_complete_aux*:
fixes $U :: nat \Rightarrow ('a \times 'a)\ set$
assumes *gen*: *countably_generated_filter* (*uniformity* :: $('a \times 'a)$ *filter*)
assumes U: $\bigwedge n.$ *eventually* $(\lambda z.\ z \in U\ n)$ *uniformity*
assumes *conv*: $\bigwedge (u :: nat \Rightarrow 'a).\ (\bigwedge N\ m\ n.\ N \leq m \Longrightarrow N \leq n \Longrightarrow (u\ m,\ u\ n) \in U\ N) \Longrightarrow$ *convergent* u
assumes *cauchy_filter* F
shows *convergent_filter* F
proof (*cases* $F = bot$)
case *False*
note $F =$ ‹*cauchy_filter* F› ‹$F \neq bot$›
from *gen* **obtain** $B :: nat \Rightarrow ('a \times 'a)\ set$ **where** B:
antimono B *uniformity* $=$ (*INF* $n.$ *principal* $(B\ n)$)
$\bigwedge P.$ *eventually* P *uniformity* $\longleftrightarrow$ $(\exists i.\ \forall x{\in}B\ i.\ P\ x)$
using *countably_generated_filter_has_antimono_basis* **by** *blast*

have *ev_B*: *eventually* $(\lambda z.\ z \in B\ n)$ *uniformity* **for** n
by (*subst* $B(3)$) *auto*
hence *ev_B'*: *eventually* $(\lambda z.\ z \in B\ n \cap U\ n)$ *uniformity* **for** n

```
    using U by (auto intro: eventually_conj)

  obtain g G where gG: antimono G ⋀n. g n ∈ G n
    ⋀n. eventually (λx. x ∈ G n) F ⋀n. G n × G n ⊆ B n ∩ U n
   using controlled_sequences_convergent_imp_complete_aux_sequence[of F λn.
B n ∩ U n, OF F ev_B'] 
    by metis

  have convergent g
  proof (rule conv)
    fix N m n :: nat
    assume mn: N ≤ m N ≤ n
    have (g m, g n) ∈ G m × G n
      using gG by auto
    also from mn have ... ⊆ G N × G N
      by (intro Sigma_mono gG antimonoD[OF gG(1)])
    also have ... ⊆ U N
      using gG by blast
    finally show (g m, g n) ∈ U N .
  qed
  then obtain L where G: g ⟶ L
    unfolding convergent_def by blast

  have F ≤ nhds L
  proof (rule le_nhds_of_cauchy_adhp_aux)
    fix P :: 'a × 'a ⇒ bool
    assume P: eventually P uniformity
    hence eventually (λn. ∀x∈B n. P x) sequentially
      using ‹antimono B› unfolding B(3) eventually_sequentially decseq_def by
blast
    moreover have eventually (λn. P (L, g n)) sequentially
      using P eventually_compose_filterlim eventually_uniformity_imp_nhds G
by blast
    ultimately have eventually (λn. (∀x∈B n. P x) ∧ P (L, g n)) sequentially
      by eventually_elim auto
    then obtain n where ∀x∈B n. P x P (L, g n)
      unfolding eventually_at_top_linorder by blast
    then show ∃X. (∀F y in F. y ∈ X) ∧ (∀z∈X × X. P z) ∧ (∃y. P (L, y) ∧
y ∈ X)
      using gG by blast+
  qed
  thus convergent_filter F
    by (auto simp: convergent_filter_iff)
qed auto

theorem (in uniform_space) controlled_sequences_convergent_imp_complete:
  fixes U :: nat ⇒ ('a × 'a) set
  assumes gen: countably_generated_filter (uniformity :: ('a × 'a) filter)
  assumes U: ⋀n. eventually (λz. z ∈ U n) uniformity
```

assumes *conv*: $\bigwedge(u :: nat \Rightarrow 'a).\ (\bigwedge N\ m\ n.\ N \leq m \Longrightarrow N \leq n \Longrightarrow (u\ m,\ u\ n) \in U\ N) \Longrightarrow convergent\ u$
shows *class.complete_uniform_space open uniformity*
by *unfold_locales* (*use assms controlled_sequences_convergent_imp_complete_aux* **in** *blast*)

lemma *filtermap_prod_filter*: $filtermap\ (map\_prod\ f\ g)\ (F \times_F G) = filtermap\ f\ F \times_F filtermap\ g\ G$
proof (*intro antisym*)
show $filtermap\ (map\_prod\ f\ g)\ (F \times_F G) \leq filtermap\ f\ F \times_F filtermap\ g\ G$
by (*auto simp*: *le_filter_def eventually_filtermap eventually_prod_filter*)
next
show $filtermap\ f\ F \times_F filtermap\ g\ G \leq filtermap\ (map\_prod\ f\ g)\ (F \times_F G)$
unfolding *le_filter_def*
proof *safe*
fix P **assume** P: $eventually\ P\ (filtermap\ (map\_prod\ f\ g)\ (F \times_F G))$
then obtain $Pf\ Pg$ **where** $*$: $eventually\ Pf\ F\ eventually\ Pg\ G\ \forall x.\ Pf\ x \longrightarrow (\forall y.\ Pg\ y \longrightarrow P\ (f\ x,\ g\ y))$
by (*auto simp*: *eventually_filtermap eventually_prod_filter*)

define Pf' **where** $Pf' = (\lambda x.\ \exists y.\ x = f\ y \wedge Pf\ y)$
define Pg' **where** $Pg' = (\lambda x.\ \exists y.\ x = g\ y \wedge Pg\ y)$

from $*(1)$ **have** $\forall_F\ x\ in\ F.\ Pf'\ (f\ x)$
by *eventually_elim* (*auto simp*: *Pf'_def*)
moreover from $*(2)$ **have** $\forall_F\ x\ in\ G.\ Pg'\ (g\ x)$
by *eventually_elim* (*auto simp*: *Pg'_def*)
moreover have $(\forall x\ y.\ Pf'\ x \longrightarrow Pg'\ y \longrightarrow P\ (x,\ y))$
using $*(3)$ **by** (*auto simp*: *Pf'_def Pg'_def*)
ultimately show $eventually\ P\ (filtermap\ f\ F \times_F filtermap\ g\ G)$
unfolding *eventually_prod_filter eventually_filtermap*
by *blast*
qed
qed

lemma (**in** *uniform_space*) *Cauchy_seq_iff_tendsto*:
$Cauchy\ f \longleftrightarrow filterlim\ (map\_prod\ f\ f)\ uniformity\ (at\_top \times_F at\_top)$
unfolding *Cauchy_uniform cauchy_filter_def filterlim_def filtermap_prod_filter*
..

theorem (**in** *uniform_space*) *controlled_seq_imp_Cauchy_seq*:
fixes $U :: nat \Rightarrow ('a \times 'a)\ set$
assumes U: $\bigwedge P.\ eventually\ P\ uniformity \Longrightarrow (\exists n.\ \forall x \in U\ n.\ P\ x)$
assumes *controlled*: $\bigwedge N\ m\ n.\ N \leq m \Longrightarrow N \leq n \Longrightarrow (f\ m,\ f\ n) \in U\ N$
shows $Cauchy\ f$
unfolding *Cauchy_seq_iff_tendsto*
proof −
show $filterlim\ (map\_prod\ f\ f)\ uniformity\ (sequentially \times_F sequentially)$

```
    unfolding filterlim_def le_filter_def
  proof safe
    fix P :: 'a × 'a ⇒ bool
    assume P: eventually P uniformity
    from U[OF this] obtain N where ∀x∈U N. P x
      by blast
    then show eventually P (filtermap (map_prod f f) (sequentially ×F sequentially))
      unfolding eventually_filtermap eventually_prod_sequentially
      by (metis controlled map_prod_simp)
  qed
qed

lemma (in uniform_space) Cauchy_seq_convergent_imp_complete_aux:
  fixes U :: nat ⇒ ('a × 'a) set
  assumes gen: countably_generated_filter (uniformity :: ('a × 'a) filter)
  assumes conv: ⋀(u :: nat ⇒ 'a). Cauchy u ⟹ convergent u
  assumes cauchy_filter F
  shows   convergent_filter F
proof −
  from gen obtain B :: nat ⇒ ('a × 'a) set where B:
    antimono B uniformity = (INF n. principal (B n))
    ⋀P. eventually P uniformity ⟷ (∃ i. ∀x∈B i. P x)
    using countably_generated_filter_has_antimono_basis by blast

  show ?thesis
   proof (rule controlled_sequences_convergent_imp_complete_aux[where U = B])
    show ∀F z in uniformity. z ∈ B n for n
      unfolding B(3) by blast
  next
    fix f :: nat ⇒ 'a
    assume f: ⋀N m n. N ≤ m ⟹ N ≤ n ⟹ (f m, f n) ∈ B N
    have Cauchy f using f B
      by (intro controlled_seq_imp_Cauchy_seq[where U = B]) auto
    with conv show convergent f
      by simp
  qed fact+
qed

theorem (in uniform_space) Cauchy_seq_convergent_imp_complete:
  fixes U :: nat ⇒ ('a × 'a) set
  assumes gen: countably_generated_filter (uniformity :: ('a × 'a) filter)
  assumes conv: ⋀(u :: nat ⇒ 'a). Cauchy u ⟹ convergent u
  shows   class.complete_uniform_space open uniformity
   by unfold_locales (use assms Cauchy_seq_convergent_imp_complete_aux in blast)

lemma (in metric_space) countably_generated_uniformity:
```

```
  countably_generated_filter uniformity
proof −
  have (INF e∈{0<..}. principal {(x, y). dist (x::'a) y < e}) =
        (INF n∈UNIV. principal {(x, y). dist x y < 1 / real (Suc n)}) (is ?F =
?G)
    unfolding uniformity_dist
  proof (intro antisym)
    have ?G = (INF e∈(λn. 1 / real (Suc n)) ‘ UNIV. principal {(x, y). dist x y
< e})
      by (simp add: image_image)
    also have ... ≥ ?F
      by (intro INF_superset_mono) auto
    finally show ?F ≤ ?G .
  next
    show ?G ≤ ?F
      unfolding le_filter_def
    proof safe
      fix P assume eventually P ?F
     then obtain ε where ε: ε > 0 eventually P (principal {(x, y). dist x y < ε})
      proof (subst (asm) eventually_INF_base, goal_cases)
        case (2 ε1 ε2)
        thus ?case
          by (intro bexI[of _ min ε1 ε2]) auto
      qed auto
      from ‹ε > 0› obtain n where 1 / real (Suc n) < ε
        using nat_approx_posE by blast
      then have eventually P (principal {(x, y). dist x y < 1 / real (Suc n)})
        using ε(2) by (auto simp: eventually_principal)
      thus eventually P ?G
        by (intro eventually_INF1) auto
    qed
  qed
  thus countably_generated_filter uniformity
    unfolding countably_generated_filter_def uniformity_dist by fast
qed

subclass (in complete_space) complete_uniform_space
proof (rule Cauchy_seq_convergent_imp_complete)
  show convergent f if Cauchy f for f
    using Cauchy_convergent that by blast
qed (fact countably_generated_uniformity)

lemma (in complete_uniform_space) complete_UNIV_cuspace [intro]: complete
UNIV
  unfolding complete_uniform using cauchy_filter_convergent
  by (auto simp: convergent_filter.simps)
```

lemma *norm_infsum_le*:
assumes (*f has_sum S*) *X*
assumes (*g has_sum T*) *X*
assumes $\bigwedge x.\ x \in X \Longrightarrow \mathit{norm}\ (f\ x) \leq g\ x$
shows *norm S* $\leq$ *T*
proof (*rule tendsto_le*)
show $((\lambda Y.\ \mathit{norm}\ (\sum x{\in}Y.\ f\ x)) \longrightarrow \mathit{norm}\ S)\ (\mathit{finite\_subsets\_at\_top}\ X)$
using *assms*(*1*) **unfolding** *has_sum_def* **by** (*intro tendsto_norm*)
show $((\lambda Y.\ \sum x{\in}Y.\ g\ x) \longrightarrow T)\ (\mathit{finite\_subsets\_at\_top}\ X)$
using *assms*(*2*) **unfolding** *has_sum_def* **.**
show $\forall_F\ x\ \mathit{in}\ \mathit{finite\_subsets\_at\_top}\ X.\ \mathit{norm}\ (\mathit{sum}\ f\ x) \leq (\sum x{\in}x.\ g\ x)$
by (*simp add*: *assms*(*3*) *eventually_finite_subsets_at_top_weakI subsetD sum_norm_le*)
qed *auto*

lemma *has_sum_imp_summable*: (*f has_sum S*) *A* $\Longrightarrow$ *f summable_on A*
by (*auto simp*: *summable_on_def*)

lemma *has_sum_reindex_bij_betw*:
assumes *bij_betw g A B*
shows $((\lambda x.\ f\ (g\ x))\ \mathit{has\_sum}\ S)\ A = (f\ \mathit{has\_sum}\ S)\ B$
proof −
have $((\lambda x.\ f\ (g\ x))\ \mathit{has\_sum}\ S)\ A \longleftrightarrow (f\ \mathit{has\_sum}\ S)\ (g\ `\ A)$
by (*subst has_sum_reindex*) (*use assms* **in** ‹*auto dest*: *bij_betw_imp_inj_on simp*: *o_def*›)
then show *?thesis*
using *assms bij_betw_imp_surj_on* **by** *blast*
qed

lemma *has_sum_reindex_bij_witness*:
assumes $\bigwedge a.\ a \in S \Longrightarrow i\ (j\ a) = a$
assumes $\bigwedge a.\ a \in S \Longrightarrow j\ a \in T$
assumes $\bigwedge b.\ b \in T \Longrightarrow j\ (i\ b) = b$
assumes $\bigwedge b.\ b \in T \Longrightarrow i\ b \in S$
assumes $\bigwedge a.\ a \in S \Longrightarrow h\ (j\ a) = g\ a$
assumes $s = s'$
shows $(g\ \mathit{has\_sum}\ s)\ S = (h\ \mathit{has\_sum}\ s')\ T$
by (*smt* (*verit*, *del_insts*) *assms bij_betwI' has_sum_cong has_sum_reindex_bij_betw*)

lemma *has_sum_homomorphism*:
assumes (*f has_sum S*) *A h* $0 = 0$ $\bigwedge a\ b.\ h\ (a + b) = h\ a + h\ b$ *continuous_on UNIV h*
shows $((\lambda x.\ h\ (f\ x))\ \mathit{has\_sum}\ (h\ S))\ A$
proof −
have *sum* $(h \circ f)\ X = h\ (\mathit{sum}\ f\ X)$ **for** *X*
by (*induction X rule*: *infinite_finite_induct*) (*simp_all add*: *assms*)
hence *sum_h*: *sum* $(h \circ f) = h \circ \mathit{sum}\ f$

by (*intro ext*) *auto*
then have ((*h* ∘ *f*) *has_sum h S*) *A*
using *assms*
by (*metis UNIV_I continuous_on_def has_sum_comm_additive_general o_apply*)
thus *?thesis*
by (*simp add*: *o_def*)
qed

lemma *summable_on_homomorphism*:
assumes *f summable_on A h 0 = 0* ⋀*a b*. *h* (*a* + *b*) = *h a* + *h b continuous_on UNIV h*
shows (λ*x*. *h* (*f x*)) *summable_on A*
proof −
from *assms*(*1*) **obtain** *S* **where** (*f has_sum S*) *A*
by (*auto simp*: *summable_on_def*)
hence ((λ*x*. *h* (*f x*)) *has_sum h S*) *A*
by (*rule has_sum_homomorphism*) (*use assms* **in** *auto*)
thus *?thesis*
by (*auto simp*: *summable_on_def*)
qed

lemma *infsum_homomorphism_strong*:
fixes *h* :: *'a* :: {*t2_space*, *topological_comm_monoid_add*} ⇒
'b :: {*t2_space*, *topological_comm_monoid_add*}
assumes (λ*x*. *h* (*f x*)) *summable_on A* ⟷ *f summable_on A*
assumes *h 0 = 0*
assumes ⋀*S*. (*f has_sum S*) *A* ⟹ ((λ*x*. *h* (*f x*)) *has_sum* (*h S*)) *A*
shows *infsum* (λ*x*. *h* (*f x*)) *A* = *h* (*infsum f A*)
by (*metis assms has_sum_infsum infsumI infsum_not_exists*)

lemma *has_sum_bounded_linear*:
assumes *bounded_linear h* **and** (*f has_sum S*) *A*
shows ((λ*x*. *h* (*f x*)) *has_sum h S*) *A*
proof −
interpret *bounded_linear h* **by** *fact*
from *assms*(*2*) **show** *?thesis*
by (*rule has_sum_homomorphism*) (*auto simp*: *add intro*!: *continuous_on*)
qed

lemma *summable_on_bounded_linear*:
assumes *bounded_linear h* **and** *f summable_on A*
shows (λ*x*. *h* (*f x*)) *summable_on A*
by (*metis assms has_sum_bounded_linear summable_on_def*)

lemma *summable_on_bounded_linear_iff*:
assumes *bounded_linear h* **and** *bounded_linear h'* **and** ⋀*x*. *h'* (*h x*) = *x*
shows (λ*x*. *h* (*f x*)) *summable_on A* ⟷ *f summable_on A*
by (*metis* (*full_types*) *assms summable_on_bounded_linear summable_on_cong*)

lemma *infsum_bounded_linear_strong*:
fixes $h :: 'a :: real\_normed\_vector \Rightarrow 'b :: real\_normed\_vector$
assumes $(\lambda x.\ h\ (f\ x))\ summable\_on\ A \longleftrightarrow f\ summable\_on\ A$
assumes *bounded_linear h*
shows $infsum\ (\lambda x.\ h\ (f\ x))\ A = h\ (infsum\ f\ A)$
proof −
interpret *bounded_linear h* **by** *fact*
show *?thesis*
by (*rule infsum_homomorphism_strong*)
(*insert assms*, *auto intro*: *add continuous_on has_sum_bounded_linear*)
qed

lemma *infsum_bounded_linear_strong'*:
fixes $mult :: 'c :: zero \Rightarrow 'a :: real\_normed\_vector \Rightarrow 'b :: real\_normed\_vector$
assumes $c \neq 0 \Longrightarrow (\lambda x.\ mult\ c\ (f\ x))\ summable\_on\ A \longleftrightarrow f\ summable\_on\ A$
assumes *bounded_linear* (*mult c*)
assumes [*simp*]: $\bigwedge x.\ mult\ 0\ x = 0$
shows $infsum\ (\lambda x.\ mult\ c\ (f\ x))\ A = mult\ c\ (infsum\ f\ A)$
by (*metis assms infsum_0 infsum_bounded_linear_strong*)

lemma *has_sum_of_nat*: $(f\ has\_sum\ S)\ A \Longrightarrow ((\lambda x.\ of\_nat\ (f\ x))\ has\_sum\ of\_nat\ S)\ A$
by (*erule has_sum_homomorphism*) (*auto intro*!: *continuous_intros*)

lemma *has_sum_of_int*: $(f\ has\_sum\ S)\ A \Longrightarrow ((\lambda x.\ of\_int\ (f\ x))\ has\_sum\ of\_int\ S)\ A$
by (*erule has_sum_homomorphism*) (*auto intro*!: *continuous_intros*)

lemma *summable_on_of_nat*: $f\ summable\_on\ A \Longrightarrow (\lambda x.\ of\_nat\ (f\ x))\ summable\_on\ A$
by (*erule summable_on_homomorphism*) (*auto intro*!: *continuous_intros*)

lemma *summable_on_of_int*: $f\ summable\_on\ A \Longrightarrow (\lambda x.\ of\_int\ (f\ x))\ summable\_on\ A$
by (*erule summable_on_homomorphism*) (*auto intro*!: *continuous_intros*)

lemma *summable_on_discrete_iff*:
fixes $f :: 'a \Rightarrow 'b :: \{discrete\_topology,\ topological\_comm\_monoid\_add,\ cancel\_comm\_monoid\_add\}$
shows $f\ summable\_on\ A \longleftrightarrow finite\ \{x{\in}A.\ f\ x \neq 0\}$
proof
assume ∗: $finite\ \{x{\in}A.\ f\ x \neq 0\}$
hence $f\ summable\_on\ \{x{\in}A.\ f\ x \neq 0\}$
by (*rule summable_on_finite*)
then show $f\ summable\_on\ A$
by (*smt* (*verit*) *DiffE mem_Collect_eq summable_on_cong_neutral*)
next
assume $f\ summable\_on\ A$
then obtain S **where** $(f\ has\_sum\ S)\ A$

```
    by (auto simp: summable_on_def)
  hence ∀F x in finite_subsets_at_top A. sum f x = S
    unfolding has_sum_def tendsto_discrete .
  then obtain X where X: finite X X ⊆ A ⋀Y. finite Y ⟹ X ⊆ Y ⟹ Y ⊆
A ⟹ sum f Y = S
    unfolding eventually_finite_subsets_at_top by metis
  have {x∈A. f x ≠ 0} ⊆ X
  proof
    fix x assume x: x ∈ {x∈A. f x ≠ 0}
    show x ∈ X
    proof (rule ccontr)
      assume [simp]: x ∉ X
      have sum f (insert x X) = S
        using X x by (intro X) auto
      then have f x = 0
        using X by auto
      with x show False
        by auto
    qed
  qed
  thus finite {x∈A. f x ≠ 0}
    using X(1) finite_subset by blast
qed

lemma has_sum_imp_sums: (f has_sum S) (UNIV :: nat set) ⟹ f sums S
  unfolding sums_def has_sum_def by (rule filterlim_compose[OF _ filterlim_lessThan_at_top])

lemma summable_on_imp_summable: f summable_on (UNIV :: nat set) ⟹
summable f
  unfolding summable_on_def summable_def by (auto dest: has_sum_imp_sums)

lemma summable_on_UNIV_nonneg_real_iff:
  assumes ⋀n. f n ≥ (0 :: real)
  shows   f summable_on UNIV ⟷ summable f
  using assms by (auto intro: norm_summable_imp_summable_on summable_on_imp_summable)

lemma summable_on_imp_bounded_partial_sums:
  fixes f :: _ ⇒ 'a :: {topological_comm_monoid_add, linorder_topology}
  assumes f: f summable_on A
  shows   ∃C. eventually (λX. sum f X ≤ C) (finite_subsets_at_top A)
proof -
  from assms obtain S where S: (sum f ⟶⟶ S) (finite_subsets_at_top A)
    unfolding summable_on_def has_sum_def by blast
  show ?thesis
  proof (cases ∃C. C > S)
    case True
    then obtain C where C: C > S
      by blast
    have ∀F X in finite_subsets_at_top A. sum f X < C
```

using *S C* **by** (*rule order_tendstoD*(*2*))
thus *?thesis*
by (*meson eventually_mono nless_le*)
next
case *False* **thus** *?thesis*
by (*meson not_eventuallyD not_le_imp_less*)
qed
qed

lemma *has_sum_mono'*:
fixes *S S'* :: *'a* :: {*linorder_topology*, *ordered_comm_monoid_add*, *topological_comm_monoid_add*}
assumes *f*: (*f has_sum S*) *A* (*f has_sum S'*) *B*
and *AB*: $A \subseteq B$ $\bigwedge x.\ x \in B - A \Longrightarrow f\ x \geq 0$
shows $S \leq S'$
using *AB has_sum_mono_neutral*[*OF f*] **by** *fastforce*

context
assumes *SORT_CONSTRAINT*(*'a* :: {*topological_comm_monoid_add*, *order_topology*, *ordered_comm_monoid_add*, *conditionally_complete_linorder*})
begin

Any family of non-negative numbers with bounded partial sums is summable, and the sum is simply the supremum of the partial sums.

lemma *nonneg_bounded_partial_sums_imp_has_sum_SUP*:
assumes *nonneg*: $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq (0::'a)$
and *bound*: *eventually* ($\lambda X.$ *sum f X* $\leq C$) (*finite_subsets_at_top A*)
shows (*f has_sum* (*SUP* $X \in \{X.\ X \subseteq A \wedge$ *finite X*$\}$. *sum f X*)) *A*
proof −
from *bound* **obtain** *X0*
where *X0*: *X0* $\subseteq$ *A finite X0* $\bigwedge X.\ X0 \subseteq X \Longrightarrow X \subseteq A \Longrightarrow$ *finite X* $\Longrightarrow$ *sum f X* $\leq C$
by (*force simp*: *eventually_finite_subsets_at_top*)
have *bound'*: *sum f X* $\leq C$ **if** $X \subseteq A$ *finite X* **for** *X*
proof −
have *sum f X* $\leq$ *sum f* ($X \cup$ *X0*)
using *that X0 assms*(*1*) **by** (*intro sum_mono2*) *auto*
also have ... $\leq C$
by (*simp add*: *X0 that*)
finally show *?thesis* **.**
qed
hence *bdd*: *bdd_above* (*sum f* ' {*X*. $X \subseteq A \wedge$ *finite X*})
by (*auto simp*: *bdd_above_def*)

show *?thesis* **unfolding** *has_sum_def*
proof (*rule increasing_tendsto*)
show $\forall_F$ *X in finite_subsets_at_top A. sum f X* $\leq$ *Sup* (*sum f* ' {*X*. $X \subseteq A \wedge$ *finite X*})

```
    by (intro eventually_finite_subsets_at_top_weakI cSUP_upper[OF _ bdd])
auto
  next
    fix y assume y < Sup (sum f ` {X. X ⊆ A ∧ finite X})
    then obtain X where X: X ⊆ A finite X y < sum f X
      by (subst (asm) less_cSUP_iff[OF _ bdd]) auto
   from X have eventually (λX'. X ⊆ X' ∧ X' ⊆ A ∧ finite X') (finite_subsets_at_top
A)
      by (auto simp: eventually_finite_subsets_at_top)
    thus eventually (λX'. y < sum f X') (finite_subsets_at_top A)
    proof eventually_elim
      case (elim X')
      note ‹y < sum f X›
      also have sum f X ≤ sum f X'
        using nonneg elim by (intro sum_mono2) auto
      finally show ?case .
    qed
  qed
qed

lemma nonneg_bounded_partial_sums_imp_summable_on:
  assumes nonneg: ⋀x. x ∈ A ⟹ f x ≥ (0::'a)
      and bound:  eventually (λX. sum f X ≤ C) (finite_subsets_at_top A)
  shows   f summable_on A
  using nonneg_bounded_partial_sums_imp_has_sum_SUP[OF assms] by (auto
simp: summable_on_def)

end

context
 assumes SORT_CONSTRAINT('a :: {topological_comm_monoid_add, linorder_topology,
          ordered_comm_monoid_add, conditionally_complete_linorder})
begin

lemma summable_on_comparison_test:
  assumes f summable_on A and ⋀x. x ∈ A ⟹ g x ≤ f x and ⋀x. x ∈ A ⟹
(0::'a) ≤ g x
  shows   g summable_on A
proof −
  obtain C where C: ∀F X in finite_subsets_at_top A. sum f X ≤ C
    using assms(1) summable_on_imp_bounded_partial_sums by blast
  show ?thesis
  proof (rule nonneg_bounded_partial_sums_imp_summable_on)
    show ∀F X in finite_subsets_at_top A. sum g X ≤ C
      using C assms
      unfolding eventually_finite_subsets_at_top
      by (smt (verit, ccfv_SIG) order_trans subsetD sum_mono)
  qed (use assms in auto)
qed
```

end

lemma *summable_on_subset*:
fixes $f :: \_ \Rightarrow 'a :: \{uniform\_topological\_group\_add, topological\_comm\_monoid\_add, ab\_group\_add, complete\_uniform\_space\}$
assumes f *summable_on* A $B \subseteq A$
shows f *summable_on* B
by (*rule summable_on_subset_aux*[*OF* _ _ *assms*]) (*auto simp*: *uniformly_continuous_add*)

lemma *summable_on_union*:
fixes $f :: \_ \Rightarrow 'a :: \{uniform\_topological\_group\_add, topological\_comm\_monoid\_add, ab\_group\_add, complete\_uniform\_space\}$
assumes f *summable_on* A f *summable_on* B
shows f *summable_on* $(A \cup B)$
proof −
have f *summable_on* $(A \cup (B - A))$
by (*meson Diff_disjoint Diff_subset assms summable_on_Un_disjoint summable_on_subset*)
also have $A \cup (B - A) = A \cup B$
by *blast*
finally show *?thesis* **.**
qed

lemma *summable_on_insert_iff*:
fixes $f :: \_ \Rightarrow 'a :: \{uniform\_topological\_group\_add, topological\_comm\_monoid\_add, ab\_group\_add, complete\_uniform\_space\}$
shows f *summable_on insert* x $A \longleftrightarrow f$ *summable_on* A
using *summable_on_union*[*of* f A $\{x\}$] **by** (*auto intro*: *summable_on_subset*)

lemma *has_sum_finiteI*: *finite* $A \Longrightarrow S = sum\ f\ A \Longrightarrow (f\ has\_sum\ S)\ A$
by *simp*

lemma *has_sum_insert*:
fixes $f :: 'a \Rightarrow 'b :: topological\_comm\_monoid\_add$
assumes $x \notin A$ **and** $(f\ has\_sum\ S)\ A$
shows $(f\ has\_sum\ (f\ x + S))\ (insert\ x\ A)$
proof −
have $(f\ has\_sum\ (f\ x + S))\ (\{x\} \cup A)$
using *assms* **by** (*intro has_sum_Un_disjoint*) (*auto intro*: *has_sum_finiteI*)
thus *?thesis* **by** *simp*
qed

lemma *infsum_insert*:
fixes $f :: \_ \Rightarrow 'a :: \{topological\_comm\_monoid\_add, t2\_space\}$
assumes f *summable_on* A $a \notin A$
shows $infsum\ f\ (insert\ a\ A) = f\ a + infsum\ f\ A$
by (*meson assms has_sum_insert infsumI summable_iff_has_sum_infsum*)

lemma *has_sum_SigmaD*:
fixes $f :: 'b \times 'c \Rightarrow 'a :: \{topological\_comm\_monoid\_add, t3\_space\}$
assumes *sum1*: $(f\ has\_sum\ S)\ (Sigma\ A\ B)$
assumes *sum2*: $\bigwedge x.\ x \in A \Longrightarrow ((\lambda y.\ f\ (x,\ y))\ has\_sum\ g\ x)\ (B\ x)$
shows $(g\ has\_sum\ S)\ A$
unfolding *has_sum_def tendsto_def eventually_finite_subsets_at_top*
proof (*safe*, *goal_cases*)
case (*1 X*)
with *nhds_closed*[*of S X*] **obtain** X'
where X': $S \in X'$ *closed* X' $X' \subseteq X$ *eventually* $(\lambda y.\ y \in X')$ (*nhds S*) **by** *blast*
from $X'(4)$ **obtain** X'' **where** X'': $S \in X''$ *open* X'' $X'' \subseteq X'$
by (*auto simp*: *eventually_nhds*)
with *sum1* **obtain** $Y :: ('b \times 'c)\ set$
where Y: $Y \subseteq Sigma\ A\ B$ *finite* Y
$\bigwedge Z.\ Y \subseteq Z \Longrightarrow Z \subseteq Sigma\ A\ B \Longrightarrow finite\ Z \Longrightarrow sum\ f\ Z \in X''$
unfolding *has_sum_def tendsto_def eventually_finite_subsets_at_top* **by** *force*
define $Y1 :: 'b\ set$ **where** $Y1 = fst\ `\ Y$
from Y **have** $Y1$: $Y1 \subseteq A$ **by** (*auto simp*: *Y1_def*)
define $Y2 :: 'b \Rightarrow 'c\ set$ **where** $Y2 = (\lambda x.\ \{y.\ (x,\ y) \in Y\})$
have *Y2*: *finite* $(Y2\ x)$ $Y2\ x \subseteq B\ x$ **if** $x \in A$ **for** x
using *that* $Y(1,2)$ **unfolding** *Y2_def*
by (*force simp*: *image_iff intro*: *finite_subset*[*of _ snd `* Y])+

show *?case*
proof (*rule exI*[*of _ Y1*], *safe*, *goal_cases*)
case (*3 Z*)
define H **where** $H = (INF\ x{\in}Z.\ filtercomap\ (\lambda p.\ p\ x)\ (finite\_subsets\_at\_top\ (B\ x)))$

have $sum\ g\ Z \in X'$
proof (*rule Lim_in_closed_set*)
show *closed* X' **by** *fact*
next
show $((\lambda B'.\ sum\ (\lambda x.\ sum\ (\lambda y.\ f\ (x,\ y))\ (B'\ x))\ Z) \longrightarrow sum\ g\ Z)\ H$
unfolding *H_def*
proof (*intro tendsto_sum filterlim_INF'*)
fix x **assume** x: $x \in Z$
with *3* **have** $x \in A$ **by** *auto*
from *sum2*[*OF this*] **have** $(sum\ (\lambda y.\ f\ (x,\ y)) \longrightarrow g\ x)\ (finite\_subsets\_at\_top\ (B\ x))$
by (*simp add*: *has_sum_def*)
thus $((\lambda B'.\ sum\ (\lambda y.\ f\ (x,\ y))\ (B'\ x)) \longrightarrow g\ x)$
$(filtercomap\ (\lambda p.\ p\ x)\ (finite\_subsets\_at\_top\ (B\ x)))$
by (*rule filterlim_compose*[*OF _ filterlim_filtercomap*])
qed *auto*
next
show $\forall_F\ h\ in\ H.\ sum\ (\lambda x.\ sum\ (\lambda y.\ f\ (x,\ y))\ (h\ x))\ Z \in X'$

```
      unfolding H_def
    proof (subst eventually_INF_finite[OF ‹finite Z›], rule exI, safe)
      fix x assume x: x ∈ Z
      hence x': x ∈ A using 3 by auto
      show eventually (λh. finite (h x) ∧ Y2 x ⊆ h x ∧ h x ⊆ B x)
            (filtercomap (λp. p x) (finite_subsets_at_top (B x))) using 3 Y2[OF
x']
        by (intro eventually_filtercomapI)
          (auto simp: eventually_finite_subsets_at_top intro: exI[of _ Y2 x])
    next
      fix h
      assume *: ∀x∈Z. finite (h x) ∧ Y2 x ⊆ h x ∧ h x ⊆ B x
      hence sum (λx. sum (λy. f (x, y)) (h x)) Z = sum f (Sigma Z h)
        using ‹finite Z› by (subst sum.Sigma) auto
      also have ... ∈ X''
        using * 3 Y(1,2) by (intro Y; force simp: Y1_def Y2_def)
      also have X'' ⊆ X' by fact
      finally show sum (λx. sum (λy. f (x, y)) (h x)) Z ∈ X' .
    qed
  next
    have H = (INF x∈SIGMA x:Z. {X. finite X ∧ X ⊆ B x}.
                principal {y. finite (y (fst x)) ∧ snd x ⊆ y (fst x) ∧ y (fst x) ⊆ B
(fst x)})
    unfolding H_def finite_subsets_at_top_def filtercomap_INF filtercomap_principal
      by (simp add: INF_Sigma)
    also have ... ≠ bot
    proof (rule INF_filter_not_bot, subst INF_principal_finite, goal_cases)
      case (2 X)
      define H' where
        H' = (⋂x∈X. {y. finite (y (fst x)) ∧ snd x ⊆ y (fst x) ∧ y (fst x) ⊆ B
(fst x)})
      from 2 have (λx. ⋃(y,Y)∈X. if x = y then Y else {}) ∈ H'
        by (force split: if_splits simp: H'_def)
      hence H' ≠ {} by blast
      thus principal H' ≠ bot by (simp add: principal_eq_bot_iff)
    qed
    finally show H ≠ bot .
  qed
  also have X' ⊆ X by fact
  finally show sum g Z ∈ X .
 qed (insert Y(1,2), auto simp: Y1_def)
qed

lemma has_sum_unique:
 fixes f :: _ ⇒ 'a :: {topological_comm_monoid_add, t2_space}
 assumes (f has_sum x) A (f has_sum y) A
 shows x = y
 using assms unfolding has_sum_def using tendsto_unique finite_subsets_at_top_neq_bot
by blast
```

lemma *has_sum_SigmaI*:
fixes $f :: \_ \Rightarrow 'a :: \{topological\_comm\_monoid\_add, t3\_space\}$
assumes *f*: $\bigwedge x.\ x \in A \Longrightarrow ((\lambda y.\ f\ (x,\ y))\ has\_sum\ g\ x)\ (B\ x)$
assumes *g*: $(g\ has\_sum\ S)\ A$
assumes *summable*: *f summable_on Sigma A B*
shows $(f\ has\_sum\ S)\ (Sigma\ A\ B)$
by (*metis f g has_sum_SigmaD has_sum_infsum has_sum_unique local.summable*)

lemma *summable_on_SigmaD1*:
fixes $f :: \_ \Rightarrow \_ \Rightarrow 'a :: \{complete\_uniform\_space, uniform\_topological\_group\_add, ab\_group\_add, topological\_comm\_monoid\_add\}$
assumes *f*: $(\lambda(x,y).\ f\ x\ y)$ *summable_on Sigma A B*
assumes *x*: $x \in A$
shows *f x summable_on B x*
proof −
have $(\lambda(x,y).\ f\ x\ y)$ *summable_on Sigma* $\{x\}$ *B*
using *f* **by** (*rule summable_on_subset*) (*use x* **in** *auto*)
also have *?this* $\longleftrightarrow$ $((\lambda y.\ f\ x\ y) \circ snd)$ *summable_on Sigma* $\{x\}$ *B*
by (*intro summable_on_cong*) *auto*
also have ... $\longleftrightarrow$ $(\lambda y.\ f\ x\ y)$ *summable_on snd ‘ Sigma* $\{x\}$ *B*
by (*intro summable_on_reindex* [*symmetric*] *inj_onI*) *auto*
also have *snd ‘ Sigma* $\{x\}$ $B = B\ x$
by (*force simp*: *Sigma_def*)
finally show *?thesis* **.**
qed

lemma *has_sum_swap*:
$(f\ has\_sum\ S)\ (A \times B) \longleftrightarrow ((\lambda(x,y).\ f\ (y,x))\ has\_sum\ S)\ (B \times A)$
proof −
have *bij_betw* $(\lambda(x,y).\ (y,x))\ (B \times A)\ (A \times B)$
by (*rule bij_betwI*[*of* _ _ _ $\lambda(x,y).\ (y,x)$]) *auto*
from *has_sum_reindex_bij_betw*[*OF this*, **where** $f = f$] **show** *?thesis*
by (*simp add*: *case_prod_unfold*)
qed

lemma *summable_on_swap*:
f summable_on $(A \times B) \longleftrightarrow (\lambda(x,y).\ f\ (y,x))$ *summable_on* $(B \times A)$
by (*metis has_sum_swap summable_on_def*)

lemma *has_sum_cmult_right_iff*:
fixes $c :: 'a :: \{topological\_semigroup\_mult, field\}$
assumes $c \neq 0$
shows $((\lambda x.\ c * f\ x)\ has\_sum\ S)\ A \longleftrightarrow (f\ has\_sum\ (S\ /\ c))\ A$
using *has_sum_cmult_right*[*of f A S*/*c c*]
has_sum_cmult_right[*of* $\lambda x.\ c * f\ x$ *A S inverse c*] *assms*
by (*auto simp*: *field_simps*)

lemma *has_sum_cmult_left_iff*:
fixes $c :: 'a :: \{topological\_semigroup\_mult, field\}$
assumes $c \neq 0$
shows $((\lambda x.\ f\ x * c)\ has\_sum\ S)\ A \longleftrightarrow (f\ has\_sum\ (S\ /\ c))\ A$
by (*smt* (*verit*, *best*) *assms has_sum_cmult_right_iff has_sum_cong mult.commute*)

lemma *finite_nonzero_values_imp_summable_on*:
assumes *finite* $\{x \in X.\ f\ x \neq 0\}$
shows *f summable_on X*
by (*smt* (*verit*, *del_insts*) *Diff_iff assms mem_Collect_eq summable_on_cong_neutral summable_on_finite*)

lemma *summable_on_of_int_iff*:
$(\lambda x{::}'a.\ of\_int\ (f\ x) :: 'b :: real\_normed\_algebra\_1)\ summable\_on\ A \longleftrightarrow f\ summable\_on\ A$
proof
assume *f summable_on A*
thus $(\lambda x.\ of\_int\ (f\ x))\ summable\_on\ A$
by (*rule summable_on_homomorphism*) *auto*
next
assume $(\lambda x.\ of\_int\ (f\ x) :: 'b)\ summable\_on\ A$
then obtain *S* **where** $((\lambda x.\ of\_int\ (f\ x) :: 'b)\ has\_sum\ S)\ A$
by (*auto simp*: *summable_on_def*)
hence $(sum\ (\lambda x.\ of\_int\ (f\ x) :: 'b) \longrightarrow S)\ (finite\_subsets\_at\_top\ A)$
unfolding *has_sum_def* **.**
moreover have $1/2 > (0 :: real)$
by *auto*
ultimately have *eventually* $(\lambda X.\ dist\ (sum\ (\lambda x.\ of\_int\ (f\ x) :: 'b)\ X)\ S < 1/2)$
$(finite\_subsets\_at\_top\ A)$
unfolding *tendsto_iff* **by** *blast*
then obtain *X* **where** *X*: *finite X* $X \subseteq A$
$\bigwedge Y.\ finite\ Y \Longrightarrow X \subseteq Y \Longrightarrow Y \subseteq A \Longrightarrow dist\ (sum\ (\lambda x.\ of\_int\ (f\ x))\ Y)\ S < 1/2$
unfolding *eventually_finite_subsets_at_top* **by** *metis*

have *sum f Y* = *sum f X* **if** *finite Y X* $\subseteq$ *Y Y* $\subseteq$ *A* **for** *Y*
proof −
have $dist\ (sum\ (\lambda x.\ of\_int\ (f\ x))\ X)\ S < 1/2$
by (*intro X*) *auto*
moreover have $dist\ (sum\ (\lambda x.\ of\_int\ (f\ x))\ Y)\ S < 1/2$
by (*intro X that*)
ultimately have $dist\ (sum\ (\lambda x.\ of\_int\ (f\ x))\ X)\ (sum\ (\lambda x.\ of\_int\ (f\ x) :: 'b)\ Y) <$
$1/2 + 1/2$
using *dist_triangle_less_add* **by** *blast*
thus *?thesis*
by (*simp add*: *dist_norm flip*: *of_int_sum of_int_diff*)
qed
then have $\{x \in A.\ f\ x \neq 0\} \subseteq X$

by (*smt* (*verit*) *X finite_insert insert_iff mem_Collect_eq subset_eq sum.insert*)
with ‹*finite X*› **have** *finite* {*x*∈*A*. *f x* ≠ *0*}
using *finite_subset* **by** *blast*
thus *f summable_on A*
by (*rule finite_nonzero_values_imp_summable_on*)
qed

lemma *summable_on_of_nat_iff*:
(λ*x*::′*a*. *of_nat* (*f x*) :: ′*b* :: *real_normed_algebra_1*) *summable_on A* ⟷ *f summable_on A*
proof
assume *f summable_on A*
thus (λ*x*. *of_nat* (*f x*) :: ′*b*) *summable_on A*
by (*rule summable_on_homomorphism*) *auto*
next
assume (λ*x*. *of_nat* (*f x*) :: ′*b*) *summable_on A*
hence (λ*x*. *of_int* (*int* (*f x*)) :: ′*b*) *summable_on A*
by *simp*
also have *?this* ⟷ (λ*x*. *int* (*f x*)) *summable_on A*
by (*rule summable_on_of_int_iff*)
also have ... ⟷ *f summable_on A*
by (*simp add*: *summable_on_discrete_iff*)
finally show *f summable_on A* **.**
qed

lemma *infsum_of_nat*:
infsum (λ*x*::′*a*. *of_nat* (*f x*) :: ′*b* :: {*real_normed_algebra_1*}) *A* = *of_nat* (*infsum f A*)
by (*metis has_sum_infsum has_sum_of_nat infsumI infsum_def of_nat_0 summable_on_of_nat_iff*)

lemma *infsum_of_int*:
infsum (λ*x*::′*a*. *of_int* (*f x*) :: ′*b* :: {*real_normed_algebra_1*}) *A* = *of_int* (*infsum f A*)
by (*metis has_sum_infsum has_sum_of_int infsumI infsum_not_exists of_int_0 summable_on_of_int_iff*)

lemma *summable_on_SigmaI*:
fixes *f* :: _ ⇒ ′*a* :: {*linorder_topology*, *ordered_comm_monoid_add*, *topological_comm_monoid_add*,
conditionally_complete_linorder}
assumes *f*: ⋀*x*. *x* ∈ *A* ⟹ ((λ*y*. *f* (*x*, *y*)) *has_sum g x*) (*B x*)
assumes *g*: *g summable_on A*
assumes *f_nonneg*: ⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *B x* ⟹ *f* (*x*, *y*) ≥ (*0* :: ′*a*)
shows *f summable_on Sigma A B*
proof −
have *g_nonneg*: *g x* ≥ *0* **if** *x* ∈ *A* **for** *x*
using *f* **by** (*rule has_sum_nonneg*) (*use f_nonneg that* **in** *auto*)

```
  obtain C where C: eventually (λX. sum g X ≤ C) (finite_subsets_at_top A)
    using summable_on_imp_bounded_partial_sums[OF g] by blast

  have sum_g_le: sum g X ≤ C if X: finite X X ⊆ A for X
  proof −
    from C obtain X' where X':
      finite X' X' ⊆ A ⋀Y. finite Y ⟹ X' ⊆ Y ⟹ Y ⊆ A ⟹ sum g Y ≤ C
      unfolding eventually_finite_subsets_at_top by metis
    have sum g X ≤ sum g (X ∪ X')
      using X X' by (intro sum_mono2 g_nonneg) auto
    also have ... ≤ C
      using X X'(1,2) by (intro X'(3)) auto
    finally show ?thesis .
  qed

  have sum f Y ≤ C if Y: finite Y Y ⊆ Sigma A B for Y
  proof −
    define Y1 and Y2 where Y1 = fst ‘ Y and Y2 = (λx. snd ‘ {z∈Y. fst z =
x})
    have Y12: Y = Sigma Y1 Y2
      unfolding Y1_def Y2_def by force
    have [intro]: finite Y1 ⋀x. x ∈ Y1 ⟹ finite (Y2 x)
      using Y unfolding Y1_def Y2_def by auto
    have Y12_subset: Y1 ⊆ A ⋀x. Y2 x ⊆ B x
      using Y by (auto simp: Y1_def Y2_def)

    have sum f Y = sum f (Sigma Y1 Y2)
      by (simp add: Y12)
    also have ... = (∑ x∈Y1. ∑ y∈Y2 x. f (x, y))
      by (subst sum.Sigma) auto
    also have ... ≤ (∑ x∈Y1. g x)
    proof (rule sum_mono)
      fix x assume x: x ∈ Y1
      show (∑ y∈Y2 x. f (x, y)) ≤ g x
      proof (rule has_sum_mono')
        show ((λy. f (x, y)) has_sum (∑ y∈Y2 x. f (x, y))) (Y2 x)
          using x by (intro has_sum_finite) auto
        show ((λy. f (x, y)) has_sum g x) (B x)
          by (rule f) (use x Y12_subset in auto)
        show f (x, y) ≥ 0 if y ∈ B x − Y2 x for y
          using x that Y12_subset by (intro f_nonneg) auto
      qed (use Y12_subset in auto)
    qed
    also have ... ≤ C
      using Y12_subset by (intro sum_g_le) auto
    finally show ?thesis .
  qed

  hence ∀F X in finite_subsets_at_top (Sigma A B). sum f X ≤ C
```

unfolding *eventually_finite_subsets_at_top* **by** *auto*
thus *?thesis*
by (*metis SigmaE f_nonneg nonneg_bounded_partial_sums_imp_summable_on*)
qed

lemma *summable_on_UnionI*:
fixes $f :: \_ \Rightarrow 'a :: \{$*linorder_topology*, *ordered_comm_monoid_add*, *topological_comm_monoid_add*,
conditionally_complete_linorder$\}$
assumes *f*: $\bigwedge x.\ x \in A \Longrightarrow (f\ \text{has\_sum}\ g\ x)\ (B\ x)$
assumes *g*: *g summable_on A*
assumes *f_nonneg*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in B\ x \Longrightarrow f\ y \geq (0 :: 'a)$
assumes *disj*: *disjoint_family_on B A*
shows *f summable_on* $(\bigcup x{\in}A.\ B\ x)$
proof −
have *f* ∘ *snd summable_on Sigma A B*
using *assms* **by** (*intro summable_on_SigmaI*[**where** $g = g$]) *auto*
also have *?this* ⟷ *f summable_on* (*snd* ‘ *Sigma A B*) **using** *assms*
by (*subst summable_on_reindex*; *force simp*: *disjoint_family_on_def inj_on_def*)
also have *snd* ‘ (*Sigma A B*) = $(\bigcup x{\in}A.\ B\ x)$
by *force*
finally show *?thesis* **.**
qed

lemma *summable_on_SigmaD*:
fixes $f :: 'a \times 'b \Rightarrow 'c :: \{$*topological_comm_monoid_add*,*t3_space*$\}$
assumes *sum1*: *f summable_on* (*Sigma A B*)
assumes *sum2*: $\bigwedge x.\ x \in A \Longrightarrow (\lambda y.\ f\ (x,\ y))$ *summable_on* $(B\ x)$
shows $(\lambda x.\ \text{infsum}\ (\lambda y.\ f\ (x,\ y))\ (B\ x))$ *summable_on A*
using *assms* **unfolding** *summable_on_def*
by (*smt* (*verit*, *del_insts*) *assms has_sum_SigmaD has_sum_cong has_sum_infsum*)

lemma *summable_on_UnionD*:
fixes $f :: 'a \Rightarrow 'c :: \{$*topological_comm_monoid_add*,*t3_space*$\}$
assumes *sum1*: *f summable_on* $(\bigcup x{\in}A.\ B\ x)$
assumes *sum2*: $\bigwedge x.\ x \in A \Longrightarrow f$ *summable_on* $(B\ x)$
assumes *disj*: *disjoint_family_on B A*
shows $(\lambda x.\ \text{infsum}\ f\ (B\ x))$ *summable_on A*
proof −
have $(\bigcup x{\in}A.\ B\ x)$ = *snd* ‘ *Sigma A B*
by (*force simp*: *Sigma_def*)
with *sum1* **have** *f summable_on* (*snd* ‘ *Sigma A B*)
by *simp*
also have *?this* ⟷ (*f* ∘ *snd*) *summable_on* (*Sigma A B*)
using *disj* **by** (*intro summable_on_reindex inj_onI*) (*force simp*: *disjoint_family_on_def*)
finally show $(\lambda x.\ \text{infsum}\ f\ (B\ x))$ *summable_on A*
using *summable_on_SigmaD*[*of f* ∘ *snd A B*] *sum2* **by** *simp*
qed

lemma *summable_on_Union_iff*:
fixes $f :: \_ \Rightarrow 'a :: \{linorder\_topology, ordered\_comm\_monoid\_add, topological\_comm\_monoid\_add,$
$conditionally\_complete\_linorder, t3\_space\}$
assumes f: $\bigwedge x.\ x \in A \Longrightarrow (f\ has\_sum\ g\ x)\ (B\ x)$
assumes *f_nonneg*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in B\ x \Longrightarrow f\ y \geq 0$
assumes *disj*: *disjoint_family_on B A*
shows $f\ summable\_on\ (\bigcup x{\in}A.\ B\ x) \longleftrightarrow g\ summable\_on\ A$
proof
assume *g summable_on A*
thus $f\ summable\_on\ (\bigcup x{\in}A.\ B\ x)$
using *summable_on_UnionI*[*of A f B g*] *assms* **by** *auto*
next
assume $f\ summable\_on\ (\bigcup x{\in}A.\ B\ x)$
hence $(\lambda x.\ infsum\ f\ (B\ x))\ summable\_on\ A$
using *assms* **by** (*intro summable_on_UnionD*) (*auto dest*: *has_sum_imp_summable*)
also have *?this* $\longleftrightarrow$ *g summable_on A*
using *assms* **by** (*intro summable_on_cong*) (*auto simp*: *infsumI*)
finally show *g summable_on A* **.**
qed

lemma *has_sum_Sigma'*:
fixes $A :: 'a\ set$ **and** $B :: 'a \Rightarrow 'b\ set$
and $f :: \langle 'a \times 'b \Rightarrow 'c :: \{comm\_monoid\_add, uniform\_space, uniform\_topological\_group\_add\}\rangle$
assumes *summableAB*: $(f\ has\_sum\ a)\ (Sigma\ A\ B)$
assumes *summableB*: $\langle \bigwedge x.\ x{\in}A \Longrightarrow ((\lambda y.\ f\ (x, y))\ has\_sum\ (b\ x))\ (B\ x)\rangle$
shows $(b\ has\_sum\ a)\ A$
by (*intro has_sum_Sigma*[*OF* _ *assms*] *uniformly_continuous_add*)

lemma *abs_summable_on_comparison_test'*:
assumes *g summable_on A*
assumes $\bigwedge x.\ x \in A \Longrightarrow norm\ (f\ x) \leq g\ x$
shows $(\lambda x.\ norm\ (f\ x))\ summable\_on\ A$
proof (*rule Infinite_Sum.abs_summable_on_comparison_test*)
have $g\ summable\_on\ A \longleftrightarrow (\lambda x.\ norm\ (g\ x))\ summable\_on\ A$
by (*metis summable_on_iff_abs_summable_on_real*)
with *assms* **show** $(\lambda x.\ norm\ (g\ x))\ summable\_on\ A$ **by** *blast*
qed (*use assms* **in** *fastforce*)

lemma *has_sum_geometric_from_1*:
fixes $z :: 'a :: \{real\_normed\_field, banach\}$
assumes $norm\ z < 1$
shows $((\lambda n.\ z\ \widehat{}\ n)\ has\_sum\ (z\ /\ (1 - z)))\ \{1..\}$
proof −
have [*simp*]: $z \neq 1$
using *assms* **by** *auto*
have $(\lambda n.\ z\ \widehat{}\ Suc\ n)\ sums\ (1\ /\ (1 - z) - 1)$
using *geometric_sums*[*of z*] *assms* **by** (*subst sums_Suc_iff*) *auto*
also have $1\ /\ (1 - z) - 1 = z\ /\ (1 - z)$

by (*auto simp*: *field_simps*)
finally have (λ*n*. *z* ^ *Suc n*) *sums* (*z* / (1 − *z*)) **.**
moreover have *summable* (λ*n*. *norm* (*z* ^ *Suc n*))
using *assms*
by (*subst summable_Suc_iff*) (*auto simp*: *norm_power intro*!: *summable_geometric*)
ultimately have ((λ*n*. *z* ^ *Suc n*) *has_sum* (*z* / (1 − *z*))) *UNIV*
by (*intro norm_summable_imp_has_sum*)
also have *?this* ⟷ *?thesis*
by (*intro has_sum_reindex_bij_witness*[*of* _ λ*n*. *n*−1 λ*n*. *n*+1]) *auto*
finally show *?thesis* **.**
qed

lemma *has_sum_divide_const*:
fixes *f* :: 'a ⇒ 'b :: {*topological_semigroup_mult*, *field*, *semiring_0*}
shows (*f has_sum S*) *A* ⟹ ((λ*x*. *f x* / *c*) *has_sum* (*S* / *c*)) *A*
using *has_sum_cmult_right*[*of f A S inverse c*] **by** (*simp add*: *field_simps*)

lemma *has_sum_uminusI*:
fixes *f* :: 'a ⇒ 'b :: {*topological_semigroup_mult*, *ring_1*}
shows (*f has_sum S*) *A* ⟹ ((λ*x*. −*f x*) *has_sum* (−*S*)) *A*
using *has_sum_cmult_right*[*of f A S* −1] **by** *simp*

end

6.9 Ordered Euclidean Space

theory *Ordered_Euclidean_Space*
imports
Convex_Euclidean_Space Abstract_Limits
HOL−*Library.Product_Order*
begin

An ordering on euclidean spaces that will allow us to talk about intervals

class *ordered_euclidean_space* = *ord* + *inf* + *sup* + *abs* + *Inf* + *Sup* + *euclidean_space* +
assumes *eucl_le*: *x* ≤ *y* ⟷ (∀ *i*∈*Basis*. *x* ∙ *i* ≤ *y* ∙ *i*)
assumes *eucl_less_le_not_le*: *x* < *y* ⟷ *x* ≤ *y* ∧ ¬ *y* ≤ *x*
assumes *eucl_inf*: *inf x y* = (∑ *i*∈*Basis*. *inf* (*x* ∙ *i*) (*y* ∙ *i*) ∗$_R$ *i*)
assumes *eucl_sup*: *sup x y* = (∑ *i*∈*Basis*. *sup* (*x* ∙ *i*) (*y* ∙ *i*) ∗$_R$ *i*)
assumes *eucl_Inf*: *Inf X* = (∑ *i*∈*Basis*. (*INF x*∈*X*. *x* ∙ *i*) ∗$_R$ *i*)
assumes *eucl_Sup*: *Sup X* = (∑ *i*∈*Basis*. (*SUP x*∈*X*. *x* ∙ *i*) ∗$_R$ *i*)
assumes *eucl_abs*: |*x*| = (∑ *i*∈*Basis*. |*x* ∙ *i*| ∗$_R$ *i*)
begin

subclass *order*
by *standard*
(*auto simp*: *eucl_le eucl_less_le_not_le intro*!: *euclidean_eqI antisym intro*: *order.trans*)

subclass *ordered_ab_group_add_abs*
by *standard* (*auto simp*: *eucl_le inner_add_left eucl_abs abs_leI*)

subclass *ordered_real_vector*
by *standard* (*auto simp*: *eucl_le intro*!: *mult_left_mono mult_right_mono*)

subclass *lattice*
by *standard* (*auto simp*: *eucl_inf eucl_sup eucl_le*)

subclass *distrib_lattice*
by *standard* (*auto simp*: *eucl_inf eucl_sup sup_inf_distrib1 intro*!: *euclidean_eqI*)

subclass *conditionally_complete_lattice*
proof
fix $z::'a$ **and** $X::'a$ *set*
assume $X \neq \{\}$
hence $\bigwedge i.\ (\lambda x.\ x \cdot i)\ `\ X \neq \{\}$ **by** *simp*
thus $(\bigwedge x.\ x \in X \Longrightarrow z \leq x) \Longrightarrow z \leq Inf\ X\ (\bigwedge x.\ x \in X \Longrightarrow x \leq z) \Longrightarrow Sup\ X \leq z$
by (*auto simp*: *eucl_Inf eucl_Sup eucl_le*
intro!: *cInf_greatest cSup_least*)
qed (*force intro*!: *cInf_lower cSup_upper*
simp: *bdd_below_def bdd_above_def preorder_class.bdd_below_def preorder_class.bdd_above_def*
eucl_Inf eucl_Sup eucl_le)+

lemma *inner_Basis_inf_left*: $i \in Basis \Longrightarrow inf\ x\ y \cdot i = inf\ (x \cdot i)\ (y \cdot i)$
and *inner_Basis_sup_left*: $i \in Basis \Longrightarrow sup\ x\ y \cdot i = sup\ (x \cdot i)\ (y \cdot i)$
by (*simp_all add*: *eucl_inf eucl_sup inner_sum_left inner_Basis if_distrib*
cong: *if_cong*)

lemma *inner_Basis_INF_left*: $i \in Basis \Longrightarrow (INF\ x{\in}X.\ f\ x) \cdot i = (INF\ x{\in}X.\ f\ x \cdot i)$
and *inner_Basis_SUP_left*: $i \in Basis \Longrightarrow (SUP\ x{\in}X.\ f\ x) \cdot i = (SUP\ x{\in}X.\ f\ x \cdot i)$
using *eucl_Sup* [*of f ` X*] *eucl_Inf* [*of f ` X*] **by** (*simp_all add*: *image_comp*)

lemma *abs_inner*: $i \in Basis \Longrightarrow |x| \cdot i = |x \cdot i|$
by (*auto simp*: *eucl_abs*)

lemma
abs_scaleR: $|a *_R b| = |a| *_R |b|$
by (*auto simp*: *eucl_abs abs_mult intro*!: *euclidean_eqI*)

lemma *interval_inner_leI*:
assumes $x \in \{a\ ..\ b\}\ 0 \leq i$
shows $a{\cdot}i \leq x{\cdot}i\ x{\cdot}i \leq b{\cdot}i$
using *assms*
unfolding *euclidean_inner*[*of a i*] *euclidean_inner*[*of x i*] *euclidean_inner*[*of b i*]

by (*auto intro*!: *ordered_comm_monoid_add_class.sum_mono mult_right_mono simp*: *eucl_le*)

lemma *inner_nonneg_nonneg*:
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a \cdot b$
using *interval_inner_leI*[*of a 0 a b*]
by *auto*

lemma *inner_Basis_mono*:
shows $a \leq b \Longrightarrow c \in Basis \Longrightarrow a \cdot c \leq b \cdot c$
by (*simp add*: *eucl_le*)

lemma *Basis_nonneg*[*intro*, *simp*]: $i \in Basis \Longrightarrow 0 \leq i$
by (*auto simp*: *eucl_le inner_Basis*)

lemma *Sup_eq_maximum_componentwise*:
fixes *s*::*'a set*
assumes *i*: $\bigwedge b.\ b \in Basis \Longrightarrow X \cdot b = i\ b \cdot b$
assumes *sup*: $\bigwedge b\ x.\ b \in Basis \Longrightarrow x \in s \Longrightarrow x \cdot b \leq X \cdot b$
assumes *i_s*: $\bigwedge b.\ b \in Basis \Longrightarrow (i\ b \cdot b) \in (\lambda x.\ x \cdot b)\ `\ s$
shows *Sup s* $= X$
using *assms*
unfolding *eucl_Sup euclidean_representation_sum*
by (*auto intro*!: *conditionally_complete_lattice_class.cSup_eq_maximum*)

lemma *Inf_eq_minimum_componentwise*:
assumes *i*: $\bigwedge b.\ b \in Basis \Longrightarrow X \cdot b = i\ b \cdot b$
assumes *sup*: $\bigwedge b\ x.\ b \in Basis \Longrightarrow x \in s \Longrightarrow X \cdot b \leq x \cdot b$
assumes *i_s*: $\bigwedge b.\ b \in Basis \Longrightarrow (i\ b \cdot b) \in (\lambda x.\ x \cdot b)\ `\ s$
shows *Inf s* $= X$
using *assms*
unfolding *eucl_Inf euclidean_representation_sum*
by (*auto intro*!: *conditionally_complete_lattice_class.cInf_eq_minimum*)

end

proposition *compact_attains_Inf_componentwise*:
fixes *b*::*'a*::*ordered_euclidean_space*
assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
obtains *x* **where** $x \in X\ x \cdot b = Inf\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$
proof *atomize_elim*
let *?proj* $= (\lambda x.\ x \cdot b)\ `\ X$
from *assms* **have** *compact ?proj* $?proj \neq \{\}$
by (*auto intro*!: *compact_continuous_image continuous_intros*)
from *compact_attains_inf*[*OF this*]
obtain *s x*
where *s*: $s \in (\lambda x.\ x \cdot b)\ `\ X\ \bigwedge t.\ t \in (\lambda x.\ x \cdot b)\ `\ X \Longrightarrow s \leq t$
and *x*: $x \in X\ s = x \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$
by *auto*

hence *Inf ?proj = x · b*
by (*auto intro*!: *conditionally_complete_lattice_class.cInf_eq_minimum*)
hence *x · b = Inf X · b*
by (*auto simp*: *eucl_Inf inner_sum_left inner_Basis if_distrib* ‹*b ∈ Basis*›
cong: *if_cong*)
with *x* **show** ∃*x*. *x ∈ X ∧ x · b = Inf X · b ∧* (∀*y*. *y ∈ X ⟶ x · b ≤ y · b*)
by *blast*
qed

proposition
compact_attains_Sup_componentwise:
fixes *b*::*'a*::*ordered_euclidean_space*
assumes *b ∈ Basis* **assumes** *X ≠ {}* *compact X*
obtains *x* **where** *x ∈ X x · b = Sup X · b* ⋀*y*. *y ∈ X ⟹ y · b ≤ x · b*
proof *atomize_elim*
let *?proj* = (λ*x*. *x · b*) ‘ *X*
from *assms* **have** *compact ?proj ?proj ≠ {}*
by (*auto intro*!: *compact_continuous_image continuous_intros*)
from *compact_attains_sup*[*OF this*]
obtain *s x*
where *s*: *s*∈(λ*x*. *x · b*) ‘ *X* ⋀*t*. *t*∈(λ*x*. *x · b*) ‘ *X ⟹ t ≤ s*
and *x*: *x ∈ X s = x · b* ⋀*y*. *y ∈ X ⟹ y · b ≤ x · b*
by *auto*
hence *Sup ?proj = x · b*
by (*auto intro*!: *cSup_eq_maximum*)
hence *x · b = Sup X · b*
by (*auto simp*: *eucl_Sup*[**where** *'a*=*'a*] *inner_sum_left inner_Basis if_distrib*
‹*b ∈ Basis*›
cong: *if_cong*)
with *x* **show** ∃*x*. *x ∈ X ∧ x · b = Sup X · b ∧* (∀*y*. *y ∈ X ⟶ y · b ≤ x · b*)
by *blast*
qed

lemma *tendsto_sup*[*tendsto_intros*]:
fixes *X* :: *'a* ⇒ *'b*::*ordered_euclidean_space*
assumes (*X ⟶ x*) *net* (*Y ⟶ y*) *net*
shows ((λ*i*. *sup* (*X i*) (*Y i*)) ⟶ *sup x y*) *net*
unfolding *sup_max eucl_sup* **by** (*intro assms tendsto_intros*)

lemma *tendsto_inf*[*tendsto_intros*]:
fixes *X* :: *'a* ⇒ *'b*::*ordered_euclidean_space*
assumes (*X ⟶ x*) *net* (*Y ⟶ y*) *net*
shows ((λ*i*. *inf* (*X i*) (*Y i*)) ⟶ *inf x y*) *net*
unfolding *inf_min eucl_inf* **by** (*intro assms tendsto_intros*)

lemma *tendsto_Inf*[*tendsto_intros*]:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c*::*ordered_euclidean_space*
assumes *finite K* ⋀*i*. *i ∈ K ⟹* ((λ*x*. *f x i*) ⟶ *l i*) *F*
shows ((λ*x*. *Inf* (*f x* ‘ *K*)) ⟶ *Inf* (*l* ‘ *K*)) *F*

using *assms*
by (*induction K rule*: *finite_induct*) (*auto simp*: *cInf_insert_If tendsto_inf*)

lemma *tendsto_Sup*[*tendsto_intros*]:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c*::*ordered_euclidean_space*
assumes *finite K* ⋀*i*. *i* ∈ *K* ⟹ ((λ*x*. *f x i*) ⟶ *l i*) *F*
shows ((λ*x*. *Sup* (*f x* ‘ *K*)) ⟶ *Sup* (*l* ‘ *K*)) *F*
using *assms*
by (*induction K rule*: *finite_induct*) (*auto simp*: *cSup_insert_If tendsto_sup*)

lemma *continuous_map_Inf* [*continuous_intros*]:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c*::*ordered_euclidean_space*
assumes *finite K* ⋀*i*. *i* ∈ *K* ⟹ *continuous_map X euclidean* (λ*x*. *f x i*)
shows *continuous_map X euclidean* (λ*x*. *INF i*∈*K*. *f x i*)
using *assms* **by** (*simp add*: *continuous_map_atin tendsto_Inf*)

lemma *continuous_map_Sup* [*continuous_intros*]:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c*::*ordered_euclidean_space*
assumes *finite K* ⋀*i*. *i* ∈ *K* ⟹ *continuous_map X euclidean* (λ*x*. *f x i*)
shows *continuous_map X euclidean* (λ*x*. *SUP i*∈*K*. *f x i*)
using *assms* **by** (*simp add*: *continuous_map_atin tendsto_Sup*)

lemma *tendsto_componentwise_max*:
assumes *f*: (*f* ⟶ *l*) *F* **and** *g*: (*g* ⟶ *m*) *F*
shows ((λ*x*. (∑ *i*∈*Basis*. *max* (*f x* ∙ *i*) (*g x* ∙ *i*) ∗$_R$ *i*)) ⟶ (∑ *i*∈*Basis*. *max* (*l* ∙ *i*) (*m* ∙ *i*) ∗$_R$ *i*)) *F*
by (*intro tendsto_intros assms*)

lemma *tendsto_componentwise_min*:
assumes *f*: (*f* ⟶ *l*) *F* **and** *g*: (*g* ⟶ *m*) *F*
shows ((λ*x*. (∑ *i*∈*Basis*. *min* (*f x* ∙ *i*) (*g x* ∙ *i*) ∗$_R$ *i*)) ⟶ (∑ *i*∈*Basis*. *min* (*l* ∙ *i*) (*m* ∙ *i*) ∗$_R$ *i*)) *F*
by (*intro tendsto_intros assms*)

instance *real* :: *ordered_euclidean_space*
by *standard auto*

lemma *in_Basis_prod_iff*:
fixes *i*::*'a*::*euclidean_space*∗*'b*::*euclidean_space*
shows *i* ∈ *Basis* ⟷ *fst i* = *0* ∧ *snd i* ∈ *Basis* ∨ *snd i* = *0* ∧ *fst i* ∈ *Basis*
by (*cases i*) (*auto simp*: *Basis_prod_def*)

instantiation *prod* :: (*abs*, *abs*) *abs*
begin

definition |*x*| = (|*fst x*|, |*snd x*|)

instance ..

end

instance *prod* :: (*ordered_euclidean_space*, *ordered_euclidean_space*) *ordered_euclidean_space*
by *standard*
(*auto intro*!: *add_mono simp add*: *euclidean_representation_sum'* *Ball_def inner_prod_def*
in_Basis_prod_iff inner_Basis_inf_left inner_Basis_sup_left inner_Basis_INF_left Inf_prod_def
inner_Basis_SUP_left Sup_prod_def less_prod_def less_eq_prod_def eucl_le[**where** *'a='a*]
eucl_le[**where** *'a='b*] *abs_prod_def abs_inner*)

Instantiation for intervals on *ordered_euclidean_space*

proposition
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *cbox_interval*: *cbox a b* = {*a*..*b*}
and *interval_cbox*: {*a*..*b*} = *cbox a b*
and *eucl_le_atMost*: {*x*. ∀ *i*∈*Basis*. *x* ∙ *i* <= *a* ∙ *i*} = {..*a*}
and *eucl_le_atLeast*: {*x*. ∀ *i*∈*Basis*. *a* ∙ *i* <= *x* ∙ *i*} = {*a*..}
by (*auto simp*: *eucl_le*[**where** *'a='a*] *eucl_less_def box_def cbox_def*)

lemma *sums_vec_nth* :
assumes *f sums a*
shows (λ*x*. *f x* $ *i*) *sums a* $ *i*
using *assms* **unfolding** *sums_def*
by (*auto dest*: *tendsto_vec_nth* [**where** *i=i*])

lemma *summable_vec_nth* :
assumes *summable f*
shows *summable* (λ*x*. *f x* $ *i*)
using *assms* **unfolding** *summable_def*
by (*blast intro*: *sums_vec_nth*)

lemma *closed_eucl_atLeastAtMost*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *closed* {*a*..*b*}
by (*simp add*: *cbox_interval*[*symmetric*] *closed_cbox*)

lemma *closed_eucl_atMost*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *closed* {..*a*}
by (*simp add*: *closed_interval_left eucl_le_atMost*[*symmetric*])

lemma *closed_eucl_atLeast*[*simp*, *intro*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *closed* {*a*..}
by (*simp add*: *closed_interval_right eucl_le_atLeast*[*symmetric*])

lemma *bounded_closed_interval* [*simp*]:

fixes *a* :: *'a*::*ordered_euclidean_space*
shows *bounded* {*a* .. *b*}
using *bounded_cbox*[*of a b*]
by (*metis interval_cbox*)

lemma *convex_closed_interval* [*simp*]:
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *convex* {*a* .. *b*}
using *convex_box*[*of a b*]
by (*metis interval_cbox*)

lemma *image_smult_interval*:(λ*x*. *m* $*_R$ (*x*::_::*ordered_euclidean_space*)) ' {*a* .. *b*} =
(*if* {*a* .. *b*} = {} *then* {} *else if* 0 ≤ *m then* {*m* $*_R$ *a* .. *m* $*_R$ *b*} *else* {*m* $*_R$ *b* .. *m* $*_R$ *a*})
using *image_smult_cbox*[*of m a b*]
by (*simp add*: *cbox_interval*)

lemma [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *is_interval_ic*: *is_interval* {..*a*}
and *is_interval_ci*: *is_interval* {*a*..}
and *is_interval_cc*: *is_interval* {*b*..*a*}
by (*force simp*: *is_interval_def eucl_le*[**where** *'a*=*'a*])+

lemma *connected_interval* [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *connected* {*a*..*b*}
using *is_interval_cc is_interval_connected* **by** *blast*

lemma *compact_interval* [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *compact* {*a* .. *b*}
by (*metis compact_cbox interval_cbox*)

no_notation *eucl_less* (**infix** ‹<*e*› *50*)

lemma *One_nonneg*: *0* ≤ (∑ *Basis*::*'a*::*ordered_euclidean_space*)
by (*auto intro*: *sum_nonneg*)

lemma
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *bdd_above_cbox*[*intro*, *simp*]: *bdd_above* (*cbox a b*)
and *bdd_below_cbox*[*intro*, *simp*]: *bdd_below* (*cbox a b*)
and *bdd_above_box*[*intro*, *simp*]: *bdd_above* (*box a b*)
and *bdd_below_box*[*intro*, *simp*]: *bdd_below* (*box a b*)
unfolding *atomize_conj*
by (*metis bdd_above_Icc bdd_above_mono bdd_below_Icc bdd_below_mono bounded_box*

bounded_subset_cbox_symmetric interval_cbox)

instantiation *vec* :: (*ordered_euclidean_space*, *finite*) *ordered_euclidean_space*
begin

definition *inf x y* = (*χ i*. *inf* (*x $ i*) (*y $ i*))
definition *sup x y* = (*χ i*. *sup* (*x $ i*) (*y $ i*))
definition *Inf X* = (*χ i*. (*INF x∈X*. *x $ i*))
definition *Sup X* = (*χ i*. (*SUP x∈X*. *x $ i*))
definition |*x*| = (*χ i*. |*x $ i*|)

instance
apply *standard*
unfolding *euclidean_representation_sum'*
apply (*auto simp*: *less_eq_vec_def inf_vec_def sup_vec_def Inf_vec_def Sup_vec_def inner_axis*
Basis_vec_def inner_Basis_inf_left inner_Basis_sup_left inner_Basis_INF_left
inner_Basis_SUP_left eucl_le[**where** *'a*=*'a*] *less_le_not_le abs_vec_def abs_inner*)
done

end

end

6.10 Arcwise-Connected Sets

theory *Arcwise_Connected*
imports *Path_Connected Ordered_Euclidean_Space HOL−Computational_Algebra.Primes*
begin

lemma *path_connected_interval* [*simp*]:
fixes *a b*::*'a*::*ordered_euclidean_space*
shows *path_connected* {*a*..*b*}
using *is_interval_cc is_interval_path_connected* **by** *blast*

lemma *segment_to_closest_point*:
fixes *S* :: *'a* :: *euclidean_space set*
shows ⟦*closed S*; *S* ≠ {}⟧ ⟹ *open_segment a* (*closest_point S a*) ∩ *S* = {}
unfolding *disjoint_iff*
by (*metis closest_point_le dist_commute dist_in_open_segment not_le*)

lemma *segment_to_point_exists*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *closed S S* ≠ {}
obtains *b* **where** *b* ∈ *S open_segment a b* ∩ *S* = {}
by (*metis assms segment_to_closest_point closest_point_exists that*)

6.10.1 The Brouwer reduction theorem

theorem *Brouwer_reduction_theorem_gen*:
fixes S :: *'a::euclidean_space set*
assumes *closed* S φ S
and φ: $\bigwedge F$. $[\![\bigwedge n$. *closed*$(F\ n)$; $\bigwedge n$. $\varphi(F\ n)$; $\bigwedge n$. $F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow \varphi(\bigcap(range\ F))$
obtains T **where** $T \subseteq S$ *closed* T φ T $\bigwedge U$. $[\![U \subseteq S$; *closed* U; $\varphi\ U]\!] \Longrightarrow \neg\ (U \subset T)$
proof −
obtain B :: *nat* ⇒ *'a set*
where *inj* B $\bigwedge n$. *open*$(B\ n)$ **and** *open_cov*: $\bigwedge S$. *open* $S \Longrightarrow \exists K$. $S = \bigcup(B\ `\ K)$
by (*metis Setcompr_eq_image that univ_second_countable_sequence*)
define A **where** $A \equiv rec\_nat\ S\ (\lambda n\ a.$ *if* $\exists U.\ U \subseteq a \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
then SOME $U.\ U \subseteq a \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
else $a)$
have [*simp*]: $A\ 0 = S$
by (*simp add*: *A_def*)
have *ASuc*: $A(Suc\ n) =$ (*if* $\exists U.\ U \subseteq A\ n \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
then SOME $U.\ U \subseteq A\ n \wedge$ *closed* $U \wedge \varphi\ U \wedge U \cap (B\ n) = \{\}$
else $A\ n$) **for** n
by (*auto simp*: *A_def*)
have *sub*: $\bigwedge n$. $A(Suc\ n) \subseteq A\ n$
by (*auto simp*: *ASuc dest*!: *someI_ex*)
have *subS*: $A\ n \subseteq S$ **for** n
by (*induction n*) (*use sub* **in** *auto*)
have *clo*: *closed* $(A\ n) \wedge \varphi\ (A\ n)$ **for** n
by (*induction n*) (*auto simp*: *assms ASuc dest*!: *someI_ex*)
show *?thesis*
proof
show $\bigcap(range\ A) \subseteq S$
using ‹$\bigwedge n$. $A\ n \subseteq S$› **by** *blast*
show *closed* $(\bigcap(A\ `\ UNIV))$
using *clo* **by** *blast*
show $\varphi\ (\bigcap(A\ `\ UNIV))$
by (*simp add*: *clo* φ *sub*)
show $\neg\ U \subset \bigcap(A\ `\ UNIV)$ **if** $U \subseteq S$ *closed* U φ U **for** U
proof −
have $\exists y.\ x \notin A\ y$ **if** $x \notin U$ **and** *Usub*: $U \subseteq (\bigcap x.\ A\ x)$ **for** x
proof −
obtain e **where** $e > 0$ **and** e: *ball* $x\ e \subseteq -U$
using ‹*closed* U› ‹$x \notin U$› *openE* [*of* $-U$] **by** *blast*
moreover obtain K **where** K: *ball* $x\ e = \bigcup(B\ `\ K)$
using *open_cov* [*of ball x e*] **by** *auto*
ultimately have $\bigcup(B\ `\ K) \subseteq -U$
by *blast*
have $K \neq \{\}$

```
        using ‹0 < e› ‹ball x e = ⋃(B ‘ K)› by auto
      then obtain n where n ∈ K x ∈ B n
        by (metis K UN_E ‹0 < e› centre_in_ball)
      then have U ∩ B n = {}
        using K e by auto
      show ?thesis
      proof (cases ∃ U⊆A n. closed U ∧ φ U ∧ U ∩ B n = {})
        case True
        then show ?thesis
          apply (rule_tac x=Suc n in exI)
          apply (simp add: ASuc)
          apply (erule someI2_ex)
          using ‹x ∈ B n› by blast
      next
        case False
        then show ?thesis
         by (meson Inf_lower Usub ‹U ∩ B n = {}› ‹φ U› ‹closed U› range_eqI
subset_trans)
      qed
    qed
    with that show ?thesis
      by (meson Inter_iff psubsetE rangeI subsetI)
  qed
 qed
qed

corollary Brouwer_reduction_theorem:
 fixes S :: 'a::euclidean_space set
 assumes compact S φ S S ≠ {}
    and φ: ⋀F. ⟦⋀n. compact(F n); ⋀n. F n ≠ {}; ⋀n. φ(F n); ⋀n. F(Suc n)
⊆ F n⟧ ⟹ φ(⋂(range F))
 obtains T where T ⊆ S compact T T ≠ {} φ T
                 ⋀U. ⟦U ⊆ S; closed U; U ≠ {}; φ U⟧ ⟹ ¬ (U ⊂ T)
proof (rule Brouwer_reduction_theorem_gen [of S λT. T ≠ {} ∧ T ⊆ S ∧ φ
T])
 fix F
 assume cloF: ⋀n. closed (F n)
   and F: ⋀n. F n ≠ {} ∧ F n ⊆ S ∧ φ (F n) and Fsub: ⋀n. F (Suc n) ⊆ F n
 show ⋂(F ‘ UNIV) ≠ {} ∧ ⋂(F ‘ UNIV) ⊆ S ∧ φ (⋂(F ‘ UNIV))
 proof (intro conjI)
   show ⋂(F ‘ UNIV) ≠ {}
     by (metis F Fsub ‹compact S› cloF closed_Int_compact compact_nest
inf.orderE lift_Suc_antimono_le)
   show ⋂(F ‘ UNIV) ⊆ S
     using F by blast
   show φ (⋂(F ‘ UNIV))
     by (metis F Fsub φ ‹compact S› cloF closed_Int_compact inf.orderE)
 qed
next
```

show $S \neq \{\} \wedge S \subseteq S \wedge \varphi\ S$
by (*simp add*: *assms*)
qed (*meson assms compact_imp_closed seq_compact_closed_subset seq_compact_eq_compact*)+

6.10.2 Arcwise Connections

6.10.3 Density of points with dyadic rational coordinates

proposition *closure_dyadic_rationals*:
$\mathit{closure}\ (\bigcup k.\ \bigcup f \in \mathit{Basis} \to \mathbb{Z}.$
$\{\ \sum i :: 'a :: \mathit{euclidean\_space} \in \mathit{Basis}.\ (f\ i\ /\ 2\hat{}k) *_R i\ \}) = \mathit{UNIV}$
proof −
have $x \in \mathit{closure}\ (\bigcup k.\ \bigcup f \in \mathit{Basis} \to \mathbb{Z}.\ \{\sum i \in \mathit{Basis}.\ (f\ i\ /\ 2\hat{}k) *_R i\})$ **for** $x::'a$
proof (*clarsimp simp*: *closure_approachable*)
fix $e::\mathit{real}$
assume $e > 0$
then obtain k **where** k: $(1/2)\hat{}k < e/\mathit{DIM}('a)$
by (*meson DIM_positive divide_less_eq_1_pos of_nat_0_less_iff one_less_numeral_iff real_arch_pow_inv semiring_norm(76) zero_less_divide_iff zero_less_numeral*)
have $\mathit{dist}\ (\sum i{\in}\mathit{Basis}.\ (\mathit{real\_of\_int}\ \lfloor 2\hat{}k*(x \cdot i)\rfloor\ /\ 2\hat{}k) *_R i)\ x =$
$\mathit{dist}\ (\sum i{\in}\mathit{Basis}.\ (\mathit{real\_of\_int}\ \lfloor 2\hat{}k*(x \cdot i)\rfloor\ /\ 2\hat{}k) *_R i)\ (\sum i{\in}\mathit{Basis}.\ (x \cdot i) *_R i)$
by (*simp add*: *euclidean_representation*)
also have $\ldots = \mathit{norm}\ ((\sum i{\in}\mathit{Basis}.\ (\mathit{real\_of\_int}\ \lfloor 2\hat{}k*(x \cdot i)\rfloor\ /\ 2\hat{}k) *_R i - (x \cdot i) *_R i))$
by (*simp add*: *dist_norm sum_subtractf*)
also have $\ldots \leq \mathit{DIM}('a)*((1/2)\hat{}k)$
proof (*rule sum_norm_bound*, *simp add*: *algebra_simps*)
fix $i::'a$
assume $i \in \mathit{Basis}$
then have $\mathit{norm}\ ((\mathit{real\_of\_int}\ \lfloor x \cdot i*2\hat{}k\rfloor\ /\ 2\hat{}k) *_R i - (x \cdot i) *_R i) =$
$|\mathit{real\_of\_int}\ \lfloor x \cdot i*2\hat{}k\rfloor\ /\ 2\hat{}k - x \cdot i|$
by (*simp add*: *scaleR_left_diff_distrib* [*symmetric*])
also have $\ldots \leq (1/2)\ \hat{}\ k$
by (*simp add*: *divide_simps*) *linarith*
finally show $\mathit{norm}\ ((\mathit{real\_of\_int}\ \lfloor x \cdot i*2\hat{}k\rfloor\ /\ 2\hat{}k) *_R i - (x \cdot i) *_R i) \leq (1/2)\ \hat{}\ k$.
qed
also have $\ldots < \mathit{DIM}('a)*(e/\mathit{DIM}('a))$
using *DIM_positive k linordered_comm_semiring_strict_class.comm_mult_strict_left_mono of_nat_0_less_iff* **by** *blast*
also have $\ldots = e$
by *simp*
finally have $\mathit{dist}\ (\sum i{\in}\mathit{Basis}.\ (\lfloor 2\hat{}k*(x \cdot i)\rfloor\ /\ 2\hat{}k) *_R i)\ x < e$.
with *Ints_of_int*
show $\exists k.\ \exists f \in \mathit{Basis} \to \mathbb{Z}.\ \mathit{dist}\ (\sum b{\in}\mathit{Basis}.\ (f\ b\ /\ 2\hat{}k) *_R b)\ x < e$
by *fastforce*
qed
then show *?thesis* **by** *auto*

qed

corollary *closure_rational_coordinates*:
closure $(\bigcup f \in Basis \to \mathbb{Q}.\ \{\ \sum i :: 'a :: euclidean\_space \in Basis.\ f\ i *_R\ i\ \}) = UNIV$
proof −
have $*$: $(\bigcup k.\ \bigcup f \in Basis \to \mathbb{Z}.\ \{\ \sum i::'a \in Basis.\ (f\ i\ /\ 2\hat{}k) *_R\ i\ \})$
$\subseteq (\bigcup f \in Basis \to \mathbb{Q}.\ \{\ \sum i \in Basis.\ f\ i *_R\ i\ \})$
proof *clarsimp*
fix k **and** $f :: 'a \Rightarrow real$
assume f: $f \in Basis \to \mathbb{Z}$
show $\exists x \in Basis \to \mathbb{Q}.\ (\sum i \in Basis.\ (f\ i\ /\ 2\hat{}k) *_R\ i) = (\sum i \in Basis.\ x\ i *_R\ i)$
apply (*rule_tac* $x=\lambda i.\ f\ i\ /\ 2\hat{}k$ **in** *bexI*)
using *Ints_subset_Rats f* **by** *auto*
qed
show *?thesis*
using *closure_dyadic_rationals closure_mono* [*OF* $*$] **by** *blast*
qed

lemma *closure_dyadic_rationals_in_convex_set*:
$[\![convex\ S;\ interior\ S \neq \{\}]\!]$
$\Longrightarrow closure(S \cap$
$(\bigcup k.\ \bigcup f \in Basis \to \mathbb{Z}.$
$\{\ \sum i :: 'a :: euclidean\_space \in Basis.\ (f\ i\ /\ 2\hat{}k) *_R\ i\ \})) =$
$closure\ S$
by (*simp add*: *closure_dyadic_rationals closure_convex_Int_superset*)

lemma *closure_rationals_in_convex_set*:
$[\![convex\ S;\ interior\ S \neq \{\}]\!]$
$\Longrightarrow closure(S \cap (\bigcup f \in Basis \to \mathbb{Q}.\ \{\ \sum i :: 'a :: euclidean\_space \in Basis.\ f\ i *_R\ i\ \})) =$
$closure\ S$
by (*simp add*: *closure_rational_coordinates closure_convex_Int_superset*)

Every path between distinct points contains an arc, and hence path connection is equivalent to arcwise connection for distinct points. The proof is based on Whyburn's "Topological Analysis".

lemma *closure_dyadic_rationals_in_convex_set_pos_1*:
fixes $S :: real\ set$
assumes *convex S* **and** *intnz*: *interior* $S \neq \{\}$ **and** *pos*: $\bigwedge x.\ x \in S \Longrightarrow 0 \leq x$
shows $closure(S \cap (\bigcup k\ m.\ \{of\_nat\ m\ /\ 2\hat{}k\})) = closure\ S$
proof −
have $\exists m.\ f\ 1/2\hat{}k = real\ m\ /\ 2\hat{}k$ **if** $(f\ 1)\ /\ 2\hat{}k \in S\ f\ 1 \in \mathbb{Z}$ **for** k **and** $f :: real \Rightarrow real$
using *that* **by** (*force simp*: *Ints_def zero_le_divide_iff power_le_zero_eq dest*: *pos zero_le_imp_eq_int*)
then have $S \cap (\bigcup k\ m.\ \{real\ m\ /\ 2\hat{}k\}) = S \cap$
$(\bigcup k.\ \bigcup f{\in}Basis \to \mathbb{Z}.\ \{\sum i{\in}Basis.\ (f\ i\ /\ 2\hat{}k) *_R\ i\})$

```
    by force
  then show ?thesis
    using closure_dyadic_rationals_in_convex_set [OF ‹convex S› intnz] by simp
qed


definition dyadics :: 'a::field_char_0 set where dyadics ≡ ⋃k m. {of_nat m /
2^k}

lemma real_in_dyadics [simp]: real m ∈ dyadics
  by (simp add: dyadics_def) (metis divide_numeral_1 numeral_One power_0)

lemma nat_neq_4k1: of_nat m ≠ (4 * of_nat k + 1) / (2 * 2^n :: 'a::field_char_0)
proof
  assume of_nat m = (4 * of_nat k + 1) / (2 * 2^n :: 'a)
  then have of_nat (m * (2 * 2^n)) = (of_nat (Suc (4 * k)) :: 'a)
    by (simp add: field_split_simps)
  then have m * (2 * 2^n) = Suc (4 * k)
    using of_nat_eq_iff by blast
  then have odd (m * (2 * 2^n))
    by simp
  then show False
    by simp
qed

lemma nat_neq_4k3: of_nat m ≠ (4 * of_nat k + 3) / (2 * 2^n :: 'a::field_char_0)
proof
  assume of_nat m = (4 * of_nat k + 3) / (2 * 2^n :: 'a)
  then have of_nat (m * (2 * 2^n)) = (of_nat (4 * k + 3) :: 'a)
    by (simp add: field_split_simps)
  then have m * (2 * 2^n) = (4 * k) + 3
    using of_nat_eq_iff by blast
  then have odd (m * (2 * 2^n))
    by simp
  then show False
    by simp
qed

lemma iff_4k:
  assumes r = real k odd k
    shows (4 * real m + r) / (2 * 2^n) = (4 * real m' + r) / (2 * 2 ^ n') ⟷
m=m' ∧ n=n'
proof −
  { assume (4 * real m + r) / (2 * 2^n) = (4 * real m' + r) / (2 * 2 ^ n')
    then have real ((4 * m + k) * (2 * 2 ^ n')) = real ((4 * m' + k) * (2 * 2^n))
      using assms by (auto simp: field_simps)
    then have (4 * m + k) * (2 * 2 ^ n') = (4 * m' + k) * (2 * 2^n)
      using of_nat_eq_iff by blast
    then have (4 * m + k) * (2 ^ n') = (4 * m' + k) * (2^n)
```

```
    by linarith
  then obtain 4*m + k = 4*m' + k n=n'
    using prime_power_cancel2 [OF two_is_prime_nat] assms
    by (metis even_mult_iff even_numeral odd_add)
  then have m=m' n=n'
    by auto
 }
 then show ?thesis by blast
qed

lemma neq_4k1_k43: (4 * real m + 1) / (2 * 2^n) ≠ (4 * real m' + 3) / (2 *
2 ^ n')
proof
 assume (4 * real m + 1) / (2 * 2^n) = (4 * real m' + 3) / (2 * 2 ^ n')
 then have real (Suc (4 * m) * (2 * 2 ^ n')) = real ((4 * m' + 3) * (2 * 2^n))
   by (auto simp: field_simps)
 then have Suc (4 * m) * (2 * 2 ^ n') = (4 * m' + 3) * (2 * 2^n)
   using of_nat_eq_iff by blast
 then have Suc (4 * m) * (2 ^ n') = (4 * m' + 3) * (2^n)
   by linarith
 then have Suc (4 * m) = (4 * m' + 3)
   by (rule prime_power_cancel2 [OF two_is_prime_nat]) auto
 then have 1 + 2 * m' = 2 * m
   using ‹Suc (4 * m) = 4 * m' + 3› by linarith
 then show False
   using even_Suc by presburger
qed

lemma dyadic_413_cases:
 obtains (of_nat m::'a::field_char_0) / 2^k ∈ Nats
 | m' k' where k' < k (of_nat m:: 'a) / 2^k = of_nat (4*m' + 1) / 2^Suc k'
 | m' k' where k' < k (of_nat m:: 'a) / 2^k = of_nat (4*m' + 3) / 2^Suc k'
proof (cases m>0)
 case False
 then have m=0 by simp
 with that show ?thesis by auto
next
 case True
 obtain k' m' where m': odd m' and k': m = m' * 2^k'
   using prime_power_canonical [OF two_is_prime_nat True] by blast
 then obtain q r where q: m' = 4*q + r and r: r < 4
   by (metis not_add_less2 split_div zero_neq_numeral)
 show ?thesis
 proof (cases k ≤ k')
   case True
   have (of_nat m:: 'a) / 2^k = of_nat m' * (2 ^ k' / 2^k)
     using k' by (simp add: field_simps)
   also have ... = (of_nat m'::'a) * 2 ^ (k'-k)
     using k' True by (simp add: power_diff)
```

```
    also have ... ∈ ℕ
      by (metis Nats_mult of_nat_in_Nats of_nat_numeral of_nat_power)
    finally show ?thesis by (auto simp: that)
  next
    case False
    then obtain kd where kd: Suc kd = k − k'
      using Suc_diff_Suc not_less by blast
    have (of_nat m:: 'a) / 2^k = of_nat m' * (2 ^ k' / 2^k)
      using k' by (simp add: field_simps)
    also have ... = (of_nat m'::'a) / 2 ^ (k−k')
      using k' False by (simp add: power_diff)
    also have ... = ((of_nat r + 4 * of_nat q)::'a) / 2 ^ (k−k')
      using q by force
    finally have meq: (of_nat m:: 'a) / 2^k = (of_nat r + 4 * of_nat q) / 2 ^ (k
− k') .
    have r ≠ 0 r ≠ 2
      using q m' by presburger+
    with r consider r = 1 | r = 3
      by linarith
    then show ?thesis
    proof cases
      assume r = 1
      with meq kd that(2) [of kd q] show ?thesis
        by simp
    next
      assume r = 3
      with meq kd that(3) [of kd q]  show ?thesis
        by simp
    qed
  qed
qed


lemma dyadics_iff:
   (dyadics :: 'a::field_char_0 set) =
    Nats ∪ (⋃k m. {of_nat (4*m + 1) / 2^Suc k}) ∪ (⋃k m. {of_nat (4*m +
3) / 2^Suc k})
          (is _ = ?rhs)
proof
  show dyadics ⊆ ?rhs
    unfolding dyadics_def
    apply clarify
    apply (rule dyadic_413_cases, force+)
    done
next
  have range of_nat ⊆ (⋃k m. {(of_nat m::'a) / 2 ^ k})
    by clarsimp (metis divide_numeral_1 numeral_One power_0)
  moreover have ⋀k m. ∃k' m'. ((1::'a) + 4 * of_nat m) / 2 ^ Suc k = of_nat
m' / 2 ^ k'
```

```
    by (metis (no_types) of_nat_Suc of_nat_mult of_nat_numeral)
  moreover have ⋀k m. ∃k' m'. (4 * of_nat m + (3::'a)) / 2 ^ Suc k = of_nat
m' / 2 ^ k'
    by (metis of_nat_add of_nat_mult of_nat_numeral)
  ultimately show ?rhs ⊆ dyadics
    by (auto simp: dyadics_def Nats_def)
qed


function (domintros) dyad_rec :: [nat ⇒ 'a, 'a⇒'a, 'a⇒'a, real] ⇒ 'a where
    dyad_rec b l r (real m) = b m
  | dyad_rec b l r ((4 * real m + 1) / 2 ^ (Suc n)) = l (dyad_rec b l r ((2*m +
1) / 2^n))
  | dyad_rec b l r ((4 * real m + 3) / 2 ^ (Suc n)) = r (dyad_rec b l r ((2*m +
1) / 2^n))
  | x ∉ dyadics ⟹ dyad_rec b l r x = undefined
  using iff_4k [of _ 1] iff_4k [of _ 3]
        apply (simp_all add: nat_neq_4k1 nat_neq_4k3 neq_4k1_k43 dyadics_iff
Nats_def)
  by (fastforce simp:  field_simps)+

lemma dyadics_levels: dyadics = (⋃K. ⋃k<K. ⋃ m. {of_nat m / 2^k})
  unfolding dyadics_def by auto

lemma dyad_rec_level_termination:
  assumes k < K
  shows dyad_rec_dom(b, l, r, real m / 2^k)
  using assms
proof (induction K arbitrary: k m)
  case 0
  then show ?case by auto
next
  case (Suc K)
  then consider k = K | k < K
    using less_antisym by blast
  then show ?case
  proof cases
    assume k = K
    show ?case
    proof (rule dyadic_413_cases [of m k, where 'a=real])
      show real m / 2^k ∈ ℕ ⟹ dyad_rec_dom (b, l, r, real m / 2^k)
      by (force simp: Nats_def nat_neq_4k1 nat_neq_4k3 intro: dyad_rec.domintros)
      show ?case if k' < k and eq: real m / 2^k = real (4 * m' + 1) / 2^Suc k'
for m' k'
      proof −
        have dyad_rec_dom (b, l, r, (4 * real m' + 1) / 2^Suc k')
        proof (rule dyad_rec.domintros)
          fix m n
          assume (4 * real m' + 1) / (2 * 2 ^ k') = (4 * real m + 1) / (2 * 2^n)
```

```
        then have m' = m k' = n using iff_4k [of _ 1]
          by auto
        have dyad_rec_dom (b, l, r, real (2 * m + 1) / 2 ^ k')
          using Suc.IH ‹k = K› ‹k' < k› by blast
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n)
          using ‹k' = n› by (auto simp: algebra_simps)
      next
        fix m n
        assume (4 * real m' + 1) / (2 * 2 ^ k') = (4 * real m + 3) / (2 * 2^n)
        then have False
          by (metis neq_4k1_k43)
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n) ..
      qed
      then show ?case by (simp add: eq add_ac)
    qed
    show ?case if k' < k and eq: real m / 2^k = real (4 * m' + 3) / 2^Suc k'
for m' k'
    proof -
      have dyad_rec_dom (b, l, r, (4 * real m' + 3) / 2^Suc k')
      proof (rule dyad_rec.domintros)
        fix m n
        assume (4 * real m' + 3) / (2 * 2 ^ k') = (4 * real m + 1) / (2 * 2^n)
        then have False
          by (metis neq_4k1_k43)
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n) ..
      next
        fix m n
        assume (4 * real m' + 3) / (2 * 2 ^ k') = (4 * real m + 3) / (2 * 2^n)
        then have m' = m k' = n using iff_4k [of _ 3]
          by auto
        have dyad_rec_dom (b, l, r, real (2 * m + 1) / 2 ^ k')
          using Suc.IH ‹k = K› ‹k' < k› by blast
        then show dyad_rec_dom (b, l, r, (2 * real m + 1) / 2^n)
          using ‹k' = n› by (auto simp: algebra_simps)
      qed
      then show ?case by (simp add: eq add_ac)
    qed
  qed
 next
  assume k < K
  then show ?case
    using Suc.IH by blast
 qed
qed


lemma dyad_rec_termination: x ∈ dyadics ⟹ dyad_rec_dom(b,l,r,x)
 by (auto simp: dyadics_levels intro: dyad_rec_level_termination)
```

lemma *dyad_rec_of_nat* [*simp*]: *dyad_rec b l r* (*real m*) = *b m*
by (*simp add*: *dyad_rec.psimps dyad_rec_termination*)

lemma *dyad_rec_41* [*simp*]: *dyad_rec b l r* ((*4* ∗ *real m* + *1*) / *2* ^ (*Suc n*)) = *l* (*dyad_rec b l r* ((*2*∗*m* + *1*) / *2*^*n*))
proof (*rule dyad_rec.psimps*)
show *dyad_rec_dom* (*b*, *l*, *r*, (*4* ∗ *real m* + *1*) / *2* ^ *Suc n*)
by (*metis add.commute dyad_rec_level_termination lessI of_nat_Suc of_nat_mult of_nat_numeral*)
qed

lemma *dyad_rec_43* [*simp*]: *dyad_rec b l r* ((*4* ∗ *real m* + *3*) / *2* ^ (*Suc n*)) = *r* (*dyad_rec b l r* ((*2*∗*m* + *1*) / *2*^*n*))
proof (*rule dyad_rec.psimps*)
show *dyad_rec_dom* (*b*, *l*, *r*, (*4* ∗ *real m* + *3*) / *2* ^ *Suc n*)
by (*metis dyad_rec_level_termination lessI of_nat_add of_nat_mult of_nat_numeral*)
qed

lemma *dyad_rec_41_times2*:
assumes *n* > *0*
shows *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *1*) / *2*^*Suc n*)) = *l* (*dyad_rec b l r* (*2* ∗ (*2* ∗ *real m* + *1*) / *2*^*n*))
proof −
obtain *n′* **where** *n′*: *n* = *Suc n′*
using *assms not0_implies_Suc* **by** *blast*
have *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *1*) / *2*^*Suc n*)) = *dyad_rec b l r* ((*2* ∗ (*4* ∗ *real m* + *1*)) / (*2* ∗ *2*^*n*))
by *auto*
also have ... = *dyad_rec b l r* ((*4* ∗ *real m* + *1*) / *2*^*n*)
by (*subst mult_divide_mult_cancel_left*) *auto*
also have ... = *l* (*dyad_rec b l r* ((*2* ∗ *real m* + *1*) / *2* ^ *n′*))
by (*simp add*: *add.commute* [*of 1*] *n′ del*: *power_Suc*)
also have ... = *l* (*dyad_rec b l r* ((*2* ∗ (*2* ∗ *real m* + *1*)) / (*2* ∗ *2* ^ *n′*)))
by (*subst mult_divide_mult_cancel_left*) *auto*
also have ... = *l* (*dyad_rec b l r* (*2* ∗ (*2* ∗ *real m* + *1*) / *2*^*n*))
by (*simp add*: *add.commute n′*)
finally show *?thesis* **.**
qed

lemma *dyad_rec_43_times2*:
assumes *n* > *0*
shows *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *3*) / *2*^*Suc n*)) = *r* (*dyad_rec b l r* (*2* ∗ (*2* ∗ *real m* + *1*) / *2*^*n*))
proof −
obtain *n′* **where** *n′*: *n* = *Suc n′*
using *assms not0_implies_Suc* **by** *blast*
have *dyad_rec b l r* (*2* ∗ ((*4* ∗ *real m* + *3*) / *2*^*Suc n*)) = *dyad_rec b l r* ((*2* ∗ (*4* ∗ *real m* + *3*)) / (*2* ∗ *2*^*n*))
by *auto*

```
  also have ... = dyad_rec b l r ((4 * real m + 3) / 2^n)
    by (subst mult_divide_mult_cancel_left) auto
  also have ... = r (dyad_rec b l r ((2 * real m + 1) / 2 ^ n'))
    by (simp add: n' del: power_Suc)
  also have ... = r (dyad_rec b l r ((2 * (2 * real m + 1)) / (2 * 2 ^ n')))
    by (subst mult_divide_mult_cancel_left) auto
  also have ... = r (dyad_rec b l r (2 * (2 * real m + 1) / 2^n))
    by (simp add: n')
  finally show ?thesis .
qed

definition dyad_rec2
    where dyad_rec2 u v lc rc x =
          dyad_rec (λz. (u,v)) (λ(a,b). (a, lc a b (midpoint a b))) (λ(a,b). (rc a b
(midpoint a b), b)) (2*x)

abbreviation leftrec where leftrec u v lc rc x ≡ fst (dyad_rec2 u v lc rc x)
abbreviation rightrec where rightrec u v lc rc x ≡ snd (dyad_rec2 u v lc rc x)

lemma leftrec_base: leftrec u v lc rc (real m / 2) = u
  by (simp add: dyad_rec2_def)

lemma leftrec_41: n > 0 ⟹ leftrec u v lc rc ((4 * real m + 1) / 2 ^ (Suc n))
= leftrec u v lc rc ((2 * real m + 1) / 2^n)
  unfolding dyad_rec2_def dyad_rec_41_times2
  by (simp add: case_prod_beta)

lemma leftrec_43: n > 0 ⟹
          leftrec u v lc rc ((4 * real m + 3) / 2 ^ (Suc n)) =
              rc (leftrec u v lc rc ((2 * real m + 1) / 2^n)) (rightrec u v lc rc ((2
* real m + 1) / 2^n))
              (midpoint (leftrec u v lc rc ((2 * real m + 1) / 2^n)) (rightrec u v lc
rc ((2 * real m + 1) / 2^n)))
  unfolding dyad_rec2_def dyad_rec_43_times2
  by (simp add: case_prod_beta)

lemma rightrec_base: rightrec u v lc rc (real m / 2) = v
  by (simp add: dyad_rec2_def)

lemma rightrec_41: n > 0 ⟹
          rightrec u v lc rc ((4 * real m + 1) / 2 ^ (Suc n)) =
              lc (leftrec u v lc rc ((2 * real m + 1) / 2^n)) (rightrec u v lc rc ((2
* real m + 1) / 2^n))
              (midpoint (leftrec u v lc rc ((2 * real m + 1) / 2^n)) (rightrec u v lc
rc ((2 * real m + 1) / 2^n)))
  unfolding dyad_rec2_def dyad_rec_41_times2
  by (simp add: case_prod_beta)

lemma rightrec_43: n > 0 ⟹ rightrec u v lc rc ((4 * real m + 3) / 2 ^ (Suc
```

```
n)) = rightrec u v lc rc ((2 * real m + 1) / 2^n)
  unfolding dyad_rec2_def dyad_rec_43_times2
  by (simp add: case_prod_beta)

lemma dyadics_in_open_unit_interval:
   {0<..<1} ∩ (⋃k m. {real m / 2^k}) = (⋃k. ⋃m ∈ {0<..<2^k}. {real m /
2^k})
  by (auto simp: field_split_simps)


lemma padic_rational_approximation_straddle:
  assumes ε > 0 p > 1
  obtains n q r
    where of_int q / p^n < x x < of_int r / p^n |q / p^n - r / p^n | < ε
proof −
  obtain n where n: 2 / ε < p ^ n
    using ‹p>1› real_arch_pow by blast
  define q where q ≡ ⌊p ^ n * x⌋ − 1
  show thesis
    proof
      show q / p ^ n < x x < real_of_int (q+2) / p ^ n
           using assms by (simp_all add: q_def divide_simps floor_less_cancel
mult.commute)
      show |q / p ^ n − real_of_int (q+2) / p ^ n| < ε
        using assms n by (simp add: q_def divide_simps mult.commute)
    qed
qed

lemma padic_rational_approximation_straddle_pos:
  assumes ε > 0 p > 1 x > 0
  obtains n q r
    where of_nat q / p^n < x x < of_nat r / p^n |q / p^n - r / p^n | < ε
proof −
  obtain n q r
    where ∗: of_int q / p^n < x x < of_int r / p^n |q / p^n - r / p^n | < ε
    using padic_rational_approximation_straddle assms by metis
  then have r ≥ 0
   using assms by (smt (verit, best) divide_nonpos_pos of_int_0_le_iff zero_less_power)
  show thesis
  proof
    show real (max 0 (nat q)) / p ^ n < x
    using ∗ by (metis assms(3) div_0 max_nat.left_neutral nat_eq_iff2 of_nat_0
of_nat_nat)
    show x < real (nat r) / p ^ n
      using ‹r ≥ 0› ∗ by force
    show |real (max 0 (nat q)) / p ^ n − real (nat r) / p ^ n| < ε
      using ∗ assms by (simp add: divide_simps)
  qed
qed
```

lemma *padic_rational_approximation_straddle_pos_le*:
assumes $\varepsilon > 0$ $p > 1$ $x \geq 0$
obtains *n q r*
where *of_nat q / p^n ≤ x x < of_nat r / p^n |q / p^n − r / p^n | < ε*
proof −
obtain *n q r*
where ∗: *of_int q / p^n < x x < of_int r / p^n |q / p^n − r / p^n | < ε*
using *padic_rational_approximation_straddle assms* **by** *metis*
then have $r \geq 0$
using *assms* **by** (*smt* (*verit*, *best*) *divide_nonpos_pos of_int_0_le_iff zero_less_power*)
show *thesis*
proof
show *real* (*max 0* (*nat q*)) / *p* ^ *n* ≤ *x*
using ∗ *assms*(*3*) *nle_le* **by** *fastforce*
show *x* < *real* (*nat r*) / *p* ^ *n*
using ‹*r* ≥ *0*› ∗ **by** *force*
show |*real* (*max 0* (*nat q*)) / *p* ^ *n* − *real* (*nat r*) / *p* ^ *n*| < *ε*
using ∗ *assms* **by** (*simp add*: *divide_simps*)
qed
qed

Definition by recursion on dyadic rationals in [0,1]

lemma *recursion_on_dyadic_fractions*:
assumes *base*: *R a b*
and *step*: ⋀*x y*. *R x y* ⟹ ∃ *z*. *R x z* ∧ *R z y* **and** *trans*: ⋀*x y z*. ⟦*R x y*; *R y z*⟧ ⟹ *R x z*
shows ∃ *f* :: *real* ⇒ *'a*. *f 0* = *a* ∧ *f 1* = *b* ∧
(∀ *x* ∈ *dyadics* ∩ {*0..1*}. ∀ *y* ∈ *dyadics* ∩ {*0..1*}. *x* < *y* ⟶ *R* (*f x*) (*f y*))
proof −
obtain *mid* **where** *mid*: *R x y* ⟹ *R x* (*mid x y*) *R x y* ⟹ *R* (*mid x y*) *y* **for** *x y*
using *step* **by** *metis*
define *g* **where** *g* ≡ *rec_nat* (λ*k*. *if k = 0 then a else b*) (λ*n r k*. *if even k then r* (*k div 2*) *else mid* (*r* ((*k − 1*) *div 2*)) (*r* ((*Suc k*) *div 2*)))
have *g0* [*simp*]: *g 0* = (λ*k*. *if k = 0 then a else b*)
by (*simp add*: *g_def*)
have *gSuc* [*simp*]: ⋀*n*. *g*(*Suc n*) = (λ*k*. *if even k then g n* (*k div 2*) *else mid* (*g n* ((*k − 1*) *div 2*)) (*g n* ((*Suc k*) *div 2*)))
by (*auto simp*: *g_def*)
have *g_eq_g*: *2* ^ *d* ∗ *k* = *k'* ⟹ *g n k* = *g* (*n* + *d*) *k'* **for** *n d k k'*
by (*induction d arbitrary*: *k k'*) *auto*
have *g n k* = *g n' k'* **if** *real k / 2^n* = *real k' / 2^n' n'* ≤ *n* **for** *k n k' n'*
proof −
have *real k* = *real k'* ∗ *2* ^ (*n*−*n'*)
using *that* **by** (*simp add*: *power_diff divide_simps*)
then have *k* = *k'* ∗ *2* ^ (*n*−*n'*)

```
    using of_nat_eq_iff by fastforce
  with g_eq_g show ?thesis
    by (metis le_add_diff_inverse mult.commute that(2))
 qed
 then have g_eq_g: g n k = g n' k' if real k / 2 ^ n = real k' / 2 ^ n' for k n
k' n'
  by (metis nat_le_linear that)
 then obtain f where (λ(k,n). g n k) = f ∘ (λ(k,n). k / 2 ^ n)
  using function_factors_left by (smt (verit, del_insts) case_prod_beta')
 then have f_eq_g: ⋀k n. f(real k / 2 ^ n) = g n k
  by (simp add: fun_eq_iff)
 show ?thesis
 proof (intro exI conjI strip)
  show f 0 = a
   by (metis f_eq_g g0 div_0 of_nat_0)
  show f 1 = b
   by (metis f_eq_g g0 div_by_1 of_nat_1_eq_iff power_0 zero_neq_one)
  show R (f x) (f y)
   if x: x ∈ dyadics ∩ {0..1} and y: y ∈ dyadics ∩ {0..1} and x < y for x y
  proof −
   obtain n1 k1 where xeq: x = real k1 / 2^n1 k1 ≤ 2^n1
    using x by (auto simp: dyadics_def)
   obtain n2 k2 where yeq: y = real k2 / 2^n2 k2 ≤ 2^n2
    using y by (auto simp: dyadics_def)
   have xcommon: x = real(2^n2 * k1) / 2 ^(n1+n2)
    using xeq by (simp add: power_add)
   have ycommon: y = real(2^n1 * k2) / 2 ^(n1+n2)
    using yeq by (simp add: power_add)
   have *: R (g n j) (g n k) if j < k k ≤ 2^n for n j k
    using that
   proof (induction n arbitrary: j k)
    case 0
    then show ?case
     by (simp add: base)
   next
    case (Suc n)
    show ?case
    proof (cases even j)
     case True
     then obtain a where [simp]: j = 2*a
      by blast
     show ?thesis
     proof (cases even k)
      case True
      with Suc show ?thesis
       by (auto elim!: evenE)
     next
      case False
      then obtain b where [simp]: k = Suc (2*b)
```

```
            using oddE by fastforce
          show ?thesis
            using Suc
            apply simp
              by (smt (verit, ccfv_SIG) less_Suc_eq linorder_not_le local.trans
mid(1) nat_mult_less_cancel1 pos2)
        qed
       next
        case False
        then obtain a where [simp]: j = Suc (2*a)
          using oddE by fastforce
        show ?thesis
        proof (cases even k)
          case True
          then obtain b where [simp]: k = 2*b
            by blast
          show ?thesis
            using Suc
            apply simp
        by (smt (verit, ccfv_SIG) Suc_leI Suc_lessD le_trans lessI linorder_neqE_nat
linorder_not_le local.trans mid(2) nat_mult_less_cancel1 pos2)
        next
          case False
          then obtain b where [simp]: k = Suc (2*b)
            using oddE by fastforce
          show ?thesis
            using Suc
            apply simp
        by (smt (verit) Suc_leI le_trans lessI less_or_eq_imp_le linorder_neqE_nat
linorder_not_le local.trans mid(1) mid(2) nat_mult_less_cancel1 pos2)
          qed
       qed
      qed
      show ?thesis
       unfolding xcommon ycommon f_eq_g
      proof (rule *)
       show 2 ^ n2 * k1 < 2 ^ n1 * k2
         using of_nat_less_iff ‹x < y› by (fastforce simp: xeq yeq field_simps)
       show 2 ^ n1 * k2 ≤ 2 ^ (n1 + n2)
         by (simp add: power_add yeq)
      qed
    qed
  qed
qed

lemma dyadics_add:
  assumes x ∈ dyadics y ∈ dyadics
  shows x+y ∈ dyadics
proof −
```

```
  obtain i j m n where x: x = of_nat i / 2 ^ m and y: y = of_nat j / 2 ^ n
    using assms by (auto simp: dyadics_def)
  have xcommon: x = of_nat(2^n * i) / 2 ^ (m+n)
    using x by (simp add: power_add)
  moreover
  have ycommon: y = of_nat(2^m * j) / 2 ^ (m+n)
    using y by (simp add: power_add)
  ultimately have x+y = (of_nat(2^n * i + 2^m * j)) / 2 ^ (m+n)
    by (simp add: field_simps)
  then show ?thesis
    unfolding dyadics_def by blast
qed

lemma dyadics_diff:
  fixes x :: 'a::linordered_field
  assumes x ∈ dyadics y ∈ dyadics y ≤ x
  shows x−y ∈ dyadics
proof −
  obtain i j m n where x: x = of_nat i / 2 ^ m and y: y = of_nat j / 2 ^ n
    using assms by (auto simp: dyadics_def)
  have j_le_i: j * 2 ^ m ≤ i * 2 ^ n
     using of_nat_le_iff ‹y ≤ x› unfolding x y by (fastforce simp add: di-
vide_simps)
  have xcommon: x = of_nat(2^n * i) / 2 ^ (m+n)
    using x by (simp add: power_add)
  moreover
  have ycommon: y = of_nat(2^m * j) / 2 ^ (m+n)
    using y by (simp add: power_add)
  ultimately have x−y = (of_nat(2^n * i − 2^m * j)) / 2 ^ (m+n)
    by (simp add: xcommon ycommon field_simps j_le_i of_nat_diff)
  then show ?thesis
    unfolding dyadics_def by blast
qed



theorem homeomorphic_monotone_image_interval:
  fixes f :: real ⇒ 'a::{real_normed_vector,complete_space}
  assumes cont_f: continuous_on {0..1} f
     and conn: ⋀y. connected ({0..1} ∩ f −‘ {y})
     and f_1not0: f 1 ≠ f 0
   shows (f ‘ {0..1}) homeomorphic {0..1::real}
proof −
  have ∃ c d. a ≤ c ∧ c ≤ m ∧ m ≤ d ∧ d ≤ b ∧
              (∀ x ∈ {c..d}. f x = f m) ∧
              (∀ x ∈ {a..<c}. (f x ≠ f m)) ∧
              (∀ x ∈ {d<..b}. (f x ≠ f m)) ∧
              (∀ x ∈ {a..<c}. ∀ y ∈ {d<..b}. f x ≠ f y)
    if m: m ∈ {a..b} and ab01: {a..b} ⊆ {0..1} for a b m
```

```
  proof −
    have comp: compact (f −‘ {f m} ∩ {0..1})
      by (simp add: compact_eq_bounded_closed bounded_Int closed_vimage_Int
cont_f)
    obtain c0 d0 where cd0: {0..1} ∩ f −‘ {f m} = {c0..d0}
      using connected_compact_interval_1 [of {0..1} ∩ f −‘ {f m}] conn comp
      by (metis Int_commute)
    with that have m ∈ cbox c0 d0
      by auto
    obtain c d where cd: {a..b} ∩ f −‘ {f m} = {c..d}
      using ab01 cd0
      by (rule_tac c=max a c0 and d=min b d0 in that) auto
    then have cdab: {c..d} ⊆ {a..b}
      by blast
    show ?thesis
    proof (intro exI conjI ballI)
      show a ≤ c d ≤ b
        using cdab cd m by auto
      show c ≤ m m ≤ d
        using cd m by auto
      show ⋀x. x ∈ {c..d} ⟹ f x = f m
        using cd by blast
      show f x ≠ f m if x ∈ {a..<c} for x
        using that m cd [THEN equalityD1, THEN subsetD] ‹c ≤ m› by force
      show f x ≠ f m if x ∈ {d<..b} for x
       using that m cd [THEN equalityD1, THEN subsetD, of x] ‹m ≤ d› by force
      show f x ≠ f y if x ∈ {a..<c} y ∈ {d<..b} for x y
      proof (cases f x = f m ∨ f y = f m)
        case True
        then show ?thesis
          using ‹⋀x. x ∈ {a..<c} ⟹ f x ≠ f m› that by auto
      next
        case False
        have False if f x = f y
        proof −
          have x ≤ m m ≤ y
            using ‹c ≤ m› ‹x ∈ {a..<c}›  ‹m ≤ d› ‹y ∈ {d<..b}› by auto
          then have x ∈ ({0..1} ∩ f −‘ {f y}) y ∈ ({0..1} ∩ f −‘ {f y})
            using ‹x ∈ {a..<c}› ‹y ∈ {d<..b}› ab01 by (auto simp: that)
          then have m ∈ ({0..1} ∩ f −‘ {f y})
              by (meson ‹m ≤ y› ‹x ≤ m› is_interval_connected_1 conn [of f y]
is_interval_1)
          with False show False by auto
        qed
        then show ?thesis by auto
      qed
    qed
  qed
  then obtain leftcut rightcut where LR:
```

$\bigwedge a\ b\ m.\ [\![m \in \{a..b\};\ \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow$
$(a \le leftcut\ a\ b\ m \wedge leftcut\ a\ b\ m \le m \wedge m \le rightcut\ a\ b\ m \wedge rightcut\ a\ b\ m \le b \wedge$
$(\forall x \in \{leftcut\ a\ b\ m..rightcut\ a\ b\ m\}.\ f\ x = f\ m) \wedge$
$(\forall x \in \{a..<leftcut\ a\ b\ m\}.\ f\ x \ne f\ m) \wedge$
$(\forall x \in \{rightcut\ a\ b\ m<..b\}.\ f\ x \ne f\ m) \wedge$
$(\forall x \in \{a..<leftcut\ a\ b\ m\}.\ \forall y \in \{rightcut\ a\ b\ m<..b\}.\ f\ x \ne f\ y))$
apply *atomize*
apply (*clarsimp simp only*: *imp_conjL* [*symmetric*] *choice_iff choice_iff'*)
apply (*rule that*, *blast*)
done
then have *left_right*: $\bigwedge a\ b\ m.\ [\![m \in \{a..b\};\ \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow a \le leftcut\ a\ b\ m \wedge rightcut\ a\ b\ m \le b$
and *left_right_m*: $\bigwedge a\ b\ m.\ [\![m \in \{a..b\};\ \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow leftcut\ a\ b\ m \le m \wedge m \le rightcut\ a\ b\ m$
by *auto*
have *left_neq*: $[\![a \le x;\ x < leftcut\ a\ b\ m;\ a \le m;\ m \le b;\ \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow f\ x \ne f\ m$
and *right_neq*: $[\![rightcut\ a\ b\ m < x;\ x \le b;\ a \le m;\ m \le b;\ \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow f\ x \ne f\ m$
and *left_right_neq*: $[\![a \le x;\ x < leftcut\ a\ b\ m;\ rightcut\ a\ b\ m < y;\ y \le b;\ a \le m;\ m \le b;\ \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow f\ x \ne f\ m$
and *feqm*: $[\![leftcut\ a\ b\ m \le x;\ x \le rightcut\ a\ b\ m;\ a \le m;\ m \le b;\ \{a..b\} \subseteq \{0..1\}]\!]$
$\Longrightarrow f\ x = f\ m$ **for** *a b m x y*
by (*meson atLeastAtMost_iff greaterThanAtMost_iff atLeastLessThan_iff LR*)+
have *f_eqI*: $\bigwedge a\ b\ m\ x\ y.\ [\![leftcut\ a\ b\ m \le x;\ x \le rightcut\ a\ b\ m;\ leftcut\ a\ b\ m \le y;\ y \le rightcut\ a\ b\ m;$
$a \le m;\ m \le b;\ \{a..b\} \subseteq \{0..1\}]\!] \Longrightarrow f\ x = f\ y$
by (*metis feqm*)
define *u* **where** $u \equiv rightcut\ 0\ 1\ 0$
have *lc*[*simp*]: $leftcut\ 0\ 1\ 0 = 0$ **and** *u01*: $0 \le u\ u \le 1$
using *LR* [*of 0 0 1*] **by** (*auto simp*: *u_def*)
have *f0u*: $\bigwedge x.\ x \in \{0..u\} \Longrightarrow f\ x = f\ 0$
using *LR* [*of 0 0 1*] **unfolding** *u_def* [*symmetric*]
by (*metis ‹leftcut 0 1 0 = 0› atLeastAtMost_iff order_refl zero_le_one*)
have *fu1*: $\bigwedge x.\ x \in \{u<..1\} \Longrightarrow f\ x \ne f\ 0$
using *LR* [*of 0 0 1*] **unfolding** *u_def* [*symmetric*] **by** *fastforce*
define *v* **where** $v \equiv leftcut\ u\ 1\ 1$
have *rc*[*simp*]: $rightcut\ u\ 1\ 1 = 1$ **and** *v01*: $u \le v\ v \le 1$
using *LR* [*of 1 u 1*] *u01* **by** (*auto simp*: *v_def*)
have *fuv*: $\bigwedge x.\ x \in \{u..<v\} \Longrightarrow f\ x \ne f\ 1$
using *LR* [*of 1 u 1*] *u01 v_def* **by** *fastforce*
have *f0v*: $\bigwedge x.\ x \in \{0..<v\} \Longrightarrow f\ x \ne f\ 1$
by (*metis f_1not0 atLeastAtMost_iff atLeastLessThan_iff f0u fuv linear*)
have *fv1*: $\bigwedge x.\ x \in \{v..1\} \Longrightarrow f\ x = f\ 1$
using *LR* [*of 1 u 1*] *u01 v_def* **by** (*metis atLeastAtMost_iff atLeastatMost_subset_iff order_refl rc*)
define *a* **where** $a \equiv leftrec\ u\ v\ leftcut\ rightcut$

```
  define b where b ≡ rightrec u v leftcut rightcut
  define c where c ≡ λx. midpoint (a x) (b x)
  have a_real [simp]: a (real j) = u for j
    using a_def leftrec_base
   by (metis nonzero_mult_div_cancel_right of_nat_mult of_nat_numeral zero_neq_numeral)
  have b_real [simp]: b (real j) = v for j
    using b_def rightrec_base
   by (metis nonzero_mult_div_cancel_right of_nat_mult of_nat_numeral zero_neq_numeral)
  have a41: a ((4 * real m + 1) / 2^Suc n) = a ((2 * real m + 1) / 2^n) if n >
0 for m n
    using that a_def leftrec_41 by blast
  have b41: b ((4 * real m + 1) / 2^Suc n) =
              leftcut (a ((2 * real m + 1) / 2^n))
                      (b ((2 * real m + 1) / 2^n))
                      (c ((2 * real m + 1) / 2^n)) if n > 0 for m n
    using that a_def b_def c_def rightrec_41 by blast
  have a43: a ((4 * real m + 3) / 2^Suc n) =
              rightcut (a ((2 * real m + 1) / 2^n))
                       (b ((2 * real m + 1) / 2^n))
                       (c ((2 * real m + 1) / 2^n)) if n > 0 for m n
    using that a_def b_def c_def leftrec_43 by blast
  have b43: b ((4 * real m + 3) / 2^Suc n) = b ((2 * real m + 1) / 2^n) if n >
0 for m n
    using that b_def rightrec_43 by blast
  have uabv: u ≤ a (real m / 2 ^ n) ∧ a (real m / 2 ^ n) ≤ b (real m / 2 ^ n) ∧
b (real m / 2 ^ n) ≤ v for m n
  proof (induction n arbitrary: m)
    case 0
    then show ?case by (simp add: v01)
  next
    case (Suc n p)
    show ?case
    proof (cases even p)
      case True
      then obtain m where p = 2*m by (metis evenE)
      then show ?thesis
        by (simp add: Suc.IH)
    next
      case False
      then obtain m where m: p = 2*m + 1 by (metis oddE)
      show ?thesis
      proof (cases n)
        case 0
        then show ?thesis
          by (simp add: a_def b_def leftrec_base rightrec_base v01)
      next
        case (Suc n')
        then have n > 0 by simp
        have a_le_c: a (real m / 2^n) ≤ c (real m / 2^n) for m
```

```
      unfolding c_def by (metis Suc.IH ge_midpoint_1)
    have c_le_b: c (real m / 2^n) ≤ b (real m / 2^n) for m
      unfolding c_def by (metis Suc.IH le_midpoint_1)
    have c_ge_u: c (real m / 2^n) ≥ u for m
      using Suc.IH a_le_c order_trans by blast
    have c_le_v: c (real m / 2^n) ≤ v for m
      using Suc.IH c_le_b order_trans by blast
    have a_ge_0: 0 ≤ a (real m / 2^n) for m
      using Suc.IH order_trans u01(1) by blast
    have b_le_1: b (real m / 2^n) ≤ 1 for m
      using Suc.IH order_trans v01(2) by blast
    have left_le: leftcut (a ((real m) / 2^n)) (b ((real m) / 2^n)) (c ((real m)
/ 2^n)) ≤ c ((real m) / 2^n) for m
      by (simp add: LR a_ge_0 a_le_c b_le_1 c_le_b)
    have right_ge: rightcut (a ((real m) / 2^n)) (b ((real m) / 2^n)) (c ((real
m) / 2^n)) ≥ c ((real m) / 2^n) for m
      by (simp add: LR a_ge_0 a_le_c b_le_1 c_le_b)
    show ?thesis
    proof (cases even m)
      case True
      then obtain r where r: m = 2*r by (metis evenE)
      show ?thesis
        using order_trans [OF left_le c_le_v, of 1+2*r] Suc.IH [of m+1]
        using a_le_c [of m+1] c_le_b [of m+1] a_ge_0 [of m+1] b_le_1 [of
m+1] left_right ‹n > 0›
        by (simp_all add: r m add.commute [of 1]  a41 b41 del: power_Suc)
    next
      case False
      then obtain r where r: m = 2*r + 1 by (metis oddE)
      show ?thesis
        using order_trans [OF c_ge_u right_ge, of 1+2*r] Suc.IH [of m]
          using a_le_c [of m] c_le_b [of m] a_ge_0 [of m] b_le_1 [of m]
left_right ‹n > 0›
        apply (simp_all add: r m add.commute [of 3] a43 b43 del: power_Suc)
        by (simp add: add.commute)
    qed
  qed
 qed
 have a_ge_0 [simp]: 0 ≤ a(m / 2^n) and b_le_1 [simp]: b(m / 2^n) ≤ 1 for
m::nat and n
  using uabv order_trans u01 v01 by blast+
 then have b_ge_0 [simp]: 0 ≤ b(m / 2^n) and a_le_1 [simp]: a(m / 2^n) ≤
1 for m::nat and n
  using uabv order_trans by blast+
 have alec [simp]: a(m / 2^n) ≤ c(m / 2^n) and cleb [simp]: c(m / 2^n) ≤ b(m
/ 2^n) for m::nat and n
  by (auto simp: c_def ge_midpoint_1 le_midpoint_1 uabv)
 have c_ge_0 [simp]: 0 ≤ c(m / 2^n) and c_le_1 [simp]: c(m / 2^n) ≤ 1 for
```

```
m::nat and n
    using a_ge_0 alec b_le_1 cleb order_trans by blast+
  have ⟦d = m−n; odd j; |real i / 2^m − real j / 2^n| < 1/2 ^ n⟧
       ⟹ (a(j / 2^n)) ≤ (c(i / 2^m)) ∧ (c(i / 2^m)) ≤ (b(j / 2^n)) for d i j m
n
  proof (induction d arbitrary: j n rule: less_induct)
    case (less d j n)
    show ?case
    proof (cases m ≤ n)
      case True
      have |2^n| * |real i / 2^m − real j / 2^n| = 0
      proof (rule Ints_nonzero_abs_less1)
        have (real i * 2^n − real j * 2^m) / 2^m = (real i * 2^n) / 2^m − (real j
* 2^m) / 2^m
          using diff_divide_distrib by blast
        also have ... = (real i * 2 ^ (n−m)) − (real j)
          using True by (auto simp: power_diff field_simps)
        also have ... ∈ ℤ
          by simp
        finally have (real i * 2^n − real j * 2^m) / 2^m ∈ ℤ .
        with True Ints_abs show |2^n| * |real i / 2^m − real j / 2^n| ∈ ℤ
          by (fastforce simp: field_split_simps)
        show ||2^n| * |real i / 2^m − real j / 2^n|| < 1
          using less.prems by (auto simp: field_split_simps)
      qed
      then have real i / 2^m = real j / 2^n
        by auto
      then show ?thesis
        by auto
    next
      case False
      then have n < m by auto
      obtain k where k: j = Suc (2*k)
        using ‹odd j› oddE by fastforce
      show ?thesis
      proof (cases n > 0)
        case False
        then have a (real j / 2^n) = u
          by simp
        also have ... ≤ c (real i / 2^m)
          using alec uabv by (blast intro: order_trans)
        finally have ac: a (real j / 2^n) ≤ c (real i / 2^m) .
        have c (real i / 2^m) ≤ v
          using cleb uabv by (blast intro: order_trans)
        also have ... = b (real j / 2^n)
          using False by simp
        finally show ?thesis
          by (auto simp: ac)
      next
```

```
      case True show ?thesis
      proof (cases i / 2^m j / 2^n rule: linorder_cases)
        case less
        moreover have real (4 * k + 1) / 2 ^ Suc n + 1 / (2 ^ Suc n) = real j
/ 2 ^ n
          using k by (force simp: field_split_simps)
        moreover have |real i / 2 ^ m - j / 2 ^ n| < 2 / (2 ^ Suc n)
          using less.prems by simp
        ultimately have closer: |real i / 2 ^ m - real (4 * k + 1) / 2 ^ Suc n|
< 1 / (2 ^ Suc n)
          using less.prems by linarith
        have a (real (4 * k + 1) / 2 ^ Suc n) ≤ c (i / 2 ^ m) ∧
              c (real i / 2 ^ m) ≤ b (real (4 * k + 1) / 2 ^ Suc n)
        proof (rule less.IH [OF _ refl])
          show m - Suc n < d
            using ‹n < m› diff_less_mono2 less.prems(1) lessI by presburger
          show |real i / 2 ^ m - real (4 * k + 1) / 2 ^ Suc n| < 1 / 2 ^ Suc n
            using closer ‹n < m› ‹d = m - n› by (auto simp: field_split_simps
‹n < m› diff_less_mono2)
        qed auto
        then show ?thesis
          using LR [of c((2*k + 1) / 2^n) a((2*k + 1) / 2^n) b((2*k + 1) /
2^n)]
          using alec [of 2*k+1] cleb [of 2*k+1] a_ge_0 [of 2*k+1] b_le_1 [of
2*k+1]
          using k a41 b41 ‹0 < n›
          by (simp add: add.commute)
      next
        case equal then show ?thesis by simp
      next
        case greater
        moreover have real (4 * k + 3) / 2 ^ Suc n - 1 / (2 ^ Suc n) = real j
/ 2 ^ n
          using k by (force simp: field_split_simps)
        moreover have |real i / 2 ^ m - real j / 2 ^ n| < 2 * 1 / (2 ^ Suc n)
          using less.prems by simp
        ultimately have closer: |real i / 2 ^ m - real (4 * k + 3) / 2 ^ Suc n|
< 1 / (2 ^ Suc n)
          using less.prems by linarith
        have a (real (4 * k + 3) / 2 ^ Suc n) ≤ c (real i / 2 ^ m) ∧
              c (real i / 2 ^ m) ≤ b (real (4 * k + 3) / 2 ^ Suc n)
        proof (rule less.IH [OF _ refl])
          show m - Suc n < d
            using ‹n < m› diff_less_mono2 less.prems(1) by blast
          show |real i / 2 ^ m - real (4 * k + 3) / 2 ^ Suc n| < 1 / 2 ^ Suc n
            using closer ‹n < m› ‹d = m - n› by (auto simp: field_split_simps
‹n < m› diff_less_mono2)
        qed auto
        then show ?thesis
```

```
          using LR [of c((2*k + 1) / 2^n) a((2*k + 1) / 2^n) b((2*k + 1) /
2^n)]
          using alec [of 2*k+1] cleb [of 2*k+1] a_ge_0 [of 2*k+1] b_le_1 [of
2*k+1]
          using k a43 b43 ‹0 < n›
          by (simp add: add.commute)
       qed
      qed
    qed
  qed
  then have aj_le_ci: a (real j / 2 ^ n) ≤ c (real i / 2 ^ m)
    and ci_le_bj: c (real i / 2 ^ m) ≤ b (real j / 2 ^ n) if odd j |real i / 2^m −
real j / 2^n| < 1/2 ^ n for i j m n
    using that by blast+
  have close_ab: odd m ⟹ |a (real m / 2 ^ n) − b (real m / 2 ^ n)| ≤ 2 / 2^n
for m n
  proof (induction n arbitrary: m)
    case 0
    with u01 v01 show ?case by auto
  next
    case (Suc n m)
    with oddE obtain k where k: m = Suc (2*k) by fastforce
    show ?case
    proof (cases n > 0)
      case False
      with u01 v01 show ?thesis
        by (simp add: a_def b_def leftrec_base rightrec_base)
    next
      case True
      show ?thesis
      proof (cases even k)
        case True
        then obtain j where j: k = 2*j by (metis evenE)
        have |a ((2 * real j + 1) / 2 ^ n) − (b ((2 * real j + 1) / 2 ^ n))| ≤ 2/2
^ n
        proof −
          have odd (Suc k)
            using True by auto
          then show ?thesis
        by (metis (no_types) Groups.add_ac(2) Suc.IH j of_nat_Suc of_nat_mult
of_nat_numeral)
        qed
        moreover have a ((2 * real j + 1) / 2 ^ n) ≤
                      leftcut (a ((2 * real j + 1) / 2 ^ n)) (b ((2 * real j + 1) / 2 ^
n)) (c ((2 * real j + 1) / 2 ^ n))
          using alec [of 2*j+1] cleb [of 2*j+1] a_ge_0 [of 2*j+1]  b_le_1 [of
2*j+1]
          by (auto simp: add.commute left_right)
        moreover have leftcut (a ((2 * real j + 1) / 2 ^ n)) (b ((2 * real j + 1)
```

```
/ 2 ^ n)) (c ((2 * real j + 1) / 2 ^ n)) ≤
                    c ((2 * real j + 1) / 2 ^ n)
          using alec [of 2*j+1] cleb [of 2*j+1] a_ge_0 [of 2*j+1]  b_le_1 [of
2*j+1]
          by (auto simp: add.commute left_right_m)
        ultimately have |a ((2 * real j + 1) / 2 ^ n) −
                     leftcut (a ((2 * real j + 1) / 2 ^ n)) (b ((2 * real j + 1) / 2
^ n)) (c ((2 * real j + 1) / 2 ^ n))|
                ≤ 2/2 ^ Suc n
          by (simp add: c_def midpoint_def)
        with j k ‹n > 0› show ?thesis
          by (simp add: add.commute [of 1] a41 b41 del: power_Suc)
      next
        case False
        then obtain j where j: k = 2*j + 1 by (metis oddE)
        have |a ((2 * real j + 1) / 2 ^ n) − (b ((2 * real j + 1) / 2 ^ n))| ≤ 2/2
^ n
          using Suc.IH [OF False] j by (auto simp: algebra_simps)
        moreover have c ((2 * real j + 1) / 2 ^ n) ≤
                    rightcut (a ((2 * real j + 1) / 2 ^ n)) (b ((2 * real j + 1) / 2
^ n)) (c ((2 * real j + 1) / 2 ^ n))
          using alec [of 2*j+1] cleb [of 2*j+1] a_ge_0 [of 2*j+1]  b_le_1 [of
2*j+1]
          by (auto simp: add.commute left_right_m)
        moreover have rightcut (a ((2 * real j + 1) / 2 ^ n)) (b ((2 * real j + 1)
/ 2 ^ n)) (c ((2 * real j + 1) / 2 ^ n)) ≤
                     b ((2 * real j + 1) / 2 ^ n)
          using alec [of 2*j+1] cleb [of 2*j+1] a_ge_0 [of 2*j+1]  b_le_1 [of
2*j+1]
          by (auto simp: add.commute left_right)
        ultimately have |rightcut (a ((2 * real j + 1) / 2 ^ n)) (b ((2 * real j +
1) / 2 ^ n)) (c ((2 * real j + 1) / 2 ^ n)) −
                      b ((2 * real j + 1) / 2 ^ n)|  ≤  2/2 ^ Suc n
          by (simp add: c_def midpoint_def)
        with j k ‹n > 0› show ?thesis
          by (simp add: add.commute [of 3] a43 b43 del: power_Suc)
      qed
    qed
  qed
  have m1_to_3: 4 * real k − 1 = real (4 * (k−1)) + 3 if 0 < k for k
    using that by auto
  have fb_eq_fa: ⟦0 < j; 2*j < 2 ^ n⟧ ⟹ f(b((2 * real j − 1) / 2^n)) = f(a((2
* real j + 1) / 2^n)) for n j
  proof (induction n arbitrary: j)
    case 0
    then show ?case by auto
  next
    case (Suc n j) show ?case
    proof (cases n > 0)
```

```
    case False
    with Suc.prems show ?thesis by auto
  next
    case True
    show ?thesis
    proof (cases even j)
      case True
      then obtain k where k: j = 2*k by (metis evenE)
      with ‹0 < j› have k > 0 2 * k < 2 ^ n
        using Suc.prems(2) k by auto
      with k ‹0 < n› Suc.IH [of k] show ?thesis
        apply (simp add: m1_to_3 a41 b43 del: power_Suc of_nat_diff)
        by simp
    next
      case False
      then obtain k where k: j = 2*k + 1 by (metis oddE)
      have f (leftcut (a ((2 * k + 1) / 2^n)) (b ((2 * k + 1) / 2^n)) (c ((2 * k
+ 1) / 2^n)))
            = f (c ((2 * k + 1) / 2^n))
        f (c ((2 * k + 1) / 2^n))
            = f (rightcut (a ((2 * k + 1) / 2^n)) (b ((2 * k + 1) / 2^n)) (c ((2
* k + 1) / 2^n)))
        using alec [of 2*k+1 n] cleb [of 2*k+1 n] a_ge_0 [of 2*k+1 n]  b_le_1
[of 2*k+1 n] k
        using left_right_m [of c((2*k + 1) / 2^n) a((2*k + 1) / 2^n) b((2*k +
1) / 2^n)]
        by (auto simp: add.commute  feqm [OF order_refl]  feqm [OF _ order_refl,
symmetric])
        then
        show ?thesis
          by (simp add: k add.commute [of 1] add.commute [of 3] a43 b41 ‹0 < n›
del: power_Suc)
      qed
    qed
  qed
  have f_eq_fc: ⟦0 < j; j < 2 ^ n⟧
              ⟹ f(b((2*j - 1) / 2 ^ (Suc n))) = f(c(j / 2^n)) ∧
                 f(a((2*j + 1) / 2 ^ (Suc n))) = f(c(j / 2^n)) for n and j::nat
  proof (induction n arbitrary: j)
    case 0
    then show ?case by auto
  next
    case (Suc n)
    show ?case
    proof (cases even j)
      case True
      then obtain k where k: j = 2*k by (metis evenE)
      then have less2n: k < 2 ^ n
        using Suc.prems(2) by auto
```

```
      have 0 < k using ‹0 < j› k by linarith
      then have m1_to_3: real (4 * k - Suc 0) = real (4 * (k-1)) + 3
        by auto
      then show ?thesis
        using Suc.IH [of k] k ‹0 < k›
       by (simp add: less2n add.commute [of 1] m1_to_3 a41 b43 del: power_Suc
of_nat_diff) auto
    next
      case False
      then obtain k where k: j = 2*k + 1 by (metis oddE)
      with Suc.prems have k < 2^n by auto
      show ?thesis
        using alec [of 2*k+1 Suc n] cleb [of 2*k+1 Suc n] a_ge_0 [of 2*k+1 Suc
n]  b_le_1 [of 2*k+1 Suc n] k
       using left_right_m [of c((2*k + 1) / 2 ^ Suc n) a((2*k + 1) / 2 ^ Suc n)
b((2*k + 1) / 2 ^ Suc n)]
       apply (simp add: add.commute [of 1] add.commute [of 3] m1_to_3 b41 a43
del: power_Suc)
        apply (force intro: feqm)
        done
    qed
  qed
  define D01 where D01 ≡ {0<..<1} ∩ (⋃k m. {real m / 2^k})
  have cloD01 [simp]: closure D01 = {0..1}
    unfolding D01_def
    by (subst closure_dyadic_rationals_in_convex_set_pos_1) auto
  have uniformly_continuous_on D01 (f ∘ c)
  proof (clarsimp simp: uniformly_continuous_on_def)
    fix e::real
    assume 0 < e
    have ucontf: uniformly_continuous_on {0..1} f
      by (simp add: compact_uniformly_continuous [OF cont_f])
    then obtain d where 0 < d and d: ⋀x x'. ⟦x ∈ {0..1}; x' ∈ {0..1}; norm
(x' - x) < d⟧ ⟹ norm (f x' - f x) < e/2
      unfolding uniformly_continuous_on_def dist_norm
      by (metis ‹0 < e› less_divide_eq_numeral1(1) mult_zero_left)
    obtain n where n: 1/2^n < min d 1
    by (metis ‹0 < d› divide_less_eq_1 less_numeral_extra(1) min_def one_less_numeral_iff
power_one_over real_arch_pow_inv semiring_norm(76) zero_less_numeral)
    with gr0I have n > 0
      by (force simp: field_split_simps)
    show ∃ d>0. ∀ x∈D01. ∀ x'∈D01. dist x' x < d ⟶ dist (f (c x')) (f (c x)) <
e
    proof (intro exI ballI impI conjI)
      show (0::real) < 1/2^n by auto
    next
      have dist_fc_close: dist (f(c(real i / 2^m))) (f(c(real j / 2^n))) < e/2
        if i: 0 < i i < 2 ^ m and j: 0 < j j < 2 ^ n and clo: abs(i / 2^m - j /
2^n) < 1/2 ^ n for i j m
```

```
    proof −
      have abs3: |x − a| < e ⟹ x = a ∨ |x − (a − e/2)| < e/2 ∨ |x − (a +
e/2)| < e/2 for x a e::real
        by linarith
      consider i / 2 ^ m = j / 2 ^ n
        | |i / 2 ^ m − (2 ∗ j − 1) / 2 ^ Suc n| < 1/2 ^ Suc n
        | |i / 2 ^ m − (2 ∗ j + 1) / 2 ^ Suc n| < 1/2 ^ Suc n
        using abs3 [OF clo] j by (auto simp: field_simps of_nat_diff)
      then show ?thesis
      proof cases
        case 1 with ‹0 < e› show ?thesis by auto
      next
        case 2
        have ∗: abs(a − b) ≤ 1/2 ^ n ∧ 1/2 ^ n < d ∧ a ≤ c ∧ c ≤ b ⟹ b −
c < d for a b c
          by auto
        have norm (c (real i / 2 ^ m) − b (real (2 ∗ j − 1) / 2 ^ Suc n)) < d
          using 2 j n close_ab [of 2∗j−1 Suc n]
          using b_ge_0 [of 2∗j−1 Suc n] b_le_1 [of 2∗j−1 Suc n]
          using aj_le_ci [of 2∗j−1 i m Suc n]
          using ci_le_bj [of 2∗j−1 i m Suc n]
          apply (simp add: divide_simps of_nat_diff del: power_Suc)
          apply (auto simp: divide_simps intro!: ∗)
          done
        moreover have f(c(j / 2^n)) = f(b ((2∗j − 1) / 2 ^ (Suc n)))
          using f_eq_fc [OF j] by metis
        ultimately show ?thesis
          by (metis dist_norm atLeastAtMost_iff b_ge_0 b_le_1 c_ge_0 c_le_1
d)
      next
        case 3
        have ∗: abs(a − b) ≤ 1/2 ^ n ∧ 1/2 ^ n < d ∧ a ≤ c ∧ c ≤ b ⟹ c −
a < d for a b c
          by auto
        have norm (c (real i / 2 ^ m) − a (real (2 ∗ j + 1) / 2 ^ Suc n)) < d
          using 3 j n close_ab [of 2∗j+1 Suc n]
          using b_ge_0 [of 2∗j+1 Suc n] b_le_1 [of 2∗j+1 Suc n]
          using aj_le_ci [of 2∗j+1 i m Suc n]
          using ci_le_bj [of 2∗j+1 i m Suc n]
          apply (simp add: divide_simps of_nat_diff del: power_Suc)
          apply (auto simp: divide_simps intro!: ∗)
          done
        moreover have f(c(j / 2^n)) = f(a ((2∗j + 1) / 2 ^ (Suc n)))
          using f_eq_fc [OF j] by metis
        ultimately show ?thesis
          by (metis dist_norm a_ge_0 atLeastAtMost_iff a_ge_0 a_le_1 c_ge_0
c_le_1 d)
      qed
    qed
```

```
    show dist (f (c x')) (f (c x)) < e
      if x ∈ D01 x' ∈ D01 dist x' x < 1/2^n for x x'
      using that unfolding D01_def dyadics_in_open_unit_interval
    proof clarsimp
      fix i k::nat and m p
      assume i: 0 < i i < 2 ^ m and k: 0<k k < 2 ^ p
      assume clo: dist (real k / 2 ^ p) (real i / 2 ^ m) < 1/2 ^ n
      obtain j::nat where 0 < j j < 2 ^ n
        and clo_ij: abs(i / 2^m - j / 2^n) < 1/2 ^ n
        and clo_kj: abs(k / 2^p - j / 2^n) < 1/2 ^ n
      proof -
        have max (2^n * i / 2^m) (2^n * k / 2^p) ≥ 0
          by (auto simp: le_max_iff_disj)
        then obtain j where floor (max (2^n*i / 2^m) (2^n*k / 2^p)) = int j
          using zero_le_floor zero_le_imp_eq_int by blast
        then have j_le: real j ≤ max (2^n * i / 2^m) (2^n * k / 2^p)
          and less_j1: max (2^n * i / 2^m) (2^n * k / 2^p) < real j + 1
            using floor_correct [of max (2^n * i / 2^m) (2^n * k / 2^p)] by
linarith+
        show thesis
        proof (cases j = 0)
          case True
          show thesis
          proof
            show (1::nat) < 2 ^ n
              by (metis Suc_1 ‹0 < n› lessI one_less_power)
            show |real i / 2 ^ m - real 1/2 ^ n| < 1/2 ^ n
              using i less_j1 by (simp add: dist_norm field_simps True)
            show |real k / 2 ^ p - real 1/2 ^ n| < 1/2 ^ n
              using k less_j1 by (simp add: dist_norm field_simps True)
          qed simp
        next
          case False
          have 1: real j * 2 ^ m < real i * 2 ^ n
            if j: real j * 2 ^ p ≤ real k * 2 ^ n and k: real k * 2 ^ m < real i * 2
^ p
            for i k m p
          proof -
            have real j * 2 ^ p * 2 ^ m ≤ real k * 2 ^ n * 2 ^ m
              using j by simp
            moreover have real k * 2 ^ m * 2 ^ n < real i * 2 ^ p * 2 ^ n
              using k by simp
            ultimately have real j * 2 ^ p * 2 ^ m < real i * 2 ^ p * 2 ^ n
              by (simp only: mult_ac)
            then show ?thesis
              by simp
          qed
          have 2: real j * 2 ^ m < 2 ^ m + real i * 2 ^ n
            if j: real j * 2 ^ p ≤ real k * 2 ^ n and k: real k * (2 ^ m * 2 ^ n) <
```

```
2 ^ m * 2 ^ p + real i * (2 ^ n * 2 ^ p)
          for i k m p
        proof -
          have real j * 2 ^ p * 2 ^ m ≤ real k * (2 ^ m * 2 ^ n)
            using j by simp
          also have ... < 2 ^ m * 2 ^ p + real i * (2 ^ n * 2 ^ p)
            by (rule k)
         finally have (real j * 2 ^ m) * 2 ^ p < (2 ^ m + real i * 2 ^ n) * 2 ^ p
            by (simp add: algebra_simps)
          then show ?thesis
            by simp
        qed
        have 3: real j * 2 ^ p < 2 ^ p + real k * 2 ^ n
         if j: real j * 2 ^ m ≤ real i * 2 ^ n and i: real i * 2 ^ p ≤ real k * 2 ^
m
        proof -
          have real j * 2 ^ m * 2 ^ p ≤ real i * 2 ^ n * 2 ^ p
            using j by simp
          moreover have real i * 2 ^ p * 2 ^ n ≤ real k * 2 ^ m * 2 ^ n
            using i by simp
          ultimately have real j * 2 ^ m * 2 ^ p ≤ real k * 2 ^ m * 2 ^ n
            by (simp only: mult_ac)
          then have real j * 2 ^ p ≤ real k * 2 ^ n
            by simp
          also have ... < 2 ^ p + real k * 2 ^ n
            by auto
          finally show ?thesis by simp
        qed
        show ?thesis
        proof
          have 2 ^ n * real i / 2 ^ m < 2 ^ n 2 ^ n * real k / 2 ^ p < 2 ^ n
            using i k by (auto simp: field_simps)
          then have max (2^n * i / 2^m) (2^n * k / 2^p) < 2^n
            by simp
          with j_le have real j < 2 ^ n by linarith
          then show j < 2 ^ n
            by auto
          have |real i * 2 ^ n - real j * 2 ^ m| < 2 ^ m
            using clo less_j1 j_le
             by (auto simp: le_max_iff_disj field_split_simps dist_norm abs_if
split: if_split_asm dest: 1 2)
          then show |real i / 2 ^ m - real j / 2 ^ n| < 1/2 ^ n
            by (auto simp: field_split_simps)
          have |real k * 2 ^ n - real j * 2 ^ p| < 2 ^ p
            using clo less_j1 j_le
             by (auto simp: le_max_iff_disj field_split_simps dist_norm abs_if
split: if_split_asm dest: 3 2)
          then show |real k / 2 ^ p - real j / 2 ^ n| < 1/2 ^ n
            by (auto simp: le_max_iff_disj field_split_simps dist_norm)
```

```
          qed (use False in simp)
        qed
      qed
      show dist (f (c (real k / 2 ^ p))) (f (c (real i / 2 ^ m))) < e
      proof (rule dist_triangle_half_l)
        show dist (f (c (real k / 2 ^ p))) (f(c(j / 2^n))) < e/2
          using ‹0 < j› ‹j < 2 ^ n› k clo_kj
          by (intro dist_fc_close) auto
        show dist (f (c (real i / 2 ^ m))) (f (c (real j / 2 ^ n))) < e/2
          using ‹0 < j› ‹j < 2 ^ n› i clo_ij
          by (intro dist_fc_close) auto
      qed
    qed
  qed
 qed
 then obtain h where ucont_h: uniformly_continuous_on {0..1} h
   and fc_eq: ⋀x. x ∈ D01 ⟹ (f ∘ c) x = h x
 proof (rule uniformly_continuous_on_extension_on_closure [of D01 f ∘ c])
 qed (use closure_subset [of D01] in ‹auto intro!: that›)
 then have cont_h: continuous_on {0..1} h
   using uniformly_continuous_imp_continuous by blast
 have h_eq: h (real k / 2 ^ m) = f (c (real k / 2 ^ m)) if 0 < k k < 2^m for k
m
   using fc_eq that by (force simp: D01_def)
 have h ‘ {0..1} = f ‘ {0..1}
 proof
   have h ‘ (closure D01) ⊆ f ‘ {0..1}
   proof (rule image_closure_subset)
     show continuous_on (closure D01) h
       using cont_h by simp
     show closed (f ‘ {0..1})
       using compact_continuous_image [OF cont_f] compact_imp_closed by
blast
     show h ‘ D01 ⊆ f ‘ {0..1}
       by (force simp: dyadics_in_open_unit_interval D01_def h_eq)
   qed
   with cloD01 show h ‘ {0..1} ⊆ f ‘ {0..1} by simp
   have a12 [simp]: a (1/2) = u
     by (metis a_def leftrec_base numeral_One of_nat_numeral)
   have b12 [simp]: b (1/2) = v
     by (metis b_def rightrec_base numeral_One of_nat_numeral)
   have f ‘ {0..1} ⊆ closure(h ‘ D01)
    proof (clarsimp simp: closure_approachable dyadics_in_open_unit_interval
D01_def)
     fix x e::real
     assume 0 ≤ x x ≤ 1 0 < e
     have ucont_f: uniformly_continuous_on {0..1} f
       using compact_uniformly_continuous cont_f by blast
     then obtain δ where δ > 0
```

```
      and δ: ⋀x x'. ⟦x ∈ {0..1}; x' ∈ {0..1}; dist x' x < δ⟧ ⟹ norm (f x' − f
x) < e
       using ‹0 < e› by (auto simp: uniformly_continuous_on_def dist_norm)
      have ∗: ∃m::nat. ∃y. odd m ∧ 0 < m ∧ m < 2 ^ n ∧ y ∈ {a(m / 2^n) ..
b(m / 2^n)} ∧ f y = f x
       if n ≠ 0 for n
       using that
      proof (induction n)
       case 0 then show ?case by auto
      next
       case (Suc n)
       show ?case
       proof (cases n=0)
        case True
        consider x ∈ {0..u} | x ∈ {u..v} | x ∈ {v..1}
         using ‹0 ≤ x› ‹x ≤ 1› by force
        then have ∃y≥a (real 1/2). y ≤ b (real 1/2) ∧ f y = f x
        proof cases
         case 1
         then show ?thesis
          using uabv [of 1 1] f0u [of u] f0u [of x] by force
        next
         case 2
         then show ?thesis
          by (rule_tac x=x in exI) auto
        next
         case 3
         then show ?thesis
          using uabv [of 1 1] fv1 [of v] fv1 [of x] by force
        qed
        with ‹n=0› show ?thesis
         by (rule_tac x=1 in exI) auto
       next
        case False
        with Suc obtain m y
         where odd m 0 < m and mless: m < 2 ^ n
          and y: y ∈ {a (real m / 2 ^ n)..b (real m / 2 ^ n)} and feq: f y = f x
         by metis
        then obtain j where j: m = 2∗j + 1 by (metis oddE)
        have j4: 4 ∗ j + 1 < 2 ^ Suc n
         using mless j by (simp add: algebra_simps)

        consider y ∈ {a((2∗j + 1) / 2^n) .. b((4∗j + 1) / 2 ^ (Suc n))}
         | y ∈ {b((4∗j + 1) / 2 ^ (Suc n)) .. a((4∗j + 3) / 2 ^ (Suc n))}
         | y ∈ {a((4∗j + 3) / 2 ^ (Suc n)) .. b((2∗j + 1) / 2^n)}
         using y j by force
        then show ?thesis
        proof cases
         case 1
```

```
          show ?thesis
          proof (intro exI conjI)
            show y ∈ {a (real (4 * j + 1) / 2 ^ Suc n)..b (real (4 * j + 1) / 2 ^
Suc n)}
              using mless j ‹n ≠ 0› 1 by (simp add: a41 b41 add.commute [of 1]
del: power_Suc)
          qed (use feq j4 in auto)
        next
          case 2
          show ?thesis
          proof (intro exI conjI)
            show b (real (4 * j + 1) / 2 ^ Suc n) ∈ {a (real (4 * j + 1) / 2 ^
Suc n)..b (real (4 * j + 1) / 2 ^ Suc n)}
              using ‹n ≠ 0› alec [of 2*j+1 n] cleb [of 2*j+1 n] a_ge_0 [of 2*j+1
n] b_le_1 [of 2*j+1 n]
               using left_right [of c((2*j + 1) / 2^n) a((2*j + 1) / 2^n) b((2*j
+ 1) / 2^n)]
               by (simp add: a41 b41 add.commute [of 1] del: power_Suc)
            show f (b (real (4 * j + 1) / 2 ^ Suc n)) = f x
              using ‹n ≠ 0› 2
                using alec [of 2*j+1 n] cleb [of 2*j+1 n] a_ge_0 [of 2*j+1 n]
b_le_1 [of 2*j+1 n]
              by (force simp add: b41 a43 add.commute [of 1] feq [symmetric] simp
del: power_Suc intro: f_eqI)
          qed (use j4 in auto)
        next
          case 3
          show ?thesis
          proof (intro exI conjI)
            show 4 * j + 3 < 2 ^ Suc n
              using mless j by simp
            show f y = f x
              by fact
            show y ∈ {a (real (4 * j + 3) / 2 ^ Suc n) .. b (real (4 * j + 3) / 2
^ Suc n)}
              using 3 False b43 [of n j] by (simp add: add.commute)
          qed (use 3 in auto)
        qed
      qed
    qed
    obtain n where n: 1/2^n < min (δ / 2) 1
    by (metis ‹0 < δ› divide_less_eq_1 less_numeral_extra(1) min_less_iff_conj
one_less_numeral_iff power_one_over real_arch_pow_inv semiring_norm(76)
zero_less_divide_iff zero_less_numeral)
    with gr0I have n ≠ 0
      by fastforce
    with * obtain m::nat and y
      where odd m 0 < m and mless: m < 2 ^ n
        and y: a(m / 2^n) ≤ y ∧ y ≤ b(m / 2^n) and feq: f x = f y
```

```
      by (metis atLeastAtMost_iff)
    then have 0 ≤ y y ≤ 1
      by (meson a_ge_0 b_le_1 order.trans)+
    moreover have y < δ + c (real m / 2 ^ n) c (real m / 2 ^ n) < δ + y
      using y alec [of m n] cleb [of m n] n field_sum_of_halves close_ab [OF ‹odd m›, of n]
      by linarith+
    moreover note ‹0 < m› mless ‹0 ≤ x› ‹x ≤ 1›
    ultimately have dist (h (real m / 2 ^ n)) (f x) < e
      by (auto simp: dist_norm h_eq feq δ)
    then show ∃k. ∃m∈{0<..<2 ^ k}. dist (h (real m / 2 ^ k)) (f x) < e
      using ‹0 < m› greaterThanLessThan_iff mless by blast
  qed
  also have ... ⊆ h ‘ {0..1}
  proof (rule closure_minimal)
    show h ‘ D01 ⊆ h ‘ {0..1}
      using cloD01 closure_subset by blast
    show closed (h ‘ {0..1})
      using compact_continuous_image [OF cont_h] compact_imp_closed by auto
  qed
  finally show f ‘ {0..1} ⊆ h ‘ {0..1} .
 qed
 moreover have inj_on h {0..1}
 proof −
  have u < v
   by (metis atLeastAtMost_iff f0u f_1not0 fv1 order.not_eq_order_implies_strict u01(1) u01(2) v01(1))
  have f_not_fu: ⋀x. ⟦u < x; x ≤ v⟧ ⟹ f x ≠ f u
    by (metis atLeastAtMost_iff f0u fu1 greaterThanAtMost_iff order_refl order_trans u01(1) v01(2))
  have f_not_fv: ⋀x. ⟦u ≤ x; x < v⟧ ⟹ f x ≠ f v
   by (metis atLeastAtMost_iff order_refl order_trans v01(2) atLeastLessThan_iff fuv fv1)
  have a_less_b:
      a(j / 2^n) < b(j / 2^n) ∧
       (∀x. a(j / 2^n) < x ⟶ x ≤ b(j / 2^n) ⟶ f x ≠ f(a(j / 2^n))) ∧
        (∀x. a(j / 2^n) ≤ x ⟶ x < b(j / 2^n) ⟶ f x ≠ f(b(j / 2^n))) for n and j::nat
  proof (induction n arbitrary: j)
    case 0 then show ?case
      by (simp add: ‹u < v› f_not_fu f_not_fv)
  next
    case (Suc n j) show ?case
    proof (cases n > 0)
      case False then show ?thesis
        by (auto simp: a_def b_def leftrec_base rightrec_base ‹u < v› f_not_fu f_not_fv)
    next
```

```
      case True show ?thesis
      proof (cases even j)
        case True
        with ‹0 < n› Suc.IH show ?thesis
          by (auto elim!: evenE)
      next
        case False
        then obtain k where k: j = 2*k + 1 by (metis oddE)
        then show ?thesis
        proof (cases even k)
          case True
          then obtain m where m: k = 2*m by (metis evenE)
          have fleft: f (leftcut (a ((2*m + 1) / 2^n)) (b ((2*m + 1) / 2^n)) (c
((2*m + 1) / 2^n))) =
                       f (c((2*m + 1) / 2^n))
            using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
            using left_right_m [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n)
b((2*m + 1) / 2^n)]
            by (auto intro: f_eqI)
          show ?thesis
          proof (intro conjI impI notI allI)
            have False if b (real j / 2 ^ Suc n) ≤ a (real j / 2 ^ Suc n)
            proof -
              have f (c ((1 + real m * 2) / 2 ^ n)) = f (a ((1 + real m * 2) / 2
^ n))
                using k m ‹0 < n› fleft that a41 [of n m] b41 [of n m]
                using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
                  using left_right [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n)
b((2*m + 1) / 2^n)]
                by (auto simp: algebra_simps)
              moreover have a (real (1 + m * 2) / 2 ^ n) < c (real (1 + m *
2) / 2 ^ n)
                using Suc.IH [of 1 + m * 2] by (simp add: c_def midpoint_def)
              moreover have c (real (1 + m * 2) / 2 ^ n) ≤ b (real (1 + m * 2)
/ 2 ^ n)
                using cleb by blast
              ultimately show ?thesis
                using Suc.IH [of 1 + m * 2] by force
            qed
            then show a (real j / 2 ^ Suc n) < b (real j / 2 ^ Suc n) by force
          next
            fix x
            assume a (real j / 2 ^ Suc n) < x x ≤ b (real j / 2 ^ Suc n) f x = f
(a (real j / 2 ^ Suc n))
            then show False
              using Suc.IH [of 1 + m * 2, THEN conjunct2, THEN conjunct1]
              using k m ‹0 < n› a41 [of n m] b41 [of n m]
```

```
            using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
            using left_right_m [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n)
b((2*m + 1) / 2^n)]
           by (auto simp: algebra_simps)
         next
          fix x
          assume a (real j / 2 ^ Suc n) ≤ x x < b (real j / 2 ^ Suc n) f x = f
(b (real j / 2 ^ Suc n))
         then show False
          using k m ‹0 < n› a41 [of n m] b41 [of n m] fleft left_neq
           using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
          by (auto simp: algebra_simps)
         qed
        next
         case False
         with oddE obtain m where m: k = Suc (2*m) by fastforce
         have fright: f (rightcut (a ((2*m + 1) / 2^n)) (b ((2*m + 1) / 2^n))
(c ((2*m + 1) / 2^n))) = f (c((2*m + 1) / 2^n))
          using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
           using left_right_m [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n)
b((2*m + 1) / 2^n)]
         by (auto intro: f_eqI [OF _ order_refl])
         show ?thesis
         proof (intro conjI impI notI allI)
          have False if b (real j / 2 ^ Suc n) ≤ a (real j / 2 ^ Suc n)
          proof −
           have f (c ((1 + real m * 2) / 2 ^ n)) = f (b ((1 + real m * 2) / 2
^ n))
            using k m ‹0 < n› fright that a43 [of n m] b43 [of n m]
            using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
              using left_right [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n)
b((2*m + 1) / 2^n)]
            by (auto simp: algebra_simps)
           moreover have a (real (1 + m * 2) / 2 ^ n) ≤ c (real (1 + m *
2) / 2 ^ n)
            using alec by blast
          moreover have c (real (1 + m * 2) / 2 ^ n) < b (real (1 + m * 2)
/ 2 ^ n)
            using Suc.IH [of 1 + m * 2] by (simp add: c_def midpoint_def)
           ultimately show ?thesis
            using Suc.IH [of 1 + m * 2] by force
          qed
          then show a (real j / 2 ^ Suc n) < b (real j / 2 ^ Suc n) by force
         next
          fix x
```

```
            assume a (real j / 2 ^ Suc n) < x x ≤ b (real j / 2 ^ Suc n) f x = f
(a (real j / 2 ^ Suc n))
            then show False
              using k m ‹0 < n› a43 [of n m] b43 [of n m] fright right_neq
               using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
              by (auto simp: algebra_simps)
          next
            fix x
            assume a (real j / 2 ^ Suc n) ≤ x x < b (real j / 2 ^ Suc n) f x = f
(b (real j / 2 ^ Suc n))
            then show False
              using Suc.IH [of 1 + m * 2, THEN conjunct2, THEN conjunct2]
              using k m ‹0 < n› a43 [of n m] b43 [of n m]
               using alec [of 2*m+1 n] cleb [of 2*m+1 n] a_ge_0 [of 2*m+1 n]
b_le_1 [of 2*m+1 n]
                using left_right_m [of c((2*m + 1) / 2^n) a((2*m + 1) / 2^n)
b((2*m + 1) / 2^n)]
              by (auto simp: algebra_simps fright simp del: power_Suc)
          qed
        qed
      qed
    qed
   have c_gt_0 [simp]: 0 < c(m / 2^n) and c_less_1 [simp]: c(m / 2^n) < 1
for m::nat and n
     using a_less_b [of m n] apply (simp_all add: c_def midpoint_def)
     using a_ge_0 [of m n] b_le_1 [of m n] by linarith+
   have approx: ∃j n. odd j ∧ n ≠ 0 ∧
                      real i / 2^m ≤ real j / 2^n ∧
                      real j / 2^n ≤ real k / 2^p ∧
                      |real i / 2 ^ m − real j / 2 ^ n| < 1/2^n ∧
                      |real k / 2 ^ p − real j / 2 ^ n| < 1/2^n
     if 0 < i i < 2 ^ m 0 < k k < 2 ^ p i / 2^m < k / 2^p m + p = N for N m
p i k
     using that
   proof (induction N arbitrary: m p i k rule: less_induct)
     case (less N)
     then consider i / 2^m ≤ 1/2 1/2 ≤ k / 2^p | k / 2^p < 1/2 | k / 2^p ≥
1/2 1/2 < i / 2^m
       by linarith
     then show ?case
     proof cases
       case 1
       with less.prems show ?thesis
         by (rule_tac x=1 in exI)+ (fastforce simp: field_split_simps)
     next
       case 2 show ?thesis
       proof (cases m)
```

```
        case 0 with less.prems show ?thesis
          by auto
      next
        case (Suc m') show ?thesis
        proof (cases p)
          case 0 with less.prems show ?thesis by auto
        next
          case (Suc p')
          have §: False if real i * 2 ^ p' < real k * 2 ^ m' k < 2 ^ p' 2 ^ m' ≤ i
          proof −
            have real k * 2 ^ m' < 2 ^ p' * 2 ^ m'
              using that by simp
            then have real i * 2 ^ p' < 2 ^ p' * 2 ^ m'
              using that by linarith
            with that show ?thesis by simp
          qed
          moreover have ∗: real i / 2 ^ m' < real k / 2^p'  k < 2 ^ p'
         using less.prems ‹m = Suc m'› 2 Suc by (force simp: field_split_simps)+
          moreover have i < 2 ^ m'
            using § ∗ by (clarsimp simp: divide_simps linorder_not_le) (meson
linorder_not_le)
          ultimately show ?thesis
            using less.IH [of m'+p' i m' k p'] less.prems ‹m = Suc m'› 2 Suc
            by (force simp: field_split_simps)
        qed
      qed
    next
      case 3 show ?thesis
      proof (cases m)
        case 0 with less.prems show ?thesis
          by auto
      next
        case (Suc m') show ?thesis
        proof (cases p)
          case 0 with less.prems show ?thesis by auto
        next
          case (Suc p')
          have real (i − 2 ^ m') / 2 ^ m' < real (k − 2 ^ p') / 2 ^ p'
              using less.prems ‹m = Suc m'› Suc 3  by (auto simp: field_simps
of_nat_diff)
          moreover have k − 2 ^ p' < 2 ^ p' i − 2 ^ m' < 2 ^ m'
            using less.prems Suc ‹m = Suc m'› by auto
          moreover
          have 2 ^ p' ≤ k 2 ^ p' ≠ k
            using less.prems ‹m = Suc m'› Suc 3 by auto
          then have 2 ^ p' < k
            by linarith
          ultimately show ?thesis
           using less.IH [of m'+p' i − 2^m' m' k − 2 ^ p' p'] less.prems ‹m =
```

```
Suc m'› Suc 3
            apply (clarsimp simp: field_simps of_nat_diff)
            apply (rule_tac x=2 ^ n + j in exI, simp)
            apply (rule_tac x=Suc n in exI)
            apply (auto simp: field_simps)
            done
         qed
       qed
     qed
   qed
   have clec: c(real i / 2^m) ≤ c(real j / 2^n)
     if i: 0 < i i < 2 ^ m and j: 0 < j j < 2 ^ n and ij: i / 2^m < j / 2^n for
m i n j
   proof −
     obtain j' n' where odd j' n' ≠ 0
       and i_le_j: real i / 2 ^ m ≤ real j' / 2 ^ n'
       and j_le_j: real j' / 2 ^ n' ≤ real j / 2 ^ n
       and clo_ij: |real i / 2 ^ m − real j' / 2 ^ n'| < 1/2 ^ n'
       and clo_jj: |real j / 2 ^ n − real j' / 2 ^ n'| < 1/2 ^ n'
       using approx [of i m j n m+n] that i j ij by auto
     with oddE obtain q where q: j' = Suc (2*q) by fastforce
     have c (real i / 2 ^ m) ≤ c((2*q + 1) / 2^n')
     proof (cases i / 2^m = (2*q + 1) / 2^n')
       case True then show ?thesis by simp
     next
       case False
       with i_le_j clo_ij q have |real i / 2 ^ m − real (4 * q + 1) / 2 ^ Suc n'|
< 1 / 2 ^ Suc n'
         by (auto simp: field_split_simps)
       then have c(i / 2^m) ≤ b(real(4 * q + 1) / 2 ^ (Suc n'))
         by (meson ci_le_bj even_mult_iff even_numeral even_plus_one_iff)
       then show ?thesis
        using alec [of 2*q+1 n'] cleb [of 2*q+1 n'] a_ge_0 [of 2*q+1 n'] b_le_1
[of 2*q+1 n'] b41 [of n' q] ‹n' ≠ 0›
         using left_right_m [of c((2*q + 1) / 2^n') a((2*q + 1) / 2^n') b((2*q
+ 1) / 2^n')]
         by (auto simp: algebra_simps)
     qed
     also have ... ≤ c(real j / 2^n)
     proof (cases j / 2^n = (2*q + 1) / 2^n')
       case True
       then show ?thesis by simp
     next
       case False
       with j_le_j q have less: (2*q + 1) / 2^n' < j / 2^n
         by auto
       have ∗: ⟦q < i; abs(i − q) < s*2; r = q + s⟧ ⟹ abs(i − r) < s for i q s
r::real
         by auto
```

```
      have |real j / 2 ^ n - real (4 * q + 3) / 2 ^ Suc n'| < 1 / 2 ^ Suc n'
      by (rule * [OF less]) (use j_le_j clo_jj q in ‹auto simp: field_split_simps›)
      then have a(real(4*q + 3) / 2 ^ (Suc n')) ≤ c(j / 2^n)
      by (metis Suc3_eq_add_3 add.commute aj_le_ci even_Suc even_mult_iff
even_numeral)
      then show ?thesis
       using alec [of 2*q+1 n'] cleb [of 2*q+1 n'] a_ge_0 [of 2*q+1 n'] b_le_1
[of 2*q+1 n'] a43 [of n' q] ‹n' ≠ 0›
       using left_right_m [of c((2*q + 1) / 2^n') a((2*q + 1) / 2^n') b((2*q
+ 1) / 2^n')]
       by (auto simp: algebra_simps)
     qed
     finally show ?thesis .
   qed
   have x = y if 0 ≤ x x ≤ 1 0 ≤ y y ≤ 1 h x = h y for x y
    using that
   proof (induction x y rule: linorder_class.linorder_less_wlog)
    case (less x1 x2)
    obtain m n where m: 0 < m m < 2 ^ n
     and x12: x1 < m / 2^n m / 2^n < x2
     and neq: h x1 ≠ h (real m / 2^n)
    proof -
     have (x1 + x2) / 2 ∈ closure D01
      using cloD01 less.hyps less.prems by auto
     with less obtain y where y ∈ D01 and dist_y: dist y ((x1 + x2) / 2) <
(x2 - x1) / 64
      unfolding closure_approachable
      by (metis diff_gt_0_iff_gt less_divide_eq_numeral1(1) mult_zero_left)
     obtain m n where m: 0 < m m < 2 ^ n
                 and clo: |real m / 2 ^ n - (x1 + x2) / 2| < (x2 - x1) / 64
                 and n: 1/2^n < (x2 - x1) / 128
     proof -
      have min 1 ((x2 - x1) / 128) > 0 1/2 < (1::real)
       using less by auto
      then obtain N where N: 1/2^N < min 1 ((x2 - x1) / 128)
       by (metis power_one_over real_arch_pow_inv)
      then have N > 0
       using less_divide_eq_1 by force
      obtain p q where p: p < 2 ^ q p ≠ 0 and yeq: y = real p / 2 ^ q
       using ‹y ∈ D01› by (auto simp: zero_less_divide_iff D01_def)
      show ?thesis
      proof
       show 0 < 2^N * p
        using p by auto
       show 2 ^ N * p < 2 ^ (N+q)
        by (simp add: p power_add)
       have |real (2 ^ N * p) / 2 ^ (N + q) - (x1 + x2) / 2| = |real p / 2 ^
q - (x1 + x2) / 2|
        by (simp add: power_add)
```

```
        also have ... = |y − (x1 + x2) / 2|
          by (simp add: yeq)
        also have ... < (x2 − x1) / 64
          using dist_y by (simp add: dist_norm)
        finally show |real (2 ^ N * p) / 2 ^ (N + q) − (x1 + x2) / 2| < (x2
− x1) / 64 .
        have (1::real) / 2 ^ (N + q) ≤ 1/2^N
          by (simp add: field_simps)
        also have ... < (x2 − x1) / 128
          using N by force
        finally show 1/2 ^ (N + q) < (x2 − x1) / 128 .
      qed
    qed
    obtain m' n' m'' n'' where 0 < m' m' < 2 ^ n' x1 < m' / 2^n' m' / 2^n'
< x2
      and 0 < m'' m'' < 2 ^ n'' x1 < m'' / 2^n'' m'' / 2^n'' < x2
      and neq: h (real m'' / 2^n'') ≠ h (real m' / 2^n')
    proof
      show 0 < Suc (2*m)
        by simp
      show m21: Suc (2*m) < 2 ^ Suc n
        using m by auto
      show x1 < real (Suc (2 * m)) / 2 ^ Suc n
        using clo by (simp add: field_simps abs_if split: if_split_asm)
      show real (Suc (2 * m)) / 2 ^ Suc n < x2
        using n clo by (simp add: field_simps abs_if split: if_split_asm)
      show 0 < 4*m + 3
        by simp
      have m+1 ≤ 2 ^ n
        using m by simp
      then have 4 * (m+1) ≤ 4 * (2 ^ n)
        by simp
      then show m43: 4*m + 3 < 2 ^ (n+2)
        by (simp add: algebra_simps)
      show x1 < real (4 * m + 3) / 2 ^ (n + 2)
        using clo by (simp add: field_simps abs_if split: if_split_asm)
      show real (4 * m + 3) / 2 ^ (n + 2) < x2
        using n clo by (simp add: field_simps abs_if split: if_split_asm)
      have c_fold: midpoint (a ((2 * real m + 1) / 2 ^ Suc n)) (b ((2 * real m
+ 1) / 2 ^ Suc n)) = c ((2 * real m + 1) / 2 ^ Suc n)
        by (simp add: c_def)
      define R where R ≡ rightcut (a ((2 * real m + 1) / 2 ^ Suc n)) (b ((2
* real m + 1) / 2 ^ Suc n))  (c ((2 * real m + 1) / 2 ^ Suc n))
      have R < b ((2 * real m + 1) / 2 ^ Suc n)
        unfolding R_def  using a_less_b [of 4*m + 3 n+2] a43 [of Suc n m]
b43 [of Suc n m]
        by simp
      then have Rless: R < midpoint R (b ((2 * real m + 1) / 2 ^ Suc n))
        by (simp add: midpoint_def)
```

```
      have midR_le: midpoint R (b ((2 * real m + 1) / 2 ^ Suc n)) ≤ b ((2 *
real m + 1) / (2 * 2 ^ n))
        using ‹R < b ((2 * real m + 1) / 2 ^ Suc n)›
        by (simp add: midpoint_def)
      have (real (Suc (2 * m)) / 2 ^ Suc n) ∈ D01  real (4 * m + 3) / 2 ^(n
+ 2) ∈ D01
          by (simp_all add: D01_def m21 m43 del: power_Suc of_nat_Suc
of_nat_add add_2_eq_Suc') blast+
      then show h (real (4 * m + 3) / 2 ^(n + 2)) ≠ h (real (Suc (2 * m))
/ 2 ^ Suc n)
        using a_less_b [of 4*m + 3 n+2, THEN conjunct1]
        using a43 [of Suc n m] b43 [of Suc n m]
       using alec [of 2*m+1 Suc n] cleb [of 2*m+1 Suc n] a_ge_0 [of 2*m+1
Suc n]  b_le_1 [of 2*m+1 Suc n]
        apply (simp add: fc_eq [symmetric] c_def del: power_Suc)
        apply (simp only: add.commute [of 1] c_fold R_def [symmetric])
        apply (rule right_neq)
        using Rless apply (simp add: R_def)
          apply (rule midR_le, auto)
        done
    qed
    then show ?thesis by (metis that)
  qed
  have m_div: 0 < m / 2^n m / 2^n < 1
    using m  by (auto simp: field_split_simps)
  have closure0m: {0..m / 2^n} = closure ({0<..< m / 2^n} ∩ (⋃k m. {real
m / 2 ^ k}))
    by (subst closure_dyadic_rationals_in_convex_set_pos_1, simp_all add:
not_le m)
  have 2^n > m
    by (simp add: m(2) not_le)
  then have closurem1: {m / 2^n .. 1} = closure ({m / 2^n <..< 1} ∩ (⋃k
m. {real m / 2 ^ k}))
    using closure_dyadic_rationals_in_convex_set_pos_1 m_div(1) by fast-
force
  have cont_h': continuous_on (closure ({u<..<v} ∩ (⋃k m. {real m / 2 ^
k}))) h
    if 0 ≤ u v ≤ 1 for u v
    using that by (intro continuous_on_subset [OF cont_h] closure_minimal
[OF subsetI]) auto
  have closed_f': closed (f ‘ {u..v}) if 0 ≤ u v ≤ 1 for u v
    by (metis compact_continuous_image cont_f compact_interval atLeastat-
Most_subset_iff
        compact_imp_closed continuous_on_subset that)
  have less_2I: ⋀k i. real i / 2 ^ k < 1 ⟹ i < 2 ^ k
    by simp
  have h ‘ ({0<..<m / 2 ^ n} ∩ (⋃q p. {real p / 2 ^ q})) ⊆ f ‘ {0..c (m / 2
^ n)}
  proof clarsimp
```

```
      fix p q
      assume p: 0 < real p / 2 ^ q real p / 2 ^ q < real m / 2 ^ n
      then have [simp]: 0 < p
        by (simp add: field_split_simps)
      have [simp]: p < 2 ^ q
        by (blast intro: p less_2I m_div less_trans)
      have f (c (real p / 2 ^ q)) ∈ f ‘ {0..c (real m / 2 ^ n)}
        by (auto simp: clec p m)
      then show h (real p / 2 ^ q) ∈ f ‘ {0..c (real m / 2 ^ n)}
        by (simp add: h_eq)
    qed
    with m_div have h ‘ {0 .. m / 2^n} ⊆ f ‘ {0 .. c(m / 2^n)}
      apply (subst closure0m)
      by (rule image_closure_subset [OF cont_h′ closed_f′]) auto
    then have hx1: h x1 ∈ f ‘ {0 .. c(m / 2^n)}
      using x12 less.prems(1) by auto
    then obtain t1 where t1: h x1 = f t1 0 ≤ t1 t1 ≤ c (m / 2 ^ n)
      by auto
    have h ‘ ({m / 2 ^ n<..<1} ∩ (⋃ q p. {real p / 2 ^ q})) ⊆ f ‘ {c (m / 2 ^
n)..1}
    proof clarsimp
      fix p q
      assume p: real m / 2 ^ n < real p / 2 ^ q and [simp]: p < 2 ^ q
      then have [simp]: 0 < p
        using gr_zeroI m_div by fastforce
      have f (c (real p / 2 ^ q)) ∈ f ‘ {c (m / 2 ^ n)..1}
        by (auto simp: clec p m)
      then show h (real p / 2 ^ q) ∈ f ‘ {c (real m / 2 ^ n)..1}
        by (simp add: h_eq)
    qed
    with m have h ‘ {m / 2^n .. 1} ⊆ f ‘ {c(m / 2^n) .. 1}
      apply (subst closurem1)
      by (rule image_closure_subset [OF cont_h′ closed_f′]) auto
    then have hx2: h x2 ∈ f ‘ {c(m / 2^n)..1}
      using x12 less.prems by auto
    then obtain t2 where t2: h x2 = f t2 c (m / 2 ^ n) ≤ t2 t2 ≤ 1
      by auto
    with t1 less neq have False
    using conn [of h x2, unfolded is_interval_connected_1 [symmetric] is_interval_1,
rule_format, of t1 t2 c(m / 2^n)]
      by (simp add: h_eq m)
    then show ?case by blast
  qed auto
  then show ?thesis
    by (auto simp: inj_on_def)
qed
ultimately have {0..1::real} homeomorphic f ‘ {0..1}
  using homeomorphic_compact [OF _ cont_h] by blast
then show ?thesis
```

```
    using homeomorphic_sym by blast
qed


theorem path_contains_arc:
  fixes p :: real ⇒ 'a::{complete_space,real_normed_vector}
  assumes path p and a: pathstart p = a and b: pathfinish p = b and a ≠ b
  obtains q where arc q path_image q ⊆ path_image p pathstart q = a pathfinish
q = b
proof −
  have ucont_p: uniformly_continuous_on {0..1} p
    using ‹path p› unfolding path_def
    by (metis compact_Icc compact_uniformly_continuous)
  define φ where φ ≡ λS. S ⊆ {0..1} ∧ 0 ∈ S ∧ 1 ∈ S ∧
                       (∀ x ∈ S. ∀ y ∈ S. open_segment x y ∩ S = {} ⟶ p x = p y)
  obtain T where closed T φ T and T: ⋀U. ⟦closed U; φ U⟧ ⟹ ¬ (U ⊂ T)
  proof (rule Brouwer_reduction_theorem_gen [of {0..1} φ])
    have ∗: {x<..<y} ∩ {0..1} = {x<..<y} if 0 ≤ x y ≤ 1 x ≤ y for x y::real
      using that by auto
    show φ {0..1}
      by (auto simp: φ_def open_segment_eq_real_ivl ∗)
    show φ (⋂(F ‘ UNIV))
     if ⋀n. closed (F n) and φ: ⋀n. φ (F n) and Fsub: ⋀n. F (Suc n) ⊆ F n for
F
    proof −
      have F01: ⋀n. F n ⊆ {0..1} ∧ 0 ∈ F n ∧ 1 ∈ F n
        and peq: ⋀n x y. ⟦x ∈ F n; y ∈ F n; open_segment x y ∩ F n = {}⟧ ⟹ p
x = p y
        by (metis φ φ_def)+
      have pqF: False if ∀ u. x ∈ F u ∀ x. y ∈ F x open_segment x y ∩ (⋂x. F x)
= {} and neg: p x ≠ p y
        for x y
        using that
      proof (induction x y rule: linorder_class.linorder_less_wlog)
        case (less x y)
        have xy: x ∈ {0..1} y ∈ {0..1}
          by (metis less.prems subsetCE F01)+
        have norm(p x − p y) / 2 > 0
          using less by auto
        then obtain e where e > 0
          and e: ⋀u v. ⟦u ∈ {0..1}; v ∈ {0..1}; dist v u < e⟧ ⟹ dist (p v) (p u)
< norm(p x − p y) / 2
          by (metis uniformly_continuous_onE [OF ucont_p])
        have minxy: min e (y − x) < (y − x) ∗ (3 / 2)
          by (subst min_less_iff_disj) (simp add: less)
        define w where w ≡ x + (min e (y − x) / 3)
        define z where z ≡y − (min e (y − x) / 3)
        have w < z and w: w ∈ {x<..<y} and z: z ∈ {x<..<y}
          and wxe: norm(w − x) < e and zye: norm(z − y) < e
```

```
      using minxy ‹0 < e› less unfolding w_def z_def by auto
    have Fclo: ⋀T. T ∈ range F ⟹ closed T
      by (metis ‹⋀n. closed (F n)› image_iff)
    have eq: {w..z} ∩ ⋂(F ‘ UNIV) = {}
      using less w z by (simp add: open_segment_eq_real_ivl disjoint_iff)
    then obtain K where finite K and K: {w..z} ∩ (⋂ (F ‘ K)) = {}
      by (metis finite_subset_image compact_imp_fip [OF compact_interval
Fclo])
    then have K ≠ {}
      using ‹w < z› ‹{w..z} ∩ ⋂(F ‘ K) = {}› by auto
    define n where n ≡ Max K
    have n ∈ K unfolding n_def by (metis ‹K ≠ {}› ‹finite K› Max_in)
    have F n ⊆ ⋂ (F ‘ K)
   unfolding n_def by (metis Fsub Max_ge ‹K ≠ {}› ‹finite K› cINF_greatest
lift_Suc_antimono_le)
    with K have wzF_null: {w..z} ∩ F n = {}
      by (metis disjoint_iff_not_equal subset_eq)
    obtain u where u: u ∈ F n u ∈ {x..w} ({u..w} - {u}) ∩ F n = {}
    proof (cases w ∈ F n)
      case True
      then show ?thesis
        by (metis wzF_null ‹w < z› atLeastAtMost_iff disjoint_iff_not_equal
less_eq_real_def)
    next
      case False
      obtain u where u ∈ F n u ∈ {x..w} {u<..<w} ∩ F n = {}
      proof (rule segment_to_point_exists [of F n ∩ {x..w} w])
        show closed (F n ∩ {x..w})
          by (metis ‹⋀n. closed (F n)› closed_Int closed_real_atLeastAtMost)
        show F n ∩ {x..w} ≠ {}
      by (metis atLeastAtMost_iff disjoint_iff_not_equal greaterThanLessThan_iff
less.prems(1) less_eq_real_def w)
      qed (auto simp: open_segment_eq_real_ivl intro!: that)
      with False show thesis
        by (auto simp add: disjoint_iff less_eq_real_def intro!: that)
    qed
    obtain v where v: v ∈ F n v ∈ {z..y} ({z..v} - {v}) ∩ F n = {}
    proof (cases z ∈ F n)
      case True
      have z ∈ {w..z}
        using ‹w < z› by auto
      then show ?thesis
        by (metis wzF_null Int_iff True empty_iff)
    next
      case False
      show ?thesis
      proof (rule segment_to_point_exists [of F n ∩ {z..y} z])
        show closed (F n ∩ {z..y})
          by (metis ‹⋀n. closed (F n)› closed_Int closed_atLeastAtMost)
```

```
          show F n ∩ {z..y} ≠ {}
        by (metis atLeastAtMost_iff disjoint_iff_not_equal greaterThanLessThan_iff
less.prems(2) less_eq_real_def z)
          show ⋀b. ⟦b ∈ F n ∩ {z..y}; open_segment z b ∩ (F n ∩ {z..y}) = {}⟧
⟹ thesis
           proof
            show ⋀b. ⟦b ∈ F n ∩ {z..y}; open_segment z b ∩ (F n ∩ {z..y}) = {}⟧
⟹ ({z..b} − {b}) ∩ F n = {}
           using False by (auto simp: open_segment_eq_real_ivl less_eq_real_def)
           qed auto
         qed
       obtain u v where u ∈ {0..1} v ∈ {0..1} norm(u − x) < e norm(v − y)
< e p u = p v
       proof
         show u ∈ {0..1} v ∈ {0..1}
           by (metis F01 ‹u ∈ F n› ‹v ∈ F n› subsetD)+
         show norm(u − x) < e norm (v − y) < e
           using ‹u ∈ {x..w}› ‹v ∈ {z..y}› atLeastAtMost_iff real_norm_def wxe
zye by auto
         show p u = p v
         proof (rule peq)
           show u ∈ F n v ∈ F n
             by (auto simp: u v)
           have False if ξ ∈ F n u < ξ ξ < v for ξ
           proof −
             have ξ ∉ {z..v}
             by (metis DiffI disjoint_iff_not_equal less_irrefl singletonD that(1,3)
v(3))
             moreover have ξ ∉ {w..z} ∩ F n
               by (metis equals0D wzF_null)
             ultimately have ξ ∈ {u..w}
               using that by auto
             then show ?thesis
                   by (metis DiffI disjoint_iff_not_equal less_eq_real_def not_le
singletonD that(1,2) u(3))
           qed
           moreover
           have ⟦ξ ∈ F n; v < ξ; ξ < u⟧ ⟹ False for ξ
             using ‹u ∈ {x..w}› ‹v ∈ {z..y}› ‹w < z› by simp
           ultimately
           show open_segment u v ∩ F n = {}
             by (force simp: open_segment_eq_real_ivl)
         qed
       qed
       then show ?case
         using e [of x u] e [of y v] xy
         by (metis dist_norm dist_triangle_half_r order_less_irrefl)
     qed (auto simp: open_segment_commute)
```

show *?thesis*
unfolding φ*_def* **by** (*metis* (*no_types*, *opaque_lifting*) *INT_I Inf_lower2 rangeI that*(*3*) *F01 subsetCE pqF*)
qed
show *closed* {*0..1*::*real*} **by** *auto*
qed (*meson* φ*_def*)
then have $T \subseteq \{0..1\}$ $0 \in T$ $1 \in T$
and *peq*: $\bigwedge x\ y.\ [\![x \in T;\ y \in T;\ \textit{open\_segment}\ x\ y \cap T = \{\}]\!] \Longrightarrow p\ x = p\ y$
unfolding φ*_def* **by** *metis*+
then have $T \neq \{\}$ **by** *auto*
define h **where** $h \equiv \lambda x.\ p(\textit{SOME}\ y.\ y \in T \wedge \textit{open\_segment}\ x\ y \cap T = \{\})$
have $p\ y = p\ z$ **if** $y \in T$ $z \in T$ **and** *xyT*: *open_segment x y* $\cap$ $T = \{\}$ **and** *xzT*: *open_segment x z* $\cap$ $T = \{\}$
for $x\ y\ z$
proof (*cases* $x \in T$)
case *True*
with *that* **show** *?thesis* **by** (*metis* ‹φ T› φ*_def*)
next
case *False*
have *insert x* (*open_segment x y* $\cup$ *open_segment x z*) $\cap$ $T = \{\}$
by (*metis False Int_Un_distrib2 Int_insert_left Un_empty_right xyT xzT*)
moreover have *open_segment y z* $\cap$ $T \subseteq$ *insert x* (*open_segment x y* $\cup$ *open_segment x z*) $\cap$ T
by (*auto simp*: *open_segment_eq_real_ivl*)
ultimately have *open_segment y z* $\cap$ $T = \{\}$
by *blast*
with *that peq* **show** *?thesis* **by** *metis*
qed
then have *h_eq_p_gen*: $h\ x = p\ y$ **if** $y \in T$ *open_segment x y* $\cap$ $T = \{\}$ **for** $x\ y$
using *that* **unfolding** *h_def*
by (*metis* (*mono_tags*, *lifting*) *some_eq_ex*)
then have *h_eq_p*: $\bigwedge x.\ x \in T \Longrightarrow h\ x = p\ x$
by *simp*
have *disjoint*: $\bigwedge x.\ \exists y.\ y \in T \wedge \textit{open\_segment}\ x\ y \cap T = \{\}$
by (*meson* ‹$T \neq \{\}$› ‹*closed T*› *segment_to_point_exists*)
have *heq*: $h\ x = h\ x'$ **if** *open_segment x x'* $\cap$ $T = \{\}$ **for** $x\ x'$
proof (*cases* $x \in T \vee x' \in T$)
case *True*
then show *?thesis*
by (*metis h_eq_p h_eq_p_gen open_segment_commute that*)
next
case *False*
obtain $y\ y'$ **where** $y \in T$ *open_segment x y* $\cap$ $T = \{\}$ $h\ x = p\ y$
$y' \in T$ *open_segment x' y'* $\cap$ $T = \{\}$ $h\ x' = p\ y'$
by (*meson disjoint h_eq_p_gen*)
moreover have *open_segment y y'* $\subseteq$ (*insert x* (*insert x'* (*open_segment x y* $\cup$ *open_segment x' y'* $\cup$ *open_segment x x'*)))
by (*auto simp*: *open_segment_eq_real_ivl*)

```
    ultimately show ?thesis
      using False that by (fastforce simp add: h_eq_p intro!: peq)
  qed
  have h ‘ {0..1} homeomorphic {0..1::real}
  proof (rule homeomorphic_monotone_image_interval)
    show continuous_on {0..1} h
    proof (clarsimp simp add: continuous_on_iff)
      fix u ε::real
      assume 0 < ε 0 ≤ u u ≤ 1
       then obtain δ where δ > 0 and δ: ⋀v. v ∈ {0..1} ⟹ dist v u < δ ⟶
dist (p v) (p u) < ε / 2
        using ucont_p [unfolded uniformly_continuous_on_def]
        by (metis atLeastAtMost_iff half_gt_zero_iff)
      then have dist (h v) (h u) < ε if v ∈ {0..1} dist v u < δ for v
      proof (cases open_segment u v ∩ T = {})
        case True
        then show ?thesis
          using ‹0 < ε› heq by auto
      next
        case False
        have uvT: closed (closed_segment u v ∩ T) closed_segment u v ∩ T ≠ {}
          using False open_closed_segment by (auto simp: ‹closed T› closed_Int)
         obtain w where w ∈ T and w: w ∈ closed_segment u v open_segment u
w ∩ T = {}
        proof (rule segment_to_point_exists [OF uvT])
          fix b
         assume b ∈ closed_segment u v ∩ T open_segment u b ∩ (closed_segment
u v ∩ T) = {}
          then show thesis
              by (metis IntD1 IntD2 ends_in_segment(1) inf.orderE inf_assoc
subset_oc_segment that)
        qed
        then have puw: dist (p u) (p w) < ε / 2
            by (metis (no_types) ‹T ⊆ {0..1}› ‹dist v u < δ› δ dist_commute
dist_in_closed_segment le_less_trans subsetCE)
         obtain z where z ∈ T and z: z ∈ closed_segment u v open_segment v z
∩ T = {}
        proof (rule segment_to_point_exists [OF uvT])
          fix b
         assume b ∈ closed_segment u v ∩ T open_segment v b ∩ (closed_segment
u v ∩ T) = {}
          then show thesis
              by (metis IntD1 IntD2 ends_in_segment(2) inf.orderE inf_assoc
subset_oc_segment that)
        qed
        then have dist (p u) (p z) < ε / 2
       by (metis ‹T ⊆ {0..1}› ‹dist v u < δ› δ dist_commute dist_in_closed_segment
le_less_trans subsetCE)
        then show ?thesis
```

```
          using puw by (metis (no_types) ‹w ∈ T› ‹z ∈ T› dist_commute
dist_triangle_half_l h_eq_p_gen w(2) z(2))
    qed
    with ‹0 < δ› show ∃δ>0. ∀v∈{0..1}. dist v u < δ ⟶ dist (h v) (h u) < ε
by blast
  qed
  show connected ({0..1} ∩ h -` {z}) for z
  proof (clarsimp simp add: connected_iff_connected_component)
    fix u v
    assume huv_eq: h v = h u and uv: 0 ≤ u u ≤ 1 0 ≤ v v ≤ 1
    have ∃ T. connected T ∧ T ⊆ {0..1} ∧ T ⊆ h -` {h u} ∧ u ∈ T ∧ v ∈ T
    proof (intro exI conjI)
      show connected (closed_segment u v)
        by simp
      show closed_segment u v ⊆ {0..1}
        by (simp add: uv closed_segment_eq_real_ivl)
      have pxy: p x = p y
        if T ⊆ {0..1} 0 ∈ T 1 ∈ T x ∈ T y ∈ T
        and disjT: open_segment x y ∩ (T - open_segment u v) = {}
        and xynot: x ∉ open_segment u v y ∉ open_segment u v
        for x y
      proof (cases open_segment x y ∩ open_segment u v = {})
        case True
        then show ?thesis
          by (metis Diff_Int_distrib Diff_empty peq disjT ‹x ∈ T› ‹y ∈ T›)
      next
        case False
         then have open_segment x u ∪ open_segment y v ⊆ open_segment x y
- open_segment u v ∨
                   open_segment y u ∪ open_segment x v ⊆ open_segment x y -
open_segment u v (is ?xuyv ∨ ?yuxv)
          using xynot by (fastforce simp add: open_segment_eq_real_ivl not_le
not_less split: if_split_asm)
        then show p x = p y
        proof
          assume ?xuyv
          then have open_segment x u ∩ T = {} open_segment y v ∩ T = {}
            using disjT by auto
          then have h x = h y
            using heq huv_eq by auto
          then show ?thesis
            using h_eq_p ‹x ∈ T› ‹y ∈ T› by auto
        next
          assume ?yuxv
          then have open_segment y u ∩ T = {} open_segment x v ∩ T = {}
            using disjT by auto
          then have h x = h y
            using heq [of y u]  heq [of x v] huv_eq by auto
          then show ?thesis
```

```
          using h_eq_p ‹x ∈ T› ‹y ∈ T› by auto
        qed
      qed
      have ¬ T − open_segment u v ⊂ T
      proof (rule T)
        show closed (T − open_segment u v)
          by (simp add: closed_Diff [OF ‹closed T›] open_segment_eq_real_ivl)
        have 0 ∉ open_segment u v 1 ∉ open_segment u v
          using open_segment_eq_real_ivl uv by auto
        then show φ (T − open_segment u v)
          using ‹T ⊆ {0..1}› ‹0 ∈ T› ‹1 ∈ T›
          by (auto simp: φ_def) (meson peq pxy)
      qed
      then have open_segment u v ∩ T = {}
        by blast
      then show closed_segment u v ⊆ h −‘ {h u}
     by (force intro: heq simp: open_segment_eq_real_ivl closed_segment_eq_real_ivl
split: if_split_asm)+
    qed auto
    then show connected_component ({0..1} ∩ h −‘ {h u}) u v
      by (simp add: connected_component_def)
  qed
  show h 1 ≠ h 0
    by (metis ‹φ T› φ_def a ‹a ≠ b› b h_eq_p pathfinish_def pathstart_def)
 qed
 then obtain f and g :: real ⇒ 'a
   where gfeq: (∀x∈h ‘ {0..1}. (g(f x) = x)) and fhim: f ‘ h ‘ {0..1} = {0..1}
and contf: continuous_on (h ‘ {0..1}) f
     and fgeq: (∀y∈{0..1}. (f(g y) = y)) and pag: path_image g = h ‘ {0..1}
and contg: continuous_on {0..1} g
   by (auto simp: homeomorphic_def homeomorphism_def path_image_def)
 then have arc g
   by (metis arc_def path_def inj_on_def)
 obtain u v where u ∈ {0..1} a = g u v ∈ {0..1} b = g v
   by (metis (mono_tags, opaque_lifting) ‹φ T› φ_def a b fhim gfeq h_eq_p im-
ageI path_image_def pathfinish_def pathfinish_in_path_image pathstart_def path-
start_in_path_image)
 then have a ∈ path_image g b ∈ path_image g
   using path_image_def by blast+
 have ph: path_image h ⊆ path_image p
  by (metis image_mono image_subset_iff path_image_def disjoint h_eq_p_gen
‹T ⊆ {0..1}›)
 show ?thesis
 proof
   show pathstart (subpath u v g) = a pathfinish (subpath u v g) = b
     by (simp_all add: ‹a = g u› ‹b = g v›)
   show path_image (subpath u v g) ⊆ path_image p
       by (metis ‹u ∈ {0..1}› ‹v ∈ {0..1}› order_trans pag path_image_def
path_image_subpath_subset ph)
```

```
    show arc (subpath u v g)
     using ‹arc g› ‹a = g u› ‹b = g v› ‹u ∈ {0..1}› ‹v ∈ {0..1}› arc_subpath_arc
‹a ≠ b› by blast
  qed
qed


corollary path_connected_arcwise:
  fixes S :: 'a::{complete_space,real_normed_vector} set
  shows path_connected S ⟷
        (∀ x ∈ S. ∀ y ∈ S. x ≠ y ⟶ (∃ g. arc g ∧ path_image g ⊆ S ∧ pathstart g
= x ∧ pathfinish g = y))
        (is ?lhs = ?rhs)
proof (intro iffI impI ballI)
  fix x y
  assume path_connected S x ∈ S y ∈ S x ≠ y
  then obtain p where p: path p path_image p ⊆ S pathstart p = x pathfinish p
= y
    by (force simp: path_connected_def)
  then show ∃ g. arc g ∧ path_image g ⊆ S ∧ pathstart g = x ∧ pathfinish g = y
    by (metis ‹x ≠ y› order_trans path_contains_arc)
next
  assume R [rule_format]: ?rhs
  show ?lhs
    unfolding path_connected_def
  proof (intro ballI)
    fix x y
    assume x ∈ S y ∈ S
    show ∃ g. path g ∧ path_image g ⊆ S ∧ pathstart g = x ∧ pathfinish g = y
    proof (cases x = y)
      case True with ‹x ∈ S› path_component_def path_component_refl show
?thesis
        by blast
    next
      case False with R [OF ‹x ∈ S› ‹y ∈ S›] show ?thesis
        by (auto intro: arc_imp_path)
    qed
  qed
qed


corollary arc_connected_trans:
  fixes g :: real ⇒ 'a::{complete_space,real_normed_vector}
  assumes arc g arc h pathfinish g = pathstart h pathstart g ≠ pathfinish h
  obtains i where arc i path_image i ⊆ path_image g ∪ path_image h
                  pathstart i = pathstart g pathfinish i = pathfinish h
  by (metis (no_types, opaque_lifting) arc_imp_path assms path_contains_arc
path_image_join path_join pathfinish_join pathstart_join)
```

6.10.4 Accessibility of frontier points

lemma *dense_accessible_frontier_points*:
fixes S :: *'a*::{*complete_space*,*real_normed_vector*} *set*
assumes *open S* **and** *openSV*: *openin* (*top_of_set* (*frontier S*)) V **and** $V \neq \{\}$
obtains *g* **where** *arc g g* ‘ $\{0..<1\} \subseteq S$ *pathstart g* $\in S$ *pathfinish g* $\in V$
proof −
obtain z **where** $z \in V$
using ‹$V \neq \{\}$› **by** *auto*
then obtain r **where** $r > 0$ **and** r: *ball z r* ∩ *frontier S* $\subseteq V$
by (*metis openin_contains_ball openSV*)
then have $z \in$ *frontier S*
using ‹$z \in V$› *openSV openin_contains_ball* **by** *blast*
then have $z \in$ *closure S z* $\notin S$
by (*simp_all add*: *frontier_def assms interior_open*)
with ‹$r > 0$› **have** *infinite* (*S* ∩ *ball z r*)
by (*auto simp*: *closure_def islimpt_eq_infinite_ball*)
then obtain y **where** $y \in S$ **and** y: $y \in$ *ball z r*
using *infinite_imp_nonempty* **by** *force*
then have $y \notin$ *frontier S*
by (*meson* ‹*open S*› *disjoint_iff_not_equal frontier_disjoint_eq*)
have $y \neq z$
using ‹$y \in S$› ‹$z \notin S$› **by** *blast*
have *path_connected*(*ball z r*)
by (*simp add*: *convex_imp_path_connected*)
with *y* ‹$r > 0$› **obtain** *g* **where** *arc g* **and** *pig*: *path_image g* $\subseteq$ *ball z r*
and *g*: *pathstart g* $= y$ *pathfinish g* $= z$
using ‹$y \neq z$› **by** (*force simp*: *path_connected_arcwise*)
have *continuous_on* $\{0..1\}$ *g*
using ‹*arc g*› *arc_imp_path path_def* **by** *blast*
then have *compact* (*g* −‘ *frontier S* ∩ $\{0..1\}$)
by (*simp add*: *bounded_Int closed_Diff closed_vimage_Int compact_eq_bounded_closed*)
moreover have *g* −‘ *frontier S* ∩ $\{0..1\} \neq \{\}$
proof −
have $\exists r.\ r \in g$ −‘ *frontier S* $\wedge\ r \in \{0..1\}$
by (*metis* ‹$z \in$ *frontier S*› *g*(*2*) *imageE path_image_def pathfinish_in_path_image vimageI2*)
then show *?thesis*
by *blast*
qed
ultimately obtain t **where** *gt*: *g t* $\in$ *frontier S* **and** $0 \leq t\ t \leq 1$
and t: $\bigwedge u.\ [\![g\ u \in$ *frontier S*; $0 \leq u$; $u \leq 1]\!] \Longrightarrow t \leq u$
by (*force simp*: *dest*!: *compact_attains_inf*)
moreover have $t \neq 0$
by (*metis* ‹$y \notin$ *frontier S*› *g*(*1*) *gt pathstart_def*)
ultimately have *t01*: $0 < t\ t \leq 1$
by *auto*
have *V* $\subseteq$ *frontier S*
using *openSV openin_contains_ball* **by** *blast*
show *?thesis*

```
  proof
    show arc (subpath 0 t g)
      by (simp add: ‹0 ≤ t› ‹t ≤ 1› ‹arc g› ‹t ≠ 0› arc_subpath_arc)
    have g 0 ∈ S
      by (metis ‹y ∈ S› g(1) pathstart_def)
    then show pathstart (subpath 0 t g) ∈ S
      by auto
    have g t ∈ V
     by (metis IntI atLeastAtMost_iff gt image_eqI path_image_def pig r subsetCE
‹0 ≤ t› ‹t ≤ 1›)
    then show pathfinish (subpath 0 t g) ∈ V
      by auto
    then have inj_on (subpath 0 t g) {0..1}
      using t01 ‹arc (subpath 0 t g)› arc_imp_inj_on by blast
    then have subpath 0 t g ` {0..<1} ⊆ subpath 0 t g ` {0..1} − {subpath 0 t g
1}
      by (force simp: dest: inj_onD)
    moreover have False if subpath 0 t g ` ({0..<1}) − S ≠ {}
    proof −
      have contg: continuous_on {0..1} g
        using ‹arc g› by (auto simp: arc_def path_def)
      have subpath 0 t g ` {0..<1} ∩ frontier S ≠ {}
      proof (rule connected_Int_frontier [OF _ _ that])
        show connected (subpath 0 t g ` {0..<1})
        proof (rule connected_continuous_image)
          show continuous_on {0..<1} (subpath 0 t g)
         by (meson ‹arc (subpath 0 t g)› arc_def atLeastLessThan_subseteq_atLeastAtMost_iff
continuous_on_subset order_refl path_def)
        qed auto
        show subpath 0 t g ` {0..<1} ∩ S ≠ {}
         using ‹y ∈ S› g(1) by (force simp: subpath_def image_def pathstart_def)
      qed
      then obtain x where x ∈ subpath 0 t g ` {0..<1} x ∈ frontier S
        by blast
      with t01 ‹0 ≤ t› mult_le_one t show False
        by (fastforce simp: subpath_def)
    qed
    then have subpath 0 t g ` {0..1} − {subpath 0 t g 1} ⊆ S
      using subsetD by fastforce
    ultimately  show subpath 0 t g ` {0..<1} ⊆ S
      by auto
  qed
qed


lemma dense_accessible_frontier_points_connected:
  fixes S :: 'a::{complete_space,real_normed_vector} set
  assumes open S connected S x ∈ S V ≠ {}
      and ope: openin (top_of_set (frontier S)) V
```

```
  obtains g where arc g g ' {0..<1} ⊆ S pathstart g = x pathfinish g ∈ V
proof −
  have V ⊆ frontier S
    using ope openin_imp_subset by blast
  with ‹open S› ‹x ∈ S› have x ∉ V
    using interior_open by (auto simp: frontier_def)
  obtain g where arc g and g: g ' {0..<1} ⊆ S pathstart g ∈ S pathfinish g ∈ V
    by (metis dense_accessible_frontier_points [OF ‹open S› ope ‹V ≠ {}›])
  then have path_connected S
    by (simp add: assms connected_open_path_connected)
  with ‹pathstart g ∈ S› ‹x ∈ S› have path_component S x (pathstart g)
    by (simp add: path_connected_component)
  then obtain f where path f and f: path_image f ⊆ S pathstart f = x pathfinish
f = pathstart g
    by (auto simp: path_component_def)
  then have path (f +++ g)
    by (simp add: ‹arc g› arc_imp_path)
  then obtain h where arc h
                  and h: path_image h ⊆ path_image (f +++ g) pathstart h = x
pathfinish h = pathfinish g
    using path_contains_arc [of f +++ g x pathfinish g] ‹x ∉ V› ‹pathfinish g ∈
V› f
    by (metis pathfinish_join pathstart_join)
  have path_image h ⊆ path_image f ∪ path_image g
    using h(1) path_image_join_subset by auto
  then have h ' {0..1} − {h 1} ⊆ S
    using f g h
   apply (simp add: path_image_def pathfinish_def subset_iff image_def Bex_def)
    by (metis le_less)
  then have h ' {0..<1} ⊆ S
    using ‹arc h› by (force simp: arc_def dest: inj_onD)
  then show thesis
    using ‹arc h› g(3) h that by presburger
qed

lemma dense_access_fp_aux:
  fixes S :: 'a::{complete_space,real_normed_vector} set
  assumes S: open S connected S
      and opeSU: openin (top_of_set (frontier S)) U
      and opeSV: openin (top_of_set (frontier S)) V
      and V ≠ {} ¬ U ⊆ V
  obtains g where arc g pathstart g ∈ U pathfinish g ∈ V g ' {0<..<1} ⊆ S
proof −
  have S ≠ {}
    using opeSV ‹V ≠ {}› by (metis frontier_empty openin_subtopology_empty)
  then obtain x where x ∈ S by auto
  obtain g where arc g and g: g ' {0..<1} ⊆ S pathstart g = x pathfinish g ∈ V
    using dense_accessible_frontier_points_connected [OF S ‹x ∈ S› ‹V ≠ {}›
opeSV] by blast
```

```
  obtain h where arc h and h: h ‘ {0..<1} ⊆ S pathstart h = x pathfinish h ∈ U
− {pathfinish g}
  proof (rule dense_accessible_frontier_points_connected [OF S ‹x ∈ S›])
    show U − {pathfinish g} ≠ {}
      using ‹pathfinish g ∈ V› ‹¬ U ⊆ V› by blast
    show openin (top_of_set (frontier S)) (U − {pathfinish g})
      by (simp add: opeSU openin_delete)
  qed auto
  obtain γ where arc γ
            and γ: path_image γ ⊆ path_image (reversepath h +++ g)
                   pathstart γ = pathfinish h pathfinish γ = pathfinish g
  proof (rule path_contains_arc [of (reversepath h +++ g) pathfinish h pathfinish
g])
    show path (reversepath h +++ g)
     by (simp add: ‹arc g› ‹arc h› ‹pathstart g = x› ‹pathstart h = x› arc_imp_path)
    show pathstart (reversepath h +++ g) = pathfinish h
         pathfinish (reversepath h +++ g) = pathfinish g
      by auto
    show pathfinish h ≠ pathfinish g
      using ‹pathfinish h ∈ U − {pathfinish g}› by auto
  qed auto
  show ?thesis
  proof
    show arc γ pathstart γ ∈ U pathfinish γ ∈ V
      using γ ‹arc γ› ‹pathfinish h ∈ U − {pathfinish g}› ‹pathfinish g ∈ V› by
auto
    have path_image γ ⊆ path_image h ∪ path_image g
      by (metis γ(1) g(2) h(2) path_image_join path_image_reversepath pathfin-
ish_reversepath)
    then have γ ‘ {0..1} − {γ 0, γ 1} ⊆ S
      using γ g h
       apply (simp add: path_image_def pathstart_def pathfinish_def subset_iff
image_def Bex_def)
      by (metis linorder_neqE_linordered_idom not_less)
    then show γ ‘ {0<..<1} ⊆ S
      using ‹arc h› ‹arc γ›
     by (metis arc_imp_simple_path path_image_def pathfinish_def pathstart_def
simple_path_endless)
  qed
qed

lemma dense_accessible_frontier_point_pairs:
  fixes S :: 'a::{complete_space,real_normed_vector} set
  assumes S: open S connected S
      and opeSU: openin (top_of_set (frontier S)) U
      and opeSV: openin (top_of_set (frontier S)) V
      and U ≠ {} V ≠ {} U ≠ V
    obtains g where arc g pathstart g ∈ U pathfinish g ∈ V g ‘ {0<..<1} ⊆ S
proof −
```

```
  consider ¬ U ⊆ V | ¬ V ⊆ U
    using ‹U ≠ V› by blast
  then show ?thesis
  proof cases
    case 1 then show ?thesis
      using assms dense_access_fp_aux [OF S opeSU opeSV] that by blast
  next
    case 2
    obtain g where arc g and g: pathstart g ∈ V pathfinish g ∈ U g ‘ {0<..<1}
⊆ S
      using assms dense_access_fp_aux [OF S opeSV opeSU] 2 by blast
    show ?thesis
    proof
      show arc (reversepath g)
        by (simp add: ‹arc g› arc_reversepath)
      show pathstart (reversepath g) ∈ U pathfinish (reversepath g) ∈ V
        using g by auto
      show reversepath g ‘ {0<..<1} ⊆ S
        using g by (auto simp: reversepath_def)
    qed
  qed
qed

end
```

6.11 The Urysohn lemma, its consequences and other advanced material about metric spaces

```
theory Urysohn
imports Abstract_Topological_Spaces Abstract_Metric_Spaces Infinite_Sum Ar-
cwise_Connected
begin
```

6.11.1 Urysohn lemma and Tietze's theorem

```
proposition Urysohn_lemma:
  fixes a b :: real
  assumes normal_space X closedin X S closedin X T disjnt S T a ≤ b
  obtains f where continuous_map X (top_of_set {a..b}) f f ‘ S ⊆ {a} f ‘ T ⊆
{b}
proof −
  obtain U where openin X U S ⊆ U X closure_of U ⊆ topspace X − T
    using assms unfolding normal_space_alt disjnt_def
    by (metis Diff_mono Un_Diff_Int closedin_def subset_eq sup_bot_right)
  have ∃ G :: real ⇒ 'a set. G 0 = U ∧ G 1 = topspace X − T ∧
            (∀ x ∈ dyadics ∩ {0..1}. ∀ y ∈ dyadics ∩ {0..1}. x < y ⟶ openin X
(G x) ∧ openin X (G y) ∧ X closure_of (G x) ⊆ G y)
  proof (rule recursion_on_dyadic_fractions)
```

show *openin X U ∧ openin X (topspace X − T) ∧ X closure_of U ⊆ topspace X − T*
using ‹*X closure_of U ⊆ topspace X − T*› ‹*openin X U*› ‹*closedin X T*› **by** *blast*
show ∃*z*. (*openin X x ∧ openin X z ∧ X closure_of x ⊆ z*) ∧ *openin X z ∧ openin X y ∧ X closure_of z ⊆ y*
if *openin X x ∧ openin X y ∧ X closure_of x ⊆ y* **for** *x y*
by (*meson that closedin_closure_of normal_space_alt* ‹*normal_space X*›)
show *openin X x ∧ openin X z ∧ X closure_of x ⊆ z*
if *openin X x ∧ openin X y ∧ X closure_of x ⊆ y* **and** *openin X y ∧ openin X z ∧ X closure_of y ⊆ z* **for** *x y z*
by (*meson that closure_of_subset openin_subset subset_trans*)
qed
then obtain *G* :: *real* ⇒ *'a set*
where *G0*: *G 0 = U* **and** *G1*: *G 1 = topspace X − T*
and *G*: ⋀*x y*. ⟦*x ∈ dyadics*; *y ∈ dyadics*; *0 ≤ x*; *x < y*; *y ≤ 1*⟧
⟹ *openin X (G x) ∧ openin X (G y) ∧ X closure_of (G x) ⊆ G y*
by (*smt* (*verit*, *del_insts*) *Int_iff atLeastAtMost_iff*)
define *f* **where** *f* ≡ λ*x*. *Inf*(*insert 1* {*r*. *r ∈ dyadics ∩ {0..1} ∧ x ∈ G r*})
have *f_ge*: *f x ≥ 0* **if** *x ∈ topspace X* **for** *x*
unfolding *f_def* **by** (*force intro*: *cInf_greatest*)
moreover have *f_le1*: *f x ≤ 1* **if** *x ∈ topspace X* **for** *x*
proof −
have *bdd_below* {*r ∈ dyadics ∩ {0..1}*. *x ∈ G r*}
by (*auto simp*: *bdd_below_def*)
then show *?thesis*
by (*auto simp*: *f_def cInf_lower*)
qed
ultimately have *fim*: *f ‘ topspace X ⊆ {0..1}*
by (*auto simp*: *f_def*)
have *0*: *0 ∈ dyadics ∩ {0..1*::*real}* **and** *1*: *1 ∈ dyadics ∩ {0..1*::*real}*
by (*force simp*: *dyadics_def*)+
then have *opeG*: *openin X (G r)* **if** *r ∈ dyadics ∩ {0..1}* **for** *r*
using *G G0* ‹*openin X U*› *less_eq_real_def that* **by** *auto*
have *x ∈ G 0* **if** *x ∈ S* **for** *x*
using *G0* ‹*S ⊆ U*› *that* **by** *blast*
with *0* **have** *fimS*: *f ‘ S ⊆ {0}*
unfolding *f_def* **by** (*force intro*!: *cInf_eq_minimum*)
have *False* **if** *r ∈ dyadics 0 ≤ r r < 1 x ∈ G r x ∈ T* **for** *r x*
using *G* [*of r 1*] *1*
by (*smt* (*verit*, *best*) *DiffD2 G1 Int_iff closure_of_subset inf.orderE openin_subset that*)
then have *r≥1* **if** *r ∈ dyadics 0 ≤ r r ≤ 1 x ∈ G r x ∈ T* **for** *r x*
using *linorder_not_le that* **by** *blast*
then have *fimT*: *f ‘ T ⊆ {1}*
unfolding *f_def* **by** (*force intro*!: *cInf_eq_minimum*)
have *fle1*: *f z ≤ 1* **for** *z*
by (*force simp*: *f_def intro*: *cInf_lower*)

```
  have fle: f z ≤ x if x ∈ dyadics ∩ {0..1} z ∈ G x for z x
    using that by (force simp: f_def intro: cInf_lower)
  have ∗: b ≤ f z if b ≤ 1 ⋀x. ⟦x ∈ dyadics ∩ {0..1}; z ∈ G x⟧ ⟹ b ≤ x for z b
    using that by (force simp: f_def intro: cInf_greatest)
  have ∗∗: r ≤ f x if r: r ∈ dyadics ∩ {0..1} x ∉ G r for r x
  proof (rule ∗)
    show r ≤ s if s ∈ dyadics ∩ {0..1} and x ∈ G s for s :: real
     using that r G [of s r] by (force simp: dest: closure_of_subset openin_subset)
  qed (use that in force)

  have ∃ U. openin X U ∧ x ∈ U ∧ (∀ y ∈ U. |f y − f x| < ε)
    if x ∈ topspace X and 0 < ε for x ε
  proof −
    have A: ∃ r. r ∈ dyadics ∩ {0..1} ∧ r < y ∧ |r − y| < d if 0 < y y ≤ 1 0 <
d for y d::real
    proof −
      obtain n q r
        where of_nat q / 2^n < y y < of_nat r / 2^n |q / 2^n − r / 2^n | < d
        by (smt (verit, del_insts) padic_rational_approximation_straddle_pos ‹0
< d› ‹0 < y›)
      then show ?thesis
        unfolding dyadics_def
        using divide_eq_0_iff that(2) by fastforce
    qed
    have B: ∃ r. r ∈ dyadics ∩ {0..1} ∧ y < r ∧ |r − y| < d if 0 ≤ y y < 1 0 <
d for y d::real
    proof −
      obtain n q r
        where of_nat q / 2^n ≤ y y < of_nat r / 2^n |q / 2^n − r / 2^n | < d
        using padic_rational_approximation_straddle_pos_le
        by (smt (verit, del_insts) ‹0 < d› ‹0 ≤ y›)
      then show ?thesis
        apply (clarsimp simp: dyadics_def)
        using divide_eq_0_iff ‹y < 1›
      by (smt (verit) divide_nonneg_nonneg divide_self of_nat_0_le_iff of_nat_1
power_0 zero_le_power)
    qed
    show ?thesis
    proof (cases f x = 0)
      case True
      with B[of 0] obtain r where r: r ∈ dyadics ∩ {0..1} 0 < r |r| < ε/2
        by (smt (verit) ‹0 < ε› half_gt_zero)
      show ?thesis
      proof (intro exI conjI)
        show openin X (G r)
          using opeG r(1) by blast
        show x ∈ G r
          using True ∗∗ r by force
        show ∀ y ∈ G r. |f y − f x| < ε
```

```
      using f_ge ‹openin X (G r)› fle openin_subset r by (fastforce simp: True)
    qed
  next
    case False
    show ?thesis
    proof (cases f x = 1)
      case True
      with A[of 1] obtain r where r: r ∈ dyadics ∩ {0..1} r < 1 |r−1| < ε/2
        by (smt (verit) ‹0 < ε› half_gt_zero)
      define G' where G' ≡ topspace X − X closure_of G r
      show ?thesis
      proof (intro exI conjI)
        show openin X G'
          unfolding G'_def by fastforce
        obtain r' where r' ∈ dyadics ∧ 0 ≤ r' ∧ r' ≤ 1 ∧ r < r' ∧ |r' − r| <
1 − r
          using B r by force
        moreover
        have 1 − r ∈ dyadics 0 ≤ r
          using 1 r dyadics_diff by force+
        ultimately have x ∉ X closure_of G r
          using G True r fle by force
        then show x ∈ G'
          by (simp add: G'_def that)
        show ∀ y ∈ G'. |f y − f x| < ε
          using ∗∗ f_le1 in_closure_of r by (fastforce simp: True G'_def)
      qed
    next
      case False
      have 0 < f x f x < 1
        using fle1 f_ge that(1) ‹f x ≠ 0› ‹f x ≠ 1› by (metis order_le_less) +
      obtain r where r: r ∈ dyadics ∩ {0..1} r < f x |r − f x| < ε / 2
        using A ‹0 < ε› ‹0 < f x› ‹f x < 1› by (smt (verit, best) half_gt_zero)
      obtain r' where r': r' ∈ dyadics ∩ {0..1} f x < r' |r' − f x| < ε / 2
        using B ‹0 < ε› ‹0 < f x› ‹f x < 1› by (smt (verit, best) half_gt_zero)
      have r < 1
        using ‹f x < 1› r(2) by force
      show ?thesis
      proof (intro conjI exI)
        show openin X (G r' − X closure_of G r)
          using closedin_closure_of opeG r' by blast
        have x ∈ X closure_of G r ⟹ False
          using B [of r f x − r] r ‹r < 1› G [of r] fle by force
        then show x ∈ G r' − X closure_of G r
          using ∗∗ r' by fastforce
        show ∀ y∈G r' − X closure_of G r. |f y − f x| < ε
        using r r' ∗∗ G closure_of_subset field_sum_of_halves fle openin_subset
subset_eq
          by (smt (verit) DiffE opeG)
```

```
      qed
    qed
  qed
 qed
 then have contf: continuous_map X (top_of_set {0..1}) f
   by (force simp: Met_TC.continuous_map_to_metric dist_real_def continuous_map_in_subtopology fim simp flip: mtopology_is_euclidean)
 define g where g ≡ λx. a + (b − a) ∗ f x
 show thesis
 proof
   have continuous_map X euclideanreal g
     using contf ‹a ≤ b› unfolding g_def by (auto simp: continuous_intros continuous_map_in_subtopology)
   moreover have g ‘ (topspace X) ⊆ {a..b}
     using mult_left_le [of f _ b−a] contf ‹a ≤ b›
     by (simp add: g_def add.commute continuous_map_in_subtopology image_subset_iff le_diff_eq)
   ultimately show continuous_map X (top_of_set {a..b}) g
     by (meson continuous_map_in_subtopology)
   show g ‘ S ⊆ {a} g ‘ T ⊆ {b}
     using fimS fimT by (auto simp: g_def)
 qed
qed

lemma Urysohn_lemma_alt:
 fixes a b :: real
 assumes normal_space X closedin X S closedin X T disjnt S T
 obtains f where continuous_map X euclideanreal f f ‘ S ⊆ {a} f ‘ T ⊆ {b}
 by (metis Urysohn_lemma assms continuous_map_in_subtopology disjnt_sym linear)

lemma normal_space_iff_Urysohn_gen_alt:
 assumes a ≠ b
 shows normal_space X ⟷
       (∀ S T. closedin X S ∧ closedin X T ∧ disjnt S T
            ⟶ (∃ f. continuous_map X euclideanreal f ∧ f ‘ S ⊆ {a} ∧ f ‘ T ⊆ {b}))
 (is ?lhs=?rhs)
proof
 show ?lhs ⟹ ?rhs
   by (metis Urysohn_lemma_alt)
next
 assume R: ?rhs
 show ?lhs
   unfolding normal_space_def
 proof clarify
   fix S T
   assume closedin X S and closedin X T and disjnt S T
   with R obtain f where contf: continuous_map X euclideanreal f and f ‘ S
```

$\subseteq \{a\}$ f ‘ $T \subseteq \{b\}$
by *meson*
show $\exists\, U\ V.$ *openin* $X\ U \wedge$ *openin* $X\ V \wedge S \subseteq U \wedge T \subseteq V \wedge$ *disjnt* $U\ V$
proof (*intro conjI exI*)
show *openin* X $\{x \in$ *topspace* $X.$ $f\ x \in$ *ball* a $(|a - b|\ /\ 2)\}$
by (*force intro*!: *openin_continuous_map_preimage* [*OF contf*])
show *openin* X $\{x \in$ *topspace* $X.$ $f\ x \in$ *ball* b $(|a - b|\ /\ 2)\}$
by (*force intro*!: *openin_continuous_map_preimage* [*OF contf*])
show $S \subseteq \{x \in$ *topspace* $X.$ $f\ x \in$ *ball* a $(|a - b|\ /\ 2)\}$
using ‹*closedin* $X\ S$› *closedin_subset* ‹f ‘ $S \subseteq \{a\}$› *assms* **by** *force*
show $T \subseteq \{x \in$ *topspace* $X.$ $f\ x \in$ *ball* b $(|a - b|\ /\ 2)\}$
using ‹*closedin* $X\ T$› *closedin_subset* ‹f ‘ $T \subseteq \{b\}$› *assms* **by** *force*
have $\bigwedge x.$ $[\![x \in$ *topspace* X; *dist* a $(f\ x) < |a-b|/2$; *dist* b $(f\ x) < |a-b|/2]\!]$
$\Longrightarrow$ *False*
by (*smt* (*verit*, *best*) *dist_real_def dist_triangle_half_l*)
then show *disjnt* $\{x \in$ *topspace* $X.$ $f\ x \in$ *ball* a $(|a-b|\ /\ 2)\}$ $\{x \in$ *topspace* $X.$ $f\ x \in$ *ball* b $(|a-b|\ /\ 2)\}$
using *disjnt_iff* **by** *fastforce*
qed
qed
qed

lemma *normal_space_iff_Urysohn_gen*:
fixes a b::*real*
shows
$a < b \Longrightarrow$
normal_space $X \longleftrightarrow$
$(\forall\, S\ T.$ *closedin* $X\ S \wedge$ *closedin* $X\ T \wedge$ *disjnt* $S\ T$
$\longrightarrow (\exists f.$ *continuous_map* X (*top_of_set* $\{a..b\}$) $f \wedge$
f ‘ $S \subseteq \{a\} \wedge f$ ‘ $T \subseteq \{b\}))$
by (*metis linear not_le Urysohn_lemma normal_space_iff_Urysohn_gen_alt continuous_map_in_subtopology*)

lemma *normal_space_iff_Urysohn_alt*:
normal_space $X \longleftrightarrow$
$(\forall\, S\ T.$ *closedin* $X\ S \wedge$ *closedin* $X\ T \wedge$ *disjnt* $S\ T$
$\longrightarrow (\exists f.$ *continuous_map* X *euclideanreal* $f \wedge$
f ‘ $S \subseteq \{0\} \wedge f$ ‘ $T \subseteq \{1\}))$
by (*rule normal_space_iff_Urysohn_gen_alt*) *auto*

lemma *normal_space_iff_Urysohn*:
normal_space $X \longleftrightarrow$
$(\forall\, S\ T.$ *closedin* $X\ S \wedge$ *closedin* $X\ T \wedge$ *disjnt* $S\ T$
$\longrightarrow (\exists f$::$'a \Rightarrow$*real*. *continuous_map* X (*top_of_set* $\{0..1\}$) $f \wedge$
f ‘ $S \subseteq \{0\} \wedge f$ ‘ $T \subseteq \{1\}))$
by (*rule normal_space_iff_Urysohn_gen*) *auto*

lemma *normal_space_perfect_map_image*:
$[\![$*normal_space* X; *perfect_map* $X\ Y\ f]\!] \Longrightarrow$ *normal_space* Y

unfolding *perfect_map_def proper_map_def*
using *normal_space_continuous_closed_map_image* **by** *fastforce*

lemma *Hausdorff_normal_space_closed_continuous_map_image*:
⟦*normal_space X*; *closed_map X Y f*; *continuous_map X Y f*;
f ‘ topspace X = topspace Y; *t1_space Y*⟧
⟹ *Hausdorff_space Y*
by (*metis normal_space_continuous_closed_map_image normal_t1_imp_Hausdorff_space*)

lemma *normal_Hausdorff_space_closed_continuous_map_image*:
⟦*normal_space X*; *Hausdorff_space X*; *closed_map X Y f*;
continuous_map X Y f; *f ‘ topspace X = topspace Y*⟧
⟹ *normal_space Y* ∧ *Hausdorff_space Y*
by (*meson normal_space_continuous_closed_map_image normal_t1_eq_Hausdorff_space t1_space_closed_map_image*)

lemma *Lindelof_cover*:
assumes *regular_space X* **and** *Lindelof_space X* **and** *S* ≠ {}
and *clo*: *closedin X S closedin X T disjnt S T*
obtains *h* :: *nat* ⇒ *'a set* **where**
⋀*n*. *openin X* (*h n*) ⋀*n*. *disjnt T* (*X closure_of* (*h n*)) **and** *S* ⊆ ⋃(*range h*)
proof −
have ∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ *disjnt T* (*X closure_of U*)
if *x* ∈ *S* **for** *x*
using ‹*regular_space X*› **unfolding** *regular_space*
by (*metis* (*full_types*) *Diff_iff* ‹*disjnt S T*› *clo closure_of_eq disjnt_iff in_closure_of that*)
then obtain *h* **where** *oh*: ⋀*x*. *x* ∈ *S* ⟹ *openin X* (*h x*)
and *xh*: ⋀*x*. *x* ∈ *S* ⟹ *x* ∈ *h x*
and *dh*: ⋀*x*. *x* ∈ *S* ⟹ *disjnt T* (*X closure_of h x*)
by *metis*
have *Lindelof_space*(*subtopology X S*)
by (*simp add*: *Lindelof_space_closedin_subtopology* ‹*Lindelof_space X*› ‹*closedin X S*›)
then obtain *U* **where** *U*: *countable U* ∧ *U* ⊆ *h ‘ S* ∧ *S* ⊆ ⋃*U*
unfolding *Lindelof_space_subtopology_subset* [*OF closedin_subset* [*OF* ‹*closedin X S*›]]
by (*smt* (*verit*, *del_insts*) *oh xh UN_I image_iff subsetI*)
with ‹*S* ≠ {}› **have** *U* ≠ {}
by *blast*
show *?thesis*
proof
show *openin X* (*from_nat_into U n*) **for** *n*
by (*metis U from_nat_into image_iff* ‹*U* ≠ {}› *oh subsetD*)
show *disjnt T* (*X closure_of* (*from_nat_into U*) *n*) **for** *n*
using *dh from_nat_into* [*OF* ‹*U* ≠ {}›]
by (*metis U f_inv_into_f inv_into_into subset_eq*)
show *S* ⊆ ⋃(*range* (*from_nat_into U*))
by (*simp add*: *U* ‹*U* ≠ {}›)

```
  qed
qed

lemma regular_Lindelof_imp_normal_space:
  assumes regular_space X and Lindelof_space X
  shows normal_space X
  unfolding normal_space_def
proof clarify
  fix S T
  assume clo: closedin X S closedin X T and disjnt S T
  show ∃ U V. openin X U ∧ openin X V ∧ S ⊆ U ∧ T ⊆ V ∧ disjnt U V
  proof (cases S={} ∨ T={})
    case True
    with clo show ?thesis
    by (meson closedin_def disjnt_empty1 disjnt_empty2 openin_empty openin_topspace
subset_empty)
  next
    case False
    obtain h :: nat ⇒ 'a set where
      openh: ⋀n. openin X (h n) and dish: ⋀n. disjnt T (X closure_of (h n))
      and Sh: S ⊆ ⋃(range h)
      by (metis Lindelof_cover False ‹disjnt S T› assms clo)
    obtain k :: nat ⇒ 'a set where
      openk: ⋀n. openin X (k n) and disk: ⋀n. disjnt S (X closure_of (k n))
      and Tk: T ⊆ ⋃(range k)
      by (metis Lindelof_cover False ‹disjnt S T› assms clo disjnt_sym)
    define U where U ≡ ⋃i. h i − (⋃j<i. X closure_of k j)
    define V where V ≡ ⋃i. k i − (⋃j≤i. X closure_of h j)
    show ?thesis
    proof (intro exI conjI)
      show openin X U openin X V
        unfolding U_def V_def
        by (force intro!: openk openh closedin_Union closedin_closure_of)+
      show S ⊆ U T ⊆ V
        using Sh Tk dish disk by (fastforce simp: U_def V_def disjnt_iff)+
      have ⋀x i j. ⟦x ∈ k i; x ∈ h j; ∀j≤i. x ∉ X closure_of h j⟧
                ⟹ ∃i<j. x ∈ X closure_of k i
        by (metis in_closure_of linorder_not_less openk openin_subset subsetD)
      then show disjnt U V
        by (force simp: U_def V_def disjnt_iff)
    qed
  qed
qed

theorem Tietze_extension_closed_real_interval:
  assumes normal_space X and closedin X S
    and contf: continuous_map (subtopology X S) euclideanreal f
    and fim: f ‘ S ⊆ {a..b} and a ≤ b
  obtains g
```

where *continuous_map X euclideanreal g*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x\ g\ `\ topspace\ X \subseteq \{a..b\}$
proof −
define *c* **where** $c \equiv max\ |a|\ |b| + 1$
have $0 < c$ **and** *c*: $\bigwedge x.\ x \in S \Longrightarrow |f\ x| \le c$
using *fim* **by** (*auto simp*: *c_def image_subset_iff*)
define *good* **where**
good $\equiv \lambda g\ n.$ *continuous_map X euclideanreal g* $\wedge$ $(\forall x \in S.\ |f\ x - g\ x| \le c * (2/3)\hat{}n)$
have *step*: $\exists g.$ *good g* (*Suc n*) $\wedge$
$(\forall x \in topspace\ X.\ |g\ x - h\ x| \le c * (2/3)\hat{}n\ /\ 3)$
if *h*: *good h n* **for** *n h*
proof −
have *pos*: $0 < c * (2/3)\ \hat{}\ n$
by (*simp add*: ‹$0 < c$›)
have *S_eq*: $S = topspace(subtopology\ X\ S)$ **and** $S \subseteq topspace\ X$
using ‹*closedin X S*› *closedin_subset* **by** *auto*
define *d* **where** $d \equiv c/3 * (2/3)\ \hat{}\ n$
define *SA* **where** $SA \equiv \{x \in S.\ f\ x - h\ x \in \{..-d\}\}$
define *SB* **where** $SB \equiv \{x \in S.\ f\ x - h\ x \in \{d..\}\}$
have *contfh*: *continuous_map* (*subtopology X S*) *euclideanreal* $(\lambda x.\ f\ x - h\ x)$
using *that*
by (*simp add*: *contf good_def continuous_map_diff continuous_map_from_subtopology*)
then have *closedin* (*subtopology X S*) *SA*
unfolding *SA_def*
by (*smt* (*verit*, *del_insts*) *closed_closedin continuous_map_closedin Collect_cong S_eq closed_real_atMost*)
then have *closedin X SA*
using ‹*closedin X S*› *closedin_trans_full* **by** *blast*
moreover have *closedin* (*subtopology X S*) *SB*
unfolding *SB_def*
using *closedin_continuous_map_preimage_gen* [*OF contfh*]
by (*metis* (*full_types*) *S_eq closed_atLeast closed_closedin closedin_topspace*)
then have *closedin X SB*
using ‹*closedin X S*› *closedin_trans_full* **by** *blast*
moreover have *disjnt SA SB*
using *pos* **by** (*auto simp*: *d_def disjnt_def SA_def SB_def*)
moreover have $-d \le d$
using *pos* **by** (*auto simp*: *d_def*)
ultimately
obtain *g* **where** *contg*: *continuous_map X* (*top_of_set* $\{-d..d\}$) *g*
and *ga*: $g\ `\ SA \subseteq \{-d\}$ **and** *gb*: $g\ `\ SB \subseteq \{d\}$
using *Urysohn_lemma* ‹*normal_space X*› **by** *metis*
then have *g_le_d*: $\bigwedge x.\ x \in topspace\ X \Longrightarrow |g\ x| \le d$
by (*fastforce simp*: *abs_le_iff continuous_map_def minus_le_iff*)
have *g_eq_d*: $\bigwedge x.\ [\![x \in S; f\ x - h\ x \le -d]\!] \Longrightarrow g\ x = -d$
using *ga* **by** (*auto simp*: *SA_def*)
have *g_eq_negd*: $\bigwedge x.\ [\![x \in S; f\ x - h\ x \ge d]\!] \Longrightarrow g\ x = d$
using *gb* **by** (*auto simp*: *SB_def*)

```
    show ?thesis
      unfolding good_def
    proof (intro conjI strip exI)
      show continuous_map X euclideanreal (λx. h x + g x)
        using contg continuous_map_add continuous_map_in_subtopology that
        unfolding good_def by blast
      show |f x − (h x + g x)| ≤ c * (2 / 3) ^ Suc n if x ∈ S for x
      proof −
        have x: x ∈ topspace X
          using ‹S ⊆ topspace X› that by auto
        have |f x − h x| ≤ c * (2/3) ^ n
          using good_def h that by blast
        with g_eq_d [OF that] g_eq_negd [OF that] g_le_d [OF x]
        have |f x − (h x + g x)| ≤ d + d
          unfolding d_def by linarith
        then show ?thesis
          by (simp add: d_def)
      qed
      show |h x + g x − h x| ≤ c * (2 / 3) ^ n / 3 if x ∈ topspace X for x
        using that d_def g_le_d by auto
    qed
  qed
  then obtain nxtg where nxtg: ⋀h n. good h n ⟹
        good (nxtg h n) (Suc n) ∧ (∀x ∈ topspace X. |nxtg h n x − h x| ≤ c *
(2/3)^n / 3)
    by metis
  define g where g ≡ rec_nat (λx. 0) (λn r. nxtg r n)
  have [simp]: g 0 x = 0 for x
    by (auto simp: g_def)
  have g_Suc: g(Suc n) = nxtg (g n) n for n
    by (auto simp: g_def)
  have good: good (g n) n for n
  proof (induction n)
    case 0
    with c show ?case
      by (auto simp: good_def)
  qed (simp add: g_Suc nxtg)
  have ∗: ⋀n x. x ∈ topspace X ⟹ |g(Suc n) x − g n x| ≤ c * (2/3) ^ n / 3
    using nxtg g_Suc good by force
  obtain h where conth: continuous_map X euclideanreal h
    and h: ⋀ε. 0 < ε ⟹ ∀F n in sequentially. ∀x∈topspace X. dist (g n x) (h x)
< ε
  proof (rule Met_TC.continuous_map_uniformly_Cauchy_limit)
    show ∀F n in sequentially. continuous_map X (Met_TC.mtopology) (g n)
      using good good_def by fastforce
    show ∃N. ∀m n x. N ≤ m ⟶ N ≤ n ⟶ x ∈ topspace X ⟶ dist (g m x)
(g n x) < ε
      if ε > 0 for ε
    proof −
```

```
    have ∀F n in sequentially. |(2/3) ^ n| < ε/c
    proof (rule Archimedean_eventually_pow_inverse)
      show 0 < ε / c
        by (simp add: ‹0 < c› that)
    qed auto
    then obtain N where N: ⋀n. n ≥ N ⟹ |(2/3) ^ n| < ε/c
      by (meson eventually_sequentially order_le_less_trans)
    have |g m x - g n x| < ε
      if N ≤ m N ≤ n and x: x ∈ topspace X m ≤ n for m n x
    proof (cases m < n)
      case True
      have 23: (∑k = m..<n. (2/3)^k) = 3 * ((2/3) ^ m - (2/3::real) ^ n)
        using ‹m ≤ n›
        by (induction n) (auto simp: le_Suc_eq)
      have |g m x - g n x| ≤ |∑k = m..<n. g (Suc k) x - g k x|
        by (subst sum_Suc_diff' [OF ‹m ≤ n›]) linarith
      also have ... ≤ (∑k = m..<n. |g (Suc k) x - g k x|)
        by (rule sum_abs)
      also have ... ≤ (∑k = m..<n. c * (2/3)^k / 3)
        by (meson * sum_mono x(1))
      also have ... = (c/3) * (∑k = m..<n. (2/3)^k)
        by (simp add: sum_distrib_left)
      also have ... = (c/3) * 3 * ((2/3) ^ m - (2/3) ^ n)
        by (simp add: sum_distrib_left 23)
      also have ... < (c/3) * 3 * ((2/3) ^ m)
        using ‹0 < c› by auto
      also have ... < ε
        using N [OF ‹N ≤ m›] ‹0 < c› by (simp add: field_simps)
      finally show ?thesis .
    qed (use ‹0 < ε› ‹m ≤ n› in auto)
    then show ?thesis
      by (metis dist_commute_lessI dist_real_def nle_le)
  qed
qed auto
define φ where φ ≡ λx. max a (min (h x) b)
show thesis
proof
  show continuous_map X euclidean φ
    unfolding φ_def using conth by (intro continuous_intros) auto
  show φ x = f x if x ∈ S for x
  proof -
    have x: x ∈ topspace X
      using ‹closedin X S› closedin_subset that by blast
    have h x = f x
    proof (rule Met_TC.limitin_metric_unique)
      show limitin Met_TC.mtopology (λn. g n x) (h x) sequentially
        using h x by (force simp: tendsto_iff eventually_sequentially)
      show limitin Met_TC.mtopology (λn. g n x) (f x) sequentially
      proof (clarsimp simp: tendsto_iff)
```

```
        fix ε::real
        assume ε > 0
        then have ∀F n in sequentially. |(2/3) ^ n| < ε/c
          by (intro Archimedean_eventually_pow_inverse) (auto simp: ‹c > 0›)
        then show ∀F n in sequentially. dist (g n x) (f x) < ε
          apply eventually_elim
           by (smt (verit) good x good_def ‹c > 0› dist_real_def mult.commute
pos_less_divide_eq that)
       qed
      qed auto
      then show ?thesis
        using that fim by (auto simp: φ_def)
    qed
    then show φ ‘ topspace X ⊆ {a..b}
      using fim ‹a ≤ b› by (auto simp: φ_def)
  qed
qed


theorem Tietze_extension_realinterval:
  assumes XS: normal_space X closedin X S and T: is_interval T T ≠ {}
    and contf: continuous_map (subtopology X S) euclideanreal f
    and f ‘ S ⊆ T
  obtains g where continuous_map X euclideanreal g  g ‘ topspace X ⊆ T  ⋀x.
x ∈ S ⟹ g x = f x
proof −
  define Φ where
        Φ ≡ λT::real set. ∀f. continuous_map (subtopology X S) euclidean f ⟶
f‘S ⊆ T
              ⟶ (∃g. continuous_map X euclidean g ∧ g ‘ topspace X ⊆ T ∧ (∀x
∈ S. g x = f x))
  have Φ T
    if ∗: ⋀T. ⟦bounded T; is_interval T; T ≠ {}⟧ ⟹ Φ T
      and is_interval T T ≠ {} for T
    unfolding Φ_def
  proof (intro strip)
    fix f
    assume contf: continuous_map (subtopology X S) euclideanreal f
      and f ‘ S ⊆ T
    have ΦT: Φ ((λx. x / (1 + |x|)) ‘ T)
    proof (rule ∗)
      show bounded ((λx. x / (1 + |x|)) ‘ T)
        using shrink_range [of T] by (force intro: boundedI [where B=1])
      show is_interval ((λx. x / (1 + |x|)) ‘ T)
        using connected_shrink that(2) is_interval_connected_1 by blast
      show (λx. x / (1 + |x|)) ‘ T ≠ {}
        using ‹T ≠ {}› by auto
    qed
    moreover have continuous_map (subtopology X S) euclidean ((λx. x / (1 +
```

```
|x|)) ∘ f)
    by (metis contf continuous_map_compose continuous_map_into_fulltopology
continuous_map_real_shrink)
   moreover have ((λx. x / (1 + |x|)) ∘ f) ‘ S ⊆ (λx. x / (1 + |x|)) ‘ T
    using ‹f ‘ S ⊆ T› by auto
   ultimately obtain g
     where contg: continuous_map X euclidean g
      and gim: g ‘ topspace X ⊆ (λx. x / (1 + |x|)) ‘ T
      and geq: ⋀x. x ∈ S ⟹ g x = ((λx. x / (1 + |x|)) ∘ f) x
    using Φ T unfolding Φ_def by force
   show ∃g. continuous_map X euclideanreal g ∧ g ‘ topspace X ⊆ T ∧ (∀x∈S.
g x = f x)
   proof (intro conjI exI)
    have continuous_map X (top_of_set {−1<..<1}) g
     using contg continuous_map_in_subtopology gim shrink_range by blast
    then show continuous_map X euclideanreal ((λx. x / (1 − |x|)) ∘ g)
     by (rule continuous_map_compose) (auto simp: continuous_on_real_grow)
    show ((λx. x / (1 − |x|)) ∘ g) ‘ topspace X ⊆ T
     using gim real_grow_shrink by fastforce
    show ∀x∈S. ((λx. x / (1 − |x|)) ∘ g) x = f x
     using geq real_grow_shrink by force
   qed
  qed
  moreover have Φ T
   if bounded T is_interval T T ≠ {} for T
   unfolding Φ_def
  proof (intro strip)
   fix f
   assume contf: continuous_map (subtopology X S) euclideanreal f
    and f ‘ S ⊆ T
   obtain a b where ab: closure T = {a..b}
   by (meson ‹bounded T› ‹is_interval T› compact_closure connected_compact_interval_1

          connected_imp_connected_closure is_interval_connected)
   with ‹T ≠ {}› have a ≤ b by auto
   have f ‘ S ⊆ {a..b}
    using ‹f ‘ S ⊆ T› ab closure_subset by auto
   then obtain g where contg: continuous_map X euclideanreal g
    and gf: ⋀x. x ∈ S ⟹ g x = f x and gim: g ‘ topspace X ⊆ {a..b}
     using Tietze_extension_closed_real_interval [OF XS contf _ ‹a ≤ b›] by
metis
   define W where W ≡ {x ∈ topspace X. g x ∈ closure T − T}
   have {a..b} − {a, b} ⊆ T
    using that
   by (metis ab atLeastAtMost_diff_ends convex_interior_closure interior_atLeastAtMost_real

       interior_subset is_interval_convex)
   with finite_imp_compact have compact (closure T − T)
    by (metis Diff_eq_empty_iff Diff_insert2 ab finite.emptyI finite_Diff_insert)
```

```
  then have closedin X W
    unfolding W_def using closedin_continuous_map_preimage [OF contg]
compact_imp_closed by force
  moreover have disjnt W S
   unfolding W_def disjnt_iff using ‹f ‘ S ⊆ T› gf by blast
  ultimately obtain h :: 'a ⇒ real
   where conth: continuous_map X (top_of_set {0..1}) h
       and him: h ‘ W ⊆ {0} h ‘ S ⊆ {1}
   by (metis XS normal_space_iff_Urysohn)
  then have him01: h ‘ topspace X ⊆ {0..1}
   by (meson continuous_map_in_subtopology)
  obtain z where z ∈ T
   using ‹T ≠ {}› by blast
  define g' where g' ≡ λx. z + h x * (g x − z)
  show ∃ g. continuous_map X euclidean g ∧ g ‘ topspace X ⊆ T ∧ (∀ x∈S. g x
= f x)
  proof (intro exI conjI)
   show continuous_map X euclideanreal g'
    unfolding g'_def using contg conth continuous_map_in_subtopology
    by (intro continuous_intros) auto
   show g' ‘ topspace X ⊆ T
    unfolding g'_def
   proof clarify
    fix x
    assume x ∈ topspace X
    show z + h x * (g x − z) ∈ T
    proof (cases g x ∈ T)
     case True
     define w where w ≡ z + h x * (g x − z)
     have |h x| * |g x − z| ≤ |g x − z| |1 − h x| * |g x − z| ≤ |g x − z|
     using him01 ‹x ∈ topspace X› by (force simp: intro: mult_left_le_one_le)+
     then consider z ≤ w ∧ w ≤ g x | g x ≤ w ∧ w ≤ z
       unfolding w_def by (smt (verit) left_diff_distrib mult_cancel_right2
mult_minus_right zero_less_mult_iff)
     then show ?thesis
       using ‹is_interval T› unfolding w_def is_interval_1 by (metis True
‹z ∈ T›)
    next
     case False
     then have g x ∈ closure T
       using ‹x ∈ topspace X› ab gim by blast
     then have h x = 0
     using him False ‹x ∈ topspace X› by (auto simp: W_def image_subset_iff)
     then show ?thesis
       by (simp add: ‹z ∈ T›)
    qed
   qed
   show ∀ x∈S. g' x = f x
    using gf him by (auto simp: W_def g'_def)
```

```
    qed
  qed
  ultimately show thesis
    using assms that unfolding Φ_def by best
qed

lemma normal_space_iff_Tietze:
   normal_space X ⟷
    (∀f S. closedin X S ∧
          continuous_map (subtopology X S) euclidean f
          ⟶ (∃g::'a ⇒ real. continuous_map X euclidean g ∧ (∀x ∈ S. g x = f
x)))
   (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  then show ?rhs
     by (metis Tietze_extension_realinterval empty_not_UNIV is_interval_univ
subset_UNIV)
next
  assume R: ?rhs
  show ?lhs
    unfolding normal_space_iff_Urysohn_alt
  proof clarify
    fix S T
    assume closedin X S
      and closedin X T
      and disjnt S T
    then have cloST: closedin X (S ∪ T)
      by (simp add: closedin_Un)
    moreover
    have continuous_map (subtopology X (S ∪ T)) euclideanreal (λx. if x ∈ S then
0 else 1)
      unfolding continuous_map_closedin
    proof (intro conjI strip)
      fix C :: real set
      define D where D ≡ {x ∈ topspace X. if x ∈ S then 0 ∈ C else x ∈ T ∧ 1
∈ C}
      have D ∈ {{}, S, T, S ∪ T}
        unfolding D_def
       using closedin_subset [OF ‹closedin X S›] closedin_subset [OF ‹closedin X
T›] ‹disjnt S T›
        by (auto simp: disjnt_iff)
      then have closedin X D
        using ‹closedin X S› ‹closedin X T› closedin_empty by blast
      with closedin_subset_topspace
      show closedin (subtopology X (S ∪ T)) {x ∈ topspace (subtopology X (S ∪
T)). (if x ∈ S then 0::real else 1) ∈ C}
        apply (simp add: D_def)
      by (smt (verit, best) Collect_cong Collect_mono_iff Un_def closedin_subset_topspace)
```

qed *auto*
ultimately obtain $g :: 'a \Rightarrow real$ **where**
contg: *continuous_map X euclidean g* **and** *gf*: $\forall x \in S \cup T.\ g\ x = (if\ x \in S\ then\ 0\ else\ 1)$
using *R* **by** *blast*
then show $\exists f.$ *continuous_map X euclideanreal f* $\wedge$ *f ' S* $\subseteq \{0\} \wedge$ *f ' T* $\subseteq \{1\}$
by (*smt* (*verit*) *Un_iff* ‹*disjnt S T*› *disjnt_iff image_subset_iff insert_iff*)
qed
qed

6.11.2 Random metric space stuff

lemma *metrizable_imp_k_space*:
assumes *metrizable_space X*
shows *k_space X*
proof −
obtain *M d* **where** *Metric_space M d* **and** *Xeq*: *X* = *Metric_space.mtopology M d*
using *assms* **unfolding** *metrizable_space_def* **by** *metis*
then interpret *Metric_space M d*
by *blast*
show *?thesis*
unfolding *k_space Xeq*
proof *clarsimp*
fix *S*
assume *S* $\subseteq$ *M* **and** *S*: $\forall K.$ *compactin mtopology K* $\longrightarrow$ *closedin* (*subtopology mtopology K*) (*K* $\cap$ *S*)
have *l* $\in$ *S*
if σ: *range* $\sigma \subseteq S$ **and** *l*: *limitin mtopology* σ *l sequentially* **for** σ *l*
proof −
define *K* **where** *K* $\equiv$ *insert l* (*range* σ)
interpret *Submetric M d K*
proof
show *K* $\subseteq$ *M*
unfolding *K_def* **using** *l limitin_mspace* ‹*S* $\subseteq$ *M*› σ **by** *blast*
qed
have *compactin mtopology K*
unfolding *K_def* **using** ‹*S* $\subseteq$ *M*› σ
by (*force intro*: *compactin_sequence_with_limit* [*OF l*])
then have $*$: *closedin sub.mtopology* (*K* $\cap$ *S*)
by (*simp add*: *S mtopology_submetric*)
have σ *n* $\in$ *K* $\cap$ *S* **for** *n*
by (*simp add*: *K_def range_subsetD* σ)
moreover have *limitin sub.mtopology* σ *l sequentially*
using *l*
unfolding *sub.limit_metric_sequentially limit_metric_sequentially*
by (*force simp*: *K_def*)
ultimately have *l* $\in$ *K* $\cap$ *S*

by (*meson* ∗ *σ image_subsetI sub.metric_closedin_iff_sequentially_closed*)

then show *?thesis*
by *simp*
qed
then show *closedin mtopology S*
unfolding *metric_closedin_iff_sequentially_closed*
using ‹*S* ⊆ *M*› **by** *blast*
qed
qed

lemma (**in** *Metric_space*) *k_space_mtopology*: *k_space mtopology*
by (*simp add*: *metrizable_imp_k_space metrizable_space_mtopology*)

lemma *k_space_euclideanreal*: *k_space* (*euclidean* :: *'a*::*metric_space topology*)
using *metrizable_imp_k_space metrizable_space_euclidean* **by** *auto*

6.11.3 Hereditarily normal spaces

lemma *hereditarily_B*:
assumes ⋀*S T*. *separatedin X S T*
⟹ ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*
shows *hereditarily normal_space X*
unfolding *hereditarily_def*
proof (*intro strip*)
fix *W*
assume *W* ⊆ *topspace X*
show *normal_space* (*subtopology X W*)
unfolding *normal_space_def*
proof *clarify*
fix *S T*
assume *clo*: *closedin* (*subtopology X W*) *S closedin* (*subtopology X W*) *T*
and *disjnt S T*
then have *separatedin* (*subtopology X W*) *S T*
by (*simp add*: *separatedin_closed_sets*)
then obtain *U V* **where** *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*
using *assms* [*of S T*] **by** (*meson separatedin_subtopology*)
then show ∃ *U V*. *openin* (*subtopology X W*) *U* ∧ *openin* (*subtopology X W*) *V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*
apply (*simp add*: *openin_subtopology_alt*)
by (*meson clo closedin_imp_subset disjnt_subset1 disjnt_subset2 inf_le2*)
qed
qed

lemma *hereditarily_C*:
assumes *separatedin X S T* **and** *norm*: ⋀*U*. *openin X U* ⟹ *normal_space* (*subtopology X U*)
shows ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*

proof −
define ST **where** $ST \equiv X\ closure\_of\ S \cap X\ closure\_of\ T$
have *subX*: $S \subseteq topspace\ X\ T \subseteq topspace\ X$
by (*meson* ‹*separatedin X S T*› *separation_closedin_Un_gen*)+
have *sub*: $S \subseteq topspace\ X - ST\ T \subseteq topspace\ X - ST$
unfolding *ST_def*
by (*metis Diff_mono Diff_triv* ‹*separatedin X S T*› *Int_lower1 Int_lower2 separatedin_def*)+
have *normal_space* (*subtopology X* (*topspace X* − *ST*))
by (*simp add*: *ST_def norm closedin_Int openin_diff*)
moreover have *disjnt* (*subtopology X* (*topspace X* − *ST*) *closure_of S*) (*subtopology X* (*topspace X* − *ST*) *closure_of T*)
using *Int_absorb1 ST_def sub* **by** (*fastforce simp*: *disjnt_iff closure_of_subtopology*)
ultimately show *?thesis*
using *sub subX*
apply (*simp add*: *normal_space_closures*)
by (*metis ST_def closedin_Int closedin_closure_of closedin_def openin_trans_full*)
qed

lemma *hereditarily_normal_space*:
hereditarily normal_space X ⟷ (∀ *U*. *openin X U* ⟶ *normal_space*(*subtopology X U*))
by (*metis hereditarily_B hereditarily_C hereditarily*)

lemma *hereditarily_normal_separation*:
hereditarily normal_space X ⟷
(∀ *S T*. *separatedin X S T*
⟶ (∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*))
by (*metis hereditarily_B hereditarily_C hereditarily*)

lemma *metrizable_imp_hereditarily_normal_space*:
metrizable_space X ⟹ *hereditarily normal_space X*
by (*simp add*: *hereditarily metrizable_imp_normal_space metrizable_space_subtopology*)

lemma *metrizable_space_separation*:
⟦*metrizable_space X*; *separatedin X S T*⟧
⟹ ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *S* ⊆ *U* ∧ *T* ⊆ *V* ∧ *disjnt U V*
by (*metis hereditarily hereditarily_C metrizable_imp_hereditarily_normal_space*)

lemma *hereditarily_normal_separation_pairwise*:
hereditarily normal_space X ⟷
(∀ $\mathcal{U}$. *finite* $\mathcal{U}$ ∧ (∀ *S* ∈ $\mathcal{U}$. *S* ⊆ *topspace X*) ∧ *pairwise* (*separatedin X*) $\mathcal{U}$
⟶ (∃ $\mathcal{F}$. (∀ *S* ∈ $\mathcal{U}$. *openin X* ($\mathcal{F}$ *S*) ∧ *S* ⊆ $\mathcal{F}$ *S*) ∧
pairwise (λ*S T*. *disjnt* ($\mathcal{F}$ *S*) ($\mathcal{F}$ *T*)) $\mathcal{U}$))
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *L*: *?lhs*

```
  show ?rhs
  proof clarify
    fix 𝒰
    assume finite 𝒰 and 𝒰: ∀ S∈𝒰. S ⊆ topspace X
      and pw𝒰: pairwise (separatedin X) 𝒰
    have ∃ V W. openin X V ∧ openin X W ∧ S ⊆ V ∧
                  (∀ T. T ∈ 𝒰 ∧ T ≠ S ⟶ T ⊆ W) ∧ disjnt V W
      if S ∈ 𝒰 for S
    proof −
      have separatedin X S (⋃ (𝒰 − {S}))
        by (metis 𝒰 ‹finite 𝒰› pw𝒰 finite_Diff pairwise_alt separatedin_Union(1)
that)
      with L show ?thesis
        unfolding hereditarily_normal_separation
        by (smt (verit) Diff_iff UnionI empty_iff insert_iff subset_iff)
    qed
    then obtain ℱ 𝒢
      where ∗: ⋀S. S ∈ 𝒰 ⟹ S ⊆ ℱ S ∧ (∀ T. T ∈ 𝒰 ∧ T ≠ S ⟶ T ⊆ 𝒢 S)
        and ope: ⋀S. S ∈ 𝒰 ⟹ openin X (ℱ S) ∧ openin X (𝒢 S)
        and dis: ⋀S. S ∈ 𝒰 ⟹ disjnt (ℱ S) (𝒢 S)
      by metis
    define ℋ where ℋ ≡ λS. ℱ S ∩ (⋂ T ∈ 𝒰 − {S}. 𝒢 T)
    show ∃ℱ. (∀ S∈𝒰. openin X (ℱ S) ∧ S ⊆ ℱ S) ∧ pairwise (λS T. disjnt (ℱ
S) (ℱ T)) 𝒰
    proof (intro exI conjI strip)
      show openin X (ℋ S) if S ∈ 𝒰 for S
        unfolding ℋ_def
        by (smt (verit) ope that DiffD1 ‹finite 𝒰› finite_Diff finite_imageI imageE
openin_Int_Inter)
      show S ⊆ ℋ S if S ∈ 𝒰 for S
        unfolding ℋ_def using ∗ that by auto
    show pairwise (λS T. disjnt (ℋ S) (ℋ T)) 𝒰
      using dis by (fastforce simp: disjnt_iff pairwise_alt ℋ_def)
    qed
  qed
next
  assume R: ?rhs
  show ?lhs
    unfolding hereditarily_normal_separation
  proof (intro strip)
    fix S T
    assume separatedin X S T
    show ∃ U V. openin X U ∧ openin X V ∧ S ⊆ U ∧ T ⊆ V ∧ disjnt U V
    proof (cases T=S)
      case True
      then show ?thesis
        using ‹separatedin X S T› by force
    next
      case False
```

```
      have pairwise (separatedin X) {S, T}
        by (simp add: ‹separatedin X S T› pairwise_insert separatedin_sym)
      moreover have ∀S∈{S, T}. S ⊆ topspace X
        by (metis ‹separatedin X S T› insertE separatedin_def singletonD)
        ultimately show ?thesis
       using R by (smt (verit) False finite.emptyI finite.insertI insertCI pairwiseD)
    qed
  qed
qed

lemma hereditarily_normal_space_perfect_map_image:
  ⟦hereditarily normal_space X; perfect_map X Y f⟧ ⟹ hereditarily normal_space
Y
  unfolding perfect_map_def proper_map_def
  by (meson hereditarily_normal_space_continuous_closed_map_image)

lemma regular_second_countable_imp_hereditarily_normal_space:
  assumes regular_space X ∧ second_countable X
  shows  hereditarily normal_space X
  unfolding hereditarily
  proof (intro regular_Lindelof_imp_normal_space strip)
  show regular_space (subtopology X S) for S
    by (simp add: assms regular_space_subtopology)
  show Lindelof_space (subtopology X S) for S
   using assms by (simp add: second_countable_imp_Lindelof_space second_countable_subtopology)
qed
```

6.11.4 Completely regular spaces

```
definition completely_regular_space where
 completely_regular_space X ≡
    ∀ S x. closedin X S ∧ x ∈ topspace X − S
          ⟶ (∃ f::'a⇒real. continuous_map X (top_of_set {0..1}) f ∧
                f x = 0 ∧ (f ‘ S ⊆ {1}))

lemma homeomorphic_completely_regular_space_aux:
  assumes X: completely_regular_space X and hom: X homeomorphic_space Y
  shows completely_regular_space Y
proof −
 obtain f g where hmf: homeomorphic_map X Y f and hmg: homeomorphic_map
Y X g
    and fg: (∀ x ∈ topspace X. g(f x) = x) ∧ (∀ y ∈ topspace Y. f(g y) = y)
    using assms X homeomorphic_maps_map homeomorphic_space_def by fast-
force
  show ?thesis
    unfolding completely_regular_space_def
  proof clarify
    fix S x
    assume A: closedin Y S and x: x ∈ topspace Y and x ∉ S
```

then have *closedin X (g‘S)*
using *hmg homeomorphic_map_closedness_eq* **by** *blast*
moreover have *g x ∉ g‘S*
by (*meson A x ‹x ∉ S› closedin_subset hmg homeomorphic_imp_injective_map inj_on_image_mem_iff*)
ultimately obtain *φ* **where** *φ*: *continuous_map X (top_of_set {0..1::real}) φ ∧ φ (g x) = 0 ∧ φ ‘ g‘S ⊆ {1}*
by (*metis DiffI X completely_regular_space_def hmg homeomorphic_imp_surjective_map image_eqI x*)
then have *continuous_map Y (top_of_set {0..1::real}) (φ ∘ g)*
by (*meson continuous_map_compose hmg homeomorphic_imp_continuous_map*)
then show *∃ψ. continuous_map Y (top_of_set {0..1::real}) ψ ∧ ψ x = 0 ∧ ψ ‘ S ⊆ {1}*
by (*metis φ comp_apply image_comp*)
qed
qed

lemma *homeomorphic_completely_regular_space*:
assumes *X homeomorphic_space Y*
shows *completely_regular_space X ⟷ completely_regular_space Y*
by (*meson assms homeomorphic_completely_regular_space_aux homeomorphic_space_sym*)

lemma *completely_regular_space_alt*:
completely_regular_space X ⟷
(∀ S x. closedin X S ⟶ x ∈ topspace X − S
⟶ (∃f. continuous_map X euclideanreal f ∧ f x = 0 ∧ f ‘ S ⊆ {1}))
proof −
have *∃f. continuous_map X (top_of_set {0..1::real}) f ∧ f x = 0 ∧ f ‘ S ⊆ {1}*
if *closedin X S x ∈ topspace X − S* **and** *f*: *continuous_map X euclideanreal f ∧ f x = 0 ∧ f ‘ S ⊆ {1}*
for *S x f*
proof (*intro exI conjI*)
show *continuous_map X (top_of_set {0..1}) (λx. max 0 (min (f x) 1))*
using *that*
by (*auto simp*: *continuous_map_in_subtopology intro*!: *continuous_map_real_max continuous_map_real_min*)
qed (*use that* **in** *auto*)
with *continuous_map_in_subtopology* **show** *?thesis*
unfolding *completely_regular_space_def* **by** *metis*
qed

As above, but with *openin*

lemma *completely_regular_space_alt'*:
completely_regular_space X ⟷
(∀ S x. openin X S ⟶ x ∈ S
⟶ (∃f. continuous_map X euclideanreal f ∧ f x = 0 ∧ f ‘ (topspace X − S) ⊆ {1}))
apply (*simp add*: *completely_regular_space_alt all_closedin*)

```
  by (meson openin_subset subsetD)

lemma completely_regular_space_gen_alt:
  fixes a b::real
  assumes a ≠ b
  shows completely_regular_space X ⟷
        (∀ S x. closedin X S ⟶ x ∈ topspace X − S
              ⟶ (∃ f. continuous_map X euclidean f ∧ f x = a ∧ (f ‘ S ⊆ {b})))
proof −
  have ∃ f. continuous_map X euclideanreal f ∧ f x = 0 ∧ f ‘ S ⊆ {1}
    if closedin X S x ∈ topspace X − S
       and f: continuous_map X euclidean f ∧ f x = a ∧ f ‘ S ⊆ {b}
    for S x f
  proof (intro exI conjI)
    show continuous_map X euclideanreal ((λx. inverse(b − a) ∗ (x − a)) ∘ f)
      using that by (intro continuous_intros) auto
  qed (use that assms in auto)
  moreover
  have ∃ f. continuous_map X euclidean f ∧ f x = a ∧ f ‘ S ⊆ {b}
    if closedin X S x ∈ topspace X − S
       and f: continuous_map X euclideanreal f ∧ f x = 0 ∧ f ‘ S ⊆ {1}
    for S x f
  proof (intro exI conjI)
    show continuous_map X euclideanreal ((λx. a + (b − a) ∗ x) ∘ f)
      using that by (intro continuous_intros) auto
  qed (use that in auto)
  ultimately show ?thesis
    unfolding completely_regular_space_alt by meson
qed
```

As above, but with *openin*

```
lemma completely_regular_space_gen_alt':
  fixes a b::real
  assumes a ≠ b
  shows completely_regular_space X ⟷
        (∀ S x. openin X S ⟶ x ∈ S
              ⟶ (∃ f. continuous_map X euclidean f ∧ f x = a ∧ (f ‘ (topspace X
− S) ⊆ {b})))
  apply (simp add: completely_regular_space_gen_alt[OF assms] all_closedin)
  by (meson openin_subset subsetD)

lemma completely_regular_space_gen:
  fixes a b::real
  assumes a < b
  shows completely_regular_space X ⟷
        (∀ S x. closedin X S ∧ x ∈ topspace X − S
              ⟶ (∃ f. continuous_map X (top_of_set {a..b}) f ∧ f x = a ∧ f ‘ S
⊆ {b}))
proof −
```

have $\exists f.$ *continuous_map X* (*top_of_set* $\{a..b\}$) $f \wedge f\ x = a \wedge f\ `\ S \subseteq \{b\}$
if *closedin X S x* $\in$ *topspace X* $-$ *S*
and *f*: *continuous_map X euclidean f* $\wedge$ *f x* $=$ *a* $\wedge$ *f ` S* $\subseteq$ $\{b\}$
for *S x f*
proof (*intro exI conjI*)
show *continuous_map X* (*top_of_set* $\{a..b\}$) ($\lambda x.$ *max a* (*min* (*f x*) *b*))
using *that assms*
by (*auto simp*: *continuous_map_in_subtopology intro*!: *continuous_map_real_max continuous_map_real_min*)
qed (*use that assms* **in** *auto*)
with *continuous_map_in_subtopology assms* **show** *?thesis*
using *completely_regular_space_gen_alt* [*of a b*]
by (*smt* (*verit*) *atLeastAtMost_singleton atLeastatMost_empty singletonI*)
qed

lemma *normal_imp_completely_regular_space_A*:
assumes *normal_space X t1_space X*
shows *completely_regular_space X*
unfolding *completely_regular_space_alt*
proof *clarify*
fix *x S*
assume *A*: *closedin X S x* $\notin$ *S*
assume *x* $\in$ *topspace X*
then have *closedin X* $\{x\}$
by (*simp add*: ‹*t1_space X*› *closedin_t1_singleton*)
with *A* ‹*normal_space X*› **have** $\exists f.$ *continuous_map X euclideanreal f* $\wedge$ *f ` {x}* $\subseteq$ $\{0\}$ $\wedge$ *f ` S* $\subseteq$ $\{1\}$
using *assms* **unfolding** *normal_space_iff_Urysohn_alt disjnt_iff* **by** *blast*
then show $\exists f.$ *continuous_map X euclideanreal f* $\wedge$ *f x* $=$ *0* $\wedge$ *f ` S* $\subseteq$ $\{1\}$
by *auto*
qed

lemma *normal_imp_completely_regular_space_B*:
assumes *normal_space X regular_space X*
shows *completely_regular_space X*
unfolding *completely_regular_space_alt*
proof *clarify*
fix *x S*
assume *closedin X S x* $\notin$ *S x* $\in$ *topspace X*
then obtain *U C* **where** *openin X U closedin X C x* $\in$ *U U* $\subseteq$ *C C* $\subseteq$ *topspace X* $-$ *S*
using *assms*
unfolding *neighbourhood_base_of_closedin* [*symmetric*] *neighbourhood_base_of closedin_def* **by** (*metis Diff_iff*)
then obtain *f* **where** *continuous_map X euclideanreal f* $\wedge$ *f ` C* $\subseteq$ $\{0\}$ $\wedge$ *f ` S* $\subseteq$ $\{1\}$
using *assms* **unfolding** *normal_space_iff_Urysohn_alt*
by (*metis Diff_iff* ‹*closedin X S*› *disjnt_iff subsetD*)
then show $\exists f.$ *continuous_map X euclideanreal f* $\wedge$ *f x* $=$ *0* $\wedge$ *f ` S* $\subseteq$ $\{1\}$

by (*meson* ‹$U \subseteq C$› ‹$x \in U$› *image_subset_iff singletonD subsetD*)
qed

lemma *normal_imp_completely_regular_space_gen*:
⟦*normal_space X*; *t1_space X* ∨ *Hausdorff_space X* ∨ *regular_space X*⟧ ⟹ *completely_regular_space X*
using *normal_imp_completely_regular_space_A normal_imp_completely_regular_space_B t1_or_Hausdorff_space* **by** *blast*

lemma *normal_imp_completely_regular_space*:
⟦*normal_space X*; *Hausdorff_space X* ∨ *regular_space X*⟧ ⟹ *completely_regular_space X*
by (*simp add*: *normal_imp_completely_regular_space_gen*)

lemma (**in** *Metric_space*) *completely_regular_space_mtopology*:
completely_regular_space mtopology
by (*simp add*: *normal_imp_completely_regular_space normal_space_mtopology regular_space_mtopology*)

lemma *metrizable_imp_completely_regular_space*:
metrizable_space X ⟹ *completely_regular_space X*
by (*simp add*: *metrizable_imp_normal_space metrizable_imp_regular_space normal_imp_completely_regular_space*)

lemma *completely_regular_space_discrete_topology*:
completely_regular_space(*discrete_topology U*)
by (*simp add*: *normal_imp_completely_regular_space normal_space_discrete_topology*)

lemma *completely_regular_space_subtopology*:
assumes *completely_regular_space X*
shows *completely_regular_space* (*subtopology X S*)
unfolding *completely_regular_space_def*
proof *clarify*
fix *A x*
assume *closedin* (*subtopology X S*) *A* **and** *x*: *x* ∈ *topspace* (*subtopology X S*) **and** *x* ∉ *A*
then obtain *T* **where** *closedin X T A* = *S* ∩ *T x* ∈ *topspace X x* ∈ *S*
by (*force simp*: *closedin_subtopology_alt image_iff*)
then show ∃*f*. *continuous_map* (*subtopology X S*) (*top_of_set* {*0*::*real*..*1*}) *f* ∧ *f x* = *0* ∧ *f* ‘ *A* ⊆ {*1*}
using *assms* ‹*x* ∉ *A*›
apply (*simp add*: *completely_regular_space_def continuous_map_from_subtopology*)
using *continuous_map_from_subtopology* **by** *fastforce*
qed

lemma *completely_regular_space_retraction_map_image*:
⟦*retraction_map X Y r*; *completely_regular_space X*⟧ ⟹ *completely_regular_space Y*
using *completely_regular_space_subtopology hereditary_imp_retractive_property*

homeomorphic_completely_regular_space **by** *blast*

lemma *completely_regular_imp_regular_space*:
assumes *completely_regular_space X*
shows *regular_space X*
proof −
have ∗: ∃ *U V*. *openin X U* ∧ *openin X V* ∧ *a* ∈ *U* ∧ *C* ⊆ *V* ∧ *disjnt U V*
if *contf*: *continuous_map X euclideanreal f* **and** *a*: *a* ∈ *topspace X* − *C* **and** *closedin X C*
and *fim*: *f* ‘ *topspace X* ⊆ {*0..1*} **and** *f0*: *f a* = *0* **and** *f1*: *f* ‘ *C* ⊆ {*1*}
for *C a f*
proof (*intro exI conjI*)
show *openin X* {*x* ∈ *topspace X*. *f x* ∈ {..<*1* / *2*}} *openin X* {*x* ∈ *topspace X*. *f x* ∈ {*1* / *2*<..}}
using *openin_continuous_map_preimage* [*OF contf*]
by (*meson open_lessThan open_greaterThan open_openin*)+
show *a* ∈ {*x* ∈ *topspace X*. *f x* ∈ {..<*1* / *2*}}
using *a f0* **by** *auto*
show *C* ⊆ {*x* ∈ *topspace X*. *f x* ∈ {*1* / *2*<..}}
using ‹*closedin X C*› *f1 closedin_subset* **by** *auto*
qed (*auto simp*: *disjnt_iff*)
show *?thesis*
using *assms*
unfolding *completely_regular_space_def regular_space_def continuous_map_in_subtopology*
by (*meson* ∗)
qed

proposition *locally_compact_regular_imp_completely_regular_space*:
assumes *locally_compact_space X Hausdorff_space X* ∨ *regular_space X*
shows *completely_regular_space X*
unfolding *completely_regular_space_def*
proof *clarify*
fix *S x*
assume *closedin X S* **and** *x* ∈ *topspace X* **and** *x* ∉ *S*
have *regular_space X*
using *assms locally_compact_Hausdorff_imp_regular_space* **by** *blast*
then have *nbase*: *neighbourhood_base_of* (λ*C*. *compactin X C* ∧ *closedin X C*) *X*
using *assms*(*1*) *locally_compact_regular_space_neighbourhood_base* **by** *blast*
then obtain *U M* **where** *openin X U compactin X M closedin X M x* ∈ *U U* ⊆ *M M* ⊆ *topspace X* − *S*
unfolding *neighbourhood_base_of* **by** (*metis* (*no_types*, *lifting*) *Diff_iff* ‹*closedin X S*› ‹*x* ∈ *topspace X*› ‹*x* ∉ *S*› *closedin_def*)
then have *M* ⊆ *topspace X*
by *blast*
obtain *V K* **where** *openin X V closedin X K x* ∈ *V V* ⊆ *K K* ⊆ *U*
by (*metis* (*no_types*, *lifting*) ‹*openin X U*› ‹*x* ∈ *U*› *neighbourhood_base_of nbase*)

have *compact_space* (*subtopology X M*)
by (*simp add*: ‹*compactin X M*› *compact_space_subtopology*)
then have *normal_space* (*subtopology X M*)
by (*simp add*: ‹*regular_space X*› *compact_Hausdorff_or_regular_imp_normal_space regular_space_subtopology*)
moreover have *closedin* (*subtopology X M*) *K*
using ‹*K* ⊆ *U*› ‹*U* ⊆ *M*› ‹*closedin X K*› *closedin_subset_topspace* **by** *fastforce*
moreover have *closedin* (*subtopology X M*) (*M* − *U*)
by (*simp add*: ‹*closedin X M*› ‹*openin X U*› *closedin_diff closedin_subset_topspace*)
moreover have *disjnt K* (*M* − *U*)
by (*meson DiffD2* ‹*K* ⊆ *U*› *disjnt_iff subsetD*)
ultimately obtain *f*::*'a*⇒*real* **where** *contf*: *continuous_map* (*subtopology X M*) (*top_of_set* {*0..1*}) *f*
and *f0*: *f* ‘ *K* ⊆ {*0*} **and** *f1*: *f* ‘ (*M* − *U*) ⊆ {*1*}
using *Urysohn_lemma* [*of subtopology X M K M*−*U 0 1*] **by** *auto*
then obtain *g*::*'a*⇒*real* **where** *contg*: *continuous_map* (*subtopology X M*) *euclidean g* **and** *gim*: *g* ‘ *M* ⊆ {*0..1*}
and *g0*: ⋀*x*. *x* ∈ *K* ⟹ *g x* = *0* **and** *g1*: ⋀*x*. ⟦*x* ∈ *M*; *x* ∉ *U*⟧ ⟹ *g x* = *1*
using ‹*M* ⊆ *topspace X*› **by** (*force simp*: *continuous_map_in_subtopology image_subset_iff*)
show ∃*f*::*'a*⇒*real*. *continuous_map X* (*top_of_set* {*0..1*}) *f* ∧ *f x* = *0* ∧ *f* ‘ *S* ⊆ {*1*}
proof (*intro exI conjI*)
show *continuous_map X* (*top_of_set* {*0..1*}) (λ*x*. *if x* ∈ *M then g x else 1*)
unfolding *continuous_map_closedin*
proof (*intro strip conjI*)
fix *C*
assume *C*: *closedin* (*top_of_set* {*0*::*real..1*}) *C*
have *eq*: {*x* ∈ *topspace X*. (*if x* ∈ *M then g x else 1*) ∈ *C*} = {*x* ∈ *M*. *g x* ∈ *C*} ∪ (*if 1* ∈ *C then topspace X* − *U else* {})
using ‹*U* ⊆ *M*› ‹*M* ⊆ *topspace X*› *g1* **by** *auto*
show *closedin X* {*x* ∈ *topspace X*. (*if x* ∈ *M then g x else 1*) ∈ *C*}
unfolding *eq*
proof (*intro closedin_Un*)
have *closedin euclidean C*
using *C closed_closedin closedin_closed_trans* **by** *blast*
then have *closedin* (*subtopology X M*) {*x* ∈ *M*. *g x* ∈ *C*}
using *closedin_continuous_map_preimage_gen* [*OF contg*] ‹*M* ⊆ *topspace X*› **by** *auto*
then show *closedin X* {*x* ∈ *M*. *g x* ∈ *C*}
using ‹*closedin X M*› *closedin_trans_full* **by** *blast*
qed (*use* ‹*openin X U*› **in** *force*)
qed (*use gim* **in** *force*)
show (*if x* ∈ *M then g x else 1*) = *0*
using ‹*U* ⊆ *M*› ‹*V* ⊆ *K*› *g0* ‹*x* ∈ *U*› ‹*x* ∈ *V*› **by** *auto*
show (λ*x*. *if x* ∈ *M then g x else 1*) ‘ *S* ⊆ {*1*}
using ‹*M* ⊆ *topspace X* − *S*› **by** *auto*
qed
qed

lemma *completely_regular_eq_regular_space*:
locally_compact_space X
$\Longrightarrow$ (*completely_regular_space X* $\longleftrightarrow$ *regular_space X*)
using *completely_regular_imp_regular_space locally_compact_regular_imp_completely_regular_space*

by *blast*

lemma *completely_regular_space_prod_topology*:
completely_regular_space (*prod_topology X Y*) $\longleftrightarrow$
(*prod_topology X Y*) = *trivial_topology* $\vee$
completely_regular_space X $\wedge$ *completely_regular_space Y* (**is** *?lhs=?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*rule topological_property_of_prod_component*)
(*auto simp*: *completely_regular_space_subtopology homeomorphic_completely_regular_space*)
next
assume *R*: *?rhs*
show *?lhs*
proof (*cases* (*prod_topology X Y*) = *trivial_topology*)
case *False*
then have *X*: *completely_regular_space X* **and** *Y*: *completely_regular_space Y*
using *R* **by** *blast+*
show *?thesis*
unfolding *completely_regular_space_alt'*
proof *clarify*
fix *W x y*
assume *openin* (*prod_topology X Y*) *W* **and** (*x*, *y*) $\in$ *W*
then obtain *U V* **where** *openin X U openin Y V x* $\in$ *U y* $\in$ *V U*$\times$*V* $\subseteq$ *W*
by (*force simp*: *openin_prod_topology_alt*)
then have *x* $\in$ *topspace X y* $\in$ *topspace Y*
using *openin_subset* **by** *fastforce+*
obtain *f* **where** *contf*: *continuous_map X euclideanreal f* **and** *f x* = *0*
and *f1*: $\bigwedge$*x*. *x* $\in$ *topspace X* $\Longrightarrow$ *x* $\notin$ *U* $\Longrightarrow$ *f x* = *1*
using *X* ‹*openin X U*› ‹*x* $\in$ *U*› **unfolding** *completely_regular_space_alt'*
by (*smt* (*verit*, *best*) *Diff_iff image_subset_iff singletonD*)
obtain *g* **where** *contg*: *continuous_map Y euclideanreal g* **and** *g y* = *0*
and *g1*: $\bigwedge$*y*. *y* $\in$ *topspace Y* $\Longrightarrow$ *y* $\notin$ *V* $\Longrightarrow$ *g y* = *1*
using *Y* ‹*openin Y V*› ‹*y* $\in$ *V*› **unfolding** *completely_regular_space_alt'*
by (*smt* (*verit*, *best*) *Diff_iff image_subset_iff singletonD*)
define *h* **where** *h* $\equiv$ λ(*x*,*y*). *1* $-$ (*1* $-$ *f x*) $*$ (*1* $-$ *g y*)
show $\exists$ *h*. *continuous_map* (*prod_topology X Y*) *euclideanreal h* $\wedge$ *h* (*x*,*y*) = *0* $\wedge$ *h* ‘ (*topspace* (*prod_topology X Y*) $-$ *W*) $\subseteq$ {*1*}
proof (*intro exI conjI*)
have *continuous_map* (*prod_topology X Y*) *euclideanreal* (*f* $\circ$ *fst*)
using *contf continuous_map_of_fst* **by** *blast*
moreover
have *continuous_map* (*prod_topology X Y*) *euclideanreal* (*g* $\circ$ *snd*)

```
        using contg continuous_map_of_snd by blast
      ultimately
      show continuous_map (prod_topology X Y) euclideanreal h
        unfolding o_def h_def case_prod_unfold
        by (intro continuous_intros) auto
      show h (x, y) = 0
        by (simp add: h_def ‹f x = 0› ‹g y = 0›)
      show h ‘ (topspace (prod_topology X Y) − W) ⊆ {1}
        using ‹U × V ⊆ W› f1 g1 by (force simp: h_def)
    qed
  qed
 qed (force simp: completely_regular_space_def)
qed


proposition completely_regular_space_product_topology:
  completely_regular_space (product_topology X I) ⟷
   (∃ i∈I. X i = trivial_topology) ∨ (∀ i ∈ I. completely_regular_space (X i))
  (is ?lhs ⟷ ?rhs)
proof
 assume ?lhs then show ?rhs
   by (rule topological_property_of_product_component)
    (auto simp: completely_regular_space_subtopology homeomorphic_completely_regular_space)
next
 assume R: ?rhs
 show ?lhs
 proof (cases ∃ i∈I. X i = trivial_topology)
   case False
   show ?thesis
     unfolding completely_regular_space_alt'
   proof clarify
     fix W x
     assume W: openin (product_topology X I) W and x ∈ W
     then obtain U where finU: finite {i ∈ I. U i ≠ topspace (X i)}
           and ope: ⋀i. i∈I ⟹ openin (X i) (U i)
           and x: x ∈ Pi_E I U and Pi_E I U ⊆ W
       by (auto simp: openin_product_topology_alt)
     have ∀ i ∈ I. openin (X i) (U i) ∧ x i ∈ U i
           ⟶ (∃f. continuous_map (X i) euclideanreal f ∧
                  f (x i) = 0 ∧ (∀ x ∈ topspace(X i). x ∉ U i ⟶ f x = 1))
       using R unfolding completely_regular_space_alt'
       by (smt (verit) DiffI False image_subset_iff singletonD)
     with ope x have ⋀i. ∃f. i ∈ I ⟶ continuous_map (X i) euclideanreal f ∧
           f (x i) = 0 ∧ (∀ x ∈ topspace (X i). x ∉ U i ⟶ f x = 1)
       by auto
     then obtain f where f: ⋀i. i ∈ I ⟹ continuous_map (X i) euclideanreal
(f i) ∧
                                        f i (x i) = 0 ∧ (∀ x ∈ topspace (X i). x ∉ U i
⟶ f i x = 1)
```

```
      by metis
    define h where h ≡ λz. 1 − prod (λi. 1 − f i (z i)) {i∈I. U i ≠ topspace(X
i)}
    show ∃h. continuous_map (product_topology X I) euclideanreal h ∧ h x = 0
∧
                  h ‘ (topspace (product_topology X I) − W) ⊆ {1}
    proof (intro conjI exI)
      have continuous_map (product_topology X I) euclidean (f i ∘ (λx. x i)) if
i∈I for i
        using f that
        by (blast intro: continuous_intros continuous_map_product_projection)
      then show continuous_map (product_topology X I) euclideanreal h
        unfolding h_def o_def by (intro continuous_intros) (auto simp: finU)
      show h x = 0
        by (simp add: h_def f)
      show h ‘ (topspace (product_topology X I) − W) ⊆ {1}
        proof −
        have ∃i. i ∈ I ∧ U i ≠ topspace (X i) ∧ f i (x′ i) = 1
          if x′ ∈ (Π_E i∈I. topspace (X i)) x′ ∉ W for x′
          using that ‹Pi_E I U ⊆ W› by (smt (verit, best) PiE_iff f in_mono)
        then show ?thesis
          by (auto simp: f h_def finU)
      qed
    qed
  qed
  qed (force simp: completely_regular_space_def)
qed


lemma zero_dimensional_imp_completely_regular_space:
  assumes X dim_le 0
  shows completely_regular_space X
proof (clarsimp simp: completely_regular_space_def)
  fix C a
  assume closedin X C and a ∈ topspace X and a ∉ C
  then obtain U where closedin X U openin X U a ∈ U and U: U ⊆ topspace
X − C
   using assms by (force simp add: closedin_def dimension_le_0_neighbourhood_base_of_clopen
open_neighbourhood_base_of)
  show ∃f. continuous_map X (top_of_set {0::real..1}) f ∧ f a = 0 ∧ f ‘ C ⊆
{1}
  proof (intro exI conjI)
    have continuous_map X euclideanreal (λx. if x ∈ U then 0 else 1)
    using ‹closedin X U› ‹openin X U› apply (clarsimp simp: continuous_map_def
closedin_def)
      by (smt (verit) Diff_iff mem_Collect_eq openin_subopen subset_eq)
    then show continuous_map X (top_of_set {0::real..1}) (λx. if x ∈ U then 0
else 1)
      by (auto simp: continuous_map_in_subtopology)
```

qed (*use U ‹a ∈ U›* **in** *auto*)
qed

lemma *zero_dimensional_imp_regular_space*: *X dim_le 0 ⟹ regular_space X*
by (*simp add*: *completely_regular_imp_regular_space zero_dimensional_imp_completely_regular_spa*

lemma (**in** *Metric_space*) *t1_space_mtopology*:
t1_space mtopology
using *Hausdorff_space_mtopology t1_or_Hausdorff_space* **by** *blast*

6.11.5 More generally, the k-ification functor

definition *kification_open*
where *kification_open ≡*
λX S. S ⊆ topspace X ∧ (∀K. compactin X K ⟶ openin (subtopology X
K) (K ∩ S))

definition *kification*
where *kification X ≡ topology (kification_open X)*

lemma *istopology_kification_open*: *istopology (kification_open X)*
unfolding *istopology_def*
proof (*intro conjI strip*)
show *kification_open X (S ∩ T)*
if *kification_open X S* **and** *kification_open X T* **for** *S T*
using *that* **unfolding** *kification_open_def*
by (*smt* (*verit*, *best*) *inf.idem inf_commute inf_left_commute le_infI2 openin_Int*)
show *kification_open X (⋃ 𝒦)* **if** *∀K∈𝒦. kification_open X K* **for** *𝒦*
using *that* **unfolding** *kification_open_def Int_Union* **by** *blast*
qed

lemma *openin_kification*:
openin (kification X) U ⟷
U ⊆ topspace X ∧
(∀K. compactin X K ⟶ openin (subtopology X K) (K ∩ U))
by (*metis topology_inverse' kification_def istopology_kification_open kification_open_def*)

lemma *openin_kification_finer*:
openin X S ⟹ openin (kification X) S
by (*simp add*: *openin_kification openin_subset openin_subtopology_Int2*)

lemma *topspace_kification* [*simp*]:
topspace(kification X) = topspace X
by (*meson openin_kification openin_kification_finer openin_topspace subset_antisym*)

lemma *closedin_kification*:
closedin (kification X) U ⟷
U ⊆ topspace X ∧
(∀K. compactin X K ⟶ closedin (subtopology X K) (K ∩ U))

proof (*cases* $U \subseteq$ *topspace* X)
case *True*
then show *?thesis*
by (*simp add*: *closedin_def Diff_Int_distrib inf_commute le_infI2 openin_kification*)
qed (*simp add*: *closedin_def*)

lemma *closedin_kification_finer*: *closedin* X $S \Longrightarrow$ *closedin* (*kification* X) S
by (*simp add*: *closedin_def openin_kification_finer*)

lemma *kification_eq_self*: *kification* $X = X \longleftrightarrow$ *k_space* X
unfolding *fun_eq_iff openin_kification k_space_alt openin_inject* [*symmetric*]
by (*metis openin_closedin_eq*)

lemma *compactin_kification* [*simp*]:
compactin (*kification* X) $K \longleftrightarrow$ *compactin* X K (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*simp add*: *compactin_contractive openin_kification_finer*)
next
assume *R*: *?rhs*
show *?lhs*
unfolding *compactin_def*
proof (*intro conjI strip*)
show $K \subseteq$ *topspace* (*kification* X)
by (*simp add*: *R compactin_subset_topspace*)
fix $\mathcal{U}$
assume $\mathcal{U}$: *Ball* $\mathcal{U}$ (*openin* (*kification* X)) $\wedge$ $K \subseteq \bigcup \mathcal{U}$
then have $*$: $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow U \subseteq$ *topspace* $X \wedge$ *openin* (*subtopology* X K) ($K \cap U$)
by (*simp add*: *R openin_kification*)
have $K \subseteq$ *topspace* X *compact_space* (*subtopology* X K)
using *R compactin_subspace* **by** *force*+
then have $\exists V.$ *finite* $V \wedge V \subseteq (\lambda U.\ K \cap U)$ ' $\mathcal{U} \wedge \bigcup V =$ *topspace* (*subtopology* X K)
unfolding *compact_space*
by (*smt* (*verit*, *del_insts*) *Int_Union* $\mathcal{U}$ $*$ *image_iff inf.order_iff inf_commute topspace_subtopology*)
then show $\exists \mathcal{F}.$ *finite* $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \wedge K \subseteq \bigcup \mathcal{F}$
by (*metis Int_Union* ‹$K \subseteq$ *topspace* X› *finite_subset_image inf.orderI topspace_subtopology_subset*)
qed
qed

lemma *compact_space_kification* [*simp*]:
compact_space(*kification* X) $\longleftrightarrow$ *compact_space* X
by (*simp add*: *compact_space_def*)

lemma *kification_kification* [*simp*]:
kification(*kification* X) = *kification* X

```
  unfolding openin_inject [symmetric]
proof
  fix U
  show openin (kification (kification X)) U = openin (kification X) U
  proof
    show openin (kification (kification X)) U ⟹ openin (kification X) U
    by (metis compactin_kification inf_commute openin_kification openin_subtopology
topspace_kification)
  qed (simp add: openin_kification_finer)
qed

lemma k_space_kification [iff]: k_space(kification X)
  using kification_eq_self by fastforce

lemma continuous_map_into_kification:
  assumes k_space X
  shows continuous_map X (kification Y) f ⟷ continuous_map X Y f (is ?lhs
⟷ ?rhs)
proof
  assume ?lhs then show ?rhs
    by (simp add: continuous_map_def openin_kification_finer)
next
  assume R: ?rhs
  have openin X {x ∈ topspace X. f x ∈ V} if V: openin (kification Y) V for V
  proof −
    have openin (subtopology X K) (K ∩ {x ∈ topspace X. f x ∈ V})
      if compactin X K for K
    proof −
      have compactin Y (f ‘ K)
        using R image_compactin that by blast
      then have openin (subtopology Y (f ‘ K)) (f ‘ K ∩ V)
        by (meson V openin_kification)
      then obtain U where U: openin Y U f‘K ∩ V = U ∩ f‘K
        by (meson openin_subtopology)
      show ?thesis
        unfolding openin_subtopology
      proof (intro conjI exI)
        show openin X {x ∈ topspace X. f x ∈ U}
          using R U openin_continuous_map_preimage_gen by (simp add: con-
tinuous_map_def)
      qed (use U in auto)
    qed
    then show ?thesis
      by (metis (full_types) Collect_subset assms k_space_open)
  qed
  with R show ?lhs
    by (simp add: continuous_map_def)
qed
```

lemma *continuous_map_from_kification*:
continuous_map X Y f ⟹ *continuous_map* (*kification X*) *Y f*
by (*simp add*: *continuous_map_openin_preimage_eq openin_kification_finer*)

lemma *continuous_map_kification*:
continuous_map X Y f ⟹ *continuous_map* (*kification X*) (*kification Y*) *f*
by (*simp add*: *continuous_map_from_kification continuous_map_into_kification*)

lemma *subtopology_kification_compact*:
assumes *compactin X K*
shows *subtopology* (*kification X*) *K* = *subtopology X K*
unfolding *openin_inject* [*symmetric*]
proof
fix *U*
show *openin* (*subtopology* (*kification X*) *K*) *U* = *openin* (*subtopology X K*) *U*
by (*metis assms inf_commute openin_kification openin_subset openin_subtopology*)
qed

lemma *subtopology_kification_finer*:
assumes *openin* (*subtopology* (*kification X*) *S*) *U*
shows *openin* (*kification* (*subtopology X S*)) *U*
using *assms*
by (*fastforce simp*: *openin_subtopology_alt image_iff openin_kification subtopology_subtopology compactin_subtopology*)

lemma *proper_map_from_kification*:
assumes *k_space Y*
shows *proper_map* (*kification X*) *Y f* ⟷ *proper_map X Y f* (**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*simp add*: *closed_map_def closedin_kification_finer proper_map_alt*)
next
assume *R*: *?rhs*
have *compactin Y K* ⟹ *compactin X* {*x* ∈ *topspace X*. *f x* ∈ *K*} **for** *K*
using *R proper_map_alt* **by** *auto*
with *R* **show** *?lhs*
by (*simp add*: *assms proper_map_into_k_space_eq subtopology_kification_compact*)
qed

lemma *perfect_map_from_kification*:
⟦*k_space Y*; *perfect_map X Y f*⟧ ⟹ *perfect_map*(*kification X*) *Y f*
by (*simp add*: *continuous_map_from_kification perfect_map_def proper_map_from_kification*)

lemma *k_space_perfect_map_image_eq*:
assumes *Hausdorff_space X perfect_map X Y f*
shows *k_space X* ⟷ *k_space Y*
proof

show *k_space X* $\Longrightarrow$ *k_space Y*
using *k_space_perfect_map_image assms* **by** *blast*
assume *k_space Y*
have *homeomorphic_map* (*kification X*) *X id*
unfolding *homeomorphic_eq_injective_perfect_map*
proof (*intro conjI perfect_map_from_composition_right* [**where** *f* = *id*])
show *perfect_map* (*kification X*) *Y* (*f* $\circ$ *id*)
by (*simp add*: ‹*k_space Y*› *assms*(*2*) *perfect_map_from_kification*)
show *continuous_map* (*kification X*) *X id*
by (*simp add*: *continuous_map_from_kification*)
qed (*use assms perfect_map_def* **in** *auto*)
then show *k_space X*
using *homeomorphic_k_space homeomorphic_space* **by** *blast*
qed

6.11.6 One-point compactifications and the Alexandroff extension construction

lemma *one_point_compactification_dense*:
⟦*compact_space X*; ¬ *compactin X* (*topspace X* − {*a*})⟧ $\Longrightarrow$ *X closure_of* (*topspace X* − {*a*}) = *topspace X*
unfolding *closure_of_complement*
by (*metis Diff_empty closedin_compact_space interior_of_eq_empty openin_closedin_eq subset_singletonD*)

lemma *one_point_compactification_interior*:
⟦*compact_space X*; ¬ *compactin X* (*topspace X* − {*a*})⟧ $\Longrightarrow$ *X interior_of* {*a*} = {}
by (*simp add*: *interior_of_eq_empty_complement one_point_compactification_dense*)

proposition *kc_space_one_point_compactification_gen*:
assumes *compact_space X*
shows *kc_space X* $\longleftrightarrow$
openin X (*topspace X* − {*a*}) ∧ (∀ *K*. *compactin X K* ∧ *a*∉*K* $\longrightarrow$ *closedin X K*) ∧
k_space (*subtopology X* (*topspace X* − {*a*})) ∧ *kc_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *L*: *?lhs* **show** *?rhs*
proof (*intro conjI strip*)
show *openin X* (*topspace X* − {*a*})
using *L kc_imp_t1_space t1_space_openin_delete_alt* **by** *auto*
then show *k_space* (*subtopology X* (*topspace X* − {*a*}))
by (*simp add*: *L assms k_space_open_subtopology_aux*)
show *closedin X k* **if** *compactin X k* ∧ *a* ∉ *k* **for** *k* :: *'a set*
using *L kc_space_def that* **by** *blast*
show *kc_space* (*subtopology X* (*topspace X* − {*a*}))
by (*simp add*: *L kc_space_subtopology*)

```
  qed
next
  assume R: ?rhs
  show ?lhs
    unfolding kc_space_def
  proof (intro strip)
    fix S
    assume compactin X S
    then have S ⊆topspace X
      by (simp add: compactin_subset_topspace)
    show closedin X S
    proof (cases a ∈ S)
      case True
      then have topspace X − S = topspace X − {a} − (S − {a})
        by auto
      moreover have openin X (topspace X − {a} − (S − {a}))
      proof (rule openin_trans_full)
        show openin (subtopology X (topspace X − {a})) (topspace X − {a} − (S
− {a}))
        proof
          show openin (subtopology X (topspace X − {a})) (topspace X − {a})
            using R openin_open_subtopology by blast
          have closedin (subtopology X ((topspace X − {a}) ∩ K)) (K ∩ (S − {a}))
            if compactin X K K ⊆ topspace X − {a} for K
          proof (intro closedin_subset_topspace)
            show closedin X (K ∩ (S − {a}))
              using that
                by (metis IntD1 Int_insert_right_if0 R True ‹compactin X S›
closed_Int_compactin insert_Diff subset_Diff_insert)
          qed (use that in auto)
          moreover have k_space (subtopology X (topspace X − {a}))
            using R by blast
          moreover have S−{a} ⊆ topspace X ∧ S−{a} ⊆ topspace X − {a}
            using ‹S ⊆ topspace X› by auto
          ultimately show closedin (subtopology X (topspace X − {a})) (S − {a})
            using ‹S ⊆ topspace X› True
          by (simp add: k_space_def compactin_subtopology subtopology_subtopology)
        qed
        show openin X (topspace X − {a})
          by (simp add: R)
      qed
      ultimately show ?thesis
        by (simp add: ‹S ⊆ topspace X› closedin_def)
    next
      case False
      then show ?thesis
        by (simp add: R ‹compactin X S›)
    qed
  qed
```

qed

inductive *Alexandroff_open* **for** *X* **where**
base: *openin X U* $\Longrightarrow$ *Alexandroff_open X* (*Some* ‘ *U*)
| *ext*: ⟦*compactin X C*; *closedin X C*⟧ $\Longrightarrow$ *Alexandroff_open X* (*insert None* (*Some* ‘ (*topspace X* − *C*)))

hide_fact (**open**) *base ext*

lemma *Alexandroff_open_iff*: *Alexandroff_open X S* $\longleftrightarrow$
(∃ *U*. (*S* = *Some* ‘ *U* ∧ *openin X U*) ∨ (*S* = *insert None* (*Some* ‘ (*topspace X* − *U*)) ∧ *compactin X U* ∧ *closedin X U*))
by (*meson Alexandroff_open.cases Alexandroff_open.ext Alexandroff_open.base*)

lemma *Alexandroff_open_Un_aux*:
assumes *U*: *openin X U* **and** *Alexandroff_open X T*
shows *Alexandroff_open X* (*Some* ‘ *U* ∪ *T*)
using ‹*Alexandroff_open X T*›
proof (*induction rule*: *Alexandroff_open.induct*)
case (*base V*)
then show *?case*
by (*metis Alexandroff_open.base U image_Un openin_Un*)
next
case (*ext C*)
have *U* ⊆ *topspace X*
by (*simp add*: *U openin_subset*)
then have *eq*: *Some* ‘ *U* ∪ *insert None* (*Some* ‘ (*topspace X* − *C*)) = *insert None* (*Some* ‘ (*topspace X* − (*C* ∩ (*topspace X* − *U*))))
by *force*
have *closedin X* (*C* ∩ (*topspace X* − *U*))
using *U ext.hyps*(*2*) **by** *blast*
moreover
have *compactin X* (*C* ∩ (*topspace X* − *U*))
using *U compact_Int_closedin ext.hyps*(*1*) **by** *blast*
ultimately show *?case*
unfolding *eq* **using** *Alexandroff_open.ext* **by** *blast*
qed

lemma *Alexandroff_open_Un*:
assumes *Alexandroff_open X S* **and** *Alexandroff_open X T*
shows *Alexandroff_open X* (*S* ∪ *T*)
using *assms*
proof (*induction rule*: *Alexandroff_open.induct*)
case (*base U*)
then show *?case*
by (*simp add*: *Alexandroff_open_Un_aux*)
next
case (*ext C*)

```
  then show ?case
   by (smt (verit, best) Alexandroff_open_Un_aux Alexandroff_open_iff Un_commute
Un_insert_left closedin_def insert_absorb2)
qed

lemma Alexandroff_open_Int_aux:
  assumes U: openin X U and Alexandroff_open X T
  shows  Alexandroff_open X (Some ` U ∩ T)
  using ‹Alexandroff_open X T›
proof (induction rule: Alexandroff_open.induct)
  case (base V)
  then show ?case
    by (metis Alexandroff_open.base U image_Int inj_Some openin_Int)
next
  case (ext C)
  have eq: Some ` U ∩ insert None (Some ` (topspace X − C)) = Some ` (topspace
X − (C ∪ (topspace X − U)))
    by force
  have openin X (topspace X − (C ∪ (topspace X − U)))
    using U ext.hyps(2) by blast
  then show ?case
    unfolding eq using Alexandroff_open.base by blast
qed

proposition istopology_Alexandroff_open: istopology (Alexandroff_open X)
  unfolding istopology_def
proof (intro conjI strip)
  fix S T
  assume Alexandroff_open X S and Alexandroff_open X T
  then show Alexandroff_open X (S ∩ T)
  proof (induction rule: Alexandroff_open.induct)
    case (base U)
    then show ?case
      using Alexandroff_open_Int_aux by blast
  next
    case EC: (ext C)
    show ?case
      using ‹Alexandroff_open X T›
    proof (induction rule: Alexandroff_open.induct)
      case (base V)
      then show ?case
     by (metis Alexandroff_open.ext Alexandroff_open_Int_aux EC.hyps inf_commute)
    next
      case (ext D)
       have eq: insert None (Some ` (topspace X − C)) ∩ insert None (Some `
(topspace X − D))
             = insert None (Some ` (topspace X − (C ∪ D)))
        by auto
      show ?case
```

```
      unfolding eq
       by (simp add: Alexandroff_open.ext EC.hyps closedin_Un compactin_Un
ext.hyps)
   qed
  qed
next
  fix 𝒦
  assume §: ∀K∈𝒦. Alexandroff_open X K
  show Alexandroff_open X (⋃𝒦)
  proof (cases None ∈ ⋃𝒦)
    case True
    have ∀K∈𝒦. ∃U. (openin X U ∧ K = Some ‘ U) ∨ (K = insert None (Some
‘ (topspace X − U)) ∧ compactin X U ∧ closedin X U)
      by (metis § Alexandroff_open_iff)
    then obtain U where U:
      ⋀K. K ∈ 𝒦 ⟹ openin X (U K) ∧ K = Some ‘ (U K)
                 ∨ (K = insert None (Some ‘ (topspace X − U K)) ∧ compactin X
(U K) ∧ closedin X (U K))
      by metis
    define 𝒦N where 𝒦N ≡ {K ∈ 𝒦. None ∈ K}
    define A where A ≡ ⋃K∈𝒦−𝒦N. U K
    define B where B ≡ ⋂K∈𝒦N. U K
    have U1: ⋀K. K ∈ 𝒦−𝒦N ⟹ openin X (U K) ∧ K = Some ‘ (U K)
      using U 𝒦N_def by auto
    have U2: ⋀K. K ∈ 𝒦N ⟹ K = insert None (Some ‘ (topspace X − U K))
∧ compactin X (U K) ∧ closedin X (U K)
      using U 𝒦N_def by auto
    have eqA: ⋃(𝒦−𝒦N) = Some ‘ A
    proof
      show ⋃ (𝒦 − 𝒦N) ⊆ Some ‘ A
        by (metis A_def Sup_le_iff U1 UN_upper subset_image_iff)
      show Some ‘ A ⊆ ⋃ (𝒦 − 𝒦N)
        using A_def U1 by blast
    qed
    have eqB: ⋃𝒦N = insert None (Some ‘ (topspace X − B))
      using U2 True
      by (auto simp: B_def image_iff 𝒦N_def)
    have ⋃𝒦 = ⋃𝒦N ∪ ⋃(𝒦−𝒦N)
      by (auto simp: 𝒦N_def)
    then have eq: ⋃𝒦 = (Some ‘ A) ∪ (insert None (Some ‘ (topspace X − B)))
      by (simp add: eqA eqB Un_commute)
    show ?thesis
      unfolding eq
    proof (intro Alexandroff_open_Un Alexandroff_open.intros)
      show openin X A
        using A_def U1 by blast
      show closedin X B
        unfolding B_def using U2 True 𝒦N_def by auto
      show compactin X B
```

by (*metis B_def U2 eqB Inf_lower Union_iff* ‹*closedin X B*› *closed_compactin imageI insertI1*)
qed
next
case *False*
then have ∀ *K*∈*𝒦*. ∃ *U*. *openin X U* ∧ *K* = *Some* ‘ *U*
by (*metis Alexandroff_open.simps UnionI* § *insertCI*)
then obtain *U* **where** *U*: ∀ *K*∈*𝒦*. *openin X* (*U K*) ∧ *K* = *Some* ‘ (*U K*)
by *metis*
then have *eq*: ⋃*𝒦* = *Some* ‘ (⋃ *K*∈*𝒦*. *U K*)
using *image_iff* **by** *fastforce*
show *?thesis*
unfolding *eq* **by** (*simp add*: *U Alexandroff_open.base openin_clauses*(*3*))
qed
qed

definition *Alexandroff_compactification* **where**
Alexandroff_compactification X ≡ *topology* (*Alexandroff_open X*)

lemma *openin_Alexandroff_compactification*:
openin(*Alexandroff_compactification X*) *V* ⟷
(∃ *U*. *openin X U* ∧ *V* = *Some* ‘ *U*) ∨
(∃ *C*. *compactin X C* ∧ *closedin X C* ∧ *V* = *insert None* (*Some* ‘ (*topspace X* − *C*)))
by (*auto simp*: *Alexandroff_compactification_def istopology_Alexandroff_open Alexandroff_open.simps*)

lemma *topspace_Alexandroff_compactification* [*simp*]:
topspace(*Alexandroff_compactification X*) = *insert None* (*Some* ‘ *topspace X*)
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
by (*force simp*: *topspace_def openin_Alexandroff_compactification*)
have *None* ∈ *topspace* (*Alexandroff_compactification X*)
by (*meson closedin_empty compactin_empty insert_subset openin_Alexandroff_compactification openin_subset*)
moreover have *Some x* ∈ *topspace* (*Alexandroff_compactification X*)
if *x* ∈ *topspace X* **for** *x*
by (*meson that imageI openin_Alexandroff_compactification openin_subset openin_topspace subsetD*)
ultimately show *?rhs* ⊆ *?lhs*
by (*auto simp*: *image_subset_iff*)
qed

lemma *closedin_Alexandroff_compactification*:
closedin (*Alexandroff_compactification X*) *C* ⟷
(∃ *K*. *compactin X K* ∧ *closedin X K* ∧ *C* = *Some* ‘ *K*) ∨

```
      (∃ U. openin X U ∧ C = topspace(Alexandroff_compactification X) − Some
‘ U)
   (is ?lhs ⟷ ?rhs)
proof
  show ?lhs ⟹ ?rhs
    apply (clarsimp simp: closedin_def openin_Alexandroff_compactification)
   by (smt (verit) Diff_Diff_Int None_notin_image_Some image_set_diff inf.absorb_iff2
inj_Some insert_Diff_if subset_insert)
  show ?rhs ⟹ ?lhs
    using openin_subset
    by (fastforce simp: closedin_def openin_Alexandroff_compactification)
qed

lemma openin_Alexandroff_compactification_image_Some [simp]:
   openin(Alexandroff_compactification X) (Some ‘ U) ⟷ openin X U
  by (auto simp: openin_Alexandroff_compactification inj_image_eq_iff)

lemma closedin_Alexandroff_compactification_image_Some [simp]:
    closedin (Alexandroff_compactification X) (Some ‘ K) ⟷ compactin X K ∧
closedin X K
  by (auto simp: closedin_Alexandroff_compactification inj_image_eq_iff)

lemma open_map_Some: open_map X (Alexandroff_compactification X) Some
  using open_map_def openin_Alexandroff_compactification by blast

lemma continuous_map_Some: continuous_map X (Alexandroff_compactification
X) Some
  unfolding continuous_map_def
proof (intro conjI strip)
  fix U
  assume openin (Alexandroff_compactification X) U
  then consider V where openin X V U = Some ‘ V
    | C where compactin X C closedin X C U = insert None (Some ‘ (topspace X
− C))
    by (auto simp: openin_Alexandroff_compactification)
  then show openin X {x ∈ topspace X. Some x ∈ U}
  proof cases
    case 1
    then show ?thesis
      using openin_subopen openin_subset by fastforce
  next
    case 2
    then show ?thesis
      by (simp add: closedin_def image_iff set_diff_eq)
  qed
qed auto


lemma embedding_map_Some: embedding_map X (Alexandroff_compactification
```

```
X) Some
  by (simp add: continuous_map_Some injective_open_imp_embedding_map open_map_Some)

lemma compact_space_Alexandroff_compactification [simp]:
   compact_space(Alexandroff_compactification X)
proof (clarsimp simp: compact_space_alt image_subset_iff)
  fix 𝒰 U
  assume ope [rule_format]: ∀ U∈𝒰. openin (Alexandroff_compactification X) U
      and cover: ∀ x∈topspace X. ∃ X∈𝒰. Some x ∈ X
      and U ∈ 𝒰 None ∈ U
  then have Usub: U ⊆ insert None (Some ‘ topspace X)
    by (metis openin_subset topspace_Alexandroff_compactification)
  with ope [OF ‹U ∈ 𝒰›] ‹None ∈ U›
  obtain C where C: compactin X C ∧ closedin X C ∧
         insert None (Some ‘ topspace X) − U = Some ‘ C
    by (auto simp: openin_closedin closedin_Alexandroff_compactification)
  then have D: compactin (Alexandroff_compactification X) (insert None (Some
‘ topspace X) − U)
    by (metis continuous_map_Some image_compactin)
  consider V where openin X V U = Some ‘ V
    | C where compactin X C closedin X C U = insert None (Some ‘ (topspace X
− C))
    using ope [OF ‹U ∈ 𝒰›] by (auto simp: openin_Alexandroff_compactification)
  then show ∃ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ∧ (∃ X∈ℱ. None ∈ X) ∧ (∀ x∈topspace X.
∃ X∈ℱ. Some x ∈ X)
  proof cases
    case 1
    then show ?thesis
      using ‹None ∈ U› by blast
  next
    case 2
    obtain ℱ where finite ℱ ℱ ⊆ 𝒰 insert None (Some ‘ topspace X) − U ⊆ ⋃ℱ
    by (smt (verit, del_insts) C D Union_iff compactinD compactin_subset_topspace
cover image_subset_iff ope subsetD)
    with ‹U ∈ 𝒰› show ?thesis
      by (rule_tac x=insert U ℱ in exI) auto
  qed
qed

lemma topspace_Alexandroff_compactification_delete:
   topspace(Alexandroff_compactification X) − {None} = Some ‘ topspace X
  by simp

lemma Alexandroff_compactification_dense:
  assumes ¬ compact_space X
  shows (Alexandroff_compactification X) closure_of (Some ‘ topspace X) =
        topspace(Alexandroff_compactification X)
  unfolding topspace_Alexandroff_compactification_delete [symmetric]
proof (intro one_point_compactification_dense)
```

show ¬ *compactin* (*Alexandroff_compactification X*) (*topspace* (*Alexandroff_compactification X*) − {*None*})
using *assms compact_space_proper_map_preimage compact_space_subtopology embedding_map_Some embedding_map_def homeomorphic_imp_proper_map* **by** *fastforce*
qed *auto*

lemma *t0_space_one_point_compactification*:
assumes *compact_space X* ∧ *openin X* (*topspace X* − {*a*})
shows *t0_space X* ⟷ *t0_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
using *t0_space_subtopology* **by** *blast*
show *?rhs* ⟹ *?lhs*
using *assms*
unfolding *t0_space_def* **by** (*bestsimp simp flip*: *Int_Diff dest*: *openin_trans_full*)
qed

lemma *t0_space_Alexandroff_compactification* [*simp*]:
t0_space (*Alexandroff_compactification X*) ⟷ *t0_space X*
using *t0_space_one_point_compactification* [*of Alexandroff_compactification X None*]
using *embedding_map_Some embedding_map_imp_homeomorphic_space homeomorphic_t0_space* **by** *fastforce*

lemma *t1_space_one_point_compactification*:
assumes *Xa*: *openin X* (*topspace X* − {*a*})
and §: ⋀*K*. ⟦*compactin* (*subtopology X* (*topspace X* − {*a*})) *K*; *closedin* (*subtopology X* (*topspace X* − {*a*})) *K*⟧ ⟹ *closedin X K*
shows *t1_space X* ⟷ *t1_space* (*subtopology X* (*topspace X* − {*a*})) (**is** *?lhs* ⟷ *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
using *t1_space_subtopology* **by** *blast*
assume *R*: *?rhs*
show *?lhs*
unfolding *t1_space_closedin_singleton*
proof (*intro strip*)
fix *x*
assume *x* ∈ *topspace X*
show *closedin X* {*x*}
proof (*cases x*=*a*)
case *True*
then show *?thesis*
using ‹*x* ∈ *topspace X*› *Xa closedin_def* **by** *blast*
next
case *False*

```
      show ?thesis
        by (simp add: § False R ‹x ∈ topspace X› closedin_t1_singleton)
    qed
  qed
qed

lemma closedin_Alexandroff_I:
  assumes compactin (Alexandroff_compactification X) K K ⊆ Some ‘ topspace
X
          closedin (Alexandroff_compactification X) T K = T ∩ Some ‘ topspace X
  shows closedin (Alexandroff_compactification X) K
proof −
  obtain S where S: S ⊆ topspace X K = Some ‘ S
    by (meson ‹K ⊆ Some ‘ topspace X› subset_imageE)
  with assms have compactin X S
    by (metis compactin_subtopology embedding_map_Some embedding_map_def
homeomorphic_map_compactness)
  moreover have closedin X S
    using assms S
     by (metis closedin_subtopology embedding_map_Some embedding_map_def
homeomorphic_map_closedness)
  ultimately show ?thesis
    by (simp add: S)
qed


lemma t1_space_Alexandroff_compactification [simp]:
  t1_space(Alexandroff_compactification X) ⟷ t1_space X
proof −
  have openin (Alexandroff_compactification X) (topspace (Alexandroff_compactification
X) − {None})
    by auto
  then show ?thesis
    using t1_space_one_point_compactification [of Alexandroff_compactification
X None]
   using embedding_map_Some embedding_map_imp_homeomorphic_space home-
omorphic_t1_space
   by (fastforce simp: compactin_subtopology closedin_Alexandroff_I closedin_subtopology)
qed


lemma kc_space_one_point_compactification:
  assumes compact_space X
    and ope: openin X (topspace X − {a})
      and §: ⋀K. ⟦compactin (subtopology X (topspace X − {a})) K; closedin
(subtopology X (topspace X − {a})) K⟧
             ⟹ closedin X K
  shows kc_space X ⟷
        k_space (subtopology X (topspace X − {a})) ∧ kc_space (subtopology X
```

(*topspace X − {a}*))
proof −
have *closedin X K*
if *kc_space* (*subtopology X* (*topspace X − {a}*)) **and** *compactin X K a ∉ K* **for** *K*
using *that* **unfolding** *kc_space_def*
by (*metis* § *Diff_empty compactin_subspace compactin_subtopology subset_Diff_insert*)
then show *?thesis*
by (*metis ‹compact_space X› kc_space_one_point_compactification_gen ope*)
qed

lemma *kc_space_Alexandroff_compactification*:
kc_space(*Alexandroff_compactification X*) ⟷ (*k_space X ∧ kc_space X*) (**is** *kc_space ?Y = _*)
proof −
have *kc_space* (*Alexandroff_compactification X*) ⟷
(*k_space* (*subtopology ?Y* (*topspace ?Y − {None}*)) ∧ *kc_space* (*subtopology ?Y* (*topspace ?Y − {None}*)))
by (*rule kc_space_one_point_compactification*) (*auto simp*: *compactin_subtopology closedin_subtopology closedin_Alexandroff_I*)
also have ... ⟷ *k_space X ∧ kc_space X*
using *embedding_map_Some embedding_map_imp_homeomorphic_space homeomorphic_k_space homeomorphic_kc_space* **by** *simp blast*
finally show *?thesis* **.**
qed

proposition *regular_space_one_point_compactification*:
assumes *compact_space X* **and** *ope*: *openin X* (*topspace X − {a}*)
and §: ⋀*K*. ⟦*compactin* (*subtopology X* (*topspace X − {a}*)) *K*; *closedin* (*subtopology X* (*topspace X − {a}*)) *K*⟧ ⟹ *closedin X K*
shows *regular_space X* ⟷
regular_space (*subtopology X* (*topspace X − {a}*)) ∧ *locally_compact_space* (*subtopology X* (*topspace X − {a}*))
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
using *assms*(*1*) *compact_imp_locally_compact_space locally_compact_space_open_subset ope regular_space_subtopology* **by** *blast*
assume *R*: *?rhs*
let *?Xa* = *subtopology X* (*topspace X − {a}*)
show *?lhs*
unfolding *neighbourhood_base_of_closedin* [*symmetric*] *neighbourhood_base_of imp_conjL*
proof (*intro strip*)
fix *W x*
assume *openin X W* **and** *x ∈ W*
show ∃ *U V*. *openin X U ∧ closedin X V ∧ x ∈ U ∧ U ⊆ V ∧ V ⊆ W*
proof (*cases x=a*)

```
      case True
      have compactin ?Xa (topspace X − W)
        using ‹openin X W› assms(1) closedin_compact_space
        by (metis Diff_mono True ‹x ∈ W› compactin_subtopology empty_subsetI
insert_subset openin_closedin_eq order_refl)
     then obtain V K where V: openin ?Xa V and K: compactin ?Xa K closedin
?Xa K and topspace X − W ⊆ V V ⊆ K
        by (metis locally_compact_space_compact_closed_compact R)
      show ?thesis
      proof (intro exI conjI)
        show openin X (topspace X − K)
          using § K by blast
        show closedin X (topspace X − V)
          using V ope openin_trans_full by blast
        show x ∈ topspace X − K
        proof (rule)
          show x ∈ topspace X
            using ‹openin X W› ‹x ∈ W› openin_subset by blast
          show x ∉ K
            using K True closedin_imp_subset by blast
        qed
        show topspace X − K ⊆ topspace X − V
          by (simp add: Diff_mono ‹V ⊆ K›)
        show topspace X − V ⊆ W
          using ‹topspace X − W ⊆ V› by auto
      qed
    next
      case False
      have openin ?Xa ((topspace X − {a}) ∩ W)
        using ‹openin X W› openin_subtopology_Int2 by blast
      moreover have x ∈ (topspace X − {a}) ∩ W
        using ‹openin X W› ‹x ∈ W› openin_subset False by blast
     ultimately obtain U V where openin ?Xa U compactin ?Xa V closedin ?Xa
V
              x ∈ U U ⊆ V V ⊆ (topspace X − {a}) ∩ W
          using R locally_compact_regular_space_neighbourhood_base neighbour-
hood_base_of
        by (metis (no_types, lifting))
      then show ?thesis
        by (meson § le_infE ope openin_trans_full)
    qed
  qed
qed


lemma regular_space_Alexandroff_compactification:
  regular_space(Alexandroff_compactification X) ⟷ regular_space X ∧ locally_compact_space
X
  (is regular_space ?Y = ?rhs)
```

proof −
have *regular_space ?Y* ⟷
regular_space (*subtopology ?Y* (*topspace ?Y* − {*None*})) ∧ *locally_compact_space* (*subtopology ?Y* (*topspace ?Y* − {*None*}))
by (*rule regular_space_one_point_compactification*) (*auto simp*: *compactin_subtopology closedin_subtopology closedin_Alexandroff_I*)
also have ... ⟷ *regular_space X* ∧ *locally_compact_space X*
by (*metis embedding_map_Some embedding_map_imp_homeomorphic_space homeomorphic_locally_compact_space*
homeomorphic_regular_space topspace_Alexandroff_compactification_delete)
finally show *?thesis* **.**
qed

lemma *Hausdorff_space_one_point_compactification*:
assumes *compact_space X* **and** *openin X* (*topspace X* − {*a*})
and ⋀*K*. ⟦*compactin* (*subtopology X* (*topspace X* − {*a*})) *K*; *closedin* (*subtopology X* (*topspace X* − {*a*})) *K*⟧ ⟹ *closedin X K*
shows *Hausdorff_space X* ⟷
Hausdorff_space (*subtopology X* (*topspace X* − {*a*})) ∧ *locally_compact_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?rhs* **if** *?lhs*
proof −
have *locally_compact_space* (*subtopology X* (*topspace X* − {*a*}))
using *assms that compact_imp_locally_compact_space locally_compact_space_open_subset*
by *blast*
with *that* **show** *?rhs*
by (*simp add*: *Hausdorff_space_subtopology*)
qed
next
show *?rhs* ⟹ *?lhs*
by (*metis assms locally_compact_Hausdorff_or_regular regular_space_one_point_compactification*
regular_t1_eq_Hausdorff_space t1_space_one_point_compactification)
qed

lemma *Hausdorff_space_Alexandroff_compactification*:
Hausdorff_space(*Alexandroff_compactification X*) ⟷ *Hausdorff_space X* ∧ *locally_compact_space X*
by (*meson compact_Hausdorff_imp_regular_space compact_space_Alexandroff_compactification*
locally_compact_Hausdorff_or_regular regular_space_Alexandroff_compactification
regular_t1_eq_Hausdorff_space t1_space_Alexandroff_compactification)

lemma *completely_regular_space_Alexandroff_compactification*:
completely_regular_space(*Alexandroff_compactification X*) ⟷

```
        completely_regular_space X ∧ locally_compact_space X
  by (metis regular_space_Alexandroff_compactification completely_regular_eq_regular_space
     compact_imp_locally_compact_space compact_space_Alexandroff_compactification)

proposition Hausdorff_space_one_point_compactification_asymmetric_prod:
  assumes compact_space X
  shows Hausdorff_space X ⟷
        kc_space (prod_topology X (subtopology X (topspace X - {a}))) ∧
         k_space (prod_topology X (subtopology X (topspace X - {a})))  (is ?lhs
⟷ ?rhs)
proof (cases a ∈ topspace X)
  case True
  show ?thesis
  proof
    show ?rhs if ?lhs
    proof
      show kc_space (prod_topology X (subtopology X (topspace X - {a})))
      using Hausdorff_imp_kc_space kc_space_prod_topology_right kc_space_subtopology
that by blast
      show k_space (prod_topology X (subtopology X (topspace X - {a})))
      by (meson Hausdorff_imp_kc_space assms compact_imp_locally_compact_space
k_space_prod_topology_left
            kc_space_one_point_compactification_gen that)
    qed
  next
    assume R: ?rhs
    define D where D ≡ (λx. (x,x)) ` (topspace X - {a})
    show ?lhs
    proof (cases topspace X = {a})
      case True
      then show ?thesis
        by (simp add: Hausdorff_space_def)
    next
      case False
      with R True have kc_space X
       using kc_space_retraction_map_image [of prod_topology X (subtopology X
(topspace X - {a})) X fst]
      by (metis Diff_subset insert_Diff retraction_map_fst topspace_discrete_topology
topspace_subtopology_subset)
       have closedin (subtopology (prod_topology X (subtopology X (topspace X -
{a}))) K) (K ∩ D)
        if compactin (prod_topology X (subtopology X (topspace X - {a}))) K for
K
      proof (intro closedin_subtopology_Int_subset[where V=K] closedin_subset_topspace)
        show fst ` K × snd ` K ∩ D ⊆ fst ` K × snd ` K K ⊆ fst ` K × snd ` K
          by force+
        have eq: (fst ` K × snd ` K ∩ D) = ((λx. (x,x)) ` (fst ` K ∩ snd ` K))
          using compactin_subset_topspace that by (force simp: D_def image_iff)
        have compactin (prod_topology X (subtopology X (topspace X - {a}))) (fst
```

‘ $K \times$ *snd* ‘ $K \cap D$)
unfolding *eq*
proof (*rule image_compactin* [*of subtopology X* (*topspace X* − {*a*})])
have *compactin X* (*fst* ‘ *K*) *compactin X* (*snd* ‘ *K*)
by (*meson compactin_subtopology continuous_map_fst continuous_map_snd image_compactin that*)+
moreover have *fst* ‘ *K* ∩ *snd* ‘ *K* ⊆ *topspace X* − {*a*}
using *compactin_subset_topspace that* **by** *force*
ultimately
show *compactin* (*subtopology X* (*topspace X* − {*a*})) (*fst* ‘ *K* ∩ *snd* ‘ *K*)
unfolding *compactin_subtopology*
by (*meson* ‹*kc_space X*› *closed_Int_compactin kc_space_def*)
show *continuous_map* (*subtopology X* (*topspace X* − {*a*})) (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) (λ*x*. (*x*,*x*))
by (*simp add*: *continuous_map_paired*)
qed
then show *closedin* (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) (*fst* ‘ *K* × *snd* ‘ *K* ∩ *D*)
using *R compactin_imp_closedin_gen* **by** *blast*
qed
moreover have *D* ⊆ *topspace X* × (*topspace X* ∩ (*topspace X* − {*a*}))
by (*auto simp*: *D_def*)
ultimately have ∗: *closedin* (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) *D*
using *R* **by** (*auto simp*: *k_space*)
have *x*=*y*
if *x* ∈ *topspace X y* ∈ *topspace X*
and §: ⋀*T*. ⟦(*x*,*y*) ∈ *T*; *openin* (*prod_topology X X*) *T*⟧ ⟹ ∃ *z* ∈ *topspace X*. (*z*,*z*) ∈ *T* **for** *x y*
proof (*cases x*=*a* ∧ *y*=*a*)
case *False*
then consider *x*≠*a* | *y*≠*a*
by *blast*
then show *?thesis*
proof *cases*
case *1*
have ∃ *z* ∈ *topspace X* − {*a*}. (*z*,*z*) ∈ *T*
if (*y*,*x*) ∈ *T openin* (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) *T* **for** *T*
proof −
have (*x*,*y*) ∈ {*z* ∈ *topspace* (*prod_topology X X*). (*snd z*,*fst z*) ∈ *T* ∩ *topspace X* × (*topspace X* − {*a*})}
by (*simp add*: *1* ‹*x* ∈ *topspace X*› ‹*y* ∈ *topspace X*› *that*)
moreover have *openin* (*prod_topology X X*) {*z* ∈ *topspace* (*prod_topology X X*). (*snd z*,*fst z*) ∈ *T* ∩ *topspace X* × (*topspace X* − {*a*})}
proof (*rule openin_continuous_map_preimage*)
show *continuous_map* (*prod_topology X X*) (*prod_topology X X*) (λ*x*. (*snd x*, *fst x*))
by (*simp add*: *continuous_map_fst continuous_map_pairedI contin-*

uous_map_snd)
have *openin* (*prod_topology X X*) (*topspace X* × (*topspace X* − {*a*}))
using ‹*kc_space X*› *assms kc_space_one_point_compactification_gen openin_prod_Times_iff* **by** *fastforce*
moreover have *openin* (*prod_topology X X*) *T*
using *kc_space_one_point_compactification_gen* [*OF* ‹*compact_space X*›] ‹*kc_space X*› *that*
by (*metis openin_prod_Times_iff openin_topspace openin_trans_full prod_topology_subtopology*(*2*))
ultimately show *openin* (*prod_topology X X*) (*T* ∩ *topspace X* × (*topspace X* − {*a*}))
by *blast*
qed
ultimately show *?thesis*
by (*smt* (*verit*) § *Int_iff fst_conv mem_Collect_eq mem_Sigma_iff snd_conv*)
qed
then have (*y*,*x*) ∈ *prod_topology X* (*subtopology X* (*topspace X* − {*a*})) *closure_of D*
by (*simp add*: *1 D_def in_closure_of that*)
then show *?thesis*
using *that* ∗ *D_def closure_of_closedin* **by** *fastforce*
next
case *2*
have ∃ *z* ∈ *topspace X* − {*a*}. (*z*,*z*) ∈ *T*
if (*x*,*y*) ∈ *T openin* (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) *T* **for** *T*
proof −
have *openin* (*prod_topology X X*) (*topspace X* × (*topspace X* − {*a*}))
using ‹*kc_space X*› *assms kc_space_one_point_compactification_gen openin_prod_Times_iff* **by** *fastforce*
moreover have *XXT*: *openin* (*prod_topology X X*) *T*
using *kc_space_one_point_compactification_gen* [*OF* ‹*compact_space X*›] ‹*kc_space X*› *that*
by (*metis openin_prod_Times_iff openin_topspace openin_trans_full prod_topology_subtopology*(*2*))
ultimately have *openin* (*prod_topology X X*) (*T* ∩ *topspace X* × (*topspace X* − {*a*}))
by *blast*
then show *?thesis*
by (*smt* (*verit*) § *Diff_subset XXT mem_Sigma_iff openin_subset subsetD that topspace_prod_topology topspace_subtopology_subset*)
qed
then have (*x*,*y*) ∈ *prod_topology X* (*subtopology X* (*topspace X* − {*a*})) *closure_of D*
by (*simp add*: *2 D_def in_closure_of that*)
then show *?thesis*
using *that* ∗ *D_def closure_of_closedin* **by** *fastforce*
qed

```
      qed auto
      then show ?thesis
      unfolding Hausdorff_space_closedin_diagonal closure_of_subset_eq [symmetric]

        by (force simp: closure_of_def)
    qed
  qed
next
  case False
  then show ?thesis
    by (simp add: assms compact_imp_k_space compact_space_prod_topology
kc_space_compact_prod_topology)
qed


lemma Hausdorff_space_Alexandroff_compactification_asymmetric_prod:
   Hausdorff_space(Alexandroff_compactification X) ⟷
       kc_space(prod_topology (Alexandroff_compactification X) X) ∧
       k_space(prod_topology (Alexandroff_compactification X) X)
    (is Hausdorff_space ?Y = ?rhs)
proof −
  have ∗: subtopology (Alexandroff_compactification X)
     (topspace (Alexandroff_compactification X) −
     {None}) homeomorphic_space X
   using embedding_map_Some embedding_map_imp_homeomorphic_space homeomorphic_space_sym by fastforce
  have Hausdorff_space (Alexandroff_compactification X) ⟷
     (kc_space (prod_topology ?Y (subtopology ?Y (topspace ?Y − {None}))) ∧
      k_space (prod_topology ?Y (subtopology ?Y (topspace ?Y − {None}))))
   by (rule Hausdorff_space_one_point_compactification_asymmetric_prod) (auto
simp: compactin_subtopology closedin_subtopology closedin_Alexandroff_I)
  also have ... ⟷ ?rhs
   using homeomorphic_k_space homeomorphic_kc_space homeomorphic_space_prod_topology

         homeomorphic_space_refl ∗ by blast
  finally show ?thesis .
qed

lemma kc_space_as_compactification_unique:
  assumes kc_space X compact_space X
  shows openin X U ⟷
        (if a ∈ U then U ⊆ topspace X ∧ compactin X (topspace X − U)
                  else openin (subtopology X (topspace X − {a})) U)
proof (cases a ∈ U)
  case True
  then show ?thesis
   by (meson assms closedin_compact_space compactin_imp_closedin_gen openin_closedin_eq)
next
  case False
```

then show *?thesis*
by (*metis Diff_empty kc_space_one_point_compactification_gen openin_open_subtopology openin_subset subset_Diff_insert assms*)
qed

lemma *kc_space_as_compactification_unique_explicit*:
assumes *kc_space X compact_space X*
shows *openin X U* ⟷
(*if a* ∈ *U then U* ⊆ *topspace X* ∧
compactin (*subtopology X* (*topspace X* − {*a*})) (*topspace X* − *U*)
∧
closedin (*subtopology X* (*topspace X* − {*a*})) (*topspace X* − *U*)
else openin (*subtopology X* (*topspace X* − {*a*})) *U*)
apply (*simp add*: *kc_space_subtopology compactin_imp_closedin_gen assms compactin_subtopology cong*: *conj_cong*)
by (*metis Diff_mono assms bot.extremum insert_subset kc_space_as_compactification_unique subset_refl*)

lemma *Alexandroff_compactification_unique*:
assumes *kc_space X compact_space X* **and** *a*: *a* ∈ *topspace X*
shows *Alexandroff_compactification* (*subtopology X* (*topspace X* − {*a*})) *homeomorphic_space X*
(**is** *?Y homeomorphic_space X*)
proof −
have [*simp*]: *topspace X* ∩ (*topspace X* − {*a*}) = *topspace X* − {*a*}
by *auto*
have [*simp*]: *insert None* (*Some* ‘ *A*) = *insert None* (*Some* ‘ *B*) ⟷ *A* = *B*
insert None (*Some* ‘ *A*) ≠ *Some* ‘ *B* **for** *A B*
by *auto*
have *quotient_map X ?Y* (λ*x*. *if x* = *a then None else Some x*)
unfolding *quotient_map_def*
proof (*intro conjI strip*)
show (λ*x*. *if x* = *a then None else Some x*) ‘ *topspace X* = *topspace ?Y*
using ‹*a* ∈ *topspace X*› **by** *force*
show *openin X* {*x* ∈ *topspace X*. (*if x* = *a then None else Some x*) ∈ *U*} = *openin ?Y U* (**is** *?L* = *?R*)
if *U* ⊆ *topspace ?Y* **for** *U*
proof (*cases None* ∈ *U*)
case *True*
then obtain *T* **where** *T*[*simp*]: *U* = *insert None* (*Some* ‘ *T*)
by (*metis Int_insert_right UNIV_I UNIV_option_conv inf.orderE inf_le2 subsetI subset_imageE*)
have *Tsub*: *T* ⊆ *topspace X* − {*a*}
using *in_these_eq that* **by** *auto*
then have {*x* ∈ *topspace X*. (*if x* = *a then None else Some x*) ∈ *U*} = *insert a T*
by (*auto simp*: *a image_iff cong*: *conj_cong*)
then have *?L* ⟷ *openin X* (*insert a T*)
by *metis*

also have ... $\longleftrightarrow$ *?R*
using *Tsub assms*
apply (*simp add*: *openin_Alexandroff_compactification kc_space_as_compactification_unique_exp*
[**where** *a*=*a*])
by (*smt* (*verit*, *best*) *Diff_insert2 Diff_subset closedin_imp_subset dou-*
ble_diff)
finally show *?thesis* **.**
next
case *False*
then obtain *T* **where** [*simp*]: *U* = *Some* ' *T*
by (*metis Int_insert_right UNIV_I UNIV_option_conv inf.orderE inf_le2*
subsetI subset_imageE)
have **: $\bigwedge V$. *openin X V* $\Longrightarrow$ *openin X* (*V* − {*a*})
by (*simp add*: *assms compactin_imp_closedin_gen openin_diff*)
have *Tsub*: *T* ⊆ *topspace X* − {*a*}
using *in_these_eq that* **by** *auto*
then have {*x* ∈ *topspace X*. (*if x* = *a then None else Some x*) ∈ *U*} = *T*
by (*auto simp*: *image_iff cong*: *conj_cong*)
then show *?thesis*
by (*simp add*: ** *Tsub openin_open_subtopology*)
qed
qed
moreover have *inj_on* (λ*x*. *if x* = *a then None else Some x*) (*topspace X*)
by (*auto simp*: *inj_on_def*)
ultimately show *?thesis*
using *homeomorphic_space_sym homeomorphic_space homeomorphic_map_def*
by *blast*
qed

6.11.7 Extending continuous maps "pointwise" in a regular space

lemma *continuous_map_on_intermediate_closure_of*:
assumes *Y*: *regular_space Y*
and *T*: *T* ⊆ *X closure_of S*
and *f*: $\bigwedge t$. *t* ∈ *T* $\Longrightarrow$ *limitin Y f* (*f t*) (*atin_within X t S*)
shows *continuous_map* (*subtopology X T*) *Y f*
proof (*clarsimp simp add*: *continuous_map_atin*)
fix *a*
assume *a* ∈ *topspace X* **and** *a* ∈ *T*
have *f* ' *T* ⊆ *topspace Y*
by (*metis f image_subsetI limitin_topspace*)
have $\forall_F$ *x in atin_within X a T*. *f x* ∈ *W*
if *W*: *openin Y W f a* ∈ *W* **for** *W*
proof −
obtain *V C* **where** *openin Y V closedin Y C f a* ∈ *V V* ⊆ *C C* ⊆ *W*
by (*metis Y W neighbourhood_base_of neighbourhood_base_of_closedin*)
have $\forall_F$ *x in atin_within X a S*. *f x* ∈ *V*
by (*metis* ‹*a* ∈ *T*› ‹*f a* ∈ *V*› ‹*openin Y V*› *f limitin_def*)

then obtain U **where** *openin* X U $a \in U$ **and** U: $\forall x \in U - \{a\}.\ x \in S \longrightarrow f\ x \in V$
by (*smt* (*verit*) *Diff_iff* ‹$a \in$ *topspace* X› *eventually_atin_within insert_iff*)
moreover have $f\ z \in W$ **if** $z \in U$ $z \neq a$ $z \in T$ **for** z
proof −
have $z \in$ *topspace* X
using ‹*openin* X U› *openin_subset* ‹$z \in U$› **by** *blast*
then have $f\ z \in$ *topspace* Y
using ‹f ‘ $T \subseteq$ *topspace* Y› ‹$z \in T$› **by** *blast*
{ assume $f\ z \in$ *topspace* Y $f\ z \notin C$
then have $\forall_F\ x$ *in atin_within* X z S. $f\ x \in$ *topspace* $Y - C$
by (*metis Diff_iff* ‹*closedin* Y C› *closedin_def f limitinD* ‹$z \in T$›)
then obtain U' **where** U': *openin* X U' $z \in U'$
$\bigwedge x.\ x \in U' - \{z\} \Longrightarrow x \in S \Longrightarrow f\ x \notin C$
by (*smt* (*verit*) *Diff_iff* ‹$z \in$ *topspace* X› *eventually_atin_within insertCI*)
then have $*$: $\bigwedge D.\ z \in D \wedge$ *openin* X $D \Longrightarrow \exists y.\ y \in S \wedge y \in D$
by (*meson* T *in_closure_of subsetD* ‹$z \in T$›)
have *False*
using $*$ [*of* $U \cap U'$] U' U ‹$V \subseteq C$› ‹$f\ a \in V$› ‹$f\ z \notin C$› ‹*openin* X U› *that*
by *blast*
}
then show *?thesis*
using ‹$C \subseteq W$› ‹$f\ z \in$ *topspace* Y› **by** *auto*
qed
ultimately have $\exists U.$ *openin* X $U \wedge a \in U \wedge (\forall x \in U - \{a\}.\ x \in T \longrightarrow f\ x \in W)$
by *blast*
then show *?thesis*
using *eventually_atin_within* **by** *fastforce*
qed
then show *limitin* Y f (f a) (*atin* (*subtopology* X T) a)
by (*metis* ‹$a \in T$› *atin_subtopology_within f limitin_def*)
qed

lemma *continuous_map_on_intermediate_closure_of_eq*:
assumes *regular_space* Y $S \subseteq T$ **and** *Tsub*: $T \subseteq X$ *closure_of* S
shows *continuous_map* (*subtopology* X T) Y $f \longleftrightarrow$ ($\forall t \in T.$ *limitin* Y f (f t) (*atin_within* X t S))
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume L: *?lhs*
show *?rhs*
proof (*clarsimp simp add*: *continuous_map_atin*)
fix x
assume $x \in T$
with L *Tsub closure_of_subset_topspace*
have *limitin* Y f (f x) (*atin* (*subtopology* X T) x)

```
    by (fastforce simp: continuous_map_atin)
   then show limitin Y f (f x) (atin_within X x S)
    using ‹x ∈ T› ‹S ⊆ T›
    by (force simp: limitin_def atin_subtopology_within eventually_atin_within)
  qed
next
  show ?rhs ⟹ ?lhs
   using assms continuous_map_on_intermediate_closure_of by blast
qed

lemma continuous_map_extension_pointwise_alt:
  assumes §: regular_space Y S ⊆ T T ⊆ X closure_of S
    and f: continuous_map (subtopology X S) Y f
    and lim: ⋀t. t ∈ T−S ⟹ ∃ l. limitin Y f l (atin_within X t S)
  obtains g where continuous_map (subtopology X T) Y g ⋀x. x ∈ S ⟹ g x =
f x
proof −
  obtain g where g: ⋀t. t ∈ T ∧ t ∉ S ⟹ limitin Y f (g t) (atin_within X t S)
    by (metis Diff_iff lim)
  let ?h = λx. if x ∈ S then f x else g x
  show thesis
  proof
    have T: T ⊆ topspace X
      using § closure_of_subset_topspace by fastforce
    have limitin Y ?h (f t) (atin_within X t S) if t ∈ T t ∈ S for t
    proof −
      have limitin Y f (f t) (atin_within X t S)
        by (meson T f limit_continuous_map_within subset_eq that)
      then show ?thesis
        by (simp add: eventually_atin_within limitin_def)
    qed
    moreover have limitin Y ?h (g t) (atin_within X t S) if t ∈ T t ∉ S for t
      by (smt (verit, del_insts) eventually_atin_within g limitin_def that)
    ultimately show continuous_map (subtopology X T) Y ?h
      unfolding continuous_map_on_intermediate_closure_of_eq [OF §]
      by (auto simp: § atin_subtopology_within)
  qed auto
qed


lemma continuous_map_extension_pointwise:
  assumes regular_space Y S ⊆ T and Tsub: T ⊆ X closure_of S
    and ex: ⋀x. x ∈ T ⟹ ∃ g. continuous_map (subtopology X (insert x S)) Y
g ∧
                    (∀ x ∈ S. g x = f x)
  obtains g where continuous_map (subtopology X T) Y g ⋀x. x ∈ S ⟹ g x =
f x
proof (rule continuous_map_extension_pointwise_alt)
  show continuous_map (subtopology X S) Y f
```

proof (*clarsimp simp add*: *continuous_map_atin*)
fix *t*
assume $t \in$ *topspace X* **and** $t \in S$
then obtain *g* **where** *g*: *limitin Y g* (*g t*) (*atin* (*subtopology X* (*insert t S*)) *t*) **and** *gf*: $\forall x \in S.\ g\ x = f\ x$
by (*metis Int_iff* ‹$S \subseteq T$› *continuous_map_atin ex inf.orderE insert_absorb topspace_subtopology*)
with ‹$t \in S$› **show** *limitin Y f* (*f t*) (*atin* (*subtopology X S*) *t*)
by (*simp add*: *limitin_def atin_subtopology_within_if eventually_atin_within gf insert_absorb*)
qed
show $\exists l.$ *limitin Y f l* (*atin_within X t S*) **if** $t \in T - S$ **for** *t*
proof −
obtain *g* **where** *g*: *continuous_map* (*subtopology X* (*insert t S*)) *Y g* **and** *gf*: $\forall x \in S.\ g\ x = f\ x$
using ‹$S \subseteq T$› *ex* ‹$t \in T - S$› **by** *force*
then have *limitin Y g* (*g t*) (*atin_within X t* (*insert t S*))
using *Tsub in_closure_of limit_continuous_map_within that* **by** *fastforce*
then show *?thesis*
unfolding *limitin_def*
by (*smt* (*verit*) *eventually_atin_within gf subsetD subset_insertI*)
qed
qed (*use assms* **in** *auto*)

6.11.8 Extending Cauchy continuous functions to the closure

lemma *Cauchy_continuous_map_extends_to_continuous_closure_of_aux*:
assumes *m2*: *mcomplete_of m2* **and** *f*: *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
and $S \subseteq$ *mspace m1*
obtains *g*
where *continuous_map* (*subtopology* (*mtopology_of m1*) (*mtopology_of m1 closure_of S*)) (*mtopology_of m2*) *g* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof (*rule continuous_map_extension_pointwise_alt*)
interpret *L*: *Metric_space12 mspace m1 mdist m1 mspace m2 mdist m2*
by (*simp add*: *Metric_space12_mspace_mdist*)
interpret *S*: *Metric_space* $S \cap$ *mspace m1 mdist m1*
by (*simp add*: *L.M1.subspace*)
show *regular_space* (*mtopology_of m2*)
by (*simp add*: *Metric_space.regular_space_mtopology mtopology_of_def*)
show $S \subseteq$ *mtopology_of m1 closure_of S*
by (*simp add*: *assms*(*3*) *closure_of_subset*)
show *continuous_map* (*subtopology* (*mtopology_of m1*) *S*) (*mtopology_of m2*) *f*
by (*metis Cauchy_continuous_imp_continuous_map f mtopology_of_submetric*)
fix *a*
assume *a*: $a \in$ *mtopology_of m1 closure_of S* − *S*
then obtain σ **where** *ran*σ: *range* $\sigma \subseteq S$ *range* $\sigma \subseteq$ *mspace m1*
and *lim*σ: *limitin L.M1.mtopology* σ *a sequentially*

by (*force simp*: *mtopology_of_def L.M1.closure_of_sequentially*)
then have *L.M1.MCauchy σ*
by (*simp add*: *L.M1.convergent_imp_MCauchy mtopology_of_def*)
then have *L.M2.MCauchy* (*f* ∘ *σ*)
using *f ranσ* **by** (*simp add*: *Cauchy_continuous_map_def L.M1.subspace Metric_space.MCauchy_def*)
then obtain *l* **where** *l*: *limitin L.M2.mtopology* (*f* ∘ *σ*) *l sequentially*
by (*meson L.M2.mcomplete_def m2 mcomplete_of_def*)
have *limitin L.M2.mtopology f l* (*atin_within L.M1.mtopology a S*)
unfolding *L.limit_atin_sequentially_within imp_conjL*
proof (*intro conjI strip*)
show *l* ∈ *mspace m2*
using *L.M2.limitin_mspace l* **by** *blast*
fix *ϱ*
assume *range ϱ* ⊆ *S* ∩ *mspace m1* − {*a*} **and** *limϱ*: *limitin L.M1.mtopology ϱ a sequentially*
then have *ranϱ*: *range ϱ* ⊆ *S range ϱ* ⊆ *mspace m1* ⋀*n*. *ϱ n* ≠ *a*
by *auto*
have *a* ∈ *mspace m1*
using *L.M1.limitin_mspace limϱ* **by** *auto*
have *S.MCauchy σ S.MCauchy ϱ*
using *L.M1.convergent_imp_MCauchy L.M1.MCauchy_def S.MCauchy_def limσ ranσ limϱ ranϱ* **by** *force*+
then have *L.M2.MCauchy* (*f* ∘ *ϱ*) *L.M2.MCauchy* (*f* ∘ *σ*)
using *f* **by** (*auto simp*: *Cauchy_continuous_map_def*)
then have *ran_f*: *range* (λ*x*. *f* (*ϱ x*)) ⊆ *mspace m2 range* (λ*x*. *f* (*σ x*)) ⊆ *mspace m2*
by (*auto simp*: *L.M2.MCauchy_def*)
have (λ*n*. *mdist m2* (*f* (*ϱ n*)) *l*) ⟶ *0*
proof (*rule Lim_null_comparison*)
have *mdist m2* (*f* (*ϱ n*)) *l* ≤ *mdist m2* (*f* (*σ n*)) *l* + *mdist m2* (*f* (*σ n*)) (*f* (*ϱ n*)) **for** *n*
using ‹*l* ∈ *mspace m2*› *ran_f L.M2.triangle″* **by** (*smt* (*verit*, *best*) *range_subsetD*)
then show ∀ *F n in sequentially*. *norm* (*mdist m2* (*f* (*ϱ n*)) *l*) ≤ *mdist m2* (*f* (*σ n*)) *l* + *mdist m2* (*f* (*σ n*)) (*f* (*ϱ n*))
by *force*
define *ψ* **where** *ψ* ≡ λ*n*. *if even n then σ* (*n div 2*) *else ϱ* (*n div 2*)
have (λ*n*. *mdist m1* (*σ n*) (*ϱ n*)) ⟶ *0*
proof (*rule Lim_null_comparison*)
show ∀ *F n in sequentially*. *norm* (*mdist m1* (*σ n*) (*ϱ n*)) ≤ *mdist m1* (*σ n*) *a* + *mdist m1* (*ϱ n*) *a*
using *L.M1.triangle′* [*of* _ *a*] *ranσ ranϱ* ‹*a* ∈ *mspace m1*› **by** (*simp add*: *range_subsetD*)
have (λ*n*. *mdist m1* (*σ n*) *a*) ⟶ *0*
using *L.M1.limitin_metric_dist_null limσ* **by** *blast*
moreover have (λ*n*. *mdist m1* (*ϱ n*) *a*) ⟶ *0*
using *L.M1.limitin_metric_dist_null limϱ* **by** *blast*
ultimately show (λ*n*. *mdist m1* (*σ n*) *a* + *mdist m1* (*ϱ n*) *a*) ⟶ *0*

```
      by (simp add: tendsto_add_zero)
    qed
    with ‹S.MCauchy σ› ‹S.MCauchy ϱ› have S.MCauchy ψ
      by (simp add: S.MCauchy_interleaving_gen ψ_def)
    then have L.M2.MCauchy (f ∘ ψ)
    by (metis Cauchy_continuous_map_def f mdist_submetric mspace_submetric)
    then have (λn. mdist m2 (f (σ n)) (f (ϱ n))) ⟶ 0
      using L.M2.MCauchy_interleaving_gen [of f ∘ σ f ∘ ϱ]
      by (simp add: if_distrib ψ_def o_def cong: if_cong)
    moreover have ∀F n in sequentially. f (σ n) ∈ mspace m2 ∧ (λx. mdist m2
(f (σ x)) l) ⟶ 0
      using l by (auto simp: L.M2.limitin_metric_dist_null ‹l ∈ mspace m2›)
    ultimately show (λn. mdist m2 (f (σ n)) l + mdist m2 (f (σ n)) (f (ϱ n)))
⟶ 0
    by (metis (mono_tags) tendsto_add_zero eventually_sequentially order_refl)
  qed
  with ran_f show limitin L.M2.mtopology (f ∘ ϱ) l sequentially
    by (auto simp: L.M2.limitin_metric_dist_null eventually_sequentially ‹l ∈
mspace m2›)
 qed
 then show ∃l. limitin (mtopology_of m2) f l (atin_within (mtopology_of m1)
a S)
  by (force simp: mtopology_of_def)
qed auto


lemma Cauchy_continuous_map_extends_to_continuous_closure_of:
 assumes mcomplete_of m2
   and f: Cauchy_continuous_map (submetric m1 S) m2 f
 obtains g
  where continuous_map (subtopology (mtopology_of m1) ((mtopology_of m1)
closure_of S))
                       (mtopology_of m2) g  ⋀x. x ∈ S ⟹ g x = f x
proof −
 obtain g where cmg:
     continuous_map (subtopology (mtopology_of m1) ((mtopology_of m1) clo-
sure_of (mspace m1 ∩ S)))
                    (mtopology_of m2) g
   and gf: (∀x ∈ mspace m1 ∩ S. g x = f x)
  using Cauchy_continuous_map_extends_to_continuous_closure_of_aux assms
   by (metis inf_commute inf_le2 submetric_restrict)
 define h where h ≡ λx. if x ∈ topspace(mtopology_of m1) then g x else f x
 show thesis
 proof
   show continuous_map (subtopology (mtopology_of m1) ((mtopology_of m1)
closure_of S))
                       (mtopology_of m2) h
    unfolding h_def
   proof (rule continuous_map_eq)
```

show *continuous_map* (*subtopology* (*mtopology_of m1*) (*mtopology_of m1 closure_of S*)) (*mtopology_of m2*) *g*
by (*metis closure_of_restrict cmg topspace_mtopology_of*)
qed *auto*
qed (*auto simp*: *gf h_def*)
qed

lemma *Cauchy_continuous_map_extends_to_continuous_intermediate_closure_of*:
assumes *mcomplete_of m2*
and *f*: *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
and *T*: $T \subseteq$ *mtopology_of m1 closure_of S*
obtains *g*
where *continuous_map* (*subtopology* (*mtopology_of m1*) *T*) (*mtopology_of m2*) *g*
$(\forall x \in S.\ g\ x = f\ x)$
by (*metis Cauchy_continuous_map_extends_to_continuous_closure_of T assms*(*1*) *continuous_map_from_subtopology_mono f*)

Technical lemma helpful for porting particularly ugly HOL Light proofs

lemma *all_in_closure_of*:
assumes *P*: $\forall x \in S.\ P\ x$ **and** *clo*: *closedin X* $\{x \in$ *topspace X*. *P x*$\}$
shows $\forall x \in X$ *closure_of S*. *P x*
proof −
have ∗: *topspace X* $\cap$ *S* $\subseteq$ $\{x \in$ *topspace X*. *P x*$\}$
using *P* **by** *auto*
show *?thesis*
using *closure_of_minimal* [*OF* ∗ *clo*] *closure_of_restrict* **by** *fastforce*
qed

lemma *Lipschitz_continuous_map_on_intermediate_closure_aux*:
assumes *lcf*: *Lipschitz_continuous_map* (*submetric m1 S*) *m2 f*
and *S* $\subseteq$ *T* **and** *Tsub*: *T* $\subseteq$ (*mtopology_of m1*) *closure_of S*
and *cmf*: *continuous_map* (*subtopology* (*mtopology_of m1*) *T*) (*mtopology_of m2*) *f*
and *S* $\subseteq$ *mspace m1*
shows *Lipschitz_continuous_map* (*submetric m1 T*) *m2 f*
proof −
interpret *L*: *Metric_space12 mspace m1 mdist m1 mspace m2 mdist m2*
by (*simp add*: *Metric_space12_mspace_mdist*)
interpret *S*: *Metric_space S* $\cap$ *mspace m1 mdist m1*
by (*simp add*: *L.M1.subspace*)
have *T* $\subseteq$ *mspace m1*
using *Tsub* **by** (*auto simp*: *mtopology_of_def closure_of_def*)
show *?thesis*
unfolding *Lipschitz_continuous_map_pos*
proof
show *f* $\in$ *mspace* (*submetric m1 T*) $\to$ *mspace m2*
by (*metis cmf Metric_space.metric_continuous_map Metric_space_mspace_mdist*

mtopology_of_def
mtopology_of_submetric image_subset_iff_funcset)
define X **where** $X \equiv$ *prod_topology* (*subtopology L.M1.mtopology T*) (*subtopology L.M1.mtopology T*)
obtain B::*real* **where** $B > 0$ **and** B: $\forall (x,y) \in S \times S$. *mdist m2* (*f x*) (*f y*) $\leq B *$ *mdist m1 x y*
using *lcf* ‹$S \subseteq$ *mspace m1*› **by** (*force simp*: *Lipschitz_continuous_map_pos*)
have *eq*: $\{z \in A$. *case z of* $(x,y) \Rightarrow p\ x\ y \leq B * q\ x\ y\} = \{z \in A$. $((\lambda(x,y).\ B * q\ x\ y - p\ x\ y)z) \in \{0..\}\}$
for *p q* **and** A::$('a*'a)$*set*
by *auto*
have *clo*: *closedin X* $\{z \in$ *topspace X*. *case z of* $(x, y) \Rightarrow$ *mdist m2* (*f x*) (*f y*) $\leq B *$ *mdist m1 x y*$\}$
unfolding *eq*
proof (*rule closedin_continuous_map_preimage*)
have $*$: *continuous_map X L.M2.mtopology* (*f* $\circ$ *fst*) *continuous_map X L.M2.mtopology* (*f* $\circ$ *snd*)
using *cmf* **by** (*auto simp*: *mtopology_of_def X_def intro*: *continuous_map_compose continuous_map_fst continuous_map_snd*)
then show *continuous_map X euclidean* (λx. *case x of* $(x, y) \Rightarrow B *$ *mdist m1 x y* $-$ *mdist m2* (*f x*) (*f y*))
unfolding *case_prod_unfold*
proof (*intro continuous_intros*; *simp add*: *mtopology_of_def o_def*)
show *continuous_map X L.M1.mtopology fst continuous_map X L.M1.mtopology snd*
by (*simp_all add*: *X_def continuous_map_subtopology_fst continuous_map_subtopology_snd flip*: *subtopology_Times*)
qed
qed *auto*
have *mdist m2* (*f x*) (*f y*) $\leq B *$ *mdist m1 x y* **if** $x \in T$ $y \in T$ **for** *x y*
using *all_in_closure_of* [*OF B clo*] ‹$S \subseteq T$› *Tsub*
by (*fastforce simp*: *X_def subset_iff closure_of_Times closure_of_subtopology inf.absorb2*
mtopology_of_def that)
then show $\exists B{>}0.\ \forall x{\in}$*mspace* (*submetric m1 T*).
$\forall y{\in}$*mspace* (*submetric m1 T*).
mdist m2 (*f x*) (*f y*) $\leq B *$ *mdist* (*submetric m1 T*) *x y*
using ‹$0 < B$› **by** *auto*
qed
qed

lemma *Lipschitz_continuous_map_on_intermediate_closure*:
assumes *Lipschitz_continuous_map* (*submetric m1 S*) *m2 f*
and $S \subseteq T$ $T \subseteq$ (*mtopology_of m1*) *closure_of S*
and *continuous_map* (*subtopology* (*mtopology_of m1*) *T*) (*mtopology_of m2*) *f*
shows *Lipschitz_continuous_map* (*submetric m1 T*) *m2 f*
by (*metis Lipschitz_continuous_map_on_intermediate_closure_aux assms clo-*

sure_of_subset_topspace subset_trans topspace_mtopology_of)

lemma *Lipschitz_continuous_map_extends_to_closure_of*:
assumes *m2*: *mcomplete_of m2*
and *f*: *Lipschitz_continuous_map* (*submetric m1 S*) *m2 f*
obtains *g*
where *Lipschitz_continuous_map* (*submetric m1* (*mtopology_of m1 closure_of S*)) *m2 g*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof −
obtain *g*
where *g*: *continuous_map* (*subtopology* (*mtopology_of m1*) ((*mtopology_of m1*) *closure_of S*))
(*mtopology_of m2*) *g* (∀ *x* ∈ *S*. *g x* = *f x*)
by (*metis Cauchy_continuous_map_extends_to_continuous_closure_of Lipschitz_imp_Cauchy_continuous_map f m2*)
have *Lipschitz_continuous_map* (*submetric m1* (*mtopology_of m1 closure_of S*)) *m2 g*
proof (*rule Lipschitz_continuous_map_on_intermediate_closure*)
show *Lipschitz_continuous_map* (*submetric m1* (*mspace m1* ∩ *S*)) *m2 g*
by (*smt* (*verit*, *best*) *IntD2 Lipschitz_continuous_map_eq f g*(*2*) *inf_commute mspace_submetric submetric_restrict*)
show *mspace m1* ∩ *S* ⊆ *mtopology_of m1 closure_of S*
using *closure_of_subset_Int* **by** *force*
show *mtopology_of m1 closure_of S* ⊆ *mtopology_of m1 closure_of* (*mspace m1* ∩ *S*)
by (*metis closure_of_restrict subset_refl topspace_mtopology_of*)
show *continuous_map* (*subtopology* (*mtopology_of m1*) (*mtopology_of m1 closure_of S*)) (*mtopology_of m2*) *g*
by (*simp add*: *g*)
qed
with *g that* **show** *thesis*
by *metis*
qed

lemma *Lipschitz_continuous_map_extends_to_intermediate_closure_of*:
assumes *mcomplete_of m2*
and *Lipschitz_continuous_map* (*submetric m1 S*) *m2 f*
and *T* ⊆ *mtopology_of m1 closure_of S*
obtains *g*
where *Lipschitz_continuous_map* (*submetric m1 T*) *m2 g* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
by (*metis Lipschitz_continuous_map_extends_to_closure_of Lipschitz_continuous_map_from_subm assms*)

This proof uses the same trick to extend the function's domain to its closure

lemma *uniformly_continuous_map_on_intermediate_closure_aux*:

assumes *ucf*: *uniformly_continuous_map* (*submetric m1 S*) *m2 f*
and $S \subseteq T$ **and** *Tsub*: $T \subseteq$ (*mtopology_of m1*) *closure_of S*
and *cmf*: *continuous_map* (*subtopology* (*mtopology_of m1*) *T*) (*mtopology_of m2*) *f*
and $S \subseteq$ *mspace m1*
shows *uniformly_continuous_map* (*submetric m1 T*) *m2 f*
proof −
interpret *L*: *Metric_space12 mspace m1 mdist m1 mspace m2 mdist m2*
by (*simp add*: *Metric_space12_mspace_mdist*)
interpret *S*: *Metric_space S* ∩ *mspace m1 mdist m1*
by (*simp add*: *L.M1.subspace*)
have $T \subseteq$ *mspace m1*
using *Tsub* **by** (*auto simp*: *mtopology_of_def closure_of_def*)
show *?thesis*
unfolding *uniformly_continuous_map_def*
proof (*intro conjI strip*)
show $f \in$ *mspace* (*submetric m1 T*) → *mspace m2*
by (*metis cmf Metric_space.metric_continuous_map Metric_space_mspace_mdist mtopology_of_def mtopology_of_submetric image_subset_iff_funcset*)
fix ε::*real*
assume $\varepsilon > 0$
then obtain δ **where** $\delta{>}0$ **and** δ: $\forall (x,y) \in S{\times}S$. *mdist m1 x y* $< \delta \longrightarrow$ *mdist m2* (*f x*) (*f y*) $\leq \varepsilon/2$
using *ucf* ‹$S \subseteq$ *mspace m1*› **unfolding** *uniformly_continuous_map_def mspace_submetric*
apply (*simp add*: *Ball_def del*: *divide_const_simps*)
by (*metis IntD2 half_gt_zero inf.orderE less_eq_real_def*)
define *X* **where** *X* ≡ *prod_topology* (*subtopology L.M1.mtopology T*) (*subtopology L.M1.mtopology T*)

A clever construction involving the union of two closed sets

have *eq*: {*z* ∈ *A*. *case z of* (*x*,*y*) ⇒ *p x y* < *d* ⟶ *q x y* ≤ *e*}
= {*z* ∈ *A*. ((λ(*x*,*y*). *p x y* − *d*)*z*) ∈ {*0*..}} ∪ {*z* ∈ *A*. ((λ(*x*,*y*). *e* − *q x y*)*z*) ∈ {*0*..}}
for *p q* **and** *d e*::*real* **and** *A*::(*'a*∗*'a*)*set*
by *auto*
have *clo*: *closedin X* {*z* ∈ *topspace X*. *case z of* (*x*, *y*) ⇒ *mdist m1 x y* < δ ⟶ *mdist m2* (*f x*) (*f y*) ≤ $\varepsilon/2$}
unfolding *eq*
proof (*intro closedin_Un closedin_continuous_map_preimage*)
have ∗: *continuous_map X L.M1.mtopology fst continuous_map X L.M1.mtopology snd*
by (*metis X_def continuous_map_subtopology_fst subtopology_Times continuous_map_subtopology_snd*)+
show *continuous_map X euclidean* (λ*x*. *case x of* (*x*, *y*) ⇒ *mdist m1 x y* − δ)
unfolding *case_prod_unfold*
by (*intro continuous_intros*; *simp add*: *mtopology_of_def* ∗)

have $*$: *continuous_map X L.M2.mtopology* (*f* ∘ *fst*) *continuous_map X L.M2.mtopology* (*f* ∘ *snd*)
using *cmf* **by** (*auto simp*: *mtopology_of_def X_def intro*: *continuous_map_compose continuous_map_fst continuous_map_snd*)
then show *continuous_map X euclidean* (λ*x*. *case x of* (*x*, *y*) ⇒ *ε* / 2 − *mdist m2* (*f x*) (*f y*))
unfolding *case_prod_unfold*
by (*intro continuous_intros*; *simp add*: *mtopology_of_def o_def*)
qed *auto*
have *mdist m2* (*f x*) (*f y*) ≤ *ε*/2 **if** *x* ∈ *T y* ∈ *T mdist m1 x y* < *δ* **for** *x y*
using *all_in_closure_of* [*OF δ clo*] ‹*S* ⊆ *T*› *Tsub*
by (*fastforce simp*: *X_def subset_iff closure_of_Times closure_of_subtopology inf.absorb2*
mtopology_of_def that)
then show ∃*δ*>0. ∀*x*∈*mspace* (*submetric m1 T*). ∀*y*∈*mspace* (*submetric m1 T*). *mdist* (*submetric m1 T*) *y x* < *δ* ⟶ *mdist m2* (*f y*) (*f x*) < *ε*
using ‹0 < *δ*› ‹0 < *ε*› **by** *fastforce*
qed
qed

lemma *uniformly_continuous_map_on_intermediate_closure*:
assumes *uniformly_continuous_map* (*submetric m1 S*) *m2 f*
and *S* ⊆ *T* **and** *T* ⊆ (*mtopology_of m1*) *closure_of S*
and *continuous_map* (*subtopology* (*mtopology_of m1*) *T*) (*mtopology_of m2*) *f*
shows *uniformly_continuous_map* (*submetric m1 T*) *m2 f*
by (*metis assms closure_of_subset_topspace subset_trans topspace_mtopology_of*

uniformly_continuous_map_on_intermediate_closure_aux)

lemma *uniformly_continuous_map_extends_to_closure_of*:
assumes *m2*: *mcomplete_of m2*
and *f*: *uniformly_continuous_map* (*submetric m1 S*) *m2 f*
obtains *g*
where *uniformly_continuous_map* (*submetric m1* (*mtopology_of m1 closure_of S*)) *m2 g*
⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
proof −
obtain *g*
where *g*: *continuous_map* (*subtopology* (*mtopology_of m1*) ((*mtopology_of m1*) *closure_of S*))
(*mtopology_of m2*) *g* (∀ *x* ∈ *S*. *g x* = *f x*)
by (*metis Cauchy_continuous_map_extends_to_continuous_closure_of uniformly_imp_Cauchy_continuous_map f m2*)
have *uniformly_continuous_map* (*submetric m1* (*mtopology_of m1 closure_of S*)) *m2 g*
proof (*rule uniformly_continuous_map_on_intermediate_closure*)
show *uniformly_continuous_map* (*submetric m1* (*mspace m1* ∩ *S*)) *m2 g*
by (*smt* (*verit*, *best*) *IntD2 uniformly_continuous_map_eq f g*(2) *inf_commute*

mspace_submetric submetric_restrict)
show *mspace m1 ∩ S ⊆ mtopology_of m1 closure_of S*
using *closure_of_subset_Int* **by** *force*
show *mtopology_of m1 closure_of S ⊆ mtopology_of m1 closure_of* (*mspace m1 ∩ S*)
by (*metis closure_of_restrict subset_refl topspace_mtopology_of*)
show *continuous_map* (*subtopology* (*mtopology_of m1*) (*mtopology_of m1 closure_of S*)) (*mtopology_of m2*) *g*
by (*simp add*: *g*)
qed
with *g that* **show** *thesis*
by *metis*
qed

lemma *uniformly_continuous_map_extends_to_intermediate_closure_of*:
assumes *mcomplete_of m2*
and *uniformly_continuous_map* (*submetric m1 S*) *m2 f*
and *T ⊆ mtopology_of m1 closure_of S*
obtains *g*
where *uniformly_continuous_map* (*submetric m1 T*) *m2 g* ⋀*x*. *x ∈ S ⟹ g x = f x*
by (*metis uniformly_continuous_map_extends_to_closure_of uniformly_continuous_map_from_submetric_mo assms*)

lemma *Cauchy_continuous_map_on_intermediate_closure_aux*:
assumes *ucf*: *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
and *S ⊆ T* **and** *Tsub*: *T ⊆* (*mtopology_of m1*) *closure_of S*
and *cmf*: *continuous_map* (*subtopology* (*mtopology_of m1*) *T*) (*mtopology_of m2*) *f*
and *S ⊆ mspace m1*
shows *Cauchy_continuous_map* (*submetric m1 T*) *m2 f*
proof −
interpret *L*: *Metric_space12 mspace m1 mdist m1 mspace m2 mdist m2*
by (*simp add*: *Metric_space12_mspace_mdist*)
interpret *S*: *Metric_space S ∩ mspace m1 mdist m1*
by (*simp add*: *L.M1.subspace*)
interpret *T*: *Metric_space T mdist m1*
by (*metis L.M1.subspace Tsub closure_of_subset_topspace dual_order.trans topspace_mtopology_of*)
have *T ⊆ mspace m1*
using *Tsub* **by** (*auto simp*: *mtopology_of_def closure_of_def*)
then show *?thesis*
proof (*clarsimp simp*: *Cauchy_continuous_map_def Int_absorb2*)
fix *σ*
assume *σ*: *T.MCauchy σ*
have ∃*y*∈*S*. *mdist m1* (*σ n*) *y* < *inverse* (*Suc n*) ∧ *mdist m2* (*f* (*σ n*)) (*f y*) < *inverse* (*Suc n*) **for** *n*

proof −
have $\sigma\ n \in T$
using σ **by** (*force simp*: *T.MCauchy_def*)
moreover have *continuous_map* (*mtopology_of* (*submetric m1 T*)) *L.M2.mtopology f*
by (*metis cmf mtopology_of_def mtopology_of_submetric*)
ultimately obtain δ **where** $\delta{>}0$ **and** δ: $\forall x \in T.\ mdist\ m1\ (\sigma\ n)\ x < \delta \longrightarrow mdist\ m2\ (f(\sigma\ n))\ (f\ x) < inverse\ (Suc\ n)$
using ‹$T \subseteq mspace\ m1$›
apply (*simp add*: *mtopology_of_def Metric_space.metric_continuous_map L.M1.subspace Int_absorb2*)
by (*metis inverse_Suc of_nat_Suc*)
have $\exists y \in S.\ mdist\ m1\ (\sigma\ n)\ y < min\ \delta\ (inverse\ (Suc\ n))$
using ‹$\sigma\ n \in T$› *Tsub* ‹$\delta{>}0$›
unfolding *mtopology_of_def L.M1.metric_closure_of subset_iff mem_Collect_eq L.M1.in_mball*
by (*smt* (*verit*, *del_insts*) *inverse_Suc*)
with δ ‹$S \subseteq T$› **show** *?thesis*
by *auto*
qed
then obtain ϱ **where** ϱS: $\bigwedge n.\ \varrho\ n \in S$ **and** $\varrho 1$: $\bigwedge n.\ mdist\ m1\ (\sigma\ n)\ (\varrho\ n) < inverse\ (Suc\ n)$
and $\varrho 2$: $\bigwedge n.\ mdist\ m2\ (f\ (\sigma\ n))\ (f\ (\varrho\ n)) < inverse\ (Suc\ n)$
by *metis*
have *S.MCauchy* ϱ
unfolding *S.MCauchy_def*
proof (*intro conjI strip*)
show $range\ \varrho \subseteq S \cap mspace\ m1$
using ‹$S \subseteq mspace\ m1$› **by** (*auto simp*: ϱS)
fix ε :: *real*
assume $\varepsilon{>}0$
then obtain M **where** M: $\bigwedge n\ n'.\ M \le n \Longrightarrow M \le n' \Longrightarrow mdist\ m1\ (\sigma\ n)\ (\sigma\ n') < \varepsilon/2$
using σ **unfolding** *T.MCauchy_def* **by** (*meson half_gt_zero*)
have $\forall_F\ n$ *in sequentially*. $inverse\ (Suc\ n) < \varepsilon/4$
using *Archimedean_eventually_inverse* ‹$0 < \varepsilon$› *divide_pos_pos zero_less_numeral* **by** *blast*
then obtain N **where** N: $\bigwedge n.\ N \le n \Longrightarrow inverse\ (Suc\ n) < \varepsilon/4$
by (*meson eventually_sequentially*)
have $mdist\ m1\ (\varrho\ n)\ (\varrho\ n') < \varepsilon$ **if** $n \ge max\ M\ N\ n' \ge max\ M\ N$ **for** $n\ n'$
proof −
have $mdist\ m1\ (\varrho\ n)\ (\varrho\ n') \le mdist\ m1\ (\varrho\ n)\ (\sigma\ n) + mdist\ m1\ (\sigma\ n)\ (\varrho\ n')$
by (*meson T.MCauchy_def T.triangle* ϱS σ ‹$S \subseteq T$› *rangeI subset_iff*)
also have $\ldots \le mdist\ m1\ (\varrho\ n)\ (\sigma\ n) + mdist\ m1\ (\sigma\ n)\ (\sigma\ n') + mdist\ m1\ (\sigma\ n')\ (\varrho\ n')$
by (*smt* (*verit*, *best*) *T.MCauchy_def T.triangle* ϱS σ ‹$S \subseteq T$› *in_mono rangeI*)
also have $\ldots < \varepsilon/4 + \varepsilon/2 + \varepsilon/4$

using *ϱ1[of n] ϱ1[of n'] N[of n] N[of n'] that M[of n n']* **by** (*simp add*: *T.commute*)
also have ... ≤ *ε*
by *simp*
finally show *?thesis* **.**
qed
then show ∃ *N*. ∀ *n n'*. *N* ≤ *n* ⟶ *N* ≤ *n'* ⟶ *mdist m1* (*ϱ n*) (*ϱ n'*) < *ε*
by *blast*
qed
then have *fϱ*: *L.M2.MCauchy* (*f* ∘ *ϱ*)
using *ucf* **by** (*simp add*: *Cauchy_continuous_map_def*)
show *L.M2.MCauchy* (*f* ∘ *σ*)
unfolding *L.M2.MCauchy_def*
proof (*intro conjI strip*)
show *range* (*f* ∘ *σ*) ⊆ *mspace m2*
using ‹*T* ⊆ *mspace m1*› *σ cmf*
apply (*auto simp*:)
by (*metis Metric_space.metric_continuous_map Metric_space_mspace_mdist T.MCauchy_def image_eqI inf.absorb1 mspace_submetric mtopology_of_def mtopology_of_submetric range_subsetD subset_iff*)
fix *ε* :: *real*
assume *ε*>*0*
then obtain *M* **where** *M*: ⋀*n n'*. *M* ≤ *n* ⟹ *M* ≤ *n'* ⟹ *mdist m2* ((*f* ∘ *ϱ*) *n*) ((*f* ∘ *ϱ*) *n'*) < *ε*/*2*
using *fϱ* **unfolding** *L.M2.MCauchy_def* **by** (*meson half_gt_zero*)
have ∀ *F n in sequentially*. *inverse* (*Suc n*) < *ε*/*4*
using *Archimedean_eventually_inverse* ‹*0* < *ε*› *divide_pos_pos zero_less_numeral* **by** *blast*
then obtain *N* **where** *N*: ⋀*n*. *N* ≤ *n* ⟹ *inverse* (*Suc n*) < *ε*/*4*
by (*meson eventually_sequentially*)
have *mdist m2* ((*f* ∘ *σ*) *n*) ((*f* ∘ *σ*) *n'*) < *ε* **if** *n* ≥ *max M N n'* ≥ *max M N* **for** *n n'*
proof −
have *mdist m2* ((*f* ∘ *σ*) *n*) ((*f* ∘ *σ*) *n'*) ≤ *mdist m2* ((*f* ∘ *σ*) *n*) ((*f* ∘ *ϱ*) *n*) + *mdist m2* ((*f* ∘ *ϱ*) *n*) ((*f* ∘ *σ*) *n'*)
by (*meson L.M2.MCauchy_def* ‹*range* (*f* ∘ *σ*) ⊆ *mspace m2*› *fϱ mdist_triangle rangeI subset_eq*)
also have ... ≤ *mdist m2* ((*f* ∘ *σ*) *n*) ((*f* ∘ *ϱ*) *n*) + *mdist m2* ((*f* ∘ *ϱ*) *n*) ((*f* ∘ *ϱ*) *n'*) + *mdist m2* ((*f* ∘ *ϱ*) *n'*) ((*f* ∘ *σ*) *n'*)
by (*smt* (*verit*) *L.M2.MCauchy_def L.M2.triangle* ‹*range* (*f* ∘ *σ*) ⊆ *mspace m2*› *fϱ range_subsetD*)
also have ... < *ε*/*4* + *ε*/*2* + *ε*/*4*
using *ϱ2[of n] ϱ2[of n'] N[of n] N[of n'] that M[of n n']* **by** (*simp add*: *L.M2.commute*)
also have ... ≤ *ε*
by *simp*
finally show *?thesis* **.**
qed
then show ∃ *N*. ∀ *n n'*. *N* ≤ *n* ⟶ *N* ≤ *n'* ⟶ *mdist m2* ((*f* ∘ *σ*) *n*) ((*f* ∘

$\sigma)\ n') < \varepsilon$
by *blast*
qed
qed
qed

lemma *Cauchy_continuous_map_on_intermediate_closure*:
assumes *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
and $S \subseteq T$ **and** $T \subseteq$ (*mtopology_of m1*) *closure_of S*
and *continuous_map* (*subtopology* (*mtopology_of m1*) *T*) (*mtopology_of m2*) *f*
shows *Cauchy_continuous_map* (*submetric m1 T*) *m2 f*
by (*metis Cauchy_continuous_map_on_intermediate_closure_aux assms closure_of_subset_topspace order.trans topspace_mtopology_of*)

lemma *Cauchy_continuous_map_extends_to_closure_of*:
assumes *m2*: *mcomplete_of m2*
and *f*: *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
obtains *g*
where *Cauchy_continuous_map* (*submetric m1* (*mtopology_of m1 closure_of S*)) *m2 g*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof −
obtain *g*
where *g*: *continuous_map* (*subtopology* (*mtopology_of m1*) ((*mtopology_of m1*) *closure_of S*))
(*mtopology_of m2*) *g* $(\forall x \in S.\ g\ x = f\ x)$
by (*metis Cauchy_continuous_map_extends_to_continuous_closure_of f m2*)
have *Cauchy_continuous_map* (*submetric m1* (*mtopology_of m1 closure_of S*)) *m2 g*
proof (*rule Cauchy_continuous_map_on_intermediate_closure*)
show *Cauchy_continuous_map* (*submetric m1* (*mspace m1* $\cap$ *S*)) *m2 g*
by (*smt* (*verit*, *best*) *IntD2 Cauchy_continuous_map_eq f g*(*2*) *inf_commute mspace_submetric submetric_restrict*)
show *mspace m1* $\cap$ *S* $\subseteq$ *mtopology_of m1 closure_of S*
using *closure_of_subset_Int* **by** *force*
show *mtopology_of m1 closure_of S* $\subseteq$ *mtopology_of m1 closure_of* (*mspace m1* $\cap$ *S*)
by (*metis closure_of_restrict subset_refl topspace_mtopology_of*)
show *continuous_map* (*subtopology* (*mtopology_of m1*) (*mtopology_of m1 closure_of S*)) (*mtopology_of m2*) *g*
by (*simp add*: *g*)
qed
with *g that* **show** *thesis*
by *metis*
qed

lemma *Cauchy_continuous_map_extends_to_intermediate_closure_of*:
assumes *mcomplete_of m2*
and *Cauchy_continuous_map* (*submetric m1 S*) *m2 f*
and $T \subseteq$ *mtopology_of m1 closure_of S*
obtains *g*
where *Cauchy_continuous_map* (*submetric m1 T*) *m2 g* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
by (*metis Cauchy_continuous_map_extends_to_closure_of Cauchy_continuous_map_from_submetric_mono assms*)

6.11.9 Metric space of bounded functions

context *Metric_space*
begin

definition *fspace* :: $'b\ set \Rightarrow ('b \Rightarrow 'a)\ set$ **where**
fspace $\equiv \lambda S.\ \{f.\ f`S \subseteq M \wedge f \in extensional\ S \wedge mbounded\ (f`S)\}$

definition *fdist* :: $['b\ set, 'b \Rightarrow 'a, 'b \Rightarrow 'a] \Rightarrow real$ **where**
fdist $\equiv \lambda S\ f\ g.$ *if* $f \in$ *fspace* $S \wedge g \in$ *fspace* $S \wedge S \neq \{\}$
then Sup $((\lambda x.\ d\ (f\ x)\ (g\ x)) `\ S)$ *else* 0

lemma *fspace_empty* [*simp*]: *fspace* $\{\} = \{\lambda x.\ undefined\}$
by (*auto simp*: *fspace_def*)

lemma *fdist_empty* [*simp*]: *fdist* $\{\} = (\lambda x\ y.\ 0)$
by (*auto simp*: *fdist_def*)

lemma *fspace_in_M*: $[\![f \in$ *fspace* $S;\ x \in S]\!] \Longrightarrow f\ x \in M$
by (*auto simp*: *fspace_def*)

lemma *bdd_above_dist*:
assumes *f*: $f \in$ *fspace* S **and** *g*: $g \in$ *fspace* S **and** $S \neq \{\}$
shows *bdd_above* $((\lambda u.\ d\ (f\ u)\ (g\ u)) `\ S)$
proof −
obtain a **where** $a \in S$
using ‹$S \neq \{\}$› **by** *blast*
obtain $B\ x\ C\ y$ **where** $B{>}0$ **and** *B*: $f`S \subseteq$ *mcball* $x\ B$
and $C{>}0$ **and** *C*: $g`S \subseteq$ *mcball* $y\ C$
using *f g mbounded_pos* **by** (*auto simp*: *fspace_def*)
have $d\ (f\ u)\ (g\ u) \leq B + d\ x\ y + C$ **if** $u{\in}S$ **for** u
proof −
have $f\ u \in M$
by (*meson B image_eqI mbounded_mcball mbounded_subset_mspace subsetD that*)
have $g\ u \in M$
by (*meson C image_eqI mbounded_mcball mbounded_subset_mspace subsetD that*)
have $x \in M\ y \in M$

```
    using B C that by auto
  have d (f u) (g u) ≤ d (f u) x + d x (g u)
    by (simp add: ‹f u ∈ M› ‹g u ∈ M› ‹x ∈ M› triangle)
  also have ... ≤ d (f u) x + d x y + d y (g u)
    by (simp add: ‹f u ∈ M› ‹g u ∈ M› ‹x ∈ M› ‹y ∈ M› triangle)
  also have ... ≤ B + d x y + C
    using B C commute that by fastforce
  finally show ?thesis .
 qed
 then show ?thesis
  by (meson bdd_above.I2)
qed


lemma Metric_space_funspace: Metric_space (fspace S) (fdist S)
proof
 show ∗: 0 ≤ fdist S f g for f g
  by (auto simp: fdist_def intro: cSUP_upper2 [OF bdd_above_dist])
 show fdist S f g = fdist S g f for f g
  by (auto simp: fdist_def commute)
 show (fdist S f g = 0) = (f = g)
  if fg: f ∈ fspace S g ∈ fspace S for f g
 proof
  assume 0: fdist S f g = 0
  show f = g
  proof (cases S={})
   case True
   with 0 that show ?thesis
    by (simp add: fdist_def fspace_def)
  next
   case False
   with 0 fg have Sup0: (SUP x∈S. d (f x) (g x)) = 0
    by (simp add: fdist_def)
   have d (f x) (g x) = 0 if x∈S for x
     by (smt (verit) False Sup0 ‹x∈S› bdd_above_dist [OF fg] less_cSUP_iff
nonneg)
   with fg show f=g
    by (simp add: fspace_def extensionalityI image_subset_iff)
  qed
 next
  show f = g ⟹ fdist S f g = 0
   using fspace_in_M [OF ‹g ∈ fspace S›] by (auto simp: fdist_def)
 qed
 show fdist S f h ≤ fdist S f g + fdist S g h
  if fgh: f ∈ fspace S g ∈ fspace S h ∈ fspace S for f g h
 proof (clarsimp simp add: fdist_def that)
  assume S ≠ {}
  have dfh: d (f x) (h x) ≤ d (f x) (g x) + d (g x) (h x) if x∈S for x
   by (meson fgh fspace_in_M that triangle)
```

have *bdd_fgh*: *bdd_above* ((λ*x*. *d* (*f x*) (*g x*)) ‘ *S*) *bdd_above* ((λ*x*. *d* (*g x*) (*h x*)) ‘ *S*)
by (*simp_all add*: ‹*S* ≠ {}› *bdd_above_dist that*)
then obtain *B C* **where** *B*: ⋀*x*. *x*∈*S* ⟹ *d* (*f x*) (*g x*) ≤ *B* **and** *C*: ⋀*x*. *x*∈*S* ⟹ *d* (*g x*) (*h x*) ≤ *C*
by (*auto simp*: *bdd_above_def*)
then have ⋀*x*. *x*∈*S* ⟹ *d* (*f x*) (*g x*) + *d* (*g x*) (*h x*) ≤ *B*+*C*
by *force*
then have *bdd*: *bdd_above* ((λ*x*. *d* (*f x*) (*g x*) + *d* (*g x*) (*h x*)) ‘ *S*)
by (*auto simp*: *bdd_above_def*)
then have (*SUP x*∈*S*. *d* (*f x*) (*h x*)) ≤ (*SUP x*∈*S*. *d* (*f x*) (*g x*) + *d* (*g x*) (*h x*))
by (*metis* (*mono_tags*, *lifting*) *cSUP_mono* ‹*S* ≠ {}› *dfh*)
also have ... ≤ (*SUP x*∈*S*. *d* (*f x*) (*g x*)) + (*SUP x*∈*S*. *d* (*g x*) (*h x*))
by (*simp add*: ‹*S* ≠ {}› *bdd cSUP_le_iff bdd_fgh add_mono cSup_upper*)
finally show (*SUP x*∈*S*. *d* (*f x*) (*h x*)) ≤ (*SUP x*∈*S*. *d* (*f x*) (*g x*)) + (*SUP x*∈*S*. *d* (*g x*) (*h x*)) **.**
qed
qed

end

definition *funspace* **where**
funspace S m ≡ *metric* (*Metric_space.fspace* (*mspace m*) (*mdist m*) *S*,
Metric_space.fdist (*mspace m*) (*mdist m*) *S*)

lemma *mspace_funspace* [*simp*]:
mspace (*funspace S m*) = *Metric_space.fspace* (*mspace m*) (*mdist m*) *S*
by (*simp add*: *Metric_space.Metric_space_funspace Metric_space.mspace_metric funspace_def*)

lemma *mdist_funspace* [*simp*]:
mdist (*funspace S m*) = *Metric_space.fdist* (*mspace m*) (*mdist m*) *S*
by (*simp add*: *Metric_space.Metric_space_funspace Metric_space.mdist_metric funspace_def*)

lemma *funspace_imp_welldefined*:
⟦*f* ∈ *mspace* (*funspace S m*); *x* ∈ *S*⟧ ⟹ *f x* ∈ *mspace m*
by (*simp add*: *Metric_space.fspace_def subset_iff*)

lemma *funspace_imp_extensional*:
f ∈ *mspace* (*funspace S m*) ⟹ *f* ∈ *extensional S*
by (*simp add*: *Metric_space.fspace_def*)

lemma *funspace_imp_bounded_image*:
f ∈ *mspace* (*funspace S m*) ⟹ *Metric_space.mbounded* (*mspace m*) (*mdist m*) (*f* ‘ *S*)
by (*simp add*: *Metric_space.fspace_def*)

lemma *funspace_imp_bounded*:
  $f \in mspace\ (funspace\ S\ m) \Longrightarrow S = \{\} \vee (\exists\, c\ B.\ \forall\, x \in S.\ mdist\ m\ c\ (f\ x) \leq B)$
  by (*auto simp*: *Metric_space.fspace_def Metric_space.mbounded*)

lemma (**in** *Metric_space*) *funspace_imp_bounded2*:
  assumes $f \in fspace\ S\ g \in fspace\ S$
  obtains B **where** $\bigwedge x.\ x \in S \Longrightarrow d\ (f\ x)\ (g\ x) \leq B$
proof −
  have *mbounded* $(f\ `\ S \cup g\ `\ S)$
    using *mbounded_Un assms* **by** (*force simp*: *fspace_def*)
  then show *thesis*
    by (*metis UnCI imageI mbounded_alt that*)
qed

lemma *funspace_imp_bounded2*:
  assumes $f \in mspace\ (funspace\ S\ m)\ g \in mspace\ (funspace\ S\ m)$
  obtains B **where** $\bigwedge x.\ x \in S \Longrightarrow mdist\ m\ (f\ x)\ (g\ x) \leq B$
  by (*metis Metric_space_mspace_mdist assms mspace_funspace Metric_space.funspace_imp_bounded2*

lemma (**in** *Metric_space*) *funspace_mdist_le*:
  assumes *fg*: $f \in fspace\ S\ g \in fspace\ S$ **and** $S \neq \{\}$
  shows $fdist\ S\ f\ g \leq a \longleftrightarrow (\forall\, x \in S.\ d\ (f\ x)\ (g\ x) \leq a)$
    using *assms bdd_above_dist* [*OF fg*] **by** (*simp add*: *fdist_def cSUP_le_iff*)

lemma *funspace_mdist_le*:
  assumes $f \in mspace\ (funspace\ S\ m)\ g \in mspace\ (funspace\ S\ m)$ **and** $S \neq \{\}$
  shows $mdist\ (funspace\ S\ m)\ f\ g \leq a \longleftrightarrow (\forall\, x \in S.\ mdist\ m\ (f\ x)\ (g\ x) \leq a)$
  using *assms* **by** (*simp add*: *Metric_space.funspace_mdist_le*)

lemma (**in** *Metric_space*) *mcomplete_funspace*:
  assumes *mcomplete*
  shows *mcomplete_of* (*funspace S Self*)
proof −
  interpret *F*: *Metric_space fspace S fdist S*
    by (*simp add*: *Metric_space_funspace*)
  show *?thesis*
  proof (*cases S*={})
    case *True*
    then show *?thesis*
      by (*simp add*: *mcomplete_of_def mcomplete_trivial_singleton*)
  next
    case *False*
    show *?thesis*
    proof (*clarsimp simp*: *mcomplete_of_def Metric_space.mcomplete_def*)
      fix σ
      assume σ: *F.MCauchy* σ

then have σM: $\bigwedge n\ x.\ x \in S \Longrightarrow \sigma\ n\ x \in M$
by (*auto simp*: *F.MCauchy_def intro*: *fspace_in_M*)
have *fdist_less*: $\exists N.\ \forall n\ n'.\ N \leq n \longrightarrow N \leq n' \longrightarrow$ *fdist S* (σ *n*) (σ *n'*) < ε **if** ε>*0* **for** ε
using σ *that* **by** (*auto simp*: *F.MCauchy_def*)
have σ*ext*: $\bigwedge n.\ \sigma\ n \in$ *extensional S*
using σ **unfolding** *F.MCauchy_def* **by** (*auto simp*: *fspace_def*)
have σ*bd*: $\bigwedge n.$ *mbounded* (σ *n* ‘ *S*)
using σ **unfolding** *F.MCauchy_def* **by** (*simp add*: *fspace_def image_subset_iff*)
have σ*in*[*simp*]: σ *n* ∈ *fspace S* **for** *n*
using *F.MCauchy_def* σ **by** *blast*
have *bd2*: $\bigwedge n\ n'.\ \exists B.\ \forall x \in S.\ d\ (\sigma\ n\ x)\ (\sigma\ n'\ x) \leq B$
using σ **unfolding** *F.MCauchy_def* **by** (*metis range_subsetD funspace_imp_bounded2*)
have *sup*: $\bigwedge n\ n'\ x0.\ x0 \in S \Longrightarrow d\ (\sigma\ n\ x0)\ (\sigma\ n'\ x0) \leq Sup\ ((\lambda x.\ d\ (\sigma\ n\ x)\ (\sigma\ n'\ x))$ ‘ *S*)
proof (*rule cSup_upper*)
show *bdd_above* ($(\lambda x.\ d\ (\sigma\ n\ x)\ (\sigma\ n'\ x))$ ‘ *S*) **if** *x0* ∈ *S* **for** *n n' x0*
using *that bd2* **by** (*meson bdd_above.I2*)
qed *auto*
have *pcy*: *MCauchy* ($\lambda n.\ \sigma\ n\ x$) **if** $x \in S$ **for** *x*
unfolding *MCauchy_def*
proof (*intro conjI strip*)
show *range* ($\lambda n.\ \sigma\ n\ x$) ⊆ *M*
using σ*M that* **by** *blast*
fix ε :: *real*
assume $\varepsilon > 0$
then obtain *N* **where** *N*: $\bigwedge n\ n'.\ N \leq n \longrightarrow N \leq n' \longrightarrow$ *fdist S* (σ *n*) (σ *n'*) < ε
using σ **by** (*force simp*: *F.MCauchy_def*)
{ fix *n n'*
assume *n*: $N \leq n\ N \leq n'$
have *d* (σ *n x*) (σ *n' x*) ≤ (*SUP* $x \in S.$ *d* (σ *n x*) (σ *n' x*))
using *that sup* **by** *presburger*
then have *d* (σ *n x*) (σ *n' x*) ≤ *fdist S* (σ *n*) (σ *n'*)
by (*simp add*: *fdist_def* ‹*S* ≠ {}›)
with *N n* **have** *d* (σ *n x*) (σ *n' x*) < ε
by *fastforce*
} then show $\exists N.\ \forall n\ n'.\ N \leq n \longrightarrow N \leq n' \longrightarrow d\ (\sigma\ n\ x)\ (\sigma\ n'\ x) < \varepsilon$
by *blast*
qed
have $\exists l.$ *limitin mtopology* ($\lambda n.\ \sigma\ n\ x$) *l sequentially* **if** $x \in S$ **for** *x*
using *assms mcomplete_def pcy* ‹$x \in S$› **by** *presburger*
then obtain *g0* **where** *g0*: $\bigwedge x.\ x \in S \Longrightarrow$ *limitin mtopology* ($\lambda n.\ \sigma\ n\ x$) (*g0 x*) *sequentially*
by *metis*
define *g* **where** $g \equiv$ *restrict g0 S*
have *gext*: $g \in$ *extensional S*
and *glim*: $\bigwedge x.\ x \in S \Longrightarrow$ *limitin mtopology* ($\lambda n.\ \sigma\ n\ x$) (*g x*) *sequentially*
by (*auto simp*: *g_def g0*)

```
    have gwd: g x ∈ M if x ∈ S for x
      using glim limitin_metric that by blast
    have unif: ∃N. ∀x n. x ∈ S ⟶ N ≤ n ⟶ d (σ n x) (g x) < ε if ε>0 for
ε
    proof −
      obtain N where N: ⋀n n'. N ≤ n ∧ N ≤ n' ⟹ Sup ((λx. d (σ n x) (σ
n' x)) ‘ S) < ε/2
        using ‹S≠{}› ‹ε>0› fdist_less [of ε/2]
        by (metis (mono_tags) σin fdist_def half_gt_zero)
      show ?thesis
      proof (intro exI strip)
        fix x n
        assume x ∈ S and N ≤ n
        obtain N' where N': ⋀n. N' ≤ n ⟹ σ n x ∈ M ∧ d (σ n x) (g x) <
ε/2
          by (metis ‹0 < ε› ‹x ∈ S› glim half_gt_zero limit_metric_sequentially)
        have d (σ n x) (g x) ≤ d (σ n x) (σ (max N N') x) + d (σ (max N N')
x) (g x)
          using ‹x ∈ S› σM gwd triangle by presburger
        also have ... < ε/2 + ε/2
          by (smt (verit) N N' ‹N ≤ n› ‹x ∈ S› max.cobounded1 max.cobounded2
sup)
        finally show d (σ n x) (g x) < ε by simp
      qed
    qed
    have limitin F.mtopology σ g sequentially
      unfolding F.limit_metric_sequentially
    proof (intro conjI strip)
      obtain N where N: ⋀n n'. N ≤ n ∧ N ≤ n' ⟹ Sup ((λx. d (σ n x) (σ
n' x)) ‘ S) < 1
        using fdist_less [of 1] ‹S≠{}› by (auto simp: fdist_def)
      have ⋀x. x ∈ σ N ‘ S ⟹ x ∈ M
        using σM by blast
      obtain a B where a ∈ M and B: ⋀x. x ∈ (σ N) ‘ S ⟹ d a x ≤ B
        by (metis False σM σbd ex_in_conv imageI mbounded_alt_pos)
      have d a (g x) ≤ B+1 if x∈S for x
      proof −
        have d a (g x) ≤ d a (σ N x) + d (σ N x) (g x)
          by (simp add: ‹a ∈ M› σM gwd that triangle)
        also have ... ≤ B+1
        proof −
          have d a (σ N x) ≤ B
            by (simp add: B that)
          moreover
          have False if 1: d (σ N x) (g x) > 1
          proof −
            obtain r where 1 < r and r: r < d (σ N x) (g x)
              using 1 dense by blast
            then obtain N' where N': ⋀n. N' ≤ n ⟹ σ n x ∈ M ∧ d (σ n x)
```

```
(g x) < r−1
            using glim [OF ‹x∈S›] by (fastforce simp: limit_metric_sequentially)
            have d (σ N x) (g x) ≤ d (σ N x) (σ (max N N′) x) + d (σ (max N
N′) x) (g x)
              by (metis ‹x ∈ S› σM commute gwd triangle′)
            also have ... < 1 + (r−1)
                 by (smt (verit) N N′ ‹x ∈ S› max.cobounded1 max.cobounded2
max.idem sup)
            finally have d (σ N x) (g x) < r
              by simp
            with r show False
              by linarith
          qed
          ultimately show ?thesis
            by force
        qed
        finally show ?thesis .
      qed
      with gwd ‹a ∈ M› have mbounded (g ‘ S)
        unfolding mbounded by blast
      with gwd gext show g ∈ fspace S
        by (auto simp: fspace_def)
      fix ε::real
      assume ε>0
      then obtain N where ⋀x n. x ∈ S ⟹ N ≤ n ⟹ d (σ n x) (g x) < ε/2
        by (meson unif half_gt_zero)
      then have fdist S (σ n) g ≤ ε/2 if N ≤ n for n
        using ‹g ∈ fspace S› False that
        by (force simp: funspace_mdist_le simp del: divide_const_simps)
      then show ∃N. ∀n≥N. σ n ∈ fspace S ∧ fdist S (σ n) g < ε
           by (metis ‹0 < ε› σin add_strict_increasing field_sum_of_halves
half_gt_zero)
    qed
    then show ∃x. limitin F.mtopology σ x sequentially
      by blast
  qed
 qed
qed
```

6.11.10 Metric space of continuous bounded functions

```
definition cfunspace where
  cfunspace X m ≡ submetric (funspace (topspace X) m) {f. continuous_map X
(mtopology_of m) f}

lemma mspace_cfunspace [simp]:
  mspace (cfunspace X m) =
    {f. f ∈ topspace X → mspace m ∧ f ∈ extensional (topspace X) ∧
        Metric_space.mbounded (mspace m) (mdist m) (f ‘ (topspace X)) ∧
```

continuous_map X (mtopology_of m) f}
by (*auto simp*: *cfunspace_def Metric_space.fspace_def*)

lemma *mdist_cfunspace_eq_mdist_funspace*:
mdist (cfunspace X m) = mdist (funspace (topspace X) m)
by (*auto simp*: *cfunspace_def*)

lemma *cfunspace_subset_funspace*:
mspace (cfunspace X m) ⊆ mspace (funspace (topspace X) m)
by (*simp add*: *cfunspace_def*)

lemma *cfunspace_mdist_le*:
⟦*f ∈ mspace (cfunspace X m)*; *g ∈ mspace (cfunspace X m)*; *topspace X ≠ {}*⟧
⟹ *mdist (cfunspace X m) f g ≤ a ⟷ (∀ x ∈ topspace X. mdist m (f x) (g x) ≤ a)*
by (*simp add*: *cfunspace_def Metric_space.funspace_mdist_le*)

lemma *cfunspace_imp_bounded2*:
assumes *f ∈ mspace (cfunspace X m) g ∈ mspace (cfunspace X m)*
obtains *B* **where** *⋀x. x ∈ topspace X ⟹ mdist m (f x) (g x) ≤ B*
by (*metis assms all_not_in_conv cfunspace_mdist_le nle_le*)

lemma *cfunspace_mdist_lt*:
⟦*compactin X (topspace X)*; *f ∈ mspace (cfunspace X m)*;
g ∈ mspace (cfunspace X m); *mdist (cfunspace X m) f g < a*;
x ∈ topspace X⟧
⟹ *mdist m (f x) (g x) < a*
by (*metis* (*full_types*) *cfunspace_mdist_le empty_iff less_eq_real_def less_le_not_le*)

lemma *mdist_cfunspace_le*:
assumes *0 ≤ B* **and** *B*: *⋀x. x ∈ topspace X ⟹ mdist m (f x) (g x) ≤ B*
shows *mdist (cfunspace X m) f g ≤ B*
proof (*cases X = trivial_topology*)
case *True*
then show *?thesis*
by (*simp add*: *Metric_space.fdist_empty ‹B≥0› cfunspace_def*)
next
case *False*
have *bdd*: *bdd_above ((λu. mdist m (f u) (g u)) ‘ topspace X)*
by (*meson B bdd_above.I2*)
with *assms bdd* **show** *?thesis*
by (*simp add*: *mdist_cfunspace_eq_mdist_funspace Metric_space.fdist_def cSUP_le_iff*)
qed

lemma *mdist_cfunspace_imp_mdist_le*:
⟦*f ∈ mspace (cfunspace X m)*; *g ∈ mspace (cfunspace X m)*;
mdist (cfunspace X m) f g ≤ a; *x ∈ topspace X*⟧ ⟹ *mdist m (f x) (g x) ≤ a*

using *cfunspace_mdist_le* **by** *blast*

lemma *compactin_mspace_cfunspace*:
compactin X (topspace X)
⟹ mspace (cfunspace X m) =
{f. (∀ x ∈ topspace X. f x ∈ mspace m) ∧
f ∈ extensional (topspace X) ∧
continuous_map X (mtopology_of m) f}
by (*auto simp*: *Metric_space.compactin_imp_mbounded image_compactin mtopology_of_def*)

lemma (**in** *Metric_space*) *mcomplete_cfunspace*:
assumes *mcomplete*
shows *mcomplete_of (cfunspace X Self)*
proof −
interpret *F*: *Metric_space fspace (topspace X) fdist (topspace X)*
by (*simp add*: *Metric_space_funspace*)
interpret *S*: *Submetric fspace (topspace X) fdist (topspace X) mspace (cfunspace X Self)*
proof
show *mspace (cfunspace X Self) ⊆ fspace (topspace X)*
by (*metis cfunspace_subset_funspace mdist_Self mspace_Self mspace_funspace*)
qed
show *?thesis*
proof (*cases X = trivial_topology*)
case *True*
then show *?thesis*
by (*simp add*: *mcomplete_of_def mcomplete_trivial_singleton mdist_cfunspace_eq_mdist_funspace cong*: *conj_cong*)
next
case *False*
have ∗: *continuous_map X mtopology g*
if *range σ ⊆ mspace (cfunspace X Self)*
and *g*: *limitin F.mtopology σ g sequentially* **for** *σ g*
unfolding *continuous_map_to_metric*
proof (*intro strip*)
have *σ*: *⋀n. continuous_map X mtopology (σ n)*
using *that* **by** (*auto simp*: *mtopology_of_def*)
fix *x* **and** *ε*::*real*
assume *x ∈ topspace X* **and** *0 < ε*
then obtain *N* **where** *N*: *⋀n. N ≤ n ⟹ σ n ∈ fspace (topspace X) ∧ fdist (topspace X) (σ n) g < ε/3*
unfolding *mtopology_of_def F.limitin_metric*
by (*metis F.limit_metric_sequentially divide_pos_pos g zero_less_numeral*)

then obtain *U* **where** *openin X U x ∈ U*
and *U*: *⋀y. y ∈ U ⟹ σ N y ∈ mball (σ N x) (ε/3)*
by (*metis Metric_space.continuous_map_to_metric Metric_space_axioms ‹0 < ε› ‹x ∈ topspace X› σ divide_pos_pos zero_less_numeral*)

```
    moreover
    have g y ∈ mball (g x) ε if y∈U for y
    proof −
      have U ⊆ topspace X
        using ‹openin X U› by (simp add: openin_subset)
      have gx: g x ∈ M
        by (meson F.limitin_mspace ‹x ∈ topspace X› fspace_in_M g)
      have y ∈ topspace X
        using ‹U ⊆ topspace X› that by auto
      have gy: g y ∈ M
      by (meson F.limitin_mspace[OF g] ‹U ⊆ topspace X› fspace_in_M subsetD
that)
      have d (g x) (g y) < ε
      proof −
        have ∗: d (σ N x0) (g x0) ≤ ε/3 if x0 ∈ topspace X for x0
        proof −
          have g ∈ fspace (topspace X)
            using F.limit_metric_sequentially g by blast
          with N that have bdd_above ((λx. d (σ N x) (g x)) ‘ topspace X)
            by (force intro: bdd_above_dist)
         then have d (σ N x0) (g x0) ≤ Sup ((λx. d (σ N x) (g x)) ‘ topspace X)
            by (simp add: cSup_upper that)
          also have ... ≤ ε/3
            using g False N ‹g ∈ fspace (topspace X)›
              by (fastforce simp: F.limit_metric_sequentially fdist_def)
          finally show ?thesis .
        qed
        have d (g x) (g y) ≤ d (g x) (σ N x) + d (σ N x) (g y)
          using U gx gy that triangle by force
        also have ... < ε/3 + ε/3 + ε/3
            by (smt (verit) ∗ U gy ‹x ∈ topspace X› ‹y ∈ topspace X› commute
in_mball that triangle)
        finally show ?thesis by simp
      qed
      with gx gy show ?thesis by simp
    qed
    ultimately show ∃ U. openin X U ∧ x ∈ U ∧ (∀ y∈U. g y ∈ mball (g x) ε)
      by blast
  qed

  have S.sub.mcomplete
  proof (rule S.sequentially_closedin_mcomplete_imp_mcomplete)
    show F.mcomplete
    by (metis assms mcomplete_funspace mcomplete_of_def mdist_Self mdist_funspace
mspace_Self mspace_funspace)
    fix σ g
    assume g: range σ ⊆ mspace (cfunspace X Self) ∧ limitin F.mtopology σ g
sequentially
    show g ∈ mspace (cfunspace X Self)
```

proof (*simp add*: *mtopology_of_def*, *intro conjI*)
show *g* ∈ *topspace X* → *M g* ∈ *extensional* (*topspace X*) *mbounded* (*g* ‘ *topspace X*)
using *g F.limitin_mspace* **by** (*force simp*: *fspace_def*)+
show *continuous_map X mtopology g*
using ∗ *g* **by** *blast*
qed
qed
then show *?thesis*
by (*simp add*: *mcomplete_of_def mdist_cfunspace_eq_mdist_funspace*)
qed
qed

6.11.11 Existence of completion for any metric space M as a subspace of M=>R

lemma (**in** *Metric_space*) *metric_completion_explicit*:
obtains *f* :: [′*a*,′*a*] ⇒ *real* **and** *S* **where**
S ⊆ *mspace*(*funspace M euclidean_metric*)
mcomplete_of (*submetric* (*funspace M euclidean_metric*) *S*)
f ∈ *M* → *S*
mtopology_of(*funspace M euclidean_metric*) *closure_of f* ‘ *M* = *S*
⋀*x y*. ⟦*x* ∈ *M*; *y* ∈ *M*⟧
⟹ *mdist* (*funspace M euclidean_metric*) (*f x*) (*f y*) = *d x y*
proof −
define *m*′:: (′*a*⇒*real*) *metric* **where** *m*′ ≡ *funspace M euclidean_metric*
show *thesis*
proof (*cases M* = {})
case *True*
then show *?thesis*
using *that* **by** (*simp add*: *mcomplete_of_def mcomplete_trivial*)
next
case *False*
then obtain *a* **where** *a* ∈ *M*
by *auto*
define *f* **where** *f* ≡ λ*x*. (λ*u* ∈ *M*. *d x u* − *d a u*)
define *S* **where** *S* ≡ *mtopology_of*(*funspace M euclidean_metric*) *closure_of* (*f* ‘ *M*)
interpret *S*: *Submetric Met_TC.fspace M Met_TC.fdist M S* ∩ *Met_TC.fspace M*
by (*simp add*: *Met_TC.Metric_space_funspace Submetric.intro Submetric_axioms_def*)

have *fim*: *f* ‘ *M* ⊆ *mspace m*′
proof (*clarsimp simp*: *m*′*_def Met_TC.fspace_def*)
fix *b*
assume *b* ∈ *M*
then have ⋀*c*. ⟦*c* ∈ *M*⟧ ⟹ |*d b c* − *d a c*| ≤ *d a b*
by (*smt* (*verit*, *best*) ‹*a* ∈ *M*› *commute triangle*′′)

```
      then have (λx. d b x − d a x) ‘ M ⊆ cball 0 (d a b)
        by force
      then show f b ∈ extensional M ∧ bounded (f b ‘ M)
          by (metis bounded_cball bounded_subset f_def image_restrict_eq re-
strict_extensional_sub set_eq_subset)
    qed
    show thesis
    proof
      show S ⊆ mspace (funspace M euclidean_metric)
        by (simp add: S_def in_closure_of subset_iff)
      have closedin S.mtopology (S ∩ Met_TC.fspace M)
        by (simp add: S_def closedin_Int funspace_def)
      moreover have S.mcomplete
        using Metric_space.mcomplete_funspace Met_TC.Metric_space_axioms
by (fastforce simp: mcomplete_of_def)
      ultimately show mcomplete_of (submetric (funspace M euclidean_metric)
S)
        by (simp add: S.closedin_eq_mcomplete mcomplete_of_def)
      show f ∈ M → S
        using S_def fim in_closure_of m′_def by fastforce
      show mtopology_of (funspace M euclidean_metric) closure_of f ‘ M = S
        by (auto simp: f_def S_def mtopology_of_def)
      show mdist (funspace M euclidean_metric) (f x) (f y) = d x y
        if x ∈ M y ∈ M for x y
      proof −
        have ∀ c∈M. dist (f x c) (f y c) ≤ r ⟹ d x y ≤ r for r
          using that by (auto simp: f_def dist_real_def)
        moreover have dist (f x z) (f y z) ≤ r if d x y ≤ r and z ∈ M for r z
          using that ‹x ∈ M› ‹y ∈ M›
          apply (simp add: f_def Met_TC.fdist_def dist_real_def)
          by (smt (verit, best) commute triangle′)
        ultimately have (SUP c ∈ M. dist (f x c) (f y c)) = d x y
          by (intro cSup_unique) auto
        with that fim show ?thesis
            using that fim by (simp add: Met_TC.fdist_def False m′_def im-
age_subset_iff)
      qed
    qed
  qed
qed


lemma (in Metric_space) metric_completion:
  obtains f :: ['a,'a] ⇒ real and m′ where
    mcomplete_of m′
    f ∈ M → mspace m′
    mtopology_of m′ closure_of f ‘ M = mspace m′
    ⋀x y. ⟦x ∈ M; y ∈ M⟧ ⟹ mdist m′ (f x) (f y) = d x y
proof −
```

```
  obtain f :: ['a,'a] ⇒ real and S where
    Ssub: S ⊆ mspace(funspace M euclidean_metric)
    and mcom: mcomplete_of (submetric (funspace M euclidean_metric) S)
    and fim: f ∈ M → S
    and eqS: mtopology_of(funspace M euclidean_metric) closure_of f ‘ M = S
    and eqd: ⋀x y. ⟦x ∈ M; y ∈ M⟧ ⟹ mdist (funspace M euclidean_metric) (f
x) (f y) = d x y
    using metric_completion_explicit by metis
  define m' where m' ≡ submetric (funspace M euclidean_metric) S
  show thesis
  proof
    show mcomplete_of m'
      by (simp add: mcom m'_def)
    show f ∈ M → mspace m'
      using Ssub fim m'_def by auto
    show mtopology_of m' closure_of f ‘ M = mspace m'
      using eqS fim Ssub
        by (force simp: m'_def mtopology_of_submetric closure_of_subtopology
Int_absorb1 image_subset_iff_funcset)
    show mdist m' (f x) (f y) = d x y if x ∈ M and y ∈ M for x y
      using that eqd m'_def by force
  qed
qed

lemma metrizable_space_completion:
  assumes metrizable_space X
  obtains f :: ['a,'a] ⇒ real and Y where
       completely_metrizable_space Y embedding_map X Y f
              Y closure_of (f ‘ (topspace X)) = topspace Y
proof −
  obtain M d where Metric_space M d and Xeq: X = Metric_space.mtopology
M d
    using assms metrizable_space_def by blast
  then interpret Metric_space M d by simp
  obtain f :: ['a,'a] ⇒ real and m' where
    mcomplete_of m'
    and fim: f ∈ M → mspace m'
    and m': mtopology_of m' closure_of f ‘ M = mspace m'
    and eqd: ⋀x y. ⟦x ∈ M; y ∈ M⟧ ⟹ mdist m' (f x) (f y) = d x y
    by (metis metric_completion)
  show thesis
  proof
    show completely_metrizable_space (mtopology_of m')
      using ‹mcomplete_of m'›
    unfolding completely_metrizable_space_def mcomplete_of_def mtopology_of_def
      by (metis Metric_space_mspace_mdist)
    show embedding_map X (mtopology_of m') f
      using Metric_space12.isometry_imp_embedding_map
    by (metis Metric_space12_def Metric_space_axioms Metric_space_mspace_mdist
```

Xeq eqd fim
mtopology_of_def)
show (*mtopology_of m'*) *closure_of f ' topspace X = topspace* (*mtopology_of m'*)
by (*simp add*: *Xeq m'*)
qed
qed

6.11.12 Contractions

lemma (**in** *Metric_space*) *contraction_imp_unique_fixpoint*:
assumes $f\ x = x\ f\ y = y$
and $f \in M \to M$
and $k < 1$
and $\bigwedge x\ y.\ [\![x \in M;\ y \in M]\!] \Longrightarrow d\ (f\ x)\ (f\ y) \leq k * d\ x\ y$
and $x \in M\ y \in M$
shows $x = y$
by (*smt* (*verit*, *ccfv_SIG*) *mdist_pos_less mult_le_cancel_right1 assms*)

Banach Fixed-Point Theorem (aka, Contraction Mapping Principle)

lemma (**in** *Metric_space*) *Banach_fixedpoint_thm*:
assumes *mcomplete* **and** $M \neq \{\}$ **and** *fim*: $f \in M \to M$
and $k < 1$
and *con*: $\bigwedge x\ y.\ [\![x \in M;\ y \in M]\!] \Longrightarrow d\ (f\ x)\ (f\ y) \leq k * d\ x\ y$
obtains x **where** $x \in M\ f\ x = x$
proof −
obtain a **where** $a \in M$
using ‹$M \neq \{\}$› **by** *blast*
show *thesis*
proof (*cases* $\forall x \in M.\ f\ x = f\ a$)
case *True*
then show *?thesis*
by (*metis* ‹$a \in M$› *fim image_subset_iff image_subset_iff_funcset that*)
next
case *False*
then obtain b **where** $b \in M$ **and** *b*: $f\ b \neq f\ a$
by *blast*
have *k>0*
using *Lipschitz_coefficient_pos* [**where** *f=f*]
by (*metis False* ‹$a \in M$› *con fim mdist_Self mspace_Self*)
define σ **where** $\sigma \equiv \lambda n.\ (f\widehat{\ }n)\ a$
have *f_iter*: $\sigma\ n \in M$ **for** n
unfolding *σ_def* **by** (*induction n*) (*use* ‹$a \in M$› *fim* **in** *auto*)
show *?thesis*
proof (*cases f a = a*)
case *True*
then show *?thesis*
using ‹$a \in M$› *that* **by** *blast*
next

```
    case False
    have MCauchy σ
    proof −
      show ?thesis
        unfolding MCauchy_def
      proof (intro conjI strip)
        show range σ ⊆ M
          using f_iter by blast
        fix ε::real
        assume ε>0
        with ‹k < 1› ‹f a ≠ a› ‹a ∈ M› fim have gt0: ((1 − k) ∗ ε) / d a (f a)
> 0
          by (fastforce simp: divide_simps Pi_iff)
        obtain N where k^N < ((1−k) ∗ ε) / d a (f a)
          using real_arch_pow_inv [OF gt0 ‹k < 1›] by blast
        then have N: ⋀n. n ≥ N ⟹ k^n < ((1−k) ∗ ε) / d a (f a)
          by (smt (verit) ‹0 < k› assms(4) power_decreasing)
        have ∀ n n′. n<n′ ⟶ N ≤ n ⟶ N ≤ n′ ⟶ d (σ n) (σ n′) < ε
        proof (intro exI strip)
          fix n n′
          assume n<n′ N ≤ n N ≤ n′
          have d (σ n) (σ n′) ≤ (∑ i=n..<n′. d (σ i) (σ (Suc i)))
          proof −
            have n < m ⟹ d (σ n) (σ m) ≤ (∑ i=n..<m. d (σ i) (σ (Suc i)))
for m
            proof (induction m)
              case 0
              then show ?case
                by simp
            next
              case (Suc m)
              then consider n<m | m=n
                by linarith
              then show ?case
              proof cases
                case 1
                have d (σ n) (σ (Suc m)) ≤ d (σ n) (σ m) + d (σ m) (σ (Suc m))
                  by (simp add: f_iter triangle)
                 also have ... ≤ (∑ i=n..<m. d (σ i) (σ (Suc i))) + d (σ m) (σ
(Suc m))
                  using Suc 1 by linarith
                also have ... = (∑ i = n..<Suc m. d (σ i) (σ (Suc i)))
                  using 1 by force
                finally show ?thesis .
              qed auto
            qed
            with ‹n < n′› show ?thesis by blast
          qed
          also have ... ≤ (∑ i=n..<n′. d a (f a) ∗ k^i)
```

```
          proof (rule sum_mono)
            fix i
            assume i ∈ {n..<n'}
            show d (σ i) (σ (Suc i)) ≤ d a (f a) * k ^ i
            proof (induction i)
              case 0
              then show ?case
                by (auto simp: σ_def)
            next
              case (Suc i)
              have d (σ (Suc i)) (σ (Suc (Suc i))) ≤ k * d (σ i) (σ (Suc i))
                using con σ_def f_iter fim by fastforce
              also have ... ≤ d a (f a) * k ^ Suc i
                using Suc ‹0 < k› by auto
              finally show ?case .
            qed
          qed
          also have ... = d a (f a) * (∑ i=n..<n'. k^i)
            by (simp add: sum_distrib_left)
          also have ... = d a (f a) * (∑ i=0..<n'-n. k^(i+n))
             using sum.shift_bounds_nat_ivl [of power k 0 n n'-n] ‹n < n'› by
simp
          also have ... = d a (f a) * k^n * (∑ i<n'-n. k^i)
            by (simp add: power_add lessThan_atLeast0 flip: sum_distrib_right)
          also have ... = d a (f a) * (k ^ n - k ^ n') / (1 - k)
            using ‹k < 1› ‹n < n'› apply (simp add: sum_gp_strict)
            by (simp add: algebra_simps flip: power_add)
          also have ... < ε
            using N ‹k < 1› ‹0 < ε› ‹0 < k› ‹N ≤ n›
            apply (simp add: field_simps)
                by (smt (verit) nonneg pos_less_divide_eq zero_less_divide_iff
zero_less_power)
          finally show d (σ n) (σ n') < ε .
        qed
        then show ∃ N. ∀ n n'. N ≤ n ⟶ N ≤ n' ⟶ d (σ n) (σ n') < ε
           by (metis ‹0 < ε› commute f_iter linorder_not_le local.mdist_zero
nat_less_le)
      qed
    qed
    then obtain l where l: limitin mtopology σ l sequentially
      using ‹mcomplete› mcomplete_def by blast
    show ?thesis
    proof
      show l ∈ M
        using l limitin_mspace by blast
      show f l = l
      proof (rule limitin_metric_unique)
        have limitin mtopology (f ∘ σ) (f l) sequentially
        proof (rule continuous_map_limit)
```

```
        have Lipschitz_continuous_map Self Self f
          using con by (auto simp: Lipschitz_continuous_map_def fim)
        then show continuous_map mtopology mtopology f
          using Lipschitz_continuous_imp_continuous_map Self_def by force
      qed (use l in auto)
      moreover have (f ∘ σ) = (λi. σ(i+1))
        by (auto simp: σ_def)
      ultimately show limitin mtopology (λn. (f^^n)a) (f l) sequentially
        using limitin_sequentially_offset_rev [of mtopology σ 1]
        by (simp add: σ_def)
    qed (use l in ‹auto simp: σ_def›)
  qed
 qed
qed
```

6.11.13 The Baire Category Theorem

Possibly relevant to the theorem "Baire" in Elementary Normed Spaces

```
lemma (in Metric_space) metric_Baire_category:
 assumes mcomplete countable 𝒢
 and ⋀T. T ∈ 𝒢 ⟹ openin mtopology T ∧ mtopology closure_of T = M
shows mtopology closure_of ⋂𝒢 = M
proof (cases 𝒢={})
 case False
 then obtain U :: nat ⇒ 'a set where U: range U = 𝒢
  by (metis ‹countable 𝒢› uncountable_def)
  with assms have u_open: ⋀n. openin mtopology (U n) and u_dense: ⋀n.
mtopology closure_of (U n) = M
  by auto
 have ⋂(range U) ∩ W ≠ {} if W: openin mtopology W W ≠ {} for W
 proof −
  have W ⊆ M
   using openin_mtopology W by blast
  have ∃r' x'. 0 < r' ∧ r' < r/2 ∧ x' ∈ M ∧ mcball x' r' ⊆ mball x r ∩ U n
   if r>0 x ∈ M for x r n
  proof −
   obtain z where z: z ∈ U n ∩ mball x r
    using u_dense [of n] ‹r>0› ‹x ∈ M›
   by (metis dense_intersects_open centre_in_mball_iff empty_iff openin_mball
topspace_mtopology equals0I)
   then have z ∈ M by auto
   have openin mtopology (U n ∩ mball x r)
    by (simp add: openin_Int u_open)
   with ‹z ∈ M› z obtain e where e>0 and e: mcball z e ⊆ U n ∩ mball x r
    by (meson openin_mtopology_mcball)
   define r' where r' ≡ min e (r/4)
   show ?thesis
   proof (intro exI conjI)
```

```
      show 0 < r' r' < r / 2 z ∈ M
        using ‹e>0› ‹r>0› ‹z ∈ M› by (auto simp: r'_def)
      show mcball z r' ⊆ mball x r ∩ U n
        using Metric_space.mcball_subset_concentric e r'_def by auto
    qed
  qed
  then obtain nextr nextx
    where nextr: ⋀r x n. ⟦r>0; x∈M⟧ ⟹ 0 < nextr r x n ∧ nextr r x n < r/2
      and nextx: ⋀r x n. ⟦r>0; x∈M⟧ ⟹ nextx r x n ∈ M
      and nextsub: ⋀r x n. ⟦r>0; x∈M⟧ ⟹ mcball (nextx r x n) (nextr r x n) ⊆
mball x r ∩ U n
    by metis
  obtain x0 where x0: x0 ∈ U 0 ∩ W
    by (metis W dense_intersects_open topspace_mtopology all_not_in_conv
u_dense)
  then have x0 ∈ M
    using ‹W ⊆ M› by fastforce
  obtain r0 where 0 < r0 r0 < 1 and sub: mcball x0 r0 ⊆ U 0 ∩ W
  proof −
    have openin mtopology (U 0 ∩ W)
      using W u_open by blast
    then obtain r where r>0 and r: mball x0 r ⊆ U 0 mball x0 r ⊆ W
      by (meson Int_subset_iff openin_mtopology x0)
    define r0 where r0 ≡ (min r 1) / 2
    show thesis
    proof
      show 0 < r0 r0 < 1
        using ‹r>0› by (auto simp: r0_def)
      show mcball x0 r0 ⊆ U 0 ∩ W
        using r ‹0 < r0› r0_def by auto
    qed
  qed
  define b where b ≡ rec_nat (x0,r0) (λn (x,r). (nextx r x n, nextr r x n))
  have b0[simp]: b 0 = (x0,r0)
    by (simp add: b_def)
  have bSuc[simp]: b (Suc n) = (let (x,r) = b n in (nextx r x n, nextr r x n))
for n
    by (simp add: b_def)
  define xf where xf ≡ fst ∘ b
  define rf where rf ≡ snd ∘ b
  have rfxf: 0 < rf n ∧ xf n ∈ M for n
  proof (induction n)
    case 0
    with ‹0 < r0› ‹x0 ∈ M› show ?case
      by (auto simp: rf_def xf_def)
  next
    case (Suc n)
    then show ?case
      by (auto simp: rf_def xf_def case_prod_unfold nextr nextx Let_def)
```

```
  qed
  have mcball_sub: mcball (xf (Suc n)) (rf (Suc n)) ⊆ mball (xf n) (rf n) ∩ U
n for n
    using rfxf nextsub by (auto simp: xf_def rf_def case_prod_unfold Let_def)
  have half: rf (Suc n) < rf n / 2 for n
    using rfxf nextr by (auto simp: xf_def rf_def case_prod_unfold Let_def)
  then have decseq rf
   using rfxf by (smt (verit, ccfv_threshold) decseq_SucI field_sum_of_halves)
  have nested: mball (xf n) (rf n) ⊆ mball (xf m) (rf m) if m ≤ n for m n
    using that
  proof (induction n)
    case (Suc n)
    then show ?case
   by (metis mcball_sub order.trans inf.boundedE le_Suc_eq mball_subset_mcball
order.refl)
  qed auto
  have MCauchy xf
    unfolding MCauchy_def
  proof (intro conjI strip)
    show range xf ⊆ M
      using rfxf by blast
    fix ε :: real
    assume ε>0
    then obtain N where N: inverse (2^N) < ε
      using real_arch_pow_inv by (force simp flip: power_inverse)
    have d (xf n) (xf n') < ε if n ≤ n' N ≤ n N ≤ n' for n n'
    proof −
      have ∗: rf n < inverse (2 ^ n) for n
      proof (induction n)
        case 0
        then show ?case
          by (simp add: ‹r0 < 1› rf_def)
      next
        case (Suc n)
        with half show ?case
          by simp (smt (verit))
      qed
      have rf n ≤ rf N
        using ‹decseq rf› ‹N ≤ n› by (simp add: decseqD)
      moreover
      have xf n' ∈ mball (xf n) (rf n)
        using nested rfxf ‹n ≤ n'› by blast
      ultimately have d (xf n) (xf n') < rf N
        by auto
      also have ... < ε
        using ∗ N order.strict_trans by blast
      finally show ?thesis .
    qed
    then show ∃ N. ∀ n n'. N ≤ n ⟶ N ≤ n' ⟶ d (xf n) (xf n') < ε
```

```
      by (metis commute linorder_le_cases)
    qed
    then obtain l where l: limitin mtopology xf l sequentially
      using ‹mcomplete› mcomplete_alt by blast
    have l_in: l ∈ mcball (xf n) (rf n) for n
    proof −
      have ∀⇩F m in sequentially. xf m ∈ mcball (xf n) (rf n)
        unfolding eventually_sequentially
       by (meson nested rfxf centre_in_mball_iff mball_subset_mcball subset_iff)
      with l limitin_closedin show ?thesis
        by (metis closedin_mcball trivial_limit_sequentially)
    qed
    then have ⋀n. l ∈ U n
      using mcball_sub by blast
    moreover have l ∈ W
      using l_in[of 0] sub  by (auto simp: xf_def rf_def)
    ultimately show ?thesis by auto
  qed
  with U show ?thesis
    by (metis dense_intersects_open topspace_mtopology)
qed auto



lemma (in Metric_space) metric_Baire_category_alt:
  assumes mcomplete countable 𝒢
    and empty: ⋀T. T ∈ 𝒢 ⟹ closedin mtopology T ∧ mtopology interior_of T
= {}
  shows mtopology interior_of ⋃𝒢 = {}
proof −
  have ∗: mtopology closure_of ⋂((−)M ‘ 𝒢) = M
  proof (intro metric_Baire_category conjI ‹mcomplete›)
    show countable ((−) M ‘ 𝒢)
      using ‹countable 𝒢› by blast
    fix T
    assume T ∈ (−) M ‘ 𝒢
    then obtain U where U: U ∈ 𝒢 T = M−U U ⊆ M
      using empty metric_closedin_iff_sequentially_closed by force
    with empty show openin mtopology T by blast
    show mtopology closure_of T = M
      using U by (simp add: closure_of_interior_of double_diff empty)
  qed
  with closure_of_eq show ?thesis
    by (fastforce simp: interior_of_closure_of split: if_split_asm)
qed
```

Since all locally compact Hausdorff spaces are regular, the disjunction in the HOL Light version is redundant.

```
lemma Baire_category_aux:
```

```
  assumes locally_compact_space X regular_space X
    and countable 𝒢
    and empty: ⋀G. G ∈ 𝒢 ⟹ closedin X G ∧ X interior_of G = {}
  shows X interior_of ⋃𝒢 = {}
proof (cases 𝒢 = {})
  case True
  then show ?thesis
    by simp
next
  case False
  then obtain T :: nat ⇒ 'a set where T: 𝒢 = range T
    by (metis ‹countable 𝒢› uncountable_def)
  with empty have Tempty: ⋀n. X interior_of (T n) = {}
    by auto
  show ?thesis
  proof (clarsimp simp: T interior_of_def)
    fix z U
    assume z ∈ U and openA: openin X U and Asub: U ⊆ ⋃ (range T)
    with openin_subset have z ∈ topspace X
      by blast
    have neighbourhood_base_of (λC. compactin X C ∧ closedin X C) X
      using assms locally_compact_regular_space_neighbourhood_base by auto
    then obtain V K where openin X V compactin X K closedin X K z ∈ V V
⊆ K K ⊆ U
      by (metis (no_types, lifting) ‹z ∈ U› neighbourhood_base_of openA)
    have nb_closedin: neighbourhood_base_of (closedin X) X
      using ‹regular_space X› neighbourhood_base_of_closedin by auto
    have ∃Φ. ∀n. (Φ n ⊆ K ∧ closedin X (Φ n) ∧ X interior_of Φ n ≠ {} ∧
disjnt (Φ n) (T n)) ∧
        Φ (Suc n) ⊆ Φ n
    proof (rule dependent_nat_choice)
      show ∃x⊆K. closedin X x ∧ X interior_of x ≠ {} ∧ disjnt x (T 0)
      proof −
        have False if V ⊆ T 0
        using Tempty ‹openin X V› ‹z ∈ V› interior_of_maximal that by fastforce
        then obtain x where openin X (V − T 0) ∧ x ∈ V − T 0
          using T ‹openin X V› empty by blast
        with nb_closedin
        obtain N C where openin X N closedin X C x ∈ N N ⊆ C C ⊆ V − T 0
          unfolding neighbourhood_base_of by metis
        show ?thesis
        proof (intro exI conjI)
          show C ⊆ K
            using ‹C ⊆ V − T 0› ‹V ⊆ K› by auto
          show X interior_of C ≠ {}
          by (metis ‹N ⊆ C› ‹openin X N› ‹x ∈ N› empty_iff interior_of_eq_empty)
          show disjnt C (T 0)
            using ‹C ⊆ V − T 0› disjnt_iff by fastforce
        qed (use ‹closedin X C› in auto)
```

```
    qed
   show ∃L. (L ⊆ K ∧ closedin X L ∧ X interior_of L ≠ {} ∧ disjnt L (T (Suc
n))) ∧ L ⊆ C
     if §: C ⊆ K ∧ closedin X C ∧ X interior_of C ≠ {} ∧ disjnt C (T n)
     for C n
    proof −
     have False if X interior_of C ⊆ T (Suc n)
      by (metis Tempty interior_of_eq_empty § openin_interior_of that)
     then obtain x where openin X (X interior_of C − T (Suc n)) ∧ x ∈ X
interior_of C − T (Suc n)
      using T empty by fastforce
     with nb_closedin
     obtain N D where openin X N closedin X D x ∈ N N ⊆ D and D: D ⊆
X interior_of C − T(Suc n)
      unfolding neighbourhood_base_of by metis
     show ?thesis
     proof (intro conjI exI)
      show D ⊆ K
       using D interior_of_subset § by fastforce
      show X interior_of D ≠ {}
     by (metis ‹N ⊆ D› ‹openin X N› ‹x ∈ N› empty_iff interior_of_eq_empty)
      show disjnt D (T (Suc n))
       using D disjnt_iff by fastforce
      show D ⊆ C
       using interior_of_subset [of X C] D by blast
     qed (use ‹closedin X D› in auto)
    qed
   qed
   then obtain Φ where Φ: ⋀n. Φ n ⊆ K ∧ closedin X (Φ n) ∧ X interior_of
Φ n ≠ {} ∧ disjnt (Φ n) (T n)
    and ⋀n. Φ (Suc n) ⊆ Φ n by metis
   then have decseq Φ
    by (simp add: decseq_SucI)
   moreover have ⋀n. Φ n ≠ {}
    by (metis Φ bot.extremum_uniqueI interior_of_subset)
   ultimately have ⋂(range Φ) ≠ {}
    by (metis Φ compact_space_imp_nest ‹compactin X K› compactin_subspace
closedin_subset_topspace)
   moreover have U ⊆ {y. ∃x. y ∈ T x}
    using Asub by auto
   with T have {a. ∀n.  a ∈ Φ n} ⊆ {}
    by (smt (verit) Asub Φ Collect_empty_eq UN_iff ‹K ⊆ U› disjnt_iff sub-
set_iff)
   ultimately show False
    by blast
  qed
qed
```

lemma *Baire_category_alt*:
assumes *completely_metrizable_space X* ∨ *locally_compact_space X* ∧ *regular_space X*
and *countable* $\mathcal{G}$
and $\bigwedge T.\ T \in \mathcal{G} \Longrightarrow$ *closedin X T* ∧ *X interior_of T* = {}
shows *X interior_of* $\bigcup \mathcal{G}$ = {}
using *Baire_category_aux* [*of X* $\mathcal{G}$] *Metric_space.metric_Baire_category_alt*
by (*metis assms completely_metrizable_space_def*)

lemma *Baire_category*:
assumes *completely_metrizable_space X* ∨ *locally_compact_space X* ∧ *regular_space X*
and *countable* $\mathcal{G}$
and *top*: $\bigwedge T.\ T \in \mathcal{G} \Longrightarrow$ *openin X T* ∧ *X closure_of T* = *topspace X*
shows *X closure_of* $\bigcap \mathcal{G}$ = *topspace X*
proof (*cases* $\mathcal{G}$={})
case *False*
have ∗: *X interior_of* $\bigcup$((−)(*topspace X*) ‘ $\mathcal{G}$) = {}
proof (*intro Baire_category_alt conjI assms*)
show *countable* ((−) (*topspace X*) ‘ $\mathcal{G}$)
using *assms* **by** *blast*
fix *T*
assume *T* ∈ (−) (*topspace X*) ‘ $\mathcal{G}$
then obtain *U* **where** *U*: *U* ∈ $\mathcal{G}$ *T* = (*topspace X*) − *U U* ⊆ (*topspace X*)
by (*meson top image_iff openin_subset*)
then show *closedin X T*
by (*simp add*: *closedin_diff top*)
show *X interior_of T* = {}
using *U top* **by** (*simp add*: *interior_of_closure_of double_diff*)
qed
then show *?thesis*
by (*simp add*: *closure_of_eq_topspace interior_of_complement*)
qed *auto*

6.11.14 Sierpinski-Hausdorff type results about countable closed unions

lemma *locally_connected_not_countable_closed_union*:
assumes *topspace X* ≠ {} **and** *csX*: *connected_space X*
and *lcX*: *locally_connected_space X*
and *X*: *completely_metrizable_space X* ∨ *locally_compact_space X* ∧ *Hausdorff_space X*
and *countable* $\mathcal{U}$ **and** *pwU*: *pairwise disjnt* $\mathcal{U}$
and *clo*: $\bigwedge C.\ C \in \mathcal{U} \Longrightarrow$ *closedin X C* ∧ *C* ≠ {}
and *UU_eq*: $\bigcup \mathcal{U}$ = *topspace X*
shows $\mathcal{U}$ = {*topspace X*}
proof −
define $\mathcal{V}$ **where** $\mathcal{V}$ ≡ (*frontier_of*) *X* ‘ $\mathcal{U}$

define B **where** $B \equiv \bigcup \mathcal{V}$
then have *Bsub*: $B \subseteq$ *topspace* X
by (*simp add*: *Sup_le_iff* $\mathcal{V}$*_def closedin_frontier_of closedin_subset*)
have *allsub*: $A \subseteq$ *topspace* X **if** $A \in \mathcal{U}$ **for** A
by (*meson clo closedin_def that*)
show *?thesis*
proof (*rule ccontr*)
assume $\mathcal{U} \neq \{$*topspace* $X\}$
with *assms* **have** $\exists A{\in}\mathcal{U}.\ \neg$ (*closedin* X $A \wedge$ *openin* X A)
by (*metis Union_empty connected_space_clopen_in singletonI subsetI subset_singleton_iff*)
then have $B \neq \{\}$
by (*auto simp*: B*_def* $\mathcal{V}$*_def frontier_of_eq_empty allsub*)
moreover
have *subtopology* X B *interior_of* $B = B$
by (*simp add*: *Bsub interior_of_openin openin_subtopology_refl*)
ultimately have *int_B_nonempty*: *subtopology* X B *interior_of* $B \neq \{\}$
by *auto*
have *subtopology* X B *interior_of* $\bigcup \mathcal{V} = \{\}$
proof (*intro Baire_category_alt conjI*)
have $\bigcup \mathcal{U} \subseteq B \cup \bigcup$ ((*interior_of*) X ‘ $\mathcal{U}$)
using *clo closure_of_closedin* **by** (*fastforce simp*: B*_def* $\mathcal{V}$*_def frontier_of_def*)
moreover have $B \cup \bigcup$ ((*interior_of*) X ‘ $\mathcal{U}$) $\subseteq \bigcup \mathcal{U}$
using *allsub clo frontier_of_subset_eq interior_of_subset* **by** (*fastforce simp*: B*_def* $\mathcal{V}$*_def*)
moreover have *disjnt* B ($\bigcup$ ((*interior_of*) X ‘ $\mathcal{U}$))
using *pwU*
apply (*clarsimp simp*: B*_def* $\mathcal{V}$*_def frontier_of_def pairwise_def disjnt_iff*)
by (*metis clo closure_of_eq interior_of_subset subsetD*)
ultimately have $B =$ *topspace* $X - \bigcup$ ((*interior_of*) X ‘ $\mathcal{U}$)
by (*auto simp*: *UU_eq disjnt_iff*)
then have *closedin* X B
by *fastforce*
with X **show** *completely_metrizable_space* (*subtopology* X B) $\vee$ *locally_compact_space* (*subtopology* X B) $\wedge$ *regular_space* (*subtopology* X B)
by (*metis completely_metrizable_space_closedin locally_compact_Hausdorff_or_regular*
locally_compact_space_closed_subset regular_space_subtopology)
show *countable* $\mathcal{V}$
by (*simp add*: $\mathcal{V}$*_def* ‹*countable* $\mathcal{U}$›)
fix V
assume $V \in \mathcal{V}$
then obtain S **where** S: $S \in \mathcal{U}$ $V = X$ *frontier_of* S
by (*auto simp*: $\mathcal{V}$*_def*)
show *closedin* (*subtopology* X B) V
by (*metis B_def Sup_upper* $\mathcal{V}$*_def* ‹$V \in \mathcal{V}$› *closedin_frontier_of closedin_subset_topspace image_iff*)
have *subtopology* X B *interior_of* (X *frontier_of* S) $= \{\}$

```
      proof (clarsimp simp: interior_of_def openin_subtopology_alt)
        fix a U
         assume a ∈ B a ∈ U and openU: openin X U and BUsub: B ∩ U ⊆ X
frontier_of S
        then have a ∈ S
          by (meson IntI ‹S ∈ 𝒰› clo frontier_of_subset_closedin subsetD)
        then obtain W C where openin X W connectedin X C a ∈ W W ⊆ C C
⊆ U
          by (metis ‹a ∈ U› lcX locally_connected_space openU)
        have W ∩ X frontier_of S ≠ {}
          using ‹B ∩ U ⊆ X frontier_of S› ‹a ∈ B› ‹a ∈ U› ‹a ∈ W› by auto
        with frontier_of_openin_straddle_Int
        obtain W ∩ S ≠ {} W − S ≠ {} W ⊆ topspace X
          using ‹openin X W› by (metis openin_subset)
        then obtain b where b ∈ topspace X b ∈ W−S
          by blast
        with UU_eq obtain T where T ∈ 𝒰 T ≠ S W ∩ T ≠ {}
          by auto
        then have disjnt S T
          by (metis ‹S ∈ 𝒰› pairwise_def pwU)
        then have C − T ≠ {}
           by (meson Diff_eq_empty_iff ‹W ⊆ C› ‹a ∈ S› ‹a ∈ W› disjnt_iff
subsetD)
        then have C ∩ X frontier_of T ≠ {}
       using ‹W ∩ T ≠ {}› ‹W ⊆ C› ‹connectedin X C› connectedin_Int_frontier_of
by blast
        moreover have C ∩ X frontier_of T = {}
        proof −
          have X frontier_of S ⊆ S X frontier_of T ⊆ T
            using frontier_of_subset_closedin ‹S ∈ 𝒰› ‹T ∈ 𝒰› clo by blast+
          moreover have X frontier_of T ∪ B = B
            using B_def 𝒱_def ‹T ∈ 𝒰› by blast
          ultimately show ?thesis
            using BUsub ‹C ⊆ U› ‹disjnt S T› unfolding disjnt_def by blast
        qed
        ultimately show False
          by simp
      qed
      with S show subtopology X B interior_of V = {}
        by meson
    qed
    then show False
      using B_def int_B_nonempty by blast
  qed
qed

lemma real_Sierpinski_lemma:
  fixes a b::real
  assumes a ≤ b
```

```
    and countable 𝒰 and pwU: pairwise disjnt 𝒰
    and clo: ⋀C. C ∈ 𝒰 ⟹ closed C ∧ C ≠ {}
    and ⋃𝒰 = {a..b}
  shows 𝒰 = {{a..b}}
proof −
  have locally_connected_space (top_of_set {a..b})
    by (simp add: locally_connected_real_interval)
  moreover
  have completely_metrizable_space (top_of_set {a..b})
    by (metis box_real(2) completely_metrizable_space_cbox)
  ultimately
  show ?thesis
   using locally_connected_not_countable_closed_union [of subtopology euclidean
{a..b}] assms
    apply (simp add: closedin_subtopology)
    by (metis Union_upper inf.orderE)
qed
```

6.11.15 The Tychonoff embedding

```
lemma completely_regular_space_cube_embedding_explicit:
  assumes completely_regular_space X Hausdorff_space X
  shows embedding_map X
           (product_topology (λf. top_of_set {0..1::real})
             (mspace (submetric (cfunspace X euclidean_metric)
               {f. f ∈ topspace X → {0..1}})) )
           (λx. λf ∈ mspace (submetric (cfunspace X euclidean_metric) {f. f ∈
topspace X → {0..1}}).
              f x)
proof −
  define K where K ≡ mspace(submetric (cfunspace X euclidean_metric) {f. f
∈ topspace X → {0..1::real}})
  define e where e ≡ λx. λf∈K. f x
  have e x ≠ e y if xy: x ≠ y x ∈ topspace X y ∈ topspace X for x y
  proof −
    have closedin X {x}
      by (simp add: ‹Hausdorff_space X› closedin_Hausdorff_singleton ‹x ∈
topspace X›)
    then obtain f where contf: continuous_map X euclideanreal f
      and f01: ⋀x. x ∈ topspace X ⟹ f x ∈ {0..1} and fxy: f y = 0 f x = 1
    using ‹completely_regular_space X› xy unfolding completely_regular_space_def
      by (smt (verit, ccfv_threshold) Diff_iff continuous_map_in_subtopology
image_subset_iff singleton_iff)
    then have bounded (f ‘ topspace X)
      by (meson bounded_closed_interval bounded_subset image_subset_iff)
    with contf f01 have restrict f (topspace X) ∈ K
      by (auto simp: K_def)
    with fxy xy show ?thesis
      unfolding e_def by (metis restrict_apply' zero_neq_one)
```

qed
then have *inj_on e* (*topspace X*)
by (*meson inj_onI*)
then obtain e' **where** e': $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow$ e' (*e x*) = *x*
by (*metis inv_into_f_f*)
have *continuous_map* (*subtopology* (*product_topology* (λ*f*. *top_of_set* {*0..1*}) *K*) (*e* ‘ *topspace X*)) *X* e'
proof (*clarsimp simp add*: *continuous_map_atin limitin_atin openin_subtopology_alt* e')
fix *x U*
assume *e x* ∈ *K* →$_E$ {*0..1*} **and** *x* ∈ *topspace X* **and** *openin X U* **and** *x* ∈ *U*
then obtain *g* **where** *contg*: *continuous_map X* (*top_of_set* {*0..1*}) *g* **and** *g x* = *0*
and *gim*: *g* ‘ (*topspace X* − *U*) ⊆ {*1*::*real*}
using ‹*completely_regular_space X*› **unfolding** *completely_regular_space_def*

by (*metis Diff_iff openin_closedin_eq*)
then have *bounded* (*g* ‘ *topspace X*)
by (*meson bounded_closed_interval bounded_subset continuous_map_in_subtopology*)
moreover have *g* ∈ *topspace X* → {*0..1*}
using *contg* **by** (*simp add*: *continuous_map_def*)
ultimately have *g_in_K*: *restrict g* (*topspace X*) ∈ *K*
using *contg* **by** (*force simp add*: *K_def continuous_map_in_subtopology*)
have *openin* (*top_of_set* {*0..1*}) {*0..<1*::*real*}
using *open_real_greaterThanLessThan*[*of* −*1 1*] **by** (*force simp*: *openin_open*)
moreover have *e x* ∈ (Π$_E$ *f*∈*K*. *if f* = *restrict g* (*topspace X*) *then* {*0..<1*} *else* {*0..1*})
using ‹*e x* ∈ *K* →$_E$ {*0..1*}› **by** (*simp add*: *e_def* ‹*g x* = *0*› ‹*x* ∈ *topspace X*› *PiE_iff*)
moreover have *e y* = *e x*
if *y* ∉ *U* **and** *ey*: *e y* ∈ (Π$_E$ *f*∈*K*. *if f* = *restrict g* (*topspace X*) *then* {*0..<1*} *else* {*0..1*})
and *y*: *y* ∈ *topspace X* **for** *y*
proof −
have *e y* (*restrict g* (*topspace X*)) ∈ {*0..<1*}
using *ey* **by** (*smt* (*verit*, *ccfv_SIG*) *PiE_mem g_in_K*)
with *gim g_in_K y* ‹*y* ∉ *U*› **show** *?thesis*
by (*fastforce simp*: *e_def*)
qed
ultimately
show ∃ *W*. *openin* (*product_topology* (λ*f*. *top_of_set* {*0..1*}) *K*) *W* ∧ *e x* ∈ *W* ∧ e' ‘ (*e* ‘ *topspace X* ∩ *W* − {*e x*}) ⊆ *U*
apply (*rule_tac x*=*PiE K* (λ*f*. *if f* = *restrict g* (*topspace X*) *then* {*0..<1*} *else* {*0..1*}) **in** *exI*)
by (*auto simp*: *openin_PiE_gen* e')
qed
with e' **have** *embedding_map X* (*product_topology* (λ*f*. *top_of_set* {*0..1*}) *K*) *e*
unfolding *embedding_map_def homeomorphic_map_maps homeomorphic_maps_def*

by (*fastforce simp*: *e_def K_def continuous_map_in_subtopology continuous_map_componentwise*)
then show *?thesis*
by (*simp add*: *K_def e_def*)
qed

lemma *completely_regular_space_cube_embedding*:
fixes *X* :: *'a topology*
assumes *completely_regular_space X Hausdorff_space X*
obtains *K*:: (*'a⇒real*)*set* **and** *e*
where *embedding_map X* (*product_topology* (*λf*. *top_of_set* {*0..1*::*real*}) *K*) *e*
using *completely_regular_space_cube_embedding_explicit* [*OF assms*] **by** *metis*

6.11.16 Urysohn and Tietze analogs for completely regular spaces

"Urysohn and Tietze analogs for completely regular spaces if (()) set is assumed compact instead of closed. Note that Hausdorffness is *not* required: inside () proof we factor through the Kolmogorov quotient." – John Harrison

lemma *Urysohn_completely_regular_closed_compact*:
fixes *a b*::*real*
assumes *a* ≤ *b completely_regular_space X closedin X S compactin X T disjnt S T*
obtains *f* **where** *continuous_map X* (*subtopology euclidean* {*a..b*}) *f f* ‘ *T* ⊆ {*a*} *f* ‘ *S* ⊆ {*b*}
proof −
obtain *f* **where** *contf*: *continuous_map X* (*subtopology euclideanreal* {*0..1*}) *f*
and *f0*: *f* ‘ *T* ⊆ {*0*} **and** *f1*: *f* ‘ *S* ⊆ {*1*}
proof (*cases T*={})
case *True*
show *thesis*
proof
show *continuous_map X* (*top_of_set* {*0..1*}) (*λx*. *1*::*real*) (*λx*. *1*::*real*) ‘ *T* ⊆ {*0*} (*λx*. *1*::*real*) ‘ *S* ⊆ {*1*}
using *True* **by** *auto*
qed
next
case *False*
have ⋀*t*. *t* ∈ *T* ⟹ ∃*f*. *continuous_map X* (*subtopology euclideanreal* ({*0..1*})) *f* ∧ *f t* = *0* ∧ *f* ‘ *S* ⊆ {*1*}
using *assms* **unfolding** *completely_regular_space_def*
by (*meson DiffI compactin_subset_topspace disjnt_iff subset_eq*)
then obtain *g* **where** *contg*: ⋀*t*. *t* ∈ *T* ⟹ *continuous_map X* (*subtopology euclideanreal* {*0..1*}) (*g t*)
and *g0*: ⋀*t*. *t* ∈ *T* ⟹ *g t t* = *0*
and *g1*: ⋀*t*. *t* ∈ *T* ⟹ *g t* ‘ *S* ⊆ {*1*}

by *metis*
then have *g01*: $\bigwedge t.\ t \in T \Longrightarrow g\ t\ `\ topspace\ X \subseteq \{0..1\}$
by (*meson continuous_map_in_subtopology*)
define *G* **where** $G \equiv \lambda t.\ \{x \in topspace\ X.\ g\ t\ x \in \{..<1/2\}\}$
have *Ball* (*G`T*) (*openin X*)
using *contg* **unfolding** *G_def continuous_map_in_subtopology*
by (*smt* (*verit*, *best*) *Collect_cong openin_continuous_map_preimage image_iff open_lessThan open_openin*)
moreover have $T \subseteq \bigcup (G`T)$
using ‹*compactin X T*› *g0 compactin_subset_topspace* **by** (*force simp*: *G_def*)
ultimately have $\exists \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq G`T \wedge T \subseteq \bigcup\ \mathcal{F}$
using ‹*compactin X T*› **unfolding** *compactin_def* **by** *blast*
then obtain *K* **where** *K*: *finite K K* $\subseteq T$ *T* $\subseteq \bigcup (G`K)$
by (*metis finite_subset_image*)
with *False* **have** $K \neq \{\}$
by *fastforce*
define *f* **where** $f \equiv \lambda x.\ 2 * max\ 0\ (Inf\ ((\lambda t.\ g\ t\ x)\ `\ K) - 1/2)$
have [*simp*]: $max\ 0\ (x - 1/2) = 0 \longleftrightarrow x \leq 1/2$ **for** *x*::*real*
by *force*
have [*simp*]: $2 * max\ 0\ (x - 1/2) = 1 \longleftrightarrow x = 1$ **for** *x*::*real*
by (*simp add*: *max_def_raw*)
show *thesis*
proof
have $g\ t\ s = 1$ **if** $s \in S\ t \in K$ **for** *s t*
using ‹$K \subseteq T$› *g1 that* **by** *auto*
then show $f\ `\ S \subseteq \{1\}$
using ‹$K \neq \{\}$› **by** (*simp add*: *f_def image_subset_iff*)
have $(INF\ t{\in}K.\ g\ t\ x) \leq 1/2$ **if** $x \in T$ **for** *x*
proof −
obtain *k* **where** $k \in K\ g\ k\ x < 1/2$
using *K* ‹$x \in T$› **by** (*auto simp*: *G_def*)
then show *?thesis*
by (*meson* ‹*finite K*› *cInf_le_finite dual_order.trans finite_imageI imageI less_le_not_le*)
qed
then show $f\ `\ T \subseteq \{0\}$
by (*force simp*: *f_def*)
have $\bigwedge t.\ t \in K \Longrightarrow continuous\_map\ X\ euclideanreal\ (g\ t)$
using ‹$K \subseteq T$› *contg continuous_map_in_subtopology* **by** *blast*
moreover have $2 * max\ 0\ ((INF\ t{\in}K.\ g\ t\ x) - 1/2) \leq 1$ **if** $x \in topspace\ X$ **for** *x*
proof −
obtain *k* **where** $k \in K\ g\ k\ x \leq 1$
using *K* ‹$x \in topspace\ X$› ‹$K \neq \{\}$› *g01* **by** (*fastforce simp*: *G_def*)
then have $(INF\ t{\in}K.\ g\ t\ x) \leq 1$
by (*meson* ‹*finite K*› *cInf_le_finite dual_order.trans finite_imageI imageI*)
then show *?thesis*
by (*simp add*: *max_def_raw*)
qed

ultimately show *continuous_map X* (*top_of_set* {*0..1*}) *f*
by (*force simp*: *f_def continuous_map_in_subtopology intro*!: ‹*finite K*› *continuous_intros*)
qed
qed
define *g* **where** $g \equiv \lambda x.\ a + (b - a) * f\ x$
show *thesis*
proof
have $\forall\, x{\in}\mathit{topspace}\ X.\ a + (b - a) * f\ x \leq b$
using *contf* ‹$a \leq b$› **apply** (*simp add*: *continuous_map_in_subtopology image_subset_iff*)
by (*smt* (*verit*, *best*) *mult_right_le_one_le*)
then show *continuous_map X* (*top_of_set* {*a..b*}) *g*
using *contf* ‹$a \leq b$› **unfolding** *g_def continuous_map_in_subtopology image_subset_iff*
by (*intro conjI continuous_intros*; *simp*)
show *g* ‘ *T* ⊆ {*a*} *g* ‘ *S* ⊆ {*b*}
using *f0 f1* **by** (*auto simp*: *g_def*)
qed
qed

lemma *Urysohn_completely_regular_compact_closed*:
fixes *a b*::*real*
assumes $a \leq b$ *completely_regular_space X compactin X S closedin X T disjnt S T*
obtains *f* **where** *continuous_map X* (*subtopology euclidean* {*a..b*}) *f f* ‘ *T* ⊆ {*a*} *f* ‘ *S* ⊆ {*b*}
proof −
obtain *f* **where** *contf*: *continuous_map X* (*subtopology euclidean* {−*b*..−*a*}) *f* **and** *fim*: *f* ‘ *T* ⊆ {−*a*} *f* ‘ *S* ⊆ {−*b*}
by (*meson Urysohn_completely_regular_closed_compact assms disjnt_sym neg_le_iff_le*)
show *thesis*
proof
show *continuous_map X* (*top_of_set* {*a..b*}) (*uminus* ∘ *f*)
using *contf* **by** (*auto simp*: *continuous_map_in_subtopology o_def*)
show (*uminus o f*) ‘ *T* ⊆ {*a*} (*uminus o f*) ‘ *S* ⊆ {*b*}
using *fim* **by** *fastforce*+
qed
qed

lemma *Urysohn_completely_regular_compact_closed_alt*:
fixes *a b*::*real*
assumes *completely_regular_space X compactin X S closedin X T disjnt S T*
obtains *f* **where** *continuous_map X euclideanreal f f* ‘ *T* ⊆ {*a*} *f* ‘ *S* ⊆ {*b*}
proof (*cases a b rule*: *le_cases*)
case *le*
then show *?thesis*

by (*meson Urysohn_completely_regular_compact_closed assms continuous_map_into_fulltopology that*)
next
case *ge*
then show *?thesis*
using *Urysohn_completely_regular_closed_compact assms*
by (*metis Urysohn_completely_regular_closed_compact assms continuous_map_into_fulltopology disjnt_sym that*)
qed

lemma *Tietze_extension_comp_reg_aux*:
fixes *T* :: *real set*
assumes *completely_regular_space X Hausdorff_space X compactin X S*
and *T*: *is_interval T T≠{}*
and *contf*: *continuous_map* (*subtopology X S*) *euclidean f* **and** *fim*: *f‘S* ⊆ *T*
obtains *g* **where** *continuous_map X euclidean g g ‘ topspace X* ⊆ *T* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
proof −
obtain *K*:: (*'a*⇒*real*)*set* **and** *e*
where *e0*: *embedding_map X* (*product_topology* (λ*f*. *top_of_set* {*0..1*::*real*}) *K*) *e*
using *assms completely_regular_space_cube_embedding* **by** *blast*
define *cube* **where** *cube* ≡ *product_topology* (λ*f*. *top_of_set* {*0..1*::*real*}) *K*
have *e*: *embedding_map X cube e*
using *e0* **by** (*simp add*: *cube_def*)
obtain *e'* **where** *e'*: *homeomorphic_maps X* (*subtopology cube* (*e ‘ topspace X*)) *e e'*
using *e* **by** (*force simp*: *cube_def embedding_map_def homeomorphic_map_maps*)
then have *conte*: *continuous_map X* (*subtopology cube* (*e ‘ topspace X*)) *e*
and *conte'*: *continuous_map* (*subtopology cube* (*e ‘ topspace X*)) *X e'*
and *e'e*: ∀ *x*∈*topspace X*. *e'* (*e x*) = *x*
by (*auto simp*: *homeomorphic_maps_def*)
have *Hausdorff_space cube*
unfolding *cube_def*
using *Hausdorff_space_euclidean Hausdorff_space_product_topology Hausdorff_space_subtopology* **by** *blast*
have *normal_space cube*
proof (*rule compact_Hausdorff_or_regular_imp_normal_space*)
show *compact_space cube*
unfolding *cube_def*
using *compact_space_product_topology compact_space_subtopology compactin_euclidean_iff* **by** *blast*
qed (*use* ‹*Hausdorff_space cube*› **in** *auto*)
moreover
have *comp*: *compactin cube* (*e ‘ S*)
by (*meson* ‹*compactin X S*› *conte continuous_map_in_subtopology image_compactin*)
then have *closedin cube* (*e ‘ S*)
by (*intro compactin_imp_closedin* ‹*Hausdorff_space cube*›)

```
  moreover
  have continuous_map (subtopology cube (e ‘ S)) euclideanreal (f ∘ e′)
  proof (intro continuous_map_compose)
    show continuous_map (subtopology cube (e ‘ S)) (subtopology X S) e′
      unfolding continuous_map_in_subtopology
    proof
      show continuous_map (subtopology cube (e ‘ S)) X e′
        by (meson ‹compactin X S› compactin_subset_topspace conte′ continuous_map_from_subtopology_mono image_mono)
      show e′ ‘ topspace (subtopology cube (e ‘ S)) ⊆ S
        using ‹compactin X S› compactin_subset_topspace e′e by fastforce
    qed
  qed (simp add: contf)
  moreover
  have (f ∘ e′) ‘ e ‘ S ⊆ T
    using ‹compactin X S› compactin_subset_topspace e′e fim by fastforce
  ultimately
  obtain g where contg: continuous_map cube euclidean g and gsub: g ‘ topspace cube ⊆ T
                and gf: ⋀x. x ∈ e‘S ⟹ g x = (f ∘ e′) x
    using Tietze_extension_realinterval T by metis
  show thesis
  proof
    show continuous_map X euclideanreal (g ∘ e)
    by (meson contg conte continuous_map_compose continuous_map_in_subtopology)
    show (g ∘ e) ‘ topspace X ⊆ T
      using gsub conte continuous_map_image_subset_topspace by fastforce
    fix x
    assume x ∈ S
    then show (g ∘ e) x = f x
      using gf ‹compactin X S› compactin_subset_topspace e′e by fastforce
  qed
qed


lemma Tietze_extension_completely_regular:
  assumes completely_regular_space X compactin X S is_interval T T ≠ {}
    and contf: continuous_map (subtopology X S) euclidean f and fim: f‘S ⊆ T
  obtains g where continuous_map X euclideanreal g g ‘ topspace X ⊆ T
    ⋀x. x ∈ S ⟹ g x = f x
proof −
  define Q where Q ≡ Kolmogorov_quotient X ‘ (topspace X)
  obtain g where contg: continuous_map (subtopology X (Kolmogorov_quotient X ‘ S)) euclidean g
    and gf: ⋀x. x ∈ S ⟹ g(Kolmogorov_quotient X x) = f x
    using Kolmogorov_quotient_lift_exists
      by (metis ‹compactin X S› contf compactin_subset_topspace open_openin t0_space_def t1_space)
  have S ⊆ topspace X
```

```
    by (simp add: ‹compactin X S› compactin_subset_topspace)
  then have [simp]: Q ∩ Kolmogorov_quotient X ‘ S = Kolmogorov_quotient X ‘
S
    using Q_def by blast
  have creg: completely_regular_space (subtopology X Q)
   by (simp add: ‹completely_regular_space X› completely_regular_space_subtopology)
  then have regular_space (subtopology X Q)
    by (simp add: completely_regular_imp_regular_space)
  then have Hausdorff_space (subtopology X Q)
    using Q_def regular_t0_eq_Hausdorff_space t0_space_Kolmogorov_quotient
by blast
  moreover
  have compactin (subtopology X Q) (Kolmogorov_quotient X ‘ S)
   by (metis Q_def ‹compactin X S› image_compactin quotient_imp_continuous_map
quotient_map_Kolmogorov_quotient)
  ultimately obtain h where conth: continuous_map (subtopology X Q) euclidean
h
              and him: h ‘ topspace (subtopology X Q) ⊆ T
              and hg: ⋀x. x ∈ Kolmogorov_quotient X ‘ S ⟹ h x = g x
   using Tietze_extension_comp_reg_aux [of subtopology X Q Kolmogorov_quotient
X ‘ S T g]
    apply (simp add: subtopology_subtopology creg contg assms)
    using fim gf by blast
  show thesis
  proof
    show continuous_map X euclideanreal (h ∘ Kolmogorov_quotient X)
    by (metis Q_def conth continuous_map_compose quotient_imp_continuous_map
quotient_map_Kolmogorov_quotient)
    show (h ∘ Kolmogorov_quotient X) ‘ topspace X ⊆ T
    using Q_def continuous_map_Kolmogorov_quotient continuous_map_image_subset_topspace
him by fastforce
    fix x
    assume x ∈ S then show (h ∘ Kolmogorov_quotient X) x = f x
      by (simp add: gf hg)
  qed
qed
```

6.11.17 Size bounds on connected or path-connected spaces

```
lemma connected_space_imp_card_ge_alt:
  assumes connected_space X completely_regular_space X closedin X S S ≠ {} S
≠ topspace X
  shows (UNIV::real set) ≲ topspace X
proof −
  have S ⊆ topspace X
    using ‹closedin X S› closedin_subset by blast
  then obtain a where a ∈ topspace X a ∉ S
    using ‹S ≠ topspace X› by blast
  have (UNIV::real set) ≲ {0..1::real}
```

```
    using eqpoll_real_subset
    by (meson atLeastAtMost_iff eqpoll_imp_lepoll eqpoll_sym less_eq_real_def
zero_less_one)
  also have ... ≲ topspace X
  proof −
    obtain f where contf: continuous_map X euclidean f
      and fim: f ∈ (topspace X) → {0..1::real}
      and f0: f a = 0 and f1: f ‘ S ⊆ {1}
      using ‹completely_regular_space X›
      unfolding completely_regular_space_def
        by (metis Diff_iff ‹a ∈ topspace X› ‹a ∉ S› ‹closedin X S› continu-
ous_map_in_subtopology image_subset_iff_funcset)
    have ∃ y∈topspace X. x = f y if 0 ≤ x and x ≤ 1 for x
    proof −
      have connectedin euclidean (f ‘ topspace X)
        using ‹connected_space X› connectedin_continuous_map_image connecte-
din_topspace contf by blast
      moreover have ∃ y. 0 = f y ∧ y ∈ topspace X
        using ‹a ∈ topspace X› f0 by auto
      moreover have ∃ y. 1 = f y ∧ y ∈ topspace X
        using ‹S ⊆ topspace X› ‹S ≠ {}› f1 by fastforce
      ultimately show ?thesis
      using that by (fastforce simp: is_interval_1 simp flip: is_interval_connected_1)
    qed
    then show ?thesis
      unfolding lepoll_iff using atLeastAtMost_iff by blast
  qed
  finally show ?thesis .
qed


lemma connected_space_imp_card_ge_gen:
  assumes connected_space X normal_space X closedin X S closedin X T S ≠ {}
T ≠ {} disjnt S T
  shows (UNIV::real set) ≲ topspace X
proof −
  have (UNIV::real set) ≲ {0..1::real}
    by (metis atLeastAtMost_iff eqpoll_real_subset eqpoll_imp_lepoll eqpoll_sym
less_le_not_le zero_less_one)
  also have ...≲ topspace X
  proof −
    obtain f where contf: continuous_map X euclidean f
      and fim: f ∈ (topspace X) → {0..1::real}
      and f0: f ‘ S ⊆ {0} and f1: f ‘ T ⊆ {1}
    using assms by (metis continuous_map_in_subtopology normal_space_iff_Urysohn
image_subset_iff_funcset)
    have ∃ y∈topspace X. x = f y if 0 ≤ x and x ≤ 1 for x
    proof −
      have connectedin euclidean (f ‘ topspace X)
```

using ‹*connected_space X*› *connectedin_continuous_map_image connectedin_topspace contf* **by** *blast*
moreover have $\exists y.\ 0 = f\ y \wedge y \in$ *topspace X*
using ‹*closedin X S*› ‹$S \neq \{\}$› *closedin_subset f0* **by** *fastforce*
moreover have $\exists y.\ 1 = f\ y \wedge y \in$ *topspace X*
using ‹*closedin X T*› ‹$T \neq \{\}$› *closedin_subset f1* **by** *fastforce*
ultimately show *?thesis*
using *that* **by** (*fastforce simp*: *is_interval_1 simp flip*: *is_interval_connected_1*)
qed
then show *?thesis*
unfolding *lepoll_iff* **using** *atLeastAtMost_iff* **by** *blast*
qed
finally show *?thesis* **.**
qed

lemma *connected_space_imp_card_ge*:
assumes *connected_space X normal_space X t1_space X* **and** *nosing*: $\neg$ ($\exists a.$ *topspace X* $\subseteq \{a\}$)
shows (*UNIV*::*real set*) $\lesssim$ *topspace X*
proof −
obtain *a b* **where** $a \in$ *topspace X* $b \in$ *topspace X* $a \neq b$
by (*metis nosing singletonI subset_iff*)
then have $\{a\} \neq$ *topspace X*
by *force*
with *connected_space_imp_card_ge_alt assms* **show** *?thesis*
by (*metis* ‹$a \in$ *topspace X*› *closedin_t1_singleton insert_not_empty normal_imp_completely_regular_space_A*)
qed

lemma *connected_space_imp_infinite_gen*:
⟦*connected_space X*; *t1_space X*; $\nexists a.$ *topspace X* $\subseteq \{a\}$⟧ $\Longrightarrow$ *infinite*(*topspace X*)
by (*metis connected_space_discrete_topology finite_t1_space_imp_discrete_topology*)

lemma *connected_space_imp_infinite*:
⟦*connected_space X*; *Hausdorff_space X*; $\nexists a.$ *topspace X* $\subseteq \{a\}$⟧ $\Longrightarrow$ *infinite*(*topspace X*)
by (*simp add*: *Hausdorff_imp_t1_space connected_space_imp_infinite_gen*)

lemma *connected_space_imp_infinite_alt*:
assumes *connected_space X regular_space X closedin X S S* $\neq \{\}$ *S* $\neq$ *topspace X*
shows *infinite*(*topspace X*)
proof −
have *S* $\subseteq$ *topspace X*
using ‹*closedin X S*› *closedin_subset* **by** *blast*
then obtain *a* **where** *a*: $a \in$ *topspace X* $a \notin S$
using ‹$S \neq$ *topspace X*› **by** *blast*
have $\exists \Phi.\ \forall n.$ (*disjnt* (Φ *n*) *S* $\wedge$ $a \in \Phi$ *n* $\wedge$ *openin X* (Φ *n*)) $\wedge$ Φ(*Suc n*) $\subset \Phi$ *n*

proof (*rule dependent_nat_choice*)
show $\exists T.$ *disjnt* T $S \wedge a \in T \wedge$ *openin* X T
by (*metis Diff_iff a* ‹*closedin X S*› *closedin_def disjnt_iff*)
fix V n
assume §: *disjnt* V $S \wedge a \in V \wedge$ *openin* X V
then obtain U C **where** U: *openin* X U *closedin* X C $a \in U$ $U \subseteq C$ $C \subseteq V$
using ‹*regular_space X*› **by** (*metis neighbourhood_base_of neighbourhood_base_of_closedin*)
with *assms* **have** $U \subset V$
by (*metis* § ‹$S \subseteq$ *topspace* X› *connected_space_clopen_in disjnt_def empty_iff inf.absorb_iff2 inf.orderE psubsetI subset_trans*)
with U **show** $\exists U.$ (*disjnt* U $S \wedge a \in U \wedge$ *openin* X U) $\wedge$ $U \subset V$
using § *disjnt_subset1* **by** *blast*
qed
then obtain Φ **where** Φ: $\bigwedge n.$ *disjnt* (Φ n) $S \wedge a \in \Phi$ $n \wedge$ *openin* X (Φ n)
and Φ*sub*: $\bigwedge n.$ Φ (*Suc* n) $\subset$ Φ n **by** *metis*
then have *decseq* Φ
by (*simp add*: *decseq_SucI psubset_eq*)
have $\forall n.$ $\exists x.$ $x \in \Phi$ $n \wedge x \notin \Phi$(*Suc* n)
by (*meson* Φ*sub psubsetE subsetI*)
then obtain f **where** *fin*: $\bigwedge n.$ f $n \in \Phi$ n **and** *fout*: $\bigwedge n.$ f $n \notin \Phi$(*Suc* n)
by *metis*
have *range* $f \subseteq$ *topspace* X
by (*meson* Φ *fin image_subset_iff openin_subset subset_iff*)
moreover have *inj* f
by (*metis Suc_le_eq* ‹*decseq* Φ› *decseq_def fin fout linorder_injI subsetD*)
ultimately show *?thesis*
using *infinite_iff_countable_subset* **by** *blast*
qed

lemma *path_connected_space_imp_card_ge*:
assumes *path_connected_space X Hausdorff_space X* **and** *nosing*: $\neg$ ($\exists x.$ *topspace* $X \subseteq \{x\}$)
shows (*UNIV*::*real set*) $\lesssim$ *topspace* X
proof −
obtain a b **where** $a \in$ *topspace* X $b \in$ *topspace* X $a \neq b$
by (*metis nosing singletonI subset_iff*)
then obtain γ **where** γ: *pathin* X γ γ $0 = a$ γ $1 = b$
by (*meson* ‹$a \in$ *topspace* X› ‹$b \in$ *topspace* X› ‹*path_connected_space* X› *path_connected_space_def*)
let *?Y* = *subtopology* X (γ ‘ (*topspace* (*subtopology euclidean* {*0..1*})))
have (*UNIV*::*real set*) $\lesssim$ *topspace ?Y*
proof (*intro compact_Hausdorff_or_regular_imp_normal_space connected_space_imp_card_ge*)
show *connected_space ?Y*
using ‹*pathin X* γ› *connectedin_def connectedin_path_image* **by** *auto*
show *Hausdorff_space ?Y* $\vee$ *regular_space ?Y*
using *Hausdorff_space_subtopology* ‹*Hausdorff_space X*› **by** *blast*
show *t1_space ?Y*
using *Hausdorff_imp_t1_space* ‹*Hausdorff_space X*› *t1_space_subtopology*
by *blast*

show *compact_space ?Y*
by (*simp add*: ‹*pathin X γ*› *compact_space_subtopology compactin_path_image*)
have *a* ∈ *topspace ?Y b* ∈ *topspace ?Y*
using *γ pathin_subtopology* **by** *fastforce+*
with ‹*a* ≠ *b*› **show** ∄*x*. *topspace ?Y* ⊆ {*x*}
by *blast*
qed
also have ... ≲ *γ* ‘ {*0..1*}
by (*simp add*: *subset_imp_lepoll*)
also have ... ≲ *topspace X*
by (*meson γ path_image_subset_topspace subset_imp_lepoll image_subset_iff_funcset*)
finally show *?thesis* **.**
qed

lemma *connected_space_imp_uncountable*:
assumes *connected_space X regular_space X Hausdorff_space X* ¬ (∃ *a*. *topspace X* ⊆ {*a*})
shows ¬ *countable*(*topspace X*)
proof
assume *coX*: *countable* (*topspace X*)
with ‹*regular_space X*› **have** *normal_space X*
using *countable_imp_Lindelof_space regular_Lindelof_imp_normal_space* **by** *blast*
then have (*UNIV*::*real set*) ≲ *topspace X*
by (*simp add*: *Hausdorff_imp_t1_space assms connected_space_imp_card_ge*)
with *coX* **show** *False*
using *countable_lepoll uncountable_UNIV_real* **by** *blast*
qed

lemma *path_connected_space_imp_uncountable*:
assumes *path_connected_space X t1_space X* **and** *nosing*: ¬ (∃ *a*. *topspace X* ⊆ {*a*})
shows ¬ *countable*(*topspace X*)
proof
assume *coX*: *countable* (*topspace X*)
obtain *a b* **where** *a* ∈ *topspace X b* ∈ *topspace X a* ≠ *b*
by (*metis nosing singletonI subset_iff*)
then obtain *γ* **where** *pathin X γ γ 0* = *a γ 1* = *b*
by (*meson* ‹*a* ∈ *topspace X*› ‹*b* ∈ *topspace X*› ‹*path_connected_space X*› *path_connected_space_def*)
then have *γ* ‘ {*0..1*} ≲ *topspace X*
by (*meson path_image_subset_topspace subset_imp_lepoll image_subset_iff_funcset*)
define *𝒜* **where** *𝒜* ≡ ((λ*a*. {*x* ∈ {*0..1*}. *γ x* ∈ {*a*}}) ‘ *topspace X*) − {{}}
have *𝒜01*: *𝒜* = {{*0..1*}}
proof (*rule real_Sierpinski_lemma*)
show *countable 𝒜*
using *𝒜_def coX* **by** *blast*
show *disjoint 𝒜*
by (*auto simp*: *𝒜_def disjnt_iff pairwise_def*)

show $\bigcup \mathcal{A} = \{0..1\}$
using ‹*pathin X γ*› *path_image_subset_topspace* **by** (*fastforce simp*: $\mathcal{A}$*_def Bex_def*)
fix *C*
assume $C \in \mathcal{A}$
then obtain *a* **where** $a \in$ *topspace X* **and** *C*: $C = \{x \in \{0..1\}.\ \gamma\ x \in \{a\}\}$ $C \neq \{\}$
by (*auto simp*: $\mathcal{A}$*_def*)
then have *closedin X* $\{a\}$
by (*meson* ‹*t1_space X*› *closedin_t1_singleton*)
then have *closedin* (*top_of_set* $\{0..1\}$) *C*
using *C* ‹*pathin X γ*› *closedin_continuous_map_preimage pathin_def* **by** *fastforce*
then show *closed C* $\wedge$ $C \neq \{\}$
using *C closedin_closed_trans* **by** *blast*
qed *auto*
then have $\{0..1\} \in \mathcal{A}$
by *blast*
then have $\exists\, a \in$ *topspace X*. $\{0..1\} \subseteq \{x.\ \gamma\ x = a\}$
using $\mathcal{A}$*_def image_iff* **by** *auto*
then show *False*
using ‹$\gamma\ 0 = a$› ‹$\gamma\ 1 = b$› ‹$a \neq b$› *atLeastAtMost_iff zero_less_one_class.zero_le_one* **by** *blast*
qed

6.11.18 Lavrentiev extension etc

lemma (**in** *Metric_space*) *convergent_eq_zero_oscillation_gen*:
assumes *mcomplete* **and** *fim*: $f \in$ (*topspace X* $\cap$ *S*) $\to$ *M*
shows ($\exists\, l.$ *limitin mtopology f l* (*atin_within X a S*)) $\longleftrightarrow$
$M \neq \{\} \wedge$
($a \in$ *topspace X*
$\longrightarrow$ ($\forall\, \varepsilon{>}0.\ \exists\, U.$ *openin X U* $\wedge$ $a \in U \wedge$
($\forall\, x \in (S \cap U) - \{a\}.\ \forall\, y \in (S \cap U) - \{a\}.$ *d* (*f x*) (*f y*) $<$ ε)))
proof (*cases* $M = \{\}$)
case *True*
with *limitin_mspace* **show** *?thesis*
by *blast*
next
case *False*
show *?thesis*
proof (*cases* $a \in$ *topspace X*)
case *True*
let *?R* = $\forall\, \varepsilon{>}0.\ \exists\, U.$ *openin X U* $\wedge$ $a \in U \wedge$ ($\forall\, x{\in}S \cap U - \{a\}.\ \forall\, y{\in}S \cap U - \{a\}.$ *d* (*f x*) (*f y*) $< \varepsilon$)
show *?thesis*
proof (*cases* $a \in X$ *derived_set_of S*)
case *True*

have *?R*
if *limitin mtopology f l* (*atin_within X a S*) **for** *l*
proof (*intro strip*)
fix *ε*::*real*
assume *ε*>*0*
with *that* ‹*a* ∈ *topspace X*›
obtain *U* **where** *U*: *openin X U a* ∈ *U l* ∈ *M*
and *Uless*: ∀ *x*∈*U* − {*a*}. *x* ∈ *S* ⟶ *f x* ∈ *M* ∧ *d* (*f x*) *l* < *ε*/*2*
unfolding *limitin_metric eventually_atin_within* **by** (*metis Diff_iff insertI1 half_gt_zero* [*OF* ‹*ε*>*0*›])
show ∃ *U*. *openin X U* ∧ *a* ∈ *U* ∧ (∀ *x*∈*S* ∩ *U* − {*a*}. ∀ *y*∈*S* ∩ *U* − {*a*}. *d* (*f x*) (*f y*) < *ε*)
proof (*intro exI strip conjI*)
fix *x y*
assume *x*: *x* ∈ *S* ∩ *U* − {*a*} **and** *y*: *y* ∈ *S* ∩ *U* − {*a*}
then have *d* (*f x*) *l* < *ε*/*2* *d* (*f y*) *l* < *ε*/*2* *f x* ∈ *M f y* ∈ *M*
using *Uless* **by** *auto*
then show *d* (*f x*) (*f y*) < *ε*
using *triangle′* ‹*l* ∈ *M*› **by** *fastforce*
qed (*auto simp add*: *U*)
qed
moreover have ∃ *l*. *limitin mtopology f l* (*atin_within X a S*)
if *R* [*rule_format*]: *?R*
proof −
define *F* **where** *F* ≡ λ*U*. *mtopology closure_of f* ‘ (*S* ∩ *U* − {*a*})
define *C* **where** *C* ≡ *F* ‘ {*U*. *openin X U* ∧ *a* ∈ *U*}
have *C_clo*: ∀ *C* ∈ *C*. *closedin mtopology C*
by (*force simp add*: *C_def F_def*)
moreover have *sub_mcball*: ∃ *C a*. *C* ∈ *C* ∧ *C* ⊆ *mcball a ε* **if** *ε*>*0* **for** *ε*
proof −
obtain *U* **where** *U*: *openin X U a* ∈ *U*
and *Uless*: ∀ *x*∈*S* ∩ *U* − {*a*}. ∀ *y*∈*S* ∩ *U* − {*a*}. *d* (*f x*) (*f y*) < *ε*
using *R* [*OF* ‹*ε*>*0*›] **by** *blast*
then obtain *b* **where** *b*: *b* ≠ *a b* ∈ *S b* ∈ *U*
using *True* **by** (*auto simp add*: *in_derived_set_of*)
have *U* ⊆ *topspace X*
by (*simp add*: *U*(*1*) *openin_subset*)
have *f b* ∈ *M*
using *b* ‹*openin X U*› **by** (*metis image_subset_iff_funcset Int_iff fim image_eqI openin_subset subsetD*)
moreover
have *mtopology closure_of f* ‘ ((*S* ∩ *U*) − {*a*}) ⊆ *mcball* (*f b*) *ε*
proof (*rule closure_of_minimal*)
have *f y* ∈ *M* **if** *y* ∈ *S* **and** *y* ∈ *U* **for** *y*
using ‹*U* ⊆ *topspace X*› *fim that* **by** (*auto simp*: *Pi_iff*)
moreover
have *d* (*f b*) (*f y*) ≤ *ε* **if** *y* ∈ *S y* ∈ *U y* ≠ *a* **for** *y*
using *that Uless b* **by** *force*
ultimately show *f* ‘ (*S* ∩ *U* − {*a*}) ⊆ *mcball* (*f b*) *ε*

```
            by (force simp: ‹f b ∈ M›)
          qed auto
          ultimately show ?thesis
            using U by (auto simp add: C_def F_def)
        qed
        moreover have ⋂F ≠ {} if finite F F ⊆ C for F
        proof −
          obtain G where sub: G ⊆ {U. openin X U ∧ a ∈ U} and eq: F = F ‘ G
and finite G
          by (metis (no_types, lifting) C_def ‹F ⊆ C› ‹finite F› finite_subset_image)
          then have U ⊆ topspace X if U ∈ G for U
            using openin_subset that by auto
          then have T ⊆ mtopology closure_of T
            if T ∈ (λU. f ‘ (S ∩ U − {a})) ‘ G for T
            using that fim by (fastforce simp add: intro!: closure_of_subset)
          moreover
          have ain: a ∈ ⋂ (insert (topspace X) G) openin X (⋂ (insert (topspace
X) G))
              using True in_derived_set_of sub ‹finite G› by (fastforce intro!:
openin_Inter)+
          then obtain y where y ≠ a y ∈ S and y: y ∈ ⋂ (insert (topspace X) G)
            by (meson ‹a ∈ X derived_set_of S› sub in_derived_set_of)
          then have f y ∈ ⋂F
              using eq that ain fim by (auto simp add: F_def image_subset_iff
in_closure_of)
          then show ?thesis by blast
        qed
        ultimately have ⋂C ≠ {}
          using ‹mcomplete› mcomplete_fip by metis
        then obtain b where b ∈ ⋂C
          by auto
        then have b ∈ M
            using sub_mcball C_clo mbounded_alt_pos mbounded_empty met-
ric_closedin_iff_sequentially_closed by force
        have limitin mtopology f b (atin_within X a S)
        proof (clarsimp simp: limitin_metric ‹b ∈ M›)
          fix ε :: real
          assume ε > 0
          then obtain U where U: openin X U a ∈ U and subU: U ⊆ topspace X
            and Uless: ∀x∈S ∩ U − {a}. ∀y∈S ∩ U − {a}. d (f x) (f y) < ε/2
            by (metis R half_gt_zero openin_subset)
          then obtain x where x: x ∈ S x ∈ U x ≠ a and fx: f x ∈ mball b (ε/2)
            using ‹b ∈ ⋂C›
            apply (simp add: C_def F_def closure_of_def del: divide_const_simps)
            by (metis Diff_iff Int_iff centre_in_mball_iff in_mball openin_mball
singletonI zero_less_numeral)
          moreover
          have d (f y) b < ε if y ∈ U y ≠ a y ∈ S for y
          proof −
```

```
          have d (f x) (f y) < ε/2
            using Uless that x by force
          moreover have d b (f x)  < ε/2
            using fx by simp
          ultimately show ?thesis
              using triangle [of b f x f y] subU that ‹b ∈ M› commute fim fx by
fastforce
        qed
        ultimately show ∀F x in atin_within X a S. f x ∈ M ∧ d (f x) b < ε
          using fim U
          apply (simp add: eventually_atin_within del: divide_const_simps flip:
image_subset_iff_funcset)
          by (smt (verit, del_insts) Diff_iff Int_iff imageI insertI1 openin_subset
subsetD)
       qed
       then show ?thesis ..
     qed
     ultimately
     show ?thesis
       by (meson True ‹M ≠ {}› in_derived_set_of)
   next
     case False
     have (∃l. limitin mtopology f l (atin_within X a S))
         by (metis ‹M ≠ {}› False derived_set_of_trivial_limit equals0I lim-
itin_trivial topspace_mtopology)
      moreover have ∀e>0. ∃U. openin X U ∧ a ∈ U ∧ (∀x∈S ∩ U − {a}.
∀y∈S ∩ U − {a}. d (f x) (f y) < e)
       by (metis Diff_iff False IntE True in_derived_set_of insert_iff)
     ultimately show ?thesis
       using limitin_mspace by blast
   qed
 next
   case False
   then show ?thesis
     by (metis derived_set_of_trivial_limit ex_in_conv in_derived_set_of lim-
itin_mspace limitin_trivial topspace_mtopology)
 qed
qed
```

The HOL Light proof uses some ugly tricks to share common parts of what are two separate proofs for the two cases

```
lemma (in Metric_space) gdelta_in_points_of_convergence_within:
  assumes mcomplete
    and f: continuous_map (subtopology X S) mtopology f ∨ t1_space X ∧ f ∈ S
→ M
  shows gdelta_in X {x ∈ topspace X. ∃l. limitin mtopology f l (atin_within X x
S)}
proof −
  have fim: f ∈ (topspace X ∩ S) → M
```

using *continuous_map_image_subset_topspace f* **by** *force*
show *?thesis*
proof (*cases M={}*)
case *True*
then show *?thesis*
by (*smt* (*verit*) *Collect_cong empty_def empty_iff gdelta_in_empty limitin_mspace*)
next
case *False*
define *A* **where** $A \equiv \{a \in \textit{topspace } X.\ \forall \varepsilon{>}0.\ \exists U.\ \textit{openin } X\ U \wedge a \in U \wedge (\forall x{\in}S \cap U - \{a\}.\ \forall y{\in}S \cap U - \{a\}.\ d\ (f\ x)\ (f\ y) < \varepsilon)\}$
have *gdelta_in X A*
using *f*
proof (*elim disjE conjE*)
assume *cm*: *continuous_map* (*subtopology X S*) *mtopology f*
define *C* **where** $C \equiv \lambda r.\ \bigcup \{U.\ \textit{openin } X\ U \wedge (\forall x \in S{\cap}U.\ \forall y \in S{\cap}U.\ d\ (f\ x)\ (f\ y) < r)\}$
define *B* **where** $B \equiv (\bigcap n.\ C(\textit{inverse}(\textit{Suc } n)))$
define *D* **where** $D \equiv (\bigcap\ (C\ `\ \{0{<}..\}))$
have *D=B*
unfolding *B_def C_def D_def*
apply (*intro Inter_eq_Inter_inverse_Suc Sup_subset_mono*)
by (*smt* (*verit*, *ccfv_threshold*) *Collect_mono_iff*)
have $B \subseteq \textit{topspace } X$
using *openin_subset* **by** (*force simp add*: *B_def C_def*)
have (*countable intersection_of openin X*) *B*
unfolding *B_def C_def*
by (*intro relative_to_inc countable_intersection_of_Inter countable_intersection_of_inc*) *auto*
then have *gdelta_in X B*
unfolding *gdelta_in_def* **by** (*intro relative_to_subset_inc* ‹$B \subseteq \textit{topspace } X$›)
moreover have *A=D*
proof (*intro equalityI subsetI*)
fix *a*
assume *x*: $a \in A$
then have $a \in \textit{topspace } X$
using *A_def* **by** *blast*
show $a \in D$
proof (*clarsimp simp*: *D_def C_def* ‹$a \in \textit{topspace } X$›)
fix ε::*real* **assume** $\varepsilon > 0$
then obtain *U* **where** *openin X U a* $\in$ *U* **and** *U*: $(\forall x{\in}S \cap U - \{a\}.\ \forall y{\in}S \cap U - \{a\}.\ d\ (f\ x)\ (f\ y) < \varepsilon)$
using *x* **by** (*force simp*: *A_def*)
show $\exists T.\ \textit{openin } X\ T \wedge (\forall x{\in}S \cap T.\ \forall y{\in}S \cap T.\ d\ (f\ x)\ (f\ y) < \varepsilon) \wedge a \in T$
proof (*cases* $a \in S$)
case *True*
then obtain *V* **where** *openin X V a* $\in$ *V* **and** *V*: $\forall x.\ x \in S \wedge x \in V$

```
⟶ f a ∈ M ∧ f x ∈ M ∧ d (f a) (f x) < ε
          using ‹a ∈ topspace X› ‹ε > 0› cm
         by (force simp add: continuous_map_to_metric openin_subtopology_alt
Ball_def)
          show ?thesis
          proof (intro exI conjI strip)
           show openin X (U ∩ V)
            using ‹openin X U› ‹openin X V› by blast
           show a ∈ U ∩ V
            using ‹a ∈ U› ‹a ∈ V› by blast
          show ⋀x y. ⟦x ∈ S ∩ (U ∩ V); y ∈ S ∩ (U ∩ V)⟧ ⟹ d (f x) (f y) <
ε
            by (metis DiffI Int_iff U V commute singletonD)
          qed
         next
          case False then show ?thesis
           using U ‹a ∈ U› ‹openin X U› by auto
         qed
        qed
       next
        fix x
        assume x: x ∈ D
        then have x ∈ topspace X
         using ‹B ⊆ topspace X› ‹D=B› by blast
        with x show x ∈ A
         apply (clarsimp simp: D_def C_def A_def)
         by (meson DiffD1 greaterThan_iff)
       qed
       ultimately show ?thesis
        by (simp add: ‹D=B›)
      next
       assume t1_space X f ∈ S → M
       define C where C ≡ λr. ⋃{U. openin X U ∧
                          (∃b ∈ topspace X. ∀x ∈ S∩U − {b}. ∀y ∈ S∩U − {b}. d (f
x) (f y) < r)}
       define B where B ≡ (⋂n. C(inverse(Suc n)))
       define D where D ≡ (⋂ (C ‘ {0<..}))
       have D=B
        unfolding B_def C_def D_def
        apply (intro Inter_eq_Inter_inverse_Suc Sup_subset_mono)
        by (smt (verit, ccfv_threshold) Collect_mono_iff)
       have B ⊆ topspace X
        using openin_subset by (force simp add: B_def C_def)
       have (countable intersection_of openin X) B
        unfolding B_def C_def
       by (intro relative_to_inc countable_intersection_of_Inter countable_intersection_of_inc)
auto
       then have gdelta_in X B
        unfolding gdelta_in_def by (intro relative_to_subset_inc ‹B ⊆ topspace
```

```
X›)
      moreover have A=D
      proof (intro equalityI subsetI)
        fix x
        assume x: x ∈ D
        then have x ∈ topspace X
          using ‹B ⊆ topspace X› ‹D=B› by blast
        show x ∈ A
        proof (clarsimp simp: A_def ‹x ∈ topspace X›)
          fix ε :: real
          assume ε>0
          then obtain U b where openin X U b ∈ topspace X
                and U: ∀ x∈S ∩ U − {b}. ∀ y∈S ∩ U − {b}. d (f x) (f y) < ε and
x ∈ U
            using x by (auto simp: D_def C_def A_def Ball_def)
          then have openin X (U−{b})
            by (meson ‹t1_space X› t1_space_openin_delete_alt)
          then show ∃ U. openin X U ∧ x ∈ U ∧ (∀ xa∈S ∩ U − {x}. ∀ y∈S ∩ U
− {x}. d (f xa) (f y) < ε)
            using U ‹openin X U› ‹x ∈ U› by auto
        qed
      next
        show ⋀x. x ∈ A ⟹ x ∈ D
          unfolding A_def D_def C_def
          by clarsimp meson
      qed
      ultimately show ?thesis
        by (simp add: ‹D=B›)
    qed
    then show ?thesis
      by (simp add: A_def convergent_eq_zero_oscillation_gen False fim ‹mcom-
plete› cong: conj_cong)
  qed
qed


lemma gdelta_in_points_of_convergence_within:
  assumes Y: completely_metrizable_space Y
      and f: continuous_map (subtopology X S) Y f ∨ t1_space X ∧ f ∈ S →
topspace Y
  shows gdelta_in X {x ∈ topspace X. ∃ l. limitin Y f l (atin_within X x S)}
  using assms
  unfolding completely_metrizable_space_def
 using Metric_space.gdelta_in_points_of_convergence_within Metric_space.topspace_mtopology
by fastforce


lemma Lavrentiev_extension_gen:
  assumes S ⊆ topspace X and Y: completely_metrizable_space Y
```

```
    and contf: continuous_map (subtopology X S) Y f
  obtains U g where gdelta_in X U S ⊆ U
          continuous_map (subtopology X (X closure_of S ∩ U)) Y g
          ⋀x. x ∈ S ⟹ g x = f x
proof −
  define U where U ≡ {x ∈ topspace X. ∃ l. limitin Y f l (atin_within X x S)}
  have S ⊆ U
   using that contf limit_continuous_map_within subsetD [OF ‹S ⊆ topspace X›]

    by (fastforce simp: U_def)
  then have S ⊆ X closure_of S ∩ U
    by (simp add: ‹S ⊆ topspace X› closure_of_subset)
  moreover
  have ⋀t. t ∈ X closure_of S ∩ U − S ⟹ ∃ l. limitin Y f l (atin_within X t S)
    using U_def by blast
  moreover have regular_space Y
   by (simp add: Y completely_metrizable_imp_metrizable_space metrizable_imp_regular_space)
  ultimately
  obtain g where g: continuous_map (subtopology X (X closure_of S ∩ U)) Y g
    and gf: ⋀x. x ∈ S ⟹ g x = f x
    using continuous_map_extension_pointwise_alt assms by blast
  show thesis
  proof
    show gdelta_in X U
      by (simp add: U_def Y contf gdelta_in_points_of_convergence_within)
    show continuous_map (subtopology X (X closure_of S ∩ U)) Y g
      by (simp add: g)
  qed (use ‹S ⊆ U› gf in auto)
qed

lemma Lavrentiev_extension:
  assumes S ⊆ topspace X
    and X: metrizable_space X ∨ topspace X ⊆ X closure_of S
    and Y: completely_metrizable_space Y
    and contf: continuous_map (subtopology X S) Y f
  obtains U g where gdelta_in X U S ⊆ U U ⊆ X closure_of S
          continuous_map (subtopology X U) Y g ⋀x. x ∈ S ⟹ g x = f x
proof −
  obtain U g where gdelta_in X U S ⊆ U
    and contg: continuous_map (subtopology X (X closure_of S ∩ U)) Y g
    and gf: ⋀x. x ∈ S ⟹ g x = f x
    using Lavrentiev_extension_gen Y assms(1) contf by blast
  define V where V ≡ X closure_of S ∩ U
  show thesis
  proof
    show gdelta_in X V
     by (metis V_def X ‹gdelta_in X U› closed_imp_gdelta_in closedin_closure_of
closure_of_subset_topspace gdelta_in_Int gdelta_in_topspace subset_antisym)
    show S ⊆ V
```

by (*simp add*: *V_def* ‹*S* ⊆ *U*› *assms*(*1*) *closure_of_subset*)
show *continuous_map* (*subtopology X V*) *Y g*
by (*simp add*: *V_def contg*)
qed (*auto simp*: *gf V_def*)
qed

6.11.19 Embedding in products and hence more about completely metrizable spaces

lemma (**in** *Metric_space*) *gdelta_homeomorphic_space_closedin_product*:
assumes *S*: ⋀*i*. *i* ∈ *I* ⟹ *openin mtopology* (*S i*)
obtains *T* **where** *closedin* (*prod_topology mtopology* (*powertop_real I*)) *T*
subtopology mtopology (⋂ *i* ∈ *I*. *S i*) *homeomorphic_space*
subtopology (*prod_topology mtopology* (*powertop_real I*)) *T*
proof (*cases I*={})
case *True*
then have *top*: *topspace* (*prod_topology mtopology* (*powertop_real I*)) = (*M* × {(λ*x*. *undefined*)})
by *simp*
show *?thesis*
proof
show *closedin* (*prod_topology mtopology* (*powertop_real I*)) (*M* × {(λ*x*. *undefined*)})
by (*metis top closedin_topspace*)
have *subtopology mtopology* (⋂ (*S* ‘ *I*)) *homeomorphic_space mtopology*
by (*simp add*: *True product_topology_empty_discrete*)
also have … *homeomorphic_space* (*prod_topology mtopology* (*discrete_topology* {λ*x*. *undefined*}))
by (*meson homeomorphic_space_sym prod_topology_homeomorphic_space_left*)
finally
show *subtopology mtopology* (⋂ (*S* ‘ *I*)) *homeomorphic_space subtopology* (*prod_topology mtopology* (*powertop_real I*)) (*M* × {(λ*x*. *undefined*)})
by (*smt* (*verit*, *ccfv_SIG*) *True product_topology_empty_discrete subtopology_topspace top*)
qed
next
case *False*
have *SM*: ⋀*i*. *i* ∈ *I* ⟹ *S i* ⊆ *M*
using *assms openin_mtopology* **by** *blast*
then have (⋂ *i* ∈ *I*. *S i*) ⊆ *M*
using *False* **by** *blast*
define *dd* **where** *dd* ≡ λ*i*. *if i*∉*I* ∨ *S i* = *M then* λ*u*. *1 else* (λ*u*. *INF x* ∈ *M* − *S i*. *d u x*)
have [*simp*]: *bdd_below* (*d u* ‘ *A*) **for** *u A*
by (*meson bdd_belowI2 nonneg*)
have *cont_dd*: *continuous_map* (*subtopology mtopology* (*S i*)) *euclidean* (*dd i*) **if** *i* ∈ *I* **for** *i*
proof −
have *dist* (*Inf* (*d x* ‘ (*M* − *S i*))) (*Inf* (*d y* ‘ (*M* − *S i*))) ≤ *d x y*

```
    if x ∈ S i x ∈ M y ∈ S i y ∈ M S i ≠ M for x y
   proof −
    have [simp]: ¬ M ⊆ S i
     using SM ‹S i ≠ M› ‹i ∈ I› by auto
    have ⋀u. ⟦u ∈ M; u ∉ S i⟧ ⟹ Inf (d x ‘ (M − S i)) ≤ d x y + d y u
     apply (clarsimp simp add: cInf_le_iff_less)
     by (smt (verit) DiffI triangle ‹x ∈ M› ‹y ∈ M›)
    then have Inf (d x ‘ (M − S i)) − d x y ≤ Inf (d y ‘ (M − S i))
     by (force simp add: le_cInf_iff)
    moreover
    have ⋀u. ⟦u ∈ M; u ∉ S i⟧ ⟹ Inf (d y ‘ (M − S i)) ≤ d x u + d x y
     apply (clarsimp simp add: cInf_le_iff_less)
     by (smt (verit) DiffI triangle'' ‹x ∈ M› ‹y ∈ M›)
    then have Inf (d y ‘ (M − S i)) − d x y ≤ Inf (d x ‘ (M − S i))
     by (force simp add: le_cInf_iff)
    ultimately show ?thesis
     by (simp add: dist_real_def abs_le_iff)
   qed
  then have ∗: Lipschitz_continuous_map (submetric Self (S i)) euclidean_metric
(λu. Inf (d u ‘ (M − S i)))
    unfolding Lipschitz_continuous_map_def by (force intro!: exI [where x=1])
   then show ?thesis
    using Lipschitz_continuous_imp_continuous_map [OF ∗]
    by (simp add: dd_def Self_def mtopology_of_submetric )
  qed
  have dd_pos: 0 < dd i x if x ∈ S i for i x
  proof (clarsimp simp add: dd_def)
   assume i ∈ I and S i ≠ M
   have opeS: openin mtopology (S i)
    by (simp add: ‹i ∈ I› assms)
   then obtain r where r>0 and r: ⋀y. ⟦y ∈ M; d x y < r⟧ ⟹ y ∈ S i
    by (meson ‹x ∈ S i› in_mball openin_mtopology subsetD)
   then have ⋀y. y ∈ M − S i ⟹ d x y ≥ r
    by (meson Diff_iff linorder_not_le)
   then have Inf (d x ‘ (M − S i)) ≥ r
   by (meson Diff_eq_empty_iff SM ‹S i ≠ M› ‹i ∈ I› cINF_greatest set_eq_subset)
   with ‹r>0› show 0 < Inf (d x ‘ (M − S i)) by simp
  qed
  define f where f ≡ λx. (x, λi∈I. inverse(dd i x))
  define T where T ≡ f ‘ (⋂ i ∈ I. S i)
  show ?thesis
  proof
   show closedin (prod_topology mtopology (powertop_real I)) T
    unfolding closure_of_subset_eq [symmetric]
   proof
    show T ⊆ topspace (prod_topology mtopology (powertop_real I))
     using False SM by (auto simp: T_def f_def)

    have (x,ds) ∈ T
```

```
      if §: ⋀U. ⟦(x, ds) ∈ U; openin (prod_topology mtopology (powertop_real I))
U⟧ ⟹ ∃y∈T. y ∈ U
        and x ∈ M and ds: ds ∈ I →E UNIV for x ds
      proof −
       have ope: ∃x. x ∈ ⋂(S ‘ I) ∧ f x ∈ U × V
        if x ∈ U and ds ∈ V and openin mtopology U and openin (powertop_real
I) V for U V
        using § [of U×V] that by (force simp add: T_def openin_prod_Times_iff)
        have x_in_INT: x ∈ ⋂(S ‘ I)
        proof clarify
         fix i
         assume i ∈ I
         show x ∈ S i
         proof (rule ccontr)
          assume x ∉ S i
           have openin (powertop_real I) {z ∈ topspace (powertop_real I). z i ∈
{ds i − 1 <..< ds i + 1}}
          proof (rule openin_continuous_map_preimage)
           show continuous_map (powertop_real I) euclidean (λx. x i)
            by (metis ‹i ∈ I› continuous_map_product_projection)
          qed auto
          then obtain y where y ∈ S i y ∈ M and dxy: d x y < inverse (|ds i|
+ 1)
                        and intvl: inverse (dd i y) ∈ {ds i − 1 <..< ds i + 1}
         using ope [of mball x (inverse(abs(ds i) + 1)) {z ∈ topspace(powertop_real
I). z i ∈ {ds i − 1<..<ds i + 1}}]
               ‹x ∈ M› ‹ds ∈ I →E UNIV› ‹i ∈ I›
          by (fastforce simp add: f_def)
          have ¬ M ⊆ S i
           using ‹x ∉ S i› ‹x ∈ M› by blast
          have inverse (|ds i| + 1) ≤ dd i y
           using intvl ‹y ∈ S i› dd_pos [of y i]
         by (smt (verit, ccfv_threshold) greaterThanLessThan_iff inverse_inverse_eq
le_imp_inverse_le)
          also have ... ≤ d x y
           using ‹i ∈ I› ‹¬ M ⊆ S i› ‹x ∉ S i› ‹x ∈ M›
           apply (simp add: dd_def cInf_le_iff_less)
           using commute by force
          finally show False
           using dxy by linarith
         qed
        qed
        moreover have ds = (λi∈I. inverse (dd i x))
        proof (rule PiE_ext [OF ds])
         fix i
         assume i ∈ I
         define e where e ≡ |ds i − inverse (dd i x)|
         { assume con: e > 0
         have continuous_map (subtopology mtopology (S i)) euclidean (λx. inverse
```

```
(dd i x))
          using dd_pos cont_dd ‹i ∈ I›
          by (fastforce simp: intro!: continuous_map_real_inverse)
        then have openin (subtopology mtopology (S i))
                  {z ∈ topspace (subtopology mtopology (S i)).
                   inverse (dd i z) ∈ {inverse(dd i x) - e/2<..<inverse(dd i x)
+ e/2}}
          using openin_continuous_map_preimage open_greaterThanLessThan
open_openin by blast
        then obtain U where openin mtopology U
            and U: {z ∈ S i. inverse (dd i x) - e/2 < inverse (dd i z) ∧
                     inverse (dd i z) < inverse (dd i x) + e/2}
                   = U ∩ S i
          using SM ‹i ∈ I›  by (auto simp: openin_subtopology)
        have x ∈ U
          using U x_in_INT ‹i ∈ I› con by fastforce
        have ds ∈ {z ∈ topspace (powertop_real I). z i ∈ {ds i - e / 2<..<ds
i + e/2}}
          by (simp add: con ds)
        moreover
        have  openin (powertop_real I) {z ∈ topspace (powertop_real I). z i ∈
{ds i - e / 2<..<ds i + e/2}}
        proof (rule openin_continuous_map_preimage)
          show continuous_map (powertop_real I) euclidean (λx. x i)
            by (metis ‹i ∈ I› continuous_map_product_projection)
        qed auto
        ultimately obtain y where y ∈ ⋂(S ` I) ∧ f y ∈ U × {z ∈ topspace
(powertop_real I). z i ∈ {ds i - e / 2<..<ds i + e/2}}
          using ope ‹x ∈ U› ‹openin mtopology U› ‹x ∈ U›
          by presburger
        with ‹i ∈ I› U
          have False unfolding set_eq_iff f_def e_def by simp (smt (verit)
field_sum_of_halves)
      }
      then show ds i = (λi∈I. inverse (dd i x)) i
        using ‹i ∈ I› by (force simp: e_def)
    qed auto
    ultimately show ?thesis
      by (auto simp: T_def f_def)
  qed
  then show prod_topology mtopology (powertop_real I) closure_of T ⊆ T
    by (auto simp: closure_of_def)
qed
have eq: (⋂(S ` I) × (I →E UNIV) ∩ f ` (M ∩ ⋂(S ` I))) = (f ` ⋂(S ` I))
  using False SM by (force simp: f_def image_iff)
have continuous_map (subtopology mtopology (⋂(S ` I))) euclidean (dd i) if i
∈ I for i
  by (meson INT_lower cont_dd continuous_map_from_subtopology_mono
that)
```

then have *continuous_map* (*subtopology mtopology* (⋂(*S* ‘ *I*))) (*powertop_real I*) (λ*x*. λ*i*∈*I*. *inverse* (*dd i x*))
using *dd_pos* **by** (*fastforce simp*: *continuous_map_componentwise intro*!: *continuous_map_real_inverse*)
then have *embedding_map* (*subtopology mtopology* (⋂(*S* ‘ *I*))) (*prod_topology* (*subtopology mtopology* (⋂(*S* ‘ *I*))) (*powertop_real I*)) *f*
by (*simp add*: *embedding_map_graph f_def*)
moreover have *subtopology* (*prod_topology* (*subtopology mtopology* (⋂(*S* ‘ *I*))) (*powertop_real I*))
(*f* ‘ *topspace* (*subtopology mtopology* (⋂(*S* ‘ *I*)))) =
subtopology (*prod_topology mtopology* (*powertop_real I*)) *T*
by (*simp add*: *prod_topology_subtopology subtopology_subtopology T_def eq*)
ultimately
show *subtopology mtopology* (⋂(*S* ‘ *I*)) *homeomorphic_space subtopology* (*prod_topology mtopology* (*powertop_real I*)) *T*
by (*metis embedding_map_imp_homeomorphic_space*)
qed
qed

lemma *gdelta_homeomorphic_space_closedin_product*:
assumes *metrizable_space X* **and** ⋀*i*. *i* ∈ *I* ⟹ *openin X* (*S i*)
obtains *T* **where** *closedin* (*prod_topology X* (*powertop_real I*)) *T*
subtopology X (⋂*i* ∈ *I*. *S i*) *homeomorphic_space*
subtopology (*prod_topology X* (*powertop_real I*)) *T*
using *Metric_space.gdelta_homeomorphic_space_closedin_product*
by (*metis assms metrizable_space_def*)

lemma *open_homeomorphic_space_closedin_product*:
assumes *metrizable_space X* **and** *openin X S*
obtains *T* **where** *closedin* (*prod_topology X euclideanreal*) *T*
subtopology X S homeomorphic_space
subtopology (*prod_topology X euclideanreal*) *T*
proof −
obtain *T* **where** *cloT*: *closedin* (*prod_topology X* (*powertop_real* {()})) *T*
and *homT*: *subtopology X S homeomorphic_space*
subtopology (*prod_topology X* (*powertop_real* {()})) *T*
using *gdelta_homeomorphic_space_closedin_product* [*of X* {()} λ*i*. *S*] *assms*
by *auto*
have *prod_topology X* (*powertop_real* {()}) *homeomorphic_space prod_topology X euclideanreal*
by (*meson homeomorphic_space_prod_topology homeomorphic_space_refl homeomorphic_space_singleton_product*)
then obtain *f* **where** *f*: *homeomorphic_map* (*prod_topology X* (*powertop_real* {()})) (*prod_topology X euclideanreal*) *f*
unfolding *homeomorphic_space* **by** *metis*
show *thesis*
proof
show *closedin* (*prod_topology X euclideanreal*) (*f* ‘ *T*)

using *cloT f homeomorphic_map_closedness_eq* **by** *blast*
moreover have T = *topspace* (*subtopology* (*prod_topology X* (*powertop_real* {()})) *T*)
by (*metis cloT closedin_subset topspace_subtopology_subset*)
ultimately show *subtopology X S homeomorphic_space subtopology* (*prod_topology X euclideanreal*) (*f* ‘ *T*)
by (*smt* (*verit*, *best*) *closedin_subset f homT homeomorphic_map_subtopologies homeomorphic_space homeomorphic_space_trans topspace_subtopology topspace_subtopology_subset*)
qed
qed

lemma *completely_metrizable_space_gdelta_in_alt*:
assumes *X*: *completely_metrizable_space X*
and *S*: (*countable intersection_of openin X*) *S*
shows *completely_metrizable_space* (*subtopology X S*)
proof −
obtain $\mathcal{U}$ **where** *countable* $\mathcal{U}$ $S = \bigcap \mathcal{U}$ **and** *ope*: $\bigwedge U.\ U \in \mathcal{U} \Longrightarrow$ *openin X U*
using *S* **by** (*force simp add*: *intersection_of_def*)
then have $\mathcal{U}$: *completely_metrizable_space* (*powertop_real* $\mathcal{U}$)
by (*simp add*: *completely_metrizable_space_euclidean completely_metrizable_space_product_topology*)
obtain *C* **where** *closedin* (*prod_topology X* (*powertop_real* $\mathcal{U}$)) *C*
and *sub*: *subtopology X* ($\bigcap \mathcal{U}$) *homeomorphic_space*
subtopology (*prod_topology X* (*powertop_real* $\mathcal{U}$)) *C*
by (*metis gdelta_homeomorphic_space_closedin_product X completely_metrizable_imp_metrizable_space ope INF_identity_eq*)
moreover have *completely_metrizable_space* (*prod_topology X* (*powertop_real* $\mathcal{U}$))
by (*simp add*: *completely_metrizable_space_prod_topology X* $\mathcal{U}$)
ultimately have *completely_metrizable_space* (*subtopology* (*prod_topology X* (*powertop_real* $\mathcal{U}$)) *C*)
using *completely_metrizable_space_closedin* **by** *blast*
then show *?thesis*
using ‹$S = \bigcap \mathcal{U}$› *sub homeomorphic_completely_metrizable_space* **by** *blast*
qed

lemma *completely_metrizable_space_gdelta_in*:
⟦*completely_metrizable_space X*; *gdelta_in X S*⟧
⟹ *completely_metrizable_space* (*subtopology X S*)
by (*simp add*: *completely_metrizable_space_gdelta_in_alt gdelta_in_alt*)

lemma *completely_metrizable_space_openin*:
⟦*completely_metrizable_space X*; *openin X S*⟧
⟹ *completely_metrizable_space* (*subtopology X S*)
by (*simp add*: *completely_metrizable_space_gdelta_in open_imp_gdelta_in*)

lemma (**in** *Metric_space*) *locally_compact_imp_completely_metrizable_space*:
assumes *locally_compact_space mtopology*

shows *completely_metrizable_space mtopology*
proof −
obtain $f :: ['a,'a] \Rightarrow real$ **and** m' **where**
mcomplete_of m′ **and** *fim*: $f \in M \to mspace\ m'$
and *clo*: *mtopology_of m′ closure_of f ‘ M = mspace m′*
and *d*: $\bigwedge x\ y.\ [\![x \in M;\ y \in M]\!] \Longrightarrow mdist\ m'\ (f\ x)\ (f\ y) = d\ x\ y$
by (*metis metric_completion*)
then have *embedding_map mtopology* (*mtopology_of m′*) *f*
unfolding *mtopology_of_def*
by (*metis Metric_space12.isometry_imp_embedding_map Metric_space12_mspace_mdist mdist_metric mspace_metric*)
then have *hom*: *mtopology homeomorphic_space subtopology* (*mtopology_of m′*) (*f ‘ M*)
by (*metis embedding_map_imp_homeomorphic_space topspace_mtopology*)
have *locally_compact_space* (*subtopology* (*mtopology_of m′*) (*f ‘ M*))
using *assms hom homeomorphic_locally_compact_space* **by** *blast*
moreover have *Hausdorff_space* (*mtopology_of m′*)
by (*simp add*: *Metric_space.Hausdorff_space_mtopology mtopology_of_def*)
ultimately have *openin* (*mtopology_of m′*) (*f ‘ M*)
by (*simp add*: *clo dense_locally_compact_openin_Hausdorff_space fim image_subset_iff_funcset*)
then
have *completely_metrizable_space* (*subtopology* (*mtopology_of m′*) (*f ‘ M*))
using ‹*mcomplete_of m′*› **unfolding** *mcomplete_of_def mtopology_of_def*
by (*metis Metric_space.completely_metrizable_space_mtopology Metric_space_mspace_mdist completely_metrizable_space_openin*)
then show *?thesis*
using *hom homeomorphic_completely_metrizable_space* **by** *blast*
qed

lemma *locally_compact_imp_completely_metrizable_space*:
assumes *metrizable_space X* **and** *locally_compact_space X*
shows *completely_metrizable_space X*
by (*metis Metric_space.locally_compact_imp_completely_metrizable_space assms metrizable_space_def*)

lemma *completely_metrizable_space_imp_gdelta_in*:
assumes *X*: *metrizable_space X* **and** $S \subseteq topspace\ X$
and *XS*: *completely_metrizable_space* (*subtopology X S*)
shows *gdelta_in X S*
proof −
obtain *U f* **where** *gdelta_in X U S* $\subseteq$ *U* **and** *U*: $U \subseteq X$ *closure_of S*
and *contf*: *continuous_map* (*subtopology X U*) (*subtopology X S*) *f*
and *fid*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = x$
using *Lavrentiev_extension*[*of S X subtopology X S id*] *assms* **by** *auto*
then have *f ‘ topspace* (*subtopology X U*) $\subseteq$ *topspace* (*subtopology X S*)
using *continuous_map_image_subset_topspace* **by** *blast*
then have $f`U \subseteq S$

by (*metis* ‹*gdelta_in X U*› ‹*S ⊆ topspace X*› *gdelta_in_subset topspace_subtopology_subset*)
moreover
have *Hausdorff_space* (*subtopology X U*)
by (*simp add*: *Hausdorff_space_subtopology X metrizable_imp_Hausdorff_space*)
then have ⋀*x*. *x* ∈ *U* ⟹ *f x* = *x*
using *U fid contf forall_in_closure_of_eq* [*of _ subtopology X U S subtopology X U f id*]
by (*metis* ‹*S ⊆ U*› *closure_of_subtopology_open continuous_map_id continuous_map_in_subtopology id_apply inf.orderE subtopology_subtopology*)
ultimately have *U ⊆ S*
by *auto*
then show *?thesis*
using ‹*S ⊆ U*› ‹*gdelta_in X U*› **by** *auto*
qed

lemma *completely_metrizable_space_eq_gdelta_in*:
⟦*completely_metrizable_space X*; *S ⊆ topspace X*⟧
⟹ *completely_metrizable_space* (*subtopology X S*) ⟷ *gdelta_in X S*
using *completely_metrizable_imp_metrizable_space completely_metrizable_space_gdelta_in completely_metrizable_space_imp_gdelta_in* **by** *blast*

lemma *gdelta_in_eq_completely_metrizable_space*:
completely_metrizable_space X
⟹ *gdelta_in X S* ⟷ *S ⊆ topspace X* ∧ *completely_metrizable_space* (*subtopology X S*)
by (*metis completely_metrizable_space_eq_gdelta_in gdelta_in_alt*)

6.11.20 Theorems from Kuratowski

Kuratowski, Remark on an Invariance Theorem, *Fundamenta Mathematicae* **37** (1950), pp. 251-252. The idea is that in suitable spaces, to show "number of components of the complement" (without distinguishing orders of infinity) is a homeomorphic invariant, it suffices to show it for closed subsets. Kuratowski states the main result for a "locally connected continuum", and seems clearly to be implicitly assuming that means metrizable. We call out the general topological hypotheses more explicitly, which do not however include connectedness.

lemma *separation_by_closed_intermediates_count*:
assumes *X*: *hereditarily normal_space X*
and *finite* $\mathcal{U}$
and *pwU*: *pairwise* (*separatedin X*) $\mathcal{U}$
and *nonempty*: {} ∉ $\mathcal{U}$
and *UU*: ⋃$\mathcal{U}$ = *topspace X* − *S*
obtains *C* **where** *closedin X C C ⊆ S*
⋀*D*. ⟦*closedin X D*; *C ⊆ D*; *D ⊆ S*⟧
⟹ ∃$\mathcal{V}$. $\mathcal{V}$ ≈ $\mathcal{U}$ ∧ *pairwise* (*separatedin X*) $\mathcal{V}$ ∧ {} ∉ $\mathcal{V}$ ∧ ⋃$\mathcal{V}$ = *topspace X* − *D*
proof −

```
  obtain F where F: ⋀S. S ∈ 𝒰 ⟹ openin X (F S) ∧ S ⊆ F S
    and pwF: pairwise (λS T. disjnt (F S) (F T)) 𝒰
   using assms by (smt (verit, best) Diff_subset Sup_le_iff hereditarily_normal_separation_pairwise)
  show thesis
  proof
    show closedin X (topspace X - ⋃(F ` 𝒰))
      using F by blast
    show topspace X - ⋃(F ` 𝒰) ⊆ S
      using UU F by auto
    show ∃𝒱. 𝒱 ≈ 𝒰 ∧ pairwise (separatedin X) 𝒱 ∧ {} ∉ 𝒱 ∧ ⋃𝒱 = topspace X
- C
      if closedin X C C ⊆ S and C: topspace X - ⋃(F ` 𝒰) ⊆ C for C
    proof (intro exI conjI strip)
      have inj_on (λS. F S - C) 𝒰
        using pwF F
        unfolding inj_on_def pairwise_def disjnt_iff
        by (metis Diff_iff UU UnionI nonempty subset_empty subset_eq ‹C ⊆ S›)
      then show (λS. F S - C) ` 𝒰 ≈ 𝒰
        by simp
      show pairwise (separatedin X) ((λS. F S - C) ` 𝒰)
          using ‹closedin X C› F pwF by (force simp: pairwise_def openin_diff
separatedin_open_sets disjnt_iff)
      show {} ∉ (λS. F S - C) ` 𝒰
        using nonempty UU ‹C ⊆ S› F
        by clarify (metis DiffD2 Diff_eq_empty_iff F UnionI subset_empty sub-
set_eq)
      show (⋃S∈𝒰. F S - C) = topspace X - C
        using UU F C openin_subset by fastforce
    qed
  qed
qed

lemma separation_by_closed_intermediates_gen:
  assumes X: hereditarily normal_space X
    and discon: ¬ connectedin X (topspace X - S)
  obtains C where closedin X C C ⊆ S
                ⋀D. ⟦closedin X D; C ⊆ D; D ⊆ S⟧ ⟹ ¬ connectedin X (topspace
X - D)
proof -
  obtain C1 C2 where Ueq: C1 ∪ C2 = topspace X - S and C1 ≠ {} C2 ≠ {}
    and sep: separatedin X C1 C2 and C1 ≠ C2
    by (metis Diff_subset connectedin_eq_not_separated discon separatedin_refl)
  then obtain C where closedin X C C ⊆ S
    and C: ⋀D. ⟦closedin X D; C ⊆ D; D ⊆ S⟧
                  ⟹ ∃𝒱. 𝒱 ≈ {C1,C2} ∧ pairwise (separatedin X) 𝒱 ∧ {} ∉ 𝒱 ∧
⋃𝒱 = topspace X - D
    using separation_by_closed_intermediates_count [of X {C1,C2} S] X
    apply (simp add: pairwise_insert separatedin_sym)
    by metis
```

```
  have ¬ connectedin X (topspace X − D)
    if D: closedin X D C ⊆ D D ⊆ S for D
  proof −
    obtain V1 V2 where ∗: pairwise (separatedin X) {V1,V2} {} ∉ {V1,V2}
                         ⋃{V1,V2} = topspace X − D V1≠V2
      by (metis C [OF D] ‹C1 ≠ C2› eqpoll_doubleton_iff)
    then have disjnt V1 V2
      by (metis pairwise_insert separatedin_imp_disjoint singleton_iff)
      with ∗ show ?thesis
        by (auto simp add: connectedin_eq_not_separated pairwise_insert)
    qed
  then show thesis
    using ‹C ⊆ S› ‹closedin X C› that by auto
qed

lemma separation_by_closed_intermediates_eq_count:
  fixes n::nat
  assumes lcX: locally_connected_space X and hnX: hereditarily normal_space
X
 shows (∃𝒰. 𝒰 ≈ {..<n} ∧ pairwise (separatedin X) 𝒰 ∧ {} ∉ 𝒰 ∧ ⋃𝒰 = topspace
X − S) ⟷
        (∃ C. closedin X C ∧ C ⊆ S ∧
            (∀ D. closedin X D ∧ C ⊆ D ∧ D ⊆ S
                ⟶ (∃𝒰. 𝒰 ≈ {..<n} ∧ pairwise (separatedin X) 𝒰 ∧ {} ∉ 𝒰 ∧
⋃𝒰 = topspace X − D)))
        (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
   by (smt (verit, best) hnX separation_by_closed_intermediates_count eqpoll_iff_finite_card
eqpoll_trans)
next
  assume R: ?rhs
  show ?lhs
  proof (cases n=0)
    case True
    with R show ?thesis
      by fastforce
  next
    case False
    obtain C where closedin X C C ⊆ S
          and C: ⋀D. ⟦closedin X D; C ⊆ D; D ⊆ S⟧
                 ⟹ ∃𝒰. 𝒰 ≈ {..<n} ∧ pairwise (separatedin X) 𝒰 ∧ {} ∉ 𝒰 ∧
⋃𝒰 = topspace X − D
      using R by force
    then have C ⊆ topspace X
      by (simp add: closedin_subset)
   define 𝒰 where 𝒰 ≡ {D ∈ connected_components_of (subtopology X (topspace
X − C)). D−S ≠ {}}
    have opeU: openin X U if U ∈ 𝒰 for U
```

using *that* ‹*closedin X C*› *lcX locally_connected_space_open_connected_components*

by (*fastforce simp add*: *closedin_def* $\mathcal{U}$*_def*)
have $\{\} \notin \mathcal{U}$
by (*auto simp*: $\mathcal{U}$*_def*)
have *pairwise disjnt* $\mathcal{U}$
using *connected_components_of_disjoint* **by** (*fastforce simp add*: *pairwise_def* $\mathcal{U}$*_def*)
show *?lhs*
proof (*rule ccontr*)
assume *con*: $\nexists\mathcal{U}.\ \mathcal{U} \approx \{..<n\} \wedge$ *pairwise* (*separatedin X*) $\mathcal{U} \wedge \{\} \notin \mathcal{U} \wedge \bigcup\mathcal{U}$ = *topspace X* − *S*
have *card*$\mathcal{U}$: *finite* $\mathcal{U} \wedge$ *card* $\mathcal{U} < n$
proof (*rule ccontr*)
assume ¬ (*finite* $\mathcal{U} \wedge$ *card* $\mathcal{U} < n$)
then obtain $\mathcal{V}$ **where** $\mathcal{V} \subseteq \mathcal{U}$ *finite* $\mathcal{V}$ *card* $\mathcal{V} = n$
by (*metis infinite_arbitrarily_large linorder_not_less obtain_subset_with_card_n*)
then obtain T **where** $T \in \mathcal{V}$
using *False* **by** *force*
define $\mathcal{W}$ **where** $\mathcal{W} \equiv$ *insert* (*topspace X* − *S* − $\bigcup(\mathcal{V} - \{T\})$) ((λ*D*. *D* − *S*) ‘ ($\mathcal{V} - \{T\}$))
have $\bigcup\mathcal{W}$ = *topspace X* − *S*
using ‹$\bigwedge U.\ U \in \mathcal{U} \Longrightarrow$ *openin X U*› ‹$\mathcal{V} \subseteq \mathcal{U}$› *topspace_def* **by** (*fastforce simp*: $\mathcal{W}$*_def*)
moreover have $\{\} \notin \mathcal{W}$
proof −
obtain *a* **where** $a \in T$ $a \notin S$
using $\mathcal{U}$*_def* ‹$T \in \mathcal{V}$› ‹$\mathcal{V} \subseteq \mathcal{U}$› **by** *blast*
then have $a \in$ *topspace X*
using ‹$T \in \mathcal{V}$› *ope*$\mathcal{U}$ ‹$\mathcal{V} \subseteq \mathcal{U}$› *openin_subset* **by** *blast*
moreover have $a \notin \bigcup(\mathcal{V} - \{T\})$
using *diff_Union_pairwise_disjoint* [*of* $\mathcal{V}$ $\{T\}$] ‹*disjoint* $\mathcal{U}$› *pairwise_subset* ‹$T \in \mathcal{V}$› ‹$\mathcal{V} \subseteq \mathcal{U}$› ‹$a \in T$›
by *auto*
ultimately have *topspace X* − *S* − $\bigcup(\mathcal{V} - \{T\}) \neq \{\}$
using ‹$a \notin S$› **by** *blast*
moreover have $\bigwedge V.\ V \in \mathcal{V} - \{T\} \Longrightarrow V - S \neq \{\}$
using $\mathcal{U}$*_def* ‹$\mathcal{V} \subseteq \mathcal{U}$› **by** *blast*
ultimately show *?thesis*
by (*metis* (*no_types*, *lifting*) $\mathcal{W}$*_def image_iff insert_iff*)
qed
moreover have *disjoint* $\mathcal{V}$
using ‹$\mathcal{V} \subseteq \mathcal{U}$› ‹*disjoint* $\mathcal{U}$› *pairwise_subset* **by** *blast*
then have *inj*: *inj_on* (λ*D*. *D* − *S*) ($\mathcal{V} - \{T\}$)
unfolding *inj_on_def* **using** ‹$\mathcal{V} \subseteq \mathcal{U}$› *disjointD* $\mathcal{U}$*_def inf_commute* **by** *blast*
have *finite* $\mathcal{W}$ *card* $\mathcal{W} = n$
using ‹$\{\} \notin \mathcal{W}$› ‹$n \neq 0$› ‹$T \in \mathcal{V}$›
by (*auto simp add*: $\mathcal{W}$*_def* ‹*finite* $\mathcal{V}$› *card_insert_if card_image inj* ‹*card*

$\mathcal{V} = n\rangle$)
moreover have *pairwise* (*separatedin X*) $\mathcal{W}$
proof −
have *disjoint* $\mathcal{W}$
using ‹*disjoint* $\mathcal{V}$› **by** (*auto simp*: $\mathcal{W}$_*def pairwise_def disjnt_iff*)
have *pairwise* (*separatedin* (*subtopology X* (*topspace X* − *S*))) $\mathcal{W}$
proof (*intro pairwiseI*)
fix *A B*
assume §: $A \in \mathcal{W}$ $B \in \mathcal{W}$ $A \neq B$
then have *disjnt A B*
by (*meson* ‹*disjoint* $\mathcal{W}$› *pairwiseD*)
have *closedin* (*subtopology X* (*topspace X* − *C*)) ($\bigcup(\mathcal{V} - \{T\})$)
using $\mathcal{U}$_*def* ‹$\mathcal{V} \subseteq \mathcal{U}$› *closedin_connected_components_of* ‹*finite* $\mathcal{V}$›
by (*force simp add*: *intro*!: *closedin_Union*)
with ‹$C \subseteq S$› **have** *openin* (*subtopology X* (*topspace X* − *S*)) (*topspace X* − *S* − $\bigcup(\mathcal{V} - \{T\})$)
by (*fastforce simp add*: *openin_closedin_eq closedin_subtopology Int_absorb1*)
moreover have $\bigwedge V.\ V \in \mathcal{V} \wedge V \neq T \Longrightarrow$ *openin* (*subtopology X* (*topspace X* − *S*)) (*V* − *S*)
using ‹$\mathcal{V} \subseteq \mathcal{U}$› *ope*$\mathcal{U}$
by (*metis IntD2 Int_Diff inf.orderE openin_subset openin_subtopology*)
ultimately have *openin* (*subtopology X* (*topspace X* − *S*)) *A openin* (*subtopology X* (*topspace X* − *S*)) *B*
using § $\mathcal{W}$_*def* **by** *blast*+
with ‹*disjnt A B*› **show** *separatedin* (*subtopology X* (*topspace X* − *S*)) *A B*
using *separatedin_open_sets* **by** *blast*
qed
then show *?thesis*
by (*simp add*: *pairwise_def separatedin_subtopology*)
qed
ultimately show *False*
by (*metis con eqpoll_iff_finite_card*)
qed
obtain $\mathcal{V}$ **where** $\mathcal{V} \approx \{..<n\}$ $\{\} \notin \mathcal{V}$
and *pw*$\mathcal{V}$: *pairwise* (*separatedin X*) $\mathcal{V}$ **and** *UV*: $\bigcup\mathcal{V}$ = *topspace X* − (*topspace X* − $\bigcup\mathcal{U}$)
proof −
have *closedin X* (*topspace X* − $\bigcup\mathcal{U}$)
using *ope*$\mathcal{U}$ **by** *blast*
moreover have *C* ⊆ *topspace X* − $\bigcup\mathcal{U}$
using ‹*C* ⊆ *topspace X*› *connected_components_of_subset* **by** (*fastforce simp*: $\mathcal{U}$_*def*)
moreover have *topspace X* − $\bigcup\mathcal{U}$ ⊆ *S*
using *Union_connected_components_of* [*of subtopology X* (*topspace X* − *C*)] ‹*C* ⊆ *S*›
by (*auto simp*: $\mathcal{U}$_*def*)

```
      ultimately show thesis
        by (metis C that)
    qed
    have 𝒱 ≲ 𝒰
    proof (rule lepoll_relational_full)
      have ⋃𝒱 = ⋃𝒰
        by (simp add: Sup_le_iff UV double_diff ope𝒰 openin_subset)
      then show ∃ U. U ∈ 𝒰 ∧ ¬ disjnt U V if V ∈ 𝒱 for V
        using that
        by (metis ‹{} ∉ 𝒱› disjnt_Union1 disjnt_self_iff_empty)
      show C1 = C2
        if T ∈ 𝒰 and C1 ∈ 𝒱 and C2 ∈ 𝒱 and ¬ disjnt T C1 and ¬ disjnt T
C2 for T C1 C2
      proof (cases C1=C2)
        case False
        then have connectedin X T
       using 𝒰_def connectedin_connected_components_of connectedin_subtopology
‹T ∈ 𝒰› by blast
        have T ⊆ C1 ∪ ⋃(𝒱 - {C1})
          using ‹⋃𝒱 = ⋃𝒰› ‹T ∈ 𝒰› by auto
        with ‹connectedin X T›
        have ¬ separatedin X C1 (⋃(𝒱 - {C1}))
          unfolding connectedin_eq_not_separated_subset
        by (smt (verit) that False disjnt_def UnionI disjnt_iff insertE insert_Diff)
        with that show ?thesis
              by (metis (no_types, lifting) ‹𝒱 ≈ {..<n}› eqpoll_iff_finite_card
finite_Diff pairwiseD pairwise_alt pw𝒱 separatedin_Union(1) separatedin_def)
      qed auto
    qed
    then show False
     by (metis ‹𝒱 ≈ {..<n}› card𝒰 eqpoll_iff_finite_card leD lepoll_iff_card_le)
  qed
 qed
qed

lemma separation_by_closed_intermediates_eq_gen:
 assumes locally_connected_space X hereditarily normal_space X
 shows ¬ connectedin X (topspace X - S) ⟷
       (∃ C. closedin X C ∧ C ⊆ S ∧
            (∀ D. closedin X D ∧ C ⊆ D ∧ D ⊆ S ⟶ ¬ connectedin X (topspace
X - D)))
   (is ?lhs = ?rhs)
proof -
 have *: (∃𝒰::'a set set. 𝒰 ≈ {..<Suc (Suc 0)} ∧ P 𝒰) ⟷ (∃ A B. A≠B ∧
P{A,B}) for P
  by (metis One_nat_def eqpoll_doubleton_iff lessThan_Suc lessThan_empty_iff
zero_neq_one)
 have *: (∃ C1 C2. separatedin X C1 C2 ∧ C1≠C2 ∧ C1≠{} ∧ C2≠{} ∧ C1 ∪
C2 = topspace X - S) ⟷
```

```
        (∃ C. closedin X C ∧ C ⊆ S ∧
            (∀ D. closedin X D ∧ C ⊆ D ∧ D ⊆ S
                ⟶ (∃ C1 C2. separatedin X C1 C2 ∧ C1≠C2 ∧ C1≠{} ∧ C2≠{}
∧ C1 ∪ C2 = topspace X − D)))
    using separation_by_closed_intermediates_eq_count [OF assms, of Suc(Suc
0) S]
    apply (simp add: ∗ pairwise_insert separatedin_sym cong: conj_cong)
    apply (simp add: eq_sym_conv conj_ac)
    done
  with separatedin_refl
  show ?thesis
    apply (simp add: connectedin_eq_not_separated)
    by (smt (verit, best) separatedin_refl)
qed



lemma lepoll_connnected_components_connectedin:
  assumes ⋀C. C ∈ 𝒰 ⟹ connectedin X C ⋃𝒰 = topspace X
  shows connected_components_of X ≲ 𝒰
proof −
  have connected_components_of X ≲ 𝒰 − {{}}
  proof (rule lepoll_relational_full)
    show ∃ U. U ∈ 𝒰 − {{}} ∧ U ⊆ V
      if V ∈ connected_components_of X for V
      using that unfolding connected_components_of_def image_iff
        by (metis Union_iff assms connected_component_of_maximal empty_iff
insert_Diff_single insert_iff)
    show V = V'
    if U ∈ 𝒰 − {{}} V ∈ connected_components_of X V' ∈ connected_components_of
X U ⊆ V U ⊆ V'
      for U V V'
    by (metis DiffD2 disjointD insertCI le_inf_iff pairwise_disjoint_connected_components_of
subset_empty that)
  qed
  also have ... ≲ 𝒰
    by (simp add: subset_imp_lepoll)
  finally show ?thesis .
qed

lemma lepoll_connected_components_alt:
  {..<n::nat} ≲ connected_components_of X ⟷
    n = 0 ∨ (∃𝒰. 𝒰 ≈ {..<n} ∧ pairwise (separatedin X) 𝒰 ∧ {} ∉ 𝒰 ∧ ⋃𝒰 =
topspace X)
  (is ?lhs ⟷ ?rhs)
proof (cases n=0)
next
  case False
  show ?thesis
```

```
proof
  assume L: ?lhs
  with False show ?rhs
  proof (induction n rule: less_induct)
    case (less n)
    show ?case
    proof (cases n≤1)
      case True
      with less.prems have topspace X ≠ {} n=1
        by (fastforce simp add: connected_components_of_def)+
      then have {} ∉ {topspace X}
        by blast
      with ‹n=1› show ?thesis
        by (simp add: eqpoll_iff_finite_card card_Suc_eq flip: ex_simps)
    next
      case False
      then have n−1 ≠ 0
        by linarith
      have n1_lesspoll: {..<n−1} ≺ {..<n}
        using False lesspoll_iff_finite_card by fastforce
      also have ... ≲ connected_components_of X
        using less by blast
      finally have {..<n−1} ≲ connected_components_of X
        using lesspoll_imp_lepoll by blast
      then obtain 𝒰 where Ueq: 𝒰 ≈ {..<n−1} and {} ∉ 𝒰
        and pwU: pairwise (separatedin X) 𝒰 and UU: ⋃𝒰 = topspace X
        by (meson ‹n − 1 ≠ 0› diff_less gr0I less zero_less_one)
      show ?thesis
      proof (cases ∀ C ∈ 𝒰. connectedin X C)
        case True
        then show ?thesis
          using lepoll_connnected_components_connectedin [of 𝒰 X] less.prems
          by (metis UU Ueq lepoll_antisym lepoll_trans lepoll_trans2 lesspoll_def
n1_lesspoll)
        next
          case False
          with UU obtain C A B where ABC: C ∈ 𝒰 A ∪ B = C A ≠ {} B ≠
{} and sep: separatedin X A B
            by (fastforce simp add: connectedin_eq_not_separated)
          define 𝒱 where 𝒱 ≡ insert A (insert B (𝒰 − {C}))
          have 𝒱 ≈ {..<n}
          proof −
            have A ≠ B
              using ‹B ≠ {}› sep by auto
            moreover obtain A ∉ 𝒰 B ∉ 𝒰
              using pwU unfolding pairwise_def
                by (metis ABC sep separatedin_Un(1) separatedin_refl separate-
din_sym)
            moreover have card 𝒰 = n−1 finite 𝒰
```

```
            using Ueq eqpoll_iff_finite_card by blast+
          ultimately
          have card (insert A (insert B (𝒰 − {C}))) = n
            using ‹C ∈ 𝒰› by (auto simp add: card_insert_if)
          then show ?thesis
            using 𝒱_def ‹finite 𝒰› eqpoll_iff_finite_card by blast
        qed
        moreover have {} ∉ 𝒱
          using ABC 𝒱_def ‹{} ∉ 𝒰› by blast
        moreover have ⋃𝒱 = topspace X
          using ABC UU 𝒱_def by auto
        moreover have pairwise (separatedin X) 𝒱
          using pwU sep ABC unfolding 𝒱_def
          apply (simp add: separatedin_sym pairwise_def)
          by (metis member_remove remove_def separatedin_Un(1))
        ultimately show ?thesis
          by blast
      qed
    qed
  qed
 next
  assume ?rhs
  then obtain 𝒰 where 𝒰 ≈ {..<n} {} ∉ 𝒰 and pwU: pairwise (separatedin X)
𝒰 and UU: ⋃𝒰 = topspace X
    using False by force
  have card (connected_components_of X) ≥ n if finite (connected_components_of
X)
  proof −
    have 𝒰 ≲ connected_components_of X
    proof (rule lepoll_relational_full)
      show ∃T. T ∈ connected_components_of X ∧ ¬ disjnt T C if C ∈ 𝒰 for
C
        by (metis that UU Union_connected_components_of Union_iff ‹{} ∉ 𝒰›
disjnt_iff equals0I)
      show (C1::'a set) = C2
        if T ∈ connected_components_of X and C1 ∈ 𝒰 C2 ∈ 𝒰 ¬ disjnt T C1
¬ disjnt T C2 for T C1 C2
      proof (rule ccontr)
        assume C1 ≠ C2
        then have connectedin X T
          by (simp add: connectedin_connected_components_of that(1))
        moreover have ¬ separatedin X C1 (⋃(𝒰 − {C1}))
          using ‹connectedin X T› pwU unfolding pairwise_def
          by (smt (verit) Sup_upper UU Union_connected_components_of ‹C1 ≠
C2› complete_lattice_class.Sup_insert connectedin_subset_separated_union dis-
jnt_subset2 disjnt_sym insert_Diff separatedin_imp_disjoint that)
        ultimately show False
          using ‹𝒰 ≈ {..<n}›
        apply (simp add: connectedin_eq_not_separated_subset eqpoll_iff_finite_card)
```

by (*metis Sup_upper UU finite_Diff pairwise_alt pwU separate-din_Union*(*1*) *that*(*2*))
qed
qed
then show *?thesis*
by (*metis* ‹$\mathcal{U} \approx \{..<n\}$› *eqpoll_iff_finite_card lepoll_iff_card_le that*)
qed
then show *?lhs*
by (*metis card_lessThan finite_lepoll_infinite finite_lessThan lepoll_iff_card_le*)
qed
qed *auto*

6.11.21 A perfect set in common cases must have at least the cardinality of the continuum

lemma (**in** *Metric_space*) *lepoll_perfect_set*:
assumes *mcomplete*
and *mtopology derived_set_of S* = *S S* ≠ {}
shows (*UNIV*::*real set*) ≲ *S*
proof −
have *S* ⊆ *M*
using *assms*(*2*) *derived_set_of_infinite_mball* **by** *blast*
have (*UNIV*::*real set*) ≲ (*UNIV*::*nat set set*)
using *eqpoll_imp_lepoll eqpoll_sym nat_sets_eqpoll_reals* **by** *blast*
also have ... ≲ *S*
proof −
have ∃ *y z* δ. *y* ∈ *S* ∧ *z* ∈ *S* ∧ 0 < δ ∧ δ < ε/2 ∧
mcball y δ ⊆ *mcball x* ε ∧ *mcball z* δ ⊆ *mcball x* ε ∧ *disjnt* (*mcball y* δ) (*mcball z* δ)
if *x* ∈ *S* 0 < ε **for** *x* ε
proof −
define *S'* **where** *S'* ≡ *S* ∩ *mball x* (ε/*4*)
have *infinite S'*
using *derived_set_of_infinite_mball* [*of S*] *assms that S'_def*
by (*smt* (*verit*, *ccfv_SIG*) *mem_Collect_eq zero_less_divide_iff*)
then have ⋀*x y z*. ¬ (*S'* ⊆ {*x*,*y*,*z*})
using *finite_subset* **by** *auto*
then obtain *l r* **where** *lr*: *l* ∈ *S' r* ∈ *S' l*≠*r l*≠*x r*≠*x*
by (*metis insert_iff subsetI*)
show *?thesis*
proof (*intro exI conjI*)
show *l* ∈ *S r* ∈ *S d l r* / *3* > *0*
using *lr* **by** (*auto simp*: *S'_def*)
show *d l r* / *3* < ε/*2 mcball l* (*d l r* / *3*) ⊆ *mcball x* ε *mcball r* (*d l r* / *3*) ⊆ *mcball x* ε
using *lr* **by** (*clarsimp simp*: *S'_def*, *smt* (*verit*) *commute triangle''*)+
show *disjnt* (*mcball l* (*d l r* / *3*)) (*mcball r* (*d l r* / *3*))
using *lr* **by** (*simp add*: *S'_def disjnt_iff*) (*smt* (*verit*, *best*) *mdist_pos_less triangle'*)

qed
qed
then obtain $l\ r\ \delta$
where *lrS*: $\bigwedge x\ \varepsilon.\ [\![x \in S;\ 0 < \varepsilon]\!] \Longrightarrow l\ x\ \varepsilon \in S \wedge r\ x\ \varepsilon \in S$
and δ: $\bigwedge x\ \varepsilon.\ [\![x \in S;\ 0 < \varepsilon]\!] \Longrightarrow 0 < \delta\ x\ \varepsilon \wedge \delta\ x\ \varepsilon < \varepsilon/2$
and $\bigwedge x\ \varepsilon.\ [\![x \in S;\ 0 < \varepsilon]\!] \Longrightarrow$ *mcball* $(l\ x\ \varepsilon)\ (\delta\ x\ \varepsilon) \subseteq$ *mcball* $x\ \varepsilon \wedge$ *mcball* $(r\ x\ \varepsilon)\ (\delta\ x\ \varepsilon) \subseteq$ *mcball* $x\ \varepsilon \wedge$ *disjnt* (*mcball* $(l\ x\ \varepsilon)\ (\delta\ x\ \varepsilon)$) (*mcball* $(r\ x\ \varepsilon)\ (\delta\ x\ \varepsilon)$)
by *metis*
then have *lr_mcball*: $\bigwedge x\ \varepsilon.\ [\![x \in S;\ 0 < \varepsilon]\!] \Longrightarrow$ *mcball* $(l\ x\ \varepsilon)\ (\delta\ x\ \varepsilon) \subseteq$ *mcball* $x\ \varepsilon \wedge$ *mcball* $(r\ x\ \varepsilon)\ (\delta\ x\ \varepsilon) \subseteq$ *mcball* $x\ \varepsilon$
and *lr_disjnt*: $\bigwedge x\ \varepsilon.\ [\![x \in S;\ 0 < \varepsilon]\!] \Longrightarrow$ *disjnt* (*mcball* $(l\ x\ \varepsilon)\ (\delta\ x\ \varepsilon)$) (*mcball* $(r\ x\ \varepsilon)\ (\delta\ x\ \varepsilon)$)
by *metis+*
obtain a **where** $a \in S$
using ‹$S \neq \{\}$› **by** *blast*
define *xe* **where** *xe* $\equiv$
$\lambda B.$ *rec_nat* $(a,1)$ $(\lambda n\ (x,\gamma).$ $((\textit{if}\ n{\in}B\ \textit{then}\ r\ \textit{else}\ l)\ x\ \gamma,\ \delta\ x\ \gamma))$
have [*simp*]: *xe* $b\ 0 = (a,1)$ **for** b
by (*simp add*: *xe_def*)
have *xe* B (*Suc* n) = (*let* (x,γ) = *xe* $B\ n$ *in* $((\textit{if}\ n{\in}B\ \textit{then}\ r\ \textit{else}\ l)\ x\ \gamma,\ \delta\ x\ \gamma)$) **for** $B\ n$
by (*simp add*: *xe_def*)
define x **where** $x \equiv \lambda B\ n.$ *fst* (*xe* $B\ n$)
define γ **where** $\gamma \equiv \lambda B\ n.$ *snd* (*xe* $B\ n$)
have [*simp*]: $x\ B\ 0 = a\ \gamma\ B\ 0 = 1$ **for** B
by (*simp_all add*: *x_def* γ*_def xe_def*)
have *x_Suc*[*simp*]: $x\ B$ (*Suc* n) = $((\textit{if}\ n{\in}B\ \textit{then}\ r\ \textit{else}\ l)\ (x\ B\ n)\ (\gamma\ B\ n))$
and γ*_Suc*[*simp*]: $\gamma\ B$ (*Suc* n) = $\delta\ (x\ B\ n)\ (\gamma\ B\ n)$ **for** $B\ n$
by (*simp_all add*: *x_def* γ*_def xe_def split*: *prod.split*)
interpret *Submetric M d S*
proof qed (*use* ‹$S \subseteq M$› **in** *metis*)
have *closedin mtopology S*
by (*metis assms*(2) *closure_of closure_of_eq inf.absorb_iff2 subset subset_Un_eq subset_refl topspace_mtopology*)
with ‹*mcomplete*›
have *sub.mcomplete*
by (*metis closedin_mcomplete_imp_mcomplete*)
have $*$: $x\ B\ n \in S \wedge \gamma\ B\ n > 0$ **for** $B\ n$
by (*induction n*) (*auto simp*: ‹$a \in S$› *lrS* δ)
with *subset* **have** *E*: $x\ B\ n \in M$ **for** $B\ n$
by *blast*
have γ*_le*: $\gamma\ B\ n \leq (1/2)\hat{}n$ **for** $B\ n$
proof(*induction n*)
case *0* **then show** *?case* **by** *auto*
next
case (*Suc n*)
then show *?case*
by *simp* (*smt* (*verit*) $*$ δ *field_sum_of_halves*)

```
  qed
  { fix B
   have ⋀n. sub.mcball (x B (Suc n)) (γ B (Suc n)) ⊆ sub.mcball (x B n) (γ B
n)
    by (smt (verit, best) * Int_iff γ_Suc x_Suc in_mono lr_mcball mcball_submetric_eq
subsetI)
     then have mon: monotone (≤) (λx y. y ⊆ x) (λn. sub.mcball (x B n) (γ B
n))
      by (simp add: decseq_SucI)
     have ∃ n a. sub.mcball (x B n) (γ B n) ⊆ sub.mcball a ε if ε>0 for ε
     proof −
      obtain n where (1/2)^n < ε
       using ‹0 < ε› real_arch_pow_inv by force
      with γ_le have ε: γ B n ≤ ε
       by (smt (verit))
      show ?thesis
      proof (intro exI)
       show sub.mcball (x B n) (γ B n) ⊆ sub.mcball (x B n) ε
        by (simp add: ε sub.mcball_subset_concentric)
      qed
     qed
     then have ∃ l. l ∈ S ∧ (⋂ n. sub.mcball (x B n) (γ B n)) = {l}
      using ‹sub.mcomplete› mon
      unfolding sub.mcomplete_nest_sing
      apply (drule_tac x=λn. sub.mcball (x B n) (γ B n) in spec)
      by (meson * order.asym sub.closedin_mcball sub.mcball_eq_empty)
  }
  then obtain z where z: ⋀B. z B ∈ S ∧ (⋂ n. sub.mcball (x B n) (γ B n)) =
{z B}
    by metis
  show ?thesis
   unfolding lepoll_def
  proof (intro exI conjI)
   show inj z
   proof (rule inj_onCI)
    fix B C
    assume eq: z B = z C and B ≠ C
    then have ne: sym_diff B C ≠ {}
     by blast
    define n where n ≡ LEAST k. k ∈ (sym_diff B C)
    with ne have n: n ∈ sym_diff B C
     by (metis Inf_nat_def1 LeastI)
    then have non: n ∈ B ⟷ n ∉ C
     by blast
    have H: z C ∈ sub.mcball (x B (Suc n)) (γ B (Suc n)) ∧ z C ∈ sub.mcball
(x C (Suc n)) (γ C (Suc n))
     using z [of B] z [of C] apply (simp add: lrS set_eq_iff non *)
     by (smt (verit, best) γ_Suc eq non x_Suc)
    have k ∈ B ⟷ k ∈ C if k<n for k
```

```
        using that unfolding n_def by (meson DiffI UnCI not_less_Least)
      moreover have (∀m. m < p ⟶ (m ∈ B ⟷ m ∈ C)) ⟹ x B p = x C
p ∧ γ B p = γ C p for p
        by (induction p) auto
      ultimately have x B n = x C n γ B n = γ C n
        by blast+
      then show False
        using lr_disjnt ∗ H non
        by (smt (verit) IntD2 γ_Suc disjnt_iff mcball_submetric_eq x_Suc)
    qed
    show range z ⊆ S
      using z by blast
  qed
 qed
 finally show ?thesis .
qed

lemma lepoll_perfect_set_aux:
 assumes lcX: locally_compact_space X and hsX: Hausdorff_space X
   and eq: X derived_set_of topspace X = topspace X and topspace X ≠ {}
 shows (UNIV::real set) ≲ topspace X
proof −
 have (UNIV::real set) ≲ (UNIV::nat set set)
   using eqpoll_imp_lepoll eqpoll_sym nat_sets_eqpoll_reals by blast
 also have ... ≲ topspace X
 proof −
   obtain z where z: z ∈ topspace X
     using assms by blast
   then obtain U K where openin X U compactin X K U ≠ {} U ⊆ K
     by (metis emptyE lcX locally_compact_space_def)
   then have closedin X K
     by (simp add: compactin_imp_closedin hsX)
   have intK_ne: X interior_of K ≠ {}
     using ‹U ≠ {}› ‹U ⊆ K› ‹openin X U› interior_of_eq_empty by blast
   have ∃D E. closedin X D ∧ D ⊆ K ∧ X interior_of D ≠ {} ∧
              closedin X E ∧ E ⊆ K ∧ X interior_of E ≠ {} ∧
              disjnt D E ∧ D ⊆ C ∧ E ⊆ C
     if closedin X C C ⊆ K and C: X interior_of C ≠ {} for C
   proof −
     obtain z where z: z ∈ X interior_of C z ∈ topspace X
       using C interior_of_subset_topspace by fastforce
     obtain x y where x ∈ X interior_of C y ∈ X interior_of C x≠y
       by (metis z eq in_derived_set_of openin_interior_of)
     then have x ∈ topspace X y ∈ topspace X
       using interior_of_subset_topspace by force+
     with hsX obtain V W where openin X V openin X W x ∈ V y ∈ W disjnt
V W
       by (metis Hausdorff_space_def ‹x ≠ y›)
     have ∗: ⋀W x. openin X W ∧ x ∈ W
```

```
        ⟹ ∃ U V. openin X U ∧ closedin X V ∧ x ∈ U ∧ U ⊆ V ∧ V ⊆ W
       using lcX hsX locally_compact_Hausdorff_imp_regular_space neighbourhood_base_of_closedin neighbourhood_base_of
       by metis
     obtain M D where MD: openin X M closedin X D y ∈ M M ⊆ D D ⊆ X interior_of C ∩ W
       using ∗ [of X interior_of C ∩ W y]
       using ‹openin X W› ‹y ∈ W› ‹y ∈ X interior_of C› by fastforce
     obtain N E where NE: openin X N closedin X E x ∈ N N ⊆ E E ⊆ X interior_of C ∩ V
       using ∗ [of X interior_of C ∩ V x]
       using ‹openin X V› ‹x ∈ V› ‹x ∈ X interior_of C› by fastforce
     show ?thesis
     proof (intro exI conjI)
       show X interior_of D ≠ {} X interior_of E ≠ {}
         using MD NE by (fastforce simp: interior_of_def)+
       show disjnt D E
           by (meson MD(5) NE(5) ‹disjnt V W› disjnt_subset1 disjnt_sym le_inf_iff)
     qed (use MD NE ‹C ⊆ K› interior_of_subset in force)+
   qed
   then obtain L R where
    LR: ⋀C. ⟦closedin X C; C ⊆ K; X interior_of C ≠ {}⟧
     ⟹ closedin X (L C) ∧ (L C) ⊆ K ∧ X interior_of (L C) ≠ {} ∧
           closedin X (R C) ∧ (R C) ⊆ K ∧ X interior_of (R C) ≠ {}
    and disjLR: ⋀C. ⟦closedin X C; C ⊆ K; X interior_of C ≠ {}⟧
     ⟹ disjnt (L C) (R C) ∧ (L C) ⊆ C ∧ (R C) ⊆ C
     by metis
   define d where d ≡ λB. rec_nat K (λn. if n ∈ B then R else L)
   have d0[simp]: d B 0 = K for B
     by (simp add: d_def)
   have [simp]: d B (Suc n) = (if n ∈ B then R else L) (d B n) for B n
     by (simp add: d_def)
   have d_correct: closedin X (d B n) ∧ d B n ⊆ K ∧ X interior_of (d B n) ≠ {} for B n
   proof (induction n)
     case 0
     then show ?case by (auto simp: ‹closedin X K› intK_ne)
   next
     case (Suc n) with LR show ?case by auto
   qed
   have (⋂ n. d B n) ≠ {} for B
   proof (rule compact_space_imp_nest)
     show compact_space (subtopology X K)
       by (simp add: ‹compactin X K› compact_space_subtopology)
     show closedin (subtopology X K) (d B n) for n :: nat
       by (simp add: closedin_subset_topspace d_correct)
     show d B n ≠ {} for n :: nat
       by (metis d_correct interior_of_empty)
```

```
      show antimono (d B)
      proof (rule antimonoI [OF transitive_stepwise_le])
        fix n
        show d B (Suc n) ⊆ d B n
        by (simp add: d_correct disjLR)
      qed auto
    qed
    then obtain x where x: ⋀B. x B ∈ (⋂n. d B n)
      unfolding set_eq_iff by (metis empty_iff)
    show ?thesis
      unfolding lepoll_def
    proof (intro exI conjI)
      show inj x
      proof (rule inj_onCI)
        fix B C
        assume eq: x B = x C and B≠C
        then have ne: sym_diff B C ≠ {}
          by blast
        define n where n ≡ LEAST k. k ∈ (sym_diff B C)
        with ne have n: n ∈ sym_diff B C
          by (metis Inf_nat_def1 LeastI)
        then have non: n ∈ B ⟷ n ∉ C
          by blast
        have k ∈ B ⟷ k ∈ C if k<n for k
          using that unfolding n_def by (meson DiffI UnCI not_less_Least)
        moreover have (∀ m. m < p ⟶ (m ∈ B ⟷ m ∈ C)) ⟹ d B p = d C
p for p
          by (induction p) auto
        ultimately have d B n = d C n
          by blast
        then have disjnt (d B (Suc n)) (d C (Suc n))
          by (simp add: d_correct disjLR disjnt_sym non)
        then show False
          by (metis InterE disjnt_iff eq rangeI x)
      qed
      show range x ⊆ topspace X
          using x d0 ‹compactin X K› compactin_subset_topspace d_correct by
fastforce
    qed
  qed
  finally show ?thesis .
qed

lemma lepoll_perfect_set:
  assumes X: completely_metrizable_space X ∨ locally_compact_space X ∧ Haus-
dorff_space X
    and S: X derived_set_of S = S S ≠ {}
  shows (UNIV::real set) ≲ S
  using X
```

proof
assume *completely_metrizable_space X*
with *assms* **show** (*UNIV*::*real set*) $\lesssim$ *S*
by (*metis Metric_space.lepoll_perfect_set completely_metrizable_space_def*)
next
assume *locally_compact_space X* $\wedge$ *Hausdorff_space X*
then show (*UNIV*::*real set*) $\lesssim$ *S*
using *lepoll_perfect_set_aux* [*of subtopology X S*]
by (*metis Hausdorff_space_subtopology S closedin_derived_set_of closedin_subset derived_set_of_subtopology*
locally_compact_space_closed_subset subtopology_topspace topspace_subtopology topspace_subtopology_subset)
qed

lemma *Kuratowski_aux1*:
assumes $\bigwedge S\ T.\ R\ S\ T \Longrightarrow R\ T\ S$
shows $(\forall\, S\ T\ n.\ R\ S\ T \longrightarrow (f\ S \approx \{..<n::nat\} \longleftrightarrow f\ T \approx \{..<n::nat\})) \longleftrightarrow$
$(\forall\, n\ S\ T.\ R\ S\ T \longrightarrow \{..<n::nat\} \lesssim f\ S \longrightarrow \{..<n::nat\} \lesssim f\ T)$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*meson eqpoll_iff_finite_card eqpoll_sym finite_lepoll_infinite finite_lessThan lepoll_trans2*)
next
assume *?rhs* **then show** *?lhs*
by (*smt* (*verit*, *best*) *lepoll_iff_finite_card assms eqpoll_iff_finite_card finite_lepoll_infinite*
finite_lessThan le_Suc_eq lepoll_antisym lepoll_iff_card_le not_less_eq_eq)
qed

lemma *Kuratowski_aux2*:
pairwise (*separatedin* (*subtopology X* (*topspace X* $-$ *S*))) $\mathcal{U} \wedge \{\} \notin \mathcal{U} \wedge$
$\bigcup\mathcal{U}$ = *topspace*(*subtopology X* (*topspace X* $-$ *S*)) $\longleftrightarrow$
pairwise (*separatedin X*) $\mathcal{U} \wedge \{\} \notin \mathcal{U} \wedge \bigcup\mathcal{U}$ = *topspace X* $-$ *S*
by (*auto simp*: *pairwise_def separatedin_subtopology*)

proposition *Kuratowski_component_number_invariance_aux*:
assumes *compact_space X* **and** *HsX*: *Hausdorff_space X*
and *lcX*: *locally_connected_space X* **and** *hnX*: *hereditarily normal_space X*
and *hom*: (*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
and *leXS*: {..<*n*::*nat*} $\lesssim$ *connected_components_of* (*subtopology X* (*topspace X* $-$ *S*))
assumes §: $\bigwedge S\ T.$
$[\![$*closedin X S*; *closedin X T*; (*subtopology X S*) *homeomorphic_space* (*subtopology X T*);
{..<*n*::*nat*} $\lesssim$ *connected_components_of* (*subtopology X* (*topspace X*

```
− S))⟧
          ⟹ {..<n::nat} ≲ connected_components_of (subtopology X (topspace
X − T))
  shows {..<n::nat} ≲ connected_components_of (subtopology X (topspace X −
T))
proof (cases n=0)
  case False
  obtain f g where homf: homeomorphic_map (subtopology X S) (subtopology X
T) f
      and homg: homeomorphic_map (subtopology X T) (subtopology X S) g
    and gf: ⋀x. x ∈ topspace (subtopology X S) ⟹ g(f x) = x
    and fg: ⋀y. y ∈ topspace (subtopology X T) ⟹ f(g y) = y
    and f: f ∈ topspace (subtopology X S) → topspace (subtopology X T)
    and g: g ∈ topspace (subtopology X T) → topspace (subtopology X S)
    using homeomorphic_space_unfold hom by metis
  obtain C where closedin X C C ⊆ S
     and C: ⋀D. ⟦closedin X D; C ⊆ D; D ⊆ S⟧
          ⟹ ∃𝒰. 𝒰 ≈ {..<n} ∧ pairwise (separatedin X) 𝒰 ∧ {} ∉ 𝒰 ∧ ⋃𝒰 =
topspace X − D
    using separation_by_closed_intermediates_eq_count [of X n S] assms
   by (smt (verit, ccfv_threshold) False Kuratowski_aux2 lepoll_connected_components_alt)
  have ∃ C. closedin X C ∧ C ⊆ T ∧
         (∀ D. closedin X D ∧ C ⊆ D ∧ D ⊆ T
             ⟶ (∃𝒰. 𝒰 ≈ {..<n} ∧ pairwise (separatedin X) 𝒰 ∧
                    {} ∉ 𝒰 ∧ ⋃𝒰 = topspace X − D))
  proof (intro exI, intro conjI strip)
    have compactin X (f ‘ C)
      by (meson ‹C ⊆ S› ‹closedin X C› assms(1) closedin_compact_space com-
pactin_subtopology homeomorphic_map_compactness_eq homf)
    then show closedin X (f ‘ C)
      using ‹Hausdorff_space X› compactin_imp_closedin by blast
    show f ‘ C ⊆ T
    by (meson ‹C ⊆ S› ‹closedin X C› closedin_imp_subset closedin_subset_topspace
homeomorphic_map_closedness_eq homf)
    fix D′
    assume D′: closedin X D′ ∧ f ‘ C ⊆ D′ ∧ D′ ⊆ T
    define D where D ≡ g ‘ D′
    have compactin X D
      unfolding D_def
    by (meson D′ ‹compact_space X› closedin_compact_space compactin_subtopology
homeomorphic_map_compactness_eq homg)
    then have closedin X D
      by (simp add: assms(2) compactin_imp_closedin)
    moreover have C ⊆ D
      using D′ D_def ‹C ⊆ S› ‹closedin X C› closedin_subset gf image_iff by
fastforce
    moreover have D ⊆ S
     by (metis D′ D_def assms(1) closedin_compact_space compactin_subtopology
homeomorphic_map_compactness_eq homg)
```

ultimately obtain $\mathcal{U}$ **where** $\mathcal{U} \approx \{..<n\}$ *pairwise* (*separatedin X*) $\mathcal{U}$ $\{\} \notin \mathcal{U}$ $\bigcup\mathcal{U} = topspace\ X - D$
using C **by** *meson*
moreover have (*subtopology X D*) *homeomorphic_space* (*subtopology X D'*)
unfolding *homeomorphic_space_def*
proof (*intro exI*)
have *subtopology X D* = *subtopology* (*subtopology X S*) *D*
by (*simp add*: ‹*D* ⊆ *S*› *inf.absorb2 subtopology_subtopology*)
moreover have *subtopology X D'* = *subtopology* (*subtopology X T*) *D'*
by (*simp add*: *D' inf.absorb2 subtopology_subtopology*)
moreover have *homeomorphic_maps* (*subtopology X T*) (*subtopology X S*) *g f*
by (*simp add*: *fg gf homeomorphic_maps_map homf homg*)
ultimately
have *homeomorphic_maps* (*subtopology X D'*) (*subtopology X D*) *g f*
by (*metis D' D_def* ‹*closedin X D*› *closedin_subset homeomorphic_maps_subtopologies topspace_subtopology Int_absorb1*)
then show *homeomorphic_maps* (*subtopology X D*) (*subtopology X D'*) *f g*
using *homeomorphic_maps_sym* **by** *blast*
qed
ultimately show $\exists\mathcal{U}.\ \mathcal{U} \approx \{..<n\} \wedge$ *pairwise* (*separatedin X*) $\mathcal{U} \wedge \{\} \notin \mathcal{U} \wedge \bigcup\ \mathcal{U} = topspace\ X - D'$
by (*smt* (*verit*, *ccfv_SIG*) § *D' False* ‹*closedin X D*› *Kuratowski_aux2 lepoll_connected_components_alt*)
qed
then have $\exists\mathcal{U}.\ \mathcal{U} \approx \{..<n\} \wedge$
pairwise (*separatedin* (*subtopology X* (*topspace X* − *T*))) $\mathcal{U} \wedge \{\} \notin \mathcal{U} \wedge \bigcup\mathcal{U} = topspace\ X - T$
using *separation_by_closed_intermediates_eq_count* [*of X n T*] *Kuratowski_aux2 lcX hnX* **by** *auto*
with *False* **show** *?thesis*
using *lepoll_connected_components_alt* **by** *fastforce*
qed *auto*

theorem *Kuratowski_component_number_invariance*:
assumes *compact_space X Hausdorff_space X locally_connected_space X hereditarily normal_space X*
shows ((∀ *S T n*.
closedin X S ∧ *closedin X T* ∧
(*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
⟶ (*connected_components_of*
(*subtopology X* (*topspace X* − *S*)) ≈ {..<*n*::*nat*} ⟷
connected_components_of
(*subtopology X* (*topspace X* − *T*)) ≈ {..<*n*::*nat*})) ⟷
(∀ *S T n*.
(*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
⟶ (*connected_components_of*
(*subtopology X* (*topspace X* − *S*)) ≈ {..<*n*::*nat*} ⟷

connected_components_of
(*subtopology X* (*topspace X − T*)) ≈ {..<*n*::*nat*}))))
(**is** *?lhs = ?rhs*)
proof
assume *L*: *?lhs*
then show *?rhs*
apply (*subst* (*asm*) *Kuratowski_aux1*, *use homeomorphic_space_sym* **in** *blast*)
apply (*subst Kuratowski_aux1*, *use homeomorphic_space_sym* **in** *blast*)
apply (*blast intro*: *Kuratowski_component_number_invariance_aux assms*)
done
qed *blast*

end
theory *Sparse_In*
imports *Homotopy*

begin

6.11.22 A set of points sparse in another set

definition *sparse_in*:: *'a* :: *topological_space set* ⇒ *'a set* ⇒ *bool*
(**infixl** ‹(*sparse'_in*)› *50*)
where
pts sparse_in A = (∀ *x*∈*A*. ∃ *B*. *x*∈*B* ∧ *open B* ∧ (∀ *y*∈*B*. ¬ *y islimpt pts*))

lemma *sparse_in_empty*[*simp*]: {} *sparse_in A*
by (*meson UNIV_I empty_iff islimpt_def open_UNIV sparse_in_def*)

lemma *finite_imp_sparse*:
fixes *pts*::*'a*:: *t1_space set*
shows *finite pts* ⟹ *pts sparse_in S*
by (*meson UNIV_I islimpt_finite open_UNIV sparse_in_def*)

lemma *sparse_in_singleton*[*simp*]: {*x*} *sparse_in* (*A*::*'a*:: *t1_space set*)
by (*rule finite_imp_sparse*) *auto*

lemma *sparse_in_ball_def*:
pts sparse_in D ⟷ (∀ *x*∈*D*. ∃ *e*>*0*. ∀ *y*∈*ball x e*. ¬ *y islimpt pts*)
unfolding *sparse_in_def*
by (*meson Elementary_Metric_Spaces.open_ball open_contains_ball_eq subset_eq*)

lemma *get_sparse_in_cover*:
assumes *pts sparse_in A*
obtains *B* **where** *open B A* ⊆ *B* ∀ *y*∈*B*. ¬ *y islimpt pts*
proof −
obtain *getB* **where** *getB*:*x*∈*getB x open* (*getB x*) ∀ *y*∈*getB x*. ¬ *y islimpt pts*
if *x*∈*A* **for** *x*
using *assms*(*1*) **unfolding** *sparse_in_def* **by** *metis*

define B **where** $B = \mathit{Union}\ (\mathit{getB}\ `\ A)$
have *open* B **unfolding** B_def **using** $\mathit{getB}(2)$ **by** *blast*
moreover have $A \subseteq B$ **unfolding** B_def **using** $\mathit{getB}(1)$ **by** *auto*
moreover have $\forall\, y{\in}B.\ \neg\ y\ \mathit{islimpt}\ \mathit{pts}$ **unfolding** B_def **by** (*meson UN_iff* $\mathit{getB}(3)$)
ultimately show *?thesis* **using** *that* **by** *blast*
qed

lemma *sparse_in_open*:
assumes *open A*
shows *pts sparse_in A* $\longleftrightarrow$ $(\forall\, y{\in}A.\ \neg y\ \mathit{islimpt}\ \mathit{pts})$
using *assms* **unfolding** *sparse_in_def* **by** *auto*

lemma *sparse_in_not_in*:
assumes *pts sparse_in A* $x{\in}A$
obtains B **where** *open* B $x{\in}B$ $\forall\, y{\in}B.\ y{\neq}x \longrightarrow y{\notin}\mathit{pts}$
using *assms* **unfolding** *sparse_in_def*
by (*metis islimptI*)

lemma *sparse_in_subset*:
assumes *pts sparse_in A* $B \subseteq A$
shows *pts sparse_in B*
using *assms* **unfolding** *sparse_in_def* **by** *auto*

lemma *sparse_in_subset2*:
assumes *pts1 sparse_in D pts2* $\subseteq$ *pts1*
shows *pts2 sparse_in D*
by (*meson assms*(1) *assms*(2) *islimpt_subset sparse_in_def*)

lemma *sparse_in_union*:
assumes *pts1 sparse_in D1 pts2 sparse_in D1*
shows (*pts1* $\cup$ *pts2*) *sparse_in* (*D1* $\cap$ *D2*)
using *assms* **unfolding** *sparse_in_def islimpt_Un*
by (*metis Int_iff open_Int*)

lemma *sparse_in_compact_finite*:
assumes *pts sparse_in A compact A*
shows *finite* $(A \cap \mathit{pts})$
apply (*rule finite_not_islimpt_in_compact*[*OF* ‹*compact A*›])
using *assms* **unfolding** *sparse_in_def* **by** *blast*

lemma *sparse_imp_closedin_pts*:
assumes *pts sparse_in D*
shows *closedin* (*top_of_set D*) (*D* $\cap$ *pts*)
using *assms islimpt_subset* **unfolding** *closedin_limpt sparse_in_def*
by *fastforce*

lemma *open_diff_sparse_pts*:
assumes *open D pts sparse_in D*

shows *open* (*D* − *pts*)
using *assms sparse_imp_closedin_pts*
by (*metis Diff_Diff_Int Diff_cancel Diff_eq_empty_iff Diff_subset closedin_def double_diff openin_open_eq topspace_euclidean_subtopology*)

lemma *sparse_in_UNIV_imp_closed*: *X sparse_in UNIV* ⟹ *closed X*
by (*simp add*: *Compl_eq_Diff_UNIV closed_open open_diff_sparse_pts*)

lemma *sparse_imp_countable*:
fixes *D*::′*a* ::*euclidean_space set*
assumes *open D pts sparse_in D*
shows *countable* (*D* ∩ *pts*)
proof −
obtain *K* :: *nat* ⇒ ′*a* ::*euclidean_space set*
where *K*: *D* = (⋃ *n*. *K n*) ⋀*n*. *compact* (*K n*)
using *assms* **by** (*metis open_Union_compact_subsets*)
then have *D*∩ *pts* = (⋃ *n*. *K n* ∩ *pts*)
by *blast*
moreover have ⋀*n*. *finite* (*K n* ∩ *pts*)
by (*metis K*(*1*) *K*(*2*) *Union_iff assms*(*2*) *rangeI sparse_in_compact_finite sparse_in_subset subsetI*)
ultimately show *?thesis*
by (*metis countableI_type countable_UN countable_finite*)
qed

lemma *sparse_imp_connected*:
fixes *D*::′*a* ::*euclidean_space set*
assumes *2* ≤ *DIM* (′*a*) *connected D open D pts sparse_in D*
shows *connected* (*D* − *pts*)
using *assms*
by (*metis Diff_Compl Diff_Diff_Int Diff_eq connected_open_diff_countable sparse_imp_countable*)

lemma *sparse_in_eventually_iff*:
assumes *open A*
shows *pts sparse_in A* ⟷ (∀ *y*∈*A*. (∀ *F y in at y*. *y* ∉ *pts*))
unfolding *sparse_in_open*[*OF* ‹*open A*›] *islimpt_iff_eventually*
by *simp*

lemma *get_sparse_from_eventually*:
fixes *A*::′*a*::*topological_space set*
assumes ∀ *x*∈*A*. ∀ *F z in at x*. *P z open A*
obtains *pts* **where** *pts sparse_in A* ∀ *x*∈*A* − *pts*. *P x*
proof −
define *pts*::′*a set* **where** *pts*={*x*. ¬*P x*}
have *pts sparse_in A* ∀ *x*∈*A* − *pts*. *P x*
unfolding *sparse_in_eventually_iff*[*OF* ‹*open A*›] *pts_def*
using *assms*(*1*) **by** *simp_all*

then show *?thesis* **using** *that* **by** *blast*
qed

lemma *sparse_disjoint*:
assumes *pts* ∩ *A* = {} *open A*
shows *pts sparse_in A*
using *assms* **unfolding** *sparse_in_eventually_iff*[*OF* ‹*open A*›]
eventually_at_topological
by *blast*

6.11.23 Co-sparseness filter

The co-sparseness filter allows us to talk about properties that hold on a given set except for an "insignificant" number of points that are sparse in that set.

lemma *is_filter_cosparse*: *is_filter* (λ*P*. {*x*. ¬*P x*} *sparse_in A*)
proof (*standard*, *goal_cases*)
case *1*
thus *?case* **by** *auto*
next
case (*2 P Q*)
from *sparse_in_union*[*OF this*, *of UNIV*] **show** *?case*
by (*auto simp*: *Un_def*)
next
case (*3 P Q*)
from *3*(*2*) **show** *?case*
by (*rule sparse_in_subset2*) (*use 3*(*1*) **in** *auto*)
qed

definition *cosparse* :: *'a set* ⇒ *'a* :: *topological_space filter* **where**
cosparse A = *Abs_filter* (λ*P*. {*x*. ¬*P x*} *sparse_in A*)

syntax
_eventually_cosparse :: *pttrn* => *'a set* => *bool* => *bool* (‹(‹*indent=3 notation=‹binder* ∀≈››∀≈_∈_./ _)› [*0*, *0*, *10*] *10*)
syntax_consts
_eventually_cosparse == *eventually*
translations
∀≈*x*∈*A*. *P* == *CONST eventually* (λ*x*. *P*) (*CONST cosparse A*)

syntax
_qeventually_cosparse :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* (‹(‹*indent=3 notation=‹binder* ∀≈››∀≈_ | (_)./ _)› [*0*, *0*, *10*] *10*)
syntax_consts
_qeventually_cosparse == *eventually*
translations
∀≈*x*|*P*. *t* => *CONST eventually* (λ*x*. *t*) (*CONST cosparse* {*x*. *P*})
print_translation ‹
[(***const_syntax*** ‹*eventually*›, *K* (*Collect_binder_tr'* ***syntax_const*** ‹*_qeventually_cosparse*›))]

›

lemma *eventually_cosparse*: *eventually P* (*cosparse A*) ⟷ {*x*. ¬*P x*} *sparse_in A*
unfolding *cosparse_def* **by** (*rule eventually_Abs_filter*[*OF is_filter_cosparse*])

lemma *eventually_not_in_cosparse*:
assumes *X sparse_in A*
shows *eventually* (λ*x*. *x* ∉ *X*) (*cosparse A*)
using *assms* **by** (*auto simp*: *eventually_cosparse*)

lemma *eventually_cosparse_open_eq*:
open A ⟹ *eventually P* (*cosparse A*) ⟷ (∀ *x*∈*A*. *eventually P* (*at x*))
unfolding *eventually_cosparse*
by (*subst sparse_in_open*) (*auto simp*: *islimpt_conv_frequently_at frequently_def*)

lemma *eventually_cosparse_imp_eventually_at*:
eventually P (*cosparse A*) ⟹ *x* ∈ *A* ⟹ *eventually P* (*at x within B*)
unfolding *eventually_cosparse sparse_in_def islimpt_def eventually_at_topological*

by *fastforce*

lemma *eventually_in_cosparse*:
assumes *A* ⊆ *X open A*
shows *eventually* (λ*x*. *x* ∈ *X*) (*cosparse A*)
proof −
have *eventually* (λ*x*. *x* ∈ *A*) (*cosparse A*)
using *assms* **by** (*auto simp*: *eventually_cosparse_open_eq intro*: *eventually_at_in_open'*)
thus *?thesis*
by *eventually_elim* (*use assms*(*1*) **in** *blast*)
qed

lemma *cosparse_eq_bot_iff*: *cosparse A* = *bot* ⟷ (∀ *x*∈*A*. *open* {*x*})
proof −
have *cosparse A* = *bot* ⟷ *eventually* (λ_. *False*) (*cosparse A*)
by (*simp add*: *trivial_limit_def*)
also have ... ⟷ (∀ *x*∈*A*. *open* {*x*})
unfolding *eventually_cosparse sparse_in_def*
by (*auto simp*: *islimpt_UNIV_iff*)
finally show *?thesis* **.**
qed

lemma *cosparse_empty* [*simp*]: *cosparse* {} = *bot*
by (*rule filter_eqI*) (*auto simp*: *eventually_cosparse sparse_in_def*)

lemma *cosparse_eq_bot_iff'* [*simp*]: *cosparse* (*A* :: *'a* :: *perfect_space set*) = *bot* ⟷ *A* = {}
by (*auto simp*: *cosparse_eq_bot_iff not_open_singleton*)

end
theory *Isolated*
imports *Elementary_Metric_Spaces Sparse_In*

begin

6.11.24 Isolate and discrete

definition (**in** *topological_space*) *isolated_in*:: *'a* ⇒ *'a set* ⇒ *bool* (**infixr** ‹*isolated'_in*› *60*)
where *x isolated_in S* ⟷ (*x*∈*S* ∧ (∃ *T*. *open T* ∧ *T* ∩ *S* = {*x*}))

definition (**in** *topological_space*) *discrete*:: *'a set* ⇒ *bool*
where *discrete S* ⟷ (∀ *x*∈*S*. *x isolated_in S*)

definition (**in** *metric_space*) *uniform_discrete* :: *'a set* ⇒ *bool* **where**
uniform_discrete S ⟷ (∃ *e*>*0*. ∀ *x*∈*S*. ∀ *y*∈*S*. *dist x y* < *e* ⟶ *x* = *y*)

lemma *discreteI*: (⋀*x*. *x* ∈ *X* ⟹ *x isolated_in X*) ⟹ *discrete X*
unfolding *discrete_def* **by** *auto*

lemma *discreteD*: *discrete X* ⟹ *x* ∈ *X* ⟹ *x isolated_in X*
unfolding *discrete_def* **by** *auto*

lemma *uniformI1*:
assumes *e*>*0* ⋀*x y*. ⟦*x*∈*S*;*y*∈*S*;*dist x y*<*e*⟧ ⟹ *x* =*y*
shows *uniform_discrete S*
unfolding *uniform_discrete_def* **using** *assms* **by** *auto*

lemma *uniformI2*:
assumes *e*>*0* ⋀*x y*. ⟦*x*∈*S*;*y*∈*S*;*x*≠*y*⟧ ⟹ *dist x y*≥*e*
shows *uniform_discrete S*
unfolding *uniform_discrete_def* **using** *assms not_less* **by** *blast*

lemma *isolated_in_islimpt_iff*:(*x isolated_in S*) ⟷ (¬ (*x islimpt S*) ∧ *x*∈*S*)
unfolding *isolated_in_def islimpt_def* **by** *auto*

lemma *isolated_in_dist_Ex_iff*:
fixes *x*::*'a*::*metric_space*
shows *x isolated_in S* ⟷ (*x*∈*S* ∧ (∃ *e*>*0*. ∀ *y*∈*S*. *dist x y* < *e* ⟶ *y*=*x*))
unfolding *isolated_in_islimpt_iff islimpt_approachable* **by** (*metis dist_commute*)

lemma *discrete_empty*[*simp*]: *discrete* {}
unfolding *discrete_def* **by** *auto*

lemma *uniform_discrete_empty*[*simp*]: *uniform_discrete* {}
unfolding *uniform_discrete_def* **by** (*simp add*: *gt_ex*)

lemma *isolated_in_insert*:

fixes $x :: 'a::t1\_space$
shows x *isolated_in* (*insert* a S) $\longleftrightarrow$ x *isolated_in* $S \vee (x{=}a \wedge \neg\ (x$ *islimpt* $S))$
by (*meson insert_iff islimpt_insert isolated_in_islimpt_iff*)

lemma *isolated_inI*:
assumes $x{\in}S$ *open* T $T \cap S = \{x\}$
shows x *isolated_in* S
using *assms* **unfolding** *isolated_in_def* **by** *auto*

lemma *isolated_inE*:
assumes x *isolated_in* S
obtains T **where** $x \in S$ *open* T $T \cap S = \{x\}$
using *assms that* **unfolding** *isolated_in_def* **by** *force*

lemma *isolated_inE_dist*:
assumes x *isolated_in* S
obtains d **where** $d > 0$ $\bigwedge y.\ y \in S \Longrightarrow$ *dist* $x\ y < d \Longrightarrow y = x$
by (*meson assms isolated_in_dist_Ex_iff*)

lemma *isolated_in_altdef*:
x *isolated_in* $S \longleftrightarrow (x{\in}S \wedge$ *eventually* $(\lambda y.\ y \notin S)$ (*at* $x))$
proof
assume x *isolated_in* S
from *isolated_inE*[*OF this*]
obtain T **where** $x \in S$ **and** T:*open* T $T \cap S = \{x\}$
by *metis*
have $\forall_F\ y$ *in nhds* $x.\ y \in T$
apply (*rule eventually_nhds_in_open*)
using T **by** *auto*
then have *eventually* $(\lambda y.\ y \in T - \{x\})$ (*at* x)
unfolding *eventually_at_filter* **by** *eventually_elim auto*
then have *eventually* $(\lambda y.\ y \notin S)$ (*at* x)
by *eventually_elim* (*use* T **in** *auto*)
then show $x \in S \wedge (\forall_F\ y$ *in at* $x.\ y \notin S)$ **using** ‹$x \in S$› **by** *auto*
next
assume $x \in S \wedge (\forall_F\ y$ *in at* $x.\ y \notin S)$
then have $\forall_F\ y$ *in at* $x.\ y \notin S$ $x{\in}S$ **by** *auto*
from *this*(*1*) **have** *eventually* $(\lambda y.\ y \notin S \vee y = x)$ (*nhds* x)
unfolding *eventually_at_filter* **by** *eventually_elim auto*
then obtain T **where** T:*open* T $x \in T$ $(\forall y{\in}T.\ y \notin S \vee y = x)$
unfolding *eventually_nhds* **by** *auto*
with ‹$x \in S$› **have** $T \cap S = \{x\}$
by *fastforce*
with ‹$x{\in}S$› ‹*open* T›
show x *isolated_in* S
unfolding *isolated_in_def* **by** *auto*
qed

lemma *discrete_altdef*:

discrete S $\longleftrightarrow$ ($\forall$ *x*$\in$*S*. $\forall_F$ *y in at x*. *y* $\notin$ *S*)
unfolding *discrete_def isolated_in_altdef* **by** *auto*

lemma *uniform_discrete_imp_closed*:
uniform_discrete S $\Longrightarrow$ *closed S*
by (*meson discrete_imp_closed uniform_discrete_def*)

lemma *uniform_discrete_imp_discrete*:
uniform_discrete S $\Longrightarrow$ *discrete S*
by (*metis discrete_def isolated_in_dist_Ex_iff uniform_discrete_def*)

lemma *isolated_in_subset*:*x isolated_in S* $\Longrightarrow$ *T* $\subseteq$ *S* $\Longrightarrow$ *x*$\in$*T* $\Longrightarrow$ *x isolated_in T*
unfolding *isolated_in_def* **by** *fastforce*

lemma *discrete_subset*[*elim*]: *discrete S* $\Longrightarrow$ *T* $\subseteq$ *S* $\Longrightarrow$ *discrete T*
unfolding *discrete_def* **using** *islimpt_subset isolated_in_islimpt_iff* **by** *blast*

lemma *uniform_discrete_subset*[*elim*]: *uniform_discrete S* $\Longrightarrow$ *T* $\subseteq$ *S* $\Longrightarrow$ *uniform_discrete T*
by (*meson subsetD uniform_discrete_def*)

lemma *continuous_on_discrete*: *discrete S* $\Longrightarrow$ *continuous_on S f*
unfolding *continuous_on_topological* **by** (*metis discrete_def islimptI isolated_in_islimpt_iff*)

lemma *uniform_discrete_insert*: *uniform_discrete* (*insert a S*) $\longleftrightarrow$ *uniform_discrete S*
proof
assume *asm*:*uniform_discrete S*
let *?thesis* = *uniform_discrete* (*insert a S*)
have *?thesis* **when** *a*$\in$*S* **using** *that asm* **by** (*simp add*: *insert_absorb*)
moreover have *?thesis* **when** *S*={} **using** *that asm* **by** (*simp add*: *uniform_discrete_def*)
moreover have *?thesis* **when** *a*$\notin$*S S*$\neq${}
proof −
obtain *e1* **where** *e1*>*0* **and** *e1_dist*:$\forall$ *x*$\in$*S*. $\forall$ *y*$\in$*S*. *dist y x* < *e1* $\longrightarrow$ *y* = *x*
using *asm* **unfolding** *uniform_discrete_def* **by** *auto*
define *e2* **where** *e2* $\equiv$ *min* (*setdist* {*a*} *S*) *e1*
have *closed S* **using** *asm uniform_discrete_imp_closed* **by** *auto*
then have *e2*>*0*
by (*smt* (*verit*) ‹*0* < *e1*› *e2_def infdist_eq_setdist infdist_pos_not_in_closed that*)
moreover have *x* = *y* **if** *x*$\in$*insert a S y*$\in$*insert a S dist x y* < *e2* **for** *x y*
proof (*cases x*=*a* $\vee$ *y*=*a*)
case *True* **then show** *?thesis*
by (*smt* (*verit*, *best*) *dist_commute e2_def infdist_eq_setdist infdist_le insertE that*)
next

```
      case False then show ?thesis
        using e1_dist e2_def that by force
    qed
    ultimately show ?thesis unfolding uniform_discrete_def by meson
  qed
  ultimately show ?thesis by auto
qed (simp add: subset_insertI uniform_discrete_subset)

lemma discrete_compact_finite_iff:
  fixes S :: 'a::t1_space set
  shows discrete S ∧ compact S ⟷ finite S
proof
  assume finite S
  then have compact S using finite_imp_compact by auto
  moreover have discrete S
   unfolding discrete_def using isolated_in_islimpt_iff islimpt_finite[OF ‹finite
S›] by auto
  ultimately show discrete S ∧ compact S by auto
next
  assume discrete S ∧ compact S
  then show finite S
   by (meson discrete_def Heine_Borel_imp_Bolzano_Weierstrass isolated_in_islimpt_iff
order_refl)
qed

lemma uniform_discrete_finite_iff:
  fixes S :: 'a::heine_borel set
  shows uniform_discrete S ∧ bounded S ⟷ finite S
proof
  assume uniform_discrete S ∧ bounded S
  then have discrete S compact S
   using uniform_discrete_imp_discrete uniform_discrete_imp_closed compact_eq_bounded_closed
    by auto
  then show finite S using discrete_compact_finite_iff by auto
next
  assume asm:finite S
  let ?thesis = uniform_discrete S ∧ bounded S
  have ?thesis when S={} using that by auto
  moreover have ?thesis when S≠{}
  proof −
    have ∀ x. ∃ d>0. ∀ y∈S. y ≠ x ⟶ d ≤ dist x y
      using finite_set_avoid[OF ‹finite S›] by auto
    then obtain f where f_pos:f x>0
       and f_dist: ∀ y∈S. y ≠ x ⟶ f x ≤ dist x y
       if x∈S for x
      by metis
    define f_min where f_min ≡ Min (f ‘ S)
    have f_min > 0
      unfolding f_min_def
```

```
    by (simp add: asm f_pos that)
   moreover have ∀ x∈S. ∀ y∈S. f_min > dist x y ⟶ x=y
    using f_dist unfolding f_min_def
    by (metis Min_le asm finite_imageI imageI le_less_trans linorder_not_less)
   ultimately have uniform_discrete S
    unfolding uniform_discrete_def by auto
   moreover have bounded S using ‹finite S› by auto
   ultimately show ?thesis by auto
  qed
  ultimately show ?thesis by blast
qed

lemma uniform_discrete_image_scale:
  assumes uniform_discrete S and dist:∀ x∈S. ∀ y∈S. dist x y = c * dist (f x) (f
y)
  shows uniform_discrete (f ‘ S)
proof −
  have ?thesis when S={} using that by auto
  moreover have ?thesis when S≠{} c≤0
  proof −
   obtain x1 where x1∈S using ‹S≠{}› by auto
   have ?thesis when S−{x1} = {}
    using ‹x1 ∈ S› subset_antisym that uniform_discrete_insert by fastforce
   moreover have ?thesis when S−{x1} ≠ {}
   proof −
    obtain x2 where x2∈S−{x1} using ‹S−{x1} ≠ {}› by auto
    then have x2∈S x1≠x2 by auto
    then have dist x1 x2 > 0 by auto
    moreover have dist x1 x2 = c * dist (f x1) (f x2)
     by (simp add: ‹x1 ∈ S› ‹x2 ∈ S› dist)
    moreover have dist (f x2) (f x2) ≥ 0 by auto
    ultimately have False using ‹c≤0› by (simp add: zero_less_mult_iff)
    then show ?thesis by auto
   qed
   ultimately show ?thesis by auto
  qed
  moreover have ?thesis when S≠{} c>0
  proof −
   obtain e1 where e1>0 and e1_dist:∀ x∈S. ∀ y∈S. dist y x < e1 ⟶ y = x
    using ‹uniform_discrete S› unfolding uniform_discrete_def by auto
   define e where e ≡ e1/c
   have x1 = x2 when x1 ∈ f ‘ S x2 ∈ f ‘ S and d: dist x1 x2 < e for x1 x2
     by (smt (verit) ‹0 < c› d dist divide_right_mono e1_dist e_def imageE
nonzero_mult_div_cancel_left that)
   moreover have e>0 using ‹e1>0› ‹c>0› unfolding e_def by auto
   ultimately show ?thesis unfolding uniform_discrete_def by meson
  qed
  ultimately show ?thesis by fastforce
qed
```

definition *sparse* :: *real* ⇒ *'a* :: *metric_space set* ⇒ *bool*
where *sparse ε X* ⟷ (∀ *x*∈*X*. ∀ *y*∈*X*−{*x*}. *dist x y* > *ε*)

lemma *sparse_empty* [*simp*, *intro*]: *sparse ε* {}
by (*auto simp*: *sparse_def*)

lemma *sparseI* [*intro?*]:
(⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ *x* ≠ *y* ⟹ *dist x y* > *ε*) ⟹ *sparse ε X*
unfolding *sparse_def* **by** *auto*

lemma *sparseD*:
sparse ε X ⟹ *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ *x* ≠ *y* ⟹ *dist x y* > *ε*
unfolding *sparse_def* **by** *auto*

lemma *sparseD'*:
sparse ε X ⟹ *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ *dist x y* ≤ *ε* ⟹ *x* = *y*
unfolding *sparse_def* **by** *force*

lemma *sparse_singleton* [*simp*, *intro*]: *sparse ε* {*x*}
by (*auto simp*: *sparse_def*)

definition *setdist_gt* **where** *setdist_gt ε X Y* ⟷ (∀ *x*∈*X*. ∀ *y*∈*Y*. *dist x y* > *ε*)

lemma *setdist_gt_empty* [*simp*]: *setdist_gt ε* {} *Y setdist_gt ε X* {}
by (*auto simp*: *setdist_gt_def*)

lemma *setdist_gtI*: (⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *Y* ⟹ *dist x y* > *ε*) ⟹ *setdist_gt ε X Y*
unfolding *setdist_gt_def* **by** *auto*

lemma *setdist_gtD*: *setdist_gt ε X Y* ⟹ *x* ∈ *X* ⟹ *y* ∈ *Y* ⟹ *dist x y* > *ε*
unfolding *setdist_gt_def* **by** *auto*

lemma *setdist_gt_setdist*: *ε* < *setdist A B* ⟹ *setdist_gt ε A B*
unfolding *setdist_gt_def* **using** *setdist_le_dist* **by** *fastforce*

lemma *setdist_gt_mono*: *setdist_gt ε' A B* ⟹ *ε* ≤ *ε'* ⟹ *A'* ⊆ *A* ⟹ *B'* ⊆ *B* ⟹ *setdist_gt ε A' B'*
by (*force simp*: *setdist_gt_def*)

lemma *setdist_gt_Un_left*: *setdist_gt ε* (*A* ∪ *B*) *C* ⟷ *setdist_gt ε A C* ∧ *setdist_gt ε B C*
by (*auto simp*: *setdist_gt_def*)

lemma *setdist_gt_Un_right*: *setdist_gt ε C* (*A* ∪ *B*) ⟷ *setdist_gt ε C A* ∧ *setdist_gt ε C B*
by (*auto simp*: *setdist_gt_def*)

lemma *compact_closed_imp_eventually_setdist_gt_at_right_0*:
assumes *compact A closed B A* ∩ *B* = {}
shows *eventually* (λε. *setdist_gt* ε *A B*) (*at_right 0*)
proof (*cases A* = {} ∨ *B* = {})
case *False*
hence *setdist A B* > *0*
by (*metis IntI assms empty_iff in_closed_iff_infdist_zero order_less_le setdist_attains_inf setdist_pos_le setdist_sym*)
hence *eventually* (λε. ε < *setdist A B*) (*at_right 0*)
using *eventually_at_right_field* **by** *blast*
thus *?thesis*
by *eventually_elim* (*auto intro*: *setdist_gt_setdist*)
qed *auto*

lemma *setdist_gt_symI*: *setdist_gt* ε *A B* ⟹ *setdist_gt* ε *B A*
by (*force simp*: *setdist_gt_def dist_commute*)

lemma *setdist_gt_sym*: *setdist_gt* ε *A B* ⟷ *setdist_gt* ε *B A*
by (*force simp*: *setdist_gt_def dist_commute*)

lemma *eventually_setdist_gt_at_right_0_mult_iff*:
assumes *c* > *0*
shows *eventually* (λε. *setdist_gt* (*c* ∗ ε) *A B*) (*at_right 0*) ⟷
eventually (λε. *setdist_gt* ε *A B*) (*at_right 0*)
proof −
have *eventually* (λε. *setdist_gt* (*c* ∗ ε) *A B*) (*at_right 0*) ⟷
eventually (λε. *setdist_gt* ε *A B*) (*filtermap* ((∗) *c*) (*at_right 0*))
by (*simp add*: *eventually_filtermap*)
also have *filtermap* ((∗) *c*) (*at_right 0*) = *at_right 0*
by (*subst filtermap_times_pos_at_right*) (*use assms* **in** *auto*)
finally show *?thesis* **.**
qed

lemma *uniform_discrete_imp_sparse*:
assumes *uniform_discrete X*
shows *X sparse_in A*
using *assms* **unfolding** *uniform_discrete_def sparse_in_ball_def*
by (*auto simp*: *discrete_imp_not_islimpt*)

end

6.12 Operator Norm

theory *Operator_Norm*
imports *Complex_Main*
begin

This formulation yields zero if $'a$ is the trivial vector space.

definition
onorm :: (*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*) ⇒ *real* **where**
onorm f = (*SUP x*. *norm* (*f x*) / *norm x*)

proposition *onorm_bound*:
assumes *0* ≤ *b* **and** ⋀*x*. *norm* (*f x*) ≤ *b* * *norm x*
shows *onorm f* ≤ *b*
unfolding *onorm_def*
proof (*rule cSUP_least*)
fix *x*
show *norm* (*f x*) / *norm x* ≤ *b*
using *assms* **by** (*cases x* = *0*) (*simp_all add*: *pos_divide_le_eq*)
qed *simp*

In non-trivial vector spaces, the first assumption is redundant.

lemma *onorm_le*:
fixes *f* :: *'a*::{*real_normed_vector*, *perfect_space*} ⇒ *'b*::*real_normed_vector*
assumes ⋀*x*. *norm* (*f x*) ≤ *b* * *norm x*
shows *onorm f* ≤ *b*
proof (*rule onorm_bound* [*OF* _ *assms*])
have {*0*::*'a*} ≠ *UNIV* **by** (*metis not_open_singleton open_UNIV*)
then obtain *a* :: *'a* **where** *a* ≠ *0* **by** *fast*
have *0* ≤ *b* * *norm a*
by (*rule order_trans* [*OF norm_ge_zero assms*])
with ‹*a* ≠ *0*› **show** *0* ≤ *b*
by (*simp add*: *zero_le_mult_iff*)
qed

lemma *le_onorm*:
assumes *bounded_linear f*
shows *norm* (*f x*) / *norm x* ≤ *onorm f*
proof −
interpret *f*: *bounded_linear f* **by** *fact*
obtain *b* **where** *0* ≤ *b* **and** ∀ *x*. *norm* (*f x*) ≤ *norm x* * *b*
using *f.nonneg_bounded* **by** *auto*
then have ∀ *x*. *norm* (*f x*) / *norm x* ≤ *b*
by (*clarify*, *case_tac x* = *0*,
simp_all add: *f.zero pos_divide_le_eq mult.commute*)
then have *bdd_above* (*range* (λ*x*. *norm* (*f x*) / *norm x*))
unfolding *bdd_above_def* **by** *fast*
with *UNIV_I* **show** *?thesis*
unfolding *onorm_def* **by** (*rule cSUP_upper*)
qed

lemma *onorm*:
assumes *bounded_linear f*
shows *norm* (*f x*) ≤ *onorm f* * *norm x*
proof −
interpret *f*: *bounded_linear f* **by** *fact*

```
  show ?thesis
  proof (cases)
    assume x = 0
    then show ?thesis by (simp add: f.zero)
  next
    assume x ≠ 0
    have norm (f x) / norm x ≤ onorm f
      by (rule le_onorm [OF assms])
    then show norm (f x) ≤ onorm f * norm x
      by (simp add: pos_divide_le_eq ‹x ≠ 0›)
  qed
qed

lemma onorm_pos_le:
  assumes f: bounded_linear f
  shows 0 ≤ onorm f
  using le_onorm [OF f, where x=0] by simp

lemma onorm_zero: onorm (λx. 0) = 0
proof (rule order_antisym)
  show onorm (λx. 0) ≤ 0
    by (simp add: onorm_bound)
  show 0 ≤ onorm (λx. 0)
    using bounded_linear_zero by (rule onorm_pos_le)
qed

lemma onorm_eq_0:
  assumes f: bounded_linear f
  shows onorm f = 0 ⟷ (∀ x. f x = 0)
  using onorm [OF f] by (auto simp: fun_eq_iff [symmetric] onorm_zero)

lemma onorm_pos_lt:
  assumes f: bounded_linear f
  shows 0 < onorm f ⟷ ¬ (∀ x. f x = 0)
  by (simp add: less_le onorm_pos_le [OF f] onorm_eq_0 [OF f])

lemma onorm_id_le: onorm (λx. x) ≤ 1
  by (rule onorm_bound) simp_all

lemma onorm_id: onorm (λx. x::'a::{real_normed_vector, perfect_space}) = 1
proof (rule antisym[OF onorm_id_le])
  have {0::'a} ≠ UNIV by (metis not_open_singleton open_UNIV)
  then obtain x :: 'a where x ≠ 0 by fast
  hence 1 ≤ norm x / norm x
    by simp
  also have ... ≤ onorm (λx::'a. x)
    by (rule le_onorm) (rule bounded_linear_ident)
  finally show 1 ≤ onorm (λx::'a. x) .
qed
```

lemma *onorm_compose*:
assumes *f*: *bounded_linear f*
assumes *g*: *bounded_linear g*
shows *onorm* (*f* ∘ *g*) ≤ *onorm f* ∗ *onorm g*
proof (*rule onorm_bound*)
show *0* ≤ *onorm f* ∗ *onorm g*
by (*intro mult_nonneg_nonneg onorm_pos_le f g*)
next
fix *x*
have *norm* (*f* (*g x*)) ≤ *onorm f* ∗ *norm* (*g x*)
by (*rule onorm* [*OF f*])
also have *onorm f* ∗ *norm* (*g x*) ≤ *onorm f* ∗ (*onorm g* ∗ *norm x*)
by (*rule mult_left_mono* [*OF onorm* [*OF g*] *onorm_pos_le* [*OF f*]])
finally show *norm* ((*f* ∘ *g*) *x*) ≤ *onorm f* ∗ *onorm g* ∗ *norm x*
by (*simp add*: *mult.assoc*)
qed

lemma *onorm_scaleR_lemma*:
assumes *f*: *bounded_linear f*
shows *onorm* (λ*x*. *r* ∗*R* *f x*) ≤ |*r*| ∗ *onorm f*
proof (*rule onorm_bound*)
show *0* ≤ |*r*| ∗ *onorm f*
by (*intro mult_nonneg_nonneg onorm_pos_le abs_ge_zero f*)
next
fix *x*
have |*r*| ∗ *norm* (*f x*) ≤ |*r*| ∗ (*onorm f* ∗ *norm x*)
by (*intro mult_left_mono onorm abs_ge_zero f*)
then show *norm* (*r* ∗*R* *f x*) ≤ |*r*| ∗ *onorm f* ∗ *norm x*
by (*simp only*: *norm_scaleR mult.assoc*)
qed

lemma *onorm_scaleR*:
assumes *f*: *bounded_linear f*
shows *onorm* (λ*x*. *r* ∗*R* *f x*) = |*r*| ∗ *onorm f*
proof (*cases r* = *0*)
assume *r* ≠ *0*
show *?thesis*
proof (*rule order_antisym*)
show *onorm* (λ*x*. *r* ∗*R* *f x*) ≤ |*r*| ∗ *onorm f*
using *f* **by** (*rule onorm_scaleR_lemma*)
next
have *bounded_linear* (λ*x*. *r* ∗*R* *f x*)
using *bounded_linear_scaleR_right f* **by** (*rule bounded_linear_compose*)
then have *onorm* (λ*x*. *inverse r* ∗*R* *r* ∗*R* *f x*) ≤ |*inverse r*| ∗ *onorm* (λ*x*. *r* ∗*R* *f x*)
by (*rule onorm_scaleR_lemma*)
with ‹*r* ≠ *0*› **show** |*r*| ∗ *onorm f* ≤ *onorm* (λ*x*. *r* ∗*R* *f x*)
by (*simp add*: *inverse_eq_divide pos_le_divide_eq mult.commute*)

```
  qed
qed (simp add: onorm_zero)

lemma onorm_scaleR_left_lemma:
  assumes r: bounded_linear r
  shows onorm (λx. r x *R f) ≤ onorm r * norm f
proof (rule onorm_bound)
  fix x
  have norm (r x *R f) = norm (r x) * norm f
    by simp
  also have ... ≤ onorm r * norm x * norm f
    by (intro mult_right_mono onorm r norm_ge_zero)
  finally show norm (r x *R f) ≤ onorm r * norm f * norm x
    by (simp add: ac_simps)
qed (intro mult_nonneg_nonneg norm_ge_zero onorm_pos_le r)

lemma onorm_scaleR_left:
  assumes f: bounded_linear r
  shows onorm (λx. r x *R f) = onorm r * norm f
proof (cases f = 0)
  assume f ≠ 0
  show ?thesis
  proof (rule order_antisym)
    show onorm (λx. r x *R f) ≤ onorm r * norm f
      using f by (rule onorm_scaleR_left_lemma)
  next
    have bl1: bounded_linear (λx. r x *R f)
      by (metis bounded_linear_scaleR_const f)
    have bounded_linear (λx. r x * norm f)
      by (metis bounded_linear_mult_const f)
    from onorm_scaleR_left_lemma[OF this, of inverse (norm f)]
    have onorm r ≤ onorm (λx. r x * norm f) * inverse (norm f)
      using ‹f ≠ 0›
      by (simp add: inverse_eq_divide)
    also have onorm (λx. r x * norm f) ≤ onorm (λx. r x *R f)
      by (rule onorm_bound)
        (auto simp: abs_mult bl1 onorm_pos_le intro!: order_trans[OF _ onorm])
    finally show onorm r * norm f ≤ onorm (λx. r x *R f)
      using ‹f ≠ 0›
      by (simp add: inverse_eq_divide pos_le_divide_eq mult.commute)
  qed
qed (simp add: onorm_zero)

lemma onorm_neg:
  shows onorm (λx. − f x) = onorm f
  unfolding onorm_def by simp

lemma onorm_triangle:
  assumes f: bounded_linear f
```

assumes *g*: *bounded_linear g*
shows *onorm* (λ*x*. *f x* + *g x*) ≤ *onorm f* + *onorm g*
proof (*rule onorm_bound*)
show *0* ≤ *onorm f* + *onorm g*
by (*intro add_nonneg_nonneg onorm_pos_le f g*)
next
fix *x*
have *norm* (*f x* + *g x*) ≤ *norm* (*f x*) + *norm* (*g x*)
by (*rule norm_triangle_ineq*)
also have *norm* (*f x*) + *norm* (*g x*) ≤ *onorm f* ∗ *norm x* + *onorm g* ∗ *norm x*
by (*intro add_mono onorm f g*)
finally show *norm* (*f x* + *g x*) ≤ (*onorm f* + *onorm g*) ∗ *norm x*
by (*simp only*: *distrib_right*)
qed

lemma *onorm_triangle_le*:
assumes *bounded_linear f*
assumes *bounded_linear g*
assumes *onorm f* + *onorm g* ≤ *e*
shows *onorm* (λ*x*. *f x* + *g x*) ≤ *e*
using *assms* **by** (*rule onorm_triangle* [*THEN order_trans*])

lemma *onorm_triangle_lt*:
assumes *bounded_linear f*
assumes *bounded_linear g*
assumes *onorm f* + *onorm g* < *e*
shows *onorm* (λ*x*. *f x* + *g x*) < *e*
using *assms* **by** (*rule onorm_triangle* [*THEN order_le_less_trans*])

lemma *onorm_sum*:
assumes *finite S*
assumes ⋀*s*. *s* ∈ *S* ⟹ *bounded_linear* (*f s*)
shows *onorm* (λ*x*. *sum* (λ*s*. *f s x*) *S*) ≤ *sum* (λ*s*. *onorm* (*f s*)) *S*
using *assms*
by (*induction*) (*auto simp*: *onorm_zero intro*!: *onorm_triangle_le bounded_linear_sum*)

lemmas *onorm_sum_le* = *onorm_sum*[*THEN order_trans*]

end

6.13 Limits on the Extended Real Number Line

theory *Extended_Real_Limits*
imports
Topology_Euclidean_Space
HOL−Library.Extended_Real
HOL−Library.Extended_Nonnegative_Real
HOL−Library.Indicator_Function
begin

lemma *compact_UNIV*:
compact (*UNIV* :: *'a*::{*complete_linorder*,*linorder_topology*,*second_countable_topology*} *set*)
using *compact_complete_linorder*
by (*auto simp*: *seq_compact_eq_compact*[*symmetric*] *seq_compact_def*)

lemma *compact_eq_closed*:
fixes *S* :: *'a*::{*complete_linorder*,*linorder_topology*,*second_countable_topology*} *set*
shows *compact S* ⟷ *closed S*
using *closed_Int_compact*[*of S*, *OF* _ *compact_UNIV*] *compact_imp_closed*
by *auto*

lemma *closed_contains_Sup_cl*:
fixes *S* :: *'a*::{*complete_linorder*,*linorder_topology*,*second_countable_topology*} *set*
assumes *closed S*
and *S* ≠ {}
shows *Sup S* ∈ *S*
proof −
from *compact_eq_closed*[*of S*] *compact_attains_sup*[*of S*] *assms*
obtain *s* **where** *S*: *s* ∈ *S* ∀ *t*∈*S*. *t* ≤ *s*
by *auto*
then have *Sup S* = *s*
by (*auto intro*!: *Sup_eqI*)
with *S* **show** *?thesis*
by *simp*
qed

lemma *closed_contains_Inf_cl*:
fixes *S* :: *'a*::{*complete_linorder*,*linorder_topology*,*second_countable_topology*} *set*
assumes *closed S*
and *S* ≠ {}
shows *Inf S* ∈ *S*
proof −
from *compact_eq_closed*[*of S*] *compact_attains_inf*[*of S*] *assms*
obtain *s* **where** *S*: *s* ∈ *S* ∀ *t*∈*S*. *s* ≤ *t*
by *auto*
then have *Inf S* = *s*
by (*auto intro*!: *Inf_eqI*)
with *S* **show** *?thesis*
by *simp*
qed

instance *enat* :: *second_countable_topology*
proof
show ∃ *B*::*enat set set*. *countable B* ∧ *open* = *generate_topology B*

```
  proof (intro exI conjI)
    show countable (range lessThan ∪ range greaterThan::enat set set)
      by auto
  qed (simp add: open_enat_def)
qed

instance ereal :: second_countable_topology
proof (standard, intro exI conjI)
  let ?B = (⋃r∈ℚ. {{..< r}, {r <..}} :: ereal set set)
  show countable ?B
    by (auto intro: countable_rat)
  show open = generate_topology ?B
  proof (intro ext iffI)
    fix S :: ereal set
    assume open S
    then show generate_topology ?B S
      unfolding open_generated_order
    proof induct
      case (Basis b)
      then obtain e where b = {..<e} ∨ b = {e<..}
        by auto
     moreover have {..<e} = ⋃{{..<x}|x. x ∈ ℚ ∧ x < e} {e<..} = ⋃{{x<..}|x.
x ∈ ℚ ∧ e < x}
        by (auto dest: ereal_dense3
                 simp del: ex_simps
                 simp add: ex_simps[symmetric] conj_commute Rats_def image_iff)
      ultimately show ?case
        by (auto intro: generate_topology.intros)
    qed (auto intro: generate_topology.intros)
  next
    fix S
    assume generate_topology ?B S
    then show open S
      by induct auto
  qed
qed
```

This is a copy from *ereal :: second_countable_topology*. Maybe find a common super class of topological spaces where the rational numbers are densely embedded ?

```
instance ennreal :: second_countable_topology
proof (standard, intro exI conjI)
  let ?B = (⋃r∈ℚ. {{..< r}, {r <..}} :: ennreal set set)
  show countable ?B
    by (auto intro: countable_rat)
  show open = generate_topology ?B
  proof (intro ext iffI)
    fix S :: ennreal set
    assume open S
```

```
    then show generate_topology ?B S
      unfolding open_generated_order
    proof induct
      case (Basis b)
      then obtain e where b = {..<e} ∨ b = {e<..}
        by auto
     moreover have {..<e} = ⋃{{..<x}|x. x ∈ ℚ ∧ x < e} {e<..} = ⋃{{x<..}|x.
x ∈ ℚ ∧ e < x}
        by (auto dest: ennreal_rat_dense
                 simp del: ex_simps
                simp add: ex_simps[symmetric] conj_commute Rats_def image_iff)
      ultimately show ?case
        by (auto intro: generate_topology.intros)
    qed (auto intro: generate_topology.intros)
  next
    fix S
    assume generate_topology ?B S
    then show open S
      by induct auto
  qed
qed

lemma ereal_open_closed_aux:
  fixes S :: ereal set
  assumes open S
    and closed S
    and S: (−∞) ∉ S
  shows S = {}
proof (rule ccontr)
  assume ¬ ?thesis
  then have ∗: Inf S ∈ S

    by (metis assms(2) closed_contains_Inf_cl)
  {
    assume Inf S = −∞
    then have False
      using ∗ assms(3) by auto
  }
  moreover
  {
    assume Inf S = ∞
    then have S = {∞}
      by (metis Inf_eq_PInfty ‹S ≠ {}›)
    then have False
      by (metis assms(1) not_open_singleton)
  }
  moreover
  {
    assume fin: |Inf S| ≠ ∞
```

```
    from ereal_open_cont_interval[OF assms(1) * fin]
    obtain e where e: e > 0 {Inf S - e<..<Inf S + e} ⊆ S .
    then obtain b where b: Inf S - e < b b < Inf S
      using fin ereal_between[of Inf S e] dense[of Inf S - e]
      by auto
    then have b ∈ {Inf S - e <..< Inf S + e}
      using e fin ereal_between[of Inf S e]
      by auto
    then have b ∈ S
      using e by auto
    then have False
      using b by (metis complete_lattice_class.Inf_lower leD)
  }
  ultimately show False
    by auto
qed

lemma ereal_open_closed:
  fixes S :: ereal set
  shows open S ∧ closed S ⟷ S = {} ∨ S = UNIV
  using ereal_open_closed_aux open_closed by auto

lemma ereal_open_atLeast:
  fixes x :: ereal
  shows open {x..} ⟷ x = -∞
  by (metis atLeast_eq_UNIV_iff bot_ereal_def closed_atLeast ereal_open_closed
not_Ici_eq_empty)

lemma mono_closed_real:
  fixes S :: real set
  assumes mono: ∀ y z. y ∈ S ∧ y ≤ z ⟶ z ∈ S
    and closed S
  shows S = {} ∨ S = UNIV ∨ (∃ a. S = {a..})
proof -
  {
    assume S ≠ {}
    { assume ex: ∃ B. ∀ x∈S. B ≤ x
      then have *: ∀ x∈S. Inf S ≤ x
        using cInf_lower[of _ S] ex by (metis bdd_below_def)
      then have Inf S ∈ S
        by (meson ‹S ≠ {}› assms(2) bdd_belowI closed_contains_Inf)
      then have ∀ x. Inf S ≤ x ⟷ x ∈ S
        using mono[rule_format, of Inf S] *
        by auto
      then have S = {Inf S ..}
        by auto
      then have ∃ a. S = {a ..}
        by auto
    }
```

```
    moreover
    {
      assume ¬ (∃ B. ∀ x∈S. B ≤ x)
      then have nex: ∀ B. ∃ x∈S. x < B
        by (simp add: not_le)
      {
        fix y
        obtain x where x∈S and x < y
          using nex by auto
        then have y ∈ S
          using mono[rule_format, of x y] by auto
      }
      then have S = UNIV
        by auto
    }
    ultimately have S = UNIV ∨ (∃ a. S = {a ..})
      by blast
  }
  then show ?thesis
    by blast
qed

lemma mono_closed_ereal:
  fixes S :: real set
  assumes mono: ∀ y z. y ∈ S ∧ y ≤ z ⟶ z ∈ S
    and closed S
  shows ∃ a. S = {x. a ≤ ereal x}
proof −
  consider S = {} ∨ S = UNIV | (∃ a. S = {a..})
    using assms(2) mono mono_closed_real by blast
  then show ?thesis
  proof cases
    case 1
    then show ?thesis
     by (meson PInfty_neq_ereal(1) UNIV_eq_I bot.extremum empty_Collect_eq
ereal_infty_less_eq(1) mem_Collect_eq)
  next
    case 2
    then show ?thesis
    by (metis atLeast_iff ereal_less_eq(3) mem_Collect_eq subsetI subset_antisym)
  qed
qed

lemma Liminf_within:
  fixes f :: 'a::metric_space ⇒ 'b::complete_lattice
  shows Liminf (at x within S) f = (SUP e∈{0<..}. INF y∈(S ∩ ball x e − {x}).
f y)
  unfolding Liminf_def eventually_at
proof (rule SUP_eq, simp_all add: Ball_def Bex_def, safe)
```

```
  fix P d
  assume 0 < d and ∀ y. y ∈ S ⟶ y ≠ x ∧ dist y x < d ⟶ P y
  then have S ∩ ball x d − {x} ⊆ {x. P x}
    by (auto simp: dist_commute)
  then show ∃ r>0. Inf (f ‘ (Collect P)) ≤ Inf (f ‘ (S ∩ ball x r − {x}))
    by (intro exI[of _ d] INF_mono conjI ‹0 < d›) auto
next
  fix d :: real
  assume 0 < d
  then show ∃ P. (∃ d>0. ∀ xa. xa ∈ S ⟶ xa ≠ x ∧ dist xa x < d ⟶ P xa) ∧
    Inf (f ‘ (S ∩ ball x d − {x})) ≤ Inf (f ‘ (Collect P))
    by (intro exI[of _ λy. y ∈ S ∩ ball x d − {x}])
      (auto intro!: INF_mono exI[of _ d] simp: dist_commute)
qed

lemma Limsup_within:
  fixes f :: 'a::metric_space ⇒ 'b::complete_lattice
  shows Limsup (at x within S) f = (INF e∈{0<..}. SUP y∈(S ∩ ball x e − {x}).
f y)
  unfolding Limsup_def eventually_at
proof (rule INF_eq, simp_all add: Ball_def Bex_def, safe)
  fix P d
  assume 0 < d and ∀ y. y ∈ S ⟶ y ≠ x ∧ dist y x < d ⟶ P y
  then have S ∩ ball x d − {x} ⊆ {x. P x}
    by (auto simp: dist_commute)
  then show ∃ r>0. Sup (f ‘ (S ∩ ball x r − {x})) ≤ Sup (f ‘ (Collect P))
    by (intro exI[of _ d] SUP_mono conjI ‹0 < d›) auto
next
  fix d :: real
  assume 0 < d
  then show ∃ P. (∃ d>0. ∀ xa. xa ∈ S ⟶ xa ≠ x ∧ dist xa x < d ⟶ P xa) ∧
    Sup (f ‘ (Collect P)) ≤ Sup (f ‘ (S ∩ ball x d − {x}))
    by (intro exI[of _ λy. y ∈ S ∩ ball x d − {x}])
      (auto intro!: SUP_mono exI[of _ d] simp: dist_commute)
qed

lemma Liminf_at:
  fixes f :: 'a::metric_space ⇒ 'b::complete_lattice
  shows Liminf (at x) f = (SUP e∈{0<..}. INF y∈(ball x e − {x}). f y)
  using Liminf_within[of x UNIV f] by simp

lemma Limsup_at:
  fixes f :: 'a::metric_space ⇒ 'b::complete_lattice
  shows Limsup (at x) f = (INF e∈{0<..}. SUP y∈(ball x e − {x}). f y)
  using Limsup_within[of x UNIV f] by simp

lemma min_Liminf_at:
  fixes f :: 'a::metric_space ⇒ 'b::complete_linorder
  shows min (f x) (Liminf (at x) f) = (SUP e∈{0<..}. INF y∈ball x e. f y)
```

apply (*simp add*: *inf_min* [*symmetric*] *Liminf_at inf_commute* [*of f x*] *SUP_inf*)
apply (*metis* (*no_types*, *lifting*) *INF_insert centre_in_ball greaterThan_iff image_cong inf_commute insert_Diff*)
done

6.13.1 Extended-Real.thy

lemma *sum_constant_ereal*:
fixes *a*::*ereal*
shows $(\sum i{\in}I.\ a) = a * card\ I$
proof (*induction I rule*: *infinite_finite_induct*)
case (*insert i I*)
then show *?case*
by (*simp add*: *ereal_right_distrib flip*: *plus_ereal.simps*)
qed *auto*

lemma *real_lim_then_eventually_real*:
assumes $(u \longrightarrow ereal\ l)\ F$
shows *eventually* $(\lambda n.\ u\ n = ereal(real\_of\_ereal(u\ n)))\ F$
proof −
have $ereal\ l \in \{-\infty<..<(\infty::ereal)\}$ **by** *simp*
moreover have *open* $\{-\infty<..<(\infty::ereal)\}$ **by** *simp*
ultimately have *eventually* $(\lambda n.\ u\ n \in \{-\infty<..<(\infty::ereal)\})\ F$ **using** *assms tendsto_def* **by** *blast*
moreover have $\bigwedge x.\ x \in \{-\infty<..<(\infty::ereal)\} \Longrightarrow x = ereal(real\_of\_ereal\ x)$
using *ereal_real* **by** *auto*
ultimately show *?thesis* **by** (*metis* (*mono_tags*, *lifting*) *eventually_mono*)
qed

lemma *ereal_Inf_cmult*:
assumes $c>(0::real)$
shows $Inf\ \{ereal\ c * x\ |x.\ P\ x\} = ereal\ c * Inf\ \{x.\ P\ x\}$
proof −
have *bij* ((∗) (*ereal c*))
apply (*rule bij_betw_byWitness*[*of* _ $\lambda x.\ (x::ereal)$ / *c*], *auto simp*: *assms ereal_mult_divide*)
using *assms ereal_divide_eq* **by** *auto*
then have $ereal\ c * Inf\ \{x.\ P\ x\} = Inf\ ((*)\ (ereal\ c)\ `\ \{x.\ P\ x\})$
by (*simp add*: *assms ereal_mult_left_mono less_imp_le mono_def mono_bij_Inf*)
then show *?thesis*
by (*simp add*: *setcompr_eq_image*)
qed

Continuity of addition

The next few lemmas remove an unnecessary assumption in *tendsto_add_ereal*, culminating in *tendsto_add_ereal_general* which essentially says that the addition is continuous on ereal times ereal, except at $(-\infty, \infty)$ and $(\infty, -\infty)$. It is much more convenient in many situations, see for instance the

proof of *tendsto_sum_ereal* below.

lemma *tendsto_add_ereal_PInf*:
fixes y :: *ereal*
assumes *y*: $y \neq -\infty$
assumes *f*: $(f \longrightarrow \infty)\ F$ **and** *g*: $(g \longrightarrow y)\ F$
shows $((\lambda x.\ f\ x + g\ x) \longrightarrow \infty)\ F$
proof −
have $\exists C.$ *eventually* $(\lambda x.\ g\ x > ereal\ C)\ F$
proof (*cases y*)
case (*real r*)
have $y > y-1$ **using** *y real* **by** (*simp add*: *ereal_between*(1))
then have *eventually* $(\lambda x.\ g\ x > y - 1)\ F$ **using** *g y order_tendsto_iff* **by** *auto*
moreover have $y-1 = ereal(real\_of\_ereal(y-1))$
by (*metis real ereal_eq_1*(1) *ereal_minus*(1) *real_of_ereal.simps*(1))
ultimately have *eventually* $(\lambda x.\ g\ x > ereal(real\_of\_ereal(y - 1)))\ F$ **by** *simp*
then show *?thesis* **by** *auto*
next
case (*PInf*)
have *eventually* $(\lambda x.\ g\ x > ereal\ 0)\ F$ **using** *g PInf* **by** (*simp add*: *tendsto_PInfty*)
then show *?thesis* **by** *auto*
qed (*simp add*: *y*)
then obtain *C*::*real* **where** *ge*: *eventually* $(\lambda x.\ g\ x > ereal\ C)\ F$ **by** *auto*

{
fix *M*::*real*
have *eventually* $(\lambda x.\ f\ x > ereal(M - C))\ F$ **using** *f* **by** (*simp add*: *tendsto_PInfty*)
then have *eventually* $(\lambda x.\ (f\ x > ereal\ (M-C)) \wedge (g\ x > ereal\ C))\ F$
by (*auto simp*: *ge eventually_conj_iff*)
moreover have $\bigwedge x.\ ((f\ x > ereal\ (M-C)) \wedge (g\ x > ereal\ C)) \Longrightarrow (f\ x + g\ x > ereal\ M)$
using *ereal_add_strict_mono2* **by** *fastforce*
ultimately have *eventually* $(\lambda x.\ f\ x + g\ x > ereal\ M)\ F$ **using** *eventually_mono* **by** *force*
}
then show *?thesis* **by** (*simp add*: *tendsto_PInfty*)
qed

One would like to deduce the next lemma from the previous one, but the fact that $-(x + y)$ is in general different from $(-x) + (-y)$ in ereal creates difficulties, so it is more efficient to copy the previous proof.

lemma *tendsto_add_ereal_MInf*:
fixes y :: *ereal*
assumes *y*: $y \neq \infty$
assumes *f*: $(f \longrightarrow -\infty)\ F$ **and** *g*: $(g \longrightarrow y)\ F$
shows $((\lambda x.\ f\ x + g\ x) \longrightarrow -\infty)\ F$

```
proof −
  have ∃ C. eventually (λx. g x < ereal C) F
  proof (cases y)
    case (real r)
    have y < y+1 using y real by (simp add: ereal_between(1))
     then have eventually (λx. g x < y + 1) F using g y order_tendsto_iff by
force
    moreover have y+1 = ereal(real_of_ereal (y+1)) by (simp add: real)
     ultimately have eventually (λx. g x < ereal(real_of_ereal(y + 1))) F by
simp
    then show ?thesis by auto
  next
    case (MInf)
     have eventually (λx. g x < ereal 0) F using g MInf by (simp add: tend-
sto_MInfty)
    then show ?thesis by auto
  qed (simp add: y)
  then obtain C::real where ge: eventually (λx. g x < ereal C) F by auto

  {
    fix M::real
     have eventually (λx. f x < ereal(M − C)) F using f by (simp add: tend-
sto_MInfty)
    then have eventually (λx. (f x < ereal (M− C)) ∧ (g x < ereal C)) F
      by (auto simp: ge eventually_conj_iff)
    moreover have ⋀x. ((f x < ereal (M−C)) ∧ (g x < ereal C)) ⟹ (f x + g x
< ereal M)
      using ereal_add_strict_mono2 by fastforce
     ultimately have eventually (λx. f x + g x < ereal M) F using eventu-
ally_mono by force
  }
  then show ?thesis by (simp add: tendsto_MInfty)
qed

lemma tendsto_add_ereal_general1:
  fixes x y :: ereal
  assumes y: |y| ≠ ∞
  assumes f: (f ⟶ x) F and g: (g ⟶ y) F
  shows ((λx. f x + g x) ⟶ x + y) F
proof (cases x)
  case (real r)
  have a: |x| ≠ ∞ by (simp add: real)
  show ?thesis by (rule tendsto_add_ereal[OF a, OF y, OF f, OF g])
next
  case PInf
  then show ?thesis using tendsto_add_ereal_PInf assms by force
next
  case MInf
  then show ?thesis using tendsto_add_ereal_MInf assms
```

```
  by (metis abs_ereal.simps(3) ereal_MInfty_eq_plus)
qed

lemma tendsto_add_ereal_general2:
  fixes x y :: ereal
  assumes x: |x| ≠ ∞
     and f: (f ⟶ x) F and g: (g ⟶ y) F
  shows ((λx. f x + g x) ⟶ x + y) F
proof −
  have ((λx. g x + f x) ⟶ x + y) F
    by (metis (full_types) add.commute f g tendsto_add_ereal_general1 x)
  moreover have ⋀x. g x + f x = f x + g x using add.commute by auto
  ultimately show ?thesis by simp
qed
```

The next lemma says that the addition is continuous on *ereal*, except at the pairs $(-\infty, \infty)$ and $(\infty, -\infty)$.

```
lemma tendsto_add_ereal_general [tendsto_intros]:
  fixes x y :: ereal
  assumes ¬((x=∞ ∧ y=−∞) ∨ (x=−∞ ∧ y=∞))
     and f: (f ⟶ x) F and g: (g ⟶ y) F
  shows ((λx. f x + g x) ⟶ x + y) F
proof (cases x)
  case (real r)
  show ?thesis
    using real assms
    by (intro tendsto_add_ereal_general2; auto)
next
  case (PInf)
  then have y ≠ −∞ using assms by simp
  then show ?thesis using tendsto_add_ereal_PInf PInf assms by auto
next
  case (MInf)
  then have y ≠ ∞ using assms by simp
  then show ?thesis
    by (metis ereal_MInfty_eq_plus tendsto_add_ereal_MInf MInf f g)
qed
```

Continuity of multiplication

In the same way as for addition, we prove that the multiplication is continuous on ereal times ereal, except at $(\infty, 0)$ and $(-\infty, 0)$ and $(0, \infty)$ and $(0, -\infty)$, starting with specific situations.

```
lemma tendsto_mult_real_ereal:
  assumes (u ⟶ ereal l) F (v ⟶ ereal m) F
  shows ((λn. u n * v n) ⟶ ereal l * ereal m) F
proof −
```

```
 have ureal: eventually (λn. u n = ereal(real_of_ereal(u n))) F by (rule real_lim_then_eventually_real
assms(1)])
  then have ((λn. ereal(real_of_ereal(u n))) ⟶ ereal l) F using assms by
auto
 then have limu: ((λn. real_of_ereal(u n)) ⟶ l) F by auto
 have vreal: eventually (λn. v n = ereal(real_of_ereal(v n))) F by (rule real_lim_then_eventually_real[
assms(2)])
  then have ((λn. ereal(real_of_ereal(v n))) ⟶ ereal m) F using assms by
auto
  then have limv: ((λn. real_of_ereal(v n)) ⟶ m) F by auto

  {
   fix n assume u n = ereal(real_of_ereal(u n)) v n = ereal(real_of_ereal(v n))
   then have ereal(real_of_ereal(u n) * real_of_ereal(v n)) = u n * v n
     by (metis times_ereal.simps(1))
  }
  then have ∗: eventually (λn. ereal(real_of_ereal(u n) * real_of_ereal(v n)) =
u n * v n) F
    using eventually_elim2[OF ureal vreal] by auto

  have ((λn. real_of_ereal(u n) * real_of_ereal(v n)) ⟶ l * m) F
    using tendsto_mult[OF limu limv] by auto
  then have ((λn. ereal(real_of_ereal(u n)) * real_of_ereal(v n)) ⟶ ereal(l *
m)) F
    by auto
  then show ?thesis using ∗ filterlim_cong by fastforce
qed

lemma tendsto_mult_ereal_PInf:
  fixes f g::_ ⇒ ereal
  assumes (f ⟶ l) F l>0 (g ⟶ ∞) F
  shows ((λx. f x * g x) ⟶ ∞) F
proof −
  obtain a::real where 0 < ereal a a < l
    using assms(2) using ereal_dense2 by blast
  have ∗: eventually (λx. f x > a) F
    using ‹a < l› assms(1) by (simp add: order_tendsto_iff)
  {
    fix K::real
    define M where M = max K 1
    then have M > 0 by simp
    then have ereal(M/a) > 0 using ‹ereal a > 0› by simp
   then have ⋀x. ((f x > a) ∧ (g x > M/a)) ⟹ (f x * g x > ereal a * ereal(M/a))
       using ereal_mult_mono_strict'[where ?c = M/a, OF ‹0 < ereal a›] by
auto
    moreover have ereal a * ereal(M/a) = M using ‹ereal a > 0› by simp
    ultimately have ⋀x. ((f x > a) ∧ (g x > M/a)) ⟹ (f x * g x > M) by simp
    moreover have M ≥ K unfolding M_def by simp
    ultimately have Imp: ⋀x. ((f x > a) ∧ (g x > M/a)) ⟹ (f x * g x > K)
```

using *ereal_less_eq*(3) *le_less_trans* **by** *blast*

have *eventually* (λx. *g x* > *M*/*a*) *F* **using** *assms*(3) **by** (*simp add*: *tendsto_PInfty*)
then have *eventually* (λx. (*f x* > *a*) ∧ (*g x* > *M*/*a*)) *F*
using ∗ **by** (*auto simp*: *eventually_conj_iff*)
then have *eventually* (λx. *f x* ∗ *g x* > *K*) *F* **using** *eventually_mono Imp* **by** *force*
}
then show *?thesis* **by** (*auto simp*: *tendsto_PInfty*)
qed

lemma *tendsto_mult_ereal_pos*:
fixes *f g*::_ ⇒ *ereal*
assumes (*f* ⟶ *l*) *F* (*g* ⟶ *m*) *F l*>*0* *m*>*0*
shows ((λx. *f x* ∗ *g x*) ⟶ *l* ∗ *m*) *F*
proof (*cases*)
assume ∗: *l* = ∞ ∨ *m* = ∞
then show *?thesis*
proof (*cases*)
assume *m* = ∞
then show *?thesis* **using** *tendsto_mult_ereal_PInf assms* **by** *auto*
next
assume ¬(*m* = ∞)
then have *l* = ∞ **using** ∗ **by** *simp*
then have ((λx. *g x* ∗ *f x*) ⟶ *l* ∗ *m*) *F* **using** *tendsto_mult_ereal_PInf assms* **by** *auto*
moreover have ⋀x. *g x* ∗ *f x* = *f x* ∗ *g x* **using** *mult.commute* **by** *auto*
ultimately show *?thesis* **by** *simp*
qed
next
assume ¬(*l* = ∞ ∨ *m* = ∞)
then have *l* < ∞ *m* < ∞ **by** *auto*
then obtain *lr mr* **where** *l* = *ereal lr m* = *ereal mr*
using ‹*l*>*0*› ‹*m*>*0*› **by** (*metis ereal_cases ereal_less*(6) *not_less_iff_gr_or_eq*)
then show *?thesis* **using** *tendsto_mult_real_ereal assms* **by** *auto*
qed

We reduce the general situation to the positive case by multiplying by suitable signs. Unfortunately, as ereal is not a ring, all the neat sign lemmas are not available there. We give the bare minimum we need.

lemma *ereal_sgn_abs*:
fixes *l*::*ereal*
shows *sgn*(*l*) ∗ *l* = *abs*(*l*)
by (*cases l*, *auto simp*: *sgn_if ereal_less_uminus_reorder*)

lemma *sgn_squared_ereal*:
assumes *l* ≠ (*0*::*ereal*)
shows *sgn*(*l*) ∗ *sgn*(*l*) = *1*

using *assms* **by** (*cases l*, *auto simp*: *one_ereal_def sgn_if*)

lemma *tendsto_mult_ereal* [*tendsto_intros*]:
fixes *f g*::_ ⇒ *ereal*
assumes (*f* ⟶ *l*) *F* (*g* ⟶ *m*) *F* ¬((*l*=*0* ∧ *abs*(*m*) = ∞) ∨ (*m*=*0* ∧ *abs*(*l*) = ∞))
shows ((λ*x*. *f x* ∗ *g x*) ⟶ *l* ∗ *m*) *F*
proof (*cases*)
assume *l*=*0* ∨ *m*=*0*
then have *abs*(*l*) ≠ ∞ *abs*(*m*) ≠ ∞ **using** *assms*(*3*) **by** *auto*
then obtain *lr mr* **where** *l* = *ereal lr m* = *ereal mr* **by** *auto*
then show *?thesis* **using** *tendsto_mult_real_ereal assms* **by** *auto*
next
have *sgn_finite*: ⋀*a*::*ereal*. *abs*(*sgn a*) ≠ ∞
by (*metis MInfty_neq_ereal*(*2*) *PInfty_neq_ereal*(*2*) *abs_eq_infinity_cases ereal_times*(*1*) *ereal_times*(*3*) *ereal_uminus_eq_reorder sgn_ereal.elims*)
then have *sgn_finite2*: ⋀*a b*::*ereal*. *abs*(*sgn a* ∗ *sgn b*) ≠ ∞
by (*metis abs_eq_infinity_cases abs_ereal.simps*(*2*) *abs_ereal.simps*(*3*) *ereal_mult_eq_MInfty ereal_mult_eq_PInfty*)
assume ¬(*l*=*0* ∨ *m*=*0*)
then have *l* ≠ *0 m* ≠ *0* **by** *auto*
then have *abs*(*l*) > *0 abs*(*m*) > *0*
by (*metis abs_ereal_ge0 abs_ereal_less0 abs_ereal_pos ereal_uminus_uminus ereal_uminus_zero less_le not_less*)+
then have *sgn*(*l*) ∗ *l* > *0 sgn*(*m*) ∗ *m* > *0* **using** *ereal_sgn_abs* **by** *auto*
moreover have ((λ*x*. *sgn*(*l*) ∗ *f x*) ⟶ (*sgn*(*l*) ∗ *l*)) *F*
by (*rule tendsto_cmult_ereal*, *auto simp*: *sgn_finite assms*(*1*))
moreover have ((λ*x*. *sgn*(*m*) ∗ *g x*) ⟶ (*sgn*(*m*) ∗ *m*)) *F*
by (*rule tendsto_cmult_ereal*, *auto simp*: *sgn_finite assms*(*2*))
ultimately have ∗: ((λ*x*. (*sgn*(*l*) ∗ *f x*) ∗ (*sgn*(*m*) ∗ *g x*)) ⟶ (*sgn*(*l*) ∗ *l*) ∗ (*sgn*(*m*) ∗ *m*)) *F*
using *tendsto_mult_ereal_pos* **by** *force*
have ((λ*x*. (*sgn*(*l*) ∗ *sgn*(*m*)) ∗ ((*sgn*(*l*) ∗ *f x*) ∗ (*sgn*(*m*) ∗ *g x*))) ⟶ (*sgn*(*l*) ∗ *sgn*(*m*)) ∗ ((*sgn*(*l*) ∗ *l*) ∗ (*sgn*(*m*) ∗ *m*))) *F*
by (*rule tendsto_cmult_ereal*, *auto simp*: *sgn_finite2* ∗)
moreover have ⋀*x*. (*sgn*(*l*) ∗ *sgn*(*m*)) ∗ ((*sgn*(*l*) ∗ *f x*) ∗ (*sgn*(*m*) ∗ *g x*)) = *f x* ∗ *g x*
by (*metis mult.left_neutral sgn_squared_ereal*[*OF* ‹*l* ≠ *0*›] *sgn_squared_ereal*[*OF* ‹*m* ≠ *0*›] *mult.assoc mult.commute*)
moreover have (*sgn*(*l*) ∗ *sgn*(*m*)) ∗ ((*sgn*(*l*) ∗ *l*) ∗ (*sgn*(*m*) ∗ *m*)) = *l* ∗ *m*
by (*metis mult.left_neutral sgn_squared_ereal*[*OF* ‹*l* ≠ *0*›] *sgn_squared_ereal*[*OF* ‹*m* ≠ *0*›] *mult.assoc mult.commute*)
ultimately show *?thesis* **by** *auto*
qed

lemma *tendsto_cmult_ereal_general* [*tendsto_intros*]:
fixes *f*::_ ⇒ *ereal* **and** *c*::*ereal*
assumes (*f* ⟶ *l*) *F* ¬ (*l*=*0* ∧ *abs*(*c*) = ∞)
shows ((λ*x*. *c* ∗ *f x*) ⟶ *c* ∗ *l*) *F*

by (*cases c = 0*, *auto simp*: *assms tendsto_mult_ereal*)

Continuity of division

lemma *tendsto_inverse_ereal_PInf*:
fixes *u*::_ ⇒ *ereal*
assumes (*u* ⟶ ∞) *F*
shows ((λ*x*. *1*/ *u x*) ⟶ *0*) *F*
proof −
{
fix *e*::*real* **assume** *e*>*0*
have *1*/*e* < ∞ **by** *auto*
then have *eventually* (λ*n*. *u n* > *1*/*e*) *F* **using** *assms*(*1*) **by** (*simp add*: *tendsto_PInfty*)
moreover
{
fix *z*::*ereal* **assume** *z*>*1*/*e*
then have *z*>*0* **using** ‹*e*>*0*› **using** *less_le_trans not_le* **by** *fastforce*
then have *1*/*z* ≥ *0* **by** *auto*
moreover have *1*/*z* < *e*
proof (*cases z*)
case (*real r*)
then show *?thesis*
using ‹*0* < *e*› ‹*0* < *z*› ‹*ereal* (*1* / *e*) < *z*› *divide_less_eq ereal_divide_less_pos* **by** *fastforce*
qed (*use* ‹*0* < *e*› ‹*0* < *z*› **in** *auto*)
ultimately have *1*/*z* ≥ *0* *1*/*z* < *e* **by** *auto*
}
ultimately have *eventually* (λ*n*. *1*/*u n*<*e*) *F* *eventually* (λ*n*. *1*/*u n*≥*0*) *F* **by** (*auto simp*: *eventually_mono*)
} note ∗ = *this*
show *?thesis*
proof (*subst order_tendsto_iff*, *auto*)
fix *a*::*ereal* **assume** *a*<*0*
then show *eventually* (λ*n*. *1*/*u n* > *a*) *F* **using** ∗(*2*) *eventually_mono less_le_trans linordered_field_no_ub* **by** *fastforce*
next
fix *a*::*ereal* **assume** *a*>*0*
then obtain *e*::*real* **where** *e*>*0* *a*>*e* **using** *ereal_dense2 ereal_less*(*2*) **by** *blast*
then have *eventually* (λ*n*. *1*/*u n* < *e*) *F* **using** ∗(*1*) **by** *auto*
then show *eventually* (λ*n*. *1*/*u n* < *a*) *F* **using** ‹*a*>*e*› **by** (*metis* (*mono_tags*, *lifting*) *eventually_mono less_trans*)
qed
qed

The next lemma deserves to exist by itself, as it is so common and useful.

lemma *tendsto_inverse_real* [*tendsto_intros*]:
fixes *u*::_ ⇒ *real*
shows (*u* ⟶ *l*) *F* ⟹ *l* ≠ *0* ⟹ ((λ*x*. *1*/ *u x*) ⟶ *1*/*l*) *F*

```
  using tendsto_inverse unfolding inverse_eq_divide .

lemma tendsto_inverse_ereal [tendsto_intros]:
  fixes u::_ ⇒ ereal
  assumes (u ⟶ l) F l ≠ 0
  shows ((λx. 1/ u x) ⟶ 1/l) F
proof (cases l)
  case (real r)
  then have r ≠ 0 using assms(2) by auto
  then have 1/l = ereal(1/r) using real by (simp add: one_ereal_def)
  define v where v = (λn. real_of_ereal(u n))
 have ureal: eventually (λn. u n = ereal(v n)) F unfolding v_def using real_lim_then_eventually_real
assms(1) real by auto
  then have ((λn. ereal(v n)) ⟶ ereal r) F using assms real v_def by auto
  then have ∗: ((λn. v n) ⟶ r) F by auto
  then have ((λn. 1/v n) ⟶ 1/r) F using ‹r ≠ 0› tendsto_inverse_real by
auto
  then have lim: ((λn. ereal(1/v n)) ⟶ 1/l) F using ‹1/l = ereal(1/r)› by
auto

  have r ∈ −{0} open (−{(0::real)}) using ‹r ≠ 0› by auto
  then have eventually (λn. v n ∈ −{0}) F using ∗ using topological_tendstoD
by blast
  then have eventually (λn. v n ≠ 0) F by auto
  moreover
  {
    fix n assume H: v n ≠ 0 u n = ereal(v n)
    then have ereal(1/v n) = 1/ereal(v n) by (simp add: one_ereal_def)
    then have ereal(1/v n) = 1/u n using H(2) by simp
  }
  ultimately have eventually (λn. ereal(1/v n) = 1/u n) F using ureal eventu-
ally_elim2 by force
  with Lim_transform_eventually[OF lim this] show ?thesis by simp
next
  case (PInf)
  then have 1/l = 0 by auto
  then show ?thesis using tendsto_inverse_ereal_PInf assms PInf by auto
next
  case (MInf)
  then have 1/l = 0 by auto
  have 1/z = −1/ −z if z < 0 for z::ereal
    apply (cases z) using divide_ereal_def ‹ z < 0 › by auto
 moreover have eventually (λn. u n < 0) F by (metis (no_types) MInf assms(1)
tendsto_MInfty zero_ereal_def)
  ultimately have ∗: eventually (λn. −1/−u n = 1/u n) F by (simp add: even-
tually_mono)

  define v where v = (λn. − u n)
  have (v ⟶ ∞) F unfolding v_def using MInf assms(1) tendsto_uminus_ereal
```

by *fastforce*
then have $((\lambda n.\ 1/v\ n) \longrightarrow 0)\ F$ **using** *tendsto_inverse_ereal_PInf* **by** *auto*
then have $((\lambda n.\ -1/v\ n) \longrightarrow 0)\ F$ **using** *tendsto_uminus_ereal* **by** *fastforce*
then show *?thesis* **unfolding** *v_def* **using** *Lim_transform_eventually*[*OF* _ ∗]
‹ $1/l = 0$ › **by** *auto*
qed

lemma *tendsto_divide_ereal* [*tendsto_intros*]:
fixes $f\ g$::_ ⇒ *ereal*
assumes $(f \longrightarrow l)\ F\ (g \longrightarrow m)\ F\ m \neq 0\ \neg(abs(l) = \infty \wedge abs(m) = \infty)$
shows $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow l\ /\ m)\ F$
proof −
define h **where** $h = (\lambda x.\ 1/\ g\ x)$
have ∗: $(h \longrightarrow 1/m)\ F$ **unfolding** *h_def* **using** *assms*(2) *assms*(3) *tendsto_inverse_ereal* **by** *auto*
have $((\lambda x.\ f\ x * h\ x) \longrightarrow l * (1/m))\ F$
apply (*rule tendsto_mult_ereal*[*OF assms*(1) ∗]) **using** *assms*(3) *assms*(4) **by** (*auto simp*: *divide_ereal_def*)
moreover have $f\ x * h\ x = f\ x\ /\ g\ x$ **for** x **unfolding** *h_def* **by** (*simp add*: *divide_ereal_def*)
moreover have $l * (1/m) = l/m$ **by** (*simp add*: *divide_ereal_def*)
ultimately show *?thesis* **unfolding** *h_def* **using** *Lim_transform_eventually* **by** *auto*
qed

Further limits

The assumptions of $[\![|?x| \neq \infty;\ |?y| \neq \infty;\ (?f \longrightarrow ?x)\ ?F;\ (?g \longrightarrow ?y)\ ?F]\!] \Longrightarrow ((\lambda x.\ ?f\ x - ?g\ x) \longrightarrow ?x - ?y)\ ?F$ are too strong, we weaken them here.

lemma *tendsto_diff_ereal_general* [*tendsto_intros*]:
fixes $u\ v$::$'a$ ⇒ *ereal*
assumes $(u \longrightarrow l)\ F\ (v \longrightarrow m)\ F\ \neg((l = \infty \wedge m = \infty) \vee (l = -\infty \wedge m = -\infty))$
shows $((\lambda n.\ u\ n - v\ n) \longrightarrow l - m)\ F$
proof −
have $\infty = l \longrightarrow ((\lambda a.\ u\ a + - v\ a) \longrightarrow l + - m)\ F$
by (*metis* (*no_types*) *assms ereal_uminus_uminus tendsto_add_ereal_general tendsto_uminus_ereal*)
then have $((\lambda n.\ u\ n + (-v\ n)) \longrightarrow l + (-m))\ F$
by (*simp add*: *assms ereal_uminus_eq_reorder tendsto_add_ereal_general*)
then show *?thesis*
by (*simp add*: *minus_ereal_def*)
qed

lemma *id_nat_ereal_tendsto_PInf* [*tendsto_intros*]:
$(\lambda\ n$::*nat*. *real* $n) \longrightarrow \infty$
by (*simp add*: *filterlim_real_sequentially tendsto_PInfty_eq_at_top*)

lemma *tendsto_at_top_pseudo_inverse* [*tendsto_intros*]:
fixes *u*::*nat* ⇒ *nat*
assumes *LIM n sequentially. u n :> at_top*
shows *LIM n sequentially. Inf* {*N. u N* ≥ *n*} :> *at_top*
proof −
{
fix *C*::*nat*
define *M* **where** *M* = *Max* {*u n*| *n. n* ≤ *C*}+*1*
{
fix *n* **assume** *n* ≥ *M*
have *eventually* (λ*N. u N* ≥ *n*) *sequentially* **using** *assms*
by (*simp add*: *filterlim_at_top*)
then have ∗: {*N. u N* ≥ *n*} ≠ {} **by** *force*

have *N* > *C* **if** *u N* ≥ *n* **for** *N*
proof (*rule ccontr*)
assume ¬(*N* > *C*)
then have *u N* ≤ *Max* {*u n*| *n. n* ≤ *C*}
using *Max_ge* **by** (*simp add*: *setcompr_eq_image not_less*)
then show *False* **using** ‹*u N* ≥ *n*› ‹*n* ≥ *M*› **unfolding** *M_def* **by** *auto*
qed
then have ∗∗: {*N. u N* ≥ *n*} ⊆ {*C*..} **by** *fastforce*
have *Inf* {*N. u N* ≥ *n*} ≥ *C*
by (*metis* ∗ ∗∗ *Inf_nat_def1 atLeast_iff subset_eq*)
}
then have *eventually* (λ*n. Inf* {*N. u N* ≥ *n*} ≥ *C*) *sequentially*
using *eventually_sequentially* **by** *auto*
}
then show *?thesis* **using** *filterlim_at_top* **by** *auto*
qed

lemma *pseudo_inverse_finite_set*:
fixes *u*::*nat* ⇒ *nat*
assumes *LIM n sequentially. u n :> at_top*
shows *finite* {*N. u N* ≤ *n*}
proof −
fix *n*
have *eventually* (λ*N. u N* ≥ *n*+*1*) *sequentially* **using** *assms*
by (*simp add*: *filterlim_at_top*)
then obtain *N1* **where** *N1*: ⋀*N. N* ≥ *N1* ⟹ *u N* ≥ *n* + *1*
using *eventually_sequentially* **by** *auto*
have {*N. u N* ≤ *n*} ⊆ {..<*N1*}
by (*metis* (*no_types*, *lifting*) *N1 Suc_eq_plus1 lessThan_iff less_le_not_le mem_Collect_eq nle_le not_less_eq_eq subset_eq*)
then show *finite* {*N. u N* ≤ *n*} **by** (*simp add*: *finite_subset*)
qed

lemma *tendsto_at_top_pseudo_inverse2* [*tendsto_intros*]:
fixes *u*::*nat* ⇒ *nat*

```
  assumes LIM n sequentially. u n :> at_top
  shows LIM n sequentially. Max {N. u N ≤ n} :> at_top
proof −
  {
    fix N0::nat
    have N0 ≤ Max {N. u N ≤ n} if n ≥ u N0 for n
      by (simp add: assms pseudo_inverse_finite_set that)
    then have eventually (λn. N0 ≤ Max {N. u N ≤ n}) sequentially
      using eventually_sequentially by blast
  }
  then show ?thesis using filterlim_at_top by auto
qed

lemma ereal_truncation_top [tendsto_intros]:
  fixes x::ereal
  shows (λn::nat. min x n) ⟶ x
proof (cases x)
  case (real r)
  then obtain K::nat where K>0 K > abs(r) using reals_Archimedean2 gr0I
by auto
  then have min x n = x if n ≥ K for n apply (subst real, subst real, auto)
using that eq_iff by fastforce
 then have eventually (λn. min x n = x) sequentially using eventually_at_top_linorder
by blast
  then show ?thesis by (simp add: tendsto_eventually)
next
  case (PInf)
  then have min x n = n for n::nat by (auto simp: min_def)
  then show ?thesis using id_nat_ereal_tendsto_PInf PInf by auto
next
  case (MInf)
  then have min x n = x for n::nat by (auto simp: min_def)
  then show ?thesis by auto
qed

lemma ereal_truncation_real_top [tendsto_intros]:
  fixes x::ereal
  assumes x ≠ − ∞
  shows (λn::nat. real_of_ereal(min x n)) ⟶ x
proof (cases x)
  case (real r)
  then obtain K::nat where K>0 K > abs(r) using reals_Archimedean2 gr0I
by auto
  then have min x n = x if n ≥ K for n apply (subst real, subst real, auto)
using that eq_iff by fastforce
  then have real_of_ereal(min x n) = r if n ≥ K for n using real that by auto
  then have eventually (λn. real_of_ereal(min x n) = r) sequentially using even-
tually_at_top_linorder by blast
 then have (λn. real_of_ereal(min x n)) ⟶ r by (simp add: tendsto_eventually)
```

then show *?thesis* **using** *real* **by** *auto*
next
case (*PInf*)
then have *real_of_ereal*(*min x n*) = *n* **for** *n*::*nat* **by** (*auto simp*: *min_def*)
then show *?thesis* **using** *id_nat_ereal_tendsto_PInf PInf* **by** *auto*
qed (*simp add*: *assms*)

lemma *ereal_truncation_bottom* [*tendsto_intros*]:
fixes *x*::*ereal*
shows (λ*n*::*nat*. *max x* (− *real n*)) ⟶ *x*
proof (*cases x*)
case (*real r*)
then obtain *K*::*nat* **where** *K*>*0 K* > *abs*(*r*) **using** *reals_Archimedean2 gr0I* **by** *auto*
then have *max x* (−*real n*) = *x* **if** *n* ≥ *K* **for** *n* **apply** (*subst real*, *subst real*, *auto*) **using** *that eq_iff* **by** *fastforce*
then have *eventually* (λ*n*. *max x* (−*real n*) = *x*) *sequentially* **using** *eventually_at_top_linorder* **by** *blast*
then show *?thesis* **by** (*simp add*: *tendsto_eventually*)
next
case (*MInf*)
then have *max x* (−*real n*) = (−*1*)∗ *ereal*(*real n*) **for** *n*::*nat* **by** (*auto simp*: *max_def*)
moreover have (λ*n*. (−*1*)∗ *ereal*(*real n*)) ⟶ −∞
using *tendsto_cmult_ereal*[*of* −*1*, *OF* _ *id_nat_ereal_tendsto_PInf*] **by** (*simp add*: *one_ereal_def*)
ultimately show *?thesis* **using** *MInf* **by** *auto*
next
case (*PInf*)
then have *max x* (−*real n*) = *x* **for** *n*::*nat* **by** (*auto simp*: *max_def*)
then show *?thesis* **by** *auto*
qed

lemma *ereal_truncation_real_bottom* [*tendsto_intros*]:
fixes *x*::*ereal*
assumes *x* ≠ ∞
shows (λ*n*::*nat*. *real_of_ereal*(*max x* (− *real n*))) ⟶ *x*
proof (*cases x*)
case (*real r*)
then obtain *K*::*nat* **where** *K*>*0 K* > *abs*(*r*) **using** *reals_Archimedean2 gr0I* **by** *auto*
then have *max x* (−*real n*) = *x* **if** *n* ≥ *K* **for** *n* **apply** (*subst real*, *subst real*, *auto*) **using** *that eq_iff* **by** *fastforce*
then have *real_of_ereal*(*max x* (−*real n*)) = *r* **if** *n* ≥ *K* **for** *n* **using** *real that* **by** *auto*
then have *eventually* (λ*n*. *real_of_ereal*(*max x* (−*real n*)) = *r*) *sequentially* **using** *eventually_at_top_linorder* **by** *blast*
then have (λ*n*. *real_of_ereal*(*max x* (−*real n*))) ⟶ *r* **by** (*simp add*: *tendsto_eventually*)

then show *?thesis* **using** *real* **by** *auto*
next
case (*MInf*)
then have *real_of_ereal*(*max x* (−*real n*)) = (−*1*)∗ *ereal*(*real n*) **for** *n*::*nat* **by** (*auto simp*: *max_def*)
moreover have (λ*n*. (−*1*)∗ *ereal*(*real n*)) ⟶ −∞
using *tendsto_cmult_ereal*[*of* −*1*, *OF* _ *id_nat_ereal_tendsto_PInf*] **by** (*simp add*: *one_ereal_def*)
ultimately show *?thesis* **using** *MInf* **by** *auto*
qed (*simp add*: *assms*)

the next one is copied from *tendsto_sum*.

lemma *tendsto_sum_ereal* [*tendsto_intros*]:
fixes *f* :: *'a* ⇒ *'b* ⇒ *ereal*
assumes ⋀*i*. *i* ∈ *S* ⟹ (*f i* ⟶ *a i*) *F*
⋀*i*. *abs*(*a i*) ≠ ∞
shows ((λ*x*. ∑ *i*∈*S*. *f i x*) ⟶ (∑ *i*∈*S*. *a i*)) *F*
proof (*cases finite S*)
assume *finite S* **then show** *?thesis* **using** *assms*
by (*induct*, *simp*, *simp add*: *tendsto_add_ereal_general2 assms*)
qed(*simp*)

lemma *continuous_ereal_abs*:
continuous_on (*UNIV*::*ereal set*) *abs*
proof −
have *continuous_on* ({..*0*} ∪ {(*0*::*ereal*)..}) *abs*
proof (*intro continuous_on_closed_Un continuous_intros*)
show *continuous_on* {..*0*::*ereal*} *abs*
by (*metis abs_ereal_ge0 abs_ereal_less0 continuous_on_eq antisym_conv1 atMost_iff continuous_uminus_ereal ereal_uminus_zero*)
show *continuous_on* {*0*::*ereal*..} *abs*
by (*metis abs_ereal_ge0 atLeast_iff continuous_on_eq continuous_on_id*)
qed
moreover have (*UNIV*::*ereal set*) = {..*0*} ∪ {(*0*::*ereal*)..} **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemmas *continuous_on_compose_ereal_abs*[*continuous_intros*] =
continuous_on_compose2[*OF continuous_ereal_abs* _ *subset_UNIV*]

lemma *tendsto_abs_ereal* [*tendsto_intros*]:
assumes (*u* ⟶ (*l*::*ereal*)) *F*
shows ((λ*n*. *abs*(*u n*)) ⟶ *abs l*) *F*
using *continuous_ereal_abs assms* **by** (*metis UNIV_I continuous_on tendsto_compose*)

lemma *ereal_minus_real_tendsto_MInf* [*tendsto_intros*]:
(λ*x*. *ereal* (− *real x*)) ⟶ − ∞
by (*subst uminus_ereal.simps*(*1*)[*symmetric*], *intro tendsto_intros*)

6.13.2 Extended-Nonnegative-Real.thy

lemma *tendsto_diff_ennreal_general* [*tendsto_intros*]:
fixes $u\ v::'a \Rightarrow ennreal$
assumes $(u \longrightarrow l)\ F\ (v \longrightarrow m)\ F\ \neg(l = \infty \wedge m = \infty)$
shows $((\lambda n.\ u\ n - v\ n) \longrightarrow l - m)\ F$
proof −
have $((\lambda n.\ e2ennreal(enn2ereal(u\ n) - enn2ereal(v\ n))) \longrightarrow e2ennreal(enn2ereal\ l - enn2ereal\ m))\ F$
by (*intro tendsto_intros*) (*use assms* **in** *auto*)
then show *?thesis* **by** *auto*
qed

lemma *tendsto_mult_ennreal* [*tendsto_intros*]:
fixes $l\ m::ennreal$
assumes $(u \longrightarrow l)\ F\ (v \longrightarrow m)\ F\ \neg((l = 0 \wedge m = \infty) \vee (l = \infty \wedge m = 0))$
shows $((\lambda n.\ u\ n * v\ n) \longrightarrow l * m)\ F$
proof −
have $((\lambda n.\ e2ennreal(enn2ereal\ (u\ n) * enn2ereal\ (v\ n))) \longrightarrow e2ennreal(enn2ereal\ l * enn2ereal\ m))\ F$
by (*intro tendsto_intros*) (*use assms enn2ereal_inject zero_ennreal.rep_eq* **in** *fastforce*)+
moreover have $e2ennreal(enn2ereal\ (u\ n) * enn2ereal\ (v\ n)) = u\ n * v\ n$ **for** n
by (*subst times_ennreal.abs_eq*[*symmetric*], *auto simp*: *eq_onp_same_args*)
moreover have $e2ennreal(enn2ereal\ l * enn2ereal\ m) = l * m$
by (*subst times_ennreal.abs_eq*[*symmetric*], *auto simp*: *eq_onp_same_args*)
ultimately show *?thesis*
by *auto*
qed

6.13.3 monoset

definition (**in** *order*) *mono_set*:
$mono\_set\ S \longleftrightarrow (\forall x\ y.\ x \leq y \longrightarrow x \in S \longrightarrow y \in S)$

lemma (**in** *order*) *mono_greaterThan* [*intro*, *simp*]: *mono_set* $\{B<..\}$ **unfolding** *mono_set* **by** *auto*
lemma (**in** *order*) *mono_atLeast* [*intro*, *simp*]: *mono_set* $\{B..\}$ **unfolding** *mono_set* **by** *auto*
lemma (**in** *order*) *mono_UNIV* [*intro*, *simp*]: *mono_set UNIV* **unfolding** *mono_set* **by** *auto*
lemma (**in** *order*) *mono_empty* [*intro*, *simp*]: *mono_set* $\{\}$ **unfolding** *mono_set* **by** *auto*

lemma (**in** *complete_linorder*) *mono_set_iff*:
fixes $S :: 'a\ set$
defines $a \equiv Inf\ S$
shows $mono\_set\ S \longleftrightarrow S = \{a <..\} \vee S = \{a..\}$ (**is** $\_ = ?c$)
proof

```
  assume mono_set S
  then have mono: ⋀x y. x ≤ y ⟹ x ∈ S ⟹ y ∈ S
    by (auto simp: mono_set)
  show ?c
  proof cases
    assume a ∈ S
    show ?c
      using mono[OF _ ‹a ∈ S›]
      by (auto intro: Inf_lower simp: a_def)
  next
    assume a ∉ S
    have S = {a <..}
    proof safe
      fix x assume x ∈ S
      then have a ≤ x
        unfolding a_def by (rule Inf_lower)
      then show a < x
        using ‹x ∈ S› ‹a ∉ S› by (cases a = x) auto
    next
      fix x assume a < x
      then obtain y where y < x y ∈ S
        unfolding a_def Inf_less_iff ..
      with mono[of y x] show x ∈ S
        by auto
    qed
    then show ?c ..
  qed
qed auto

lemma ereal_open_mono_set:
  fixes S :: ereal set
  shows open S ∧ mono_set S ⟷ S = UNIV ∨ S = {Inf S <..}
  by (metis Inf_UNIV atLeast_eq_UNIV_iff ereal_open_atLeast
    ereal_open_closed mono_set_iff open_ereal_greaterThan)

lemma ereal_closed_mono_set:
  fixes S :: ereal set
  shows closed S ∧ mono_set S ⟷ S = {} ∨ S = {Inf S ..}
  by (metis Inf_UNIV atLeast_eq_UNIV_iff closed_ereal_atLeast
    ereal_open_closed mono_empty mono_set_iff open_ereal_greaterThan)

lemma ereal_Liminf_Sup_monoset:
  fixes f :: 'a ⇒ ereal
  shows Liminf net f =
    Sup {l. ∀S. open S ⟶ mono_set S ⟶ l ∈ S ⟶ eventually (λx. f x ∈ S)
net}
    (is _ = Sup ?A)
proof (safe intro!: Liminf_eqI complete_lattice_class.Sup_upper complete_lattice_class.Sup_least)
  fix P
```

```
  assume P: eventually P net
  fix S
  assume S: mono_set S Inf (f ‘ (Collect P)) ∈ S
  {
    fix x
    assume P x
    then have Inf (f ‘ (Collect P)) ≤ f x
      by (intro complete_lattice_class.INF_lower) simp
    with S have f x ∈ S
      by (simp add: mono_set)
  }
  with P show eventually (λx. f x ∈ S) net
    by (auto elim: eventually_mono)
next
  fix y l
  assume S: ∀S. open S ⟶ mono_set S ⟶ l ∈ S ⟶ eventually  (λx. f x ∈ S)
net
  assume P: ∀P. eventually P net ⟶ Inf (f ‘ (Collect P)) ≤ y
  show l ≤ y
  proof (rule dense_le)
    fix B
    assume B < l
    then have eventually (λx. f x ∈ {B <..}) net
      by (intro S[rule_format]) auto
    then have Inf (f ‘ {x. B < f x}) ≤ y
      using P by auto
    moreover have B ≤ Inf (f ‘ {x. B < f x})
      by (intro INF_greatest) auto
    ultimately show B ≤ y
      by simp
  qed
qed

lemma ereal_Limsup_Inf_monoset:
  fixes f :: 'a ⇒ ereal
  shows Limsup net f =
    Inf {l. ∀S. open S ⟶ mono_set (uminus ‘ S) ⟶ l ∈ S ⟶ eventually (λx.
f x ∈ S) net}
    (is _ = Inf ?A)
proof (safe intro!: Limsup_eqI complete_lattice_class.Inf_lower complete_lattice_class.Inf_greatest)
  fix P
  assume P: eventually P net
  fix S
  assume S: mono_set (uminus‘S) Sup (f ‘ (Collect P)) ∈ S
  {
    fix x
    assume P x
    then have f x ≤ Sup (f ‘ (Collect P))
      by (intro complete_lattice_class.SUP_upper) simp
```

```
    with S(1)[unfolded mono_set, rule_format, of − Sup (f ‘ (Collect P)) − f x]
S(2)
    have f x ∈ S
      by (simp add: inj_image_mem_iff) }
  with P show eventually (λx. f x ∈ S) net
    by (auto elim: eventually_mono)
next
  fix y l
  assume S: ∀ S. open S ⟶ mono_set (uminus ‘ S) ⟶ l ∈ S ⟶ eventually
(λx. f x ∈ S) net
  assume P: ∀ P. eventually P net ⟶ y ≤ Sup (f ‘ (Collect P))
  show y ≤ l
  proof (rule dense_ge)
    fix B
    assume l < B
    then have eventually (λx. f x ∈ {..< B}) net
      by (intro S[rule_format]) auto
    then have y ≤ Sup (f ‘ {x. f x < B})
      using P by auto
    moreover have Sup (f ‘ {x. f x < B}) ≤ B
      by (intro SUP_least) auto
    ultimately show y ≤ B
      by simp
  qed
qed

lemma liminf_bounded_open:
  fixes x :: nat ⇒ ereal
  shows x0 ≤ liminf x ⟷ (∀ S. open S ⟶ mono_set S ⟶ x0 ∈ S ⟶ (∃ N.
∀ n≥N. x n ∈ S))
  (is _ ⟷ ?P x0)
proof
  assume ?P x0
  then show x0 ≤ liminf x
    unfolding ereal_Liminf_Sup_monoset eventually_sequentially
    by (intro complete_lattice_class.Sup_upper) auto
next
  assume x0 ≤ liminf x
  {
    fix S :: ereal set
    assume om: open S mono_set S x0 ∈ S
    then have ∃ N. ∀ n≥N. x n ∈ S
          by (metis ‹x0 ≤ liminf x› ereal_open_mono_set greaterThan_iff lim-
inf_bounded_iff om UNIV_I)
  }
  then show ?P x0
    by auto
qed
```

```
lemma limsup_finite_then_bounded:
  fixes u::nat ⇒ real
  assumes limsup u < ∞
  shows ∃ C. ∀ n. u n ≤ C
proof −
  obtain C where C: limsup u < C C < ∞ using assms ereal_dense2 by blast
  then have C = ereal(real_of_ereal C) using ereal_real by force
  have eventually (λn. u n < C) sequentially
    using SUP_lessD eventually_mono C(1)
    by (fastforce simp: INF_less_iff Limsup_def)
 then obtain N where N: ⋀n. n ≥ N ⟹ u n < C using eventually_sequentially
by auto
  define D where D = max (real_of_ereal C) (Max {u n |n. n ≤ N})
  have ⋀n. u n ≤ D
  proof −
    fix n show u n ≤ D
    proof (cases)
      assume ∗: n ≤ N
      have u n ≤ Max {u n |n. n ≤ N} by (rule Max_ge, auto simp: ∗)
      then show u n ≤ D unfolding D_def by linarith
    next
      assume ¬(n ≤ N)
      then have n ≥ N by simp
      then have u n < C using N by auto
        then have u n < real_of_ereal C using ‹C = ereal(real_of_ereal C)›
less_ereal.simps(1) by fastforce
      then show u n ≤ D unfolding D_def by linarith
    qed
  qed
  then show ?thesis by blast
qed

lemma liminf_finite_then_bounded_below:
  fixes u::nat ⇒ real
  assumes liminf u > −∞
  shows ∃ C. ∀ n. u n ≥ C
proof −
  obtain C where C: liminf u > C C > −∞ using assms using ereal_dense2
by blast
  then have C = ereal(real_of_ereal C) using ereal_real by force
  have eventually (λn. u n > C) sequentially
    using eventually_elim2 less_INF_D C(1)
    by (fastforce simp: less_SUP_iff Liminf_def)
 then obtain N where N: ⋀n. n ≥ N ⟹ u n > C using eventually_sequentially
by auto
  define D where D = min (real_of_ereal C) (Min {u n |n. n ≤ N})
  have ⋀n. u n ≥ D
  proof −
    fix n show u n ≥ D
```

```
    proof (cases)
      assume *: n ≤ N
      have u n ≥ Min {u n |n. n ≤ N} by (rule Min_le, auto simp: *)
      then show u n ≥ D unfolding D_def by linarith
    next
      assume ¬(n ≤ N)
      then have n ≥ N by simp
      then have u n > C using N by auto
      then have u n > real_of_ereal C
        using ‹C = ereal(real_of_ereal C)› less_ereal.simps(1) by fastforce
      then show u n ≥ D unfolding D_def by linarith
    qed
  qed
  then show ?thesis by blast
qed

lemma liminf_upper_bound:
  fixes u:: nat ⇒ ereal
  assumes liminf u < l
  shows ∃ N>k. u N < l
by (metis assms gt_ex less_le_trans liminf_bounded_iff not_less)

lemma limsup_shift:
  limsup (λn. u (n+1)) = limsup u
proof −
  have (SUP m∈{n+1..}. u m) = (SUP m∈{n..}. u (m + 1)) for n
    by (rule SUP_eq) (use Suc_le_D in auto)
  then have a: (INF n. SUP m∈{n..}. u (m + 1)) = (INF n. (SUP m∈{n+1..}.
u m)) by auto
   have b: (INF n. (SUP m∈{n+1..}. u m)) = (INF n∈{1..}. (SUP m∈{n..}. u
m))
    by (rule INF_eq) (use Suc_le_D in auto)
  have (INF n∈{1..}. v n) = (INF n. v n) if decseq v for v::nat ⇒ 'a
    by (rule INF_eq) (use ‹decseq v› decseq_Suc_iff in auto)
 moreover have decseq (λn. (SUP m∈{n..}. u m)) by (simp add: SUP_subset_mono
decseq_def)
   ultimately have c: (INF n∈{1..}. (SUP m∈{n..}. u m)) = (INF n. (SUP
m∈{n..}. u m)) by simp
  have (INF n. Sup (u ‘ {n..})) = (INF n. SUP m∈{n..}. u (m + 1)) using a b
c by simp
  then show ?thesis by (auto cong: limsup_INF_SUP)
qed

lemma limsup_shift_k:
  limsup (λn. u (n+k)) = limsup u
proof (induction k)
  case (Suc k)
 have limsup (λn. u (n+k+1)) = limsup (λn. u (n+k)) using limsup_shift[where
?u=λn. u(n+k)] by simp
```

then show *?case* **using** *Suc.IH* **by** *simp*
qed (*auto*)

lemma *liminf_shift*:
liminf (λ*n*. *u* (*n*+*1*)) = *liminf u*
proof −
have (*INF m*∈{*n*+*1*..}. *u m*) = (*INF m*∈{*n*..}. *u* (*m* + *1*)) **for** *n*
by (*rule INF_eq*) (*use Suc_le_D* **in** *auto*)
then have *a*: (*SUP n. INF m*∈{*n*..}. *u* (*m* + *1*)) = (*SUP n.* (*INF m*∈{*n*+*1*..}. *u m*)) **by** *auto*
have *b*: (*SUP n.* (*INF m*∈{*n*+*1*..}. *u m*)) = (*SUP n*∈{*1*..}. (*INF m*∈{*n*..}. *u m*))
by (*rule SUP_eq*) (*use Suc_le_D* **in** *auto*)
have (*SUP n*∈{*1*..}. *v n*) = (*SUP n. v n*) **if** *incseq v* **for** *v*::*nat* ⇒ *'a*
by (*rule SUP_eq*) (*use* ‹*incseq v*› *incseq_Suc_iff* **in** *auto*)
moreover have *incseq* (λ*n*. (*INF m*∈{*n*..}. *u m*)) **by** (*simp add*: *INF_superset_mono mono_def*)
ultimately have *c*: (*SUP n*∈{*1*..}. (*INF m*∈{*n*..}. *u m*)) = (*SUP n.* (*INF m*∈{*n*..}. *u m*)) **by** *simp*
have (*SUP n. Inf* (*u* ‘ {*n*..})) = (*SUP n. INF m*∈{*n*..}. *u* (*m* + *1*)) **using** *a b c* **by** *simp*
then show *?thesis* **by** (*auto cong*: *liminf_SUP_INF*)
qed

lemma *liminf_shift_k*:
liminf (λ*n*. *u* (*n*+*k*)) = *liminf u*
proof (*induction k*)
case (*Suc k*)
have *liminf* (λ*n*. *u* (*n*+*k*+*1*)) = *liminf* (λ*n*. *u* (*n*+*k*))
using *liminf_shift*[**where** *?u*=λ*n*. *u*(*n*+*k*)] **by** *simp*
then show *?case* **using** *Suc.IH* **by** *simp*
qed (*auto*)

lemma *Limsup_obtain*:
fixes *u*::*_* ⇒ *'a* :: *complete_linorder*
assumes *Limsup F u* > *c*
shows ∃ *i*. *u i* > *c*
proof −
have (*INF P*∈{*P. eventually P F*}. *SUP x*∈{*x. P x*}. *u x*) > *c* **using** *assms* **by** (*simp add*: *Limsup_def*)
then show *?thesis* **by** (*metis eventually_True mem_Collect_eq less_INF_D less_SUP_iff*)
qed

The next lemma is extremely useful, as it often makes it possible to reduce statements about limsups to statements about limits.

lemma *limsup_subseq_lim*:
fixes *u*::*nat* ⇒ *'a* :: {*complete_linorder*, *linorder_topology*}
shows ∃ *r*::*nat*⇒*nat. strict_mono r* ∧ (*u o r*) ⟶ *limsup u*

proof (*cases*)
assume $\forall n.\ \exists p{>}n.\ \forall m{\geq}p.\ u\ m \leq u\ p$
then have $\exists r.\ \forall n.\ (\forall m{\geq}r\ n.\ u\ m \leq u\ (r\ n)) \wedge r\ n < r\ (Suc\ n)$
by (*intro dependent_nat_choice*) (*auto simp*: *conj_commute*)
then obtain $r :: nat \Rightarrow nat$ **where** *strict_mono r* **and** *mono*: $\bigwedge n\ m.\ r\ n \leq m \Longrightarrow u\ m \leq u\ (r\ n)$
by (*auto simp*: *strict_mono_Suc_iff*)
define *umax* **where** $umax = (\lambda n.\ (SUP\ m{\in}\{n..\}.\ u\ m))$
have *decseq umax* **unfolding** *umax_def* **by** (*simp add*: *SUP_subset_mono antimono_def*)
then have $umax \longrightarrow limsup\ u$ **unfolding** *umax_def* **by** (*metis LIMSEQ_INF limsup_INF_SUP*)
then have $*$: $(umax\ o\ r) \longrightarrow limsup\ u$ **by** (*simp add*: *LIMSEQ_subseq_LIMSEQ* ‹*strict_mono r*›)
have $\bigwedge n.\ umax(r\ n) = u(r\ n)$ **unfolding** *umax_def* **using** *mono*
by (*metis SUP_le_iff antisym atLeast_def mem_Collect_eq order_refl*)
then have $umax\ o\ r = u\ o\ r$ **unfolding** *o_def* **by** *simp*
then have $(u\ o\ r) \longrightarrow limsup\ u$ **using** $*$ **by** *simp*
then show *?thesis* **using** ‹*strict_mono r*› **by** *blast*
next
assume $\neg\ (\forall n.\ \exists p{>}n.\ (\forall m{\geq}p.\ u\ m \leq u\ p))$
then obtain N **where** N: $\bigwedge p.\ p > N \Longrightarrow \exists m{>}p.\ u\ p < u\ m$ **by** (*force simp*: *not_le le_less*)
have $\exists r.\ \forall n.\ N < r\ n \wedge r\ n < r\ (Suc\ n) \wedge (\forall i{\in}\ \{N{<}..r\ (Suc\ n)\}.\ u\ i \leq u\ (r\ (Suc\ n)))$
proof (*rule dependent_nat_choice*)
fix x **assume** $N < x$
then have *a*: *finite* $\{N{<}..x\}$ $\{N{<}..x\} \neq \{\}$ **by** *simp_all*
have *Max* $\{u\ i\ |i.\ i \in \{N{<}..x\}\} \in \{u\ i\ |i.\ i \in \{N{<}..x\}\}$ **apply** (*rule Max_in*) **using** *a* **by** (*auto*)
then obtain p **where** $p \in \{N{<}..x\}$ **and** *upmax*: $u\ p = Max\{u\ i\ |i.\ i \in \{N{<}..x\}\}$ **by** *auto*
define U **where** $U = \{m.\ m > p \wedge u\ p < u\ m\}$
have $U \neq \{\}$ **unfolding** *U_def* **using** $N[of\ p]$ ‹$p \in \{N{<}..x\}$› **by** *auto*
define y **where** $y = Inf\ U$
then have $y \in U$ **using** ‹$U \neq \{\}$› **by** (*simp add*: *Inf_nat_def1*)
have *a*: $\bigwedge i.\ i \in \{N{<}..x\} \Longrightarrow u\ i \leq u\ p$
proof −
fix i **assume** $i \in \{N{<}..x\}$
then have $u\ i \in \{u\ i\ |i.\ i \in \{N{<}..x\}\}$ **by** *blast*
then show $u\ i \leq u\ p$ **using** *upmax* **by** *simp*
qed
moreover have $u\ p < u\ y$ **using** ‹$y \in U$› *U_def* **by** *auto*
ultimately have $y \notin \{N{<}..x\}$ **using** *not_le* **by** *blast*
moreover have $y > N$ **using** ‹$y \in U$› *U_def* ‹$p \in \{N{<}..x\}$› **by** *auto*
ultimately have $y > x$ **by** *auto*

have $\bigwedge i.\ i \in \{N{<}..y\} \Longrightarrow u\ i \leq u\ y$
proof −

```
    fix i assume i ∈ {N<..y} show u i ≤ u y
    proof (cases)
      assume i = y
      then show ?thesis by simp
    next
      assume ¬(i=y)
      then have i:i ∈ {N<..<y} using ‹i ∈ {N<..y}› by simp
      have u i ≤ u p
      proof (cases)
        assume i ≤ x
        then have i ∈ {N<..x} using i by simp
        then show ?thesis using a by simp
      next
        assume ¬(i ≤ x)
        then have i > x by simp
        then have ∗: i > p using ‹p ∈ {N<..x}› by simp
        have i < Inf U using i y_def by simp
        then have i ∉ U using Inf_nat_def not_less_Least by auto
        then show ?thesis using U_def ∗ by auto
      qed
      then show u i ≤ u y using ‹u p < u y› by auto
    qed
  qed
  then have N < y ∧ x < y ∧ (∀ i∈{N<..y}. u i ≤ u y) using ‹y > x› ‹y > N› by auto
  then show ∃ y>N. x < y ∧ (∀ i∈{N<..y}. u i ≤ u y) by auto
 qed (auto)
 then obtain r where r: ∀ n. N < r n ∧ r n < r (Suc n) ∧ (∀ i∈ {N<..r (Suc n)}. u i ≤ u (r (Suc n))) by auto
 have strict_mono r using r by (auto simp: strict_mono_Suc_iff)
 have incseq (u o r) unfolding o_def using r by (simp add: incseq_SucI order.strict_implies_order)
 then have (u o r) ⟶ (SUP n. (u o r) n) using LIMSEQ_SUP by blast
 then have limsup (u o r) = (SUP n. (u o r) n) by (simp add: lim_imp_Limsup)
 moreover have limsup (u o r) ≤ limsup u using ‹strict_mono r› by (simp add: limsup_subseq_mono)
 ultimately have (SUP n. (u o r) n) ≤ limsup u by simp

 {
  fix i assume i: i ∈ {N<..}
   obtain n where i < r (Suc n) using ‹strict_mono r› using Suc_le_eq seq_suble by blast
  then have i ∈ {N<..r(Suc n)} using i by simp
  then have u i ≤ u (r(Suc n)) using r by simp
 then have u i ≤ (SUP n. (u o r) n) unfolding o_def by (meson SUP_upper2 UNIV_I)
 }
 then have (SUP i∈{N<..}. u i) ≤ (SUP n. (u o r) n) using SUP_least by blast
```

then have *limsup u* ≤ (*SUP n.* (*u o r*) *n*) **unfolding** *Limsup_def*
by (*metis* (*mono_tags*, *lifting*) *INF_lower2 atLeast_Suc_greaterThan atLeast_def eventually_ge_at_top mem_Collect_eq*)
then have *limsup u* = (*SUP n.* (*u o r*) *n*) **using** ‹(*SUP n.* (*u o r*) *n*) ≤ *limsup u*› **by** *simp*
then have (*u o r*) ⟶ *limsup u* **using** ‹(*u o r*) ⟶ (*SUP n.* (*u o r*) *n*)› **by** *simp*
then show *?thesis* **using** ‹*strict_mono r*› **by** *auto*
qed

lemma *liminf_subseq_lim*:
fixes *u*::*nat* ⇒ *'a* :: {*complete_linorder*, *linorder_topology*}
shows ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*u o r*) ⟶ *liminf u*
proof (*cases*)
assume ∀ *n*. ∃ *p*>*n*. ∀ *m*≥*p*. *u m* ≥ *u p*
then have ∃ *r*. ∀ *n*. (∀ *m*≥*r n*. *u m* ≥ *u* (*r n*)) ∧ *r n* < *r* (*Suc n*)
by (*intro dependent_nat_choice*) (*auto simp*: *conj_commute*)
then obtain *r* :: *nat* ⇒ *nat* **where** *strict_mono r* **and** *mono*: ⋀*n m*. *r n* ≤ *m* ⟹ *u m* ≥ *u* (*r n*)
by (*auto simp*: *strict_mono_Suc_iff*)
define *umin* **where** *umin* = (λ*n*. (*INF m*∈{*n*..}. *u m*))
have *incseq umin* **unfolding** *umin_def* **by** (*simp add*: *INF_superset_mono incseq_def*)
then have *umin* ⟶ *liminf u* **unfolding** *umin_def* **by** (*metis LIMSEQ_SUP liminf_SUP_INF*)
then have ∗: (*umin o r*) ⟶ *liminf u* **by** (*simp add*: *LIMSEQ_subseq_LIMSEQ* ‹*strict_mono r*›)
have ⋀*n*. *umin*(*r n*) = *u*(*r n*) **unfolding** *umin_def* **using** *mono*
by (*metis le_INF_iff antisym atLeast_def mem_Collect_eq order_refl*)
then have *umin o r* = *u o r* **unfolding** *o_def* **by** *simp*
then have (*u o r*) ⟶ *liminf u* **using** ∗ **by** *simp*
then show *?thesis* **using** ‹*strict_mono r*› **by** *blast*
next
assume ¬ (∀ *n*. ∃ *p*>*n*. (∀ *m*≥*p*. *u m* ≥ *u p*))
then obtain *N* **where** *N*: ⋀*p*. *p* > *N* ⟹ ∃ *m*>*p*. *u p* > *u m* **by** (*force simp*: *not_le le_less*)
have ∃ *r*. ∀ *n*. *N* < *r n* ∧ *r n* < *r* (*Suc n*) ∧ (∀ *i*∈ {*N*<..*r* (*Suc n*)}. *u i* ≥ *u* (*r* (*Suc n*)))
proof (*rule dependent_nat_choice*)
fix *x* **assume** *N* < *x*
then have *a*: *finite* {*N*<..*x*} {*N*<..*x*} ≠ {} **by** *simp_all*
have *Min* {*u i* |*i*. *i* ∈ {*N*<..*x*}} ∈ {*u i* |*i*. *i* ∈ {*N*<..*x*}} **apply** (*rule Min_in*) **using** *a* **by** (*auto*)
then obtain *p* **where** *p* ∈ {*N*<..*x*} **and** *upmin*: *u p* = *Min*{*u i* |*i*. *i* ∈ {*N*<..*x*}} **by** *auto*
define *U* **where** *U* = {*m*. *m* > *p* ∧ *u p* > *u m*}
have *U* ≠ {} **unfolding** *U_def* **using** *N*[*of p*] ‹*p* ∈ {*N*<..*x*}› **by** *auto*
define *y* **where** *y* = *Inf U*
then have *y* ∈ *U* **using** ‹*U* ≠ {}› **by** (*simp add*: *Inf_nat_def1*)

```
  have a: ⋀i. i ∈ {N<..x} ⟹ u i ≥ u p
  proof −
    fix i assume i ∈ {N<..x}
    then have u i ∈ {u i |i. i ∈ {N<..x}} by blast
    then show u i ≥ u p using upmin by simp
  qed
  moreover have u p > u y using ‹y ∈ U› U_def by auto
  ultimately have y ∉ {N<..x} using not_le by blast
  moreover have y > N using ‹y ∈ U› U_def ‹p ∈ {N<..x}› by auto
  ultimately have y > x by auto

  have ⋀i. i ∈ {N<..y} ⟹ u i ≥ u y
  proof −
    fix i assume i ∈ {N<..y} show u i ≥ u y
    proof (cases)
      assume i = y
      then show ?thesis by simp
    next
      assume ¬(i=y)
      then have i:i ∈ {N<..<y} using ‹i ∈ {N<..y}› by simp
      have u i ≥ u p
      proof (cases)
        assume i ≤ x
        then show ?thesis using a ‹i ∈ {N<..y}› by force
      next
        assume ¬(i ≤ x)
        then have i > x by simp
        then have ∗: i > p using ‹p ∈ {N<..x}› by simp
        have i < Inf U using i y_def by simp
        then have i ∉ U using Inf_nat_def not_less_Least by auto
        then show ?thesis using U_def ∗ by auto
      qed
      then show u i ≥ u y using ‹u p > u y› by auto
    qed
  qed
  then have N < y ∧ x < y ∧ (∀ i∈{N<..y}. u i ≥ u y) using ‹y > x› ‹y >
N› by auto
  then show ∃ y>N. x < y ∧ (∀ i∈{N<..y}. u i ≥ u y) by auto
qed (auto)
then obtain r :: nat ⇒ nat
  where r: ∀ n. N < r n ∧ r n < r (Suc n) ∧ (∀ i∈ {N<..r (Suc n)}. u i ≥ u
(r (Suc n))) by auto
have strict_mono r using r by (auto simp: strict_mono_Suc_iff)
have decseq (u o r) unfolding o_def using r by (simp add: decseq_SucI or-
der.strict_implies_order)
then have (u o r) ⟶ (INF n. (u o r) n) using LIMSEQ_INF by blast
then have liminf (u o r) = (INF n. (u o r) n) by (simp add: lim_imp_Liminf)
moreover have liminf (u o r) ≥ liminf u using ‹strict_mono r› by (simp add:
liminf_subseq_mono)
```

ultimately have $(INF\ n.\ (u\ o\ r)\ n) \geq liminf\ u$ **by** *simp*

{
fix i **assume** i: $i \in \{N<..\}$
obtain n **where** $i < r\ (Suc\ n)$ **using** ‹*strict_mono r*› **using** *Suc_le_eq seq_suble* **by** *blast*
then have $i \in \{N<..r(Suc\ n)\}$ **using** i **by** *simp*
then have $u\ i \geq u\ (r(Suc\ n))$ **using** r **by** *simp*
then have $u\ i \geq (INF\ n.\ (u\ o\ r)\ n)$ **unfolding** *o_def* **by** (*meson INF_lower2 UNIV_I*)
}
then have $(INF\ i{\in}\{N<..\}.\ u\ i) \geq (INF\ n.\ (u\ o\ r)\ n)$ **using** *INF_greatest* **by** *blast*
then have $liminf\ u \geq (INF\ n.\ (u\ o\ r)\ n)$ **unfolding** *Liminf_def*
by (*metis* (*mono_tags*, *lifting*) *SUP_upper2 atLeast_Suc_greaterThan atLeast_def eventually_ge_at_top mem_Collect_eq*)
then have $liminf\ u = (INF\ n.\ (u\ o\ r)\ n)$ **using** ‹$(INF\ n.\ (u\ o\ r)\ n) \geq liminf\ u$› **by** *simp*
then have $(u\ o\ r) \longrightarrow liminf\ u$ **using** ‹$(u\ o\ r) \longrightarrow (INF\ n.\ (u\ o\ r)\ n)$› **by** *simp*
then show *?thesis* **using** ‹*strict_mono r*› **by** *auto*
qed

The following statement about limsups is reduced to a statement about limits using subsequences thanks to *limsup_subseq_lim*. The statement for limits follows for instance from *tendsto_add_ereal_general*.

lemma *ereal_limsup_add_mono*:
fixes $u\ v$::$nat \Rightarrow ereal$
shows $limsup\ (\lambda n.\ u\ n + v\ n) \leq limsup\ u + limsup\ v$
proof (*cases*)
assume $(limsup\ u = \infty) \vee (limsup\ v = \infty)$
then have $limsup\ u + limsup\ v = \infty$ **by** *simp*
then show *?thesis* **by** *auto*
next
assume $\neg((limsup\ u = \infty) \vee (limsup\ v = \infty))$
then have $limsup\ u < \infty\ limsup\ v < \infty$ **by** *auto*

define w **where** $w = (\lambda n.\ u\ n + v\ n)$
obtain r **where** r: $strict\_mono\ r\ (w\ o\ r) \longrightarrow limsup\ w$ **using** *limsup_subseq_lim* **by** *auto*
obtain s **where** s: $strict\_mono\ s\ (u\ o\ r\ o\ s) \longrightarrow limsup\ (u\ o\ r)$ **using** *limsup_subseq_lim* **by** *auto*
obtain t **where** t: $strict\_mono\ t\ (v\ o\ r\ o\ s\ o\ t) \longrightarrow limsup\ (v\ o\ r\ o\ s)$ **using** *limsup_subseq_lim* **by** *auto*

define a **where** $a = r\ o\ s\ o\ t$
have $strict\_mono\ a$ **using** $r\ s\ t$ **by** (*simp add*: *a_def strict_mono_o*)
have l:$(w\ o\ a) \longrightarrow limsup\ w$
$(u\ o\ a) \longrightarrow limsup\ (u\ o\ r)$

```
      (v o a) ⟶ limsup (v o r o s)
 apply (metis (no_types, lifting) r(2) s(1) t(1) LIMSEQ_subseq_LIMSEQ a_def comp_assoc)
  apply (metis (no_types, lifting) s(2) t(1) LIMSEQ_subseq_LIMSEQ a_def comp_assoc)
 apply (metis (no_types, lifting) t(2) a_def comp_assoc)
 done

 have limsup (u o r) ≤ limsup u by (simp add: limsup_subseq_mono r(1))
 then have a: limsup (u o r) ≠ ∞ using ‹limsup u < ∞› by auto
 have limsup (v o r o s) ≤ limsup v
  by (simp add: comp_assoc limsup_subseq_mono r(1) s(1) strict_mono_o)
 then have b: limsup (v o r o s) ≠ ∞ using ‹limsup v < ∞› by auto

 have (λn. (u o a) n + (v o a) n) ⟶ limsup (u o r) + limsup (v o r o s)
  using l tendsto_add_ereal_general a b by fastforce
 moreover have (λn. (u o a) n + (v o a) n) = (w o a) unfolding w_def by auto
 ultimately have (w o a) ⟶ limsup (u o r) + limsup (v o r o s) by simp
 then have limsup w = limsup (u o r) + limsup (v o r o s) using l(1) LIMSEQ_unique by blast
 then have limsup w ≤ limsup u + limsup v
  using ‹limsup (u o r) ≤ limsup u› ‹limsup (v o r o s) ≤ limsup v› add_mono by simp
 then show ?thesis unfolding w_def by simp
qed
```

There is an asymmetry between liminfs and limsups in *ereal*, as $\infty + (-\infty) = \infty$. This explains why there are more assumptions in the next lemma dealing with liminfs that in the previous one about limsups.

```
lemma ereal_liminf_add_mono:
 fixes u v::nat ⇒ ereal
 assumes ¬((liminf u = ∞ ∧ liminf v = -∞) ∨ (liminf u = -∞ ∧ liminf v = ∞))
 shows liminf (λn. u n + v n) ≥ liminf u + liminf v
proof (cases)
 assume (liminf u = -∞) ∨ (liminf v = -∞)
 then have *: liminf u + liminf v = -∞ using assms by auto
 show ?thesis by (simp add: *)
next
 assume ¬((liminf u = -∞) ∨ (liminf v = -∞))
 then have liminf u > -∞ liminf v > -∞ by auto

 define w where w = (λn. u n + v n)
 obtain r where r: strict_mono r (w o r) ⟶ liminf w using liminf_subseq_lim by auto
  obtain s where s: strict_mono s (u o r o s) ⟶ liminf (u o r) using liminf_subseq_lim by auto
 obtain t where t: strict_mono t (v o r o s o t) ⟶ liminf (v o r o s) using
```

liminf_subseq_lim **by** *auto*

define *a* **where** *a* = *r o s o t*
have *strict_mono a* **using** *r s t* **by** (*simp add*: *a_def strict_mono_o*)
have *l*:(*w o a*) ⟶ *liminf w*
(*u o a*) ⟶ *liminf* (*u o r*)
(*v o a*) ⟶ *liminf* (*v o r o s*)
apply (*metis* (*no_types*, *lifting*) *r*(2) *s*(1) *t*(1) *LIMSEQ_subseq_LIMSEQ a_def comp_assoc*)
apply (*metis* (*no_types*, *lifting*) *s*(2) *t*(1) *LIMSEQ_subseq_LIMSEQ a_def comp_assoc*)
apply (*metis* (*no_types*, *lifting*) *t*(2) *a_def comp_assoc*)
done

have *liminf* (*u o r*) ≥ *liminf u* **by** (*simp add*: *liminf_subseq_mono r*(1))
then have *a*: *liminf* (*u o r*) ≠ −∞ **using** ‹*liminf u* > −∞› **by** *auto*
have *liminf* (*v o r o s*) ≥ *liminf v*
by (*simp add*: *comp_assoc liminf_subseq_mono r*(1) *s*(1) *strict_mono_o*)
then have *b*: *liminf* (*v o r o s*) ≠ −∞ **using** ‹*liminf v* > −∞› **by** *auto*

have (λ*n*. (*u o a*) *n* + (*v o a*) *n*) ⟶ *liminf* (*u o r*) + *liminf* (*v o r o s*)
using *l tendsto_add_ereal_general a b* **by** *fastforce*
moreover have (λ*n*. (*u o a*) *n* + (*v o a*) *n*) = (*w o a*) **unfolding** *w_def* **by** *auto*
ultimately have (*w o a*) ⟶ *liminf* (*u o r*) + *liminf* (*v o r o s*) **by** *simp*
then have *liminf w* = *liminf* (*u o r*) + *liminf* (*v o r o s*) **using** *l*(1) *LIMSEQ_unique* **by** *blast*
then have *liminf w* ≥ *liminf u* + *liminf v*
using ‹*liminf* (*u o r*) ≥ *liminf u*› ‹*liminf* (*v o r o s*) ≥ *liminf v*› *add_mono* **by** *simp*
then show *?thesis* **unfolding** *w_def* **by** *simp*
qed

lemma *ereal_limsup_lim_add*:
fixes *u v*::*nat* ⇒ *ereal*
assumes *u* ⟶ *a abs*(*a*) ≠ ∞
shows *limsup* (λ*n*. *u n* + *v n*) = *a* + *limsup v*
proof −
have *limsup u* = *a* **using** *assms*(1) **using** *tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*
have (λ*n*. −*u n*) ⟶ −*a* **using** *assms*(1) **by** *auto*
then have *limsup* (λ*n*. −*u n*) = −*a* **using** *tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*

have *limsup* (λ*n*. *u n* + *v n*) ≤ *limsup u* + *limsup v*
by (*rule ereal_limsup_add_mono*)
then have *up*: *limsup* (λ*n*. *u n* + *v n*) ≤ *a* + *limsup v* **using** ‹*limsup u* = *a*› **by** *simp*

have *a*: *limsup* (λ*n*. (*u n* + *v n*) + (−*u n*)) ≤ *limsup* (λ*n*. *u n* + *v n*) + *limsup* (λ*n*. −*u n*)
by (*rule ereal_limsup_add_mono*)
have *eventually* (λ*n*. *u n* = *ereal*(*real_of_ereal*(*u n*))) *sequentially* **using** *assms* *real_lim_then_eventually_real* **by** *auto*
moreover have ⋀*x*. *x* = *ereal*(*real_of_ereal*(*x*)) ⟹ *x* + (−*x*) = *0*
by (*metis plus_ereal.simps*(*1*) *right_minus uminus_ereal.simps*(*1*) *zero_ereal_def*)
ultimately have *eventually* (λ*n*. *u n* + (−*u n*) = *0*) *sequentially*
by (*metis* (*mono_tags*, *lifting*) *eventually_mono*)
moreover have ⋀*n*. *u n* + (−*u n*) = *0* ⟹ *u n* + *v n* + (−*u n*) = *v n*
by (*metis add.commute add.left_commute add.left_neutral*)
ultimately have *eventually* (λ*n*. *u n* + *v n* + (−*u n*) = *v n*) *sequentially*
using *eventually_mono* **by** *force*
then have *limsup v* = *limsup* (λ*n*. *u n* + *v n* + (−*u n*)) **using** *Limsup_eq* **by** *force*
then have *limsup v* ≤ *limsup* (λ*n*. *u n* + *v n*) −*a* **using** *a* ‹*limsup* (λ*n*. −*u n*) = −*a*› **by** (*simp add*: *minus_ereal_def*)
then have *limsup* (λ*n*. *u n* + *v n*) ≥ *a* + *limsup v* **using** *assms*(*2*) **by** (*metis add.commute ereal_le_minus*)
then show *?thesis* **using** *up* **by** *simp*
qed

lemma *ereal_limsup_lim_mult*:
fixes *u v*::*nat* ⇒ *ereal*
assumes *u* ⟶ *a* *a*>*0* *a* ≠ ∞
shows *limsup* (λ*n*. *u n* ∗ *v n*) = *a* ∗ *limsup v*
proof −
define *w* **where** *w* = (λ*n*. *u n* ∗ *v n*)
obtain *r* **where** *r*: *strict_mono r* (*v o r*) ⟶ *limsup v* **using** *limsup_subseq_lim* **by** *auto*
have (*u o r*) ⟶ *a* **using** *assms*(*1*) *LIMSEQ_subseq_LIMSEQ r* **by** *auto*
with *tendsto_mult_ereal*[*OF this r*(*2*)] **have** (λ*n*. (*u o r*) *n* ∗ (*v o r*) *n*) ⟶ *a* ∗ *limsup v* **using** *assms*(*2*) *assms*(*3*) **by** *auto*
moreover have ⋀*n*. (*w o r*) *n* = (*u o r*) *n* ∗ (*v o r*) *n* **unfolding** *w_def* **by** *auto*
ultimately have (*w o r*) ⟶ *a* ∗ *limsup v* **unfolding** *w_def* **by** *presburger*
then have *limsup* (*w o r*) = *a* ∗ *limsup v* **by** (*simp add*: *tendsto_iff_Liminf_eq_Limsup*)
then have *I*: *limsup w* ≥ *a* ∗ *limsup v* **by** (*metis limsup_subseq_mono r*(*1*))

obtain *s* **where** *s*: *strict_mono s* (*w o s*) ⟶ *limsup w* **using** *limsup_subseq_lim* **by** *auto*
have ∗: (*u o s*) ⟶ *a* **using** *assms*(*1*) *LIMSEQ_subseq_LIMSEQ s* **by** *auto*
have *eventually* (λ*n*. (*u o s*) *n* > *0*) *sequentially* **using** *assms*(*2*) ∗ *order_tendsto_iff* **by** *blast*
moreover have *eventually* (λ*n*. (*u o s*) *n* < ∞) *sequentially* **using** *assms*(*3*) ∗ *order_tendsto_iff* **by** *blast*
moreover have (*w o s*) *n* / (*u o s*) *n* = (*v o s*) *n* **if** (*u o s*) *n* > *0* (*u o s*) *n* < ∞ **for** *n*
unfolding *w_def* **using** *that* **by** (*auto simp*: *ereal_divide_eq*)

ultimately have *eventually* $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n = (v\ o\ s)\ n)$ *sequentially* **using** *eventually_elim2* **by** *force*
moreover have $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n) \longrightarrow (limsup\ w)\ /\ a$
apply (*rule tendsto_divide_ereal*[*OF* $s(2)$ $*$]) **using** $assms(2)$ $assms(3)$ **by** *auto*
ultimately have $(v\ o\ s) \longrightarrow (limsup\ w)\ /\ a$ **using** *Lim_transform_eventually* **by** *fastforce*
then have $limsup\ (v\ o\ s) = (limsup\ w)\ /\ a$ **by** (*simp add*: *tendsto_iff_Liminf_eq_Limsup*)
then have $limsup\ v \geq (limsup\ w)\ /\ a$ **by** (*metis limsup_subseq_mono* $s(1)$)
then have $a * limsup\ v \geq limsup\ w$ **using** $assms(2)$ $assms(3)$ **by** (*simp add*: *ereal_divide_le_pos*)
then show *?thesis* **using** I **unfolding** w_def **by** *auto*
qed

lemma *ereal_liminf_lim_mult*:
fixes $u\ v{::}nat \Rightarrow ereal$
assumes $u \longrightarrow a\ a{>}0\ a \neq \infty$
shows $liminf\ (\lambda n.\ u\ n * v\ n) = a * liminf\ v$
proof −
define w **where** $w = (\lambda n.\ u\ n * v\ n)$
obtain r **where** r: *strict_mono* r $(v\ o\ r) \longrightarrow liminf\ v$ **using** *liminf_subseq_lim* **by** *auto*
have $(u\ o\ r) \longrightarrow a$ **using** $assms(1)$ *LIMSEQ_subseq_LIMSEQ* r **by** *auto*
with *tendsto_mult_ereal*[*OF this* $r(2)$] **have** $(\lambda n.\ (u\ o\ r)\ n * (v\ o\ r)\ n) \longrightarrow a * liminf\ v$ **using** $assms(2)$ $assms(3)$ **by** *auto*
moreover have $\bigwedge n.\ (w\ o\ r)\ n = (u\ o\ r)\ n * (v\ o\ r)\ n$ **unfolding** w_def **by** *auto*
ultimately have $(w\ o\ r) \longrightarrow a * liminf\ v$ **unfolding** w_def **by** *presburger*
then have $liminf\ (w\ o\ r) = a * liminf\ v$ **by** (*simp add*: *tendsto_iff_Liminf_eq_Limsup*)
then have I: $liminf\ w \leq a * liminf\ v$ **by** (*metis liminf_subseq_mono* $r(1)$)

obtain s **where** s: *strict_mono* s $(w\ o\ s) \longrightarrow liminf\ w$ **using** *liminf_subseq_lim* **by** *auto*
have $*$: $(u\ o\ s) \longrightarrow a$ **using** $assms(1)$ *LIMSEQ_subseq_LIMSEQ* s **by** *auto*
have *eventually* $(\lambda n.\ (u\ o\ s)\ n > 0)$ *sequentially* **using** $assms(2)$ $*$ *order_tendsto_iff* **by** *blast*
moreover have *eventually* $(\lambda n.\ (u\ o\ s)\ n < \infty)$ *sequentially* **using** $assms(3)$ $*$ *order_tendsto_iff* **by** *blast*
moreover have $(w\ o\ s)\ n\ /\ (u\ o\ s)\ n = (v\ o\ s)\ n$ **if** $(u\ o\ s)\ n > 0$ $(u\ o\ s)\ n < \infty$ **for** n
unfolding w_def **using** *that* **by** (*auto simp*: *ereal_divide_eq*)
ultimately have *eventually* $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n = (v\ o\ s)\ n)$ *sequentially* **using** *eventually_elim2* **by** *force*
moreover have $(\lambda n.\ (w\ o\ s)\ n\ /\ (u\ o\ s)\ n) \longrightarrow (liminf\ w)\ /\ a$
using $*$ *assms s tendsto_divide_ereal* **by** *fastforce*
ultimately have $(v\ o\ s) \longrightarrow (liminf\ w)\ /\ a$ **using** *Lim_transform_eventually* **by** *fastforce*
then have $liminf\ (v\ o\ s) = (liminf\ w)\ /\ a$ **by** (*simp add*: *tendsto_iff_Liminf_eq_Limsup*)
then have $liminf\ v \leq (liminf\ w)\ /\ a$ **by** (*metis liminf_subseq_mono* $s(1)$)

then have $a * liminf\ v \leq liminf\ w$ **using** *assms(2) assms(3)* **by** (*simp add*: *ereal_le_divide_pos*)
then show *?thesis* **using** *I* **unfolding** *w_def* **by** *auto*
qed

lemma *ereal_liminf_lim_add*:
fixes $u\ v{::}nat \Rightarrow ereal$
assumes $u \longrightarrow a\ abs(a) \neq \infty$
shows $liminf\ (\lambda n.\ u\ n + v\ n) = a + liminf\ v$
proof −
have $liminf\ u = a$ **using** *assms(1) tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*
then have $*$: $abs(liminf\ u) \neq \infty$ **using** *assms(2)* **by** *auto*
have $(\lambda n.\ -u\ n) \longrightarrow -a$ **using** *assms(1)* **by** *auto*
then have $liminf\ (\lambda n.\ -u\ n) = -a$ **using** *tendsto_iff_Liminf_eq_Limsup trivial_limit_at_top_linorder* **by** *blast*
then have $**$: $abs(liminf\ (\lambda n.\ -u\ n)) \neq \infty$ **using** *assms(2)* **by** *auto*

have $liminf\ (\lambda n.\ u\ n + v\ n) \geq liminf\ u + liminf\ v$
using *abs_ereal.simps* **by** (*metis* (*full_types*) $*$ *ereal_liminf_add_mono*)
then have *up*: $liminf\ (\lambda n.\ u\ n + v\ n) \geq a + liminf\ v$ **using** ‹$liminf\ u = a$› **by** *simp*

have *a*: $liminf\ (\lambda n.\ (u\ n + v\ n) + (-u\ n)) \geq liminf\ (\lambda n.\ u\ n + v\ n) + liminf\ (\lambda n.\ -u\ n)$
apply (*rule ereal_liminf_add_mono*) **using** $**$ **by** *auto*
have *eventually* $(\lambda n.\ u\ n = ereal(real\_of\_ereal(u\ n)))$ *sequentially* **using** *assms real_lim_then_eventually_real* **by** *auto*
moreover have $\bigwedge x.\ x = ereal(real\_of\_ereal(x)) \Longrightarrow x + (-x) = 0$
by (*metis plus_ereal.simps(1) right_minus uminus_ereal.simps(1) zero_ereal_def*)
ultimately have *eventually* $(\lambda n.\ u\ n + (-u\ n) = 0)$ *sequentially*
by (*metis* (*mono_tags*, *lifting*) *eventually_mono*)
moreover have $\bigwedge n.\ u\ n + (-u\ n) = 0 \Longrightarrow u\ n + v\ n + (-u\ n) = v\ n$
by (*metis add.commute add.left_commute add.left_neutral*)
ultimately have *eventually* $(\lambda n.\ u\ n + v\ n + (-u\ n) = v\ n)$ *sequentially*
using *eventually_mono* **by** *force*
then have $liminf\ v = liminf\ (\lambda n.\ u\ n + v\ n + (-u\ n))$ **using** *Liminf_eq* **by** *force*
then have $liminf\ v \geq liminf\ (\lambda n.\ u\ n + v\ n) - a$ **using** *a* ‹$liminf\ (\lambda n.\ -u\ n) = -a$› **by** (*simp add*: *minus_ereal_def*)
then have $liminf\ (\lambda n.\ u\ n + v\ n) \leq a + liminf\ v$ **using** *assms(2)* **by** (*metis add.commute ereal_minus_le*)
then show *?thesis* **using** *up* **by** *simp*
qed

lemma *ereal_liminf_limsup_add*:
fixes $u\ v{::}nat \Rightarrow ereal$
shows $liminf\ (\lambda n.\ u\ n + v\ n) \leq liminf\ u + limsup\ v$
proof (*cases*)

assume *limsup v* = ∞ ∨ *liminf u* = ∞
then show *?thesis* **by** *auto*
next
assume ¬(*limsup v* = ∞ ∨ *liminf u* = ∞)
then have *limsup v* < ∞ *liminf u* < ∞ **by** *auto*

define *w* **where** *w* = (λ*n*. *u n* + *v n*)
obtain *r* **where** *r*: *strict_mono r* (*u o r*) ⟶ *liminf u* **using** *liminf_subseq_lim* **by** *auto*
obtain *s* **where** *s*: *strict_mono s* (*w o r o s*) ⟶ *liminf* (*w o r*) **using** *liminf_subseq_lim* **by** *auto*
obtain *t* **where** *t*: *strict_mono t* (*v o r o s o t*) ⟶ *limsup* (*v o r o s*) **using** *limsup_subseq_lim* **by** *auto*

define *a* **where** *a* = *r o s o t*
have *strict_mono a* **using** *r s t* **by** (*simp add*: *a_def strict_mono_o*)
have *l*:(*u o a*) ⟶ *liminf u*
(*w o a*) ⟶ *liminf* (*w o r*)
(*v o a*) ⟶ *limsup* (*v o r o s*)
apply (*metis* (*no_types*, *lifting*) *r*(*2*) *s*(*1*) *t*(*1*) *LIMSEQ_subseq_LIMSEQ a_def comp_assoc*)
apply (*metis* (*no_types*, *lifting*) *s*(*2*) *t*(*1*) *LIMSEQ_subseq_LIMSEQ a_def comp_assoc*)
apply (*metis* (*no_types*, *lifting*) *t*(*2*) *a_def comp_assoc*)
done

have *liminf* (*w o r*) ≥ *liminf w* **by** (*simp add*: *liminf_subseq_mono r*(*1*))
have *limsup* (*v o r o s*) ≤ *limsup v*
by (*simp add*: *comp_assoc limsup_subseq_mono r*(*1*) *s*(*1*) *strict_mono_o*)
then have *b*: *limsup* (*v o r o s*) < ∞ **using** ‹*limsup v* < ∞› **by** *auto*

have (λ*n*. (*u o a*) *n* + (*v o a*) *n*) ⟶ *liminf u* + *limsup* (*v o r o s*)
apply (*rule tendsto_add_ereal_general*) **using** *b* ‹*liminf u* < ∞› *l*(*1*) *l*(*3*) **by** *force*+
moreover have (λ*n*. (*u o a*) *n* + (*v o a*) *n*) = (*w o a*) **unfolding** *w_def* **by** *auto*
ultimately have (*w o a*) ⟶ *liminf u* + *limsup* (*v o r o s*) **by** *simp*
then have *liminf* (*w o r*) = *liminf u* + *limsup* (*v o r o s*) **using** *l*(*2*) **using** *LIMSEQ_unique* **by** *blast*
then have *liminf w* ≤ *liminf u* + *limsup v*
using ‹*liminf* (*w o r*) ≥ *liminf w*› ‹*limsup* (*v o r o s*) ≤ *limsup v*›
by (*metis add_mono_thms_linordered_semiring*(*2*) *le_less_trans not_less*)
then show *?thesis* **unfolding** *w_def* **by** *simp*
qed

lemma *ereal_liminf_limsup_minus*:
fixes *u v*::*nat* ⇒ *ereal*
shows *liminf* (λ*n*. *u n* − *v n*) ≤ *limsup u* − *limsup v*
unfolding *minus_ereal_def*

apply (*subst add.commute*)
apply (*rule order_trans*[*OF ereal_liminf_limsup_add*])
using *ereal_Limsup_uminus*[*of sequentially* λ*n. − v n*]
apply (*simp add*: *add.commute*)
done

lemma *liminf_minus_ennreal*:
fixes *u v*::*nat* ⇒ *ennreal*
shows (⋀*n. v n* ≤ *u n*) ⟹ *liminf* (λ*n. u n − v n*) ≤ *limsup u − limsup v*
unfolding *liminf_SUP_INF limsup_INF_SUP*
including *ennreal.lifting*
proof (*transfer*, *clarsimp*)
fix *v u* :: *nat* ⇒ *ereal* **assume** ∗: ∀*x. 0* ≤ *v x* ∀*x. 0* ≤ *u x* ⋀*n. v n* ≤ *u n*
moreover have *0* ≤ *limsup u − limsup v*
using ∗ **by** (*intro ereal_diff_positive Limsup_mono always_eventually*) *simp*
moreover have *0* ≤ *Sup* (*u* ‘ {*x*..}) **for** *x*
using ∗ **by** (*intro SUP_upper2*[*of x*]) *auto*
moreover have *0* ≤ *Sup* (*v* ‘ {*x*..}) **for** *x*
using ∗ **by** (*intro SUP_upper2*[*of x*]) *auto*
ultimately show (*SUP n. INF n*∈{*n*..}. *max 0* (*u n − v n*))
≤ *max 0* ((*INF x. max 0* (*Sup* (*u* ‘ {*x*..}))) − (*INF x. max 0* (*Sup* (*v* ‘ {*x*..}))))
by (*auto simp*: ∗ *ereal_diff_positive max.absorb2 liminf_SUP_INF*[*symmetric*] *limsup_INF_SUP*[*symmetric*] *ereal_liminf_limsup_minus*)
qed

6.13.4 Relate extended reals and the indicator function

lemma *ereal_indicator_le_0*: (*indicator S x*::*ereal*) ≤ *0* ⟷ *x* ∉ *S*
by (*auto split*: *split_indicator simp*: *one_ereal_def*)

lemma *ereal_indicator*: *ereal* (*indicator A x*) = *indicator A x*
by (*auto simp*: *indicator_def one_ereal_def*)

lemma *ereal_mult_indicator*: *ereal* (*x* ∗ *indicator A y*) = *ereal x* ∗ *indicator A y*
by (*simp split*: *split_indicator*)

lemma *ereal_indicator_mult*: *ereal* (*indicator A y* ∗ *x*) = *indicator A y* ∗ *ereal x*
by (*simp split*: *split_indicator*)

lemma *ereal_indicator_nonneg*[*simp*, *intro*]: *0* ≤ (*indicator A x* ::*ereal*)
unfolding *indicator_def* **by** *auto*

lemma *indicator_inter_arith_ereal*: *indicator A x* ∗ *indicator B x* = (*indicator* (*A* ∩ *B*) *x* :: *ereal*)
by (*simp split*: *split_indicator*)

end

6.14 Radius of Convergence and Summation Tests

theory *Summation_Tests*
imports
Complex_Main
HOL−Library.Discrete_Functions
HOL−Library.Extended_Real
HOL−Library.Liminf_Limsup
Extended_Real_Limits
begin

The definition of the radius of convergence of a power series, various summability tests, lemmas to compute the radius of convergence etc.

6.14.1 Convergence tests for infinite sums

Root test

lemma *limsup_root_powser*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\_normed\_div\_algebra\}$
shows $limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n * z\ \hat{}\ n)))) =$
$limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n)))) * ereal\ (norm\ z)$
proof −
have *A*: $(\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n * z\ \hat{}\ n)))) =$
$(\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n)))) * ereal\ (norm\ z))$ (**is** *?g* = *?h*)
proof
fix *n* **show** *?g n* = *?h n*
by (*cases n* = *0*) (*simp_all add*: *norm_mult real_root_mult real_root_pos2 norm_power*)
qed
show *?thesis* **by** (*subst A*, *subst limsup_ereal_mult_right*) *simp_all*
qed

lemma *limsup_root_limit*:
assumes $(\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n)))) \longrightarrow l$ (**is** *?g* $\longrightarrow$ _)
shows $limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n)))) = l$
proof −
from *assms* **have** *convergent ?g lim ?g* = *l*
unfolding *convergent_def* **by** (*blast intro*: *limI*)+
with *convergent_limsup_cl* **show** *?thesis* **by** *force*
qed

lemma *limsup_root_limit′*:
assumes $(\lambda n.\ root\ n\ (norm\ (f\ n))) \longrightarrow l$
shows $limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n)))) = ereal\ l$
by (*intro limsup_root_limit tendsto_ereal assms*)

theorem *root_test_convergence′*:
fixes $f :: nat \Rightarrow 'a :: banach$
defines $l \equiv limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$

```
  assumes l: l < 1
  shows   summable f
proof −
  have 0 = limsup (λn. 0) by (simp add: Limsup_const)
   also have ... ≤ l unfolding l_def by (intro Limsup_mono) (simp_all add:
real_root_ge_zero)
  finally have l ≥ 0 by simp
  with l obtain l' where l': l = ereal l' by (cases l) simp_all

  define c where c = (1 − l') / 2
  from l and ‹l ≥ 0› have c: l + c > l l' + c ≥ 0 l' + c < 1 unfolding c_def
    by (simp_all add: field_simps l')
  have ∀ C>l. eventually (λn. ereal (root n (norm (f n))) < C) sequentially
    by (subst Limsup_le_iff[symmetric]) (simp add: l_def)
 with c have eventually (λn. ereal (root n (norm (f n))) < l + ereal c) sequentially
by simp
  with eventually_gt_at_top[of 0::nat]
    have eventually (λn. norm (f n) ≤ (l' + c) ^ n) sequentially
  proof eventually_elim
    fix n :: nat assume n: n > 0
    assume ereal (root n (norm (f n))) < l + ereal c
    hence root n (norm (f n)) ≤ l' + c by (simp add: l')
    with c n have root n (norm (f n)) ^ n ≤ (l' + c) ^ n
      by (intro power_mono) (simp_all add: real_root_ge_zero)
    also from n have root n (norm (f n)) ^ n = norm (f n) by simp
    finally show norm (f n) ≤ (l' + c) ^ n by simp
  qed
  thus ?thesis
     by (rule summable_comparison_test_ev[OF _ summable_geometric]) (simp
add: c)
qed

theorem root_test_divergence:
  fixes f :: nat ⇒ 'a :: banach
  defines l ≡ limsup (λn. ereal (root n (norm (f n))))
  assumes l: l > 1
  shows   ¬summable f
proof
  assume summable f
  hence bounded: Bseq f by (simp add: summable_imp_Bseq)

  have 0 = limsup (λn. 0) by (simp add: Limsup_const)
   also have ... ≤ l unfolding l_def by (intro Limsup_mono) (simp_all add:
real_root_ge_zero)
  finally have l_nonneg: l ≥ 0 by simp

  define c where c = (if l = ∞ then 2 else 1 + (real_of_ereal l − 1) / 2)
  from l l_nonneg consider l = ∞ | ∃ l'. l = ereal l' by (cases l) simp_all
  hence c: c > 1 ∧ ereal c < l by cases (insert l, auto simp: c_def field_simps)
```

```
  have unbounded: ¬bdd_above {n. root n (norm (f n)) > c}
  proof
    assume bdd_above {n. root n (norm (f n)) > c}
    then obtain N where ∀n. root n (norm (f n)) > c ⟶ n ≤ N unfolding
bdd_above_def by blast
    hence ∃N. ∀n≥N. root n (norm (f n)) ≤ c
      by (intro exI[of _ N + 1]) (force simp: not_less_eq_eq[symmetric])
    hence eventually (λn. root n (norm (f n)) ≤ c) sequentially
      by (auto simp: eventually_at_top_linorder)
    hence l ≤ c unfolding l_def by (intro Limsup_bounded) simp_all
    with c show False by auto
  qed

  from bounded obtain K where K: K > 0 ⋀n. norm (f n) ≤ K using BseqE
by blast
  define n where n = nat ⌈log c K⌉
 from unbounded have ∃m>n. c < root m (norm (f m)) unfolding bdd_above_def
    by (auto simp: not_le)
  then obtain m where m: n < m c < root m (norm (f m)) by auto
  from c K have K = c powr log c K by (simp add: powr_def log_def)
  also from c have c powr log c K ≤ c powr real n unfolding n_def
    by (intro powr_mono, linarith, simp)
  finally have K ≤ c ^ n using c by (simp add: powr_realpow)
  also from c m have c ^ n < c ^ m by simp
 also from c m have c ^ m < root m (norm (f m)) ^ m by (intro power_strict_mono)
simp_all
  also from m have ... = norm (f m) by simp
  finally show False using K(2)[of m]  by simp
qed
```

Cauchy's condensation test

```
context
fixes f :: nat ⇒ real
begin

private lemma condensation_inequality:
  assumes mono: ⋀m n. 0 < m ⟹ m ≤ n ⟹ f n ≤ f m
  shows   (∑k=1..<n. f k) ≥ (∑k=1..<n. f (2 * 2 ^ floor_log k)) (is ?thesis1)
          (∑k=1..<n. f k) ≤ (∑k=1..<n. f (2 ^ floor_log k)) (is ?thesis2)
  by (intro sum_mono mono floor_log_exp2_ge floor_log_exp2_le, simp, simp)+

private lemma condensation_condense1: (∑k=1..<2^n. f (2 ^ floor_log k)) =
(∑k<n. 2^k * f (2 ^ k))
proof (induction n)
  case (Suc n)
  have {1..<2^Suc n} = {1..<2^n} ∪ {2^n..<(2^Suc n :: nat)} by auto
  also have (∑k∈.... f (2 ^ floor_log k)) =
```

```
                (∑k<n. 2^k * f (2^k)) + (∑k = 2^n..<2^Suc n. f (2^floor_log k))
    by (subst sum.union_disjoint) (insert Suc, auto)
  also have floor_log k = n if k ∈ {2^n..<2^Suc n} for k using that by (intro
floor_log_eqI) simp_all
  hence (∑k = 2^n..<2^Suc n. f (2^floor_log k)) = (∑(_::nat) = 2^n..<2^Suc
n. f (2^n))
    by (intro sum.cong) simp_all
  also have ... = 2^n * f (2^n) by (simp)
  finally show ?case by simp
qed simp

private lemma condensation_condense2: (∑k=1..<2^n. f (2 * 2 ^ floor_log
k)) = (∑k<n. 2^k * f (2 ^ Suc k))
proof (induction n)
  case (Suc n)
  have {1..<2^Suc n} = {1..<2^n} ∪ {2^n..<(2^Suc n :: nat)} by auto
  also have (∑k∈.... f (2 * 2 ^ floor_log k)) =
                (∑k<n. 2^k * f (2^Suc k)) + (∑k = 2^n..<2^Suc n. f (2 *
2^floor_log k))
    by (subst sum.union_disjoint) (insert Suc, auto)
  also have floor_log k = n if k ∈ {2^n..<2^Suc n} for k using that by (intro
floor_log_eqI) simp_all
  hence (∑k = 2^n..<2^Suc n. f (2*2^floor_log k)) = (∑(_::nat) = 2^n..<2^Suc
n. f (2^Suc n))
    by (intro sum.cong) simp_all
  also have ... = 2^n * f (2^Suc n) by (simp)
  finally show ?case by simp
qed simp

theorem condensation_test:
  assumes mono: ⋀m. 0 < m ⟹ f (Suc m) ≤ f m
  assumes nonneg: ⋀n. f n ≥ 0
  shows summable f ⟷ summable (λn. 2^n * f (2^n))
proof −
  define f' where f' n = (if n = 0 then 0 else f n) for n
  from mono have mono': decseq (λn. f (Suc n)) by (intro decseq_SucI) simp
  hence mono': f n ≤ f m if m ≤ n m > 0 for m n
    using that decseqD[OF mono', of m − 1 n − 1] by simp

  have (λn. f (Suc n)) = (λn. f' (Suc n)) by (intro ext) (simp add: f'_def)
  hence summable f ⟷ summable f'
    by (subst (1 2) summable_Suc_iff [symmetric]) (simp only:)
 also have ... ⟷ convergent (λn. ∑k<n. f' k) unfolding summable_iff_convergent
..
  also have monoseq (λn. ∑k<n. f' k) unfolding f'_def
    by (intro mono_SucI1) (auto intro!: mult_nonneg_nonneg nonneg)
  hence convergent (λn. ∑k<n. f' k) ⟷ Bseq (λn. ∑k<n. f' k)
    by (rule monoseq_imp_convergent_iff_Bseq)
  also have ... ⟷ Bseq (λn. ∑k=1..<n. f' k) unfolding One_nat_def
```

```
  by (subst sum_shift_lb_Suc0_0_upt) (simp_all add: f'_def atLeast0LessThan)
 also have ... ⟷ Bseq (λn. ∑k=1..<n. f k) unfolding f'_def by simp
 also have ... ⟷ Bseq (λn. ∑k=1..<2^n. f k)
   by (rule nonneg_incseq_Bseq_subseq_iff[symmetric])
      (auto intro!: sum_nonneg incseq_SucI nonneg simp: strict_mono_def)
 also have ... ⟷ Bseq (λn. ∑k<n. 2^k * f (2^k))
 proof (intro iffI)
   assume A: Bseq (λn. ∑k=1..<2^n. f k)
  have eventually (λn. norm (∑k<n. 2^k * f (2^Suc k)) ≤ norm (∑k=1..<2^n.
f k)) sequentially
   proof (intro always_eventually allI)
     fix n :: nat
      have norm (∑k<n. 2^k * f (2^Suc k)) = (∑k<n. 2^k * f (2^Suc k))
unfolding real_norm_def
       by (intro abs_of_nonneg sum_nonneg ballI mult_nonneg_nonneg nonneg)
simp_all
     also have ... ≤ (∑k=1..<2^n. f k)
     by (subst condensation_condense2 [symmetric]) (intro condensation_inequality
mono')
     also have ... = norm ... unfolding real_norm_def
     by (intro abs_of_nonneg[symmetric] sum_nonneg ballI mult_nonneg_nonneg
nonneg)
     finally show norm (∑k<n. 2 ^ k * f (2 ^ Suc k)) ≤ norm (∑k=1..<2^n.
f k) .
   qed
    from this and A have Bseq (λn. ∑k<n. 2^k * f (2^Suc k)) by (rule
Bseq_eventually_mono)
   from Bseq_mult[OF Bfun_const[of 2] this] have Bseq (λn. ∑k<n. 2^Suc k
* f (2^Suc k))
     by (simp add: sum_distrib_left sum_distrib_right mult_ac)
   hence Bseq (λn. (∑k=Suc 0..<Suc n. 2^k * f (2^k)) + f 1)
   by (intro Bseq_add, subst sum.shift_bounds_Suc_ivl) (simp add: atLeast0LessThan)
   hence Bseq (λn. (∑k=0..<Suc n. 2^k * f (2^k)))
     by (subst sum.atLeast_Suc_lessThan) (simp_all add: add_ac)
   thus Bseq (λn. (∑k<n. 2^k * f (2^k)))
     by (subst (asm) Bseq_Suc_iff) (simp add: atLeast0LessThan)
 next
   assume A: Bseq (λn. (∑k<n. 2^k * f (2^k)))
    have eventually (λn. norm (∑k=1..<2^n. f k) ≤ norm (∑k<n. 2^k * f
(2^k))) sequentially
   proof (intro always_eventually allI)
     fix n :: nat
   have norm (∑k=1..<2^n. f k) = (∑k=1..<2^n. f k) unfolding real_norm_def
      by (intro abs_of_nonneg sum_nonneg ballI mult_nonneg_nonneg nonneg)
     also have ... ≤ (∑k<n. 2^k * f (2^k))
     by (subst condensation_condense1 [symmetric]) (intro condensation_inequality
mono')
     also have ... = norm ... unfolding real_norm_def
     by (intro abs_of_nonneg [symmetric] sum_nonneg ballI mult_nonneg_nonneg
```

nonneg) *simp_all*
finally show *norm* ($\sum k{=}1..{<}2\hat{}n.\ f\ k$) $\leq$ *norm* ($\sum k{<}n.\ 2\hat{}k * f\ (2\hat{}k)$) **.**
qed
from *this* **and** *A* **show** *Bseq* ($\lambda n.\ \sum k{=}1..{<}2\hat{}n.\ f\ k$) **by** (*rule Bseq_eventually_mono*)
qed
also have *monoseq* ($\lambda n.$ ($\sum k{<}n.\ 2\hat{}k * f\ (2\hat{}k)$))
by (*intro mono_SucI1*) (*auto intro*!: *mult_nonneg_nonneg nonneg*)
hence *Bseq* ($\lambda n.$ ($\sum k{<}n.\ 2\hat{}k * f\ (2\hat{}k)$)) $\longleftrightarrow$ *convergent* ($\lambda n.$ ($\sum k{<}n.\ 2\hat{}k *$ *f* (*2*$\hat{}$*k*)))
by (*rule monoseq_imp_convergent_iff_Bseq* [*symmetric*])
also have ... $\longleftrightarrow$ *summable* ($\lambda k.\ 2\hat{}k * f\ (2\hat{}k)$) **by** (*simp only*: *summable_iff_convergent*)
finally show *?thesis* **.**
qed

end

Summability of powers

lemma *abs_summable_complex_powr_iff*:
summable ($\lambda n.$ *norm* (*exp* (*of_real* (*ln* (*of_nat n*)) * *s*))) $\longleftrightarrow$ *Re s* < -1
proof (*cases Re s* $\leq$ *0*)
let *?l* = $\lambda n.$ *complex_of_real* (*ln* (*of_nat n*))
case *False*
have *eventually* ($\lambda n.$ *norm* (*1* :: *real*) $\leq$ *norm* (*exp* (*?l n* * *s*))) *sequentially*
apply (*rule eventually_mono* [*OF eventually_gt_at_top*[*of 0*::*nat*]])
using *False ge_one_powr_ge_zero* **by** *auto*
from *summable_comparison_test_ev*[*OF this*] *False* **show** *?thesis* **by** (*auto simp*: *summable_const_iff*)
next
let *?l* = $\lambda n.$ *complex_of_real* (*ln* (*of_nat n*))
case *True*
hence *summable* ($\lambda n.$ *norm* (*exp* (*?l n* * *s*))) $\longleftrightarrow$ *summable* ($\lambda n.$ *2*$\hat{}$*n* * *norm* (*exp* (*?l* (*2*$\hat{}$*n*) * *s*)))
by (*intro condensation_test*) (*auto intro*!: *mult_right_mono_neg*)
also have ($\lambda n.$ *2*$\hat{}$*n* * *norm* (*exp* (*?l* (*2*$\hat{}$*n*) * *s*))) = ($\lambda n.$ (*2 powr* (*Re s* + *1*)) $\hat{}$ *n*)
proof
fix *n* :: *nat*
have *2*$\hat{}$*n* * *norm* (*exp* (*?l* (*2*$\hat{}$*n*) * *s*)) = *exp* (*real n* * *ln 2*) * *exp* (*real n* * *ln 2* * *Re s*)
using *True* **by** (*subst exp_of_nat_mult*) (*simp add*: *ln_realpow algebra_simps*)
also have ... = *exp* (*real n* * (*ln 2* * (*Re s* + *1*)))
by (*simp add*: *algebra_simps exp_add*)
also have ... = *exp* (*ln 2* * (*Re s* + *1*)) $\hat{}$ *n* **by** (*subst exp_of_nat_mult*) *simp*
also have *exp* (*ln 2* * (*Re s* + *1*)) = *2 powr* (*Re s* + *1*) **by** (*simp add*: *powr_def*)
finally show *2*$\hat{}$*n* * *norm* (*exp* (*?l* (*2*$\hat{}$*n*) * *s*)) = (*2 powr* (*Re s* + *1*)) $\hat{}$ *n* **.**
qed
also have *summable* ... $\longleftrightarrow$ *2 powr* (*Re s* + *1*) $<$ *2 powr 0*

```
    by (subst summable_geometric_iff) simp
  also have ... ⟷ Re s < −1 by (subst powr_less_cancel_iff) (simp, linarith)
  finally show ?thesis .
qed

theorem summable_complex_powr_iff:
  assumes Re s < −1
  shows   summable (λn. exp (of_real (ln (of_nat n)) * s))
  by (rule summable_norm_cancel, subst abs_summable_complex_powr_iff) fact

lemma summable_real_powr_iff: summable (λn. of_nat n powr s :: real) ⟷ s
< −1
proof −
  from eventually_gt_at_top[of 0::nat]
    have summable (λn. of_nat n powr s) ⟷ summable (λn. exp (ln (of_nat n)
* s))
    by (intro summable_cong) (auto elim!: eventually_mono simp: powr_def)
  also have ... ⟷ s < −1 using abs_summable_complex_powr_iff[of of_real
s] by simp
  finally show ?thesis .
qed

lemma inverse_power_summable:
  assumes s: s ≥ 2
  shows summable (λn. inverse (of_nat n ^ s :: 'a :: {real_normed_div_algebra,banach}))
proof (rule summable_norm_cancel, subst summable_cong)
  from eventually_gt_at_top[of 0::nat]
    show eventually (λn. norm (inverse (of_nat n ^ s:: 'a)) = real_of_nat n powr
(−real s)) at_top
    by eventually_elim (simp add: norm_inverse norm_power powr_minus powr_realpow)
qed (insert s summable_real_powr_iff[of −s], simp_all)

lemma not_summable_harmonic: ¬summable (λn. inverse (of_nat n) :: 'a ::
real_normed_field)
proof
  assume summable (λn. inverse (of_nat n) :: 'a)
  hence convergent (λn. norm (of_real (∑k<n. inverse (of_nat k)) :: 'a))
    by (simp add: summable_iff_convergent convergent_norm)
  hence convergent (λn. abs (∑k<n. inverse (of_nat k)) :: real) by (simp only:
norm_of_real)
  also have (λn. abs (∑k<n. inverse (of_nat k)) :: real) = (λn. ∑k<n. inverse
(of_nat k))
    by (intro ext abs_of_nonneg sum_nonneg) auto
  also have convergent ... ⟷ summable (λk. inverse (of_nat k) :: real)
    by (simp add: summable_iff_convergent)
  finally show False using summable_real_powr_iff[of −1] by (simp add: powr_minus)
qed
```

Kummer's test

theorem *kummers_test_convergence*:
fixes $f\ p :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
assumes *nonneg_p*: *eventually* $(\lambda n.\ p\ n \geq 0)$ *sequentially*
defines $l \equiv liminf\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
assumes *l*: $l > 0$
shows *summable f*
unfolding *summable_iff_convergent'*
proof −
define *r* **where** $r = (if\ l = \infty\ then\ 1\ else\ real\_of\_ereal\ l\ /\ 2)$
from *l* **have** $r > 0 \wedge of\_real\ r < l$ **by** (*cases l*) (*simp_all add*: *r_def*)
hence *r*: $r > 0\ of\_real\ r < l$ **by** *simp_all*
hence *eventually* $(\lambda n.\ p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n) > r)$ *sequentially*
unfolding *l_def* **by** (*force dest*: *less_LiminfD*)
moreover from *pos_f* **have** *eventually* $(\lambda n.\ f\ (Suc\ n) > 0)$ *sequentially*
by (*subst eventually_sequentially_Suc*)
ultimately have *eventually* $(\lambda n.\ p\ n * f\ n - p\ (Suc\ n) * f\ (Suc\ n) > r * f\ (Suc\ n))$ *sequentially*
by *eventually_elim* (*simp add*: *field_simps*)
from *eventually_conj*[*OF pos_f eventually_conj*[*OF nonneg_p this*]]
obtain *m* **where** *m*: $\bigwedge n.\ n \geq m \Longrightarrow f\ n > 0\ \bigwedge n.\ n \geq m \Longrightarrow p\ n \geq 0$
$\bigwedge n.\ n \geq m \Longrightarrow p\ n * f\ n - p\ (Suc\ n) * f\ (Suc\ n) > r * f\ (Suc\ n)$
unfolding *eventually_at_top_linorder* **by** *blast*

let *?c* = $(norm\ (\sum k{\leq}m.\ r * f\ k) + p\ m * f\ m)\ /\ r$
have *Bseq* $(\lambda n.\ (\sum k{\leq}n + Suc\ m.\ f\ k))$
proof (*rule BseqI'*)
fix *k* :: *nat*
define *n* **where** $n = k + Suc\ m$
have *n*: $n > m$ **by** (*simp add*: *n_def*)

from *r* **have** $r * norm\ (\sum k{\leq}n.\ f\ k) = norm\ (\sum k{\leq}n.\ r * f\ k)$
by (*simp add*: *sum_distrib_left*[*symmetric*] *abs_mult*)
also from *n* **have** $\{..n\} = \{..m\} \cup \{Suc\ m..n\}$ **by** *auto*
hence $(\sum k{\leq}n.\ r * f\ k) = (\sum k{\in}\{..m\} \cup \{Suc\ m..n\}.\ r * f\ k)$ **by** (*simp only*:)
also have $\ldots = (\sum k{\leq}m.\ r * f\ k) + (\sum k{=}Suc\ m..n.\ r * f\ k)$
by (*subst sum.union_disjoint*) *auto*
also have $norm\ \ldots \leq norm\ (\sum k{\leq}m.\ r * f\ k) + norm\ (\sum k{=}Suc\ m..n.\ r * f\ k)$
by (*rule norm_triangle_ineq*)
also from *r less_imp_le*[*OF m*(*1*)] **have** $(\sum k{=}Suc\ m..n.\ r * f\ k) \geq 0$
by (*intro sum_nonneg*) *auto*
hence $norm\ (\sum k{=}Suc\ m..n.\ r * f\ k) = (\sum k{=}Suc\ m..n.\ r * f\ k)$ **by** *simp*
also have $(\sum k{=}Suc\ m..n.\ r * f\ k) = (\sum k{=}m..{<}n.\ r * f\ (Suc\ k))$
by (*subst sum.shift_bounds_Suc_ivl* [*symmetric*])
(*simp only*: *atLeastLessThanSuc_atLeastAtMost*)
also from *m* **have** $\ldots \leq (\sum k{=}m..{<}n.\ p\ k * f\ k - p\ (Suc\ k) * f\ (Suc\ k))$
by (*intro sum_mono*[*OF less_imp_le*]) *simp_all*

```
  also have ... = -(∑k=m..<n. p (Suc k) * f (Suc k) - p k * f k)
    by (simp add: sum_negf [symmetric] algebra_simps)
  also from n have ... = p m * f m - p n * f n
    by (cases n, simp, simp only: atLeastLessThanSuc_atLeastAtMost, subst
sum_Suc_diff) simp_all
  also from less_imp_le[OF m(1)] m(2) n have ... ≤ p m * f m by simp
  finally show norm (∑k≤n. f k) ≤ (norm (∑k≤m. r * f k) + p m * f m) /
r using r
    by (subst pos_le_divide_eq[OF r(1)]) (simp only: mult_ac)
 qed
 moreover have (∑k≤n. f k) ≤ (∑k≤n'. f k) if Suc m ≤ n n ≤ n' for n n'
  using less_imp_le[OF m(1)] that by (intro sum_mono2) auto
 ultimately show convergent (λn. ∑k≤n. f k) by (rule Bseq_monoseq_convergent'_inc)
qed


theorem kummers_test_divergence:
 fixes f p :: nat ⇒ real
 assumes pos_f: eventually (λn. f n > 0) sequentially
 assumes pos_p: eventually (λn. p n > 0) sequentially
 assumes divergent_p: ¬summable (λn. inverse (p n))
 defines l ≡ limsup (λn. ereal (p n * f n / f (Suc n) - p (Suc n)))
 assumes l: l < 0
 shows  ¬summable f
proof
 assume summable f
 from eventually_conj[OF pos_f eventually_conj[OF pos_p Limsup_lessD[OF
l[unfolded l_def]]]]
  obtain N where N: ⋀n. n ≥ N ⟹ p n > 0 ⋀n. n ≥ N ⟹ f n > 0
                    ⋀n. n ≥ N ⟹ p n * f n / f (Suc n) - p (Suc n) < 0
  by (auto simp: eventually_at_top_linorder)
 hence A: p n * f n < p (Suc n) * f (Suc n) if n ≥ N for n using that N[of n]
N[of Suc n]
  by (simp add: field_simps)
 have B: p n * f n ≥ p N * f N if n ≥ N for n using that and A
  by (induction n rule: dec_induct) (auto intro!: less_imp_le elim!: order.trans)
 have eventually (λn. norm (p N * f N * inverse (p n)) ≤ f n) sequentially
  apply (rule eventually_mono [OF eventually_ge_at_top[of N]])
  using B N  by (auto  simp: field_simps abs_of_pos)
 from this and ‹summable f› have summable (λn. p N * f N * inverse (p n))
  by (rule summable_comparison_test_ev)
 from summable_mult[OF this, of inverse (p N * f N)] N[OF le_refl]
  have summable (λn. inverse (p n)) by (simp add: field_split_simps)
 with divergent_p show False by contradiction
qed
```

Ratio test

```
theorem ratio_test_convergence:
```

fixes $f :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv liminf\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
assumes *l*: $l > 1$
shows *summable f*
proof (*rule kummers_test_convergence*[*OF pos_f*])
note *l*
also have $l = liminf\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n) - 1)) + 1$
by (*subst Liminf_add_ereal_right*[*symmetric*]) (*simp_all add*: *minus_ereal_def l_def one_ereal_def*)
finally show $liminf\ (\lambda n.\ ereal\ (1 * f\ n\ /\ f\ (Suc\ n) - 1)) > 0$
by (*cases liminf* $(\lambda n.\ ereal\ (1 * f\ n\ /\ f\ (Suc\ n) - 1))$ *simp_all*
qed *simp*

theorem *ratio_test_divergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv limsup\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
assumes *l*: $l < 1$
shows $\neg summable\ f$
proof (*rule kummers_test_divergence*[*OF pos_f*])
have $limsup\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n) - 1)) + 1 = l$
by (*subst Limsup_add_ereal_right*[*symmetric*]) (*simp_all add*: *minus_ereal_def l_def one_ereal_def*)
also note *l*
finally show $limsup\ (\lambda n.\ ereal\ (1 * f\ n\ /\ f\ (Suc\ n) - 1)) < 0$
by (*cases limsup* $(\lambda n.\ ereal\ (1 * f\ n\ /\ f\ (Suc\ n) - 1))$ *simp_all*
qed (*simp_all add*: *summable_const_iff*)

Raabe's test

theorem *raabes_test_convergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv liminf\ (\lambda n.\ ereal\ (of\_nat\ n * (f\ n\ /\ f\ (Suc\ n) - 1)))$
assumes *l*: $l > 1$
shows *summable f*
proof (*rule kummers_test_convergence*)
let $?l' = liminf\ (\lambda n.\ ereal\ (of\_nat\ n * f\ n\ /\ f\ (Suc\ n) - of\_nat\ (Suc\ n)))$
have $1 < l$ **by** *fact*
also have $l = liminf\ (\lambda n.\ ereal\ (of\_nat\ n * f\ n\ /\ f\ (Suc\ n) - of\_nat\ (Suc\ n)) + 1)$
by (*simp add*: *l_def algebra_simps*)
also have $\ldots = ?l' + 1$ **by** (*subst Liminf_add_ereal_right*) *simp_all*
finally show $?l' > 0$ **by** (*cases ?l'*) (*simp_all add*: *algebra_simps*)
qed (*simp_all add*: *pos*)

theorem *raabes_test_divergence*:
fixes $f :: nat \Rightarrow real$

assumes *pos*: *eventually* (λ*n*. *f n* > *0*) *sequentially*
defines *l* ≡ *limsup* (λ*n*. *ereal* (*of_nat n* ∗ (*f n* / *f* (*Suc n*) − *1*)))
assumes *l*: *l* < *1*
shows ¬*summable f*
proof (*rule kummers_test_divergence*)
let *?l′* = *limsup* (λ*n*. *ereal* (*of_nat n* ∗ *f n* / *f* (*Suc n*) − *of_nat* (*Suc n*)))
note *l*
also have *l* = *limsup* (λ*n*. *ereal* (*of_nat n* ∗ *f n* / *f* (*Suc n*) − *of_nat* (*Suc n*)) + *1*)
by (*simp add*: *l_def algebra_simps*)
also have ... = *?l′* + *1* **by** (*subst Limsup_add_ereal_right*) *simp_all*
finally show *?l′* < *0* **by** (*cases ?l′*) (*simp_all add*: *algebra_simps*)
qed (*insert pos eventually_gt_at_top*[*of 0*::*nat*] *not_summable_harmonic*, *simp_all*)

6.14.2 Radius of convergence

The radius of convergence of a power series. This value always exists, ranges from *0* to ∞, and the power series is guaranteed to converge for all inputs with a norm that is smaller than that radius and to diverge for all inputs with a norm that is greater.

definition *conv_radius* :: (*nat* ⇒ *'a* :: *banach*) ⇒ *ereal* **where**
conv_radius f = *inverse* (*limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))))

lemma *conv_radius_cong_weak* [*cong*]: (⋀*n*. *f n* = *g n*) ⟹ *conv_radius f* = *conv_radius g*
by (*drule ext*) *simp_all*

lemma *conv_radius_nonneg*: *conv_radius f* ≥ *0*
proof −
have *0* = *limsup* (λ*n*. *0*) **by** (*subst Limsup_const*) *simp_all*
also have ... ≤ *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
by (*intro Limsup_mono*) (*simp_all add*: *real_root_ge_zero*)
finally show *?thesis*
unfolding *conv_radius_def* **by** (*auto simp*: *ereal_inverse_nonneg_iff*)
qed

lemma *conv_radius_zero* [*simp*]: *conv_radius* (λ_. *0*) = ∞
by (*auto simp*: *conv_radius_def zero_ereal_def* [*symmetric*] *Limsup_const*)

lemma *conv_radius_altdef*:
conv_radius f = *liminf* (λ*n*. *inverse* (*ereal* (*root n* (*norm* (*f n*)))))
by (*subst Liminf_inverse_ereal*) (*simp_all add*: *real_root_ge_zero conv_radius_def*)

lemma *conv_radius_cong′*:
assumes *eventually* (λ*x*. *f x* = *g x*) *sequentially*
shows *conv_radius f* = *conv_radius g*
unfolding *conv_radius_altdef* **by** (*intro Liminf_eq eventually_mono* [*OF assms*])
auto

lemma *conv_radius_cong*:
assumes *eventually* (λ*x*. *norm* (*f x*) = *norm* (*g x*)) *sequentially*
shows *conv_radius f* = *conv_radius g*
unfolding *conv_radius_altdef* **by** (*intro Liminf_eq eventually_mono* [*OF assms*]) *auto*

theorem *abs_summable_in_conv_radius*:
fixes *f* :: *nat* ⇒ ′*a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) < *conv_radius f*
shows *summable* (λ*n*. *norm* (*f n* ∗ *z* ^ *n*))
proof (*rule root_test_convergence*′)
define *l* **where** *l* = *limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*))))
have *0* = *limsup* (λ*n*. *0*) **by** (*simp add*: *Limsup_const*)
also have ... ≤ *l* **unfolding** *l_def* **by** (*intro Limsup_mono*) (*simp_all add*: *real_root_ge_zero*)
finally have *l_nonneg*: *l* ≥ *0* **.**

have *limsup* (λ*n*. *root n* (*norm* (*f n* ∗ *z*^*n*))) = *l* ∗ *ereal* (*norm z*) **unfolding** *l_def*
by (*rule limsup_root_powser*)
also from *l_nonneg* **consider** *l* = *0* | *l* = ∞ | ∃ *l*′. *l* = *ereal l*′ ∧ *l*′ > *0*
by (*cases l*) (*auto simp*: *less_le*)
hence *l* ∗ *ereal* (*norm z*) < *1*
proof *cases*
assume *l* = ∞
hence *conv_radius f* = *0* **unfolding** *conv_radius_def l_def* **by** *simp*
with *assms* **show** *?thesis* **by** *simp*
next
assume ∃ *l*′. *l* = *ereal l*′ ∧ *l*′ > *0*
then obtain *l*′ **where** *l*′: *l* = *ereal l*′ *0* < *l*′ **by** *auto*
hence *l* ≠ ∞ **by** *auto*
have *l* ∗ *ereal* (*norm z*) < *l* ∗ *conv_radius f*
by (*intro ereal_mult_strict_left_mono*) (*simp_all add*: *l*′ *assms*)
also have *conv_radius f* = *inverse l* **by** (*simp add*: *conv_radius_def l_def*)
also from *l*′ **have** *l* ∗ *inverse l* = *1* **by** *simp*
finally show *?thesis* **.**
qed *simp_all*
finally show *limsup* (λ*n*. *ereal* (*root n* (*norm* (*norm* (*f n* ∗ *z* ^ *n*))))) < *1* **by** *simp*
qed

lemma *summable_in_conv_radius*:
fixes *f* :: *nat* ⇒ ′*a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) < *conv_radius f*
shows *summable* (λ*n*. *f n* ∗ *z* ^ *n*)
by (*rule summable_norm_cancel*, *rule abs_summable_in_conv_radius*) *fact*+

theorem *not_summable_outside_conv_radius*:

```
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  assumes ereal (norm z) > conv_radius f
  shows   ¬summable (λn. f n * z ^ n)
proof (rule root_test_divergence)
  define l where l = limsup (λn. ereal (root n (norm (f n))))
  have 0 = limsup (λn. 0) by (simp add: Limsup_const)
  also have ... ≤ l unfolding l_def by (intro Limsup_mono) (simp_all add:
real_root_ge_zero)
  finally have l_nonneg: l ≥ 0 .
  from assms have l_nz: l ≠ 0 unfolding conv_radius_def l_def by auto

  have limsup (λn. ereal (root n (norm (f n * z^n)))) = l * ereal (norm z)
    unfolding l_def by (rule limsup_root_powser)
  also have ... > 1
  proof (cases l)
    assume l = ∞
    with assms conv_radius_nonneg[of f] show ?thesis
      by (auto simp: zero_ereal_def[symmetric])
  next
    fix l' assume l': l = ereal l'
    from l_nonneg l_nz have 1 = l * inverse l by (auto simp: l' field_simps)
    also from l_nz have inverse l = conv_radius f
      unfolding l_def conv_radius_def by auto
    also from l' l_nz l_nonneg assms have l * ... < l * ereal (norm z)
      by (intro ereal_mult_strict_left_mono) (auto simp: l')
    finally show ?thesis .
  qed (insert l_nonneg, simp_all)
  finally show limsup (λn. ereal (root n (norm (f n * z^n)))) > 1 .
qed


lemma conv_radius_geI:
  assumes summable (λn. f n * z ^ n :: 'a :: {banach, real_normed_div_algebra})
  shows   conv_radius f ≥ norm z
  using not_summable_outside_conv_radius[of f z] assms by (force simp: not_le[symmetric])

lemma conv_radius_leI:
  assumes ¬summable (λn. norm (f n * z ^ n :: 'a :: {banach, real_normed_div_algebra}))
  shows   conv_radius f ≤ norm z
  using abs_summable_in_conv_radius[of z f] assms by (force simp: not_le[symmetric])

lemma conv_radius_leI':
  assumes ¬summable (λn. f n * z ^ n :: 'a :: {banach, real_normed_div_algebra})
  shows   conv_radius f ≤ norm z
  using summable_in_conv_radius[of z f] assms by (force simp: not_le[symmetric])

lemma conv_radius_geI_ex:
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  assumes ⋀r. 0 < r ⟹ ereal r < R ⟹ ∃z. norm z = r ∧ summable (λn. f n
```

```
* z^n)
  shows   conv_radius f ≥ R
proof (rule linorder_cases[of conv_radius f R])
  assume R: conv_radius f < R
  with conv_radius_nonneg[of f] obtain conv_radius'
    where [simp]: conv_radius f = ereal conv_radius'
    by (cases conv_radius f) simp_all
  define r where r = (if R = ∞ then conv_radius' + 1 else (real_of_ereal R +
conv_radius') / 2)
   from R conv_radius_nonneg[of f] have 0 < r ∧ ereal r < R ∧ ereal r >
conv_radius f
    unfolding r_def by (cases R) (auto simp: r_def field_simps)
  with assms(1)[of r] obtain z where norm z > conv_radius f summable (λn. f
n * z^n) by auto
  with not_summable_outside_conv_radius[of f z] show ?thesis by simp
qed simp_all

lemma conv_radius_geI_ex':
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  assumes ⋀r. 0 < r ⟹ ereal r < R ⟹ summable (λn. f n * of_real r^n)
  shows   conv_radius f ≥ R
proof (rule conv_radius_geI_ex)
  fix r assume 0 < r ereal r < R
  with assms[of r] show ∃z. norm z = r ∧ summable (λn. f n * z ^ n)
    by (intro exI[of _ of_real r :: 'a]) auto
qed

lemma conv_radius_leI_ex:
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  assumes R ≥ 0
  assumes ⋀r. 0 < r ⟹ ereal r > R ⟹ ∃z. norm z = r ∧ ¬summable (λn.
norm (f n * z^n))
  shows   conv_radius f ≤ R
proof (rule linorder_cases[of conv_radius f R])
  assume R: conv_radius f > R
  from R assms(1) obtain R' where R': R = ereal R' by (cases R) simp_all
  define r where
    r = (if conv_radius f = ∞ then R' + 1 else (R' + real_of_ereal (conv_radius
f)) / 2)
  from R conv_radius_nonneg[of f] have r > R ∧ r < conv_radius f unfolding
r_def
    by (cases conv_radius f) (auto simp: r_def field_simps R')
  with assms(1) assms(2)[of r] R'
    obtain z where norm z < conv_radius f ¬summable (λn. norm (f n * z^n))
by auto
  with abs_summable_in_conv_radius[of z f] show ?thesis by auto
qed simp_all

lemma conv_radius_leI_ex':
```

fixes $f :: nat \Rightarrow 'a :: \{banach, real\_normed\_div\_algebra\}$
assumes $R \geq 0$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r > R \Longrightarrow \neg summable\ (\lambda n.\ f\ n * of\_real\ r \hat{}\ n)$
shows *conv_radius f* $\leq R$
proof (*rule conv_radius_leI_ex*)
fix r **assume** $0 < r$ *ereal* $r > R$
with *assms*(2)[*of r*] **show** $\exists z.\ norm\ z = r \wedge \neg summable\ (\lambda n.\ norm\ (f\ n * z \hat{}\ n))$
by (*intro exI*[*of* _ *of_real r* :: $'a$]) (*auto dest*: *summable_norm_cancel*)
qed *fact*+

lemma *conv_radius_eqI*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\_normed\_div\_algebra\}$
assumes $R \geq 0$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r < R \Longrightarrow \exists z.\ norm\ z = r \wedge summable\ (\lambda n.\ f\ n * z \hat{}\ n)$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r > R \Longrightarrow \exists z.\ norm\ z = r \wedge \neg summable\ (\lambda n.\ norm\ (f\ n * z \hat{}\ n))$
shows *conv_radius f* $= R$
by (*intro antisym conv_radius_geI_ex conv_radius_leI_ex assms*)

lemma *conv_radius_eqI'*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\_normed\_div\_algebra\}$
assumes $R \geq 0$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r < R \Longrightarrow summable\ (\lambda n.\ f\ n * (of\_real\ r) \hat{}\ n)$
assumes $\bigwedge r.\ 0 < r \Longrightarrow ereal\ r > R \Longrightarrow \neg summable\ (\lambda n.\ norm\ (f\ n * (of\_real\ r) \hat{}\ n))$
shows *conv_radius f* $= R$
proof (*intro conv_radius_eqI*[*OF assms*(1)])
fix r **assume** $0 < r$ *ereal* $r < R$ **with** *assms*(2)[*OF this*]
show $\exists z.\ norm\ z = r \wedge summable\ (\lambda n.\ f\ n * z \hat{}\ n)$ **by** *force*
next
fix r **assume** $0 < r$ *ereal* $r > R$ **with** *assms*(3)[*OF this*]
show $\exists z.\ norm\ z = r \wedge \neg summable\ (\lambda n.\ norm\ (f\ n * z \hat{}\ n))$ **by** *force*
qed

lemma *conv_radius_zeroI*:
fixes $f :: nat \Rightarrow 'a :: \{banach,real\_normed\_div\_algebra\}$
assumes $\bigwedge z.\ z \neq 0 \Longrightarrow \neg summable\ (\lambda n.\ f\ n * z \hat{}\ n)$
shows *conv_radius f* $= 0$
proof (*rule ccontr*)
assume *conv_radius f* $\neq 0$
with *conv_radius_nonneg*[*of f*] **have** *pos*: *conv_radius f* > 0 **by** *simp*
define r **where** $r = (if\ conv\_radius\ f = \infty\ then\ 1\ else\ real\_of\_ereal\ (conv\_radius\ f)\ /\ 2)$
from *pos* **have** r: *ereal* $r > 0 \wedge$ *ereal* $r <$ *conv_radius f*
by (*cases conv_radius f*) (*simp_all add*: *r_def*)
hence *summable* $(\lambda n.\ f\ n * of\_real\ r \hat{}\ n)$ **by** (*intro summable_in_conv_radius*) *simp*

```
  moreover from r and assms[of of_real r] have ¬summable (λn. f n * of_real
r ^ n) by simp
  ultimately show False by contradiction
qed

lemma conv_radius_inftyI':
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes ⋀r. r > c ⟹ ∃z. norm z = r ∧ summable (λn. f n * z^n)
  shows   conv_radius f = ∞
proof −
  {
    fix r :: real
    have max r (c + 1) > c by (auto simp: max_def)
    from assms[OF this] obtain z where norm z = max r (c + 1) summable (λn.
f n * z^n) by blast
    from conv_radius_geI[OF this(2)] this(1) have conv_radius f ≥ r by simp
  }
  from this[of real_of_ereal (conv_radius f + 1)] show conv_radius f = ∞
    by (cases conv_radius f) simp_all
qed

lemma conv_radius_inftyI:
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes ⋀r. ∃z. norm z = r ∧ summable (λn. f n * z^n)
  shows   conv_radius f = ∞
  using assms by (rule conv_radius_inftyI')

lemma conv_radius_inftyI'':
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes ⋀z. summable (λn. f n * z^n)
  shows   conv_radius f = ∞
proof (rule conv_radius_inftyI')
  fix r :: real assume r > 0
  with assms show ∃z. norm z = r ∧ summable (λn. f n * z^n)
    by (intro exI[of _ of_real r]) simp
qed

lemma conv_radius_conv_Sup:
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  shows conv_radius f = Sup {r. ∀z. ereal (norm z) < r ⟶ summable (λn. f n
* z ^ n)}
proof (rule Sup_eqI [symmetric], goal_cases)
  case (1 r)
  thus ?case
    by (intro conv_radius_geI_ex') auto
next
  case (2 r)
  from 2[of 0] have r: r ≥ 0 by auto
  show ?case
```

```
  proof (intro conv_radius_leI_ex' r)
    fix R assume R: R > 0 ereal R > r
    with r obtain r' where [simp]: r = ereal r' by (cases r) auto
    show ¬summable (λn. f n * of_real R ^ n)
    proof
      assume *: summable (λn. f n * of_real R ^ n)
      define R' where R' = (R + r') / 2
      from R have R': R' > r' R' < R by (simp_all add: R'_def)
      hence ∀ z. norm z < R' ⟶ summable (λn. f n * z ^ n)
        using powser_inside[OF *] by auto
      from 2[of R'] and this have R' ≤ r' by auto
      with ‹R' > r'› show False by simp
    qed
  qed
qed

lemma conv_radius_shift:
  fixes f :: nat ⇒ 'a :: {banach, real_normed_div_algebra}
  shows   conv_radius (λn. f (n + m)) = conv_radius f
  unfolding conv_radius_conv_Sup summable_powser_ignore_initial_segment ..

lemma conv_radius_norm [simp]: conv_radius (λx. norm (f x)) = conv_radius f
  by (simp add: conv_radius_def)

lemma conv_radius_ratio_limit_ereal:
  fixes f :: nat ⇒ 'a :: {banach,real_normed_div_algebra}
  assumes nz:  eventually (λn. f n ≠ 0) sequentially
  assumes lim: (λn. ereal (norm (f n) / norm (f (Suc n)))) ——⟶ c
  shows   conv_radius f = c
proof (rule conv_radius_eqI')
  show c ≥ 0 by (intro Lim_bounded2[OF lim]) simp_all
next
  fix r assume r: 0 < r ereal r < c
  let ?l = liminf (λn. ereal (norm (f n * of_real r ^ n) / norm (f (Suc n) * of_real
r ^ Suc n)))
  have ?l = liminf (λn. ereal (norm (f n) / (norm (f (Suc n)))) * ereal (inverse
r))
    using r by (simp add: norm_mult norm_power field_split_simps)
  also from r have ... = liminf (λn. ereal (norm (f n) / (norm (f (Suc n))))) *
ereal (inverse r)
    by (intro Liminf_ereal_mult_right) simp_all
  also have liminf (λn. ereal (norm (f n) / (norm (f (Suc n))))) = c
    by (intro lim_imp_Liminf lim) simp
  finally have l: ?l = c * ereal (inverse r) by simp
  from r have l': c * ereal (inverse r) > 1 by (cases c) (simp_all add: field_simps)
  show summable (λn. f n * of_real r^n)
    by (rule summable_norm_cancel, rule ratio_test_convergence)
       (insert r nz l l', auto elim!: eventually_mono)
next
```

fix r **assume** r: $0 < r$ $ereal\ r > c$
let $?l = limsup\ (\lambda n.\ ereal\ (norm\ (f\ n * of\_real\ r\ \hat{}\ n)\ /\ norm\ (f\ (Suc\ n) * of\_real\ r\ \hat{}\ Suc\ n)))$
have $?l = limsup\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ (norm\ (f\ (Suc\ n)))) * ereal\ (inverse\ r))$
using r **by** (*simp add*: *norm_mult norm_power field_split_simps*)
also from r **have** $\ldots = limsup\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ (norm\ (f\ (Suc\ n))))) * ereal\ (inverse\ r)$
by (*intro Limsup_ereal_mult_right*) *simp_all*
also have $limsup\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ (norm\ (f\ (Suc\ n))))) = c$
by (*intro lim_imp_Limsup lim*) *simp*
finally have l: $?l = c * ereal\ (inverse\ r)$ **by** *simp*
from r **have** l': $c * ereal\ (inverse\ r) < 1$ **by** (*cases c*) (*simp_all add*: *field_simps*)
show $\neg summable\ (\lambda n.\ norm\ (f\ n * of\_real\ r\hat{}n))$
by (*rule ratio_test_divergence*) (*insert r nz l l'*, *auto elim*!: *eventually_mono*)
qed

lemma *conv_radius_ratio_limit_ereal_nonzero*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\_normed\_div\_algebra\}$
assumes nz: $c \neq 0$
assumes lim: $(\lambda n.\ ereal\ (norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n)))) \longrightarrow c$
shows $conv\_radius\ f = c$
proof (*rule conv_radius_ratio_limit_ereal*[*OF* _ *lim*], *rule ccontr*)
assume $\neg eventually\ (\lambda n.\ f\ n \neq 0)\ sequentially$
hence $frequently\ (\lambda n.\ f\ n = 0)\ sequentially$ **by** (*simp add*: *frequently_def*)
hence $frequently\ (\lambda n.\ ereal\ (norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n))) = 0)\ sequentially$
by (*force elim*!: *frequently_elim1*)
hence $c = 0$ **by** (*intro limit_frequently_eq*[*OF* _ _ *lim*]) *auto*
with nz **show** *False* **by** *contradiction*
qed

lemma *conv_radius_ratio_limit*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\_normed\_div\_algebra\}$
assumes $c' = ereal\ c$
assumes nz: $eventually\ (\lambda n.\ f\ n \neq 0)\ sequentially$
assumes lim: $(\lambda n.\ norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n))) \longrightarrow c$
shows $conv\_radius\ f = c'$
using *assms* **by** (*intro conv_radius_ratio_limit_ereal*) *simp_all*

lemma *conv_radius_ratio_limit_nonzero*:
fixes $f :: nat \Rightarrow 'a :: \{banach, real\_normed\_div\_algebra\}$
assumes $c' = ereal\ c$
assumes nz: $c \neq 0$
assumes lim: $(\lambda n.\ norm\ (f\ n)\ /\ norm\ (f\ (Suc\ n))) \longrightarrow c$
shows $conv\_radius\ f = c'$
using *assms* **by** (*intro conv_radius_ratio_limit_ereal_nonzero*) *simp_all*

lemma *conv_radius_cmult_left*:
assumes $c \neq (0 :: 'a :: \{banach, real\_normed\_div\_algebra\})$

```
  shows   conv_radius (λn. c * f n) = conv_radius f
proof −
  have conv_radius (λn. c * f n) =
         inverse (limsup (λn. ereal (root n (norm (c * f n)))))
    unfolding conv_radius_def ..
  also have (λn. ereal (root n (norm (c * f n)))) =
              (λn. ereal (root n (norm c))) * ereal (root n (norm (f n))))
    by (rule ext) (auto simp: norm_mult real_root_mult)
  also have limsup ... = ereal 1 * limsup (λn. ereal (root n (norm (f n))))
   using assms by (intro ereal_limsup_lim_mult tendsto_ereal LIMSEQ_root_const)
auto
  also have inverse ... = conv_radius f by (simp add: conv_radius_def)
  finally show ?thesis .
qed

lemma conv_radius_cmult_right:
  assumes c ≠ (0 :: 'a :: {banach, real_normed_div_algebra})
  shows   conv_radius (λn. f n * c) = conv_radius f
proof −
  have conv_radius (λn. f n * c) = conv_radius (λn. c * f n)
    by (simp add: conv_radius_def norm_mult mult.commute)
  with conv_radius_cmult_left[OF assms, of f] show ?thesis by simp
qed

lemma conv_radius_mult_power:
  assumes c ≠ (0 :: 'a :: {real_normed_div_algebra,banach})
  shows   conv_radius (λn. c ^ n * f n) = conv_radius f / ereal (norm c)
proof −
  have limsup (λn. ereal (root n (norm (c ^ n * f n)))) =
         limsup (λn. ereal (norm c) * ereal (root n (norm (f n))))
    by (intro Limsup_eq eventually_mono [OF eventually_gt_at_top[of 0::nat]])
       (auto simp: norm_mult norm_power real_root_mult real_root_power)
  also have ... = ereal (norm c) * limsup (λn. ereal (root n (norm (f n))))
    using assms by (subst Limsup_ereal_mult_left[symmetric]) simp_all
  finally have A: limsup (λn. ereal (root n (norm (c ^ n * f n)))) =
                    ereal (norm c) * limsup (λn. ereal (root n (norm (f n)))) .
  show ?thesis using assms
    apply (cases limsup (λn. ereal (root n (norm (f n)))) = 0)
    apply (simp add: A conv_radius_def)
    apply (unfold conv_radius_def A divide_ereal_def, simp add: mult.commute
ereal_inverse_mult)
    done
qed

lemma conv_radius_mult_power_right:
  assumes c ≠ (0 :: 'a :: {real_normed_div_algebra,banach})
  shows   conv_radius (λn. f n * c ^ n) = conv_radius f / ereal (norm c)
  using conv_radius_mult_power[OF assms, of f]
  unfolding conv_radius_def by (simp add: mult.commute norm_mult)
```

lemma *conv_radius_divide_power*:
assumes $c \neq (0 :: 'a :: \{real\_normed\_div\_algebra, banach\})$
shows *conv_radius* $(\lambda n.\ f\ n\ /\ c\hat{}n) =$ *conv_radius* $f *$ *ereal* $(norm\ c)$
proof −
from *assms* **have** *inverse* $c \neq 0$ **by** *simp*
from *conv_radius_mult_power_right*[*OF this*, *of f*] **show** *?thesis*
by (*simp add*: *divide_inverse divide_ereal_def assms norm_inverse power_inverse*)
qed

lemma *conv_radius_add_ge*:
min (*conv_radius f*) (*conv_radius g*) ≤
conv_radius $(\lambda x.\ f\ x + g\ x :: 'a :: \{banach, real\_normed\_div\_algebra\})$
by (*rule conv_radius_geI_ex'*)
(*auto simp*: *algebra_simps intro*!: *summable_add summable_in_conv_radius*)

lemma *conv_radius_mult_ge*:
fixes $f\ g :: nat \Rightarrow ('a :: \{banach, real\_normed\_div\_algebra\})$
shows *conv_radius* $(\lambda x. \sum i{\leq}x.\ f\ i * g\ (x - i)) \geq$ *min* (*conv_radius f*) (*conv_radius g*)
proof (*rule conv_radius_geI_ex'*)
fix *r* **assume** *r*: $r > 0$ *ereal* $r <$ *min* (*conv_radius f*) (*conv_radius g*)
from *r* **have** *summable* $(\lambda n.\ (\sum i{\leq}n.\ (f\ i * of\_real\ r\hat{}i) * (g\ (n - i) * of\_real\ r\hat{}(n - i))))$
by (*intro summable_Cauchy_product abs_summable_in_conv_radius*) *simp_all*
thus *summable* $(\lambda n.\ (\sum i{\leq}n.\ f\ i * g\ (n - i)) * of\_real\ r\ \hat{}\ n)$
by (*simp add*: *algebra_simps of_real_def power_add* [*symmetric*] *scaleR_sum_right*)
qed

lemma *le_conv_radius_iff*:
fixes $a :: nat \Rightarrow 'a :: \{real\_normed\_div\_algebra, banach\}$
shows $r \leq$ *conv_radius a* $\longleftrightarrow (\forall x.\ norm\ (x{-}\xi) < r \longrightarrow summable\ (\lambda i.\ a\ i * (x - \xi)\ \hat{}\ i))$
apply (*intro iffI allI impI summable_in_conv_radius conv_radius_geI_ex*)
apply (*meson less_ereal.simps*(1) *not_le order_trans*)
apply (*rule_tac x*=*of_real ra* **in** *exI*, *simp*)
apply (*metis abs_of_nonneg add_diff_cancel_left' less_eq_real_def norm_of_real*)
done

end

6.15 Uniform Limit and Uniform Convergence

theory *Uniform_Limit*
imports *Connected Summation_Tests Infinite_Sum*
begin

6.15.1 Definition

definition *uniformly_on* :: *'a set* ⇒ (*'a* ⇒ *'b*::*metric_space*) ⇒ (*'a* ⇒ *'b*) *filter*
where *uniformly_on S l* = (*INF e*∈{*0* <..}. *principal* {*f*. ∀*x*∈*S*. *dist* (*f x*) (*l x*) < *e*})

abbreviation
uniform_limit S f l ≡ *filterlim f* (*uniformly_on S l*)

definition *uniformly_convergent_on* **where**
uniformly_convergent_on X f ⟷ (∃*l*. *uniform_limit X f l sequentially*)

definition *uniformly_Cauchy_on* **where**
uniformly_Cauchy_on X f ⟷ (∀*e*>*0*. ∃*M*. ∀*x*∈*X*. ∀(*m*::*nat*)≥*M*. ∀*n*≥*M*. *dist* (*f m x*) (*f n x*) < *e*)

proposition *uniform_limit_iff*:
uniform_limit S f l F ⟷ (∀*e*>*0*. ∀$_F$ *n in F*. ∀*x*∈*S*. *dist* (*f n x*) (*l x*) < *e*)
unfolding *filterlim_iff uniformly_on_def*
by (*subst eventually_INF_base*)
(*fastforce*
simp: *eventually_principal uniformly_on_def*
intro: *bexI*[**where** *x*=*min a b* **for** *a b*]
elim: *eventually_mono*)+

lemma *uniform_limitD*:
uniform_limit S f l F ⟹ *e* > *0* ⟹ ∀$_F$ *n in F*. ∀*x*∈*S*. *dist* (*f n x*) (*l x*) < *e*
by (*simp add*: *uniform_limit_iff*)

lemma *uniform_limitI*:
(⋀*e*. *e* > *0* ⟹ ∀$_F$ *n in F*. ∀*x*∈*S*. *dist* (*f n x*) (*l x*) < *e*) ⟹ *uniform_limit S f l F*
by (*simp add*: *uniform_limit_iff*)

lemma *uniform_limit_sequentially_iff*:
uniform_limit S f l sequentially ⟷ (∀*e*>*0*. ∃*N*. ∀*n*≥*N*. ∀*x* ∈ *S*. *dist* (*f n x*) (*l x*) < *e*)
unfolding *uniform_limit_iff eventually_sequentially* **..**

lemma *uniform_limit_at_iff*:
uniform_limit S f l (*at x*) ⟷
(∀*e*>*0*. ∃*d*>*0*. ∀*z*. *0* < *dist z x* ∧ *dist z x* < *d* ⟶ (∀*x*∈*S*. *dist* (*f z x*) (*l x*) < *e*))
unfolding *uniform_limit_iff eventually_at* **by** *simp*

lemma *uniform_limit_at_le_iff*:
uniform_limit S f l (*at x*) ⟷
(∀*e*>*0*. ∃*d*>*0*. ∀*z*. *0* < *dist z x* ∧ *dist z x* < *d* ⟶ (∀*x*∈*S*. *dist* (*f z x*) (*l x*) ≤ *e*))
unfolding *uniform_limit_iff eventually_at*

by (*fastforce dest*: *spec*[**where** $x = e\ /\ 2$ **for** e])

lemma *uniform_limit_compose*:
assumes *ul*: *uniform_limit X g l F*
and *cont*: *uniformly_continuous_on U f*
and *g*: $\forall_F\ n$ *in F*. *g n ' X* $\subseteq$ *U* **and** *l*: *l ' X* $\subseteq$ *U*
shows *uniform_limit X* ($\lambda a\ b$. *f* (*g a b*)) (*f* $\circ$ *l*) *F*
proof (*rule uniform_limitI*)
fix ε::*real*
assume $0 < \varepsilon$
then obtain δ **where** $\delta > 0$ **and** δ: $\bigwedge u\ v$. $[\![u{\in}U;\ v{\in}U;\ dist\ u\ v < \delta]\!] \Longrightarrow dist$ (*f u*) (*f v*) $< \varepsilon$
by (*metis cont uniformly_continuous_on_def*)
show $\forall_F\ n$ *in F*. $\forall x{\in}X$. *dist* (*f* (*g n x*)) ((*f* $\circ$ *l*) *x*) $< \varepsilon$
using *uniform_limitD* [*OF ul* ‹$\delta{>}0$›] *g* **unfolding** *o_def*
by *eventually_elim* (*use* δ *l* **in** *blast*)
qed

lemma *metric_uniform_limit_imp_uniform_limit*:
assumes *f*: *uniform_limit S f a F*
assumes *le*: *eventually* (λx. $\forall y{\in}S$. *dist* (*g x y*) (*b y*) $\leq$ *dist* (*f x y*) (*a y*)) *F*
shows *uniform_limit S g b F*
proof (*rule uniform_limitI*)
fix *e* :: *real*
assume $0 < e$
from *uniform_limitD*[*OF f this*] *le*
show $\forall_F\ x$ *in F*. $\forall y{\in}S$. *dist* (*g x y*) (*b y*) $< e$
by *eventually_elim force*
qed

6.15.2 Exchange limits

proposition *swap_uniform_limit*:
assumes *f*: $\forall_F\ n$ *in F*. (*f n* $\longrightarrow$ *g n*) (*at x within S*)
assumes *g*: (*g* $\longrightarrow$ *l*) *F*
assumes *uc*: *uniform_limit S f h F*
assumes $\neg$*trivial_limit F*
shows (*h* $\longrightarrow$ *l*) (*at x within S*)
proof (*rule tendstoI*)
fix *e* :: *real*
define e' **where** $e' = e/3$
assume $0 < e$
then have $0 < e'$ **by** (*simp add*: e'*_def*)
from *uniform_limitD*[*OF uc* ‹$0 < e'$›]
have $\forall_F\ n$ *in F*. $\forall x{\in}S$. *dist* (*h x*) (*f n x*) $< e'$
by (*simp add*: *dist_commute*)
moreover
from *f*
have $\forall_F\ n$ *in F*. $\forall_F\ x$ *in at x within S*. *dist* (*g n*) (*f n x*) $< e'$

```
    by eventually_elim (auto dest!: tendstoD[OF _ ‹0 < e'›] simp: dist_commute)
  moreover
  from tendstoD[OF g ‹0 < e'›] have ∀⇩F x in F. dist l (g x) < e'
    by (simp add: dist_commute)
  ultimately
  have ∀⇩F _ in F. ∀⇩F x in at x within S. dist (h x) l < e
  proof eventually_elim
    case (elim n)
    note fh = elim(1)
    note gl = elim(3)
    have ∀⇩F x in at x within S. x ∈ S
      by (auto simp: eventually_at_filter)
    with elim(2)
    show ?case
    proof eventually_elim
      case (elim x)
      from fh[rule_format, OF ‹x ∈ S›] elim(1)
      have dist (h x) (g n) < e' + e'
        by (rule dist_triangle_lt[OF add_strict_mono])
      from dist_triangle_lt[OF add_strict_mono, OF this gl]
      show ?case by (simp add: e'_def)
    qed
  qed
  thus ∀⇩F x in at x within S. dist (h x) l < e
    using eventually_happens by (metis ‹¬trivial_limit F›)
qed
```

6.15.3 Uniform limit theorem

```
lemma tendsto_uniform_limitI:
  assumes uniform_limit S f l F
  assumes x ∈ S
  shows ((λy. f y x) ⟶ l x) F
  using assms
  by (auto intro!: tendstoI simp: eventually_mono dest!: uniform_limitD)

theorem uniform_limit_theorem:
  assumes c: ∀⇩F n in F. continuous_on A (f n)
  assumes ul: uniform_limit A f l F
  assumes ¬ trivial_limit F
  shows continuous_on A l
  unfolding continuous_on_def
proof safe
  fix x assume x ∈ A
  then have ∀⇩F n in F. (f n ⟶ f n x) (at x within A) ((λn. f n x) ⟶ l x) F
    using c ul
    by (auto simp: continuous_on_def eventually_mono tendsto_uniform_limitI)
  then show (l ⟶ l x) (at x within A)
    by (rule swap_uniform_limit) fact+
```

qed

lemma *uniformly_Cauchy_onI*:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists M.\ \forall x{\in}X.\ \forall m{\geq}M.\ \forall n{\geq}M.\ dist\ (f\ m\ x)\ (f\ n\ x) < e$
shows *uniformly_Cauchy_on X f*
using *assms* **unfolding** *uniformly_Cauchy_on_def* **by** *blast*

lemma *uniformly_Cauchy_onI'*:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists M.\ \forall x{\in}X.\ \forall m{\geq}M.\ \forall n{>}m.\ dist\ (f\ m\ x)\ (f\ n\ x) < e$
shows *uniformly_Cauchy_on X f*
proof (*rule uniformly_Cauchy_onI*)
fix *e* :: *real* **assume** *e*: $e > 0$
from *assms*[*OF this*] **obtain** *M*
where *M*: $\bigwedge x\ m\ n.\ x \in X \Longrightarrow m \geq M \Longrightarrow n > m \Longrightarrow dist\ (f\ m\ x)\ (f\ n\ x) < e$ **by** *fast*
{
fix *x m n* **assume** *x*: $x \in X$ **and** *m*: $m \geq M$ **and** *n*: $n \geq M$
with *M*[*OF this*(*1*,*2*), *of n*] *M*[*OF this*(*1*,*3*), *of m*] *e* **have** $dist\ (f\ m\ x)\ (f\ n\ x) < e$
by (*cases m n rule*: *linorder_cases*) (*simp_all add*: *dist_commute*)
}
thus $\exists M.\ \forall x{\in}X.\ \forall m{\geq}M.\ \forall n{\geq}M.\ dist\ (f\ m\ x)\ (f\ n\ x) < e$ **by** *fast*
qed

lemma *uniformly_Cauchy_imp_Cauchy*:
uniformly_Cauchy_on X f $\Longrightarrow x \in X \Longrightarrow$ *Cauchy* $(\lambda n.\ f\ n\ x)$
unfolding *Cauchy_def uniformly_Cauchy_on_def* **by** *fast*

lemma *uniform_limit_cong*:
fixes *f g* :: $'a \Rightarrow 'b \Rightarrow ('c$ :: *metric_space*) **and** *h i* :: $'b \Rightarrow 'c$
assumes *eventually* $(\lambda y.\ \forall x{\in}X.\ f\ y\ x = g\ y\ x)\ F$
assumes $\bigwedge x.\ x \in X \Longrightarrow h\ x = i\ x$
shows *uniform_limit X f h F* $\longleftrightarrow$ *uniform_limit X g i F*
proof −
{
fix *f g* :: $'a \Rightarrow 'b \Rightarrow 'c$ **and** *h i* :: $'b \Rightarrow 'c$
assume *C*: *uniform_limit X f h F* **and** *A*: *eventually* $(\lambda y.\ \forall x{\in}X.\ f\ y\ x = g\ y\ x)\ F$
and *B*: $\bigwedge x.\ x \in X \Longrightarrow h\ x = i\ x$
{
fix *e* ::*real* **assume** $e > 0$
with *C* **have** *eventually* $(\lambda y.\ \forall x{\in}X.\ dist\ (f\ y\ x)\ (h\ x) < e)\ F$
unfolding *uniform_limit_iff* **by** *blast*
with *A* **have** *eventually* $(\lambda y.\ \forall x{\in}X.\ dist\ (g\ y\ x)\ (i\ x) < e)\ F$
by *eventually_elim* (*insert B*, *simp_all*)
}
hence *uniform_limit X g i F* **unfolding** *uniform_limit_iff* **by** *blast*
} **note** *A* = *this*
show *?thesis* **by** (*rule iffI*) (*erule A*; *insert assms*; *simp add*: *eq_commute*)+

qed

lemma *uniform_limit_cong'*:
fixes $f\ g :: 'a \Rightarrow 'b \Rightarrow ('c :: metric\_space)$ **and** $h\ i :: 'b \Rightarrow 'c$
assumes $\bigwedge y\ x.\ x \in X \Longrightarrow f\ y\ x = g\ y\ x$
assumes $\bigwedge x.\ x \in X \Longrightarrow h\ x = i\ x$
shows $uniform\_limit\ X\ f\ h\ F \longleftrightarrow uniform\_limit\ X\ g\ i\ F$
using *assms* **by** (*intro uniform_limit_cong always_eventually*) *blast+*

lemma *uniformly_convergent_cong*:
assumes $eventually\ (\lambda x.\ \forall y{\in}A.\ f\ x\ y = g\ x\ y)\ sequentially\ A = B$
shows $uniformly\_convergent\_on\ A\ f \longleftrightarrow uniformly\_convergent\_on\ B\ g$
unfolding *uniformly_convergent_on_def assms(2)* [*symmetric*]
by (*intro iff_exI uniform_limit_cong eventually_mono* [*OF assms(1)*]) *auto*

lemma *uniformly_convergent_on_compose*:
assumes *uniformly_convergent_on A f*
assumes *filterlim g sequentially sequentially*
shows $uniformly\_convergent\_on\ A\ (\lambda n.\ f\ (g\ n))$
proof −
from *assms(1)* **obtain** f' **where** $uniform\_limit\ A\ f\ f'\ sequentially$
by (*auto simp*: *uniformly_convergent_on_def*)
hence $uniform\_limit\ A\ (\lambda n.\ f\ (g\ n))\ f'\ sequentially$
by (*rule filterlim_compose*) *fact*
thus *?thesis*
by (*auto simp*: *uniformly_convergent_on_def*)
qed

lemma *uniformly_convergent_uniform_limit_iff*:
$uniformly\_convergent\_on\ X\ f \longleftrightarrow uniform\_limit\ X\ f\ (\lambda x.\ lim\ (\lambda n.\ f\ n\ x))$
sequentially
proof
assume *uniformly_convergent_on X f*
then obtain *l* **where** *l*: $uniform\_limit\ X\ f\ l\ sequentially$
unfolding *uniformly_convergent_on_def* **by** *blast*
from *l* **have** $uniform\_limit\ X\ f\ (\lambda x.\ lim\ (\lambda n.\ f\ n\ x))\ sequentially \longleftrightarrow$
$uniform\_limit\ X\ f\ l\ sequentially$
by (*intro uniform_limit_cong' limI tendsto_uniform_limitI*[*of f X l*]) *simp_all*
also note *l*
finally show $uniform\_limit\ X\ f\ (\lambda x.\ lim\ (\lambda n.\ f\ n\ x))\ sequentially$ **.**
qed (*auto simp*: *uniformly_convergent_on_def*)

lemma *uniformly_convergentI*: $uniform\_limit\ X\ f\ l\ sequentially \Longrightarrow uniformly\_convergent\_on\ X\ f$
unfolding *uniformly_convergent_on_def* **by** *blast*

lemma *uniformly_convergent_on_empty* [*iff*]: $uniformly\_convergent\_on\ \{\}\ f$
by (*simp add*: *uniformly_convergent_on_def uniform_limit_sequentially_iff*)

lemma *uniformly_convergent_on_const* [*simp*,*intro*]:
uniformly_convergent_on A (λ_. *c*)
by (*auto simp*: *uniformly_convergent_on_def uniform_limit_iff intro*!: *exI*[*of* _ *c*])

Cauchy-type criteria for uniform convergence.

lemma *Cauchy_uniformly_convergent*:
fixes *f* :: *nat* ⇒ *'a* ⇒ *'b* :: *complete_space*
assumes *uniformly_Cauchy_on X f*
shows *uniformly_convergent_on X f*
unfolding *uniformly_convergent_uniform_limit_iff uniform_limit_iff*
proof *safe*
let *?f* = λ*x*. *lim* (λ*n*. *f n x*)
fix *e* :: *real* **assume** *e*: *e* > *0*
hence *e*/*2* > *0* **by** *simp*
with *assms* **obtain** *N* **where** *N*: ⋀*x m n*. *x* ∈ *X* ⟹ *m* ≥ *N* ⟹ *n* ≥ *N* ⟹ *dist* (*f m x*) (*f n x*) < *e*/*2*
unfolding *uniformly_Cauchy_on_def* **by** *fast*
show *eventually* (λ*n*. ∀ *x*∈*X*. *dist* (*f n x*) (*?f x*) < *e*) *sequentially*
using *eventually_ge_at_top*[*of N*]
proof *eventually_elim*
fix *n* **assume** *n*: *n* ≥ *N*
show ∀ *x*∈*X*. *dist* (*f n x*) (*?f x*) < *e*
proof
fix *x* **assume** *x*: *x* ∈ *X*
with *assms* **have** (λ*n*. *f n x*) ⟶ *?f x*
by (*auto dest*!: *Cauchy_convergent uniformly_Cauchy_imp_Cauchy simp*: *convergent_LIMSEQ_iff*)
with ‹*e*/*2* > *0*› **have** *eventually* (λ*m*. *m* ≥ *N* ∧ *dist* (*f m x*) (*?f x*) < *e*/*2*) *sequentially*
by (*intro tendstoD eventually_conj eventually_ge_at_top*)
then obtain *m* **where** *m*: *m* ≥ *N dist* (*f m x*) (*?f x*) < *e*/*2*
unfolding *eventually_at_top_linorder* **by** *blast*
have *dist* (*f n x*) (*?f x*) ≤ *dist* (*f n x*) (*f m x*) + *dist* (*f m x*) (*?f x*)
by (*rule dist_triangle*)
also from *x n* **have** ... < *e*/*2* + *e*/*2* **by** (*intro add_strict_mono N m*)
finally show *dist* (*f n x*) (*?f x*) < *e* **by** *simp*
qed
qed
qed

lemma *uniformly_convergent_Cauchy*:
assumes *uniformly_convergent_on X f*
shows *uniformly_Cauchy_on X f*
proof (*rule uniformly_Cauchy_onI*)
fix *e*::*real* **assume** *e* > *0*
then have *0* < *e* / *2* **by** *simp*
with *assms*[*unfolded uniformly_convergent_on_def uniform_limit_sequentially_iff*]
obtain *l N* **where** *l*:*x* ∈ *X* ⟹ *n* ≥ *N* ⟹ *dist* (*f n x*) (*l x*) < *e* / *2* **for** *n x*

by *metis*
from *l l* **have** $x \in X \Longrightarrow n \geq N \Longrightarrow m \geq N \Longrightarrow dist\ (f\ n\ x)\ (f\ m\ x) < e$ **for** *n m x*
by (*rule dist_triangle_half_l*)
then show $\exists M.\ \forall x{\in}X.\ \forall m{\geq}M.\ \forall n{\geq}M.\ dist\ (f\ m\ x)\ (f\ n\ x) < e$ **by** *blast*
qed

lemma *uniformly_convergent_eq_Cauchy*:
uniformly_convergent_on X f = *uniformly_Cauchy_on X f* **for** *f*::*nat* ⇒ *'b* ⇒ *'a*::*complete_space*
using *Cauchy_uniformly_convergent uniformly_convergent_Cauchy* **by** *blast*

lemma *uniformly_convergent_eq_cauchy*:
fixes *s*::*nat* ⇒ *'b* ⇒ *'a*::*complete_space*
shows
$(\exists l.\ \forall e{>}0.\ \exists N.\ \forall n\ x.\ N \leq n \wedge P\ x \longrightarrow dist(s\ n\ x)(l\ x) < e) \longleftrightarrow$
$(\forall e{>}0.\ \exists N.\ \forall m\ n\ x.\ N \leq m \wedge N \leq n \wedge P\ x \longrightarrow dist\ (s\ m\ x)\ (s\ n\ x) < e)$
proof −
have ∗: $(\forall n{\geq}N.\ \forall x.\ Q\ x \longrightarrow R\ n\ x) \longleftrightarrow (\forall n\ x.\ N \leq n \wedge Q\ x \longrightarrow R\ n\ x)$
$(\forall x.\ Q\ x \longrightarrow (\forall m{\geq}N.\ \forall n{\geq}N.\ S\ n\ m\ x)) \longleftrightarrow (\forall m\ n\ x.\ N \leq m \wedge N \leq n \wedge Q\ x \longrightarrow S\ n\ m\ x)$
for *N*::*nat* **and** *Q*::*'b* ⇒ *bool* **and** *R S*
by *blast*+
show *?thesis*
using *uniformly_convergent_eq_Cauchy*[*of Collect P s*]
unfolding *uniformly_convergent_on_def uniformly_Cauchy_on_def uniform_limit_sequentially_iff*
by (*simp add*: ∗)
qed

lemma *uniformly_cauchy_imp_uniformly_convergent*:
fixes *s* :: *nat* ⇒ *'a* ⇒ *'b*::*complete_space*
assumes $\forall e{>}0.\exists N.\ \forall m\ (n{::}nat)\ x.\ N \leq m \wedge N \leq n \wedge P\ x \dashrightarrow dist(s\ m\ x)(s\ n\ x) < e$
and $\forall x.\ P\ x \dashrightarrow (\forall e{>}0.\ \exists N.\ \forall n.\ N \leq n \longrightarrow dist(s\ n\ x)(l\ x) < e)$
shows $\forall e{>}0.\ \exists N.\ \forall n\ x.\ N \leq n \wedge P\ x \longrightarrow dist(s\ n\ x)(l\ x) < e$
proof −
obtain l' **where** $l{:}\forall e{>}0.\ \exists N.\ \forall n\ x.\ N \leq n \wedge P\ x \longrightarrow dist\ (s\ n\ x)\ (l'\ x) < e$
using *assms*(*1*) **unfolding** *uniformly_convergent_eq_cauchy*[*symmetric*] **by** *auto*
moreover
{
fix *x*
assume *P x*
then have $l\ x = l'\ x$
using *tendsto_unique*[*OF trivial_limit_sequentially, of* $\lambda n.\ s\ n\ x\ l\ x\ l'\ x$]
using *l* **and** *assms*(*2*) **unfolding** *lim_sequentially* **by** *blast*
}
ultimately show *?thesis* **by** *auto*
qed

lemma *uniformly_convergent_on_sum_E*:
fixes ε::*real* **and** f :: *nat* $\Rightarrow$ 'a $\Rightarrow$ 'b::{*complete_space*,*real_normed_vector*}
assumes *uconv*: *uniformly_convergent_on K* ($\lambda n\ z.\ \sum k{<}n.\ f\ k\ z$) **and** $\varepsilon{>}0$
obtains *N* **where** $\bigwedge m\ n\ z.\ [\![N \le m;\ m \le n;\ z{\in}K]\!] \Longrightarrow norm(\sum k{=}m..{<}n.\ f\ k\ z) < \varepsilon$
proof −
obtain *N* **where** *N*: $\bigwedge m\ n\ z.\ [\![N \le m;\ N \le n;\ z{\in}K]\!] \Longrightarrow dist\ (\sum k{<}m.\ f\ k\ z)\ (\sum k{<}n.\ f\ k\ z) < \varepsilon$
using *uconv* ‹$\varepsilon{>}0$› **unfolding** *uniformly_Cauchy_on_def uniformly_convergent_eq_Cauchy*
by *meson*
show *thesis*
proof
fix *m n z*
assume $N \le m\ m \le n\ z \in K$
moreover have $(\sum k = m..{<}n.\ f\ k\ z) = (\sum k{<}n.\ f\ k\ z) - (\sum k{<}m.\ f\ k\ z)$
by (*metis atLeast0LessThan le0 sum_diff_nat_ivl* ‹$m \le n$›)
ultimately show $norm(\sum k = m..{<}n.\ f\ k\ z) < \varepsilon$
using *N* **by** (*simp add*: *dist_norm*)
qed
qed

lemma *uniformly_convergent_on_sum_iff*:
fixes f :: *nat* $\Rightarrow$ 'a $\Rightarrow$ 'b::{*complete_space*,*real_normed_vector*}
shows *uniformly_convergent_on K* ($\lambda n\ z.\ \sum k{<}n.\ f\ k\ z$)
$\longleftrightarrow$ ($\forall \varepsilon{>}0.\ \exists N.\ \forall m\ n\ z.\ N{\le}m \longrightarrow m{\le}n \longrightarrow z{\in}K \longrightarrow norm\ (\sum k{=}m..{<}n.\ f\ k\ z) < \varepsilon$) (**is** *?lhs*=*?rhs*)
proof
assume *R*: *?rhs*
show *?lhs*
unfolding *uniformly_Cauchy_on_def uniformly_convergent_eq_Cauchy*
proof (*intro strip*)
fix ε::*real*
assume $\varepsilon{>}0$
then obtain *N* **where** $\bigwedge m\ n\ z.\ [\![N \le m;\ m \le n;\ z{\in}K]\!] \Longrightarrow norm(\sum k = m..{<}n.\ f\ k\ z) < \varepsilon$
using *R* **by** *blast*
then have $\forall x{\in}K.\ \forall m{\ge}N.\ \forall n{\ge}m.\ norm\ ((\sum k{<}m.\ f\ k\ x) - (\sum k{<}n.\ f\ k\ x)) < \varepsilon$
by (*metis atLeast0LessThan le0 sum_diff_nat_ivl norm_minus_commute*)
then have $\forall x{\in}K.\ \forall m{\ge}N.\ \forall n{\ge}N.\ norm\ ((\sum k{<}m.\ f\ k\ x) - (\sum k{<}n.\ f\ k\ x)) < \varepsilon$
by (*metis linorder_le_cases norm_minus_commute*)
then show $\exists M.\ \forall x{\in}K.\ \forall m{\ge}M.\ \forall n{\ge}M.\ dist\ (\sum k{<}m.\ f\ k\ x)\ (\sum k{<}n.\ f\ k\ x) < \varepsilon$
by (*metis dist_norm*)
qed
qed (*metis uniformly_convergent_on_sum_E*)

lemma *uniform_limit_suminf*:
fixes *f*:: *nat* ⇒ *'a*::{*metric_space*, *comm_monoid_add*} ⇒ *'a*
assumes *uniformly_convergent_on X* (λ*n x*. ∑ *k*<*n*. *f k x*)
shows *uniform_limit X* (λ*n x*. ∑ *k*<*n*. *f k x*) (λ*x*. ∑ *k*. *f k x*) *sequentially*
proof −
obtain *S* **where** *S*: *uniform_limit X* (λ*n x*. ∑ *k*<*n*. *f k x*) *S sequentially*
using *assms uniformly_convergent_on_def* **by** *blast*
then have (∑ *k*. *f k x*) = *S x* **if** *x* ∈ *X* **for** *x*
using *that sums_iff sums_def* **by** (*blast intro*: *tendsto_uniform_limitI* [*OF S*])
with *S* **show** *?thesis*
by (*simp cong*: *uniform_limit_cong'*)
qed

TODO: remove explicit formulations (∃ *l*. ∀ *e*>*0*. ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *?P x* ⟶ *dist* (*?s n x*) (*l x*) < *e*) = (∀ *e*>*0*. ∃ *N*. ∀ *m n x*. *N* ≤ *m* ∧ *N* ≤ *n* ∧ *?P x* ⟶ *dist* (*?s m x*) (*?s n x*) < *e*)

⟦∀ *e*>*0*. ∃ *N*. ∀ *m n x*. *N* ≤ *m* ∧ *N* ≤ *n* ∧ *?P x* ⟶ *dist* (*?s m x*) (*?s n x*) < *e*; ∀ *x*. *?P x* ⟶ (∀ *e*>*0*. ∃ *N*. ∀ *n*≥*N*. *dist* (*?s n x*) (*?l x*) < *e*)⟧ ⟹ ∀ *e*>*0*. ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *?P x* ⟶ *dist* (*?s n x*) (*?l x*) < *e*?!

lemma *uniformly_convergent_imp_convergent*:
uniformly_convergent_on X f ⟹ *x* ∈ *X* ⟹ *convergent* (λ*n*. *f n x*)
unfolding *uniformly_convergent_on_def convergent_def*
by (*auto dest*: *tendsto_uniform_limitI*)

6.15.4 Comparison Test

lemma *uniformly_summable_comparison_test*:
fixes *f* :: *nat* ⇒ *'a* ⇒ *'b* :: *banach*
assumes *uniformly_convergent_on A* (λ*N x*. ∑ *n*<*N*. *g n x*)
assumes ⋀*n x*. *x* ∈ *A* ⟹ *norm* (*f n x*) ≤ *g n x*
shows *uniformly_convergent_on A* (λ*N x*. ∑ *n*<*N*. *f n x*)
proof −
have *uniformly_Cauchy_on A* (λ*N x*. ∑ *n*<*N*. *f n x*)
proof (*rule uniformly_Cauchy_onI'*)
fix *e* :: *real* **assume** *e*: *e* > *0*
obtain *M* **where** *M*: ⋀*x m n*. *x* ∈ *A* ⟹ *m* ≥ *M* ⟹ *n* ≥ *M* ⟹ *dist* (∑ *k*<*m*. *g k x*) (∑ *k*<*n*. *g k x*) < *e*
using *assms*(*1*) *e* **unfolding** *uniformly_convergent_eq_Cauchy uniformly_Cauchy_on_def* **by** *metis*
show ∃ *M*. ∀ *x*∈*A*. ∀ *m*≥*M*. ∀ *n*>*m*. *dist* (∑ *k*<*m*. *f k x*) (∑ *k*<*n*. *f k x*) < *e*
proof (*rule exI*[*of* _ *M*], *safe*)
fix *x m n* **assume** *xmn*: *x* ∈ *A* *m* ≥ *M m* < *n*
have *nonneg*: *g k x* ≥ *0* **for** *k*
by (*rule order.trans*[*OF* _ *assms*(*2*)]) (*use xmn* **in** *auto*)
have *dist* (∑ *k*<*m*. *f k x*) (∑ *k*<*n*. *f k x*) = *norm* (∑ *k*∈{..<*n*}−{..<*m*}. *f k x*)
using *xmn* **by** (*subst sum_diff*) (*auto simp*: *dist_norm norm_minus_commute*)
also have {..<*n*}−{..<*m*} = {*m*..<*n*}

```
      by auto
    also have norm (∑k∈{m..<n}. f k x) ≤ (∑k∈{m..<n}. norm (f k x))
      using norm_sum by blast
    also have ... ≤ (∑k∈{m..<n}. g k x)
      by (intro sum_mono assms xmn)
    also have ... = |∑k∈{m..<n}. g k x|
      by (subst abs_of_nonneg) (auto simp: nonneg intro!: sum_nonneg)
    also have {m..<n} = {..<n} - {..<m}
      by auto
    also have |∑k∈.... g k x| = dist (∑k<m. g k x) (∑k<n. g k x)
    using xmn by (subst sum_diff) (auto simp: abs_minus_commute dist_norm)
    also have ... < e
      by (rule M) (use xmn in auto)
    finally show dist (∑k<m. f k x) (∑k<n. f k x) < e .
  qed
 qed
 thus ?thesis
  unfolding uniformly_convergent_eq_Cauchy .
qed

lemma uniform_limit_compose_uniformly_continuous_on:
 fixes f :: 'a :: metric_space ⇒ 'b :: metric_space
 assumes lim: uniform_limit A g g' F
 assumes cont: uniformly_continuous_on B f
 assumes ev: eventually (λx. ∀y∈A. g x y ∈ B) F and closed B
 shows   uniform_limit A (λx y. f (g x y)) (λy. f (g' y)) F
proof (cases F = bot)
 case [simp]: False
 show ?thesis
  unfolding uniform_limit_iff
 proof safe
  fix e :: real assume e: e > 0

  have g'_in_B: g' y ∈ B if y ∈ A for y
  proof (rule Lim_in_closed_set)
   show eventually (λx. g x y ∈ B) F
    using ev by eventually_elim (use that in auto)
   show ((λx. g x y) ⟶ g' y) F
    using lim that by (metis tendsto_uniform_limitI)
  qed (use ‹closed B› in auto)

  obtain d where d: d > 0 ⋀x y. x ∈ B ⟹ y ∈ B ⟹ dist x y < d ⟹ dist
(f x) (f y) < e
   using e cont unfolding uniformly_continuous_on_def by metis
  from lim have eventually (λx. ∀y∈A. dist (g x y) (g' y) < d) F
   unfolding uniform_limit_iff using ‹d > 0› by meson
  thus eventually (λx. ∀y∈A. dist (f (g x y)) (f (g' y)) < e) F
   using assms(3)
  proof eventually_elim
```

```
      case (elim x)
      show ∀ y∈A. dist (f (g x y)) (f (g' y)) < e
      proof safe
        fix y assume y: y ∈ A
        show dist (f (g x y)) (f (g' y)) < e
        proof (rule d)
          show dist (g x y) (g' y) < d
            using elim y by blast
        qed (use y elim g'_in_B in auto)
      qed
    qed
  qed
qed (auto simp: filterlim_def)

lemma uniformly_convergent_on_compose_uniformly_continuous_on:
  fixes f :: 'a :: metric_space ⇒ 'b :: metric_space
  assumes lim: uniformly_convergent_on A g
  assumes cont: uniformly_continuous_on B f
  assumes ev: eventually (λx. ∀ y∈A. g x y ∈ B) sequentially and closed B
  shows   uniformly_convergent_on A (λx y. f (g x y))
proof −
  from lim obtain g' where g': uniform_limit A g g' sequentially
    by (auto simp: uniformly_convergent_on_def)
  thus ?thesis
   using uniform_limit_compose_uniformly_continuous_on[OF g' cont ev ‹closed
B›]
    by (auto simp: uniformly_convergent_on_def)
qed
```

6.15.5 Weierstrass M-Test

```
proposition Weierstrass_m_test_ev:
fixes f :: _ ⇒ _ ⇒ _ :: banach
assumes eventually (λn. ∀ x∈A. norm (f n x) ≤ M n) sequentially
assumes summable M
shows uniform_limit A (λn x. ∑ i<n. f i x) (λx. suminf (λi. f i x)) sequentially
proof (rule uniform_limitI)
  fix e :: real
  assume 0 < e
  from suminf_exist_split[OF ‹0 < e› ‹summable M›]
  have ∀ F k in sequentially. norm (∑ i. M (i + k)) < e
    by (auto simp: eventually_sequentially)
  with eventually_all_ge_at_top[OF assms(1)]
    show ∀ F n in sequentially. ∀ x∈A. dist (∑ i<n. f i x) (∑ i. f i x) < e
  proof eventually_elim
    case (elim k)
    show ?case
    proof safe
      fix x assume x ∈ A
```

```
    have ∃ N. ∀ n≥N. norm (f n x) ≤ M n
      using assms(1) ‹x ∈ A› by (force simp: eventually_at_top_linorder)
    hence summable_norm_f: summable (λn. norm (f n x))
      by(rule summable_norm_comparison_test[OF _ ‹summable M›])
    have summable_f: summable (λn. f n x)
      using summable_norm_cancel[OF summable_norm_f] .
    have summable_norm_f_plus_k: summable (λi. norm (f (i + k) x))
      using summable_ignore_initial_segment[OF summable_norm_f]
      by auto
    have summable_M_plus_k: summable (λi. M (i + k))
      using summable_ignore_initial_segment[OF ‹summable M›]
      by auto

    have dist (∑ i<k. f i x) (∑ i. f i x) = norm ((∑ i. f i x) − (∑ i<k. f i x))
      using dist_norm dist_commute by (subst dist_commute)
    also have ... = norm (∑ i. f (i + k) x)
       using suminf_minus_initial_segment[OF summable_f, where k=k] by
simp
    also have ... ≤ (∑ i. norm (f (i + k) x))
      using summable_norm[OF summable_norm_f_plus_k] .
    also have ... ≤ (∑ i. M (i + k))
    by (rule suminf_le[OF _ summable_norm_f_plus_k summable_M_plus_k])
        (insert elim(1) ‹x ∈ A›, simp)
    finally show dist (∑ i<k. f i x) (∑ i. f i x) < e
      using elim by auto
  qed
 qed
qed
```

Alternative version, formulated as in HOL Light

```
corollary series_comparison_uniform:
 fixes f :: _ ⇒ nat ⇒ _ :: banach
 assumes g: summable g and le: ⋀n x. N ≤ n ∧ x ∈ A ⟹ norm(f x n) ≤ g n
   shows ∃ l. ∀ e. 0 < e ⟶ (∃ N. ∀ n x. N ≤ n ∧ x ∈ A ⟶ dist(sum (f x)
{..<n}) (l x) < e)
proof −
 have 1: ∀ F n in sequentially. ∀ x∈A. norm (f x n) ≤ g n
   using le eventually_sequentially by auto
 show ?thesis
   apply (rule_tac x=(λx. ∑ i. f x i) in exI)
   apply (metis (no_types, lifting) eventually_sequentially uniform_limitD [OF
Weierstrass_m_test_ev [OF 1 g]])
  done
qed

corollary Weierstrass_m_test:
 fixes f :: _ ⇒ _ ⇒ _ :: banach
 assumes ⋀n x. x ∈ A ⟹ norm (f n x) ≤ M n
 assumes summable M
```

shows *uniform_limit A* ($\lambda n\ x.\ \sum i{<}n.\ f\ i\ x$) ($\lambda x.\ suminf\ (\lambda i.\ f\ i\ x)$) *sequentially*
using *assms* **by** (*intro Weierstrass_m_test_ev always_eventually*) *auto*

corollary *Weierstrass_m_test'_ev*:
fixes $f :: \_ \Rightarrow \_ \Rightarrow \_ ::$ *banach*
assumes *eventually* ($\lambda n.\ \forall x{\in}A.\ norm\ (f\ n\ x) \leq M\ n$) *sequentially summable M*
shows *uniformly_convergent_on A* ($\lambda n\ x.\ \sum i{<}n.\ f\ i\ x$)
unfolding *uniformly_convergent_on_def* **by** (*rule exI*, *rule Weierstrass_m_test_ev*[*OF assms*])

corollary *Weierstrass_m_test'*:
fixes $f :: \_ \Rightarrow \_ \Rightarrow \_ ::$ *banach*
assumes $\bigwedge n\ x.\ x \in A \Longrightarrow norm\ (f\ n\ x) \leq M\ n$ *summable M*
shows *uniformly_convergent_on A* ($\lambda n\ x.\ \sum i{<}n.\ f\ i\ x$)
unfolding *uniformly_convergent_on_def* **by** (*rule exI*, *rule Weierstrass_m_test*[*OF assms*])

lemma *Weierstrass_m_test_general*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c ::$ *banach*
fixes $M :: 'a \Rightarrow real$
assumes *norm_le*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in Y \Longrightarrow norm\ (f\ x\ y) \leq M\ x$
assumes *summable*: *M summable_on X*
shows *uniform_limit Y* ($\lambda X\ y.\ \sum x{\in}X.\ f\ x\ y$) ($\lambda y.\ \sum_\infty x{\in}X.\ f\ x\ y$) (*finite_subsets_at_top X*)
proof (*rule uniform_limitI*)
fix $\varepsilon ::$ *real*
assume $\varepsilon > 0$
define *S* **where** $S = (\lambda y.\ \sum_\infty x{\in}X.\ f\ x\ y)$
have *S*: (($\lambda x.\ f\ x\ y$) *has_sum S y*) *X* **if** *y*: $y \in Y$ **for** *y*
unfolding *S_def*
proof (*rule has_sum_infsum*)
have ($\lambda x.\ norm\ (f\ x\ y)$) *summable_on X*
by (*rule abs_summable_on_comparison_test'*[*OF summable norm_le*]) (*use y* **in** *auto*)
thus ($\lambda x.\ f\ x\ y$) *summable_on X*
by (*metis abs_summable_summable*)
qed

define *T* **where** $T = (\sum_\infty x{\in}X.\ M\ x)$
have *T*: (*M has_sum T*) *X*
unfolding *T_def* **by** (*simp add*: *local.summable*)
have *M_summable*: *M summable_on X'* **if** $X' \subseteq X$ **for** X'
using *local.summable summable_on_subset_banach that* **by** *blast*

have *f_summable*: ($\lambda x.\ f\ x\ y$) *summable_on X'* **if** $X' \subseteq X\ y \in Y$ **for** $X'\ y$
using *S summable_on_def summable_on_subset_banach that* **by** *blast*
have *eventually* ($\lambda X'.\ dist\ (\sum x{\in}X'.\ M\ x)\ T < \varepsilon$) (*finite_subsets_at_top X*)
using *T* ‹$\varepsilon > 0$› **unfolding** *T_def has_sum_def tendsto_iff* **by** *blast*

moreover have *eventually* ($\lambda X'$. *finite* $X' \wedge X' \subseteq X$) (*finite_subsets_at_top* X)
by (*simp add*: *eventually_finite_subsets_at_top_weakI*)
ultimately show $\forall_F\ X'$ *in finite_subsets_at_top* X. $\forall y \in Y$. *dist* ($\sum x \in X'$. $f\ x\ y$) ($\sum_\infty x \in X$. $f\ x\ y$) $< \varepsilon$
proof *eventually_elim*
case X': (*elim* X')
show $\forall y \in Y$. *dist* ($\sum x \in X'$. $f\ x\ y$) ($\sum_\infty x \in X$. $f\ x\ y$) $< \varepsilon$
proof
fix y **assume** y: $y \in Y$
have *1*: ((λx. $f\ x\ y$) *has_sum* ($S\ y - (\sum x \in X'$. $f\ x\ y)$)) ($X - X'$)
using $X'\ y$ **by** (*intro has_sum_Diff S has_sum_finite*[*of* X'] *f_summable*) *auto*
have *2*: (M *has_sum* ($T - (\sum x \in X'$. $M\ x)$)) ($X - X'$)
using $X'\ y$ **by** (*intro has_sum_Diff T has_sum_finite*[*of* X'] *M_summable*) *auto*
have *dist* ($\sum x \in X'$. $f\ x\ y$) ($\sum_\infty x \in X$. $f\ x\ y$) = *norm* ($S\ y - (\sum x \in X'$. $f\ x\ y)$)
by (*simp add*: *dist_norm norm_minus_commute S_def*)
also have *norm* ($S\ y - (\sum x \in X'$. $f\ x\ y)$) $\leq$ ($\sum_\infty x \in X - X'$. $M\ x$)
using *2 y* **by** (*intro norm_infsum_le*[*OF 1 _ norm_le*]) (*auto simp*: *infsumI*)
also have $\ldots$ = $T - (\sum x \in X'$. $M\ x)$
using *2* **by** (*auto simp*: *infsumI*)
also have $\ldots$ $< \varepsilon$
using X' **by** (*simp add*: *dist_norm*)
finally show *dist* ($\sum x \in X'$. $f\ x\ y$) ($\sum_\infty x \in X$. $f\ x\ y$) $< \varepsilon$ **.**
qed
qed
qed

6.15.6 Structural introduction rules

lemma *uniform_limit_eq_rhs*: *uniform_limit* $X\ f\ l\ F \Longrightarrow l = m \Longrightarrow$ *uniform_limit* $X\ f\ m\ F$
by *simp*

named_theorems *uniform_limit_intros introduction rules for uniform_limit*
setup ‹
Global_Theory.add_thms_dynamic (***binding*** ‹*uniform_limit_eq_intros*›,
fn context =>
Named_Theorems.get (*Context.proof_of context*) ***named_theorems*** ‹*uniform_limit_intros*›
|> *map_filter* (*try* (*fn thm =>* @{*thm uniform_limit_eq_rhs*} *OF* [*thm*])))
›

lemma (**in** *bounded_linear*) *uniform_limit*[*uniform_limit_intros*]:
assumes *uniform_limit* $X\ g\ l\ F$
shows *uniform_limit* X ($\lambda a\ b$. $f\ (g\ a\ b)$) (λa. $f\ (l\ a)$) F
proof (*rule uniform_limitI*)
fix e::*real*
from *pos_bounded* **obtain** K

```
    where K: ⋀x y. dist (f x) (f y) ≤ K * dist x y K > 0
    by (auto simp: ac_simps dist_norm diff[symmetric])
  assume 0 < e with ‹K > 0› have e / K > 0 by simp
  from uniform_limitD[OF assms this]
  show ∀F n in F. ∀x∈X. dist (f (g n x)) (f (l x)) < e
   by eventually_elim (metis le_less_trans mult.commute pos_less_divide_eq K)
qed

lemma (in bounded_linear) uniformly_convergent_on:
  assumes uniformly_convergent_on A g
  shows   uniformly_convergent_on A (λx y. f (g x y))
proof −
  from assms obtain l where uniform_limit A g l sequentially
    unfolding uniformly_convergent_on_def by blast
  hence uniform_limit A (λx y. f (g x y)) (λx. f (l x)) sequentially
    by (rule uniform_limit)
  thus ?thesis unfolding uniformly_convergent_on_def by blast
qed

lemmas bounded_linear_uniform_limit_intros[uniform_limit_intros] =
  bounded_linear.uniform_limit[OF bounded_linear_Im]
  bounded_linear.uniform_limit[OF bounded_linear_Re]
  bounded_linear.uniform_limit[OF bounded_linear_cnj]
  bounded_linear.uniform_limit[OF bounded_linear_fst]
  bounded_linear.uniform_limit[OF bounded_linear_snd]
  bounded_linear.uniform_limit[OF bounded_linear_zero]
  bounded_linear.uniform_limit[OF bounded_linear_of_real]
  bounded_linear.uniform_limit[OF bounded_linear_inner_left]
  bounded_linear.uniform_limit[OF bounded_linear_inner_right]
  bounded_linear.uniform_limit[OF bounded_linear_divide]
  bounded_linear.uniform_limit[OF bounded_linear_scaleR_right]
  bounded_linear.uniform_limit[OF bounded_linear_mult_left]
  bounded_linear.uniform_limit[OF bounded_linear_mult_right]
  bounded_linear.uniform_limit[OF bounded_linear_scaleR_left]


lemmas uniform_limit_uminus[uniform_limit_intros] =
 bounded_linear.uniform_limit[OF bounded_linear_minus[OF bounded_linear_ident]]

lemma uniform_limit_const[uniform_limit_intros]: uniform_limit S (λx. c) c f
  by (auto intro!: uniform_limitI)

lemma uniform_limit_add[uniform_limit_intros]:
  fixes f g::'a ⇒ 'b ⇒ 'c::real_normed_vector
  assumes uniform_limit X f l F
  assumes uniform_limit X g m F
  shows uniform_limit X (λa b. f a b + g a b) (λa. l a + m a) F
proof (rule uniform_limitI)
  fix e::real
```

```
  assume 0 < e
  hence 0 < e / 2 by simp
  from
    uniform_limitD[OF assms(1) this]
    uniform_limitD[OF assms(2) this]
  show ∀⇩F n in F. ∀x∈X. dist (f n x + g n x) (l x + m x) < e
    by eventually_elim (simp add: dist_triangle_add_half)
qed

lemma uniform_limit_minus[uniform_limit_intros]:
  fixes f g::'a ⇒ 'b ⇒ 'c::real_normed_vector
  assumes uniform_limit X f l F
  assumes uniform_limit X g m F
  shows uniform_limit X (λa b. f a b − g a b) (λa. l a − m a) F
  unfolding diff_conv_add_uminus
  by (rule uniform_limit_intros assms)+

lemma uniform_limit_norm[uniform_limit_intros]:
  assumes uniform_limit S g l f
  shows uniform_limit S (λx y. norm (g x y)) (λx. norm (l x)) f
  using assms
  apply (rule metric_uniform_limit_imp_uniform_limit)
  apply (rule eventuallyI)
  by (metis dist_norm norm_triangle_ineq3 real_norm_def)

lemma (in bounded_bilinear) bounded_uniform_limit[uniform_limit_intros]:
  assumes uniform_limit X f l F
  assumes uniform_limit X g m F
  assumes bounded (m ‘ X)
  assumes bounded (l ‘ X)
  shows uniform_limit X (λa b. prod (f a b) (g a b)) (λa. prod (l a) (m a)) F
proof (rule uniform_limitI)
  fix e::real
  from pos_bounded obtain K where K:
    0 < K ⋀a b. norm (prod a b) ≤ norm a * norm b * K
    by auto
  hence sqrt (K*4) > 0 by simp

  from assms obtain Km Kl
  where Km: Km > 0 ⋀x. x ∈ X ⟹ norm (m x) ≤ Km
    and Kl: Kl > 0 ⋀x. x ∈ X ⟹ norm (l x) ≤ Kl
    by (auto simp: bounded_pos)
  hence K * Km * 4 > 0 K * Kl * 4 > 0
    using ‹K > 0›
    by simp_all
  assume 0 < e

  hence sqrt e > 0 by simp
  from uniform_limitD[OF assms(1) divide_pos_pos[OF this ‹sqrt (K*4) > 0›]]
```

```
    uniform_limitD[OF assms(2) divide_pos_pos[OF this ‹sqrt (K*4) > 0›]]
    uniform_limitD[OF assms(1) divide_pos_pos[OF ‹e > 0› ‹K * Km * 4 > 0›]]
    uniform_limitD[OF assms(2) divide_pos_pos[OF ‹e > 0› ‹K * Kl * 4 > 0›]]
  show ∀ F n in F. ∀ x∈X. dist (prod (f n x) (g n x)) (prod (l x) (m x)) < e
  proof eventually_elim
    case (elim n)
    show ?case
    proof safe
      fix x assume x ∈ X
      have dist (prod (f n x) (g n x)) (prod (l x) (m x)) ≤
        norm (prod (f n x − l x) (g n x − m x)) +
        norm (prod (f n x − l x) (m x)) +
        norm (prod (l x) (g n x − m x))
       by (auto simp: dist_norm prod_diff_prod intro: order_trans norm_triangle_ineq
add_mono)
      also note K(2)[of f n x − l x g n x − m x]
      also from elim(1)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
      have norm (f n x − l x) ≤ sqrt e / sqrt (K * 4)
        by simp
      also from elim(2)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
      have norm (g n x − m x) ≤ sqrt e / sqrt (K * 4)
        by simp
      also have sqrt e / sqrt (K * 4) * (sqrt e / sqrt (K * 4)) * K = e / 4
        using ‹K > 0› ‹e > 0› by auto
      also note K(2)[of f n x − l x m x]
      also note K(2)[of l x g n x − m x]
      also from elim(3)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
      have norm (f n x − l x) ≤ e / (K * Km * 4)
        by simp
      also from elim(4)[THEN bspec, OF ‹_ ∈ X›, unfolded dist_norm]
      have norm (g n x − m x) ≤ e / (K * Kl * 4)
        by simp
      also note Kl(2)[OF ‹_ ∈ X›]
      also note Km(2)[OF ‹_ ∈ X›]
      also have e / (K * Km * 4) * Km * K = e / 4
        using ‹K > 0› ‹Km > 0› by simp
      also have  Kl * (e / (K * Kl * 4)) * K = e / 4
        using ‹K > 0› ‹Kl > 0› by simp
      also have e / 4 + e / 4 + e / 4 < e using ‹e > 0› by simp
      finally show dist (prod (f n x) (g n x)) (prod (l x) (m x)) < e
        using ‹K > 0› ‹Kl > 0› ‹Km > 0› ‹e > 0›
        by (simp add: algebra_simps mult_right_mono divide_right_mono)
    qed
  qed
qed

lemmas bounded_bilinear_bounded_uniform_limit_intros[uniform_limit_intros]
=
  bounded_bilinear.bounded_uniform_limit[OF Inner_Product.bounded_bilinear_inner]
```

bounded_bilinear.bounded_uniform_limit[*OF Real_Vector_Spaces.bounded_bilinear_mult*]
bounded_bilinear.bounded_uniform_limit[*OF Real_Vector_Spaces.bounded_bilinear_scaleR*]

lemma *uniform_lim_mult*:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c*::*real_normed_algebra*
assumes *f*: *uniform_limit S f l F*
and *g*: *uniform_limit S g m F*
and *l*: *bounded* (*l* ‘ *S*)
and *m*: *bounded* (*m* ‘ *S*)
shows *uniform_limit S* (λ*a b*. *f a b* ∗ *g a b*) (λ*a*. *l a* ∗ *m a*) *F*
by (*intro bounded_bilinear_bounded_uniform_limit_intros assms*)

lemma *uniform_lim_inverse*:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c*::*real_normed_field*
assumes *f*: *uniform_limit S f l F*
and *b*: ⋀*x*. *x* ∈ *S* ⟹ *b* ≤ *norm*(*l x*)
and *b* > *0*
shows *uniform_limit S* (λ*x y*. *inverse* (*f x y*)) (*inverse* ∘ *l*) *F*
proof (*rule uniform_limitI*)
fix *e*::*real*
assume *e* > *0*
have *lte*: *dist* (*inverse* (*f x y*)) ((*inverse* ∘ *l*) *y*) < *e*
if *b*/*2* ≤ *norm* (*f x y*) *norm* (*f x y* − *l y*) < *e* ∗ *b*^{2} / *2* *y* ∈ *S*
for *x y*
proof −
have [*simp*]: *l y* ≠ *0* *f x y* ≠ *0*
using ‹*b* > *0*› *that b* [*OF* ‹*y* ∈ *S*›] **by** *fastforce*+
have *norm* (*l y* − *f x y*) < *e* ∗ *b*^{2} / *2*
by (*metis norm_minus_commute that*(*2*))
also have ... ≤ *e* ∗ (*norm* (*f x y*) ∗ *norm* (*l y*))
using ‹*e* > *0*› *that b* [*OF* ‹*y* ∈ *S*›] **apply** (*simp add*: *power2_eq_square*)
by (*metis* ‹*b* > *0*› *less_eq_real_def mult.left_commute mult_mono'*)
finally show *?thesis*
by (*auto simp*: *dist_norm field_split_simps norm_mult norm_divide*)
qed
have ∀ *F n in F*. ∀ *x*∈*S*. *dist* (*f n x*) (*l x*) < *b*/*2*
using *uniform_limitD* [*OF f*, *of b*/*2*] **by** (*simp add*: ‹*b* > *0*›)
then have ∀ *F x in F*. ∀ *y*∈*S*. *b*/*2* ≤ *norm* (*f x y*)
apply (*rule eventually_mono*)
using *b* **apply** (*simp only*: *dist_norm*)
by (*metis* (*no_types*, *opaque_lifting*) *diff_zero dist_commute dist_norm norm_triangle_half_l not_less*)
then have ∀ *F x in F*. ∀ *y*∈*S*. *b*/*2* ≤ *norm* (*f x y*) ∧ *norm* (*f x y* − *l y*) < *e* ∗ *b*^{2} / *2*
apply (*simp only*: *ball_conj_distrib dist_norm* [*symmetric*])
apply (*rule eventually_conj*, *assumption*)
apply (*rule uniform_limitD* [*OF f*, *of e* ∗ *b* ^ *2* / *2*])
using ‹*b* > *0*› ‹*e* > *0*› **by** *auto*
then show ∀ *F x in F*. ∀ *y*∈*S*. *dist* (*inverse* (*f x y*)) ((*inverse* ∘ *l*) *y*) < *e*

```
    using lte by (force intro: eventually_mono)
qed

lemma uniform_lim_divide:
  fixes f :: 'a ⇒ 'b ⇒ 'c::real_normed_field
  assumes f: uniform_limit S f l F
      and g: uniform_limit S g m F
      and l: bounded (l ‘ S)
      and b: ⋀x. x ∈ S ⟹ b ≤ norm(m x)
      and b > 0
    shows uniform_limit S (λa b. f a b / g a b) (λa. l a / m a) F
proof −
  have m: bounded ((inverse ∘ m) ‘ S)
    using b ‹b > 0›
    apply (simp add: bounded_iff)
    by (metis le_imp_inverse_le norm_inverse)
  have uniform_limit S (λa b. f a b * inverse (g a b))
        (λa. l a * (inverse ∘ m) a) F
    by (rule uniform_lim_mult [OF f uniform_lim_inverse [OF g b ‹b > 0›] l m])
  then show ?thesis
    by (simp add: field_class.field_divide_inverse)
qed

lemma uniform_limit_null_comparison:
  assumes ∀F x in F. ∀a∈S. norm (f x a) ≤ g x a
  assumes uniform_limit S g (λ_. 0) F
  shows uniform_limit S f (λ_. 0) F
  using assms(2)
proof (rule metric_uniform_limit_imp_uniform_limit)
  show ∀F x in F. ∀y∈S. dist (f x y) 0 ≤ dist (g x y) 0
    using assms(1) by (rule eventually_mono) (force simp add: dist_norm)
qed

lemma uniform_limit_on_Un:
  uniform_limit I f g F ⟹ uniform_limit J f g F ⟹ uniform_limit (I ∪ J) f g
F
  by (auto intro!: uniform_limitI dest!: uniform_limitD elim: eventually_elim2)

lemma uniform_limit_on_empty [iff]:
  uniform_limit {} f g F
  by (auto intro!: uniform_limitI)

lemma uniform_limit_on_UNION:
  assumes finite S
  assumes ⋀s. s ∈ S ⟹ uniform_limit (h s) f g F
  shows uniform_limit (⋃(h ‘ S)) f g F
  using assms
  by induct (auto intro: uniform_limit_on_empty uniform_limit_on_Un)
```

lemma *uniform_limit_on_Union*:
assumes *finite I*
assumes $\bigwedge J.\ J \in I \Longrightarrow$ *uniform_limit J f g F*
shows *uniform_limit* (*Union I*) *f g F*
by (*metis SUP_identity_eq assms uniform_limit_on_UNION*)

lemma *uniform_limit_on_subset*:
uniform_limit J f g F $\Longrightarrow$ *I* $\subseteq$ *J* $\Longrightarrow$ *uniform_limit I f g F*
by (*auto intro*!: *uniform_limitI dest*!: *uniform_limitD intro*: *eventually_mono*)

lemma *uniform_limit_bounded*:
fixes *f*::*'i* $\Rightarrow$ *'a*::*topological_space* $\Rightarrow$ *'b*::*metric_space*
assumes *l*: *uniform_limit S f l F*
assumes *bnd*: $\forall_F$ *i in F. bounded* (*f i* ‘ *S*)
assumes *F* $\neq$ *bot*
shows *bounded* (*l* ‘ *S*)
proof −
from *l* **have** $\forall_F$ *n in F.* $\forall x{\in}S.$ *dist* (*l x*) (*f n x*) < *1*
by (*auto simp*: *uniform_limit_iff dist_commute dest*!: *spec*[**where** *x=1*])
with *bnd*
have $\forall_F$ *n in F.* $\exists M.$ $\forall x{\in}S.$ *dist undefined* (*l x*) $\leq$ *M* + *1*
by *eventually_elim*
(*auto intro*!: *order_trans*[*OF dist_triangle2 add_mono*] *intro*: *less_imp_le*
simp: *bounded_any_center*[**where** *a=undefined*])
then show *?thesis* **using** *assms*
by (*auto simp*: *bounded_any_center*[**where** *a=undefined*] *dest*!: *eventually_happens*)
qed

lemma *uniformly_convergent_add*:
uniformly_convergent_on A f $\Longrightarrow$ *uniformly_convergent_on A g*$\Longrightarrow$
uniformly_convergent_on A ($\lambda k\ x.\ f\ k\ x + g\ k\ x :: 'a :: \{real\_normed\_algebra\}$)
unfolding *uniformly_convergent_on_def* **by** (*blast dest*: *uniform_limit_add*)

lemma *uniformly_convergent_minus*:
uniformly_convergent_on A f $\Longrightarrow$ *uniformly_convergent_on A g*$\Longrightarrow$
uniformly_convergent_on A ($\lambda k\ x.\ f\ k\ x - g\ k\ x :: 'a :: \{real\_normed\_algebra\}$)
unfolding *uniformly_convergent_on_def* **by** (*blast dest*: *uniform_limit_minus*)

lemma *uniformly_convergent_mult*:
uniformly_convergent_on A f $\Longrightarrow$
uniformly_convergent_on A ($\lambda k\ x.\ c * f\ k\ x :: 'a :: \{real\_normed\_algebra\}$)
unfolding *uniformly_convergent_on_def*
by (*blast dest*: *bounded_linear_uniform_limit_intros*(*13*))

6.15.7 Power series and uniform convergence

proposition *powser_uniformly_convergent*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real_normed_div_algebra*,*banach*}
assumes *r* < *conv_radius a*

shows *uniformly_convergent_on* (*cball* ξ *r*) (λn *x*. $\sum i<n$. *a i* $*$ ($x - \xi$) $\hat{}$ *i*)
proof (*cases* 0 $\leq$ *r*)
case *True*
then have $*$: *summable* (λn. *norm* (*a n*) $*$ *r* $\hat{}$ *n*)
using *abs_summable_in_conv_radius* [*of of_real r a*] *assms*
by (*simp add*: *norm_mult norm_power*)
show *?thesis*
by (*simp add*: *Weierstrass_m_test'_ev* [*OF _ **] *norm_mult norm_power*
mult_left_mono power_mono dist_norm norm_minus_commute)
next
case *False* **then show** *?thesis* **by** (*simp add*: *not_le*)
qed

lemma *powser_uniform_limit*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real_normed_div_algebra,banach*}
assumes *r* $<$ *conv_radius a*
shows *uniform_limit* (*cball* ξ *r*) (λn *x*. $\sum i<n$. *a i* $*$ ($x - \xi$) $\hat{}$ *i*) (λx. *suminf*
(λi. *a i* $*$ ($x - \xi$) $\hat{}$ *i*)) *sequentially*
using *powser_uniformly_convergent* [*OF assms*]
by (*simp add*: *Uniform_Limit.uniformly_convergent_uniform_limit_iff Series.suminf_eq_lim*)

lemma *powser_continuous_suminf*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real_normed_div_algebra,banach*}
assumes *r* $<$ *conv_radius a*
shows *continuous_on* (*cball* ξ *r*) (λx. *suminf* (λi. *a i* $*$ ($x - \xi$) $\hat{}$ *i*))
apply (*rule uniform_limit_theorem* [*OF _ powser_uniform_limit*])
apply (*rule eventuallyI continuous_intros assms*)+
apply (*simp add*:)
done

lemma *powser_continuous_sums*:
fixes *a* :: *nat* $\Rightarrow$ *'a*::{*real_normed_div_algebra,banach*}
assumes *r*: *r* $<$ *conv_radius a*
and *sm*: $\bigwedge x$. *x* $\in$ *cball* ξ *r* $\Longrightarrow$ (λn. *a n* $*$ ($x - \xi$) $\hat{}$ *n*) *sums* (*f x*)
shows *continuous_on* (*cball* ξ *r*) *f*
apply (*rule continuous_on_cong* [*THEN iffD1*, *OF refl _ powser_continuous_suminf*
[*OF r*]])
using *sm sums_unique* **by** *fastforce*

lemmas *uniform_limit_subset_union* = *uniform_limit_on_subset*[*OF uniform_limit_on_Union*]

end

6.16 Bounded Linear Function

theory *Bounded_Linear_Function*
imports
Topology_Euclidean_Space
Operator_Norm

Uniform_Limit
Function_Topology

begin

lemma *onorm_componentwise*:
assumes *bounded_linear f*
shows *onorm f* $\leq$ $(\sum i{\in}Basis.\ norm\ (f\ i))$
proof −
{
fix *i*::*'a*
assume $i \in Basis$
hence *onorm* $(\lambda x.\ (x \cdot i) *_R f\ i) \leq onorm\ (\lambda x.\ (x \cdot i)) * norm\ (f\ i)$
by (*auto intro*!: *onorm_scaleR_left_lemma bounded_linear_inner_left*)
also have $\ldots \leq$ *norm i* $*$ *norm* $(f\ i)$
by (*rule mult_right_mono*)
(*auto simp*: *ac_simps Cauchy_Schwarz_ineq2 intro*!: *onorm_le*)
finally have *onorm* $(\lambda x.\ (x \cdot i) *_R f\ i) \leq norm\ (f\ i)$ **using** ‹$i \in Basis$›
by *simp*
} **hence** *onorm* $(\lambda x.\ \sum i{\in}Basis.\ (x \cdot i) *_R f\ i) \leq (\sum i{\in}Basis.\ norm\ (f\ i))$
by (*auto intro*!: *order_trans*[*OF onorm_sum_le*] *bounded_linear_scaleR_const*
sum_mono bounded_linear_inner_left)
also have $(\lambda x.\ \sum i{\in}Basis.\ (x \cdot i) *_R f\ i) = (\lambda x.\ f\ (\sum i{\in}Basis.\ (x \cdot i) *_R i))$
by (*simp add*: *linear_sum bounded_linear.linear assms linear_simps*)
also have $\ldots = f$
by (*simp add*: *euclidean_representation*)
finally show *?thesis* **.**
qed

lemmas *onorm_componentwise_le* = *order_trans*[*OF onorm_componentwise*]

6.16.1 Intro rules for *bounded_linear*

named_theorems *bounded_linear_intros*

lemma *onorm_inner_left*:
assumes *bounded_linear r*
shows *onorm* $(\lambda x.\ r\ x \cdot f) \leq onorm\ r * norm\ f$
proof (*rule onorm_bound*)
fix *x*
have *norm* $(r\ x \cdot f) \leq norm\ (r\ x) * norm\ f$
by (*simp add*: *Cauchy_Schwarz_ineq2*)
also have $\ldots \leq$ *onorm r* $*$ *norm x* $*$ *norm f*
by (*intro mult_right_mono onorm assms norm_ge_zero*)
finally show *norm* $(r\ x \cdot f) \leq$ *onorm r* $*$ *norm f* $*$ *norm x*
by (*simp add*: *ac_simps*)
qed (*intro mult_nonneg_nonneg norm_ge_zero onorm_pos_le assms*)

lemma *onorm_inner_right*:

assumes *bounded_linear r*
shows *onorm* (λ*x*. *f* • *r x*) ≤ *norm f* ∗ *onorm r*
apply (*subst inner_commute*)
apply (*rule onorm_inner_left*[*OF assms*, *THEN order_trans*])
apply *simp*
done

lemmas [*bounded_linear_intros*] =
bounded_linear_zero
bounded_linear_add
bounded_linear_const_mult
bounded_linear_mult_const
bounded_linear_scaleR_const
bounded_linear_const_scaleR
bounded_linear_ident
bounded_linear_sum
bounded_linear_Pair
bounded_linear_sub
bounded_linear_fst_comp
bounded_linear_snd_comp
bounded_linear_inner_left_comp
bounded_linear_inner_right_comp

6.16.2 declaration of derivative/continuous/tendsto introduction rules for bounded linear functions

attribute_setup *bounded_linear* =
‹*Scan.succeed* (*Thm.declaration_attribute* (*fn thm* =>
fold (*fn* (*r*, *s*) => *Named_Theorems.add_thm s* (*thm RS r*))
[
(@{*thm bounded_linear.has_derivative*}, ***named_theorems*** ‹*derivative_intros*›),
(@{*thm bounded_linear.tendsto*}, ***named_theorems*** ‹*tendsto_intros*›),
(@{*thm bounded_linear.continuous*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear.continuous_on*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear.uniformly_continuous_on*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear_compose*}, ***named_theorems*** ‹*bounded_linear_intros*›)
]))›

attribute_setup *bounded_bilinear* =
‹*Scan.succeed* (*Thm.declaration_attribute* (*fn thm* =>
fold (*fn* (*r*, *s*) => *Named_Theorems.add_thm s* (*thm RS r*))
[
(@{*thm bounded_bilinear.FDERIV*}, ***named_theorems*** ‹*derivative_intros*›),
(@{*thm bounded_bilinear.tendsto*}, ***named_theorems*** ‹*tendsto_intros*›),
(@{*thm bounded_bilinear.continuous*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_bilinear.continuous_on*}, ***named_theorems*** ‹*continuous_intros*›),
(@{*thm bounded_linear_compose*[*OF bounded_bilinear.bounded_linear_left*]},
named_theorems ‹*bounded_linear_intros*›),
(@{*thm bounded_linear_compose*[*OF bounded_bilinear.bounded_linear_right*]},

named_theorems ‹*bounded_linear_intros*›),
(@{*thm bounded_linear.uniformly_continuous_on*[*OF bounded_bilinear.bounded_linear_left*]},
named_theorems ‹*continuous_intros*›),
(@{*thm bounded_linear.uniformly_continuous_on*[*OF bounded_bilinear.bounded_linear_right*]},
named_theorems ‹*continuous_intros*›)
]))›

6.16.3 Type of bounded linear functions

typedef (**overloaded**) ('*a*, '*b*) *blinfun* (‹(‹*notation*=‹*infix* $\Rightarrow_L$›› _ $\Rightarrow_L$ /_)› [*22*, *21*] *21*) =
{*f*::'*a*::*real_normed_vector*⇒'*b*::*real_normed_vector*. *bounded_linear f*}
morphisms *blinfun_apply Blinfun*
by (*blast intro*: *bounded_linear_intros*)

declare [[*coercion*
blinfun_apply :: ('*a*::*real_normed_vector* $\Rightarrow_L$ '*b*::*real_normed_vector*) ⇒ '*a* ⇒ '*b*]]

lemma *bounded_linear_blinfun_apply*[*bounded_linear_intros*]:
bounded_linear g ⟹ *bounded_linear* (λ*x*. *blinfun_apply f* (*g x*))
by (*metis blinfun_apply mem_Collect_eq bounded_linear_compose*)

setup_lifting *type_definition_blinfun*

lemma *blinfun_eqI*: (⋀*i*. *blinfun_apply x i* = *blinfun_apply y i*) ⟹ *x* = *y*
by *transfer auto*

lemma *bounded_linear_Blinfun_apply*: *bounded_linear f* ⟹ *blinfun_apply* (*Blinfun f*) = *f*
by (*auto simp*: *Blinfun_inverse*)

6.16.4 Type class instantiations

instantiation *blinfun* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
begin

lift_definition *norm_blinfun* :: '*a* $\Rightarrow_L$ '*b* ⇒ *real* **is** *onorm* .

lift_definition *minus_blinfun* :: '*a* $\Rightarrow_L$ '*b* ⇒ '*a* $\Rightarrow_L$ '*b* ⇒ '*a* $\Rightarrow_L$ '*b*
is λ*f g x*. *f x* − *g x*
by (*rule bounded_linear_sub*)

definition *dist_blinfun* :: '*a* $\Rightarrow_L$ '*b* ⇒ '*a* $\Rightarrow_L$ '*b* ⇒ *real*
where *dist_blinfun a b* = *norm* (*a* − *b*)

definition [*code del*]:
(*uniformity* :: (('*a* $\Rightarrow_L$ '*b*) × ('*a* $\Rightarrow_L$ '*b*)) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open_blinfun* :: (*'a* ⇒$_L$ *'b*) *set* ⇒ *bool*
where [*code del*]: *open_blinfun S* = (∀ *x*∈*S*. ∀$_F$ (*x'*, *y*) *in uniformity*. *x'* = *x* ⟶ *y* ∈ *S*)

lift_definition *uminus_blinfun* :: *'a* ⇒$_L$ *'b* ⇒ *'a* ⇒$_L$ *'b* **is** λ*f x*. − *f x*
by (*rule bounded_linear_minus*)

lift_definition *zero_blinfun* :: *'a* ⇒$_L$ *'b* **is** λ*x*. *0*
by (*rule bounded_linear_zero*)

lift_definition *plus_blinfun* :: *'a* ⇒$_L$ *'b* ⇒ *'a* ⇒$_L$ *'b* ⇒ *'a* ⇒$_L$ *'b*
is λ*f g x*. *f x* + *g x*
by (*metis bounded_linear_add*)

lift_definition *scaleR_blinfun*::*real* ⇒ *'a* ⇒$_L$ *'b* ⇒ *'a* ⇒$_L$ *'b* **is** λ*r f x*. *r* ∗$_R$ *f x*
by (*metis bounded_linear_compose bounded_linear_scaleR_right*)

definition *sgn_blinfun* :: *'a* ⇒$_L$ *'b* ⇒ *'a* ⇒$_L$ *'b*
where *sgn_blinfun x* = *scaleR* (*inverse* (*norm x*)) *x*

instance
apply *standard*
unfolding *dist_blinfun_def open_blinfun_def sgn_blinfun_def uniformity_blinfun_def*
apply (*rule refl* | (*transfer*, *force simp*: *onorm_triangle onorm_scaleR onorm_eq_0 algebra_simps*))+
done

end

declare *uniformity_Abort*[**where** *'a*=(*'a* :: *real_normed_vector*) ⇒$_L$ (*'b* :: *real_normed_vector*), *code*]

lemma *norm_blinfun_eqI*:
assumes *n* ≤ *norm* (*blinfun_apply f x*) / *norm x*
assumes ⋀*x*. *norm* (*blinfun_apply f x*) ≤ *n* ∗ *norm x*
assumes *0* ≤ *n*
shows *norm f* = *n*
by (*auto simp*: *norm_blinfun_def*
intro!: *antisym onorm_bound assms order_trans*[*OF _ le_onorm*]
bounded_linear_intros)

lemma *norm_blinfun*: *norm* (*blinfun_apply f x*) ≤ *norm f* ∗ *norm x*
by *transfer* (*rule onorm*)

lemma *norm_blinfun_bound*: *0* ≤ *b* ⟹ (⋀*x*. *norm* (*blinfun_apply f x*) ≤ *b* ∗ *norm x*) ⟹ *norm f* ≤ *b*
by *transfer* (*rule onorm_bound*)

lemma *bounded_bilinear_blinfun_apply*[*bounded_bilinear*]: *bounded_bilinear blin-*

fun_apply
proof
fix *f g*::*'a* ⇒$_L$ *'b* **and** *a b*::*'a* **and** *r*::*real*
show (*f* + *g*) *a* = *f a* + *g a* (*r* ∗$_R$ *f*) *a* = *r* ∗$_R$ *f a*
by (*transfer*, *simp*)+
interpret *bounded_linear f* **for** *f*::*'a* ⇒$_L$ *'b*
by (*auto intro*!: *bounded_linear_intros*)
show *f* (*a* + *b*) = *f a* + *f b f* (*r* ∗$_R$ *a*) = *r* ∗$_R$ *f a*
by (*simp_all add*: *add scaleR*)
show ∃ *K*. ∀ *a b*. *norm* (*blinfun_apply a b*) ≤ *norm a* ∗ *norm b* ∗ *K*
by (*auto intro*!: *exI*[**where** *x*=*1*] *norm_blinfun*)
qed

interpretation *blinfun*: *bounded_bilinear blinfun_apply*
by (*rule bounded_bilinear_blinfun_apply*)

lemmas *bounded_linear_apply_blinfun*[*intro*, *simp*] = *blinfun.bounded_linear_left*

declare *blinfun.zero_left* [*simp*] *blinfun.zero_right* [*simp*]

context *bounded_bilinear*
begin

named_theorems *bilinear_simps*

lemmas [*bilinear_simps*] =
add_left
add_right
diff_left
diff_right
minus_left
minus_right
scaleR_left
scaleR_right
zero_left
zero_right
sum_left
sum_right

end

instance *blinfun* :: (*real_normed_vector*, *banach*) *banach*
proof
fix *X*::*nat* ⇒ *'a* ⇒$_L$ *'b*
assume *Cauchy X*
{
fix *x*::*'a*

```
  {
    fix x::'a
    assume norm x ≤ 1
    have Cauchy (λn. X n x)
    proof (rule CauchyI)
      fix e::real
      assume 0 < e
      from CauchyD[OF ‹Cauchy X› ‹0 < e›] obtain M
        where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) < e
        by auto
      show ∃ M. ∀ m≥M. ∀ n≥M. norm (X m x − X n x) < e
      proof (safe intro!: exI[where x=M])
        fix m n
        assume le: M ≤ m M ≤ n
        have norm (X m x − X n x) = norm ((X m − X n) x)
          by (simp add: blinfun.bilinear_simps)
        also have ... ≤ norm (X m − X n) * norm x
          by (rule norm_blinfun)
        also have ... ≤ norm (X m − X n) * 1
          using ‹norm x ≤ 1› norm_ge_zero by (rule mult_left_mono)
        also have ... = norm (X m − X n) by simp
        also have ... < e using le by fact
        finally show norm (X m x − X n x) < e .
      qed
    qed
    hence convergent (λn. X n x)
      by (metis Cauchy_convergent_iff)
  } note convergent_norm1 = this
  define y where y = x /R norm x
  have y: norm y ≤ 1 and xy: x = norm x *R y
    by (simp_all add: y_def inverse_eq_divide)
  have convergent (λn. norm x *R X n y)
    by (intro bounded_bilinear.convergent[OF bounded_bilinear_scaleR] convergent_const
        convergent_norm1 y)
  also have (λn. norm x *R X n y) = (λn. X n x)
    by (subst xy) (simp add: blinfun.bilinear_simps)
  finally have convergent (λn. X n x) .
}
then obtain v where v: ⋀x. (λn. X n x) ⟶ v x
  unfolding convergent_def
  by metis

have Cauchy (λn. norm (X n))
proof (rule CauchyI)
  fix e::real
  assume e > 0
  from CauchyD[OF ‹Cauchy X› ‹0 < e›] obtain M
    where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) < e
```

```
    by auto
   show ∃M. ∀m≥M. ∀n≥M. norm (norm (X m) − norm (X n)) < e
   proof (safe intro!: exI[where x=M])
    fix m n assume mn: m ≥ M n ≥ M
    have norm (norm (X m) − norm (X n)) ≤ norm (X m − X n)
     by (metis norm_triangle_ineq3 real_norm_def)
    also have ... < e using mn by fact
    finally show norm (norm (X m) − norm (X n)) < e .
   qed
  qed
  then obtain K where K: (λn. norm (X n)) ⟶ K
   unfolding Cauchy_convergent_iff convergent_def
   by metis

  have bounded_linear v
  proof
   fix x y and r::real
  from tendsto_add[OF v[of x] v [of y]] v[of x + y, unfolded blinfun.bilinear_simps]
      tendsto_scaleR[OF tendsto_const[of r] v[of x]] v[of r *R x, unfolded blinfun.bilinear_simps]
   show v (x + y) = v x + v y v (r *R x) = r *R v x
    by (metis (poly_guards_query) LIMSEQ_unique)+
   show ∃K. ∀x. norm (v x) ≤ norm x * K
   proof (safe intro!: exI[where x=K])
    fix x
    have norm (v x) ≤ K * norm x
    by (rule tendsto_le[OF _ tendsto_mult[OF K tendsto_const] tendsto_norm[OF v]])
       (auto simp: norm_blinfun)
    thus norm (v x) ≤ norm x * K
     by (simp add: ac_simps)
   qed
  qed
  hence Bv: ⋀x. (λn. X n x) ⟶ Blinfun v x
   by (auto simp: bounded_linear_Blinfun_apply v)

  have X ⟶ Blinfun v
  proof (rule LIMSEQ_I)
   fix r::real assume r > 0
   define r' where r' = r / 2
   have 0 < r' r' < r using ‹r > 0› by (simp_all add: r'_def)
   from CauchyD[OF ‹Cauchy X› ‹r' > 0›]
   obtain M where M: ⋀m n. m ≥ M ⟹ n ≥ M ⟹ norm (X m − X n) < r'
    by metis
   show ∃no. ∀n≥no. norm (X n − Blinfun v) < r
   proof (safe intro!: exI[where x=M])
    fix n assume n: M ≤ n
    have norm (X n − Blinfun v) ≤ r'
    proof (rule norm_blinfun_bound)
```

```
      fix x
      have eventually (λm. m ≥ M) sequentially
        by (metis eventually_ge_at_top)
        hence ev_le: eventually (λm. norm (X n x − X m x) ≤ r' ∗ norm x)
sequentially
      proof eventually_elim
        case (elim m)
        have norm (X n x − X m x) = norm ((X n − X m) x)
          by (simp add: blinfun.bilinear_simps)
        also have ... ≤ norm ((X n − X m)) ∗ norm x
          by (rule norm_blinfun)
        also have ... ≤ r' ∗ norm x
          using M[OF n elim] by (simp add: mult_right_mono)
        finally show ?case .
      qed
       have tendsto_v: (λm. norm (X n x − X m x)) ⟶ norm (X n x −
Blinfun v x)
        by (auto intro!: tendsto_intros Bv)
      show norm ((X n − Blinfun v) x) ≤ r' ∗ norm x
     by (auto intro!: tendsto_upperbound tendsto_v ev_le simp: blinfun.bilinear_simps)
    qed (simp add: ‹0 < r'› less_imp_le)
    thus norm (X n − Blinfun v) < r
      by (metis ‹r' < r› le_less_trans)
  qed
 qed
 thus convergent X
   by (rule convergentI)
qed
```

6.16.5 On Euclidean Space

```
lemma Zfun_sum:
  assumes finite s
  assumes f: ⋀i. i ∈ s ⟹ Zfun (f i) F
  shows Zfun (λx. sum (λi. f i x) s) F
  using assms by induct (auto intro!: Zfun_zero Zfun_add)

lemma norm_blinfun_euclidean_le:
  fixes a::'a::euclidean_space ⇒_L 'b::real_normed_vector
  shows norm a ≤ sum (λx. norm (a x)) Basis
  apply (rule norm_blinfun_bound)
   apply (simp add: sum_nonneg)
  apply (subst euclidean_representation[symmetric, where 'a='a])
  apply (simp only: blinfun.bilinear_simps sum_distrib_right)
  apply (rule order.trans[OF norm_sum sum_mono])
  apply (simp add: abs_mult mult_right_mono ac_simps Basis_le_norm)
  done

lemma tendsto_componentwise1:
```

fixes a::$'a$::*euclidean_space* $\Rightarrow_L$ $'b$::*real_normed_vector*
and b::$'c \Rightarrow 'a \Rightarrow_L 'b$
assumes $(\bigwedge j.\ j \in Basis \Longrightarrow ((\lambda n.\ b\ n\ j) \longrightarrow a\ j)\ F)$
shows $(b \longrightarrow a)\ F$
proof −
have $\bigwedge j.\ j \in Basis \Longrightarrow Zfun\ (\lambda x.\ norm\ (b\ x\ j - a\ j))\ F$
using *assms* **unfolding** *tendsto_Zfun_iff Zfun_norm_iff* **.**
hence $Zfun\ (\lambda x.\ \sum j{\in}Basis.\ norm\ (b\ x\ j - a\ j))\ F$
by (*auto intro*!: *Zfun_sum*)
thus *?thesis*
unfolding *tendsto_Zfun_iff*
by (*rule Zfun_le*)
(*auto intro*!: *order_trans*[*OF norm_blinfun_euclidean_le*] *simp*: *blinfun.bilinear_simps*)
qed

lift_definition
blinfun_of_matrix::($'b$::*euclidean_space* ⇒ $'a$::*euclidean_space* ⇒ *real*) ⇒ $'a \Rightarrow_L$
$'b$
is $\lambda a\ x.\ \sum i{\in}Basis.\ \sum j{\in}Basis.\ ((x \cdot j) * a\ i\ j) *_R i$
by (*intro bounded_linear_intros*)

lemma *blinfun_of_matrix_works*:
fixes f::$'a$::*euclidean_space* $\Rightarrow_L$ $'b$::*euclidean_space*
shows *blinfun_of_matrix* $(\lambda i\ j.\ (f\ j) \cdot i) = f$
proof (*transfer*, *rule*, *rule euclidean_eqI*)
fix f::$'a \Rightarrow 'b$ **and** x::$'a$ **and** b::$'b$ **assume** *bounded_linear f* **and** b: $b \in Basis$
then interpret *bounded_linear f* **by** *simp*
have $(\sum j{\in}Basis.\ \sum i{\in}Basis.\ (x \cdot i * (f\ i \cdot j)) *_R j) \cdot b$
$= (\sum j{\in}Basis.\ if\ j = b\ then\ (\sum i{\in}Basis.\ (x \cdot i * (f\ i \cdot j)))\ else\ 0)$
using b
by (*simp add*: *inner_sum_left inner_Basis if_distrib cong*: *if_cong*) (*simp add*:
sum.swap)
also have $\ldots = (\sum i{\in}Basis.\ (x \cdot i * (f\ i \cdot b)))$
using b **by** (*simp*)
also have $\ldots = f\ x \cdot b$
by (*metis* (*mono_tags*, *lifting*) *Linear_Algebra.linear_componentwise linear_axioms*)
finally show $(\sum j{\in}Basis.\ \sum i{\in}Basis.\ (x \cdot i * (f\ i \cdot j)) *_R j) \cdot b = f\ x \cdot b$ **.**
qed

lemma *blinfun_of_matrix_apply*:
blinfun_of_matrix a x $= (\sum i{\in}Basis.\ \sum j{\in}Basis.\ ((x \cdot j) * a\ i\ j) *_R i)$
by *transfer simp*

lemma *blinfun_of_matrix_minus*: *blinfun_of_matrix x* − *blinfun_of_matrix y*
= *blinfun_of_matrix* $(x - y)$
by *transfer* (*auto simp*: *algebra_simps sum_subtractf*)

lemma *norm_blinfun_of_matrix*:
norm (*blinfun_of_matrix a*) $\leq (\sum i{\in}Basis.\ \sum j{\in}Basis.\ |a\ i\ j|)$

apply (*rule norm_blinfun_bound*)
apply (*simp add*: *sum_nonneg*)
apply (*simp only*: *blinfun_of_matrix_apply sum_distrib_right*)
apply (*rule order_trans*[*OF norm_sum sum_mono*])
apply (*rule order_trans*[*OF norm_sum sum_mono*])
apply (*simp add*: *abs_mult mult_right_mono ac_simps Basis_le_norm*)
done

lemma *tendsto_blinfun_of_matrix*:
assumes $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow ((\lambda n.\ b\ n\ i\ j) \longrightarrow a\ i\ j)\ F$
shows $((\lambda n.\ blinfun\_of\_matrix\ (b\ n)) \longrightarrow blinfun\_of\_matrix\ a)\ F$
proof −
have $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow Zfun\ (\lambda x.\ norm\ (b\ x\ i\ j - a\ i\ j))\ F$
using *assms* **unfolding** *tendsto_Zfun_iff Zfun_norm_iff* **.**
hence $Zfun\ (\lambda x.\ (\sum i{\in}Basis.\ \sum j{\in}Basis.\ |b\ x\ i\ j - a\ i\ j|))\ F$
by (*auto intro*!: *Zfun_sum*)
thus *?thesis*
unfolding *tendsto_Zfun_iff blinfun_of_matrix_minus*
by (*rule Zfun_le*) (*auto intro*!: *order_trans*[*OF norm_blinfun_of_matrix*])
qed

lemma *tendsto_componentwise*:
fixes $a::'a::euclidean\_space \Rightarrow_L 'b::euclidean\_space$
and $b::'c \Rightarrow 'a \Rightarrow_L 'b$
shows $(\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow ((\lambda n.\ b\ n\ j \cdot i) \longrightarrow a\ j \cdot i)\ F) \Longrightarrow (b \longrightarrow a)\ F$
apply (*subst blinfun_of_matrix_works*[*of a*, *symmetric*])
apply (*subst blinfun_of_matrix_works*[*of b x* **for** *x*, *symmetric*, *abs_def*])
by (*rule tendsto_blinfun_of_matrix*)

lemma
continuous_blinfun_componentwiseI:
fixes $f::'b::t2\_space \Rightarrow 'a::euclidean\_space \Rightarrow_L 'c::euclidean\_space$
assumes $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow continuous\ F\ (\lambda x.\ (f\ x)\ j \cdot i)$
shows *continuous F f*
using *assms* **by** (*auto simp*: *continuous_def intro*!: *tendsto_componentwise*)

lemma
continuous_blinfun_componentwiseI1:
fixes $f::'b::t2\_space \Rightarrow 'a::euclidean\_space \Rightarrow_L 'c::real\_normed\_vector$
assumes $\bigwedge i.\ i \in Basis \Longrightarrow continuous\ F\ (\lambda x.\ f\ x\ i)$
shows *continuous F f*
using *assms* **by** (*auto simp*: *continuous_def intro*!: *tendsto_componentwise1*)

lemma
continuous_on_blinfun_componentwise:
fixes $f::'d::t2\_space \Rightarrow 'e::euclidean\_space \Rightarrow_L 'f::real\_normed\_vector$
assumes $\bigwedge i.\ i \in Basis \Longrightarrow continuous\_on\ s\ (\lambda x.\ f\ x\ i)$
shows *continuous_on s f*

using *assms*
by (*auto intro*!: *continuous_at_imp_continuous_on intro*!: *tendsto_componentwise1*
simp: *continuous_on_eq_continuous_within continuous_def*)

lemma *bounded_linear_blinfun_matrix*: *bounded_linear* (λ*x*. (*x*::_⇒$_L$ _) *j* · *i*)
by (*auto intro*!: *bounded_linearI'* *bounded_linear_intros*)

lemma *continuous_blinfun_matrix*:
fixes *f*:: *'b*::*t2_space* ⇒ *'a*::*real_normed_vector* ⇒$_L$ *'c*::*real_inner*
assumes *continuous F f*
shows *continuous F* (λ*x*. (*f x*) *j* · *i*)
by (*rule bounded_linear.continuous*[*OF bounded_linear_blinfun_matrix assms*])

lemma *continuous_on_blinfun_matrix*:
fixes *f*::*'a*::*t2_space* ⇒ *'b*::*real_normed_vector* ⇒$_L$ *'c*::*real_inner*
assumes *continuous_on S f*
shows *continuous_on S* (λ*x*. (*f x*) *j* · *i*)
using *assms*
by (*auto simp*: *continuous_on_eq_continuous_within continuous_blinfun_matrix*)

lemma *continuous_on_blinfun_of_matrix*[*continuous_intros*]:
assumes ⋀*i j*. *i* ∈ *Basis* ⟹ *j* ∈ *Basis* ⟹ *continuous_on S* (λ*s*. *g s i j*)
shows *continuous_on S* (λ*s*. *blinfun_of_matrix* (*g s*))
using *assms*
by (*auto simp*: *continuous_on intro*!: *tendsto_blinfun_of_matrix*)

lemma *mult_if_delta*:
(*if P then* (1::*'a*::*comm_semiring_1*) *else* 0) ∗ *q* = (*if P then q else* 0)
by *auto*

lemma *compact_blinfun_lemma*:
fixes *f* :: *nat* ⇒ *'a*::*euclidean_space* ⇒$_L$ *'b*::*euclidean_space*
assumes *bounded* (*range f*)
shows ∀ *d*⊆*Basis*. ∃ *l*::*'a* ⇒$_L$ *'b*. ∃ *r*::*nat*⇒*nat*.
strict_mono r ∧ (∀ *e*>0. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) *i*) (*l i*) < *e*)
sequentially)
by (*rule compact_lemma_general*[**where** *unproj* = λ*e*. *blinfun_of_matrix* (λ*i j*.
e j · *i*)])
(*auto intro*!: *euclidean_eqI*[**where** *'a*=*'b*] *bounded_linear_image assms*
simp: *blinfun_of_matrix_works blinfun_of_matrix_apply inner_Basis mult_if_delta*
sum.delta'
scaleR_sum_left[*symmetric*])

lemma *blinfun_euclidean_eqI*: (⋀*i*. *i* ∈ *Basis* ⟹ *blinfun_apply x i* = *blinfun_apply y i*) ⟹ *x* = *y*
apply (*auto intro*!: *blinfun_eqI*)
apply (*subst* (2) *euclidean_representation*[*symmetric*, **where** *'a*=*'a*])
apply (*subst* (1) *euclidean_representation*[*symmetric*, **where** *'a*=*'a*])
apply (*simp add*: *blinfun.bilinear_simps*)

done

lemma *Blinfun_eq_matrix*: *bounded_linear f* $\Longrightarrow$ *Blinfun f* = *blinfun_of_matrix* ($\lambda i\ j.\ f\ j \cdot i$)
by (*intro blinfun_euclidean_eqI*)
(*auto simp*: *blinfun_of_matrix_apply bounded_linear_Blinfun_apply inner_Basis if_distrib*
if_distribR sum.delta' euclidean_representation
cong: *if_cong*)

TODO: generalize (via *compact_cball*)?

instance *blinfun* :: (*euclidean_space*, *euclidean_space*) *heine_borel*
proof
fix *f* :: *nat* $\Rightarrow$ *'a* $\Rightarrow_L$ *'b*
assume *f*: *bounded* (*range f*)
then obtain *l*::*'a* $\Rightarrow_L$ *'b* **and** *r* **where** *r*: *strict_mono r*
and *l*: $\forall\, e{>}0.$ *eventually* ($\lambda n.\ \forall\, i{\in}Basis.$ *dist* (*f* (*r n*) *i*) (*l i*) < *e*) *sequentially*
using *compact_blinfun_lemma* [*OF f*] **by** *blast*
{
fix *e*::*real*
let *?d* = *real_of_nat DIM*(*'a*) * *real_of_nat DIM*(*'b*)
assume *e* > *0*
hence *e* / *?d* > *0* **by** (*simp*)
with *l* **have** *eventually* ($\lambda n.\ \forall\, i{\in}Basis.$ *dist* (*f* (*r n*) *i*) (*l i*) < *e* / *?d*) *sequentially*
by *simp*
moreover
{
fix *n*
assume *n*: $\forall\, i{\in}Basis.$ *dist* (*f* (*r n*) *i*) (*l i*) < *e* / *?d*
have *norm* (*f* (*r n*) − *l*) = *norm* (*blinfun_of_matrix* ($\lambda i\ j.$ (*f* (*r n*) − *l*) *j* · *i*))
unfolding *blinfun_of_matrix_works* **..**
also note *norm_blinfun_of_matrix*
also have ($\sum i{\in}Basis.\ \sum j{\in}Basis.$ |(*f* (*r n*) − *l*) *j* · *i*|) <
($\sum i{\in}$(*Basis*::*'b set*). *e* / *real_of_nat DIM*(*'b*))
proof (*rule sum_strict_mono*)
fix *i*::*'b* **assume** *i*: *i* ∈ *Basis*
have ($\sum j$::*'a*$\in$*Basis*. |(*f* (*r n*) − *l*) *j* · *i*|) < ($\sum j$::*'a*$\in$*Basis*. *e* / *?d*)
proof (*rule sum_strict_mono*)
fix *j*::*'a* **assume** *j*: *j* ∈ *Basis*
have |(*f* (*r n*) − *l*) *j* · *i*| ≤ *norm* ((*f* (*r n*) − *l*) *j*)
by (*simp add*: *Basis_le_norm i*)
also have ... < *e* / *?d*
using *n i j* **by** (*auto simp*: *dist_norm blinfun.bilinear_simps*)
finally show |(*f* (*r n*) − *l*) *j* · *i*| < *e* / *?d* **by** *simp*
qed *simp_all*
also have ... ≤ *e* / *real_of_nat DIM*(*'b*)
by *simp*
finally show ($\sum j{\in}Basis.$ |(*f* (*r n*) − *l*) *j* · *i*|) < *e* / *real_of_nat DIM*(*'b*)

```
        by simp
    qed simp_all
    also have ... ≤ e by simp
    finally have dist (f (r n)) l < e
      by (auto simp: dist_norm)
  }
  ultimately have eventually (λn. dist (f (r n)) l < e) sequentially
    using eventually_elim2 by force
 }
 then have ∗: ((f ∘ r) ⟶ l) sequentially
  unfolding o_def tendsto_iff by simp
 with r show ∃ l r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially
  by auto
qed
```

6.16.6 concrete bounded linear functions

```
lemma transfer_bounded_bilinear_bounded_linearI:
 assumes g = (λi x. (blinfun_apply (f i) x))
 shows bounded_bilinear g = bounded_linear f
proof
 assume bounded_bilinear g
 then interpret bounded_bilinear f by (simp add: assms)
 show bounded_linear f
 proof (unfold_locales, safe intro!: blinfun_eqI)
  fix i
  show f (x + y) i = (f x + f y) i f (r ∗R x) i = (r ∗R f x) i for r x y
    by (auto intro!: blinfun_eqI simp: blinfun.bilinear_simps)
  from _ nonneg_bounded show ∃ K. ∀ x. norm (f x) ≤ norm x ∗ K
      by (rule ex_reg) (auto intro!: onorm_bound simp: norm_blinfun.rep_eq
ac_simps)
 qed
qed (auto simp: assms intro!: blinfun.comp)

lemma transfer_bounded_bilinear_bounded_linear[transfer_rule]:
 (rel_fun (rel_fun (=) (pcr_blinfun (=) (=))) (=)) bounded_bilinear bounded_linear
 by (auto simp: pcr_blinfun_def cr_blinfun_def rel_fun_def OO_def
  intro!: transfer_bounded_bilinear_bounded_linearI)

context bounded_bilinear
begin

lift_definition prod_left::'b ⇒ 'a ⇒L 'c is (λb a. prod a b)
 by (rule bounded_linear_left)
declare prod_left.rep_eq[simp]

lemma bounded_linear_prod_left[bounded_linear]: bounded_linear prod_left
 by transfer (rule flip)
```

lift_definition *prod_right*::'*a* ⇒ '*b* ⇒$_L$ '*c* **is** (λ*a b*. *prod a b*)
by (*rule bounded_linear_right*)
declare *prod_right.rep_eq*[*simp*]

lemma *bounded_linear_prod_right*[*bounded_linear*]: *bounded_linear prod_right*
by *transfer* (*rule bounded_bilinear_axioms*)

end

lift_definition *id_blinfun*::'*a*::*real_normed_vector* ⇒$_L$ '*a* **is** λ*x*. *x*
by (*rule bounded_linear_ident*)

lemmas *blinfun_apply_id_blinfun*[*simp*] = *id_blinfun.rep_eq*

lemma *norm_blinfun_id*[*simp*]:
norm (*id_blinfun*::'*a*::{*real_normed_vector*, *perfect_space*} ⇒$_L$ '*a*) = *1*
by *transfer* (*auto simp*: *onorm_id*)

lemma *norm_blinfun_id_le*:
norm (*id_blinfun*::'*a*::*real_normed_vector* ⇒$_L$ '*a*) ≤ *1*
by *transfer* (*auto simp*: *onorm_id_le*)

lift_definition *fst_blinfun*::('*a*::*real_normed_vector* × '*b*::*real_normed_vector*)
⇒$_L$ '*a* **is** *fst*
by (*rule bounded_linear_fst*)

lemma *blinfun_apply_fst_blinfun*[*simp*]: *blinfun_apply fst_blinfun* = *fst*
by *transfer* (*rule refl*)

lift_definition *snd_blinfun*::('*a*::*real_normed_vector* × '*b*::*real_normed_vector*)
⇒$_L$ '*b* **is** *snd*
by (*rule bounded_linear_snd*)

lemma *blinfun_apply_snd_blinfun*[*simp*]: *blinfun_apply snd_blinfun* = *snd*
by *transfer* (*rule refl*)

lift_definition *blinfun_compose*::
'*a*::*real_normed_vector* ⇒$_L$ '*b*::*real_normed_vector* ⇒
'*c*::*real_normed_vector* ⇒$_L$ '*a* ⇒
'*c* ⇒$_L$ '*b* (**infixl** ‹*o*$_L$› *55*) **is** (*o*)
parametric *comp_transfer*
unfolding *o_def*
by (*rule bounded_linear_compose*)

lemma *blinfun_apply_blinfun_compose*[*simp*]: (*a o*$_L$ *b*) *c* = *a* (*b c*)
by (*simp add*: *blinfun_compose.rep_eq*)

lemma *norm_blinfun_compose*:
norm (*f* o_L *g*) ≤ *norm f* ∗ *norm g*
by *transfer* (*rule onorm_compose*)

lemma *bounded_bilinear_blinfun_compose*[*bounded_bilinear*]: *bounded_bilinear* (o_L)
by *unfold_locales*
(*auto intro*!: *blinfun_eqI exI*[**where** *x*=*1*] *simp*: *blinfun.bilinear_simps norm_blinfun_compose*)

lemma *blinfun_compose_zero*[*simp*]:
blinfun_compose 0 = (λ_. *0*)
blinfun_compose x 0 = *0*
by (*auto simp*: *blinfun.bilinear_simps intro*!: *blinfun_eqI*)

lemma *blinfun_bij2*:
fixes *f*::*'a* ⇒$_L$ *'a*::*euclidean_space*
assumes *f* o_L *g* = *id_blinfun*
shows *bij* (*blinfun_apply g*)
proof (*rule bijI*)
show *inj g*
using *assms*
by (*metis blinfun_apply_id_blinfun blinfun_compose.rep_eq injI inj_on_imageI2*)
then show *surj g*
using *blinfun.bounded_linear_right bounded_linear_def linear_inj_imp_surj*
by *blast*
qed

lemma *blinfun_bij1*:
fixes *f*::*'a* ⇒$_L$ *'a*::*euclidean_space*
assumes *f* o_L *g* = *id_blinfun*
shows *bij* (*blinfun_apply f*)
proof (*rule bijI*)
show *surj* (*blinfun_apply f*)
by (*metis assms blinfun_apply_blinfun_compose blinfun_apply_id_blinfun surjI*)
then show *inj* (*blinfun_apply f*)
using *blinfun.bounded_linear_right bounded_linear_def linear_surj_imp_inj*
by *blast*
qed

lift_definition *blinfun_inner_right*::*'a*::*real_inner* ⇒ *'a* ⇒$_L$ *real* **is** (·)
by (*rule bounded_linear_inner_right*)
declare *blinfun_inner_right.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_inner_right*[*bounded_linear*]: *bounded_linear blinfun_inner_right*
by *transfer* (*rule bounded_bilinear_inner*)

lift_definition *blinfun_inner_left*::'*a*::*real_inner* ⇒ '*a* ⇒$_L$ *real* **is** λ*x y*. *y* · *x*
by (*rule bounded_linear_inner_left*)
declare *blinfun_inner_left.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_inner_left*[*bounded_linear*]: *bounded_linear blinfun_inner_left*
by *transfer* (*rule bounded_bilinear.flip*[*OF bounded_bilinear_inner*])

lift_definition *blinfun_scaleR_right*::*real* ⇒ '*a* ⇒$_L$ '*a*::*real_normed_vector* **is** (∗$_R$)
by (*rule bounded_linear_scaleR_right*)
declare *blinfun_scaleR_right.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_scaleR_right*[*bounded_linear*]: *bounded_linear blinfun_scaleR_right*
by *transfer* (*rule bounded_bilinear_scaleR*)

lift_definition *blinfun_scaleR_left*::'*a*::*real_normed_vector* ⇒ *real* ⇒$_L$ '*a* **is** λ*x y*. *y* ∗$_R$ *x*
by (*rule bounded_linear_scaleR_left*)
lemmas [*simp*] = *blinfun_scaleR_left.rep_eq*

lemma *bounded_linear_blinfun_scaleR_left*[*bounded_linear*]: *bounded_linear blinfun_scaleR_left*
by *transfer* (*rule bounded_bilinear.flip*[*OF bounded_bilinear_scaleR*])

lift_definition *blinfun_mult_right*::'*a* ⇒ '*a* ⇒$_L$ '*a*::*real_normed_algebra* **is** (∗)
by (*rule bounded_linear_mult_right*)
declare *blinfun_mult_right.rep_eq*[*simp*]

lemma *bounded_linear_blinfun_mult_right*[*bounded_linear*]: *bounded_linear blinfun_mult_right*
by *transfer* (*rule bounded_bilinear_mult*)

lift_definition *blinfun_mult_left*::'*a*::*real_normed_algebra* ⇒ '*a* ⇒$_L$ '*a* **is** λ*x y*. *y* ∗ *x*
by (*rule bounded_linear_mult_left*)
lemmas [*simp*] = *blinfun_mult_left.rep_eq*

lemma *bounded_linear_blinfun_mult_left*[*bounded_linear*]: *bounded_linear blinfun_mult_left*
by *transfer* (*rule bounded_bilinear.flip*[*OF bounded_bilinear_mult*])

lemmas *bounded_linear_function_uniform_limit_intros*[*uniform_limit_intros*] = *bounded_linear.uniform_limit*[*OF bounded_linear_apply_blinfun*]

bounded_linear.uniform_limit[*OF bounded_linear_blinfun_apply*]
bounded_linear.uniform_limit[*OF bounded_linear_blinfun_matrix*]

6.16.7 The strong operator topology on continuous linear operators

Let $'a$ and $'b$ be two normed real vector spaces. Then the space of linear continuous operators from $'a$ to $'b$ has a canonical norm, and therefore a canonical corresponding topology (the type classes instantiation are given in `Bounded_Linear_Function.thy`).

However, there is another topology on this space, the strong operator topology, where T_n tends to T iff, for all x in $'a$, then $T_n\ x$ tends to $T\ x$. This is precisely the product topology where the target space is endowed with the norm topology. It is especially useful when $'b$ is the set of real numbers, since then this topology is compact.

We can not implement it using type classes as there is already a topology, but at least we can define it as a topology.

Note that there is yet another (common and useful) topology on operator spaces, the weak operator topology, defined analogously using the product topology, but where the target space is given the weak-* topology, i.e., the pullback of the weak topology on the bidual of the space under the canonical embedding of a space into its bidual. We do not define it there, although it could also be defined analogously.

definition *strong_operator_topology*::('*a*::*real_normed_vector* ⇒$_L$ '*b*::*real_normed_vector*) *topology*
where *strong_operator_topology* = *pullback_topology UNIV blinfun_apply euclidean*

lemma *strong_operator_topology_topspace*:
topspace strong_operator_topology = *UNIV*
unfolding *strong_operator_topology_def topspace_pullback_topology topspace_euclidean*
by *auto*

lemma *strong_operator_topology_basis*:
fixes *f*::('*a*::*real_normed_vector* ⇒$_L$ '*b*::*real_normed_vector*) **and** *U*::'*i* ⇒ '*b set*
and *x*::'*i* ⇒ '*a*
assumes *finite I* ⋀*i*. *i* ∈ *I* ⟹ *open* (*U i*)
shows *openin strong_operator_topology* {*f*. ∀ *i*∈*I*. *blinfun_apply f* (*x i*) ∈ *U i*}
proof −
have *open* {*g*::('*a*⇒'*b*). ∀ *i*∈*I*. *g* (*x i*) ∈ *U i*}
by (*rule product_topology_basis'*[*OF assms*])
moreover have {*f*. ∀ *i*∈*I*. *blinfun_apply f* (*x i*) ∈ *U i*}
= *blinfun_apply*−‘{*g*::('*a*⇒'*b*). ∀ *i*∈*I*. *g* (*x i*) ∈ *U i*} ∩ *UNIV*
by *auto*
ultimately show *?thesis*
unfolding *strong_operator_topology_def* **by** (*subst openin_pullback_topology*)
auto

qed

lemma *strong_operator_topology_continuous_evaluation*:
continuous_map strong_operator_topology euclidean (λ*f*. *blinfun_apply f x*)
proof −
have *continuous_map strong_operator_topology euclidean* ((λ*f*. *f x*) *o blinfun_apply*)
unfolding *strong_operator_topology_def* **apply** (*rule continuous_map_pullback*)
using *continuous_on_product_coordinates* **by** *fastforce*
then show *?thesis* **unfolding** *comp_def* **by** *simp*
qed

lemma *continuous_on_strong_operator_topo_iff_coordinatewise*:
continuous_map T strong_operator_topology f
⟷ (∀ *x*. *continuous_map T euclidean* (λ*y*. *blinfun_apply* (*f y*) *x*))
proof (*auto*)
fix *x*::*'b*
assume *continuous_map T strong_operator_topology f*
with *continuous_map_compose*[*OF this strong_operator_topology_continuous_evaluation*]
have *continuous_map T euclidean* ((λ*z*. *blinfun_apply z x*) *o f*)
by *simp*
then show *continuous_map T euclidean* (λ*y*. *blinfun_apply* (*f y*) *x*)
unfolding *comp_def* **by** *auto*
next
assume ∗: ∀ *x*. *continuous_map T euclidean* (λ*y*. *blinfun_apply* (*f y*) *x*)
have ⋀*i*. *continuous_map T euclidean* (λ*x*. *blinfun_apply* (*f x*) *i*)
using ∗ **unfolding** *comp_def* **by** *auto*
then have *continuous_map T euclidean* (*blinfun_apply o f*)
unfolding *o_def*
by (*metis* (*no_types*) *continuous_map_componentwise_UNIV euclidean_product_topology*)
show *continuous_map T strong_operator_topology f*
unfolding *strong_operator_topology_def*
apply (*rule continuous_map_pullback'*)
by (*auto simp add*: ‹*continuous_map T euclidean* (*blinfun_apply o f*)›)
qed

lemma *strong_operator_topology_weaker_than_euclidean*:
continuous_map euclidean strong_operator_topology (λ*f*. *f*)
by (*subst continuous_on_strong_operator_topo_iff_coordinatewise*,
auto simp add: *linear_continuous_on*)

end

6.17 Derivative

theory *Derivative*
imports
Bounded_Linear_Function
Line_Segment

Convex_Euclidean_Space
begin

declare *bounded_linear_inner_left* [*intro*]

declare *has_derivative_bounded_linear*[*dest*]

6.17.1 Derivatives

lemma *has_derivative_add_const*:
(*f has_derivative f'*) *net* $\Longrightarrow$ ($\lambda x.\ f\ x + c$) *has_derivative f'*) *net*
by (*intro derivative_eq_intros*) *auto*

6.17.2 Derivative with composed bilinear function

More explicit epsilon-delta forms.

proposition *has_derivative_within'*:
(*f has_derivative f'*)(*at x within s*) $\longleftrightarrow$
bounded_linear f' $\wedge$
($\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x'{\in}s.\ 0 < norm\ (x' - x) \wedge norm\ (x' - x) < d \longrightarrow$
norm (*f x'* $-$ *f x* $-$ *f'*(*x'* $-$ *x*)) / *norm* (*x'* $-$ *x*) $<$ *e*)
unfolding *has_derivative_within Lim_within dist_norm*
by (*simp add*: *diff_diff_eq*)

lemma *has_derivative_at'*:
(*f has_derivative f'*) (*at x*)
$\longleftrightarrow$ *bounded_linear f'* $\wedge$
($\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x'.\ 0 < norm\ (x' - x) \wedge norm\ (x' - x) < d \longrightarrow$
norm (*f x'* $-$ *f x* $-$ *f'*(*x'* $-$ *x*)) / *norm* (*x'* $-$ *x*) $<$ *e*)
using *has_derivative_within'* [*of f f' x UNIV*] **by** *simp*

lemma *has_derivative_componentwise_within*:
(*f has_derivative f'*) (*at a within S*) $\longleftrightarrow$
($\forall\, i \in Basis.$ (($\lambda x.\ f\ x \cdot i$) *has_derivative* ($\lambda x.\ f'\ x \cdot i$)) (*at a within S*))
apply (*simp add*: *has_derivative_within*)
apply (*subst tendsto_componentwise_iff*)
apply (*simp add*: *ball_conj_distrib inner_diff_left inner_left_distrib flip*: *bounded_linear_component*
done

lemma *has_derivative_at_withinI*:
(*f has_derivative f'*) (*at x*) $\Longrightarrow$ (*f has_derivative f'*) (*at x within s*)
unfolding *has_derivative_within' has_derivative_at'*
by *blast*

lemma *has_derivative_right*:
fixes *f* :: *real* $\Rightarrow$ *real*
and *y* :: *real*
shows (*f has_derivative* ((∗) *y*)) (*at x within* ({*x* <..} $\cap$ *I*)) $\longleftrightarrow$
(($\lambda t.$ (*f x* $-$ *f t*) / (*x* $-$ *t*)) $\longrightarrow$ *y*) (*at x within* ({*x* <..} $\cap$ *I*))

proof −
have ((λ*t*. (*f t* − (*f x* + *y* ∗ (*t* − *x*))) / |*t* − *x*|) ⟶ *0*) (*at x within* ({*x*<..} ∩ *I*)) ⟷
((λ*t*. (*f t* − *f x*) / (*t* − *x*) − *y*) ⟶ *0*) (*at x within* ({*x*<..} ∩ *I*))
by (*intro Lim_cong_within*) (*auto simp add*: *diff_divide_distrib add_divide_distrib*)
also have ... ⟷ ((λ*t*. (*f t* − *f x*) / (*t* − *x*)) ⟶ *y*) (*at x within* ({*x*<..} ∩ *I*))
by (*simp add*: *Lim_null*[*symmetric*])
also have ... ⟷ ((λ*t*. (*f x* − *f t*) / (*x* − *t*)) ⟶ *y*) (*at x within* ({*x*<..} ∩ *I*))
by (*intro Lim_cong_within*) (*simp_all add*: *field_simps*)
finally show *?thesis*
by (*simp add*: *bounded_linear_mult_right has_derivative_within*)
qed

Caratheodory characterization

lemma *DERIV_caratheodory_within*:
(*f has_field_derivative l*) (*at x within S*) ⟷
(∃*g*. (∀*z*. *f z* − *f x* = *g z* ∗ (*z* − *x*)) ∧ *continuous* (*at x within S*) *g* ∧ *g x* = *l*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
show *?rhs*
proof (*intro exI conjI*)
let *?g* = (%*z*. *if z* = *x then l else* (*f z* − *f x*) / (*z*−*x*))
show ∀*z*. *f z* − *f x* = *?g z* ∗ (*z*−*x*) **by** *simp*
show *continuous* (*at x within S*) *?g* **using** ‹*?lhs*›
by (*auto simp add*: *continuous_within has_field_derivative_iff cong*: *Lim_cong_within*)
show *?g x* = *l* **by** *simp*
qed
next
assume *?rhs*
then obtain *g* **where**
(∀*z*. *f z* − *f x* = *g z* ∗ (*z*−*x*)) **and** *continuous* (*at x within S*) *g* **and** *g x* = *l*
by *blast*
thus *?lhs*
by (*auto simp add*: *continuous_within has_field_derivative_iff cong*: *Lim_cong_within*)
qed

6.17.3 Differentiability

definition
differentiable_on :: ('*a*::*real_normed_vector* ⇒ '*b*::*real_normed_vector*) ⇒ '*a set* ⇒ *bool*
(**infix** ‹*differentiable'_on*› *50*)
where *f differentiable_on s* ⟷ (∀*x*∈*s*. *f differentiable* (*at x within s*))

lemma *differentiableI*: (*f has_derivative f'*) *net* ⟹ *f differentiable net*
unfolding *differentiable_def*

by *auto*

lemma *differentiable_onD*: $[\![f$ *differentiable_on* $S; x \in S]\!] \Longrightarrow f$ *differentiable* (*at* x *within* S)
using *differentiable_on_def* **by** *blast*

lemma *differentiable_at_withinI*: f *differentiable* (*at* x) $\Longrightarrow f$ *differentiable* (*at* x *within* s)
unfolding *differentiable_def*
using *has_derivative_at_withinI*
by *blast*

lemma *differentiable_at_imp_differentiable_on*:
$(\bigwedge x.\ x \in s \Longrightarrow f$ *differentiable at* $x) \Longrightarrow f$ *differentiable_on* s
by (*metis differentiable_at_withinI differentiable_on_def*)

corollary *differentiable_iff_scaleR*:
fixes f :: *real* $\Rightarrow$ *'a*::*real_normed_vector*
shows f *differentiable* $F \longleftrightarrow (\exists d.\ (f$ *has_derivative* $(\lambda x.\ x *_R d))\ F)$
by (*auto simp*: *differentiable_def dest*: *has_derivative_linear linear_imp_scaleR*)

lemma *differentiable_on_eq_differentiable_at*:
open $s \Longrightarrow f$ *differentiable_on* $s \longleftrightarrow (\forall x{\in}s.\ f$ *differentiable at* $x)$
unfolding *differentiable_on_def*
by (*metis at_within_interior interior_open*)

lemma *differentiable_transform_within*:
assumes f *differentiable* (*at* x *within* s)
and $0 < d$
and $x \in s$
and $\bigwedge x'.\ [\![x'{\in}s;$ *dist* $x'\ x < d]\!] \Longrightarrow f\ x' = g\ x'$
shows g *differentiable* (*at* x *within* s)
using *assms has_derivative_transform_within* **unfolding** *differentiable_def*
by *blast*

lemma *differentiable_on_ident* [*simp*, *derivative_intros*]: $(\lambda x.\ x)$ *differentiable_on* S
by (*simp add*: *differentiable_at_imp_differentiable_on*)

lemma *differentiable_on_id* [*simp*, *derivative_intros*]: *id differentiable_on* S
by (*simp add*: *id_def*)

lemma *differentiable_on_const* [*simp*, *derivative_intros*]: $(\lambda z.\ c)$ *differentiable_on* S
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_mult* [*simp*, *derivative_intros*]:
fixes f :: *'M*::*real_normed_vector* $\Rightarrow$ *'a*::*real_normed_algebra*
shows $[\![f$ *differentiable_on* $S; g$ *differentiable_on* $S]\!] \Longrightarrow (\lambda z.\ f\ z * g\ z)$ *differen-*

tiable_on S
unfolding *differentiable_on_def differentiable_def*
using *differentiable_def differentiable_mult* **by** *blast*

lemma *differentiable_on_compose*:
⟦*g differentiable_on S*; *f differentiable_on* (*g ' S*)⟧ ⟹ (λ*x*. *f* (*g x*)) *differentiable_on S*
by (*simp add*: *differentiable_in_compose differentiable_on_def*)

lemma *bounded_linear_imp_differentiable_on*: *bounded_linear f* ⟹ *f differentiable_on S*
by (*simp add*: *differentiable_on_def bounded_linear_imp_differentiable*)

lemma *linear_imp_differentiable_on*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
shows *linear f* ⟹ *f differentiable_on S*
by (*simp add*: *differentiable_on_def linear_imp_differentiable*)

lemma *differentiable_on_minus* [*simp*, *derivative_intros*]:
f differentiable_on S ⟹ (λ*z*. −(*f z*)) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_add* [*simp*, *derivative_intros*]:
⟦*f differentiable_on S*; *g differentiable_on S*⟧ ⟹ (λ*z*. *f z* + *g z*) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_diff* [*simp*, *derivative_intros*]:
⟦*f differentiable_on S*; *g differentiable_on S*⟧ ⟹ (λ*z*. *f z* − *g z*) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_inverse* [*simp*, *derivative_intros*]:
fixes *f* :: *'a* :: *real_normed_vector* ⇒ *'b* :: *real_normed_field*
shows *f differentiable_on S* ⟹ (⋀*x*. *x* ∈ *S* ⟹ *f x* ≠ *0*) ⟹ (λ*x*. *inverse* (*f x*)) *differentiable_on S*
by (*simp add*: *differentiable_on_def*)

lemma *differentiable_on_scaleR* [*derivative_intros*, *simp*]:
⟦*f differentiable_on S*; *g differentiable_on S*⟧ ⟹ (λ*x*. *f x* ∗*R* *g x*) *differentiable_on S*
unfolding *differentiable_on_def*
by (*blast intro*: *differentiable_scaleR*)

lemma *has_derivative_sqnorm_at* [*derivative_intros*, *simp*]:
((λ*x*. (*norm x*)²) *has_derivative* (λ*x*. *2* ∗*R* (*a* • *x*))) (*at a*)
using *bounded_bilinear.FDERIV* [*of* (•) *id id a _ id id*]
by (*auto simp*: *inner_commute dot_square_norm bounded_bilinear_inner*)

lemma *differentiable_sqnorm_at* [*derivative_intros*, *simp*]:
fixes *a* :: *'a* :: {*real_normed_vector*,*real_inner*}
shows (*λx*. (*norm x*)^2) *differentiable* (*at a*)
by (*force simp add*: *differentiable_def intro*: *has_derivative_sqnorm_at*)

lemma *differentiable_on_sqnorm* [*derivative_intros*, *simp*]:
fixes *S* :: *'a* :: {*real_normed_vector*,*real_inner*} *set*
shows (*λx*. (*norm x*)^2) *differentiable_on S*
by (*simp add*: *differentiable_at_imp_differentiable_on*)

lemma *differentiable_norm_at* [*derivative_intros*, *simp*]:
fixes *a* :: *'a* :: {*real_normed_vector*,*real_inner*}
shows *a* ≠ *0* ⟹ *norm differentiable* (*at a*)
using *differentiableI has_derivative_norm* **by** *blast*

lemma *differentiable_on_norm* [*derivative_intros*, *simp*]:
fixes *S* :: *'a* :: {*real_normed_vector*,*real_inner*} *set*
shows *0* ∉ *S* ⟹ *norm differentiable_on S*
by (*metis differentiable_at_imp_differentiable_on differentiable_norm_at*)

6.17.4 Frechet derivative and Jacobian matrix

definition *frechet_derivative f net* = (*SOME f'*. (*f has_derivative f'*) *net*)

proposition *frechet_derivative_works*:
f differentiable net ⟷ (*f has_derivative* (*frechet_derivative f net*)) *net*
unfolding *frechet_derivative_def differentiable_def*
unfolding *some_eq_ex*[*of λ f'* . (*f has_derivative f'*) *net*] **..**

lemma *linear_frechet_derivative*: *f differentiable net* ⟹ *linear* (*frechet_derivative f net*)
unfolding *frechet_derivative_works has_derivative_def*
by (*auto intro*: *bounded_linear.linear*)

lemma *frechet_derivative_const* [*simp*]: *frechet_derivative* (*λx*. *c*) (*at a*) = (*λx*. *0*)
using *differentiable_const frechet_derivative_works has_derivative_const has_derivative_unique*
by *blast*

lemma *frechet_derivative_id* [*simp*]: *frechet_derivative id* (*at a*) = *id*
using *differentiable_def frechet_derivative_works has_derivative_id has_derivative_unique*
by *blast*

lemma *frechet_derivative_ident* [*simp*]: *frechet_derivative* (*λx*. *x*) (*at a*) = (*λx*. *x*)
by (*metis eq_id_iff frechet_derivative_id*)

6.17.5 Differentiability implies continuity

proposition *differentiable_imp_continuous_within*:

f differentiable (*at x within s*) ⟹ *continuous* (*at x within s*) *f*
by (*auto simp*: *differentiable_def intro*: *has_derivative_continuous*)

lemma *differentiable_imp_continuous_on*:
f differentiable_on s ⟹ *continuous_on s f*
unfolding *differentiable_on_def continuous_on_eq_continuous_within*
using *differentiable_imp_continuous_within* **by** *blast*

lemma *differentiable_on_subset*:
f differentiable_on t ⟹ *s* ⊆ *t* ⟹ *f differentiable_on s*
unfolding *differentiable_on_def*
using *differentiable_within_subset*
by *blast*

lemma *differentiable_on_empty*: *f differentiable_on* {}
unfolding *differentiable_on_def*
by *auto*

lemma *has_derivative_continuous_on*:
(⋀*x*. *x* ∈ *s* ⟹ (*f has_derivative f' x*) (*at x within s*)) ⟹ *continuous_on s f*
by (*auto intro*!: *differentiable_imp_continuous_on differentiableI simp*: *differentiable_on_def*)

Results about neighborhoods filter.

lemma *eventually_nhds_metric_le*:
eventually P (*nhds a*) = (∃ *d*>0. ∀ *x*. *dist x a* ≤ *d* ⟶ *P x*)
unfolding *eventually_nhds_metric* **by** (*safe*, *rule_tac x*=*d* / *2* **in** *exI*, *auto*)

lemma *le_nhds*: *F* ≤ *nhds a* ⟷ (∀ *S*. *open S* ∧ *a* ∈ *S* ⟶ *eventually* (λ*x*. *x* ∈ *S*) *F*)
unfolding *le_filter_def eventually_nhds* **by** (*fast elim*: *eventually_mono*)

lemma *le_nhds_metric*: *F* ≤ *nhds a* ⟷ (∀ *e*>0. *eventually* (λ*x*. *dist x a* < *e*) *F*)
unfolding *le_filter_def eventually_nhds_metric* **by** (*fast elim*: *eventually_mono*)

lemma *le_nhds_metric_le*: *F* ≤ *nhds a* ⟷ (∀ *e*>0. *eventually* (λ*x*. *dist x a* ≤ *e*) *F*)
unfolding *le_filter_def eventually_nhds_metric_le* **by** (*fast elim*: *eventually_mono*)

Several results are easier using a "multiplied-out" variant. (I got this idea from Dieudonne's proof of the chain rule).

lemma *has_derivative_within_alt*:
(*f has_derivative f'*) (*at x within s*) ⟷ *bounded_linear f'* ∧
(∀ *e*>0. ∃ *d*>0. ∀ *y*∈*s*. *norm*(*y* − *x*) < *d* ⟶ *norm* (*f y* − *f x* − *f'* (*y* − *x*)) ≤ *e* ∗ *norm* (*y* − *x*))
unfolding *has_derivative_within filterlim_def le_nhds_metric_le eventually_filtermap eventually_at dist_norm diff_diff_eq*
by (*force simp add*: *linear_0 bounded_linear.linear pos_divide_le_eq*)

lemma *has_derivative_within_alt2*:
(*f has_derivative f'*) (*at x within s*) ⟷ *bounded_linear f'* ∧
(∀ *e*>*0*. *eventually* (λ*y*. *norm* (*f y* − *f x* − *f'* (*y* − *x*)) ≤ *e* ∗ *norm* (*y* − *x*))
(*at x within s*))
unfolding *has_derivative_within filterlim_def le_nhds_metric_le eventually_filtermap*
eventually_at dist_norm diff_diff_eq
by (*force simp add*: *linear_0 bounded_linear.linear pos_divide_le_eq*)

lemma *has_derivative_at_alt*:
(*f has_derivative f'*) (*at x*) ⟷
bounded_linear f' ∧
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *y*. *norm*(*y* − *x*) < *d* ⟶ *norm* (*f y* − *f x* − *f'*(*y* − *x*)) ≤ *e* ∗
norm (*y* − *x*))
using *has_derivative_within_alt*[**where** *s*=*UNIV*]
by *simp*

6.17.6 The chain rule

proposition *diff_chain_within*[*derivative_intros*]:
assumes (*f has_derivative f'*) (*at x within s*)
and (*g has_derivative g'*) (*at* (*f x*) *within* (*f* ‘ *s*))
shows ((*g* ∘ *f*) *has_derivative* (*g'* ∘ *f'*))(*at x within s*)
using *has_derivative_in_compose*[*OF assms*]
by (*simp add*: *comp_def*)

lemma *diff_chain_at*[*derivative_intros*]:
(*f has_derivative f'*) (*at x*) ⟹
(*g has_derivative g'*) (*at* (*f x*)) ⟹ ((*g* ∘ *f*) *has_derivative* (*g'* ∘ *f'*)) (*at x*)
by (*meson diff_chain_within has_derivative_at_withinI*)

lemma *has_vector_derivative_shift*: (*f has_vector_derivative D x*) (*at x*)
⟹ ((+) *d* ∘ *f has_vector_derivative D x*) (*at x*)
using *diff_chain_at* [*OF _ shift_has_derivative_id*]
by (*simp add*: *has_derivative_iff_Ex has_vector_derivative_def*)

lemma *has_vector_derivative_within_open*:
a ∈ *S* ⟹ *open S* ⟹
(*f has_vector_derivative f'*) (*at a within S*) ⟷ (*f has_vector_derivative f'*)
(*at a*)
by (*simp only*: *at_within_interior interior_open*)

lemma *field_vector_diff_chain_within*:
assumes *Df*: (*f has_vector_derivative f'*) (*at x within S*)
and *Dg*: (*g has_field_derivative g'*) (*at* (*f x*) *within f* ‘ *S*)
shows ((*g* ∘ *f*) *has_vector_derivative* (*f'* ∗ *g'*)) (*at x within S*)
using *diff_chain_within*[*OF Df*[*unfolded has_vector_derivative_def*]
Dg [*unfolded has_field_derivative_def*]]
by (*auto simp*: *o_def mult.commute has_vector_derivative_def*)

lemma *vector_derivative_diff_chain_within*:
assumes *Df*: (*f has_vector_derivative f'*) (*at x within S*)
and *Dg*: (*g has_derivative g'*) (*at* (*f x*) *within f'S*)
shows ((*g* ∘ *f*) *has_vector_derivative* (*g' f'*)) (*at x within S*)
using *diff_chain_within*[*OF Df*[*unfolded has_vector_derivative_def*] *Dg*]
linear.scaleR[*OF has_derivative_linear*[*OF Dg*]]
unfolding *has_vector_derivative_def o_def*
by (*auto simp*: *o_def mult.commute has_vector_derivative_def*)

6.17.7 Composition rules stated just for differentiability

lemma *differentiable_chain_at*:
f differentiable (*at x*) ⟹
g differentiable (*at* (*f x*)) ⟹ (*g* ∘ *f*) *differentiable* (*at x*)
unfolding *differentiable_def*
by (*meson diff_chain_at*)

lemma *differentiable_chain_within*:
f differentiable (*at x within S*) ⟹
g differentiable (*at*(*f x*) *within* (*f ' S*)) ⟹ (*g* ∘ *f*) *differentiable* (*at x within S*)
unfolding *differentiable_def*
by (*meson diff_chain_within*)

6.17.8 Uniqueness of derivative

The general result is a bit messy because we need approachability of the limit point from any direction. But OK for nontrivial intervals etc.

proposition *frechet_derivative_unique_within*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes *1*: (*f has_derivative f'*) (*at x within S*)
and *2*: (*f has_derivative f''*) (*at x within S*)
and *S*: ⋀*i e*. ⟦*i*∈*Basis*; *e*>*0*⟧ ⟹ ∃ *d*. *0* < |*d*| ∧ |*d*| < *e* ∧ (*x* + *d* ∗$_R$ *i*) ∈ *S*
shows *f' = f''*
proof −
note *as* = *assms*(*1*,*2*)[*unfolded has_derivative_def*]
then interpret *f'*: *bounded_linear f'* **by** *auto*
from *as* **interpret** *f''*: *bounded_linear f''* **by** *auto*
have *x islimpt S* **unfolding** *islimpt_approachable*
proof (*intro allI impI*)
fix *e* :: *real*
assume *e* > *0*
obtain *d* **where** *0* < |*d*| **and** |*d*| < *e* **and** *x* + *d* ∗$_R$ (*SOME i*. *i* ∈ *Basis*) ∈ *S*
using *assms*(*3*) *SOME_Basis* ‹*e*>*0*› **by** *blast*
then show ∃ *x'*∈*S*. *x'* ≠ *x* ∧ *dist x' x* < *e*
by (*rule_tac x*=*x* + *d* ∗$_R$ (*SOME i*. *i* ∈ *Basis*) **in** *bexI*) (*auto simp*: *dist_norm SOME_Basis nonzero_Basis*) **qed**
then have ∗: *netlimit* (*at x within S*) = *x*
by (*simp add*: *Lim_ident_at trivial_limit_within*)

```
  show ?thesis
  proof (rule linear_eq_stdbasis)
    show linear f' linear f''
      unfolding linear_conv_bounded_linear using as by auto
  next
    fix i :: 'a
    assume i: i ∈ Basis
    define e where e = norm (f' i − f'' i)
    show f' i = f'' i
    proof (rule ccontr)
      assume f' i ≠ f'' i
      then have e > 0
        unfolding e_def by auto
      obtain d where d:
        0 < d
        (⋀y. y∈S ⟶ 0 < dist y x ∧ dist y x < d ⟶
          dist ((f y − f x − f' (y − x)) /R norm (y − x) −
              (f y − f x − f'' (y − x)) /R norm (y − x)) (0 − 0) < e)
        using tendsto_diff [OF as(1,2)[THEN conjunct2]]
        unfolding ∗ Lim_within
        using ‹e>0› by blast
      obtain c where c: 0 < |c| |c| < d ∧ x + c ∗R i ∈ S
        using assms(3) i d(1) by blast
      have ∗: norm (− ((1 / |c|) ∗R f' (c ∗R i)) + (1 / |c|) ∗R f'' (c ∗R i)) =
        norm ((1 / |c|) ∗R (− (f' (c ∗R i)) + f'' (c ∗R i)))
        unfolding scaleR_right_distrib by auto
      also have ... = norm ((1 / |c|) ∗R (c ∗R (− (f' i) + f'' i)))
        unfolding f'.scaleR f''.scaleR
        unfolding scaleR_right_distrib scaleR_minus_right
        by auto
      also have ... = e
        unfolding e_def
        using c(1)
        using norm_minus_cancel[of f' i − f'' i]
        by auto
      finally show False
        using c
        using d(2)[of x + c ∗R i]
        unfolding dist_norm
        unfolding f'.scaleR f''.scaleR f'.add f''.add f'.diff f''.diff
          scaleR_scaleR scaleR_right_diff_distrib scaleR_right_distrib
        using i
        by (auto simp: inverse_eq_divide)
    qed
  qed
qed

proposition frechet_derivative_unique_within_closed_interval:
  fixes f::'a::euclidean_space ⇒ 'b::real_normed_vector
```

```
  assumes ab: ⋀i. i∈Basis ⟹ a•i < b•i
    and x: x ∈ cbox a b
    and (f has_derivative f' ) (at x within cbox a b)
    and (f has_derivative f'') (at x within cbox a b)
  shows f' = f''
proof (rule frechet_derivative_unique_within)
  fix e :: real
  fix i :: 'a
  assume e > 0 and i: i ∈ Basis
  then show ∃ d. 0 < |d| ∧ |d| < e ∧ x + d *R i ∈ cbox a b
  proof (cases x•i = a•i)
    case True
    with ab[of i] ‹e>0› x i show ?thesis
      by (rule_tac x=(min (b•i − a•i) e) / 2 in exI)
        (auto simp add: mem_box field_simps inner_simps inner_Basis)
  next
    case False
    moreover have a • i < x • i
      using False i mem_box(2) x by force
    moreover {
      have a • i * 2 + min (x • i − a • i) e ≤ a•i *2 + x•i − a•i
        by auto
      also have ... = a•i + x•i
        by auto
      also have ... ≤ 2 * (x•i)
        using ‹a • i < x • i› by auto
      finally have a • i * 2 + min (x • i − a • i) e ≤ x • i * 2
        by auto
    }
    moreover have min (x • i − a • i) e ≥ 0
      by (simp add: ‹0 < e› ‹a • i < x • i› less_eq_real_def)
    then have x • i * 2 ≤ b • i * 2 + min (x • i − a • i) e
      using i mem_box(2) x by force
    ultimately show ?thesis
    using ab[of i] ‹e>0› x i
      by (rule_tac x=− (min (x•i − a•i) e) / 2 in exI)
        (auto simp add: mem_box field_simps inner_simps inner_Basis)
  qed
qed (use assms in auto)

lemma frechet_derivative_unique_within_open_interval:
  fixes f::'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes x: x ∈ box a b
    and f: (f has_derivative f' ) (at x within box a b) (f has_derivative f'') (at x
within box a b)
  shows f' = f''
  by (metis at_within_open assms has_derivative_unique open_box)

lemma frechet_derivative_at:
```

(*f has_derivative f'*) (*at x*) ⟹ *f'* = *frechet_derivative f* (*at x*)
using *differentiable_def frechet_derivative_works has_derivative_unique* **by** *blast*

lemma *frechet_derivative_compose*:
frechet_derivative (*f o g*) (*at x*) = *frechet_derivative* (*f*) (*at* (*g x*)) *o frechet_derivative g* (*at x*)
if *g differentiable at x f differentiable at* (*g x*)
by (*metis diff_chain_at frechet_derivative_at frechet_derivative_works that*)

lemma *frechet_derivative_within_cbox*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes ⋀*i*. *i*∈*Basis* ⟹ *a*•*i* < *b*•*i*
and *x* ∈ *cbox a b*
and (*f has_derivative f'*) (*at x within cbox a b*)
shows *frechet_derivative f* (*at x within cbox a b*) = *f'*
using *assms*
by (*metis Derivative.differentiableI frechet_derivative_unique_within_closed_interval frechet_derivative_works*)

lemma *frechet_derivative_transform_within_open*:
frechet_derivative f (*at x*) = *frechet_derivative g* (*at x*)
if *f differentiable at x open X x* ∈ *X* ⋀*x*. *x* ∈ *X* ⟹ *f x* = *g x*
by (*meson frechet_derivative_at frechet_derivative_works has_derivative_transform_within_open that*)

6.17.9 Derivatives of local minima and maxima are zero

lemma *has_derivative_local_min*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *real*
assumes *deriv*: (*f has_derivative f'*) (*at x*)
assumes *min*: *eventually* (λ*y*. *f x* ≤ *f y*) (*at x*)
shows *f'* = (λ*h*. *0*)
proof
fix *h* :: *'a*
interpret *f'*: *bounded_linear f'*
using *deriv* **by** (*rule has_derivative_bounded_linear*)
show *f' h* = *0*
proof (*cases h* = *0*)
case *False*
from *min* **obtain** *d* **where** *d1*: *0* < *d* **and** *d2*: ∀ *y*∈*ball x d*. *f x* ≤ *f y*
unfolding *eventually_at* **by** (*force simp*: *dist_commute*)
have *FDERIV* (λ*r*. *x* + *r* ∗*R h*) *0* :> (λ*r*. *r* ∗*R h*)
by (*intro derivative_eq_intros*) *auto*
then have *FDERIV* (λ*r*. *f* (*x* + *r* ∗*R h*)) *0* :> (λ*k*. *f'* (*k* ∗*R h*))
by (*rule has_derivative_compose*, *simp add*: *deriv*)
then have *DERIV* (λ*r*. *f* (*x* + *r* ∗*R h*)) *0* :> *f' h*
unfolding *has_field_derivative_def* **by** (*simp add*: *f'.scaleR mult_commute_abs*)
moreover have *0* < *d* / *norm h* **using** *d1* **and** ‹*h* ≠ *0*› **by** *simp*
moreover have ∀ *y*. |*0* − *y*| < *d* / *norm h* ⟶ *f* (*x* + *0* ∗*R h*) ≤ *f* (*x* + *y* ∗*R*

```
h)
      using ‹h ≠ 0› by (auto simp add: d2 dist_norm pos_less_divide_eq)
    ultimately show f' h = 0
      by (rule DERIV_local_min)
  qed simp
qed

lemma has_derivative_local_max:
  fixes f :: 'a::real_normed_vector ⇒ real
  assumes (f has_derivative f') (at x)
  assumes eventually (λy. f y ≤ f x) (at x)
  shows f' = (λh. 0)
  using has_derivative_local_min [of λx. − f x λh. − f' h x]
  using assms unfolding fun_eq_iff by simp

lemma differential_zero_maxmin:
  fixes f::'a::real_normed_vector ⇒ real
  assumes x ∈ S
    and open S
    and deriv: (f has_derivative f') (at x)
    and mono: (∀ y∈S. f y ≤ f x) ∨ (∀ y∈S. f x ≤ f y)
  shows f' = (λv. 0)
  using mono
proof
  assume ∀ y∈S. f y ≤ f x
  with ‹x ∈ S› and ‹open S› have eventually (λy. f y ≤ f x) (at x)
    unfolding eventually_at_topological by auto
  with deriv show ?thesis
    by (rule has_derivative_local_max)
next
  assume ∀ y∈S. f x ≤ f y
  with ‹x ∈ S› and ‹open S› have eventually (λy. f x ≤ f y) (at x)
    unfolding eventually_at_topological by auto
  with deriv show ?thesis
    by (rule has_derivative_local_min)
qed

lemma differential_zero_maxmin_component:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes k: k ∈ Basis
    and ball: 0 < e (∀ y ∈ ball x e. (f y)•k ≤ (f x)•k) ∨ (∀ y∈ball x e. (f x)•k ≤ (f
y)•k)
    and diff: f differentiable (at x)
  shows (∑j∈Basis. (frechet_derivative f (at x) j • k) *R j) = (0::'a) (is ?D k =
0)
proof −
  let ?f' = frechet_derivative f (at x)
  have x ∈ ball x e using ‹0 < e› by simp
  moreover have open (ball x e) by simp
```

moreover have $((\lambda x.\ f\ x \cdot k)$ *has_derivative* $(\lambda h.\ ?f'\ h \cdot k))$ $(at\ x)$
using *bounded_linear_inner_left diff*[*unfolded frechet_derivative_works*]
by (*rule bounded_linear.has_derivative*)
ultimately have $(\lambda h.$ *frechet_derivative* f $(at\ x)$ $h \cdot k) = (\lambda v.\ 0)$
using *ball*(*2*) **by** (*rule differential_zero_maxmin*)
then show *?thesis*
unfolding *fun_eq_iff* **by** *simp*
qed

6.17.10 One-dimensional mean value theorem

lemma *mvt_simple*:
fixes $f :: real \Rightarrow real$
assumes $a < b$
and *derf*: $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow (f$ *has_derivative* $f'\ x)$ $(at\ x$ *within* $\{a..b\})$
shows $\exists\, x{\in}\{a<..<b\}.\ f\ b - f\ a = f'\ x\ (b - a)$
proof (*rule mvt*)
have f *differentiable_on* $\{a..b\}$
using *derf* **unfolding** *differentiable_on_def differentiable_def* **by** *force*
then show *continuous_on* $\{a..b\}$ f
by (*rule differentiable_imp_continuous_on*)
show $(f$ *has_derivative* $f'\ x)$ $(at\ x)$ **if** $a < x\ x < b$ **for** x
by (*metis at_within_Icc_at derf leI order.asym that*)
qed (*use assms* **in** *auto*)

lemma *mvt_very_simple*:
fixes $f :: real \Rightarrow real$
assumes $a \le b$
and *derf*: $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow (f$ *has_derivative* $f'\ x)$ $(at\ x$ *within* $\{a..b\})$
shows $\exists\, x{\in}\{a..b\}.\ f\ b - f\ a = f'\ x\ (b - a)$
proof (*cases* $a = b$)
interpret *bounded_linear* $f'\ b$
using *assms* **by** *auto*
case *True*
then show *?thesis*
by *force*
next
case *False*
then show *?thesis*
using *mvt_simple*[*OF _ derf*]
by (*metis* ‹$a \le b$› *atLeastAtMost_iff dual_order.order_iff_strict greaterThanLessThan_iff*)
qed

A nice generalization (see Havin's proof of 5.19 from Rudin's book).

lemma *mvt_general*:
fixes $f :: real \Rightarrow 'a{::}real\_inner$
assumes $a < b$
and *contf*: *continuous_on* $\{a..b\}$ f
and *derf*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow (f$ *has_derivative* $f'\ x)$ $(at\ x)$

shows $\exists x{\in}\{a<..<b\}.\ norm\ (f\ b - f\ a) \leq norm\ (f'\ x\ (b - a))$
proof −
have $\exists x{\in}\{a<..<b\}.\ (f\ b - f\ a) \cdot f\ b - (f\ b - f\ a) \cdot f\ a = (f\ b - f\ a) \cdot f'\ x\ (b - a)$
apply (*rule mvt* [*OF* ‹$a < b$›, **where** $f = \lambda x.\ (f\ b - f\ a) \cdot f\ x$])
apply (*intro continuous_intros contf*)
using *derf* **apply** (*auto intro*: *has_derivative_inner_right*)
done
then obtain x **where** x: $x \in \{a<..<b\}$
$(f\ b - f\ a) \cdot f\ b - (f\ b - f\ a) \cdot f\ a = (f\ b - f\ a) \cdot f'\ x\ (b - a)$ **..**
show *?thesis*
proof (*cases* $f\ a = f\ b$)
case *False*
have $norm\ (f\ b - f\ a) * norm\ (f\ b - f\ a) = (norm\ (f\ b - f\ a))^2$
by (*simp add*: *power2_eq_square*)
also have $\ldots = (f\ b - f\ a) \cdot (f\ b - f\ a)$
unfolding *power2_norm_eq_inner* **..**
also have $\ldots = (f\ b - f\ a) \cdot f'\ x\ (b - a)$
using *x*(*2*) **by** (*simp only*: *inner_diff_right*)
also have $\ldots \leq norm\ (f\ b - f\ a) * norm\ (f'\ x\ (b - a))$
by (*rule norm_cauchy_schwarz*)
finally show *?thesis*
using *False x*(*1*)
by (*auto simp add*: *mult_left_cancel*)
next
case *True*
then show *?thesis*
using ‹$a < b$› **by** (*rule_tac* x=$(a + b)\ /2$ **in** *bexI*) *auto*
qed
qed

6.17.11 More general bound theorems

proposition *differentiable_bound_general*:
fixes f :: *real* ⇒ *'a*::*real_normed_vector*
assumes $a < b$
and *f_cont*: *continuous_on* $\{a..b\}$ f
and *phi_cont*: *continuous_on* $\{a..b\}$ φ
and *f'*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
and *phi'*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (\varphi\ has\_vector\_derivative\ \varphi'\ x)\ (at\ x)$
and *bnd*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow norm\ (f'\ x) \leq \varphi'\ x$
shows $norm\ (f\ b - f\ a) \leq \varphi\ b - \varphi\ a$
proof −
{
fix x **assume** x: $a < x\ x < b$
have $0 \leq norm\ (f'\ x)$ **by** *simp*
also have $\ldots \leq \varphi'\ x$ **using** x **by** (*auto intro*!: *bnd*)
finally have $0 \leq \varphi'\ x$ **.**
} **note** *phi'_nonneg* = *this*

```
  note f_tendsto = assms(2)[simplified continuous_on_def, rule_format]
  note phi_tendsto = assms(3)[simplified continuous_on_def, rule_format]
  {
    fix e::real assume e > 0
    define e2 where e2 = e / 2
    with ‹e > 0› have e2 > 0 by simp
    let ?le = λx1. norm (f x1 − f a) ≤ φ x1 − φ a + e ∗ (x1 − a) + e
    define A where A = {x2. a ≤ x2 ∧ x2 ≤ b ∧ (∀ x1∈{a ..< x2}. ?le x1)}
    have A_subset: A ⊆ {a..b} by (auto simp: A_def)
    {
      fix x2
      assume a: a ≤ x2 x2 ≤ b and le: ∀ x1∈{a..<x2}. ?le x1
      have ?le x2 using ‹e > 0›
      proof cases
        assume x2 ≠ a with a have a < x2 by simp
        have at x2 within {a <..<x2}≠ bot
          using ‹a < x2›
          by (auto simp: trivial_limit_within islimpt_in_closure)
        moreover
        have ((λx1. (φ x1 − φ a) + e ∗ (x1 − a) + e) ⟶ (φ x2 − φ a) + e ∗
(x2 − a) + e) (at x2 within {a <..<x2})
            ((λx1. norm (f x1 − f a)) ⟶ norm (f x2 − f a)) (at x2 within {a
<..<x2})
          using a
          by (auto intro!: tendsto_eq_intros f_tendsto phi_tendsto
            intro: tendsto_within_subset[where S={a..b}])
        moreover
        have eventually (λx. x > a) (at x2 within {a <..<x2})
          by (auto simp: eventually_at_filter)
        hence eventually ?le (at x2 within {a <..<x2})
          unfolding eventually_at_filter
          by eventually_elim (insert le, auto)
        ultimately
        show ?thesis
          by (rule tendsto_le)
      qed simp
    } note le_cont = this
    have a ∈ A
      using assms by (auto simp: A_def)
    hence [simp]: A ≠ {} by auto
    have A_ivl: ⋀x1 x2. x2 ∈ A ⟹ x1 ∈ {a ..x2} ⟹ x1 ∈ A
      by (simp add: A_def)
    have [simp]: bdd_above A by (auto simp: A_def)
    define y where y = Sup A
    have y ≤ b
      unfolding y_def
      by (simp add: cSup_le_iff) (simp add: A_def)
     have leI: ⋀x x1. a ≤ x1 ⟹ x ∈ A ⟹ x1 < x ⟹ ?le x1
       by (auto simp: A_def intro!: le_cont)
```

```
    have y_all_le: ∀x1∈{a..<y}. ?le x1
      by (auto simp: y_def less_cSup_iff leI)
    have a ≤ y
      by (metis ‹a ∈ A› ‹bdd_above A› cSup_upper y_def)
    have y ∈ A
      using y_all_le ‹a ≤ y› ‹y ≤ b›
      by (auto simp: A_def)
    hence A = {a .. y}
      using A_subset by (auto simp: subset_iff y_def cSup_upper intro: A_ivl)
    from le_cont[OF ‹a ≤ y› ‹y ≤ b› y_all_le] have le_y: ?le y .
    have y = b
    proof (cases a = y)
      case True
      with ‹a < b› have y < b by simp
      with ‹a = y› f_cont phi_cont ‹e2 > 0›
      have 1: ∀F x in at y within {y..b}. dist (f x) (f y) < e2
       and 2: ∀F x in at y within {y..b}. dist (φ x) (φ y) < e2
        by (auto simp: continuous_on_def tendsto_iff)
      have 3: eventually (λx. y < x) (at y within {y..b})
        by (auto simp: eventually_at_filter)
      have 4: eventually (λx::real. x < b) (at y within {y..b})
        using _ ‹y < b›
        by (rule order_tendstoD) (auto intro!: tendsto_eq_intros)
      from 1 2 3 4
      have eventually_le: eventually (λx. ?le x) (at y within {y .. b})
      proof eventually_elim
        case (elim x1)
        have norm (f x1 − f a) = norm (f x1 − f y)
          by (simp add: ‹a = y›)
        also have norm (f x1 − f y) ≤ e2
          using elim ‹a = y› by (auto simp : dist_norm intro!:  less_imp_le)
        also have ... ≤ e2 + (φ x1 − φ a + e2 + e * (x1 − a))
          using ‹0 < e› elim
          by (intro add_increasing2[OF add_nonneg_nonneg order.refl])
            (auto simp: ‹a = y› dist_norm intro!: mult_nonneg_nonneg)
        also have ... = φ x1 − φ a + e * (x1 − a) + e
          by (simp add: e2_def)
        finally show ?le x1 .
      qed
      from this[unfolded eventually_at_topological] ‹?le y›
      obtain S where S: open S y ∈ S ⋀x. x∈S ⟹ x ∈ {y..b} ⟹ ?le x
        by metis
      from ‹open S› obtain d where d: ⋀x. dist x y < d ⟹ x ∈ S d > 0
        by (force simp: dist_commute open_dist ball_def dest!: bspec[OF _ ‹y ∈
S›])
      define d' where d' = min b (y + (d/2))
      have d' ∈ A
        unfolding A_def
      proof safe
```

```
    show a ≤ d' using ‹a = y› ‹0 < d› ‹y < b› by (simp add: d'_def)
    show d' ≤ b by (simp add: d'_def)
    fix x1
    assume x1 ∈ {a..<d'}
    hence x1 ∈ S x1 ∈ {y..b}
      by (auto simp: ‹a = y› d'_def dist_real_def intro!: d )
    thus ?le x1
      by (rule S)
  qed
  hence d' ≤ y
    unfolding y_def
    by (rule cSup_upper) simp
  then show y = b using ‹d > 0› ‹y < b›
    by (simp add: d'_def)
next
  case False
  with ‹a ≤ y› have a < y by simp
  show y = b
  proof (rule ccontr)
    assume y ≠ b
    hence y < b using ‹y ≤ b› by simp
    let ?F = at y within {y..<b}
    from f' phi'
    have (f has_vector_derivative f' y) ?F
      and (φ has_vector_derivative φ' y) ?F
      using ‹a < y› ‹y < b›
   by (auto simp add: at_within_open[of _ {a<..<b}] has_vector_derivative_def
        intro!: has_derivative_subset[where s={a<..<b} and t={y..<b}])
    hence ∀F x1 in ?F. norm (f x1 − f y − (x1 − y) *R f' y) ≤ e2 * |x1 − y|
        ∀F x1 in ?F. norm (φ x1 − φ y − (x1 − y) *R φ' y) ≤ e2 * |x1 − y|
      using ‹e2 > 0›
      by (auto simp: has_derivative_within_alt2 has_vector_derivative_def)
    moreover
    have ∀F x1 in ?F. y ≤ x1 ∀F x1 in ?F. x1 < b
      by (auto simp: eventually_at_filter)
    ultimately
    have ∀F x1 in ?F. norm (f x1 − f y) ≤ (φ x1 − φ y) + e * |x1 − y|
      (is ∀F x1 in ?F. ?le' x1)
    proof eventually_elim
      case (elim x1)
      from norm_triangle_ineq2[THEN order_trans, OF elim(1)]
      have norm (f x1 − f y) ≤ norm (f' y) * |x1 − y| + e2 * |x1 − y|
        by (simp add: ac_simps)
      also have norm (f' y) ≤ φ' y using bnd ‹a < y› ‹y < b› by simp
      also have φ' y * |x1 − y| ≤ φ x1 − φ y + e2 * |x1 − y|
        using elim by (simp add: ac_simps)
      finally
      have norm (f x1 − f y) ≤ φ x1 − φ y + e2 * |x1 − y| + e2 * |x1 − y|
        by (auto simp: mult_right_mono)
```

```
      thus ?case by (simp add: e2_def)
    qed
    moreover have ?le' y by simp
    ultimately obtain S
    where S: open S y ∈ S ⋀x. x∈S ⟹ x ∈ {y..<b} ⟹ ?le' x
      unfolding eventually_at_topological
      by metis
    from ‹open S› obtain d where d: ⋀x. dist x y < d ⟹ x ∈ S d > 0
      by (force simp: dist_commute open_dist ball_def dest!: bspec[OF _ ‹y ∈
S›])
    define d' where d' = min ((y + b)/2) (y + (d/2))
    have d' ∈ A
      unfolding A_def
    proof safe
      show a ≤ d' using ‹a < y› ‹0 < d› ‹y < b› by (simp add: d'_def)
      show d' ≤ b using ‹y < b› by (simp add: d'_def min_def)
      fix x1
      assume x1: x1 ∈ {a..<d'}
      show ?le x1
      proof (cases x1 < y)
        case True
        then show ?thesis
          using ‹y ∈ A› local.leI x1 by auto
      next
        case False
        hence x1': x1 ∈ S x1 ∈ {y..<b} using x1
          by (auto simp: d'_def dist_real_def intro!: d)
        have norm (f x1 − f a) ≤ norm (f x1 − f y) + norm (f y − f a)
          by (rule order_trans[OF _ norm_triangle_ineq]) simp
        also note S(3)[OF x1']
        also note le_y
        finally show ?le x1
          using False by (auto simp: algebra_simps)
      qed
    qed
    hence d' ≤ y
      unfolding y_def by (rule cSup_upper) simp
    thus False using ‹d > 0› ‹y < b›
      by (simp add: d'_def min_def split: if_split_asm)
  qed
 qed
 with le_y have norm (f b − f a) ≤ φ b − φ a + e ∗ (b − a + 1)
  by (simp add: algebra_simps)
} note ∗ = this
show ?thesis
proof (rule field_le_epsilon)
 fix e::real assume e > 0
 then show norm (f b − f a) ≤ φ b − φ a + e
  using ∗[of e / (b − a + 1)] ‹a < b› by simp
```

```
  qed
qed

lemma differentiable_bound:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes convex S
    and derf: ⋀x. x∈S ⟹ (f has_derivative f' x) (at x within S)
    and B: ⋀x. x ∈ S ⟹ onorm (f' x) ≤ B
    and x: x ∈ S
    and y: y ∈ S
  shows norm (f x − f y) ≤ B * norm (x − y)
proof −
  let ?p = λu. x + u *R (y − x)
  let ?φ = λh. h * B * norm (x − y)
  have *: x + u *R (y − x) ∈ S if u ∈ {0..1} for u
  proof −
    have u *R y = u *R (y − x) + u *R x
      by (simp add: scale_right_diff_distrib)
    then show x + u *R (y − x) ∈ S
      using that ‹convex S› x y by (simp add: convex_alt)
        (metis pth_b(2) pth_c(1) scaleR_collapse)
  qed
  have ⋀z. z ∈ (λu. x + u *R (y − x)) ` {0..1} ⟹
        (f has_derivative f' z) (at z within (λu. x + u *R (y − x)) ` {0..1})
    by (auto intro: * has_derivative_subset [OF derf])
  then have continuous_on (?p ` {0..1}) f
    unfolding continuous_on_eq_continuous_within
    by (meson has_derivative_continuous)
  with * have 1: continuous_on {0 .. 1} (f ∘ ?p)
    by (intro continuous_intros)+
  {
    fix u::real assume u: u ∈{0 <..< 1}
    let ?u = ?p u
    interpret linear (f' ?u)
      using u by (auto intro!: has_derivative_linear derf *)
   have (f ∘ ?p has_derivative (f' ?u) ∘ (λu. 0 + u *R (y − x))) (at u within box
0 1)
     by (intro derivative_intros has_derivative_subset [OF derf]) (use u * in auto)
    hence ((f ∘ ?p) has_vector_derivative f' ?u (y − x)) (at u)
       by (simp add: at_within_open[OF u open_greaterThanLessThan] scaleR
has_vector_derivative_def o_def)
  } note 2 = this
  have 3: continuous_on {0..1} ?φ
    by (rule continuous_intros)+
  have 4: (?φ has_vector_derivative B * norm (x − y)) (at u) for u
    by (auto simp: has_vector_derivative_def intro!: derivative_eq_intros)
  {
    fix u::real assume u: u ∈{0 <..< 1}
    let ?u = ?p u
```

```
    interpret bounded_linear (f' ?u)
      using u by (auto intro!: has_derivative_bounded_linear derf *)
    have norm (f' ?u (y - x)) ≤ onorm (f' ?u) * norm (y - x)
      by (rule onorm) (rule bounded_linear)
    also have onorm (f' ?u) ≤ B
      using u by (auto intro!: assms(3)[rule_format] *)
    finally have norm ((f' ?u) (y - x)) ≤ B * norm (x - y)
      by (simp add: mult_right_mono norm_minus_commute)
  } note 5 = this
  have norm (f x - f y) = norm ((f ∘ (λu. x + u *R (y - x))) 1 - (f ∘ (λu. x
+ u *R (y - x))) 0)
    by (auto simp add: norm_minus_commute)
  also
  from differentiable_bound_general[OF zero_less_one 1, OF 3 2 4 5]
  have norm ((f ∘ ?p) 1 - (f ∘ ?p) 0) ≤ B * norm (x - y)
    by simp
  finally show ?thesis .
qed

lemma field_differentiable_bound:
  fixes S :: 'a::real_normed_field set
  assumes cvs: convex S
      and df: ⋀z. z ∈ S ⟹ (f has_field_derivative f' z) (at z within S)
      and dn: ⋀z. z ∈ S ⟹ norm (f' z) ≤ B
      and x ∈ S  y ∈ S
    shows norm(f x - f y) ≤ B * norm(x - y)
proof (rule differentiable_bound [OF cvs])
  show ⋀x. x ∈ S ⟹ (f has_derivative (*) (f' x)) (at x within S)
    by (simp add: df has_field_derivative_imp_has_derivative)
  show ⋀x. x ∈ S ⟹ onorm ((*) (f' x)) ≤ B
    by (metis (no_types, opaque_lifting) dn norm_mult onorm_le order.refl or-
der_trans)
qed (use assms in auto)

lemma
  differentiable_bound_segment:
  fixes f::'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes ⋀t. t ∈ {0..1} ⟹ x0 + t *R a ∈ G
  assumes f': ⋀x. x ∈ G ⟹ (f has_derivative f' x) (at x within G)
  assumes B: ⋀x. x ∈ {0..1} ⟹ onorm (f' (x0 + x *R a)) ≤ B
  shows norm (f (x0 + a) - f x0) ≤ norm a * B
proof -
  let ?G = (λx. x0 + x *R a) ` {0..1}
  have ?G = (+) x0 ` (λx. x *R a) ` {0..1} by auto
  also have convex ...
    by (intro convex_translation convex_scaled convex_real_interval)
  finally have convex ?G .
  moreover have ?G ⊆ G x0 ∈ ?G x0 + a ∈ ?G using assms by (auto intro:
image_eqI[where x=1])
```

ultimately show *?thesis*
using *has_derivative_subset*[*OF f' ‹?G ⊆ G›*] *B*
differentiable_bound[*of* (λx. *x0* + *x* *$_R$ *a*) ‘ {*0..1*} *f f' B x0* + *a x0*]
by (*force simp*: *ac_simps*)
qed

lemma *differentiable_bound_linearization*:
fixes *f*::*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*
assumes *S*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *a* + *t* *$_R$ (*b* − *a*) ∈ *S*
assumes *f'*[*derivative_intros*]: ⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f' x*) (*at x within S*)
assumes *B*: ⋀*x*. *x* ∈ *S* ⟹ *onorm* (*f' x* − *f' x0*) ≤ *B*
assumes *x0* ∈ *S*
shows *norm* (*f b* − *f a* − *f' x0* (*b* − *a*)) ≤ *norm* (*b* − *a*) ∗ *B*
proof −
define *g* **where** [*abs_def*]: *g x* = *f x* − *f' x0 x* **for** *x*
have *g*: ⋀*x*. *x* ∈ *S* ⟹ (*g has_derivative* (λ*i*. *f' x i* − *f' x0 i*)) (*at x within S*)
unfolding *g_def* **using** *assms*
by (*auto intro*!: *derivative_eq_intros*
bounded_linear.has_derivative[*OF has_derivative_bounded_linear*, *OF f'*])
from *B* **have** ∀ *x*∈{*0..1*}. *onorm* (λ*i*. *f'* (*a* + *x* *$_R$ (*b* − *a*)) *i* − *f' x0 i*) ≤ *B*
using *assms* **by** (*auto simp*: *fun_diff_def*)
with *differentiable_bound_segment*[*OF S g*] ‹*x0* ∈ *S*›
show *?thesis*
by (*simp add*: *g_def field_simps linear_diff*[*OF has_derivative_linear*[*OF f'*]])
qed

lemma *vector_differentiable_bound_linearization*:
fixes *f*::*real* ⇒ *'b*::*real_normed_vector*
assumes *f'*: ⋀*x*. *x* ∈ *S* ⟹ (*f has_vector_derivative f' x*) (*at x within S*)
assumes *closed_segment a b* ⊆ *S*
assumes *B*: ⋀*x*. *x* ∈ *S* ⟹ *norm* (*f' x* − *f' x0*) ≤ *B*
assumes *x0* ∈ *S*
shows *norm* (*f b* − *f a* − (*b* − *a*) *$_R$ *f' x0*) ≤ *norm* (*b* − *a*) ∗ *B*
using *assms*
by (*intro differentiable_bound_linearization*[*of a b S f* λ*x h*. *h* *$_R$ *f' x x0 B*])
(*force simp*: *closed_segment_real_eq has_vector_derivative_def*
scaleR_diff_right[*symmetric*] *mult.commute*[*of B*]
intro!: *onorm_le mult_left_mono*)+

In particular.

lemma *has_derivative_zero_constant*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*
assumes *convex s*
and ⋀*x*. *x* ∈ *s* ⟹ (*f has_derivative* (λ*h*. *0*)) (*at x within s*)
shows ∃ *c*. ∀ *x*∈*s*. *f x* = *c*
proof −
{ **fix** *x y* **assume** *x* ∈ *s y* ∈ *s*
then have *norm* (*f x* − *f y*) ≤ *0* ∗ *norm* (*x* − *y*)

```
    using assms by (intro differentiable_bound[of s]) (auto simp: onorm_zero)
  then have f x = f y
    by simp }
 then show ?thesis
  by metis
qed

lemma has_field_derivative_zero_constant:
 assumes convex s ⋀x. x ∈ s ⟹ (f has_field_derivative 0) (at x within s)
 shows  ∃ c. ∀ x∈s. f (x) = (c :: 'a :: real_normed_field)
proof (rule has_derivative_zero_constant)
 have A: (∗) 0 = (λ_. 0 :: 'a) by (intro ext) simp
 fix x assume x ∈ s thus (f has_derivative (λh. 0)) (at x within s)
  using assms(2)[of x] by (simp add: has_field_derivative_def A)
qed fact

lemma
 has_vector_derivative_zero_constant:
 assumes convex s
 assumes ⋀x. x ∈ s ⟹ (f has_vector_derivative 0) (at x within s)
 obtains c where ⋀x. x ∈ s ⟹ f x = c
 using has_derivative_zero_constant[of s f] assms
 by (auto simp: has_vector_derivative_def)

lemma has_derivative_zero_unique:
 fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
 assumes convex s
  and ⋀x. x ∈ s ⟹ (f has_derivative (λh. 0)) (at x within s)
  and x ∈ s y ∈ s
 shows f x = f y
 using has_derivative_zero_constant[OF assms(1,2)] assms(3−) by force

lemma has_derivative_zero_unique_connected:
 fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
 assumes open s connected s
 assumes f: ⋀x. x ∈ s ⟹ (f has_derivative (λx. 0)) (at x)
 assumes x ∈ s y ∈ s
 shows f x = f y
proof (rule connected_local_const[where f=f, OF ‹connected s› ‹x∈s› ‹y∈s›])
 show ∀ a∈s. eventually (λb. f a = f b) (at a within s)
 proof
  fix a assume a ∈ s
  with ‹open s› obtain e where 0 < e ball a e ⊆ s
   by (rule openE)
  then have ∃ c. ∀ x∈ball a e. f x = c
   by (intro has_derivative_zero_constant)
     (auto simp: at_within_open[OF _ open_ball] f)
  with ‹0<e› have ∀ x∈ball a e. f a = f x
   by auto
```

```
    then show eventually (λb. f a = f b) (at a within s)
      using ‹0<e› unfolding eventually_at_topological
      by (intro exI[of _ ball a e]) auto
  qed
qed
```

6.17.12 Differentiability of inverse function (most basic form)

```
lemma has_derivative_inverse_basic:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes derf: (f has_derivative f') (at (g y))
    and ling': bounded_linear g'
    and g' ∘ f' = id
    and contg: continuous (at y) g
    and open T
    and y ∈ T
    and fg: ⋀z. z ∈ T ⟹ f (g z) = z
  shows (g has_derivative g') (at y)
proof −
  interpret f': bounded_linear f'
    using assms unfolding has_derivative_def by auto
  interpret g': bounded_linear g'
    using assms by auto
  obtain C where C: 0 < C ⋀x. norm (g' x) ≤ norm x * C
    using bounded_linear.pos_bounded[OF assms(2)] by blast
  have lem1: ∀ e>0. ∃ d>0. ∀ z.
    norm (z − y) < d ⟶ norm (g z − g y − g'(z − y)) ≤ e * norm (g z − g y)
  proof (intro allI impI)
    fix e :: real
    assume e > 0
    with C(1) have *: e / C > 0 by auto
    obtain d0 where  0 < d0 and d0:
        ⋀u. norm (u − g y) < d0 ⟹ norm (f u − f (g y) − f' (u − g y)) ≤ e /
C * norm (u − g y)
      using derf * unfolding has_derivative_at_alt by blast
    obtain d1 where 0 < d1 and d1: ⋀x. ⟦0 < dist x y; dist x y < d1⟧ ⟹ dist
(g x) (g y) < d0
      using contg ‹0 < d0› unfolding continuous_at Lim_at by blast
    obtain d2 where 0 < d2 and d2: ⋀u. dist u y < d2 ⟹ u ∈ T
      using ‹open T› ‹y ∈ T› unfolding open_dist by blast
    obtain d where d: 0 < d d < d1 d < d2
      using field_lbound_gt_zero[OF ‹0 < d1› ‹0 < d2›] by blast
    show ∃ d>0. ∀ z. norm (z − y) < d ⟶ norm (g z − g y − g' (z − y)) ≤ e *
norm (g z − g y)
    proof (intro exI allI impI conjI)
      fix z
      assume as: norm (z − y) < d
      then have z ∈ T
        using d2 d unfolding dist_norm by auto
```

```
      have norm (g z − g y − g′ (z − y)) ≤ norm (g′ (f (g z) − y − f′ (g z − g
y)))
        unfolding g′.diff f′.diff
       unfolding assms(3)[unfolded o_def id_def, THEN fun_cong] fg[OF ‹z∈T›]
        by (simp add: norm_minus_commute)
      also have ... ≤ norm (f (g z) − y − f′ (g z − g y)) ∗ C
        by (rule C(2))
      also have ... ≤ (e / C) ∗ norm (g z − g y) ∗ C
      proof −
        have norm (g z − g y) < d0
          by (metis as cancel_comm_monoid_add_class.diff_cancel d(2) ‹0 < d0›
d1 diff_gt_0_iff_gt diff_strict_mono dist_norm dist_self zero_less_dist_iff)
        then show ?thesis
          by (metis C(1) ‹y ∈ T› d0 fg mult_le_cancel_right_pos)
      qed
      also have ... ≤ e ∗ norm (g z − g y)
        using C by (auto simp add: field_simps)
      finally show norm (g z − g y − g′ (z − y)) ≤ e ∗ norm (g z − g y)
        by simp
    qed (use d in auto)
  qed
  have ∗: (0::real) < 1 / 2
    by auto
  obtain d where 0 < d and d:
      ⋀z. norm (z − y) < d ⟹ norm (g z − g y − g′ (z − y)) ≤ 1/2 ∗ norm (g
z − g y)
    using lem1 ∗ by blast
  define B where B = C ∗ 2
  have B > 0
    unfolding B_def using C by auto
  have lem2: norm (g z − g y) ≤ B ∗ norm (z − y) if z: norm(z − y) < d for z
  proof −
    have norm (g z − g y) ≤ norm(g′ (z − y)) + norm ((g z − g y) − g′(z − y))
      by (rule norm_triangle_sub)
    also have ... ≤ norm (g′ (z − y)) + 1 / 2 ∗ norm (g z − g y)
      by (rule add_left_mono) (use d z in auto)
    also have ... ≤ norm (z − y) ∗ C + 1 / 2 ∗ norm (g z − g y)
      by (rule add_right_mono) (use C in auto)
    finally show norm (g z − g y) ≤ B ∗ norm (z − y)
      unfolding B_def
      by (auto simp add: field_simps)
  qed
  show ?thesis
    unfolding has_derivative_at_alt
  proof (intro conjI assms allI impI)
    fix e :: real
    assume e > 0
    then have ∗: e / B > 0 by (metis ‹B > 0› divide_pos_pos)
    obtain d′ where 0 < d′ and d′:
```

```
      ⋀z. norm (z − y) < d' ⟹ norm (g z − g y − g' (z − y)) ≤ e / B * norm
(g z − g y)
      using lem1 * by blast
    obtain k where k: 0 < k k < d k < d'
      using field_lbound_gt_zero[OF ‹0 < d› ‹0 < d'›] by blast
    show ∃ d>0. ∀ ya. norm (ya − y) < d ⟶ norm (g ya − g y − g' (ya − y))
≤ e * norm (ya − y)
    proof (intro exI allI impI conjI)
      fix z
      assume as: norm (z − y) < k
      then have norm (g z − g y − g' (z − y)) ≤ e / B * norm(g z − g y)
        using d' k by auto
      also have ... ≤ e * norm (z − y)
        unfolding times_divide_eq_left pos_divide_le_eq[OF ‹B>0›]
        using lem2[of z] k as ‹e > 0›
        by (auto simp add: field_simps)
      finally show norm (g z − g y − g' (z − y)) ≤ e * norm (z − y)
        by simp
    qed (use k in auto)
  qed
qed
```

Inverse function theorem for complex derivatives

```
lemma has_field_derivative_inverse_basic:
  shows DERIV f (g y) :> f' ⟹
        f' ≠ 0 ⟹
        continuous (at y) g ⟹
        open t ⟹
        y ∈ t ⟹
        (⋀z. z ∈ t ⟹ f (g z) = z)
        ⟹ DERIV g y :> inverse (f')
  unfolding has_field_derivative_def
  by (rule has_derivative_inverse_basic) (auto simp: bounded_linear_mult_right)
```

Simply rewrite that based on the domain point x.

```
lemma has_derivative_inverse_basic_x:
  fixes f :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes (f has_derivative f') (at x)
    and bounded_linear g'
    and g' ∘ f' = id
    and continuous (at (f x)) g
    and g (f x) = x
    and open T
    and f x ∈ T
    and ⋀y. y ∈ T ⟹ f (g y) = y
  shows (g has_derivative g') (at (f x))
  by (rule has_derivative_inverse_basic) (use assms in auto)
```

This is the version in Dieudonne', assuming continuity of f and g.

lemma *has_derivative_inverse_dieudonne*:
fixes f :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *open S*
and *fS*: *open* (*f* ‘ *S*)
and *A*: *continuous_on S f continuous_on* (*f* ‘ *S*) *g*
⋀*x*. *x* ∈ *S* ⟹ *g* (*f x*) = *x x* ∈ *S*
and *B*: (*f has_derivative f'*) (*at x*) *bounded_linear g' g'* ∘ *f'* = *id*
shows (*g has_derivative g'*) (*at* (*f x*))
using *A fS continuous_on_eq_continuous_at*
by (*intro has_derivative_inverse_basic_x*[*OF B* _ _ *fS*]) *force*+

Here's the simplest way of not assuming much about g.

proposition *has_derivative_inverse*:
fixes f :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *compact S*
and *x* ∈ *S*
and *fx*: *f x* ∈ *interior* (*f* ‘ *S*)
and *continuous_on S f*
and *gf*: ⋀*y*. *y* ∈ *S* ⟹ *g* (*f y*) = *y*
and *B*: (*f has_derivative f'*) (*at x*) *bounded_linear g' g'* ∘ *f'* = *id*
shows (*g has_derivative g'*) (*at* (*f x*))
proof −
have ∗: ⋀*y*. *y* ∈ *interior* (*f* ‘ *S*) ⟹ *f* (*g y*) = *y*
by (*metis gf image_iff interior_subset subsetCE*)
show *?thesis*
using *assms* ∗ *continuous_on_interior continuous_on_inv fx*
by (*intro has_derivative_inverse_basic_x*[*OF B*, **where** *T* = *interior* (*f*‘*S*)])
blast+
qed

Invertible derivative continuous at a point implies local injectivity. It's only for this we need continuity of the derivative, except of course if we want the fact that the inverse derivative is also continuous. So if we know for some other reason that the inverse function exists, it's OK.

proposition *has_derivative_locally_injective*:
fixes f :: *'n::euclidean_space* ⇒ *'m::euclidean_space*
assumes *a* ∈ *S*
and *open S*
and *bling*: *bounded_linear g'*
and *g' f' a* = *id*
and *derf*: ⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f' x*) (*at x*)
and ⋀*e*. *e* > *0* ⟹ ∃ *d*>*0*. ∀ *x*. *dist a x* < *d* ⟶ *onorm* (λ*v*. *f' x v* − *f' a v*)
< *e*
obtains *r* **where** *r* > *0 ball a r* ⊆ *S inj_on f* (*ball a r*)
proof −
interpret *bounded_linear g'*
using *assms* **by** *auto*
note *f'g'* = *assms*(*4*)[*unfolded id_def o_def*,*THEN cong*]
have *g'* (*f' a* (∑ *Basis*)) = (∑ *Basis*) (∑ *Basis*) ≠ (*0*::*'n*)

```
  using f'g' by auto
 then have *: 0 < onorm g'
  unfolding onorm_pos_lt[OF assms(3)]
  by fastforce
 define k where k = 1 / onorm g' / 2
 have *: k > 0
  unfolding k_def using * by auto
 obtain d1 where d1:
    0 < d1
    ⋀x. dist a x < d1 ⟹ onorm (λv. f' x v − f' a v) < k
  using assms(6) * by blast
 from ‹open S› obtain d2 where d2 > 0 ball a d2 ⊆ S
  using ‹a∈S› ..
 obtain d2 where d2: 0 < d2 ball a d2 ⊆ S
  using ‹0 < d2› ‹ball a d2 ⊆ S› by blast
 obtain d where d: 0 < d d < d1 d < d2
  using field_lbound_gt_zero[OF d1(1) d2(1)] by blast
 show ?thesis
 proof
  show 0 < d by (fact d)
  show ball a d ⊆ S
   using ‹d < d2› ‹ball a d2 ⊆ S› by auto
  show inj_on f (ball a d)
  unfolding inj_on_def
  proof (intro strip)
   fix x y
   assume as: x ∈ ball a d y ∈ ball a d f x = f y
   define ph where [abs_def]: ph w = w − g' (f w − f x) for w
   have ph':ph = g' ∘ (λw. f' a w − (f w − f x))
    unfolding ph_def o_def  by (simp add: diff f'g')
   have norm (ph x − ph y) ≤ (1 / 2) * norm (x − y)
   proof (rule differentiable_bound[OF convex_ball _ _ as(1−2)])
    fix u
    assume u: u ∈ ball a d
    then have u ∈ S
     using d d2 by auto
    have *: (λv. v − g' (f' u v)) = g' ∘ (λw. f' a w − f' u w)
     unfolding o_def and diff
     using f'g' by auto
    have blin: bounded_linear (f' a)
     using ‹a ∈ S› derf by blast
    show (ph has_derivative (λv. v − g' (f' u v))) (at u within ball a d)
     unfolding ph' * comp_def
      by (rule ‹u ∈ S› derivative_eq_intros has_derivative_at_withinI [OF
derf] bounded_linear.has_derivative [OF blin] bounded_linear.has_derivative [OF
bling] |simp)+
     have **: bounded_linear (λx. f' u x − f' a x) bounded_linear (λx. f' a x
− f' u x)
      using ‹u ∈ S› blin bounded_linear_sub derf by auto
```

```
      then have onorm (λv. v − g′ (f′ u v)) ≤ onorm g′ ∗ onorm (λw. f′ a w −
f′ u w)
        by (simp add: ∗ bounded_linear_axioms onorm_compose)
      also have ... ≤ onorm g′ ∗ k
        apply (rule mult_left_mono)
        using d1(2)[of u]
         using onorm_neg[where f=λx. f′ u x − f′ a x] d u onorm_pos_le[OF
bling]
         apply (auto simp: algebra_simps)
        done
      also have ... ≤ 1 / 2
        unfolding k_def by auto
      finally show onorm (λv. v − g′ (f′ u v)) ≤ 1 / 2 .
    qed
    moreover have norm (ph y − ph x) = norm (y − x)
      by (simp add: as(3) ph_def)
    ultimately show x = y
      unfolding norm_minus_commute by auto
  qed
 qed
qed
```

6.17.13 Uniformly convergent sequence of derivatives

```
lemma has_derivative_sequence_lipschitz_lemma:
 fixes f :: nat ⇒ 'a::real_normed_vector ⇒ 'b::real_normed_vector
 assumes convex S
   and derf: ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f′ n x)) (at x within S)
   and nle: ⋀n x h. ⟦n≥N; x ∈ S⟧ ⟹ norm (f′ n x h − g′ x h) ≤ e ∗ norm h
   and 0 ≤ e
 shows ∀ m≥N. ∀ n≥N. ∀ x∈S. ∀ y∈S. norm ((f m x − f n x) − (f m y − f n y))
≤ 2 ∗ e ∗ norm (x − y)
proof clarify
 fix m n x y
 assume as: N ≤ m N ≤ n x ∈ S y ∈ S
 show norm ((f m x − f n x) − (f m y − f n y)) ≤ 2 ∗ e ∗ norm (x − y)
 proof (rule differentiable_bound[where f′=λx h. f′ m x h − f′ n x h, OF ‹convex
S› _ _ as(3−4)])
   fix x
   assume x ∈ S
    show ((λa. f m a − f n a) has_derivative (λh. f′ m x h − f′ n x h)) (at x
within S)
     by (rule derivative_intros derf ‹x∈S›)+
   show onorm (λh. f′ m x h − f′ n x h) ≤ 2 ∗ e
   proof (rule onorm_bound)
     fix h
     have norm (f′ m x h − f′ n x h) ≤ norm (f′ m x h − g′ x h) + norm (f′ n
x h − g′ x h)
       using norm_triangle_ineq[of f′ m x h − g′ x h − f′ n x h + g′ x h]
```

```
      by (auto simp add: algebra_simps norm_minus_commute)
    also have ... ≤ e * norm h + e * norm h
      using nle[OF ‹N ≤ m› ‹x ∈ S›, of h] nle[OF ‹N ≤ n› ‹x ∈ S›, of h]
      by (auto simp add: field_simps)
    finally show norm (f' m x h − f' n x h) ≤ 2 * e * norm h
      by auto
  qed (simp add: ‹0 ≤ e›)
 qed
qed

lemma has_derivative_sequence_Lipschitz:
 fixes f :: nat ⇒ 'a::real_normed_vector ⇒ 'b::real_normed_vector
 assumes convex S
   and ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f' n x)) (at x within S)
   and nle: ⋀e. e > 0 ⟹ ∀F n in sequentially. ∀x∈S. ∀h. norm (f' n x h − g'
x h) ≤ e * norm h
   and e > 0
 shows ∃N. ∀m≥N. ∀n≥N. ∀x∈S. ∀y∈S.
   norm ((f m x − f n x) − (f m y − f n y)) ≤ e * norm (x − y)
proof −
 have ∗: 2 * (e/2) = e
   using ‹e > 0› by auto
 obtain N where ∀n≥N. ∀x∈S. ∀h. norm (f' n x h − g' x h) ≤ (e/2) * norm
h
   using nle ‹e > 0›
   unfolding eventually_sequentially
   by (metis less_divide_eq_numeral1(1) mult_zero_left)
 then show ∃N. ∀m≥N. ∀n≥N. ∀x∈S. ∀y∈S. norm (f m x − f n x − (f m y
− f n y)) ≤ e * norm (x − y)
   apply (rule_tac x=N in exI)
  apply (rule has_derivative_sequence_lipschitz_lemma[where e=e/2, unfolded
∗])
   using assms ‹e > 0›
   apply auto
   done
qed

proposition has_derivative_sequence:
 fixes f :: nat ⇒ 'a::real_normed_vector ⇒ 'b::banach
 assumes convex S
   and derf: ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f' n x)) (at x within S)
   and nle: ⋀e. e > 0 ⟹ ∀F n in sequentially. ∀x∈S. ∀h. norm (f' n x h − g'
x h) ≤ e * norm h
   and x0 ∈ S
   and lim: ((λn. f n x0) ⟶ l) sequentially
 shows ∃g. ∀x∈S. (λn. f n x) ⟶ g x ∧ (g has_derivative g'(x)) (at x within
S)
proof −
 have lem1: ⋀e. e > 0 ⟹ ∃N. ∀m≥N. ∀n≥N. ∀x∈S. ∀y∈S.
```

```
    norm ((f m x − f n x) − (f m y − f n y)) ≤ e ∗ norm (x − y)
   using assms(1,2,3) by (rule has_derivative_sequence_Lipschitz)
  have ∃ g. ∀ x∈S. ((λn. f n x) ⟶ g x) sequentially
  proof (intro ballI bchoice)
   fix x
   assume x ∈ S
   show ∃ y. (λn. f n x) ⟶ y
   unfolding convergent_eq_Cauchy
   proof (cases x = x0)
    case True
    then show Cauchy (λn. f n x)
     using LIMSEQ_imp_Cauchy[OF lim] by auto
   next
    case False
    show Cauchy (λn. f n x)
     unfolding Cauchy_def
    proof (intro allI impI)
     fix e :: real
     assume e > 0
     hence ∗: e / 2 > 0 e / 2 / norm (x − x0) > 0 using False by auto
     obtain M where M: ∀ m≥M. ∀ n≥M. dist (f m x0) (f n x0) < e / 2
      using LIMSEQ_imp_Cauchy[OF lim] ∗ unfolding Cauchy_def by blast
     obtain N where N:
      ∀ m≥N. ∀ n≥N.
       ∀ u∈S. ∀ y∈S. norm (f m u − f n u − (f m y − f n y)) ≤
         e / 2 / norm (x − x0) ∗ norm (u − y)
     using lem1 ∗(2) by blast
     show ∃ M. ∀ m≥M. ∀ n≥M. dist (f m x) (f n x) < e
     proof (intro exI allI impI)
      fix m n
      assume as: max M N ≤m max M N≤n
      have dist (f m x) (f n x) ≤ norm (f m x0 − f n x0) + norm (f m x − f n
x − (f m x0 − f n x0))
       unfolding dist_norm
       by (rule norm_triangle_sub)
      also have ... ≤ norm (f m x0 − f n x0) + e / 2
       using N ‹x∈S› ‹x0∈S› as False by fastforce
      also have ... < e / 2 + e / 2
       by (rule add_strict_right_mono) (use as M in ‹auto simp: dist_norm›)
      finally show dist (f m x) (f n x) < e
       by auto
     qed
    qed
   qed
  qed
  then obtain g where g: ∀ x∈S. (λn. f n x) ⟶ g x ..
  have lem2: ∃ N. ∀ n≥N. ∀ x∈S. ∀ y∈S. norm ((f n x − f n y) − (g x − g y)) ≤
e ∗ norm (x − y) if e > 0 for e
  proof −
```

```
    obtain N where
      N: ∀ m≥N. ∀ n≥N. ∀ x∈S. ∀ y∈S. norm (f m x − f n x − (f m y − f n y))
≤ e ∗ norm (x − y)
      using lem1 ‹e > 0› by blast
    show ∃ N. ∀ n≥N. ∀ x∈S. ∀ y∈S. norm (f n x − f n y − (g x − g y)) ≤ e ∗
norm (x − y)
    proof (intro exI ballI allI impI)
      fix n x y
      assume as: N ≤ n x ∈ S y ∈ S
      have ((λm. norm (f n x − f n y − (f m x − f m y))) ⟶ norm (f n x − f
n y − (g x − g y))) sequentially
        by (intro tendsto_intros g[rule_format] as)
      moreover have eventually (λm. norm (f n x − f n y − (f m x − f m y)) ≤
e ∗ norm (x − y)) sequentially
        unfolding eventually_sequentially
      proof (intro exI allI impI)
        fix m
        assume N ≤ m
        then show norm (f n x − f n y − (f m x − f m y)) ≤ e ∗ norm (x − y)
          using N as by (auto simp add: algebra_simps)
      qed
      ultimately show norm (f n x − f n y − (g x − g y)) ≤ e ∗ norm (x − y)
        by (simp add: tendsto_upperbound)
    qed
  qed
  have ∀ x∈S. ((λn. f n x) ⟶ g x) sequentially ∧ (g has_derivative g′ x) (at x
within S)
    unfolding has_derivative_within_alt2
  proof (intro ballI conjI allI impI)
    fix x
    assume x ∈ S
    then show (λn. f n x) ⟶ g x
      by (simp add: g)
    have tog′: (λn. f′ n x u) ⟶ g′ x u for u
      unfolding filterlim_def le_nhds_metric_le eventually_filtermap dist_norm
    proof (intro allI impI)
      fix e :: real
      assume e > 0
      show eventually (λn. norm (f′ n x u − g′ x u) ≤ e) sequentially
      proof (cases u = 0)
        case True
        have eventually (λn. norm (f′ n x u − g′ x u) ≤ e ∗ norm u) sequentially
          using nle ‹0 < e› ‹x ∈ S› by (fast elim: eventually_mono)
        then show ?thesis
          using ‹u = 0› ‹0 < e› by (auto elim: eventually_mono)
      next
        case False
        with ‹0 < e› have 0 < e / norm u by simp
        then have eventually (λn. norm (f′ n x u − g′ x u) ≤ e / norm u ∗ norm
```

```
u) sequentially
        using nle ‹x ∈ S› by (fast elim: eventually_mono)
      then show ?thesis
        using ‹u ≠ 0› by simp
    qed
  qed
  show bounded_linear (g' x)
  proof
    fix x' y z :: 'a
    fix c :: real
  note lin = assms(2)[rule_format,OF ‹x∈S›,THEN has_derivative_bounded_linear]
    have (λn. f' n x (c *R x')) ⟶ c *R g' x x'
      unfolding lin[THEN bounded_linear.linear, THEN linear_cmul]
      by (intro tendsto_intros tog')
    then show g' x (c *R x') = c *R g' x x'
      using LIMSEQ_unique tog' by blast
    have (λn. f' n x (y + z)) ⟶ g' x y + g' x z
      unfolding lin[THEN bounded_linear.linear, THEN linear_add]
      by (simp add: tendsto_add tog')
    then show g' x (y + z) = g' x y + g' x z
      using LIMSEQ_unique tog' by blast
    obtain N where N: ∀h. norm (f' N x h − g' x h) ≤ 1 * norm h
        using nle ‹x ∈ S› unfolding eventually_sequentially by (fast intro:
zero_less_one)
    have bounded_linear (f' N x)
      using derf ‹x ∈ S› by fast
    from bounded_linear.bounded [OF this]
    obtain K where K: ∀h. norm (f' N x h) ≤ norm h * K ..
    {
      fix h
      have norm (g' x h) = norm (f' N x h − (f' N x h − g' x h))
        by simp
      also have ... ≤ norm (f' N x h) + norm (f' N x h − g' x h)
        by (rule norm_triangle_ineq4)
      also have ... ≤ norm h * K + 1 * norm h
        using N K by (fast intro: add_mono)
      finally have norm (g' x h) ≤ norm h * (K + 1)
        by (simp add: ring_distribs)
    }
    then show ∃K. ∀h. norm (g' x h) ≤ norm h * K by fast
  qed
  show eventually (λy. norm (g y − g x − g' x (y − x)) ≤ e * norm (y − x))
(at x within S)
    if e > 0 for e
  proof −
    have *: e / 3 > 0
      using that by auto
    obtain N1 where N1: ∀n≥N1. ∀x∈S. ∀h. norm (f' n x h − g' x h) ≤ e /
3 * norm h
```

```
      using nle * unfolding eventually_sequentially by blast
    obtain N2 where
      N2[rule_format]: ∀ n≥N2. ∀ x∈S. ∀ y∈S. norm (f n x - f n y - (g x - g y)) ≤ e / 3 * norm (x - y)
      using lem2 * by blast
    let ?N = max N1 N2
    have eventually (λy. norm (f ?N y - f ?N x - f' ?N x (y - x)) ≤ e / 3 * norm (y - x)) (at x within S)
      using derf[unfolded has_derivative_within_alt2] and ‹x ∈ S› and * by fast
    moreover have eventually (λy. y ∈ S) (at x within S)
      unfolding eventually_at by (fast intro: zero_less_one)
    ultimately show ∀ F y in at x within S. norm (g y - g x - g' x (y - x)) ≤ e * norm (y - x)
    proof (rule eventually_elim2)
      fix y
      assume y ∈ S
     assume norm (f ?N y - f ?N x - f' ?N x (y - x)) ≤ e / 3 * norm (y - x)
      moreover have norm (g y - g x - (f ?N y - f ?N x)) ≤ e / 3 * norm (y - x)
        using N2[OF _ ‹y ∈ S› ‹x ∈ S›]
        by (simp add: norm_minus_commute)
      ultimately have norm (g y - g x - f' ?N x (y - x)) ≤ 2 * e / 3 * norm (y - x)
        using norm_triangle_le[of g y - g x - (f ?N y - f ?N x) f ?N y - f ?N x - f' ?N x (y - x) 2 * e / 3 * norm (y - x)]
        by (auto simp add: algebra_simps)
      moreover
      have  norm (f' ?N x (y - x) - g' x (y - x)) ≤ e / 3 * norm (y - x)
        using N1 ‹x ∈ S› by auto
      ultimately show norm (g y - g x - g' x (y - x)) ≤ e * norm (y - x)
       using norm_triangle_le[of g y - g x - f' (max N1 N2) x (y - x) f' (max N1 N2) x (y - x) - g' x (y - x)]
        by (auto simp add: algebra_simps)
    qed
  qed
 qed
 then show ?thesis by fast
qed
```

Can choose to line up antiderivatives if we want.

```
lemma has_antiderivative_sequence:
 fixes f :: nat ⇒ 'a::real_normed_vector ⇒ 'b::banach
 assumes convex S
   and der: ⋀n x. x ∈ S ⟹ ((f n) has_derivative (f' n x)) (at x within S)
   and no: ⋀e. e > 0 ⟹ ∀ F n in sequentially.
     ∀ x∈S. ∀ h. norm (f' n x h - g' x h) ≤ e * norm h
 shows ∃ g. ∀ x∈S. (g has_derivative g' x) (at x within S)
proof (cases S = {})
```

case *False*
then obtain a **where** $a \in S$
by *auto*
have $*$: $\bigwedge P\ Q.\ \exists g.\ \forall x{\in}S.\ P\ g\ x \wedge Q\ g\ x \Longrightarrow \exists g.\ \forall x{\in}S.\ Q\ g\ x$
by *auto*
show *?thesis*
apply (*rule* $*$)
apply (*rule has_derivative_sequence* [*OF* ‹*convex S*› _ *no*, *of* $\lambda n\ x.\ f\ n\ x + (f\ 0\ a - f\ n\ a)$])
apply (*metis assms*(2) *has_derivative_add_const*)
using ‹$a \in S$›
apply *auto*
done
qed *auto*

lemma *has_antiderivative_limit*:
fixes g' :: *'a*::*real_normed_vector* ⇒ *'a* ⇒ *'b*::*banach*
assumes *convex S*
and $\bigwedge e.\ e{>}0 \Longrightarrow \exists f\ f'.\ \forall x{\in}S.$
$(f$ *has_derivative* $(f'\ x))$ $(at\ x\ within\ S) \wedge (\forall h.\ norm\ (f'\ x\ h - g'\ x\ h) \leq e * norm\ h)$
shows $\exists g.\ \forall x{\in}S.$ $(g$ *has_derivative* $g'\ x)$ $(at\ x\ within\ S)$
proof −
have $*$: $\forall n.\ \exists f\ f'.\ \forall x{\in}S.$
$(f$ *has_derivative* $(f'\ x))$ $(at\ x\ within\ S) \wedge$
$(\forall h.\ norm(f'\ x\ h - g'\ x\ h) \leq inverse\ (real\ (Suc\ n)) * norm\ h)$
by (*simp add*: *assms*(2))
obtain f **where**
$*$: $\bigwedge x.\ \exists f'.\ \forall xa{\in}S.\ (f\ x$ *has_derivative* $f'\ xa)$ $(at\ xa\ within\ S) \wedge$
$(\forall h.\ norm\ (f'\ xa\ h - g'\ xa\ h) \leq inverse\ (real\ (Suc\ x)) * norm\ h)$
using $*$ **by** *metis*
obtain f' **where**
f': $\bigwedge x.\ \forall z{\in}S.\ (f\ x$ *has_derivative* $f'\ x\ z)$ $(at\ z\ within\ S) \wedge$
$(\forall h.\ norm\ (f'\ x\ z\ h - g'\ z\ h) \leq inverse\ (real\ (Suc\ x)) * norm\ h)$
using $*$ **by** *metis*
show *?thesis*
proof (*rule has_antiderivative_sequence*[*OF* ‹*convex S*›, *of f f'*])
fix e :: *real*
assume $e > 0$
obtain N **where** N: $inverse\ (real\ (Suc\ N)) < e$
using *reals_Archimedean*[*OF* ‹$e{>}0$›] **..**
show $\forall_F\ n$ *in sequentially*. $\forall x{\in}S.\ \forall h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq e * norm\ h$
unfolding *eventually_sequentially*
proof (*intro exI allI ballI impI*)
fix $n\ x\ h$
assume n: $N \leq n$ **and** x: $x \in S$
have $*$: $inverse\ (real\ (Suc\ n)) \leq e$
using $n\ N$
by (*smt* (*verit*, *best*) *le_imp_inverse_le of_nat_0_less_iff of_nat_Suc*

of_nat_le_iff zero_less_Suc)
    show *norm* $(f'\ n\ x\ h - g'\ x\ h) \leq e * norm\ h$
      by (*meson* $*$ *mult_right_mono norm_ge_zero order.trans x f'*)
  qed
 qed (*use f'* **in** *auto*)
qed

6.17.14 Differentiation of a series

proposition *has_derivative_series*:
 fixes *f* :: *nat* ⇒ *'a*::*real_normed_vector* ⇒ *'b*::*banach*
 assumes *convex S*
   and $\bigwedge n\ x.\ x \in S \Longrightarrow ((f\ n)\ has\_derivative\ (f'\ n\ x))\ (at\ x\ within\ S)$
   and $\bigwedge e.\ e>0 \Longrightarrow \forall_F\ n\ in\ sequentially.\ \forall x{\in}S.\ \forall h.\ norm\ (sum\ (\lambda i.\ f'\ i\ x\ h)\ \{..<n\} - g'\ x\ h) \leq e * norm\ h$
   and *x* ∈ *S*
   and (λ*n*. *f n x*) *sums l*
 shows $\exists g.\ \forall x{\in}S.\ (\lambda n.\ f\ n\ x)\ sums\ (g\ x) \wedge (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
 unfolding *sums_def*
 apply (*rule has_derivative_sequence*[*OF assms*(*1*) _ *assms*(*3*)])
 apply (*metis assms*(*2*) *has_derivative_sum*)
 using *assms*(*4*−*5*)
 unfolding *sums_def*
 apply *auto*
 done

lemma *has_field_derivative_series*:
 fixes *f* :: *nat* ⇒ (*'a* :: {*real_normed_field*,*banach*}) ⇒ *'a*
 assumes *convex S*
 assumes $\bigwedge n\ x.\ x \in S \Longrightarrow (f\ n\ has\_field\_derivative\ f'\ n\ x)\ (at\ x\ within\ S)$
 assumes *uniform_limit S* (λ*n x*. $\sum i{<}n.\ f'\ i\ x$) *g' sequentially*
 assumes *x0* ∈ *S summable* (λ*n*. *f n x0*)
 shows $\exists g.\ \forall x{\in}S.\ (\lambda n.\ f\ n\ x)\ sums\ g\ x \wedge (g\ has\_field\_derivative\ g'\ x)\ (at\ x\ within\ S)$
unfolding *has_field_derivative_def*
proof (*rule has_derivative_series*)
 show $\forall_F\ n\ in\ sequentially.$
     $\forall x{\in}S.\ \forall h.\ norm\ ((\sum i{<}n.\ f'\ i\ x * h) - g'\ x * h) \leq e * norm\ h$ **if** $e > 0$ **for** *e*
   unfolding *eventually_sequentially*
 proof −
   from *that assms*(*3*) **obtain** *N* **where** *N*: $\bigwedge n\ x.\ n \geq N \Longrightarrow x \in S \Longrightarrow norm\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) < e$
    unfolding *uniform_limit_iff eventually_at_top_linorder dist_norm* **by** *blast*
   {
    fix *n* :: *nat* **and** *x h* :: *'a* **assume** *nx*: $n \geq N\ x \in S$
    have $norm\ ((\sum i{<}n.\ f'\ i\ x * h) - g'\ x * h) = norm\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) * norm\ h$

by (*simp add*: *norm_mult* [*symmetric*] *ring_distribs sum_distrib_right*)
also from *N*[*OF nx*] **have** *norm* ((∑ *i*<*n*. *f'* *i* *x*) − *g'* *x*) ≤ *e* **by** *simp*
hence *norm* ((∑ *i*<*n*. *f'* *i* *x*) − *g'* *x*) ∗ *norm h* ≤ *e* ∗ *norm h*
by (*intro mult_right_mono*) *simp_all*
finally have *norm* ((∑ *i*<*n*. *f'* *i* *x* ∗ *h*) − *g'* *x* ∗ *h*) ≤ *e* ∗ *norm h* **.**
}
thus ∃ *N*. ∀ *n*≥*N*. ∀ *x*∈*S*. ∀ *h*. *norm* ((∑ *i*<*n*. *f'* *i* *x* ∗ *h*) − *g'* *x* ∗ *h*) ≤ *e* ∗ *norm h* **by** *blast*
qed
qed (*use assms* **in** ‹*auto simp*: *has_field_derivative_def*›)

lemma *has_field_derivative_series'*:
fixes *f* :: *nat* ⇒ ('*a* :: {*real_normed_field*,*banach*}) ⇒ '*a*
assumes *convex S*
assumes ⋀*n x*. *x* ∈ *S* ⟹ (*f n has_field_derivative f' n x*) (*at x within S*)
assumes *uniformly_convergent_on S* (λ*n x*. ∑ *i*<*n*. *f'* *i* *x*)
assumes *x0* ∈ *S summable* (λ*n*. *f n x0*) *x* ∈ *interior S*
shows *summable* (λ*n*. *f n x*) ((λ*x*. ∑ *n*. *f n x*) *has_field_derivative* (∑ *n*. *f'* *n x*)) (*at x*)
proof −
from ‹*x* ∈ *interior S*› **have** *x* ∈ *S* **using** *interior_subset* **by** *blast*
define *g'* **where** [*abs_def*]: *g'* *x* = (∑ *i*. *f'* *i* *x*) **for** *x*
from *assms*(*3*) **have** *uniform_limit S* (λ*n x*. ∑ *i*<*n*. *f'* *i* *x*) *g' sequentially*
by (*simp add*: *uniformly_convergent_uniform_limit_iff suminf_eq_lim g'_def*)
from *has_field_derivative_series*[*OF assms*(*1*,*2*) *this assms*(*4*,*5*)] **obtain** *g* **where** *g*:
⋀*x*. *x* ∈ *S* ⟹ (λ*n*. *f n x*) *sums g x*
⋀*x*. *x* ∈ *S* ⟹ (*g has_field_derivative g' x*) (*at x within S*) **by** *blast*
from *g*(*1*)[*OF* ‹*x* ∈ *S*›] **show** *summable* (λ*n*. *f n x*) **by** (*simp add*: *sums_iff*)
from *g*(*2*)[*OF* ‹*x* ∈ *S*›] ‹*x* ∈ *interior S*› **have** (*g has_field_derivative g' x*) (*at x*)
by (*simp add*: *at_within_interior*[*of x S*])
also have (*g has_field_derivative g' x*) (*at x*) ⟷
((λ*x*. ∑ *n*. *f n x*) *has_field_derivative g' x*) (*at x*)
using *eventually_nhds_in_nhd*[*OF* ‹*x* ∈ *interior S*›] *interior_subset*[*of S*] *g*(*1*)
by (*intro DERIV_cong_ev*) (*auto elim*!: *eventually_mono simp*: *sums_iff*)
finally show ((λ*x*. ∑ *n*. *f n x*) *has_field_derivative g' x*) (*at x*) **.**
qed

lemma *differentiable_series*:
fixes *f* :: *nat* ⇒ ('*a* :: {*real_normed_field*,*banach*}) ⇒ '*a*
assumes *convex S open S*
assumes ⋀*n x*. *x* ∈ *S* ⟹ (*f n has_field_derivative f' n x*) (*at x*)
assumes *uniformly_convergent_on S* (λ*n x*. ∑ *i*<*n*. *f'* *i* *x*)
assumes *x0* ∈ *S summable* (λ*n*. *f n x0*) **and** *x*: *x* ∈ *S*
shows *summable* (λ*n*. *f n x*) **and** (λ*x*. ∑ *n*. *f n x*) *differentiable* (*at x*)
proof −
from *assms*(*4*) **obtain** *g'* **where** *A*: *uniform_limit S* (λ*n x*. ∑ *i*<*n*. *f'* *i* *x*) *g' sequentially*

unfolding *uniformly_convergent_on_def* **by** *blast*
from *x* **and** ‹*open S*› **have** *S*: *at x within S* = *at x* **by** (*rule at_within_open*)
have ∃ *g*. ∀ *x*∈*S*. (λ*n*. *f n x*) *sums g x* ∧ (*g has_field_derivative g' x*) (*at x within S*)
by (*intro has_field_derivative_series*[*of S f f' g' x0*] *assms A has_field_derivative_at_within*)
then obtain *g* **where** *g*: ⋀*x*. *x* ∈ *S* ⟹ (λ*n*. *f n x*) *sums g x*
⋀*x*. *x* ∈ *S* ⟹ (*g has_field_derivative g' x*) (*at x within S*) **by** *blast*
from *g*[*OF x*] **show** *summable* (λ*n*. *f n x*) **by** (*auto simp*: *summable_def*)
from *g*(*2*)[*OF x*] **have** *g'*: (*g has_derivative* (∗)) (*g' x*)) (*at x*)
by (*simp add*: *has_field_derivative_def S*)
have ((λ*x*. ∑ *n*. *f n x*) *has_derivative* (∗)) (*g' x*)) (*at x*)
by (*rule has_derivative_transform_within_open*[*OF g' ‹open S› x*])
(*insert g*, *auto simp*: *sums_iff*)
thus (λ*x*. ∑ *n*. *f n x*) *differentiable* (*at x*) **unfolding** *differentiable_def*
by (*auto simp*: *summable_def differentiable_def has_field_derivative_def*)
qed

lemma *differentiable_series'*:
fixes *f* :: *nat* ⇒ ('*a* :: {*real_normed_field*,*banach*}) ⇒ '*a*
assumes *convex S open S*
assumes ⋀*n x*. *x* ∈ *S* ⟹ (*f n has_field_derivative f' n x*) (*at x*)
assumes *uniformly_convergent_on S* (λ*n x*. ∑ *i*<*n*. *f' i x*)
assumes *x0* ∈ *S summable* (λ*n*. *f n x0*)
shows (λ*x*. ∑ *n*. *f n x*) *differentiable* (*at x0*)
using *differentiable_series*[*OF assms*, *of x0*] ‹*x0* ∈ *S*› **by** *blast*+

6.17.15 Derivative as a vector

Considering derivative *real* ⇒ '*b* as a vector.

definition *vector_derivative f net* = (*SOME f'*. (*f has_vector_derivative f'*) *net*)

lemma *vector_derivative_unique_within*:
assumes *not_bot*: *at x within S* ≠ *bot*
and *f'*: (*f has_vector_derivative f'*) (*at x within S*)
and *f''*: (*f has_vector_derivative f''*) (*at x within S*)
shows *f'* = *f''*
proof −
have (λ*x*. *x* ∗*R f'*) = (λ*x*. *x* ∗*R f''*)
proof (*rule frechet_derivative_unique_within*, *simp_all*)
show ∃ *d*. *d* ≠ *0* ∧ |*d*| < *e* ∧ *x* + *d* ∈ *S* **if** *0* < *e* **for** *e*
proof −
from *that*
obtain *x'* **where** *x'* ∈ *S x'* ≠ *x* |*x'* − *x*| < *e*
using *islimpt_approachable_real*[*of x S*] *not_bot*
by (*auto simp add*: *trivial_limit_within*)
then show *?thesis*
using *eq_iff_diff_eq_0* **by** *fastforce*
qed
qed (*use f' f'' in* ‹*auto simp*: *has_vector_derivative_def*›)

then show *?thesis*
unfolding *fun_eq_iff* **by** (*metis scaleR_one*)
qed

lemma *vector_derivative_unique_at*:
(*f has_vector_derivative f'*) (*at x*) ⟹ (*f has_vector_derivative f''*) (*at x*) ⟹ *f' = f''*
by (*rule vector_derivative_unique_within*) *auto*

lemma *differentiableI_vector*: (*f has_vector_derivative y*) *F* ⟹ *f differentiable F*
by (*auto simp*: *differentiable_def has_vector_derivative_def*)

proposition *vector_derivative_works*:
f differentiable net ⟷ (*f has_vector_derivative* (*vector_derivative f net*)) *net*
(**is** *?l = ?r*)
proof
assume *?l*
obtain *f'* **where** *f'*: (*f has_derivative f'*) *net*
using ‹*?l*› **unfolding** *differentiable_def* **..**
then interpret *bounded_linear f'*
by *auto*
show *?r*
unfolding *vector_derivative_def has_vector_derivative_def*
by (*rule someI*[*of _ f' 1*]) (*simp add*: *scaleR*[*symmetric*] *f'*)
qed (*auto simp*: *vector_derivative_def has_vector_derivative_def differentiable_def*)

lemma *vector_derivative_within*:
assumes *not_bot*: *at x within S* ≠ *bot* **and** *y*: (*f has_vector_derivative y*) (*at x within S*)
shows *vector_derivative f* (*at x within S*) = *y*
using *y*
by (*intro vector_derivative_unique_within*[*OF not_bot vector_derivative_works*[*THEN iffD1*] *y*])
(*auto simp*: *differentiable_def has_vector_derivative_def*)

lemma *deriv_of_real* [*simp*]:
at x within A ≠ *bot* ⟹ *vector_derivative of_real* (*at x within A*) = *1*
by (*auto intro*!: *vector_derivative_within derivative_eq_intros*)

lemma *frechet_derivative_eq_vector_derivative*:
assumes *f differentiable* (*at x*)
shows (*frechet_derivative f* (*at x*)) = (λ*r*. *r* ∗$_R$ *vector_derivative f* (*at x*))
using *assms*
by (*auto simp*: *differentiable_iff_scaleR vector_derivative_def has_vector_derivative_def*
intro: *someI frechet_derivative_at* [*symmetric*])

lemma *has_real_derivative*:
fixes *f* :: *real* ⇒ *real*

assumes (*f has_derivative f'*) *F*
obtains *c* **where** (*f has_real_derivative c*) *F*
proof −
obtain *c* **where** $f' = (\lambda x.\ x * c)$
by (*metis assms has_derivative_bounded_linear real_bounded_linear*)
then show *?thesis*
by (*metis assms that has_field_derivative_def mult_commute_abs*)
qed

lemma *has_real_derivative_iff*:
fixes *f* :: *real* ⇒ *real*
shows $(\exists\, c.\ (f \text{ has\_real\_derivative } c)\ F) = (\exists\, D.\ (f \text{ has\_derivative } D)\ F)$
by (*metis has_field_derivative_def has_real_derivative*)

lemma *has_vector_derivative_cong_ev*:
assumes ∗: *eventually* $(\lambda x.\ x \in S \longrightarrow f\ x = g\ x)$ (*nhds x*) $f\ x = g\ x$
shows (*f has_vector_derivative f'*) (*at x within S*) = (*g has_vector_derivative f'*) (*at x within S*)
proof (*cases at x within S = bot*)
case *True*
then show *?thesis*
by (*simp add*: *has_derivative_def has_vector_derivative_def*)
next
case *False*
then show *?thesis*
unfolding *has_vector_derivative_def has_derivative_def*
using ∗
apply (*intro refl conj_cong filterlim_cong*)
apply (*auto simp*: *Lim_ident_at eventually_at_filter elim*: *eventually_mono*)
done
qed

lemma *vector_derivative_cong_eq*:
assumes *eventually* $(\lambda x.\ x \in A \longrightarrow f\ x = g\ x)$ (*nhds x*) $x = y$ $A = B$ $x \in A$
shows *vector_derivative f* (*at x within A*) = *vector_derivative g* (*at y within B*)
proof −
have $f\ x = g\ x$
using *assms eventually_nhds_x_imp_x* **by** *blast*
hence $(\lambda D.\ (f \text{ has\_vector\_derivative } D)\ (\text{at } x \text{ within } A)) =$
$(\lambda D.\ (g \text{ has\_vector\_derivative } D)\ (\text{at } x \text{ within } A))$ **using** *assms*
by (*intro ext has_vector_derivative_cong_ev refl assms*) *simp_all*
thus *?thesis* **by** (*simp add*: *vector_derivative_def assms*)
qed

lemma *islimpt_closure_open*:
fixes *s* :: *'a*::*perfect_space set*
assumes *open s* **and** *t*: $t = \text{closure } s$ $x \in t$
shows *x islimpt t*

proof *cases*
assume $x \in s$
{ fix T **assume** $x \in T$ *open* T
then have *open* $(s \cap T)$
using ‹*open s*› **by** *auto*
then have $s \cap T \neq \{x\}$
using *not_open_singleton*[*of x*] **by** *auto*
with ‹$x \in T$› ‹$x \in s$› **have** $\exists\, y{\in}t.\ y \in T \wedge y \neq x$
using *closure_subset*[*of s*] **by** (*auto simp*: *t*) **}**
then show *?thesis*
by (*auto intro*!: *islimptI*)
next
assume $x \notin s$ **with** *t* **show** *?thesis*
unfolding *t closure_def* **by** (*auto intro*: *islimpt_subset*)
qed

lemma *vector_derivative_unique_within_closed_interval*:
assumes *ab*: $a < b$ $x \in$ *cbox a b*
assumes *D*: (*f has_vector_derivative f'*) (*at x within cbox a b*) (*f has_vector_derivative f''*) (*at x within cbox a b*)
shows $f' = f''$
using *ab*
by (*intro vector_derivative_unique_within*[*OF _ D*])
(*auto simp*: *trivial_limit_within intro*!: *islimpt_closure_open*[**where** *s*={*a* <..< *b*}])

lemma *vector_derivative_at*:
(*f has_vector_derivative f'*) (*at x*) $\Longrightarrow$ *vector_derivative f* (*at x*) $= f'$
by (*intro vector_derivative_within at_neq_bot*)

lemma *has_vector_derivative_id_at* [*simp*]: *vector_derivative* ($\lambda x.\ x$) (*at a*) $= 1$
by (*simp add*: *vector_derivative_at*)

lemma *vector_derivative_minus_at* [*simp*]:
f differentiable at a
$\Longrightarrow$ *vector_derivative* ($\lambda x.\ -\ f\ x$) (*at a*) $= -$ *vector_derivative f* (*at a*)
by (*simp add*: *vector_derivative_at has_vector_derivative_minus vector_derivative_works* [*symmetric*])

lemma *vector_derivative_add_at* [*simp*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
$\Longrightarrow$ *vector_derivative* ($\lambda x.\ f\ x\ +\ g\ x$) (*at a*) $=$ *vector_derivative f* (*at a*) $+$ *vector_derivative g* (*at a*)
by (*simp add*: *vector_derivative_at has_vector_derivative_add vector_derivative_works* [*symmetric*])

lemma *vector_derivative_diff_at* [*simp*,*derivative_intros*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
$\Longrightarrow$ *vector_derivative* ($\lambda x.\ f\ x\ -\ g\ x$) (*at a*) $=$ *vector_derivative f* (*at a*) $-$

vector_derivative g (at a)
by (*simp add*: *vector_derivative_at has_vector_derivative_diff vector_derivative_works* [*symmetric*])

lemma *vector_derivative_mult_at* [*simp*]:
fixes *f g* :: *real* ⇒ *'a* :: *real_normed_algebra*
shows ⟦*f differentiable at a*; *g differentiable at a*⟧
⟹ *vector_derivative* (λ*x*. *f x* ∗ *g x*) (*at a*) = *f a* ∗ *vector_derivative g* (*at a*)
+ *vector_derivative f* (*at a*) ∗ *g a*
by (*simp add*: *vector_derivative_at has_vector_derivative_mult vector_derivative_works* [*symmetric*])

lemma *vector_derivative_scaleR_at* [*simp*]:
⟦*f differentiable at a*; *g differentiable at a*⟧
⟹ *vector_derivative* (λ*x*. *f x* ∗*R* *g x*) (*at a*) = *f a* ∗*R* *vector_derivative g* (*at a*) + *vector_derivative f* (*at a*) ∗*R* *g a*
apply (*intro vector_derivative_at has_vector_derivative_scaleR*)
apply (*auto simp*: *vector_derivative_works has_vector_derivative_def has_field_derivative_def mult_commute_abs*)
done

lemma *vector_derivative_within_cbox*:
assumes *ab*: *a* < *b x* ∈ *cbox a b*
assumes *f*: (*f has_vector_derivative f'*) (*at x within cbox a b*)
shows *vector_derivative f* (*at x within cbox a b*) = *f'*
by (*metis assms box_real*(2) *f islimpt_Icc trivial_limit_within vector_derivative_within*)

lemma *vector_derivative_within_closed_interval*:
fixes *f*::*real* ⇒ *'a*::*euclidean_space*
assumes *a* < *b* **and** *x* ∈ {*a*..*b*}
assumes (*f has_vector_derivative f'*) (*at x within* {*a*..*b*})
shows *vector_derivative f* (*at x within* {*a*..*b*}) = *f'*
using *assms vector_derivative_within_cbox*
by *fastforce*

lemma *has_vector_derivative_within_subset*:
(*f has_vector_derivative f'*) (*at x within S*) ⟹ *T* ⊆ *S* ⟹ (*f has_vector_derivative f'*) (*at x within T*)
by (*auto simp*: *has_vector_derivative_def intro*: *has_derivative_subset*)

lemma *has_vector_derivative_at_within*:
(*f has_vector_derivative f'*) (*at x*) ⟹ (*f has_vector_derivative f'*) (*at x within S*)
unfolding *has_vector_derivative_def*
by (*rule has_derivative_at_withinI*)

lemma *has_vector_derivative_weaken*:
fixes *x D* **and** *f g S T*
assumes *f*: (*f has_vector_derivative D*) (*at x within T*)

and $x \in S$ $S \subseteq T$
and $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows $(g\ has\_vector\_derivative\ D)\ (at\ x\ within\ S)$
proof −
have $(f\ has\_vector\_derivative\ D)\ (at\ x\ within\ S) \longleftrightarrow (g\ has\_vector\_derivative\ D)\ (at\ x\ within\ S)$
unfolding *has_vector_derivative_def has_derivative_iff_norm*
using *assms* **by** (*intro conj_cong Lim_cong_within refl*) *auto*
then show *?thesis*
using *has_vector_derivative_within_subset*[*OF f* ‹$S \subseteq T$›] **by** *simp*
qed

lemma *has_vector_derivative_transform_within*:
assumes $(f\ has\_vector\_derivative\ f')\ (at\ x\ within\ S)$
and $0 < d$
and $x \in S$
and $\bigwedge x'.\ [\![x' \in S;\ dist\ x'\ x < d]\!] \Longrightarrow f\ x' = g\ x'$
shows $(g\ has\_vector\_derivative\ f')\ (at\ x\ within\ S)$
using *assms*
unfolding *has_vector_derivative_def*
by (*rule has_derivative_transform_within*)

lemma *has_vector_derivative_transform_within_open*:
assumes $(f\ has\_vector\_derivative\ f')\ (at\ x)$
and *open S*
and $x \in S$
and $\bigwedge y.\ y \in S \Longrightarrow f\ y = g\ y$
shows $(g\ has\_vector\_derivative\ f')\ (at\ x)$
using *assms*
unfolding *has_vector_derivative_def*
by (*rule has_derivative_transform_within_open*)

lemma *has_vector_derivative_transform*:
assumes $x \in S$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
assumes f': $(f\ has\_vector\_derivative\ f')\ (at\ x\ within\ S)$
shows $(g\ has\_vector\_derivative\ f')\ (at\ x\ within\ S)$
using *assms*
unfolding *has_vector_derivative_def*
by (*rule has_derivative_transform*)

lemma *vector_diff_chain_at*:
assumes $(f\ has\_vector\_derivative\ f')\ (at\ x)$
and $(g\ has\_vector\_derivative\ g')\ (at\ (f\ x))$
shows $((g \circ f)\ has\_vector\_derivative\ (f' *_R g'))\ (at\ x)$
using *assms has_vector_derivative_at_within has_vector_derivative_def vector_derivative_diff_chain_within* **by** *blast*

lemma *vector_diff_chain_within*:
assumes $(f\ has\_vector\_derivative\ f')\ (at\ x\ within\ s)$

and (*g has_vector_derivative g'*) (*at* (*f x*) *within f ' s*)
shows ((*g* ∘ *f*) *has_vector_derivative* (*f'* *$_R$* *g'*)) (*at x within s*)
using *assms has_vector_derivative_def vector_derivative_diff_chain_within* **by** *blast*

lemma *vector_derivative_const_at* [*simp*]: *vector_derivative* (λ*x*. *c*) (*at a*) = *0*
by (*simp add*: *vector_derivative_at*)

lemma *vector_derivative_at_within_ivl*:
(*f has_vector_derivative f'*) (*at x*) ⟹
a ≤ *x* ⟹ *x* ≤ *b* ⟹ *a*<*b* ⟹ *vector_derivative f* (*at x within* {*a*..*b*}) = *f'*
using *has_vector_derivative_at_within vector_derivative_within_cbox* **by** *fastforce*

lemma *vector_derivative_chain_at*:
assumes *f differentiable at x* (*g differentiable at* (*f x*))
shows *vector_derivative* (*g* ∘ *f*) (*at x*) =
vector_derivative f (*at x*) *$*_R$* *vector_derivative g* (*at* (*f x*))
by (*metis vector_diff_chain_at vector_derivative_at vector_derivative_works assms*)

lemma *field_vector_diff_chain_at*:
assumes *Df*: (*f has_vector_derivative f'*) (*at x*)
and *Dg*: (*g has_field_derivative g'*) (*at* (*f x*))
shows ((*g* ∘ *f*) *has_vector_derivative* (*f'* ∗ *g'*)) (*at x*)
using *diff_chain_at*[*OF Df*[*unfolded has_vector_derivative_def*]
Dg [*unfolded has_field_derivative_def*]]
by (*auto simp*: *o_def mult.commute has_vector_derivative_def*)

lemma *vector_derivative_chain_within*:
assumes *at x within S* ≠ *bot f differentiable* (*at x within S*)
(*g has_derivative g'*) (*at* (*f x*) *within f ' S*)
shows *vector_derivative* (*g* ∘ *f*) (*at x within S*) =
g' (*vector_derivative f* (*at x within S*))
apply (*rule vector_derivative_within* [*OF* ‹*at x within S* ≠ *bot*›])
apply (*rule vector_derivative_diff_chain_within*)
using *assms*(*2*−*3*) *vector_derivative_works*
by *auto*

6.17.16 Field differentiability

definition *field_differentiable* :: [*'a* ⇒ *'a*::*real_normed_field*, *'a filter*] ⇒ *bool*
(**infixr** ‹(*field'_differentiable*)› *50*)
where *f field_differentiable F* ≡ ∃*f'*. (*f has_field_derivative f'*) *F*

lemma *field_differentiable_imp_differentiable*:
f field_differentiable F ⟹ *f differentiable F*
unfolding *field_differentiable_def differentiable_def*
using *has_field_derivative_imp_has_derivative* **by** *auto*

lemma *field_differentiable_imp_continuous_at*:
f field_differentiable (*at x within S*) ⟹ *continuous* (*at x within S*) *f*
by (*metis DERIV_continuous field_differentiable_def*)

lemma *field_differentiable_within_subset*:
⟦*f field_differentiable* (*at x within S*); *T* ⊆ *S*⟧ ⟹ *f field_differentiable* (*at x within T*)
by (*metis DERIV_subset field_differentiable_def*)

lemma *field_differentiable_at_within*:
⟦*f field_differentiable* (*at x*)⟧
⟹ *f field_differentiable* (*at x within S*)
unfolding *field_differentiable_def*
by (*metis DERIV_subset top_greatest*)

lemma *field_differentiable_linear* [*simp*,*derivative_intros*]: ((∗) *c*) *field_differentiable F*
unfolding *field_differentiable_def has_field_derivative_def mult_commute_abs*
by (*force intro*: *has_derivative_mult_right*)

lemma *field_differentiable_const* [*simp*,*derivative_intros*]: (λ*z*. *c*) *field_differentiable F*
unfolding *field_differentiable_def has_field_derivative_def*
using *DERIV_const has_field_derivative_imp_has_derivative* **by** *blast*

lemma *field_differentiable_ident* [*simp*,*derivative_intros*]: (λ*z*. *z*) *field_differentiable F*
unfolding *field_differentiable_def has_field_derivative_def*
using *DERIV_ident has_field_derivative_def* **by** *blast*

lemma *field_differentiable_id* [*simp*,*derivative_intros*]: *id field_differentiable F*
unfolding *id_def* **by** (*rule field_differentiable_ident*)

lemma *field_differentiable_minus* [*derivative_intros*]:
f field_differentiable F ⟹ (λ*z*. − (*f z*)) *field_differentiable F*
unfolding *field_differentiable_def* **by** (*metis field_differentiable_minus*)

lemma *field_differentiable_diff_const* [*simp*,*derivative_intros*]:
(−)*c field_differentiable F*
unfolding *field_differentiable_def* **by** (*rule derivative_eq_intros exI* | *force*)+

lemma *field_differentiable_add* [*derivative_intros*]:
assumes *f field_differentiable F g field_differentiable F*
shows (λ*z*. *f z* + *g z*) *field_differentiable F*
using *assms* **unfolding** *field_differentiable_def*
by (*metis field_differentiable_add*)

lemma *field_differentiable_add_const* [*simp*,*derivative_intros*]:
(+) *c field_differentiable F*

by (*simp add*: *field_differentiable_add*)

lemma *field_differentiable_sum* [*derivative_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *field_differentiable F*) ⟹ (λ*z*. ∑ *i*∈*I*. *f i z*) *field_differentiable F*
by (*induct I rule*: *infinite_finite_induct*)
(*auto intro*: *field_differentiable_add field_differentiable_const*)

lemma *field_differentiable_diff* [*derivative_intros*]:
assumes *f field_differentiable F g field_differentiable F*
shows (λ*z*. *f z* − *g z*) *field_differentiable F*
using *assms* **unfolding** *field_differentiable_def*
by (*metis field_differentiable_diff*)

lemma *field_differentiable_inverse* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*) *f a* ≠ *0*
shows (λ*z*. *inverse* (*f z*)) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_inverse_fun*)

lemma *field_differentiable_mult* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*)
g field_differentiable (*at a within S*)
shows (λ*z*. *f z* ∗ *g z*) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_mult* [*of f _ a S g*])

lemma *field_differentiable_divide* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*)
g field_differentiable (*at a within S*)
g a ≠ *0*
shows (λ*z*. *f z* / *g z*) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_divide* [*of f _ a S g*])

lemma *field_differentiable_power* [*derivative_intros*]:
assumes *f field_differentiable* (*at a within S*)
shows (λ*z*. *f z* ^ *n*) *field_differentiable* (*at a within S*)
using *assms* **unfolding** *field_differentiable_def*
by (*metis DERIV_power*)

lemma *field_differentiable_cnj_cnj*:
assumes *f field_differentiable* (*at* (*cnj z*))
shows (*cnj* ∘ *f* ∘ *cnj*) *field_differentiable* (*at z*)
using *has_field_derivative_cnj_cnj assms*
by (*auto simp*: *field_differentiable_def*)

lemma *field_differentiable_transform_within*:
0 < *d* ⟹

```
      x ∈ S ⟹
      (⋀x'. x' ∈ S ⟹ dist x' x < d ⟹ f x' = g x') ⟹
      f field_differentiable (at x within S)
      ⟹ g field_differentiable (at x within S)
  unfolding field_differentiable_def has_field_derivative_def
  by (blast intro: has_derivative_transform_within)

lemma field_differentiable_compose_within:
  assumes f field_differentiable (at a within S)
        g field_differentiable (at (f a) within f`S)
    shows (g o f) field_differentiable (at a within S)
  using assms unfolding field_differentiable_def
  by (metis DERIV_image_chain)

lemma field_differentiable_compose:
  f field_differentiable at z ⟹ g field_differentiable at (f z)
        ⟹ (g o f) field_differentiable at z
by (metis field_differentiable_at_within field_differentiable_compose_within)

lemma field_differentiable_within_open:
    ⟦a ∈ S; open S⟧ ⟹ f field_differentiable at a within S ⟷
                      f field_differentiable at a
  unfolding field_differentiable_def
  by (metis at_within_open)

lemma exp_scaleR_has_vector_derivative_right:
  ((λt. exp (t *R A)) has_vector_derivative exp (t *R A) * A) (at t within T)
  unfolding has_vector_derivative_def
proof (rule has_derivativeI)
  let ?F = at t within (T ∩ {t - 1 <..< t + 1})
  have *: at t within T = ?F
    by (rule at_within_nhd[where S={t - 1 <..< t + 1}]) auto
  let ?e = λi x. (inverse (1 + real i) * inverse (fact i) * (x - t) ^ i) *R (A * A
^ i)
  have ∀F n in sequentially.
     ∀x∈T ∩ {t - 1<..<t + 1}. norm (?e n x) ≤ norm (A ^(n + 1) /R fact (n
+ 1))
    apply (auto simp: algebra_split_simps intro!: eventuallyI)
    apply (rule mult_left_mono)
     apply (auto simp add: field_simps power_abs intro!: divide_right_mono
power_le_one)
    done
  then have uniform_limit (T ∩ {t - 1<..<t + 1}) (λn x. ∑i<n. ?e i x) (λx.
∑i. ?e i x) sequentially
     by (rule Weierstrass_m_test_ev) (intro summable_ignore_initial_segment
summable_norm_exp)
  moreover
  have ∀F x in sequentially. x > 0
    by (metis eventually_gt_at_top)
```

then have
$\forall_F\ n$ *in sequentially.* $((\lambda x.\ \sum i{<}n.\ ?e\ i\ x) \longrightarrow A)\ ?F$
by *eventually_elim*
(*auto intro*!: *tendsto_eq_intros*
simp: *power_0_left if_distrib if_distribR*
cong: *if_cong*)
ultimately
have [*tendsto_intros*]: $((\lambda x.\ \sum i.\ ?e\ i\ x) \longrightarrow A)\ ?F$
by (*auto intro*!: *swap_uniform_limit*[**where** $f{=}\lambda n\ x.\ \sum i < n.\ ?e\ i\ x$ **and** F = *sequentially*])
have [*tendsto_intros*]: $((\lambda x.\ \textit{if}\ x = t\ \textit{then}\ 0\ \textit{else}\ 1) \longrightarrow 1)\ ?F$
by (*rule tendsto_eventually*) (*simp add*: *eventually_at_filter*)
have $((\lambda y.\ ((y - t) / abs\ (y - t)) *_R ((\sum n.\ ?e\ n\ y) - A)) \longrightarrow 0)$ (*at t within* T)
unfolding $*$
by (*rule tendsto_norm_zero_cancel*) (*auto intro*!: *tendsto_eq_intros*)

moreover have $\forall_F\ x$ *in at t within* $T.\ x \neq t$
by (*simp add*: *eventually_at_filter*)
then have $\forall_F\ x$ *in at t within* $T.\ ((x - t) / |x - t|) *_R ((\sum n.\ ?e\ n\ x) - A) =$
$(exp\ ((x - t) *_R A) - 1 - (x - t) *_R A) /_R norm\ (x - t)$
proof *eventually_elim*
case (*elim x*)
have $(exp\ ((x - t) *_R A) - 1 - (x - t) *_R A) /_R norm\ (x - t) =$
$((\sum n.\ (x - t) *_R ?e\ n\ x) - (x - t) *_R A) /_R norm\ (x - t)$
unfolding *exp_first_term*
by (*simp add*: *ac_simps*)
also
have *summable* $(\lambda n.\ ?e\ n\ x)$
proof −
from *elim* **have** $?e\ n\ x = (((x - t) *_R A)\ \hat{}\ (n + 1)) /_R fact\ (n + 1) /_R (x - t)$ **for** n
by *simp*
then show *?thesis*
by (*auto simp only*:
intro!: *summable_scaleR_right summable_ignore_initial_segment summable_exp_generic*)
qed
then have $(\sum n.\ (x - t) *_R ?e\ n\ x) = (x - t) *_R (\sum n.\ ?e\ n\ x)$
by (*rule suminf_scaleR_right*[*symmetric*])
also have $(\ldots - (x - t) *_R A) /_R norm\ (x - t) = (x - t) *_R ((\sum n.\ ?e\ n\ x) - A) /_R norm\ (x - t)$
by (*simp add*: *algebra_simps*)
finally show *?case*
by *simp* (*simp add*: *field_simps*)
qed

ultimately have $((\lambda y.\ (exp\ ((y - t) *_R A) - 1 - (y - t) *_R A) /_R norm\ (y - t)) \longrightarrow 0)$ (*at t within* T)
by (*rule Lim_transform_eventually*)

from *tendsto_mult_right_zero*[*OF this*, **where** *c*=*exp* (*t* $*_R$ *A*)]
show ((λ*y*. (*exp* (*y* $*_R$ *A*) − *exp* (*t* $*_R$ *A*) − (*y* − *t*) $*_R$ (*exp* (*t* $*_R$ *A*) ∗ *A*)) /$_R$ *norm* (*y* − *t*)) ⟶ 0)
(*at t within T*)
by (*rule Lim_transform_eventually*)
(*auto simp*: *field_split_simps exp_add_commuting*[*symmetric*])
qed (*rule bounded_linear_scaleR_left*)

lemma *exp_times_scaleR_commute*: *exp* (*t* $*_R$ *A*) ∗ *A* = *A* ∗ *exp* (*t* $*_R$ *A*)
using *exp_times_arg_commute*[*symmetric*, *of t* $*_R$ *A*]
by (*auto simp*: *algebra_simps*)

lemma *exp_scaleR_has_vector_derivative_left*: ((λ*t*. *exp* (*t* $*_R$ *A*)) *has_vector_derivative* *A* ∗ *exp* (*t* $*_R$ *A*)) (*at t*)
using *exp_scaleR_has_vector_derivative_right*[*of A t*]
by (*simp add*: *exp_times_scaleR_commute*)

lemma *field_differentiable_series*:
fixes *f* :: *nat* ⇒ 'a::{*real_normed_field*,*banach*} ⇒ 'a
assumes *convex S open S*
assumes ⋀*n x*. *x* ∈ *S* ⟹ (*f n has_field_derivative f' n x*) (*at x*)
assumes *uniformly_convergent_on S* (λ*n x*. ∑ *i*<*n*. *f' i x*)
assumes *x0* ∈ *S summable* (λ*n*. *f n x0*) **and** *x*: *x* ∈ *S*
shows (λ*x*. ∑ *n*. *f n x*) *field_differentiable* (*at x*)
proof −
from *assms*(*4*) **obtain** *g'* **where** *A*: *uniform_limit S* (λ*n x*. ∑ *i*<*n*. *f' i x*) *g'* *sequentially*
unfolding *uniformly_convergent_on_def* **by** *blast*
from *x* **and** ‹*open S*› **have** *S*: *at x within S* = *at x* **by** (*rule at_within_open*)
have ∃ *g*. ∀ *x*∈*S*. (λ*n*. *f n x*) *sums g x* ∧ (*g has_field_derivative g' x*) (*at x within* *S*)
by (*intro has_field_derivative_series*[*of S f f' g' x0*] *assms A has_field_derivative_at_within*)
then obtain *g* **where** *g*: ⋀*x*. *x* ∈ *S* ⟹ (λ*n*. *f n x*) *sums g x*
⋀*x*. *x* ∈ *S* ⟹ (*g has_field_derivative g' x*) (*at x within S*) **by** *blast*
from *g*(*2*)[*OF x*] **have** *g'*: (*g has_derivative* (∗) (*g' x*)) (*at x*)
by (*simp add*: *has_field_derivative_def S*)
have ((λ*x*. ∑ *n*. *f n x*) *has_derivative* (∗) (*g' x*)) (*at x*)
by (*rule has_derivative_transform_within_open*[*OF g'* ‹*open S*› *x*])
(*insert g*, *auto simp*: *sums_iff*)
thus (λ*x*. ∑ *n*. *f n x*) *field_differentiable* (*at x*) **unfolding** *differentiable_def*
by (*auto simp*: *summable_def field_differentiable_def has_field_derivative_def*)
qed

Caratheodory characterization

lemma *field_differentiable_caratheodory_at*:
f field_differentiable (*at z*) ⟷
(∃ *g*. (∀ *w*. *f*(*w*) − *f*(*z*) = *g*(*w*) ∗ (*w* − *z*)) ∧ *continuous* (*at z*) *g*)
using *CARAT_DERIV* [*of f*]

by (*simp add*: *field_differentiable_def has_field_derivative_def*)

lemma *field_differentiable_caratheodory_within*:
f field_differentiable (*at z within s*) ⟷
(∃ *g*. (∀ *w*. *f*(*w*) − *f*(*z*) = *g*(*w*) ∗ (*w* − *z*)) ∧ *continuous* (*at z within s*) *g*)
using *DERIV_caratheodory_within* [*of f*]
by (*simp add*: *field_differentiable_def has_field_derivative_def*)

6.17.17 Field derivative

definition *deriv* :: ('*a* ⇒ '*a*::*real_normed_field*) ⇒ '*a* ⇒ '*a* **where**
deriv f x ≡ *SOME D*. *DERIV f x* :> *D*

lemma *DERIV_imp_deriv*: *DERIV f x* :> *f'* ⟹ *deriv f x* = *f'*
unfolding *deriv_def* **by** (*metis some_equality DERIV_unique*)

lemma *DERIV_deriv_iff_has_field_derivative*:
DERIV f x :> *deriv f x* ⟷ (∃ *f'*. (*f has_field_derivative f'*) (*at x*))
by (*auto simp*: *has_field_derivative_def DERIV_imp_deriv*)

lemma *DERIV_deriv_iff_real_differentiable*:
fixes *x* :: *real*
shows *DERIV f x* :> *deriv f x* ⟷ *f differentiable at x*
unfolding *differentiable_def* **by** (*metis DERIV_imp_deriv has_real_derivative_iff*)

lemma *DERIV_deriv_iff_field_differentiable*:
DERIV f x :> *deriv f x* ⟷ *f field_differentiable at x*
unfolding *field_differentiable_def* **by** (*metis DERIV_imp_deriv*)

lemma *vector_derivative_of_real_left*:
assumes *f differentiable at x*
shows *vector_derivative* (λ*x*. *of_real* (*f x*)) (*at x*) = *of_real* (*deriv f x*)
by (*metis DERIV_deriv_iff_real_differentiable assms has_vector_derivative_of_real vector_derivative_at*)

lemma *vector_derivative_of_real_right*:
assumes *f field_differentiable at* (*of_real x*)
shows *vector_derivative* (λ*x*. *f* (*of_real x*)) (*at x*) = *deriv f* (*of_real x*)
by (*metis DERIV_deriv_iff_field_differentiable assms has_vector_derivative_real_field vector_derivative_at*)

lemma *deriv_cong_ev*:
assumes *eventually* (λ*x*. *f x* = *g x*) (*nhds x*) *x* = *y*
shows *deriv f x* = *deriv g y*
proof −
have (λ*D*. (*f has_field_derivative D*) (*at x*)) = (λ*D*. (*g has_field_derivative D*) (*at y*))
by (*intro ext DERIV_cong_ev refl assms*)
thus *?thesis* **by** (*simp add*: *deriv_def assms*)

qed

lemma *higher_deriv_cong_ev*:
assumes *eventually* (λ*x*. *f x* = *g x*) (*nhds x*) *x* = *y*
shows (*deriv* ^^ *n*) *f x* = (*deriv* ^^ *n*) *g y*
proof −
from *assms*(*1*) **have** *eventually* (λ*x*. (*deriv* ^^ *n*) *f x* = (*deriv* ^^ *n*) *g x*) (*nhds x*)
proof (*induction n arbitrary*: *f g*)
case (*Suc n*)
from *Suc.prems* **have** *eventually* (λ*y*. *eventually* (λ*z*. *f z* = *g z*) (*nhds y*)) (*nhds x*)
by (*simp add*: *eventually_eventually*)
hence *eventually* (λ*x*. *deriv f x* = *deriv g x*) (*nhds x*)
by *eventually_elim* (*rule deriv_cong_ev*, *simp_all*)
thus *?case* **by** (*auto intro*!: *deriv_cong_ev Suc simp*: *funpow_Suc_right simp del*: *funpow.simps*)
qed *auto*
with ‹*x* = *y*› *eventually_nhds_x_imp_x* **show** *?thesis* **by** *blast*
qed

lemma *real_derivative_chain*:
fixes *x* :: *real*
shows *f differentiable at x* ⟹ *g differentiable at* (*f x*)
⟹ *deriv* (*g o f*) *x* = *deriv g* (*f x*) ∗ *deriv f x*
by (*metis DERIV_deriv_iff_real_differentiable DERIV_chain DERIV_imp_deriv*)
lemma *field_derivative_eq_vector_derivative*:
(*deriv f x*) = *vector_derivative f* (*at x*)
by (*simp add*: *mult.commute deriv_def vector_derivative_def has_vector_derivative_def has_field_derivative_def*)

proposition *field_differentiable_derivI*:
f field_differentiable (*at x*) ⟹ (*f has_field_derivative deriv f x*) (*at x*)
by (*simp add*: *field_differentiable_def DERIV_deriv_iff_has_field_derivative*)

lemma *vector_derivative_chain_at_general*:
assumes *f differentiable at x g field_differentiable at* (*f x*)
shows *vector_derivative* (*g* ∘ *f*) (*at x*) = *vector_derivative f* (*at x*) ∗ *deriv g* (*f x*)
using *assms field_differentiable_derivI field_vector_diff_chain_at*
vector_derivative_at vector_derivative_works **by** *blast*

lemma *deriv_chain*:
f field_differentiable at x ⟹ *g field_differentiable at* (*f x*)
⟹ *deriv* (*g o f*) *x* = *deriv g* (*f x*) ∗ *deriv f x*
by (*metis DERIV_deriv_iff_field_differentiable DERIV_chain DERIV_imp_deriv*)

lemma *deriv_linear* [*simp*]: *deriv* (λ*w*. *c* ∗ *w*) = (λ*z*. *c*)
by (*metis DERIV_imp_deriv DERIV_cmult_Id*)

lemma *deriv_uminus* [*simp*]: *deriv* (λ*w*. −*w*) = (λ*z*. −*1*)
using *deriv_linear*[*of* −*1*] **by** (*simp del*: *deriv_linear*)

lemma *deriv_ident* [*simp*]: *deriv* (λ*w*. *w*) = (λ*z*. *1*)
by (*metis DERIV_imp_deriv DERIV_ident*)

lemma *deriv_id* [*simp*]: *deriv id* = (λ*z*. *1*)
by (*simp add*: *id_def*)

lemma *deriv_const* [*simp*]: *deriv* (λ*w*. *c*) = (λ*z*. *0*)
by (*metis DERIV_imp_deriv DERIV_const*)

lemma *deriv_add* [*simp*]:
⟦*f field_differentiable at z*; *g field_differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* + *g w*) *z* = *deriv f z* + *deriv g z*
unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
by (*auto intro*!: *DERIV_imp_deriv derivative_intros*)

lemma *deriv_minus* [*simp*]:
f field_differentiable at z ⟹ *deriv* (λ*w*. − *f w*) *z* = − *deriv f z*
by (*simp add*: *DERIV_deriv_iff_field_differentiable DERIV_imp_deriv Deriv.field_differentiable_minus*)

lemma *deriv_diff* [*simp*]:
⟦*f field_differentiable at z*; *g field_differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* − *g w*) *z* = *deriv f z* − *deriv g z*
unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
by (*auto intro*!: *DERIV_imp_deriv derivative_intros*)

lemma *deriv_mult* [*simp*]:
⟦*f field_differentiable at z*; *g field_differentiable at z*⟧
⟹ *deriv* (λ*w*. *f w* ∗ *g w*) *z* = *f z* ∗ *deriv g z* + *deriv f z* ∗ *g z*
unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
by (*auto intro*!: *DERIV_imp_deriv derivative_eq_intros*)

lemma *deriv_cmult*:
f field_differentiable at z ⟹ *deriv* (λ*w*. *c* ∗ *f w*) *z* = *c* ∗ *deriv f z*
by *simp*

lemma *deriv_cmult_right*:
f field_differentiable at z ⟹ *deriv* (λ*w*. *f w* ∗ *c*) *z* = *deriv f z* ∗ *c*
by *simp*

lemma *deriv_inverse* [*simp*]:
⟦*f field_differentiable at z*; *f z* ≠ *0*⟧
⟹ *deriv* (λ*w*. *inverse* (*f w*)) *z* = − *deriv f z* / *f z* ^ *2*
unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
by (*safe intro*!: *DERIV_imp_deriv derivative_eq_intros*) (*auto simp*: *field_split_simps*

power2_eq_square)

lemma *deriv_divide* [*simp*]:
  ⟦*f field_differentiable at z*; *g field_differentiable at z*; *g z* ≠ *0*⟧
   ⟹ *deriv* (λ*w*. *f w* / *g w*) *z* = (*deriv f z* ∗ *g z* − *f z* ∗ *deriv g z*) / *g z* ^ *2*
  by (*simp add*: *field_class.field_divide_inverse field_differentiable_inverse*)
    (*simp add*: *field_split_simps power2_eq_square*)

lemma *deriv_cdivide_right*:
  f field_differentiable at z ⟹ *deriv* (λ*w*. *f w* / *c*) *z* = *deriv f z* / *c*
  by (*simp add*: *field_class.field_divide_inverse*)

lemma *deriv_pow*: ⟦*f field_differentiable at z*⟧
   ⟹ *deriv* (λ*w*. *f w* ^ *n*) *z* = (*if n=0 then 0 else n* ∗ *deriv f z* ∗ *f z* ^ (*n* − *Suc 0*))
  unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
  by (*auto intro*!: *DERIV_imp_deriv derivative_eq_intros*)

lemma *deriv_sum* [*simp*]:
  ⟦⋀*i*. *f i field_differentiable at z*⟧
   ⟹ *deriv* (λ*w*. *sum* (λ*i*. *f i w*) *S*) *z* = *sum* (λ*i*. *deriv* (*f i*) *z*) *S*
  unfolding *DERIV_deriv_iff_field_differentiable*[*symmetric*]
  by (*auto intro*!: *DERIV_imp_deriv derivative_intros*)

lemma *deriv_compose_linear'*:
  assumes *f field_differentiable at* (*c*∗*z* + *a*)
  shows *deriv* (λ*w*. *f* (*c*∗*w* + *a*)) *z* = *c* ∗ *deriv f* (*c*∗*z* + *a*)
  apply (*subst deriv_chain* [**where** *f*=λ*w*. *c*∗*w* + *a*,*unfolded comp_def*])
  using *assms* **by** (*auto intro*: *derivative_intros*)

lemma *deriv_compose_linear*:
  assumes *f field_differentiable at* (*c* ∗ *z*)
  shows *deriv* (λ*w*. *f* (*c* ∗ *w*)) *z* = *c* ∗ *deriv f* (*c* ∗ *z*)
proof −
  have *deriv* (λ*a*. *f* (*c* ∗ *a*)) *z* = *deriv f* (*c* ∗ *z*) ∗ *c*
   using *assms* **by** (*simp add*: *DERIV_chain2 DERIV_deriv_iff_field_differentiable DERIV_imp_deriv*)
  then show *?thesis*
    by *simp*
qed

lemma *nonzero_deriv_nonconstant*:
  assumes *df*: *DERIV f ξ* :> *df* **and** *S*: *open S ξ* ∈ *S* **and** *df* ≠ *0*
    shows ¬ *f constant_on S*
unfolding *constant_on_def*
by (*metis* ‹*df* ≠ *0*› *has_field_derivative_transform_within_open* [*OF df S*] *DERIV_const DERIV_unique*)

6.17.18 Relation between convexity and derivative

proposition *convex_on_imp_above_tangent*:
assumes *convex*: *convex_on A f* **and** *connected*: *connected A*
assumes *c*: $c \in$ *interior A* **and** $x : x \in A$
assumes *deriv*: $(f$ *has_field_derivative* $f')$ *(at c within A)*
shows $f\ x - f\ c \geq f' * (x - c)$
proof *(cases x c rule*: *linorder_cases)*
assume *xc*: $x > c$
let *?A′* = *interior A* $\cap$ $\{c<..\}$
from *c* **have** $c \in$ *interior A* $\cap$ *closure* $\{c<..\}$ **by** *auto*
also have ... $\subseteq$ *closure* *(interior A* $\cap$ $\{c<..\})$ **by** *(intro open_Int_closure_subset) auto*
finally have *at c within ?A′* $\neq$ *bot* **by** *(subst at_within_eq_bot_iff) auto*
moreover from *deriv* **have** $((\lambda y.\ (f\ y - f\ c)\ /\ (y - c)) \longrightarrow f')$ *(at c within ?A′)*
unfolding *has_field_derivative_iff* **using** *interior_subset[of A]* **by** *(blast intro: tendsto_mono at_le)*
moreover from *eventually_at_right_real[OF xc]*
have *eventually* $(\lambda y.\ (f\ y - f\ c)\ /\ (y - c) \leq (f\ x - f\ c)\ /\ (x - c))$ *(at_right c)*
proof *eventually_elim*
fix *y* **assume** *y*: $y \in \{c<..<x\}$
with *convex connected x c* **have** $f\ y \leq (f\ x - f\ c)\ /\ (x - c) * (y - c) + f\ c$
using *interior_subset[of A]*
by *(intro convex_onD_Icc′ convex_on_subset[OF convex] connected_contains_Icc) auto*
hence $f\ y - f\ c \leq (f\ x - f\ c)\ /\ (x - c) * (y - c)$ **by** *simp*
thus $(f\ y - f\ c)\ /\ (y - c) \leq (f\ x - f\ c)\ /\ (x - c)$ **using** *y xc* **by** *(simp add: field_split_simps)*
qed
hence *eventually* $(\lambda y.\ (f\ y - f\ c)\ /\ (y - c) \leq (f\ x - f\ c)\ /\ (x - c))$ *(at c within ?A′)*
by *(blast intro*: *filter_leD at_le)*
ultimately have $f' \leq (f\ x - f\ c)\ /\ (x - c)$ **by** *(simp add*: *tendsto_upperbound)*
thus *?thesis* **using** *xc* **by** *(simp add*: *field_simps)*
next
assume *xc*: $x < c$
let *?A′* = *interior A* $\cap$ $\{..<c\}$
from *c* **have** $c \in$ *interior A* $\cap$ *closure* $\{..<c\}$ **by** *auto*
also have ... $\subseteq$ *closure* *(interior A* $\cap$ $\{..<c\})$ **by** *(intro open_Int_closure_subset) auto*
finally have *at c within ?A′* $\neq$ *bot* **by** *(subst at_within_eq_bot_iff) auto*
moreover from *deriv* **have** $((\lambda y.\ (f\ y - f\ c)\ /\ (y - c)) \longrightarrow f')$ *(at c within ?A′)*
unfolding *has_field_derivative_iff* **using** *interior_subset[of A]* **by** *(blast intro: tendsto_mono at_le)*
moreover from *eventually_at_left_real[OF xc]*
have *eventually* $(\lambda y.\ (f\ y - f\ c)\ /\ (y - c) \geq (f\ x - f\ c)\ /\ (x - c))$ *(at_left c)*
proof *eventually_elim*
fix *y* **assume** *y*: $y \in \{x<..<c\}$

with *convex connected x c* **have** $f\ y \leq (f\ x - f\ c)\ /\ (c - x) * (c - y) + f\ c$
using *interior_subset*[*of A*]
by (*intro convex_onD_Icc'' convex_on_subset*[*OF convex*] *connected_contains_Icc*) *auto*
hence $f\ y - f\ c \leq (f\ x - f\ c) * ((c - y)\ /\ (c - x))$ **by** *simp*
also have $(c - y)\ /\ (c - x) = (y - c)\ /\ (x - c)$ **using** *y xc* **by** (*simp add*: *field_simps*)
finally show $(f\ y - f\ c)\ /\ (y - c) \geq (f\ x - f\ c)\ /\ (x - c)$ **using** *y xc*
by (*simp add*: *field_split_simps*)
qed
hence *eventually* $(\lambda y.\ (f\ y - f\ c)\ /\ (y - c) \geq (f\ x - f\ c)\ /\ (x - c))$ (*at c within ?A'*)
by (*blast intro*: *filter_leD at_le*)
ultimately have $f' \geq (f\ x - f\ c)\ /\ (x - c)$ **by** (*simp add*: *tendsto_lowerbound*)
thus *?thesis* **using** *xc* **by** (*simp add*: *field_simps*)
qed *simp_all*

6.17.19 Partial derivatives

lemma *eventually_at_Pair_within_TimesI1*:
fixes *x*::'*a*::*metric_space*
assumes $\forall_F\ x'$ *in at x within X*. *P x'*
assumes *P x*
shows $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. *P x'*
proof −
from *assms*[*unfolded eventually_at_topological*]
obtain *S* **where** *S*: *open S* $x \in S$ $\bigwedge x'.\ x' \in X \Longrightarrow x' \in S \Longrightarrow P\ x'$
by *metis*
show $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. *P x'*
unfolding *eventually_at_topological*
by (*auto intro*!: *exI*[**where** *x*=$S \times UNIV$] *S open_Times*)
qed

lemma *eventually_at_Pair_within_TimesI2*:
fixes *x*::'*a*::*metric_space*
assumes $\forall_F\ y'$ *in at y within Y*. *P y' P y*
shows $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. *P y'*
proof −
from *assms*[*unfolded eventually_at_topological*]
obtain *S* **where** *S*: *open S* $y \in S$ $\bigwedge y'.\ y' \in Y \Longrightarrow y' \in S \Longrightarrow P\ y'$
by *metis*
show $\forall_F\ (x', y')$ *in at* (x, y) *within* $X \times Y$. *P y'*
unfolding *eventually_at_topological*
by (*auto intro*!: *exI*[**where** *x*=$UNIV \times S$] *S open_Times*)
qed

proposition *has_derivative_partialsI*:
fixes *f*::'*a*::*real_normed_vector* ⇒ '*b*::*real_normed_vector* ⇒ '*c*::*real_normed_vector*
assumes *fx*: $((\lambda x.\ f\ x\ y)$ *has_derivative fx*$)$ $(at\ x\ within\ X)$

assumes *fy*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in Y \Longrightarrow ((\lambda y.\ f\ x\ y)\ has\_derivative\ blinfun\_apply\ (fy\ x\ y))\ (at\ y\ within\ Y)$
assumes *fy_cont*[*unfolded continuous_within*]: *continuous* $(at\ (x,\ y)\ within\ X \times Y)\ (\lambda(x,\ y).\ fy\ x\ y)$
assumes $y \in Y$ *convex* Y
shows $((\lambda(x,\ y).\ f\ x\ y)\ has\_derivative\ (\lambda(tx,\ ty).\ fx\ tx + fy\ x\ y\ ty))\ (at\ (x,\ y)\ within\ X \times Y)$
proof (*safe intro*!: *has_derivativeI tendstoI*, *goal_cases*)
case (*2* e')
interpret *fx*: *bounded_linear fx* **using** *fx* **by** (*rule has_derivative_bounded_linear*)
define e **where** $e = e' / 9$
have $e > 0$ **using** $\langle e' > 0 \rangle$ **by** (*simp add*: *e_def*)

from *fy_cont*[*THEN tendstoD*, *OF* $\langle e > 0 \rangle$]
have $\forall_F\ (x',\ y')\ in\ at\ (x,\ y)\ within\ X \times Y.\ dist\ (fy\ x'\ y')\ (fy\ x\ y) < e$
by (*auto simp*: *split_beta'*)
from *this*[*unfolded eventually_at*] **obtain** d' **where**
$d' > 0$
$\bigwedge x'\ y'.\ x' \in X \Longrightarrow y' \in Y \Longrightarrow (x',\ y') \neq (x,\ y) \Longrightarrow dist\ (x',\ y')\ (x,\ y) < d' \Longrightarrow$
$dist\ (fy\ x'\ y')\ (fy\ x\ y) < e$
by *auto*
then
have d': $x' \in X \Longrightarrow y' \in Y \Longrightarrow dist\ (x',\ y')\ (x,\ y) < d' \Longrightarrow dist\ (fy\ x'\ y')\ (fy\ x\ y) < e$
for $x'\ y'$
using $\langle 0 < e \rangle$
by (*cases* $(x',\ y') = (x,\ y)$) *auto*
define d **where** $d = d' / sqrt\ 2$
have $d > 0$ **using** $\langle 0 < d' \rangle$ **by** (*simp add*: *d_def*)
have d: $x' \in X \Longrightarrow y' \in Y \Longrightarrow dist\ x'\ x < d \Longrightarrow dist\ y'\ y < d \Longrightarrow dist\ (fy\ x'\ y')\ (fy\ x\ y) < e$
for $x'\ y'$
by (*auto simp*: *dist_prod_def d_def intro*!: d' *real_sqrt_sum_squares_less*)

let *?S* = $ball\ y\ d \cap Y$
have *convex ?S*
by (*auto intro*!: *convex_Int* $\langle convex\ Y \rangle$)
{
fix x'::$'a$ **and** y'::$'b$
assume x': $x' \in X$ **and** y': $y' \in Y$
assume dx': $dist\ x'\ x < d$ **and** dy': $dist\ y'\ y < d$
have $norm\ (fy\ x'\ y' - fy\ x'\ y) \leq dist\ (fy\ x'\ y')\ (fy\ x\ y) + dist\ (fy\ x'\ y)\ (fy\ x\ y)$
by *norm*
also have $dist\ (fy\ x'\ y')\ (fy\ x\ y) < e$
by (*rule d*; *fact*)
also have $dist\ (fy\ x'\ y)\ (fy\ x\ y) < e$
by (*auto intro*!: *d simp*: *dist_prod_def* x' $\langle d > 0 \rangle$ $\langle y \in Y \rangle$ dx')
finally

```
    have norm (fy x' y' - fy x' y) < e + e
      by arith
    then have onorm (blinfun_apply (fy x' y') - blinfun_apply (fy x' y)) < e + e
      by (auto simp: norm_blinfun.rep_eq blinfun.diff_left[abs_def] fun_diff_def)
  } note onorm = this

  have ev_mem: ∀F (x', y') in at (x, y) within X × Y. (x', y') ∈ X × Y
    using ‹y ∈ Y›
    by (auto simp: eventually_at intro!: zero_less_one)
  moreover
  have ev_dist: ∀F xy in at (x, y) within X × Y. dist xy (x, y) < d if d > 0 for
d
    using eventually_at_ball[OF that]
    by (rule eventually_elim2) (auto simp: dist_commute intro!: eventually_True)
  note ev_dist[OF ‹0 < d›]
  ultimately
  have ∀F (x', y') in at (x, y) within X × Y.
    norm (f x' y' - f x' y - (fy x' y) (y' - y)) ≤ norm (y' - y) * (e + e)
  proof (eventually_elim, safe)
    fix x' y'
    assume x' ∈ X and y': y' ∈ Y
    assume dist: dist (x', y') (x, y) < d
    then have dx: dist x' x < d and dy: dist y' y < d
      unfolding dist_prod_def fst_conv snd_conv atomize_conj
    by (metis le_less_trans real_sqrt_sum_squares_ge1 real_sqrt_sum_squares_ge2)
    {
      fix t::real
      assume t ∈ {0 .. 1}
      then have y + t *R (y' - y) ∈ closed_segment y y'
        by (auto simp: closed_segment_def algebra_simps intro!: exI[where x=t])
      also
      have ... ⊆ ball y d ∩ Y
        using ‹y ∈ Y› ‹0 < d› dy y'
        by (intro ‹convex ?S›[unfolded convex_contains_segment, rule_format, of
y y'])
           (auto simp: dist_commute)
      finally have y + t *R (y' - y) ∈ ?S .
    } note seg = this

    have ⋀x. x ∈ ball y d ∩ Y ⟹ onorm (blinfun_apply (fy x' x) - blinfun_apply
(fy x' y)) ≤ e + e
      by (safe intro!: onorm less_imp_le ‹x' ∈ X› dx) (auto simp: dist_commute
‹0 < d› ‹y ∈ Y›)
    with seg has_derivative_subset[OF assms(2)[OF ‹x' ∈ X›]]
    show norm (f x' y' - f x' y - (fy x' y) (y' - y)) ≤ norm (y' - y) * (e + e)
      by (rule differentiable_bound_linearization[where S=?S])
        (auto intro!: ‹0 < d› ‹y ∈ Y›)
  qed
  moreover
```

let *?le* = $\lambda x'$. *norm* (*f x' y* − *f x y* − (*fx*) (*x'* − *x*)) ≤ *norm* (*x'* − *x*) ∗ *e*
from *fx*[*unfolded has_derivative_within*, *THEN conjunct2*, *THEN tendstoD*, *OF* ‹0 < *e*›]
have ∀$_F$ *x' in at x within X*. *?le x'*
by *eventually_elim* (*simp*,
simp add: *dist_norm field_split_simps split*: *if_split_asm*)
then have ∀$_F$ (*x'*, *y'*) *in at* (*x*, *y*) *within X* × *Y*. *?le x'*
by (*rule eventually_at_Pair_within_TimesI1*)
(*simp add*: *blinfun.bilinear_simps*)
moreover have ∀$_F$ (*x'*, *y'*) *in at* (*x*, *y*) *within X* × *Y*. *norm* ((*x'*, *y'*) − (*x*, *y*)) ≠ *0*
unfolding *norm_eq_zero right_minus_eq*
by (*auto simp*: *eventually_at intro*!: *zero_less_one*)
moreover
from *fy_cont*[*THEN tendstoD*, *OF* ‹0 < *e*›]
have ∀$_F$ *x' in at x within X*. *norm* (*fy x' y* − *fy x y*) < *e*
unfolding *eventually_at*
using ‹*y* ∈ *Y*›
by (*auto simp*: *dist_prod_def dist_norm*)
then have ∀$_F$ (*x'*, *y'*) *in at* (*x*, *y*) *within X* × *Y*. *norm* (*fy x' y* − *fy x y*) < *e*
by (*rule eventually_at_Pair_within_TimesI1*)
(*simp add*: *blinfun.bilinear_simps* ‹0 < *e*›)
ultimately
have ∀$_F$ (*x'*, *y'*) *in at* (*x*, *y*) *within X* × *Y*.
norm ((*f x' y'* − *f x y* − (*fx* (*x'* − *x*) + *fy x y* (*y'* − *y*))) /$_R$
norm ((*x'*, *y'*) − (*x*, *y*)))
< *e'*
proof (*eventually_elim*, *safe*)
fix *x' y'*
have *norm* (*f x' y'* − *f x y* − (*fx* (*x'* − *x*) + *fy x y* (*y'* − *y*))) ≤
norm (*f x' y'* − *f x' y* − *fy x' y* (*y'* − *y*)) +
norm (*fy x y* (*y'* − *y*) − *fy x' y* (*y'* − *y*)) +
norm (*f x' y* − *f x y* − *fx* (*x'* − *x*))
by *norm*
also
assume *nz*: *norm* ((*x'*, *y'*) − (*x*, *y*)) ≠ *0*
and *nfy*: *norm* (*fy x' y* − *fy x y*) < *e*
assume *norm* (*f x' y'* − *f x' y* − *blinfun_apply* (*fy x' y*) (*y'* − *y*)) ≤ *norm* (*y'* − *y*) ∗ (*e* + *e*)
also assume *norm* (*f x' y* − *f x y* − (*fx*) (*x'* − *x*)) ≤ *norm* (*x'* − *x*) ∗ *e*
also
have *norm* ((*fy x y*) (*y'* − *y*) − (*fy x' y*) (*y'* − *y*)) ≤ *norm* ((*fy x y*) − (*fy x' y*)) ∗ *norm* (*y'* − *y*)
by (*auto simp*: *blinfun.bilinear_simps*[*symmetric*] *intro*!: *norm_blinfun*)
also have ... ≤ (*e* + *e*) ∗ *norm* (*y'* − *y*)
using ‹*e* > *0*› *nfy*
by (*auto simp*: *norm_minus_commute intro*!: *mult_right_mono*)
also have *norm* (*x'* − *x*) ∗ *e* ≤ *norm* (*x'* − *x*) ∗ (*e* + *e*)
using ‹*0* < *e*› **by** *simp*

```
    also have norm (y' − y) * (e + e) + (e + e) * norm (y' − y) + norm (x' −
x) * (e + e) ≤
        (norm (y' − y) + norm (x' − x)) * (4 * e)
      using ‹e > 0›
      by (simp add: algebra_simps)
    also have ... ≤ 2 * norm ((x', y') − (x, y)) * (4 * e)
      using ‹0 < e› real_sqrt_sum_squares_ge1[of norm (x' − x) norm (y' − y)]
        real_sqrt_sum_squares_ge2[of norm (y' − y) norm (x' − x)]
      by (auto intro!: mult_right_mono simp: norm_prod_def
        simp del: real_sqrt_sum_squares_ge1 real_sqrt_sum_squares_ge2)
    also have ... ≤ norm ((x', y') − (x, y)) * (8 * e)
      by simp
    also have ... < norm ((x', y') − (x, y)) * e'
      using ‹0 < e'› nz
      by (auto simp: e_def)
    finally show norm ((f x' y' − f x y − (fx (x' − x) + fy x y (y' − y))) /R norm
((x', y') − (x, y))) < e'
      by (simp add: dist_norm) (auto simp add: field_split_simps)
  qed
  then show ?case
    by eventually_elim (auto simp: dist_norm field_simps)
next
  from has_derivative_bounded_linear[OF fx]
  obtain fxb where fx = blinfun_apply fxb
    by (metis bounded_linear_Blinfun_apply)
  then show bounded_linear (λ(tx, ty). fx tx + blinfun_apply (fy x y) ty)
    by (auto intro!: bounded_linear_intros simp: split_beta')
qed
```

6.17.20 Differentiable case distinction

```
lemma has_derivative_within_If_eq:
  ((λx. if P x then f x else g x) has_derivative f') (at x within S) =
    (bounded_linear f' ∧
      ((λy.(if P y then (f y − ((if P x then f x else g x) + f' (y − x)))/R norm (y
− x)
          else (g y − ((if P x then f x else g x) + f' (y − x)))/R norm (y − x)))
      ⟶ 0) (at x within S))
  (is _ = (_ ∧ (?if ⟶ 0) _))
proof −
  have (λy. (1 / norm (y − x)) *R
          ((if P y then f y else g y) −
           ((if P x then f x else g x) + f' (y − x)))) = ?if
    by (auto simp: inverse_eq_divide)
  thus ?thesis by (auto simp: has_derivative_within)
qed

lemma has_derivative_If_within_closures:
  assumes f': x ∈ S ∪ (closure S ∩ closure T) ⟹
```

(*f has_derivative f' x*) (*at x within S ∪ (closure S ∩ closure T)*)
assumes *g'*: *x ∈ T ∪ (closure S ∩ closure T) ⟹*
(*g has_derivative g' x*) (*at x within T ∪ (closure S ∩ closure T)*)
assumes *connect*: *x ∈ closure S ⟹ x ∈ closure T ⟹ f x = g x*
assumes *connect'*: *x ∈ closure S ⟹ x ∈ closure T ⟹ f' x = g' x*
assumes *x_in*: *x ∈ S ∪ T*
shows ((*λx. if x ∈ S then f x else g x*) *has_derivative*
(*if x ∈ S then f' x else g' x*)) (*at x within (S ∪ T)*)
proof −
from *f' x_in* **interpret** *f'*: *bounded_linear if x ∈ S then f' x else (λx. 0)*
by (*auto simp add*: *has_derivative_within*)
from *g'* **interpret** *g'*: *bounded_linear if x ∈ T then g' x else (λx. 0)*
by (*auto simp add*: *has_derivative_within*)
have *bl*: *bounded_linear* (*if x ∈ S then f' x else g' x*)
using *f'.scaleR f'.bounded f'.add g'.scaleR g'.bounded g'.add x_in*
by (*unfold_locales*; *force*)
show *?thesis*
using *f' g' closure_subset*[*of T*] *closure_subset*[*of S*]
unfolding *has_derivative_within_If_eq*
by (*intro conjI bl tendsto_If_within_closures x_in*)
(*auto simp*: *has_derivative_within inverse_eq_divide connect connect' subsetD*)
qed

lemma *has_vector_derivative_If_within_closures*:
assumes *x_in*: *x ∈ S ∪ T*
assumes *u* = *S ∪ T*
assumes *f'*: *x ∈ S ∪ (closure S ∩ closure T) ⟹*
(*f has_vector_derivative f' x*) (*at x within S ∪ (closure S ∩ closure T)*)
assumes *g'*: *x ∈ T ∪ (closure S ∩ closure T) ⟹*
(*g has_vector_derivative g' x*) (*at x within T ∪ (closure S ∩ closure T)*)
assumes *connect*: *x ∈ closure S ⟹ x ∈ closure T ⟹ f x = g x*
assumes *connect'*: *x ∈ closure S ⟹ x ∈ closure T ⟹ f' x = g' x*
shows ((*λx. if x ∈ S then f x else g x*) *has_vector_derivative*
(*if x ∈ S then f' x else g' x*)) (*at x within u*)
unfolding *has_vector_derivative_def assms*
using *x_in f' g'*
by (*intro has_derivative_If_within_closures*[**where** *?f' = λx a. a ∗R f' x* **and** *?g' = λx a. a ∗R g' x*,
THEN has_derivative_eq_rhs]; *force simp*: *assms has_vector_derivative_def*)

6.17.21 The Inverse Function Theorem

lemma *linear_injective_contraction*:
assumes *linear f c < 1* **and** *le*: *⋀x. norm (f x − x) ≤ c ∗ norm x*
shows *inj f*
unfolding *linear_injective_0*[*OF ‹linear f›*]
proof *safe*
fix *x*

```
  assume f x = 0
  with le [of x] have norm x ≤ c * norm x
    by simp
  then show x = 0
    using ‹c < 1› by (simp add: mult_le_cancel_right1)
qed
```

From an online proof by J. Michael Boardman, Department of Mathematics, Johns Hopkins University

```
lemma inverse_function_theorem_scaled:
  fixes f::'a::euclidean_space ⇒ 'a
    and f'::'a ⇒ ('a ⇒_L 'a)
  assumes open U
    and derf: ⋀x. x ∈ U ⟹ (f has_derivative blinfun_apply (f' x)) (at x)
    and contf: continuous_on U f'
    and 0 ∈ U and [simp]: f 0 = 0
    and id: f' 0 = id_blinfun
  obtains U' V g g' where open U' U' ⊆ U 0 ∈ U' open V 0 ∈ V homeomorphism
U' V f g
                ⋀y. y ∈ V ⟹ (g has_derivative (g' y)) (at y)
                ⋀y. y ∈ V ⟹ g' y = inv (blinfun_apply (f'(g y)))
                ⋀y. y ∈ V ⟹ bij (blinfun_apply (f'(g y)))
proof −
  obtain d1 where cball 0 d1 ⊆ U d1 > 0
    using ‹open U› ‹0 ∈ U› open_contains_cball by blast
  obtain d2 where d2: ⋀x. ⟦x ∈ U; dist x 0 ≤ d2⟧ ⟹ dist (f' x) (f' 0) < 1/2
0 < d2
   using continuous_onE [OF contf, of 0 1/2] by (metis ‹0 ∈ U› half_gt_zero_iff
zero_less_one)
  obtain δ where le: ⋀x. norm x ≤ δ ⟹ dist (f' x) id_blinfun ≤ 1/2 and 0 <
δ
    and subU: cball 0 δ ⊆ U
  proof
    show min d1 d2 > 0
      by (simp add: ‹0 < d1› ‹0 < d2›)
    show cball 0 (min d1 d2) ⊆ U
      using ‹cball 0 d1 ⊆ U› by auto
    show dist (f' x) id_blinfun ≤ 1/2 if norm x ≤ min d1 d2 for x
      using ‹cball 0 d1 ⊆ U› d2 that id by fastforce
  qed
  let ?D = cball 0 δ
  define V:: 'a set where V ≡ ball 0 (δ/2)
  have 4: norm (f (x + h) − f x − h) ≤ 1/2 * norm h
    if x ∈ ?D x+h ∈ ?D for x h
  proof −
    let ?w = λx. f x − x
    have B: ⋀x. x ∈ ?D ⟹ onorm (blinfun_apply (f' x − id_blinfun)) ≤ 1/2
      by (metis dist_norm le mem_cball_0 norm_blinfun.rep_eq)
    have ⋀x. x ∈ ?D ⟹ (?w has_derivative (blinfun_apply (f' x − id_blinfun)))
```

```
(at x)
      by (rule derivative_eq_intros derf subsetD [OF subU] | force simp: blin-
fun.diff_left)+
    then have Dw: ⋀x. x ∈ ?D ⟹ (?w has_derivative (blinfun_apply (f' x −
id_blinfun))) (at x within ?D)
      using has_derivative_at_withinI by blast
    have norm (?w (x+h) − ?w x) ≤ (1/2) ∗ norm h
      using differentiable_bound [OF convex_cball Dw B] that by fastforce
    then show ?thesis
      by (auto simp: algebra_simps)
  qed
  have for_g: ∃!x. norm x < δ ∧ f x = y if y: norm y < δ/2 for y
  proof −
    let ?u = λx. x + (y − f x)
    have ∗: norm (?u x) < δ if x ∈ ?D for x
    proof −
      have fxx: norm (f x − x) ≤ δ/2
        using 4 [of 0 x] ‹0 < δ› ‹f 0 = 0› that by auto
      have norm (?u x) ≤ norm y + norm (f x − x)
      by (metis add.commute add_diff_eq norm_minus_commute norm_triangle_ineq)
      also have ... < δ/2 + δ/2
        using fxx y by auto
      finally show ?thesis
        by simp
    qed
    have ∃!x ∈ ?D. ?u x = x
    proof (rule banach_fix)
      show cball 0 δ ≠ {}
        using ‹0 < δ› by auto
      show (λx. x + (y − f x)) ‘ cball 0 δ ⊆ cball 0 δ
        using ∗ by force
      have dist (x + (y − f x)) (xh + (y − f xh)) ∗ 2 ≤ dist x xh
        if norm x ≤ δ and norm xh ≤ δ for x xh
       using that 4 [of x xh−x] by (auto simp: dist_norm norm_minus_commute
algebra_simps)
      then show ∀x∈cball 0 δ. ∀ya∈cball 0 δ. dist (x + (y − f x)) (ya + (y − f
ya)) ≤ (1/2) ∗ dist x ya
        by auto
    qed (auto simp: complete_eq_closed)
    then show ?thesis
     by (metis ∗ add_cancel_right_right eq_iff_diff_eq_0 le_less mem_cball_0)
  qed
  define g where g ≡ λy. THE x. norm x < δ ∧ f x = y
  have g: norm (g y) < δ ∧ f (g y) = y if norm y < δ/2 for y
    unfolding g_def using that theI' [OF for_g] by meson
  then have fg[simp]: f (g y) = y if y ∈ V for y
    using that by (auto simp: V_def)
  have 5: norm (g y' − g y) ≤ 2 ∗ norm (y' − y) if y ∈ V y' ∈ V for y y'
  proof −
```

```
    have no: norm (g y) ≤ δ norm (g y') ≤ δ and [simp]: f (g y) = y
      using that g unfolding V_def by force+
    have norm (g y' − g y) ≤ norm (g y' − g y − (y' − y)) + norm (y' − y)
      by (simp add: add.commute norm_triangle_sub)
    also have ... ≤ (1/2) * norm (g y' − g y) + norm (y' − y)
      using 4 [of g y g y' − g y] that no by (simp add: g norm_minus_commute
V_def)
    finally show ?thesis
      by auto
  qed
  have contg: continuous_on V g
  proof
    fix y::'a and e::real
    assume 0 < e and y: y ∈ V
    show ∃ d>0. ∀ x'∈V. dist x' y < d ⟶ dist (g x') (g y) ≤ e
    proof (intro exI conjI ballI impI)
      show 0 < e/2
        by (simp add: ‹0 < e›)
    qed (use 5 y in ‹force simp: dist_norm›)
  qed
  show thesis
  proof
    define U' where U' ≡ (f −‘ V) ∩ ball 0 δ
    have contf: continuous_on U f
    using derf has_derivative_at_withinI by (fast intro: has_derivative_continuous_on)
    then have continuous_on (ball 0 δ) f
      by (meson ball_subset_cball continuous_on_subset subU)
    then show open U'
      by (simp add: U'_def V_def Int_commute continuous_open_preimage)
    show 0 ∈ U' U' ⊆ U open V 0 ∈ V
      using ‹0 < δ› subU by (auto simp: U'_def V_def)
    show hom: homeomorphism U' V f g
    proof
      show continuous_on U' f
        using ‹U' ⊆ U› contf continuous_on_subset by blast
      show continuous_on V g
        using contg by blast
      show f ‘ U' ⊆ V
        using U'_def by blast
      show g ‘ V ⊆ U'
        by (simp add: U'_def V_def g image_subset_iff)
      show g (f x) = x if x ∈ U' for x
        by (metis that fg Int_iff U'_def V_def for_g g mem_ball_0 vimage_eq)
      show f (g y) = y if y ∈ V for y
        using that by (simp add: g V_def)
    qed
    show bij: bij (blinfun_apply (f'(g y))) if y ∈ V for y
    proof −
      have inj: inj (blinfun_apply (f' (g y)))
```

```
    proof (rule linear_injective_contraction)
      show linear (blinfun_apply (f' (g y)))
        using blinfun.bounded_linear_right bounded_linear_def by blast
    next
      fix x
      have norm (blinfun_apply (f' (g y)) x - x) = norm (blinfun_apply (f' (g
y) - id_blinfun) x)
        by (simp add: blinfun.diff_left)
      also have ... ≤ norm (f' (g y) - id_blinfun) * norm x
        by (rule norm_blinfun)
      also have ... ≤ (1/2) * norm x
      proof (rule mult_right_mono)
        show norm (f' (g y) - id_blinfun) ≤ 1/2
          using that g [of y] le by (auto simp: V_def dist_norm)
      qed auto
      finally show norm (blinfun_apply (f' (g y)) x - x) ≤ (1/2) * norm x .
    qed auto
    moreover
    have surj (blinfun_apply (f' (g y)))
      using blinfun.bounded_linear_right bounded_linear_def
      by (blast intro!: linear_inj_imp_surj [OF _ inj])
    ultimately show ?thesis
      using bijI by blast
  qed
  define g' where g' ≡ λy. inv (blinfun_apply (f'(g y)))
  show (g has_derivative g' y) (at y) if y ∈ V for y
  proof −
    have gy: g y ∈ U
      using g subU that unfolding V_def by fastforce
    obtain e where e: ⋀h. f (g y + h) = y + blinfun_apply (f' (g y)) h + e h
      and e0: (λh. norm (e h) / norm h) −0→ 0
      using iffD1 [OF has_derivative_iff_Ex derf [OF gy]] ‹y ∈ V› by auto
    have [simp]: e 0 = 0
      using e [of 0] that by simp
    let ?INV = inv (blinfun_apply (f' (g y)))
    have inj: inj (blinfun_apply (f' (g y)))
      using bij bij_betw_def that by blast
    have (g has_derivative g' y) (at y within V)
      unfolding has_derivative_at_within_iff_Ex [OF ‹y ∈ V› ‹open V›]
    proof
      show blinv: bounded_linear (g' y)
        unfolding g'_def using derf gy inj inj_linear_imp_inv_bounded_linear
by blast
      define eg where eg ≡ λk. − ?INV (e (g (y+k) − g y))
      have g (y+k) = g y + g' y k + eg k if y + k ∈ V for k
      proof −
        have ?INV k = ?INV (blinfun_apply (f' (g y)) (g (y+k) − g y) + e (g
(y+k) − g y))
          using e [of g(y+k) − g y] that by simp
```

```
      then have g (y+k) = g y + ?INV k - ?INV (e (g (y+k) - g y))
        using inj blinv by (simp add: linear_simps g'_def)
      then show ?thesis
        by (auto simp: eg_def g'_def)
    qed
    moreover have (λk. norm (eg k) / norm k) -0→ 0
    proof (rule Lim_null_comparison)
      let ?g = λk. 2 * onorm ?INV * norm (e (g (y+k) - g y)) / norm (g
(y+k) - g y)
      show ∀F k in at 0. norm (norm (eg k) / norm k) ≤ ?g k
        unfolding eventually_at_topological
      proof (intro exI conjI ballI impI)
        show open ((+)(-y) ‘ V)
          using ‹open V› open_translation by blast
        show 0 ∈ (+)(-y) ‘ V
          by (simp add: that)
        show norm (norm (eg k) / norm k) ≤ 2 * onorm (inv (blinfun_apply
(f' (g y)))) * norm (e (g (y+k) - g y)) / norm (g (y+k) - g y)
          if k ∈ (+)(-y) ‘ V k ≠ 0 for k
        proof -
          have y+k ∈ V
            using that by auto
          have norm (norm (eg k) / norm k) ≤ onorm ?INV * norm (e (g (y+k)
- g y)) / norm k
            using blinv g'_def onorm by (force simp: eg_def divide_simps)
          also have ... = (norm (g (y+k) - g y) / norm k) * (onorm ?INV *
(norm (e (g (y+k) - g y)) / norm (g (y+k) - g y)))
            by (simp add: divide_simps)
          also have ... ≤ 2 * (onorm ?INV * (norm (e (g (y+k) - g y)) /
norm (g (y+k) - g y)))
            apply (rule mult_right_mono)
            using 5 [of y y+k] ‹y ∈ V› ‹y + k ∈ V›  onorm_pos_le [OF blinv]
            apply (auto simp: divide_simps zero_le_mult_iff zero_le_divide_iff
g'_def)
            done
          finally show norm (norm (eg k) / norm k) ≤ 2 * onorm ?INV * norm
(e (g (y+k) - g y)) / norm (g (y+k) - g y)
            by simp
        qed
      qed
      have 1: (λh. norm (e h) / norm h) -0→ (norm (e 0) / norm 0)
        using e0 by auto
      have 2: (λk. g (y+k) - g y) -0→ 0
          using contg ‹open V› ‹y ∈ V› LIM_offset_zero_iff LIM_zero_iff
at_within_open continuous_on_def by fastforce
      from tendsto_compose [OF 1 2, simplified]
      have (λk. norm (e (g (y+k) - g y)) / norm (g (y+k) - g y)) -0→ 0 .
      from tendsto_mult_left [OF this] show ?g -0→ 0 by auto
    qed
```

ultimately show $\exists e.\ (\forall k.\ y + k \in V \longrightarrow g\ (y+k) = g\ y + g'\ y\ k + e\ k) \wedge (\lambda k.\ norm\ (e\ k)\ /\ norm\ k)\ -0\rightarrow\ 0$
by *blast*
qed
then show *?thesis*
by (*metis* ‹*open V*› *at_within_open that*)
qed
show $g'\ y = inv\ (blinfun\_apply\ (f'\ (g\ y)))$
if $y \in V$ **for** y
by (*simp add*: *g'_def*)
qed
qed

We need all this to justify the scaling and translations.

theorem *inverse_function_theorem*:
fixes $f{::}'a{::}euclidean\_space \Rightarrow\ 'a$
and $f'{::}'a \Rightarrow ('a \Rightarrow_L\ 'a)$
assumes *open U*
and *derf*: $\bigwedge x.\ x \in U \Longrightarrow (f\ has\_derivative\ (blinfun\_apply\ (f'\ x)))\ (at\ x)$
and *contf*: *continuous_on U f'*
and $x0 \in U$
and *invf*: $invf\ o_L\ f'\ x0 = id\_blinfun$
obtains $U'\ V\ g\ g'$ **where** *open U' U' ⊆ U x0 ∈ U' open V f x0 ∈ V homeomorphism U' V f g*
$\bigwedge y.\ y \in V \Longrightarrow (g\ has\_derivative\ (g'\ y))\ (at\ y)$
$\bigwedge y.\ y \in V \Longrightarrow g'\ y = inv\ (blinfun\_apply\ (f'(g\ y)))$
$\bigwedge y.\ y \in V \Longrightarrow bij\ (blinfun\_apply\ (f'(g\ y)))$
proof −
have *apply1* [*simp*]: $\bigwedge i.\ blinfun\_apply\ invf\ (blinfun\_apply\ (f'\ x0)\ i) = i$
by (*metis blinfun_apply_blinfun_compose blinfun_apply_id_blinfun invf*)
have *apply2* [*simp*]: $\bigwedge i.\ blinfun\_apply\ (f'\ x0)\ (blinfun\_apply\ invf\ i) = i$
by (*metis apply1 bij_inv_eq_iff blinfun_bij1 invf*)
have [*simp*]: $(range\ (blinfun\_apply\ invf)) = UNIV$
using *apply1 surjI* **by** *blast*
let $?f = invf \circ (\lambda x.\ (f \circ (+)x0)x - f\ x0)$
let $?f' = \lambda x.\ invf\ o_L\ (f'\ (x + x0))$
obtain $U'\ V\ g\ g'$ **where** *open U'* **and** $U'{:}\ U' \subseteq (+)(-x0)\ `\ U\ 0 \in U'$
and *open V 0 ∈ V* **and** *hom*: *homeomorphism U' V ?f g*
and *derg*: $\bigwedge y.\ y \in V \Longrightarrow (g\ has\_derivative\ (g'\ y))\ (at\ y)$
and $g'{:}\ \bigwedge y.\ y \in V \Longrightarrow g'\ y = inv\ (?f'(g\ y))$
and *bij*: $\bigwedge y.\ y \in V \Longrightarrow bij\ (?f'(g\ y))$
proof (*rule inverse_function_theorem_scaled* [*of* $(+)(-x0)\ `\ U\ ?f\ ?f'$])
show *ope*: *open* $((+)\ (-\ x0)\ `\ U)$
using ‹*open U*› *open_translation* **by** *blast*
show $(?f\ has\_derivative\ blinfun\_apply\ (?f'\ x))\ (at\ x)$
if $x \in (+)\ (-\ x0)\ `\ U$ **for** x
using *that*
apply *clarify*
apply (*rule derf derivative_eq_intros* | *simp add*: *blinfun_compose.rep_eq*)+

```
    done
  have YY: (λx. f' (x + x0)) −u−x0→ f' u
    if f' −u→ f' u u ∈ U for u
    using that LIM_offset [where k = x0] by (auto simp: algebra_simps)
  then have continuous_on ((+) (− x0) ‘ U) (λx. f' (x + x0))
    using contf ‹open U› Lim_at_imp_Lim_at_within
    by (fastforce simp: continuous_on_def at_within_open_NO_MATCH ope)
  then show continuous_on ((+) (− x0) ‘ U) ?f'
    by (intro continuous_intros) simp
 qed (auto simp: invf ‹x0 ∈ U›)
 show thesis
 proof
  let ?U' = (+)x0 ‘ U'
  let ?V = ((+)(f x0) ∘ f' x0) ‘ V
  let ?g = (+)x0 ∘ g ∘ invf ∘ (+)(− f x0)
  let ?g' = λy. inv (blinfun_apply (f' (?g y)))
  show oU': open ?U'
    by (simp add: ‹open U'› open_translation)
  show subU: ?U' ⊆ U
    using ComplI ‹U' ⊆ (+) (− x0) ‘ U› by auto
  show x0 ∈ ?U'
    by (simp add: ‹0 ∈ U'›)
  show open ?V
    using blinfun_bij2 [OF invf]
   by (metis ‹open V› bij_is_surj blinfun.bounded_linear_right bounded_linear_def
image_comp open_surjective_linear_image open_translation)
  show f x0 ∈ ?V
    using ‹0 ∈ V› image_iff by fastforce
  show homeomorphism ?U' ?V f ?g
  proof
    show continuous_on ?U' f
     by (meson subU continuous_on_eq_continuous_at derf has_derivative_continuous
oU' subsetD)
    have ?f ‘ U' ⊆ V
      using hom homeomorphism_image1 by blast
    then show f ‘ ?U' ⊆ ?V
      unfolding image_subset_iff
     by (clarsimp simp: image_def) (metis apply2 add.commute diff_add_cancel)
    show ?g ‘ ?V ⊆ ?U'
      using hom invf by (auto simp: image_def homeomorphism_def)
    show ?g (f x) = x
      if x ∈ ?U' for x
      using that hom homeomorphism_apply1 by fastforce
    have continuous_on V g
      using hom homeomorphism_def by blast
    then show continuous_on ?V ?g
      by (intro continuous_intros) (auto elim!: continuous_on_subset)
    have fg: ?f (g x) = x if x ∈ V for x
      using hom homeomorphism_apply2 that by blast
```

```
    show f (?g y) = y
      if y ∈ ?V for y
        using that fg by (simp add: image_iff) (metis apply2 add.commute
diff_add_cancel)
  qed
  show (?g has_derivative ?g' y) (at y) bij (blinfun_apply (f' (?g y)))
    if y ∈ ?V for y
  proof −
    have 1: bij (blinfun_apply invf)
      using blinfun_bij1 invf by blast
    then have 2: bij (blinfun_apply (f' (x0 + g x))) if x ∈ V for x
      by (metis add.commute bij bij_betw_comp_iff2 blinfun_compose.rep_eq
that top_greatest)
    then show bij (blinfun_apply (f' (?g y)))
      using that by auto
    have g' x ∘ blinfun_apply invf = inv (blinfun_apply (f' (x0 + g x)))
      if x ∈ V for x
      using that
      by (simp add: g' o_inv_distrib blinfun_compose.rep_eq 1 2 add.commute
bij_is_inj flip: o_assoc)
    then show (?g has_derivative ?g' y) (at y)
      using that invf
      by clarsimp (rule derg derivative_eq_intros | simp flip: id_def)+
  qed
 qed auto
qed
```

6.17.22 Piecewise differentiable functions

```
definition piecewise_differentiable_on
        (infixr ‹piecewise'_differentiable'_on› 50)
 where f piecewise_differentiable_on i ≡
        continuous_on i f ∧
        (∃ S. finite S ∧ (∀ x ∈ i − S. f differentiable (at x within i)))

lemma piecewise_differentiable_on_imp_continuous_on:
  f piecewise_differentiable_on S ⟹ continuous_on S f
by (simp add: piecewise_differentiable_on_def)

lemma piecewise_differentiable_on_subset:
  f piecewise_differentiable_on S ⟹ T ≤ S ⟹ f piecewise_differentiable_on
T
 using continuous_on_subset
 by (smt (verit) Diff_iff differentiable_within_subset in_mono piecewise_differentiable_on_def)

lemma differentiable_on_imp_piecewise_differentiable:
 fixes a:: 'a::{linorder_topology,real_normed_vector}
 shows f differentiable_on {a..b} ⟹ f piecewise_differentiable_on {a..b}
 using differentiable_imp_continuous_on differentiable_onD piecewise_differentiable_on_def
```

by *fastforce*

lemma *differentiable_imp_piecewise_differentiable*:
(⋀*x*. *x* ∈ *S* ⟹ *f differentiable* (*at x within S*))
⟹ *f piecewise_differentiable_on S*
by (*auto simp*: *piecewise_differentiable_on_def differentiable_imp_continuous_on differentiable_on_def*
intro: *differentiable_within_subset*)

lemma *piecewise_differentiable_const* [*iff*]: (λ*x*. *z*) *piecewise_differentiable_on S*
by (*simp add*: *differentiable_imp_piecewise_differentiable*)

lemma *piecewise_differentiable_compose*:
⟦*f piecewise_differentiable_on S*; *g piecewise_differentiable_on* (*f* ‘ *S*);
⋀*x*. *finite* (*S* ∩ *f*−‘{*x*})⟧
⟹ (*g* ∘ *f*) *piecewise_differentiable_on S*
apply (*simp add*: *piecewise_differentiable_on_def*, *safe*)
apply (*blast intro*: *continuous_on_compose2*)
apply (*rename_tac A B*)
apply (*rule_tac x*=*A* ∪ (⋃*x*∈*B*. *S* ∩ *f*−‘{*x*}) **in** *exI*)
apply (*blast intro*!: *differentiable_chain_within*)
done

lemma *piecewise_differentiable_affine*:
fixes *m*::*real*
assumes *f piecewise_differentiable_on* ((λ*x*. *m* ∗*R* *x* + *c*) ‘ *S*)
shows (*f* ∘ (λ*x*. *m* ∗*R* *x* + *c*)) *piecewise_differentiable_on S*
proof (*cases m* = *0*)
case *True*
then show *?thesis*
unfolding *o_def*
by (*force intro*: *differentiable_imp_piecewise_differentiable differentiable_const*)
next
case *False*
show *?thesis*
apply (*rule piecewise_differentiable_compose* [*OF differentiable_imp_piecewise_differentiable*])
apply (*rule assms derivative_intros* | *simp add*: *False vimage_def real_vector_affinity_eq*)+
done
qed

lemma *piecewise_differentiable_cases*:
fixes *c*::*real*
assumes *f piecewise_differentiable_on* {*a*..*c*}
g piecewise_differentiable_on {*c*..*b*}
a ≤ *c c* ≤ *b f c* = *g c*
shows (λ*x*. *if x* ≤ *c then f x else g x*) *piecewise_differentiable_on* {*a*..*b*}
proof −
obtain *S T* **where** *st*: *finite S finite T*
and *fd*: ⋀*x*. *x* ∈ {*a*..*c*} − *S* ⟹ *f differentiable at x within* {*a*..*c*}

```
            and gd: ⋀x. x ∈ {c..b} − T ⟹ g differentiable at x within {c..b}
    using assms
    by (auto simp: piecewise_differentiable_on_def)
  have finabc: finite ({a,b,c} ∪ (S ∪ T))
    by (metis ‹finite S› ‹finite T› finite_Un finite_insert finite.emptyI)
  have continuous_on {a..c} f continuous_on {c..b} g
    using assms piecewise_differentiable_on_def by auto
  then have continuous_on {a..b} (λx. if x ≤ c then f x else g x)
    using continuous_on_cases [OF closed_real_atLeastAtMost [of a c],
                               OF closed_real_atLeastAtMost [of c b],
                               of f g λx. x≤c]  assms
    by (force simp: ivl_disj_un_two_touch)
  moreover
  { fix x
    assume x: x ∈ {a..b} − ({a,b,c} ∪ (S ∪ T))
   have (λx. if x ≤ c then f x else g x) differentiable at x within {a..b} (is ?diff_fg)
    proof (cases x c rule: le_cases)
      case le show ?diff_fg
      proof (rule differentiable_transform_within [where d = dist x c])
        have f differentiable at x
          using x le fd [of x] at_within_interior [of x {a..c}] by simp
        then show f differentiable at x within {a..b}
          by (simp add: differentiable_at_withinI)
      qed (use x le st dist_real_def in auto)
    next
      case ge show ?diff_fg
      proof (rule differentiable_transform_within [where d = dist x c])
        have g differentiable at x
          using x ge gd [of x] at_within_interior [of x {c..b}] by simp
        then show g differentiable at x within {a..b}
          by (simp add: differentiable_at_withinI)
      qed (use x ge st dist_real_def in auto)
    qed
  }
  then have ∃S. finite S ∧
                (∀x∈{a..b} − S. (λx. if x ≤ c then f x else g x) differentiable at x
within {a..b})
    by (meson finabc)
  ultimately show ?thesis
    by (simp add: piecewise_differentiable_on_def)
qed

lemma piecewise_differentiable_neg:
   f piecewise_differentiable_on S ⟹ (λx. −(f x)) piecewise_differentiable_on S
  by (auto simp: piecewise_differentiable_on_def continuous_on_minus)

lemma piecewise_differentiable_add:
  assumes f piecewise_differentiable_on i
          g piecewise_differentiable_on i
```

shows $(\lambda x.\ f\ x + g\ x)$ *piecewise_differentiable_on i*
proof −
obtain *S T* **where** *st*: *finite S finite T*
$\forall x{\in}i - S.$ *f differentiable at x within i*
$\forall x{\in}i - T.$ *g differentiable at x within i*
using *assms* **by** (*auto simp*: *piecewise_differentiable_on_def*)
then have *finite* $(S \cup T) \wedge (\forall x{\in}i - (S \cup T).\ (\lambda x.\ f\ x + g\ x)$ *differentiable at x within i*)
by *auto*
moreover have *continuous_on i f continuous_on i g*
using *assms piecewise_differentiable_on_def* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *piecewise_differentiable_on_def continuous_on_add*)
qed

lemma *piecewise_differentiable_diff*:
$[\![f$ *piecewise_differentiable_on S*; g *piecewise_differentiable_on S*$]\!]$
$\Longrightarrow (\lambda x.\ f\ x - g\ x)$ *piecewise_differentiable_on S*
unfolding *diff_conv_add_uminus*
by (*metis piecewise_differentiable_add piecewise_differentiable_neg*)

6.17.23 The concept of continuously differentiable

John Harrison writes as follows:

"The usual assumption in complex analysis texts is that a path γ should be piecewise continuously differentiable, which ensures that the path integral exists at least for any continuous f, since all piecewise continuous functions are integrable. However, our notion of validity is weaker, just piecewise differentiability... [namely] continuity plus differentiability except on a finite set... [Our] underlying theory of integration is the Kurzweil-Henstock theory. In contrast to the Riemann or Lebesgue theory (but in common with a simple notion based on antiderivatives), this can integrate all derivatives."

"Formalizing basic complex analysis." From Insight to Proof: Festschrift in Honour of Andrzej Trybulec. Studies in Logic, Grammar and Rhetoric 10.23 (2007): 151-165.

And indeed he does not assume that his derivatives are continuous, but the penalty is unreasonably difficult proofs concerning winding numbers. We need a self-contained and straightforward theorem asserting that all derivatives can be integrated before we can adopt Harrison's choice.

definition *C1_differentiable_on* :: $(real \Rightarrow 'a{::}real\_normed\_vector) \Rightarrow real\ set \Rightarrow bool$
(**infix** ‹*C1'_differentiable'_on*› *50*)
where
f C1_differentiable_on S $\longleftrightarrow$
$(\exists D.\ (\forall x \in S.\ (f$ *has_vector_derivative* $(D\ x))\ (at\ x)) \wedge$ *continuous_on S D*$)$

lemma *C1_differentiable_on_eq*:
  f C1_differentiable_on S $\longleftrightarrow$
  $(\forall x \in S.$ *f differentiable at x*$) \wedge$ *continuous_on S* $(\lambda x.$ *vector_derivative f* (*at x*))
  (**is** *?lhs* = *?rhs*)
proof
 assume *?lhs*
 then show *?rhs*
  unfolding *C1_differentiable_on_def*
  by (*metis* (*no_types*, *lifting*) *continuous_on_eq differentiableI_vector vector_derivative_at*)
next
 assume *?rhs*
 then show *?lhs*
  using *C1_differentiable_on_def vector_derivative_works* **by** *fastforce*
qed

lemma *C1_differentiable_on_subset*:
 f C1_differentiable_on T $\Longrightarrow$ *S* $\subseteq$ *T* $\Longrightarrow$ *f C1_differentiable_on S*
 unfolding *C1_differentiable_on_def continuous_on_eq_continuous_within*
 by (*blast intro*: *continuous_within_subset*)

lemma *C1_differentiable_compose*:
 assumes *fg*: *f C1_differentiable_on S g C1_differentiable_on* (*f* ‘ *S*) **and** *fin*: $\bigwedge x.$ *finite* (*S* $\cap$ *f*−‘{*x*})
 shows (*g* $\circ$ *f*) *C1_differentiable_on S*
proof −
 have $\bigwedge x.$ *x* $\in$ *S* $\Longrightarrow$ *g* $\circ$ *f differentiable at x*
  by (*meson C1_differentiable_on_eq assms differentiable_chain_at imageI*)
 moreover have *continuous_on S* $(\lambda x.$ *vector_derivative* (*g* $\circ$ *f*) (*at x*))
  proof (*rule continuous_on_eq* [*of* _ $\lambda x.$ *vector_derivative f* (*at x*) $*_R$ *vector_derivative g* (*at* (*f x*))])
   show *continuous_on S* $(\lambda x.$ *vector_derivative f* (*at x*) $*_R$ *vector_derivative g* (*at* (*f x*)))
    using *fg*
    apply (*clarsimp simp add*: *C1_differentiable_on_eq*)
    apply (*rule Limits.continuous_on_scaleR*, *assumption*)
     by (*metis* (*mono_tags*, *lifting*) *continuous_at_imp_continuous_on continuous_on_compose continuous_on_cong differentiable_imp_continuous_within o_def*)
   show $\bigwedge x.$ *x* $\in$ *S* $\Longrightarrow$ *vector_derivative f* (*at x*) $*_R$ *vector_derivative g* (*at* (*f x*)) = *vector_derivative* (*g* $\circ$ *f*) (*at x*)
    by (*metis* (*mono_tags*, *opaque_lifting*) *C1_differentiable_on_eq fg imageI vector_derivative_chain_at*)
 qed
 ultimately show *?thesis*
  by (*simp add*: *C1_differentiable_on_eq*)
qed

lemma *C1_diff_imp_diff*: *f C1_differentiable_on S* ⟹ *f differentiable_on S*
by (*simp add*: *C1_differentiable_on_eq differentiable_at_imp_differentiable_on*)

lemma *C1_differentiable_on_ident* [*simp*, *derivative_intros*]: (λ*x*. *x*) *C1_differentiable_on S*
by (*auto simp*: *C1_differentiable_on_eq*)

lemma *C1_differentiable_on_const* [*simp*, *derivative_intros*]: (λ*z*. *a*) *C1_differentiable_on S*
by (*auto simp*: *C1_differentiable_on_eq*)

lemma *C1_differentiable_on_add* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* + *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq* **by** (*auto intro*: *continuous_intros*)

lemma *C1_differentiable_on_minus* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ (λ*x*. − *f x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq* **by** (*auto intro*: *continuous_intros*)

lemma *C1_differentiable_on_diff* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* − *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq* **by** (*auto intro*: *continuous_intros*)

lemma *C1_differentiable_on_mult* [*simp*, *derivative_intros*]:
fixes *f g* :: *real* ⇒ *'a* :: *real_normed_algebra*
shows *f C1_differentiable_on S* ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* ∗ *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq*
by (*auto simp*: *continuous_on_add continuous_on_mult continuous_at_imp_continuous_on differentiable_imp_continuous_within*)

lemma *C1_differentiable_on_scaleR* [*simp*, *derivative_intros*]:
f C1_differentiable_on S ⟹ *g C1_differentiable_on S* ⟹ (λ*x*. *f x* ∗*R* *g x*) *C1_differentiable_on S*
unfolding *C1_differentiable_on_eq*
by (*rule continuous_intros* | *simp add*: *continuous_at_imp_continuous_on differentiable_imp_continuous_within*)+

lemma *C1_differentiable_on_of_real* [*derivative_intros*]: *of_real C1_differentiable_on S*
unfolding *C1_differentiable_on_def*
using *vector_derivative_works* **by** *fastforce*

lemma *C1_differentiable_on_translation*:
f C1_differentiable_on U − *S* ⟹ (+) *d* ∘ *f C1_differentiable_on U* − *S*
by (*metis C1_differentiable_on_def has_vector_derivative_shift*)

lemma *C1_differentiable_on_translation_eq*:
fixes *d* :: *'a*::*real_normed_vector*
shows (+) *d* ∘ *f C1_differentiable_on i* − *S* ⟷ *f C1_differentiable_on i* − *S*
by (*force simp*: *o_def intro*: *C1_differentiable_on_translation dest*: *C1_differentiable_on_translation* [*of concl*: −*d*])

definition *piecewise_C1_differentiable_on*
(**infixr** ‹*piecewise'_C1'_differentiable'_on*› *50*)
where *f piecewise_C1_differentiable_on i* ≡
continuous_on i f ∧
(∃ *S*. *finite S* ∧ (*f C1_differentiable_on* (*i* − *S*)))

lemma *C1_differentiable_imp_piecewise*:
f C1_differentiable_on S ⟹ *f piecewise_C1_differentiable_on S*
by (*auto simp*: *piecewise_C1_differentiable_on_def C1_differentiable_on_eq continuous_at_imp_continuous_on differentiable_imp_continuous_within*)

lemma *piecewise_C1_imp_differentiable*:
f piecewise_C1_differentiable_on i ⟹ *f piecewise_differentiable_on i*
by (*auto simp*: *piecewise_C1_differentiable_on_def piecewise_differentiable_on_def C1_differentiable_on_def differentiable_def has_vector_derivative_def intro*: *has_derivative_at_withinI*)

lemma *piecewise_C1_differentiable_on_translation_eq*:
((+) *d* ∘ *f piecewise_C1_differentiable_on i*) ⟷ (*f piecewise_C1_differentiable_on i*)
unfolding *piecewise_C1_differentiable_on_def continuous_on_translation_eq*
by (*metis C1_differentiable_on_translation_eq*)

lemma *piecewise_C1_differentiable_compose* [*derivative_intros*]:
assumes *fg*: *f piecewise_C1_differentiable_on S g piecewise_C1_differentiable_on* (*f* ‘ *S*) **and** *fin*: ⋀*x*. *finite* (*S* ∩ *f*−‘{*x*})
shows (*g* ∘ *f*) *piecewise_C1_differentiable_on S*
proof −
have *continuous_on S* (λ*x*. *g* (*f x*))
by (*metis continuous_on_compose2 fg order_refl piecewise_C1_differentiable_on_def*)
moreover have ∃ *T*. *finite T* ∧ *g* ∘ *f C1_differentiable_on S* − *T*
proof −
obtain *F* **where** *finite F* **and** *F*: *f C1_differentiable_on S* − *F* **and** *f*: *f piecewise_C1_differentiable_on S*
using *fg* **by** (*auto simp*: *piecewise_C1_differentiable_on_def*)
obtain *G* **where** *finite G* **and** *G*: *g C1_differentiable_on f* ‘ *S* − *G* **and** *g*: *g piecewise_C1_differentiable_on f* ‘ *S*
using *fg* **by** (*auto simp*: *piecewise_C1_differentiable_on_def*)
show *?thesis*
proof (*intro exI conjI*)
show *finite* (*F* ∪ (⋃*x*∈*G*. *S* ∩ *f*−‘{*x*}))
using *fin* **by** (*auto simp only*: *Int_Union* ‹*finite F*› ‹*finite G*› *finite_UN*

finite_imageI)
show $g \circ f$ *C1_differentiable_on* $S - (F \cup (\bigcup x{\in}G.\ S \cap f -` \{x\}))$
apply (*rule C1_differentiable_compose*)
apply (*blast intro*: *C1_differentiable_on_subset* [*OF F*])
apply (*blast intro*: *C1_differentiable_on_subset* [*OF G*])
by (*simp add*: *C1_differentiable_on_subset G Diff_Int_distrib2 fin*)
qed
qed
ultimately show *?thesis*
by (*simp add*: *piecewise_C1_differentiable_on_def*)
qed

lemma *piecewise_C1_differentiable_on_subset*:
f *piecewise_C1_differentiable_on* $S \Longrightarrow T \leq S \Longrightarrow f$ *piecewise_C1_differentiable_on* T
by (*auto simp*: *piecewise_C1_differentiable_on_def elim*!: *continuous_on_subset C1_differentiable_on_subset*)

lemma *C1_differentiable_imp_continuous_on*:
f *C1_differentiable_on* $S \Longrightarrow$ *continuous_on* S f
unfolding *C1_differentiable_on_eq continuous_on_eq_continuous_within*
using *differentiable_at_withinI differentiable_imp_continuous_within* **by** *blast*

lemma *C1_differentiable_on_empty* [*iff*,*derivative_intros*]: f *C1_differentiable_on* $\{\}$
unfolding *C1_differentiable_on_def*
by *auto*

lemma *piecewise_C1_differentiable_affine*:
fixes *m*::*real*
assumes f *piecewise_C1_differentiable_on* $((\lambda x.\ m * x + c)\ `\ S)$
shows $(f \circ (\lambda x.\ m *_R x + c))$ *piecewise_C1_differentiable_on* S
proof (*cases* $m = 0$)
case *True*
then show *?thesis*
unfolding *o_def* **by** (*auto simp*: *piecewise_C1_differentiable_on_def*)
next
case *False*
have $*$: $\bigwedge x.$ *finite* $(S \cap \{y.\ m * y + c = x\})$
using *False not_finite_existsD* **by** *fastforce*
show *?thesis*
apply (*rule piecewise_C1_differentiable_compose* [*OF C1_differentiable_imp_piecewise*])
apply (*rule* $*$ *assms derivative_intros* | *simp add*: *False vimage_def*)+
done
qed

lemma *piecewise_C1_differentiable_cases* [*derivative_intros*]:
fixes *c*::*real*
assumes f *piecewise_C1_differentiable_on* $\{a..c\}$

```
        g piecewise_C1_differentiable_on {c..b}
        a ≤ c c ≤ b f c = g c
  shows (λx. if x ≤ c then f x else g x) piecewise_C1_differentiable_on {a..b}
proof −
  obtain S T where st: f C1_differentiable_on ({a..c} − S)
                      g C1_differentiable_on ({c..b} − T)
                      finite S finite T
    using assms
    by (force simp: piecewise_C1_differentiable_on_def)
  then have f_diff: f differentiable_on {a..<c} − S
       and g_diff: g differentiable_on {c<..b} − T
    by (simp_all add: C1_differentiable_on_eq differentiable_at_withinI differentiable_on_def)
  have continuous_on {a..c} f continuous_on {c..b} g
    using assms piecewise_C1_differentiable_on_def by auto
  then have cab: continuous_on {a..b} (λx. if x ≤ c then f x else g x)
    using continuous_on_cases [OF closed_real_atLeastAtMost [of a c],
                               OF closed_real_atLeastAtMost [of c b],
                               of f g λx. x≤c]  assms
    by (force simp: ivl_disj_un_two_touch)
  { fix x
    assume x: x ∈ {a..b} − insert c (S ∪ T)
    have (λx. if x ≤ c then f x else g x) differentiable at x (is ?diff_fg)
    proof (cases x c rule: le_cases)
      case le show ?diff_fg
       apply (rule differentiable_transform_within [where f=f and d = dist x c])
        using x dist_real_def le st by (auto simp: C1_differentiable_on_eq)
    next
      case ge show ?diff_fg
        apply (rule differentiable_transform_within [where f=g and d = dist x c])
     using dist_nz x dist_real_def ge st x by (auto simp: C1_differentiable_on_eq)
    qed
  }
  then have (∀ x ∈ {a..b} − insert c (S ∪ T). (λx. if x ≤ c then f x else g x) differentiable at x)
    by auto
  moreover
  { assume fcon: continuous_on ({a<..<c} − S) (λx. vector_derivative f (at x))
       and gcon: continuous_on ({c<..<b} − T) (λx. vector_derivative g (at x))
    have open ({a<..<c} − S)  open ({c<..<b} − T)
      using st by (simp_all add: open_Diff finite_imp_closed)
    moreover have continuous_on ({a<..<c} − S) (λx. vector_derivative (λx. if x ≤ c then f x else g x) (at x))
    proof −
      have ((λx. if x ≤ c then f x else g x) has_vector_derivative vector_derivative f (at x))
                (at x)
        if a < x x < c x ∉ S for x
      proof −
```

```
      have f: f differentiable at x
      by (meson C1_differentiable_on_eq Diff_iff atLeastAtMost_iff less_eq_real_def
st(1) that)
      show ?thesis
        using that
      apply (rule_tac f=f and d=dist x c in has_vector_derivative_transform_within)
          apply (auto simp: dist_norm vector_derivative_works [symmetric] f)
        done
    qed
    then show ?thesis
    by (metis (no_types, lifting) continuous_on_eq [OF fcon] DiffE greaterThanLessThan_iff
vector_derivative_at)
  qed
  moreover have continuous_on ({c<..<b} - T) (λx. vector_derivative (λx. if
x ≤ c then f x else g x) (at x))
  proof -
    have ((λx. if x ≤ c then f x else g x) has_vector_derivative vector_derivative
g (at x))            (at x)
      if c < x x < b x ∉ T for x
    proof -
      have g: g differentiable at x
       by (metis C1_differentiable_on_eq DiffD1 DiffI atLeastAtMost_diff_ends
greaterThanLessThan_iff st(2) that)
      show ?thesis
        using that
      apply (rule_tac f=g and d=dist x c in has_vector_derivative_transform_within)
          apply (auto simp: dist_norm vector_derivative_works [symmetric] g)
        done
    qed
    then show ?thesis
    by (metis (no_types, lifting) continuous_on_eq [OF gcon] DiffE greaterThanLessThan_iff
vector_derivative_at)
  qed
  ultimately have continuous_on ({a<..<b} - insert c (S ∪ T))
      (λx. vector_derivative (λx. if x ≤ c then f x else g x) (at x))
    by (rule continuous_on_subset [OF continuous_on_open_Un], auto)
 } note * = this
 have continuous_on ({a<..<b} - insert c (S ∪ T)) (λx. vector_derivative (λx.
if x ≤ c then f x else g x) (at x))
  using st
  by (auto simp: C1_differentiable_on_eq elim!: continuous_on_subset intro: *)
 ultimately have ∃ S. finite S ∧ ((λx. if x ≤ c then f x else g x) C1_differentiable_on
{a..b} - S)
  apply (rule_tac x={a,b,c} ∪ S ∪ T in exI)
 using st by (auto simp: C1_differentiable_on_eq elim!: continuous_on_subset)
 with cab show ?thesis
  by (simp add: piecewise_C1_differentiable_on_def)
qed
```

lemma *piecewise_C1_differentiable_const* [*derivative_intros*]:
(λ*x*. *c*) *piecewise_C1_differentiable_on S*
by (*simp add*: *C1_differentiable_imp_piecewise*)

lemma *piecewise_C1_differentiable_scaleR* [*derivative_intros*]:
⟦*f piecewise_C1_differentiable_on S*⟧
⟹ (λ*x*. *c* *$_R$ *f x*) *piecewise_C1_differentiable_on S*
by (*force simp add*: *piecewise_C1_differentiable_on_def continuous_on_scaleR*)

lemma *piecewise_C1_differentiable_neg* [*derivative_intros*]:
f piecewise_C1_differentiable_on S ⟹ (λ*x*. −(*f x*)) *piecewise_C1_differentiable_on S*
unfolding *piecewise_C1_differentiable_on_def*
by (*auto intro*!: *continuous_on_minus C1_differentiable_on_minus*)

lemma *piecewise_C1_differentiable_add* [*derivative_intros*]:
assumes *f piecewise_C1_differentiable_on i*
g piecewise_C1_differentiable_on i
shows (λ*x*. *f x* + *g x*) *piecewise_C1_differentiable_on i*
proof −
obtain *S t* **where** *st*: *finite S finite t*
f C1_differentiable_on (*i*−*S*)
g C1_differentiable_on (*i*−*t*)
using *assms* **by** (*auto simp*: *piecewise_C1_differentiable_on_def*)
then have *finite* (*S* ∪ *t*) ∧ (λ*x*. *f x* + *g x*) *C1_differentiable_on i* − (*S* ∪ *t*)
by (*auto intro*: *C1_differentiable_on_add elim*!: *C1_differentiable_on_subset*)
moreover have *continuous_on i f continuous_on i g*
using *assms piecewise_C1_differentiable_on_def* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *piecewise_C1_differentiable_on_def continuous_on_add*)
qed

lemma *piecewise_C1_differentiable_diff* [*derivative_intros*]:
⟦*f piecewise_C1_differentiable_on S*; *g piecewise_C1_differentiable_on S*⟧
⟹ (λ*x*. *f x* − *g x*) *piecewise_C1_differentiable_on S*
unfolding *diff_conv_add_uminus*
by (*metis piecewise_C1_differentiable_add piecewise_C1_differentiable_neg*)

lemma *piecewise_C1_differentiable_cmult_right* [*derivative_intros*]:
fixes *c*::*complex*
shows *f piecewise_C1_differentiable_on S*
⟹ (λ*x*. *f x* ∗ *c*) *piecewise_C1_differentiable_on S*
by (*force simp*: *piecewise_C1_differentiable_on_def continuous_on_mult_right*)

lemma *piecewise_C1_differentiable_cmult_left* [*derivative_intros*]:
fixes *c*::*complex*
shows *f piecewise_C1_differentiable_on S*
⟹ (λ*x*. *c* ∗ *f x*) *piecewise_C1_differentiable_on S*
using *piecewise_C1_differentiable_cmult_right* [*of f S c*] **by** (*simp add*: *mult.commute*)

lemma *piecewise_C1_differentiable_on_of_real* [*derivative_intros*]:
of_real piecewise_C1_differentiable_on S
by (*simp add*: *C1_differentiable_imp_piecewise C1_differentiable_on_of_real*)

end

6.18 Finite Cartesian Products of Euclidean Spaces

theory *Cartesian_Euclidean_Space*
imports *Derivative*
begin

lemma *subspace_special_hyperplane*: *subspace* {*x*. *x* $ *k* = *0*}
by (*simp add*: *subspace_def*)

lemma *sum_mult_product*:
sum h {..<*A* * *B* :: *nat*} = (∑ *i*∈{..<*A*}. ∑ *j*∈{..<*B*}. *h* (*j* + *i* * *B*))
unfolding *sum.nat_group*[*of h B A*, *unfolded atLeast0LessThan*, *symmetric*]
proof (*rule sum.cong*, *simp*, *rule sum.reindex_cong*)
fix *i*
show *inj_on* (λ*j*. *j* + *i* * *B*) {..<*B*} **by** (*auto intro*!: *inj_onI*)
show {*i* * *B*..<*i* * *B* + *B*} = (λ*j*. *j* + *i* * *B*) ‘ {..<*B*}
proof *safe*
fix *j* **assume** *j* ∈ {*i* * *B*..<*i* * *B* + *B*}
then show *j* ∈ (λ*j*. *j* + *i* * *B*) ‘ {..<*B*}
by (*auto intro*!: *image_eqI*[*of* _ _ *j* − *i* * *B*])
qed *simp*
qed *simp*

lemma *interval_cbox_cart*: {*a*::*real*^'*n*..*b*} = *cbox a b*
by (*auto simp add*: *less_eq_vec_def mem_box Basis_vec_def inner_axis*)

lemma *differentiable_vec*:
fixes *S* :: '*a*::*euclidean_space set*
shows *vec differentiable_on S*
by (*simp add*: *linear_linear bounded_linear_imp_differentiable_on*)

lemma *continuous_vec* [*continuous_intros*]:
fixes *x* :: '*a*::*euclidean_space*
shows *isCont vec x*
apply (*clarsimp simp add*: *continuous_def LIM_def dist_vec_def L2_set_def*)
apply (*rule_tac x*=*r* / *sqrt* (*real CARD*('*b*)) **in** *exI*)
by (*simp add*: *mult.commute pos_less_divide_eq real_sqrt_mult*)

lemma *box_vec_eq_empty* [*simp*]:
shows *cbox* (*vec a*) (*vec b*) = {} ⟷ *cbox a b* = {}
box (*vec a*) (*vec b*) = {} ⟷ *box a b* = {}
by (*auto simp*: *Basis_vec_def mem_box box_eq_empty inner_axis*)

6.18.1 Closures and interiors of halfspaces

lemma *interior_halfspace_component_le* [*simp*]:
  interior {*x*. *x*$*k* ≤ *a*} = {*x* :: (*real*^'*n*). *x*$*k* < *a*} (**is** *?LE*)
 and *interior_halfspace_component_ge* [*simp*]:
  interior {*x*. *x*$*k* ≥ *a*} = {*x* :: (*real*^'*n*). *x*$*k* > *a*} (**is** *?GE*)
proof −
 have *axis k* (*1*::*real*) ≠ *0*
  by (*simp add*: *axis_def vec_eq_iff*)
 moreover have *axis k* (*1*::*real*) · *x* = *x*$*k* **for** *x*
  by (*simp add*: *cart_eq_inner_axis inner_commute*)
 ultimately show *?LE ?GE*
  using *interior_halfspace_le* [*of axis k* (*1*::*real*) *a*]
    interior_halfspace_ge [*of axis k* (*1*::*real*) *a*] **by** *auto*
qed

lemma *closure_halfspace_component_lt* [*simp*]:
  closure {*x*. *x*$*k* < *a*} = {*x* :: (*real*^'*n*). *x*$*k* ≤ *a*} (**is** *?LE*)
 and *closure_halfspace_component_gt* [*simp*]:
  closure {*x*. *x*$*k* > *a*} = {*x* :: (*real*^'*n*). *x*$*k* ≥ *a*} (**is** *?GE*)
proof −
 have *axis k* (*1*::*real*) ≠ *0*
  by (*simp add*: *axis_def vec_eq_iff*)
 moreover have *axis k* (*1*::*real*) · *x* = *x*$*k* **for** *x*
  by (*simp add*: *cart_eq_inner_axis inner_commute*)
 ultimately show *?LE ?GE*
  using *closure_halfspace_lt* [*of axis k* (*1*::*real*) *a*]
    closure_halfspace_gt [*of axis k* (*1*::*real*) *a*] **by** *auto*
qed

lemma *interior_standard_hyperplane*:
 interior {*x* :: (*real*^'*n*). *x*$*k* = *a*} = {}
proof −
 have *axis k* (*1*::*real*) ≠ *0*
  by (*simp add*: *axis_def vec_eq_iff*)
 moreover have *axis k* (*1*::*real*) · *x* = *x*$*k* **for** *x*
  by (*simp add*: *cart_eq_inner_axis inner_commute*)
 ultimately show *?thesis*
  using *interior_hyperplane* [*of axis k* (*1*::*real*) *a*]
  by *force*
qed

lemma *matrix_vector_mul_bounded_linear*[*intro*, *simp*]: *bounded_linear* ((∗*v*) *A*)
for *A* :: *'a*::{*euclidean_space*,*real_algebra_1*}^'*n*^'*m*
 using *matrix_vector_mul_linear*[*of A*]
 by (*simp add*: *linear_conv_bounded_linear linear_matrix_vector_mul_eq*)

lemma
 fixes *A* :: *'a*::{*euclidean_space*,*real_algebra_1*}^'*n*^'*m*
 shows *matrix_vector_mult_linear_continuous_at* [*continuous_intros*]: *isCont*

$((*v)\ A)\ z$
and *matrix_vector_mult_linear_continuous_on* [*continuous_intros*]: *continuous_on S* $((*v)\ A)$
by (*simp_all add*: *linear_continuous_at linear_continuous_on*)

6.18.2 Bounds on components etc. relative to operator norm

lemma *norm_column_le_onorm*:
fixes $A :: real^{\wedge}{}'n^{\wedge}{}'m$
shows $norm(column\ i\ A) \leq onorm((*v)\ A)$
proof −
have $norm\ (\chi\ j.\ A\ \$\ j\ \$\ i) \leq norm\ (A *v\ axis\ i\ 1)$
by (*simp add*: *matrix_mult_dot cart_eq_inner_axis*)
also have $\ldots \leq onorm\ ((*v)\ A)$
using *onorm* [*OF matrix_vector_mul_bounded_linear*, *of A axis i 1*] **by** *auto*
finally have $norm\ (\chi\ j.\ A\ \$\ j\ \$\ i) \leq onorm\ ((*v)\ A)$ **.**
then show *?thesis*
unfolding *column_def* **.**
qed

lemma *matrix_component_le_onorm*:
fixes $A :: real^{\wedge}{}'n^{\wedge}{}'m$
shows $|A\ \$\ i\ \$\ j| \leq onorm((*v)\ A)$
proof −
have $|A\ \$\ i\ \$\ j| \leq norm\ (\chi\ n.\ (A\ \$\ n\ \$\ j))$
by (*metis* (*full_types*, *lifting*) *component_le_norm_cart vec_lambda_beta*)
also have $\ldots \leq onorm\ ((*v)\ A)$
by (*metis* (*no_types*) *column_def norm_column_le_onorm*)
finally show *?thesis* **.**
qed

lemma *component_le_onorm*:
fixes $f :: real^{\wedge}{}'m \Rightarrow real^{\wedge}{}'n$
shows $linear\ f \Longrightarrow |matrix\ f\ \$\ i\ \$\ j| \leq onorm\ f$
by (*metis matrix_component_le_onorm matrix_vector_mul*(*2*))

lemma *onorm_le_matrix_component_sum*:
fixes $A :: real^{\wedge}{}'n^{\wedge}{}'m$
shows $onorm((*v)\ A) \leq (\sum i{\in}UNIV.\ \sum j{\in}UNIV.\ |A\ \$\ i\ \$\ j|)$
proof (*rule onorm_le*)
fix x
have $norm\ (A *v\ x) \leq (\sum i{\in}UNIV.\ |(A *v\ x)\ \$\ i|)$
by (*rule norm_le_l1_cart*)
also have $\ldots \leq (\sum i{\in}UNIV.\ \sum j{\in}UNIV.\ |A\ \$\ i\ \$\ j| * norm\ x)$
proof (*rule sum_mono*)
fix i
have $|(A *v\ x)\ \$\ i| \leq |\sum j{\in}UNIV.\ A\ \$\ i\ \$\ j * x\ \$\ j|$
by (*simp add*: *matrix_vector_mult_def*)
also have $\ldots \leq (\sum j{\in}UNIV.\ |A\ \$\ i\ \$\ j * x\ \$\ j|)$

by (*rule sum_abs*)
also have ... $\leq (\sum j{\in}UNIV.\ |A\ \$\ i\ \$\ j| * norm\ x)$
by (*rule sum_mono*) (*simp add*: *abs_mult component_le_norm_cart mult_left_mono*)
finally show $|(A *v x)\ \$\ i| \leq (\sum j{\in}UNIV.\ |A\ \$\ i\ \$\ j| * norm\ x)$ **.**
qed
finally show $norm\ (A *v x) \leq (\sum i{\in}UNIV.\ \sum j{\in}UNIV.\ |A\ \$\ i\ \$\ j|) * norm\ x$
by (*simp add*: *sum_distrib_right*)
qed

lemma *onorm_le_matrix_component*:
fixes $A :: real^{\wedge}{}'n^{\wedge}{}'m$
assumes $\bigwedge i\ j.\ abs(A\$i\$j) \leq B$
shows $onorm((*v)\ A) \leq real\ (CARD('m)) * real\ (CARD('n)) * B$
proof (*rule onorm_le*)
fix $x :: real^{\wedge}{}'n::\_$
have $norm\ (A *v x) \leq (\sum i{\in}UNIV.\ |(A *v x)\ \$\ i|)$
by (*rule norm_le_l1_cart*)
also have ... $\leq (\sum i::'m\in UNIV.\ real\ (CARD('n)) * B * norm\ x)$
proof (*rule sum_mono*)
fix *i*
have $|(A *v x)\ \$\ i| \leq norm(A\ \$\ i) * norm\ x$
by (*simp add*: *matrix_mult_dot Cauchy_Schwarz_ineq2*)
also have ... $\leq (\sum j{\in}UNIV.\ |A\ \$\ i\ \$\ j|) * norm\ x$
by (*simp add*: *mult_right_mono norm_le_l1_cart*)
also have ... $\leq real\ (CARD('n)) * B * norm\ x$
by (*simp add*: *assms sum_bounded_above mult_right_mono*)
finally show $|(A *v x)\ \$\ i| \leq real\ (CARD('n)) * B * norm\ x$ **.**
qed
also have ... $\leq CARD('m) * real\ (CARD('n)) * B * norm\ x$
by *simp*
finally show $norm\ (A *v x) \leq CARD('m) * real\ (CARD('n)) * B * norm\ x$ **.**
qed

lemma *vector_sub_project_orthogonal_cart*: $(b::real^{\wedge}{}'n) \cdot (x - ((b \cdot x)\ /\ (b \cdot b)) *s\ b) = 0$
unfolding *inner_simps scalar_mult_eq_scaleR* **by** *auto*

lemma *infnorm_cart*:*infnorm* $(x::real^{\wedge}{}'n) = Sup\ \{|x\$i|\ |i.\ i{\in}UNIV\}$
by (*simp add*: *infnorm_def inner_axis Basis_vec_def*) (*metis* (*lifting*) *inner_axis real_inner_1_right*)

lemma *component_le_infnorm_cart*: $|x\$i| \leq infnorm\ (x::real^{\wedge}{}'n)$
using *Basis_le_infnorm*[*of axis i 1 x*]
by (*simp add*: *Basis_vec_def axis_eq_axis inner_axis*)

lemma *continuous_component*[*continuous_intros*]: *continuous F f* $\Longrightarrow$ *continuous F* $(\lambda x.\ f\ x\ \$\ i)$
unfolding *continuous_def* **by** (*rule tendsto_vec_nth*)

lemma *continuous_on_component*[*continuous_intros*]: *continuous_on s f* ⟹ *continuous_on s* (λ*x*. *f x* $ *i*)
unfolding *continuous_on_def* **by** (*fast intro*: *tendsto_vec_nth*)

lemma *continuous_on_vec_lambda*[*continuous_intros*]:
(⋀*i*. *continuous_on S* (*f i*)) ⟹ *continuous_on S* (λ*x*. χ *i*. *f i x*)
unfolding *continuous_on_def* **by** (*auto intro*: *tendsto_vec_lambda*)

lemma *closed_positive_orthant*: *closed* {*x*::*real*^'*n*. ∀ *i*. 0 ≤*x*$*i*}
by (*simp add*: *Collect_all_eq closed_INT closed_Collect_le continuous_on_component*)

lemma *bounded_component_cart*: *bounded s* ⟹ *bounded* ((λ*x*. *x* $ *i*) ' *s*)
unfolding *bounded_def*
apply *clarify*
apply (*rule_tac x*=*x* $ *i* **in** *exI*)
apply (*rule_tac x*=*e* **in** *exI*)
apply *clarify*
apply (*rule order_trans* [*OF dist_vec_nth_le*], *simp*)
done

lemma *compact_lemma_cart*:
fixes *f* :: *nat* ⇒ '*a*::*heine_borel* ^ '*n*
assumes *f*: *bounded* (*range f*)
shows ∃ *l r*. *strict_mono r* ∧
(∀ *e*>0. *eventually* (λ*n*. ∀ *i*∈*d*. *dist* (*f* (*r n*) $ *i*) (*l* $ *i*) < *e*) *sequentially*)
(**is** *?th d*)
proof −
have ∀ *d'* ⊆ *d*. *?th d'*
by (*rule compact_lemma_general*[**where** *unproj*=*vec_lambda*])
(*auto intro*!: *f bounded_component_cart*)
then show *?th d* **by** *simp*
qed

instance *vec* :: (*heine_borel*, *finite*) *heine_borel*
proof
fix *f* :: *nat* ⇒ '*a* ^ '*b*
assume *f*: *bounded* (*range f*)
then obtain *l r* **where** *r*: *strict_mono r*
and *l*: ∀ *e*>0. *eventually* (λ*n*. ∀ *i*∈*UNIV*. *dist* (*f* (*r n*) $ *i*) (*l* $ *i*) < *e*) *sequentially*
using *compact_lemma_cart* [*OF f*] **by** *blast*
let *?d* = *UNIV*::'*b set*
{ **fix** *e*::*real* **assume** *e*>0
hence 0 < *e* / (*real_of_nat* (*card ?d*))
using *zero_less_card_finite divide_pos_pos*[*of e*, *of real_of_nat* (*card ?d*)]
by *auto*
with *l* **have** *eventually* (λ*n*. ∀ *i*. *dist* (*f* (*r n*) $ *i*) (*l* $ *i*) < *e* / (*real_of_nat* (*card ?d*))) *sequentially*
by *simp*

```
    moreover
    { fix n
      assume n: ∀ i. dist (f (r n) $ i) (l $ i) < e / (real_of_nat (card ?d))
      have dist (f (r n)) l ≤ (∑ i∈?d. dist (f (r n) $ i) (l $ i))
        unfolding dist_vec_def using zero_le_dist by (rule L2_set_le_sum)
      also have ... < (∑ i∈?d. e / (real_of_nat (card ?d)))
        by (rule sum_strict_mono) (simp_all add: n)
      finally have dist (f (r n)) l < e by simp
    }
    ultimately have eventually (λn. dist (f (r n)) l < e) sequentially
      by (rule eventually_mono)
  }
  hence ((f ∘ r) ⟶ l) sequentially unfolding o_def tendsto_iff by simp
  with r show ∃ l r. strict_mono r ∧ ((f ∘ r) ⟶ l) sequentially by auto
qed

lemma interval_cart:
  fixes a :: real^'n
  shows box a b = {x::real^'n. ∀ i. a$i < x$i ∧ x$i < b$i}
    and cbox a b = {x::real^'n. ∀ i. a$i ≤ x$i ∧ x$i ≤ b$i}
   by (auto simp add: set_eq_iff less_vec_def less_eq_vec_def mem_box Basis_vec_def inner_axis)

lemma mem_box_cart:
  fixes a :: real^'n
  shows x ∈ box a b ⟷ (∀ i. a$i < x$i ∧ x$i < b$i)
    and x ∈ cbox a b ⟷ (∀ i. a$i ≤ x$i ∧ x$i ≤ b$i)
 using interval_cart[of a b] by (auto simp add: set_eq_iff less_vec_def less_eq_vec_def)

lemma interval_eq_empty_cart:
  fixes a :: real^'n
  shows (box a b = {} ⟷ (∃ i. b$i ≤ a$i)) (is ?th1)
    and (cbox a b = {} ⟷ (∃ i. b$i < a$i)) (is ?th2)
proof -
  { fix i x assume as:b$i ≤ a$i and x:x∈box a b
    hence a $ i < x $ i ∧ x $ i < b $ i unfolding mem_box_cart by auto
    hence a$i < b$i by auto
    hence False using as by auto }
  moreover
  { assume as:∀ i. ¬ (b$i ≤ a$i)
    let ?x = (1/2) *_R (a + b)
    { fix i
      have a$i < b$i using as[THEN spec[where x=i]] by auto
      hence a$i < ((1/2) *_R (a+b)) $ i ((1/2) *_R (a+b)) $ i < b$i
        unfolding vector_smult_component and vector_add_component
        by auto }
    hence box a b ≠ {} using mem_box_cart(1)[of ?x a b] by auto }
  ultimately show ?th1 by blast
```

```
  { fix i x assume as:b$i < a$i and x:x∈cbox a b
    hence a $ i ≤ x $ i ∧ x $ i ≤ b $ i unfolding mem_box_cart by auto
    hence a$i ≤ b$i by auto
    hence False using as by auto }
  moreover
  { assume as:∀ i. ¬ (b$i < a$i)
    let ?x = (1/2) *R (a + b)
    { fix i
      have a$i ≤ b$i using as[THEN spec[where x=i]] by auto
      hence a$i ≤ ((1/2) *R (a+b)) $ i ((1/2) *R (a+b)) $ i ≤ b$i
        unfolding vector_smult_component and vector_add_component
        by auto }
    hence cbox a b ≠ {} using mem_box_cart(2)[of ?x a b] by auto  }
  ultimately show ?th2 by blast
qed

lemma interval_ne_empty_cart:
  fixes a :: real^'n
  shows cbox a b ≠ {} ⟷ (∀ i. a$i ≤ b$i)
    and box a b ≠ {} ⟷ (∀ i. a$i < b$i)
  unfolding interval_eq_empty_cart[of a b] by (auto simp add: not_less not_le)


lemma subset_interval_imp_cart:
  fixes a :: real^'n
  shows (∀ i. a$i ≤ c$i ∧ d$i ≤ b$i) ⟹ cbox c d ⊆ cbox a b
    and (∀ i. a$i < c$i ∧ d$i < b$i) ⟹ cbox c d ⊆ box a b
    and (∀ i. a$i ≤ c$i ∧ d$i ≤ b$i) ⟹ box c d ⊆ cbox a b
    and (∀ i. a$i ≤ c$i ∧ d$i ≤ b$i) ⟹ box c d ⊆ box a b
  unfolding subset_eq[unfolded Ball_def] unfolding mem_box_cart
  by (auto intro: order_trans less_le_trans le_less_trans less_imp_le)

lemma interval_sing:
  fixes a :: 'a::linorder^'n
  shows {a .. a} = {a} ∧ {a<..<a} = {}
  apply (auto simp add: set_eq_iff less_vec_def less_eq_vec_def vec_eq_iff)
  done

lemma subset_interval_cart:
  fixes a :: real^'n
  shows cbox c d ⊆ cbox a b ⟷ (∀ i. c$i ≤ d$i) --> (∀ i. a$i ≤ c$i ∧ d$i ≤
b$i) (is ?th1)
    and cbox c d ⊆ box a b ⟷ (∀ i. c$i ≤ d$i) --> (∀ i. a$i < c$i ∧ d$i < b$i)
(is ?th2)
    and box c d ⊆ cbox a b ⟷ (∀ i. c$i < d$i) --> (∀ i. a$i ≤ c$i ∧ d$i ≤ b$i)
(is ?th3)
    and box c d ⊆ box a b ⟷ (∀ i. c$i < d$i) --> (∀ i. a$i ≤ c$i ∧ d$i ≤ b$i)
(is ?th4)
  using subset_box[of c d a b] by (simp_all add: Basis_vec_def inner_axis)
```

lemma *disjoint_interval_cart*:
fixes *a*::*real^'n*
shows *cbox a b ∩ cbox c d = {} ⟷ (∃ i. (b$i < a$i ∨ d$i < c$i ∨ b$i < c$i ∨ d$i < a$i))* (**is** *?th1*)
and *cbox a b ∩ box c d = {} ⟷ (∃ i. (b$i < a$i ∨ d$i ≤ c$i ∨ b$i ≤ c$i ∨ d$i ≤ a$i))* (**is** *?th2*)
and *box a b ∩ cbox c d = {} ⟷ (∃ i. (b$i ≤ a$i ∨ d$i < c$i ∨ b$i ≤ c$i ∨ d$i ≤ a$i))* (**is** *?th3*)
and *box a b ∩ box c d = {} ⟷ (∃ i. (b$i ≤ a$i ∨ d$i ≤ c$i ∨ b$i ≤ c$i ∨ d$i ≤ a$i))* (**is** *?th4*)
using *disjoint_interval*[*of a b c d*] **by** (*simp_all add*: *Basis_vec_def inner_axis*)

lemma *Int_interval_cart*:
fixes *a* :: *real^'n*
shows *cbox a b ∩ cbox c d = {(χ i. max (a$i) (c$i)) .. (χ i. min (b$i) (d$i))}*
unfolding *Int_interval*
by (*auto simp*: *mem_box less_eq_vec_def*)
(*auto simp*: *Basis_vec_def inner_axis*)

lemma *closed_interval_left_cart*:
fixes *b* :: *real^'n*
shows *closed {x::real^'n. ∀ i. x$i ≤ b$i}*
by (*simp add*: *Collect_all_eq closed_INT closed_Collect_le continuous_on_component*)

lemma *closed_interval_right_cart*:
fixes *a*::*real^'n*
shows *closed {x::real^'n. ∀ i. a$i ≤ x$i}*
by (*simp add*: *Collect_all_eq closed_INT closed_Collect_le continuous_on_component*)

lemma *is_interval_cart*:
is_interval (*s*::(*real^'n*) *set*) ⟷
(∀ *a*∈*s*. ∀ *b*∈*s*. ∀ *x*. (∀ *i*. ((*a$i* ≤ *x$i* ∧ *x$i* ≤ *b$i*) ∨ (*b$i* ≤ *x$i* ∧ *x$i* ≤ *a$i*))) ⟶ *x* ∈ *s*)
by (*simp add*: *is_interval_def Ball_def Basis_vec_def inner_axis imp_ex*)

lemma *closed_halfspace_component_le_cart*: *closed {x::real^'n. x$i ≤ a}*
by (*simp add*: *closed_Collect_le continuous_on_component*)

lemma *closed_halfspace_component_ge_cart*: *closed {x::real^'n. x$i ≥ a}*
by (*simp add*: *closed_Collect_le continuous_on_component*)

lemma *open_halfspace_component_lt_cart*: *open {x::real^'n. x$i < a}*
by (*simp add*: *open_Collect_less continuous_on_component*)

lemma *open_halfspace_component_gt_cart*: *open {x::real^'n. x$i > a}*
by (*simp add*: *open_Collect_less continuous_on_component*)

lemma *Lim_component_le_cart*:

fixes $f :: 'a \Rightarrow real^{\wedge}'n$
assumes $(f \longrightarrow l)$ *net* $\neg$ (*trivial_limit net*) *eventually* $(\lambda x.\ f\ x\ \$i \leq b)$ *net*
shows $l\$i \leq b$
by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_const tendsto_vec_nth*, *OF assms*(*1*, *3*)])

lemma *Lim_component_ge_cart*:
fixes $f :: 'a \Rightarrow real^{\wedge}'n$
assumes $(f \longrightarrow l)$ *net* $\neg$ (*trivial_limit net*) *eventually* $(\lambda x.\ b \leq (f\ x)\$i)$ *net*
shows $b \leq l\$i$
by (*rule tendsto_le*[*OF assms*(*2*) *tendsto_vec_nth tendsto_const*, *OF assms*(*1*, *3*)])

lemma *Lim_component_eq_cart*:
fixes $f :: 'a \Rightarrow real^{\wedge}'n$
assumes *net*: $(f \longrightarrow l)$ *net* $\neg$ *trivial_limit net* **and** *ev*:*eventually* $(\lambda x.\ f(x)\$i = b)$ *net*
shows $l\$i = b$
using *ev*[*unfolded order_eq_iff eventually_conj_iff*] **and**
Lim_component_ge_cart[*OF net*, *of b i*] **and**
Lim_component_le_cart[*OF net*, *of i b*] **by** *auto*

lemma *connected_ivt_component_cart*:
fixes $x :: real^{\wedge}'n$
shows *connected* $s \Longrightarrow x \in s \Longrightarrow y \in s \Longrightarrow x\$k \leq a \Longrightarrow a \leq y\$k \Longrightarrow (\exists z{\in}s.\ z\$k = a)$
using *connected_ivt_hyperplane*[*of s x y axis k 1 a*]
by (*auto simp add*: *inner_axis inner_commute*)

lemma *subspace_substandard_cart*: *vec.subspace* $\{x.\ (\forall i.\ P\ i \longrightarrow x\$i = 0)\}$
unfolding *vec.subspace_def* **by** *auto*

lemma *closed_substandard_cart*:
closed $\{x::'a::real\_normed\_vector\ ^{\wedge}\ 'n.\ \forall i.\ P\ i \longrightarrow x\$i = 0\}$
proof −
{ fix $i::'n$
have *closed* $\{x::'a\ ^{\wedge}\ 'n.\ P\ i \longrightarrow x\$i = 0\}$
by (*cases P i*) (*simp_all add*: *closed_Collect_eq continuous_on_component*)
}
thus *?thesis*
unfolding *Collect_all_eq* **by** (*simp add*: *closed_INT*)
qed

6.18.3 Convex Euclidean Space

lemma *Cart_1*:$(1::real^{\wedge}'n) = \sum Basis$
using *const_vector_cart*[*of 1*] **by** (*simp add*: *one_vec_def*)

declare *vector_add_ldistrib*[*simp*] *vector_ssub_ldistrib*[*simp*] *vector_smult_assoc*[*simp*]

vector_smult_rneg[*simp*]
declare *vector_sadd_rdistrib*[*simp*] *vector_sub_rdistrib*[*simp*]

lemmas *vector_component_simps* = *vector_minus_component vector_smult_component vector_add_component less_eq_vec_def vec_lambda_beta vector_uminus_component*

lemma *convex_box_cart*:
assumes $\bigwedge i.$ *convex* $\{x.\ P\ i\ x\}$
shows *convex* $\{x.\ \forall\, i.\ P\ i\ (x\$i)\}$
using *assms* **unfolding** *convex_def* **by** *auto*

6.18.4 Arbitrarily good rational approximations

lemma *rational_approximation*:
assumes $e > 0$
obtains *r*::*real* **where** $r \in \mathbb{Q}$ $|r - x| < e$
using *Rats_dense_in_real* [*of* $x - e/2$ $x + e/2$] *assms* **by** *auto*

lemma *Rats_closure_real*: *closure* $\mathbb{Q}$ = (*UNIV*::*real set*)
proof −
have $\bigwedge$*x*::*real*. $x \in$ *closure* $\mathbb{Q}$
by (*metis closure_approachable dist_real_def rational_approximation*)
then show *?thesis* **by** *auto*
qed

proposition *matrix_rational_approximation*:
fixes A :: *real*^'*n*^'*m*
assumes $e > 0$
obtains B **where** $\bigwedge i\ j.\ B\$i\$j \in \mathbb{Q}$ *onorm*$(\lambda x.\ (A - B) *v\ x) < e$
proof −
have $\forall\, i\ j.\ \exists\, q \in \mathbb{Q}.\ |q - A\ \$\ i\ \$\ j| < e\ /\ (2 * CARD('m) * CARD('n))$
using *assms* **by** (*force intro*: *rational_approximation* [*of* e / (2 * *CARD*('*m*) * *CARD*('*n*))])
then obtain B **where** B: $\bigwedge i\ j.\ B\$i\$j \in \mathbb{Q}$ **and** *Bclo*: $\bigwedge i\ j.\ |B\$i\$j - A\ \$\ i\ \$\ j| < e\ /\ (2 * CARD('m) * CARD('n))$
by (*auto simp*: *lambda_skolem Bex_def*)
show *?thesis*
proof
have *onorm* ((∗*v*) (*A* − *B*)) ≤ *real CARD*('*m*) ∗ *real CARD*('*n*) ∗
(*e* / (2 ∗ *real CARD*('*m*) ∗ *real CARD*('*n*)))
apply (*rule onorm_le_matrix_component*)
using *Bclo* **by** (*simp add*: *abs_minus_commute less_imp_le*)
also have ... < *e*
using ‹0 < *e*› **by** (*simp add*: *field_split_simps*)
finally show *onorm* ((∗*v*) (*A* − *B*)) < *e* **.**
qed (*use B* **in** *auto*)
qed

6.18.5 Derivative

definition *jacobian f net = matrix(frechet_derivative f net)*

proposition *jacobian_works*:
(*f*::(*real^'a*) ⇒ (*real^'b*)) *differentiable net* ⟷
(*f has_derivative* (λ*h*. (*jacobian f net*) ∗*v h*)) *net* (**is** *?lhs = ?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*simp add*: *frechet_derivative_works has_derivative_linear jacobian_def*)
next
assume *?rhs* **then show** *?lhs*
by (*rule differentiableI*)
qed

Component of the differential must be zero if it exists at a local maximum or minimum for that corresponding component

proposition *differential_zero_maxmin_cart*:
fixes *f*::*real^'a* ⇒ *real^'b*
assumes *0 < e* ((∀ *y* ∈ *ball x e*. (*f y*)$*k* ≤ (*f x*)$*k*) ∨ (∀ *y*∈*ball x e*. (*f x*)$*k* ≤ (*f y*)$*k*))
f differentiable (*at x*)
shows *jacobian f* (*at x*) $ *k = 0*
using *differential_zero_maxmin_component*[*of axis k 1 e x f*] *assms*
vector_cart[*of* λ*j*. *frechet_derivative f* (*at x*) *j* $ *k*]
by (*simp add*: *Basis_vec_def axis_eq_axis inner_axis jacobian_def matrix_def*)

6.18.6 Routine results connecting the types (*real*, *1*) *vec* and *real*

lemma *vec_cbox_1_eq* [*simp*]:
shows *vec ‘ cbox u v = cbox* (*vec u*) (*vec v* ::*real^1*)
by (*force simp*: *Basis_vec_def cart_eq_inner_axis* [*symmetric*] *mem_box*)

lemma *vec_nth_cbox_1_eq* [*simp*]:
fixes *u v* :: *'a*::*euclidean_space^1*
shows (λ*x*. *x* $ *1*) ‘ *cbox u v = cbox* (*u*$*1*) (*v*$*1*)
by (*auto simp*: *Basis_vec_def cart_eq_inner_axis* [*symmetric*] *mem_box image_iff Bex_def inner_axis*) (*metis vec_component*)

lemma *vec_nth_1_iff_cbox* [*simp*]:
fixes *a b* :: *'a*::*euclidean_space*
shows (λ*x*::*'a^1*. *x* $ *1*) ‘ *S = cbox a b* ⟷ *S = cbox* (*vec a*) (*vec b*)
(**is** *?lhs = ?rhs*)
proof
assume *L*: *?lhs* **show** *?rhs*
proof (*intro equalityI subsetI*)
fix *x*
assume *x* ∈ *S*
then have *x* $ *1* ∈ (λ*v*. *v* $ (*1*::*1*)) ‘ *cbox* (*vec a*) (*vec b*)

```
    using L by auto
  then show x ∈ cbox (vec a) (vec b)
    by (metis (no_types, lifting) imageE vector_one_nth)
 next
  fix x :: 'a^1
  assume x ∈ cbox (vec a) (vec b)
  then show x ∈ S
  by (metis (no_types, lifting) L imageE imageI vec_component vec_nth_cbox_1_eq
vector_one_nth)
 qed
qed simp

lemma vec_nth_real_1_iff_cbox [simp]:
 fixes a b :: real
 shows (λx::real^1. x $ 1) ' S = {a..b} ⟷ S = cbox (vec a) (vec b)
 using vec_nth_1_iff_cbox[of S a b]
 by simp

lemma interval_split_cart:
 {a..b::real^'n} ∩ {x. x$k ≤ c} = {a .. (χ i. if i = k then min (b$k) c else b$i)}
 cbox a b ∩ {x. x$k ≥ c} = {(χ i. if i = k then max (a$k) c else a$i) .. b}
 unfolding Int_iff mem_box_cart mem_Collect_eq interval_cbox_cart set_eq_iff
 unfolding vec_lambda_beta
 by auto

lemmas cartesian_euclidean_space_uniform_limit_intros[uniform_limit_intros]
=
 bounded_linear.uniform_limit[OF blinfun.bounded_linear_right]
 bounded_linear.uniform_limit[OF bounded_linear_vec_nth]

end
```

6.19 Complex Analysis Basics

Definitions of analytic and holomorphic functions, limit theorems, complex differentiation

```
theory Complex_Analysis_Basics
 imports Derivative HOL−Library.Nonpos_Ints Uncountable_Sets
begin
```

6.19.1 General lemmas

```
lemma nonneg_Reals_cmod_eq_Re: z ∈ ℝ≥0 ⟹ norm z = Re z
 by (simp add: complex_nonneg_Reals_iff cmod_eq_Re)

lemma fact_cancel:
 fixes c :: 'a::real_field
 shows of_nat (Suc n) * c / (fact (Suc n)) = c / (fact n)
```

using *of_nat_neq_0* **by** *force*

lemma *vector_derivative_cnj_within*:
assumes *at x within A ≠ bot* **and** *f differentiable at x within A*
shows *vector_derivative* (λ*z*. *cnj* (*f z*)) (*at x within A*) =
cnj (*vector_derivative f* (*at x within A*)) (**is** _ = *cnj ?D*)
proof −
let *?D* = *vector_derivative f* (*at x within A*)
from *assms* **have** (*f has_vector_derivative ?D*) (*at x within A*)
by (*subst* (*asm*) *vector_derivative_works*)
hence ((λ*x*. *cnj* (*f x*)) *has_vector_derivative cnj ?D*) (*at x within A*)
by (*rule has_vector_derivative_cnj*)
thus *?thesis* **using** *assms* **by** (*auto dest*: *vector_derivative_within*)
qed

lemma *vector_derivative_cnj*:
assumes *f differentiable at x*
shows *vector_derivative* (λ*z*. *cnj* (*f z*)) (*at x*) = *cnj* (*vector_derivative f* (*at x*))
using *assms* **by** (*intro vector_derivative_cnj_within*) *auto*

lemma
shows *open_halfspace_Re_lt*: *open* {*z*. *Re*(*z*) < *b*}
and *open_halfspace_Re_gt*: *open* {*z*. *Re*(*z*) > *b*}
and *closed_halfspace_Re_ge*: *closed* {*z*. *Re*(*z*) ≥ *b*}
and *closed_halfspace_Re_le*: *closed* {*z*. *Re*(*z*) ≤ *b*}
and *closed_halfspace_Re_eq*: *closed* {*z*. *Re*(*z*) = *b*}
and *open_halfspace_Im_lt*: *open* {*z*. *Im*(*z*) < *b*}
and *open_halfspace_Im_gt*: *open* {*z*. *Im*(*z*) > *b*}
and *closed_halfspace_Im_ge*: *closed* {*z*. *Im*(*z*) ≥ *b*}
and *closed_halfspace_Im_le*: *closed* {*z*. *Im*(*z*) ≤ *b*}
and *closed_halfspace_Im_eq*: *closed* {*z*. *Im*(*z*) = *b*}
by (*intro open_Collect_less closed_Collect_le closed_Collect_eq continuous_on_Re continuous_on_Im continuous_on_id continuous_on_const*)+

lemma *uncountable_halfspace_Im_gt*: *uncountable* {*z*. *Im z* > *c*}
proof −
obtain *r* **where** *r*: *r* > *0 ball* ((*c* + *1*) ∗*R* i) *r* ⊆ {*z*. *Im z* > *c*}
using *open_halfspace_Im_gt*[*of c*] **unfolding** *open_contains_ball* **by** *force*
then show *?thesis*
using *countable_subset uncountable_ball* **by** *blast*
qed

lemma *uncountable_halfspace_Im_lt*: *uncountable* {*z*. *Im z* < *c*}
proof −
obtain *r* **where** *r*: *r* > *0 ball* ((*c* − *1*) ∗*R* i) *r* ⊆ {*z*. *Im z* < *c*}
using *open_halfspace_Im_lt*[*of c*] **unfolding** *open_contains_ball* **by** *force*
then show *?thesis*
using *countable_subset uncountable_ball* **by** *blast*

qed

lemma *uncountable_halfspace_Re_gt*: *uncountable* $\{z.\ Re\ z > c\}$
proof −
  obtain *r* **where** *r*: $r > 0$ *ball* $(of\_real(c + 1))\ r \subseteq \{z.\ Re\ z > c\}$
    using *open_halfspace_Re_gt*[*of c*] **unfolding** *open_contains_ball* **by** *force*
  then show *?thesis*
    using *countable_subset uncountable_ball* **by** *blast*
qed

lemma *uncountable_halfspace_Re_lt*: *uncountable* $\{z.\ Re\ z < c\}$
proof −
  obtain *r* **where** *r*: $r > 0$ *ball* $(of\_real(c - 1))\ r \subseteq \{z.\ Re\ z < c\}$
    using *open_halfspace_Re_lt*[*of c*] **unfolding** *open_contains_ball* **by** *force*
  then show *?thesis*
    using *countable_subset uncountable_ball* **by** *blast*
qed

lemma *connected_halfspace_Im_gt* [*intro*]: *connected* $\{z.\ c < Im\ z\}$
  by (*intro convex_connected convex_halfspace_Im_gt*)

lemma *connected_halfspace_Im_lt* [*intro*]: *connected* $\{z.\ c > Im\ z\}$
  by (*intro convex_connected convex_halfspace_Im_lt*)

lemma *connected_halfspace_Re_gt* [*intro*]: *connected* $\{z.\ c < Re\ z\}$
  by (*intro convex_connected convex_halfspace_Re_gt*)

lemma *connected_halfspace_Re_lt* [*intro*]: *connected* $\{z.\ c > Re\ z\}$
  by (*intro convex_connected convex_halfspace_Re_lt*)

lemma *closed_complex_Reals*: *closed* ($\mathbb{R}$ :: *complex set*)
proof −
  have ($\mathbb{R}$ :: *complex set*) = $\{z.\ Im\ z = 0\}$
    by (*auto simp*: *complex_is_Real_iff*)
  then show *?thesis*
    by (*metis closed_halfspace_Im_eq*)
qed

lemma *closed_Real_halfspace_Re_le*: *closed* ($\mathbb{R} \cap \{w.\ Re\ w \leq x\}$)
  by (*simp add*: *closed_Int closed_complex_Reals closed_halfspace_Re_le*)

lemma *closed_nonpos_Reals_complex* [*simp*]: *closed* ($\mathbb{R}_{\leq 0}$ :: *complex set*)
proof −
  have $\mathbb{R}_{\leq 0} = \mathbb{R} \cap \{z.\ Re(z) \leq 0\}$
    using *complex_nonpos_Reals_iff complex_is_Real_iff* **by** *auto*
  then show *?thesis*
    by (*metis closed_Real_halfspace_Re_le*)
qed

lemma *closed_Real_halfspace_Re_ge*: *closed* ($\mathbb{R} \cap \{w.\ x \leq Re(w)\}$)
using *closed_halfspace_Re_ge*
by (*simp add*: *closed_Int closed_complex_Reals*)

lemma *closed_nonneg_Reals_complex* [*simp*]: *closed* ($\mathbb{R}_{\geq 0}$ :: *complex set*)
proof −
have $\mathbb{R}_{\geq 0} = \mathbb{R} \cap \{z.\ Re(z) \geq 0\}$
using *complex_nonneg_Reals_iff complex_is_Real_iff* **by** *auto*
then show *?thesis*
by (*metis closed_Real_halfspace_Re_ge*)
qed

lemma *closed_real_abs_le*: *closed* $\{w \in \mathbb{R}.\ |Re\ w| \leq r\}$
proof −
have $\{w \in \mathbb{R}.\ |Re\ w| \leq r\} = (\mathbb{R} \cap \{w.\ Re\ w \leq r\}) \cap (\mathbb{R} \cap \{w.\ Re\ w \geq -r\})$
by *auto*
then show *closed* $\{w \in \mathbb{R}.\ |Re\ w| \leq r\}$
by (*simp add*: *closed_Int closed_Real_halfspace_Re_ge closed_Real_halfspace_Re_le*)
qed

lemma *real_lim*:
fixes *l*::*complex*
assumes $(f \longrightarrow l)\ F$ **and** $\neg$ *trivial_limit F* **and** *eventually P F* **and** $\bigwedge a.\ P\ a \Longrightarrow f\ a \in \mathbb{R}$
shows $l \in \mathbb{R}$
using *Lim_in_closed_set*[*OF closed_complex_Reals*] *assms*
by (*smt* (*verit*) *eventually_mono*)

lemma *real_lim_sequentially*:
fixes *l*::*complex*
shows $(f \longrightarrow l)$ *sequentially* $\Longrightarrow (\exists N.\ \forall n \geq N.\ f\ n \in \mathbb{R}) \Longrightarrow l \in \mathbb{R}$
by (*rule real_lim* [**where** *F*=*sequentially*]) (*auto simp*: *eventually_sequentially*)

lemma *real_series*:
fixes *l*::*complex*
shows *f sums l* $\Longrightarrow (\bigwedge n.\ f\ n \in \mathbb{R}) \Longrightarrow l \in \mathbb{R}$
unfolding *sums_def*
by (*metis real_lim_sequentially sum_in_Reals*)

lemma *Lim_null_comparison_Re*:
assumes *eventually* $(\lambda x.\ norm(f\ x) \leq Re(g\ x))\ F$ $(g \longrightarrow 0)\ F$ **shows** $(f \longrightarrow 0)\ F$
using *Lim_null_comparison assms tendsto_Re* **by** *fastforce*

6.19.2 Holomorphic functions

definition *holomorphic_on* :: [*complex* ⇒ *complex*, *complex set*] ⇒ *bool*
(**infixl** ‹(*holomorphic'_on*)› *50*)
where *f holomorphic_on s* ≡ ∀ *x*∈*s*. *f field_differentiable* (*at x within s*)

named_theorems *holomorphic_intros structural introduction rules for holomorphic_on*

lemma *holomorphic_onI* [*intro?*]: $(\bigwedge x.\ x \in s \Longrightarrow f\ field\_differentiable\ (at\ x\ within\ s)) \Longrightarrow f\ holomorphic\_on\ s$
by (*simp add*: *holomorphic_on_def*)

lemma *holomorphic_onD* [*dest?*]: $[\![f\ holomorphic\_on\ s;\ x \in s]\!] \Longrightarrow f\ field\_differentiable\ (at\ x\ within\ s)$
by (*simp add*: *holomorphic_on_def*)

lemma *holomorphic_on_imp_differentiable_on*:
$f\ holomorphic\_on\ s \Longrightarrow f\ differentiable\_on\ s$
unfolding *holomorphic_on_def differentiable_on_def*
by (*simp add*: *field_differentiable_imp_differentiable*)

lemma *holomorphic_on_imp_differentiable_at*:
$[\![f\ holomorphic\_on\ s;\ open\ s;\ x \in s]\!] \Longrightarrow f\ field\_differentiable\ (at\ x)$
using *at_within_open holomorphic_on_def* **by** *fastforce*

lemma *holomorphic_on_empty* [*holomorphic_intros*]: $f\ holomorphic\_on\ \{\}$
by (*simp add*: *holomorphic_on_def*)

lemma *holomorphic_on_open*:
$open\ s \Longrightarrow f\ holomorphic\_on\ s \longleftrightarrow (\forall x \in s.\ \exists f'.\ DERIV\ f\ x :> f')$
by (*auto simp*: *holomorphic_on_def field_differentiable_def has_field_derivative_def at_within_open* [*of* _ *s*])

lemma *holomorphic_on_UN_open*:
assumes $\bigwedge n.\ n \in I \Longrightarrow f\ holomorphic\_on\ A\ n\ \bigwedge n.\ n \in I \Longrightarrow open\ (A\ n)$
shows $f\ holomorphic\_on\ (\bigcup n{\in}I.\ A\ n)$
by (*metis UN_E assms holomorphic_on_open open_UN*)

lemma *holomorphic_on_imp_continuous_on*:
$f\ holomorphic\_on\ s \Longrightarrow continuous\_on\ s\ f$
using *differentiable_imp_continuous_on holomorphic_on_imp_differentiable_on*
by *blast*

lemma *holomorphic_closedin_preimage_constant*:
assumes *f holomorphic_on D*
shows *closedin* (*top_of_set D*) $\{z{\in}D.\ f\ z = a\}$
by (*simp add*: *assms continuous_closedin_preimage_constant holomorphic_on_imp_continuous_on*)

lemma *holomorphic_closed_preimage_constant*:
assumes *f holomorphic_on UNIV*
shows *closed* $\{z.\ f\ z = a\}$
using *holomorphic_closedin_preimage_constant* [*OF assms*] **by** *simp*

lemma *holomorphic_on_subset* [*elim*]:
f holomorphic_on s $\Longrightarrow$ *t* $\subseteq$ *s* $\Longrightarrow$ *f holomorphic_on t*
unfolding *holomorphic_on_def*
by (*metis field_differentiable_within_subset subsetD*)

lemma *holomorphic_transform*: $[\![f$ *holomorphic_on s*; $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x]\!]$ $\Longrightarrow$ *g holomorphic_on s*
by (*metis field_differentiable_transform_within linordered_field_no_ub holomorphic_on_def*)

lemma *holomorphic_cong*: *s* = *t* ==> $(\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x) \Longrightarrow f$ *holomorphic_on s* $\longleftrightarrow$ *g holomorphic_on t*
by (*metis holomorphic_transform*)

lemma *holomorphic_on_linear* [*simp*, *holomorphic_intros*]: ((∗) *c*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_linear*)

lemma *holomorphic_on_const* [*simp*, *holomorphic_intros*]: (λ*z*. *c*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_const*)

lemma *holomorphic_on_ident* [*simp*, *holomorphic_intros*]: (λ*x*. *x*) *holomorphic_on s*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_ident*)

lemma *holomorphic_on_id* [*simp*, *holomorphic_intros*]: *id holomorphic_on s*
unfolding *id_def* **by** (*rule holomorphic_on_ident*)

lemma *constant_on_imp_holomorphic_on*:
assumes *f constant_on A*
shows *f holomorphic_on A*
by (*metis assms constant_on_def holomorphic_on_const holomorphic_transform*)

lemma *holomorphic_on_compose*:
f holomorphic_on s $\Longrightarrow$ *g holomorphic_on* (*f* ‘ *s*) $\Longrightarrow$ (*g* ∘ *f*) *holomorphic_on s*
using *field_differentiable_compose_within*[*of f _ s g*]
by (*auto simp*: *holomorphic_on_def*)

lemma *holomorphic_on_compose_gen*:
f holomorphic_on s $\Longrightarrow$ *g holomorphic_on t* $\Longrightarrow$ *f* ‘ *s* $\subseteq$ *t* $\Longrightarrow$ (*g* ∘ *f*) *holomorphic_on s*
by (*metis holomorphic_on_compose holomorphic_on_subset*)

lemma *holomorphic_on_balls_imp_entire*:
assumes ¬*bdd_above A* $\bigwedge r.\ r \in A \Longrightarrow f$ *holomorphic_on ball c r*
shows *f holomorphic_on B*
proof (*rule holomorphic_on_subset*)
show *f holomorphic_on UNIV* **unfolding** *holomorphic_on_def*

proof
fix z :: *complex*
from ‹¬*bdd_above A*› **obtain** r **where** r: $r \in A$ $r > norm\ (z - c)$
by (*meson bdd_aboveI not_le*)
with *assms*(*2*) **have** *f holomorphic_on ball c r* **by** *blast*
moreover from r **have** $z \in ball\ c\ r$ **by** (*auto simp*: *dist_norm norm_minus_commute*)
ultimately show *f field_differentiable at z*
by (*auto simp*: *holomorphic_on_def at_within_open*[*of _ ball c r*])
qed
qed *auto*

lemma *holomorphic_on_balls_imp_entire'*:
assumes $\bigwedge r.\ r > 0 \Longrightarrow f$ *holomorphic_on ball c r*
shows *f holomorphic_on B*
proof (*rule holomorphic_on_balls_imp_entire*)
show ¬*bdd_above* {(*0*::*real*)<..} **unfolding** *bdd_above_def*
by (*meson greaterThan_iff gt_ex less_le_not_le order_le_less_trans*)
qed (*use assms* **in** *auto*)

lemma *holomorphic_on_minus* [*holomorphic_intros*]: *f holomorphic_on A* $\Longrightarrow$ $(\lambda z.\ -(f\ z))$ *holomorphic_on A*
by (*metis field_differentiable_minus holomorphic_on_def*)

lemma *holomorphic_on_add* [*holomorphic_intros*]:
$[\![f$ *holomorphic_on A*; *g holomorphic_on A*$]\!] \Longrightarrow (\lambda z.\ f\ z + g\ z)$ *holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_add*)

lemma *holomorphic_on_diff* [*holomorphic_intros*]:
$[\![f$ *holomorphic_on A*; *g holomorphic_on A*$]\!] \Longrightarrow (\lambda z.\ f\ z - g\ z)$ *holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_diff*)

lemma *holomorphic_on_mult* [*holomorphic_intros*]:
$[\![f$ *holomorphic_on A*; *g holomorphic_on A*$]\!] \Longrightarrow (\lambda z.\ f\ z * g\ z)$ *holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_mult*)

lemma *holomorphic_on_inverse* [*holomorphic_intros*]:
$[\![f$ *holomorphic_on A*; $\bigwedge z.\ z \in A \Longrightarrow f\ z \neq 0]\!] \Longrightarrow (\lambda z.$ *inverse* $(f\ z))$ *holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_inverse*)

lemma *holomorphic_on_divide* [*holomorphic_intros*]:
$[\![f$ *holomorphic_on A*; *g holomorphic_on A*; $\bigwedge z.\ z \in A \Longrightarrow g\ z \neq 0]\!] \Longrightarrow (\lambda z.\ f\ z\ /\ g\ z)$ *holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_divide*)

lemma *holomorphic_on_power* [*holomorphic_intros*]:

f holomorphic_on A ⟹ (λ*z*. (*f z*) ^ *n*) *holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_power*)

lemma *holomorphic_on_power_int* [*holomorphic_intros*]:
assumes *nz*: *n* ≥ *0* ∨ (∀ *x*∈*A*. *f x* ≠ *0*) **and** *f*: *f holomorphic_on A*
shows (λ*x*. *f x powi n*) *holomorphic_on A*
proof (*cases n* ≥ *0*)
case *True*
have (λ*x*. *f x* ^ *nat n*) *holomorphic_on A*
by (*simp add*: *f holomorphic_on_power*)
with *True* **show** *?thesis*
by (*simp add*: *power_int_def*)
next
case *False*
hence (λ*x*. *inverse* (*f x* ^ *nat* (−*n*))) *holomorphic_on A*
using *nz* **by** (*auto intro*!: *holomorphic_intros f*)
with *False* **show** *?thesis*
by (*simp add*: *power_int_def power_inverse*)
qed

lemma *holomorphic_on_sum* [*holomorphic_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *holomorphic_on A*) ⟹ (λ*x*. *sum* (λ*i*. *f i x*) *I*) *holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*metis field_differentiable_sum*)

lemma *holomorphic_on_prod* [*holomorphic_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *holomorphic_on A*) ⟹ (λ*x*. *prod* (λ*i*. *f i x*) *I*) *holomorphic_on A*
by (*induction I rule*: *infinite_finite_induct*) (*auto intro*: *holomorphic_intros*)

lemma *holomorphic_pochhammer* [*holomorphic_intros*]:
f holomorphic_on A ⟹ (λ*s*. *pochhammer* (*f s*) *n*) *holomorphic_on A*
by (*induction n*) (*auto intro*!: *holomorphic_intros simp*: *pochhammer_Suc*)

lemma *holomorphic_on_scaleR* [*holomorphic_intros*]:
f holomorphic_on A ⟹ (λ*x*. *c* *$_R$ *f x*) *holomorphic_on A*
by (*auto simp*: *scaleR_conv_of_real intro*!: *holomorphic_intros*)

lemma *holomorphic_on_Un* [*holomorphic_intros*]:
assumes *f holomorphic_on A f holomorphic_on B open A open B*
shows *f holomorphic_on* (*A* ∪ *B*)
by (*metis Un_iff assms holomorphic_on_open open_Un*)

lemma *holomorphic_on_If_Un* [*holomorphic_intros*]:
assumes *f holomorphic_on A g holomorphic_on B open A open B*
assumes ⋀*z*. *z* ∈ *A* ⟹ *z* ∈ *B* ⟹ *f z* = *g z*
shows (λ*z*. *if z* ∈ *A then f z else g z*) *holomorphic_on* (*A* ∪ *B*) (**is** *?h holomorphic_on* _)
proof (*intro holomorphic_on_Un*)

note ‹*f holomorphic_on A*›
also have *f holomorphic_on A* ⟷ *?h holomorphic_on A*
by (*intro holomorphic_cong*) *auto*
finally show ... **.**
next
note ‹*g holomorphic_on B*›
also have *g holomorphic_on B* ⟷ *?h holomorphic_on B*
using *assms* **by** (*intro holomorphic_cong*) *auto*
finally show ... **.**
qed (*use assms* **in** *auto*)

lemma *holomorphic_derivI*:
⟦*f holomorphic_on S*; *open S*; *x* ∈ *S*⟧ ⟹ (*f has_field_derivative deriv f x*) (*at x within T*)
by (*metis DERIV_deriv_iff_field_differentiable at_within_open holomorphic_on_def has_field_derivative_at_within*)

lemma *complex_derivative_transform_within_open*:
⟦*f holomorphic_on s*; *g holomorphic_on s*; *open s*; *z* ∈ *s*; ⋀*w*. *w* ∈ *s* ⟹ *f w* = *g w*⟧
⟹ *deriv f z* = *deriv g z*
by (*smt* (*verit*) *DERIV_imp_deriv has_field_derivative_transform_within_open holomorphic_on_open*)

lemma *holomorphic_on_compose_cnj_cnj*:
assumes *f holomorphic_on cnj* ‘ *A open A*
shows *cnj* ∘ *f* ∘ *cnj holomorphic_on A*
proof −
have [*simp*]: *open* (*cnj* ‘ *A*)
unfolding *image_cnj_conv_vimage_cnj* **using** *assms* **by** (*intro open_vimage*) *auto*
show *?thesis*
using *assms* **unfolding** *holomorphic_on_def*
by (*auto intro*!: *field_differentiable_cnj_cnj simp*: *at_within_open_NO_MATCH*)
qed

lemma *holomorphic_nonconstant*:
assumes *holf*: *f holomorphic_on S* **and** *open S ξ* ∈ *S deriv f ξ* ≠ *0*
shows ¬ *f constant_on S*
by (*rule nonzero_deriv_nonconstant* [*of f deriv f ξ ξ S*])
(*use assms* **in** ‹*auto simp*: *holomorphic_derivI*›)

6.19.3 Analyticity on a set

definition *analytic_on* (**infixl** ‹(*analytic'_on*)› *50*)
where *f analytic_on S* ≡ ∀ *x* ∈ *S*. ∃ *ε*. *0* < *ε* ∧ *f holomorphic_on* (*ball x ε*)

named_theorems *analytic_intros introduction rules for proving analyticity*

lemma *analytic_imp_holomorphic*: *f analytic_on S* $\Longrightarrow$ *f holomorphic_on S*
unfolding *analytic_on_def holomorphic_on_def*
using *centre_in_ball field_differentiable_at_within field_differentiable_within_open*
by *blast*

lemma *analytic_on_open*: *open S* $\Longrightarrow$ *f analytic_on S* $\longleftrightarrow$ *f holomorphic_on S*
by (*meson analytic_imp_holomorphic analytic_on_def holomorphic_on_subset openE*)

lemma *constant_on_imp_analytic_on*:
assumes *f constant_on A open A*
shows *f analytic_on A*
by (*simp add*: *analytic_on_open assms constant_on_imp_holomorphic_on*)

lemma *analytic_on_imp_differentiable_at*:
f analytic_on S $\Longrightarrow$ *x* $\in$ *S* $\Longrightarrow$ *f field_differentiable* (*at x*)
using *analytic_on_def holomorphic_on_imp_differentiable_at* **by** *auto*

lemma *analytic_at_imp_isCont*:
assumes *f analytic_on* {*z*}
shows *isCont f z*
by (*meson analytic_on_imp_differentiable_at assms field_differentiable_imp_continuous_at insertCI*)

lemma *analytic_at_neq_imp_eventually_neq*:
assumes *f analytic_on* {*x*} *f x* $\neq$ *c*
shows *eventually* (*λy. f y* $\neq$ *c*) (*at x*)
using *analytic_at_imp_isCont assms isContD tendsto_imp_eventually_ne* **by** *blast*

lemma *analytic_on_subset*: *f analytic_on S* $\Longrightarrow$ *T* $\subseteq$ *S* $\Longrightarrow$ *f analytic_on T*
by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_Un*: *f analytic_on* (*S* $\cup$ *T*) $\longleftrightarrow$ *f analytic_on S* $\wedge$ *f analytic_on T*
by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_Union*: *f analytic_on* ($\bigcup \mathcal{T}$) $\longleftrightarrow$ ($\forall$ *T* $\in$ $\mathcal{T}$. *f analytic_on T*)
by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_UN*: *f analytic_on* ($\bigcup$ *i*$\in$*I*. *S i*) $\longleftrightarrow$ ($\forall$ *i*$\in$*I*. *f analytic_on* (*S i*))
by (*auto simp*: *analytic_on_def*)

lemma *analytic_on_holomorphic*:
f analytic_on S $\longleftrightarrow$ ($\exists$ *T*. *open T* $\wedge$ *S* $\subseteq$ *T* $\wedge$ *f holomorphic_on T*)
(**is** *?lhs* = *?rhs*)
proof −

```
  have ?lhs ⟷ (∃ T. open T ∧ S ⊆ T ∧ f analytic_on T)
  proof safe
    assume f analytic_on S
    then have ∀ x ∈ ⋃ {U. open U ∧ f analytic_on U}. ∃ ε>0. f holomorphic_on
ball x ε
      using analytic_on_def by force
    moreover have S ⊆ ⋃ {U. open U ∧ f analytic_on U}
      using ‹f analytic_on S›
     by (smt (verit, best) open_ball Union_iff analytic_on_def analytic_on_open
centre_in_ball mem_Collect_eq subsetI)
    ultimately show ∃ T. open T ∧ S ⊆ T ∧ f analytic_on T
      unfolding analytic_on_def
      by (metis (mono_tags, lifting) mem_Collect_eq open_Union)
  next
    fix T
    assume open T S ⊆ T f analytic_on T
    then show f analytic_on S
       by (metis analytic_on_subset)
  qed
  also have ... ⟷ ?rhs
    by (auto simp: analytic_on_open)
  finally show ?thesis .
qed

lemma analytic_on_linear [analytic_intros,simp]: ((∗) c) analytic_on S
  by (auto simp add: analytic_on_holomorphic)

lemma analytic_on_const [analytic_intros,simp]: (λz. c) analytic_on S
  by (metis analytic_on_def holomorphic_on_const zero_less_one)

lemma analytic_on_ident [analytic_intros,simp]: (λx. x) analytic_on S
  by (simp add: analytic_on_def gt_ex)

lemma analytic_on_id [analytic_intros]: id analytic_on S
  unfolding id_def by (rule analytic_on_ident)

lemma analytic_on_scaleR [analytic_intros]: f analytic_on A ⟹ (λw. x ∗R f w)
analytic_on A
  by (metis analytic_on_holomorphic holomorphic_on_scaleR)

lemma analytic_on_compose:
  assumes f: f analytic_on S
      and g: g analytic_on (f ‘ S)
    shows (g ∘ f) analytic_on S
unfolding analytic_on_def
proof (intro ballI)
  fix x
  assume x: x ∈ S
  then obtain e where e: 0 < e and fh: f holomorphic_on ball x e using f
```

```
    by (metis analytic_on_def)
  obtain e' where e': 0 < e' and gh: g holomorphic_on ball (f x) e' using g
    by (metis analytic_on_def g image_eqI x)
  have isCont f x
   by (metis analytic_on_imp_differentiable_at field_differentiable_imp_continuous_at
f x)
  with e' obtain d where d: 0 < d and fd: f ‘ ball x d ⊆ ball (f x) e'
     by (auto simp: continuous_at_ball)
  have g ∘ f holomorphic_on ball x (min d e)
     by (meson fd fh gh holomorphic_on_compose_gen holomorphic_on_subset
image_mono min.cobounded1 min.cobounded2 subset_ball)
  then show ∃e>0. g ∘ f holomorphic_on ball x e
    by (metis d e min_less_iff_conj)
qed

lemma analytic_on_compose_gen:
  f analytic_on S ⟹ g analytic_on T ⟹ (⋀z. z ∈ S ⟹ f z ∈ T)
          ⟹ g ∘ f analytic_on S
  by (metis analytic_on_compose analytic_on_subset image_subset_iff)

lemma analytic_on_neg [analytic_intros]:
  f analytic_on S ⟹ (λz. −(f z)) analytic_on S
  by (metis analytic_on_holomorphic holomorphic_on_minus)

lemma analytic_on_add [analytic_intros]:
  assumes f: f analytic_on S
      and g: g analytic_on S
    shows (λz. f z + g z) analytic_on S
unfolding analytic_on_def
proof (intro ballI)
  fix z
  assume z: z ∈ S
  then obtain e where e: 0 < e and fh: f holomorphic_on ball z e using f
    by (metis analytic_on_def)
  obtain e' where e': 0 < e' and gh: g holomorphic_on ball z e' using g
    by (metis analytic_on_def g z)
  have (λz. f z + g z) holomorphic_on ball z (min e e')
    by (metis fh gh holomorphic_on_add holomorphic_on_subset linorder_linear
min_def subset_ball)
  then show ∃e>0. (λz. f z + g z) holomorphic_on ball z e
    by (metis e e' min_less_iff_conj)
qed

lemma analytic_on_mult [analytic_intros]:
  assumes f: f analytic_on S
      and g: g analytic_on S
    shows (λz. f z * g z) analytic_on S
unfolding analytic_on_def
proof (intro ballI)
```

fix z
assume z: $z \in S$
then obtain e **where** e: $0 < e$ **and** fh: f *holomorphic_on ball* z e **using** f
by (*metis analytic_on_def*)
obtain e' **where** e': $0 < e'$ **and** gh: g *holomorphic_on ball* z e' **using** g
by (*metis analytic_on_def g z*)
have ($\lambda z.\ f\ z * g\ z$) *holomorphic_on ball* z (*min* e e')
by (*metis fh gh holomorphic_on_mult holomorphic_on_subset min.absorb_iff2 min_def subset_ball*)
then show $\exists e{>}0.$ ($\lambda z.\ f\ z * g\ z$) *holomorphic_on ball* z e
by (*metis e e' min_less_iff_conj*)
qed

lemma *analytic_on_diff* [*analytic_intros*]:
assumes f: f *analytic_on* S **and** g: g *analytic_on* S
shows ($\lambda z.\ f\ z - g\ z$) *analytic_on* S
proof −
have ($\lambda z.\ -\ g\ z$) *analytic_on* S
by (*simp add*: *analytic_on_neg g*)
then have ($\lambda z.\ f\ z + -\ g\ z$) *analytic_on* S
using *analytic_on_add f* **by** *blast*
then show *?thesis*
by *fastforce*
qed

lemma *analytic_on_inverse* [*analytic_intros*]:
assumes f: f *analytic_on* S
and nz: ($\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$)
shows ($\lambda z.$ *inverse* ($f\ z$)) *analytic_on* S
unfolding *analytic_on_def*
proof (*intro ballI*)
fix z
assume z: $z \in S$
then obtain e **where** e: $0 < e$ **and** fh: f *holomorphic_on ball* z e **using** f
by (*metis analytic_on_def*)
have *continuous_on* (*ball* z e) f
by (*metis fh holomorphic_on_imp_continuous_on*)
then obtain e' **where** e': $0 < e'$ **and** nz': $\bigwedge y.$ *dist* z $y < e' \Longrightarrow f\ y \neq 0$
by (*metis open_ball centre_in_ball continuous_on_open_avoid e z nz*)
have ($\lambda z.$ *inverse* ($f\ z$)) *holomorphic_on ball* z (*min* e e')
using *fh holomorphic_on_inverse holomorphic_on_open nz'* **by** *fastforce*
then show $\exists e{>}0.$ ($\lambda z.$ *inverse* ($f\ z$)) *holomorphic_on ball* z e
by (*metis e e' min_less_iff_conj*)
qed

lemma *analytic_on_divide* [*analytic_intros*]:
assumes f: f *analytic_on* S **and** g: g *analytic_on* S
and nz: ($\bigwedge z.\ z \in S \Longrightarrow g\ z \neq 0$)
shows ($\lambda z.\ f\ z\ /\ g\ z$) *analytic_on* S

unfolding *divide_inverse* **by** (*metis analytic_on_inverse analytic_on_mult f g nz*)

lemma *analytic_on_power* [*analytic_intros*]:
f *analytic_on S* ⟹ (λ*z*. (*f z*) ^ *n*) *analytic_on S*
by (*induct n*) (*auto simp*: *analytic_on_mult*)

lemma *analytic_on_power_int* [*analytic_intros*]:
assumes *nz*: *n* ≥ *0* ∨ (∀ *x*∈*A*. *f x* ≠ *0*) **and** *f*: *f analytic_on A*
shows (λ*x*. *f x powi n*) *analytic_on A*
proof (*cases n* ≥ *0*)
case *True*
have (λ*x*. *f x* ^ *nat n*) *analytic_on A*
using *analytic_on_power f* **by** *blast*
with *True* **show** *?thesis*
by (*simp add*: *power_int_def*)
next
case *False*
hence (λ*x*. *inverse* (*f x* ^ *nat* (−*n*))) *analytic_on A*
using *nz* **by** (*auto intro*!: *analytic_intros f*)
with *False* **show** *?thesis*
by (*simp add*: *power_int_def power_inverse*)
qed

lemma *analytic_on_sum* [*analytic_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *analytic_on S*) ⟹ (λ*x*. *sum* (λ*i*. *f i x*) *I*) *analytic_on S*
by (*induct I rule*: *infinite_finite_induct*) (*auto simp*: *analytic_on_add*)

lemma *analytic_on_prod* [*analytic_intros*]:
(⋀*i*. *i* ∈ *I* ⟹ (*f i*) *analytic_on S*) ⟹ (λ*x*. *prod* (λ*i*. *f i x*) *I*) *analytic_on S*
by (*induct I rule*: *infinite_finite_induct*) (*auto simp*: *analytic_on_mult*)

lemma *analytic_on_gbinomial* [*analytic_intros*]:
f *analytic_on A* ⟹ (λ*w*. *f w gchoose n*) *analytic_on A*
unfolding *gbinomial_prod_rev* **by** (*intro analytic_intros*) *auto*

lemma *deriv_left_inverse*:
assumes *f holomorphic_on S* **and** *g holomorphic_on T*
and *open S* **and** *open T*
and *f* ‘ *S* ⊆ *T*
and [*simp*]: ⋀*z*. *z* ∈ *S* ⟹ *g* (*f z*) = *z*
and *w* ∈ *S*
shows *deriv f w* ∗ *deriv g* (*f w*) = *1*
proof −
have *deriv f w* ∗ *deriv g* (*f w*) = *deriv g* (*f w*) ∗ *deriv f w*
by (*simp add*: *algebra_simps*)
also have ... = *deriv* (*g* ∘ *f*) *w*
using *assms*
by (*metis analytic_on_imp_differentiable_at analytic_on_open deriv_chain*

image_subset_iff)
also have ... = *deriv id w*
proof (*rule complex_derivative_transform_within_open* [**where** *s*=*S*])
show *g* ∘ *f holomorphic_on S*
by (*rule assms holomorphic_on_compose_gen holomorphic_intros*)+
qed (*use assms* **in** *auto*)
also have ... = *1*
by *simp*
finally show *?thesis* **.**
qed

6.19.4 Analyticity at a point

lemma *analytic_at_ball*:
f analytic_on {*z*} ⟷ (∃ *e*. 0<*e* ∧ *f holomorphic_on ball z e*)
by (*metis analytic_on_def singleton_iff*)

lemma *analytic_at*:
f analytic_on {*z*} ⟷ (∃ *s*. *open s* ∧ *z* ∈ *s* ∧ *f holomorphic_on s*)
by (*metis analytic_on_holomorphic empty_subsetI insert_subset*)

lemma *holomorphic_on_imp_analytic_at*:
assumes *f holomorphic_on A open A z* ∈ *A*
shows *f analytic_on* {*z*}
using *assms* **by** (*meson analytic_at*)

lemma *analytic_on_analytic_at*:
f analytic_on s ⟷ (∀ *z* ∈ *s*. *f analytic_on* {*z*})
by (*metis analytic_at_ball analytic_on_def*)

lemma *analytic_at_two*:
f analytic_on {*z*} ∧ *g analytic_on* {*z*} ⟷
(∃ *S*. *open S* ∧ *z* ∈ *S* ∧ *f holomorphic_on S* ∧ *g holomorphic_on S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *S T*
where *st*: *open S z* ∈ *S f holomorphic_on S*
open T z ∈ *T g holomorphic_on T*
by (*auto simp*: *analytic_at*)
then show *?rhs*
by (*metis Int_iff holomorphic_on_subset inf_le1 inf_le2 open_Int*)
next
assume *?rhs*
then show *?lhs*
by (*force simp add*: *analytic_at*)
qed

6.19.5 Combining theorems for derivative with "analytic at" hypotheses

lemma
assumes *f analytic_on {z} g analytic_on {z}*
shows *complex_derivative_add_at*: *deriv (λw. f w + g w) z = deriv f z + deriv g z*
and *complex_derivative_diff_at*: *deriv (λw. f w − g w) z = deriv f z − deriv g z*
and *complex_derivative_mult_at*: *deriv (λw. f w * g w) z =*
*f z * deriv g z + deriv f z * g z*
proof −
show *deriv (λw. f w + g w) z = deriv f z + deriv g z*
using *analytic_on_imp_differentiable_at assms* **by** *auto*
show *deriv (λw. f w − g w) z = deriv f z − deriv g z*
using *analytic_on_imp_differentiable_at assms* **by** *force*
obtain *S* **where** *open S z ∈ S f holomorphic_on S g holomorphic_on S*
using *assms* **by** (*metis analytic_at_two*)
then show *deriv (λw. f w * g w) z = f z * deriv g z + deriv f z * g z*
by (*simp add*: *DERIV_imp_deriv* [*OF DERIV_mult'*] *holomorphic_derivI*)
qed

lemma *deriv_cmult_at*:
*f analytic_on {z} ⟹ deriv (λw. c * f w) z = c * deriv f z*
by (*auto simp*: *complex_derivative_mult_at*)

lemma *deriv_cmult_right_at*:
*f analytic_on {z} ⟹ deriv (λw. f w * c) z = deriv f z * c*
by (*auto simp*: *complex_derivative_mult_at*)

6.19.6 Complex differentiation of sequences and series

lemma *has_complex_derivative_sequence*:
fixes *S* :: *complex set*
assumes *cvs*: *convex S*
and *df*: *⋀n x. x ∈ S ⟹ (f n has_field_derivative f' n x) (at x within S)*
and *conv*: *⋀e. 0 < e ⟹ ∃N. ∀n x. n ≥ N ⟶ x ∈ S ⟶ norm (f' n x − g' x) ≤ e*
and *∃x l. x ∈ S ∧ ((λn. f n x) ⟶ l) sequentially*
shows *∃g. ∀x ∈ S. ((λn. f n x) ⟶ g x) sequentially ∧*
(g has_field_derivative (g' x)) (at x within S)
proof −
from *assms* **obtain** *x l* **where** *x*: *x ∈ S* **and** *tf*: *((λn. f n x) ⟶ l) sequentially*
by *blast*
show *?thesis*
unfolding *has_field_derivative_def*
proof (*rule has_derivative_sequence* [*OF cvs _ _ x*])
show *(λn. f n x) ⟶ l*
by (*rule tf*)
next

have $**$: $\exists N.\ \forall n{\geq}N.\ \forall x{\in}S.\ \forall h.\ cmod\ (f'\ n\ x * h - g'\ x * h) \leq \varepsilon * cmod\ h$
if $\varepsilon > 0$ **for** ε::*real*
by (*metis that left_diff_distrib mult_right_mono norm_ge_zero norm_mult conv*)
show $\bigwedge e.\ e > 0 \Longrightarrow \forall_F\ n$ *in sequentially*. $\forall x{\in}S.\ \forall h.\ cmod\ (f'\ n\ x * h - g'\ x * h) \leq e * cmod\ h$
unfolding *eventually_sequentially* **by** (*blast intro*: $**$)
qed (*metis has_field_derivative_def df*)
qed

lemma *has_complex_derivative_series*:
fixes S :: *complex set*
assumes *cvs*: *convex S*
and *df*: $\bigwedge n\ x.\ x \in S \Longrightarrow (f\ n\ has\_field\_derivative\ f'\ n\ x)\ (at\ x\ within\ S)$
and *conv*: $\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall n\ x.\ n \geq N \longrightarrow x \in S \longrightarrow cmod\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) \leq e$
and $\exists x\ l.\ x \in S \wedge ((\lambda n.\ f\ n\ x)\ sums\ l)$
shows $\exists g.\ \forall x \in S.\ ((\lambda n.\ f\ n\ x)\ sums\ g\ x) \wedge ((g\ has\_field\_derivative\ g'\ x)\ (at\ x\ within\ S))$
proof −
from *assms* **obtain** $x\ l$ **where** x: $x \in S$ **and** *sf*: $((\lambda n.\ f\ n\ x)\ sums\ l)$
by *blast*
{ fix ε::*real* **assume** *e*: $\varepsilon > 0$
then obtain N **where** N: $\forall n\ x.\ n \geq N \longrightarrow x \in S \longrightarrow cmod\ ((\sum i{<}n.\ f'\ i\ x) - g'\ x) \leq \varepsilon$
by (*metis conv*)
have $\exists N.\ \forall n{\geq}N.\ \forall x{\in}S.\ \forall h.\ cmod\ ((\sum i{<}n.\ h * f'\ i\ x) - g'\ x * h) \leq \varepsilon * cmod\ h$
proof (*rule exI* [*of* _ *N*], *clarify*)
fix *n y h*
assume $N \leq n\ y \in S$
have $cmod\ h * cmod\ ((\sum i{<}n.\ f'\ i\ y) - g'\ y) \leq cmod\ h * \varepsilon$
by (*simp add*: N ‹$N \leq n$› ‹$y \in S$› *mult_le_cancel_left*)
then show $cmod\ ((\sum i{<}n.\ h * f'\ i\ y) - g'\ y * h) \leq \varepsilon * cmod\ h$
by (*simp add*: *norm_mult* [*symmetric*] *field_simps sum_distrib_left*)
qed
} note $** = this$
show *?thesis*
unfolding *has_field_derivative_def*
proof (*rule has_derivative_series* [*OF cvs* _ _ *x*])
fix *n x*
assume $x \in S$
then show $((f\ n)\ has\_derivative\ (\lambda z.\ z * f'\ n\ x))\ (at\ x\ within\ S)$
by (*metis df has_field_derivative_def mult_commute_abs*)
next show $((\lambda n.\ f\ n\ x)\ sums\ l)$
by (*rule sf*)
next show $\bigwedge e.\ e{>}0 \Longrightarrow \forall_F\ n$ *in sequentially*. $\forall x{\in}S.\ \forall h.\ cmod\ ((\sum i{<}n.\ h * f'\ i\ x) - g'\ x * h) \leq e * cmod\ h$
unfolding *eventually_sequentially* **by** (*blast intro*: $**$)

qed
qed

6.19.7 Taylor on Complex Numbers

lemma *sum_Suc_reindex*:
fixes $f :: nat \Rightarrow 'a::ab\_group\_add$
shows $sum\ f\ \{0..n\} = f\ 0 - f\ (Suc\ n) + sum\ (\lambda i.\ f\ (Suc\ i))\ \{0..n\}$
by (*induct n*) *auto*

lemma *field_Taylor*:
assumes *S*: *convex S*
and *f*: $\bigwedge i\ x.\ x \in S \Longrightarrow i \le n \Longrightarrow (f\ i\ has\_field\_derivative\ f\ (Suc\ i)\ x)\ (at\ x\ within\ S)$
and *B*: $\bigwedge x.\ x \in S \Longrightarrow norm\ (f\ (Suc\ n)\ x) \le B$
and *w*: $w \in S$
and *z*: $z \in S$
shows $norm(f\ 0\ z - (\sum i{\le}n.\ f\ i\ w * (z{-}w)\ \hat{}\ i\ /\ (fact\ i))) \le B * norm(z - w)\hat{}(Suc\ n)\ /\ fact\ n$
proof −
have *wzs*: *closed_segment w z* ⊆ *S* **using** *assms*
by (*metis convex_contains_segment*)
{ fix *u*
assume $u \in closed\_segment\ w\ z$
then have $u \in S$
by (*metis wzs subsetD*)
have $*$: $(\sum i{\le}n.\ f\ i\ u * (-\ of\_nat\ i * (z{-}u)\hat{}(i - 1))\ /\ (fact\ i) + f\ (Suc\ i)\ u * (z{-}u)\hat{}i\ /\ (fact\ i)) = f\ (Suc\ n)\ u * (z{-}u)\ \hat{}\ n\ /\ (fact\ n)$
proof (*induction n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*)
have $(\sum i{\le}Suc\ n.\ f\ i\ u * (-\ of\_nat\ i * (z{-}u)\ \hat{}\ (i - 1))\ /\ (fact\ i) + f\ (Suc\ i)\ u * (z{-}u)\ \hat{}\ i\ /\ (fact\ i)) =$
$f\ (Suc\ n)\ u * (z{-}u)\ \hat{}\ n\ /\ (fact\ n) +$
$f\ (Suc\ (Suc\ n))\ u * ((z{-}u) * (z{-}u)\ \hat{}\ n)\ /\ (fact\ (Suc\ n)) -$
$f\ (Suc\ n)\ u * ((1 + of\_nat\ n) * (z{-}u)\ \hat{}\ n)\ /\ (fact\ (Suc\ n))$
using *Suc* **by** *simp*
also have $\ldots = f\ (Suc\ (Suc\ n))\ u * (z{-}u)\ \hat{}\ Suc\ n\ /\ (fact\ (Suc\ n))$
proof −
have $(fact(Suc\ n)) *$
$(f(Suc\ n)\ u *(z{-}u)\ \hat{}\ n\ /\ (fact\ n) +$
$f(Suc(Suc\ n))\ u *((z{-}u) *(z{-}u)\ \hat{}\ n)\ /\ (fact(Suc\ n)) -$
$f(Suc\ n)\ u *((1 + of\_nat\ n) *(z{-}u)\ \hat{}\ n)\ /\ (fact(Suc\ n))) =$
$((fact(Suc\ n)) *(f(Suc\ n)\ u *(z{-}u)\ \hat{}\ n))\ /\ (fact\ n) +$
$((fact(Suc\ n)) *(f(Suc(Suc\ n))\ u *((z{-}u) *(z{-}u)\ \hat{}\ n))\ /\ (fact(Suc\ n))) -$
$((fact(Suc\ n)) *(f(Suc\ n)\ u *(of\_nat(Suc\ n) *(z{-}u)\ \hat{}\ n)))\ /\ (fact(Suc$

```
n))
        by (simp add: algebra_simps del: fact_Suc)
      also have ... = ((fact (Suc n)) * (f (Suc n) u * (z-u) ^ n)) / (fact n) +
                      (f (Suc (Suc n)) u * ((z-u) * (z-u) ^ n)) -
                      (f (Suc n) u * ((1 + of_nat n) * (z-u) ^ n))
        by (simp del: fact_Suc)
      also have ... = (of_nat (Suc n) * (f (Suc n) u * (z-u) ^ n)) +
                      (f (Suc (Suc n)) u * ((z-u) * (z-u) ^ n)) -
                      (f (Suc n) u * ((1 + of_nat n) * (z-u) ^ n))
        by (simp only: fact_Suc of_nat_mult ac_simps) simp
      also have ... = f (Suc (Suc n)) u * ((z-u) * (z-u) ^ n)
        by (simp add: algebra_simps)
      finally show ?thesis
       by (simp add: mult_left_cancel [where c = (fact (Suc n)), THEN iffD1]
del: fact_Suc)
    qed
    finally show ?case .
  qed
  have ((λv. (∑ i≤n. f i v * (z - v) ^ i / (fact i)))
          has_field_derivative f (Suc n) u * (z-u) ^ n / (fact n))
         (at u within S)
    unfolding * [symmetric]
    by (rule derivative_eq_intros assms ‹u ∈ S› refl | auto simp: field_simps)+
 } note sum_deriv = this
 { fix u
  assume u: u ∈ closed_segment w z
  then have us: u ∈ S
    by (metis wzs subsetD)
  have norm (f (Suc n) u) * norm (z - u) ^ n ≤ norm (f (Suc n) u) * norm
(u - z) ^ n
    by (metis norm_minus_commute order_refl)
  also have ... ≤ norm (f (Suc n) u) * norm (z - w) ^ n
    by (metis mult_left_mono norm_ge_zero power_mono segment_bound [OF
u])
  also have ... ≤ B * norm (z - w) ^ n
    by (metis norm_ge_zero zero_le_power mult_right_mono  B [OF us])
  finally have norm (f (Suc n) u) * norm (z - u) ^ n ≤ B * norm (z - w) ^
n .
 } note cmod_bound = this
 have (∑ i≤n. f i z * (z - z) ^ i / (fact i)) = (∑ i≤n. (f i z / (fact i)) * 0 ^ i)
   by simp
 also have ... = f 0 z / (fact 0)
   by (subst sum_zero_power) simp
 finally have norm (f 0 z - (∑ i≤n. f i w * (z - w) ^ i / (fact i)))
           ≤ norm ((∑ i≤n. f i w * (z - w) ^ i / (fact i)) -
                   (∑ i≤n. f i z * (z - z) ^ i / (fact i)))
   by (simp add: norm_minus_commute)
 also have ... ≤ B * norm (z - w) ^ n / (fact n) * norm (w - z)
 proof (rule field_differentiable_bound)
```

```
    show ⋀x. x ∈ closed_segment w z ⟹
         ((λξ. ∑ i≤n. f i ξ * (z − ξ) ^ i / fact i) has_field_derivative f (Suc n) x
* (z − x) ^ n / fact n)
          (at x within closed_segment w z)
      using DERIV_subset sum_deriv wzs by blast
  qed (auto simp: norm_divide norm_mult norm_power divide_le_cancel cmod_bound)
  also have ... ≤ B * norm (z − w) ^ Suc n / (fact n)
    by (simp add: algebra_simps norm_minus_commute)
  finally show ?thesis .
qed

lemma complex_Taylor:
  assumes S: convex S
      and f: ⋀i x. x ∈ S ⟹ i ≤ n ⟹ (f i has_field_derivative f (Suc i) x) (at
x within S)
      and B: ⋀x. x ∈ S ⟹ cmod (f (Suc n) x) ≤ B
      and w: w ∈ S
      and z: z ∈ S
    shows cmod(f 0 z − (∑ i≤n. f i w * (z−w) ^ i / (fact i))) ≤ B * cmod(z −
w)^(Suc n) / fact n
  using assms by (rule field_Taylor)
```

Something more like the traditional MVT for real components

```
lemma complex_mvt_line:
  assumes ⋀u. u ∈ closed_segment w z ⟹ (f has_field_derivative f'(u)) (at u)
    shows ∃ u. u ∈ closed_segment w z ∧ Re(f z) − Re(f w) = Re(f'(u) * (z − w))
proof −
  define φ where φ ≡ λt. (1 − t) *R w + t *R z
  have twz: ⋀t. φ t = w + t *R (z − w)
   by (simp add: φ_def real_vector.scale_left_diff_distrib real_vector.scale_right_diff_distrib)
  note assms[unfolded has_field_derivative_def, derivative_intros]
  have *: ⋀x. ⟦0 ≤ x; x ≤ 1⟧
        ⟹ (Re ∘ f ∘ φ has_derivative Re ∘ (*) (f' (φ x)) ∘ (λt. t *R (z − w)))
           (at x within {0..1})
    unfolding φ_def
    by (intro derivative_eq_intros has_derivative_at_withinI)
       (auto simp: in_segment scaleR_right_diff_distrib)
  obtain x where 0<x x<1 (Re ∘ f ∘ φ) 1 −
      (Re ∘ f ∘ φ) 0 = (Re ∘ (*) (f' (φ x)) ∘ (λt. t *R (z − w))) (1 − 0)
    using mvt_simple [OF zero_less_one *] by force
  then show ?thesis
    unfolding φ_def
    by (smt (verit) comp_apply in_segment(1) scaleR_left_distrib scaleR_one
scaleR_zero_left)
qed

lemma complex_Taylor_mvt:
  assumes ⋀i x. ⟦x ∈ closed_segment w z; i ≤ n⟧ ⟹ ((f i) has_field_derivative
f (Suc i) x) (at x)
```

```
    shows ∃u. u ∈ closed_segment w z ∧
          Re (f 0 z) =
          Re ((∑i = 0..n. f i w * (z − w) ^ i / (fact i)) +
              (f (Suc n) u * (z−u)^n / (fact n)) * (z − w))
proof −
  { fix u
    assume u: u ∈ closed_segment w z
    have (∑i = 0..n.
            (f (Suc i) u * (z−u) ^ i − of_nat i * (f i u * (z−u) ^ (i − Suc 0))) /
             (fact i)) =
         f (Suc 0) u −
            (f (Suc (Suc n)) u * ((z−u) ^ Suc n) − (of_nat (Suc n)) * (z−u) ^ n
* f (Suc n) u) /
            (fact (Suc n)) +
            (∑i = 0..n.
               (f (Suc (Suc i)) u * ((z−u) ^ Suc i) − of_nat (Suc i) * (f (Suc i)
u * (z−u) ^ i)) /
               (fact (Suc i)))
      by (subst sum_Suc_reindex) simp
    also have ... = f (Suc 0) u −
            (f (Suc (Suc n)) u * ((z−u) ^ Suc n) − (of_nat (Suc n)) * (z−u) ^ n
* f (Suc n) u) /
            (fact (Suc n)) +
            (∑i = 0..n.
               f (Suc (Suc i)) u * ((z−u) ^ Suc i) / (fact (Suc i)) −
               f (Suc i) u * (z−u) ^ i / (fact i))
      by (simp only: diff_divide_distrib fact_cancel ac_simps)
    also have ... = f (Suc 0) u −
            (f (Suc (Suc n)) u * (z−u) ^ Suc n − of_nat (Suc n) * (z−u) ^ n * f
(Suc n) u) /
            (fact (Suc n)) +
            f (Suc (Suc n)) u * (z−u) ^ Suc n / (fact (Suc n)) − f (Suc 0) u
      by (subst sum_Suc_diff) auto
    also have ... = f (Suc n) u * (z−u) ^ n / (fact n)
      by (simp only: algebra_simps diff_divide_distrib fact_cancel)
    finally have *: (∑i = 0..n. (f (Suc i) u * (z − u) ^ i
                          − of_nat i * (f i u * (z−u) ^ (i − Suc 0))) / (fact i)) =
                  f (Suc n) u * (z − u) ^ n / (fact n) .
    have ((λu. ∑i = 0..n. f i u * (z − u) ^ i / (fact i)) has_field_derivative
              f (Suc n) u * (z − u) ^ n / (fact n)) (at u)
      unfolding * [symmetric]
      by (rule derivative_eq_intros assms u refl | auto simp: field_simps)+
  }
  then show ?thesis
    apply (cut_tac complex_mvt_line [of w z λu. ∑i = 0..n. f i u * (z−u) ^ i /
(fact i)
              λu. (f (Suc n) u * (z−u)^n / (fact n))])
    apply (auto simp add: intro: open_closed_segment)
    done
```

qed

end

6.20 Complex Transcendental Functions

By John Harrison et al. Ported from HOL Light by L C Paulson (2015)

theory *Complex_Transcendental*
imports
Complex_Analysis_Basics Summation_Tests HOL−Library.Periodic_Fun
begin

6.20.1 Möbius transformations

definition *moebius a b c d* ≡ (λ*z*. (*a*∗*z*+*b*) / (*c*∗*z*+*d* :: *'a* :: *field*))

theorem *moebius_inverse*:
assumes *a* ∗ *d* ≠ *b* ∗ *c c* ∗ *z* + *d* ≠ *0*
shows *moebius d* (−*b*) (−*c*) *a* (*moebius a b c d z*) = *z*
proof −
from *assms* **have** (−*c*) ∗ *moebius a b c d z* + *a* ≠ *0* **unfolding** *moebius_def*
by (*simp add*: *field_simps*)
with *assms* **show** *?thesis*
unfolding *moebius_def* **by** (*simp add*: *moebius_def divide_simps*) (*simp add*: *algebra_simps*)*?*
qed

lemma *moebius_inverse'*:
assumes *a* ∗ *d* ≠ *b* ∗ *c c* ∗ *z* − *a* ≠ *0*
shows *moebius a b c d* (*moebius d* (−*b*) (−*c*) *a z*) = *z*
using *assms moebius_inverse*[*of d a* −*b* −*c z*]
by (*auto simp*: *algebra_simps*)

lemma *cmod_add_real_less*:
assumes *Im z* ≠ *0 r*≠*0*
shows *cmod* (*z* + *r*) < *cmod z* + |*r*|
proof (*cases z*)
case (*Complex x y*)
then have *0* < *y* ∗ *y*
using *assms mult_neg_neg* **by** *force*
with *assms* **have** *r* ∗ *x* / |*r*| < *sqrt* (*x*∗*x* + *y*∗*y*)
by (*simp add*: *real_less_rsqrt power2_eq_square*)
then show *?thesis* **using** *assms Complex*
apply (*simp add*: *cmod_def*)
apply (*rule power2_less_imp_less*, *auto*)
apply (*simp add*: *power2_eq_square field_simps*)
done

qed

lemma *cmod_diff_real_less*: *Im z* $\neq$ *0* $\Longrightarrow$ *x*$\neq$*0* $\Longrightarrow$ *cmod* (*z* $-$ *x*) $<$ *cmod z* + |*x*|
using *cmod_add_real_less* [*of z* $-$*x*]
by *simp*

lemma *cmod_square_less_1_plus*:
assumes *Im z* = *0* $\Longrightarrow$ |*Re z*| $<$ *1*
shows (*cmod z*)^2 $<$ *1* + *cmod* (*1* $-$ *z*^{2})
proof (*cases Im z* = *0* $\vee$ *Re z* = *0*)
case *True*
with *assms abs_square_less_1* **show** *?thesis*
by (*force simp add*: *Re_power2 Im_power2 cmod_def*)
next
case *False*
with *cmod_diff_real_less* [*of 1* $-$ *z*^{2} *1*] **show** *?thesis*
by (*simp add*: *norm_power Im_power2*)
qed

6.20.2 The Exponential Function

lemma *exp_npi_numeral*: *exp* (i $*$ *pi* $*$ *Num.numeral n*) = (*$-$1*) ^ *Num.numeral n*
by (*metis exp_of_nat2_mult exp_pi_i' of_nat_numeral*)

lemma *norm_exp_i_times* [*simp*]: *norm* (*exp*(i $*$ *of_real y*)) = *1*
by *simp*

lemma *norm_exp_imaginary*: *norm*(*exp z*) = *1* $\Longrightarrow$ *Re z* = *0*
by *simp*

lemma *field_differentiable_within_exp*: *exp field_differentiable* (*at z within s*)
using *DERIV_exp field_differentiable_at_within field_differentiable_def* **by** *blast*

lemma *continuous_within_exp*:
fixes *z*::'*a*::{*real_normed_field*,*banach*}
shows *continuous* (*at z within s*) *exp*
by (*simp add*: *continuous_at_imp_continuous_within*)

lemma *holomorphic_on_exp* [*holomorphic_intros*]: *exp holomorphic_on s*
by (*simp add*: *field_differentiable_within_exp holomorphic_on_def*)

lemma *holomorphic_on_exp'* [*holomorphic_intros*]:
f holomorphic_on s $\Longrightarrow$ (λ*x*. *exp* (*f x*)) *holomorphic_on s*
using *holomorphic_on_compose*[*OF* _ *holomorphic_on_exp*] **by** (*simp add*: *o_def*)

lemma *exp_analytic_on* [*analytic_intros*]:
assumes *f analytic_on A*
shows (λ*z*. *exp* (*f z*)) *analytic_on A*
by (*metis analytic_on_holomorphic assms holomorphic_on_exp'*)

lemma
assumes ⋀*w*. *w* ∈ *A* ⟹ *exp* (*f w*) = *w*
assumes *f holomorphic_on A z* ∈ *A open A*
shows *deriv_complex_logarithm*: *deriv f z* = *1* / *z*
and *has_field_derivative_complex_logarithm*: (*f has_field_derivative 1* / *z*) (*at z*)
proof −
have [*simp*]: *z* ≠ *0*
using *assms*(*1*)[*of z*] *assms*(*3*) **by** *auto*
have *deriv* [*derivative_intros*]: (*f has_field_derivative deriv f z*) (*at z*)
using *assms holomorphic_derivI* **by** *blast*
have ((λ*w*. *w*) *has_field_derivative 1*) (*at z*)
by (*intro derivative_intros*)
also have *?this* ⟷ ((λ*w*. *exp* (*f w*)) *has_field_derivative 1*) (*at z*)
by (*smt* (*verit*, *best*) *assms has_field_derivative_transform_within_open*)
finally have ((λ*w*. *exp* (*f w*)) *has_field_derivative 1*) (*at z*) **.**
moreover have ((λ*w*. *exp* (*f w*)) *has_field_derivative exp* (*f z*) ∗ *deriv f z*) (*at z*)
by (*rule derivative_eq_intros refl*)+
ultimately have *exp* (*f z*) ∗ *deriv f z* = *1*
using *DERIV_unique* **by** *blast*
with *assms* **show** *deriv f z* = *1* / *z*
by (*simp add*: *field_simps*)
with *deriv* **show** (*f has_field_derivative 1* / *z*) (*at z*)
by *simp*
qed

6.20.3 Euler and de Moivre formulas

The sine series times i

lemma *sin_i_eq*: (λ*n*. (i ∗ *sin_coeff n*) ∗ *z*^*n*) *sums* (i ∗ *sin z*)
proof −
have (λ*n*. i ∗ *sin_coeff n* ∗$_R$ *z*^*n*) *sums* (i ∗ *sin z*)
using *sin_converges sums_mult* **by** *blast*
then show *?thesis*
by (*simp add*: *scaleR_conv_of_real field_simps*)
qed

theorem *exp_Euler*: *exp*(i ∗ *z*) = *cos*(*z*) + i ∗ *sin*(*z*)
proof −
have (λ*n*. (*cos_coeff n* + i ∗ *sin_coeff n*) ∗ *z*^*n*) = (λ*n*. (i ∗ *z*) ^ *n* /$_R$ (*fact n*))
by (*force simp*: *cos_coeff_def sin_coeff_def scaleR_conv_of_real field_simps elim*!: *evenE oddE*)
also have ... *sums* (*exp* (i ∗ *z*))

by (*rule exp_converges*)
finally have (λ*n*. (*cos_coeff n* + i ∗ *sin_coeff n*) ∗ *z*^*n*) *sums* (*exp* (i ∗ *z*)) **.**
moreover have (λ*n*. (*cos_coeff n* + i ∗ *sin_coeff n*) ∗ *z*^*n*) *sums* (*cos z* + i ∗ *sin z*)
using *sums_add* [*OF cos_converges* [*of z*] *sin_i_eq* [*of z*]]
by (*simp add*: *field_simps scaleR_conv_of_real*)
ultimately show *?thesis*
using *sums_unique2* **by** *blast*
qed

corollary *exp_minus_Euler*: *exp*(−(i ∗ *z*)) = *cos*(*z*) − i ∗ *sin*(*z*)
using *exp_Euler* [*of* −*z*] **by** *simp*

lemma *sin_exp_eq*: *sin z* = (*exp*(i ∗ *z*) − *exp*(−(i ∗ *z*))) / (*2*∗i)
by (*simp add*: *exp_Euler exp_minus_Euler*)

lemma *sin_exp_eq′*: *sin z* = i ∗ (*exp*(−(i ∗ *z*)) − *exp*(i ∗ *z*)) / *2*
by (*simp add*: *exp_Euler exp_minus_Euler*)

lemma *cos_exp_eq*: *cos z* = (*exp*(i ∗ *z*) + *exp*(−(i ∗ *z*))) / *2*
by (*simp add*: *exp_Euler exp_minus_Euler*)

theorem *Euler*: *exp*(*z*) = *of_real*(*exp*(*Re z*)) ∗
(*of_real*(*cos*(*Im z*)) + i ∗ *of_real*(*sin*(*Im z*)))
by (*simp add*: *Complex_eq cis.code exp_eq_polar*)

lemma *Re_sin*: *Re*(*sin z*) = *sin*(*Re z*) ∗ (*exp*(*Im z*) + *exp*(−(*Im z*))) / *2*
by (*simp add*: *sin_exp_eq field_simps Re_divide Im_exp*)

lemma *Im_sin*: *Im*(*sin z*) = *cos*(*Re z*) ∗ (*exp*(*Im z*) − *exp*(−(*Im z*))) / *2*
by (*simp add*: *sin_exp_eq field_simps Im_divide Re_exp*)

lemma *Re_cos*: *Re*(*cos z*) = *cos*(*Re z*) ∗ (*exp*(*Im z*) + *exp*(−(*Im z*))) / *2*
by (*simp add*: *cos_exp_eq field_simps Re_divide Re_exp*)

lemma *Im_cos*: *Im*(*cos z*) = *sin*(*Re z*) ∗ (*exp*(−(*Im z*)) − *exp*(*Im z*)) / *2*
by (*simp add*: *cos_exp_eq field_simps Im_divide Im_exp*)

lemma *Re_sin_pos*: *0* < *Re z* ⟹ *Re z* < *pi* ⟹ *Re* (*sin z*) > *0*
by (*auto simp*: *Re_sin Im_sin add_pos_pos sin_gt_zero*)

lemma *Im_sin_nonneg*: *Re z* = *0* ⟹ *0* ≤ *Im z* ⟹ *0* ≤ *Im* (*sin z*)
by (*simp add*: *Re_sin Im_sin algebra_simps*)

lemma *Im_sin_nonneg2*: *Re z* = *pi* ⟹ *Im z* ≤ *0* ⟹ *0* ≤ *Im* (*sin z*)
by (*simp add*: *Re_sin Im_sin algebra_simps*)

6.20.4 Relationships between real and complex trigonometric and hyperbolic functions

lemma *real_sin_eq* [*simp*]: $Re(sin(of\_real\ x)) = sin\ x$
by (*simp add*: *sin_of_real*)

lemma *real_cos_eq* [*simp*]: $Re(cos(of\_real\ x)) = cos\ x$
by (*simp add*: *cos_of_real*)

lemma *DeMoivre*: $(cos\ z + \mathrm{i} * sin\ z)\ \hat{}\ n = cos(n * z) + \mathrm{i} * sin(n * z)$
by (*metis exp_Euler* [*symmetric*] *exp_of_nat_mult mult.left_commute*)

lemma *exp_cnj*: $cnj\ (exp\ z) = exp\ (cnj\ z)$
by (*simp add*: *cis_cnj exp_eq_polar*)

lemma *cnj_sin*: $cnj(sin\ z) = sin(cnj\ z)$
by (*simp add*: *sin_exp_eq exp_cnj field_simps*)

lemma *cnj_cos*: $cnj(cos\ z) = cos(cnj\ z)$
by (*simp add*: *cos_exp_eq exp_cnj field_simps*)

lemma *field_differentiable_at_sin*: *sin field_differentiable at z*
using *DERIV_sin field_differentiable_def* **by** *blast*

lemma *field_differentiable_within_sin*: *sin field_differentiable* (*at z within S*)
by (*simp add*: *field_differentiable_at_sin field_differentiable_at_within*)

lemma *field_differentiable_at_cos*: *cos field_differentiable at z*
using *DERIV_cos field_differentiable_def* **by** *blast*

lemma *field_differentiable_within_cos*: *cos field_differentiable* (*at z within S*)
by (*simp add*: *field_differentiable_at_cos field_differentiable_at_within*)

lemma *holomorphic_on_sin*: *sin holomorphic_on S*
by (*simp add*: *field_differentiable_within_sin holomorphic_on_def*)

lemma *holomorphic_on_cos*: *cos holomorphic_on S*
by (*simp add*: *field_differentiable_within_cos holomorphic_on_def*)

lemma *holomorphic_on_sin'* [*holomorphic_intros*]:
assumes *f holomorphic_on A*
shows (*λx. sin* (*f x*)) *holomorphic_on A*
using *holomorphic_on_compose*[*OF assms holomorphic_on_sin*] **by** (*simp add*: *o_def*)

lemma *holomorphic_on_cos'* [*holomorphic_intros*]:
assumes *f holomorphic_on A*
shows (*λx. cos* (*f x*)) *holomorphic_on A*
using *holomorphic_on_compose*[*OF assms holomorphic_on_cos*] **by** (*simp add*: *o_def*)

lemma *analytic_on_sin* [*analytic_intros*]: *f analytic_on A* $\Longrightarrow$ $(\lambda w.\ sin\ (f\ w))$ *analytic_on A*
and *analytic_on_cos* [*analytic_intros*]: *f analytic_on A* $\Longrightarrow$ $(\lambda w.\ cos\ (f\ w))$ *analytic_on A*
and *analytic_on_sinh* [*analytic_intros*]: *f analytic_on A* $\Longrightarrow$ $(\lambda w.\ sinh\ (f\ w))$ *analytic_on A*
and *analytic_on_cosh* [*analytic_intros*]: *f analytic_on A* $\Longrightarrow$ $(\lambda w.\ cosh\ (f\ w))$ *analytic_on A*
unfolding *sin_exp_eq cos_exp_eq sinh_def cosh_def*
by (*auto intro*!: *analytic_intros*)

lemma *analytic_on_tan* [*analytic_intros*]:
f *analytic_on A* $\Longrightarrow$ $(\bigwedge z.\ z \in A \Longrightarrow cos\ (f\ z) \neq 0) \Longrightarrow (\lambda w.\ tan\ (f\ w))$ *analytic_on A*
and *analytic_on_cot* [*analytic_intros*]:
f *analytic_on A* $\Longrightarrow$ $(\bigwedge z.\ z \in A \Longrightarrow sin\ (f\ z) \neq 0) \Longrightarrow (\lambda w.\ cot\ (f\ w))$ *analytic_on A*
and *analytic_on_tanh* [*analytic_intros*]:
f *analytic_on A* $\Longrightarrow$ $(\bigwedge z.\ z \in A \Longrightarrow cosh\ (f\ z) \neq 0) \Longrightarrow (\lambda w.\ tanh\ (f\ w))$ *analytic_on A*
unfolding *tan_def cot_def tanh_def* **by** (*auto intro*!: *analytic_intros*)

6.20.5 More on the Polar Representation of Complex Numbers

lemma *exp_Complex*: $exp(Complex\ r\ t) = of\_real(exp\ r) * Complex\ (cos\ t)\ (sin\ t)$
using *Complex_eq Euler complex.sel* **by** *presburger*

lemma *exp_eq_1*: $exp\ z = 1 \longleftrightarrow Re(z) = 0 \wedge (\exists\, n{::}int.\ Im(z) = of\_int\ (2 * n) * pi)$
(**is** *?lhs* = *?rhs*)
proof
assume $exp\ z = 1$
then have $Re\ z = 0$
by (*metis exp_eq_one_iff norm_exp_eq_Re norm_one*)
with ‹*?lhs*› **show** *?rhs*
by (*metis Re_exp cos_one_2pi_int exp_zero mult.commute mult_1 of_int_mult of_int_numeral one_complex.simps*(1))
next
assume *?rhs* **then show** *?lhs*
using *Im_exp Re_exp complex_eq_iff*
by (*simp add*: *cos_one_2pi_int cos_one_sin_zero mult.commute*)
qed

lemma *exp_eq*: $exp\ w = exp\ z \longleftrightarrow (\exists\, n{::}int.\ w = z + (of\_int\ (2 * n) * pi) * \mathrm{i})$
(**is** *?lhs* = *?rhs*)
proof −

```
  have exp w = exp z ⟷ exp (w−z) = 1
    by (simp add: exp_diff)
  also have ... ⟷ (Re w = Re z ∧ (∃ n::int. Im w − Im z = of_int (2 * n) *
pi))
    by (simp add: exp_eq_1)
  also have ... ⟷ ?rhs
    by (auto simp: algebra_simps intro!: complex_eqI)
  finally show ?thesis .
qed

lemma exp_complex_eqI: |Im w − Im z| < 2*pi ⟹ exp w = exp z ⟹ w = z
  by (auto simp: exp_eq abs_mult)

lemma exp_integer_2pi:
  assumes n ∈ ℤ
  shows exp((2 * n * pi) * i) = 1
  by (metis assms cis_conv_exp cis_multiple_2pi mult.assoc mult.commute)

lemma exp_plus_2pin [simp]: exp (z + i * (of_int n * (of_real pi * 2))) = exp z
  by (simp add: exp_eq)

lemma exp_integer_2pi_plus1:
  assumes n ∈ ℤ
  shows exp(((2 * n + 1) * pi) * i) = − 1
  using exp_integer_2pi [OF assms]
 by (metis cis_conv_exp cis_mult cis_pi distrib_left mult.commute mult.right_neutral)

lemma inj_on_exp_pi:
  fixes z::complex shows inj_on exp (ball z pi)
proof (clarsimp simp: inj_on_def exp_eq)
  fix y n
  assume dist z (y + 2 * of_int n * of_real pi * i) < pi
         dist z y < pi
  then have dist y (y + 2 * of_int n * of_real pi * i) < pi+pi
    using dist_commute_lessI dist_triangle_less_add by blast
  then have norm (2 * of_int n * of_real pi * i) < 2*pi
    by (simp add: dist_norm)
  then show n = 0
    by (auto simp: norm_mult)
qed

lemma cmod_add_squared:
  fixes r1 r2::real
  shows (cmod (r1 * exp (i * ϑ1) + r2 * exp (i * ϑ2)))^2 = r1^2 + r2^2 + 2 * r1
* r2 * cos (ϑ1 − ϑ2) (is (cmod (?z1 + ?z2))^2 = ?rhs)
proof −
  have (cmod (?z1 + ?z2))^2 = (?z1 + ?z2) * cnj (?z1 + ?z2)
    by (rule complex_norm_square)
  also have ... = (?z1 * cnj ?z1 + ?z2 * cnj ?z2) + (?z1 * cnj ?z2 + cnj ?z1 *
```

?z2)
  by (*simp add*: *algebra_simps*)
 also have ... = (*norm ?z1*)^2 + (*norm ?z2*)^2 + *2* ∗ *Re* (*?z1* ∗ *cnj ?z2*)
  unfolding *complex_norm_square* [*symmetric*] *cnj_add_mult_eq_Re* **by** *simp*
 also have ... = *?rhs*
  by (*simp add*: *norm_mult*) (*simp add*: *exp_Euler complex_is_Real_iff* [*THEN iffD1*] *cos_diff algebra_simps*)
 finally show *?thesis*
  using *of_real_eq_iff* **by** *blast*
qed

lemma *cmod_diff_squared*:
 fixes *r1 r2*::*real*
  shows (*cmod* (*r1* ∗ *exp* (i ∗ *ϑ1*) − *r2* ∗ *exp* (i ∗ *ϑ2*)))^2 = *r1*^{2} + *r2*^{2} − *2*∗*r1*∗*r2*∗*cos* (*ϑ1* − *ϑ2*)
 using *cmod_add_squared* [*of r1 _* −*r2*] **by** *simp*

lemma *polar_convergence*:
 fixes *R*::*real*
 assumes ⋀*j*. *r j* > *0 R* > *0*
 shows ((λ*j*. *r j* ∗ *exp* (i ∗ *ϑ j*)) ⟶ (*R* ∗ *exp* (i ∗ Θ))) ⟷
       (*r* ⟶ *R*) ∧ (∃ *k*. (λ*j*. *ϑ j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)) ⟶ Θ) (**is** (*?z* ⟶ *?Z*) = *?rhs*)
proof
 assume *L*: *?z* ⟶ *?Z*
 have *rR*: *r* ⟶ *R*
  using *tendsto_norm* [*OF L*] *assms* **by** (*auto simp*: *norm_mult abs_of_pos*)
 moreover obtain *k* **where** (λ*j*. *ϑ j* − *of_int* (*k j*) ∗ (*2* ∗ *pi*)) ⟶ Θ
 proof −
  have *cos* (*ϑ j* − Θ) = ((*r j*)^2 + *R*^{2} − (*norm*(*?z j* − *?Z*))^2) / (*2* ∗ *R* ∗ *r j*) **for** *j*
   using *assms* **by** (*auto simp*: *cmod_diff_squared less_le*)
  moreover have (λ*j*. ((*r j*)^2 + *R*^{2} − (*norm*(*?z j* − *?Z*))^2) / (*2* ∗ *R* ∗ *r j*)) ⟶ ((*R*^{2} + *R*^{2} − (*norm*(*?Z* − *?Z*))^2) / (*2* ∗ *R* ∗ *R*))
   by (*intro L rR tendsto_intros*) (*use* ‹*R* > *0*› **in** *force*)
  moreover have ((*R*^{2} + *R*^{2} − (*norm*(*?Z* − *?Z*))^2) / (*2* ∗ *R* ∗ *R*)) = *1*
   using ‹*R* > *0*› **by** (*simp add*: *power2_eq_square field_split_simps*)
  ultimately have (λ*j*. *cos* (*ϑ j* − Θ)) ⟶ *1*
   by *auto*
  then show *?thesis*
   using *that cos_diff_limit_1* **by** *blast*
 qed
 ultimately show *?rhs*
  by *metis*
next
 assume *R*: *?rhs*
 show *?z* ⟶ *?Z*
 proof (*rule tendsto_mult*)
  show (λ*x*. *complex_of_real* (*r x*)) ⟶ *of_real R*

using *R* **by** (*auto simp*: *tendsto_of_real_iff*)
obtain *k* **where** $(\lambda j.\ \vartheta\ j - of\_int\ (k\ j) * (2 * pi)) \longrightarrow \Theta$
using *R* **by** *metis*
then have $(\lambda j.\ complex\_of\_real\ (\vartheta\ j - of\_int\ (k\ j) * (2 * pi))) \longrightarrow of\_real\ \Theta$
using *tendsto_of_real_iff* **by** *force*
then have $(\lambda j.\ exp\ (\mathrm{i} * of\_real\ (\vartheta\ j - of\_int\ (k\ j) * (2 * pi)))) \longrightarrow exp\ (\mathrm{i} * \Theta)$
using *tendsto_mult* [*OF tendsto_const*] *isCont_exp isCont_tendsto_compose* **by** *blast*
moreover have $exp\ (\mathrm{i} * of\_real\ (\vartheta\ j - of\_int\ (k\ j) * (2 * pi))) = exp\ (\mathrm{i} * \vartheta\ j)$ **for** *j*
unfolding *exp_eq*
by (*rule_tac x*=$- k\ j$ **in** *exI*) (*auto simp*: *algebra_simps*)
ultimately show $(\lambda j.\ exp\ (\mathrm{i} * \vartheta\ j)) \longrightarrow exp\ (\mathrm{i} * \Theta)$
by *auto*
qed
qed

lemma *exp_i_ne_1*:
assumes $0 < x\ x < 2*pi$
shows $exp(\mathrm{i} * of\_real\ x) \neq 1$
by (*smt* (*verit*) *Im_i_times Re_complex_of_real assms exp_complex_eqI exp_zero zero_complex.sel*(2))

lemma *sin_eq_0*:
fixes *z*::*complex*
shows $sin\ z = 0 \longleftrightarrow (\exists n{::}int.\ z = of\_real(n * pi))$
by (*simp add*: *sin_exp_eq exp_eq*)

lemma *cos_eq_0*:
fixes *z*::*complex*
shows $cos\ z = 0 \longleftrightarrow (\exists n{::}int.\ z = complex\_of\_real(n * pi) + of\_real\ pi/2)$
using *sin_eq_0* [*of z* − *of_real pi*/2]
by (*simp add*: *sin_diff algebra_simps*)

lemma *cos_eq_1*:
fixes *z*::*complex*
shows $cos\ z = 1 \longleftrightarrow (\exists n{::}int.\ z = complex\_of\_real(2 * n * pi))$
by (*metis Re_complex_of_real cos_of_real cos_one_2pi_int cos_one_sin_zero mult.commute of_real_1 sin_eq_0*)

lemma *csin_eq_1*:
fixes *z*::*complex*
shows $sin\ z = 1 \longleftrightarrow (\exists n{::}int.\ z = of\_real(2 * n * pi) + of\_real\ pi/2)$
using *cos_eq_1* [*of z* − *of_real pi*/2]
by (*simp add*: *cos_diff algebra_simps*)

lemma *csin_eq_minus1*:

fixes *z*::*complex*
shows *sin z = −1 ⟷ (∃ n*::*int. z = complex_of_real(2 ∗ n ∗ pi) + 3/2∗pi)*
(**is** *_ = ?rhs*)
proof −
have *sin z = −1 ⟷ sin (−z) = 1*
by (*simp add*: *equation_minus_iff*)
also have *... ⟷ (∃ n*::*int. z = − of_real(2 ∗ n ∗ pi) − of_real pi/2*)
by (*metis* (*mono_tags*, *lifting*) *add_uminus_conv_diff csin_eq_1 equation_minus_iff minus_add_distrib*)
also have *... = ?rhs*
apply *safe*
apply (*rule_tac* [*2*] *x*=−(*x*+*1*) **in** *exI*)
apply (*rule_tac x*=−(*x*+*1*) **in** *exI*)
apply (*simp_all add*: *algebra_simps*)
done
finally show *?thesis* **.**
qed

lemma *ccos_eq_minus1*:
fixes *z*::*complex*
shows *cos z = −1 ⟷ (∃ n*::*int. z = complex_of_real(2 ∗ n ∗ pi) + pi*)
using *csin_eq_1* [*of z − of_real pi/2*]
by (*simp add*: *sin_diff algebra_simps equation_minus_iff*)

lemma *sin_eq_1*: *sin x = 1 ⟷ (∃ n*::*int. x = (2 ∗ n + 1 / 2) ∗ pi*)
(**is** *_ = ?rhs*)
proof −
have *sin x = 1 ⟷ sin (complex_of_real x) = 1*
by (*metis of_real_1 one_complex.simps*(*1*) *real_sin_eq sin_of_real*)
also have *... ⟷ (∃ n*::*int. x = of_real(2 ∗ n ∗ pi) + of_real pi/2*)
by (*metis csin_eq_1 Re_complex_of_real id_apply of_real_add of_real_divide of_real_eq_id of_real_numeral*)
also have *... = ?rhs*
by (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *sin_eq_minus1*: *sin x = −1 ⟷ (∃ n*::*int. x = (2∗n + 3/2) ∗ pi*) (**is** *_ = ?rhs*)
proof −
have *sin x = −1 ⟷ sin (complex_of_real x) = −1*
by (*metis Re_complex_of_real of_real_def scaleR_minus1_left sin_of_real*)
also have *... ⟷ (∃ n*::*int. x = of_real(2 ∗ n ∗ pi) + 3/2∗pi*)
by (*metis Re_complex_of_real csin_eq_minus1 id_apply of_real_add of_real_eq_id*)
also have *... = ?rhs*
by (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *cos_eq_minus1*: *cos x = −1 ⟷ (∃ n::int. x = (2∗n + 1) ∗ pi)*
(**is** *_ = ?rhs*)
proof −
have *cos x = −1 ⟷ cos (complex_of_real x) = −1*
by (*metis Re_complex_of_real of_real_def scaleR_minus1_left cos_of_real*)
also have *... ⟷ (∃ n::int. x = of_real(2 ∗ n ∗ pi) + pi)*
by (*metis ccos_eq_minus1 id_apply of_real_add of_real_eq_id of_real_eq_iff*)
also have *... = ?rhs*
by (*auto simp*: *algebra_simps*)
finally show *?thesis* **.**
qed

lemma *cos_gt_neg1*:
assumes (*t::real*) ∈ {*−pi<..<pi*}
shows *cos t > −1*
using *assms*
by *simp* (*metis cos_minus cos_monotone_0_pi cos_monotone_minus_pi_0 cos_pi linorder_le_cases*)

lemma *dist_exp_i_1*: *norm(exp*(i ∗ *of_real t) − 1) = 2 ∗ |sin(t / 2)|*
proof −
have *sqrt (2 − cos t ∗ 2) = 2 ∗ |sin (t / 2)|*
using *cos_double_sin* [*of t/2*] **by** (*simp add*: *real_sqrt_mult*)
then show *?thesis*
by (*simp add*: *exp_Euler cmod_def power2_diff sin_of_real cos_of_real algebra_simps*)
qed

lemma *sin_cx_2pi* [*simp*]: ⟦*z = of_int m*; *even m*⟧ ⟹ *sin (z ∗ complex_of_real pi) = 0*
by (*simp add*: *sin_eq_0*)

lemma *cos_cx_2pi* [*simp*]: ⟦*z = of_int m*; *even m*⟧ ⟹ *cos (z ∗ complex_of_real pi) = 1*
using *cos_eq_1* **by** *auto*

lemma *complex_sin_eq*:
fixes *w :: complex*
shows *sin w = sin z ⟷ (∃ n ∈ ℤ. w = z + of_real(2∗n∗pi) ∨ w = −z + of_real((2∗n + 1)∗pi))*
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then consider *sin ((w − z) / 2) = 0 | cos ((w + z) / 2) = 0*
by (*metis divide_eq_0_iff nonzero_eq_divide_eq right_minus_eq sin_diff_sin zero_neq_numeral*)
then show *?rhs*
proof *cases*
case *1*

```
    then show ?thesis
    by (simp add: sin_eq_0 algebra_simps) (metis Ints_of_int of_real_of_int_eq)
  next
    case 2
    then show ?thesis
    by (simp add: cos_eq_0 algebra_simps) (metis Ints_of_int of_real_of_int_eq)
  qed
next
  assume ?rhs
  then consider n::int where w = z + of_real (2 * of_int n * pi)
            | n::int where   w = -z + of_real ((2 * of_int n + 1) * pi)
    using Ints_cases by blast
  then show ?lhs
  proof cases
    case 1
    then show ?thesis
      using Periodic_Fun.sin.plus_of_int [of z n]
      by (auto simp: algebra_simps)
  next
    case 2
    then show ?thesis
      using Periodic_Fun.sin.plus_of_int [of -z n]
      apply (simp add: algebra_simps)
    by (metis add.commute add.inverse_inverse add_diff_cancel_left diff_add_cancel
sin_plus_pi)
  qed
qed

lemma complex_cos_eq:
  fixes w :: complex
  shows cos w = cos z ⟷ (∃n ∈ ℤ. w = z + of_real(2*n*pi) ∨ w = -z +
of_real(2*n*pi))
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then consider sin ((w + z) / 2) = 0 | sin ((z - w) / 2) = 0
    by (metis mult_eq_0_iff cos_diff_cos right_minus_eq zero_neq_numeral)
  then show ?rhs
  proof cases
    case 1
    then obtain n where w + z = of_int n * (complex_of_real pi * 2)
      by (auto simp: sin_eq_0 algebra_simps)
    then have w = -z + of_real(2 * of_int n * pi)
      by (auto simp: algebra_simps)
    then show ?thesis
      using Ints_of_int by blast
  next
    case 2
    then obtain n where z = w + of_int n * (complex_of_real pi * 2)
```

```
    by (auto simp: sin_eq_0 algebra_simps)
  then have w = z + complex_of_real (2 * of_int(-n) * pi)
    by (auto simp: algebra_simps)
  then show ?thesis
    using Ints_of_int by blast
 qed
next
 assume ?rhs
 then obtain n::int where w: w = z + of_real (2* of_int n*pi) ∨
                             w = -z + of_real(2*n*pi)
  using Ints_cases  by (metis of_int_mult of_int_numeral)
 then show ?lhs
  using Periodic_Fun.cos.plus_of_int [of z n]
  apply (simp add: algebra_simps)
  by (metis cos.plus_of_int cos_minus minus_add_cancel mult.commute)
qed

lemma sin_eq:
  sin x = sin y ⟷ (∃n ∈ ℤ. x = y + 2*n*pi ∨ x = -y + (2*n + 1)*pi)
 using complex_sin_eq [of x y]
 by (simp only: sin_of_real Re_complex_of_real of_real_add [symmetric] of_real_minus
[symmetric] of_real_mult [symmetric] of_real_eq_iff)

lemma cos_eq:
  cos x = cos y ⟷ (∃n ∈ ℤ. x = y + 2*n*pi ∨ x = -y + 2*n*pi)
 using complex_cos_eq [of x y] unfolding cos_of_real
 by (metis Re_complex_of_real of_real_add of_real_minus)

lemma sinh_complex:
 fixes z :: complex
 shows (exp z - inverse (exp z)) / 2 = -i * sin(i * z)
 by (simp add: sin_exp_eq field_split_simps exp_minus)

lemma sin_i_times:
 fixes z :: complex
 shows sin(i * z) = i * ((exp z - inverse (exp z)) / 2)
 using sinh_complex by auto

lemma sinh_real:
 fixes x :: real
 shows of_real((exp x - inverse (exp x)) / 2) = -i * sin(i * of_real x)
 by (simp add: exp_of_real sin_i_times)

lemma cosh_complex:
 fixes z :: complex
 shows (exp z + inverse (exp z)) / 2 = cos(i * z)
 by (simp add: cos_exp_eq field_split_simps exp_minus exp_of_real)

lemma cosh_real:
```

fixes $x :: real$
shows $of\_real((exp\ x + inverse\ (exp\ x))\ /\ 2) = cos(\mathrm{i} * of\_real\ x)$
by (*simp add*: *cos_exp_eq field_split_simps exp_minus exp_of_real*)

lemmas *cos_i_times* = *cosh_complex* [*symmetric*]

lemma *norm_cos_squared*:
$norm(cos\ z)\ \hat{}\ 2 = cos(Re\ z)\ \hat{}\ 2 + (exp(Im\ z) - inverse(exp(Im\ z)))\ \hat{}\ 2\ /\ 4$
proof (*cases z*)
case (*Complex x1 x2*)
then show *?thesis*
apply (*simp only*: *cos_add cmod_power2 cos_of_real sin_of_real Complex_eq*)
apply (*simp add*: *cos_exp_eq sin_exp_eq exp_minus exp_of_real Re_divide Im_divide power_divide*)
apply (*simp only*: *left_diff_distrib* [*symmetric*] *power_mult_distrib sin_squared_eq*)
apply (*simp add*: *power2_eq_square field_split_simps*)
done
qed

lemma *norm_sin_squared*:
$norm(sin\ z)\ \hat{}\ 2 = (exp(2 * Im\ z) + inverse(exp(2 * Im\ z)) - 2 * cos(2 * Re\ z))\ /\ 4$
using *cos_double_sin* [*of Re z*]
apply (*simp add*: *sin_cos_eq norm_cos_squared exp_minus mult.commute* [*of _ 2*] *exp_double*)
apply (*simp add*: *algebra_simps power2_eq_square*)
done

lemma *exp_uminus_Im*: $exp\ (-\ Im\ z) \leq exp\ (cmod\ z)$
using *abs_Im_le_cmod linear order_trans* **by** *fastforce*

lemma *norm_cos_le*:
fixes *z*::*complex*
shows $norm(cos\ z) \leq exp(norm\ z)$
proof −
have $Im\ z \leq cmod\ z$
using *abs_Im_le_cmod abs_le_D1* **by** *auto*
then have $exp\ (-\ Im\ z) + exp\ (Im\ z) \leq exp\ (cmod\ z) * 2$
by (*metis exp_uminus_Im add_mono exp_le_cancel_iff mult_2_right*)
then show *?thesis*
by (*force simp add*: *cos_exp_eq norm_divide intro*: *order_trans* [*OF norm_triangle_ineq*])
qed

lemma *norm_cos_plus1_le*:
fixes *z*::*complex*
shows $norm(1 + cos\ z) \leq 2 * exp(norm\ z)$
by (*metis mult_2 norm_cos_le norm_ge_zero norm_one norm_triangle_mono one_le_exp_iff*)

lemma *sinh_conv_sin*: *sinh z* = −i ∗ *sin* (i∗*z*)
by (*simp add*: *sinh_field_def sin_i_times exp_minus*)

lemma *cosh_conv_cos*: *cosh z* = *cos* (i∗*z*)
by (*simp add*: *cosh_field_def cos_i_times exp_minus*)

lemma *tanh_conv_tan*: *tanh z* = −i ∗ *tan* (i∗*z*)
by (*simp add*: *tanh_def sinh_conv_sin cosh_conv_cos tan_def*)

lemma *sin_conv_sinh*: *sin z* = −i ∗ *sinh* (i∗*z*)
by (*simp add*: *sinh_conv_sin*)

lemma *cos_conv_cosh*: *cos z* = *cosh* (i∗*z*)
by (*simp add*: *cosh_conv_cos*)

lemma *tan_conv_tanh*: *tan z* = −i ∗ *tanh* (i∗*z*)
by (*simp add*: *tan_def sin_conv_sinh cos_conv_cosh tanh_def*)

lemma *sinh_complex_eq_iff*:
sinh (*z* :: *complex*) = *sinh w* ⟷
(∃ *n*∈ℤ. *z* = *w* − *2* ∗ i ∗ *of_real n* ∗ *of_real pi* ∨
z = −(*2* ∗ *complex_of_real n* + *1*) ∗ i ∗ *complex_of_real pi* − *w*) (**is** _ = *?rhs*)
proof −
have *sinh z* = *sinh w* ⟷ *sin* (i ∗ *z*) = *sin* (i ∗ *w*)
by (*simp add*: *sinh_conv_sin*)
also have . . . ⟷ *?rhs*
by (*subst complex_sin_eq*) (*force simp*: *field_simps complex_eq_iff*)
finally show *?thesis* **.**
qed

6.20.6 Taylor series for complex exponential, sine and cosine

declare *power_Suc* [*simp del*]

lemma *Taylor_exp_field*:
fixes *z*::′*a*::{*banach*,*real_normed_field*}
shows *norm* (*exp z* − (∑ *i*≤*n*. *z* ^ *i* / *fact i*)) ≤ *exp* (*norm z*) ∗ (*norm z* ^ *Suc n*) / *fact n*
proof (*rule field_Taylor*[*of* _ *n* λ*k*. *exp exp* (*norm z*) *0 z*, *simplified*])
show *convex* (*closed_segment 0 z*)
by (*rule convex_closed_segment* [*of 0 z*])
next
fix *k x*
assume *x* ∈ *closed_segment 0 z k* ≤ *n*
show (*exp has_field_derivative exp x*) (*at x within closed_segment 0 z*)
using *DERIV_exp DERIV_subset* **by** *blast*
next
fix *x*

assume *x*: *x* ∈ *closed_segment 0 z*
have *norm* (*exp x*) ≤ *exp* (*norm x*)
by (*rule norm_exp*)
also have *norm x* ≤ *norm z*
using *x* **by** (*auto simp*: *closed_segment_def intro*!: *mult_left_le_one_le*)
finally show *norm* (*exp x*) ≤ *exp* (*norm z*)
by *simp*
qed *auto*

For complex *z*, a tighter bound than in the previous result

lemma *Taylor_exp*:
norm(*exp z* − (∑ *k*≤*n*. *z* ^ *k* / (*fact k*))) ≤ *exp*|*Re z*| ∗ (*norm z*) ^ (*Suc n*) / (*fact n*)
proof (*rule complex_Taylor* [*of* _ *n* λ*k*. *exp exp*|*Re z*| *0 z*, *simplified*])
show *convex* (*closed_segment 0 z*)
by (*rule convex_closed_segment* [*of 0 z*])
next
fix *k x*
assume *x* ∈ *closed_segment 0 z k* ≤ *n*
show (*exp has_field_derivative exp x*) (*at x within closed_segment 0 z*)
using *DERIV_exp DERIV_subset* **by** *blast*
next
fix *x*
assume *x* ∈ *closed_segment 0 z*
then obtain *u* **where** *u*: *x* = *complex_of_real u* ∗ *z 0* ≤ *u u* ≤ *1*
by (*auto simp*: *closed_segment_def scaleR_conv_of_real*)
then have *u* ∗ *Re z* ≤ |*Re z*|
by (*metis abs_ge_self abs_ge_zero mult.commute mult.right_neutral mult_mono*)
then show *Re x* ≤ |*Re z*|
by (*simp add*: *u*)
qed *auto*

lemma
assumes *0* ≤ *u u* ≤ *1*
shows *cmod_sin_le_exp*: *cmod* (*sin* (*u* ∗*R z*)) ≤ *exp* |*Im z*|
and *cmod_cos_le_exp*: *cmod* (*cos* (*u* ∗*R z*)) ≤ *exp* |*Im z*|
proof −
have *mono*: ⋀*u w z*::*real*. *w* ≤ *u* ⟹ *z* ≤ *u* ⟹ (*w* + *z*)/*2* ≤ *u*
by *simp*
have ∗: (*cmod* (*exp* (i ∗ (*u* ∗ *z*))) + *cmod* (*exp* (− (i ∗ (*u* ∗ *z*))))) / *2* ≤ *exp* |*Im z*|
proof (*rule mono*)
show *cmod* (*exp* (i ∗ (*u* ∗ *z*))) ≤ *exp* |*Im z*|
using *assms*
by (*auto simp*: *abs_if mult_left_le_one_le not_less intro*: *order_trans* [*of* _ *0*])
show *cmod* (*exp* (− (i ∗ (*u* ∗ *z*)))) ≤ *exp* |*Im z*|
using *assms*
by (*auto simp*: *abs_if mult_left_le_one_le mult_nonneg_nonpos intro*: *or-*

```
der_trans [of _ 0])
  qed
  have cmod (sin (u *R z)) = cmod (exp (i * (u * z)) - exp (- (i * (u * z)))) / 2
    by (auto simp: scaleR_conv_of_real norm_mult norm_power sin_exp_eq
norm_divide)
  also have ... ≤ (cmod (exp (i * (u * z))) + cmod (exp (- (i * (u * z)))) ) / 2
    by (intro divide_right_mono norm_triangle_ineq4) simp
  also have ... ≤ exp |Im z|
    by (rule *)
  finally show cmod (sin (u *R z)) ≤ exp |Im z| .
  have cmod (cos (u *R z)) = cmod (exp (i * (u * z)) + exp (- (i * (u * z)))) / 2
    by (auto simp: scaleR_conv_of_real norm_mult norm_power cos_exp_eq
norm_divide)
  also have ... ≤ (cmod (exp (i * (u * z))) + cmod (exp (- (i * (u * z)))) ) / 2
    by (intro divide_right_mono norm_triangle_ineq) simp
  also have ... ≤ exp |Im z|
    by (rule *)
  finally show cmod (cos (u *R z)) ≤ exp |Im z| .
qed

lemma Taylor_sin:
  norm(sin z - (∑ k≤n. complex_of_real (sin_coeff k) * z ^ k))
   ≤ exp|Im z| * (norm z) ^ (Suc n) / (fact n)
proof -
  have mono: ⋀u w z::real. w ≤ u ⟹ z ≤ u ⟹ w + z ≤ u*2
    by arith
  have *: cmod (sin z -
              (∑ i≤n. (-1) ^ (i div 2) * (if even i then sin 0 else cos 0) * z ^ i /
(fact i)))
         ≤ exp |Im z| * cmod z ^ Suc n / (fact n)
  proof (rule complex_Taylor [of closed_segment 0 z n
                              λk x. (-1)^(k div 2) * (if even k then sin x else cos x)
                              exp|Im z| 0 z, simplified])
    fix k x
   show ((λx. (- 1) ^ (k div 2) * (if even k then sin x else cos x)) has_field_derivative
          (- 1) ^ (Suc k div 2) * (if odd k then sin x else cos x))
          (at x within closed_segment 0 z)
     by (cases even k) (intro derivative_eq_intros | simp add: power_Suc)+
  next
    fix x
    assume x ∈ closed_segment 0 z
    then show cmod ((- 1) ^ (Suc n div 2) * (if odd n then sin x else cos x)) ≤
exp |Im z|
    by (auto simp: closed_segment_def norm_mult norm_power cmod_sin_le_exp
cmod_cos_le_exp)
  qed
  have **: ⋀k. complex_of_real (sin_coeff k) * z ^ k
         = (-1)^(k div 2) * (if even k then sin 0 else cos 0) * z^k / of_nat (fact
k)
```

```
    by (auto simp: sin_coeff_def elim!: oddE)
  show ?thesis
    by (simp add: ** order_trans [OF _ *])
qed

lemma Taylor_cos:
  norm(cos z - (∑ k≤n. complex_of_real (cos_coeff k) * z ^ k))
   ≤ exp|Im z| * (norm z) ^ Suc n / (fact n)
proof -
  have mono: ⋀u w z::real. w ≤ u ⟹ z ≤ u ⟹ w + z ≤ u*2
     by arith
  have *: cmod (cos z -
                (∑ i≤n. (-1) ^ (Suc i div 2) * (if even i then cos 0 else sin 0) * z
^ i / (fact i)))
          ≤ exp |Im z| * cmod z ^ Suc n / (fact n)
  proof (rule complex_Taylor [of closed_segment 0 z n
                                λk x. (-1)^(Suc k div 2) * (if even k then cos x else sin
x)
                                exp|Im z| 0 z, simplified])
    fix k x
    assume x ∈ closed_segment 0 z k ≤ n
      show ((λx. (- 1) ^ (Suc k div 2) * (if even k then cos x else sin x))
has_field_derivative
           (- 1) ^ Suc (k div 2) * (if odd k then cos x else sin x))
            (at x within closed_segment 0 z)
      by (cases even k) (intro derivative_eq_intros | simp add: power_Suc)+
  next
    fix x
    assume x ∈ closed_segment 0 z
    then show cmod ((- 1) ^ Suc (n div 2) * (if odd n then cos x else sin x)) ≤
exp |Im z|
    by (auto simp: closed_segment_def norm_mult norm_power cmod_sin_le_exp
cmod_cos_le_exp)
  qed
  have **: ⋀k. complex_of_real (cos_coeff k) * z ^ k
          = (-1)^(Suc k div 2) * (if even k then cos 0 else sin 0) * z^k / of_nat
(fact k)
    by (auto simp: cos_coeff_def elim!: evenE)
  show ?thesis
    by (simp add: ** order_trans [OF _ *])
qed

declare power_Suc [simp]
```

32-bit Approximation to e

```
lemma e_approx_32: |exp(1) - 5837465777 / 2147483648| ≤ (inverse(2 ^ 32)::real)
  using Taylor_exp [of 1 14] exp_le
  apply (simp add: sum_distrib_right in_Reals_norm Re_exp atMost_nat_numeral
fact_numeral)
```

apply (*simp only*: *pos_le_divide_eq* [*symmetric*])
done

lemma *e_less_272*: *exp 1* < (*272*/*100*::*real*)
using *e_approx_32*
by (*simp add*: *abs_if split*: *if_split_asm*)

lemma *ln_272_gt_1*: *ln* (*272*/*100*) > (*1*::*real*)
by (*metis e_less_272 exp_less_cancel_iff exp_ln_iff less_trans ln_exp*)

Apparently redundant. But many arguments involve integers.

lemma *ln3_gt_1*: *ln 3* > (*1*::*real*)
by (*simp add*: *less_trans* [*OF ln_272_gt_1*])

6.20.7 The argument of a complex number (HOL Light version)

definition *is_Arg* :: [*complex*,*real*] ⇒ *bool*
where *is_Arg z r* ≡ *z* = *of_real*(*norm z*) ∗ *exp*(i ∗ *of_real r*)

definition *Arg2pi* :: *complex* ⇒ *real*
where *Arg2pi z* ≡ *if z* = *0 then 0 else THE t*. *0* ≤ *t* ∧ *t* < *2*∗*pi* ∧ *is_Arg z t*

lemma *is_Arg_2pi_iff*: *is_Arg z* (*r* + *of_int k* ∗ (*2* ∗ *pi*)) ⟷ *is_Arg z r*
by (*simp add*: *algebra_simps is_Arg_def*)

lemma *is_Arg_eqI*:
assumes *is_Arg z r* **and** *is_Arg z s* **and** *abs* (*r*−*s*) < *2*∗*pi* **and** *z* ≠ *0*
shows *r* = *s*
using *assms* **unfolding** *is_Arg_def*
by (*metis Im_i_times Re_complex_of_real exp_complex_eqI mult_cancel_left mult_eq_0_iff*)

This function returns the angle of a complex number from its representation in polar coordinates. Due to periodicity, its range is arbitrary. *Arg2pi* follows HOL Light in adopting the interval $[0,2\pi)$. But we have the same periodicity issue with logarithms, and it is usual to adopt the same interval for the complex logarithm and argument functions. Further on down, we shall define both functions for the interval $(-\pi,\pi]$. The present version is provided for compatibility.

lemma *Arg2pi_0* [*simp*]: *Arg2pi*(*0*) = *0*
by (*simp add*: *Arg2pi_def*)

lemma *Arg2pi_unique_lemma*:
assumes *is_Arg z t*
and *is_Arg z t′*
and *0* ≤ *t t* < *2*∗*pi*
and *0* ≤ *t′ t′* < *2*∗*pi*

```
    and z ≠ 0
  shows t' = t
  using is_Arg_eqI assms by force

lemma Arg2pi: 0 ≤ Arg2pi z ∧ Arg2pi z < 2*pi ∧ is_Arg z (Arg2pi z)
proof (cases z=0)
  case True then show ?thesis
    by (simp add: Arg2pi_def is_Arg_def)
next
  case False
  obtain t where t: 0 ≤ t t < 2*pi
            and ReIm: Re z / cmod z = cos t Im z / cmod z = sin t
    using sincos_total_2pi [OF complex_unit_circle [OF False]]
    by blast
  then have z: is_Arg z t
    unfolding is_Arg_def
    using t False ReIm
   by (intro complex_eqI) (auto simp: exp_Euler sin_of_real cos_of_real field_split_simps)
  show ?thesis
    apply (simp add: Arg2pi_def False)
    apply (rule theI [where a=t])
    using t z False
    apply (auto intro: Arg2pi_unique_lemma)
    done
qed

corollary
  shows Arg2pi_ge_0: 0 ≤ Arg2pi z
    and Arg2pi_lt_2pi: Arg2pi z < 2*pi
    and Arg2pi_eq: z = of_real(norm z) * exp(i * of_real(Arg2pi z))
  using Arg2pi is_Arg_def by auto

lemma complex_norm_eq_1_exp: norm z = 1 ⟷ exp(i * of_real (Arg2pi z))
= z
  by (metis Arg2pi_eq cis_conv_exp mult.left_neutral norm_cis of_real_1)

lemma Arg2pi_unique: ⟦of_real r * exp(i * of_real a) = z; 0 < r; 0 ≤ a; a <
2*pi⟧ ⟹ Arg2pi z = a
  by (rule Arg2pi_unique_lemma [unfolded is_Arg_def, OF _ Arg2pi_eq]) (use
Arg2pi [of z] in ‹auto simp: norm_mult›)

lemma cos_Arg2pi: cmod z * cos (Arg2pi z) = Re z and sin_Arg2pi: cmod z *
sin (Arg2pi z) = Im z
  using Arg2pi_eq [of z] cis_conv_exp Re_rcis Im_rcis unfolding rcis_def by
metis+

lemma Arg2pi_minus:
  assumes z ≠ 0 shows Arg2pi (−z) = (if Arg2pi z < pi then Arg2pi z + pi else
Arg2pi z − pi)
```

apply (*rule Arg2pi_unique* [*of norm z*, *OF complex_eqI*])
using *cos_Arg2pi sin_Arg2pi Arg2pi_ge_0 Arg2pi_lt_2pi* [*of z*] *assms*
by (*auto simp*: *Re_exp Im_exp*)

lemma *Arg2pi_times_of_real* [*simp*]:
assumes $0 < r$ **shows** *Arg2pi* (*of_real r* $*$ *z*) = *Arg2pi z*
by (*metis* (*no_types*, *lifting*) *Arg2pi Arg2pi_eq Arg2pi_unique assms mult.assoc*

mult_eq_0_iff mult_pos_pos of_real_mult zero_less_norm_iff)

lemma *Arg2pi_times_of_real2* [*simp*]: $0 < r \Longrightarrow$ *Arg2pi* (*z* $*$ *of_real r*) = *Arg2pi z*
by (*metis Arg2pi_times_of_real mult.commute*)

lemma *Arg2pi_divide_of_real* [*simp*]: $0 < r \Longrightarrow$ *Arg2pi* (*z* / *of_real r*) = *Arg2pi z*
by (*metis Arg2pi_times_of_real2 less_irrefl nonzero_eq_divide_eq of_real_eq_0_iff*)

lemma *Arg2pi_le_pi*: *Arg2pi z* $\leq$ *pi* $\longleftrightarrow$ $0 \leq$ *Im z*
proof (*cases z=0*)
case *False*
have $0 \leq$ *Im z* $\longleftrightarrow$ $0 \leq$ *Im* (*of_real* (*cmod z*) $*$ *exp* (i $*$ *complex_of_real* (*Arg2pi z*)))
by (*metis Arg2pi_eq*)
also have ... = ($0 \leq$ *Im* (*exp* (i $*$ *complex_of_real* (*Arg2pi z*))))
using *False* **by** (*simp add*: *zero_le_mult_iff*)
also have ... $\longleftrightarrow$ *Arg2pi z* $\leq$ *pi*
by (*simp add*: *Im_exp*) (*metis Arg2pi_ge_0 Arg2pi_lt_2pi sin_lt_zero sin_ge_zero not_le*)
finally show *?thesis*
by *blast*
qed *auto*

lemma *Arg2pi_lt_pi*: $0 <$ *Arg2pi z* $\wedge$ *Arg2pi z* $<$ *pi* $\longleftrightarrow$ $0 <$ *Im z*
using *Arg2pi_le_pi* [*of z*]
by (*smt* (*verit*, *del_insts*) *Arg2pi_0 Arg2pi_le_pi Arg2pi_minus uminus_complex.simps*(2) *zero_complex.simps*(2))

lemma *Arg2pi_eq_0*: *Arg2pi z* = *0* $\longleftrightarrow$ *z* $\in \mathbb{R} \wedge 0 \leq$ *Re z*
proof (*cases z=0*)
case *False*
then have *z* $\in \mathbb{R} \wedge 0 \leq$ *Re z* $\longleftrightarrow$ *z* $\in \mathbb{R} \wedge 0 \leq$ *Re* (*exp* (i $*$ *complex_of_real* (*Arg2pi z*)))
by (*metis cis.sel*(1) *cis_conv_exp cos_Arg2pi norm_ge_zero norm_le_zero_iff zero_le_mult_iff*)
also have ... $\longleftrightarrow$ *Arg2pi z* = *0*
proof −
have [*simp*]: *Arg2pi z* = *0* $\Longrightarrow$ *z* $\in \mathbb{R}$
using *Arg2pi_eq* [*of z*] **by** (*auto simp*: *Reals_def*)

moreover have $[\![z \in \mathbb{R};\ 0 \le \cos\ (\mathit{Arg2pi}\ z)]\!] \Longrightarrow \mathit{Arg2pi}\ z = 0$
by (*smt* (*verit*, *ccfv_SIG*) *Arg2pi_ge_0 Arg2pi_le_pi Arg2pi_lt_pi complex_is_Real_iff cos_pi*)
ultimately show *?thesis*
by (*auto simp*: *Re_exp*)
qed
finally show *?thesis*
by *blast*
qed *auto*

corollary *Arg2pi_gt_0*:
assumes $z \notin \mathbb{R}_{\ge 0}$
shows *Arg2pi z* > *0*
using *Arg2pi_eq_0 Arg2pi_ge_0 assms dual_order.strict_iff_order*
unfolding *nonneg_Reals_def* **by** *fastforce*

lemma *Arg2pi_eq_pi*: $\mathit{Arg2pi}\ z = \mathit{pi} \longleftrightarrow z \in \mathbb{R} \wedge \mathit{Re}\ z < 0$
using *Arg2pi_le_pi* [*of z*] *Arg2pi_lt_pi* [*of z*] *Arg2pi_eq_0* [*of z*] *Arg2pi_ge_0* [*of z*]
by (*fastforce simp*: *complex_is_Real_iff*)

lemma *Arg2pi_eq_0_pi*: $\mathit{Arg2pi}\ z = 0 \vee \mathit{Arg2pi}\ z = \mathit{pi} \longleftrightarrow z \in \mathbb{R}$
using *Arg2pi_eq_0 Arg2pi_eq_pi not_le* **by** *auto*

lemma *Arg2pi_of_real*: *Arg2pi* (*of_real r*) = (*if r*<*0 then pi else 0*)
using *Arg2pi_eq_0_pi Arg2pi_eq_pi* **by** *fastforce*

lemma *Arg2pi_real*: $z \in \mathbb{R} \Longrightarrow \mathit{Arg2pi}\ z = (\mathit{if}\ 0 \le \mathit{Re}\ z\ \mathit{then}\ 0\ \mathit{else}\ \mathit{pi})$
using *Arg2pi_eq_0 Arg2pi_eq_0_pi* **by** *auto*

lemma *Arg2pi_inverse*: *Arg2pi*(*inverse z*) = (*if z* ∈ ℝ *then Arg2pi z else 2*∗*pi* − *Arg2pi z*)
proof (*cases z*=*0*)
case *False*
show *?thesis*
apply (*rule Arg2pi_unique* [*of inverse* (*norm z*)])
using *Arg2pi_eq False Arg2pi_ge_0* [*of z*] *Arg2pi_lt_2pi* [*of z*] *Arg2pi_eq_0* [*of z*]
by (*auto simp*: *Arg2pi_real in_Reals_norm exp_diff field_simps*)
qed *auto*

lemma *Arg2pi_eq_iff*:
assumes *w* ≠ *0 z* ≠ *0*
shows *Arg2pi w* = *Arg2pi z* ⟷ (∃ *x*. *0* < *x* ∧ *w* = *of_real x* ∗ *z*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have (*cmod w*) ∗ (*z* / *cmod z*) = *w*
by (*metis Arg2pi_eq assms*(*2*) *mult_eq_0_iff nonzero_mult_div_cancel_left*)

then show *?rhs*
by (*metis assms divide_pos_pos of_real_divide times_divide_eq_left times_divide_eq_right zero_less_norm_iff*)
qed *auto*

lemma *Arg2pi_inverse_eq_0*: *Arg2pi*(*inverse z*) = *0* ⟷ *Arg2pi z* = *0*
by (*metis Arg2pi_eq_0 Arg2pi_inverse inverse_inverse_eq*)

lemma *Arg2pi_divide*:
assumes *w* ≠ *0 z* ≠ *0 Arg2pi w* ≤ *Arg2pi z*
shows *Arg2pi*(*z* / *w*) = *Arg2pi z* − *Arg2pi w*
apply (*rule Arg2pi_unique* [*of norm*(*z* / *w*)])
using *assms Arg2pi_eq Arg2pi_ge_0* [*of w*] *Arg2pi_lt_2pi* [*of z*]
apply (*auto simp*: *exp_diff norm_divide field_simps*)
done

lemma *Arg2pi_le_div_sum*:
assumes *w* ≠ *0 z* ≠ *0 Arg2pi w* ≤ *Arg2pi z*
shows *Arg2pi z* = *Arg2pi w* + *Arg2pi*(*z* / *w*)
by (*simp add*: *Arg2pi_divide assms*)

lemma *Arg2pi_le_div_sum_eq*:
assumes *w* ≠ *0 z* ≠ *0*
shows *Arg2pi w* ≤ *Arg2pi z* ⟷ *Arg2pi z* = *Arg2pi w* + *Arg2pi*(*z* / *w*)
using *assms* **by** (*auto simp*: *Arg2pi_ge_0 intro*: *Arg2pi_le_div_sum*)

lemma *Arg2pi_diff*:
assumes *w* ≠ *0 z* ≠ *0*
shows *Arg2pi w* − *Arg2pi z* = (*if Arg2pi z* ≤ *Arg2pi w then Arg2pi*(*w* / *z*) *else Arg2pi*(*w*/*z*) − *2*∗*pi*)
using *assms Arg2pi_divide Arg2pi_inverse* [*of w*/*z*] *Arg2pi_eq_0_pi*
by (*force simp add*: *Arg2pi_ge_0 Arg2pi_divide not_le split*: *if_split_asm*)

lemma *Arg2pi_add*:
assumes *w* ≠ *0 z* ≠ *0*
shows *Arg2pi w* + *Arg2pi z* = (*if Arg2pi w* + *Arg2pi z* < *2*∗*pi then Arg2pi*(*w* ∗ *z*) *else Arg2pi*(*w* ∗ *z*) + *2*∗*pi*)
using *assms Arg2pi_diff* [*of w*∗*z z*] *Arg2pi_le_div_sum_eq* [*of z w*∗*z*] *Arg2pi* [*of w* ∗ *z*]
by *auto*

lemma *Arg2pi_times*:
assumes *w* ≠ *0 z* ≠ *0*
shows *Arg2pi* (*w* ∗ *z*) = (*if Arg2pi w* + *Arg2pi z* < *2*∗*pi then Arg2pi w* + *Arg2pi z*
else (*Arg2pi w* + *Arg2pi z*) − *2*∗*pi*)
using *Arg2pi_add* [*OF assms*] **by** *auto*

lemma *Arg2pi_cnj_eq_inverse*:

assumes $z \neq 0$ **shows** *Arg2pi* (*cnj z*) = *Arg2pi* (*inverse z*)
proof −
have $\exists r{>}0$. *of_real r* / *z* = *cnj z*
by (*metis assms complex_norm_square nonzero_mult_div_cancel_left zero_less_norm_iff zero_less_power*)
then show *?thesis*
by (*metis Arg2pi_times_of_real2 divide_inverse_commute*)
qed

lemma *Arg2pi_cnj*: *Arg2pi*(*cnj z*) = (*if z* ∈ ℝ *then Arg2pi z else* 2∗*pi* − *Arg2pi z*)
by (*metis Arg2pi_cnj_eq_inverse Arg2pi_inverse Reals_cnj_iff complex_cnj_zero*)

lemma *Arg2pi_exp*: 0 ≤ *Im z* ⟹ *Im z* < 2∗*pi* ⟹ *Arg2pi*(*exp z*) = *Im z*
by (*simp add*: *Arg2pi_unique exp_eq_polar*)

lemma *complex_split_polar*:
obtains *r a*::*real* **where** *z* = *complex_of_real r* ∗ (*cos a* + i ∗ *sin a*) 0 ≤ *r* 0 ≤ *a a* < 2∗*pi*
using *Arg2pi cis.ctr cis_conv_exp* **unfolding** *Complex_eq is_Arg_def* **by** *fastforce*

lemma *Re_Im_le_cmod*: *Im w* ∗ *sin* φ + *Re w* ∗ *cos* φ ≤ *cmod w*
proof (*cases w rule*: *complex_split_polar*)
case (*1 r a*) **with** *sin_cos_le1* [*of a* φ] **show** *?thesis*
apply (*simp add*: *norm_mult cmod_unit_one*)
by (*metis* (*no_types*, *opaque_lifting*) *abs_le_D1 distrib_left mult.commute mult.left_commute mult_left_le*)
qed

6.20.8 Analytic properties of tangent function

lemma *cnj_tan*: *cnj*(*tan z*) = *tan*(*cnj z*)
by (*simp add*: *cnj_cos cnj_sin tan_def*)

lemma *field_differentiable_at_tan*: *cos z* ≠ 0 ⟹ *tan field_differentiable at z*
unfolding *field_differentiable_def*
using *DERIV_tan* **by** *blast*

lemma *field_differentiable_within_tan*: *cos z* ≠ 0
⟹ *tan field_differentiable* (*at z within s*)
using *field_differentiable_at_tan field_differentiable_at_within* **by** *blast*

lemma *continuous_within_tan*: *cos z* ≠ 0 ⟹ *continuous* (*at z within s*) *tan*
using *continuous_at_imp_continuous_within isCont_tan* **by** *blast*

lemma *continuous_on_tan* [*continuous_intros*]: (⋀*z*. *z* ∈ *s* ⟹ *cos z* ≠ 0) ⟹ *continuous_on s tan*
by (*simp add*: *continuous_at_imp_continuous_on*)

lemma *holomorphic_on_tan*: $(\bigwedge z.\ z \in s \Longrightarrow \cos z \neq 0) \Longrightarrow \tan \text{ holomorphic\_on } s$
by (*simp add*: *field_differentiable_within_tan holomorphic_on_def*)

6.20.9 The principal branch of the Complex logarithm

instantiation *complex* :: *ln*
begin

definition *ln_complex* :: *complex* ⇒ *complex*
where $ln\_complex \equiv \lambda z.\ THE\ w.\ exp\ w = z\ \&\ -pi < Im(w)\ \&\ Im(w) \leq pi$

NOTE: within this scope, the constant Ln is not yet available!

lemma
assumes $z \neq 0$
shows *exp_Ln* [*simp*]: $exp(ln\ z) = z$
and *mpi_less_Im_Ln*: $-pi < Im(ln\ z)$
and *Im_Ln_le_pi*: $Im(ln\ z) \leq pi$
proof −
obtain ψ **where** z: $z\ /\ (cmod\ z) = Complex\ (cos\ \psi)\ (sin\ \psi)$
using *complex_unimodular_polar* [*of z / (norm z)*] *assms*
by (*auto simp*: *norm_divide field_split_simps*)
obtain φ **where** φ: $-\ pi < \varphi\ \varphi \leq pi\ sin\ \varphi = sin\ \psi\ cos\ \varphi = cos\ \psi$
using *sincos_principal_value* [*of ψ*] *assms*
by (*auto simp*: *norm_divide field_split_simps*)
have $exp(ln\ z) = z\ \&\ -pi < Im(ln\ z)\ \&\ Im(ln\ z) \leq pi$ **unfolding** *ln_complex_def*
apply (*rule theI* [**where** $a = Complex\ (ln(norm\ z))\ \varphi$])
using *z assms φ*
apply (*auto simp*: *field_simps exp_complex_eqI exp_eq_polar cis.code*)
done
then show $exp(ln\ z) = z\ -pi < Im(ln\ z)\ Im(ln\ z) \leq pi$
by *auto*
qed

lemma *Ln_exp* [*simp*]:
assumes $-pi < Im(z)\ Im(z) \leq pi$
shows $ln(exp\ z) = z$
proof (*rule exp_complex_eqI*)
show $|Im\ (ln\ (exp\ z)) - Im\ z| < 2 * pi$
using *assms mpi_less_Im_Ln* [*of exp z*] *Im_Ln_le_pi* [*of exp z*] **by** *auto*
qed *auto*

6.20.10 Relation to Real Logarithm

lemma *Ln_of_real*:
assumes $0 < z$
shows $ln(of\_real\ z::complex) = of\_real(ln\ z)$
by (*smt* (*verit*) *Im_complex_of_real Ln_exp assms exp_ln of_real_exp pi_ge_two*)

corollary *Ln_in_Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow Re\ z > 0 \Longrightarrow ln\ z \in \mathbb{R}$
by (*auto simp*: *Ln_of_real elim*: *Reals_cases*)

corollary *Im_Ln_of_real* [*simp*]: $r > 0 \Longrightarrow Im\ (ln\ (of\_real\ r)) = 0$
by (*simp add*: *Ln_of_real*)

lemma *cmod_Ln_Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow 0 < Re\ z \Longrightarrow cmod\ (ln\ z) = norm\ (ln\ (Re\ z))$
using *Ln_of_real* **by** *force*

lemma *Ln_Reals_eq*: $[\![x \in \mathbb{R}; Re\ x > 0]\!] \Longrightarrow ln\ x = of\_real\ (ln\ (Re\ x))$
using *Ln_of_real* **by** *force*

lemma *Ln_1* [*simp*]: $ln\ 1 = (0::complex)$
by (*smt* (*verit*, *best*) *Ln_of_real ln_one of_real_0 of_real_1*)

lemma *Ln_eq_zero_iff* [*simp*]: $x \notin \mathbb{R}_{\leq 0} \Longrightarrow ln\ x = 0 \longleftrightarrow x = 1$ **for** $x::complex$
by (*metis* (*mono_tags*, *lifting*) *Ln_1 exp_Ln exp_zero nonpos_Reals_zero_I*)

instance
by *intro_classes* (*rule ln_complex_def Ln_1*)

end

abbreviation $Ln :: complex \Rightarrow complex$
where $Ln \equiv ln$

lemma *Ln_eq_iff*: $w \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (Ln\ w = Ln\ z \longleftrightarrow w = z)$
by (*metis exp_Ln*)

lemma *Ln_unique*: $exp(z) = w \Longrightarrow -pi < Im(z) \Longrightarrow Im(z) \leq pi \Longrightarrow Ln\ w = z$
using *Ln_exp* **by** *blast*

lemma *Re_Ln* [*simp*]: $z \neq 0 \Longrightarrow Re(Ln\ z) = ln(norm\ z)$
by (*metis exp_Ln ln_exp norm_exp_eq_Re*)

corollary *ln_cmod_le*:
assumes *z*: $z \neq 0$
shows $ln\ (cmod\ z) \leq cmod\ (Ln\ z)$
by (*metis Re_Ln complex_Re_le_cmod z*)

proposition *exists_complex_root*:
fixes $z :: complex$
assumes $n \neq 0$ **obtains** w **where** $z = w\ \hat{}\ n$
by (*metis assms exp_Ln exp_of_nat_mult nonzero_mult_div_cancel_left of_nat_eq_0_iff power_0_left times_divide_eq_right*)

corollary *exists_complex_root_nonzero*:

fixes *z*::*complex*
assumes $z \neq 0$ $n \neq 0$
obtains *w* **where** $w \neq 0$ $z = w \mathbin{\hat{}} n$
by (*metis exists_complex_root* [*of n z*] *assms power_0_left*)

6.20.11 Derivative of Ln away from the branch cut

lemma *Im_Ln_less_pi*:
assumes $z \notin \mathbb{R}_{\leq 0}$**shows** *Im* (*Ln z*) < *pi*
proof −
have *znz* [*simp*]: $z \neq 0$
using *assms* **by** *auto*
with *Im_Ln_le_pi* [*of z*] **show** *?thesis*
by (*smt* (*verit*, *best*) *Arg2pi_eq_0_pi Arg2pi_exp Ln_in_Reals assms complex_is_Real_iff complex_nonpos_Reals_iff exp_Ln pi_ge_two*)
qed

lemma *has_field_derivative_Ln*:
assumes $z \notin \mathbb{R}_{\leq 0}$
shows (*Ln has_field_derivative inverse*(*z*)) (*at z*)
proof −
have *znz* [*simp*]: $z \neq 0$
using *assms* **by** *auto*
then have *Im* (*Ln z*) ≠ *pi*
by (*smt* (*verit*, *best*) *Arg2pi_eq_0_pi Arg2pi_exp Ln_in_Reals assms complex_is_Real_iff complex_nonpos_Reals_iff exp_Ln pi_ge_two*)
let *?U* = {*w*. −*pi* < *Im*(*w*) ∧ *Im*(*w*) < *pi*}
have *1*: *open ?U*
by (*simp add*: *open_Collect_conj open_halfspace_Im_gt open_halfspace_Im_lt*)
have *2*: ⋀*x*. *x* ∈ *?U* ⟹ (*exp has_derivative blinfun_apply*(*Blinfun* ((∗) (*exp x*)))) (*at x*)
by (*simp add*: *bounded_linear_Blinfun_apply bounded_linear_mult_right has_field_derivative_imp_has_deriv*
have *3*: *continuous_on ?U* (λ*x*. *Blinfun* ((∗) (*exp x*)))
unfolding *blinfun_mult_right.abs_eq* [*symmetric*] **by** (*intro continuous_intros*)
have *4*: *Ln z* ∈ *?U*
by (*simp add*: *Im_Ln_less_pi assms mpi_less_Im_Ln*)
have *5*: *Blinfun* ((∗) (*inverse z*)) o_L *Blinfun* ((∗) (*exp* (*Ln z*))) = *id_blinfun*
by (*rule blinfun_eqI*) (*simp add*: *bounded_linear_mult_right bounded_linear_Blinfun_apply*)
obtain *U′ V g g′* **where** *open U′* **and** *sub*: *U′* ⊆ *?U*
and *Ln z* ∈ *U′ open V z* ∈ *V*
and *hom*: *homeomorphism U′ V exp g*
and *g*: ⋀*y*. *y* ∈ *V* ⟹ (*g has_derivative* (*g′ y*)) (*at y*)
and *g′*: ⋀*y*. *y* ∈ *V* ⟹ *g′ y* = *inv* ((∗) (*exp* (*g y*)))
and *bij*: ⋀*y*. *y* ∈ *V* ⟹ *bij* ((∗) (*exp* (*g y*)))
using *inverse_function_theorem* [*OF 1 2 3 4 5*]
by (*simp add*: *bounded_linear_Blinfun_apply bounded_linear_mult_right*)
blast
show (*Ln has_field_derivative inverse*(*z*)) (*at z*)
unfolding *has_field_derivative_def*

proof (*rule has_derivative_transform_within_open*)
show *g_eq_Ln*: *g y* = *Ln y* **if** *y* ∈ *V* **for** *y*
by (*smt* (*verit*, *ccfv_threshold*) *Ln_exp hom homeomorphism_def imageI mem_Collect_eq sub subset_iff that*)
have *0* ∉ *V*
by (*meson exp_not_eq_zero hom homeomorphism_def*)
then have ⋀*y*. *y* ∈ *V* ⟹ *g*′ *y* = *inv* ((∗) *y*)
by (*metis exp_Ln g*′ *g_eq_Ln*)
then have *g*′: *g*′ *z* = (λ*x*. *x*/*z*)
by (*metis* ‹*z* ∈ *V*› *bij bij_inv_eq_iff exp_Ln g_eq_Ln nonzero_mult_div_cancel_left znz*)
show (*g has_derivative* (∗)) (*inverse z*)) (*at z*)
using *g* [*OF* ‹*z* ∈ *V*›] *g*′ **by** (*simp add*: *divide_inverse_commute*)
qed (*auto simp*: ‹*z* ∈ *V*› ‹*open V*›)
qed

declare *has_field_derivative_Ln* [*derivative_intros*]
declare *has_field_derivative_Ln* [*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_at_Ln*: *z* ∉ ℝ≤0 ⟹ *Ln field_differentiable at z*
using *field_differentiable_def has_field_derivative_Ln* **by** *blast*

lemma *field_differentiable_within_Ln*: *z* ∉ ℝ≤0
⟹ *Ln field_differentiable* (*at z within S*)
using *field_differentiable_at_Ln field_differentiable_within_subset* **by** *blast*

lemma *continuous_at_Ln*: *z* ∉ ℝ≤0 ⟹ *continuous* (*at z*) *Ln*
by (*simp add*: *field_differentiable_imp_continuous_at field_differentiable_within_Ln*)

lemma *isCont_Ln*′ [*simp*,*continuous_intros*]:
⟦*isCont f z*; *f z* ∉ ℝ≤0⟧ ⟹ *isCont* (λ*x*. *Ln* (*f x*)) *z*
by (*blast intro*: *isCont_o2* [*OF* _ *continuous_at_Ln*])

lemma *continuous_within_Ln* [*continuous_intros*]: *z* ∉ ℝ≤0 ⟹ *continuous* (*at z within S*) *Ln*
using *continuous_at_Ln continuous_at_imp_continuous_within* **by** *blast*

lemma *continuous_on_Ln* [*continuous_intros*]: (⋀*z*. *z* ∈ *S* ⟹ *z* ∉ ℝ≤0) ⟹ *continuous_on S Ln*
by (*simp add*: *continuous_at_imp_continuous_on continuous_within_Ln*)

lemma *continuous_on_Ln*′ [*continuous_intros*]:
continuous_on S f ⟹ (⋀*z*. *z* ∈ *S* ⟹ *f z* ∉ ℝ≤0) ⟹ *continuous_on S* (λ*x*. *Ln* (*f x*))
by (*rule continuous_on_compose2*[*OF continuous_on_Ln*, *of UNIV* − *nonpos_Reals S f*]) *auto*

lemma *holomorphic_on_Ln* [*holomorphic_intros*]: *S* ∩ ℝ≤0 = {} ⟹ *Ln holomorphic_on S*

by (*simp add*: *disjoint_iff field_differentiable_within_Ln holomorphic_on_def*)

lemma *holomorphic_on_Ln'* [*holomorphic_intros*]:
(⋀*z*. *z* ∈ *A* ⟹ *f z* ∉ ℝ$_{\leq 0}$) ⟹ *f holomorphic_on A* ⟹ (λ*z*. *Ln* (*f z*)) *holomorphic_on A*
using *holomorphic_on_compose_gen*[*OF* _ *holomorphic_on_Ln*, *of f A* − ℝ$_{\leq 0}$]
by (*auto simp*: *o_def*)

lemma *analytic_on_ln* [*analytic_intros*]:
assumes *f analytic_on A f* ‘ *A* ∩ ℝ$_{\leq 0}$ = {}
shows (λ*w*. *ln* (*f w*)) *analytic_on A*
proof −
have ∗: *ln analytic_on* (−ℝ$_{\leq 0}$)
by (*subst analytic_on_open*) (*auto intro*!: *holomorphic_intros*)
have (*ln* ∘ *f*) *analytic_on A*
by (*rule analytic_on_compose_gen*[*OF assms*(*1*) ∗]) (*use assms*(*2*) **in** *auto*)
thus *?thesis*
by (*simp add*: *o_def*)
qed

lemma *tendsto_Ln* [*tendsto_intros*]:
assumes (*f* ⟶ *L*) *F L* ∉ ℝ$_{\leq 0}$
shows ((λ*x*. *Ln* (*f x*)) ⟶ *Ln L*) *F*
by (*simp add*: *assms isCont_tendsto_compose*)

lemma *divide_ln_mono*:
fixes *x y*::*real*
assumes *3* ≤ *x x* ≤ *y*
shows *x* / *ln x* ≤ *y* / *ln y*
proof −
have ⋀*u*. ⟦*x* ≤ *u*; *u* ≤ *y*⟧ ⟹ ((λ*z*. *z* / *Ln z*) *has_field_derivative 1* / *Ln u* − *1* / (*Ln u*)^2) (*at u*)
using ‹*3* ≤ *x*› **by** (*force intro*!: *derivative_eq_intros simp*: *field_simps power_eq_if*)
moreover
have *x* / *ln x* ≤ *y* / *ln y*
if *Re* (*y* / *Ln y*) − *Re* (*x* / *Ln x*) = (*Re* (*1* / *Ln u*) − *Re* (*1* / (*Ln u*)^2)) ∗ (*y* − *x*)
and *x*: *x* ≤ *u u* ≤ *y* **for** *u*
proof −
have *eq*: *y* / *ln y* = (*1* / *ln u* − *1* / (*ln u*)^2) ∗ (*y* − *x*) + *x* / *ln x*
using *that* ‹*3* ≤ *x*› **by** (*auto simp*: *Ln_Reals_eq in_Reals_norm group_add_class.diff_eq_eq*)
show *?thesis*
using *exp_le* ‹*3* ≤ *x*› *x* **by** (*simp add*: *eq*) (*simp add*: *power_eq_if divide_simps ln_ge_iff*)
qed
ultimately show *?thesis*
using *complex_mvt_line* [*of x y* λ*z*. *z* / *Ln z* λ*z*. *1*/(*Ln z*) − *1*/(*Ln z*)^*2*] *assms*
by (*force simp add*: *closed_segment_Reals closed_segment_eq_real_ivl*)

qed

theorem *Ln_series*:
fixes *z* :: *complex*
assumes *norm z < 1*
shows (λ*n*. (−1)^*Suc n* / *of_nat n* ∗ *z*^*n*) *sums ln* (1 + *z*) (**is** (λ*n*. *?f n* ∗ *z*^*n*) *sums* _)
proof −
let *?F* = λ*z*. ∑ *n*. *?f n* ∗ *z*^*n* **and** *?F′* = λ*z*. ∑ *n*. *diffs ?f n* ∗ *z*^*n*
have *r*: *conv_radius ?f* = 1
by (*intro conv_radius_ratio_limit_nonzero*[*of* _ 1])
(*simp_all add*: *norm_divide LIMSEQ_Suc_n_over_n del*: *of_nat_Suc*)

have ∃ *c*. ∀ *z*∈*ball 0 1*. *ln* (1 + *z*) − *?F z* = *c*
proof (*rule has_field_derivative_zero_constant*)
fix *z* :: *complex* **assume** *z′*: *z* ∈ *ball 0 1*
hence *z*: *norm z < 1* **by** *simp*
define *t* :: *complex* **where** *t* = *of_real* (1 + *norm z*) / 2
from *z* **have** *t*: *norm z < norm t norm t < 1* **unfolding** *t_def*
by (*simp_all add*: *field_simps norm_divide del*: *of_real_add*)

have *Re* (−*z*) ≤ *norm* (−*z*) **by** (*rule complex_Re_le_cmod*)
also from *z* **have** ... < *1* **by** *simp*
finally have ((λ*z*. *ln* (1 + *z*)) *has_field_derivative inverse* (1+*z*)) (*at z*)
by (*auto intro*!: *derivative_eq_intros simp*: *complex_nonpos_Reals_iff*)
moreover have (*?F has_field_derivative ?F′ z*) (*at z*) **using** *t r*
by (*intro termdiffs_strong*[*of* _ *t*] *summable_in_conv_radius*) *simp_all*
ultimately have ((λ*z*. *ln* (1 + *z*) − *?F z*) *has_field_derivative* (*inverse* (1 + *z*) − *?F′ z*))
(*at z within ball 0 1*)
by (*intro derivative_intros*) (*simp_all add*: *at_within_open*[*OF z′*])
also have (λ*n*. *of_nat n* ∗ *?f n* ∗ *z* ^ (*n* − *Suc 0*)) *sums ?F′ z* **using** *t r*
by (*intro diffs_equiv termdiff_converges*[*OF t*(*1*)] *summable_in_conv_radius*) *simp_all*
from *sums_split_initial_segment*[*OF this*, *of 1*]
have (λ*i*. (−*z*) ^ *i*) *sums ?F′ z* **by** (*simp add*: *power_minus*[*of z*] *del*: *of_nat_Suc*)
hence *?F′ z* = *inverse* (1 + *z*) **using** *z* **by** (*simp add*: *sums_iff suminf_geometric divide_inverse*)
also have *inverse* (1 + *z*) − *inverse* (1 + *z*) = *0* **by** *simp*
finally show ((λ*z*. *ln* (1 + *z*) − *?F z*) *has_field_derivative 0*) (*at z within ball 0 1*) .
qed *simp_all*
then obtain *c* **where** *c*: ⋀*z*. *z* ∈ *ball 0 1* ⟹ *ln* (1 + *z*) − *?F z* = *c* **by** *blast*
from *c*[*of 0*] **have** *c* = *0* **by** (*simp only*: *powser_zero*) *simp*
with *c*[*of z*] *assms* **have** *ln* (1 + *z*) = *?F z* **by** *simp*
moreover have *summable* (λ*n*. *?f n* ∗ *z*^*n*) **using** *assms r*
by (*intro summable_in_conv_radius*) *simp_all*
ultimately show *?thesis* **by** (*simp add*: *sums_iff*)

qed

lemma *Ln_series'*: *cmod z < 1 ⟹ (λn. − ((−z)^n) / of_nat n) sums ln (1 + z)*
by (*drule Ln_series*) (*simp add*: *power_minus'*)

lemma *ln_series'*:
fixes *x*::*real*
assumes *|x| < 1*
shows *(λn. − ((−x)^n) / of_nat n) sums ln (1 + x)*
proof −
from *assms* **have** *(λn. − ((−of_real x)^n) / of_nat n) sums ln (1 + complex_of_real x)*
by (*intro Ln_series'*) *simp_all*
also have *(λn. − ((−of_real x)^n) / of_nat n) = (λn. complex_of_real (− ((−x)^n) / of_nat n))*
by (*rule ext*) *simp*
also from *assms* **have** *ln (1 + complex_of_real x) = of_real (ln (1 + x))*
by (*smt* (*verit*) *Ln_of_real of_real_1 of_real_add*)
finally show *?thesis* **by** (*subst* (*asm*) *sums_of_real_iff*)
qed

lemma *Ln_approx_linear*:
fixes *z* :: *complex*
assumes *norm z < 1*
shows *norm (ln (1 + z) − z) ≤ norm z^2 / (1 − norm z)*
proof −
let *?f = λn. (−1)^Suc n / of_nat n*
from *assms* **have** *(λn. ?f n * z^n) sums ln (1 + z)* **using** *Ln_series* **by** *simp*
moreover have *(λn. (if n = 1 then 1 else 0) * z^n) sums z* **using** *powser_sums_if*[*of 1*] **by** *simp*
ultimately have *(λn. (?f n − (if n = 1 then 1 else 0)) * z^n) sums (ln (1 + z) − z)*
by (*subst left_diff_distrib*, *intro sums_diff*) *simp_all*
from *sums_split_initial_segment*[*OF this*, *of Suc 1*]
have *(λi. (−(z^2)) * inverse (2 + of_nat i) * (− z)^i) sums (Ln (1 + z) − z)*
by (*simp add*: *power2_eq_square mult_ac power_minus*[*of z*] *divide_inverse*)
hence *(Ln (1 + z) − z) = (∑ i. (−(z^2)) * inverse (of_nat (i+2)) * (−z)^i)*
by (*simp add*: *sums_iff*)
also have *A*: *summable (λn. norm z^2 * (inverse (real_of_nat (Suc (Suc n))) * cmod z ^ n))*
by (*rule summable_mult*, *rule summable_comparison_test_ev*[*OF _ summable_geometric*[*of norm z*]])
(*auto simp*: *assms field_simps intro*!: *always_eventually*)
hence *norm (∑ i. (−(z^2)) * inverse (of_nat (i+2)) * (−z)^i)*
*≤ (∑ i. norm (−(z^2) * inverse (of_nat (i+2)) * (−z)^i))*
by (*intro summable_norm*)
(*auto simp*: *norm_power norm_inverse norm_mult mult_ac simp del*:

of_nat_add of_nat_Suc)
also have *norm* ((−*z*)^2 ∗ (−*z*)^*i*) ∗ *inverse* (*of_nat* (*i*+2)) ≤ *norm* ((−*z*)^2 ∗ (−*z*)^*i*) ∗ *1* **for** *i*
by (*intro mult_left_mono*) (*simp_all add*: *field_split_simps*)
hence (∑ *i*. *norm* (−(*z*^2) ∗ *inverse* (*of_nat* (*i*+2)) ∗ (−*z*)^*i*))
≤ (∑ *i*. *norm* (−(*z*^2) ∗ (−*z*)^*i*))
using *A assms*
unfolding *norm_power norm_inverse norm_divide norm_mult*
apply (*intro suminf_le summable_mult summable_geometric*)
apply (*auto simp*: *norm_power field_simps simp del*: *of_nat_add of_nat_Suc*)
done
also have ... = *norm z*^2 ∗ (∑ *i*. *norm z*^*i*) **using** *assms*
by (*subst suminf_mult* [*symmetric*]) (*auto intro*!: *summable_geometric simp*: *norm_mult norm_power*)
also have (∑ *i*. *norm z*^*i*) = *inverse* (*1* − *norm z*) **using** *assms*
by (*subst suminf_geometric*) (*simp_all add*: *divide_inverse*)
also have *norm z*^2 ∗ ... = *norm z*^2 / (*1* − *norm z*) **by** (*simp add*: *divide_inverse*)
finally show *?thesis* **.**
qed

lemma *norm_Ln_le*:
fixes *z* :: *complex*
assumes *norm z* < *1*/*2*
shows *norm* (*Ln*(*1*+*z*)) ≤ *2* ∗ *norm z*
proof −
have *sums*: (λ*n*. − ((− *z*) ^ *n*) / *of_nat n*) *sums ln* (*1* + *z*)
by (*intro Ln_series'*) (*use assms* **in** *auto*)
have *summable*: *summable* (λ*n*. *norm* (− ((− *z*) ^ *n* / *of_nat n*)))
using *ln_series'*[*of* −*norm z*] *assms*
by (*simp add*: *sums_iff summable_minus_iff norm_power norm_divide*)

have *norm* (*ln* (*1* + *z*)) = *norm* (∑ *n*. −((−*z*) ^ *n* / *of_nat n*))
using *sums* **by** (*simp add*: *sums_iff*)
also have ... ≤ (∑ *n*. *norm* (−((−*z*) ^ *n* / *of_nat n*)))
using *summable* **by** (*rule summable_norm*)
also have ... = (∑ *n*. *norm* (−((−*z*) ^ *Suc n* / *of_nat* (*Suc n*))))
using *summable* **by** (*subst suminf_split_head*) *auto*
also have ... ≤ (∑ *n*. *norm z* ∗ (*1* / *2*) ^ *n*)
proof (*rule suminf_le*)
show *summable* (λ*n*. *norm z* ∗ (*1* / *2*) ^ *n*)
by (*intro summable_mult summable_geometric*) *auto*
next
show *summable* (λ*n*. *norm* (− ((− *z*) ^ *Suc n* / *of_nat* (*Suc n*))))
using *summable* **by** (*subst summable_Suc_iff*)
next
fix *n*
have *norm* (−((−*z*) ^ *Suc n* / *of_nat* (*Suc n*))) = *norm z* ∗ (*norm z* ^ *n* /

real (Suc n))
by (*simp add*: *norm_power norm_divide norm_mult del*: *of_nat_Suc*)
also have ... ≤ *norm z* ∗ ((*1 / 2*) ^ *n / 1*)
using *assms* **by** (*intro mult_left_mono frac_le power_mono*) *auto*
finally show *norm* (− ((− *z*) ^ *Suc n / of_nat* (*Suc n*))) ≤ *norm z* ∗ (*1 / 2*) ^ *n*
by *simp*
qed
also have ... = *norm z* ∗ (∑ *n*. (*1 / 2*) ^ *n*)
by (*subst suminf_mult*) (*auto intro*: *summable_geometric*)
also have (∑ *n*. (*1 / 2* :: *real*) ^ *n*) = *2*
using *geometric_sums*[*of 1 / 2* :: *real*] **by** (*simp add*: *sums_iff*)
finally show *?thesis*
by (*simp add*: *mult_ac*)
qed

6.20.12 Quadrant-type results for Ln

lemma *cos_lt_zero_pi*: *pi/2* < *x* ⟹ *x* < *3∗pi/2* ⟹ *cos x* < *0*
using *cos_minus_pi cos_gt_zero_pi* [*of x−pi*]
by *simp*

lemma *Re_Ln_pos_le*:
assumes *z* ≠ *0*
shows |*Im*(*Ln z*)| ≤ *pi/2* ⟷ *0* ≤ *Re*(*z*)
proof −
{ fix *w*
assume *w* = *Ln z*
then have *w*: *Im w* ≤ *pi* − *pi* < *Im w*
using *Im_Ln_le_pi* [*of z*] *mpi_less_Im_Ln* [*of z*] *assms*
by *auto*
then have |*Im w*| ≤ *pi/2* ⟷ *0* ≤ *Re*(*exp w*)
using *cos_lt_zero_pi* [*of* − (*Im w*)] *cos_lt_zero_pi* [*of* (*Im w*)] *not_le*
by (*auto simp*: *Re_exp zero_le_mult_iff abs_if intro*: *cos_ge_zero*)
}
then show *?thesis* **using** *assms*
by *auto*
qed

lemma *Re_Ln_pos_lt*:
assumes *z* ≠ *0*
shows |*Im*(*Ln z*)| < *pi/2* ⟷ *0* < *Re*(*z*)
using *Re_Ln_pos_le assms*
by (*smt* (*verit*) *Re_exp arccos_cos cos_minus cos_pi_half exp_Ln exp_gt_zero field_sum_of_halves mult_eq_0_iff*)

lemma *Im_Ln_pos_le*:
assumes *z* ≠ *0*
shows *0* ≤ *Im*(*Ln z*) ∧ *Im*(*Ln z*) ≤ *pi* ⟷ *0* ≤ *Im*(*z*)

proof −
 { fix *w*
  assume *w* = *Ln z*
  then have *w*: *Im w* ≤ *pi* − *pi* < *Im w*
   using *Im_Ln_le_pi* [*of z*] *mpi_less_Im_Ln* [*of z*] *assms*
   by *auto*
  then have *0* ≤ *Im w* ∧ *Im w* ≤ *pi* ⟷ *0* ≤ *Im*(*exp w*)
   using *sin_ge_zero* [*of* − (*Im w*)] *sin_ge_zero* [*of abs*(*Im w*)] *sin_zero_pi_iff* [*of Im w*]
   by (*force simp*: *Im_exp zero_le_mult_iff sin_ge_zero*) **}**
 then show *?thesis* **using** *assms*
  by *auto*
qed

lemma *Im_Ln_pos_lt*:
 assumes *z* ≠ *0*
 shows *0* < *Im*(*Ln z*) ∧ *Im*(*Ln z*) < *pi* ⟷ *0* < *Im*(*z*)
 using *Im_Ln_pos_le* [*OF assms*] *assms*
 by (*smt* (*verit*, *best*) *Arg2pi_exp Arg2pi_lt_pi exp_Ln*)

lemma *Re_Ln_pos_lt_imp*: *0* < *Re*(*z*) ⟹ |*Im*(*Ln z*)| < *pi*/*2*
 by (*metis Re_Ln_pos_lt less_irrefl zero_complex.simps*(*1*))

lemma *Im_Ln_pos_lt_imp*: *0* < *Im*(*z*) ⟹ *0* < *Im*(*Ln z*) ∧ *Im*(*Ln z*) < *pi*
 by (*metis Im_Ln_pos_lt not_le order_refl zero_complex.simps*(*2*))

A reference to the set of positive real numbers

lemma *Im_Ln_eq_0*: *z* ≠ *0* ⟹ (*Im*(*Ln z*) = *0* ⟷ *0* < *Re*(*z*) ∧ *Im*(*z*) = *0*)
 using *Im_Ln_pos_le Im_Ln_pos_lt Re_Ln_pos_lt* **by** *fastforce*

lemma *Im_Ln_eq_pi*: *z* ≠ *0* ⟹ (*Im*(*Ln z*) = *pi* ⟷ *Re*(*z*) < *0* ∧ *Im*(*z*) = *0*)
 using *Im_Ln_eq_0 Im_Ln_pos_le Im_Ln_pos_lt complex.expand* **by** *fastforce*

6.20.13 More Properties of Ln

lemma *cnj_Ln*: **assumes** *z* ∉ ℝ≤0 **shows** *cnj*(*Ln z*) = *Ln*(*cnj z*)
proof (*cases z=0*)
 case *False*
 show *?thesis*
  by (*smt* (*verit*) *False Im_Ln_less_pi Ln_exp assms cnj.sel*(*2*) *exp_Ln exp_cnj mpi_less_Im_Ln*)
qed (*use assms* **in** *auto*)

lemma *Ln_inverse*: **assumes** *z* ∉ ℝ≤0 **shows** *Ln*(*inverse z*) = −(*Ln z*)
proof (*cases z=0*)
 case *False*
 show *?thesis*
  by (*smt* (*verit*) *False Im_Ln_less_pi Ln_exp assms exp_Ln exp_minus mpi_less_Im_Ln*

uminus_complex.sel(2))
qed (*use assms* **in** *auto*)

lemma *Ln_minus1* [*simp*]: *Ln*(−*1*) = i ∗ *pi*
proof (*rule exp_complex_eqI*)
show |*Im* (*Ln* (− *1*)) − *Im* (i ∗ *pi*)| < *2* ∗ *pi*
using *Im_Ln_le_pi* [*of* −*1*] *mpi_less_Im_Ln* [*of* −*1*] **by** *auto*
qed *auto*

lemma *Ln_ii* [*simp*]: *Ln* i = i ∗ *of_real pi/2*
using *Ln_exp* [*of* i ∗ (*of_real pi/2*)]
unfolding *exp_Euler*
by *simp*

lemma *Ln_minus_ii* [*simp*]: *Ln*(−i) = − (i ∗ *pi/2*)
using *Ln_inverse* **by** *fastforce*

lemma *Ln_times*:
assumes *w* ≠ *0 z* ≠ *0*
shows *Ln*(*w* ∗ *z*) =
(*if Im*(*Ln w* + *Ln z*) ≤ −*pi then* (*Ln*(*w*) + *Ln*(*z*)) + i ∗ *of_real*(*2*∗*pi*)
else if Im(*Ln w* + *Ln z*) > *pi then* (*Ln*(*w*) + *Ln*(*z*)) − i ∗ *of_real*(*2*∗*pi*)
else Ln(*w*) + *Ln*(*z*))
using *pi_ge_zero Im_Ln_le_pi* [*of w*] *Im_Ln_le_pi* [*of z*]
using *assms mpi_less_Im_Ln* [*of w*] *mpi_less_Im_Ln* [*of z*]
by (*auto simp*: *exp_add exp_diff sin_double cos_double exp_Euler intro*!: *Ln_unique*)

corollary *Ln_times_simple*:
⟦*w* ≠ *0*; *z* ≠ *0*; −*pi* < *Im*(*Ln w*) + *Im*(*Ln z*); *Im*(*Ln w*) + *Im*(*Ln z*) ≤ *pi*⟧
⟹ *Ln*(*w* ∗ *z*) = *Ln*(*w*) + *Ln*(*z*)
by (*simp add*: *Ln_times*)

corollary *Ln_times_of_real*:
⟦*r* > *0*; *z* ≠ *0*⟧ ⟹ *Ln*(*of_real r* ∗ *z*) = *ln r* + *Ln*(*z*)
using *mpi_less_Im_Ln Im_Ln_le_pi*
by (*force simp*: *Ln_times*)

corollary *Ln_times_of_nat*:
⟦*r* > *0*; *z* ≠ *0*⟧ ⟹ *Ln*(*of_nat r* ∗ *z* :: *complex*) = *ln* (*of_nat r*) + *Ln*(*z*)
using *Ln_times_of_real*[*of of_nat r z*] **by** *simp*

corollary *Ln_times_Reals*:
⟦*r* ∈ *Reals*; *Re r* > *0*; *z* ≠ *0*⟧ ⟹ *Ln*(*r* ∗ *z*) = *ln* (*Re r*) + *Ln*(*z*)
using *Ln_Reals_eq Ln_times_of_real* **by** *fastforce*

corollary *Ln_divide_of_real*:
⟦*r* > *0*; *z* ≠ *0*⟧ ⟹ *Ln*(*z* / *of_real r*) = *Ln*(*z*) − *ln r*
using *Ln_times_of_real* [*of inverse r z*]
by (*simp add*: *ln_inverse Ln_of_real mult.commute divide_inverse flip*: *of_real_inverse*)

corollary *Ln_prod*:
fixes $f :: 'a \Rightarrow complex$
assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
shows $\exists\, n.\ Ln\ (prod\ f\ A) = (\sum x \in A.\ Ln\ (f\ x) + (of\_int\ (n\ x) * (2 * pi)) * \mathrm{i})$
using *assms*
proof (*induction A*)
case (*insert x A*)
then obtain *n* **where** *n*: $Ln\ (prod\ f\ A) = (\sum x{\in}A.\ Ln\ (f\ x) + of\_real\ (of\_int\ (n\ x) * (2 * pi)) * \mathrm{i})$
by *auto*
define *D* **where** $D \equiv Im\ (Ln\ (f\ x)) + Im\ (Ln\ (prod\ f\ A))$
define *q*::*int* **where** $q \equiv (if\ D \leq -pi\ then\ 1\ else\ if\ D > pi\ then\ -1\ else\ 0)$
have $prod\ f\ A \neq 0\ f\ x \neq 0$
by (*auto simp*: *insert.hyps insert.prems*)
with *insert.hyps pi_ge_zero* **show** *?case*
by (*rule_tac x*=*n*(*x*:=*q*) **in** *exI*) (*force simp*: *Ln_times q_def D_def n intro*!: *sum.cong*)
qed *auto*

lemma *Ln_minus*:
assumes $z \neq 0$
shows $Ln(-z) = (if\ Im(z) \leq 0 \wedge \neg(Re(z) < 0 \wedge Im(z) = 0)$
$then\ Ln(z) + \mathrm{i} * pi$
$else\ Ln(z) - \mathrm{i} * pi)$
using *Im_Ln_le_pi* [*of z*] *mpi_less_Im_Ln* [*of z*] *assms*
Im_Ln_eq_pi [*of z*] *Im_Ln_pos_lt* [*of z*]
by (*intro Ln_unique*) (*auto simp*: *exp_add exp_diff*)

lemma *Ln_inverse_if*:
assumes $z \neq 0$
shows $Ln\ (inverse\ z) = (if\ z \in \mathbb{R}_{\leq 0}\ then\ -(Ln\ z) + \mathrm{i} * 2 * complex\_of\_real\ pi\ else\ -(Ln\ z))$
proof (*cases* $z \in \mathbb{R}_{\leq 0}$)
case *False* **then show** *?thesis*
by (*simp add*: *Ln_inverse*)
next
case *True*
then have *z*: $Im\ z = 0\ Re\ z < 0 - z \notin \mathbb{R}_{\leq 0}$
using *assms complex_eq_iff complex_nonpos_Reals_iff* **by** *auto*
have $Ln(inverse\ z) = Ln(-\ (inverse\ (-z)))$
by *simp*
also have $\ldots = Ln\ (inverse\ (-z)) + \mathrm{i} * complex\_of\_real\ pi$
using *assms z* **by** (*simp add*: *Ln_minus divide_less_0_iff*)
also have $\ldots = -\ Ln\ (-\ z) + \mathrm{i} * complex\_of\_real\ pi$
using *z Ln_inverse* **by** *presburger*
also have $\ldots = -\ (Ln\ z) + \mathrm{i} * 2 * complex\_of\_real\ pi$
using *Ln_minus assms z* **by** *auto*
finally show *?thesis* **by** (*simp add*: *True*)

qed

lemma *Ln_times_ii*:
assumes $z \neq 0$
shows $Ln(\mathrm{i} * z) = (\textit{if } 0 \leq Re(z) \mid Im(z) < 0$
$\textit{then } Ln(z) + \mathrm{i} * \textit{of\_real } pi/2$
$\textit{else } Ln(z) - \mathrm{i} * \textit{of\_real}(3 * pi/2))$
using *Im_Ln_le_pi* [*of z*] *mpi_less_Im_Ln* [*of z*] *assms*
Im_Ln_eq_pi [*of z*] *Im_Ln_pos_lt* [*of z*] *Re_Ln_pos_le* [*of z*]
by (*simp add*: *Ln_times*) *auto*

lemma *Ln_of_nat* [*simp*]: $0 < n \Longrightarrow Ln\ (\textit{of\_nat } n) = \textit{of\_real}\ (ln\ (\textit{of\_nat } n))$
by (*metis Ln_of_real of_nat_0_less_iff of_real_of_nat_eq*)

lemma *Ln_of_nat_over_of_nat*:
assumes $m > 0\ n > 0$
shows $Ln\ (\textit{of\_nat } m\ /\ \textit{of\_nat } n) = \textit{of\_real}\ (ln\ (\textit{of\_nat } m) - ln\ (\textit{of\_nat } n))$
proof −
have $\textit{of\_nat } m\ /\ \textit{of\_nat } n = (\textit{of\_real}\ (\textit{of\_nat } m\ /\ \textit{of\_nat } n) :: \textit{complex})$ **by** *simp*
also from *assms* **have** $Ln \ldots = \textit{of\_real}\ (ln\ (\textit{of\_nat } m\ /\ \textit{of\_nat } n))$
by (*simp add*: *Ln_of_real*[*symmetric*])
also from *assms* **have** $\ldots = \textit{of\_real}\ (ln\ (\textit{of\_nat } m) - ln\ (\textit{of\_nat } n))$
by (*simp add*: *ln_div*)
finally show *?thesis* **.**
qed

lemma *norm_Ln_times_le*:
assumes $w \neq 0\ z \neq 0$
shows $\textit{cmod}\ (Ln(w * z)) \leq \textit{cmod}\ (Ln(w) + Ln(z))$
proof (*cases* $- pi < Im(Ln\ w + Ln\ z) \wedge Im(Ln\ w + Ln\ z) \leq pi$)
case *True*
then show *?thesis*
by (*simp add*: *Ln_times_simple assms*)
next
case *False*
then show *?thesis*
by (*smt* (*verit*) *Im_Ln_le_pi assms cmod_Im_le_iff exp_Ln exp_add ln_unique mpi_less_Im_Ln mult_eq_0_iff norm_exp_eq_Re*)
qed

corollary *norm_Ln_prod_le*:
fixes $f :: 'a \Rightarrow \textit{complex}$
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
shows $\textit{cmod}\ (Ln\ (\textit{prod } f\ A)) \leq (\sum x \in A.\ \textit{cmod}\ (Ln\ (f\ x)))$
using *assms*
proof (*induction A rule*: *infinite_finite_induct*)
case (*insert x A*)
then show *?case*

by *simp* (*smt* (*verit*) *norm_Ln_times_le norm_triangle_ineq prod_zero_iff*)
qed *auto*

lemma *norm_Ln_exp_le*: *norm* (*Ln* (*exp z*)) ≤ *norm z*
by (*smt* (*verit*) *Im_Ln_le_pi Ln_exp Re_Ln cmod_Im_le_iff exp_not_eq_zero ln_exp mpi_less_Im_Ln norm_exp_eq_Re*)

6.20.14 Uniform convergence and products

lemma *uniformly_convergent_on_prod_aux*:
fixes *f* :: *nat* ⇒ *complex* ⇒ *complex*
assumes *norm_f*: ⋀*n x*. *x* ∈ *A* ⟹ *norm* (*f n x*) < *1*
assumes *cont*: ⋀*n*. *continuous_on A* (*f n*)
assumes *conv*: *uniformly_convergent_on A* (λ*N x*. ∑ *n*<*N*. *ln* (*1* + *f n x*))
assumes *A*: *compact A*
shows *uniformly_convergent_on A* (λ*N x*. ∏ *n*<*N*. *1* + *f n x*)
proof −
from *conv* **obtain** *S* **where** *S*: *uniform_limit A* (λ*N x*. ∑ *n*<*N*. *ln* (*1* + *f n x*)) *S sequentially*
by (*auto simp*: *uniformly_convergent_on_def*)
have *cont'*: *continuous_on A S*
proof (*rule uniform_limit_theorem*[*OF* _ *S*])
have *f n x* + *1* ∉ ℝ≤0 **if** *x* ∈ *A* **for** *n x*
proof
assume *f n x* + *1* ∈ ℝ≤0
then obtain *t* **where** *t*: *t* ≤ *0 f n x* = *of_real* (*t* − *1*)
by (*metis add_diff_cancel nonpos_Reals_cases of_real_1 of_real_diff*)
moreover have *norm* ... ≥ *1*
using *t* **by** (*subst norm_of_real*) *auto*
ultimately show *False*
using *norm_f*[*of x n*] *that* **by** *auto*
qed
thus ∀ *F n in sequentially*. *continuous_on A* (λ*x*. ∑ *n*<*n*. *Ln* (*1* + *f n x*))
by (*auto intro*!: *always_eventually continuous_intros cont simp*: *add_ac*)
qed *auto*

define *B* **where** *B* = {*x* + *y* |*x y*. *x* ∈ *S* ‘ *A* ∧ *y* ∈ *cball 0 1*}
have *compact B*
unfolding *B_def* **by** (*intro compact_sums compact_continuous_image cont' A*) *auto*

have *uniformly_convergent_on A* (λ*N x*. *exp* ((∑ *n*<*N*. *ln* (*1* + *f n x*))))
using *conv*
proof (*rule uniformly_convergent_on_compose_uniformly_continuous_on*)
show *closed B*
using ‹*compact B*› **by** (*auto dest*: *compact_imp_closed*)
show *uniformly_continuous_on B exp*
by (*intro compact_uniformly_continuous continuous_intros* ‹*compact B*›)

```
    have eventually (λN. ∀x∈A. dist (∑n<N. Ln (1 + f n x)) (S x) < 1)
sequentially
      using S unfolding uniform_limit_iff by simp
    thus eventually (λN. ∀x∈A. (∑n<N. Ln (1 + f n x)) ∈ B) sequentially
    proof eventually_elim
      case (elim N)
      show ∀x∈A. (∑n<N. Ln (1 + f n x)) ∈ B
      proof safe
        fix x assume x: x ∈ A
        have (∑n<N. Ln (1 + f n x)) = S x + ((∑n<N. Ln (1 + f n x)) − S x)
          by auto
        moreover have ((∑n<N. Ln (1 + f n x)) − S x) ∈ ball 0 1
          using elim x by (auto simp: dist_norm norm_minus_commute)
        ultimately show (∑n<N. Ln (1 + f n x)) ∈ B
          unfolding B_def using x by fastforce
      qed
    qed
  qed
  also have ?this ⟷ uniformly_convergent_on A (λN x. ∏n<N. 1 + f n x)
  proof (intro uniformly_convergent_cong refl always_eventually allI ballI)
    fix N :: nat and x assume x: x ∈ A
    have exp (∑n<N. ln (1 + f n x)) = (∏n<N. exp (ln (1 + f n x)))
      by (simp add: exp_sum)
    also have ... = (∏n<N. 1 + f n x)
      using norm_f[of x] x
    by (smt (verit, best) add.right_neutral add_diff_cancel exp_Ln norm_minus_commute
norm_one prod.cong)
    finally show exp (∑n<N. ln (1 + f n x)) = (∏n<N. 1 + f n x) .
  qed
  finally show ?thesis .
qed
```

Theorem 17.6 by Bak and Newman, Complex Analysis [roughly]

```
lemma uniformly_convergent_on_prod:
  fixes f :: nat ⇒ complex ⇒ complex
  assumes cont: ⋀n. continuous_on A (f n)
  assumes A: compact A
  assumes conv_sum: uniformly_convergent_on A (λN x. ∑n<N. norm (f n x))
  shows   uniformly_convergent_on A (λN x. ∏n<N. 1 + f n x)
proof −
  obtain M where M: ⋀n x. n ≥ M ⟹ x ∈ A ⟹ norm (f n x) < 1 / 2
  proof −
    from conv_sum have uniformly_Cauchy_on A (λN x. ∑n<N. norm (f n x))
      using uniformly_convergent_Cauchy by blast
    moreover have (1 / 2 :: real) > 0
      by simp
    ultimately obtain M where M:
      ⋀x m n. x ∈ A ⟹ m ≥ M ⟹ n ≥ M ⟹ dist (∑k<m. norm (f k x))
(∑k<n. norm (f k x)) < 1 / 2
```

```
    unfolding uniformly_Cauchy_on_def by fast
  show ?thesis
  proof (rule that[of M])
    fix n x assume nx: n ≥ M x ∈ A
    have dist (∑ k<Suc n. norm (f k x)) (∑ k<n. norm (f k x)) < 1 / 2
      by (rule M) (use nx in auto)
    also have dist (∑ k<Suc n. norm (f k x)) (∑ k<n. norm (f k x)) = norm (f
n x)
      by (simp add: dist_norm)
    finally show norm (f n x) < 1 / 2 .
  qed
 qed

 have conv: uniformly_convergent_on A (λN x. ∑ n<N. ln (1 + f (n + M) x))
 proof (rule uniformly_summable_comparison_test)
  show norm (ln (1 + f (n + M) x)) ≤ 2 * norm (f (n + M) x) if x ∈ A for
n x
    by (rule norm_Ln_le) (use M[of n + M x] that in auto)
  have *: filterlim (λn. n + M) at_top at_top
    by (rule filterlim_add_const_nat_at_top)
  have uniformly_convergent_on A (λN x. 2 * ((∑ n<N+M. norm (f n x)) −
(∑ n<M. norm (f n x))))
    by (intro uniformly_convergent_mult uniformly_convergent_minus
        uniformly_convergent_on_compose[OF conv_sum *] uniformly_convergent_on_const)
   also have (λN x. 2 * ((∑ n<N+M. norm (f n x)) − (∑ n<M. norm (f n
x)))) =
            (λN x. ∑ n<N. 2 * norm (f (n + M) x)) (is ?lhs = ?rhs)
  proof (intro ext)
    fix N x
   have (∑ n<N+M. norm (f n x)) − (∑ n<M. norm (f n x)) = (∑ n∈{..<N+M}−{..<M}.
norm (f n x))
      by (subst sum_diff) auto
    also have ... = (∑ n<N. norm (f (n + M) x))
      by (intro sum.reindex_bij_witness[of _ λn. n + M λn. n − M]) auto
    finally show ?lhs N x = ?rhs N x
      by (simp add: sum_distrib_left)
  qed
  finally show uniformly_convergent_on A (λN x. ∑ n<N. 2 * cmod (f (n +
M) x)) .
 qed

 have conv': uniformly_convergent_on A (λN x. ∏ n<N. 1 + f (n + M) x)
 proof (rule uniformly_convergent_on_prod_aux)
  show norm (f (n + M) x) < 1 if x ∈ A for n x
    using M[of n + M x] that by simp
 qed (use M assms conv in auto)

 then obtain S where S: uniform_limit A (λN x. ∏ n<N. 1 + f (n + M) x)
S sequentially
```

by (*auto simp*: *uniformly_convergent_on_def*)
have *cont'*: *continuous_on A S*
by (*intro uniform_limit_theorem*[*OF _ S*] *always_eventually ballI allI continuous_intros cont*) *auto*

have *uniform_limit A* (λ*N x*. (∏ *n*<*M*. *1* + *f n x*) * (∏ *n*<*N*. *1* + *f* (*n* + *M*) *x*)) (λ*x*. (∏ *n*<*M*. *1* + *f n x*) * *S x*) *sequentially*
proof (*rule uniform_lim_mult*[*OF uniform_limit_const S*])
show *bounded* (*S* ‘ *A*)
by (*intro compact_imp_bounded compact_continuous_image A cont'*)
show *bounded* ((λ*x*. ∏ *n*<*M*. *1* + *f n x*) ‘ *A*)
by (*intro compact_imp_bounded compact_continuous_image A continuous_intros cont*)
qed
hence *uniformly_convergent_on A* (λ*N x*. (∏ *n*<*M*. *1* + *f n x*) * (∏ *n*<*N*. *1* + *f* (*n* + *M*) *x*))
by (*auto simp*: *uniformly_convergent_on_def*)
also have (λ*N x*. (∏ *n*<*M*. *1* + *f n x*) * (∏ *n*<*N*. *1* + *f* (*n* + *M*) *x*)) = (λ*N x*. (∏ *n*<*M*+*N*. *1* + *f n x*))
proof (*intro ext*)
fix *N* :: *nat* **and** *x* :: *complex*
have (∏ *n*<*N*. *1* + *f* (*n* + *M*) *x*) = (∏ *n*∈{*M*..<*M*+*N*}. *1* + *f n x*)
by (*intro prod.reindex_bij_witness*[*of _* λ*n*. *n* − *M* λ*n*. *n* + *M*]) *auto*
also have (∏ *n*<*M*. *1* + *f n x*) * … = (∏ *n*∈{..<*M*}∪{*M*..<*M*+*N*}. *1* + *f n x*)
by (*subst prod.union_disjoint*) *auto*
also have {..<*M*} ∪ {*M*..<*M*+*N*} = {..<*M*+*N*}
by *auto*
finally show (∏ *n*<*M*. *1* + *f n x*) * (∏ *n*<*N*. *1* + *f* (*n* + *M*) *x*) = (∏ *n*<*M*+*N*. *1* + *f n x*) **.**
qed
finally have *uniformly_convergent_on A* (λ*N x*. ∏ *n*<*M* + *N*. *1* + *f n x*)
by *simp*
hence *uniformly_convergent_on A* (λ*N x*. ∏ *n*<*M* + (*N* − *M*). *1* + *f n x*)
by (*rule uniformly_convergent_on_compose*) (*rule filterlim_minus_const_nat_at_top*)
also have *?this* ⟷ *?thesis*
proof (*rule uniformly_convergent_cong*)
show *eventually* (λ*x*. ∀ *y*∈*A*. (∏ *n*<*M* + (*x* − *M*). *1* + *f n y*) = (∏ *n*<*x*. *1* + *f n y*)) *at_top*
using *eventually_ge_at_top*[*of M*] **by** *eventually_elim auto*
qed *auto*
finally show *?thesis* **.**
qed

lemma *uniformly_convergent_on_prod'*:
fixes *f* :: *nat* ⇒ *complex* ⇒ *complex*
assumes *cont*: ⋀*n*. *continuous_on A* (*f n*)
assumes *A*: *compact A*
assumes *conv_sum*: *uniformly_convergent_on A* (λ*N x*. ∑ *n*<*N*. *norm* (*f n x*

− 1))
shows *uniformly_convergent_on A* (*λN x.* $\prod n{<}N.$ *f n x*)
proof −
have *uniformly_convergent_on A* (*λN x.* $\prod n{<}N.$ *1 +* (*f n x − 1*))
by (*rule uniformly_convergent_on_prod*) (*use assms* **in** ‹*auto intro*!: *continuous_intros*›)
thus *?thesis*
by *simp*
qed

Prop 17.6 of Bak and Newman, Complex Analysis, p. 243. Only this version is for the reals. Can the two proofs be consolidated?

lemma *uniformly_convergent_on_prod_real*:
fixes *u* :: *nat* ⇒ *real* ⇒ *real*
assumes *contu*: $\bigwedge k.$ *continuous_on K* (*u k*)
and *uconv*: *uniformly_convergent_on K* (*λn x.* $\sum k{<}n.$ |*u k x*|)
and *K*: *compact K*
shows *uniformly_convergent_on K* (*λn x.* $\prod k{<}n.$ *1 + u k x*)
proof −
define *f* **where** *f* ≡ *λk. complex_of_real* ∘ *u k* ∘ *Re*
define *L* **where** *L* ≡ *complex_of_real* ‘ *K*
have *compact L*
by (*simp add*: ‹*compact K*› *L_def compact_continuous_image*)
have *Re* ‘ *complex_of_real* ‘ *X* = *X* **for** *X*
by (*auto simp*: *image_iff*)
with *contu* **have** *contf*: $\bigwedge k.$ *continuous_on L* (*f k*)
unfolding *f_def L_def* **by** (*intro continuous_intros*) *auto*
obtain *S* **where** *S*: $\bigwedge \varepsilon.$ *ε>0* ⟹ $\forall_F$ *n in sequentially.* ∀ *x*∈*K. dist* ($\sum k{<}n.$ |*u k x*|) (*S x*) < *ε*
using *uconv* **unfolding** *uniformly_convergent_on_def uniform_limit_iff* **by** *presburger*
have $\forall_F$ *n in sequentially.* ∀ *z*∈*L. dist* ($\sum k{<}n.$ *cmod* (*f k z*)) ((*of_real* ∘ *S* ∘ *Re*) *z*) < *ε*
if *ε>0* **for** *ε*
using *S* [*OF that*] **by** *eventually_elim* (*simp add*: *L_def f_def*)
then have *uconvf*: *uniformly_convergent_on L* (*λn z.* $\sum k{<}n.$ *norm* (*f k z*))
unfolding *uniformly_convergent_on_def uniform_limit_iff* **by** *blast*
obtain *P* **where** *P*: $\bigwedge \varepsilon.$ *ε>0* ⟹ $\forall_F$ *n in sequentially.* ∀ *z*∈*L. dist* ($\prod k{<}n.$ *1 + f k z*) (*P z*) < *ε*
using *uniformly_convergent_on_prod* [*OF contf* ‹*compact L*› *uconvf*]
unfolding *uniformly_convergent_on_def uniform_limit_iff* **by** *blast*
have §: |($\prod k{<}n.$ *1 + u k x*) − *Re* (*P x*)| ≤ *cmod* (($\prod k{<}n.$ *1 + of_real* (*u k x*)) − *P x*) **for** *n x*
proof −
have ($\prod k{\in}N.$ *of_real* (*1 + u k x*)) = ($\prod k{\in}N.$ *1 + of_real* (*u k x*)) **for** *N*
by *force*
then show *?thesis*
by (*metis Re_complex_of_real abs_Re_le_cmod minus_complex.sel*(*1*) *of_real_prod*)

```
  qed
  have ∀F n in sequentially. ∀x∈K. dist (∏k<n. 1 + u k x) ((Re ∘ P ∘ of_real)
x) < ε
    if ε>0 for ε
     using P [OF that] by eventually_elim (simp add: L_def f_def dist_norm
le_less_trans [OF §])
  then show ?thesis
    unfolding uniformly_convergent_on_def uniform_limit_iff by blast
qed
```

6.20.15 The Argument of a Complex Number

Unlike in HOL Light, it's defined for the same interval as the complex logarithm: $(-\pi,\pi]$.

```
lemma Arg_eq_Im_Ln:
  assumes z ≠ 0 shows Arg z = Im (Ln z)
proof (rule cis_Arg_unique)
  show sgn z = cis (Im (Ln z))
   by (metis assms exp_Ln exp_eq_polar nonzero_mult_div_cancel_left norm_eq_zero
            norm_exp_eq_Re of_real_eq_0_iff sgn_eq)
  show − pi < Im (Ln z)
    by (simp add: assms mpi_less_Im_Ln)
  show Im (Ln z) ≤ pi
    by (simp add: Im_Ln_le_pi assms)
qed
```

The 1990s definition of argument coincides with the more recent one

```
lemma Arg_def:
  shows Arg z = (if z = 0 then 0 else Im (Ln z))
  by (simp add: Arg_eq_Im_Ln Arg_zero)

lemma Arg_of_real [simp]: Arg (of_real r) = (if r<0 then pi else 0)
  by (simp add: Im_Ln_eq_pi Arg_def)

lemma mpi_less_Arg: −pi < Arg z and Arg_le_pi: Arg z ≤ pi
  by (auto simp: Arg_def mpi_less_Im_Ln Im_Ln_le_pi)

lemma Arg_eq:
  assumes z ≠ 0
  shows z = of_real(norm z) * exp(i * Arg z)
  using cis_conv_exp rcis_cmod_Arg rcis_def by force

lemma is_Arg_Arg: z ≠ 0 ⟹ is_Arg z (Arg z)
  by (simp add: Arg_eq is_Arg_def)

lemma Argument_exists:
  assumes z ≠ 0 and R: R = {r−pi<..r+pi}
  obtains s where is_Arg z s s∈R
```

proof −
let *?rp* = *r* − *Arg z* + *pi*
define *k* **where** *k* ≡ ⌊*?rp* / (*2* ∗ *pi*)⌋
have (*Arg z* + *of_int k* ∗ (*2* ∗ *pi*)) ∈ *R*
using *floor_divide_lower* [*of 2*∗*pi ?rp*] *floor_divide_upper* [*of 2*∗*pi ?rp*]
by (*auto simp*: *k_def algebra_simps R*)
then show *?thesis*
using *Arg_eq* ‹*z* ≠ *0*› *is_Arg_2pi_iff is_Arg_def that* **by** *blast*
qed

lemma *Argument_exists_unique*:
assumes *z* ≠ *0* **and** *R*: *R* = {*r*−*pi*<..*r*+*pi*}
obtains *s* **where** *is_Arg z s s*∈*R* ⋀*t*. ⟦*is_Arg z t*; *t*∈*R*⟧ ⟹ *s*=*t*
proof −
obtain *s* **where** *s*: *is_Arg z s s*∈*R*
using *Argument_exists* [*OF assms*] **.**
moreover have ⋀*t*. ⟦*is_Arg z t*; *t*∈*R*⟧ ⟹ *s*=*t*
using *assms s* **by** (*auto simp*: *is_Arg_eqI*)
ultimately show *thesis*
using *that* **by** *blast*
qed

lemma *Argument_Ex1*:
assumes *z* ≠ *0* **and** *R*: *R* = {*r*−*pi*<..*r*+*pi*}
shows ∃!*s*. *is_Arg z s* ∧ *s* ∈ *R*
using *Argument_exists_unique* [*OF assms*] **by** *metis*

lemma *Arg_divide*:
assumes *w* ≠ *0 z* ≠ *0*
shows *is_Arg* (*z* / *w*) (*Arg z* − *Arg w*)
using *Arg_eq* [*of z*] *Arg_eq* [*of w*] *Arg_eq* [*of norm*(*z* / *w*)] *assms*
by (*auto simp*: *is_Arg_def norm_divide field_simps exp_diff Arg_of_real*)

lemma *Arg_unique_lemma*:
assumes *is_Arg z t is_Arg z t′*
and − *pi* < *t* *t* ≤ *pi*
and − *pi* < *t′* *t′* ≤ *pi*
and *z* ≠ *0*
shows *t′* = *t*
using *is_Arg_eqI assms* **by** *force*

lemma *complex_norm_eq_1_exp_eq*: *norm z* = *1* ⟷ *exp*(i ∗ (*Arg z*)) = *z*
by (*metis Arg2pi_eq Arg_eq complex_norm_eq_1_exp norm_eq_zero norm_exp_i_times*)

lemma *Arg_unique*: ⟦*of_real r* ∗ *exp*(i ∗ *a*) = *z*; *0* < *r*; −*pi* < *a*; *a* ≤ *pi*⟧ ⟹
Arg z = *a*
by (*rule Arg_unique_lemma* [*unfolded is_Arg_def*, *OF* _ *Arg_eq*])
(*use mpi_less_Arg Arg_le_pi* **in** ‹*auto simp*: *norm_mult*›)

lemma *Arg_minus*:
assumes $z \neq 0$
shows $Arg\ (-z) = (if\ Arg\ z \leq 0\ then\ Arg\ z + pi\ else\ Arg\ z - pi)$
proof −
have [*simp*]: $cmod\ z * cos\ (Arg\ z) = Re\ z$
using *assms Arg_eq* [*of z*] **by** (*metis Re_exp exp_Ln norm_exp_eq_Re Arg_def*)
have [*simp*]: $cmod\ z * sin\ (Arg\ z) = Im\ z$
using *assms Arg_eq* [*of z*] **by** (*metis Im_exp exp_Ln norm_exp_eq_Re Arg_def*)
show *?thesis*
using *mpi_less_Arg* [*of z*] *Arg_le_pi* [*of z*] *assms*
by (*intro Arg_unique* [*of norm z, OF complex_eqI*]) (*auto simp*: *Re_exp Im_exp*)
qed

lemma *Arg_1* [*simp*]: $Arg\ 1 = 0$
by (*rule Arg_unique*[*of 1*]) *auto*

lemma *Arg_numeral* [*simp*]: $Arg\ (numeral\ n) = 0$
by (*rule Arg_unique*[*of numeral n*]) *auto*

lemma *Arg_times_of_real* [*simp*]:
assumes $0 < r$ **shows** $Arg\ (of\_real\ r * z) = Arg\ z$
using *Arg_def Ln_times_of_real assms* **by** *auto*

lemma *Arg_times_of_real2* [*simp*]: $0 < r \Longrightarrow Arg\ (z * of\_real\ r) = Arg\ z$
by (*metis Arg_times_of_real mult.commute*)

lemma *Arg_divide_of_real* [*simp*]: $0 < r \Longrightarrow Arg\ (z\ /\ of\_real\ r) = Arg\ z$
by (*metis Arg_times_of_real2 less_irrefl nonzero_eq_divide_eq of_real_eq_0_iff*)

lemma *Arg_less_0*: $0 \leq Arg\ z \longleftrightarrow 0 \leq Im\ z$
using *Im_Ln_le_pi Im_Ln_pos_le*
by (*simp add*: *Arg_def*)

converse fails because the argument can equal π.

lemma *Arg_uminus*: $Arg\ z < 0 \Longrightarrow Arg\ (-z) > 0$
by (*smt* (*verit*) *Arg_bounded Arg_minus Complex.Arg_def*)

lemma *Arg_eq_pi*: $Arg\ z = pi \longleftrightarrow Re\ z < 0 \wedge Im\ z = 0$
by (*auto simp*: *Arg_def Im_Ln_eq_pi*)

lemma *Arg_lt_pi*: $0 < Arg\ z \wedge Arg\ z < pi \longleftrightarrow 0 < Im\ z$
using *Arg_less_0* [*of z*] *Im_Ln_pos_lt*
by (*auto simp*: *order.order_iff_strict Arg_def*)

lemma *Arg_eq_0*: $Arg\ z = 0 \longleftrightarrow z \in \mathbb{R} \wedge 0 \leq Re\ z$
using *Arg_def Im_Ln_eq_0 complex_eq_iff complex_is_Real_iff* **by** *auto*

corollary *Arg_ne_0*: **assumes** $z \notin \mathbb{R}_{\geq 0}$ **shows** $Arg\ z \neq 0$
using *assms* **by** (*auto simp*: *nonneg_Reals_def Arg_eq_0*)

lemma *Arg_eq_pi_iff*: $Arg\ z = pi \longleftrightarrow z \in \mathbb{R} \wedge Re\ z < 0$
using *Arg_eq_pi complex_is_Real_iff* **by** *blast*

lemma *Arg_eq_0_pi*: $Arg\ z = 0 \vee Arg\ z = pi \longleftrightarrow z \in \mathbb{R}$
using *Arg_eq_pi_iff Arg_eq_0* **by** *force*

lemma *Arg_real*: $z \in \mathbb{R} \Longrightarrow Arg\ z = (if\ 0 \leq Re\ z\ then\ 0\ else\ pi)$
using *Arg_eq_0 Arg_eq_0_pi* **by** *auto*

lemma *Arg_inverse*: $Arg(inverse\ z) = (if\ z \in \mathbb{R}\ then\ Arg\ z\ else - Arg\ z)$
proof (*cases* $z \in \mathbb{R}$)
case *False*
then show *?thesis*
by (*simp add*: *Arg_def Ln_inverse complex_is_Real_iff complex_nonpos_Reals_iff*)
qed (*use Arg_real Re_inverse* **in** *auto*)

lemma *Arg_eq_iff*:
assumes $w \neq 0\ z \neq 0$
shows $Arg\ w = Arg\ z \longleftrightarrow (\exists x.\ 0 < x \wedge w = of\_real\ x * z)$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then have $w = (cmod\ w\ /\ cmod\ z) * z$
by (*metis Arg_eq assms divide_divide_eq_right eq_divide_eq exp_not_eq_zero of_real_divide*)
then show *?rhs*
using *assms divide_pos_pos zero_less_norm_iff* **by** *blast*
qed *auto*

lemma *Arg_inverse_eq_0*: $Arg(inverse\ z) = 0 \longleftrightarrow Arg\ z = 0$
by (*metis Arg_eq_0 Arg_inverse inverse_inverse_eq*)

lemma *Arg_cnj_eq_inverse*: $z{\neq}0 \Longrightarrow Arg\ (cnj\ z) = Arg\ (inverse\ z)$
using *Arg2pi_cnj_eq_inverse Arg2pi_eq_iff Arg_eq_iff* **by** *auto*

lemma *Arg_cnj*: $Arg(cnj\ z) = (if\ z \in \mathbb{R}\ then\ Arg\ z\ else - Arg\ z)$
by (*metis Arg_cnj_eq_inverse Arg_inverse Reals_0 complex_cnj_zero*)

lemma *Arg_exp*: $-pi < Im\ z \Longrightarrow Im\ z \leq pi \Longrightarrow Arg(exp\ z) = Im\ z$
by (*simp add*: *Arg_eq_Im_Ln*)

lemma *Arg_cis*: $x \in \{-pi{<}..pi\} \Longrightarrow Arg\ (cis\ x) = x$
unfolding *cis_conv_exp* **by** (*subst Arg_exp*) *auto*

lemma *Arg_rcis*: $x \in \{-pi{<}..pi\} \Longrightarrow r > 0 \Longrightarrow Arg\ (rcis\ r\ x) = x$
unfolding *rcis_def* **by** (*subst Arg_times_of_real*) (*auto simp*: *Arg_cis*)

lemma *Ln_Arg*: $z{\neq}0 \Longrightarrow Ln(z) = ln(norm\ z) + \mathrm{i} * Arg(z)$
by (*metis Arg_def Re_Ln complex_eq*)

lemma *continuous_at_Arg*:
assumes $z \notin \mathbb{R}_{\leq 0}$
shows *continuous* (*at z*) *Arg*
proof −
have $(\lambda z.\ Im\ (Ln\ z)) \longrightarrow z \rightarrow Arg\ z$
using *Arg_def assms continuous_at* **by** *fastforce*
then show *?thesis*
unfolding *continuous_at*
by (*smt* (*verit*, *del_insts*) *Arg_eq_Im_Ln Lim_transform_away_at assms nonpos_Reals_zero_I*)
qed

lemma *continuous_within_Arg*: $z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *continuous* (*at z within S*) *Arg*
using *continuous_at_Arg continuous_at_imp_continuous_within* **by** *blast*

lemma *Arg_Re_pos*: $|Arg\ z| < pi\ /\ 2 \longleftrightarrow Re\ z > 0 \vee z = 0$
using *Arg_def Re_Ln_pos_lt* **by** *auto*

lemma *Arg_Re_nonneg*: $|Arg\ z| \leq pi\ /\ 2 \longleftrightarrow Re\ z \geq 0$
using *Re_Ln_pos_le*[*of z*] **by** (*cases z = 0*) (*auto simp*: *Arg_eq_Im_Ln Arg_zero*)

lemma *Arg_times*:
assumes $Arg\ z + Arg\ w \in \{-pi<..pi\}$ $z \neq 0$ $w \neq 0$
shows $Arg\ (z * w) = Arg\ z + Arg\ w$
using *Arg_eq_Im_Ln Ln_times_simple assms* **by** *auto*

6.20.16 The Unwinding Number and the Ln product Formula

Note that in this special case the unwinding number is -1, 0 or 1. But it's always an integer.

lemma *is_Arg_exp_Im*: *is_Arg* (*exp z*) (*Im z*)
using *exp_eq_polar is_Arg_def norm_exp_eq_Re* **by** *auto*

lemma *is_Arg_exp_diff_2pi*:
assumes *is_Arg* (*exp z*) ϑ
shows $\exists k.\ Im\ z - of\_int\ k * (2 * pi) = \vartheta$
proof (*intro exI is_Arg_eqI*)
let $?k = \lfloor (Im\ z - \vartheta)\ /\ (2 * pi) \rfloor$
show *is_Arg* (*exp z*) (*Im z* − *real_of_int ?k* ∗ (2 ∗ *pi*))
by (*metis diff_add_cancel is_Arg_2pi_iff is_Arg_exp_Im*)
show $|Im\ z - real\_of\_int\ ?k * (2 * pi) - \vartheta| < 2 * pi$
using *floor_divide_upper* [*of* 2∗*pi Im z* − ϑ] *floor_divide_lower* [*of* 2∗*pi Im z* − ϑ]

by (*auto simp*: *algebra_simps abs_if*)
qed (*auto simp*: *is_Arg_exp_Im assms*)

lemma *Arg_exp_diff_2pi*: $\exists k.\ Im\ z - of\_int\ k * (2 * pi) = Arg\ (exp\ z)$
using *is_Arg_exp_diff_2pi* [*OF is_Arg_Arg*] **by** *auto*

lemma *unwinding_in_Ints*: $(z - Ln(exp\ z)) / (of\_real(2*pi) * \mathrm{i}) \in \mathbb{Z}$
using *Arg_exp_diff_2pi* [*of z*]
by (*force simp*: *Ints_def image_def field_simps Arg_def intro*!: *complex_eqI*)

definition *unwinding* :: *complex* ⇒ *int* **where**
unwinding z ≡ *THE k. of_int k* = $(z - Ln(exp\ z)) / (of\_real(2*pi) * \mathrm{i})$

lemma *unwinding*: *of_int* (*unwinding z*) = $(z - Ln(exp\ z)) / (of\_real(2*pi) * \mathrm{i})$
using *unwinding_in_Ints* [*of z*]
unfolding *unwinding_def Ints_def* **by** *force*

lemma *unwinding_2pi*: $(2*pi) * \mathrm{i} * unwinding(z) = z - Ln(exp\ z)$
by (*simp add*: *unwinding*)

lemma *Ln_times_unwinding*:
$w \neq 0 \Longrightarrow z \neq 0 \Longrightarrow Ln(w * z) = Ln(w) + Ln(z) - (2*pi) * \mathrm{i} * unwinding(Ln\ w + Ln\ z)$
using *unwinding_2pi* **by** (*simp add*: *exp_add*)

lemma *arg_conv_arctan*:
assumes *Re z* > *0*
shows *Arg z* = *arctan* (*Im z* / *Re z*)
proof (*rule cis_Arg_unique*)
show *sgn z* = *cis* (*arctan* (*Im z* / *Re z*))
proof (*rule complex_eqI*)
have *Re* (*cis* (*arctan* (*Im z* / *Re z*))) = *1* / *sqrt* (*1* + (*Im z*)^2 / (*Re z*)^2)
by (*simp add*: *cos_arctan power_divide*)
also have *1* + *Im z* ^ *2* / *Re z* ^ *2* = *norm z* ^ *2* / *Re z* ^ *2*
using *assms* **by** (*simp add*: *cmod_def field_simps*)
also have *1* / *sqrt* ... = *Re z* / *norm z*
using *assms* **by** (*simp add*: *real_sqrt_divide*)
finally show *Re* (*sgn z*) = *Re* (*cis* (*arctan* (*Im z* / *Re z*)))
by *simp*
next
have *Im* (*cis* (*arctan* (*Im z* / *Re z*))) = *Im z* / (*Re z* * *sqrt* (*1* + (*Im z*)^2 / (*Re z*)^2))
by (*simp add*: *sin_arctan field_simps*)
also have *1* + *Im z* ^ *2* / *Re z* ^ *2* = *norm z* ^ *2* / *Re z* ^ *2*
using *assms* **by** (*simp add*: *cmod_def field_simps*)
also have *Im z* / (*Re z* * *sqrt* ...) = *Im z* / *norm z*
using *assms* **by** (*simp add*: *real_sqrt_divide*)
finally show *Im* (*sgn z*) = *Im* (*cis* (*arctan* (*Im z* / *Re z*)))

```
    by simp
 qed
next
 show arctan (Im z / Re z) > −pi
   by (smt (verit, ccfv_SIG) arctan field_sum_of_halves)
next
 show arctan (Im z / Re z) ≤ pi
  by (smt (verit, best) arctan field_sum_of_halves)
qed
```

6.20.17 Characterisation of *Im (Ln z)* (Wenda Li)

```
lemma Im_Ln_eq_pi_half:
   z ≠ 0 ⟹ (Im(Ln z) = pi/2 ⟷ 0 < Im(z) ∧ Re(z) = 0)
   z ≠ 0 ⟹ (Im(Ln z) = −pi/2 ⟷ Im(z) < 0 ∧ Re(z) = 0)
 using Im_Ln_pos_lt Im_Ln_pos_le Re_Ln_pos_le Re_Ln_pos_lt pi_ge_two
by fastforce+

lemma Im_Ln_eq:
 assumes z≠0
 shows Im (Ln z) = (if Re z≠0 then
                      if Re z>0 then
                        arctan (Im z/Re z)
                      else if Im z≥0 then
                        arctan (Im z/Re z) + pi
                      else
                        arctan (Im z/Re z) − pi
                    else
                      if Im z>0 then pi/2 else −pi/2)
proof −
 have eq_arctan_pos: Im (Ln z) = arctan (Im z/Re z) when Re z>0 for z
  by (metis Arg_eq_Im_Ln arg_conv_arctan order_less_irrefl that zero_complex.simps(1))
 have ?thesis when Re z=0
   using Im_Ln_eq_pi_half[OF ‹z≠0›] that
   using assms complex_eq_iff by auto
 moreover have ?thesis when Re z>0
   using eq_arctan_pos[OF that] that by auto
 moreover have ?thesis when Re z<0 Im z≥0
 proof −
   have Im (Ln (− z)) = arctan (Im (− z) / Re (− z))
     by (simp add: eq_arctan_pos that(1))
   moreover have Ln (− z) = Ln z − i * complex_of_real pi
     using Ln_minus assms that by fastforce
   ultimately show ?thesis using that by auto
 qed
 moreover have ?thesis when Re z<0 Im z<0
 proof −
   have Im (Ln (− z)) = arctan (Im (− z) / Re (− z))
     by (simp add: eq_arctan_pos that(1))
```

moreover have *Ln* (− *z*) = *Ln z* + i ∗ *complex_of_real pi*
using *Ln_minus assms that* **by** *auto*
ultimately show *?thesis* **using** *that* **by** *auto*
qed
ultimately show *?thesis* **by** *linarith*
qed

6.20.18 Relation between Ln and Arg2pi, and hence continuity of Arg2pi

lemma *Arg2pi_Ln*: *0* < *Arg2pi z* ⟹ *Arg2pi z* = *Im*(*Ln*(−*z*)) + *pi*
by (*smt* (*verit*, *best*) *Arg2pi_0 Arg2pi_exp Arg2pi_minus Arg_exp Arg_minus Im_Ln_le_pi*
exp_Ln mpi_less_Im_Ln neg_equal_0_iff_equal)

lemma *continuous_at_Arg2pi*:
assumes *z* ∉ $\mathbb{R}_{\geq 0}$
shows *continuous* (*at z*) *Arg2pi*
proof −
have *isCont* (λ*z*. *Im* (*Ln* (− *z*)) + *pi*) *z*
by (*rule Complex.isCont_Im isCont_Ln′ continuous_intros* | *simp add*: *assms complex_is_Real_iff*)+
moreover consider *Re z* < *0* | *Im z* ≠ *0* **using** *assms*
using *complex_nonneg_Reals_iff not_le* **by** *blast*
ultimately have (λ*z*. *Im* (*Ln* (− *z*)) + *pi*) −*z*→ *Arg2pi z*
by (*simp add*: *Arg2pi_Ln Arg2pi_gt_0 assms continuous_within*)
then show *?thesis*
unfolding *continuous_at*
by (*metis* (*mono_tags*, *lifting*) *Arg2pi_Ln Arg2pi_gt_0 Compl_iff Lim_transform_within_open assms*
closed_nonneg_Reals_complex open_Compl)
qed

Relation between Arg2pi and arctangent in upper halfplane

lemma *Arg2pi_arctan_upperhalf*:
assumes *0* < *Im z*
shows *Arg2pi z* = *pi*/*2* − *arctan*(*Re z* / *Im z*)
proof (*cases z* = *0*)
case *False*
show *?thesis*
proof (*rule Arg2pi_unique* [*of norm z*])
show (*cmod z*) ∗ *exp* (i ∗ (*pi* / *2* − *arctan* (*Re z* / *Im z*))) = *z*
apply (*rule complex_eqI*)
using *assms norm_complex_def* [*of z*, *symmetric*]
unfolding *exp_Euler cos_diff sin_diff sin_of_real cos_of_real*
by (*simp_all add*: *field_simps real_sqrt_divide sin_arctan cos_arctan*)
qed (*use False arctan* [*of Re z* / *Im z*] **in** *auto*)
qed (*use assms* **in** *auto*)

lemma *Arg2pi_eq_Im_Ln*:
assumes $0 \leq Im\ z\ 0 < Re\ z$
shows $Arg2pi\ z = Im\ (Ln\ z)$
by (*smt* (*verit*, *ccfv_SIG*) *Arg2pi_exp Im_Ln_pos_le assms exp_Ln pi_neq_zero zero_complex.simps*(1))

lemma *continuous_within_upperhalf_Arg2pi*:
assumes $z \neq 0$
shows *continuous* (*at z within* $\{z.\ 0 \leq Im\ z\}$) *Arg2pi*
proof (*cases* $z \in \mathbb{R}_{\geq 0}$)
case *False* **then show** *?thesis*
using *continuous_at_Arg2pi continuous_at_imp_continuous_within* **by** *auto*
next
case *True*
then have *z*: $z \in \mathbb{R}\ 0 < Re\ z$
using *assms* **by** (*auto simp*: *complex_nonneg_Reals_iff complex_is_Real_iff complex_neq_0*)
then have [*simp*]: $Arg2pi\ z = 0\ Im\ (Ln\ z) = 0$
by (*auto simp*: *Arg2pi_eq_0 Im_Ln_eq_0 assms complex_is_Real_iff*)
show *?thesis*
proof (*clarsimp simp add*: *continuous_within Lim_within dist_norm*)
fix *e*::*real*
assume $0 < e$
moreover have *continuous* (*at z*) ($\lambda x.\ Im\ (Ln\ x)$)
using *z* **by** (*simp add*: *continuous_at_Ln complex_nonpos_Reals_iff*)
ultimately
obtain *d* **where** *d*: $d{>}0\ \bigwedge x.\ x \neq z \Longrightarrow cmod\ (x - z) < d \Longrightarrow |Im\ (Ln\ x)| < e$
by (*auto simp*: *continuous_within Lim_within dist_norm*)
{ **fix** *x*
assume $cmod\ (x - z) < Re\ z\ /\ 2$
then have $|Re\ x - Re\ z| < Re\ z\ /\ 2$
by (*metis le_less_trans abs_Re_le_cmod minus_complex.simps*(1))
then have $0 < Re\ x$
using *z* **by** *linarith*
}
then show $\exists d{>}0.\ \forall x.\ 0 \leq Im\ x \longrightarrow x \neq z \wedge cmod\ (x - z) < d \longrightarrow |Arg2pi\ x| < e$
apply (*rule_tac x*=*min d* (*Re z* / *2*) **in** *exI*)
using *z d* **by** (*auto simp*: *Arg2pi_eq_Im_Ln*)
qed
qed

lemma *continuous_on_upperhalf_Arg2pi*: *continuous_on* ($\{z.\ 0 \leq Im\ z\} - \{0\}$) *Arg2pi*
unfolding *continuous_on_eq_continuous_within*
by (*metis DiffE Diff_subset continuous_within_subset continuous_within_upperhalf_Arg2pi insertCI*)

lemma *open_Arg2pi2pi_less_Int*:
assumes $0 \leq s\ t \leq 2{*}pi$
shows *open* $(\{y.\ s < Arg2pi\ y\} \cap \{y.\ Arg2pi\ y < t\})$
proof −
have *1*: *continuous_on* $(UNIV - \mathbb{R}_{\geq 0})$ *Arg2pi*
using *continuous_at_Arg2pi continuous_at_imp_continuous_within*
by (*auto simp*: *continuous_on_eq_continuous_within*)
have *2*: *open* $(UNIV - \mathbb{R}_{\geq 0} :: complex\ set)$ **by** (*simp add*: *open_Diff*)
have *open* $(\{z.\ s < z\} \cap \{z.\ z < t\})$
using *open_lessThan* [*of t*] *open_greaterThan* [*of s*]
by (*metis greaterThan_def lessThan_def open_Int*)
moreover have $\{y.\ s < Arg2pi\ y\} \cap \{y.\ Arg2pi\ y < t\} \subseteq - \mathbb{R}_{\geq 0}$
using *assms* **by** (*auto simp*: *Arg2pi_real complex_nonneg_Reals_iff complex_is_Real_iff*)
ultimately show *?thesis*
using *continuous_imp_open_vimage* [*OF 1 2*, *of* $\{z.\ Re\ z > s\} \cap \{z.\ Re\ z < t\}$]
by *auto*
qed

lemma *open_Arg2pi2pi_gt*: *open* $\{z.\ t < Arg2pi\ z\}$
proof (*cases* $t < 0$)
case *True* **then have** $\{z.\ t < Arg2pi\ z\} = UNIV$
using *Arg2pi_ge_0 less_le_trans* **by** *auto*
then show *?thesis*
by *simp*
next
case *False* **then show** *?thesis*
using *open_Arg2pi2pi_less_Int* [*of t 2∗pi*] *Arg2pi_lt_2pi*
by *auto*
qed

lemma *closed_Arg2pi2pi_le*: *closed* $\{z.\ Arg2pi\ z \leq t\}$
using *open_Arg2pi2pi_gt* [*of t*]
by (*simp add*: *closed_def Set.Collect_neg_eq* [*symmetric*] *not_le*)

6.20.19 Complex Powers

lemma *powr_to_1* [*simp*]: *z powr 1* = (*z*::*complex*)
by (*simp add*: *powr_def*)

lemma *powr_nat*:
fixes *n*::*nat* **and** *z*::*complex* **shows** *z powr n* = (*if z* = *0 then 0 else z*^*n*)
by (*simp add*: *exp_of_nat_mult powr_def*)

lemma *powr_nat′*: $(z :: complex) \neq 0 \vee n \neq 0 \Longrightarrow z\ powr\ of\_nat\ n = z \hat{}\ n$
by (*cases z* = *0*) (*auto simp*: *powr_nat*)

lemma *norm_powr_real*: $w \in \mathbb{R} \Longrightarrow 0 < Re\ w \Longrightarrow norm(w\ powr\ z) = exp(Re\ z$

$* \ ln(Re\ w))$
using *Ln_Reals_eq norm_exp_eq_Re* **by** (*auto simp*: *Im_Ln_eq_0 powr_def norm_complex_def*)

lemma *norm_powr_real_powr'*: $w \in \mathbb{R} \Longrightarrow norm\ (z\ powr\ w) = norm\ z\ powr\ Re\ w$
by (*auto simp*: *powr_def Reals_def*)

lemma *powr_complexpow* [*simp*]:
fixes *x*::*complex* **shows** $x \neq 0 \Longrightarrow x\ powr\ (of\_nat\ n) = x\hat{}n$
by (*simp add*: *powr_nat'*)

lemma *powr_complexnumeral* [*simp*]:
fixes *x*::*complex* **shows** $x\ powr\ (numeral\ n) = x\ \hat{}\ (numeral\ n)$
by (*metis of_nat_numeral power_zero_numeral powr_nat*)

lemma *cnj_powr*:
assumes $Im\ a = 0 \Longrightarrow Re\ a \geq 0$
shows $cnj\ (a\ powr\ b) = cnj\ a\ powr\ cnj\ b$
proof (*cases* $a = 0$)
case *False*
with *assms* **have** $a \notin \mathbb{R}_{\leq 0}$ **by** (*auto simp*: *complex_eq_iff complex_nonpos_Reals_iff*)
with *False* **show** *?thesis* **by** (*simp add*: *powr_def exp_cnj cnj_Ln*)
qed *simp*

lemma *powr_real_real*:
assumes $w \in \mathbb{R}$ $z \in \mathbb{R}$ $0 < Re\ w$
shows $w\ powr\ z = exp(Re\ z * ln(Re\ w))$
proof −
have $w \neq 0$
using *assms* **by** *auto*
with *assms* **show** *?thesis*
by (*simp add*: *powr_def Ln_Reals_eq of_real_exp*)
qed

lemma *powr_of_real*:
fixes *x*::*real* **and** *y*::*real*
shows $0 \leq x \Longrightarrow of\_real\ x\ powr\ (of\_real\ y{::}complex) = of\_real\ (x\ powr\ y)$
by (*simp_all add*: *powr_def exp_eq_polar*)

lemma *powr_of_int*:
fixes *z*::*complex* **and** *n*::*int*
assumes $z{\neq}(0{::}complex)$
shows $z\ powr\ of\_int\ n = (if\ n{\geq}0\ then\ z\hat{}nat\ n\ else\ inverse\ (z\hat{}nat\ (-n)))$
by (*metis assms not_le of_int_of_nat powr_complexpow powr_minus*)

lemma *complex_powr_of_int*: $z \neq 0 \vee n \neq 0 \Longrightarrow z\ powr\ of\_int\ n = (z :: complex)\ powi\ n$
by (*cases* $z = 0 \vee n = 0$)

(*auto simp*: *power_int_def powr_minus powr_nat powr_of_int power_0_left power_inverse*)

lemma *powr_Reals_eq*: $[\![x \in \mathbb{R}; y \in \mathbb{R}; Re\ x \geq 0]\!] \Longrightarrow x\ powr\ y = of\_real\ (Re\ x\ powr\ Re\ y)$
by (*metis of_real_Re powr_of_real*)

lemma *norm_powr_real_mono*:
$[\![w \in \mathbb{R}; 1 < Re\ w]\!] \Longrightarrow cmod(w\ powr\ z1) \leq cmod(w\ powr\ z2) \longleftrightarrow Re\ z1 \leq Re\ z2$
by (*auto simp*: *powr_def algebra_simps Reals_def Ln_of_real*)

lemma *powr_times_real*:
$[\![x \in \mathbb{R}; y \in \mathbb{R}; 0 \leq Re\ x; 0 \leq Re\ y]\!] \Longrightarrow (x * y)\ powr\ z = x\ powr\ z * y\ powr\ z$
by (*auto simp*: *Reals_def powr_def Ln_times exp_add algebra_simps less_eq_real_def Ln_of_real*)

lemma *Re_powr_le*: $r \in \mathbb{R}_{\geq 0} \Longrightarrow Re\ (r\ powr\ z) \leq Re\ r\ powr\ Re\ z$
by (*auto simp*: *powr_def nonneg_Reals_def order_trans* [*OF complex_Re_le_cmod*])

lemma
fixes *w*::*complex*
assumes $w \in \mathbb{R}_{\geq 0}$ $z \in \mathbb{R}$
shows *Reals_powr* [*simp*]: *w powr z* $\in \mathbb{R}$ **and** *nonneg_Reals_powr* [*simp*]: *w powr z* $\in \mathbb{R}_{\geq 0}$
using *assms* **by** (*auto simp*: *nonneg_Reals_def Reals_def powr_of_real*)

lemma *exp_powr_complex*:
fixes *x*::*complex*
assumes $-pi < Im(x)$ $Im(x) \leq pi$
shows *exp x powr y* = *exp* (*x*∗*y*)
using *assms* **by** (*simp add*: *powr_def mult.commute*)

lemma *powr_neg_real_complex*:
fixes *w*::*complex*
shows (− *of_real x*) *powr w* = (−1) *powr* (*of_real* (*sgn x*) ∗ *w*) ∗ *of_real x powr w*
proof (*cases x* = *0*)
assume *x*: $x \neq 0$
hence (−*x*) *powr w* = *exp* (*w* ∗ *ln* (−*of_real x*)) **by** (*simp add*: *powr_def*)
also from *x* **have** *ln* (−*of_real x*) = *Ln* (*of_real x*) + *of_real* (*sgn x*) ∗ *pi* ∗ i
by (*simp add*: *Ln_minus Ln_of_real*)
also from *x* **have** *exp* (*w* ∗ . . .) = *cis pi powr* (*of_real* (*sgn x*) ∗ *w*) ∗ *of_real x powr w*
by (*simp add*: *powr_def exp_add algebra_simps Ln_of_real cis_conv_exp*)
also note *cis_pi*
finally show *?thesis* **by** *simp*
qed *simp_all*

lemma *has_field_derivative_powr*:
fixes z :: *complex*
assumes $z \notin \mathbb{R}_{\leq 0}$
shows $((\lambda z.\ z\ powr\ s)\ has\_field\_derivative\ (s * z\ powr\ (s - 1)))\ (at\ z)$
proof (*cases z=0*)
case *False*
then have §: $exp\ (s * Ln\ z) * inverse\ z = exp\ ((s - 1) * Ln\ z)$
by (*simp add*: *divide_complex_def exp_diff left_diff_distrib'*)
show *?thesis*
unfolding *powr_def*
proof (*rule has_field_derivative_transform_within*)
show $((\lambda z.\ exp\ (s * Ln\ z))\ has\_field\_derivative\ s * (if\ z = 0\ then\ 0\ else\ exp\ ((s - 1) * Ln\ z)))$
$(at\ z)$
by (*intro derivative_eq_intros | simp add*: *assms False* §)+
qed (*use False* **in** *auto*)
qed (*use assms* **in** *auto*)

declare *has_field_derivative_powr*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *has_field_derivative_powr_of_int*:
fixes z :: *complex*
assumes *gderiv*: $(g\ has\_field\_derivative\ gd)\ (at\ z\ within\ S)$ **and** $g\ z \neq 0$
shows $((\lambda z.\ g\ z\ powr\ of\_int\ n)\ has\_field\_derivative\ (n * g\ z\ powr\ (of\_int\ n - 1) * gd))\ (at\ z\ within\ S)$
proof −
obtain e **where** $e>0$ **and** *e_dist*: $\forall\, y{\in}S.\ dist\ z\ y < e \longrightarrow g\ y \neq 0$
using *DERIV_continuous assms continuous_within_avoid gderiv* **by** *blast*
define D **where** $D = of\_int\ n * g\ z\ powr\ (of\_int\ (n - 1)) * gd$
define E **where** $E = of\_int\ n * g\ z\ powi\ (n - 1) * gd$
have $((\lambda z.\ g\ z\ powr\ of\_int\ n)\ has\_field\_derivative\ D)\ (at\ z\ within\ S)$
$\longleftrightarrow ((\lambda z.\ g\ z\ powr\ of\_int\ n)\ has\_field\_derivative\ E)\ (at\ z\ within\ S)$
using *assms complex_powr_of_int D_def E_def* **by** *presburger*
also have $\ldots \longleftrightarrow ((\lambda z.\ g\ z\ powi\ n)\ has\_field\_derivative\ E)\ (at\ z\ within\ S)$
proof (*rule has_field_derivative_cong_eventually*)
show $\forall_F\ x\ in\ at\ z\ within\ S.\ g\ x\ powr\ of\_int\ n = g\ x\ powi\ n$
unfolding *eventually_at* **by** (*metis ‹0 < e› complex_powr_of_int dist_commute e_dist*)
qed (*simp add*: *assms complex_powr_of_int*)
also have $((\lambda z.\ g\ z\ powi\ n)\ has\_field\_derivative\ E)\ (at\ z\ within\ S)$
unfolding *E_def* **using** *gderiv assms* **by** (*auto intro*!: *derivative_eq_intros*)
finally show *?thesis*
by (*simp add*: *D_def*)
qed

lemma *field_differentiable_powr_of_int*:
fixes z :: *complex*
assumes $g\ field\_differentiable\ (at\ z\ within\ S)$ **and** $g\ z \neq 0$

shows $(\lambda z.\ g\ z\ powr\ of\_int\ n)$ *field_differentiable* $(at\ z\ within\ S)$
using *has_field_derivative_powr_of_int assms field_differentiable_def* **by** *blast*

lemma *holomorphic_on_powr_of_int* [*holomorphic_intros*]:
assumes *f holomorphic_on S* **and** $\bigwedge z.\ z{\in}S \Longrightarrow f\ z \neq 0$
shows $(\lambda z.\ (f\ z)\ powr\ of\_int\ n)$ *holomorphic_on S*
using *assms field_differentiable_powr_of_int holomorphic_on_def* **by** *auto*

lemma *has_field_derivative_powr_right* [*derivative_intros*]:
$w \neq 0 \Longrightarrow ((\lambda z.\ w\ powr\ z)\ has\_field\_derivative\ Ln\ w * w\ powr\ z)\ (at\ z)$
unfolding *powr_def* **by** (*intro derivative_eq_intros* | *simp*)+

lemma *field_differentiable_powr_right* [*derivative_intros*]:
fixes *w*::*complex*
shows $w \neq 0 \Longrightarrow (\lambda z.\ w\ powr\ z)$ *field_differentiable* $(at\ z)$
using *field_differentiable_def has_field_derivative_powr_right* **by** *blast*

lemma *holomorphic_on_powr_right* [*holomorphic_intros*]:
assumes *f holomorphic_on S*
shows $(\lambda z.\ w\ powr\ (f\ z))$ *holomorphic_on S*
proof (*cases* $w = 0$)
case *False*
with *assms* **show** *?thesis*
unfolding *holomorphic_on_def field_differentiable_def*
by (*metis* (*full_types*) *DERIV_chain'' has_field_derivative_powr_right*)
qed *simp*

lemma *holomorphic_on_divide_gen* [*holomorphic_intros*]:
assumes *f holomorphic_on S g holomorphic_on S* **and** $\bigwedge z\ z'.\ [\![z \in S;\ z' \in S]\!] \Longrightarrow g\ z = 0 \longleftrightarrow g\ z' = 0$
shows $(\lambda z.\ f\ z\ /\ g\ z)$ *holomorphic_on S*
by (*metis* (*no_types*, *lifting*) *assms division_ring_divide_zero holomorphic_on_divide holomorphic_transform*)

lemma *norm_powr_real_powr*:
$w \in \mathbb{R} \Longrightarrow 0 \leq Re\ w \Longrightarrow cmod\ (w\ powr\ z) = Re\ w\ powr\ Re\ z$
by (*metis dual_order.order_iff_strict norm_powr_real norm_zero of_real_0 of_real_Re powr_def*)

lemma *tendsto_powr_complex*:
fixes *f g* :: _ ⇒ *complex*
assumes *a*: $a \notin \mathbb{R}_{\leq 0}$
assumes *f*: $(f \longrightarrow a)\ F$ **and** *g*: $(g \longrightarrow b)\ F$
shows $((\lambda z.\ f\ z\ powr\ g\ z) \longrightarrow a\ powr\ b)\ F$
proof −
from *a* **have** [*simp*]: $a \neq 0$ **by** *auto*
from *f g a* **have** $((\lambda z.\ exp\ (g\ z * ln\ (f\ z))) \longrightarrow a\ powr\ b)\ F$ (**is** *?P*)
by (*auto intro*!: *tendsto_intros simp*: *powr_def*)
also **{**

```
    have eventually (λz. z ≠ 0) (nhds a)
      by (intro t1_space_nhds) simp_all
    with f have eventually (λz. f z ≠ 0) F using filterlim_iff by blast
  }
  hence ?P ⟷ ((λz. f z powr g z) ⟶ a powr b) F
    by (intro tendsto_cong refl) (simp_all add: powr_def mult_ac)
  finally show ?thesis .
qed

lemma tendsto_powr_complex_0:
  fixes f g :: 'a ⇒ complex
  assumes f: (f ⟶ 0) F and g: (g ⟶ b) F and b: Re b > 0
  shows   ((λz. f z powr g z) ⟶ 0) F
proof (rule tendsto_norm_zero_cancel)
  define h where
    h = (λz. if f z = 0 then 0 else exp (Re (g z) * ln (cmod (f z)) + abs (Im (g z))
* pi))
  {
    fix z :: 'a assume z: f z ≠ 0
    define c where c = abs (Im (g z)) * pi
    from mpi_less_Im_Ln[OF z] Im_Ln_le_pi[OF z]
      have abs (Im (Ln (f z))) ≤ pi by simp
    from mult_left_mono[OF this, of abs (Im (g z))]
      have abs (Im (g z) * Im (ln (f z))) ≤ c by (simp add: abs_mult c_def)
    hence −Im (g z) * Im (ln (f z)) ≤ c by simp
     hence norm (f z powr g z) ≤ h z by (simp add: powr_def field_simps h_def
c_def)
  }
  hence le: norm (f z powr g z) ≤ h z for z
    by (simp add: h_def)

  have g': (g ⟶ b) (inf F (principal {z. f z ≠ 0}))
    by (rule tendsto_mono[OF _ g]) simp_all
  have ((λx. norm (f x)) ⟶ 0) (inf F (principal {z. f z ≠ 0}))
    by (subst tendsto_norm_zero_iff, rule tendsto_mono[OF _ f]) simp_all
  moreover {
    have filterlim (λx. norm (f x)) (principal {0<..}) (principal {z. f z ≠ 0})
      by (auto simp: filterlim_def)
    hence filterlim (λx. norm (f x)) (principal {0<..}) (inf F (principal {z. f z ≠
0}))
      by (rule filterlim_mono) simp_all
  }
  ultimately have norm: filterlim (λx. norm (f x)) (at_right 0) (inf F (principal
{z. f z ≠ 0}))
    by (simp add: filterlim_inf at_within_def)

  have A: LIM x inf F (principal {z. f z ≠ 0}). Re (g x) * −ln (cmod (f x)) :>
at_top
    by (rule filterlim_tendsto_pos_mult_at_top tendsto_intros g' b
```

filterlim_compose[*OF filterlim_uminus_at_top_at_bot*] *filterlim_compose*[*OF ln_at_0*] *norm*)+
have *B*: *LIM x inf F* (*principal* {*z*. *f z* ≠ *0*}).
−|*Im* (*g x*)| ∗ *pi* + −(*Re* (*g x*) ∗ *ln* (*cmod* (*f x*))) :> *at_top*
by (*rule filterlim_tendsto_add_at_top tendsto_intros g'*)+ (*insert A*, *simp_all*)
have *C*: (*h* ⟶ *0*) *F* **unfolding** *h_def*
by (*intro filterlim_If tendsto_const filterlim_compose*[*OF exp_at_bot*])
(*insert B*, *auto simp*: *filterlim_uminus_at_bot algebra_simps*)
show ((λ*x*. *norm* (*f x powr g x*)) ⟶ *0*) *F*
by (*rule Lim_null_comparison*[*OF always_eventually C*]) (*insert le*, *auto*)
qed

lemma *tendsto_powr_complex'* [*tendsto_intros*]:
fixes *f g* :: _ ⇒ *complex*
assumes *a* ∉ ℝ≤0 ∨ (*a* = *0* ∧ *Re b* > *0*) **and** (*f* ⟶ *a*) *F* (*g* ⟶ *b*) *F*
shows ((λ*z*. *f z powr g z*) ⟶ *a powr b*) *F*
using *assms tendsto_powr_complex tendsto_powr_complex_0* **by** *fastforce*

lemma *tendsto_neg_powr_complex_of_real*:
assumes *filterlim f at_top F* **and** *Re s* < *0*
shows ((λ*x*. *complex_of_real* (*f x*) *powr s*) ⟶ *0*) *F*
proof −
have ((λ*x*. *norm* (*complex_of_real* (*f x*) *powr s*)) ⟶ *0*) *F*
proof (*rule Lim_transform_eventually*)
from *assms*(*1*) **have** *eventually* (λ*x*. *f x* ≥ *0*) *F*
by (*auto simp*: *filterlim_at_top*)
thus *eventually* (λ*x*. *f x powr Re s* = *norm* (*of_real* (*f x*) *powr s*)) *F*
by *eventually_elim* (*simp add*: *norm_powr_real_powr*)
from *assms* **show** ((λ*x*. *f x powr Re s*) ⟶ *0*) *F*
by (*intro tendsto_neg_powr*)
qed
thus *?thesis* **by** (*simp add*: *tendsto_norm_zero_iff*)
qed

lemma *tendsto_neg_powr_complex_of_nat*:
assumes *filterlim f at_top F* **and** *Re s* < *0*
shows ((λ*x*. *of_nat* (*f x*) *powr s*) ⟶ *0*) *F*
using *tendsto_neg_powr_complex_of_real* [*of real o f F s*]
proof −
have ((λ*x*. *of_real* (*real* (*f x*)) *powr s*) ⟶ *0*) *F* **using** *assms*(*2*)
by (*intro filterlim_compose*[*OF _ tendsto_neg_powr_complex_of_real*]
filterlim_compose[*OF _ assms*(*1*)] *filterlim_real_sequentially filterlim_ident*) *auto*
thus *?thesis* **by** *simp*
qed

lemma *continuous_powr_complex*:
assumes *f* (*netlimit F*) ∉ ℝ≤0 *continuous F f continuous F g*
shows *continuous F* (λ*z*. *f z powr g z* :: *complex*)

using *assms* **unfolding** *continuous_def* **by** (*intro tendsto_powr_complex*) *simp_all*

lemma *isCont_powr_complex* [*continuous_intros*]:
assumes $f\ z \notin \mathbb{R}_{\leq 0}$ *isCont f z isCont g z*
shows *isCont* ($\lambda z.\ f\ z\ powr\ g\ z :: complex$) *z*
using *assms* **unfolding** *isCont_def* **by** (*intro tendsto_powr_complex*) *simp_all*

lemma *continuous_on_powr_complex* [*continuous_intros*]:
assumes $A \subseteq \{z.\ Re\ (f\ z) \geq 0 \vee Im\ (f\ z) \neq 0\}$
assumes $\bigwedge z.\ z \in A \Longrightarrow f\ z = 0 \Longrightarrow Re\ (g\ z) > 0$
assumes *continuous_on A f continuous_on A g*
shows *continuous_on A* ($\lambda z.\ f\ z\ powr\ g\ z$)
unfolding *continuous_on_def*
proof
fix *z* **assume** *z*: $z \in A$
show $((\lambda z.\ f\ z\ powr\ g\ z) \longrightarrow f\ z\ powr\ g\ z)\ (at\ z\ within\ A)$
proof (*cases* $f\ z = 0$)
case *False*
from *assms*(*1*,*2*) *z* **have** $Re\ (f\ z) \geq 0 \vee Im\ (f\ z) \neq 0\ f\ z = 0 \longrightarrow Re\ (g\ z) > 0$ **by** *auto*
with *assms*(*3*,*4*) *z* **show** *?thesis*
by (*intro tendsto_powr_complex'*)
(*auto elim*!: *nonpos_Reals_cases simp*: *complex_eq_iff continuous_on_def*)
next
case *True*
with *assms z* **show** *?thesis*
by (*auto intro*!: *tendsto_powr_complex_0 simp*: *continuous_on_def*)
qed
qed

6.20.20 Some Limits involving Logarithms

lemma *lim_Ln_over_power*:
fixes *s*::*complex*
assumes $0 < Re\ s$
shows ($\lambda n.\ Ln\ (of\_nat\ n)\ /\ of\_nat\ n\ powr\ s$) $\longrightarrow 0$
proof (*simp add*: *lim_sequentially dist_norm*, *clarify*)
fix *e*::*real*
assume *e*: $0 < e$
have $\exists xo{>}0.\ \forall x{\geq}xo.\ 0 < e * 2 + (e * Re\ s * 2 - 2) * x + e * (Re\ s)^2 * x^2$
proof (*rule_tac x*=$2/(e * (Re\ s)^2)$ **in** *exI*, *safe*)
show $0 < 2\ /\ (e * (Re\ s)^2)$
using *e assms* **by** (*simp add*: *field_simps*)
next
fix *x*::*real*
assume *x*: $2\ /\ (e * (Re\ s)^2) \leq x$
have $2\ /\ (e * (Re\ s)^2) > 0$
using *e assms* **by** *simp*
with *x* **have** $x > 0$

```
    by linarith
  then have x * 2 ≤ e * (x^2 * (Re s)^2)
    using e assms x by (auto simp: power2_eq_square field_simps)
  also have ... < e * (2 + (x * (Re s * 2) + x^2 * (Re s)^2))
    using e assms ‹x > 0›
    by (auto simp: power2_eq_square field_simps add_pos_pos)
  finally show 0 < e * 2 + (e * Re s * 2 - 2) * x + e * (Re s)^2 * x^2
    by (auto simp: algebra_simps)
 qed
 then have ∃xo>0. ∀x≥xo. x / e < 1 + (Re s * x) + (1/2) * (Re s * x)^2
  using e  by (simp add: field_simps)
 then have ∃xo>0. ∀x≥xo. x / e < exp (Re s * x)
  using assms
  by (force intro: less_le_trans [OF _ exp_lower_Taylor_quadratic])
 then obtain xo where xo > 0 and xo: ⋀x. x ≥ xo ⟹ x < e * exp (Re s * x)
  using e  by (auto simp: field_simps)
 have norm (Ln (of_nat n) / of_nat n powr s) < e if n ≥ nat ⌈exp xo⌉ for n
 proof -
  have ln (real n) ≥ xo
   using that exp_gt_zero ln_ge_iff [of n] nat_ceiling_le_eq by fastforce
  then show ?thesis
  using e xo [of ln n]  by (auto simp: norm_divide norm_powr_real field_split_simps)
 qed
 then show ∃no. ∀n≥no. norm (Ln (of_nat n) / of_nat n powr s) < e
  by blast
qed

lemma lim_Ln_over_n: ((λn. Ln(of_nat n) / of_nat n) ⟶ 0) sequentially
 using lim_Ln_over_power [of 1] by simp

lemma lim_ln_over_power:
 fixes s :: real
 assumes 0 < s
 shows (λn. ln (real n) / real n powr s) ⟶ 0
proof -
 have (λn. ln (Suc n) / (Suc n) powr s) ⟶ 0
    using lim_Ln_over_power [of of_real s, THEN filterlim_sequentially_Suc
[THEN iffD2]] assms
  by (simp add: lim_sequentially dist_norm Ln_Reals_eq norm_powr_real_powr
norm_divide)
 then show ?thesis
   using filterlim_sequentially_Suc[of λn::nat. ln n / n powr s] by auto
qed

lemma lim_ln_over_n [tendsto_intros]: ((λn. ln(real_of_nat n) / of_nat n)
⟶ 0) sequentially
 using lim_ln_over_power [of 1] by auto

lemma lim_log_over_n [tendsto_intros]:
```

```
  (λn. log k n/n) ⟶ 0
proof −
  have ∗: log k n/n = (1/ln k) ∗ (ln n / n) for n
    unfolding log_def by auto
  have (λn. (1/ln k) ∗ (ln n / n)) ⟶ (1/ln k) ∗ 0
    by (intro tendsto_intros)
  then show ?thesis
    unfolding ∗ by auto
qed

lemma lim_1_over_complex_power:
  assumes 0 < Re s
  shows (λn. 1 / of_nat n powr s) ⟶ 0
proof (rule Lim_null_comparison)
  have ∀n>0. 3 ≤ n ⟶ 1 ≤ ln (real_of_nat n)
    using ln_272_gt_1
    by (force intro: order_trans [of _ ln (272/100)])
 then show ∀F x in sequentially. cmod (1 / of_nat x powr s) ≤ cmod (Ln (of_nat
x) / of_nat x powr s)
    by (auto simp: norm_divide field_split_simps eventually_sequentially)
  show (λn. cmod (Ln (of_nat n) / of_nat n powr s)) ⟶ 0
    using lim_Ln_over_power [OF assms] by (metis tendsto_norm_zero_iff)
qed

lemma lim_1_over_real_power:
  fixes s :: real
  assumes 0 < s
  shows ((λn. 1 / (of_nat n powr s)) ⟶ 0) sequentially
 using lim_1_over_complex_power [of of_real s, THEN filterlim_sequentially_Suc
[THEN iffD2]] assms
  apply (subst filterlim_sequentially_Suc [symmetric])
  by (simp add: lim_sequentially dist_norm Ln_Reals_eq norm_powr_real_powr
norm_divide)

lemma lim_1_over_Ln: (λn. 1 / Ln (complex_of_nat n)) ⟶ 0
proof (clarsimp simp add: lim_sequentially dist_norm norm_divide field_split_simps)
  fix r::real
  assume 0 < r
  have ir: inverse (exp (inverse r)) > 0
    by simp
  obtain n where n: 1 < of_nat n ∗ inverse (exp (inverse r))
    using ex_less_of_nat_mult [of _ 1, OF ir]
    by auto
  then have exp (inverse r) < of_nat n
    by (simp add: field_split_simps)
  then have ln (exp (inverse r)) < ln (of_nat n)
    by (metis exp_gt_zero less_trans ln_exp ln_less_cancel_iff)
  with ‹0 < r› have 1 < r ∗ ln (real_of_nat n)
    by (simp add: field_simps)
```

```
  moreover have n > 0 using n
    using neq0_conv by fastforce
  ultimately show ∃no. ∀k. Ln (of_nat k) ≠ 0 ⟶ no ≤ k ⟶ 1 < r * cmod
(Ln (of_nat k))
    using n ‹0 < r›
    by (rule_tac x=n in exI) (force simp: field_split_simps intro: less_le_trans)
qed

lemma lim_1_over_ln: (λn. 1 / ln (real n)) ⟶⟶ 0
  using lim_1_over_Ln [THEN filterlim_sequentially_Suc [THEN iffD2]]
  apply (subst filterlim_sequentially_Suc [symmetric])
  by (simp add: lim_sequentially dist_norm Ln_Reals_eq norm_powr_real_powr
norm_divide)

lemma lim_ln1_over_ln: (λn. ln(Suc n) / ln n) ⟶⟶ 1
proof (rule Lim_transform_eventually)
  have (λn. ln(1 + 1/n) / ln n) ⟶⟶ 0
  proof (rule Lim_transform_bound)
    show (inverse o real) ⟶⟶ 0
      by (metis comp_def lim_inverse_n lim_explicit)
    show ∀F n in sequentially. norm (ln (1 + 1 / n) / ln n) ≤ norm ((inverse ∘
real) n)
    proof
      fix n::nat
      assume n: 3 ≤ n
      then have ln 3 ≤ ln n and ln0: 0 ≤ ln n
        by auto
      with ln3_gt_1 have 1/ ln n ≤ 1
        by (simp add: field_split_simps)
      moreover have ln (1 + 1 / real n) ≤ 1/n
        by (simp add: ln_add_one_self_le_self)
      ultimately have ln (1 + 1 / real n) * (1 / ln n) ≤ (1/n) * 1
        by (intro mult_mono) (use n in auto)
      then show norm (ln (1 + 1 / n) / ln n) ≤ norm ((inverse ∘ real) n)
        by (simp add: field_simps ln0)
      qed
  qed
  then show (λn. 1 + ln(1 + 1/n) / ln n) ⟶⟶ 1
    by (metis (full_types) add.right_neutral tendsto_add_const_iff)
  show ∀F k in sequentially. 1 + ln (1 + 1 / k) / ln k = ln(Suc k) / ln k
    by (simp add: field_split_simps ln_div eventually_sequentiallyI [of 2])
qed

lemma lim_ln_over_ln1: (λn. ln n / ln(Suc n)) ⟶⟶ 1
  using tendsto_inverse [OF lim_ln1_over_ln] by force
```

6.20.21 Relation between Square Root and exp/ln, hence its derivative

lemma *csqrt_exp_Ln*:
assumes $z \neq 0$
shows *csqrt z* = *exp*(*Ln z* / *2*)
proof −
have (*exp* (*Ln z* / *2*))^2 = (*exp* (*Ln z*))
by (*metis exp_double nonzero_mult_div_cancel_left times_divide_eq_right zero_neq_numeral*)
also have ... = *z*
using *assms exp_Ln* **by** *blast*
finally have *csqrt z* = *csqrt* ((*exp* (*Ln z* / *2*))^2)
by *simp*
also have ... = *exp* (*Ln z* / *2*)
apply (*rule csqrt_square*)
using *cos_gt_zero_pi* [*of* (*Im* (*Ln z*) / *2*)] *Im_Ln_le_pi mpi_less_Im_Ln assms*
by (*fastforce simp*: *Re_exp Im_exp*)
finally show *?thesis* **using** *assms csqrt_square*
by *simp*
qed

lemma *csqrt_conv_powr*: *csqrt z* = *z powr* (*1*/*2*)
by (*auto simp*: *csqrt_exp_Ln powr_def*)

lemma *csqrt_mult*:
assumes *Arg z* + *Arg w* ∈ {−*pi*<..*pi*}
shows *csqrt* (*z* ∗ *w*) = *csqrt z* ∗ *csqrt w*
proof (*cases z* = *0* ∨ *w* = *0*)
case *False*
have *csqrt* (*z* ∗ *w*) = *exp* ((*ln* (*z* ∗ *w*)) / *2*)
using *False* **by** (*intro csqrt_exp_Ln*) *auto*
also have ... = *exp* ((*Ln z* + *Ln w*) / *2*)
using *False assms* **by** (*subst Ln_times_simple*) (*auto simp*: *Arg_eq_Im_Ln*)
also have (*Ln z* + *Ln w*) / *2* = *Ln z* / *2* + *Ln w* / *2*
by (*simp add*: *add_divide_distrib*)
also have *exp* ... = *csqrt z* ∗ *csqrt w*
using *False* **by** (*simp add*: *exp_add csqrt_exp_Ln*)
finally show *?thesis* **.**
qed *auto*

lemma *Arg_csqrt* [*simp*]: *Arg* (*csqrt z*) = *Arg z* / *2*
proof (*cases z* = *0*)
case *False*
have *Im* (*Ln z*) ∈ {−*pi*<..*pi*}
by (*simp add*: *False Im_Ln_le_pi mpi_less_Im_Ln*)
also have ... ⊆ {−*2*∗*pi*<..*2*∗*pi*}
by *auto*
finally show *?thesis*

using *False* **by** (*auto simp*: *csqrt_exp_Ln Arg_exp Arg_eq_Im_Ln*)
qed (*auto simp*: *Arg_zero*)

lemma *csqrt_inverse*:
z ∉ $\mathbb{R}_{\leq 0}$ ⟹ *csqrt* (*inverse z*) = *inverse* (*csqrt z*)
by (*metis Ln_inverse csqrt_eq_0 csqrt_exp_Ln divide_minus_left exp_minus inverse_nonzero_iff_nonzero*)

lemma *cnj_csqrt*: *z* ∉ $\mathbb{R}_{\leq 0}$ ⟹ *cnj*(*csqrt z*) = *csqrt*(*cnj z*)
by (*metis cnj_Ln complex_cnj_divide complex_cnj_numeral complex_cnj_zero_iff csqrt_eq_0 csqrt_exp_Ln exp_cnj*)

lemma *has_field_derivative_csqrt*:
assumes *z* ∉ $\mathbb{R}_{\leq 0}$
shows (*csqrt has_field_derivative inverse*(2 ∗ *csqrt z*)) (*at z*)
proof −
have *z*: *z* ≠ *0*
using *assms* **by** *auto*
then have ∗: *inverse z* = *inverse* (*2∗z*) ∗ *2*
by (*simp add*: *field_split_simps*)
have [*simp*]: *exp* (*Ln z* / *2*) ∗ *inverse z* = *inverse* (*csqrt z*)
by (*simp add*: *z field_simps csqrt_exp_Ln* [*symmetric*]) (*metis power2_csqrt power2_eq_square*)
have *Im z* = *0* ⟹ *0* < *Re z*
using *assms complex_nonpos_Reals_iff not_less* **by** *blast*
with *z* **have** ((λ*z*. *exp* (*Ln z* / *2*)) *has_field_derivative inverse* (*2* ∗ *csqrt z*)) (*at z*)
by (*force intro*: *derivative_eq_intros* ∗ *simp add*: *assms*)
then show *?thesis*
proof (*rule has_field_derivative_transform_within*)
show ⋀*x*. *dist x z* < *cmod z* ⟹ *exp* (*Ln x* / *2*) = *csqrt x*
by (*metis csqrt_exp_Ln dist_0_norm less_irrefl*)
qed (*use z* **in** *auto*)
qed

lemma *field_differentiable_at_csqrt*:
z ∉ $\mathbb{R}_{\leq 0}$ ⟹ *csqrt field_differentiable at z*
using *field_differentiable_def has_field_derivative_csqrt* **by** *blast*

lemma *field_differentiable_within_csqrt*:
z ∉ $\mathbb{R}_{\leq 0}$ ⟹ *csqrt field_differentiable* (*at z within s*)
using *field_differentiable_at_csqrt field_differentiable_within_subset* **by** *blast*

lemma *continuous_at_csqrt*:
z ∉ $\mathbb{R}_{\leq 0}$ ⟹ *continuous* (*at z*) *csqrt*
by (*simp add*: *field_differentiable_within_csqrt field_differentiable_imp_continuous_at*)

corollary *isCont_csqrt′* [*simp*]:
⟦*isCont f z*; *f z* ∉ $\mathbb{R}_{\leq 0}$⟧ ⟹ *isCont* (λ*x*. *csqrt* (*f x*)) *z*

by (*blast intro*: *isCont_o2* [*OF* _ *continuous_at_csqrt*])

lemma *continuous_within_csqrt*:
  $z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *continuous* (*at z within s*) *csqrt*
by (*simp add*: *field_differentiable_imp_continuous_at field_differentiable_within_csqrt*)

lemma *continuous_on_csqrt* [*continuous_intros*]:
  continuous_on $(-\mathbb{R}_{\leq 0})$ *csqrt*
by (*simp add*: *continuous_at_imp_continuous_on continuous_within_csqrt*)

lemma *holomorphic_on_csqrt* [*holomorphic_intros*]: *csqrt holomorphic_on* $-\mathbb{R}_{\leq 0}$
by (*simp add*: *field_differentiable_within_csqrt holomorphic_on_def*)

lemma *holomorphic_on_csqrt'* [*holomorphic_intros*]:
  f holomorphic_on A $\Longrightarrow$ $(\bigwedge z.\ z \in A \Longrightarrow f\ z \notin \mathbb{R}_{\leq 0}) \Longrightarrow$ $(\lambda z.\ csqrt\ (f\ z))$ *holomorphic_on A*
using *holomorphic_on_compose_gen*[*OF* _ *holomorphic_on_csqrt*, *of f A*] **by** (*auto simp*: *o_def*)

lemma *analytic_on_csqrt* [*analytic_intros*]: *csqrt analytic_on* $-\mathbb{R}_{\leq 0}$
using *holomorphic_on_csqrt* **by** (*subst analytic_on_open*) *auto*

lemma *analytic_on_csqrt'* [*analytic_intros*]:
f analytic_on A $\Longrightarrow$ $(\bigwedge z.\ z \in A \Longrightarrow f\ z \notin \mathbb{R}_{\leq 0}) \Longrightarrow$ $(\lambda z.\ csqrt\ (f\ z))$ *analytic_on A*
using *analytic_on_compose_gen*[*OF* _ *analytic_on_csqrt*, *of f A*] **by** (*auto simp*: *o_def*)

lemma *continuous_within_closed_nontrivial*:
  closed s $\Longrightarrow$ *a* $\notin$ *s* ==> *continuous* (*at a within s*) *f*
using *Compl_iff continuous_within_topological open_Compl* **by** *fastforce*

lemma *continuous_within_csqrt_posreal*:
  continuous (*at z within* $(\mathbb{R} \cap \{w.\ 0 \leq Re(w)\})$) *csqrt*
proof (*cases* $z \in \mathbb{R}_{\leq 0}$)
case *True*
then have [*simp*]: *Im z* = *0* **and** *0*: *Re z* < *0* $\vee$ *z* = *0*
using *complex_nonpos_Reals_iff complex_eq_iff* **by** *force+*
show *?thesis*
using *0*
proof
assume *Re z* < *0*
then show *?thesis*
by (*auto simp*: *continuous_within_closed_nontrivial* [*OF closed_Real_halfspace_Re_ge*])
next
assume *z* = *0*
moreover
have $\bigwedge e.\ 0 < e$
  $\Longrightarrow \forall x' \in \mathbb{R} \cap \{w.\ 0 \leq Re\ w\}.\ cmod\ x' < e\hat{}2 \longrightarrow cmod\ (csqrt\ x') < e$

by (*auto simp*: *Reals_def real_less_lsqrt*)
ultimately show *?thesis*
using *zero_less_power* **by** (*fastforce simp*: *continuous_within_eps_delta*)
qed
qed (*blast intro*: *continuous_within_csqrt*)

6.20.22 Complex arctangent

The branch cut gives standard bounds in the real case.

definition *Arctan* :: *complex* ⇒ *complex* **where**
Arctan ≡ λ*z*. (i/*2*) ∗ *Ln*((*1* − i∗*z*) / (*1* + i∗*z*))

lemma *Arctan_def_moebius*: *Arctan z* = i/*2* ∗ *Ln* (*moebius* (−i) *1* i *1 z*)
by (*simp add*: *Arctan_def moebius_def add_ac*)

lemma *Ln_conv_Arctan*:
assumes *z* ≠ −*1*
shows *Ln z* = −*2*∗i ∗ *Arctan* (*moebius 1* (− *1*) (− i) (− i) *z*)
proof −
have *Arctan* (*moebius 1* (− *1*) (− i) (− i) *z*) =
i/*2* ∗ *Ln* (*moebius* (− i) *1* i *1* (*moebius 1* (− *1*) (− i) (− i) *z*))
by (*simp add*: *Arctan_def_moebius*)
also from *assms* **have** i ∗ *z* ≠ i ∗ (−*1*) **by** (*subst mult_left_cancel*) *simp*
hence i ∗ *z* − −i ≠ *0* **by** (*simp add*: *eq_neg_iff_add_eq_0*)
from *moebius_inverse*′[*OF* _ *this*, *of 1 1*]
have *moebius* (− i) *1* i *1* (*moebius 1* (− *1*) (− i) (− i) *z*) = *z* **by** *simp*
finally show *?thesis* **by** (*simp add*: *field_simps*)
qed

lemma *Arctan_0* [*simp*]: *Arctan 0* = *0*
by (*simp add*: *Arctan_def*)

lemma *Im_complex_div_lemma*: *Im*((*1* − i∗*z*) / (*1* + i∗*z*)) = *0* ⟷ *Re z* = *0*
by (*auto simp*: *Im_complex_div_eq_0 algebra_simps*)

lemma *Re_complex_div_lemma*: *0* < *Re*((*1* − i∗*z*) / (*1* + i∗*z*)) ⟷ *norm z* < *1*
by (*simp add*: *Re_complex_div_gt_0 algebra_simps cmod_def power2_eq_square*)

lemma *tan_Arctan*:
assumes $z^2 \neq -1$
shows [*simp*]: *tan*(*Arctan z*) = *z*
proof −
obtain *1* + i∗*z* ≠ *0* *1* − i∗*z* ≠ *0*
by (*metis add_diff_cancel_left*′ *assms diff_0 i_times_eq_iff mult_cancel_left2 power2_i power2_minus right_minus_eq*)
then show *?thesis*
by (*simp add*: *Arctan_def tan_def sin_exp_eq cos_exp_eq exp_minus divide_simps*

```
      flip: csqrt_exp_Ln power2_eq_square)
qed

lemma Arctan_tan [simp]:
  assumes |Re z| < pi/2
    shows Arctan(tan z) = z
proof −
  have Ln ((1 − i ∗ tan z) / (1 + i ∗ tan z)) = 2 ∗ z / i
  proof (rule Ln_unique)
    have ge_pi2: ⋀n::int. |of_int (2∗n + 1) ∗ pi/2| ≥ pi/2
      by (case_tac n rule: int_cases) (auto simp: abs_mult)
    have exp (i∗z)∗exp (i∗z) = −1 ⟷ exp (2∗i∗z) = −1
      by (metis distrib_right exp_add mult_2)
    also have ... ⟷ exp (2∗i∗z) = exp (i∗pi)
      using cis_conv_exp cis_pi by auto
    also have ... ⟷ exp (2∗i∗z − i∗pi) = 1
    by (metis (no_types) diff_add_cancel diff_minus_eq_add exp_add exp_minus_inverse
mult.commute)
     also have ... ⟷ Re(i∗2∗z − i∗pi) = 0 ∧ (∃ n::int. Im(i∗2∗z − i∗pi) =
of_int (2 ∗ n) ∗ pi)
      by (simp add: exp_eq_1)
    also have ... ⟷ Im z = 0 ∧ (∃ n::int. 2 ∗ Re z = of_int (2∗n + 1) ∗ pi)
      by (simp add: algebra_simps)
    also have ... ⟷ False
      using assms ge_pi2
     by (metis eq_divide_eq linorder_not_less mult.commute zero_neq_numeral)
    finally have exp (i∗z)∗exp (i∗z) + 1 ≠ 0
      by (auto simp: add.commute minus_unique)
    then show exp (2 ∗ z / i) = (1 − i ∗ tan z) / (1 + i ∗ tan z)
     apply (simp add: tan_def sin_exp_eq cos_exp_eq exp_minus divide_simps)
      by (simp add: algebra_simps flip: power2_eq_square exp_double)
  qed (use assms in auto)
  then show ?thesis
    by (auto simp: Arctan_def)
qed

lemma
  assumes Re z = 0 ⟹ |Im z| < 1
  shows Re_Arctan_bounds: |Re(Arctan z)| < pi/2
    and has_field_derivative_Arctan: (Arctan has_field_derivative inverse(1 +
z^2)) (at z)
proof −
  have nz0: 1 + i∗z ≠ 0
    using assms
   by (metis abs_one add_diff_cancel_left' complex_i_mult_minus diff_0 i_squared
imaginary_unit.simps
              less_asym neg_equal_iff_equal)
  have z ≠ −i using assms
    by auto
```

```
  then have zz: 1 + z * z ≠ 0
     by (metis abs_one assms i_squared imaginary_unit.simps less_irrefl mi-
nus_unique square_eq_iff)
  have nz1: 1 − i*z ≠ 0
    using assms by (force simp add: i_times_eq_iff)
  have nz2: inverse (1 + i*z) ≠ 0
    using assms
    by (metis Im_complex_div_lemma Re_complex_div_lemma cmod_eq_Im di-
vide_complex_def
         less_irrefl mult_zero_right zero_complex.simps(1) zero_complex.simps(2))
  have nzi: ((1 − i*z) * inverse (1 + i*z)) ≠ 0
    using nz1 nz2 by auto
  have Im ((1 − i*z) / (1 + i*z)) = 0 ⟹ 0 < Re ((1 − i*z) / (1 + i*z))
   by (simp add: Im_complex_div_lemma Re_complex_div_lemma assms cmod_eq_Im)
  then have *: ((1 − i*z) / (1 + i*z)) ∉ ℝ≤0
    by (auto simp add: complex_nonpos_Reals_iff)
  show |Re(Arctan z)| < pi/2
    unfolding Arctan_def divide_complex_def
    using mpi_less_Im_Ln [OF nzi]
    by (auto simp: abs_if intro!: Im_Ln_less_pi * [unfolded divide_complex_def])
  show (Arctan has_field_derivative inverse(1 + z^2)) (at z)
    unfolding Arctan_def scaleR_conv_of_real
    apply (intro derivative_eq_intros | simp add: nz0 *)+
    using nz1 zz
    apply (simp add: field_split_simps power2_eq_square)
    apply algebra
    done
qed

lemma field_differentiable_at_Arctan: (Re z = 0 ⟹ |Im z| < 1) ⟹ Arctan
field_differentiable at z
  using has_field_derivative_Arctan
  by (auto simp: field_differentiable_def)

lemma field_differentiable_within_Arctan:
    (Re z = 0 ⟹ |Im z| < 1) ⟹ Arctan field_differentiable (at z within s)
  using field_differentiable_at_Arctan field_differentiable_at_within by blast

declare has_field_derivative_Arctan [derivative_intros]
declare has_field_derivative_Arctan [THEN DERIV_chain2, derivative_intros]

lemma continuous_at_Arctan:
    (Re z = 0 ⟹ |Im z| < 1) ⟹ continuous (at z) Arctan
 by (simp add: field_differentiable_imp_continuous_at field_differentiable_within_Arctan)

lemma continuous_within_Arctan:
    (Re z = 0 ⟹ |Im z| < 1) ⟹ continuous (at z within s) Arctan
  using continuous_at_Arctan continuous_at_imp_continuous_within by blast
```

lemma *continuous_on_Arctan* [*continuous_intros*]:
(⋀*z*. *z* ∈ *s* ⟹ *Re z* = *0* ⟹ |*Im z*| < *1*) ⟹ *continuous_on s Arctan*
by (*auto simp*: *continuous_at_imp_continuous_on continuous_within_Arctan*)

lemma *holomorphic_on_Arctan*:
(⋀*z*. *z* ∈ *s* ⟹ *Re z* = *0* ⟹ |*Im z*| < *1*) ⟹ *Arctan holomorphic_on s*
by (*simp add*: *field_differentiable_within_Arctan holomorphic_on_def*)

theorem *Arctan_series*:
assumes *z*: *norm* (*z* :: *complex*) < *1*
defines *g* ≡ λ*n*. *if odd n then* −i∗i^*n* / *n else 0*
defines *h* ≡ λ*z n*. (−*1*)^*n* / *of_nat* (*2*∗*n*+*1*) ∗ (*z*::*complex*)^(*2*∗*n*+*1*)
shows (λ*n*. *g n* ∗ *z*^*n*) *sums Arctan z*
and *h z sums Arctan z*
proof −
define *G* **where** [*abs_def*]: *G z* = (∑ *n*. *g n* ∗ *z*^*n*) **for** *z*
have *summable*: *summable* (λ*n*. *g n* ∗ *u*^*n*) **if** *norm u* < *1* **for** *u*
proof (*cases u* = *0*)
case *False*
have (λ*n*. *ereal* (*norm* (*h u n*) / *norm* (*h u* (*Suc n*)))) = (λ*n*. *ereal* (*inverse* (*norm u*)^*2*) ∗
ereal ((*2* + *inverse* (*real* (*Suc n*))) / (*2* − *inverse* (*real* (*Suc n*)))))
proof
fix *n*
have *ereal* (*norm* (*h u n*) / *norm* (*h u* (*Suc n*))) =
ereal (*inverse* (*norm u*)^*2*) ∗ *ereal* (((*2*∗*Suc n*+*1*) / (*Suc n*)) /
((*2*∗*Suc n*−*1*) / (*Suc n*)))
by (*simp add*: *h_def norm_mult norm_power norm_divide field_split_simps*
power2_eq_square eval_nat_numeral del: *of_nat_add of_nat_Suc*)
also have *of_nat* (*2*∗*Suc n*+*1*) / *of_nat* (*Suc n*) = (*2*::*real*) + *inverse* (*real* (*Suc n*))
by (*auto simp*: *field_split_simps simp del*: *of_nat_Suc*) *simp_all?*
also have *of_nat* (*2*∗*Suc n*−*1*) / *of_nat* (*Suc n*) = (*2*::*real*) − *inverse* (*real* (*Suc n*))
by (*auto simp*: *field_split_simps simp del*: *of_nat_Suc*) *simp_all?*
finally show *ereal* (*norm* (*h u n*) / *norm* (*h u* (*Suc n*))) = *ereal* (*inverse* (*norm u*)^*2*) ∗
ereal ((*2* + *inverse* (*real* (*Suc n*))) / (*2* − *inverse* (*real* (*Suc n*)))) .
qed
also have ... ⟶ *ereal* (*inverse* (*norm u*)^*2*) ∗ *ereal* ((*2* + *0*) / (*2* − *0*))
by (*intro tendsto_intros LIMSEQ_inverse_real_of_nat*) *simp_all*
finally have *liminf* (λ*n*. *ereal* (*cmod* (*h u n*) / *cmod* (*h u* (*Suc n*)))) = *inverse* (*norm u*)^*2*
by (*intro lim_imp_Liminf*) *simp_all*
moreover from *power_strict_mono*[*OF that*, *of 2*] *False* **have** *inverse* (*norm u*)^*2* > *1*
by (*simp add*: *field_split_simps*)
ultimately have *A*: *liminf* (λ*n*. *ereal* (*cmod* (*h u n*) / *cmod* (*h u* (*Suc n*)))) > *1* **by** *simp*

```
    from False have summable (h u)
      by (intro summable_norm_cancel[OF ratio_test_convergence[OF _ A]])
       (auto simp: h_def norm_divide norm_mult norm_power simp del: of_nat_Suc
             intro!: mult_pos_pos divide_pos_pos always_eventually)
    thus summable (λn. g n * u^n)
      by (subst summable_mono_reindex[of λn. 2*n+1, symmetric])
        (auto simp: power_mult strict_mono_def g_def h_def elim!: oddE)
  qed (simp add: h_def)

  have ∃ c. ∀ u∈ball 0 1. Arctan u - G u = c
  proof (rule has_field_derivative_zero_constant)
    fix u :: complex assume u ∈ ball 0 1
    hence u: norm u < 1 by (simp)
    define K where K = (norm u + 1) / 2
    from u and abs_Im_le_cmod[of u] have Im_u: |Im u| < 1 by linarith
    from u have K: 0 ≤ K norm u < K K < 1 by (simp_all add: K_def)
     hence (G has_field_derivative (∑ n. diffs g n * u ^ n)) (at u) unfolding
G_def
      by (intro termdiffs_strong[of _ of_real K] summable) simp_all
    also have (λn. diffs g n * u^n) = (λn. if even n then (i*u)^n else 0)
     by (intro ext) (simp_all del: of_nat_Suc add: g_def diffs_def power_mult_distrib)
    also have suminf ... = (∑ n. (-(u^2))^n)
      by (subst suminf_mono_reindex[of λn. 2*n, symmetric])
       (auto elim!: evenE simp: strict_mono_def power_mult power_mult_distrib)
    also from u have norm u^2 < 1^2 by (intro power_strict_mono) simp_all
    hence (∑ n. (-(u^2))^n) = inverse (1 + u^2)
     by (subst suminf_geometric) (simp_all add: norm_power inverse_eq_divide)
    finally have (G has_field_derivative inverse (1 + u^2)) (at u) .
    from DERIV_diff[OF has_field_derivative_Arctan this] Im_u u
      show ((λu. Arctan u - G u) has_field_derivative 0) (at u within ball 0 1)
      by (simp_all add: at_within_open[OF _ open_ball])
  qed simp_all
  then obtain c where c: ⋀u. norm u < 1 ⟹ Arctan u - G u = c by auto
  from this[of 0] have c = 0 by (simp add: G_def g_def)
  with c z have Arctan z = G z by simp
  with summable[OF z] show (λn. g n * z^n) sums Arctan z unfolding G_def
by (simp add: sums_iff)
  thus h z sums Arctan z by (subst (asm) sums_mono_reindex[of λn. 2*n+1,
symmetric])
                            (auto elim!: oddE simp: strict_mono_def power_mult
g_def h_def)
qed
```

A quickly-converging series for the logarithm, based on the arctangent.

```
theorem ln_series_quadratic:
  assumes x: x > (0::real)
  shows (λn. (2*((x - 1) / (x + 1)) ^(2*n+1) / of_nat (2*n+1))) sums ln x
proof -
  define y :: complex where y = of_real ((x-1)/(x+1))
```

from x **have** x': *complex_of_real* $x \neq$ *of_real* (-1) **by** (*subst of_real_eq_iff*) *auto*
from x **have** $|x - 1| < |x + 1|$ **by** *linarith*
hence *norm* (*complex_of_real* $(x - 1)$ / *complex_of_real* $(x + 1)) < 1$
by (*simp add*: *norm_divide del*: *of_real_add of_real_diff*)
hence *norm* $(\mathrm{i} * y) < 1$ **unfolding** *y_def* **by** (*subst norm_mult*) *simp*
hence $(\lambda n.\ (-2{*}\mathrm{i}) * ((-1)\hat{}n\ /\ of\_nat\ (2{*}n{+}1) * (\mathrm{i}{*}y)\hat{}(2{*}n{+}1)))$ *sums* $((-2{*}\mathrm{i}) * Arctan\ (\mathrm{i}{*}y))$
by (*intro Arctan_series sums_mult*) *simp_all*
also have $(\lambda n.\ (-2{*}\mathrm{i}) * ((-1)\hat{}n\ /\ of\_nat\ (2{*}n{+}1) * (\mathrm{i}{*}y)\hat{}(2{*}n{+}1))) =$
$(\lambda n.\ (-2{*}\mathrm{i}) * ((-1)\hat{}n * (\mathrm{i}{*}y{*}(-y^2)\hat{}n)/of\_nat\ (2{*}n{+}1)))$
by (*intro ext*) (*simp_all add*: *power_mult power_mult_distrib*)
also have $\ldots = (\lambda n.\ 2{*}y{*}\ ((-1) * (-y^2))\hat{}n/of\_nat\ (2{*}n{+}1))$
by (*intro ext*, *subst power_mult_distrib*) (*simp add*: *algebra_simps power_mult*)
also have $\ldots = (\lambda n.\ 2{*}y\hat{}(2{*}n{+}1)\ /\ of\_nat\ (2{*}n{+}1))$
by (*subst power_add*, *subst power_mult*) (*simp add*: *mult_ac*)
also have $\ldots = (\lambda n.\ of\_real\ (2{*}((x{-}1)/(x{+}1))\hat{}(2{*}n{+}1)\ /\ of\_nat\ (2{*}n{+}1)))$
by (*intro ext*) (*simp add*: *y_def*)
also have $\mathrm{i} * y = (of\_real\ x - 1)\ /\ (-\mathrm{i} * (of\_real\ x + 1))$
by (*subst divide_divide_eq_left* [*symmetric*]) (*simp add*: *y_def*)
also have $\ldots = moebius\ 1\ (-1)\ (-\mathrm{i})\ (-\mathrm{i})\ (of\_real\ x)$ **by** (*simp add*: *moebius_def algebra_simps*)
also from x' **have** $-2{*}\mathrm{i}{*}Arctan\ \ldots = Ln\ (of\_real\ x)$ **by** (*intro Ln_conv_Arctan* [*symmetric*]) *simp_all*
also from x **have** $\ldots = ln\ x$ **by** (*rule Ln_of_real*)
finally show *?thesis* **by** (*subst* (*asm*) *sums_of_real_iff*)
qed

6.20.23 Real arctangent

lemma *Im_Arctan_of_real* [*simp*]: *Im* (*Arctan* (*of_real x*)) = *0*
proof −
have *ne*: $1 + x^2 \neq 0$
by (*metis power_one sum_power2_eq_zero_iff zero_neq_one*)
have *ne1*: $1 + \mathrm{i} * of\_complex\ x \neq 0$
using *Complex_eq complex_eq_cancel_iff2* **by** *fastforce*
have *Re* (*Ln* $((1 - \mathrm{i} * x) * inverse\ (1 + \mathrm{i} * x))) = 0$
apply (*rule norm_exp_imaginary*)
using *ne*
apply (*simp add*: *ne1 cmod_def*)
apply (*auto simp*: *field_split_simps*)
apply *algebra*
done
then show *?thesis*
unfolding *Arctan_def divide_complex_def* **by** (*simp add*: *complex_eq_iff*)
qed

lemma *arctan_eq_Re_Arctan*: *arctan x* = *Re* (*Arctan* (*of_real x*))
proof (*rule arctan_unique*)

```
  have (1 − i ∗ x) / (1 + i ∗ x) ∉ ℝ≤0
    by (auto simp: Im_complex_div_lemma complex_nonpos_Reals_iff)
  then show − (pi / 2) < Re (Arctan (complex_of_real x))
    by (simp add: Arctan_def Im_Ln_less_pi)
next
  have ∗: (1 − i∗x) / (1 + i∗x) ≠ 0
    by (simp add: field_split_simps) ( simp add: complex_eq_iff)
  show Re (Arctan (complex_of_real x)) < pi / 2
    using mpi_less_Im_Ln [OF ∗]
    by (simp add: Arctan_def)
next
  have tan (Re (Arctan (of_real x))) = Re (tan (Arctan (of_real x)))
   by (metis Im_Arctan_of_real Re_complex_of_real complex_is_Real_iff of_real_Re
tan_of_real)
  also have ... = x
  proof −
    have (complex_of_real x)^2 ≠ − 1
    by (smt (verit, best) Im_complex_of_real imaginary_unit.sel(2) of_real_minus
power2_eq_iff power2_i)
    then show ?thesis
      by simp
  qed
  finally show tan (Re (Arctan (complex_of_real x))) = x .
qed

lemma Arctan_of_real: Arctan (of_real x) = of_real (arctan x)
  unfolding arctan_eq_Re_Arctan divide_complex_def
  by (simp add: complex_eq_iff)

lemma Arctan_in_Reals [simp]: z ∈ ℝ ⟹ Arctan z ∈ ℝ
  by (metis Reals_cases Reals_of_real Arctan_of_real)

declare arctan_one [simp]

lemma arctan_less_pi4_pos: x < 1 ⟹ arctan x < pi/4
  by (metis arctan_less_iff arctan_one)

lemma arctan_less_pi4_neg: −1 < x ⟹ −(pi/4) < arctan x
  by (metis arctan_less_iff arctan_minus arctan_one)

lemma arctan_less_pi4: |x| < 1 ⟹ |arctan x| < pi/4
  by (metis abs_less_iff arctan_less_pi4_pos arctan_minus)

lemma arctan_le_pi4: |x| ≤ 1 ⟹ |arctan x| ≤ pi/4
  by (metis abs_le_iff arctan_le_iff arctan_minus arctan_one)

lemma abs_arctan: |arctan x| = arctan |x|
  by (simp add: abs_if arctan_minus)
```

lemma *arctan_add_raw*:
assumes $|arctan\ x + arctan\ y| < pi/2$
shows $arctan\ x + arctan\ y = arctan((x + y) / (1 - x * y))$
proof (*rule arctan_unique* [*symmetric*])
show $-(pi / 2) < arctan\ x + arctan\ y\ arctan\ x + arctan\ y < pi / 2$
using *assms* **by** *linarith*+
show $tan\ (arctan\ x + arctan\ y) = (x + y) / (1 - x * y)$
using *cos_gt_zero_pi* [*OF 12*] **by** (*simp add*: *arctan tan_add*)
qed

lemma *arctan_inverse*:
$0 < x \Longrightarrow arctan(inverse\ x) = pi/2 - arctan\ x$
by (*smt* (*verit*, *del_insts*) *arctan arctan_unique tan_cot zero_less_arctan_iff*)

lemma *arctan_add_small*:
assumes $|x * y| < 1$
shows $(arctan\ x + arctan\ y = arctan((x + y) / (1 - x * y)))$
proof (*cases* $x = 0 \vee y = 0$)
case *False*
with *assms* **have** $|x| < inverse\ |y|$
by (*simp add*: *field_split_simps abs_mult*)
with *False* **have** $|arctan\ x| < pi / 2 - |arctan\ y|$ **using** *assms*
by (*auto simp add*: *abs_arctan arctan_inverse* [*symmetric*] *arctan_less_iff*)
then show *?thesis*
by (*intro arctan_add_raw*) *linarith*
qed *auto*

lemma *abs_arctan_le*:
fixes x::*real* **shows** $|arctan\ x| \leq |x|$
proof −
have *1*: $\bigwedge x.\ x \in \mathbb{R} \Longrightarrow cmod\ (inverse\ (1 + x^2)) \leq 1$
by (*simp add*: *norm_divide divide_simps in_Reals_norm complex_is_Real_iff power2_eq_square*)
have *cmod* (*Arctan w − Arctan z*) $\leq 1 * cmod\ (w-z)$ **if** $w \in \mathbb{R}$ $z \in \mathbb{R}$ **for** $w\ z$
apply (*rule field_differentiable_bound* [*OF convex_Reals*, *of Arctan _ 1*])
apply (*rule has_field_derivative_at_within* [*OF has_field_derivative_Arctan*])
using *1 that* **by** (*auto simp*: *Reals_def*)
then have *cmod* (*Arctan* (*of_real x*) − *Arctan 0*) $\leq 1 * cmod\ (of\_real\ x - 0)$
using *Reals_0 Reals_of_real* **by** *blast*
then show *?thesis*
by (*simp add*: *Arctan_of_real*)
qed

lemma *arctan_le_self*: $0 \leq x \Longrightarrow arctan\ x \leq x$
by (*metis abs_arctan_le abs_of_nonneg zero_le_arctan_iff*)

lemma *abs_tan_ge*: $|x| < pi/2 \Longrightarrow |x| \leq |tan\ x|$
by (*metis abs_arctan_le abs_less_iff arctan_tan minus_less_iff*)

lemma *arctan_bounds*:
  assumes $0 \leq x$ $x < 1$
  shows *arctan_lower_bound*:
    $(\sum k<2 * n.\ (-1)\ \hat{}\ k * (1\ /\ real\ (k * 2 + 1) * x\ \hat{}\ (k * 2 + 1))) \leq arctan\ x$ (**is** $(\sum k<\_.\ \_ * ?a\ k) \leq \_$)
    and *arctan_upper_bound*:
    $arctan\ x \leq (\sum k<2 * n + 1.\ (-1)\ \hat{}\ k * (1\ /\ real\ (k * 2 + 1) * x\ \hat{}\ (k * 2 + 1)))$
proof −
  have *tendsto_zero*: $?a \longrightarrow 0$
  proof (*rule tendsto_eq_rhs*)
    show $(\lambda k.\ 1\ /\ real\ (k * 2 + 1) * x\ \hat{}\ (k * 2 + 1)) \longrightarrow 0 * 0$
      using *assms*
      by (*intro tendsto_mult real_tendsto_divide_at_top*)
        (*auto simp*: *filterlim_sequentially_iff_filterlim_real*
       intro!: *real_tendsto_divide_at_top tendsto_power_zero filterlim_real_sequentially*
       tendsto_eq_intros filterlim_at_top_mult_tendsto_pos filterlim_tendsto_add_at_top)
  qed *simp*
  have *nonneg*: $0 \leq ?a\ n$ **for** *n*
    by (*force intro*!: *divide_nonneg_nonneg mult_nonneg_nonneg zero_le_power assms*)
  have *le*: $?a\ (Suc\ n) \leq ?a\ n$ **for** *n*
    by (*rule mult_mono*[*OF _ power_decreasing*]) (*auto simp*: *field_split_simps assms less_imp_le*)
  from *summable_Leibniz'*(*4*)[*of ?a*, *OF tendsto_zero nonneg le*, *of n*]
    summable_Leibniz'(*2*)[*of ?a*, *OF tendsto_zero nonneg le*, *of n*]
    assms
  show $(\sum k<2*n.\ (-1)\ \hat{}\ k * ?a\ k) \leq arctan\ x\ arctan\ x \leq (\sum k<2 * n + 1.\ (-1)\ \hat{}\ k * ?a\ k)$
    by (*auto simp*: *arctan_series*)
qed

6.20.24 Bounds on pi using real arctangent

lemma *pi_machin*: $pi = 16 * arctan\ (1\ /\ 5) - 4 * arctan\ (1\ /\ 239)$
  using *machin* **by** *simp*

lemma *pi_approx*: $3.141592653588 \leq pi\ pi \leq 3.1415926535899$
  unfolding *pi_machin*
  using *arctan_bounds*[*of 1/5 4*]
      arctan_bounds[*of 1/239 4*]
  by (*simp_all add*: *eval_nat_numeral*)

lemma *pi_gt3*: $pi > 3$
  using *pi_approx* **by** *simp*

6.20.25 Inverse Sine

definition *Arcsin* :: *complex* ⇒ *complex* **where**
  $Arcsin \equiv \lambda z.\ -\mathrm{i} * Ln(\mathrm{i} * z + csqrt(1 - z^2))$

lemma *Arcsin_body_lemma*: $\mathrm{i} * z + csqrt(1 - z^2) \neq 0$
using *power2_csqrt* [*of* $1 - z^2$]
by (*metis add.inverse_unique diff_0 diff_add_cancel mult.left_commute mult_minus1_right power2_i power2_minus power_mult_distrib zero_neq_one*)

lemma *Arcsin_range_lemma*: $|Re\ z| < 1 \Longrightarrow 0 < Re(\mathrm{i} * z + csqrt(1 - z^2))$
using *Complex.cmod_power2* [*of z*, *symmetric*]
by (*simp add*: *real_less_rsqrt algebra_simps Re_power2 cmod_square_less_1_plus*)

lemma *Re_Arcsin*: $Re(Arcsin\ z) = Im\ (Ln\ (\mathrm{i} * z + csqrt(1 - z^2)))$
by (*simp add*: *Arcsin_def*)

lemma *Im_Arcsin*: $Im(Arcsin\ z) = -\ ln\ (cmod\ (\mathrm{i} * z + csqrt\ (1 - z^2)))$
by (*simp add*: *Arcsin_def Arcsin_body_lemma*)

lemma *one_minus_z2_notin_nonpos_Reals*:
assumes $Im\ z = 0 \Longrightarrow |Re\ z| < 1$
shows $1 - z^2 \notin \mathbb{R}_{\leq 0}$
proof (*cases Im z = 0*)
case *True*
with *assms* **show** *?thesis*
by (*simp add*: *complex_nonpos_Reals_iff flip*: *abs_square_less_1*)
next
case *False*
have $\neg\ (Im\ z)^2 \leq -\ 1$
using *False power2_less_eq_zero_iff* **by** *fastforce*
with *False* **show** *?thesis*
by (*auto simp add*: *complex_nonpos_Reals_iff Re_power2 Im_power2*)
qed

lemma *isCont_Arcsin_lemma*:
assumes *le0*: $Re\ (\mathrm{i} * z + csqrt\ (1 - z^2)) \leq 0$ **and** $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *False*
proof (*cases Im z = 0*)
case *True*
then show *?thesis*
using *assms* **by** (*fastforce simp*: *cmod_def abs_square_less_1* [*symmetric*])
next
case *False*
have *leim*: $(cmod\ (1 - z^2) + (1 - Re\ (z^2))) / 2 \leq (Im\ z)^2$
using *le0 sqrt_le_D* **by** *fastforce*
have *neq*: $(cmod\ z)^2 \neq 1 + cmod\ (1 - z^2)$
proof (*clarsimp simp add*: *cmod_def*)
assume $(Re\ z)^2 + (Im\ z)^2 = 1 + sqrt\ ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
then have $((Re\ z)^2 + (Im\ z)^2 - 1)^2 = ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
by *simp*
then show *False* **using** *False*
by (*simp add*: *power2_eq_square algebra_simps*)

qed
moreover have *2*: $(Im\ z)^2 = (1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2)\ /\ 2$
using *leim cmod_power2* [*of z*] *norm_triangle_ineq2* [*of z^2 1*]
by (*simp add*: *norm_power Re_power2 norm_minus_commute* [*of 1*])
ultimately show *False*
by (*simp add*: *Re_power2 Im_power2 cmod_power2*)
qed

lemma *isCont_Arcsin*:
assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *isCont Arcsin z*
proof −
have *1*: $\mathrm{i} * z + csqrt\ (1 - z^2) \notin \mathbb{R}_{\leq 0}$
by (*metis isCont_Arcsin_lemma assms complex_nonpos_Reals_iff*)
have *2*: $1 - z^2 \notin \mathbb{R}_{\leq 0}$
by (*simp add*: *one_minus_z2_notin_nonpos_Reals assms*)
show *?thesis*
using *assms* **unfolding** *Arcsin_def* **by** (*intro isCont_Ln′ isCont_csqrt′ continuous_intros 1 2*)
qed

lemma *isCont_Arcsin′* [*simp*]:
shows $isCont\ f\ z \Longrightarrow (Im\ (f\ z) = 0 \Longrightarrow |Re\ (f\ z)| < 1) \Longrightarrow isCont\ (\lambda x.\ Arcsin\ (f\ x))\ z$
by (*blast intro*: *isCont_o2* [*OF _ isCont_Arcsin*])

lemma *sin_Arcsin* [*simp*]: $sin(Arcsin\ z) = z$
proof −
have $\mathrm{i}*z*2 + csqrt\ (1 - z^2)*2 = 0 \longleftrightarrow (\mathrm{i}*z)*2 + csqrt\ (1 - z^2)*2 = 0$
by (*simp add*: *algebra_simps*) — Cancelling a factor of 2
moreover have $\ldots \longleftrightarrow (\mathrm{i}*z) + csqrt\ (1 - z^2) = 0$
by (*metis Arcsin_body_lemma distrib_right no_zero_divisors zero_neq_numeral*)
ultimately show *?thesis*
apply (*simp add*: *sin_exp_eq Arcsin_def Arcsin_body_lemma exp_minus divide_simps*)
apply (*simp add*: *algebra_simps*)
apply (*simp add*: *right_diff_distrib flip*: *power2_eq_square*)
done
qed

lemma *Re_eq_pihalf_lemma*:
$|Re\ z| = pi/2 \Longrightarrow Im\ z = 0 \Longrightarrow$
$Re\ ((exp\ (\mathrm{i}*z) + inverse\ (exp\ (\mathrm{i}*z)))\ /\ 2) = 0 \wedge 0 \leq Im\ ((exp\ (\mathrm{i}*z) + inverse\ (exp\ (\mathrm{i}*z)))\ /\ 2)$
apply (*simp add*: *cos_i_times* [*symmetric*] *Re_cos Im_cos abs_if del*: *eq_divide_eq_numeral1*)
by (*metis cos_minus cos_pi_half*)

lemma *Re_less_pihalf_lemma*:
assumes $|Re\ z| < pi\ /\ 2$

shows *0 < Re ((exp (i∗z) + inverse (exp (i∗z))) / 2)*
proof −
have *0 < cos (Re z)* **using** *assms*
using *cos_gt_zero_pi* **by** *auto*
then show *?thesis*
by (*simp add*: *cos_i_times* [*symmetric*] *Re_cos Im_cos add_pos_pos*)
qed

lemma *Arcsin_sin*:
assumes *|Re z| < pi/2 ∨ (|Re z| = pi/2 ∧ Im z = 0)*
shows *Arcsin(sin z) = z*
proof −
have *Arcsin(sin z) = − (i ∗ Ln (csqrt (1 − (i ∗ (exp (i∗z) − inverse (exp (i∗z))))^2 / 4) − (inverse (exp (i∗z)) − exp (i∗z)) / 2))*
by (*simp add*: *sin_exp_eq Arcsin_def exp_minus power_divide*)
also have *... = − (i ∗ Ln (csqrt (((exp (i∗z) + inverse (exp (i∗z)))/2)^2) − (inverse (exp (i∗z)) − exp (i∗z)) / 2))*
by (*simp add*: *field_simps power2_eq_square*)
also have *... = − (i ∗ Ln (((exp (i∗z) + inverse (exp (i∗z)))/2) − (inverse (exp (i∗z)) − exp (i∗z)) / 2))*
apply (*subst csqrt_square*)
using *assms Re_eq_pihalf_lemma Re_less_pihalf_lemma* **by** *auto*
also have *... = − (i ∗ Ln (exp (i∗z)))*
by (*simp add*: *field_simps power2_eq_square*)
also have *... = z*
using *assms* **by** (*auto simp*: *abs_if simp del*: *eq_divide_eq_numeral1 split*: *if_split_asm*)
finally show *?thesis* **.**
qed

lemma *Arcsin_unique*:
⟦*sin z = w*; *|Re z| < pi/2 ∨ (|Re z| = pi/2 ∧ Im z = 0)*⟧ ⟹ *Arcsin w = z*
by (*metis Arcsin_sin*)

lemma *Arcsin_0* [*simp*]: *Arcsin 0 = 0*
by (*simp add*: *Arcsin_unique*)

lemma *Arcsin_1* [*simp*]: *Arcsin 1 = pi/2*
using *Arcsin_unique sin_of_real_pi_half* **by** *fastforce*

lemma *Arcsin_minus_1* [*simp*]: *Arcsin(−1) = − (pi/2)*
by (*simp add*: *Arcsin_unique*)

lemma *has_field_derivative_Arcsin*:
assumes *Im z = 0 ⟹ |Re z| < 1*
shows (*Arcsin has_field_derivative inverse(cos(Arcsin z))*) (*at z*)
proof −
have *(sin (Arcsin z))^2 ≠ 1*
using *assms one_minus_z2_notin_nonpos_Reals* **by** *force*

then have *cos (Arcsin z) ≠ 0*
by (*metis diff_0_right power_zero_numeral sin_squared_eq*)
then show *?thesis*
by (*rule has_field_derivative_inverse_basic* [*OF DERIV_sin _ _ open_ball* [*of z 1*]]) (*auto intro*: *isCont_Arcsin assms*)
qed

declare *has_field_derivative_Arcsin* [*derivative_intros*]
declare *has_field_derivative_Arcsin* [*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_at_Arcsin*:
(Im z = 0 ⟹ |Re z| < 1) ⟹ Arcsin field_differentiable at z
using *field_differentiable_def has_field_derivative_Arcsin* **by** *blast*

lemma *field_differentiable_within_Arcsin*:
(Im z = 0 ⟹ |Re z| < 1) ⟹ Arcsin field_differentiable (at z within s)
using *field_differentiable_at_Arcsin field_differentiable_within_subset* **by** *blast*

lemma *continuous_within_Arcsin*:
(Im z = 0 ⟹ |Re z| < 1) ⟹ continuous (at z within s) Arcsin
using *continuous_at_imp_continuous_within isCont_Arcsin* **by** *blast*

lemma *continuous_on_Arcsin* [*continuous_intros*]:
(⋀z. z ∈ s ⟹ Im z = 0 ⟹ |Re z| < 1) ⟹ continuous_on s Arcsin
by (*simp add*: *continuous_at_imp_continuous_on*)

lemma *holomorphic_on_Arcsin*: *(⋀z. z ∈ s ⟹ Im z = 0 ⟹ |Re z| < 1) ⟹ Arcsin holomorphic_on s*
by (*simp add*: *field_differentiable_within_Arcsin holomorphic_on_def*)

6.20.26 Inverse Cosine

definition *Arccos* :: *complex ⇒ complex* **where**
Arccos ≡ λz. −i ∗ Ln(z + i ∗ csqrt(1 − z²))

lemma *Arccos_range_lemma*: *|Re z| < 1 ⟹ 0 < Im(z + i ∗ csqrt(1 − z²))*
using *Arcsin_range_lemma* [*of −z*] **by** *simp*

lemma *Arccos_body_lemma*: *z + i ∗ csqrt(1 − z²) ≠ 0*
by (*metis Arcsin_body_lemma complex_i_mult_minus diff_0 diff_eq_eq power2_minus*)

lemma *Re_Arccos*: *Re(Arccos z) = Im (Ln (z + i ∗ csqrt(1 − z²)))*
by (*simp add*: *Arccos_def*)

lemma *Im_Arccos*: *Im(Arccos z) = − ln (cmod (z + i ∗ csqrt (1 − z²)))*
by (*simp add*: *Arccos_def Arccos_body_lemma*)

A very tricky argument to find!

lemma *isCont_Arccos_lemma*:

assumes *eq0*: $Im\ (z + \mathrm{i} * csqrt\ (1 - z^2)) = 0$ **and** $Im\ z = 0 \Longrightarrow |Re\ z| < 1$
shows *False*
proof (*cases* $Im\ z = 0$)
case *True*
then show *?thesis*
using *assms* **by** (*fastforce simp add*: *cmod_def abs_square_less_1* [*symmetric*])
next
case *False*
have *Imz*: $Im\ z = -\ sqrt\ ((1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2)\ /\ 2)$
using *eq0 abs_Re_le_cmod* [*of* $1-z^2$]
by (*simp add*: *Re_power2 algebra_simps*)
have $(cmod\ z)^2 - 1 \neq cmod\ (1 - z^2)$
proof (*clarsimp simp add*: *cmod_def*)
assume $(Re\ z)^2 + (Im\ z)^2 - 1 = sqrt\ ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
then have $((Re\ z)^2 + (Im\ z)^2 - 1)^2 = ((1 - Re\ (z^2))^2 + (Im\ (z^2))^2)$
by *simp*
then show *False* **using** *False*
by (*simp add*: *power2_eq_square algebra_simps*)
qed
moreover have $(Im\ z)^2 = (1 + ((Im\ z)^2 + cmod\ (1 - z^2)) - (Re\ z)^2)\ /\ 2$
using *abs_Re_le_cmod* [*of* $1-z^2$] **by** (*subst Imz*) (*simp add*: *Re_power2*)
ultimately show *False*
by (*simp add*: *cmod_power2*)
qed

lemma *isCont_Arccos*:
assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
shows *isCont Arccos z*
proof −
have $z + \mathrm{i} * csqrt\ (1 - z^2) \notin \mathbb{R}_{\leq 0}$
by (*metis complex_nonpos_Reals_iff isCont_Arccos_lemma assms*)
with *assms* **show** *?thesis*
unfolding *Arccos_def*
by (*simp_all add*: *one_minus_z2_notin_nonpos_Reals assms*)
qed

lemma *isCont_Arccos'* [*simp*]:
$isCont\ f\ z \Longrightarrow (Im\ (f\ z) = 0 \Longrightarrow |Re\ (f\ z)| < 1) \Longrightarrow isCont\ (\lambda x.\ Arccos\ (f\ x))\ z$
by (*blast intro*: *isCont_o2* [*OF* _ *isCont_Arccos*])

lemma *cos_Arccos* [*simp*]: $cos(Arccos\ z) = z$
proof −
have $z*2 + \mathrm{i} * (2 * csqrt\ (1 - z^2)) = 0 \longleftrightarrow z*2 + \mathrm{i} * csqrt\ (1 - z^2)*2 = 0$
by (*simp add*: *algebra_simps*) — Cancelling a factor of 2
moreover have $\ldots \longleftrightarrow z + \mathrm{i} * csqrt\ (1 - z^2) = 0$
by (*metis distrib_right mult_eq_0_iff zero_neq_numeral*)
ultimately show *?thesis*
by (*simp add*: *cos_exp_eq Arccos_def Arccos_body_lemma exp_minus field_simps flip*: *power2_eq_square*)

qed

lemma *Arccos_cos*:
  assumes $0 < Re\ z \wedge Re\ z < pi \vee$
    $Re\ z = 0 \wedge 0 \leq Im\ z \vee$
    $Re\ z = pi \wedge Im\ z \leq 0$
  shows $Arccos(cos\ z) = z$
proof −
 have $*$: $((\mathrm{i} - (exp\ (\mathrm{i} * z))^2 * \mathrm{i})\ /\ (2 * exp\ (\mathrm{i} * z))) = sin\ z$
  by (*simp add*: *sin_exp_eq exp_minus field_simps power2_eq_square*)
 have $1 - (exp\ (\mathrm{i} * z) + inverse\ (exp\ (\mathrm{i} * z)))^2\ /\ 4 = ((\mathrm{i} - (exp\ (\mathrm{i} * z))^2 * \mathrm{i})\ /$
$(2 * exp\ (\mathrm{i} * z)))^2$
  by (*simp add*: *field_simps power2_eq_square*)
 then have $Arccos(cos\ z) = -\ (\mathrm{i} * Ln\ ((exp\ (\mathrm{i} * z) + inverse\ (exp\ (\mathrm{i} * z)))\ /\ 2$
$+$
    $\mathrm{i} * csqrt\ (((\mathrm{i} - (exp\ (\mathrm{i} * z))^2 * \mathrm{i})\ /\ (2 * exp\ (\mathrm{i} * z)))^2)))$
  by (*simp add*: *cos_exp_eq Arccos_def exp_minus power_divide*)
 also have $\ldots = -\ (\mathrm{i} * Ln\ ((exp\ (\mathrm{i} * z) + inverse\ (exp\ (\mathrm{i} * z)))\ /\ 2 +$
    $\mathrm{i} * ((\mathrm{i} - (exp\ (\mathrm{i} * z))^2 * \mathrm{i})\ /\ (2 * exp\ (\mathrm{i} * z)))))$
  apply (*subst csqrt_square*)
  using *assms Re_sin_pos* [*of z*] *Im_sin_nonneg* [*of z*] *Im_sin_nonneg2* [*of z*]
  by (*auto simp*: $*$ *Re_sin Im_sin*)
 also have $\ldots = -\ (\mathrm{i} * Ln\ (exp\ (\mathrm{i}{*}z)))$
  by (*simp add*: *field_simps power2_eq_square*)
 also have $\ldots = z$
  using *assms*
  by (*subst Complex_Transcendental.Ln_exp*, *auto*)
 finally show *?thesis* **.**
qed

lemma *Arccos_unique*:
  $[\![cos\ z = w;$
    $0 < Re\ z \wedge Re\ z < pi \vee$
    $Re\ z = 0 \wedge 0 \leq Im\ z \vee$
    $Re\ z = pi \wedge Im\ z \leq 0]\!] \Longrightarrow Arccos\ w = z$
 using *Arccos_cos* **by** *blast*

lemma *Arccos_0* [*simp*]: $Arccos\ 0 = pi/2$
 by (*rule Arccos_unique*) *auto*

lemma *Arccos_1* [*simp*]: $Arccos\ 1 = 0$
 by (*rule Arccos_unique*) *auto*

lemma *Arccos_minus1*: $Arccos(-1) = pi$
 by (*rule Arccos_unique*) *auto*

lemma *has_field_derivative_Arccos*:
 assumes $(Im\ z = 0 \Longrightarrow |Re\ z| < 1)$
  shows $(Arccos\ has\_field\_derivative\ -\ inverse(sin(Arccos\ z)))\ (at\ z)$

proof −
have $x^2 \neq -1$ **for** x::*real*
by (*sos* ((*R*<1 + (([~1] ∗ *A*=0) + (*R*<1 ∗ (*R*<1 ∗ [*x*___]^2))))))
with *assms* **have** (*cos* (*Arccos z*))$^2 \neq 1$
by (*auto simp*: *complex_eq_iff Re_power2 Im_power2 abs_square_eq_1*)
then have − *sin* (*Arccos z*) ≠ *0*
by (*metis cos_squared_eq diff_0_right mult_zero_left neg_0_equal_iff_equal power2_eq_square*)
then have (*Arccos has_field_derivative inverse*(− *sin*(*Arccos z*))) (*at z*)
by (*rule has_field_derivative_inverse_basic* [*OF DERIV_cos _ _ open_ball* [*of z 1*]])
(*auto intro*: *isCont_Arccos assms*)
then show *?thesis*
by *simp*
qed

declare *has_field_derivative_Arcsin* [*derivative_intros*]
declare *has_field_derivative_Arcsin* [*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_at_Arccos*:
(*Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *Arccos field_differentiable at z*
using *field_differentiable_def has_field_derivative_Arccos* **by** *blast*

lemma *field_differentiable_within_Arccos*:
(*Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *Arccos field_differentiable* (*at z within s*)
using *field_differentiable_at_Arccos field_differentiable_within_subset* **by** *blast*

lemma *continuous_within_Arccos*:
(*Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *continuous* (*at z within s*) *Arccos*
using *continuous_at_imp_continuous_within isCont_Arccos* **by** *blast*

lemma *continuous_on_Arccos* [*continuous_intros*]:
(⋀*z*. *z* ∈ *s* ⟹ *Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *continuous_on s Arccos*
by (*simp add*: *continuous_at_imp_continuous_on*)

lemma *holomorphic_on_Arccos*: (⋀*z*. *z* ∈ *s* ⟹ *Im z* = *0* ⟹ |*Re z*| < *1*) ⟹ *Arccos holomorphic_on s*
by (*simp add*: *field_differentiable_within_Arccos holomorphic_on_def*)

6.20.27 Upper and Lower Bounds for Inverse Sine and Cosine

lemma *Arcsin_bounds*: |*Re z*| < *1* ⟹ |*Re*(*Arcsin z*)| < *pi*/*2*
unfolding *Re_Arcsin*
by (*blast intro*: *Re_Ln_pos_lt_imp Arcsin_range_lemma*)

lemma *Arccos_bounds*: |*Re z*| < *1* ⟹ *0* < *Re*(*Arccos z*) ∧ *Re*(*Arccos z*) < *pi*
unfolding *Re_Arccos*
by (*blast intro*!: *Im_Ln_pos_lt_imp Arccos_range_lemma*)

lemma *Re_Arccos_bounds*: $-pi < Re(Arccos\ z) \wedge Re(Arccos\ z) \leq pi$
unfolding *Re_Arccos*
by (*blast intro*!: *mpi_less_Im_Ln Im_Ln_le_pi Arccos_body_lemma*)

lemma *Re_Arccos_bound*: $|Re(Arccos\ z)| \leq pi$
by (*meson Re_Arccos_bounds abs_le_iff less_eq_real_def minus_less_iff*)

lemma *Im_Arccos_bound*: $|Im\ (Arccos\ w)| \leq cmod\ w$
proof −
have $(Im\ (Arccos\ w))^2 \leq (cmod\ (cos\ (Arccos\ w)))^2 - (cos\ (Re\ (Arccos\ w)))^2$
using *norm_cos_squared* [*of Arccos w*] *real_le_abs_sinh* [*of Im* (*Arccos w*)]
by (*simp only*: *abs_le_square_iff*) (*simp add*: *field_split_simps*)
also have $\ldots \leq (cmod\ w)^2$
by (*auto simp*: *cmod_power2*)
finally show *?thesis*
using *abs_le_square_iff* **by** *force*
qed

lemma *Re_Arcsin_bounds*: $-pi < Re(Arcsin\ z)$ & $Re(Arcsin\ z) \leq pi$
unfolding *Re_Arcsin*
by (*blast intro*!: *mpi_less_Im_Ln Im_Ln_le_pi Arcsin_body_lemma*)

lemma *Re_Arcsin_bound*: $|Re(Arcsin\ z)| \leq pi$
by (*meson Re_Arcsin_bounds abs_le_iff less_eq_real_def minus_less_iff*)

lemma *norm_Arccos_bounded*:
fixes $w :: complex$
shows $norm\ (Arccos\ w) \leq pi + norm\ w$
proof −
have *Re*: $(Re\ (Arccos\ w))^2 \leq pi^2$ $(Im\ (Arccos\ w))^2 \leq (cmod\ w)^2$
using *Re_Arccos_bound* [*of w*] *Im_Arccos_bound* [*of w*] *abs_le_square_iff* **by** *force+*
have $Arccos\ w \cdot Arccos\ w \leq pi^2 + (cmod\ w)^2$
using *Re* **by** (*simp add*: *dot_square_norm cmod_power2* [*of Arccos w*])
then have $cmod\ (Arccos\ w) \leq pi + cmod\ (cos\ (Arccos\ w))$
by (*smt* (*verit*) *Im_Arccos_bound Re_Arccos_bound cmod_le cos_Arccos*)
then show $cmod\ (Arccos\ w) \leq pi + cmod\ w$
by *auto*
qed

6.20.28 Interrelations between Arcsin and Arccos

lemma *cos_Arcsin_nonzero*: $z^2 \neq 1 \Longrightarrow cos(Arcsin\ z) \neq 0$
by (*metis diff_0_right power_zero_numeral sin_Arcsin sin_squared_eq*)

lemma *sin_Arccos_nonzero*: $z^2 \neq 1 \Longrightarrow sin(Arccos\ z) \neq 0$
by (*metis add.right_neutral cos_Arccos power2_eq_square power_zero_numeral sin_cos_squared_add3*)

lemma *cos_sin_csqrt*:
assumes $0 < cos(Re\ z) \ \vee \ cos(Re\ z) = 0 \wedge Im\ z * sin(Re\ z) \leq 0$
shows $cos\ z = csqrt(1 - (sin\ z)^2)$
proof (*rule csqrt_unique* [*THEN sym*])
show $(cos\ z)^2 = 1 - (sin\ z)^2$
by (*simp add*: *cos_squared_eq*)
qed (*use assms* **in** ‹*auto simp*: *Re_cos Im_cos add_pos_pos mult_le_0_iff zero_le_mult_iff*›)

lemma *sin_cos_csqrt*:
assumes $0 < sin(Re\ z) \ \vee \ sin(Re\ z) = 0 \wedge 0 \leq Im\ z * cos(Re\ z)$
shows $sin\ z = csqrt(1 - (cos\ z)^2)$
proof (*rule csqrt_unique* [*THEN sym*])
show $(sin\ z)^2 = 1 - (cos\ z)^2$
by (*simp add*: *sin_squared_eq*)
qed (*use assms* **in** ‹*auto simp*: *Re_sin Im_sin add_pos_pos mult_le_0_iff zero_le_mult_iff*›)

lemma *Arcsin_Arccos_csqrt_pos*:
$(0 < Re\ z \vee Re\ z = 0 \wedge 0 \leq Im\ z) \Longrightarrow Arcsin\ z = Arccos(csqrt(1 - z^2))$
by (*simp add*: *Arcsin_def Arccos_def Complex.csqrt_square add.commute*)

lemma *Arccos_Arcsin_csqrt_pos*:
$(0 < Re\ z \vee Re\ z = 0 \wedge 0 \leq Im\ z) \Longrightarrow Arccos\ z = Arcsin(csqrt(1 - z^2))$
by (*simp add*: *Arcsin_def Arccos_def Complex.csqrt_square add.commute*)

lemma *sin_Arccos*:
$0 < Re\ z \vee Re\ z = 0 \wedge 0 \leq Im\ z \Longrightarrow sin(Arccos\ z) = csqrt(1 - z^2)$
by (*simp add*: *Arccos_Arcsin_csqrt_pos*)

lemma *cos_Arcsin*:
$0 < Re\ z \vee Re\ z = 0 \wedge 0 \leq Im\ z \Longrightarrow cos(Arcsin\ z) = csqrt(1 - z^2)$
by (*simp add*: *Arcsin_Arccos_csqrt_pos*)

6.20.29 Relationship with Arcsin on the Real Numbers

lemma *of_real_arcsin*: $|x| \leq 1 \Longrightarrow of\_real(arcsin\ x) = Arcsin(of\_real\ x)$
by (*smt* (*verit*, *best*) *Arcsin_sin Im_complex_of_real Re_complex_of_real arcsin sin_of_real*)

lemma *Im_Arcsin_of_real*: $|x| \leq 1 \Longrightarrow Im\ (Arcsin\ (of\_real\ x)) = 0$
by (*metis Im_complex_of_real of_real_arcsin*)

corollary *Arcsin_in_Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow |Re\ z| \leq 1 \Longrightarrow Arcsin\ z \in \mathbb{R}$
by (*metis Im_Arcsin_of_real Re_complex_of_real Reals_cases complex_is_Real_iff*)

lemma *arcsin_eq_Re_Arcsin*: $|x| \leq 1 \Longrightarrow arcsin\ x = Re\ (Arcsin\ (of\_real\ x))$
by (*metis Re_complex_of_real of_real_arcsin*)

6.20.30 Relationship with Arccos on the Real Numbers

lemma *of_real_arccos*: $|x| \leq 1 \Longrightarrow$ *of_real*(*arccos x*) = *Arccos*(*of_real x*)
by (*smt* (*verit*, *del_insts*) *Arccos_unique Im_complex_of_real Re_complex_of_real arccos_lbound*
arccos_ubound cos_arccos_abs cos_of_real)

lemma *Im_Arccos_of_real*: $|x| \leq 1 \Longrightarrow$ *Im* (*Arccos* (*of_real x*)) = *0*
by (*metis Im_complex_of_real of_real_arccos*)

corollary *Arccos_in_Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow |Re\ z| \leq 1 \Longrightarrow Arccos\ z \in \mathbb{R}$
by (*metis Im_Arccos_of_real complex_is_Real_iff of_real_Re*)

lemma *arccos_eq_Re_Arccos*: $|x| \leq 1 \Longrightarrow$ *arccos x* = *Re* (*Arccos* (*of_real x*))
by (*metis Re_complex_of_real of_real_arccos*)

6.20.31 Continuity results for arcsin and arccos

lemma *continuous_on_Arcsin_real* [*continuous_intros*]:
continuous_on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ *Arcsin*
proof −
have *continuous_on* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ ($\lambda x.$ *complex_of_real* (*arcsin* (*Re x*))) =
continuous_on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ ($\lambda x.$ *complex_of_real* (*Re* (*Arcsin* (*of_real* (*Re x*)))))
by (*rule continuous_on_cong* [*OF refl*]) (*simp add*: *arcsin_eq_Re_Arcsin*)
also have ... = *?thesis*
by (*rule continuous_on_cong* [*OF refl*]) *simp*
finally show *?thesis*
using *continuous_on_arcsin* [*OF continuous_on_Re* [*OF continuous_on_id*], *of* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$]
continuous_on_of_real
by *fastforce*
qed

lemma *continuous_within_Arcsin_real*:
continuous (*at z within* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$) *Arcsin*
using *closed_real_abs_le continuous_on_Arcsin_real continuous_on_eq_continuous_within*
continuous_within_closed_nontrivial **by** *blast*

lemma *continuous_on_Arccos_real*:
continuous_on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ *Arccos*
proof −
have *continuous_on* $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ ($\lambda x.$ *complex_of_real* (*arccos* (*Re x*))) =
continuous_on $\{w \in \mathbb{R}.\ |Re\ w| \leq 1\}$ ($\lambda x.$ *complex_of_real* (*Re* (*Arccos* (*of_real* (*Re x*)))))
by (*rule continuous_on_cong* [*OF refl*]) (*simp add*: *arccos_eq_Re_Arccos*)
also have ... = *?thesis*

by (*rule continuous_on_cong* [*OF refl*]) *simp*
finally show *?thesis*
using *continuous_on_arccos* [*OF continuous_on_Re* [*OF continuous_on_id*], *of* {$w \in \mathbb{R}$. |*Re w*| ≤ *1*}]
continuous_on_of_real
by *fastforce*
qed

lemma *continuous_within_Arccos_real*:
continuous (*at z within* {$w \in \mathbb{R}$. |*Re w*| ≤ *1*}) *Arccos*
using *closed_real_abs_le continuous_on_Arccos_real continuous_on_eq_continuous_within*

continuous_within_closed_nontrivial **by** *blast*

lemma *sinh_ln_complex*: *x* ≠ *0* ⟹ *sinh* (*ln x* :: *complex*) = (*x* − *inverse x*) / *2*
by (*simp add*: *sinh_def exp_minus scaleR_conv_of_real exp_of_real*)

lemma *cosh_ln_complex*: *x* ≠ *0* ⟹ *cosh* (*ln x* :: *complex*) = (*x* + *inverse x*) / *2*
by (*simp add*: *cosh_def exp_minus scaleR_conv_of_real*)

lemma *tanh_ln_complex*: *x* ≠ *0* ⟹ *tanh* (*ln x* :: *complex*) = (*x* ^ *2* − *1*) / (*x* ^ *2* + *1*)
by (*simp add*: *tanh_def sinh_ln_complex cosh_ln_complex divide_simps power2_eq_square*)

6.20.32 Roots of unity

theorem *complex_root_unity*:
fixes *j*::*nat*
assumes *n* ≠ *0*
shows *exp*(*2* * *of_real pi* * i * *of_nat j* / *of_nat n*)^*n* = *1*
by (*metis assms bot_nat_0.not_eq_extremum exp_divide_power_eq exp_of_nat2_mult exp_two_pi_i power_one*)

lemma *complex_root_unity_eq*:
fixes *j*::*nat* **and** *k*::*nat*
assumes *1* ≤ *n*
shows (*exp*(*2* * *of_real pi* * i * *of_nat j* / *of_nat n*) = *exp*(*2* * *of_real pi* * i * *of_nat k* / *of_nat n*)
⟷ *j mod n* = *k mod n*)
proof −
have (∃ *z*::*int*. i * (*of_nat j* * (*of_real pi* * *2*)) =
i * (*of_nat k* * (*of_real pi* * *2*)) + i * (*of_int z* * (*of_nat n* * (*of_real pi* * *2*)))) ⟷
(∃ *z*::*int*. *of_nat j* * (i * (*of_real pi* * *2*)) =
(*of_nat k* + *of_nat n* * *of_int z*) * (i * (*of_real pi* * *2*)))
by (*simp add*: *algebra_simps*)
also have ... ⟷ (∃ *z*::*int*. *of_nat j* = *of_nat k* + *of_nat n* * (*of_int z* :: *complex*))
by *simp*

```
  also have ... ⟷ (∃ z::int. of_nat j = of_nat k + of_nat n * z)
   by (metis (mono_tags, opaque_lifting) of_int_add of_int_eq_iff of_int_mult
of_int_of_nat_eq)
  also have ... ⟷ int j mod int n = int k mod int n
    by (auto simp: mod_eq_dvd_iff dvd_def algebra_simps)
  also have ... ⟷ j mod n = k mod n
    by (metis of_nat_eq_iff zmod_int)
  finally have (∃ z. i * (of_nat j * (of_real pi * 2)) =
        i * (of_nat k * (of_real pi * 2)) + i * (of_int z * (of_nat n * (of_real
pi * 2)))) ⟷ j mod n = k mod n .
 note * = this
 show ?thesis
  using assms by (simp add: exp_eq field_split_simps *)
qed

corollary bij_betw_roots_unity:
  bij_betw (λj. exp(2 * of_real pi * i * of_nat j / of_nat n))
        {..<n}  {exp(2 * of_real pi * i * of_nat j / of_nat n) | j. j < n}
 by (auto simp: bij_betw_def inj_on_def complex_root_unity_eq)

lemma complex_root_unity_eq_1:
 fixes j::nat and k::nat
 assumes 1 ≤ n
  shows exp(2 * of_real pi * i * of_nat j / of_nat n) = 1 ⟷ n dvd j
proof −
 have 1 = exp(2 * of_real pi * i * (of_nat n / of_nat n))
  using assms by simp
 then have exp(2 * of_real pi * i * (of_nat j / of_nat n)) = 1 ⟷ j mod n =
n mod n
   using complex_root_unity_eq [of n j n] assms
   by simp
 then show ?thesis
  by auto
qed

lemma finite_complex_roots_unity_explicit:
 finite {exp(2 * of_real pi * i * of_nat j / of_nat n) | j::nat. j < n}
 by simp

lemma card_complex_roots_unity_explicit:
   card {exp(2 * of_real pi * i * of_nat j / of_nat n) | j::nat. j < n} = n
 by (simp add: Finite_Set.bij_betw_same_card [OF bij_betw_roots_unity, sym-
metric])

lemma complex_roots_unity:
 assumes 1 ≤ n
  shows {z::complex. z^n = 1} = {exp(2 * of_real pi * i * of_nat j / of_nat
n) | j. j < n}
 apply (rule Finite_Set.card_seteq [symmetric])
```

using *assms*
apply (*auto simp*: *card_complex_roots_unity_explicit finite_roots_unity complex_root_unity card_roots_unity*)
done

lemma *card_complex_roots_unity*: $1 \leq n \Longrightarrow$ *card* $\{z$::*complex*. $z\hat{}n = 1\} = n$
by (*simp add*: *card_complex_roots_unity_explicit complex_roots_unity*)

lemma *complex_not_root_unity*:
$1 \leq n \Longrightarrow \exists u$::*complex*. *norm* $u = 1 \wedge u\hat{}n \neq 1$
apply (*rule_tac x*=*exp* (*of_real pi* $*$ i $*$ *of_real* (*1* / *n*)) **in** *exI*)
apply (*auto simp*: *Re_complex_div_eq_0 exp_of_nat_mult* [*symmetric*] *mult_ac exp_Euler*)
done

end

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Chapter 7

Measure and Integration Theory

theory *Sigma_Algebra*
imports
Complex_Main
HOL−Library.Countable_Set
HOL−Library.FuncSet
HOL−Library.Indicator_Function
HOL−Library.Extended_Nonnegative_Real
HOL−Library.Disjoint_Sets
begin

7.1 Sigma Algebra

Sigma algebras are an elementary concept in measure theory. To measure — that is to integrate — functions, we first have to measure sets. Unfortunately, when dealing with a large universe, it is often not possible to consistently assign a measure to every subset. Therefore it is necessary to define the set of measurable subsets of the universe. A sigma algebra is such a set that has three very natural and desirable properties.

7.1.1 Families of sets

locale *subset_class* =
fixes Ω :: *'a set* **and** M :: *'a set set*
assumes *space_closed*: $M \subseteq Pow\ \Omega$

lemma (**in** *subset_class*) *sets_into_space*: $x \in M \Longrightarrow x \subseteq \Omega$
by (*metis PowD contra_subsetD space_closed*)

Semiring of sets

locale *semiring_of_sets* = *subset_class* +

assumes *empty_sets*[*iff*]: $\{\} \in M$
assumes *Int*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cap b \in M$
assumes *Diff_cover*:
$\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow \exists\, C \subseteq M.\ \textit{finite}\ C \wedge \textit{disjoint}\ C \wedge a - b = \bigcup C$

lemma (**in** *semiring_of_sets*) *finite_INT*[*intro*]:
assumes *finite* I $I \neq \{\}$ $\bigwedge i.\ i \in I \Longrightarrow A\ i \in M$
shows $(\bigcap i \in I.\ A\ i) \in M$
using *assms* **by** (*induct rule*: *finite_ne_induct*) *auto*

lemma (**in** *semiring_of_sets*) *Int_space_eq1* [*simp*]: $x \in M \Longrightarrow \Omega \cap x = x$
by (*metis Int_absorb1 sets_into_space*)

lemma (**in** *semiring_of_sets*) *Int_space_eq2* [*simp*]: $x \in M \Longrightarrow x \cap \Omega = x$
by (*metis Int_absorb2 sets_into_space*)

lemma (**in** *semiring_of_sets*) *sets_Collect_conj*:
assumes $\{x \in \Omega.\ P\ x\} \in M$ $\{x \in \Omega.\ Q\ x\} \in M$
shows $\{x \in \Omega.\ Q\ x \wedge P\ x\} \in M$
proof −
have $\{x \in \Omega.\ Q\ x \wedge P\ x\} = \{x \in \Omega.\ Q\ x\} \cap \{x \in \Omega.\ P\ x\}$
by *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *semiring_of_sets*) *sets_Collect_finite_All′*:
assumes $\bigwedge i.\ i \in S \Longrightarrow \{x \in \Omega.\ P\ i\ x\} \in M$ *finite* S $S \neq \{\}$
shows $\{x \in \Omega.\ \forall\, i \in S.\ P\ i\ x\} \in M$
proof −
have $\{x \in \Omega.\ \forall\, i \in S.\ P\ i\ x\} = (\bigcap i \in S.\ \{x \in \Omega.\ P\ i\ x\})$
using ‹$S \neq \{\}$› **by** *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

Ring of sets

locale *ring_of_sets* = *semiring_of_sets* +
assumes *Un* [*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cup b \in M$

lemma (**in** *ring_of_sets*) *finite_Union* [*intro*]:
finite $X \Longrightarrow X \subseteq M \Longrightarrow \bigcup X \in M$
by (*induct set*: *finite*) (*auto simp add*: *Un*)

lemma (**in** *ring_of_sets*) *finite_UN*[*intro*]:
assumes *finite* I **and** $\bigwedge i.\ i \in I \Longrightarrow A\ i \in M$
shows $(\bigcup i \in I.\ A\ i) \in M$
using *assms* **by** *induct auto*

lemma (**in** *ring_of_sets*) *Diff* [*intro*]:

assumes $a \in M$ $b \in M$ **shows** $a - b \in M$
using *Diff_cover*[*OF assms*] **by** *auto*

lemma *ring_of_setsI*:
assumes *space_closed*: $M \subseteq Pow\ \Omega$
assumes *empty_sets*[*iff*]: $\{\} \in M$
assumes *Un*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cup b \in M$
assumes *Diff*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a - b \in M$
shows *ring_of_sets* $\Omega\ M$
proof
fix $a\ b$ **assume** *ab*: $a \in M\ b \in M$
from *ab* **show** $\exists\, C \subseteq M.\ finite\ C \wedge disjoint\ C \wedge a - b = \bigcup C$
by (*intro exI*[*of* _ $\{a - b\}$]) (*auto simp*: *disjoint_def*)
have $a \cap b = a - (a - b)$ **by** *auto*
also have $\ldots \in M$ **using** *ab* **by** *auto*
finally show $a \cap b \in M$ **.**
qed *fact*+

lemma *ring_of_sets_iff*: *ring_of_sets* $\Omega\ M \longleftrightarrow M \subseteq Pow\ \Omega \wedge \{\} \in M \wedge (\forall\, a{\in}M.\ \forall\, b{\in}M.\ a \cup b \in M) \wedge (\forall\, a{\in}M.\ \forall\, b{\in}M.\ a - b \in M)$
proof
assume *ring_of_sets* $\Omega\ M$
then interpret *ring_of_sets* $\Omega\ M$ **.**
show $M \subseteq Pow\ \Omega \wedge \{\} \in M \wedge (\forall\, a{\in}M.\ \forall\, b{\in}M.\ a \cup b \in M) \wedge (\forall\, a{\in}M.\ \forall\, b{\in}M.\ a - b \in M)$
using *space_closed* **by** *auto*
qed (*auto intro*!: *ring_of_setsI*)

lemma (**in** *ring_of_sets*) *insert_in_sets*:
assumes $\{x\} \in M$ $A \in M$ **shows** *insert* $x\ A \in M$
proof −
have $\{x\} \cup A \in M$ **using** *assms* **by** (*rule Un*)
thus *?thesis* **by** *auto*
qed

lemma (**in** *ring_of_sets*) *sets_Collect_disj*:
assumes $\{x{\in}\Omega.\ P\ x\} \in M$ $\{x{\in}\Omega.\ Q\ x\} \in M$
shows $\{x{\in}\Omega.\ Q\ x \vee P\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ Q\ x \vee P\ x\} = \{x{\in}\Omega.\ Q\ x\} \cup \{x{\in}\Omega.\ P\ x\}$
by *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *ring_of_sets*) *sets_Collect_finite_Ex*:
assumes $\bigwedge i.\ i \in S \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M\ finite\ S$
shows $\{x{\in}\Omega.\ \exists\, i{\in}S.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \exists\, i{\in}S.\ P\ i\ x\} = (\bigcup i{\in}S.\ \{x{\in}\Omega.\ P\ i\ x\})$

by *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

Algebra of sets

locale *algebra* = *ring_of_sets* +
assumes *top* [*iff*]: $\Omega \in M$

lemma (**in** *algebra*) *compl_sets* [*intro*]:
$a \in M \Longrightarrow \Omega - a \in M$
by *auto*

proposition *algebra_iff_Un*:
algebra Ω $M \longleftrightarrow$
$M \subseteq Pow\ \Omega \wedge$
$\{\} \in M \wedge$
$(\forall a \in M.\ \Omega - a \in M) \wedge$
$(\forall a \in M.\ \forall\ b \in M.\ a \cup b \in M)$ (**is** _ $\longleftrightarrow$ *?Un*)
proof
assume *algebra* Ω M
then interpret *algebra* Ω M **.**
show *?Un* **using** *sets_into_space* **by** *auto*
next
assume *?Un*
then have $\Omega \in M$ **by** *auto*
interpret *ring_of_sets* Ω M
proof (*rule ring_of_setsI*)
show Ω: $M \subseteq Pow\ \Omega\ \{\} \in M$
using ‹*?Un*› **by** *auto*
fix a b **assume** a: $a \in M$ **and** b: $b \in M$
then show $a \cup b \in M$ **using** ‹*?Un*› **by** *auto*
have $a - b = \Omega - ((\Omega - a) \cup b)$
using Ω a b **by** *auto*
then show $a - b \in M$
using a b ‹*?Un*› **by** *auto*
qed
show *algebra* Ω M **proof qed** *fact*
qed

proposition *algebra_iff_Int*:
algebra Ω $M \longleftrightarrow$
$M \subseteq Pow\ \Omega$ & $\{\} \in M$ &
$(\forall a \in M.\ \Omega - a \in M)$ &
$(\forall a \in M.\ \forall\ b \in M.\ a \cap b \in M)$ (**is** _ $\longleftrightarrow$ *?Int*)
proof
assume *algebra* Ω M
then interpret *algebra* Ω M **.**
show *?Int* **using** *sets_into_space* **by** *auto*

next
assume *?Int*
show *algebra* Ω *M*
proof (*unfold algebra_iff_Un*, *intro conjI ballI*)
show Ω: $M \subseteq Pow\ \Omega\ \{\} \in M$
using ‹*?Int*› **by** *auto*
from ‹*?Int*› **show** $\bigwedge a.\ a \in M \Longrightarrow \Omega - a \in M$ **by** *auto*
fix *a b* **assume** *M*: $a \in M\ b \in M$
hence $a \cup b = \Omega - ((\Omega - a) \cap (\Omega - b))$
using Ω **by** *blast*
also have $... \in M$
using *M* ‹*?Int*› **by** *auto*
finally show $a \cup b \in M$ **.**
qed
qed

lemma (**in** *algebra*) *sets_Collect_neg*:
assumes $\{x \in \Omega.\ P\ x\} \in M$
shows $\{x \in \Omega.\ \neg\ P\ x\} \in M$
proof −
have $\{x \in \Omega.\ \neg\ P\ x\} = \Omega - \{x \in \Omega.\ P\ x\}$ **by** *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *algebra*) *sets_Collect_imp*:
$\{x \in \Omega.\ P\ x\} \in M \Longrightarrow \{x \in \Omega.\ Q\ x\} \in M \Longrightarrow \{x \in \Omega.\ Q\ x \longrightarrow P\ x\} \in M$
unfolding *imp_conv_disj* **by** (*intro sets_Collect_disj sets_Collect_neg*)

lemma (**in** *algebra*) *sets_Collect_const*:
$\{x \in \Omega.\ P\} \in M$
by (*cases P*) *auto*

lemma *algebra_single_set*:
$X \subseteq S \Longrightarrow$ *algebra* S { {}, X, $S - X$, S }
by (*auto simp*: *algebra_iff_Int*)

Restricted algebras

abbreviation (**in** *algebra*)
restricted_space $A \equiv ((\cap)\ A)$ ‘ M

lemma (**in** *algebra*) *restricted_algebra*:
assumes $A \in M$ **shows** *algebra* A (*restricted_space* A)
using *assms* **by** (*auto simp*: *algebra_iff_Int*)

Sigma Algebras

locale *sigma_algebra* = *algebra* +
assumes *countable_nat_UN* [*intro*]: $\bigwedge A.\ range\ A \subseteq M \Longrightarrow (\bigcup i{::}nat.\ A\ i) \in M$

lemma (**in** *algebra*) *is_sigma_algebra*:
assumes *finite M*
shows *sigma_algebra* Ω *M*
proof
fix *A* :: *nat* ⇒ *'a set* **assume** *range A* ⊆ *M*
then have (⋃ *i*. *A i*) = (⋃ *s*∈*M* ∩ *range A*. *s*)
by *auto*
also have (⋃ *s*∈*M* ∩ *range A*. *s*) ∈ *M*
using ‹*finite M*› **by** *auto*
finally show (⋃ *i*. *A i*) ∈ *M* **.**
qed

lemma *countable_UN_eq*:
fixes *A* :: *'i*::*countable* ⇒ *'a set*
shows (*range A* ⊆ *M* ⟶ (⋃ *i*. *A i*) ∈ *M*) ⟷
(*range* (*A* ∘ *from_nat*) ⊆ *M* ⟶ (⋃ *i*. (*A* ∘ *from_nat*) *i*) ∈ *M*)
proof −
let *?A'* = *A* ∘ *from_nat*
have ∗: (⋃ *i*. *?A' i*) = (⋃ *i*. *A i*) (**is** *?l* = *?r*)
proof *safe*
fix *x i* **assume** *x* ∈ *A i* **thus** *x* ∈ *?l*
by (*auto intro*!: *exI*[*of* _ *to_nat i*])
next
fix *x i* **assume** *x* ∈ *?A' i* **thus** *x* ∈ *?r*
by (*auto intro*!: *exI*[*of* _ *from_nat i*])
qed
have *A* ‘ *range from_nat* = *range A*
using *surj_from_nat* **by** *simp*
then have ∗∗: *range ?A'* = *range A*
by (*simp only*: *image_comp* [*symmetric*])
show *?thesis* **unfolding** ∗ ∗∗ **..**
qed

lemma (**in** *sigma_algebra*) *countable_Union* [*intro*]:
assumes *countable X X* ⊆ *M* **shows** ⋃ *X* ∈ *M*
proof *cases*
assume *X* ≠ {}
hence ⋃ *X* = (⋃ *n*. *from_nat_into X n*)
using *assms* **by** (*auto cong del*: *SUP_cong*)
also have . . . ∈ *M* **using** *assms*
by (*auto intro*!: *countable_nat_UN*) (*metis* ‹*X* ≠ {}› *from_nat_into subsetD*)
finally show *?thesis* **.**
qed *simp*

lemma (**in** *sigma_algebra*) *countable_UN*[*intro*]:
fixes *A* :: *'i*::*countable* ⇒ *'a set*
assumes *A*‘*X* ⊆ *M*
shows (⋃ *x*∈*X*. *A x*) ∈ *M*
proof −

let *?A* = λ*i*. *if i* ∈ *X then A i else* {}
from *assms* **have** *range ?A* ⊆ *M* **by** *auto*
with *countable_nat_UN*[*of ?A* ∘ *from_nat*] *countable_UN_eq*[*of ?A M*]
have (⋃*x*. *?A x*) ∈ *M* **by** *auto*
moreover have (⋃*x*. *?A x*) = (⋃*x*∈*X*. *A x*) **by** (*auto split*: *if_split_asm*)
ultimately show *?thesis* **by** *simp*
qed

lemma (**in** *sigma_algebra*) *countable_UN'*:
fixes *A* :: *'i* ⇒ *'a set*
assumes *X*: *countable X*
assumes *A*: *A*‘*X* ⊆ *M*
shows (⋃*x*∈*X*. *A x*) ∈ *M*
proof −
have (⋃*x*∈*X*. *A x*) = (⋃*i*∈*to_nat_on X* ‘ *X*. *A* (*from_nat_into X i*))
using *X* **by** *auto*
also have ... ∈ *M*
using *A X*
by (*intro countable_UN*) *auto*
finally show *?thesis* **.**
qed

lemma (**in** *sigma_algebra*) *countable_UN''*:
⟦ *countable X*; ⋀*x y*. *x* ∈ *X* ⟹ *A x* ∈ *M* ⟧ ⟹ (⋃*x*∈*X*. *A x*) ∈ *M*
by(*erule countable_UN'*)(*auto*)

lemma (**in** *sigma_algebra*) *countable_INT* [*intro*]:
fixes *A* :: *'i*::*countable* ⇒ *'a set*
assumes *A*: *A*‘*X* ⊆ *M X* ≠ {}
shows (⋂*i*∈*X*. *A i*) ∈ *M*
proof −
from *A* **have** ∀*i*∈*X*. *A i* ∈ *M* **by** *fast*
hence Ω − (⋃*i*∈*X*. Ω − *A i*) ∈ *M* **by** *blast*
moreover
have (⋂*i*∈*X*. *A i*) = Ω − (⋃*i*∈*X*. Ω − *A i*) **using** *space_closed A*
by *blast*
ultimately show *?thesis* **by** *metis*
qed

lemma (**in** *sigma_algebra*) *countable_INT'*:
fixes *A* :: *'i* ⇒ *'a set*
assumes *X*: *countable X X* ≠ {}
assumes *A*: *A*‘*X* ⊆ *M*
shows (⋂*x*∈*X*. *A x*) ∈ *M*
proof −
have (⋂*x*∈*X*. *A x*) = (⋂*i*∈*to_nat_on X* ‘ *X*. *A* (*from_nat_into X i*))
using *X* **by** *auto*
also have ... ∈ *M*
using *A X*

by (*intro countable_INT*) *auto*
finally show *?thesis* **.**
qed

lemma (**in** *sigma_algebra*) *countable_INT''*:
$UNIV \in M \Longrightarrow countable\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow F\ i \in M) \Longrightarrow (\bigcap i{\in}I.\ F\ i) \in M$
by (*cases* $I = \{\}$) (*auto intro*: *countable_INT'*)

lemma (**in** *sigma_algebra*) *countable*:
assumes $\bigwedge a.\ a \in A \Longrightarrow \{a\} \in M\ countable\ A$
shows $A \in M$
proof −
have $(\bigcup a{\in}A.\ \{a\}) \in M$
using *assms* **by** (*intro countable_UN'*) *auto*
also have $(\bigcup a{\in}A.\ \{a\}) = A$ **by** *auto*
finally show *?thesis* **by** *auto*
qed

lemma *ring_of_sets_Pow*: *ring_of_sets sp* (*Pow sp*)
by (*auto simp*: *ring_of_sets_iff*)

lemma *algebra_Pow*: *algebra sp* (*Pow sp*)
by (*auto simp*: *algebra_iff_Un*)

lemma *sigma_algebra_iff*:
sigma_algebra $\Omega\ M \longleftrightarrow$
algebra $\Omega\ M \wedge (\forall A.\ range\ A \subseteq M \longrightarrow (\bigcup i{::}nat.\ A\ i) \in M)$
by (*simp add*: *sigma_algebra_def sigma_algebra_axioms_def*)

lemma *sigma_algebra_Pow*: *sigma_algebra sp* (*Pow sp*)
by (*auto simp*: *sigma_algebra_iff algebra_iff_Int*)

lemma (**in** *sigma_algebra*) *sets_Collect_countable_All*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \forall i{::}'i{::}countable.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \forall i{::}'i{::}countable.\ P\ i\ x\} = (\bigcap i.\ \{x{\in}\Omega.\ P\ i\ x\})$ **by** *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ex*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \exists i{::}'i{::}countable.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \exists i{::}'i{::}countable.\ P\ i\ x\} = (\bigcup i.\ \{x{\in}\Omega.\ P\ i\ x\})$ **by** *auto*
with *assms* **show** *?thesis* **by** *auto*
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ex'*:

assumes $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$
assumes *countable* I
shows $\{x{\in}\Omega.\ \exists\, i{\in}I.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \exists\, i{\in}I.\ P\ i\ x\} = (\bigcup i{\in}I.\ \{x{\in}\Omega.\ P\ i\ x\})$ **by** *auto*
with *assms* **show** *?thesis*
by (*auto intro*!: *countable_UN'*)
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_All'*:
assumes $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$
assumes *countable* I
shows $\{x{\in}\Omega.\ \forall\, i{\in}I.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \forall\, i{\in}I.\ P\ i\ x\} = (\bigcap i{\in}I.\ \{x{\in}\Omega.\ P\ i\ x\}) \cap \Omega$ **by** *auto*
with *assms* **show** *?thesis*
by (*cases* $I = \{\}$) (*auto intro*!: *countable_INT'*)
qed

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ex1'*:
assumes $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}\Omega.\ P\ i\ x\} \in M$
assumes *countable* I
shows $\{x{\in}\Omega.\ \exists!\, i{\in}I.\ P\ i\ x\} \in M$
proof −
have $\{x{\in}\Omega.\ \exists!\, i{\in}I.\ P\ i\ x\} = \{x{\in}\Omega.\ \exists\, i{\in}I.\ P\ i\ x \wedge (\forall\, j{\in}I.\ P\ j\ x \longrightarrow i = j)\}$
by *auto*
with *assms* **show** *?thesis*
by (*auto intro*!: *sets_Collect_countable_All' sets_Collect_countable_Ex' sets_Collect_conj sets_Collect_imp sets_Collect_const*)
qed

lemmas (**in** *sigma_algebra*) *sets_Collect* =
sets_Collect_imp sets_Collect_disj sets_Collect_conj sets_Collect_neg sets_Collect_const sets_Collect_countable_All sets_Collect_countable_Ex sets_Collect_countable_All

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Ball*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \forall\, i{::}'i{::}countable{\in}X.\ P\ i\ x\} \in M$
unfolding *Ball_def* **by** (*intro sets_Collect assms*)

lemma (**in** *sigma_algebra*) *sets_Collect_countable_Bex*:
assumes $\bigwedge i.\ \{x{\in}\Omega.\ P\ i\ x\} \in M$
shows $\{x{\in}\Omega.\ \exists\, i{::}'i{::}countable{\in}X.\ P\ i\ x\} \in M$
unfolding *Bex_def* **by** (*intro sets_Collect assms*)

lemma *sigma_algebra_single_set*:
assumes $X \subseteq S$
shows *sigma_algebra* S $\{\ \{\},\ X,\ S - X,\ S\ \}$
using *algebra.is_sigma_algebra*[*OF algebra_single_set*[*OF* ‹$X \subseteq S$›]] **by** *simp*

Binary Unions

definition *binary* :: *'a* ⇒ *'a* ⇒ *nat* ⇒ *'a*
  where *binary a b* = (λ*x*. *b*)(*0* := *a*)

lemma *range_binary_eq*: *range*(*binary a b*) = {*a*,*b*}
  by (*auto simp add*: *binary_def*)

lemma *Un_range_binary*: *a* ∪ *b* = (⋃ *i*::*nat*. *binary a b i*)
  by (*simp add*: *range_binary_eq cong del*: *SUP_cong_simp*)

lemma *Int_range_binary*: *a* ∩ *b* = (⋂ *i*::*nat*. *binary a b i*)
  by (*simp add*: *range_binary_eq cong del*: *INF_cong_simp*)

lemma *sigma_algebra_iff2*:
  sigma_algebra Ω *M* ⟷
    M ⊆ *Pow* Ω ∧ {} ∈ *M* ∧ (∀ *s* ∈ *M*. Ω − *s* ∈ *M*)
    ∧ (∀ *A*. *range A* ⊆ *M* ⟶(⋃ *i*::*nat*. *A i*) ∈ *M*) (**is** *?P* ⟷ *?R* ∧ *?S* ∧ *?V* ∧ *?W*)
proof
  assume *?P*
  then interpret *sigma_algebra* Ω *M* **.**
  from *space_closed* **show** *?R* ∧ *?S* ∧ *?V* ∧ *?W*
    by *auto*
next
  assume *?R* ∧ *?S* ∧ *?V* ∧ *?W*
  then have *?R ?S ?V ?W*
    by *simp_all*
  show *?P*
  proof (*rule sigma_algebra.intro*)
    show *sigma_algebra_axioms M*
      by *standard* (*use* ‹*?W*› **in** *simp*)
    from ‹*?W*› **have** ∗: *range* (*binary a b*) ⊆ *M* ⟹ ⋃ (*range* (*binary a b*)) ∈ *M*
for *a b*
      by *auto*
    show *algebra* Ω *M*
      unfolding *algebra_iff_Un* **using** ‹*?R*› ‹*?S*› ‹*?V*› ∗
      by (*auto simp add*: *range_binary_eq*)
  qed
qed

Initial Sigma Algebra

Sigma algebras can naturally be created as the closure of any set of M with regard to the properties just postulated.

inductive_set *sigma_sets* :: *'a set* ⇒ *'a set set* ⇒ *'a set set*
  for *sp* :: *'a set* **and** *A* :: *'a set set*
  where
    Basic[*intro*, *simp*]: *a* ∈ *A* ⟹ *a* ∈ *sigma_sets sp A*

| *Empty*: $\{\} \in$ *sigma_sets sp A*
| *Compl*: $a \in$ *sigma_sets sp A* $\Longrightarrow$ *sp* $-$ *a* $\in$ *sigma_sets sp A*
| *Union*: $(\bigwedge i{::}nat.\ a\ i \in$ *sigma_sets sp A*$) \Longrightarrow (\bigcup i.\ a\ i) \in$ *sigma_sets sp A*

lemma (**in** *sigma_algebra*) *sigma_sets_subset*:
assumes *a*: $a \subseteq M$
shows *sigma_sets* $\Omega\ a \subseteq M$
proof
fix *x*
assume $x \in$ *sigma_sets* $\Omega\ a$
from *this* **show** $x \in M$
by (*induct rule*: *sigma_sets.induct*, *auto*) (*metis a subsetD*)
qed

lemma *sigma_sets_into_sp*: $A \subseteq$ *Pow sp* $\Longrightarrow x \in$ *sigma_sets sp A* $\Longrightarrow x \subseteq$ *sp*
by (*erule sigma_sets.induct*, *auto*)

lemma *sigma_sets_finite*: $[\![x \in$ *sigma_sets* Ω (*Pow* Ω); *finite* $\Omega]\!] \Longrightarrow$ *finite x*
by (*meson finite_subset order.refl sigma_sets_into_sp*)

lemma *sigma_algebra_sigma_sets*:
$a \subseteq$ *Pow* $\Omega \Longrightarrow$ *sigma_algebra* Ω (*sigma_sets* $\Omega\ a$)
by (*auto simp add*: *sigma_algebra_iff2 dest*: *sigma_sets_into_sp*
intro!: *sigma_sets.Union sigma_sets.Empty sigma_sets.Compl*)

lemma *sigma_sets_least_sigma_algebra*:
assumes $A \subseteq$ *Pow S*
shows *sigma_sets S A* $= \bigcap\{B.\ A \subseteq B \wedge$ *sigma_algebra S B*$\}$
proof *safe*
fix *B X* **assume** $A \subseteq B$ **and** *sa*: *sigma_algebra S B*
and *X*: $X \in$ *sigma_sets S A*
from *sigma_algebra.sigma_sets_subset*[*OF sa*, *simplified*, *OF* ‹$A \subseteq B$›] *X*
show $X \in B$ **by** *auto*
next
fix *X* **assume** $X \in \bigcap\{B.\ A \subseteq B \wedge$ *sigma_algebra S B*$\}$
then have [*intro*!]: $\bigwedge B.\ A \subseteq B \Longrightarrow$ *sigma_algebra S B* $\Longrightarrow X \in B$
by *simp*
have $A \subseteq$ *sigma_sets S A* **using** *assms* **by** *auto*
moreover have *sigma_algebra S* (*sigma_sets S A*)
using *assms* **by** (*intro sigma_algebra_sigma_sets*[*of A*]) *auto*
ultimately show $X \in$ *sigma_sets S A* **by** *auto*
qed

lemma *sigma_sets_top*: *sp* $\in$ *sigma_sets sp A*
by (*metis Diff_empty sigma_sets.Compl sigma_sets.Empty*)

lemma *binary_in_sigma_sets*:
binary a b i $\in$ *sigma_sets sp A* **if** $a \in$ *sigma_sets sp A* **and** $b \in$ *sigma_sets sp A*
using *that* **by** (*simp add*: *binary_def*)

lemma *sigma_sets_Un*:
  $a \cup b \in$ *sigma_sets sp A* **if** $a \in$ *sigma_sets sp A* **and** $b \in$ *sigma_sets sp A*
  using *that* **by** (*simp add*: *Un_range_binary binary_in_sigma_sets Union*)

lemma *sigma_sets_Inter*:
  assumes *Asb*: $A \subseteq$ *Pow sp*
  shows $(\bigwedge i{::}nat.\ a\ i \in$ *sigma_sets sp A*$) \Longrightarrow (\bigcap i.\ a\ i) \in$ *sigma_sets sp A*
proof −
  assume *ai*: $\bigwedge i{::}nat.\ a\ i \in$ *sigma_sets sp A*
  hence $\bigwedge i{::}nat.\ sp{-}(a\ i) \in$ *sigma_sets sp A*
    by (*rule sigma_sets.Compl*)
  hence $(\bigcup i.\ sp{-}(a\ i)) \in$ *sigma_sets sp A*
    by (*rule sigma_sets.Union*)
  hence $sp{-}(\bigcup i.\ sp{-}(a\ i)) \in$ *sigma_sets sp A*
    by (*rule sigma_sets.Compl*)
  also have $sp{-}(\bigcup i.\ sp{-}(a\ i)) =$ *sp Int* $(\bigcap i.\ a\ i)$
    by *auto*
  also have $\ldots = (\bigcap i.\ a\ i)$ **using** *ai*
    by (*blast dest*: *sigma_sets_into_sp* [*OF Asb*])
  finally show *?thesis* **.**
qed

lemma *sigma_sets_INTER*:
  assumes *Asb*: $A \subseteq$ *Pow sp*
    and *ai*: $\bigwedge i{::}nat.\ i \in S \Longrightarrow a\ i \in$ *sigma_sets sp A* **and** *non*: $S \neq \{\}$
  shows $(\bigcap i{\in}S.\ a\ i) \in$ *sigma_sets sp A*
proof −
  from *ai* **have** $\bigwedge i.$ (*if* $i{\in}S$ *then a i else sp*) $\in$ *sigma_sets sp A*
    by (*simp add*: *sigma_sets.intros*(2−) *sigma_sets_top*)
  hence $(\bigcap i.$ (*if* $i{\in}S$ *then a i else sp*)$) \in$ *sigma_sets sp A*
    by (*rule sigma_sets_Inter* [*OF Asb*])
  also have $(\bigcap i.$ (*if* $i{\in}S$ *then a i else sp*)$) = (\bigcap i{\in}S.\ a\ i)$
    by *auto* (*metis ai non sigma_sets_into_sp subset_empty subset_iff Asb*)+
  finally show *?thesis* **.**
qed

lemma *sigma_sets_UNION*:
  countable B $\Longrightarrow (\bigwedge b.\ b \in B \Longrightarrow b \in$ *sigma_sets X A*$) \Longrightarrow \bigcup B \in$ *sigma_sets X A*
  using *from_nat_into* [*of B*] *range_from_nat_into* [*of B*] *sigma_sets.Union* [*of from_nat_into B X A*]
  by (*cases* $B = \{\}$) (*simp_all add*: *sigma_sets.Empty cong del*: *SUP_cong*)

lemma (**in** *sigma_algebra*) *sigma_sets_eq*:
    sigma_sets $\Omega\ M = M$
proof
  show $M \subseteq$ *sigma_sets* $\Omega\ M$
    by (*metis Set.subsetI sigma_sets.Basic*)

next
show *sigma_sets* Ω *M* ⊆ *M*
by (*metis sigma_sets_subset subset_refl*)
qed

lemma *sigma_sets_eqI*:
assumes *A*: ⋀*a*. *a* ∈ *A* ⟹ *a* ∈ *sigma_sets M B*
assumes *B*: ⋀*b*. *b* ∈ *B* ⟹ *b* ∈ *sigma_sets M A*
shows *sigma_sets M A* = *sigma_sets M B*
proof (*intro set_eqI iffI*)
fix *a* **assume** *a* ∈ *sigma_sets M A*
from *this A* **show** *a* ∈ *sigma_sets M B*
by *induct* (*auto intro*!: *sigma_sets.intros*(2−) *del*: *sigma_sets.Basic*)
next
fix *b* **assume** *b* ∈ *sigma_sets M B*
from *this B* **show** *b* ∈ *sigma_sets M A*
by *induct* (*auto intro*!: *sigma_sets.intros*(2−) *del*: *sigma_sets.Basic*)
qed

lemma *sigma_sets_subseteq*: **assumes** *A* ⊆ *B* **shows** *sigma_sets X A* ⊆ *sigma_sets X B*
proof
fix *x* **assume** *x* ∈ *sigma_sets X A* **then show** *x* ∈ *sigma_sets X B*
by *induct* (*insert* ‹*A* ⊆ *B*›, *auto intro*: *sigma_sets.intros*(2−))
qed

lemma *sigma_sets_mono*: **assumes** *A* ⊆ *sigma_sets X B* **shows** *sigma_sets X A* ⊆ *sigma_sets X B*
proof
fix *x* **assume** *x* ∈ *sigma_sets X A* **then show** *x* ∈ *sigma_sets X B*
by *induct* (*insert* ‹*A* ⊆ *sigma_sets X B*›, *auto intro*: *sigma_sets.intros*(2−))
qed

lemma *sigma_sets_mono'*: **assumes** *A* ⊆ *B* **shows** *sigma_sets X A* ⊆ *sigma_sets X B*
proof
fix *x* **assume** *x* ∈ *sigma_sets X A* **then show** *x* ∈ *sigma_sets X B*
by *induct* (*insert* ‹*A* ⊆ *B*›, *auto intro*: *sigma_sets.intros*(2−))
qed

lemma *sigma_sets_superset_generator*: *A* ⊆ *sigma_sets X A*
by (*auto intro*: *sigma_sets.Basic*)

lemma (**in** *sigma_algebra*) *restriction_in_sets*:
fixes *A* :: *nat* ⇒ *'a set*
assumes *S* ∈ *M*
and ∗: *range A* ⊆ (λ*A*. *S* ∩ *A*) ‘ *M* (**is** _ ⊆ *?r*)
shows *range A* ⊆ *M* (⋃ *i*. *A i*) ∈ (λ*A*. *S* ∩ *A*) ‘ *M*
proof −

```
  { fix i have A i ∈ ?r using ∗ by auto
    hence ∃ B. A i = B ∩ S ∧ B ∈ M by auto
    hence A i ⊆ S A i ∈ M using ‹S ∈ M› by auto }
  thus range A ⊆ M (⋃ i. A i) ∈ (λA. S ∩ A) ‘ M
    by (auto intro!: image_eqI[of _ _ (⋃ i. A i)])
qed

lemma (in sigma_algebra) restricted_sigma_algebra:
  assumes S ∈ M
  shows sigma_algebra S (restricted_space S)
  unfolding sigma_algebra_def sigma_algebra_axioms_def
proof safe
  show algebra S (restricted_space S) using restricted_algebra[OF assms] .
next
  fix A :: nat ⇒ 'a set assume range A ⊆ restricted_space S
  from restriction_in_sets[OF assms this[simplified]]
  show (⋃ i. A i) ∈ restricted_space S by simp
qed

lemma sigma_sets_Int:
  assumes A ∈ sigma_sets sp st A ⊆ sp
  shows (∩) A ‘ sigma_sets sp st = sigma_sets A ((∩) A ‘ st)
proof (intro equalityI subsetI)
  fix x assume x ∈ (∩) A ‘ sigma_sets sp st
  then obtain y where y ∈ sigma_sets sp st x = y ∩ A by auto
  then have x ∈ sigma_sets (A ∩ sp) ((∩) A ‘ st)
  proof (induct arbitrary: x)
    case (Compl a)
    then show ?case
      by (force intro!: sigma_sets.Compl simp: Diff_Int_distrib ac_simps)
  next
    case (Union a)
    then show ?case
      by (auto intro!: sigma_sets.Union
               simp add: UN_extend_simps simp del: UN_simps)
  qed (auto intro!: sigma_sets.intros(2−))
  then show x ∈ sigma_sets A ((∩) A ‘ st)
    using ‹A ⊆ sp› by (simp add: Int_absorb2)
next
  fix x assume x ∈ sigma_sets A ((∩) A ‘ st)
  then show x ∈ (∩) A ‘ sigma_sets sp st
  proof induct
    case (Compl a)
    then obtain x where a = A ∩ x x ∈ sigma_sets sp st by auto
    then show ?case using ‹A ⊆ sp›
      by (force simp add: image_iff intro!: bexI[of _ sp − x] sigma_sets.Compl)
  next
    case (Union a)
    then have ∀ i. ∃ x. x ∈ sigma_sets sp st ∧ a i = A ∩ x
```

```
    by (auto simp: image_iff Bex_def)
  then obtain f where ∀x. f x ∈ sigma_sets sp st ∧ a x = A ∩ f x
    by metis
  then show ?case
    by (auto intro!: bexI[of _ (⋃x. f x)] sigma_sets.Union
          simp add: image_iff)
 qed (auto intro!: sigma_sets.intros(2−))
qed

lemma sigma_sets_empty_eq: sigma_sets A {} = {{}, A}
proof (intro set_eqI iffI)
 fix a assume a ∈ sigma_sets A {} then show a ∈ {{}, A}
   by induct blast+
qed (auto intro: sigma_sets.Empty sigma_sets_top)

lemma sigma_sets_single[simp]: sigma_sets A {A} = {{}, A}
proof (intro set_eqI iffI)
 fix x assume x ∈ sigma_sets A {A}
 then show x ∈ {{}, A}
   by induct blast+
next
 fix x assume x ∈ {{}, A}
 then show x ∈ sigma_sets A {A}
   by (auto intro: sigma_sets.Empty sigma_sets_top)
qed

lemma sigma_sets_sigma_sets_eq:
 M ⊆ Pow S ⟹ sigma_sets S (sigma_sets S M) = sigma_sets S M
 by (rule sigma_algebra.sigma_sets_eq[OF sigma_algebra_sigma_sets, of M S])
auto

lemma sigma_sets_singleton:
 assumes X ⊆ S
 shows sigma_sets S { X } = { {}, X, S − X, S }
proof −
 interpret sigma_algebra S { {}, X, S − X, S }
   by (rule sigma_algebra_single_set) fact
 have sigma_sets S { X } ⊆ sigma_sets S { {}, X, S − X, S }
   by (rule sigma_sets_subseteq) simp
 moreover have ... = { {}, X, S − X, S }
   using sigma_sets_eq by simp
 moreover
 { fix A assume A ∈ { {}, X, S − X, S }
   then have A ∈ sigma_sets S { X }
     by (auto intro: sigma_sets.intros(2−) sigma_sets_top) }
 ultimately have sigma_sets S { X } = sigma_sets S { {}, X, S − X, S }
   by (intro antisym) auto
 with sigma_sets_eq show ?thesis by simp
qed
```

lemma *restricted_sigma*:
assumes *S*: $S \in$ *sigma_sets* Ω M **and** *M*: $M \subseteq$ *Pow* Ω
shows *algebra.restricted_space* (*sigma_sets* Ω M) S =
sigma_sets S (*algebra.restricted_space* M S)
proof −
from S *sigma_sets_into_sp*[*OF* M]
have $S \in$ *sigma_sets* Ω M $S \subseteq \Omega$ **by** *auto*
from *sigma_sets_Int*[*OF this*]
show *?thesis* **by** *simp*
qed

lemma *sigma_sets_vimage_commute*:
assumes *X*: $X \in \Omega \to \Omega'$
shows $\{X -` A \cap \Omega \mid A.\ A \in$ *sigma_sets* Ω' $M'\}$
$=$ *sigma_sets* Ω $\{X -` A \cap \Omega \mid A.\ A \in M'\}$ (**is** *?L* = *?R*)
proof
show *?L* $\subseteq$ *?R*
proof *clarify*
fix A **assume** $A \in$ *sigma_sets* Ω' M'
then show $X -` A \cap \Omega \in$ *?R*
proof *induct*
case *Empty* **then show** *?case*
by (*auto intro*!: *sigma_sets.Empty*)
next
case (*Compl* B)
have [*simp*]: $X -` (\Omega' - B) \cap \Omega = \Omega - (X -` B \cap \Omega)$
by (*auto simp add*: *funcset_mem* [*OF* X])
with *Compl* **show** *?case*
by (*auto intro*!: *sigma_sets.Compl*)
next
case (*Union* F)
then show *?case*
by (*auto simp add*: *vimage_UN UN_extend_simps*(*4*) *simp del*: *UN_simps*
intro!: *sigma_sets.Union*)
qed *auto*
qed
show *?R* $\subseteq$ *?L*
proof *clarify*
fix A **assume** $A \in$ *?R*
then show $\exists B.\ A = X -` B \cap \Omega \wedge B \in$ *sigma_sets* Ω' M'
proof *induct*
case (*Basic* B) **then show** *?case* **by** *auto*
next
case *Empty* **then show** *?case*
by (*auto intro*!: *sigma_sets.Empty exI*[*of* _ {}])
next
case (*Compl* B)
then obtain A **where** A: $B = X -` A \cap \Omega$ $A \in$ *sigma_sets* Ω' M' **by** *auto*

```
      then have [simp]: Ω − B = X −‘ (Ω' − A) ∩ Ω
        by (auto simp add: funcset_mem [OF X])
      with A(2) show ?case
        by (auto intro: sigma_sets.Compl)
    next
      case (Union F)
      then have ∀ i. ∃ B. F i = X −‘ B ∩ Ω ∧ B ∈ sigma_sets Ω' M' by auto
      then obtain A where ∀ x. F x = X −‘ A x ∩ Ω ∧ A x ∈ sigma_sets Ω' M'
        by metis
      then show ?case
        by (auto simp: vimage_UN[symmetric] intro: sigma_sets.Union)
    qed
  qed
qed

lemma (in ring_of_sets) UNION_in_sets:
  fixes A:: nat ⇒ 'a set
  assumes A: range A ⊆ M
  shows  (⋃ i∈{0..<n}. A i) ∈ M
proof (induct n)
  case 0 show ?case by simp
next
  case (Suc n)
  thus ?case
    by (simp add: atLeastLessThanSuc) (metis A Un UNIV_I image_subset_iff)
qed

lemma (in ring_of_sets) range_disjointed_sets:
  assumes A: range A ⊆ M
  shows  range (disjointed A) ⊆ M
proof (auto simp add: disjointed_def)
  fix n
  show A n − (⋃ i∈{0..<n}. A i) ∈ M using UNION_in_sets
    by (metis A Diff UNIV_I image_subset_iff)
qed

lemma (in algebra) range_disjointed_sets':
  range A ⊆ M ⟹ range (disjointed A) ⊆ M
  using range_disjointed_sets .

lemma sigma_algebra_disjoint_iff:
  sigma_algebra Ω M ⟷ algebra Ω M ∧
    (∀ A. range A ⊆ M ⟶ disjoint_family A ⟶ (⋃ i::nat. A i) ∈ M)
proof (auto simp add: sigma_algebra_iff)
  fix A :: nat ⇒ 'a set
  assume M: algebra Ω M
     and A: range A ⊆ M
     and UnA: ∀ A. range A ⊆ M ⟶ disjoint_family A ⟶ (⋃ i::nat. A i) ∈ M
  hence range (disjointed A) ⊆ M ⟶
```

disjoint_family (*disjointed A*) ⟶
(⋃ *i*. *disjointed A i*) ∈ *M* **by** *blast*
hence (⋃ *i*. *disjointed A i*) ∈ *M*
by (*simp add*: *algebra.range_disjointed_sets'*[*of* Ω] *M A disjoint_family_disjointed*)
thus (⋃ *i*::*nat*. *A i*) ∈ *M* **by** (*simp add*: *UN_disjointed_eq*)
qed

Ring generated by a semiring

definition (**in** *semiring_of_sets*) *generated_ring* :: *'a set set* **where**
generated_ring = { ⋃ *C* | *C*. *C* ⊆ *M* ∧ *finite C* ∧ *disjoint C* }

lemma (**in** *semiring_of_sets*) *generated_ringE*[*elim?*]:
assumes *a* ∈ *generated_ring*
obtains *C* **where** *finite C disjoint C C* ⊆ *M a* = ⋃ *C*
using *assms* **unfolding** *generated_ring_def* **by** *auto*

lemma (**in** *semiring_of_sets*) *generated_ringI*[*intro?*]:
assumes *finite C disjoint C C* ⊆ *M a* = ⋃ *C*
shows *a* ∈ *generated_ring*
using *assms* **unfolding** *generated_ring_def* **by** *auto*

lemma (**in** *semiring_of_sets*) *generated_ringI_Basic*:
A ∈ *M* ⟹ *A* ∈ *generated_ring*
by (*rule generated_ringI*[*of* {*A*}]) (*auto simp*: *disjoint_def*)

lemma (**in** *semiring_of_sets*) *generated_ring_disjoint_Un*[*intro*]:
assumes *a*: *a* ∈ *generated_ring* **and** *b*: *b* ∈ *generated_ring*
and *a* ∩ *b* = {}
shows *a* ∪ *b* ∈ *generated_ring*
proof −
from *a b* **obtain** *Ca Cb*
where *Ca*: *finite Ca disjoint Ca Ca* ⊆ *M a* = ⋃ *Ca*
and *Cb*: *finite Cb disjoint Cb Cb* ⊆ *M b* = ⋃ *Cb*
using *generated_ringE* **by** *metis*
show *?thesis*
proof
from ‹*a* ∩ *b* = {}› *Ca Cb* **show** *disjoint* (*Ca* ∪ *Cb*)
by (*auto intro*!: *disjoint_union*)
qed (*use Ca Cb* **in** *auto*)
qed

lemma (**in** *semiring_of_sets*) *generated_ring_empty*: {} ∈ *generated_ring*
by (*auto simp*: *generated_ring_def disjoint_def*)

lemma (**in** *semiring_of_sets*) *generated_ring_disjoint_Union*:
assumes *finite A* **shows** *A* ⊆ *generated_ring* ⟹ *disjoint A* ⟹ ⋃ *A* ∈ *generated_ring*
using *assms* **by** (*induct A*) (*auto simp*: *disjoint_def intro*!: *generated_ring_disjoint_Un*

generated_ring_empty)

lemma (**in** *semiring_of_sets*) *generated_ring_disjoint_UNION*:
finite I ⟹ *disjoint* (*A* ‘ *I*) ⟹ (⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *generated_ring*) ⟹ ⋃(*A* ‘ *I*) ∈ *generated_ring*
by (*intro generated_ring_disjoint_Union*) *auto*

lemma (**in** *semiring_of_sets*) *generated_ring_Int*:
assumes *a*: *a* ∈ *generated_ring* **and** *b*: *b* ∈ *generated_ring*
shows *a* ∩ *b* ∈ *generated_ring*
proof −
from *a b* **obtain** *Ca Cb*
where *Ca*: *finite Ca disjoint Ca Ca* ⊆ *M a* = ⋃ *Ca*
and *Cb*: *finite Cb disjoint Cb Cb* ⊆ *M b* = ⋃ *Cb*
using *generated_ringE* **by** *metis*
define *C* **where** *C* = (λ(*a*,*b*). *a* ∩ *b*)‘ (*Ca*×*Cb*)
show *?thesis*
proof
show *disjoint C*
proof (*simp add*: *disjoint_def C_def*, *intro ballI impI*)
fix *a1 b1 a2 b2* **assume** *sets*: *a1* ∈ *Ca b1* ∈ *Cb a2* ∈ *Ca b2* ∈ *Cb*
assume *a1* ∩ *b1* ≠ *a2* ∩ *b2*
then have *a1* ≠ *a2* ∨ *b1* ≠ *b2* **by** *auto*
then show (*a1* ∩ *b1*) ∩ (*a2* ∩ *b2*) = {}
proof
assume *a1* ≠ *a2*
with *sets Ca* **have** *a1* ∩ *a2* = {}
by (*auto simp*: *disjoint_def*)
then show *?thesis* **by** *auto*
next
assume *b1* ≠ *b2*
with *sets Cb* **have** *b1* ∩ *b2* = {}
by (*auto simp*: *disjoint_def*)
then show *?thesis* **by** *auto*
qed
qed
qed (*use Ca Cb* **in** ‹*auto simp*: *C_def*›)
qed

lemma (**in** *semiring_of_sets*) *generated_ring_Inter*:
assumes *finite A A* ≠ {} **shows** *A* ⊆ *generated_ring* ⟹ ⋂ *A* ∈ *generated_ring*
using *assms* **by** (*induct A rule*: *finite_ne_induct*) (*auto intro*: *generated_ring_Int*)

lemma (**in** *semiring_of_sets*) *generated_ring_INTER*:
finite I ⟹ *I* ≠ {} ⟹ (⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *generated_ring*) ⟹ ⋂(*A* ‘ *I*) ∈ *generated_ring*
by (*intro generated_ring_Inter*) *auto*

lemma (**in** *semiring_of_sets*) *generating_ring*:

```
  ring_of_sets Ω generated_ring
proof (rule ring_of_setsI)
  let ?R = generated_ring
  show ?R ⊆ Pow Ω
   using sets_into_space by (auto simp: generated_ring_def generated_ring_empty)
  show {} ∈ ?R by (rule generated_ring_empty)

  {
    fix a b assume a ∈ ?R b ∈ ?R
    then obtain Ca Cb
      where Ca: finite Ca disjoint Ca Ca ⊆ M a = ⋃ Ca
        and Cb: finite Cb disjoint Cb Cb ⊆ M b = ⋃ Cb
      using generated_ringE by metis
    show a − b ∈ ?R
    proof cases
      assume Cb = {} with Cb ‹a ∈ ?R› show ?thesis
        by simp
    next
      assume Cb ≠ {}
      with Ca Cb have a − b = (⋃a'∈Ca. ⋂b'∈Cb. a' − b') by auto
      also have ... ∈ ?R
      proof (intro generated_ring_INTER generated_ring_disjoint_UNION)
        fix a b assume a ∈ Ca b ∈ Cb
        with Ca Cb Diff_cover[of a b] show a − b ∈ ?R
          by (auto simp add: generated_ring_def)
            (metis DiffI Diff_eq_empty_iff empty_iff)
      next
        show disjoint ((λa'. ⋂b'∈Cb. a' − b')‘Ca)
          using Ca by (auto simp add: disjoint_def ‹Cb ≠ {}›)
      next
        show finite Ca finite Cb Cb ≠ {} by fact+
      qed
      finally show a − b ∈ ?R .
    qed
  }
  note Diff = this

  fix a b assume sets: a ∈ ?R b ∈ ?R
  have a ∪ b = (a − b) ∪ (a ∩ b) ∪ (b − a) by auto
  also have ... ∈ ?R
    by (intro sets generated_ring_disjoint_Un generated_ring_Int Diff) auto
  finally show a ∪ b ∈ ?R .
qed

lemma (in semiring_of_sets) sigma_sets_generated_ring_eq: sigma_sets Ω generated_ring = sigma_sets Ω M
proof
  interpret M: sigma_algebra Ω sigma_sets Ω M
    using space_closed by (rule sigma_algebra_sigma_sets)
```

show *sigma_sets* Ω *generated_ring* ⊆ *sigma_sets* Ω *M*
by (*blast intro*!: *sigma_sets_mono elim*: *generated_ringE*)
qed (*auto intro*!: *generated_ringI_Basic sigma_sets_mono*)

A Two-Element Series

definition *binaryset* :: *'a set* ⇒ *'a set* ⇒ *nat* ⇒ *'a set*
where *binaryset A B* = (λ*x*. {})(*0* := *A*, *Suc 0* := *B*)

lemma *range_binaryset_eq*: *range*(*binaryset A B*) = {*A*,*B*,{}}
apply (*simp add*: *binaryset_def*)
apply (*rule set_eqI*)
apply (*auto simp add*: *image_iff*)
done

lemma *UN_binaryset_eq*: (⋃ *i*. *binaryset A B i*) = *A* ∪ *B*
by (*simp add*: *range_binaryset_eq cong del*: *SUP_cong_simp*)

Closed CDI

definition *closed_cdi* :: *'a set* ⇒ *'a set set* ⇒ *bool* **where**
closed_cdi Ω *M* ⟷
M ⊆ *Pow* Ω &
(∀ *s* ∈ *M*. Ω − *s* ∈ *M*) &
(∀ *A*. (*range A* ⊆ *M*) & (*A 0* = {}) & (∀ *n*. *A n* ⊆ *A* (*Suc n*)) ⟶
(⋃ *i*. *A i*) ∈ *M*) &
(∀ *A*. (*range A* ⊆ *M*) & *disjoint_family A* ⟶ (⋃ *i*::*nat*. *A i*) ∈ *M*)

inductive_set
smallest_ccdi_sets :: *'a set* ⇒ *'a set set* ⇒ *'a set set*
for Ω *M*
where
Basic [*intro*]:
a ∈ *M* ⟹ *a* ∈ *smallest_ccdi_sets* Ω *M*
| *Compl* [*intro*]:
a ∈ *smallest_ccdi_sets* Ω *M* ⟹ Ω − *a* ∈ *smallest_ccdi_sets* Ω *M*
| *Inc*:
range A ∈ *Pow*(*smallest_ccdi_sets* Ω *M*) ⟹ *A 0* = {} ⟹ (⋀*n*. *A n* ⊆ *A*
(*Suc n*))
⟹ (⋃ *i*. *A i*) ∈ *smallest_ccdi_sets* Ω *M*
| *Disj*:
range A ∈ *Pow*(*smallest_ccdi_sets* Ω *M*) ⟹ *disjoint_family A*
⟹ (⋃ *i*::*nat*. *A i*) ∈ *smallest_ccdi_sets* Ω *M*

lemma (**in** *subset_class*) *smallest_closed_cdi1*: *M* ⊆ *smallest_ccdi_sets* Ω *M*
by *auto*

lemma (**in** *subset_class*) *smallest_ccdi_sets*: *smallest_ccdi_sets* Ω *M* ⊆ *Pow* Ω
apply (*rule subsetI*)
apply (*erule smallest_ccdi_sets.induct*)

apply (*auto intro*: *range_subsetD dest*: *sets_into_space*)
done

lemma (**in** *subset_class*) *smallest_closed_cdi2*: *closed_cdi* Ω (*smallest_ccdi_sets* Ω *M*)
apply (*auto simp add*: *closed_cdi_def smallest_ccdi_sets*)
apply (*blast intro*: *smallest_ccdi_sets.Inc smallest_ccdi_sets.Disj*) +
done

lemma *closed_cdi_subset*: *closed_cdi* Ω *M* ⟹ *M* ⊆ *Pow* Ω
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Compl*: *closed_cdi* Ω *M* ⟹ *s* ∈ *M* ⟹ Ω − *s* ∈ *M*
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Inc*:
closed_cdi Ω *M* ⟹ *range A* ⊆ *M* ⟹ *A 0* = {} ⟹ (!!*n*. *A n* ⊆ *A* (*Suc n*)) ⟹ (⋃ *i*. *A i*) ∈ *M*
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Disj*:
closed_cdi Ω *M* ⟹ *range A* ⊆ *M* ⟹ *disjoint_family A* ⟹ (⋃ *i*::*nat*. *A i*) ∈ *M*
by (*simp add*: *closed_cdi_def*)

lemma *closed_cdi_Un*:
assumes *cdi*: *closed_cdi* Ω *M* **and** *empty*: {} ∈ *M*
and *A*: *A* ∈ *M* **and** *B*: *B* ∈ *M*
and *disj*: *A* ∩ *B* = {}
shows *A* ∪ *B* ∈ *M*
proof −
have *ra*: *range* (*binaryset A B*) ⊆ *M*
by (*simp add*: *range_binaryset_eq empty A B*)
have *di*: *disjoint_family* (*binaryset A B*) **using** *disj*
by (*simp add*: *disjoint_family_on_def binaryset_def Int_commute*)
from *closed_cdi_Disj* [*OF cdi ra di*]
show *?thesis*
by (*simp add*: *UN_binaryset_eq*)
qed

lemma (**in** *algebra*) *smallest_ccdi_sets_Un*:
assumes *A*: *A* ∈ *smallest_ccdi_sets* Ω *M* **and** *B*: *B* ∈ *smallest_ccdi_sets* Ω *M*
and *disj*: *A* ∩ *B* = {}
shows *A* ∪ *B* ∈ *smallest_ccdi_sets* Ω *M*
proof −
have *ra*: *range* (*binaryset A B*) ∈ *Pow* (*smallest_ccdi_sets* Ω *M*)
by (*simp add*: *range_binaryset_eq A B smallest_ccdi_sets.Basic*)
have *di*: *disjoint_family* (*binaryset A B*) **using** *disj*
by (*simp add*: *disjoint_family_on_def binaryset_def Int_commute*)

```
  from Disj [OF ra di]
  show ?thesis
    by (simp add: UN_binaryset_eq)
qed

lemma (in algebra) smallest_ccdi_sets_Int1:
  assumes a: a ∈ M
  shows b ∈ smallest_ccdi_sets Ω M ⟹ a ∩ b ∈ smallest_ccdi_sets Ω M
proof (induct rule: smallest_ccdi_sets.induct)
  case (Basic x)
  thus ?case
    by (metis a Int smallest_ccdi_sets.Basic)
next
  case (Compl x)
  have a ∩ (Ω − x) = Ω − ((Ω − a) ∪ (a ∩ x))
    by blast
  also have ... ∈ smallest_ccdi_sets Ω M
    by (metis smallest_ccdi_sets.Compl a Compl(2) Diff_Int2 Diff_Int_distrib2
          Diff_disjoint Int_Diff Int_empty_right smallest_ccdi_sets_Un
          smallest_ccdi_sets.Basic smallest_ccdi_sets.Compl)
  finally show ?case .
next
  case (Inc A)
  have 1: (⋃i. (λi. a ∩ A i) i) = a ∩ (⋃i. A i)
    by blast
  have range (λi. a ∩ A i) ∈ Pow(smallest_ccdi_sets Ω M) using Inc
    by blast
  moreover have (λi. a ∩ A i) 0 = {}
    by (simp add: Inc)
  moreover have !!n. (λi. a ∩ A i) n ⊆ (λi. a ∩ A i) (Suc n) using Inc
    by blast
  ultimately have 2: (⋃i. (λi. a ∩ A i) i) ∈ smallest_ccdi_sets Ω M
    by (rule smallest_ccdi_sets.Inc)
  show ?case
    by (metis 1 2)
next
  case (Disj A)
  have 1: (⋃i. (λi. a ∩ A i) i) = a ∩ (⋃i. A i)
    by blast
  have range (λi. a ∩ A i) ∈ Pow(smallest_ccdi_sets Ω M) using Disj
    by blast
  moreover have disjoint_family (λi. a ∩ A i) using Disj
    by (auto simp add: disjoint_family_on_def)
  ultimately have 2: (⋃i. (λi. a ∩ A i) i) ∈ smallest_ccdi_sets Ω M
    by (rule smallest_ccdi_sets.Disj)
  show ?case
    by (metis 1 2)
qed
```

lemma (**in** *algebra*) *smallest_ccdi_sets_Int*:
assumes *b*: $b \in$ *smallest_ccdi_sets* Ω *M*
shows $a \in$ *smallest_ccdi_sets* Ω *M* $\Longrightarrow a \cap b \in$ *smallest_ccdi_sets* Ω *M*
proof (*induct rule*: *smallest_ccdi_sets.induct*)
case (*Basic x*)
thus *?case*
by (*metis b smallest_ccdi_sets_Int1*)
next
case (*Compl x*)
have $(\Omega - x) \cap b = \Omega - (x \cap b \cup (\Omega - b))$
by *blast*
also have ... $\in$ *smallest_ccdi_sets* Ω *M*
by (*metis Compl*(*2*) *Diff_disjoint Int_Diff Int_commute Int_empty_right b*
smallest_ccdi_sets.Compl smallest_ccdi_sets_Un)
finally show *?case* **.**
next
case (*Inc A*)
have *1*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) = (\bigcup i.\ A\ i) \cap b$
by *blast*
have *range* $(\lambda i.\ A\ i \cap b) \in$ *Pow*(*smallest_ccdi_sets* Ω *M*) **using** *Inc*
by *blast*
moreover have $(\lambda i.\ A\ i \cap b)\ 0 = \{\}$
by (*simp add*: *Inc*)
moreover have $!!n.\ (\lambda i.\ A\ i \cap b)\ n \subseteq (\lambda i.\ A\ i \cap b)\ (Suc\ n)$ **using** *Inc*
by *blast*
ultimately have *2*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) \in$ *smallest_ccdi_sets* Ω *M*
by (*rule smallest_ccdi_sets.Inc*)
show *?case*
by (*metis 1 2*)
next
case (*Disj A*)
have *1*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) = (\bigcup i.\ A\ i) \cap b$
by *blast*
have *range* $(\lambda i.\ A\ i \cap b) \in$ *Pow*(*smallest_ccdi_sets* Ω *M*) **using** *Disj*
by *blast*
moreover have *disjoint_family* $(\lambda i.\ A\ i \cap b)$ **using** *Disj*
by (*auto simp add*: *disjoint_family_on_def*)
ultimately have *2*: $(\bigcup i.\ (\lambda i.\ A\ i \cap b)\ i) \in$ *smallest_ccdi_sets* Ω *M*
by (*rule smallest_ccdi_sets.Disj*)
show *?case*
by (*metis 1 2*)
qed

lemma (**in** *algebra*) *sigma_property_disjoint_lemma*:
assumes *sbC*: $M \subseteq C$
and *ccdi*: *closed_cdi* Ω *C*
shows *sigma_sets* Ω *M* $\subseteq C$
proof −

```
  have smallest_ccdi_sets Ω M ∈ {B . M ⊆ B ∧ sigma_algebra Ω B}
    apply (auto simp add: sigma_algebra_disjoint_iff algebra_iff_Int
          smallest_ccdi_sets_Int)
    apply (metis Union_Pow_eq Union_upper subsetD smallest_ccdi_sets)
    apply (blast intro: smallest_ccdi_sets.Disj)
    done
  hence sigma_sets (Ω) (M) ⊆ smallest_ccdi_sets Ω M
    by clarsimp
       (drule sigma_algebra.sigma_sets_subset [where a=M], auto)
  also have ... ⊆ C
    proof
      fix x
      assume x: x ∈ smallest_ccdi_sets Ω M
      thus x ∈ C
        proof (induct rule: smallest_ccdi_sets.induct)
          case (Basic x)
          thus ?case
            by (metis Basic subsetD sbC)
        next
          case (Compl x)
          thus ?case
            by (blast intro: closed_cdi_Compl [OF ccdi, simplified])
        next
          case (Inc A)
          thus ?case
              by (auto intro: closed_cdi_Inc [OF ccdi, simplified])
        next
          case (Disj A)
          thus ?case
              by (auto intro: closed_cdi_Disj [OF ccdi, simplified])
        qed
    qed
  finally show ?thesis .
qed

lemma (in algebra) sigma_property_disjoint:
  assumes sbC: M ⊆ C
      and compl: !!s. s ∈ C ∩ sigma_sets (Ω) (M) ⟹ Ω − s ∈ C
      and inc: !!A. range A ⊆ C ∩ sigma_sets (Ω) (M)
                    ⟹ A 0 = {} ⟹ (!!n. A n ⊆ A (Suc n))
                    ⟹ (⋃ i. A i) ∈ C
      and disj: !!A. range A ⊆ C ∩ sigma_sets (Ω) (M)
                     ⟹ disjoint_family A ⟹ (⋃ i::nat. A i) ∈ C
  shows sigma_sets (Ω) (M) ⊆ C
proof −
  have sigma_sets (Ω) (M) ⊆ C ∩ sigma_sets (Ω) (M)
    proof (rule sigma_property_disjoint_lemma)
      show M ⊆ C ∩ sigma_sets (Ω) (M)
        by (metis Int_greatest Set.subsetI sbC sigma_sets.Basic)
```

next
show *closed_cdi* Ω (*C* ∩ *sigma_sets* (Ω) (*M*))
by (*simp add*: *closed_cdi_def compl inc disj*)
(*metis PowI Set.subsetI le_infI2 sigma_sets_into_sp space_closed*
IntE sigma_sets.Compl range_subsetD sigma_sets.Union)
qed
thus *?thesis*
by *blast*
qed

Dynkin systems

locale *Dynkin_system* = *subset_class* +
assumes *space*: Ω ∈ *M*
and *compl*[*intro*!]: ⋀*A*. *A* ∈ *M* ⟹ Ω − *A* ∈ *M*
and *UN*[*intro*!]: ⋀*A*. *disjoint_family A* ⟹ *range A* ⊆ *M*
⟹ (⋃ *i*::*nat*. *A i*) ∈ *M*

lemma (**in** *Dynkin_system*) *empty*[*intro*, *simp*]: {} ∈ *M*
using *space compl*[*of* Ω] **by** *simp*

lemma (**in** *Dynkin_system*) *diff*:
assumes *sets*: *D* ∈ *M E* ∈ *M* **and** *D* ⊆ *E*
shows *E* − *D* ∈ *M*
proof −
let *?f* = λ*x*. *if x* = *0 then D else if x* = *Suc 0 then* Ω − *E else* {}
have *range ?f* = {*D*, Ω − *E*, {}}
by (*auto simp*: *image_iff*)
moreover have *D* ∪ (Ω − *E*) = (⋃ *i*. *?f i*)
by (*auto simp*: *image_iff split*: *if_split_asm*)
moreover
have *disjoint_family ?f* **unfolding** *disjoint_family_on_def*
using ‹*D* ∈ *M*›[*THEN sets_into_space*] ‹*D* ⊆ *E*› **by** *auto*
ultimately have Ω − (*D* ∪ (Ω − *E*)) ∈ *M*
using *sets UN* **by** *auto fastforce*
also have Ω − (*D* ∪ (Ω − *E*)) = *E* − *D*
using *assms sets_into_space* **by** *auto*
finally show *?thesis* **.**
qed

lemma *Dynkin_systemI*:
assumes ⋀ *A*. *A* ∈ *M* ⟹ *A* ⊆ Ω Ω ∈ *M*
assumes ⋀ *A*. *A* ∈ *M* ⟹ Ω − *A* ∈ *M*
assumes ⋀ *A*. *disjoint_family A* ⟹ *range A* ⊆ *M*
⟹ (⋃ *i*::*nat*. *A i*) ∈ *M*
shows *Dynkin_system* Ω *M*
using *assms* **by** (*auto simp*: *Dynkin_system_def Dynkin_system_axioms_def*
subset_class_def)

lemma *Dynkin_systemI'*:
assumes *1*: $\bigwedge A.\ A \in M \Longrightarrow A \subseteq \Omega$
assumes *empty*: $\{\} \in M$
assumes *Diff*: $\bigwedge A.\ A \in M \Longrightarrow \Omega - A \in M$
assumes *2*: $\bigwedge A.\ \mathit{disjoint\_family}\ A \Longrightarrow \mathit{range}\ A \subseteq M \Longrightarrow (\bigcup i{::}\mathit{nat}.\ A\ i) \in M$
shows *Dynkin_system* Ω M
proof −
from *Diff*[*OF empty*] **have** $\Omega \in M$ **by** *auto*
from *1 this Diff 2* **show** *?thesis*
by (*intro Dynkin_systemI*) *auto*
qed

lemma *Dynkin_system_trivial*:
shows *Dynkin_system* A (*Pow* A)
by (*rule Dynkin_systemI*) *auto*

lemma *sigma_algebra_imp_Dynkin_system*:
assumes *sigma_algebra* Ω M **shows** *Dynkin_system* Ω M
proof −
interpret *sigma_algebra* Ω M **by** *fact*
show *?thesis* **using** *sets_into_space* **by** (*fastforce intro*!: *Dynkin_systemI*)
qed

Intersection sets systems

definition *Int_stable* :: $'a\ set\ set \Rightarrow bool$ **where**
Int_stable $M \longleftrightarrow (\forall\ a \in M.\ \forall\ b \in M.\ a \cap b \in M)$

lemma (**in** *algebra*) *Int_stable*: *Int_stable* M
unfolding *Int_stable_def* **by** *auto*

lemma *Int_stableI_image*:
$(\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow \exists k{\in}I.\ A\ i \cap A\ j = A\ k) \Longrightarrow$ *Int_stable* $(A\ `\ I)$
by (*auto simp*: *Int_stable_def image_def*)

lemma *Int_stableI*:
$(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow a \cap b \in A) \Longrightarrow$ *Int_stable* A
unfolding *Int_stable_def* **by** *auto*

lemma *Int_stableD*:
Int_stable $M \Longrightarrow a \in M \Longrightarrow b \in M \Longrightarrow a \cap b \in M$
unfolding *Int_stable_def* **by** *auto*

lemma (**in** *Dynkin_system*) *sigma_algebra_eq_Int_stable*:
sigma_algebra Ω $M \longleftrightarrow$ *Int_stable* M
proof
assume *sigma_algebra* Ω M **then show** *Int_stable* M
unfolding *sigma_algebra_def* **using** *algebra.Int_stable* **by** *auto*

next
assume *Int_stable M*
show *sigma_algebra* Ω *M*
unfolding *sigma_algebra_disjoint_iff algebra_iff_Un*
proof (*intro conjI ballI allI impI*)
show *M* ⊆ *Pow* (Ω) **using** *sets_into_space* **by** *auto*
next
fix *A B* **assume** *A* ∈ *M B* ∈ *M*
then have *A* ∪ *B* = Ω − ((Ω − *A*) ∩ (Ω − *B*))
Ω − *A* ∈ *M* Ω − *B* ∈ *M*
using *sets_into_space* **by** *auto*
then show *A* ∪ *B* ∈ *M*
using ‹*Int_stable M*› **unfolding** *Int_stable_def* **by** *auto*
qed *auto*
qed

Smallest Dynkin systems

definition *Dynkin* :: *'a set* ⇒ *'a set set* ⇒ *'a set set* **where**
Dynkin Ω *M* = (⋂{*D*. *Dynkin_system* Ω *D* ∧ *M* ⊆ *D*})

lemma *Dynkin_system_Dynkin*:
assumes *M* ⊆ *Pow* (Ω)
shows *Dynkin_system* Ω (*Dynkin* Ω *M*)
proof (*rule Dynkin_systemI*)
fix *A* **assume** *A* ∈ *Dynkin* Ω *M*
moreover
{ fix *D* **assume** *A* ∈ *D* **and** *d*: *Dynkin_system* Ω *D*
then have *A* ⊆ Ω **by** (*auto simp*: *Dynkin_system_def subset_class_def*) **}**
moreover have {*D*. *Dynkin_system* Ω *D* ∧ *M* ⊆ *D*} ≠ {}
using *assms Dynkin_system_trivial* **by** *fastforce*
ultimately show *A* ⊆ Ω
unfolding *Dynkin_def* **using** *assms*
by *auto*
next
show Ω ∈ *Dynkin* Ω *M*
unfolding *Dynkin_def* **using** *Dynkin_system.space* **by** *fastforce*
next
fix *A* **assume** *A* ∈ *Dynkin* Ω *M*
then show Ω − *A* ∈ *Dynkin* Ω *M*
unfolding *Dynkin_def* **using** *Dynkin_system.compl* **by** *force*
next
fix *A* :: *nat* ⇒ *'a set*
assume *A*: *disjoint_family A range A* ⊆ *Dynkin* Ω *M*
show (⋃ *i*. *A i*) ∈ *Dynkin* Ω *M* **unfolding** *Dynkin_def*
proof (*simp*, *safe*)
fix *D* **assume** *Dynkin_system* Ω *D M* ⊆ *D*
with *A* **have** (⋃ *i*. *A i*) ∈ *D*
by (*intro Dynkin_system.UN*) (*auto simp*: *Dynkin_def*)

```
    then show (⋃ i. A i) ∈ D by auto
  qed
qed

lemma Dynkin_Basic[intro]: A ∈ M ⟹ A ∈ Dynkin Ω M
  unfolding Dynkin_def by auto

lemma (in Dynkin_system) restricted_Dynkin_system:
  assumes D ∈ M
  shows Dynkin_system Ω {Q. Q ⊆ Ω ∧ Q ∩ D ∈ M}
proof (rule Dynkin_systemI, simp_all)
  have Ω ∩ D = D
    using ‹D ∈ M› sets_into_space by auto
  then show Ω ∩ D ∈ M
    using ‹D ∈ M› by auto
next
  fix A assume A ⊆ Ω ∧ A ∩ D ∈ M
  moreover have (Ω − A) ∩ D = (Ω − (A ∩ D)) − (Ω − D)
    by auto
  ultimately show (Ω − A) ∩ D ∈ M
    using ‹D ∈ M› by (auto intro: diff)
next
  fix A :: nat ⇒ 'a set
  assume disjoint_family A range A ⊆ {Q. Q ⊆ Ω ∧ Q ∩ D ∈ M}
  then have ⋀i. A i ⊆ Ω disjoint_family (λi. A i ∩ D)
    range (λi. A i ∩ D) ⊆ M (⋃ x. A x) ∩ D = (⋃ x. A x ∩ D)
    by ((fastforce simp: disjoint_family_on_def)+)
  then show (⋃ x. A x) ⊆ Ω ∧ (⋃ x. A x) ∩ D ∈ M
    by (auto simp del: UN_simps)
qed

lemma (in Dynkin_system) Dynkin_subset:
  assumes N ⊆ M
  shows Dynkin Ω N ⊆ M
proof −
  have Dynkin_system Ω M ..
  then have Dynkin_system Ω M
    using assms unfolding Dynkin_system_def Dynkin_system_axioms_def sub-
set_class_def by simp
  with ‹N ⊆ M› show ?thesis by (auto simp add: Dynkin_def)
qed

lemma sigma_eq_Dynkin:
  assumes sets: M ⊆ Pow Ω
  assumes Int_stable M
  shows sigma_sets Ω M = Dynkin Ω M
proof −
  have Dynkin Ω M ⊆ sigma_sets (Ω) (M)
    using sigma_algebra_imp_Dynkin_system
```

```
    unfolding Dynkin_def sigma_sets_least_sigma_algebra[OF sets] by auto
  moreover
  interpret Dynkin_system Ω Dynkin Ω M
    using Dynkin_system_Dynkin[OF sets] .
  have sigma_algebra Ω (Dynkin Ω M)
    unfolding sigma_algebra_eq_Int_stable Int_stable_def
  proof (intro ballI)
    fix A B assume A ∈ Dynkin Ω M B ∈ Dynkin Ω M
    let ?D = λE. {Q. Q ⊆ Ω ∧ Q ∩ E ∈ Dynkin Ω M}
    have M ⊆ ?D B
    proof
      fix E assume E ∈ M
      then have M ⊆ ?D E E ∈ Dynkin Ω M
        using sets_into_space ‹Int_stable M› by (auto simp: Int_stable_def)
      then have Dynkin Ω M ⊆ ?D E
        using restricted_Dynkin_system ‹E ∈ Dynkin Ω M›
        by (intro Dynkin_system.Dynkin_subset) simp_all
      then have B ∈ ?D E
        using ‹B ∈ Dynkin Ω M› by auto
      then have E ∩ B ∈ Dynkin Ω M
        by (subst Int_commute) simp
      then show E ∈ ?D B
        using sets ‹E ∈ M› by auto
    qed
    then have Dynkin Ω M ⊆ ?D B
      using restricted_Dynkin_system ‹B ∈ Dynkin Ω M›
      by (intro Dynkin_system.Dynkin_subset) simp_all
    then show A ∩ B ∈ Dynkin Ω M
      using ‹A ∈ Dynkin Ω M› sets_into_space by auto
  qed
  from sigma_algebra.sigma_sets_subset[OF this, of M]
  have sigma_sets (Ω) (M) ⊆ Dynkin Ω M by auto
  ultimately have sigma_sets (Ω) (M) = Dynkin Ω M by auto
  then show ?thesis
    by (auto simp: Dynkin_def)
qed

lemma (in Dynkin_system) Dynkin_idem:
  Dynkin Ω M = M
proof −
  have Dynkin Ω M = M
  proof
    show M ⊆ Dynkin Ω M
      using Dynkin_Basic by auto
    show Dynkin Ω M ⊆ M
      by (intro Dynkin_subset) auto
  qed
  then show ?thesis
    by (auto simp: Dynkin_def)
```

qed

lemma (**in** *Dynkin_system*) *Dynkin_lemma*:
  assumes *Int_stable E*
  and *E*: $E \subseteq M$ $M \subseteq$ *sigma_sets* Ω E
  shows *sigma_sets* Ω $E = M$
proof −
  have $E \subseteq$ *Pow* Ω
    using *E sets_into_space* **by** *force*
  then have ∗: *sigma_sets* Ω E = *Dynkin* Ω E
    using ‹*Int_stable E*› **by** (*rule sigma_eq_Dynkin*)
  then have *Dynkin* Ω $E = M$
    using *assms Dynkin_subset*[*OF E*(*1*)] **by** *simp*
  with ∗ **show** *?thesis*
    using *assms* **by** (*auto simp*: *Dynkin_def*)
qed

Induction rule for intersection-stable generators

The reason to introduce Dynkin-systems is the following induction rules for σ-algebras generated by a generator closed under intersection.

proposition *sigma_sets_induct_disjoint*[*consumes 3*, *case_names basic empty compl union*]:
  assumes *Int_stable G*
    and *closed*: $G \subseteq$ *Pow* Ω
    and *A*: $A \in$ *sigma_sets* Ω G
  assumes *basic*: $\bigwedge A.\ A \in G \Longrightarrow P\ A$
    and *empty*: $P\ \{\}$
    and *compl*: $\bigwedge A.\ A \in$ *sigma_sets* Ω $G \Longrightarrow P\ A \Longrightarrow P\ (\Omega - A)$
    and *union*: $\bigwedge A.$ *disjoint_family* $A \Longrightarrow$ *range* $A \subseteq$ *sigma_sets* Ω $G \Longrightarrow (\bigwedge i.\ P\ (A\ i)) \Longrightarrow P\ (\bigcup i{::}nat.\ A\ i)$
  shows $P\ A$
proof −
  let *?D* = { $A \in$ *sigma_sets* Ω G. $P\ A$ }
  interpret *sigma_algebra* Ω *sigma_sets* Ω G
    using *closed* **by** (*rule sigma_algebra_sigma_sets*)
  from *compl*[*OF _ empty*] *closed* **have** *space*: $P\ \Omega$ **by** *simp*
  interpret *Dynkin_system* Ω *?D*
    by *standard* (*auto dest*: *sets_into_space intro*!: *space compl union*)
  have *sigma_sets* Ω G = *?D*
    by (*rule Dynkin_lemma*) (*auto simp*: *basic* ‹*Int_stable G*›)
  with *A* **show** *?thesis* **by** *auto*
qed

7.1.2 Measure type

definition *positive* :: *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *bool* **where**
  positive M $\mu \longleftrightarrow \mu\ \{\} = 0$

definition *countably_additive* :: *'a set set ⇒ ('a set ⇒ ennreal) ⇒ bool* **where**
countably_additive M f ⟷
(∀ *A*. *range A ⊆ M ⟶ disjoint_family A ⟶ (⋃ i*. *A i) ∈ M ⟶*
(∑ *i*. *f* (*A i*)) = *f* (⋃ *i*. *A i*))

definition *measure_space* :: *'a set ⇒ 'a set set ⇒ ('a set ⇒ ennreal) ⇒ bool* **where**
measure_space Ω A μ ⟷
sigma_algebra Ω A ∧ positive A μ ∧ countably_additive A μ

typedef *'a measure* =
{(Ω::*'a set*, *A*, *μ*). (∀ *a*∈−*A*. *μ a* = *0*) ∧ *measure_space Ω A μ* }
proof
have *sigma_algebra UNIV* {{}, *UNIV*}
by (*auto simp*: *sigma_algebra_iff2*)
then show (*UNIV*, {{}, *UNIV*}, λ*A*. *0*) ∈ {(Ω, *A*, *μ*). (∀ *a*∈−*A*. *μ a* = *0*) ∧
measure_space Ω A μ}
by (*auto simp*: *measure_space_def positive_def countably_additive_def*)
qed

definition *space* :: *'a measure ⇒ 'a set* **where**
space M = *fst* (*Rep_measure M*)

definition *sets* :: *'a measure ⇒ 'a set set* **where**
sets M = *fst* (*snd* (*Rep_measure M*))

definition *emeasure* :: *'a measure ⇒ 'a set ⇒ ennreal* **where**
emeasure M = *snd* (*snd* (*Rep_measure M*))

definition *measure* :: *'a measure ⇒ 'a set ⇒ real* **where**
measure M A = *enn2real* (*emeasure M A*)

declare [[*coercion sets*]]

declare [[*coercion measure*]]

declare [[*coercion emeasure*]]

lemma *measure_space*: *measure_space* (*space M*) (*sets M*) (*emeasure M*)
by (*cases M*) (*auto simp*: *space_def sets_def emeasure_def Abs_measure_inverse*)

interpretation *sets*: *sigma_algebra space M sets M* **for** *M* :: *'a measure*
using *measure_space*[*of M*] **by** (*auto simp*: *measure_space_def*)

definition *measure_of* :: *'a set ⇒ 'a set set ⇒ ('a set ⇒ ennreal) ⇒ 'a measure*
where
measure_of Ω A μ =
Abs_measure (Ω, *if A ⊆ Pow Ω then sigma_sets Ω A else* {{}, Ω},
λ*a*. *if a ∈ sigma_sets Ω A ∧ measure_space Ω* (*sigma_sets Ω A*) *μ then μ a*

else 0)

abbreviation *sigma* Ω *A* ≡ *measure_of* Ω *A* (λ*x*. *0*)

lemma *measure_space_0*: *A* ⊆ *Pow* Ω ⟹ *measure_space* Ω (*sigma_sets* Ω *A*)
(λ*x*. *0*)
unfolding *measure_space_def*
by (*auto intro*!: *sigma_algebra_sigma_sets simp*: *positive_def countably_additive_def*)

lemma *sigma_algebra_trivial*: *sigma_algebra* Ω {{}, Ω}
by *unfold_locales*(*fastforce intro*: *exI*[**where** *x*={{}}] *exI*[**where** *x*={Ω}])+

lemma *measure_space_0'*: *measure_space* Ω {{}, Ω} (λ*x*. *0*)
by(*simp add*: *measure_space_def positive_def countably_additive_def sigma_algebra_trivial*)

lemma *measure_space_closed*:
assumes *measure_space* Ω *M* μ
shows *M* ⊆ *Pow* Ω
proof −
interpret *sigma_algebra* Ω *M* **using** *assms* **by**(*simp add*: *measure_space_def*)
show *?thesis* **by**(*rule space_closed*)
qed

lemma (**in** *ring_of_sets*) *positive_cong_eq*:
(⋀*a*. *a* ∈ *M* ⟹ μ' *a* = μ *a*) ⟹ *positive M* μ' = *positive M* μ
by (*auto simp add*: *positive_def*)

lemma (**in** *sigma_algebra*) *countably_additive_eq*:
(⋀*a*. *a* ∈ *M* ⟹ μ' *a* = μ *a*) ⟹ *countably_additive M* μ' = *countably_additive*
M μ
unfolding *countably_additive_def*
by (*intro arg_cong*[**where** *f*=*All*] *ext*) (*auto simp add*: *countably_additive_def*
subset_eq)

lemma *measure_space_eq*:
assumes *closed*: *A* ⊆ *Pow* Ω **and** *eq*: ⋀*a*. *a* ∈ *sigma_sets* Ω *A* ⟹ μ *a* = μ' *a*
shows *measure_space* Ω (*sigma_sets* Ω *A*) μ = *measure_space* Ω (*sigma_sets*
Ω *A*) μ'
proof −
interpret *sigma_algebra* Ω *sigma_sets* Ω *A* **using** *closed* **by** (*rule sigma_algebra_sigma_sets*)
from *positive_cong_eq*[*OF eq*, *of* λ*i*. *i*] *countably_additive_eq*[*OF eq*, *of* λ*i*. *i*]
show *?thesis*
by (*auto simp*: *measure_space_def*)
qed

lemma *measure_of_eq*:
assumes *closed*: *A* ⊆ *Pow* Ω **and** *eq*: (⋀*a*. *a* ∈ *sigma_sets* Ω *A* ⟹ μ *a* = μ' *a*)
shows *measure_of* Ω *A* μ = *measure_of* Ω *A* μ'
proof −

have *measure_space* Ω (*sigma_sets* Ω *A*) μ = *measure_space* Ω (*sigma_sets* Ω *A*) μ′
using *assms* **by** (*rule measure_space_eq*)
with *eq* **show** *?thesis*
by (*auto simp add*: *measure_of_def intro*!: *arg_cong*[**where** *f*=*Abs_measure*])
qed

lemma *measure_space_Pow_eq*:
assumes ⋀*X*. *X* ∈ *Pow* Ω ⟹ μ *X* = μ′ *X*
shows *measure_space* Ω (*Pow* Ω) μ = *measure_space* Ω (*Pow* Ω) μ′
by (*smt* (*verit*, *best*) *assms measure_space_eq sigma_algebra.sigma_sets_eq sigma_algebra_Pow subset_eq*)

lemma
shows *space_measure_of_conv*: *space* (*measure_of* Ω *A* μ) = Ω (**is** *?space*)
and *sets_measure_of_conv*:
sets (*measure_of* Ω *A* μ) = (*if A* ⊆ *Pow* Ω *then sigma_sets* Ω *A else* {{}, Ω}) (**is** *?sets*)
and *emeasure_measure_of_conv*:
emeasure (*measure_of* Ω *A* μ) =
(λ*B*. *if B* ∈ *sigma_sets* Ω *A* ∧ *measure_space* Ω (*sigma_sets* Ω *A*) μ *then* μ *B else 0*) (**is** *?emeasure*)
proof −
have *?space* ∧ *?sets* ∧ *?emeasure*
proof(*cases measure_space* Ω (*sigma_sets* Ω *A*) μ)
case *True*
from *measure_space_closed*[*OF this*] *sigma_sets_superset_generator*[*of A* Ω]
have *A* ⊆ *Pow* Ω **by** *simp*
hence *measure_space* Ω (*sigma_sets* Ω *A*) μ = *measure_space* Ω (*sigma_sets* Ω *A*)
(λ*a*. *if a* ∈ *sigma_sets* Ω *A then* μ *a else 0*)
by(*rule measure_space_eq*) *auto*
with *True* ‹*A* ⊆ *Pow* Ω› **show** *?thesis*
by(*simp add*: *measure_of_def space_def sets_def emeasure_def Abs_measure_inverse*)
next
case *False* **thus** *?thesis*
by(*cases A* ⊆ *Pow* Ω)(*simp_all add*: *Abs_measure_inverse measure_of_def sets_def space_def emeasure_def measure_space_0 measure_space_0 ′*)
qed
thus *?space ?sets ?emeasure* **by** *simp_all*
qed

lemma [*simp*]:
assumes *A*: *A* ⊆ *Pow* Ω
shows *sets_measure_of*: *sets* (*measure_of* Ω *A* μ) = *sigma_sets* Ω *A*
and *space_measure_of*: *space* (*measure_of* Ω *A* μ) = Ω
using *assms*
by(*simp_all add*: *sets_measure_of_conv space_measure_of_conv*)

lemma *space_in_measure_of*[*simp*]: $\Omega \in$ *sets* (*measure_of* Ω M μ)
by (*subst sets_measure_of_conv*) (*auto simp*: *sigma_sets_top*)

lemma (**in** *sigma_algebra*) *sets_measure_of_eq*[*simp*]: *sets* (*measure_of* Ω M μ) = M
using *space_closed* **by** (*auto intro*!: *sigma_sets_eq*)

lemma (**in** *sigma_algebra*) *space_measure_of_eq*[*simp*]: *space* (*measure_of* Ω M μ) = Ω
by (*rule space_measure_of_conv*)

lemma *measure_of_subset*: $M \subseteq$ *Pow* $\Omega \Longrightarrow M' \subseteq M \Longrightarrow$ *sets* (*measure_of* Ω M' μ) $\subseteq$ *sets* (*measure_of* Ω M μ')
by (*auto intro*!: *sigma_sets_subseteq*)

lemma *emeasure_sigma*: *emeasure* (*sigma* Ω A) = $(\lambda x.\ 0)$
unfolding *measure_of_def emeasure_def*
by (*subst Abs_measure_inverse*)
(*auto simp*: *measure_space_def positive_def countably_additive_def*
intro!: *sigma_algebra_sigma_sets sigma_algebra_trivial*)

lemma *sigma_sets_mono''*:
assumes $A \in$ *sigma_sets* C D
assumes $B \subseteq D$
assumes $D \subseteq$ *Pow* C
shows *sigma_sets* A $B \subseteq$ *sigma_sets* C D
proof
fix x **assume** $x \in$ *sigma_sets* A B
thus $x \in$ *sigma_sets* C D
proof *induct*
case (*Basic a*) **with** *assms* **have** $a \in D$ **by** *auto*
thus *?case* **..**
next
case *Empty* **show** *?case* **by** (*rule sigma_sets.Empty*)
next
from *assms* **have** $A \in$ *sets* (*sigma* C D) **by** (*subst sets_measure_of*[*OF* ‹$D \subseteq$ *Pow* C›])
moreover case (*Compl a*) **hence** $a \in$ *sets* (*sigma* C D) **by** (*subst sets_measure_of*[*OF* ‹$D \subseteq$ *Pow* C›])
ultimately have $A - a \in$ *sets* (*sigma* C D) **..**
thus *?case* **by** (*subst* (*asm*) *sets_measure_of*[*OF* ‹$D \subseteq$ *Pow* C›])
next
case (*Union a*)
thus *?case* **by** (*intro sigma_sets.Union*)
qed
qed

lemma *in_measure_of*[*intro*, *simp*]: $M \subseteq$ *Pow* $\Omega \Longrightarrow A \in M \Longrightarrow A \in$ *sets* (*measure_of* Ω M μ)

by *auto*

lemma *space_empty_iff*: *space N* = {} ⟷ *sets N* = {{}}
by (*metis Pow_empty Sup_bot_conv*(1) *cSup_singleton empty_iff*
sets.sigma_sets_eq sets.space_closed sigma_sets_top subset_singletonD)

Constructing simple *'a measure*

proposition *emeasure_measure_of*:
assumes *M*: *M* = *measure_of* Ω *A* μ
assumes *ms*: *A* ⊆ *Pow* Ω *positive* (*sets M*) μ *countably_additive* (*sets M*) μ
assumes *X*: *X* ∈ *sets M*
shows *emeasure M X* = μ *X*
proof −
interpret *sigma_algebra* Ω *sigma_sets* Ω *A* **by** (*rule sigma_algebra_sigma_sets*)
fact
have *measure_space* Ω (*sigma_sets* Ω *A*) μ
using *ms M* **by** (*simp add*: *measure_space_def sigma_algebra_sigma_sets*)
thus *?thesis* **using** *X ms*
by(*simp add*: *M emeasure_measure_of_conv sets_measure_of_conv*)
qed

lemma *emeasure_measure_of_sigma*:
assumes *ms*: *sigma_algebra* Ω *M positive M* μ *countably_additive M* μ
assumes *A*: *A* ∈ *M*
shows *emeasure* (*measure_of* Ω *M* μ) *A* = μ *A*
proof −
interpret *sigma_algebra* Ω *M* **by** *fact*
have *measure_space* Ω (*sigma_sets* Ω *M*) μ
using *ms sigma_sets_eq* **by** (*simp add*: *measure_space_def*)
thus *?thesis* **by**(*simp add*: *emeasure_measure_of_conv A*)
qed

lemma *measure_cases*[*cases type*: *measure*]:
obtains (*measure*) Ω *A* μ **where** *x* = *Abs_measure* (Ω, *A*, μ) ∀ *a*∈−*A*. μ *a* = *0*
measure_space Ω *A* μ
by *atomize_elim* (*cases x*, *auto*)

lemma *sets_le_imp_space_le*: *sets A* ⊆ *sets B* ⟹ *space A* ⊆ *space B*
by (*auto dest*: *sets.sets_into_space*)

lemma *sets_eq_imp_space_eq*: *sets M* = *sets M′* ⟹ *space M* = *space M′*
by (*auto intro*!: *antisym sets_le_imp_space_le*)

lemma *emeasure_notin_sets*: *A* ∉ *sets M* ⟹ *emeasure M A* = *0*
by (*cases M*) (*auto simp*: *sets_def emeasure_def Abs_measure_inverse measure_space_def*)

lemma *emeasure_neq_0_sets*: *emeasure M A* ≠ *0* ⟹ *A* ∈ *sets M*

using *emeasure_notin_sets*[*of A M*] **by** *blast*

lemma *measure_notin_sets*: $A \notin sets\ M \Longrightarrow measure\ M\ A = 0$
by (*simp add*: *measure_def emeasure_notin_sets zero_ennreal.rep_eq*)

lemma *measure_eqI*:
fixes $M\ N :: 'a\ measure$
assumes $sets\ M = sets\ N$ **and** *eq*: $\bigwedge A.\ A \in sets\ M \Longrightarrow emeasure\ M\ A = emeasure\ N\ A$
shows $M = N$
proof (*cases M N rule*: *measure_cases*[*case_product measure_cases*])
case (*measure_measure* $\Omega\ A\ \mu\ \Omega'\ A'\ \mu'$)
interpret *M*: *sigma_algebra* $\Omega\ A$ **using** *measure_measure* **by** (*auto simp*: *measure_space_def*)
interpret *N*: *sigma_algebra* $\Omega'\ A'$ **using** *measure_measure* **by** (*auto simp*: *measure_space_def*)
have $A = sets\ M\ A' = sets\ N$
using *measure_measure* **by** (*simp_all add*: *sets_def Abs_measure_inverse*)
with ‹*sets M* = *sets N*› **have** *AA'*: $A = A'$ **by** *simp*
moreover from *M.top N.top M.space_closed N.space_closed AA'* **have** $\Omega = \Omega'$ **by** *auto*
moreover { **fix** *B* **have** $\mu\ B = \mu'\ B$
proof *cases*
assume $B \in A$
with *eq* ‹$A = sets\ M$› **have** $emeasure\ M\ B = emeasure\ N\ B$ **by** *simp*
with *measure_measure* **show** $\mu\ B = \mu'\ B$
by (*simp add*: *emeasure_def Abs_measure_inverse*)
next
assume $B \notin A$
with ‹$A = sets\ M$› ‹$A' = sets\ N$› ‹$A = A'$› **have** $B \notin sets\ M\ B \notin sets\ N$
by *auto*
then have $emeasure\ M\ B = 0\ emeasure\ N\ B = 0$
by (*simp_all add*: *emeasure_notin_sets*)
with *measure_measure* **show** $\mu\ B = \mu'\ B$
by (*simp add*: *emeasure_def Abs_measure_inverse*)
qed }
then have $\mu = \mu'$ **by** *auto*
ultimately show $M = N$
by (*simp add*: *measure_measure*)
qed

lemma *sigma_eqI*:
assumes [*simp*]: $M \subseteq Pow\ \Omega\ N \subseteq Pow\ \Omega\ sigma\_sets\ \Omega\ M = sigma\_sets\ \Omega\ N$
shows $sigma\ \Omega\ M = sigma\ \Omega\ N$
by (*rule measure_eqI*) (*simp_all add*: *emeasure_sigma*)

Measurable functions

definition *measurable* :: $'a\ measure \Rightarrow 'b\ measure \Rightarrow ('a \Rightarrow 'b)\ set$

(**infixr** ‹$\rightarrow_M$› *60*) **where**
measurable A B = {*f* ∈ *space A* → *space B*. ∀ *y* ∈ *sets B*. *f* −‘ *y* ∩ *space A* ∈ *sets A*}

lemma *measurableI*:
(⋀*x*. *x* ∈ *space M* ⟹ *f x* ∈ *space N*) ⟹ (⋀*A*. *A* ∈ *sets N* ⟹ *f* −‘ *A* ∩ *space M* ∈ *sets M*) ⟹
f ∈ *measurable M N*
by (*auto simp*: *measurable_def*)

lemma *measurable_space*:
f ∈ *measurable M A* ⟹ *x* ∈ *space M* ⟹ *f x* ∈ *space A*
unfolding *measurable_def* **by** *auto*

lemma *measurable_sets*:
f ∈ *measurable M A* ⟹ *S* ∈ *sets A* ⟹ *f* −‘ *S* ∩ *space M* ∈ *sets M*
unfolding *measurable_def* **by** *auto*

lemma *measurable_sets_Collect*:
assumes *f*: *f* ∈ *measurable M N* **and** *P*: {*x*∈*space N*. *P x*} ∈ *sets N* **shows** {*x*∈*space M*. *P* (*f x*)} ∈ *sets M*
proof −
have *f* −‘ {*x* ∈ *space N*. *P x*} ∩ *space M* = {*x*∈*space M*. *P* (*f x*)}
using *measurable_space*[*OF f*] **by** *auto*
with *measurable_sets*[*OF f P*] **show** *?thesis*
by *simp*
qed

lemma *measurable_sigma_sets*:
assumes *B*: *sets N* = *sigma_sets* Ω *A* *A* ⊆ *Pow* Ω
and *f*: *f* ∈ *space M* → Ω
and *ba*: ⋀*y*. *y* ∈ *A* ⟹ (*f* −‘ *y*) ∩ *space M* ∈ *sets M*
shows *f* ∈ *measurable M N*
proof −
interpret *A*: *sigma_algebra* Ω *sigma_sets* Ω *A* **using** *B*(*2*) **by** (*rule sigma_algebra_sigma_sets*)
from *B sets.top*[*of N*] *A.top sets.space_closed*[*of N*] *A.space_closed* **have** Ω: Ω = *space N* **by** *force*

{ **fix** *X* **assume** *X* ∈ *sigma_sets* Ω *A*
then have *f* −‘ *X* ∩ *space M* ∈ *sets M* ∧ *X* ⊆ Ω
proof *induct*
case (*Basic a*) **then show** *?case*
by (*auto simp add*: *ba*) (*metis B*(*2*) *subsetD PowD*)
next
case (*Compl a*)
have [*simp*]: *f* −‘ Ω ∩ *space M* = *space M*
by (*auto simp add*: *funcset_mem* [*OF f*])
then show *?case*
by (*auto simp add*: *vimage_Diff Diff_Int_distrib2 sets.compl_sets Compl*)

```
    next
      case (Union a)
      then show ?case
        by (simp add: vimage_UN, simp only: UN_extend_simps(4)) blast
    qed auto }
  with f show ?thesis
    by (auto simp add: measurable_def B Ω)
qed

lemma measurable_measure_of:
  assumes B: N ⊆ Pow Ω
      and f: f ∈ space M → Ω
      and ba: ⋀y. y ∈ N ⟹ (f -` y) ∩ space M ∈ sets M
  shows f ∈ measurable M (measure_of Ω N μ)
proof -
  have sets (measure_of Ω N μ) = sigma_sets Ω N
    using B by (rule sets_measure_of)
  from this assms show ?thesis by (rule measurable_sigma_sets)
qed

lemma measurable_iff_measure_of:
  assumes N ⊆ Pow Ω f ∈ space M → Ω
  shows f ∈ measurable M (measure_of Ω N μ) ⟷ (∀A∈N. f -` A ∩ space M
∈ sets M)
  by (metis assms in_measure_of measurable_measure_of assms measurable_sets)

lemma measurable_cong_sets:
  assumes sets: sets M = sets M' sets N = sets N'
  shows measurable M N = measurable M' N'
  using sets[THEN sets_eq_imp_space_eq] sets by (simp add: measurable_def)

lemma measurable_cong:
  assumes ⋀w. w ∈ space M ⟹ f w = g w
  shows f ∈ measurable M M' ⟷ g ∈ measurable M M'
  unfolding measurable_def using assms
  by (simp cong: vimage_inter_cong Pi_cong)

lemma measurable_cong':
  assumes ⋀w. w ∈ space M =simp=> f w = g w
  shows f ∈ measurable M M' ⟷ g ∈ measurable M M'
  unfolding measurable_def using assms
  by (simp cong: vimage_inter_cong Pi_cong add: simp_implies_def)

lemma measurable_cong_simp:
  M = N ⟹ M' = N' ⟹ (⋀w. w ∈ space M ⟹ f w = g w) ⟹
    f ∈ measurable M M' ⟷ g ∈ measurable N N'
  by (metis measurable_cong)

lemma measurable_compose:
```

assumes f: $f \in$ *measurable M N* **and** g: $g \in$ *measurable N L*
shows $(\lambda x.\ g\ (f\ x)) \in$ *measurable M L*
proof −
have $\bigwedge A.\ (\lambda x.\ g\ (f\ x))\ -`\ A \cap$ *space M* $= f\ -`\ (g\ -`\ A \cap$ *space N*$) \cap$ *space M*
using *measurable_space*[*OF f*] **by** *auto*
with *measurable_space*[*OF f*] *measurable_space*[*OF g*] **show** *?thesis*
by (*auto intro*: *measurable_sets*[*OF f*] *measurable_sets*[*OF g*]
simp del: *vimage_Int simp add*: *measurable_def*)
qed

lemma *measurable_comp*:
$f \in$ *measurable M N* $\Longrightarrow g \in$ *measurable N L* $\Longrightarrow g \circ f \in$ *measurable M L*
using *measurable_compose*[*of f M N g L*] **by** (*simp add*: *comp_def*)

lemma *measurable_const*:
$c \in$ *space* $M' \Longrightarrow (\lambda x.\ c) \in$ *measurable M* M'
by (*auto simp add*: *measurable_def*)

lemma *measurable_ident*: $id \in$ *measurable M M*
by (*auto simp add*: *measurable_def*)

lemma *measurable_id*: $(\lambda x.\ x) \in$ *measurable M M*
by (*simp add*: *measurable_def*)

lemma *measurable_ident_sets*:
assumes *eq*: *sets M* = *sets* M' **shows** $(\lambda x.\ x) \in$ *measurable M* M'
using *measurable_ident*[*of M*]
unfolding *id_def measurable_def eq sets_eq_imp_space_eq*[*OF eq*] **.**

lemma *sets_Least*:
assumes *meas*: $\bigwedge i$::*nat*. $\{x \in$*space M*. *P i x*$\} \in$ *M*
shows $(\lambda x.$ *LEAST j*. *P j x*$)\ -`\ A \cap$ *space M* $\in$ *sets M*
proof −
{ fix *i* **have** $(\lambda x.$ *LEAST j*. *P j x*$)\ -`\ \{i\} \cap$ *space M* $\in$ *sets M*
proof *cases*
assume *i*: (*LEAST j*. *False*) = *i*
have $(\lambda x.$ *LEAST j*. *P j x*$)\ -`\ \{i\} \cap$ *space M* =
$\{x \in$*space M*. *P i x*$\} \cap$ (*space M* − ($\bigcup j{<}i.\ \{x \in$*space M*. *P j x*$\}$)) ∪ (*space M* − ($\bigcup i.\ \{x \in$*space M*. *P i x*$\}$))
by (*simp add*: *set_eq_iff*, *safe*)
(*insert i*, *auto dest*: *Least_le intro*: *LeastI intro*!: *Least_equality*)
with *meas* **show** *?thesis*
by (*auto intro*!: *sets.Int*)
next
assume *i*: (*LEAST j*. *False*) ≠ *i*
then have $(\lambda x.$ *LEAST j*. *P j x*$)\ -`\ \{i\} \cap$ *space M* =
$\{x \in$*space M*. *P i x*$\} \cap$ (*space M* − ($\bigcup j{<}i.\ \{x \in$*space M*. *P j x*$\}$))
proof (*simp add*: *set_eq_iff*, *safe*)
fix *x* **assume** *neq*: (*LEAST j*. *False*) ≠ (*LEAST j*. *P j x*)

```
      have ∃j. P j x
        by (rule ccontr) (insert neq, auto)
      then show P (LEAST j. P j x) x by (rule LeastI_ex)
    qed (auto dest: Least_le intro!: Least_equality)
    with meas show ?thesis
      by auto
  qed }
 then have (⋃i∈A. (λx. LEAST j. P j x) -` {i} ∩ space M) ∈ sets M
  by (intro sets.countable_UN) auto
 moreover have (⋃i∈A. (λx. LEAST j. P j x) -` {i} ∩ space M) =
  (λx. LEAST j. P j x) -` A ∩ space M by auto
 ultimately show ?thesis by auto
qed

lemma measurable_mono1:
 M' ⊆ Pow Ω ⟹ M ⊆ M' ⟹
   measurable (measure_of Ω M μ) N ⊆ measurable (measure_of Ω M' μ') N
 using measure_of_subset[of M' Ω M] by (auto simp add: measurable_def)
```

Counting space

```
definition count_space :: 'a set ⇒ 'a measure where
count_space Ω = measure_of Ω (Pow Ω) (λA. if finite A then of_nat (card A)
else ∞)

lemma
 shows space_count_space[simp]: space (count_space Ω) = Ω
   and sets_count_space[simp]: sets (count_space Ω) = Pow Ω
 using sigma_sets_into_sp[of Pow Ω Ω]
 by (auto simp: count_space_def)

lemma measurable_count_space_eq1[simp]:
 f ∈ measurable (count_space A) M ⟷ f ∈ A → space M
unfolding measurable_def by simp

lemma finite_count_space: finite Ω ⟹ count_space Ω = measure_of Ω (Pow
Ω) card
 unfolding count_space_def
by (smt (verit, best) PowD Pow_top count_space_def finite_subset measure_of_eq
sets_count_space sets_measure_of)

lemma measurable_compose_countable':
 assumes f: ⋀i. i ∈ I ⟹ (λx. f i x) ∈ measurable M N
 and g: g ∈ measurable M (count_space I) and I: countable I
 shows (λx. f (g x) x) ∈ measurable M N
 unfolding measurable_def
proof safe
 fix x assume x ∈ space M then show f (g x) x ∈ space N
   using measurable_space[OF f] g[THEN measurable_space] by auto
```

next
fix A **assume** A: $A \in sets\ N$
have $(\lambda x.\ f\ (g\ x)\ x) -` A \cap space\ M = (\bigcup i{\in}I.\ (g -` \{i\} \cap space\ M) \cap (f\ i -` A \cap space\ M))$
using *measurable_space*[*OF g*] **by** *auto*
also have $\ldots \in sets\ M$
using *f*[*THEN measurable_sets*, *OF _ A*] *g*[*THEN measurable_sets*]
by (*auto intro*!: *sets.countable_UN' I intro*: *sets.Int*[*OF measurable_sets measurable_sets*])
finally show $(\lambda x.\ f\ (g\ x)\ x) -` A \cap space\ M \in sets\ M$ **.**
qed

lemma *measurable_count_space_eq_countable*:
assumes *countable A*
shows $f \in measurable\ M\ (count\_space\ A) \longleftrightarrow (f \in space\ M \to A \wedge (\forall a{\in}A.\ f -` \{a\} \cap space\ M \in sets\ M))$
proof −
{ fix X **assume** $X \subseteq A\ f \in space\ M \to A$
with ‹*countable A*› **have** $f -` X \cap space\ M = (\bigcup a{\in}X.\ f -` \{a\} \cap space\ M)$ *countable X*
by (*auto dest*: *countable_subset*)
moreover assume $\forall a{\in}A.\ f -` \{a\} \cap space\ M \in sets\ M$
ultimately have $f -` X \cap space\ M \in sets\ M$
using ‹$X \subseteq A$› **by** (*auto intro*!: *sets.countable_UN' simp del*: *UN_simps*) **}**
then show *?thesis*
unfolding *measurable_def* **by** *auto*
qed

lemma *measurable_count_space_eq2*:
finite A $\Longrightarrow f \in measurable\ M\ (count\_space\ A) \longleftrightarrow (f \in space\ M \to A \wedge (\forall a{\in}A.\ f -` \{a\} \cap space\ M \in sets\ M))$
by (*intro measurable_count_space_eq_countable countable_finite*)

lemma *measurable_count_space_eq2_countable*:
fixes $f :: 'a => 'c{::}countable$
shows $f \in measurable\ M\ (count\_space\ A) \longleftrightarrow (f \in space\ M \to A \wedge (\forall a{\in}A.\ f -` \{a\} \cap space\ M \in sets\ M))$
by (*intro measurable_count_space_eq_countable countableI_type*)

lemma *measurable_compose_countable*:
assumes *f*: $\bigwedge i{::}'i{::}countable.\ (\lambda x.\ f\ i\ x) \in measurable\ M\ N$ **and** *g*: $g \in measurable\ M\ (count\_space\ UNIV)$
shows $(\lambda x.\ f\ (g\ x)\ x) \in measurable\ M\ N$
by (*rule measurable_compose_countable'*[*OF assms*]) *auto*

lemma *measurable_count_space_const*:
$(\lambda x.\ c) \in measurable\ M\ (count\_space\ UNIV)$
by (*simp add*: *measurable_const*)

lemma *measurable_count_space*:
f ∈ *measurable* (*count_space A*) (*count_space UNIV*)
by *simp*

lemma *measurable_compose_rev*:
assumes *f*: *f* ∈ *measurable L N* **and** *g*: *g* ∈ *measurable M L*
shows (λ*x*. *f* (*g x*)) ∈ *measurable M N*
using *measurable_compose*[*OF g f*] **.**

lemma *measurable_empty_iff*:
space N = {} ⟹ *f* ∈ *measurable M N* ⟷ *space M* = {}
by (*auto simp add*: *measurable_def Pi_iff*)

Extend measure

definition *extend_measure* :: *'a set* ⇒ *'b set* ⇒ (*'b* ⇒ *'a set*) ⇒ (*'b* ⇒ *ennreal*) ⇒ *'a measure*
where
extend_measure Ω *I G* μ =
(*if* (∃ μ'. (∀ *i*∈*I*. μ' (*G i*) = μ *i*) ∧ *measure_space* Ω (*sigma_sets* Ω (*G*‘*I*)) μ') ∧ ¬ (∀ *i*∈*I*. μ *i* = *0*)
then measure_of Ω (*G*‘*I*) (*SOME* μ'. (∀ *i*∈*I*. μ' (*G i*) = μ *i*) ∧ *measure_space* Ω (*sigma_sets* Ω (*G*‘*I*)) μ')
else measure_of Ω (*G*‘*I*) (λ_. *0*))

lemma *space_extend_measure*: *G* ‘ *I* ⊆ *Pow* Ω ⟹ *space* (*extend_measure* Ω *I G* μ) = Ω
unfolding *extend_measure_def* **by** *simp*

lemma *sets_extend_measure*: *G* ‘ *I* ⊆ *Pow* Ω ⟹ *sets* (*extend_measure* Ω *I G* μ) = *sigma_sets* Ω (*G*‘*I*)
unfolding *extend_measure_def* **by** *simp*

lemma *emeasure_extend_measure*:
assumes *M*: *M* = *extend_measure* Ω *I G* μ
and *eq*: ⋀*i*. *i* ∈ *I* ⟹ μ' (*G i*) = μ *i*
and *ms*: *G* ‘ *I* ⊆ *Pow* Ω *positive* (*sets M*) μ' *countably_additive* (*sets M*) μ'
and *i* ∈ *I*
shows *emeasure M* (*G i*) = μ *i*
proof *cases*
assume ∗: (∀ *i*∈*I*. μ *i* = *0*)
with *M* **have** *M_eq*: *M* = *measure_of* Ω (*G*‘*I*) (λ_. *0*)
by (*simp add*: *extend_measure_def*)
from *measure_space_0*[*OF ms*(*1*)] *ms* ‹*i*∈*I*›
have *emeasure M* (*G i*) = *0*
by (*intro emeasure_measure_of*[*OF M_eq*]) (*auto simp add*: *M measure_space_def sets_extend_measure*)
with ‹*i*∈*I*› ∗ **show** *?thesis*
by *simp*

next
define P **where** $P\ \mu' \longleftrightarrow (\forall\, i{\in}I.\ \mu'\ (G\ i) = \mu\ i) \wedge \mathit{measure\_space}\ \Omega\ (\mathit{sigma\_sets}\ \Omega\ (G`I))\ \mu'$ **for** μ'
assume $\neg\ (\forall\, i{\in}I.\ \mu\ i = 0)$
moreover
have $\mathit{measure\_space}\ (\mathit{space}\ M)\ (\mathit{sets}\ M)\ \mu'$
using ms **unfolding** $\mathit{measure\_space\_def}$ **by** $\mathit{auto}\ \mathit{standard}$
with $\mathit{ms}\ \mathit{eq}$ **have** $\exists\, \mu'.\ P\ \mu'$
unfolding P_def
by $(\mathit{intro}\ \mathit{exI}[\mathit{of}\_\ \mu'])\ (\mathit{auto}\ \mathit{simp}\ \mathit{add}\colon M.\mathit{space\_extend\_measure}\ M.\mathit{sets\_extend\_measure})$
ultimately have $M\_\mathit{eq}\colon M = \mathit{measure\_of}\ \Omega\ (G`I)\ (\mathit{Eps}\ P)$
by $(\mathit{simp}\ \mathit{add}\colon M.\mathit{extend\_measure\_def}\ P\_\mathit{def}[\mathit{symmetric}])$

from $\langle \exists\, \mu'.\ P\ \mu' \rangle$ **have** $P\colon P\ (\mathit{Eps}\ P)$ **by** $(\mathit{rule}\ \mathit{someI\_ex})$
show $\mathit{emeasure}\ M\ (G\ i) = \mu\ i$
proof $(\mathit{subst}\ \mathit{emeasure\_measure\_of}[\mathit{OF}\ M\_\mathit{eq}])$
have $\mathit{sets\_M}\colon \mathit{sets}\ M = \mathit{sigma\_sets}\ \Omega\ (G`I)$
using $M\_\mathit{eq}\ \mathit{ms}$ **by** $(\mathit{auto}\ \mathit{simp}\colon \mathit{sets\_extend\_measure})$
then show $G\ i \in \mathit{sets}\ M$ **using** $\langle i \in I \rangle$ **by** auto
show $\mathit{positive}\ (\mathit{sets}\ M)\ (\mathit{Eps}\ P)\ \mathit{countably\_additive}\ (\mathit{sets}\ M)\ (\mathit{Eps}\ P)\ \mathit{Eps}\ P\ (G\ i) = \mu\ i$
using $P\ \langle i{\in}I \rangle$ **by** $(\mathit{auto}\ \mathit{simp}\ \mathit{add}\colon \mathit{sets\_M}\ \mathit{measure\_space\_def}\ P\_\mathit{def})$
qed fact
qed

lemma $\mathit{emeasure\_extend\_measure\_Pair}$:
assumes $M\colon M = \mathit{extend\_measure}\ \Omega\ \{(i, j).\ I\ i\ j\}\ (\lambda(i, j).\ G\ i\ j)\ (\lambda(i, j).\ \mu\ i\ j)$
and $\mathit{eq}\colon \bigwedge i\ j.\ I\ i\ j \Longrightarrow \mu'\ (G\ i\ j) = \mu\ i\ j$
and $\mathit{ms}\colon \bigwedge i\ j.\ I\ i\ j \Longrightarrow G\ i\ j \in \mathit{Pow}\ \Omega\ \mathit{positive}\ (\mathit{sets}\ M)\ \mu'\ \mathit{countably\_additive}\ (\mathit{sets}\ M)\ \mu'$
and $I\ i\ j$
shows $\mathit{emeasure}\ M\ (G\ i\ j) = \mu\ i\ j$
using $\mathit{emeasure\_extend\_measure}[\mathit{OF}\ M\ \_\ \_\ \mathit{ms}(2,3),\ \mathit{of}\ (i,j)]\ \mathit{eq}\ \mathit{ms}(1)\ \langle I\ i\ j \rangle$
by $(\mathit{auto}\ \mathit{simp}\colon \mathit{subset\_eq})$

7.1.3 The smallest σ-algebra regarding a function

definition $\mathit{vimage\_algebra} :: {'a}\ \mathit{set} \Rightarrow ({'a} \Rightarrow {'b}) \Rightarrow {'b}\ \mathit{measure} \Rightarrow {'a}\ \mathit{measure}$ **where**
$\mathit{vimage\_algebra}\ X\ f\ M = \mathit{sigma}\ X\ \{f -` A \cap X \mid A.\ A \in \mathit{sets}\ M\}$

lemma $\mathit{space\_vimage\_algebra}[\mathit{simp}]\colon \mathit{space}\ (\mathit{vimage\_algebra}\ X\ f\ M) = X$
unfolding $\mathit{vimage\_algebra\_def}$ **by** $(\mathit{rule}\ \mathit{space\_measure\_of})\ \mathit{auto}$

lemma $\mathit{sets\_vimage\_algebra}\colon \mathit{sets}\ (\mathit{vimage\_algebra}\ X\ f\ M) = \mathit{sigma\_sets}\ X\ \{f -` A \cap X \mid A.\ A \in \mathit{sets}\ M\}$
unfolding $\mathit{vimage\_algebra\_def}$ **by** $(\mathit{rule}\ \mathit{sets\_measure\_of})\ \mathit{auto}$

lemma $\mathit{sets\_vimage\_algebra2}$:

$f \in X \to space\ M \Longrightarrow sets\ (vimage\_algebra\ X\ f\ M) = \{f -` A \cap X \mid A.\ A \in sets\ M\}$
using *sigma_sets_vimage_commute*[*of f X space M sets M*]
unfolding *sets_vimage_algebra sets.sigma_sets_eq* **by** *simp*

lemma *sets_vimage_algebra_cong*: $sets\ M = sets\ N \Longrightarrow sets\ (vimage\_algebra\ X\ f\ M) = sets\ (vimage\_algebra\ X\ f\ N)$
by (*simp add*: *sets_vimage_algebra*)

lemma *vimage_algebra_cong*:
assumes $X = Y$
assumes $\bigwedge x.\ x \in Y \Longrightarrow f\ x = g\ x$
assumes $sets\ M = sets\ N$
shows $vimage\_algebra\ X\ f\ M = vimage\_algebra\ Y\ g\ N$
by (*auto simp*: *vimage_algebra_def assms intro*!: *arg_cong2*[**where** *f*=*sigma*])

lemma *in_vimage_algebra*: $A \in sets\ M \Longrightarrow f -` A \cap X \in sets\ (vimage\_algebra\ X\ f\ M)$
by (*auto simp*: *vimage_algebra_def*)

lemma *sets_image_in_sets*:
assumes *N*: *space N* = *X*
assumes *f*: *f* ∈ *measurable N M*
shows *sets* (*vimage_algebra X f M*) ⊆ *sets N*
unfolding *sets_vimage_algebra N*[*symmetric*]
by (*rule sets.sigma_sets_subset*) (*auto intro*!: *measurable_sets f*)

lemma *measurable_vimage_algebra1*: $f \in X \to space\ M \Longrightarrow f \in measurable\ (vimage\_algebra\ X\ f\ M)\ M$
unfolding *measurable_def* **by** (*auto intro*: *in_vimage_algebra*)

lemma *measurable_vimage_algebra2*:
assumes *g*: *g* ∈ *space N* → *X* **and** *f*: (λ*x*. *f* (*g x*)) ∈ *measurable N M*
shows *g* ∈ *measurable N* (*vimage_algebra X f M*)
unfolding *vimage_algebra_def*
proof (*rule measurable_measure_of*)
fix *A* **assume** *A* ∈ {*f* −` *A* ∩ *X* | *A*. *A* ∈ *sets M*}
then obtain *Y* **where** *Y*: *Y* ∈ *sets M* **and** *A*: *A* = *f* −` *Y* ∩ *X*
by *auto*
then have *g* −` *A* ∩ *space N* = (λ*x*. *f* (*g x*)) −` *Y* ∩ *space N*
using *g* **by** *auto*
also have ... ∈ *sets N*
using *f Y* **by** (*rule measurable_sets*)
finally show *g* −` *A* ∩ *space N* ∈ *sets N* **.**
qed (*insert g*, *auto*)

lemma *vimage_algebra_sigma*:
assumes *X*: *X* ⊆ *Pow* Ω′ **and** *f*: *f* ∈ Ω → Ω′
shows *vimage_algebra* Ω *f* (*sigma* Ω′ *X*) = *sigma* Ω {*f* −` *A* ∩ Ω | *A*. *A* ∈ *X* }

(**is** *?V = ?S*)
proof (*rule measure_eqI*)
have Ω: {*f* −‘ *A* ∩ Ω |*A*. *A* ∈ *X*} ⊆ *Pow* Ω **by** *auto*
show *sets ?V = sets ?S*
using *sigma_sets_vimage_commute*[*OF f*, *of X*]
by (*simp add*: *space_measure_of_conv f sets_vimage_algebra2* Ω *X*)
qed (*simp add*: *vimage_algebra_def emeasure_sigma*)

lemma *vimage_algebra_vimage_algebra_eq*:
assumes ∗: *f* ∈ *X* → *Y g* ∈ *Y* → *space M*
shows *vimage_algebra X f* (*vimage_algebra Y g M*) = *vimage_algebra X* (λ*x*. *g* (*f x*)) *M*
(**is** *?VV = ?V*)
proof (*rule measure_eqI*)
have (λ*x*. *g* (*f x*)) ∈ *X* → *space M* ⋀*A*. *A* ∩ *f* −‘ *Y* ∩ *X* = *A* ∩ *X*
using ∗ **by** *auto*
with ∗ **show** *sets ?VV = sets ?V*
by (*simp add*: *sets_vimage_algebra2 vimage_comp comp_def flip*: *ex_simps*)
qed (*simp add*: *vimage_algebra_def emeasure_sigma*)

Restricted Space Sigma Algebra

definition *restrict_space* :: ′*a measure* ⇒ ′*a set* ⇒ ′*a measure* **where**
restrict_space M Ω = *measure_of* (Ω ∩ *space M*) (((∩) Ω) ‘ *sets M*) (*emeasure M*)

lemma *space_restrict_space*: *space* (*restrict_space M* Ω) = Ω ∩ *space M*
using *sets.sets_into_space* **unfolding** *restrict_space_def* **by** (*subst space_measure_of*) *auto*

lemma *space_restrict_space2* [*simp*]: Ω ∈ *sets M* ⟹ *space* (*restrict_space M* Ω) = Ω
by (*simp add*: *space_restrict_space sets.sets_into_space*)

lemma *sets_restrict_space*: *sets* (*restrict_space M* Ω) = ((∩) Ω) ‘ *sets M*
unfolding *restrict_space_def*
proof (*subst sets_measure_of*)
show (∩) Ω ‘ *sets M* ⊆ *Pow* (Ω ∩ *space M*)
by (*auto dest*: *sets.sets_into_space*)
have *sigma_sets* (Ω ∩ *space M*) {((λ*x*. *x*) −‘ *X*) ∩ (Ω ∩ *space M*) | *X*. *X* ∈ *sets M*} =
(λ*X*. *X* ∩ (Ω ∩ *space M*)) ‘ *sets M*
by (*subst sigma_sets_vimage_commute*[*symmetric*, **where** Ω′ = *space M*])
(*auto simp add*: *sets.sigma_sets_eq*)
moreover have {((λ*x*. *x*) −‘ *X*) ∩ (Ω ∩ *space M*) | *X*. *X* ∈ *sets M*} = (λ*X*. *X* ∩ (Ω ∩ *space M*)) ‘ *sets M*
by *auto*
moreover have (λ*X*. *X* ∩ (Ω ∩ *space M*)) ‘ *sets M* = ((∩) Ω) ‘ *sets M*
by (*intro image_cong*) (*auto dest*: *sets.sets_into_space*)

ultimately show *sigma_sets* ($\Omega \cap$ *space* M) (($\cap$) Ω ' *sets* M) = ($\cap$) Ω ' *sets* M
by *simp*
qed

lemma *restrict_space_sets_cong*:
$A = B \Longrightarrow$ *sets* M = *sets* $N \Longrightarrow$ *sets* (*restrict_space* M A) = *sets* (*restrict_space* N B)
by (*auto simp*: *sets_restrict_space*)

lemma *sets_restrict_space_count_space* :
sets (*restrict_space* (*count_space* A) B) = *sets* (*count_space* ($A \cap B$))
by(*auto simp add*: *sets_restrict_space*)

lemma *sets_restrict_UNIV*[*simp*]: *sets* (*restrict_space* M *UNIV*) = *sets* M
by (*auto simp add*: *sets_restrict_space*)

lemma *sets_restrict_restrict_space*:
sets (*restrict_space* (*restrict_space* M A) B) = *sets* (*restrict_space* M ($A \cap B$))
unfolding *sets_restrict_space image_comp* **by** (*intro image_cong*) *auto*

lemma *sets_restrict_space_iff*:
$\Omega \cap$ *space* $M \in$ *sets* $M \Longrightarrow A \in$ *sets* (*restrict_space* M Ω) $\longleftrightarrow$ ($A \subseteq \Omega \wedge A \in$ *sets* M)
proof (*subst sets_restrict_space*, *safe*)
fix A **assume** $\Omega \cap$ *space* $M \in$ *sets* M **and** A: $A \in$ *sets* M
then have ($\Omega \cap$ *space* M) $\cap A \in$ *sets* M
by *rule*
also have ($\Omega \cap$ *space* M) $\cap A = \Omega \cap A$
using *sets.sets_into_space*[*OF A*] **by** *auto*
finally show $\Omega \cap A \in$ *sets* M
by *auto*
qed *auto*

lemma *sets_restrict_space_cong*: *sets* M = *sets* $N \Longrightarrow$ *sets* (*restrict_space* M Ω) = *sets* (*restrict_space* N Ω)
by (*simp add*: *sets_restrict_space*)

lemma *restrict_space_eq_vimage_algebra*:
$\Omega \subseteq$ *space* $M \Longrightarrow$ *sets* (*restrict_space* M Ω) = *sets* (*vimage_algebra* Ω ($\lambda x.\ x$) M)
unfolding *restrict_space_def*
apply (*subst sets_measure_of*)
apply (*auto simp add*: *image_subset_iff dest*: *sets.sets_into_space*) []
apply (*auto simp add*: *sets_vimage_algebra intro*!: *arg_cong2*[**where** f=*sigma_sets*])
done

lemma *sets_Collect_restrict_space_iff*:
assumes $S \in$ *sets* M
shows {x∈*space* (*restrict_space* M S). P x} $\in$ *sets* (*restrict_space* M S) $\longleftrightarrow$

$\{x{\in}space\ M.\ x \in S \wedge P\ x\} \in sets\ M$
proof −
have $\{x{\in}S.\ P\ x\} = \{x{\in}space\ M.\ x \in S \wedge P\ x\}$
using *sets.sets_into_space*[*OF assms*] **by** *auto*
then show *?thesis*
by (*subst sets_restrict_space_iff*) (*auto simp add*: *space_restrict_space assms*)
qed

lemma *measurable_restrict_space1*:
assumes *f*: $f \in measurable\ M\ N$
shows $f \in measurable\ (restrict\_space\ M\ \Omega)\ N$
unfolding *measurable_def*
proof (*intro CollectI conjI ballI*)
show *sp*: $f \in space\ (restrict\_space\ M\ \Omega) \to space\ N$
using *measurable_space*[*OF f*] **by** (*auto simp*: *space_restrict_space*)

fix *A* **assume** $A \in sets\ N$
have $f -` A \cap space\ (restrict\_space\ M\ \Omega) = (f -` A \cap space\ M) \cap (\Omega \cap space\ M)$
by (*auto simp*: *space_restrict_space*)
also have $\ldots \in sets\ (restrict\_space\ M\ \Omega)$
unfolding *sets_restrict_space*
using *measurable_sets*[*OF f* ‹$A \in sets\ N$›] **by** *blast*
finally show $f -` A \cap space\ (restrict\_space\ M\ \Omega) \in sets\ (restrict\_space\ M\ \Omega)$
.
qed

lemma *measurable_restrict_space2_iff*:
$f \in measurable\ M\ (restrict\_space\ N\ \Omega) \longleftrightarrow (f \in measurable\ M\ N \wedge f \in space\ M \to \Omega)$
proof −
have $\bigwedge A.\ f \in space\ M \to \Omega \Longrightarrow f -` \Omega \cap f -` A \cap space\ M = f -` A \cap space\ M$
by *auto*
then show *?thesis*
by (*auto simp*: *measurable_def space_restrict_space Pi_Int*[*symmetric*] *sets_restrict_space*)
qed

lemma *measurable_restrict_space2*:
$f \in space\ M \to \Omega \Longrightarrow f \in measurable\ M\ N \Longrightarrow f \in measurable\ M\ (restrict\_space\ N\ \Omega)$
by (*simp add*: *measurable_restrict_space2_iff*)

lemma *measurable_piecewise_restrict*:
assumes *I*: *countable C*
and *X*: $\bigwedge\Omega.\ \Omega \in C \Longrightarrow \Omega \cap space\ M \in sets\ M\ space\ M \subseteq \bigcup C$
and *f*: $\bigwedge\Omega.\ \Omega \in C \Longrightarrow f \in measurable\ (restrict\_space\ M\ \Omega)\ N$
shows $f \in measurable\ M\ N$
proof (*rule measurableI*)

fix x **assume** $x \in$ *space* M
with X **obtain** Ω **where** $\Omega \in C\ x \in \Omega\ x \in$ *space* M **by** *auto*
then show $f\ x \in$ *space* N
by (*auto simp*: *space_restrict_space intro*: *f measurable_space*)
next
fix A **assume** A: $A \in$ *sets* N
have $f -` A \cap$ *space* $M = (\bigcup \Omega \in C.\ (f -` A \cap (\Omega \cap$ *space* $M)))$
using X **by** (*auto simp*: *subset_eq*)
also have $\ldots \in$ *sets* M
using *measurable_sets*[*OF f A*] X I
by (*intro sets.countable_UN'*) (*auto simp*: *sets_restrict_space_iff space_restrict_space*)
finally show $f -` A \cap$ *space* $M \in$ *sets* M **.**
qed

lemma *measurable_piecewise_restrict_iff*:
countable $C \Longrightarrow (\bigwedge \Omega.\ \Omega \in C \Longrightarrow \Omega \cap$ *space* $M \in$ *sets* $M) \Longrightarrow$ *space* $M \subseteq (\bigcup C)$
$\Longrightarrow$
$f \in$ *measurable* $M\ N \longleftrightarrow (\forall \Omega \in C.\ f \in$ *measurable* (*restrict_space* $M\ \Omega$) $N)$
by (*auto intro*: *measurable_piecewise_restrict measurable_restrict_space1*)

lemma *measurable_If_restrict_space_iff*:
$\{x \in$ *space* $M.\ P\ x\} \in$ *sets* $M \Longrightarrow$
$(\lambda x.$ *if* $P\ x$ *then* $f\ x$ *else* $g\ x) \in$ *measurable* $M\ N \longleftrightarrow$
$(f \in$ *measurable* (*restrict_space* $M\ \{x.\ P\ x\}$) $N \wedge g \in$ *measurable* (*restrict_space* $M\ \{x.\ \neg\ P\ x\}$) $N)$
by (*subst measurable_piecewise_restrict_iff*[**where** C=$\{\{x.\ P\ x\}, \{x.\ \neg\ P\ x\}\}$])
(*auto simp*: *Int_def sets.sets_Collect_neg space_restrict_space conj_commute*[*of* _ $x \in$ *space* M **for** x]
cong: *measurable_cong'*)

lemma *measurable_If*:
$f \in$ *measurable* $M\ M' \Longrightarrow g \in$ *measurable* $M\ M' \Longrightarrow \{x \in$ *space* $M.\ P\ x\} \in$ *sets* $M \Longrightarrow$
$(\lambda x.$ *if* $P\ x$ *then* $f\ x$ *else* $g\ x) \in$ *measurable* $M\ M'$
unfolding *measurable_If_restrict_space_iff* **by** (*auto intro*: *measurable_restrict_space1*)

lemma *measurable_If_set*:
assumes *measure*: $f \in$ *measurable* $M\ M'\ g \in$ *measurable* $M\ M'$
assumes P: $A \cap$ *space* $M \in$ *sets* M
shows $(\lambda x.$ *if* $x \in A$ *then* $f\ x$ *else* $g\ x) \in$ *measurable* $M\ M'$
proof (*rule measurable_If*[*OF measure*])
have $\{x \in$ *space* $M.\ x \in A\} = A \cap$ *space* M **by** *auto*
thus $\{x \in$ *space* $M.\ x \in A\} \in$ *sets* M **using** ‹$A \cap$ *space* $M \in$ *sets* M› **by** *auto*
qed

lemma *measurable_restrict_space_iff*:
$\Omega \cap$ *space* $M \in$ *sets* $M \Longrightarrow c \in$ *space* $N \Longrightarrow$
$f \in$ *measurable* (*restrict_space* $M\ \Omega$) $N \longleftrightarrow (\lambda x.$ *if* $x \in \Omega$ *then* $f\ x$ *else* $c) \in$ *measurable* $M\ N$

by (*subst measurable_If_restrict_space_iff*)
(*simp_all add*: *Int_def conj_commute measurable_const*)

lemma *restrict_space_singleton*: $\{x\} \in sets\ M \Longrightarrow sets\ (restrict\_space\ M\ \{x\}) = sets\ (count\_space\ \{x\})$
using *sets_restrict_space_iff*[*of* $\{x\}$ *M*]
by (*auto simp add*: *sets_restrict_space_iff dest*!: *subset_singletonD*)

lemma *measurable_restrict_countable*:
assumes *X*[*intro*]: *countable X*
assumes *sets*[*simp*]: $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in sets\ M$
assumes *space*[*simp*]: $\bigwedge x.\ x \in X \Longrightarrow f\ x \in space\ N$
assumes *f*: $f \in measurable\ (restrict\_space\ M\ (-\ X))\ N$
shows $f \in measurable\ M\ N$
using *f sets.countable*[*OF sets X*]
by (*intro measurable_piecewise_restrict*[**where** *M*=*M* **and** *C*=$\{-\ X\} \cup ((\lambda x.\ \{x\})\ `\ X)$])
(*auto simp*: *Diff_Int_distrib2 Compl_eq_Diff_UNIV Int_insert_left sets.Diff restrict_space_singleton*
simp del: *sets_count_space cong*: *measurable_cong_sets*)

lemma *measurable_discrete_difference*:
assumes *f*: $f \in measurable\ M\ N$
assumes *X*: *countable X* $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in sets\ M$ $\bigwedge x.\ x \in X \Longrightarrow g\ x \in space\ N$
assumes *eq*: $\bigwedge x.\ x \in space\ M \Longrightarrow x \notin X \Longrightarrow f\ x = g\ x$
shows $g \in measurable\ M\ N$
by (*rule measurable_restrict_countable*[*OF X*])
(*auto simp*: *eq*[*symmetric*] *space_restrict_space cong*: *measurable_cong′ intro*: *f measurable_restrict_space1*)

lemma *measurable_count_space_extend*: $A \subseteq B \Longrightarrow f \in space\ M \to A \Longrightarrow f \in M \to_M count\_space\ B \Longrightarrow f \in M \to_M count\_space\ A$
by (*auto simp*: *measurable_def*)

end

7.2 Measurability Prover

theory *Measurable*
imports
Sigma_Algebra
HOL−Library.Order_Continuity
begin

lemma (**in** *algebra*) *sets_Collect_finite_All*:
assumes $\bigwedge i.\ i \in S \Longrightarrow \{x \in \Omega.\ P\ i\ x\} \in M$ *finite S*
shows $\{x \in \Omega.\ \forall i \in S.\ P\ i\ x\} \in M$

proof −
have $\{x{\in}\Omega.\ \forall i{\in}S.\ P\ i\ x\} = (\mathit{if}\ S = \{\}\ \mathit{then}\ \Omega\ \mathit{else}\ \bigcap i{\in}S.\ \{x{\in}\Omega.\ P\ i\ x\})$
by *auto*
with *assms* **show** *?thesis* **by** (*auto intro*!: *sets_Collect_finite_All'*)
qed

abbreviation *pred M P* ≡ $P \in \mathit{measurable}\ M\ (\mathit{count\_space}\ (\mathit{UNIV}{::}\mathit{bool\ set}))$

lemma *pred_def*: *pred M P* ⟷ $\{x{\in}\mathit{space}\ M.\ P\ x\} \in \mathit{sets}\ M$
proof
assume *pred M P*
then have $P -` \{\mathit{True}\} \cap \mathit{space}\ M \in \mathit{sets}\ M$
by (*auto simp*: *measurable_count_space_eq2*)
also have $P -` \{\mathit{True}\} \cap \mathit{space}\ M = \{x{\in}\mathit{space}\ M.\ P\ x\}$ **by** *auto*
finally show $\{x{\in}\mathit{space}\ M.\ P\ x\} \in \mathit{sets}\ M$ **.**
next
assume *P*: $\{x{\in}\mathit{space}\ M.\ P\ x\} \in \mathit{sets}\ M$
moreover
{ fix *X*
have $X \in \mathit{Pow}\ (\mathit{UNIV} :: \mathit{bool\ set})$ **by** *simp*
then have $P -` X \cap \mathit{space}\ M = \{x{\in}\mathit{space}\ M.\ ((X = \{\mathit{True}\} \longrightarrow P\ x) \wedge (X = \{\mathit{False}\} \longrightarrow \neg\ P\ x) \wedge X \neq \{\})\}$
unfolding *UNIV_bool Pow_insert Pow_empty* **by** *auto*
then have $P -` X \cap \mathit{space}\ M \in \mathit{sets}\ M$
by (*auto intro*!: *sets.sets_Collect_neg sets.sets_Collect_imp sets.sets_Collect_conj sets.sets_Collect_const P*) **}**
then show *pred M P*
by (*auto simp*: *measurable_def*)
qed

lemma *pred_sets1*: $\{x{\in}\mathit{space}\ M.\ P\ x\} \in \mathit{sets}\ M \Longrightarrow f \in \mathit{measurable}\ N\ M \Longrightarrow \mathit{pred}\ N\ (\lambda x.\ P\ (f\ x))$
by (*rule measurable_compose*[**where** *f*=*f* **and** *N*=*M*]) (*auto simp*: *pred_def*)

lemma *pred_sets2*: $A \in \mathit{sets}\ N \Longrightarrow f \in \mathit{measurable}\ M\ N \Longrightarrow \mathit{pred}\ M\ (\lambda x.\ f\ x \in A)$
by (*rule measurable_compose*[**where** *f*=*f* **and** *N*=*N*]) (*auto simp*: *pred_def Int_def*[*symmetric*])

ML_file ‹*measurable.ML*›

attribute_setup *measurable* = ‹
Scan.lift (
(*Args.add* >> *K true* || *Args.del* >> *K false* || *Scan.succeed true*) −−
Scan.optional (*Args.parens* (
Scan.optional (*Args.$$$ raw* >> *K true*) *false* −−
Scan.optional (*Args.$$$ generic* >> *K Measurable.Generic*) *Measurable.Concrete*))
(*false*, *Measurable.Concrete*) >>
Measurable.measurable_thm_attr)

› *declaration of measurability theorems*

attribute_setup *measurable_dest = Measurable.dest_thm_attr*
add dest rule to measurability prover

attribute_setup *measurable_cong = Measurable.cong_thm_attr*
add congurence rules to measurability prover

method_setup *measurable = ‹ Scan.lift (Scan.succeed (METHOD o Measurable.measurable_tac)) ›*
measurability prover

simproc_setup *measurable (A ∈ sets M | f ∈ measurable M N) =*
‹K Measurable.proc›

setup ‹
*Global_Theory.add_thms_dynamic (**binding** ‹measurable›, Measurable.get_all)*
›

declare
pred_sets1[measurable_dest]
pred_sets2[measurable_dest]
sets.sets_into_space[measurable_dest]

declare
sets.top[measurable]
sets.empty_sets[measurable (raw)]
sets.Un[measurable (raw)]
sets.Diff[measurable (raw)]

declare
measurable_count_space[measurable (raw)]
measurable_ident[measurable (raw)]
measurable_id[measurable (raw)]
measurable_const[measurable (raw)]
measurable_If[measurable (raw)]
measurable_comp[measurable (raw)]
measurable_sets[measurable (raw)]

declare *measurable_cong_sets[measurable_cong]*
declare *sets_restrict_space_cong[measurable_cong]*
declare *sets_restrict_UNIV[measurable_cong]*

lemma *predE[measurable (raw)]*:
pred M P ⟹ {x∈space M. P x} ∈ sets M
unfolding *pred_def* **.**

lemma *pred_intros_imp'[measurable (raw)]*:
(K ⟹ pred M (λx. P x)) ⟹ pred M (λx. K ⟶ P x)

by (*cases K*) *auto*

lemma *pred_intros_conj1'*[*measurable* (*raw*)]:
(*K* ⟹ *pred M* (λ*x*. *P x*)) ⟹ *pred M* (λ*x*. *K* ∧ *P x*)
by (*cases K*) *auto*

lemma *pred_intros_conj2'*[*measurable* (*raw*)]:
(*K* ⟹ *pred M* (λ*x*. *P x*)) ⟹ *pred M* (λ*x*. *P x* ∧ *K*)
by (*cases K*) *auto*

lemma *pred_intros_disj1'*[*measurable* (*raw*)]:
(¬ *K* ⟹ *pred M* (λ*x*. *P x*)) ⟹ *pred M* (λ*x*. *K* ∨ *P x*)
by (*cases K*) *auto*

lemma *pred_intros_disj2'*[*measurable* (*raw*)]:
(¬ *K* ⟹ *pred M* (λ*x*. *P x*)) ⟹ *pred M* (λ*x*. *P x* ∨ *K*)
by (*cases K*) *auto*

lemma *pred_intros_logic*[*measurable* (*raw*)]:
pred M (λ*x*. *x* ∈ *space M*)
pred M (λ*x*. *P x*) ⟹ *pred M* (λ*x*. ¬ *P x*)
pred M (λ*x*. *Q x*) ⟹ *pred M* (λ*x*. *P x*) ⟹ *pred M* (λ*x*. *Q x* ∧ *P x*)
pred M (λ*x*. *Q x*) ⟹ *pred M* (λ*x*. *P x*) ⟹ *pred M* (λ*x*. *Q x* ⟶ *P x*)
pred M (λ*x*. *Q x*) ⟹ *pred M* (λ*x*. *P x*) ⟹ *pred M* (λ*x*. *Q x* ∨ *P x*)
pred M (λ*x*. *Q x*) ⟹ *pred M* (λ*x*. *P x*) ⟹ *pred M* (λ*x*. *Q x* = *P x*)
pred M (λ*x*. *f x* ∈ *UNIV*)
pred M (λ*x*. *f x* ∈ {})
pred M (λ*x*. *P'* (*f x*) *x*) ⟹ *pred M* (λ*x*. *f x* ∈ {*y*. *P'* *y x*})
pred M (λ*x*. *f x* ∈ (*B x*)) ⟹ *pred M* (λ*x*. *f x* ∈ − (*B x*))
pred M (λ*x*. *f x* ∈ (*A x*)) ⟹ *pred M* (λ*x*. *f x* ∈ (*B x*)) ⟹ *pred M* (λ*x*. *f x* ∈ (*A x*) − (*B x*))
pred M (λ*x*. *f x* ∈ (*A x*)) ⟹ *pred M* (λ*x*. *f x* ∈ (*B x*)) ⟹ *pred M* (λ*x*. *f x* ∈ (*A x*) ∩ (*B x*))
pred M (λ*x*. *f x* ∈ (*A x*)) ⟹ *pred M* (λ*x*. *f x* ∈ (*B x*)) ⟹ *pred M* (λ*x*. *f x* ∈ (*A x*) ∪ (*B x*))
pred M (λ*x*. *g x* (*f x*) ∈ (*X x*)) ⟹ *pred M* (λ*x*. *f x* ∈ (*g x*) −‘ (*X x*))
by (*auto simp*: *iff_conv_conj_imp pred_def*)

lemma *pred_intros_countable*[*measurable* (*raw*)]:
fixes *P* :: *'a* ⇒ *'i* :: *countable* ⇒ *bool*
shows
(⋀*i*. *pred M* (λ*x*. *P x i*)) ⟹ *pred M* (λ*x*. ∀ *i*. *P x i*)
(⋀*i*. *pred M* (λ*x*. *P x i*)) ⟹ *pred M* (λ*x*. ∃ *i*. *P x i*)
by (*auto intro*!: *sets.sets_Collect_countable_All sets.sets_Collect_countable_Ex simp*: *pred_def*)

lemma *pred_intros_countable_bounded*[*measurable* (*raw*)]:
fixes *X* :: *'i* :: *countable set*
shows

$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcap i{\in}X.\ N\ x\ i))$
$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcup i{\in}X.\ N\ x\ i))$
$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \forall i{\in}X.\ P\ x\ i)$
$(\bigwedge i.\ i \in X \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \exists i{\in}X.\ P\ x\ i)$
by *simp_all* (*auto simp*: *Bex_def Ball_def*)

lemma *pred_intros_finite*[*measurable* (*raw*)]:
finite I $\Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcap i{\in}I.\ N\ x\ i))$
finite I $\Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ x \in N\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ x \in (\bigcup i{\in}I.\ N\ x\ i))$
finite I $\Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \forall i{\in}I.\ P\ x\ i)$
finite I $\Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow pred\ M\ (\lambda x.\ \exists i{\in}I.\ P\ x\ i)$
by (*auto intro*!: *sets.sets_Collect_finite_Ex sets.sets_Collect_finite_All simp*: *iff_conv_conj_imp pred_def*)

lemma *countable_Un_Int*[*measurable* (*raw*)]:
$(\bigwedge i :: 'i :: countable.\ i \in I \Longrightarrow N\ i \in sets\ M) \Longrightarrow (\bigcup i{\in}I.\ N\ i) \in sets\ M$
$I \neq \{\} \Longrightarrow (\bigwedge i :: 'i :: countable.\ i \in I \Longrightarrow N\ i \in sets\ M) \Longrightarrow (\bigcap i{\in}I.\ N\ i) \in sets\ M$
by *auto*

declare
finite_UN[*measurable* (*raw*)]
finite_INT[*measurable* (*raw*)]

lemma *sets_Int_pred*[*measurable* (*raw*)]:
assumes *space*: $A \cap B \subseteq space\ M$ **and** [*measurable*]: *pred M* $(\lambda x.\ x \in A)$ *pred M* $(\lambda x.\ x \in B)$
shows $A \cap B \in sets\ M$
proof −
have $\{x{\in}space\ M.\ x \in A \cap B\} \in sets\ M$ **by** *auto*
also have $\{x{\in}space\ M.\ x \in A \cap B\} = A \cap B$
using *space* **by** *auto*
finally show *?thesis* **.**
qed

lemma [*measurable* (*raw generic*)]:
assumes *f*: $f \in measurable\ M\ N$ **and** *c*: $c \in space\ N \Longrightarrow \{c\} \in sets\ N$
shows *pred_eq_const1*: *pred M* $(\lambda x.\ f\ x = c)$
and *pred_eq_const2*: *pred M* $(\lambda x.\ c = f\ x)$
proof −
show *pred M* $(\lambda x.\ f\ x = c)$
proof *cases*
assume $c \in space\ N$
with *measurable_sets*[*OF f c*] **show** *?thesis*
by (*auto simp*: *Int_def conj_commute pred_def*)
next
assume $c \notin space\ N$

with *f*[*THEN measurable_space*] **have** $\{x \in space\ M.\ f\ x = c\} = \{\}$ **by** *auto*
then show *?thesis* **by** (*auto simp*: *pred_def cong*: *conj_cong*)
qed
then show *pred M* ($\lambda x.\ c = f\ x$)
by (*simp add*: *eq_commute*)
qed

lemma *pred_count_space_const1*[*measurable* (*raw*)]:
$f \in measurable\ M\ (count\_space\ UNIV) \Longrightarrow Measurable.pred\ M\ (\lambda x.\ f\ x = c)$
by (*intro pred_eq_const1*[**where** *N*=*count_space UNIV*]) (*auto*)

lemma *pred_count_space_const2*[*measurable* (*raw*)]:
$f \in measurable\ M\ (count\_space\ UNIV) \Longrightarrow Measurable.pred\ M\ (\lambda x.\ c = f\ x)$
by (*intro pred_eq_const2*[**where** *N*=*count_space UNIV*]) (*auto*)

lemma *pred_le_const*[*measurable* (*raw generic*)]:
assumes *f*: $f \in measurable\ M\ N$ **and** *c*: $\{..\ c\} \in sets\ N$ **shows** $pred\ M\ (\lambda x.\ f\ x \leq c)$
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

lemma *pred_const_le*[*measurable* (*raw generic*)]:
assumes *f*: $f \in measurable\ M\ N$ **and** *c*: $\{c\ ..\} \in sets\ N$ **shows** $pred\ M\ (\lambda x.\ c \leq f\ x)$
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

lemma *pred_less_const*[*measurable* (*raw generic*)]:
assumes *f*: $f \in measurable\ M\ N$ **and** *c*: $\{..< c\} \in sets\ N$ **shows** $pred\ M\ (\lambda x.\ f\ x < c)$
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

lemma *pred_const_less*[*measurable* (*raw generic*)]:
assumes *f*: $f \in measurable\ M\ N$ **and** *c*: $\{c <..\} \in sets\ N$ **shows** $pred\ M\ (\lambda x.\ c < f\ x)$
using *measurable_sets*[*OF f c*]
by (*auto simp*: *Int_def conj_commute eq_commute pred_def*)

declare
sets.Int[*measurable* (*raw*)]

lemma *pred_in_If*[*measurable* (*raw*)]:
$(P \Longrightarrow pred\ M\ (\lambda x.\ x \in A\ x)) \Longrightarrow (\neg\ P \Longrightarrow pred\ M\ (\lambda x.\ x \in B\ x)) \Longrightarrow$
$pred\ M\ (\lambda x.\ x \in (if\ P\ then\ A\ x\ else\ B\ x))$
by *auto*

lemma *sets_range*[*measurable_dest*]:
$A\ `\ I \subseteq sets\ M \Longrightarrow i \in I \Longrightarrow A\ i \in sets\ M$

by *auto*

lemma *pred_sets_range*[*measurable_dest*]:
A ‘ I ⊆ *sets N* ⟹ *i* ∈ *I* ⟹ *f* ∈ *measurable M N* ⟹ *pred M* (λ*x*. *f x* ∈ *A i*)
using *pred_sets2*[*OF sets_range*] **by** *auto*

lemma *sets_All*[*measurable_dest*]:
∀ *i*. *A i* ∈ *sets* (*M i*) ⟹ *A i* ∈ *sets* (*M i*)
by *auto*

lemma *pred_sets_All*[*measurable_dest*]:
∀ *i*. *A i* ∈ *sets* (*N i*) ⟹ *f* ∈ *measurable M* (*N i*) ⟹ *pred M* (λ*x*. *f x* ∈ *A i*)
using *pred_sets2*[*OF sets_All*, *of A N f*] **by** *auto*

lemma *sets_Ball*[*measurable_dest*]:
∀ *i*∈*I*. *A i* ∈ *sets* (*M i*) ⟹ *i*∈*I* ⟹ *A i* ∈ *sets* (*M i*)
by *auto*

lemma *pred_sets_Ball*[*measurable_dest*]:
∀ *i*∈*I*. *A i* ∈ *sets* (*N i*) ⟹ *i*∈*I* ⟹ *f* ∈ *measurable M* (*N i*) ⟹ *pred M* (λ*x*. *f x* ∈ *A i*)
using *pred_sets2*[*OF sets_Ball*, *of* _ _ _ *f*] **by** *auto*

lemma *measurable_finite*[*measurable* (*raw*)]:
fixes *S* :: *'a* ⇒ *nat set*
assumes [*measurable*]: ⋀*i*. {*x*∈*space M*. *i* ∈ *S x*} ∈ *sets M*
shows *pred M* (λ*x*. *finite* (*S x*))
unfolding *finite_nat_set_iff_bounded* **by** (*simp add*: *Ball_def*)

lemma *measurable_Least*[*measurable*]:
assumes [*measurable*]: (⋀*i*::*nat*. (λ*x*. *P i x*) ∈ *measurable M* (*count_space UNIV*))
shows (λ*x*. *LEAST i*. *P i x*) ∈ *measurable M* (*count_space UNIV*)
unfolding *measurable_def* **by** (*safe intro*!: *sets_Least*) *simp_all*

lemma *measurable_Max_nat*[*measurable* (*raw*)]:
fixes *P* :: *nat* ⇒ *'a* ⇒ *bool*
assumes [*measurable*]: ⋀*i*. *Measurable.pred M* (*P i*)
shows (λ*x*. *Max* {*i*. *P i x*}) ∈ *measurable M* (*count_space UNIV*)
unfolding *measurable_count_space_eq2_countable*
proof *safe*
fix *n*

{ fix *x* **assume** ∀ *i*. ∃ *n*≥*i*. *P n x*
then have *infinite* {*i*. *P i x*}
unfolding *infinite_nat_iff_unbounded_le* **by** *auto*
then have *Max* {*i*. *P i x*} = *the None*
by (*rule Max.infinite*) **}**
note *1* = *this*

```
  { fix x i j assume P i x ∀ n≥j. ¬ P n x
    then have finite {i. P i x}
      by (auto simp: subset_eq not_le[symmetric] finite_nat_iff_bounded)
    with ‹P i x› have P (Max {i. P i x}) x i ≤ Max {i. P i x} finite {i. P i x}
      using Max_in[of {i. P i x}] by auto }
  note 2 = this

  have (λx. Max {i. P i x}) -` {n} ∩ space M = {x∈space M. Max {i. P i x} =
n}
    by auto
  also have ... =
    {x∈space M. if (∀ i. ∃ n≥i. P n x) then the None = n else
      if (∃ i. P i x) then P n x ∧ (∀ i>n. ¬ P i x)
      else Max {} = n}
    by (intro arg_cong[where f=Collect] ext conj_cong)
       (auto simp add: 1 2 not_le[symmetric] intro!: Max_eqI)
  also have ... ∈ sets M
    by measurable
  finally show (λx. Max {i. P i x}) -` {n} ∩ space M ∈ sets M .
qed simp

lemma measurable_Min_nat[measurable (raw)]:
  fixes P :: nat ⇒ 'a ⇒ bool
  assumes [measurable]: ⋀i. Measurable.pred M (P i)
  shows (λx. Min {i. P i x}) ∈ measurable M (count_space UNIV)
  unfolding measurable_count_space_eq2_countable
proof safe
  fix n

  { fix x assume ∀ i. ∃ n≥i. P n x
    then have infinite {i. P i x}
      unfolding infinite_nat_iff_unbounded_le by auto
    then have Min {i. P i x} = the None
      by (rule Min.infinite) }
  note 1 = this

  { fix x i j assume P i x ∀ n≥j. ¬ P n x
    then have finite {i. P i x}
      by (auto simp: subset_eq not_le[symmetric] finite_nat_iff_bounded)
    with ‹P i x› have P (Min {i. P i x}) x Min {i. P i x} ≤ i finite {i. P i x}
      using Min_in[of {i. P i x}] by auto }
  note 2 = this

  have (λx. Min {i. P i x}) -` {n} ∩ space M = {x∈space M. Min {i. P i x} =
n}
    by auto
  also have ... =
    {x∈space M. if (∀ i. ∃ n≥i. P n x) then the None = n else
```

if $(\exists\, i.\ P\ i\ x)$ *then* $P\ n\ x \wedge (\forall\, i{<}n.\ \neg\ P\ i\ x)$
else $Min\ \{\} = n\}$
by (*intro arg_cong*[**where** *f*=*Collect*] *ext conj_cong*)
(*auto simp add*: *1 2 not_le*[*symmetric*] *intro*!: *Min_eqI*)
also have ... $\in$ *sets M*
by *measurable*
finally show $(\lambda x.\ Min\ \{i.\ P\ i\ x\}) -` \{n\} \cap space\ M \in sets\ M$ **.**
qed *simp*

lemma *measurable_count_space_insert*[*measurable* (*raw*)]:
$s \in S \Longrightarrow A \in sets\ (count\_space\ S) \Longrightarrow insert\ s\ A \in sets\ (count\_space\ S)$
by *simp*

lemma *sets_UNIV* [*measurable* (*raw*)]: $A \in sets\ (count\_space\ UNIV)$
by *simp*

lemma *measurable_card*[*measurable*]:
fixes $S :: 'a \Rightarrow nat\ set$
assumes [*measurable*]: $\bigwedge i.\ \{x{\in}space\ M.\ i \in S\ x\} \in sets\ M$
shows $(\lambda x.\ card\ (S\ x)) \in measurable\ M\ (count\_space\ UNIV)$
unfolding *measurable_count_space_eq2_countable*
proof *safe*
fix *n* **show** $(\lambda x.\ card\ (S\ x)) -` \{n\} \cap space\ M \in sets\ M$
proof (*cases n*)
case *0*
then have $(\lambda x.\ card\ (S\ x)) -` \{n\} \cap space\ M = \{x{\in}space\ M.\ infinite\ (S\ x) \vee (\forall\, i.\ i \notin S\ x)\}$
by *auto*
also have ... $\in$ *sets M*
by *measurable*
finally show *?thesis* **.**
next
case (*Suc i*)
then have $(\lambda x.\ card\ (S\ x)) -` \{n\} \cap space\ M =$
$(\bigcup F{\in}\{A{\in}\{A.\ finite\ A\}.\ card\ A = n\}.\ \{x{\in}space\ M.\ (\forall\, i.\ i \in S\ x \longleftrightarrow i \in F)\})$
unfolding *set_eq_iff*[*symmetric*] *Collect_bex_eq*[*symmetric*] **by** (*auto intro*: *card_ge_0_finite*)
also have ... $\in$ *sets M*
by (*intro sets.countable_UN' countable_Collect countable_Collect_finite*) *auto*
finally show *?thesis* **.**
qed
qed *rule*

lemma *measurable_pred_countable*[*measurable* (*raw*)]:
assumes *countable X*
shows
$(\bigwedge i.\ i \in X \Longrightarrow Measurable.pred\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow Measurable.pred\ M\ (\lambda x.$

$\forall i{\in}X.\ P\ x\ i)$
$(\bigwedge i.\ i \in X \Longrightarrow \mathit{Measurable.pred}\ M\ (\lambda x.\ P\ x\ i)) \Longrightarrow \mathit{Measurable.pred}\ M\ (\lambda x.\ \exists i{\in}X.\ P\ x\ i)$
unfolding *pred_def*
by (*auto intro*!: *sets.sets_Collect_countable_All' sets.sets_Collect_countable_Ex' assms*)

7.2.1 Measurability for (co)inductive predicates

lemma *measurable_bot*[*measurable*]: $\mathit{bot} \in \mathit{measurable}\ M\ (\mathit{count\_space}\ \mathit{UNIV})$
by (*simp add*: *bot_fun_def*)

lemma *measurable_top*[*measurable*]: $\mathit{top} \in \mathit{measurable}\ M\ (\mathit{count\_space}\ \mathit{UNIV})$
by (*simp add*: *top_fun_def*)

lemma *measurable_SUP*[*measurable*]:
fixes $F :: 'i \Rightarrow 'a \Rightarrow 'b{::}\{\mathit{complete\_lattice}, \mathit{countable}\}$
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in \mathit{measurable}\ M\ (\mathit{count\_space}\ \mathit{UNIV})$
shows $(\lambda x.\ \mathit{SUP}\ i{\in}I.\ F\ i\ x) \in \mathit{measurable}\ M\ (\mathit{count\_space}\ \mathit{UNIV})$
unfolding *measurable_count_space_eq2_countable*
proof (*safe intro*!: *UNIV_I*)
fix *a*
have $(\lambda x.\ \mathit{SUP}\ i{\in}I.\ F\ i\ x) -` \{a\} \cap \mathit{space}\ M =$
$\{x{\in}\mathit{space}\ M.\ (\forall i{\in}I.\ F\ i\ x \le a) \wedge (\forall b.\ (\forall i{\in}I.\ F\ i\ x \le b) \longrightarrow a \le b)\}$
unfolding *SUP_le_iff*[*symmetric*] **by** *auto*
also have $\ldots \in \mathit{sets}\ M$
by *measurable*
finally show $(\lambda x.\ \mathit{SUP}\ i{\in}I.\ F\ i\ x) -` \{a\} \cap \mathit{space}\ M \in \mathit{sets}\ M$.
qed

lemma *measurable_INF*[*measurable*]:
fixes $F :: 'i \Rightarrow 'a \Rightarrow 'b{::}\{\mathit{complete\_lattice}, \mathit{countable}\}$
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in \mathit{measurable}\ M\ (\mathit{count\_space}\ \mathit{UNIV})$
shows $(\lambda x.\ \mathit{INF}\ i{\in}I.\ F\ i\ x) \in \mathit{measurable}\ M\ (\mathit{count\_space}\ \mathit{UNIV})$
unfolding *measurable_count_space_eq2_countable*
proof (*safe intro*!: *UNIV_I*)
fix *a*
have $(\lambda x.\ \mathit{INF}\ i{\in}I.\ F\ i\ x) -` \{a\} \cap \mathit{space}\ M =$
$\{x{\in}\mathit{space}\ M.\ (\forall i{\in}I.\ a \le F\ i\ x) \wedge (\forall b.\ (\forall i{\in}I.\ b \le F\ i\ x) \longrightarrow b \le a)\}$
unfolding *le_INF_iff*[*symmetric*] **by** *auto*
also have $\ldots \in \mathit{sets}\ M$
by *measurable*
finally show $(\lambda x.\ \mathit{INF}\ i{\in}I.\ F\ i\ x) -` \{a\} \cap \mathit{space}\ M \in \mathit{sets}\ M$.
qed

lemma *measurable_lfp_coinduct*[*consumes 1*, *case_names continuity step*]:
fixes $F :: ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b{::}\{\mathit{complete\_lattice}, \mathit{countable}\})$

assumes *P M*
assumes *F*: *sup_continuous F*
assumes ∗: ⋀*M A*. *P M* ⟹ (⋀*N*. *P N* ⟹ *A* ∈ *measurable N* (*count_space UNIV*)) ⟹ *F A* ∈ *measurable M* (*count_space UNIV*)
shows *lfp F* ∈ *measurable M* (*count_space UNIV*)
proof −
{ fix *i* **from** ‹*P M*› **have** ((*F* ^^ *i*) *bot*) ∈ *measurable M* (*count_space UNIV*)
by (*induct i arbitrary*: *M*) (*auto intro*!: ∗) **}**
then have (λ*x*. *SUP i*. (*F* ^^ *i*) *bot x*) ∈ *measurable M* (*count_space UNIV*)
by *measurable*
also have (λ*x*. *SUP i*. (*F* ^^ *i*) *bot x*) = *lfp F*
by (*subst sup_continuous_lfp*) (*auto intro*: *F simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_lfp*:
fixes *F* :: ('*a* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*::{*complete_lattice*, *countable*})
assumes *F*: *sup_continuous F*
assumes ∗: ⋀*A*. *A* ∈ *measurable M* (*count_space UNIV*) ⟹ *F A* ∈ *measurable M* (*count_space UNIV*)
shows *lfp F* ∈ *measurable M* (*count_space UNIV*)
by (*coinduction rule*: *measurable_lfp_coinduct*[*OF* _ *F*]) (*blast intro*: ∗)

lemma *measurable_gfp_coinduct*[*consumes 1*, *case_names continuity step*]:
fixes *F* :: ('*a* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*::{*complete_lattice*, *countable*})
assumes *P M*
assumes *F*: *inf_continuous F*
assumes ∗: ⋀*M A*. *P M* ⟹ (⋀*N*. *P N* ⟹ *A* ∈ *measurable N* (*count_space UNIV*)) ⟹ *F A* ∈ *measurable M* (*count_space UNIV*)
shows *gfp F* ∈ *measurable M* (*count_space UNIV*)
proof −
{ fix *i* **from** ‹*P M*› **have** ((*F* ^^ *i*) *top*) ∈ *measurable M* (*count_space UNIV*)
by (*induct i arbitrary*: *M*) (*auto intro*!: ∗) **}**
then have (λ*x*. *INF i*. (*F* ^^ *i*) *top x*) ∈ *measurable M* (*count_space UNIV*)
by *measurable*
also have (λ*x*. *INF i*. (*F* ^^ *i*) *top x*) = *gfp F*
by (*subst inf_continuous_gfp*) (*auto intro*: *F simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_gfp*:
fixes *F* :: ('*a* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*::{*complete_lattice*, *countable*})
assumes *F*: *inf_continuous F*
assumes ∗: ⋀*A*. *A* ∈ *measurable M* (*count_space UNIV*) ⟹ *F A* ∈ *measurable M* (*count_space UNIV*)
shows *gfp F* ∈ *measurable M* (*count_space UNIV*)
by (*coinduction rule*: *measurable_gfp_coinduct*[*OF* _ *F*]) (*blast intro*: ∗)

lemma *measurable_lfp2_coinduct*[*consumes 1*, *case_names continuity step*]:

fixes $F :: ('a \Rightarrow 'c \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'c \Rightarrow 'b::\{complete\_lattice, countable\})$
assumes $P\ M\ s$
assumes F: *sup_continuous F*
assumes $*$: $\bigwedge M\ A\ s.\ P\ M\ s \Longrightarrow (\bigwedge N\ t.\ P\ N\ t \Longrightarrow A\ t \in measurable\ N\ (count\_space\ UNIV)) \Longrightarrow F\ A\ s \in measurable\ M\ (count\_space\ UNIV)$
shows *lfp F s* $\in$ *measurable M* (*count_space UNIV*)
proof −
{ fix i **from** ‹$P\ M\ s$› **have** $(\lambda x.\ (F\ \widehat{\ }\widehat{\ }\ i)\ bot\ s\ x) \in measurable\ M\ (count\_space\ UNIV)$
by (*induct i arbitrary*: *M s*) (*auto intro*!: $*$) **}**
then have $(\lambda x.\ SUP\ i.\ (F\ \widehat{\ }\widehat{\ }\ i)\ bot\ s\ x) \in measurable\ M\ (count\_space\ UNIV)$
by *measurable*
also have $(\lambda x.\ SUP\ i.\ (F\ \widehat{\ }\widehat{\ }\ i)\ bot\ s\ x) = lfp\ F\ s$
by (*subst sup_continuous_lfp*) (*auto simp*: *F simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_gfp2_coinduct*[*consumes 1*, *case_names continuity step*]:
fixes $F :: ('a \Rightarrow 'c \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'c \Rightarrow 'b::\{complete\_lattice, countable\})$
assumes $P\ M\ s$
assumes F: *inf_continuous F*
assumes $*$: $\bigwedge M\ A\ s.\ P\ M\ s \Longrightarrow (\bigwedge N\ t.\ P\ N\ t \Longrightarrow A\ t \in measurable\ N\ (count\_space\ UNIV)) \Longrightarrow F\ A\ s \in measurable\ M\ (count\_space\ UNIV)$
shows *gfp F s* $\in$ *measurable M* (*count_space UNIV*)
proof −
{ fix i **from** ‹$P\ M\ s$› **have** $(\lambda x.\ (F\ \widehat{\ }\widehat{\ }\ i)\ top\ s\ x) \in measurable\ M\ (count\_space\ UNIV)$
by (*induct i arbitrary*: *M s*) (*auto intro*!: $*$) **}**
then have $(\lambda x.\ INF\ i.\ (F\ \widehat{\ }\widehat{\ }\ i)\ top\ s\ x) \in measurable\ M\ (count\_space\ UNIV)$
by *measurable*
also have $(\lambda x.\ INF\ i.\ (F\ \widehat{\ }\widehat{\ }\ i)\ top\ s\ x) = gfp\ F\ s$
by (*subst inf_continuous_gfp*) (*auto simp*: *F simp*: *image_comp*)
finally show *?thesis* **.**
qed

lemma *measurable_enat_coinduct*:
fixes $f :: 'a \Rightarrow enat$
assumes $R\ f$
assumes $*$: $\bigwedge f.\ R\ f \Longrightarrow \exists g\ h\ i\ P.\ R\ g \wedge f = (\lambda x.\ if\ P\ x\ then\ h\ x\ else\ eSuc\ (g\ (i\ x))) \wedge$
Measurable.pred M P $\wedge$
$i \in measurable\ M\ M\ \wedge$
$h \in measurable\ M\ (count\_space\ UNIV)$
shows $f \in measurable\ M\ (count\_space\ UNIV)$
proof (*simp add*: *measurable_count_space_eq2_countable*, *rule*)
fix $a :: enat$
have $f -` \{a\} \cap space\ M = \{x \in space\ M.\ f\ x = a\}$
by *auto*
{ fix $i :: nat$

```
    from ‹R f› have Measurable.pred M (λx. f x = enat i)
    proof (induction i arbitrary: f)
      case 0
      from ∗[OF this] obtain g h i P
        where f: f = (λx. if P x then h x else eSuc (g (i x))) and
          [measurable]: Measurable.pred M P i ∈ measurable M M h ∈ measurable M
(count_space UNIV)
        by auto
      have Measurable.pred M (λx. P x ∧ h x = 0)
        by measurable
      also have (λx. P x ∧ h x = 0) = (λx. f x = enat 0)
        by (auto simp: f zero_enat_def[symmetric])
      finally show ?case .
    next
      case (Suc n)
      from ∗[OF Suc.prems] obtain g h i P
        where f: f = (λx. if P x then h x else eSuc (g (i x))) and R g and
          M[measurable]: Measurable.pred M P i ∈ measurable M M h ∈ measurable
M (count_space UNIV)
        by auto
      have (λx. f x = enat (Suc n)) =
        (λx. (P x ⟶ h x = enat (Suc n)) ∧ (¬ P x ⟶ g (i x) = enat n))
        by (auto simp: f zero_enat_def[symmetric] eSuc_enat[symmetric])
      also have Measurable.pred M ...
          by (intro pred_intros_logic measurable_compose[OF M(2)] Suc ‹R g›)
measurable
      finally show ?case .
    qed
    then have f −‘ {enat i} ∩ space M ∈ sets M
      by (simp add: pred_def Int_def conj_commute) }
  note fin = this
  show f −‘ {a} ∩ space M ∈ sets M
  proof (cases a)
    case infinity
    then have f −‘ {a} ∩ space M = space M − (⋃n. f −‘ {enat n} ∩ space M)
      by auto
    also have ... ∈ sets M
      by (intro sets.Diff sets.top sets.Un sets.countable_UN) (auto intro!: fin)
    finally show ?thesis .
  qed (simp add: fin)
qed

lemma measurable_THE:
  fixes P :: 'a ⇒ 'b ⇒ bool
  assumes [measurable]: ⋀i. Measurable.pred M (P i)
  assumes I[simp]: countable I ⋀i x. x ∈ space M ⟹ P i x ⟹ i ∈ I
  assumes unique: ⋀x i j. x ∈ space M ⟹ P i x ⟹ P j x ⟹ i = j
  shows (λx. THE i. P i x) ∈ measurable M (count_space UNIV)
  unfolding measurable_def
```

proof *safe*
fix X
define f **where** $f\ x = (THE\ i.\ P\ i\ x)$ **for** x
define *undef* **where** $undef = (THE\ i{::}'a.\ False)$
{ fix $i\ x$ **assume** $x \in space\ M\ P\ i\ x$ **then have** $f\ x = i$
unfolding *f_def* **using** *unique* **by** *auto* **}**
note *f_eq* = *this*
{ fix x **assume** $x \in space\ M\ \forall i{\in}I.\ \neg\ P\ i\ x$
then have $\bigwedge i.\ \neg\ P\ i\ x$
using *I*(2)[*of x*] **by** *auto*
then have $f\ x = undef$
by (*auto simp*: *undef_def f_def*) **}**
then have $f\ -`\ X \cap space\ M = (\bigcup i{\in}I \cap X.\ \{x{\in}space\ M.\ P\ i\ x\}) \cup$
(*if undef* $\in X$ *then space M* $-$ $(\bigcup i{\in}I.\ \{x{\in}space\ M.\ P\ i\ x\})$ *else* $\{\}$)
by (*auto dest*: *f_eq*)
also have $\ldots \in sets\ M$
by (*auto intro*!: *sets.Diff sets.countable_UN'*)
finally show $f\ -`\ X \cap space\ M \in sets\ M$ **.**
qed *simp*

lemma *measurable_Ex1*[*measurable* (*raw*)]:
assumes [*simp*]: *countable I* **and** [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow$ *Measurable.pred M* (*P i*)
shows *Measurable.pred M* $(\lambda x.\ \exists!i{\in}I.\ P\ i\ x)$
unfolding *bex1_def* **by** *measurable*

lemma *measurable_Sup_nat*[*measurable* (*raw*)]:
fixes $F :: 'a \Rightarrow nat\ set$
assumes [*measurable*]: $\bigwedge i.$ *Measurable.pred M* $(\lambda x.\ i \in F\ x)$
shows $(\lambda x.\ Sup\ (F\ x)) \in M \rightarrow_M count\_space\ UNIV$
proof (*clarsimp simp add*: *measurable_count_space_eq2_countable*)
fix a
have *F_empty_iff*: $F\ x = \{\} \longleftrightarrow (\forall i.\ i \notin F\ x)$ **for** x
by *auto*
have *Measurable.pred M* $(\lambda x.$ *if finite* $(F\ x)$ *then if* $F\ x = \{\}$ *then* $a = 0$
else $a \in F\ x \wedge (\forall j.\ j \in F\ x \longrightarrow j \leq a)$ *else* $a =$ *the None*)
unfolding *finite_nat_set_iff_bounded Ball_def F_empty_iff* **by** *measurable*
moreover have $(\lambda x.\ Sup\ (F\ x)) -`\ \{a\} \cap space\ M =$
$\{x{\in}space\ M.$ *if finite* $(F\ x)$ *then if* $F\ x = \{\}$ *then* $a = 0$
else $a \in F\ x \wedge (\forall j.\ j \in F\ x \longrightarrow j \leq a)$ *else* $a =$ *the None*$\}$
by (*intro set_eqI*)
(*auto simp*: *Sup_nat_def Max.infinite intro*!: *Max_in Max_eqI*)
ultimately show $(\lambda x.\ Sup\ (F\ x)) -`\ \{a\} \cap space\ M \in sets\ M$
by *auto*
qed

lemma *measurable_if_split*[*measurable* (*raw*)]:
$(c \Longrightarrow$ *Measurable.pred M f*$) \Longrightarrow (\neg\ c \Longrightarrow$ *Measurable.pred M g*$) \Longrightarrow$
Measurable.pred M (*if c then f else g*)

by *simp*

lemma *pred_restrict_space*:
assumes $S \in$ *sets M*
shows *Measurable.pred* (*restrict_space M S*) $P \longleftrightarrow$ *Measurable.pred M* ($\lambda x.\ x \in S \wedge P\ x$)
unfolding *pred_def sets_Collect_restrict_space_iff*[*OF assms*] **..**

lemma *measurable_predpow*[*measurable*]:
assumes *Measurable.pred M T*
assumes $\bigwedge Q.$ *Measurable.pred M Q* $\Longrightarrow$ *Measurable.pred M* (*R Q*)
shows *Measurable.pred M* ((*R* $\widehat{\ }\widehat{\ }$ *n*) *T*)
by (*induct n*) (*auto intro*: *assms*)

lemma *measurable_compose_countable_restrict*:
assumes *P*: *countable* $\{i.\ P\ i\}$
and *f*: $f \in M \to_M$ *count_space UNIV*
and *Q*: $\bigwedge i.\ P\ i \Longrightarrow$ *pred M* (*Q i*)
shows *pred M* ($\lambda x.\ P\ (f\ x) \wedge Q\ (f\ x)\ x$)
proof −
have *P_f*: $\{x \in$ *space M*. *P* (*f x*)$\} \in$ *sets M*
unfolding *pred_def*[*symmetric*] **by** (*rule measurable_compose*[*OF f*]) *simp*
have *pred* (*restrict_space M* $\{x \in$*space M*. *P* (*f x*)$\}$) ($\lambda x.\ Q\ (f\ x)\ x$)
proof (*rule measurable_compose_countable'*[**where** *g*=*f*, *OF _ _ P*])
show $f \in$ *restrict_space M* $\{x \in$*space M*. *P* (*f x*)$\} \to_M$ *count_space* $\{i.\ P\ i\}$
by (*rule measurable_count_space_extend*[*OF subset_UNIV*])
(*auto simp*: *space_restrict_space intro*!: *measurable_restrict_space1 f*)
qed (*auto intro*!: *measurable_restrict_space1 Q*)
then show *?thesis*
unfolding *pred_restrict_space*[*OF P_f*] **by** (*simp cong*: *measurable_cong*)
qed

lemma *measurable_limsup* [*measurable* (*raw*)]:
assumes [*measurable*]: $\bigwedge n.\ A\ n \in$ *sets M*
shows *limsup A* $\in$ *sets M*
by (*subst limsup_INF_SUP*, *auto*)

lemma *measurable_liminf* [*measurable* (*raw*)]:
assumes [*measurable*]: $\bigwedge n.\ A\ n \in$ *sets M*
shows *liminf A* $\in$ *sets M*
by (*subst liminf_SUP_INF*, *auto*)

lemma *measurable_case_enat*[*measurable* (*raw*)]:
assumes *f*: $f \in M \to_M$ *count_space UNIV* **and** *g*: $\bigwedge i.\ g\ i \in M \to_M N$ **and** *h*: $h \in M \to_M N$
shows ($\lambda x.$ *case f x of enat i* $\Rightarrow$ *g i x* | $\infty \Rightarrow$ *h x*) $\in M \to_M N$
apply (*rule measurable_compose_countable*[*OF _ f*])
subgoal for *i*
by (*cases i*) (*auto intro*: *g h*)

done

hide_const (**open**) *pred*

end

7.3 Measure Spaces

theory *Measure_Space*
imports
Measurable HOL−Library.Extended_Nonnegative_Real
begin

7.3.1 Relate extended reals and the indicator function

lemma *suminf_cmult_indicator*:
fixes $f :: nat \Rightarrow ennreal$
assumes *disjoint_family A* $x \in A\ i$
shows $(\sum n.\ f\ n * indicator\ (A\ n)\ x) = f\ i$
proof −
have $**$: $\bigwedge n.\ f\ n * indicator\ (A\ n)\ x = (if\ n = i\ then\ f\ n\ else\ 0 :: ennreal)$
using ‹$x \in A\ i$› *assms* **unfolding** *disjoint_family_on_def indicator_def* **by** *auto*
then have $\bigwedge n.\ (\sum j{<}n.\ f\ j * indicator\ (A\ j)\ x) = (if\ i < n\ then\ f\ i\ else\ 0 :: ennreal)$
by (*auto simp*: *sum.If_cases*)
moreover have $(SUP\ n.\ if\ i < n\ then\ f\ i\ else\ 0) = (f\ i :: ennreal)$
proof (*rule SUP_eqI*)
fix $y :: ennreal$ **assume** $\bigwedge n.\ n \in UNIV \Longrightarrow (if\ i < n\ then\ f\ i\ else\ 0) \le y$
from *this*[*of Suc i*] **show** $f\ i \le y$ **by** *auto*
qed (*use assms* **in** *simp*)
ultimately show *?thesis* **using** *assms*
by (*simp add*: *suminf_eq_SUP*)
qed

lemma *suminf_indicator*:
assumes *disjoint_family A*
shows $(\sum n.\ indicator\ (A\ n)\ x :: ennreal) = indicator\ (\bigcup i.\ A\ i)\ x$
proof *cases*
assume $*$: $x \in (\bigcup i.\ A\ i)$
then obtain i **where** $x \in A\ i$ **by** *auto*
from *suminf_cmult_indicator*[*OF assms*(*1*), *OF* ‹$x \in A\ i$›, *of* $\lambda k.\ 1$]
show *?thesis* **using** $*$ **by** *simp*
qed *simp*

lemma *sum_indicator_disjoint_family*:
fixes $f :: 'd \Rightarrow 'e$::*semiring_1*
assumes *d*: *disjoint_family_on A P* **and** $x \in A\ j$ **and** *finite P* **and** $j \in P$
shows $(\sum i{\in}P.\ f\ i * indicator\ (A\ i)\ x) = f\ j$

proof −
  have $P \cap \{i.\ x \in A\ i\} = \{j\}$
    using d ‹$x \in A\ j$› ‹$j \in P$› **unfolding** *disjoint_family_on_def*
    by *auto*
  with ‹*finite P*› **show** *?thesis*
    by (*simp add*: *indicator_def*)
qed

The type for emeasure spaces is already defined in *HOL−Analysis.Sigma_Algebra*, as it is also used to represent sigma algebras (with an arbitrary emeasure).

7.3.2 Extend binary sets

lemma *LIMSEQ_binaryset*:
  assumes f: $f\ \{\} = 0$
  shows $(\lambda n.\ \sum i{<}n.\ f\ (\mathit{binaryset}\ A\ B\ i)) \longrightarrow f\ A + f\ B$
proof −
  have $(\lambda n.\ \sum i < \mathit{Suc}\ (\mathit{Suc}\ n).\ f\ (\mathit{binaryset}\ A\ B\ i)) = (\lambda n.\ f\ A + f\ B)$
    proof
      fix n
      show $(\sum i < \mathit{Suc}\ (\mathit{Suc}\ n).\ f\ (\mathit{binaryset}\ A\ B\ i)) = f\ A + f\ B$
        by (*induct n*) (*auto simp add*: *binaryset_def f*)
    qed
  moreover
  have $\ldots \longrightarrow f\ A + f\ B$ **by** (*rule tendsto_const*)
  ultimately have $(\lambda n.\ \sum i{<}\ n{+}2.\ f\ (\mathit{binaryset}\ A\ B\ i)) \longrightarrow f\ A + f\ B$
    by *simp*
  thus *?thesis* **by** (*rule LIMSEQ_offset* [**where** *k=2*])
qed

lemma *binaryset_sums*:
  assumes f: $f\ \{\} = 0$
  shows $(\lambda n.\ f\ (\mathit{binaryset}\ A\ B\ n))\ \mathit{sums}\ (f\ A + f\ B)$
  using *LIMSEQ_binaryset f sums_def* **by** *blast*

lemma *suminf_binaryset_eq*:
  fixes $f :: 'a\ \mathit{set} \Rightarrow 'b{::}\{\mathit{comm\_monoid\_add}, \mathit{t2\_space}\}$
  shows $f\ \{\} = 0 \Longrightarrow (\sum n.\ f\ (\mathit{binaryset}\ A\ B\ n)) = f\ A + f\ B$
  by (*metis binaryset_sums sums_unique*)

7.3.3 Properties of a premeasure μ

The definitions for *positive* and *countably_additive* should be here, by they are necessary to define *'a measure* in *HOL−Analysis.Sigma_Algebra*.

definition *subadditive* **where**
  $\mathit{subadditive}\ M\ f \longleftrightarrow (\forall x{\in}M.\ \forall y{\in}M.\ x \cap y = \{\} \longrightarrow f\ (x \cup y) \le f\ x + f\ y)$

lemma *subadditiveD*: $\mathit{subadditive}\ M\ f \Longrightarrow x \cap y = \{\} \Longrightarrow x \in M \Longrightarrow y \in M \Longrightarrow f\ (x \cup y) \le f\ x + f\ y$

by (*auto simp add*: *subadditive_def*)

definition *countably_subadditive* **where**
countably_subadditive M f ⟷
(∀ *A*. *range A* ⊆ *M* ⟶ *disjoint_family A* ⟶ (⋃ *i*. *A i*) ∈ *M* ⟶ (*f* (⋃ *i*. *A i*) ≤ (∑ *i*. *f* (*A i*))))

lemma (**in** *ring_of_sets*) *countably_subadditive_subadditive*:
fixes *f* :: *'a set* ⇒ *ennreal*
assumes *f*: *positive M f* **and** *cs*: *countably_subadditive M f*
shows *subadditive M f*
proof (*auto simp add*: *subadditive_def*)
fix *x y*
assume *x*: *x* ∈ *M* **and** *y*: *y* ∈ *M* **and** *x* ∩ *y* = {}
hence *disjoint_family* (*binaryset x y*)
by (*auto simp add*: *disjoint_family_on_def binaryset_def*)
hence *range* (*binaryset x y*) ⊆ *M* ⟶
(⋃ *i*. *binaryset x y i*) ∈ *M* ⟶
f (⋃ *i*. *binaryset x y i*) ≤ (∑ *n*. *f* (*binaryset x y n*))
using *cs* **by** (*auto simp add*: *countably_subadditive_def*)
hence {*x*,*y*,{}} ⊆ *M* ⟶ *x* ∪ *y* ∈ *M* ⟶
f (*x* ∪ *y*) ≤ (∑ *n*. *f* (*binaryset x y n*))
by (*simp add*: *range_binaryset_eq UN_binaryset_eq*)
thus *f* (*x* ∪ *y*) ≤ *f x* + *f y* **using** *f x y*
by (*auto simp add*: *Un o_def suminf_binaryset_eq positive_def*)
qed

definition *additive* **where**
additive M μ ⟷ (∀ *x*∈*M*. ∀ *y*∈*M*. *x* ∩ *y* = {} ⟶ μ (*x* ∪ *y*) = μ *x* + μ *y*)

definition *increasing* **where**
increasing M μ ⟷ (∀ *x*∈*M*. ∀ *y*∈*M*. *x* ⊆ *y* ⟶ μ *x* ≤ μ *y*)

lemma *positiveD1*: *positive M f* ⟹ *f* {} = *0* **by** (*auto simp*: *positive_def*)

lemma *positiveD_empty*:
positive M f ⟹ *f* {} = *0*
by (*auto simp add*: *positive_def*)

lemma *additiveD*:
additive M f ⟹ *x* ∩ *y* = {} ⟹ *x* ∈ *M* ⟹ *y* ∈ *M* ⟹ *f* (*x* ∪ *y*) = *f x* + *f y*
by (*auto simp add*: *additive_def*)

lemma *increasingD*:
increasing M f ⟹ *x* ⊆ *y* ⟹ *x*∈*M* ⟹ *y*∈*M* ⟹ *f x* ≤ *f y*
by (*auto simp add*: *increasing_def*)

lemma *countably_additiveI*[*case_names countably*]:
(⋀*A*. ⟦*range A* ⊆ *M*; *disjoint_family A*; (⋃ *i*. *A i*) ∈ *M*⟧ ⟹ (∑ *i*. *f*(*A i*)) =

$f(\bigcup i.\ A\ i))$
  $\Longrightarrow$ *countably_additive M f*
  by (*simp add*: *countably_additive_def*)

lemma (**in** *ring_of_sets*) *disjointed_additive*:
  assumes *f*: *positive M f additive M f* **and** *A*: *range A* $\subseteq$ *M incseq A*
  shows $(\sum i{\leq}n.\ f\ (disjointed\ A\ i)) = f\ (A\ n)$
proof (*induct n*)
  case (*Suc n*)
  then have $(\sum i{\leq}Suc\ n.\ f\ (disjointed\ A\ i)) = f\ (A\ n) + f\ (disjointed\ A\ (Suc\ n))$
    by *simp*
  also have $\ldots = f\ (A\ n \cup disjointed\ A\ (Suc\ n))$
    using *A* **by** (*subst f*(*2*)[*THEN additiveD*]) (*auto simp*: *disjointed_mono*)
  also have *A n* $\cup$ *disjointed A* (*Suc n*) = *A* (*Suc n*)
    using ‹*incseq A*› **by** (*auto dest*: *incseq_SucD simp*: *disjointed_mono*)
  finally show *?case* **.**
qed *simp*

lemma (**in** *ring_of_sets*) *additive_sum*:
  fixes *A*:: *'i* $\Rightarrow$ *'a set*
  assumes *f*: *positive M f* **and** *ad*: *additive M f* **and** *finite S*
      and *A*: *A*‘*S* $\subseteq$ *M*
      and *disj*: *disjoint_family_on A S*
  shows $(\sum i{\in}S.\ f\ (A\ i)) = f\ (\bigcup i{\in}S.\ A\ i)$
  using ‹*finite S*› *disj A*
proof *induct*
  case *empty* **show** *?case* **using** *f* **by** (*simp add*: *positive_def*)
next
  case (*insert s S*)
  then have *A s* $\cap$ $(\bigcup i{\in}S.\ A\ i) = \{\}$
    by (*auto simp add*: *disjoint_family_on_def neq_iff*)
  moreover
  have *A s* $\in$ *M* **using** *insert* **by** *blast*
  moreover have $(\bigcup i{\in}S.\ A\ i) \in M$
    using *insert* ‹*finite S*› **by** *auto*
  ultimately have *f* (*A s* $\cup$ $(\bigcup i{\in}S.\ A\ i)) = f\ (A\ s) + f(\bigcup i{\in}S.\ A\ i)$
    using *ad UNION_in_sets A* **by** (*auto simp add*: *additive_def*)
  with *insert* **show** *?case* **using** *ad disjoint_family_on_mono*[*of S insert s S A*]
    by (*auto simp add*: *additive_def subset_insertI*)
qed

lemma (**in** *ring_of_sets*) *additive_increasing*:
  fixes *f* :: *'a set* $\Rightarrow$ *ennreal*
  assumes *posf*: *positive M f* **and** *addf*: *additive M f*
  shows *increasing M f*
proof (*auto simp add*: *increasing_def*)
  fix *x y*
  assume *xy*: *x* $\in$ *M y* $\in$ *M x* $\subseteq$ *y*
  then have *y* $-$ *x* $\in$ *M* **by** *auto*

```
  then have f x + 0 ≤ f x + f (y−x) by (intro add_left_mono zero_le)
  also have ... = f (x ∪ (y−x))
    by (metis addf Diff_disjoint ‹y − x ∈ M› additiveD xy(1))
  also have ... = f y
    by (metis Un_Diff_cancel Un_absorb1 xy(3))
  finally show f x ≤ f y by simp
qed

lemma (in ring_of_sets) subadditive:
  fixes f :: 'a set ⇒ ennreal
  assumes f: positive M f additive M f and A: A`S ⊆ M and S: finite S
  shows f (⋃i∈S. A i) ≤ (∑i∈S. f (A i))
using S A
proof (induct S)
  case empty thus ?case using f by (auto simp: positive_def)
next
  case (insert x F)
  hence in_M: A x ∈ M (⋃i∈F. A i) ∈ M (⋃i∈F. A i) − A x ∈ M using A
by force+
  have subs: (⋃i∈F. A i) − A x ⊆ (⋃i∈F. A i) by auto
  have (⋃i∈(insert x F). A i) = A x ∪ ((⋃i∈F. A i) − A x) by auto
  hence f (⋃i∈(insert x F). A i) = f (A x ∪ ((⋃i∈F. A i) − A x))
    by simp
  also have ... = f (A x) + f ((⋃i∈F. A i) − A x)
    using f(2) by (rule additiveD) (insert in_M, auto)
  also have ... ≤ f (A x) + f (⋃i∈F. A i)
    using additive_increasing[OF f] in_M subs
    by (simp add: increasingD)
  also have ... ≤ f (A x) + (∑i∈F. f (A i))
    using insert by (auto intro: add_left_mono)
  finally show f (⋃i∈(insert x F). A i) ≤ (∑i∈(insert x F). f (A i))
    by (simp add: insert)
qed

lemma (in ring_of_sets) countably_additive_additive:
  fixes f :: 'a set ⇒ ennreal
  assumes posf: positive M f and ca: countably_additive M f
  shows additive M f
proof (auto simp add: additive_def)
  fix x y
  assume x: x ∈ M and y: y ∈ M and x ∩ y = {}
  hence disjoint_family (binaryset x y)
    by (auto simp add: disjoint_family_on_def binaryset_def)
  hence range (binaryset x y) ⊆ M ⟶
        (⋃i. binaryset x y i) ∈ M ⟶
        f (⋃i. binaryset x y i) = (∑ n. f (binaryset x y n))
    using ca by (simp add: countably_additive_def)
  hence {x,y,{}} ⊆ M ⟶ x ∪ y ∈ M ⟶ f (x ∪ y) = (∑n. f (binaryset x y n))
    by (simp add: range_binaryset_eq UN_binaryset_eq)
```

```
  thus f (x ∪ y) = f x + f y using posf x y
    by (auto simp add: Un suminf_binaryset_eq positive_def)
qed

lemma (in algebra) increasing_additive_bound:
  fixes A:: nat ⇒ 'a set and  f :: 'a set ⇒ ennreal
  assumes f: positive M f and ad: additive M f
      and inc: increasing M f
      and A: range A ⊆ M
      and disj: disjoint_family A
  shows  (∑ i. f (A i)) ≤ f Ω
proof (safe intro!: suminf_le_const)
  fix N
  note disj_N = disjoint_family_on_mono[OF _ disj, of {..<N}]
  have (∑ i<N. f (A i)) = f (⋃ i∈{..<N}. A i)
    using A by (intro additive_sum [OF f ad]) (auto simp: disj_N)
  also have ... ≤ f Ω using space_closed A
    by (intro increasingD[OF inc] finite_UN) auto
  finally show (∑ i<N. f (A i)) ≤ f Ω by simp
qed (insert f A, auto simp: positive_def)

lemma (in ring_of_sets) countably_additiveI_finite:
  fixes μ :: 'a set ⇒ ennreal
  assumes finite Ω positive M μ additive M μ
  shows countably_additive M μ
proof (rule countably_additiveI)
  fix F :: nat ⇒ 'a set assume F: range F ⊆ M (⋃ i. F i) ∈ M and disj:
disjoint_family F

  have ∀ i. F i ≠ {} ⟶ (∃ x. x ∈ F i) by auto
  then obtain f where f: ⋀ i. F i ≠ {} ⟹ f i ∈ F i by metis

  have finU: finite (⋃ i. F i)
    by (metis F(2) assms(1) infinite_super sets_into_space)

  have F_subset: {i. μ (F i) ≠ 0} ⊆ {i. F i ≠ {}}
    by (auto simp: positiveD_empty[OF ‹positive M μ›])
  moreover have fin_not_empty: finite {i. F i ≠ {}}
  proof (rule finite_imageD)
    from f have f`{i. F i ≠ {}} ⊆ (⋃ i. F i) by auto
    then show finite (f`{i. F i ≠ {}})
      by (simp add: finU finite_subset)
    show inj_f: inj_on f {i. F i ≠ {}}
      using f disj
      by (simp add: inj_on_def disjoint_family_on_def disjoint_iff) metis
  qed
  ultimately have fin_not_0: finite {i. μ (F i) ≠ 0}
    by (rule finite_subset)
```

have *disj_not_empty*: *disjoint_family_on F* {*i*. *F i* ≠ {}}
using *disj* **by** (*auto simp*: *disjoint_family_on_def*)

from *fin_not_0* **have** (∑ *i*. *μ* (*F i*)) = (∑ *i* | *μ* (*F i*) ≠ *0*. *μ* (*F i*))
by (*rule suminf_finite*) *auto*
also have ... = (∑ *i* | *F i* ≠ {}. *μ* (*F i*))
using *fin_not_empty F_subset* **by** (*rule sum.mono_neutral_left*) *auto*
also have ... = *μ* (⋃ *i*∈{*i*. *F i* ≠ {}}. *F i*)
using ‹*positive M μ*› ‹*additive M μ*› *fin_not_empty disj_not_empty F* **by** (*intro additive_sum*) *auto*
also have ... = *μ* (⋃ *i*. *F i*)
by (*rule arg_cong*[**where** *f*=*μ*]) *auto*
finally show (∑ *i*. *μ* (*F i*)) = *μ* (⋃ *i*. *F i*) **.**
qed

lemma (**in** *ring_of_sets*) *countably_additive_iff_continuous_from_below*:
fixes *f* :: *'a set* ⇒ *ennreal*
assumes *f*: *positive M f additive M f*
shows *countably_additive M f* ⟷
(∀ *A*. *range A* ⊆ *M* ⟶ *incseq A* ⟶ (⋃ *i*. *A i*) ∈ *M* ⟶ (λ*i*. *f* (*A i*)) ⟶ *f* (⋃ *i*. *A i*))
unfolding *countably_additive_def*
proof *safe*
assume *count_sum*: ∀ *A*. *range A* ⊆ *M* ⟶ *disjoint_family A* ⟶ ⋃ (*A* ‘ *UNIV*) ∈ *M* ⟶ (∑ *i*. *f* (*A i*)) = *f* (⋃ (*A* ‘ *UNIV*))
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *M incseq A* (⋃ *i*. *A i*) ∈ *M*
then have *dA*: *range* (*disjointed A*) ⊆ *M* **by** (*auto simp*: *range_disjointed_sets*)
with *count_sum*[*THEN spec*, *of disjointed A*] *A*(*3*)
have *f_UN*: (∑ *i*. *f* (*disjointed A i*)) = *f* (⋃ *i*. *A i*)
by (*auto simp*: *UN_disjointed_eq disjoint_family_disjointed*)
moreover have (λ*n*. (∑ *i*<*n*. *f* (*disjointed A i*))) ⟶ (∑ *i*. *f* (*disjointed A i*))
by (*simp add*: *summable_LIMSEQ*)
from *LIMSEQ_Suc*[*OF this*]
have (λ*n*. (∑ *i*≤*n*. *f* (*disjointed A i*))) ⟶ (∑ *i*. *f* (*disjointed A i*))
unfolding *lessThan_Suc_atMost* **.**
moreover have ⋀*n*. (∑ *i*≤*n*. *f* (*disjointed A i*)) = *f* (*A n*)
using *disjointed_additive*[*OF f A*(*1*,*2*)] **.**
ultimately show (λ*i*. *f* (*A i*)) ⟶ *f* (⋃ *i*. *A i*) **by** *simp*
next
assume *cont*[*rule_format*]: ∀ *A*. *range A* ⊆ *M* ⟶ *incseq A* ⟶ (⋃ *i*. *A i*) ∈ *M* ⟶ (λ*i*. *f* (*A i*)) ⟶ *f* (⋃ *i*. *A i*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *M disjoint_family A* (⋃ *i*. *A i*) ∈ *M*
have ∗: (⋃ *n*. (⋃ *i*<*n*. *A i*)) = (⋃ *i*. *A i*) **by** *auto*
have *range* (λ*i*. ⋃ *i*<*i*. *A i*) ⊆ *M* (⋃ *i*. ⋃ *i*<*i*. *A i*) ∈ *M*
using *A* ∗ **by** *auto*
then have (λ*n*. *f* (⋃ *i*<*n*. *A i*)) ⟶ *f* (⋃ *i*. *A i*)
unfolding ∗[*symmetric*] **by** (*force intro*!: *cont incseq_SucI*)+
moreover have ⋀*n*. *f* (⋃ *i*<*n*. *A i*) = (∑ *i*<*n*. *f* (*A i*))

```
    using A
   by (intro additive_sum[OF f, symmetric]) (auto intro: disjoint_family_on_mono)
  ultimately
  have (λi. f (A i)) sums f (⋃ i. A i)
    unfolding sums_def by simp
  then show (∑ i. f (A i)) = f (⋃ i. A i)
    by (metis sums_unique)
qed

lemma (in ring_of_sets) continuous_from_above_iff_empty_continuous:
  fixes f :: 'a set ⇒ ennreal
  assumes f: positive M f additive M f
  shows (∀ A. range A ⊆ M ⟶ decseq A ⟶ (⋂ i. A i) ∈ M ⟶ (∀ i. f (A i) ≠
∞) ⟶ (λi. f (A i)) ⟶⟶ f (⋂ i. A i))
     ⟷ (∀ A. range A ⊆ M ⟶ decseq A ⟶ (⋂ i. A i) = {} ⟶ (∀ i. f (A i) ≠
∞) ⟶ (λi. f (A i)) ⟶⟶ 0)
proof safe
  assume cont[rule_format]: (∀ A. range A ⊆ M ⟶ decseq A ⟶ (⋂ i. A i) ∈
M ⟶ (∀ i. f (A i) ≠ ∞) ⟶ (λi. f (A i)) ⟶⟶ f (⋂ i. A i))
  fix A :: nat ⇒ 'a set
  assume A: range A ⊆ M decseq A (⋂ i. A i) = {} ∀ i. f (A i) ≠ ∞
  with cont[of A] show (λi. f (A i)) ⟶⟶ 0
    using ‹positive M f›[unfolded positive_def] by auto
next
  assume cont[rule_format]: ∀ A. range A ⊆ M ⟶ decseq A ⟶ (⋂ i. A i) = {}
⟶ (∀ i. f (A i) ≠ ∞) ⟶ (λi. f (A i)) ⟶⟶ 0
  fix A :: nat ⇒ 'a set
  assume A: range A ⊆ M decseq A (⋂ i. A i) ∈ M ∀ i. f (A i) ≠ ∞

  have f_mono: ⋀a b. a ∈ M ⟹ b ∈ M ⟹ a ⊆ b ⟹ f a ≤ f b
    using additive_increasing[OF f] unfolding increasing_def by simp

  have decseq_fA: decseq (λi. f (A i))
    using A by (auto simp: decseq_def intro!: f_mono)
  have decseq: decseq (λi. A i - (⋂ i. A i))
    using A by (auto simp: decseq_def)
  then have decseq_f: decseq (λi. f (A i - (⋂ i. A i)))
    using A unfolding decseq_def by (auto intro!: f_mono Diff)
  have f (⋂ x. A x) ≤ f (A 0)
    using A by (auto intro!: f_mono)
  then have f_Int_fin: f (⋂ x. A x) ≠ ∞
    using A by (auto simp: top_unique)
  have f_fin: f (A i - (⋂ i. A i)) ≠ ∞ for i
   using A by (metis Diff Diff_subset f_mono infinity_ennreal_def range_subsetD
top_unique)
  have (λi. f (A i - (⋂ i. A i))) ⟶⟶ 0
  proof (intro cont[ OF _ decseq _ f_fin])
    show range (λi. A i - (⋂ i. A i)) ⊆ M (⋂ i. A i - (⋂ i. A i)) = {}
      using A by auto
```

```
  qed
  with INF_Lim decseq_f have (INF n. f (A n - (⋂i. A i))) = 0 by metis
  moreover have (INF n. f (⋂i. A i)) = f (⋂i. A i)
    by auto
  ultimately have (INF n. f (A n - (⋂i. A i)) + f (⋂i. A i)) = 0 + f (⋂i. A
i)
    using A(4) f_fin f_Int_fin
    using INF_ennreal_add_const by presburger
  moreover {
    fix n
    have f (A n - (⋂i. A i)) + f (⋂i. A i) = f ((A n - (⋂i. A i)) ∪ (⋂i. A i))
      using A by (subst f(2)[THEN additiveD]) auto
    also have (A n - (⋂i. A i)) ∪ (⋂i. A i) = A n
      by auto
    finally have f (A n - (⋂i. A i)) + f (⋂i. A i) = f (A n) . }
  ultimately have (INF n. f (A n)) = f (⋂i. A i)
    by simp
  with LIMSEQ_INF[OF decseq_fA]
  show (λi. f (A i)) ⟶ f (⋂i. A i) by simp
qed

lemma (in ring_of_sets) empty_continuous_imp_continuous_from_below:
  fixes f :: 'a set ⇒ ennreal
  assumes f: positive M f additive M f ∀A∈M. f A ≠ ∞
  assumes cont: ∀A. range A ⊆ M ⟶ decseq A ⟶ (⋂i. A i) = {} ⟶ (λi. f
(A i)) ⟶ 0
  assumes A: range A ⊆ M incseq A (⋃i. A i) ∈ M
  shows (λi. f (A i)) ⟶ f (⋃i. A i)
proof -
  from A have (λi. f ((⋃i. A i) - A i)) ⟶ 0
    by (intro cont[rule_format]) (auto simp: decseq_def incseq_def)
  moreover
  { fix i
    have f ((⋃i. A i) - A i ∪ A i) = f ((⋃i. A i) - A i) + f (A i)
      using A by (intro f(2)[THEN additiveD]) auto
    also have ((⋃i. A i) - A i) ∪ A i = (⋃i. A i)
      by auto
    finally have f ((⋃i. A i) - A i) = f (⋃i. A i) - f (A i)
      using assms f by fastforce
  }
  moreover have ∀F i in sequentially. f (A i) ≤ f (⋃i. A i)
    using increasingD[OF additive_increasing[OF f(1, 2)], of A _ ⋃i. A i] A
    by (auto intro!: always_eventually simp: subset_eq)
  ultimately show (λi. f (A i)) ⟶ f (⋃i. A i)
    by (auto intro: ennreal_tendsto_const_minus)
qed

lemma (in ring_of_sets) empty_continuous_imp_countably_additive:
  fixes f :: 'a set ⇒ ennreal
```

assumes f: *positive M f additive M f* **and** *fin*: $\forall A{\in}M.\ f\ A \neq \infty$
assumes *cont*: $\bigwedge A.$ *range A* $\subseteq M \Longrightarrow$ *decseq A* $\Longrightarrow (\bigcap i.\ A\ i) = \{\} \Longrightarrow (\lambda i.\ f\ (A\ i)) \longrightarrow 0$
shows *countably_additive M f*
using *countably_additive_iff_continuous_from_below*[*OF f*]
using *empty_continuous_imp_continuous_from_below*[*OF f fin*] *cont*
by *blast*

7.3.4 Properties of *emeasure*

lemma *emeasure_positive*: *positive* (*sets M*) (*emeasure M*)
by (*cases M*) (*auto simp*: *sets_def emeasure_def Abs_measure_inverse measure_space_def*)

lemma *emeasure_empty*[*simp*, *intro*]: *emeasure M* $\{\} = 0$
using *emeasure_positive*[*of M*] **by** (*simp add*: *positive_def*)

lemma *emeasure_single_in_space*: *emeasure M* $\{x\} \neq 0 \Longrightarrow x \in$ *space M*
using *emeasure_notin_sets*[*of* $\{x\}$ *M*] **by** (*auto dest*: *sets.sets_into_space zero_less_iff_neq_zero*[*TH iffD2*])

lemma *emeasure_countably_additive*: *countably_additive* (*sets M*) (*emeasure M*)
by (*cases M*) (*auto simp*: *sets_def emeasure_def Abs_measure_inverse measure_space_def*)

lemma *suminf_emeasure*:
range A $\subseteq$ *sets M* $\Longrightarrow$ *disjoint_family A* $\Longrightarrow (\sum i.$ *emeasure M* $(A\ i)) =$ *emeasure M* $(\bigcup i.\ A\ i)$
using *sets.countable_UN*[*of A UNIV M*] *emeasure_countably_additive*[*of M*]
by (*simp add*: *countably_additive_def*)

lemma *sums_emeasure*:
disjoint_family F $\Longrightarrow (\bigwedge i.\ F\ i \in$ *sets M*$) \Longrightarrow (\lambda i.$ *emeasure M* $(F\ i))$ *sums emeasure M* $(\bigcup i.\ F\ i)$
unfolding *sums_iff* **by** (*intro conjI suminf_emeasure*) *auto*

lemma *emeasure_additive*: *additive* (*sets M*) (*emeasure M*)
by (*metis sets.countably_additive_additive emeasure_positive emeasure_countably_additive*)

lemma *plus_emeasure*:
$a \in$ *sets M* $\Longrightarrow b \in$ *sets M* $\Longrightarrow a \cap b = \{\} \Longrightarrow$ *emeasure M a* + *emeasure M b* = *emeasure M* $(a \cup b)$
using *additiveD*[*OF emeasure_additive*] **..**

lemma *emeasure_Un*:
$A \in$ *sets M* $\Longrightarrow B \in$ *sets M* $\Longrightarrow$ *emeasure M* $(A \cup B) =$ *emeasure M A* + *emeasure M* $(B - A)$
using *plus_emeasure*[*of A M B* $-$ *A*] **by** *auto*

lemma *emeasure_Un_Int*:
assumes $A \in sets\ M\ B \in sets\ M$
shows *emeasure M A + emeasure M B = emeasure M (A ∪ B) + emeasure M (A ∩ B)*
proof −
have $A = (A-B) \cup (A \cap B)$ **by** *auto*
then have *emeasure M A = emeasure M (A−B) + emeasure M (A ∩ B)*
by (*metis Diff_Diff_Int Diff_disjoint assms plus_emeasure sets.Diff*)
moreover have $A \cup B = (A-B) \cup B$ **by** *auto*
then have *emeasure M (A ∪ B) = emeasure M (A−B) + emeasure M B*
by (*metis Diff_disjoint Int_commute assms plus_emeasure sets.Diff*)
ultimately show *?thesis* **by** (*metis add.assoc add.commute*)
qed

lemma *sum_emeasure*:
$F`I \subseteq sets\ M \Longrightarrow$ *disjoint_family_on F I* $\Longrightarrow$ *finite I* $\Longrightarrow$
$(\sum i \in I.\ emeasure\ M\ (F\ i)) = emeasure\ M\ (\bigcup i \in I.\ F\ i)$
by (*metis sets.additive_sum emeasure_positive emeasure_additive*)

lemma *emeasure_mono*:
$a \subseteq b \Longrightarrow b \in sets\ M \Longrightarrow emeasure\ M\ a \leq emeasure\ M\ b$
by (*metis zero_le sets.additive_increasing emeasure_additive emeasure_notin_sets emeasure_positive increasingD*)

lemma *emeasure_space*:
emeasure M A ≤ emeasure M (space M)
by (*metis emeasure_mono emeasure_notin_sets sets.sets_into_space sets.top zero_le*)

lemma *emeasure_Diff*:
assumes $emeasure\ M\ B \neq \infty$
and $A \in sets\ M\ B \in sets\ M$ **and** $B \subseteq A$
shows *emeasure M (A − B) = emeasure M A − emeasure M B*
by (*smt* (*verit*, *best*) *add_diff_self_ennreal assms emeasure_Un emeasure_mono*

ennreal_add_left_cancel le_iff_sup)

lemma *emeasure_compl*:
$s \in sets\ M \Longrightarrow emeasure\ M\ s \neq \infty \Longrightarrow$ *emeasure M (space M − s) = emeasure M (space M) − emeasure M s*
by (*rule emeasure_Diff*) (*auto dest*: *sets.sets_into_space*)

lemma *Lim_emeasure_incseq*:
range A ⊆ sets M $\Longrightarrow$ *incseq A* $\Longrightarrow$ $(\lambda i.\ (emeasure\ M\ (A\ i))) \longrightarrow emeasure\ M\ (\bigcup i.\ A\ i)$
using *emeasure_countably_additive*
by (*auto simp add*: *sets.countably_additive_iff_continuous_from_below emeasure_positive*
emeasure_additive)

lemma *incseq_emeasure*:
assumes *range B ⊆ sets M incseq B*
shows *incseq (λi. emeasure M (B i))*
using *assms* **by** (*auto simp*: *incseq_def intro*!: *emeasure_mono*)

lemma *SUP_emeasure_incseq*:
assumes *A*: *range A ⊆ sets M incseq A*
shows (*SUP n. emeasure M (A n)*) = *emeasure M* (⋃ *i. A i*)
using *LIMSEQ_SUP*[*OF incseq_emeasure*, *OF A*] *Lim_emeasure_incseq*[*OF A*]
by (*simp add*: *LIMSEQ_unique*)

lemma *decseq_emeasure*:
assumes *range B ⊆ sets M decseq B*
shows *decseq (λi. emeasure M (B i))*
using *assms* **by** (*auto simp*: *decseq_def intro*!: *emeasure_mono*)

lemma *INF_emeasure_decseq*:
assumes *A*: *range A ⊆ sets M* **and** *decseq A*
and *finite*: ⋀*i. emeasure M (A i) ≠ ∞*
shows (*INF n. emeasure M (A n)*) = *emeasure M* (⋂ *i. A i*)
proof −
have *le_MI*: *emeasure M* (⋂ *i. A i*) ≤ *emeasure M (A 0)*
using *A* **by** (*auto intro*!: *emeasure_mono*)
hence ∗: *emeasure M* (⋂ *i. A i*) ≠ ∞ **using** *finite*[*of 0*] **by** (*auto simp*: *top_unique*)

have *emeasure M (A 0)* − (*INF n. emeasure M (A n)*) = (*SUP n. emeasure M (A 0)* − *emeasure M (A n)*)
by (*simp add*: *ennreal_INF_const_minus*)
also have ... = (*SUP n. emeasure M (A 0* − *A n)*)
using *A finite* ‹*decseq A*›[*unfolded decseq_def*] **by** (*subst emeasure_Diff*) *auto*
also have ... = *emeasure M* (⋃ *i. A 0* − *A i*)
proof (*rule SUP_emeasure_incseq*)
show *range (λn. A 0* − *A n) ⊆ sets M*
using *A* **by** *auto*
show *incseq (λn. A 0* − *A n)*
using ‹*decseq A*› **by** (*auto simp add*: *incseq_def decseq_def*)
qed
also have ... = *emeasure M (A 0)* − *emeasure M* (⋂ *i. A i*)
using *A finite* ∗ **by** (*simp*, *subst emeasure_Diff*) *auto*
finally show *?thesis*
by (*smt* (*verit*, *best*) *Inf_lower diff_diff_ennreal le_MI finite range_eqI*)
qed

lemma *INF_emeasure_decseq′*:
assumes *A*: ⋀*i. A i ∈ sets M* **and** *decseq A*
and *finite*: ∃ *i. emeasure M (A i) ≠ ∞*
shows (*INF n. emeasure M (A n)*) = *emeasure M* (⋂ *i. A i*)
proof −

```
  from finite obtain i where i: emeasure M (A i) < ∞
    by (auto simp: less_top)
  have fin: i ≤ j ⟹ emeasure M (A j) < ∞ for j
    by (rule le_less_trans[OF emeasure_mono i]) (auto intro!: decseqD[OF ‹decseq
A›] A)

  have (INF n. emeasure M (A n)) = (INF n. emeasure M (A (n + i)))
  proof (rule INF_eq)
    show ∃j∈UNIV. emeasure M (A (j + i)) ≤ emeasure M (A i') for i'
      by (meson A ‹decseq A› decseq_def emeasure_mono iso_tuple_UNIV_I
nat_le_iff_add)
  qed auto
  also have ... = emeasure M (INF n. (A (n + i)))
    using A ‹decseq A› fin by (intro INF_emeasure_decseq) (auto simp: decseq_def
less_top)
  also have (INF n. (A (n + i))) = (INF n. A n)
    by (meson INF_eq UNIV_I assms(2) decseqD le_add1)
  finally show ?thesis .
qed

lemma emeasure_INT_decseq_subset:
  fixes F :: nat ⇒ 'a set
  assumes I: I ≠ {} and F: ⋀i j. i ∈ I ⟹ j ∈ I ⟹ i ≤ j ⟹ F j ⊆ F i
  assumes F_sets[measurable]: ⋀i. i ∈ I ⟹ F i ∈ sets M
    and fin: ⋀i. i ∈ I ⟹ emeasure M (F i) ≠ ∞
  shows emeasure M (⋂i∈I. F i) = (INF i∈I. emeasure M (F i))
proof cases
  assume finite I
  have (⋂i∈I. F i) = F (Max I)
    using I ‹finite I› by (intro antisym INF_lower INF_greatest F) auto
  moreover have (INF i∈I. emeasure M (F i)) = emeasure M (F (Max I))
   using I ‹finite I› by (intro antisym INF_lower INF_greatest F emeasure_mono)
auto
  ultimately show ?thesis
    by simp
next
  assume infinite I
  define L where L n = (LEAST i. i ∈ I ∧ i ≥ n) for n
  have L: L n ∈ I ∧ n ≤ L n for n
    unfolding L_def
  proof (rule LeastI_ex)
    show ∃x. x ∈ I ∧ n ≤ x
      using ‹infinite I› finite_subset[of I {..< n}]
      by (rule_tac ccontr) (auto simp: not_le)
  qed
  have L_eq[simp]: i ∈ I ⟹ L i = i for i
    unfolding L_def by (intro Least_equality) auto
  have L_mono: i ≤ j ⟹ L i ≤ L j for i j
    using L[of j] unfolding L_def by (intro Least_le) (auto simp: L_def)
```

```
  have emeasure M (⋂ i. F (L i)) = (INF i. emeasure M (F (L i)))
  proof (intro INF_emeasure_decseq[symmetric])
    show decseq (λi. F (L i))
      using L by (intro antimonoI F L_mono) auto
  qed (insert L fin, auto)
  also have ... = (INF i∈I. emeasure M (F i))
  proof (intro antisym INF_greatest)
    show i ∈ I ⟹ (INF i. emeasure M (F (L i))) ≤ emeasure M (F i) for i
      by (intro INF_lower2[of i]) auto
  qed (insert L, auto intro: INF_lower)
  also have (⋂ i. F (L i)) = (⋂ i∈I. F i)
  proof (intro antisym INF_greatest)
    show i ∈ I ⟹ (⋂ i. F (L i)) ⊆ F i for i
      by (intro INF_lower2[of i]) auto
  qed (insert L, auto)
  finally show ?thesis .
qed

lemma Lim_emeasure_decseq:
  assumes A: range A ⊆ sets M decseq A and fin: ⋀i. emeasure M (A i) ≠ ∞
  shows (λi. emeasure M (A i)) ⟶ emeasure M (⋂ i. A i)
  using LIMSEQ_INF[OF decseq_emeasure, OF A]
  using INF_emeasure_decseq[OF A fin] by simp

lemma emeasure_lfp'[consumes 1, case_names cont measurable]:
  assumes P M
  assumes cont: sup_continuous F
  assumes ∗: ⋀M A. P M ⟹ (⋀N. P N ⟹ Measurable.pred N A) ⟹ Measurable.pred M (F A)
  shows emeasure M {x∈space M. lfp F x} = (SUP i. emeasure M {x∈space M. (F ^^ i) (λx. False) x})
proof −
  have emeasure M {x∈space M. lfp F x} = emeasure M (⋃ i. {x∈space M. (F ^^ i) (λx. False) x})
    using sup_continuous_lfp[OF cont] by (auto simp add: bot_fun_def intro!: arg_cong2[where f=emeasure])
  moreover { fix i from ‹P M› have {x∈space M. (F ^^ i) (λx. False) x} ∈ sets M
    by (induct i arbitrary: M) (auto simp add: pred_def[symmetric] intro: ∗) }
  moreover have incseq (λi. {x∈space M. (F ^^ i) (λx. False) x})
  proof (rule incseq_SucI)
    fix i
    have (F ^^ i) (λx. False) ≤ (F ^^ (Suc i)) (λx. False)
    proof (induct i)
      case 0 show ?case by (simp add: le_fun_def)
    next
      case Suc thus ?case using monoD[OF sup_continuous_mono[OF cont] Suc]
by auto
```

```
    qed
    then show {x ∈ space M. (F ^^ i) (λx. False) x} ⊆ {x ∈ space M. (F ^^ Suc
i) (λx. False) x}
      by auto
  qed
  ultimately show ?thesis
    by (subst SUP_emeasure_incseq) auto
qed

lemma emeasure_lfp:
  assumes [simp]: ⋀s. sets (M s) = sets N
  assumes cont: sup_continuous F sup_continuous f
  assumes meas: ⋀P. Measurable.pred N P ⟹ Measurable.pred N (F P)
  assumes iter: ⋀P s. Measurable.pred N P ⟹ P ≤ lfp F ⟹ emeasure (M s)
{x∈space N. F P x} = f (λs. emeasure (M s) {x∈space N. P x}) s
  shows emeasure (M s) {x∈space N. lfp F x} = lfp f s
proof (subst lfp_transfer_bounded[where α=λF s. emeasure (M s) {x∈space N.
F x} and f=F , symmetric])
  fix C assume incseq C ⋀i. Measurable.pred N (C i)
  then show (λs. emeasure (M s) {x ∈ space N. (SUP i. C i) x}) = (SUP i. (λs.
emeasure (M s) {x ∈ space N. C i x}))
    unfolding SUP_apply
     by (subst SUP_emeasure_incseq) (auto simp: mono_def fun_eq_iff intro!:
arg_cong2[where f=emeasure])
qed (auto simp add: iter le_fun_def SUP_apply intro!: meas cont)

lemma emeasure_subadditive_finite:
  finite I ⟹ A ‘ I ⊆ sets M ⟹ emeasure M (⋃i∈I. A i) ≤ (∑ i∈I. emeasure
M (A i))
  by (rule sets.subadditive[OF emeasure_positive emeasure_additive]) auto

lemma emeasure_subadditive:
   A ∈ sets M ⟹ B ∈ sets M ⟹ emeasure M (A ∪ B) ≤ emeasure M A +
emeasure M B
  using emeasure_subadditive_finite[of {True, False} λTrue ⇒ A | False ⇒ B M]
by simp

lemma emeasure_subadditive_countably:
  assumes range f ⊆ sets M
  shows emeasure M (⋃ i. f i) ≤ (∑ i. emeasure M (f i))
proof −
  have emeasure M (⋃ i. f i) = emeasure M (⋃ i. disjointed f i)
    unfolding UN_disjointed_eq ..
  also have ... = (∑ i. emeasure M (disjointed f i))
    using sets.range_disjointed_sets[OF assms] suminf_emeasure[of disjointed f]
    by (simp add:  disjoint_family_disjointed comp_def)
  also have ... ≤ (∑ i. emeasure M (f i))
    using sets.range_disjointed_sets[OF assms] assms
    by (auto intro!: suminf_le emeasure_mono disjointed_subset)
```

finally show *?thesis* **.**
qed

lemma *emeasure_insert*:
assumes *sets*: $\{x\} \in$ *sets* M $A \in$ *sets* M **and** $x \notin A$
shows *emeasure* M (*insert* x A) = *emeasure* M $\{x\}$ + *emeasure* M A
proof −
have $\{x\} \cap A = \{\}$ **using** ‹$x \notin A$› **by** *auto*
from *plus_emeasure*[*OF sets this*] **show** *?thesis* **by** *simp*
qed

lemma *emeasure_insert_ne*:
$A \neq \{\} \Longrightarrow \{x\} \in$ *sets* $M \Longrightarrow A \in$ *sets* $M \Longrightarrow x \notin A \Longrightarrow$ *emeasure* M (*insert* x A) = *emeasure* M $\{x\}$ + *emeasure* M A
by (*rule emeasure_insert*)

lemma *emeasure_eq_sum_singleton*:
assumes *finite* S $\bigwedge x.\ x \in S \Longrightarrow \{x\} \in$ *sets* M
shows *emeasure* M S = $(\sum x{\in}S.$ *emeasure* M $\{x\})$
using *sum_emeasure*[*of* $\lambda x.\ \{x\}$ S M] *assms*
by (*auto simp*: *disjoint_family_on_def subset_eq*)

lemma *sum_emeasure_cover*:
assumes *finite* S **and** $A \in$ *sets* M **and** *br_in_M*: B ‘ $S \subseteq$ *sets* M
assumes A: $A \subseteq (\bigcup i{\in}S.\ B\ i)$
assumes *disj*: *disjoint_family_on* B S
shows *emeasure* M A = $(\sum i{\in}S.$ *emeasure* M $(A \cap (B\ i)))$
proof −
have $(\sum i{\in}S.$ *emeasure* M $(A \cap (B\ i))) =$ *emeasure* M $(\bigcup i{\in}S.\ A \cap (B\ i))$
proof (*rule sum_emeasure*)
show *disjoint_family_on* $(\lambda i.\ A \cap B\ i)$ S
using ‹*disjoint_family_on* B S›
unfolding *disjoint_family_on_def* **by** *auto*
qed (*insert assms*, *auto*)
also have $(\bigcup i{\in}S.\ A \cap (B\ i)) = A$
using A **by** *auto*
finally show *?thesis* **by** *simp*
qed

lemma *emeasure_eq_0*:
$N \in$ *sets* $M \Longrightarrow$ *emeasure* M $N = 0 \Longrightarrow K \subseteq N \Longrightarrow$ *emeasure* M $K = 0$
by (*metis emeasure_mono order_eq_iff zero_le*)

lemma *emeasure_UN_eq_0*:
assumes $\bigwedge i{::}nat.$ *emeasure* M $(N\ i) = 0$ **and** *range* $N \subseteq$ *sets* M
shows *emeasure* M $(\bigcup i.\ N\ i) = 0$
proof −
have *emeasure* M $(\bigcup i.\ N\ i) \leq 0$
using *emeasure_subadditive_countably*[*OF assms*(*2*)] *assms*(*1*) **by** *simp*

then show *?thesis*
by (*auto intro*: *antisym zero_le*)
qed

lemma *measure_eqI_finite*:
assumes [*simp*]: *sets M* = *Pow A sets N* = *Pow A* **and** *finite A*
assumes *eq*: ⋀*a*. *a* ∈ *A* ⟹ *emeasure M* {*a*} = *emeasure N* {*a*}
shows *M* = *N*
proof (*rule measure_eqI*)
fix *X* **assume** *X* ∈ *sets M*
then have *X*: *X* ⊆ *A* **by** *auto*
then have *emeasure M X* = (∑ *a*∈*X*. *emeasure M* {*a*})
using ‹*finite A*› **by** (*subst emeasure_eq_sum_singleton*) (*auto dest*: *finite_subset*)
also have ... = (∑ *a*∈*X*. *emeasure N* {*a*})
using *X eq* **by** (*auto intro*!: *sum.cong*)
also have ... = *emeasure N X*
using *X* ‹*finite A*› **by** (*subst emeasure_eq_sum_singleton*) (*auto dest*: *finite_subset*)
finally show *emeasure M X* = *emeasure N X* **.**
qed *simp*

lemma *measure_eqI_generator_eq*:
fixes *M N* :: *'a measure* **and** *E* :: *'a set set* **and** *A* :: *nat* ⇒ *'a set*
assumes *Int_stable E E* ⊆ *Pow* Ω
and *eq*: ⋀*X*. *X* ∈ *E* ⟹ *emeasure M X* = *emeasure N X*
and *M*: *sets M* = *sigma_sets* Ω *E*
and *N*: *sets N* = *sigma_sets* Ω *E*
and *A*: *range A* ⊆ *E* (⋃ *i*. *A i*) = Ω ⋀*i*. *emeasure M* (*A i*) ≠ ∞
shows *M* = *N*
proof −
let *?μ* = *emeasure M* **and** *?ν* = *emeasure N*
interpret *S*: *sigma_algebra* Ω *sigma_sets* Ω *E* **by** (*rule sigma_algebra_sigma_sets*) *fact*
have *space M* = Ω
using *sets.top*[*of M*] *sets.space_closed*[*of M*] *S.top S.space_closed* ‹*sets M* = *sigma_sets* Ω *E*›
by *blast*

{ fix *F D* **assume** *F* ∈ *E* **and** *?μ F* ≠ ∞
then have [*intro*]: *F* ∈ *sigma_sets* Ω *E* **by** *auto*
have *?ν F* ≠ ∞ **using** ‹*?μ F* ≠ ∞› ‹*F* ∈ *E*› *eq* **by** *simp*
assume *D* ∈ *sets M*
with ‹*Int_stable E*› ‹*E* ⊆ *Pow* Ω› **have** *emeasure M* (*F* ∩ *D*) = *emeasure N* (*F* ∩ *D*)
unfolding *M*
proof (*induct rule*: *sigma_sets_induct_disjoint*)
case (*basic A*)
then have *F* ∩ *A* ∈ *E* **using** ‹*Int_stable E*› ‹*F* ∈ *E*› **by** (*auto simp*: *Int_stable_def*)

```
    then show ?case using eq by auto
  next
    case empty then show ?case by simp
  next
    case (compl A)
    then have **: F ∩ (Ω − A) = F − (F ∩ A)
      and [intro]: F ∩ A ∈ sigma_sets Ω E
      using ‹F ∈ E› S.sets_into_space by (auto simp: M)
    have ?ν (F ∩ A) ≤ ?ν F by (auto intro!: emeasure_mono simp: M N)
   then have ?ν (F ∩ A) ≠ ∞ using ‹?ν F ≠ ∞› by (auto simp: top_unique)
    have ?μ (F ∩ A) ≤ ?μ F by (auto intro!: emeasure_mono simp: M N)
   then have ?μ (F ∩ A) ≠ ∞ using ‹?μ F ≠ ∞› by (auto simp: top_unique)
    then have ?μ (F ∩ (Ω − A)) = ?μ F − ?μ (F ∩ A) unfolding **
      using ‹F ∩ A ∈ sigma_sets Ω E› by (auto intro!: emeasure_Diff simp: M
N)
    also have ... = ?ν F − ?ν (F ∩ A) using eq ‹F ∈ E› compl by simp
    also have ... = ?ν (F ∩ (Ω − A)) unfolding **
      using ‹F ∩ A ∈ sigma_sets Ω E› ‹?ν (F ∩ A) ≠ ∞›
      by (auto intro!: emeasure_Diff[symmetric] simp: M N)
    finally show ?case
      using ‹space M = Ω› by auto
  next
    case (union A)
    then have ?μ (⋃ x. F ∩ A x) = ?ν (⋃ x. F ∩ A x)
    by (subst (1 2) suminf_emeasure[symmetric]) (auto simp: disjoint_family_on_def
subset_eq M N)
    with A show ?case
      by auto
  qed }
 note * = this
 show M = N
 proof (rule measure_eqI)
  show sets M = sets N
    using M N by simp
  have [simp, intro]: ⋀i. A i ∈ sets M
    using A(1) by (auto simp: subset_eq M)
  fix F assume F ∈ sets M
  let ?D = disjointed (λi. F ∩ A i)
  from ‹space M = Ω› have F_eq: F = (⋃ i. ?D i)
    using ‹F ∈ sets M›[THEN sets.sets_into_space] A(2)[symmetric] by (auto
simp: UN_disjointed_eq)
  have [simp, intro]: ⋀i. ?D i ∈ sets M
    using sets.range_disjointed_sets[of λi. F ∩ A i M] ‹F ∈ sets M›
    by (auto simp: subset_eq)
  have disjoint_family ?D
    by (auto simp: disjoint_family_disjointed)
  moreover
  have (∑ i. emeasure M (?D i)) = (∑ i. emeasure N (?D i))
  proof (intro arg_cong[where f=suminf] ext)
```

```
    fix i
    have A i ∩ ?D i = ?D i
      by (auto simp: disjointed_def)
    then show emeasure M (?D i) = emeasure N (?D i)
      using ∗[of A i ?D i, OF _ A(3)] A(1) by auto
  qed
  ultimately show emeasure M F = emeasure N F
  by (simp add: image_subset_iff ‹sets M = sets N›[symmetric] F_eq[symmetric]
suminf_emeasure)
 qed
qed

lemma space_empty: space M = {} ⟹ M = count_space {}
 by (rule measure_eqI) (simp_all add: space_empty_iff)

lemma measure_eqI_generator_eq_countable:
 fixes M N :: 'a measure and E :: 'a set set and A :: 'a set set
 assumes E: Int_stable E E ⊆ Pow Ω ⋀X. X ∈ E ⟹ emeasure M X = emeasure
N X
   and sets: sets M = sigma_sets Ω E sets N = sigma_sets Ω E
 and A: A ⊆ E (⋃ A) = Ω countable A ⋀a. a ∈ A ⟹ emeasure M a ≠ ∞
 shows M = N
proof cases
 assume Ω = {}
 have ∗: sigma_sets Ω E = sets (sigma Ω E)
   using E(2) by simp
 have space M = Ω space N = Ω
   using sets E(2) unfolding ∗ by (auto dest: sets_eq_imp_space_eq simp del:
sets_measure_of)
 then show M = N
   unfolding ‹Ω = {}› by (auto dest: space_empty)
next
 assume Ω ≠ {} with ‹⋃ A = Ω› have A ≠ {} by auto
 from this ‹countable A› have rng: range (from_nat_into A) = A
   by (rule range_from_nat_into)
 show M = N
 proof (rule measure_eqI_generator_eq[OF E sets])
   show range (from_nat_into A) ⊆ E
     unfolding rng using ‹A ⊆ E› .
   show (⋃ i. from_nat_into A i) = Ω
     unfolding rng using ‹⋃ A = Ω› .
   show emeasure M (from_nat_into A i) ≠ ∞ for i
     using rng by (intro A) auto
 qed
qed

lemma measure_of_of_measure: measure_of (space M) (sets M) (emeasure M)
= M
proof (intro measure_eqI emeasure_measure_of_sigma)
```

show *sigma_algebra* (*space M*) (*sets M*) **..**
show *positive* (*sets M*) (*emeasure M*)
by (*simp add*: *positive_def*)
show *countably_additive* (*sets M*) (*emeasure M*)
by (*simp add*: *emeasure_countably_additive*)
qed *simp_all*

7.3.5 μ-null sets

definition *null_sets* :: *'a measure* ⇒ *'a set set* **where**
null_sets M = {*N*∈*sets M*. *emeasure M N* = *0*}

lemma *null_setsD1*[*dest*]: *A* ∈ *null_sets M* ⟹ *emeasure M A* = *0*
by (*simp add*: *null_sets_def*)

lemma *null_setsD2*[*dest*]: *A* ∈ *null_sets M* ⟹ *A* ∈ *sets M*
unfolding *null_sets_def* **by** *simp*

lemma *null_setsI*[*intro*]: *emeasure M A* = *0* ⟹ *A* ∈ *sets M* ⟹ *A* ∈ *null_sets M*
unfolding *null_sets_def* **by** *simp*

interpretation *null_sets*: *ring_of_sets space M null_sets M* **for** *M*
proof (*rule ring_of_setsI*)
show *null_sets M* ⊆ *Pow* (*space M*)
using *sets.sets_into_space* **by** *auto*
show {} ∈ *null_sets M*
by *auto*
fix *A B* **assume** *null_sets*: *A* ∈ *null_sets M B* ∈ *null_sets M*
then have *sets*: *A* ∈ *sets M B* ∈ *sets M*
by *auto*
then have ∗: *emeasure M* (*A* ∪ *B*) ≤ *emeasure M A* + *emeasure M B*
emeasure M (*A* − *B*) ≤ *emeasure M A*
by (*auto intro*!: *emeasure_subadditive emeasure_mono*)
then have *emeasure M B* = *0 emeasure M A* = *0*
using *null_sets* **by** *auto*
with *sets* ∗ **show** *A* − *B* ∈ *null_sets M A* ∪ *B* ∈ *null_sets M*
by (*auto intro*!: *antisym zero_le*)
qed

lemma *UN_from_nat_into*:
assumes *I*: *countable I I* ≠ {}
shows (⋃*i*∈*I*. *N i*) = (⋃*i*. *N* (*from_nat_into I i*))
proof −
have (⋃*i*∈*I*. *N i*) = ⋃(*N* ‘ *range* (*from_nat_into I*))
using *I* **by** *simp*
also have ... = (⋃*i*. (*N* ∘ *from_nat_into I*) *i*)
by *simp*
finally show *?thesis* **by** *simp*

qed

lemma *null_sets_UN'*:
assumes *countable I*
assumes $\bigwedge i.\ i \in I \Longrightarrow N\ i \in null\_sets\ M$
shows $(\bigcup i{\in}I.\ N\ i) \in null\_sets\ M$
proof *cases*
assume $I = \{\}$ **then show** *?thesis* **by** *simp*
next
assume $I \neq \{\}$
show *?thesis*
proof (*intro conjI CollectI null_setsI*)
show $(\bigcup i{\in}I.\ N\ i) \in sets\ M$
using *assms* **by** (*intro sets.countable_UN'*) *auto*
have *emeasure M* $(\bigcup i{\in}I.\ N\ i) \leq (\sum n.\ emeasure\ M\ (N\ (from\_nat\_into\ I\ n)))$
unfolding *UN_from_nat_into*[*OF* ‹*countable I*› ‹$I \neq \{\}$›]
using *assms* ‹$I \neq \{\}$› **by** (*intro emeasure_subadditive_countably*) (*auto intro*: *from_nat_into*)
also have $(\lambda n.\ emeasure\ M\ (N\ (from\_nat\_into\ I\ n))) = (\lambda\_.\ 0)$
using *assms* ‹$I \neq \{\}$› **by** (*auto intro*: *from_nat_into*)
finally show *emeasure M* $(\bigcup i{\in}I.\ N\ i) = 0$
by (*intro antisym zero_le*) *simp*
qed
qed

lemma *null_sets_UN*[*intro*]:
$(\bigwedge i::'i::countable.\ N\ i \in null\_sets\ M) \Longrightarrow (\bigcup i.\ N\ i) \in null\_sets\ M$
by (*rule null_sets_UN'*) *auto*

lemma *null_set_Int1*:
assumes $B \in null\_sets\ M\ A \in sets\ M$ **shows** $A \cap B \in null\_sets\ M$
proof (*intro CollectI conjI null_setsI*)
show *emeasure M* $(A \cap B) = 0$ **using** *assms*
by (*intro emeasure_eq_0*[*of B _ A ∩ B*]) *auto*
qed (*insert assms*, *auto*)

lemma *null_set_Int2*:
assumes $B \in null\_sets\ M\ A \in sets\ M$ **shows** $B \cap A \in null\_sets\ M$
using *assms* **by** (*subst Int_commute*) (*rule null_set_Int1*)

lemma *emeasure_Diff_null_set*:
assumes $B \in null\_sets\ M\ A \in sets\ M$
shows $emeasure\ M\ (A - B) = emeasure\ M\ A$
proof −
have $*$: $A - B = (A - (A \cap B))$ **by** *auto*
have $A \cap B \in null\_sets\ M$ **using** *assms* **by** (*rule null_set_Int1*)
then show *?thesis*
unfolding $*$ **using** *assms*
by (*subst emeasure_Diff*) *auto*

qed

lemma *null_set_Diff*:
assumes $B \in$ *null_sets* M $A \in$ *sets* M **shows** $B - A \in$ *null_sets* M
proof (*intro CollectI conjI null_setsI*)
show *emeasure* M $(B - A) = 0$ **using** *assms* **by** (*intro emeasure_eq_0*[*of* B _ $B - A$]) *auto*
qed (*insert assms*, *auto*)

lemma *emeasure_Un_null_set*:
assumes $A \in$ *sets* M $B \in$ *null_sets* M
shows *emeasure* M $(A \cup B) =$ *emeasure* M A
proof −
have $*$: $A \cup B = A \cup (B - A)$ **by** *auto*
have $B - A \in$ *null_sets* M **using** *assms*(*2*,*1*) **by** (*rule null_set_Diff*)
then show *?thesis*
unfolding $*$ **using** *assms*
by (*subst plus_emeasure*[*symmetric*]) *auto*
qed

lemma *emeasure_Un'*:
assumes $A \in$ *sets* M $B \in$ *sets* M $A \cap B \in$ *null_sets* M
shows *emeasure* M $(A \cup B) =$ *emeasure* M $A +$ *emeasure* M B
proof −
have $A \cup B = A \cup (B - A \cap B)$ **by** *blast*
also have *emeasure* M ... = *emeasure* M $A +$ *emeasure* M $(B - A \cap B)$
using *assms* **by** (*subst plus_emeasure*) *auto*
also have *emeasure* M $(B - A \cap B) =$ *emeasure* M B
using *assms* **by** (*intro emeasure_Diff_null_set*) *auto*
finally show *?thesis* **.**
qed

7.3.6 The almost everywhere filter (i.e. quantifier)

definition *ae_filter* :: *'a measure* ⇒ *'a filter* **where**
ae_filter M = (*INF* N∈*null_sets* M. *principal* (*space* $M - N$))

abbreviation *almost_everywhere* :: *'a measure* ⇒ (*'a* ⇒ *bool*) ⇒ *bool* **where**
almost_everywhere M $P \equiv$ *eventually* P (*ae_filter* M)

syntax
_almost_everywhere :: *pttrn* ⇒ *'a* ⇒ *bool* ⇒ *bool*
(‹(‹*open_block notation*=‹*binder AE*›› *AE* _ *in* _. _)› [*0*,*0*,*10*] *10*)
syntax_consts
_almost_everywhere ⇌ *almost_everywhere*
translations
AE x in M. P ⇌ *CONST almost_everywhere* M (λx. P)

abbreviation

set_almost_everywhere A M P ≡ AE x in M. x ∈ A ⟶ P x

syntax
__set_almost_everywhere :: pttrn ⇒ 'a set ⇒ 'a ⇒ bool ⇒ bool
(‹(‹open_block notation=‹binder AE››AE _∈_ in _./ _)› [0,0,0,10] 10)
syntax_consts
__set_almost_everywhere ⇌ set_almost_everywhere
translations
AE x∈A in M. P ⇌ CONST set_almost_everywhere A M (λx. P)

lemma *eventually_ae_filter*: *eventually P (ae_filter M) ⟷ (∃N∈null_sets M. {x ∈ space M. ¬ P x} ⊆ N)*
unfolding *ae_filter_def* **by** (*subst eventually_INF_base*) (*auto simp*: *eventually_principal subset_eq*)

lemma *AE_I'*:
N ∈ null_sets M ⟹ {x∈space M. ¬ P x} ⊆ N ⟹ (AE x in M. P x)
unfolding *eventually_ae_filter* **by** *auto*

lemma *AE_iff_null*:
assumes *{x∈space M. ¬ P x} ∈ sets M* (**is** *?P ∈ sets M*)
shows *(AE x in M. P x) ⟷ {x∈space M. ¬ P x} ∈ null_sets M*
proof
assume *AE x in M. P x* **then obtain** *N* **where** *N*: *N ∈ sets M ?P ⊆ N emeasure M N = 0*
unfolding *eventually_ae_filter* **by** *auto*
have *emeasure M ?P ≤ emeasure M N*
using *assms N*(*1*,*2*) **by** (*auto intro*: *emeasure_mono*)
then have *emeasure M ?P = 0*
unfolding ‹*emeasure M N = 0*› **by** *auto*
then show *?P ∈ null_sets M* **using** *assms* **by** *auto*
next
assume *?P ∈ null_sets M* **with** *assms* **show** *AE x in M. P x* **by** (*auto intro*: *AE_I'*)
qed

lemma *AE_iff_null_sets*:
N ∈ sets M ⟹ N ∈ null_sets M ⟷ (AE x in M. x ∉ N)
using *Int_absorb1*[*OF sets.sets_into_space*, *of N M*]
by (*subst AE_iff_null*) (*auto simp*: *Int_def*[*symmetric*])

lemma *ae_filter_eq_bot_iff*: *ae_filter M = bot ⟷ emeasure M (space M) = 0*
proof −
have *ae_filter M = bot ⟷ (AE x in M. False)*
using *trivial_limit_def* **by** *blast*
also have *... ⟷ space M ∈ null_sets M*
by (*simp add*: *AE_iff_null_sets eventually_ae_filter*)
also have *... ⟷ emeasure M (space M) = 0*
by *auto*

finally show *?thesis* **.**
qed

lemma *AE_not_in*:
N ∈ *null_sets M* ⟹ *AE x in M*. *x* ∉ *N*
by (*metis AE_iff_null_sets null_setsD2*)

lemma *AE_iff_measurable*:
N ∈ *sets M* ⟹ {*x*∈*space M*. ¬ *P x*} = *N* ⟹ (*AE x in M*. *P x*) ⟷ *emeasure M N* = *0*
using *AE_iff_null*[*of* _ *P*] **by** *auto*

lemma *AE_E*[*consumes 1*]:
assumes *AE x in M*. *P x*
obtains *N* **where** {*x* ∈ *space M*. ¬ *P x*} ⊆ *N emeasure M N* = *0 N* ∈ *sets M*
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*

lemma *AE_E2*:
assumes *AE x in M*. *P x*
shows *emeasure M* {*x*∈*space M*. ¬ *P x*} = *0*
by (*metis* (*mono_tags*, *lifting*) *AE_iff_null assms emeasure_notin_sets null_setsD1*)

lemma *AE_E3*:
assumes *AE x in M*. *P x*
obtains *N* **where** ⋀*x*. *x* ∈ *space M* − *N* ⟹ *P x N* ∈ *null_sets M*
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*

lemma *AE_I*:
assumes {*x* ∈ *space M*. ¬ *P x*} ⊆ *N emeasure M N* = *0 N* ∈ *sets M*
shows *AE x in M*. *P x*
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*

lemma *AE_mp*[*elim*!]:
assumes *AE_P*: *AE x in M*. *P x* **and** *AE_imp*: *AE x in M*. *P x* ⟶ *Q x*
shows *AE x in M*. *Q x*
using *assms* **by** (*fact eventually_rev_mp*)

The next lemma is convenient to combine with a lemma whose conclusion is of the form *AE x in M*. *P x* = *Q x*: for such a lemma, there is no [*symmetric*] variant, but using *AE_symmetric*[*OF*...] will replace it.

lemma
shows *AE_iffI*: *AE x in M*. *P x* ⟹ *AE x in M*. *P x* ⟷ *Q x* ⟹ *AE x in M*. *Q x*
and *AE_disjI1*: *AE x in M*. *P x* ⟹ *AE x in M*. *P x* ∨ *Q x*
and *AE_disjI2*: *AE x in M*. *Q x* ⟹ *AE x in M*. *P x* ∨ *Q x*
and *AE_conjI*: *AE x in M*. *P x* ⟹ *AE x in M*. *Q x* ⟹ *AE x in M*. *P x* ∧ *Q x*
and *AE_conj_iff*[*simp*]: (*AE x in M*. *P x* ∧ *Q x*) ⟷ (*AE x in M*. *P x*) ∧ (*AE x in M*. *Q x*)

by *auto*

lemma *AE_symmetric*:
assumes *AE x in M. P x = Q x*
shows *AE x in M. Q x = P x*
using *assms* **by** *auto*

lemma *AE_impI*:
(P ⟹ AE x in M. Q x) ⟹ AE x in M. P ⟶ Q x
by *fastforce*

lemma *AE_measure*:
assumes *AE*: *AE x in M. P x* **and** *sets*: *{x∈space M. P x} ∈ sets M* (**is** *?P ∈ sets M*)
shows *emeasure M {x∈space M. P x} = emeasure M (space M)*
proof −
from *AE_E[OF AE]* **obtain** *N*
where *N*: *{x ∈ space M. ¬ P x} ⊆ N emeasure M N = 0 N ∈ sets M*
by *auto*
with *sets* **have** *emeasure M (space M) ≤ emeasure M (?P ∪ N)*
by (*intro emeasure_mono*) *auto*
also have *... ≤ emeasure M ?P + emeasure M N*
using *sets N* **by** (*intro emeasure_subadditive*) *auto*
also have *... = emeasure M ?P* **using** *N* **by** *simp*
finally show *emeasure M ?P = emeasure M (space M)*
using *emeasure_space[of M ?P]* **by** *auto*
qed

lemma *AE_space*: *AE x in M. x ∈ space M*
by (*rule AE_I*[**where** *N*={}]) *auto*

lemma *AE_I2*[*simp*, *intro*]:
(⋀x. x ∈ space M ⟹ P x) ⟹ AE x in M. P x
using *AE_space* **by** *force*

lemma *AE_Ball_mp*:
∀ x∈space M. P x ⟹ AE x in M. P x ⟶ Q x ⟹ AE x in M. Q x
by *auto*

lemma *AE_cong*[*cong*]:
(⋀x. x ∈ space M ⟹ P x ⟷ Q x) ⟹ (AE x in M. P x) ⟷ (AE x in M. Q x)
by *auto*

lemma *AE_cong_simp*: *M = N ⟹ (⋀x. x ∈ space N =simp=> P x = Q x) ⟹ (AE x in M. P x) ⟷ (AE x in N. Q x)*
by (*auto simp*: *simp_implies_def*)

lemma *AE_all_countable*:

(*AE x in M. ∀ i. P i x*) ⟷ (∀ *i*::*'i*::*countable. AE x in M. P i x*)
proof
assume ∀ *i. AE x in M. P i x*
from *this*[*unfolded eventually_ae_filter Bex_def*, *THEN choice*]
obtain *N* **where** *N*: ⋀*i. N i* ∈ *null_sets M* ⋀*i.* {*x*∈*space M.* ¬ *P i x*} ⊆ *N i*
by *auto*
have {*x*∈*space M.* ¬ (∀ *i. P i x*)} ⊆ (⋃ *i.* {*x*∈*space M.* ¬ *P i x*}) **by** *auto*
also have ... ⊆ (⋃ *i. N i*) **using** *N* **by** *auto*
finally have {*x*∈*space M.* ¬ (∀ *i. P i x*)} ⊆ (⋃ *i. N i*) **.**
moreover from *N* **have** (⋃ *i. N i*) ∈ *null_sets M*
by (*intro null_sets_UN*) *auto*
ultimately show *AE x in M.* ∀ *i. P i x*
unfolding *eventually_ae_filter* **by** *auto*
qed *auto*

lemma *AE_ball_countable*:
assumes [*intro*]: *countable X*
shows (*AE x in M.* ∀ *y*∈*X. P x y*) ⟷ (∀ *y*∈*X. AE x in M. P x y*)
proof
assume ∀ *y*∈*X. AE x in M. P x y*
from *this*[*unfolded eventually_ae_filter Bex_def*, *THEN bchoice*]
obtain *N* **where** *N*: ⋀*y. y* ∈ *X* ⟹ *N y* ∈ *null_sets M* ⋀*y. y* ∈ *X* ⟹ {*x*∈*space M.* ¬ *P x y*} ⊆ *N y*
by *auto*
have {*x*∈*space M.* ¬ (∀ *y*∈*X. P x y*)} ⊆ (⋃ *y*∈*X.* {*x*∈*space M.* ¬ *P x y*})
by *auto*
also have ... ⊆ (⋃ *y*∈*X. N y*)
using *N* **by** *auto*
finally have {*x*∈*space M.* ¬ (∀ *y*∈*X. P x y*)} ⊆ (⋃ *y*∈*X. N y*) **.**
moreover from *N* **have** (⋃ *y*∈*X. N y*) ∈ *null_sets M*
by (*intro null_sets_UN'*) *auto*
ultimately show *AE x in M.* ∀ *y*∈*X. P x y*
unfolding *eventually_ae_filter* **by** *auto*
qed *auto*

lemma *AE_ball_countable'*:
(⋀*N. N* ∈ *I* ⟹ *AE x in M. P N x*) ⟹ *countable I* ⟹ *AE x in M.* ∀ *N* ∈ *I. P N x*
unfolding *AE_ball_countable* **by** *simp*

lemma *AE_pairwise*: *countable F* ⟹ *pairwise* (λ*A B. AE x in M. R x A B*) *F* ⟷ (*AE x in M. pairwise* (*R x*) *F*)
unfolding *pairwise_alt* **by** (*simp add*: *AE_ball_countable*)

lemma *AE_discrete_difference*:
assumes *X*: *countable X*
assumes *null*: ⋀*x. x* ∈ *X* ⟹ *emeasure M* {*x*} = *0*
assumes *sets*: ⋀*x. x* ∈ *X* ⟹ {*x*} ∈ *sets M*
shows *AE x in M. x* ∉ *X*

proof −
have $(\bigcup x{\in}X.\ \{x\}) \in$ *null_sets M*
using *assms* **by** (*intro null_sets_UN′*) *auto*
from *AE_not_in*[*OF this*] **show** *AE x in M*. $x \notin X$
by *auto*
qed

lemma *AE_finite_all*:
assumes *f*: *finite S* **shows** (*AE x in M*. $\forall i{\in}S.$ *P i x*) $\longleftrightarrow$ ($\forall i{\in}S.$ *AE x in M*. *P i x*)
using *f* **by** *induct auto*

lemma *AE_finite_allI*:
assumes *finite S*
shows ($\bigwedge s.$ $s \in S \Longrightarrow$ *AE x in M*. *Q s x*) $\Longrightarrow$ *AE x in M*. $\forall s{\in}S.$ *Q s x*
using *AE_finite_all*[*OF* ‹*finite S*›] **by** *auto*

lemma *emeasure_mono_AE*:
assumes *imp*: *AE x in M*. $x \in A \longrightarrow x \in B$
and *B*: $B \in$ *sets M*
shows *emeasure M A* $\leq$ *emeasure M B*
proof *cases*
assume *A*: $A \in$ *sets M*
from *imp* **obtain** *N* **where** *N*: $\{x{\in}$*space M*. $\neg\ (x \in A \longrightarrow x \in B)\} \subseteq N$ $N \in$ *null_sets M*
by (*auto simp*: *eventually_ae_filter*)
have *emeasure M A* = *emeasure M* $(A - N)$
using *N A* **by** (*subst emeasure_Diff_null_set*) *auto*
also have *emeasure M* $(A - N) \leq$ *emeasure M* $(B - N)$
using *N A B sets.sets_into_space* **by** (*auto intro*!: *emeasure_mono*)
also have *emeasure M* $(B - N)$ = *emeasure M B*
using *N B* **by** (*subst emeasure_Diff_null_set*) *auto*
finally show *?thesis* **.**
qed (*simp add*: *emeasure_notin_sets*)

lemma *emeasure_eq_AE*:
assumes *iff*: *AE x in M*. $x \in A \longleftrightarrow x \in B$
assumes *A*: $A \in$ *sets M* **and** *B*: $B \in$ *sets M*
shows *emeasure M A* = *emeasure M B*
using *assms* **by** (*safe intro*!: *antisym emeasure_mono_AE*) *auto*

lemma *emeasure_Collect_eq_AE*:
AE x in M. *P x* $\longleftrightarrow$ *Q x* $\Longrightarrow$ *Measurable.pred M Q* $\Longrightarrow$ *Measurable.pred M P* $\Longrightarrow$
emeasure M $\{x{\in}$*space M*. *P x*$\}$ = *emeasure M* $\{x{\in}$*space M*. *Q x*$\}$
by (*intro emeasure_eq_AE*) *auto*

lemma *emeasure_eq_0_AE*: *AE x in M*. $\neg$ *P x* $\Longrightarrow$ *emeasure M* $\{x{\in}$*space M*. *P x*$\}$ = *0*

using *AE_iff_measurable*[*OF* _ *refl*, *of M* λ*x*. ¬ *P x*]
by (*cases* {*x*∈*space M*. *P x*} ∈ *sets M*) (*simp_all add*: *emeasure_notin_sets*)

lemma *emeasure_0_AE*:
assumes *emeasure M* (*space M*) = *0*
shows *AE x in M*. *P x*
using *eventually_ae_filter assms* **by** *blast*

lemma *emeasure_add_AE*:
assumes [*measurable*]: *A* ∈ *sets M B* ∈ *sets M C* ∈ *sets M*
assumes *1*: *AE x in M*. *x* ∈ *C* ⟷ *x* ∈ *A* ∨ *x* ∈ *B*
assumes *2*: *AE x in M*. ¬ (*x* ∈ *A* ∧ *x* ∈ *B*)
shows *emeasure M C* = *emeasure M A* + *emeasure M B*
proof −
have *emeasure M C* = *emeasure M* (*A* ∪ *B*)
by (*rule emeasure_eq_AE*) (*insert 1*, *auto*)
also have ... = *emeasure M A* + *emeasure M* (*B* − *A*)
by (*subst plus_emeasure*) *auto*
also have *emeasure M* (*B* − *A*) = *emeasure M B*
by (*rule emeasure_eq_AE*) (*insert 2*, *auto*)
finally show *?thesis* **.**
qed

7.3.7 σ-finite Measures

locale *sigma_finite_measure* =
fixes *M* :: *'a measure*
assumes *sigma_finite_countable*:
∃ *A*::*'a set set*. *countable A* ∧ *A* ⊆ *sets M* ∧ (⋃ *A*) = *space M* ∧ (∀ *a*∈*A*. *emeasure M a* ≠ ∞)

lemma (**in** *sigma_finite_measure*) *sigma_finite*:
obtains *A* :: *nat* ⇒ *'a set*
where *range A* ⊆ *sets M* (⋃ *i*. *A i*) = *space M* ⋀*i*. *emeasure M* (*A i*) ≠ ∞
proof −
obtain *A* :: *'a set set* **where**
[*simp*]: *countable A* **and**
A: *A* ⊆ *sets M* (⋃ *A*) = *space M* ⋀*a*. *a* ∈ *A* ⟹ *emeasure M a* ≠ ∞
using *sigma_finite_countable* **by** *metis*
show *thesis*
proof *cases*
assume *A* = {} **with** ‹(⋃ *A*) = *space M*› **show** *thesis*
by (*intro that*[*of* λ_. {}]) *auto*
next
assume *A* ≠ {}
show *thesis*
proof
show *range* (*from_nat_into A*) ⊆ *sets M*
using ‹*A* ≠ {}› *A* **by** *auto*

```
    have (⋃i. from_nat_into A i) = ⋃A
      using range_from_nat_into[OF ‹A ≠ {}› ‹countable A›] by auto
    with A show (⋃i. from_nat_into A i) = space M
      by auto
  qed (intro A from_nat_into ‹A ≠ {}›)
 qed
qed

lemma (in sigma_finite_measure) sigma_finite_disjoint:
 obtains A :: nat ⇒ 'a set
 where range A ⊆ sets M (⋃i. A i) = space M ⋀i. emeasure M (A i) ≠ ∞
disjoint_family A
proof −
 obtain A :: nat ⇒ 'a set where
   range: range A ⊆ sets M and
   space: (⋃i. A i) = space M and
   measure: ⋀i. emeasure M (A i) ≠ ∞
   using sigma_finite by blast
 show thesis
 proof (rule that[of disjointed A])
   show range (disjointed A) ⊆ sets M
     by (rule sets.range_disjointed_sets[OF range])
   show (⋃i. disjointed A i) = space M
     and disjoint_family (disjointed A)
     using disjoint_family_disjointed UN_disjointed_eq[of A] space range
     by auto
   show emeasure M (disjointed A i) ≠ ∞ for i
   proof −
     have emeasure M (disjointed A i) ≤ emeasure M (A i)
       using range disjointed_subset[of A i] by (auto intro!: emeasure_mono)
     then show ?thesis using measure[of i] by (auto simp: top_unique)
   qed
 qed
qed

lemma (in sigma_finite_measure) sigma_finite_incseq:
 obtains A :: nat ⇒ 'a set
 where range A ⊆ sets M (⋃i. A i) = space M ⋀i. emeasure M (A i) ≠ ∞
incseq A
proof −
 obtain F :: nat ⇒ 'a set where
   F: range F ⊆ sets M (⋃i. F i) = space M ⋀i. emeasure M (F i) ≠ ∞
   using sigma_finite by blast
 show thesis
 proof (rule that[of λn. ⋃i≤n. F i])
   show range (λn. ⋃i≤n. F i) ⊆ sets M
     using F by (force simp: incseq_def)
   show (⋃n. ⋃i≤n. F i) = space M
   proof −
```

```
    from F have ⋀x. x ∈ space M ⟹ ∃ i. x ∈ F i by auto
    with F show ?thesis by fastforce
  qed
  show emeasure M (⋃ i≤n. F i) ≠ ∞ for n
  proof −
    have emeasure M (⋃ i≤n. F i) ≤ (∑ i≤n. emeasure M (F i))
      using F by (auto intro!: emeasure_subadditive_finite)
    also have ... < ∞
      using F by (auto simp: sum_Pinfty less_top)
    finally show ?thesis by simp
  qed
  show incseq (λn. ⋃ i≤n. F i)
    by (force simp: incseq_def)
 qed
qed

lemma (in sigma_finite_measure) approx_PInf_emeasure_with_finite:
 fixes C::real
 assumes W_meas: W ∈ sets M
    and W_inf: emeasure M W = ∞
 obtains Z where Z ∈ sets M Z ⊆ W emeasure M Z < ∞ emeasure M Z > C
proof −
 obtain A :: nat ⇒ 'a set
   where A: range A ⊆ sets M (⋃ i. A i) = space M ⋀i. emeasure M (A i) ≠ ∞
incseq A
   using sigma_finite_incseq by blast
 define B where B = (λi. W ∩ A i)
 have B_meas: ⋀i. B i ∈ sets M using W_meas ‹range A ⊆ sets M› B_def by
blast
 have b: ⋀i. B i ⊆ W using B_def by blast

 { fix i
   have emeasure M (B i) ≤ emeasure M (A i)
     using A by (intro emeasure_mono) (auto simp: B_def)
   also have emeasure M (A i) < ∞
     using ‹⋀i. emeasure M (A i) ≠ ∞› by (simp add: less_top)
   finally have emeasure M (B i) < ∞ . }
 note c = this

 have W = (⋃ i. B i) using B_def ‹(⋃ i. A i) = space M› W_meas by auto
  moreover have incseq B using B_def ‹incseq A› by (simp add: incseq_def
subset_eq)
 ultimately have (λi. emeasure M (B i)) ⟶ emeasure M W using W_meas
B_meas
   by (simp add: B_meas Lim_emeasure_incseq image_subset_iff)
 then have (λi. emeasure M (B i)) ⟶ ∞ using W_inf by simp
 from order_tendstoD(1)[OF this, of C]
 obtain i where d: emeasure M (B i) > C
   by (auto simp: eventually_sequentially)
```

have $B\ i \in sets\ M\ B\ i \subseteq W\ emeasure\ M\ (B\ i) < \infty\ emeasure\ M\ (B\ i) > C$
using $B\_meas\ b\ c\ d$ **by** *auto*
then show *?thesis* **using** *that* **by** *blast*
qed

7.3.8 Measure space induced by distribution of $(\rightarrow_M)$-functions

definition $distr :: 'a\ measure \Rightarrow 'b\ measure \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b\ measure$ **where**
$distr\ M\ N\ f =$
$measure\_of\ (space\ N)\ (sets\ N)\ (\lambda A.\ emeasure\ M\ (f -` A \cap space\ M))$

lemma
shows $sets\_distr[simp, measurable\_cong]: sets\ (distr\ M\ N\ f) = sets\ N$
and $space\_distr[simp]: space\ (distr\ M\ N\ f) = space\ N$
by $(auto\ simp: distr\_def)$

lemma
shows $measurable\_distr\_eq1[simp]: measurable\ (distr\ Mf\ Nf\ f)\ Mf' = measurable\ Nf\ Mf'$
and $measurable\_distr\_eq2[simp]: measurable\ Mg'\ (distr\ Mg\ Ng\ g) = measurable\ Mg'\ Ng$
by $(auto\ simp: measurable\_def)$

lemma $distr\_cong$:
$M = K \Longrightarrow sets\ N = sets\ L \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow distr\ M\ N\ f = distr\ K\ L\ g$
using $sets\_eq\_imp\_space\_eq[of\ N\ L]$ **by** $(simp\ add: distr\_def\ Int\_def\ cong: rev\_conj\_cong)$

lemma $emeasure\_distr$:
fixes $f :: 'a \Rightarrow 'b$
assumes $f: f \in measurable\ M\ N$ **and** $A: A \in sets\ N$
shows $emeasure\ (distr\ M\ N\ f)\ A = emeasure\ M\ (f -` A \cap space\ M)$ **(is** $\_ = ?\mu\ A)$
unfolding $distr\_def$
proof $(rule\ emeasure\_measure\_of\_sigma)$
show $positive\ (sets\ N)\ ?\mu$
by $(auto\ simp: positive\_def)$

show $countably\_additive\ (sets\ N)\ ?\mu$
proof $(intro\ countably\_additiveI)$
fix $A :: nat \Rightarrow 'b\ set$ **assume** $range\ A \subseteq sets\ N\ disjoint\_family\ A$
then have $A: \bigwedge i.\ A\ i \in sets\ N\ (\bigcup i.\ A\ i) \in sets\ N$ **by** *auto*
then have $*: range\ (\lambda i.\ f -` (A\ i) \cap space\ M) \subseteq sets\ M$
using *f* **by** $(auto\ simp: measurable\_def)$
moreover have $(\bigcup i.\ f -`\ A\ i \cap space\ M) \in sets\ M$
using $*$ **by** *blast*
moreover have $**: disjoint\_family\ (\lambda i.\ f -` A\ i \cap space\ M)$
using $\langle disjoint\_family\ A \rangle$ **by** $(auto\ simp: disjoint\_family\_on\_def)$

```
    ultimately show (∑ i. ?μ (A i)) = ?μ (⋃ i. A i)
      using suminf_emeasure[OF _ **] A f
      by (auto simp: comp_def vimage_UN)
  qed
  show sigma_algebra (space N) (sets N) ..
qed fact

lemma emeasure_Collect_distr:
  assumes X[measurable]: X ∈ measurable M N Measurable.pred N P
  shows emeasure (distr M N X) {x∈space N. P x} = emeasure M {x∈space M.
P (X x)}
  by (subst emeasure_distr)
     (auto intro!: arg_cong2[where f=emeasure] X(1)[THEN measurable_space])

lemma emeasure_lfp2[consumes 1, case_names cont f measurable]:
  assumes P M
  assumes cont: sup_continuous F
  assumes f: ⋀M. P M ⟹ f ∈ measurable M' M
  assumes *: ⋀M A. P M ⟹ (⋀N. P N ⟹ Measurable.pred N A) ⟹ Measur-
able.pred M (F A)
  shows emeasure M' {x∈space M'. lfp F (f x)} = (SUP i. emeasure M' {x∈space
M'. (F ^^ i) (λx. False) (f x)})
proof (subst (1 2) emeasure_Collect_distr[symmetric, where X=f])
  show f ∈ measurable M' M  f ∈ measurable M' M
    using f[OF ‹P M›] by auto
  { fix i show Measurable.pred M ((F ^^ i) (λx. False))
    using ‹P M› by (induction i arbitrary: M) (auto intro!: *) }
  show Measurable.pred M (lfp F)
    using ‹P M› cont * by (rule measurable_lfp_coinduct[of P])

  have emeasure (distr M' M f) {x ∈ space (distr M' M f). lfp F x} =
     (SUP i. emeasure (distr M' M f) {x ∈ space (distr M' M f). (F ^^ i) (λx.
False) x})
    using ‹P M›
  proof (coinduction arbitrary: M rule: emeasure_lfp')
    case (measurable A N) then have ⋀N. P N ⟹ Measurable.pred (distr M' N
f) A
      by metis
    then have ⋀N. P N ⟹ Measurable.pred N A
      by simp
    with ‹P N›[THEN *] show ?case
      by auto
  qed fact
  then show emeasure (distr M' M f) {x ∈ space M. lfp F x} =
    (SUP i. emeasure (distr M' M f) {x ∈ space M. (F ^^ i) (λx. False) x})
   by simp
qed

lemma distr_id[simp]: distr N N (λx. x) = N
```

by (*rule measure_eqI*) (*auto simp*: *emeasure_distr*)

lemma *distr_id2*: *sets M* = *sets N* ⟹ *distr N M* (λ*x*. *x*) = *N*
by (*rule measure_eqI*) (*auto simp*: *emeasure_distr*)

lemma *measure_distr*:
f ∈ *measurable M N* ⟹ *S* ∈ *sets N* ⟹ *measure* (*distr M N f*) *S* = *measure M* (*f* −‘ *S* ∩ *space M*)
by (*simp add*: *emeasure_distr measure_def*)

lemma *distr_cong_AE*:
assumes *1*: *M* = *K sets N* = *sets L* **and**
2: (*AE x in M*. *f x* = *g x*) **and** *f* ∈ *measurable M N* **and** *g* ∈ *measurable K L*
shows *distr M N f* = *distr K L g*
proof (*rule measure_eqI*)
fix *A* **assume** *A* ∈ *sets* (*distr M N f*)
with *assms* **show** *emeasure* (*distr M N f*) *A* = *emeasure* (*distr K L g*) *A*
by (*auto simp add*: *emeasure_distr intro*!: *emeasure_eq_AE measurable_sets*)
qed (*insert 1*, *simp*)

lemma *AE_distrD*:
assumes *f*: *f* ∈ *measurable M M′*
and *AE*: *AE x in distr M M′ f*. *P x*
shows *AE x in M*. *P* (*f x*)
proof −
from *AE*[*THEN AE_E*] **obtain** *N*
where {*x* ∈ *space* (*distr M M′ f*). ¬ *P x*} ⊆ *N*
emeasure (*distr M M′ f*) *N* = *0*
N ∈ *sets* (*distr M M′ f*)
by *auto*
with *f* **show** *?thesis*
by (*simp add*: *eventually_ae_filter*, *intro bexI*[*of* _ *f* −‘ *N* ∩ *space M*])
(*auto simp*: *emeasure_distr measurable_def*)
qed

lemma *AE_distr_iff*:
assumes *f*[*measurable*]: *f* ∈ *measurable M N* **and** *P*[*measurable*]: {*x* ∈ *space N*. *P x*} ∈ *sets N*
shows (*AE x in distr M N f*. *P x*) ⟷ (*AE x in M*. *P* (*f x*))
proof (*subst* (*1 2*) *AE_iff_measurable*[*OF* _ *refl*])
have *f* −‘ {*x*∈*space N*. ¬ *P x*} ∩ *space M* = {*x* ∈ *space M*. ¬ *P* (*f x*)}
using *f*[*THEN measurable_space*] **by** *auto*
then show (*emeasure* (*distr M N f*) {*x* ∈ *space* (*distr M N f*). ¬ *P x*} = *0*) =
(*emeasure M* {*x* ∈ *space M*. ¬ *P* (*f x*)} = *0*)
by (*simp add*: *emeasure_distr*)
qed *auto*

lemma *null_sets_distr_iff*:
f ∈ *measurable M N* ⟹ *A* ∈ *null_sets* (*distr M N f*) ⟷ *f* −‘ *A* ∩ *space M* ∈

null_sets M ∧ A ∈ sets N
by (*auto simp add*: *null_sets_def emeasure_distr*)

proposition *distr_distr*:
g ∈ measurable N L ⟹ f ∈ measurable M N ⟹ distr (*distr M N f*) *L g* = *distr M L* (*g ∘ f*)
by (*auto simp add*: *emeasure_distr measurable_space*
intro!: *arg_cong*[**where** *f*=*emeasure M*] *measure_eqI*)

7.3.9 Real measure values

lemma *ring_of_finite_sets*: *ring_of_sets* (*space M*) {*A*∈*sets M*. *emeasure M A* ≠ *top*}
proof (*rule ring_of_setsI*)
show *a ∈* {*A ∈ sets M*. *emeasure M A ≠ top*} *⟹ b ∈* {*A ∈ sets M*. *emeasure M A ≠ top*} *⟹*
a ∪ b ∈ {*A ∈ sets M*. *emeasure M A ≠ top*} **for** *a b*
using *emeasure_subadditive*[*of a M b*] **by** (*auto simp*: *top_unique*)

show *a ∈* {*A ∈ sets M*. *emeasure M A ≠ top*} *⟹ b ∈* {*A ∈ sets M*. *emeasure M A ≠ top*} *⟹*
a − b ∈ {*A ∈ sets M*. *emeasure M A ≠ top*} **for** *a b*
using *emeasure_mono*[*of a − b a M*] **by** (*auto simp*: *top_unique*)
qed (*auto dest*: *sets.sets_into_space*)

lemma *measure_nonneg*[*simp*]: *0 ≤ measure M A*
unfolding *measure_def* **by** *auto*

lemma *measure_nonneg′* [*simp*]: *¬ measure M A < 0*
using *measure_nonneg not_le* **by** *blast*

lemma *zero_less_measure_iff*: *0 < measure M A ⟷ measure M A ≠ 0*
using *measure_nonneg*[*of M A*] **by** (*auto simp add*: *le_less*)

lemma *measure_le_0_iff*: *measure M X ≤ 0 ⟷ measure M X = 0*
using *measure_nonneg*[*of M X*] **by** *linarith*

lemma *measure_empty*[*simp*]: *measure M {} = 0*
unfolding *measure_def* **by** (*simp add*: *zero_ennreal.rep_eq*)

lemma *emeasure_eq_ennreal_measure*:
emeasure M A ≠ top ⟹ emeasure M A = ennreal (*measure M A*)
by (*cases emeasure M A rule*: *ennreal_cases*) (*auto simp*: *measure_def*)

lemma *measure_zero_top*: *emeasure M A = top ⟹ measure M A = 0*
by (*simp add*: *measure_def*)

lemma *measure_eq_emeasure_eq_ennreal*: *0 ≤ x ⟹ emeasure M A = ennreal x ⟹ measure M A = x*

using *emeasure_eq_ennreal_measure*[*of M A*]
by (*cases A ∈ M*) (*auto simp*: *measure_notin_sets emeasure_notin_sets*)

lemma *enn2real_plus*:*a < top ⟹ b < top ⟹ enn2real* (*a + b*) = *enn2real a + enn2real b*
by (*simp add*: *enn2real_def plus_ennreal.rep_eq real_of_ereal_add less_top*
del: *real_of_ereal_enn2ereal*)

lemma *enn2real_sum*:(*⋀i. i ∈ I ⟹ f i < top*) *⟹ enn2real* (*sum f I*) = *sum* (*enn2real ∘ f*) *I*
by (*induction I rule*: *infinite_finite_induct*) (*auto simp*: *enn2real_plus*)

lemma *measure_eq_AE*:
assumes *iff*: *AE x in M. x ∈ A ⟷ x ∈ B*
assumes *A*: *A ∈ sets M* **and** *B*: *B ∈ sets M*
shows *measure M A* = *measure M B*
using *assms emeasure_eq_AE*[*OF assms*] **by** (*simp add*: *measure_def*)

lemma *measure_Union*:
emeasure M A ≠ ∞ ⟹ emeasure M B ≠ ∞ ⟹ A ∈ sets M ⟹ B ∈ sets M ⟹ A ∩ B = {} *⟹*
measure M (*A ∪ B*) = *measure M A + measure M B*
by (*simp add*: *measure_def plus_emeasure*[*symmetric*] *enn2real_plus less_top*)

lemma *measure_finite_Union*:
finite S ⟹ A‘S ⊆ sets M ⟹ disjoint_family_on A S ⟹ (*⋀i. i ∈ S ⟹ emeasure M* (*A i*) *≠ ∞*) *⟹*
measure M (*⋃i∈S. A i*) = (*∑ i∈S. measure M* (*A i*))
by (*induction S rule*: *finite_induct*)
(*auto simp*: *disjoint_family_on_insert measure_Union sum_emeasure*[*symmetric*] *sets.countable_UN'*[*OF countable_finite*])

lemma *measure_Diff*:
assumes *finite*: *emeasure M A ≠ ∞*
and *measurable*: *A ∈ sets M B ∈ sets M B ⊆ A*
shows *measure M* (*A − B*) = *measure M A − measure M B*
proof −
have *emeasure M* (*A − B*) *≤ emeasure M A emeasure M B ≤ emeasure M A*
using *measurable* **by** (*auto intro*!: *emeasure_mono*)
hence *measure M* ((*A − B*) *∪ B*) = *measure M* (*A − B*) + *measure M B*
using *measurable finite* **by** (*rule_tac measure_Union*) (*auto simp*: *top_unique*)
thus *?thesis* **using** ‹*B ⊆ A*› **by** (*auto simp*: *Un_absorb2*)
qed

lemma *measure_UNION*:
assumes *measurable*: *range A ⊆ sets M disjoint_family A*
assumes *finite*: *emeasure M* (*⋃ i. A i*) *≠ ∞*
shows (*λi. measure M* (*A i*)) *sums* (*measure M* (*⋃ i. A i*))
proof −

```
  have (λi. emeasure M (A i)) sums (emeasure M (⋃ i. A i))
   unfolding suminf_emeasure[OF measurable, symmetric] by (simp add: summable_sums)
  moreover
  { fix i
    have emeasure M (A i) ≤ emeasure M (⋃ i. A i)
      using measurable by (auto intro!: emeasure_mono)
    then have emeasure M (A i) = ennreal ((measure M (A i)))
    using finite by (intro emeasure_eq_ennreal_measure) (auto simp: top_unique)
}
  ultimately show ?thesis using finite
    by (subst (asm) (2) emeasure_eq_ennreal_measure) simp_all
qed

lemma measure_subadditive:
  assumes measurable: A ∈ sets M B ∈ sets M
  and fin: emeasure M A ≠ ∞ emeasure M B ≠ ∞
  shows measure M (A ∪ B) ≤ measure M A + measure M B
proof −
  have emeasure M (A ∪ B) ≠ ∞
    using emeasure_subadditive[OF measurable] fin by (auto simp: top_unique)
  then show (measure M (A ∪ B)) ≤ (measure M A) + (measure M B)
    unfolding measure_def
   by (metis emeasure_subadditive[OF measurable] fin   enn2real_mono enn2real_plus

       ennreal_add_less_top infinity_ennreal_def less_top)
qed

lemma measure_subadditive_finite:
  assumes A: finite I A‘I ⊆ sets M and fin: ⋀i. i ∈ I ⟹ emeasure M (A i) ≠
∞
  shows measure M (⋃i∈I. A i) ≤ (∑ i∈I. measure M (A i))
proof −
  { have emeasure M (⋃i∈I. A i) ≤ (∑ i∈I. emeasure M (A i))
      using emeasure_subadditive_finite[OF A] .
    also have ... < ∞
      using fin by (simp add: less_top A)
    finally have emeasure M (⋃i∈I. A i) ≠ top by simp }
  note ∗ = this
  show ?thesis
    using emeasure_subadditive_finite[OF A] fin
    unfolding emeasure_eq_ennreal_measure[OF ∗]
    by (simp_all add: sum_nonneg emeasure_eq_ennreal_measure)
qed

lemma measure_subadditive_countably:
  assumes A: range A ⊆ sets M and fin: (∑ i. emeasure M (A i)) ≠ ∞
  shows measure M (⋃ i. A i) ≤ (∑ i. measure M (A i))
proof −
  have ∗∗: ⋀i. emeasure M (A i) ≠ top
```

using *fin ennreal_suminf_lessD*[*of* $\lambda i.$ *emeasure M* (*A i*)] **by** (*simp add*: *less_top*)
have *ge0*: $(\sum i.$ *Sigma_Algebra.measure M* (*A i*)) ≥ 0
using *fin emeasure_eq_ennreal_measure*[*OF* $**$]
by (*metis infinity_ennreal_def measure_nonneg suminf_cong suminf_nonneg summable_suminf_not_top*)
have *emeasure M* $(\bigcup i.$ *A i*$) \neq$ *top*
by (*metis A emeasure_subadditive_countably fin infinity_ennreal_def neq_top_trans*)
then have *ennreal* (*measure M* $(\bigcup i.$ *A i*)) = *emeasure M* $(\bigcup i.$ *A i*)
by (*rule emeasure_eq_ennreal_measure*[*symmetric*])
also have $\ldots \leq (\sum i.$ *emeasure M* (*A i*))
using *emeasure_subadditive_countably*[*OF A*] **.**
also have $\ldots =$ *ennreal* $(\sum i.$ *measure M* (*A i*))
using *fin* **unfolding** *emeasure_eq_ennreal_measure*[*OF* $**$]
by (*subst suminf_ennreal*) (*auto simp*: $**$)
finally show *?thesis*
using *ge0 ennreal_le_iff* **by** *blast*
qed

lemma *measure_Un_null_set*: *A* $\in$ *sets M* $\Longrightarrow$ *B* $\in$ *null_sets M* $\Longrightarrow$ *measure M* (*A* $\cup$ *B*) = *measure M A*
by (*simp add*: *measure_def emeasure_Un_null_set*)

lemma *measure_Diff_null_set*: *A* $\in$ *sets M* $\Longrightarrow$ *B* $\in$ *null_sets M* $\Longrightarrow$ *measure M* (*A* $-$ *B*) = *measure M A*
by (*simp add*: *measure_def emeasure_Diff_null_set*)

lemma *measure_eq_sum_singleton*:
finite S $\Longrightarrow$ $(\bigwedge x.\ x \in S \Longrightarrow \{x\} \in$ *sets M*$) \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow$ *emeasure M* $\{x\}$ $\neq \infty) \Longrightarrow$
measure M S $= (\sum x \in S.$ *measure M* $\{x\})$
using *emeasure_eq_sum_singleton*[*of S M*]
by (*intro measure_eq_emeasure_eq_ennreal*) (*auto simp*: *sum_nonneg emeasure_eq_ennreal_measure*)

lemma *Lim_measure_incseq*:
assumes *A*: *range A* $\subseteq$ *sets M incseq A* **and** *fin*: *emeasure M* $(\bigcup i.$ *A i*$) \neq \infty$
shows $(\lambda i.$ *measure M* (*A i*)) $\longrightarrow$ *measure M* $(\bigcup i.$ *A i*)
proof (*rule tendsto_ennrealD*)
have *ennreal* (*measure M* $(\bigcup i.$ *A i*)) = *emeasure M* $(\bigcup i.$ *A i*)
using *fin* **by** (*auto simp*: *emeasure_eq_ennreal_measure*)
moreover have *ennreal* (*measure M* (*A i*)) = *emeasure M* (*A i*) **for** *i*
using *assms emeasure_mono*[*of A* _ $\bigcup i.$ *A i M*]
by (*intro emeasure_eq_ennreal_measure*[*symmetric*]) (*auto simp*: *less_top UN_upper intro*: *le_less_trans*)
ultimately show $(\lambda x.$ *ennreal* (*measure M* (*A x*))) $\longrightarrow$ *ennreal* (*measure M* $(\bigcup i.$ *A i*))
using *A* **by** (*auto intro*!: *Lim_emeasure_incseq*)
qed *auto*

lemma *Lim_measure_decseq*:
assumes *A*: *range A* ⊆ *sets M decseq A* **and** *fin*: ⋀*i*. *emeasure M* (*A i*) ≠ ∞
shows (λ*n*. *measure M* (*A n*)) ⟶ *measure M* (⋂ *i*. *A i*)
proof (*rule tendsto_ennrealD*)
have *ennreal* (*measure M* (⋂ *i*. *A i*)) = *emeasure M* (⋂ *i*. *A i*)
using *fin*[*of 0*] *A emeasure_mono*[*of* ⋂ *i*. *A i A 0 M*]
by (*auto intro*!: *emeasure_eq_ennreal_measure*[*symmetric*] *simp*: *INT_lower less_top intro*: *le_less_trans*)
moreover have *ennreal* (*measure M* (*A i*)) = *emeasure M* (*A i*) **for** *i*
using *A fin*[*of i*] **by** (*intro emeasure_eq_ennreal_measure*[*symmetric*]) *auto*
ultimately show (λ*x*. *ennreal* (*measure M* (*A x*))) ⟶ *ennreal* (*measure M* (⋂ *i*. *A i*))
using *fin A* **by** (*auto intro*!: *Lim_emeasure_decseq*)
qed *auto*

7.3.10 Set of measurable sets with finite measure

definition *fmeasurable* :: *'a measure* ⇒ *'a set set* **where**
fmeasurable M = {*A*∈*sets M*. *emeasure M A* < ∞}

lemma *fmeasurableD*[*dest*, *measurable_dest*]: *A* ∈ *fmeasurable M* ⟹ *A* ∈ *sets M*
by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableD2*: *A* ∈ *fmeasurable M* ⟹ *emeasure M A* ≠ *top*
by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableI*: *A* ∈ *sets M* ⟹ *emeasure M A* < ∞ ⟹ *A* ∈ *fmeasurable M*
by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableI_null_sets*: *A* ∈ *null_sets M* ⟹ *A* ∈ *fmeasurable M*
by (*auto simp*: *fmeasurable_def*)

lemma *fmeasurableI2*: *A* ∈ *fmeasurable M* ⟹ *B* ⊆ *A* ⟹ *B* ∈ *sets M* ⟹ *B* ∈ *fmeasurable M*
using *emeasure_mono*[*of B A M*] **by** (*auto simp*: *fmeasurable_def*)

lemma *measure_mono_fmeasurable*:
A ⊆ *B* ⟹ *A* ∈ *sets M* ⟹ *B* ∈ *fmeasurable M* ⟹ *measure M A* ≤ *measure M B*
by (*auto simp*: *measure_def fmeasurable_def intro*!: *emeasure_mono enn2real_mono*)

lemma *emeasure_eq_measure2*: *A* ∈ *fmeasurable M* ⟹ *emeasure M A* = *measure M A*
by (*simp add*: *emeasure_eq_ennreal_measure fmeasurable_def less_top*)

interpretation *fmeasurable*: *ring_of_sets space M fmeasurable M*
proof (*rule ring_of_setsI*)

show *fmeasurable M* ⊆ *Pow* (*space M*) {} ∈ *fmeasurable M*
by (*auto simp*: *fmeasurable_def dest*: *sets.sets_into_space*)
fix *a b* **assume** ∗: *a* ∈ *fmeasurable M b* ∈ *fmeasurable M*
then have *emeasure M* (*a* ∪ *b*) ≤ *emeasure M a* + *emeasure M b*
by (*intro emeasure_subadditive*) *auto*
also have ... < *top*
using ∗ **by** (*auto simp*: *fmeasurable_def*)
finally show *a* ∪ *b* ∈ *fmeasurable M*
using ∗ **by** (*auto intro*: *fmeasurableI*)
show *a* − *b* ∈ *fmeasurable M*
using *emeasure_mono*[*of a* − *b a M*] ∗ **by** (*auto simp*: *fmeasurable_def*)
qed

7.3.11 Measurable sets formed by unions and intersections

lemma *fmeasurable_Diff*: *A* ∈ *fmeasurable M* ⟹ *B* ∈ *sets M* ⟹ *A* − *B* ∈ *fmeasurable M*
using *fmeasurableI2*[*of A M A* − *B*] **by** *auto*

lemma *fmeasurable_Int_fmeasurable*:
⟦*S* ∈ *fmeasurable M*; *T* ∈ *sets M*⟧ ⟹ (*S* ∩ *T*) ∈ *fmeasurable M*
by (*meson fmeasurableD fmeasurableI2 inf_le1 sets.Int*)

lemma *fmeasurable_UN*:
assumes *countable I* ⋀*i*. *i* ∈ *I* ⟹ *F i* ⊆ *A* ⋀*i*. *i* ∈ *I* ⟹ *F i* ∈ *sets M A* ∈ *fmeasurable M*
shows (⋃*i*∈*I*. *F i*) ∈ *fmeasurable M*
proof (*rule fmeasurableI2*)
show *A* ∈ *fmeasurable M* (⋃*i*∈*I*. *F i*) ⊆ *A* **using** *assms* **by** *auto*
show (⋃*i*∈*I*. *F i*) ∈ *sets M*
using *assms* **by** (*intro sets.countable_UN′*) *auto*
qed

lemma *fmeasurable_INT*:
assumes *countable I i* ∈ *I* ⋀*i*. *i* ∈ *I* ⟹ *F i* ∈ *sets M F i* ∈ *fmeasurable M*
shows (⋂*i*∈*I*. *F i*) ∈ *fmeasurable M*
proof (*rule fmeasurableI2*)
show *F i* ∈ *fmeasurable M* (⋂*i*∈*I*. *F i*) ⊆ *F i*
using *assms* **by** *auto*
show (⋂*i*∈*I*. *F i*) ∈ *sets M*
using *assms* **by** (*intro sets.countable_INT′*) *auto*
qed

lemma *measurable_measure_Diff*:
assumes *A* ∈ *fmeasurable M B* ∈ *sets M B* ⊆ *A*
shows *measure M* (*A* − *B*) = *measure M A* − *measure M B*
by (*simp add*: *assms fmeasurableD fmeasurableD2 measure_Diff*)

lemma *measurable_Un_null_set*:

assumes $B \in$ *null_sets M*
shows $(A \cup B \in$ *fmeasurable M* $\wedge$ $A \in$ *sets M*$) \longleftrightarrow A \in$ *fmeasurable M*
using *assms* **by** (*fastforce simp add*: *fmeasurable.Un fmeasurableI_null_sets intro*: *fmeasurableI2*)

lemma *measurable_Diff_null_set*:
assumes $B \in$ *null_sets M*
shows $(A - B) \in$ *fmeasurable M* $\wedge$ $A \in$ *sets M* $\longleftrightarrow A \in$ *fmeasurable M*
using *assms*
by (*metis Un_Diff_cancel2 fmeasurable.Diff fmeasurableD fmeasurableI_null_sets measurable_Un_null_set*)

lemma *fmeasurable_Diff_D*:
assumes *m*: $T - S \in$ *fmeasurable M* $S \in$ *fmeasurable M* **and** *sub*: $S \subseteq T$
shows $T \in$ *fmeasurable M*
proof −
have $T = S \cup (T - S)$
using *assms* **by** *blast*
then show *?thesis*
by (*metis m fmeasurable.Un*)
qed

lemma *measure_Un2*:
$A \in$ *fmeasurable M* $\Longrightarrow B \in$ *fmeasurable M* $\Longrightarrow$ *measure M* $(A \cup B) =$ *measure M A* + *measure M* $(B - A)$
using *measure_Union*[*of M A* $B - A$] **by** (*auto simp*: *fmeasurableD2 fmeasurable.Diff*)

lemma *measure_Un3*:
assumes $A \in$ *fmeasurable M* $B \in$ *fmeasurable M*
shows *measure M* $(A \cup B) =$ *measure M A* + *measure M B* − *measure M* $(A \cap B)$
proof −
have *measure M* $(A \cup B) =$ *measure M A* + *measure M* $(B - A)$
using *assms* **by** (*rule measure_Un2*)
also have $B - A = B - (A \cap B)$
by *auto*
also have *measure M* $(B - (A \cap B)) =$ *measure M B* − *measure M* $(A \cap B)$
using *assms* **by** (*intro measure_Diff*) (*auto simp*: *fmeasurable_def*)
finally show *?thesis*
by *simp*
qed

lemma *measure_Un_AE*:
AE x in M. $x \notin A \vee x \notin B \Longrightarrow A \in$ *fmeasurable M* $\Longrightarrow B \in$ *fmeasurable M* $\Longrightarrow$
measure M $(A \cup B) =$ *measure M A* + *measure M B*
by (*subst measure_Un2*) (*auto intro*!: *measure_eq_AE*)

lemma *measure_UNION_AE*:

assumes *I*: *finite I*
shows $(\bigwedge i.\ i \in I \Longrightarrow F\ i \in fmeasurable\ M) \Longrightarrow pairwise\ (\lambda i\ j.\ AE\ x\ in\ M.\ x \notin F\ i \vee x \notin F\ j)\ I \Longrightarrow$
measure M $(\bigcup i{\in}I.\ F\ i) = (\sum i{\in}I.\ measure\ M\ (F\ i))$
unfolding *AE_pairwise*[*OF countable_finite*, *OF I*]
using *I*
proof (*induction I rule*: *finite_induct*)
case (*insert x I*)
have *measure M* $(F\ x \cup \bigcup(F\ `\ I)) = measure\ M\ (F\ x) + measure\ M\ (\bigcup(F\ `\ I))$
by (*rule measure_Un_AE*) (*use insert* **in** ‹*auto simp*: *pairwise_insert*›)
with *insert* **show** *?case*
by (*simp add*: *pairwise_insert*)
qed *simp*

lemma *measure_UNION′*:
$finite\ I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow F\ i \in fmeasurable\ M) \Longrightarrow pairwise\ (\lambda i\ j.\ disjnt\ (F\ i)\ (F\ j))\ I \Longrightarrow$
measure M $(\bigcup i{\in}I.\ F\ i) = (\sum i{\in}I.\ measure\ M\ (F\ i))$
by (*intro measure_UNION_AE*) (*auto simp*: *disjnt_def elim*!: *pairwise_mono intro*!: *always_eventually*)

lemma *measure_Union_AE*:
$finite\ F \Longrightarrow (\bigwedge S.\ S \in F \Longrightarrow S \in fmeasurable\ M) \Longrightarrow pairwise\ (\lambda S\ T.\ AE\ x\ in\ M.\ x \notin S \vee x \notin T)\ F \Longrightarrow$
measure M $(\bigcup F) = (\sum S{\in}F.\ measure\ M\ S)$
using *measure_UNION_AE*[*of F* $\lambda x.\ x$ *M*] **by** *simp*

lemma *measure_Union′*:
$finite\ F \Longrightarrow (\bigwedge S.\ S \in F \Longrightarrow S \in fmeasurable\ M) \Longrightarrow pairwise\ disjnt\ F \Longrightarrow measure\ M\ (\bigcup F) = (\sum S{\in}F.\ measure\ M\ S)$
using *measure_UNION′*[*of F* $\lambda x.\ x$ *M*] **by** *simp*

lemma *measure_Un_le*:
assumes $A \in sets\ M\ B \in sets\ M$ **shows** $measure\ M\ (A \cup B) \leq measure\ M\ A + measure\ M\ B$
proof *cases*
assume $A \in fmeasurable\ M \wedge B \in fmeasurable\ M$
with *measure_subadditive*[*of A M B*] *assms* **show** *?thesis*
by (*auto simp*: *fmeasurableD2*)
next
assume $\neg\ (A \in fmeasurable\ M \wedge B \in fmeasurable\ M)$
then have $A \cup B \notin fmeasurable\ M$
using *fmeasurableI2*[*of* $A \cup B$ *M A*] *fmeasurableI2*[*of* $A \cup B$ *M B*] *assms* **by** *auto*
with *assms* **show** *?thesis*
by (*auto simp*: *fmeasurable_def measure_def less_top*[*symmetric*])
qed

lemma *measure_UNION_le*:
finite I $\Longrightarrow$ ($\bigwedge i.\ i \in I \Longrightarrow F\ i \in sets\ M$) $\Longrightarrow$ *measure M* ($\bigcup i{\in}I.\ F\ i$) $\leq$ ($\sum i{\in}I.$ *measure M* (*F i*))
proof (*induction I rule*: *finite_induct*)
case (*insert i I*)
then have *measure M* ($\bigcup i{\in}$*insert i I. F i*) = *measure M* (*F i* $\cup$ $\bigcup$ (*F ' I*))
by *simp*
also from *insert* **have** *measure M* (*F i* $\cup$ $\bigcup$ (*F ' I*)) $\leq$ *measure M* (*F i*) + *measure M* ($\bigcup$ (*F ' I*))
by (*intro measure_Un_le sets.finite_Union*) *auto*
also have *measure M* ($\bigcup i{\in}I.\ F\ i$) $\leq$ ($\sum i{\in}I.$ *measure M* (*F i*))
using *insert* **by** *auto*
finally show *?case*
using *insert* **by** *simp*
qed *simp*

lemma *measure_Union_le*:
finite F $\Longrightarrow$ ($\bigwedge S.\ S \in F \Longrightarrow S \in sets\ M$) $\Longrightarrow$ *measure M* ($\bigcup F$) $\leq$ ($\sum S{\in}F.$ *measure M S*)
using *measure_UNION_le*[*of F* $\lambda x.\ x$ *M*] **by** *simp*

Version for indexed union over a countable set

lemma
assumes *countable I* **and** *I*: $\bigwedge i.\ i \in I \Longrightarrow A\ i \in fmeasurable\ M$
and *bound*: $\bigwedge I'.\ I' \subseteq I \Longrightarrow$ *finite I'* $\Longrightarrow$ *measure M* ($\bigcup i{\in}I'.\ A\ i$) $\leq B$
shows *fmeasurable_UN_bound*: ($\bigcup i{\in}I.\ A\ i$) $\in$ *fmeasurable M* (**is** *?fm*)
and *measure_UN_bound*: *measure M* ($\bigcup i{\in}I.\ A\ i$) $\leq B$ (**is** *?m*)
proof −
have $B \geq 0$
using *bound* **by** *force*
have *?fm* $\wedge$ *?m*
proof *cases*
assume $I = \{\}$
with ‹$B \geq 0$› **show** *?thesis*
by *simp*
next
assume $I \neq \{\}$
have ($\bigcup i{\in}I.\ A\ i$) = ($\bigcup i.$ ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*)))
by (*subst range_from_nat_into*[*symmetric*, *OF* ‹$I \neq \{\}$› ‹*countable I*›]) *auto*
then have *emeasure M* ($\bigcup i{\in}I.\ A\ i$) = *emeasure M* ($\bigcup i.$ ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*))) **by** *simp*
also have ... = (*SUP i. emeasure M* ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*)))
using *I* ‹$I \neq \{\}$›[*THEN from_nat_into*] **by** (*intro SUP_emeasure_incseq*[*symmetric*]) (*fastforce simp*: *incseq_Suc_iff*)+
also have ... $\leq B$
proof (*intro SUP_least*)
fix *i* :: *nat*
have *emeasure M* ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*)) = *measure M* ($\bigcup n{\leq}i.\ A$ (*from_nat_into I n*))

using I ‹$I \neq \{\}$›[*THEN from_nat_into*] **by** (*intro emeasure_eq_measure2 fmeasurable.finite_UN*) *auto*
also have ... = *measure M* ($\bigcup n{\in}$*from_nat_into I* ‘ {..*i*}. *A n*)
by *simp*
also have ... $\leq$ *B*
by (*intro ennreal_leI bound*) (*auto intro*: *from_nat_into*[*OF* ‹$I \neq \{\}$›])
finally show *emeasure M* ($\bigcup n{\leq}i$. *A* (*from_nat_into I n*)) $\leq$ *ennreal B* **.**
qed
finally have $*$: *emeasure M* ($\bigcup i{\in}I$. *A i*) $\leq$ *B* **.**
then have *?fm*
using *I* ‹*countable I*› **by** (*intro fmeasurableI conjI*) (*auto simp*: *less_top*[*symmetric*] *top_unique*)
with $*$ ‹$0{\leq}B$› **show** *?thesis*
by (*simp add*: *emeasure_eq_measure2*)
qed
then show *?fm ?m* **by** *auto*
qed

Version for big union of a countable set

lemma
assumes *countable* $\mathcal{D}$
and *meas*: $\bigwedge D.\ D \in \mathcal{D} \Longrightarrow D \in$ *fmeasurable M*
and *bound*: $\bigwedge \mathcal{E}.\ [\![\mathcal{E} \subseteq \mathcal{D}; \textit{finite } \mathcal{E}]\!] \Longrightarrow \textit{measure } M\ (\bigcup \mathcal{E}) \leq B$
shows *fmeasurable_Union_bound*: $\bigcup \mathcal{D} \in$ *fmeasurable M* (**is** *?fm*)
and *measure_Union_bound*: *measure M* ($\bigcup \mathcal{D}$) $\leq$ *B* (**is** *?m*)
proof −
have $B \geq 0$
using *bound* **by** *force*
have *?fm* $\wedge$ *?m*
proof (*cases* $\mathcal{D} = \{\}$)
case *True*
with ‹$B \geq 0$› **show** *?thesis*
by *auto*
next
case *False*
then obtain *D* :: *nat* $\Rightarrow$ *'a set* **where** *D*: $\mathcal{D}$ = *range D*
using ‹*countable* $\mathcal{D}$› *uncountable_def* **by** *force*
have *1*: $\bigwedge i.$ *D i* $\in$ *fmeasurable M*
by (*simp add*: *D meas*)
have *2*: $\bigwedge I'.$ *finite* $I' \Longrightarrow$ *measure M* ($\bigcup x{\in}I'.$ *D x*) $\leq$ *B*
by (*simp add*: *D bound image_subset_iff*)
show *?thesis*
unfolding *D*
by (*intro conjI fmeasurable_UN_bound* [*OF _ 1 2*] *measure_UN_bound* [*OF _ 1 2*]) *auto*
qed
then show *?fm ?m* **by** *auto*
qed

Version for indexed union over the type of naturals

lemma
fixes $S :: nat \Rightarrow 'a\ set$
assumes S: $\bigwedge i.\ S\ i \in fmeasurable\ M$ **and** B: $\bigwedge n.\ measure\ M\ (\bigcup i{\leq}n.\ S\ i) \leq B$
shows *fmeasurable_countable_Union*: $(\bigcup i.\ S\ i) \in fmeasurable\ M$
and *measure_countable_Union_le*: $measure\ M\ (\bigcup i.\ S\ i) \leq B$
proof −
have *mB*: $measure\ M\ (\bigcup i{\in}I.\ S\ i) \leq B$ **if** *finite I* **for** I
proof −
have $(\bigcup i{\in}I.\ S\ i) \subseteq (\bigcup i{\leq}Max\ I.\ S\ i)$
using *Max_ge that* **by** *force*
then have $measure\ M\ (\bigcup i{\in}I.\ S\ i) \leq measure\ M\ (\bigcup i \leq Max\ I.\ S\ i)$
by (*rule measure_mono_fmeasurable*) (*use S* **in** ‹*blast*+›)
then show *?thesis*
using *B order_trans* **by** *blast*
qed
show $(\bigcup i.\ S\ i) \in fmeasurable\ M$
by (*auto intro*: *fmeasurable_UN_bound* [*OF _ S mB*])
show $measure\ M\ (\bigcup n.\ S\ n) \leq B$
by (*auto intro*: *measure_UN_bound* [*OF _ S mB*])
qed

lemma *measure_diff_le_measure_setdiff*:
assumes $S \in fmeasurable\ M$ $T \in fmeasurable\ M$
shows $measure\ M\ S - measure\ M\ T \leq measure\ M\ (S - T)$
proof −
have $measure\ M\ S \leq measure\ M\ ((S - T) \cup T)$
by (*simp add*: *assms fmeasurable.Un fmeasurableD measure_mono_fmeasurable*)
also have $\ldots \leq measure\ M\ (S - T) + measure\ M\ T$
using *assms* **by** (*blast intro*: *measure_Un_le*)
finally show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *suminf_exist_split2*:
fixes $f :: nat \Rightarrow 'a{::}real\_normed\_vector$
assumes *summable f*
shows $(\lambda n.\ (\sum k.\ f(k+n))) \longrightarrow 0$
by (*subst lim_sequentially*, *auto simp add*: *dist_norm suminf_exist_split*[*OF _ assms*])

lemma *emeasure_union_summable*:
assumes [*measurable*]: $\bigwedge n.\ A\ n \in sets\ M$
and $\bigwedge n.\ emeasure\ M\ (A\ n) < \infty$ *summable* $(\lambda n.\ measure\ M\ (A\ n))$
shows $emeasure\ M\ (\bigcup n.\ A\ n) < \infty$ $emeasure\ M\ (\bigcup n.\ A\ n) \leq (\sum n.\ measure\ M\ (A\ n))$
proof −
define B **where** $B = (\lambda N.\ (\bigcup n{\in}\{..{<}N\}.\ A\ n))$
have [*measurable*]: $B\ N \in sets\ M$ **for** N **unfolding** *B_def* **by** *auto*
have $(\lambda N.\ emeasure\ M\ (B\ N)) \longrightarrow emeasure\ M\ (\bigcup N.\ B\ N)$

apply (*rule Lim_emeasure_incseq*) **unfolding** *B_def* **by** (*auto simp add*: *SUP_subset_mono incseq_def*)
moreover have *emeasure M* (*B N*) ≤ *ennreal* (∑ *n*. *measure M* (*A n*)) **for** *N*
proof −
have ∗: (∑ *n*<*N*. *measure M* (*A n*)) ≤ (∑ *n*. *measure M* (*A n*))
using *assms*(*3*) *measure_nonneg sum_le_suminf* **by** *blast*

have *emeasure M* (*B N*) ≤ (∑ *n*<*N*. *emeasure M* (*A n*))
unfolding *B_def* **by** (*rule emeasure_subadditive_finite*, *auto*)
also have ... = (∑ *n*<*N*. *ennreal*(*measure M* (*A n*)))
using *assms*(*2*) **by** (*simp add*: *emeasure_eq_ennreal_measure less_top*)
also have ... = *ennreal* (∑ *n*<*N*. *measure M* (*A n*))
by *auto*
also have ... ≤ *ennreal* (∑ *n*. *measure M* (*A n*))
using ∗ **by** (*auto simp*: *ennreal_leI*)
finally show *?thesis* **by** *simp*
qed
ultimately have *emeasure M* (⋃ *N*. *B N*) ≤ *ennreal* (∑ *n*. *measure M* (*A n*))
by (*simp add*: *Lim_bounded*)
then show *emeasure M* (⋃ *n*. *A n*) ≤ (∑ *n*. *measure M* (*A n*))
unfolding *B_def* **by** (*metis UN_UN_flatten UN_lessThan_UNIV*)
then show *emeasure M* (⋃ *n*. *A n*) < ∞
by (*auto simp*: *less_top*[*symmetric*] *top_unique*)
qed

lemma *borel_cantelli_limsup1*:
assumes [*measurable*]: ⋀*n*. *A n* ∈ *sets M*
and ⋀*n*. *emeasure M* (*A n*) < ∞ *summable* (λ*n*. *measure M* (*A n*))
shows *limsup A* ∈ *null_sets M*
proof −
have *emeasure M* (*limsup A*) ≤ *0*
proof (*rule LIMSEQ_le_const*)
have (λ*n*. (∑ *k*. *measure M* (*A* (*k*+*n*)))) ⟶ *0* **by** (*rule suminf_exist_split2*[*OF assms*(*3*)])
then show (λ*n*. *ennreal* (∑ *k*. *measure M* (*A* (*k*+*n*)))) ⟶ *0*
unfolding *ennreal_0*[*symmetric*] **by** (*intro tendsto_ennrealI*)
have *emeasure M* (*limsup A*) ≤ (∑ *k*. *measure M* (*A* (*k*+*n*))) **for** *n*
proof −
have *I*: (⋃ *k*∈{*n*..}. *A k*) = (⋃ *k*. *A* (*k*+*n*)) **by** (*auto*, *metis le_add_diff_inverse2*, *fastforce*)
have *emeasure M* (*limsup A*) ≤ *emeasure M* (⋃ *k*∈{*n*..}. *A k*)
by (*rule emeasure_mono*, *auto simp add*: *limsup_INF_SUP*)
also have ... = *emeasure M* (⋃ *k*. *A* (*k*+*n*))
using *I* **by** *auto*
also have ... ≤ (∑ *k*. *measure M* (*A* (*k*+*n*)))
apply (*rule emeasure_union_summable*)
using *assms summable_ignore_initial_segment*[*OF assms*(*3*), *of n*] **by** *auto*
finally show *?thesis* **by** *simp*
qed

then show $\exists N.\ \forall n{\geq}N.\ emeasure\ M\ (limsup\ A) \leq (\sum k.\ measure\ M\ (A\ (k{+}n)))$
by *auto*
qed
then show *?thesis* **using** *assms*(*1*) *measurable_limsup* **by** *auto*
qed

lemma *borel_cantelli_AE1*:
assumes [*measurable*]: $\bigwedge n.\ A\ n \in sets\ M$
and $\bigwedge n.\ emeasure\ M\ (A\ n) < \infty\ summable\ (\lambda n.\ measure\ M\ (A\ n))$
shows *AE x in M. eventually* $(\lambda n.\ x \in space\ M - A\ n)$ *sequentially*
proof −
have *AE x in M.* $x \notin limsup\ A$
using *borel_cantelli_limsup1*[*OF assms*] **unfolding** *eventually_ae_filter* **by** *auto*
moreover have $\forall_F\ n$ *in sequentially.* $x \notin A\ n$ **if** $x \notin limsup\ A$ **for** x
using *that* **by** (*auto simp*: *limsup_INF_SUP eventually_sequentially*)
ultimately show *?thesis* **by** *auto*
qed

7.3.12 Measure spaces with *emeasure M (space M)* $< \infty$

locale *finite_measure* = *sigma_finite_measure M* **for** *M* +
assumes *finite_emeasure_space*: *emeasure M (space M)* $\neq$ *top*

lemma *finite_measureI*[*Pure.intro*!]:
emeasure M (space M) $\neq \infty \Longrightarrow$ *finite_measure M*
proof qed (*auto intro*!: *exI*[*of* _ {*space M*}])

lemma (**in** *finite_measure*) *emeasure_finite*[*simp*, *intro*]: *emeasure M A* $\neq$ *top*
using *finite_emeasure_space emeasure_space*[*of M A*] **by** (*auto simp*: *top_unique*)

lemma (**in** *finite_measure*) *fmeasurable_eq_sets*: *fmeasurable M* = *sets M*
by (*auto simp*: *fmeasurable_def less_top*[*symmetric*])

lemma (**in** *finite_measure*) *emeasure_eq_measure*: *emeasure M A* = *ennreal* (*measure M A*)
by (*intro emeasure_eq_ennreal_measure*) *simp*

lemma (**in** *finite_measure*) *emeasure_real*: $\exists r.\ 0 \leq r \wedge$ *emeasure M A* = *ennreal r*
using *emeasure_finite*[*of A*] **by** (*cases emeasure M A rule*: *ennreal_cases*) *auto*

lemma (**in** *finite_measure*) *bounded_measure*: *measure M A* $\leq$ *measure M* (*space M*)
using *emeasure_space*[*of M A*] *emeasure_real*[*of A*] *emeasure_real*[*of space M*]
by (*auto simp*: *measure_def*)

lemma (**in** *finite_measure*) *finite_measure_Diff*:

assumes *sets*: $A \in$ *sets* M $B \in$ *sets* M **and** $B \subseteq A$
shows *measure* M $(A - B) =$ *measure* M $A -$ *measure* M B
using *measure_Diff*[*OF* _ *assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_Union*:
assumes *sets*: $A \in$ *sets* M $B \in$ *sets* M **and** $A \cap B = \{\}$
shows *measure* M $(A \cup B) =$ *measure* M $A +$ *measure* M B
using *measure_Union*[*OF* _ _ *assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_finite_Union*:
assumes *measurable*: *finite* S A'$S \subseteq$ *sets* M *disjoint_family_on* A S
shows *measure* M $(\bigcup i{\in}S.\ A\ i) = (\sum i{\in}S.$ *measure* M $(A\ i))$
using *measure_finite_Union*[*OF assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_UNION*:
assumes A: *range* $A \subseteq$ *sets* M *disjoint_family* A
shows $(\lambda i.$ *measure* M $(A\ i))$ *sums* (*measure* M $(\bigcup i.\ A\ i))$
using *measure_UNION*[*OF* A] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_mono*:
assumes $A \subseteq B$ $B \in$ *sets* M **shows** *measure* M $A \leq$ *measure* M B
using *emeasure_mono*[*OF assms*] *emeasure_real*[*of* A] *emeasure_real*[*of* B] **by** (*auto simp*: *measure_def*)

lemma (**in** *finite_measure*) *finite_measure_subadditive*:
assumes *m*: $A \in$ *sets* M $B \in$ *sets* M
shows *measure* M $(A \cup B) \leq$ *measure* M $A +$ *measure* M B
using *measure_subadditive*[*OF m*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_subadditive_finite*:
assumes *finite* I A'$I \subseteq$ *sets* M **shows** *measure* M $(\bigcup i{\in}I.\ A\ i) \leq (\sum i{\in}I.$ *measure* M $(A\ i))$
using *measure_subadditive_finite*[*OF assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_subadditive_countably*:
range $A \subseteq$ *sets* $M \Longrightarrow$ *summable* $(\lambda i.$ *measure* M $(A\ i)) \Longrightarrow$ *measure* M $(\bigcup i.\ A\ i) \leq (\sum i.$ *measure* M $(A\ i))$
by (*rule measure_subadditive_countably*)
(*simp_all add*: *ennreal_suminf_neq_top emeasure_eq_measure*)

lemma (**in** *finite_measure*) *finite_measure_eq_sum_singleton*:
assumes *finite* S **and** $*$: $\bigwedge x.\ x \in S \Longrightarrow \{x\} \in$ *sets* M
shows *measure* M $S = (\sum x{\in}S.$ *measure* M $\{x\})$
using *measure_eq_sum_singleton*[*OF assms*] **by** *simp*

lemma (**in** *finite_measure*) *finite_Lim_measure_incseq*:
assumes A: *range* $A \subseteq$ *sets* M *incseq* A
shows $(\lambda i.$ *measure* M $(A\ i)) \longrightarrow$ *measure* M $(\bigcup i.\ A\ i)$
using *Lim_measure_incseq*[*OF* A] **by** *simp*

lemma (**in** *finite_measure*) *finite_Lim_measure_decseq*:
assumes *A*: *range A* ⊆ *sets M decseq A*
shows (λ*n*. *measure M* (*A n*)) ⟶ *measure M* (⋂ *i*. *A i*)
using *Lim_measure_decseq*[*OF A*] **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_compl*:
assumes *S*: *S* ∈ *sets M*
shows *measure M* (*space M* − *S*) = *measure M* (*space M*) − *measure M S*
using *measure_Diff*[*OF _ sets.top S sets.sets_into_space*] *S* **by** *simp*

lemma (**in** *finite_measure*) *finite_measure_mono_AE*:
assumes *imp*: *AE x in M*. *x* ∈ *A* ⟶ *x* ∈ *B* **and** *B*: *B* ∈ *sets M*
shows *measure M A* ≤ *measure M B*
using *assms emeasure_mono_AE*[*OF imp B*]
by (*simp add*: *emeasure_eq_measure*)

lemma (**in** *finite_measure*) *finite_measure_eq_AE*:
assumes *iff*: *AE x in M*. *x* ∈ *A* ⟷ *x* ∈ *B*
assumes *A*: *A* ∈ *sets M* **and** *B*: *B* ∈ *sets M*
shows *measure M A* = *measure M B*
using *assms emeasure_eq_AE*[*OF assms*] **by** (*simp add*: *emeasure_eq_measure*)

lemma (**in** *finite_measure*) *measure_increasing*: *increasing M* (*measure M*)
by (*auto intro*!: *finite_measure_mono simp*: *increasing_def*)

lemma (**in** *finite_measure*) *measure_zero_union*:
assumes *s* ∈ *sets M t* ∈ *sets M measure M t* = *0*
shows *measure M* (*s* ∪ *t*) = *measure M s*
using *assms*
proof −
have *measure M* (*s* ∪ *t*) ≤ *measure M s*
using *finite_measure_subadditive*[*of s t*] *assms* **by** *auto*
moreover have *measure M* (*s* ∪ *t*) ≥ *measure M s*
using *assms* **by** (*blast intro*: *finite_measure_mono*)
ultimately show *?thesis* **by** *simp*
qed

lemma (**in** *finite_measure*) *measure_eq_compl*:
assumes *s* ∈ *sets M t* ∈ *sets M*
assumes *measure M* (*space M* − *s*) = *measure M* (*space M* − *t*)
shows *measure M s* = *measure M t*
using *assms finite_measure_compl* **by** *auto*

lemma (**in** *finite_measure*) *measure_eq_bigunion_image*:
assumes *range f* ⊆ *sets M range g* ⊆ *sets M*
assumes *disjoint_family f disjoint_family g*
assumes ⋀ *n* :: *nat*. *measure M* (*f n*) = *measure M* (*g n*)
shows *measure M* (⋃ *i*. *f i*) = *measure M* (⋃ *i*. *g i*)

using *assms*
proof −
have *a*: (λ *i*. *measure M* (*f i*)) *sums* (*measure M* (⋃ *i*. *f i*))
by (*rule finite_measure_UNION*[*OF assms*(*1*,*3*)])
have *b*: (λ *i*. *measure M* (*g i*)) *sums* (*measure M* (⋃ *i*. *g i*))
by (*rule finite_measure_UNION*[*OF assms*(*2*,*4*)])
show *?thesis* **using** *sums_unique*[*OF b*] *sums_unique*[*OF a*] *assms* **by** *simp*
qed

lemma (**in** *finite_measure*) *measure_countably_zero*:
assumes *range c* ⊆ *sets M*
assumes ⋀ *i*. *measure M* (*c i*) = *0*
shows *measure M* (⋃ *i* :: *nat*. *c i*) = *0*
proof (*rule antisym*)
show *measure M* (⋃ *i* :: *nat*. *c i*) ≤ *0*
using *finite_measure_subadditive_countably*[*OF assms*(*1*)] **by** (*simp add*: *assms*(*2*))
qed *simp*

lemma (**in** *finite_measure*) *measure_space_inter*:
assumes *events*:*s* ∈ *sets M t* ∈ *sets M*
assumes *measure M t* = *measure M* (*space M*)
shows *measure M* (*s* ∩ *t*) = *measure M s*
proof −
have *measure M* ((*space M* − *s*) ∪ (*space M* − *t*)) = *measure M* (*space M* − *s*)
using *events assms finite_measure_compl*[*of t*] **by** (*auto intro*!: *measure_zero_union*)
also have (*space M* − *s*) ∪ (*space M* − *t*) = *space M* − (*s* ∩ *t*)
by *blast*
finally show *measure M* (*s* ∩ *t*) = *measure M s*
using *events* **by** (*auto intro*!: *measure_eq_compl*[*of s* ∩ *t s*])
qed

lemma (**in** *finite_measure*) *measure_equiprobable_finite_unions*:
assumes *s*: *finite s* ⋀*x*. *x* ∈ *s* ⟹ {*x*} ∈ *sets M*
assumes ⋀ *x y*. ⟦*x* ∈ *s*; *y* ∈ *s*⟧ ⟹ *measure M* {*x*} = *measure M* {*y*}
shows *measure M s* = *real* (*card s*) ∗ *measure M* {*SOME x*. *x* ∈ *s*}
proof *cases*
assume *s* ≠ {}
then have ∃ *x*. *x* ∈ *s* **by** *blast*
from *someI_ex*[*OF this*] *assms*
have *prob_some*: ⋀ *x*. *x* ∈ *s* ⟹ *measure M* {*x*} = *measure M* {*SOME y*. *y* ∈
s} **by** *blast*
have *measure M s* = (∑ *x* ∈ *s*. *measure M* {*x*})
using *finite_measure_eq_sum_singleton*[*OF s*] **by** *simp*
also have ... = (∑ *x* ∈ *s*. *measure M* {*SOME y*. *y* ∈ *s*}) **using** *prob_some* **by**
auto
also have ... = *real* (*card s*) ∗ *measure M* {(*SOME x*. *x* ∈ *s*)}
using *sum_constant assms* **by** *simp*
finally show *?thesis* **by** *simp*
qed *simp*

lemma (**in** *finite_measure*) *measure_real_sum_image_fn*:
assumes $e \in sets\ M$
assumes $\bigwedge x.\ x \in s \Longrightarrow e \cap f\ x \in sets\ M$
assumes *finite s*
assumes *disjoint*: $\bigwedge x\ y.\ [\![x \in s\ ;\ y \in s\ ;\ x \neq y]\!] \Longrightarrow f\ x \cap f\ y = \{\}$
assumes *upper*: $space\ M \subseteq (\bigcup i \in s.\ f\ i)$
shows $measure\ M\ e = (\sum\ x \in s.\ measure\ M\ (e \cap f\ x))$
proof −
have $e \subseteq (\bigcup i{\in}s.\ f\ i)$
using ‹$e \in sets\ M$› *sets.sets_into_space upper* **by** *blast*
then have *e*: $e = (\bigcup i \in s.\ e \cap f\ i)$
by *auto*
hence $measure\ M\ e = measure\ M\ (\bigcup i \in s.\ e \cap f\ i)$ **by** *simp*
also have $\ldots = (\sum\ x \in s.\ measure\ M\ (e \cap f\ x))$
proof (*rule finite_measure_finite_Union*)
show *finite s* **by** *fact*
show $(\lambda i.\ e \cap f\ i)`s \subseteq sets\ M$ **using** *assms*(*2*) **by** *auto*
show *disjoint_family_on* $(\lambda i.\ e \cap f\ i)\ s$
using *disjoint* **by** (*auto simp*: *disjoint_family_on_def*)
qed
finally show *?thesis* **.**
qed

lemma (**in** *finite_measure*) *measure_exclude*:
assumes $A \in sets\ M\ B \in sets\ M$
assumes $measure\ M\ A = measure\ M\ (space\ M)\ A \cap B = \{\}$
shows $measure\ M\ B = 0$
using *measure_space_inter*[*of B A*] *assms* **by** (*auto simp*: *ac_simps*)
lemma (**in** *finite_measure*) *finite_measure_distr*:
assumes *f*: $f \in measurable\ M\ M'$
shows *finite_measure* (*distr M M' f*)
proof (*rule finite_measureI*)
have $f\ -`\ space\ M' \cap space\ M = space\ M$ **using** *f* **by** (*auto dest*: *measurable_space*)
with *f* **show** *emeasure* (*distr M M' f*) (*space* (*distr M M' f*)) $\neq \infty$ **by** (*auto simp*: *emeasure_distr*)
qed

lemma *emeasure_gfp*[*consumes 1*, *case_names cont measurable*]:
assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
assumes $\bigwedge s.\ finite\_measure\ (M\ s)$
assumes *cont*: *inf_continuous F inf_continuous f*
assumes *meas*: $\bigwedge P.\ Measurable.pred\ N\ P \Longrightarrow Measurable.pred\ N\ (F\ P)$
assumes *iter*: $\bigwedge P\ s.\ Measurable.pred\ N\ P \Longrightarrow emeasure\ (M\ s)\ \{x{\in}space\ N.\ F\ P\ x\} = f\ (\lambda s.\ emeasure\ (M\ s)\ \{x{\in}space\ N.\ P\ x\})\ s$
assumes *bound*: $\bigwedge P.\ f\ P \leq f\ (\lambda s.\ emeasure\ (M\ s)\ (space\ (M\ s)))$
shows $emeasure\ (M\ s)\ \{x{\in}space\ N.\ gfp\ F\ x\} = gfp\ f\ s$
proof (*subst gfp_transfer_bounded*[**where** $\alpha{=}\lambda F\ s.\ emeasure\ (M\ s)\ \{x{\in}space\ N.$

F x} **and** *P*=*Measurable.pred N*, *symmetric*])
interpret *finite_measure M s* **for** *s* **by** *fact*
fix *C* **assume** *decseq C* $\bigwedge i$. *Measurable.pred N* (*C i*)
then show (λs. *emeasure* (*M s*) {*x* ∈ *space N*. (*INF i*. *C i*) *x*}) = (*INF i*. (λs. *emeasure* (*M s*) {*x* ∈ *space N*. *C i x*}))
unfolding *INF_apply*
by (*subst INF_emeasure_decseq*) (*auto simp*: *antimono_def fun_eq_iff intro*!: *arg_cong2*[**where** *f*=*emeasure*])
next
show *f x* ≤ (λs. *emeasure* (*M s*) {*x* ∈ *space N*. *F top x*}) **for** *x*
using *bound*[*of x*] *sets_eq_imp_space_eq*[*OF sets*] **by** (*simp add*: *iter*)
qed (*auto simp add*: *iter le_fun_def INF_apply*[*abs_def*] *intro*!: *meas cont*)

7.3.13 Counting space

lemma *strict_monoI_Suc*:
assumes ($\bigwedge n$. *f n* < *f* (*Suc n*)) **shows** *strict_mono f*
by (*simp add*: *assms strict_mono_Suc_iff*)

lemma *emeasure_count_space*:
assumes *X* ⊆ *A* **shows** *emeasure* (*count_space A*) *X* = (*if finite X then of_nat* (*card X*) *else* ∞)
(**is** *_* = *?M X*)
unfolding *count_space_def*
proof (*rule emeasure_measure_of_sigma*)
show *X* ∈ *Pow A* **using** ‹*X* ⊆ *A*› **by** *auto*
show *sigma_algebra A* (*Pow A*) **by** (*rule sigma_algebra_Pow*)
show *positive*: *positive* (*Pow A*) *?M*
by (*auto simp*: *positive_def*)
have *additive*: *additive* (*Pow A*) *?M*
by (*auto simp*: *card_Un_disjoint additive_def*)

interpret *ring_of_sets A Pow A*
by (*rule ring_of_setsI*) *auto*
show *countably_additive* (*Pow A*) *?M*
unfolding *countably_additive_iff_continuous_from_below*[*OF positive additive*]
proof *safe*
fix *F* :: *nat* ⇒ *'a set* **assume** *incseq F*
show (λi. *?M* (*F i*)) ⟶ *?M* ($\bigcup i$. *F i*)
proof *cases*
assume ∃ *i*. ∀ *j*≥*i*. *F i* = *F j*
then obtain *i* **where** *i*: ∀ *j*≥*i*. *F i* = *F j* **..**
with ‹*incseq F*› **have** *F j* ⊆ *F i* **for** *j*
by (*cases i* ≤ *j*) (*auto simp*: *incseq_def*)
then have *eq*: ($\bigcup i$. *F i*) = *F i*
by *auto*
with *i* **show** *?thesis*
by (*auto intro*!: *Lim_transform_eventually*[*OF tendsto_const*] *eventu-*

```
ally_sequentiallyI[where c=i])
    next
      assume ¬ (∃ i. ∀ j≥i. F i = F j)
      then obtain f where f: ⋀i. i ≤ f i ⋀i. F i ≠ F (f i) by metis
      then have ⋀i. F i ⊆ F (f i) using ‹incseq F› by (auto simp: incseq_def)
      with f have ∗: ⋀i. F i ⊂ F (f i) by auto

      have incseq (λi. ?M (F i))
        using ‹incseq F› unfolding incseq_def by (auto simp: card_mono dest:
finite_subset)
      then have (λi. ?M (F i)) ⟶ (SUP n. ?M (F n))
        by (rule LIMSEQ_SUP)

      moreover have (SUP n. ?M (F n)) = top
      proof (rule ennreal_SUP_eq_top)
        fix n :: nat show ∃ k::nat∈UNIV. of_nat n ≤ ?M (F k)
        proof (induct n)
          case (Suc n)
          then obtain k where of_nat n ≤ ?M (F k) ..
         moreover have finite (F k) ⟹ finite (F (f k)) ⟹ card (F k) < card (F
(f k))
            using ‹F k ⊂ F (f k)› by (simp add: psubset_card_mono)
          moreover have finite (F (f k)) ⟹ finite (F k)
           using ‹k ≤ f k› ‹incseq F› by (auto simp: incseq_def dest: finite_subset)
          ultimately show ?case
            by (auto intro!: exI[of _ f k] simp del: of_nat_Suc)
        qed auto
      qed

      moreover
      have inj (λn. F ((f ^^ n) 0))
        by (intro strict_mono_imp_inj_on strict_monoI_Suc) (simp add: ∗)
      then have 1: infinite (range (λi. F ((f ^^ i) 0)))
        by (rule range_inj_infinite)
      have infinite (Pow (⋃ i. F i))
        by (rule infinite_super[OF _ 1]) auto
      then have infinite (⋃ i. F i)
        by auto
      ultimately show ?thesis by (simp only:) simp

    qed
  qed
qed

lemma distr_bij_count_space:
  assumes f: bij_betw f A B
  shows distr (count_space A) (count_space B) f = count_space B
proof (rule measure_eqI)
  have f': f ∈ measurable (count_space A) (count_space B)
```

using *f* **unfolding** *Pi_def bij_betw_def* **by** *auto*
fix *X* **assume** *X* ∈ *sets* (*distr* (*count_space A*) (*count_space B*) *f*)
then have *X*: *X* ∈ *sets* (*count_space B*) **by** *auto*
moreover from *X* **have** *f* −‘ *X* ∩ *A* = *the_inv_into A f* ‘ *X*
using *f* **by** (*auto simp*: *bij_betw_def subset_image_iff image_iff the_inv_into_f_f intro*: *the_inv_into_f_f*[*symmetric*])
moreover have *inj_on* (*the_inv_into A f*) *B*
using *X f* **by** (*auto simp*: *bij_betw_def inj_on_the_inv_into*)
with *X* **have** *inj_on* (*the_inv_into A f*) *X*
by (*auto intro*: *subset_inj_on*)
ultimately show *emeasure* (*distr* (*count_space A*) (*count_space B*) *f*) *X* = *emeasure* (*count_space B*) *X*
using *f* **unfolding** *emeasure_distr*[*OF f′ X*]
by (*subst* (*1 2*) *emeasure_count_space*) (*auto simp*: *card_image dest*: *finite_imageD*)
qed *simp*

lemma *emeasure_count_space_finite*[*simp*]:
X ⊆ *A* ⟹ *finite X* ⟹ *emeasure* (*count_space A*) *X* = *of_nat* (*card X*)
using *emeasure_count_space*[*of X A*] **by** *simp*

lemma *emeasure_count_space_infinite*[*simp*]:
X ⊆ *A* ⟹ *infinite X* ⟹ *emeasure* (*count_space A*) *X* = ∞
using *emeasure_count_space*[*of X A*] **by** *simp*

lemma *measure_count_space*: *measure* (*count_space A*) *X* = (*if X* ⊆ *A then of_nat* (*card X*) *else 0*)
by (*cases finite X*) (*auto simp*: *measure_notin_sets ennreal_of_nat_eq_real_of_nat measure_zero_top measure_eq_emeasure_eq_ennreal*)

lemma *emeasure_count_space_eq_0*:
emeasure (*count_space A*) *X* = *0* ⟷ (*X* ⊆ *A* ⟶ *X* = {})
proof *cases*
assume *X*: *X* ⊆ *A*
then show *?thesis*
proof (*intro iffI impI*)
assume *emeasure* (*count_space A*) *X* = *0*
with *X* **show** *X* = {}
by (*subst* (*asm*) *emeasure_count_space*) (*auto split*: *if_split_asm*)
qed *simp*
qed (*simp add*: *emeasure_notin_sets*)

lemma *null_sets_count_space*: *null_sets* (*count_space A*) = { {} }
unfolding *null_sets_def* **by** (*auto simp add*: *emeasure_count_space_eq_0*)

lemma *AE_count_space*: (*AE x in count_space A*. *P x*) ⟷ (∀ *x*∈*A*. *P x*)
unfolding *eventually_ae_filter* **by** (*auto simp add*: *null_sets_count_space*)

lemma *sigma_finite_measure_count_space_countable*:

assumes *A*: *countable A*
shows *sigma_finite_measure* (*count_space A*)
proof qed (*insert A*, *auto intro*!: *exI*[*of* _ (λ*a*. {*a*}) ‘ *A*])

lemma *sigma_finite_measure_count_space*:
fixes *A* :: *'a*::*countable set* **shows** *sigma_finite_measure* (*count_space A*)
by (*rule sigma_finite_measure_count_space_countable*) *auto*

lemma *finite_measure_count_space*:
assumes [*simp*]: *finite A*
shows *finite_measure* (*count_space A*)
by *rule simp*

lemma *sigma_finite_measure_count_space_finite*:
assumes *A*: *finite A* **shows** *sigma_finite_measure* (*count_space A*)
proof −
interpret *finite_measure count_space A* **using** *A* **by** (*rule finite_measure_count_space*)
show *sigma_finite_measure* (*count_space A*) **..**
qed

7.3.14 Measure restricted to space

lemma *emeasure_restrict_space*:
assumes Ω ∩ *space M* ∈ *sets M A* ⊆ Ω
shows *emeasure* (*restrict_space M* Ω) *A* = *emeasure M A*
proof (*cases A* ∈ *sets M*)
case *True*
show *?thesis*
proof (*rule emeasure_measure_of*[*OF restrict_space_def*])
show (∩) Ω ‘ *sets M* ⊆ *Pow* (Ω ∩ *space M*) *A* ∈ *sets* (*restrict_space M* Ω)
using ‹*A* ⊆ Ω› ‹*A* ∈ *sets M*› *sets.space_closed* **by** (*auto simp*: *sets_restrict_space*)
show *positive* (*sets* (*restrict_space M* Ω)) (*emeasure M*)
by (*auto simp*: *positive_def*)
show *countably_additive* (*sets* (*restrict_space M* Ω)) (*emeasure M*)
proof (*rule countably_additiveI*)
fix *A* :: *nat* ⇒ _ **assume** *range A* ⊆ *sets* (*restrict_space M* Ω) *disjoint_family A*
with *assms* **have** ⋀*i*. *A i* ∈ *sets M* ⋀*i*. *A i* ⊆ *space M disjoint_family A*
by (*fastforce simp*: *sets_restrict_space_iff*[*OF assms*(*1*)] *image_subset_iff*
dest: *sets.sets_into_space*)+
then show (∑ *i*. *emeasure M* (*A i*)) = *emeasure M* (⋃ *i*. *A i*)
by (*subst suminf_emeasure*) (*auto simp*: *disjoint_family_subset*)
qed
qed
next
case *False*
with *assms* **have** *A* ∉ *sets* (*restrict_space M* Ω)
by (*simp add*: *sets_restrict_space_iff*)
with *False* **show** *?thesis*

```
    by (simp add: emeasure_notin_sets)
qed

lemma measure_restrict_space:
  assumes Ω ∩ space M ∈ sets M A ⊆ Ω
  shows measure (restrict_space M Ω) A = measure M A
  using emeasure_restrict_space[OF assms] by (simp add: measure_def)

lemma AE_restrict_space_iff:
  assumes Ω ∩ space M ∈ sets M
  shows (AE x in restrict_space M Ω. P x) ⟷ (AE x in M. x ∈ Ω ⟶ P x)
proof −
  have ex_cong: ⋀P Q f. (⋀x. P x ⟹ Q x) ⟹ (⋀x. Q x ⟹ P (f x)) ⟹ (∃ x.
P x) ⟷ (∃ x. Q x)
    by auto
  { fix X assume X: X ∈ sets M emeasure M X = 0
    then have emeasure M (Ω ∩ space M ∩ X) ≤ emeasure M X
      by (intro emeasure_mono) auto
    then have emeasure M (Ω ∩ space M ∩ X) = 0
      using X by (auto intro!: antisym) }
  with assms show ?thesis
    unfolding eventually_ae_filter
    by (auto simp add: space_restrict_space null_sets_def sets_restrict_space_iff
                       emeasure_restrict_space cong: conj_cong
             intro!: ex_cong[where f=λX. (Ω ∩ space M) ∩ X])
qed

lemma restrict_restrict_space:
  assumes A ∩ space M ∈ sets M B ∩ space M ∈ sets M
  shows restrict_space (restrict_space M A) B = restrict_space M (A ∩ B) (is ?l
= ?r)
proof (rule measure_eqI[symmetric])
  show sets ?r = sets ?l
    unfolding sets_restrict_space image_comp by (intro image_cong) auto
next
  fix X assume X ∈ sets (restrict_space M (A ∩ B))
  then obtain Y where Y ∈ sets M X = Y ∩ A ∩ B
    by (auto simp: sets_restrict_space)
  with assms sets.Int[OF assms] show emeasure ?r X = emeasure ?l X
    by (subst (1 2) emeasure_restrict_space)
     (auto simp: space_restrict_space sets_restrict_space_iff emeasure_restrict_space
ac_simps)
qed

lemma restrict_count_space: restrict_space (count_space B) A = count_space
(A ∩ B)
proof (rule measure_eqI)
  show sets (restrict_space (count_space B) A) = sets (count_space (A ∩ B))
    by (subst sets_restrict_space) auto
```

moreover fix X **assume** $X \in sets\ (restrict\_space\ (count\_space\ B)\ A)$
ultimately have $X \subseteq A \cap B$ **by** *auto*
then show $emeasure\ (restrict\_space\ (count\_space\ B)\ A)\ X = emeasure\ (count\_space\ (A \cap B))\ X$
by (*cases finite X*) (*auto simp add*: *emeasure_restrict_space*)
qed

lemma *sigma_finite_measure_restrict_space*:
assumes *sigma_finite_measure M*
and *A*: $A \in sets\ M$
shows *sigma_finite_measure* (*restrict_space M A*)
proof −
interpret *sigma_finite_measure M* **by** *fact*
from *sigma_finite_countable* **obtain** *C*
where *C*: *countable C* $C \subseteq sets\ M$ $(\bigcup C) = space\ M$ $\forall a{\in}C.\ emeasure\ M\ a \neq \infty$
by *blast*
let *?C* = $(\cap)\ A$ ‘ *C*
from *C* **have** *countable ?C* $?C \subseteq sets\ (restrict\_space\ M\ A)$ $(\bigcup ?C) = space\ (restrict\_space\ M\ A)$
by(*auto simp add*: *sets_restrict_space space_restrict_space*)
moreover {
fix *a*
assume $a \in ?C$
then obtain a' **where** $a = A \cap a'$ $a' \in C$ **..**
then have $emeasure\ (restrict\_space\ M\ A)\ a \leq emeasure\ M\ a'$
using *A C* **by**(*auto simp add*: *emeasure_restrict_space intro*: *emeasure_mono*)
also have $\ldots < \infty$ **using** *C*(*4*)[*rule_format*, *of a'*] ‹$a' \in C$› **by** (*simp add*: *less_top*)
finally have $emeasure\ (restrict\_space\ M\ A)\ a \neq \infty$ **by** *simp* **}**
ultimately show *?thesis*
by *unfold_locales* (*rule exI conjI*|*assumption*|*blast*)+
qed

lemma *finite_measure_restrict_space*:
assumes *finite_measure M*
and *A*: $A \in sets\ M$
shows *finite_measure* (*restrict_space M A*)
proof −
interpret *finite_measure M* **by** *fact*
show *?thesis*
by(*rule finite_measureI*)(*simp add*: *emeasure_restrict_space A space_restrict_space*)
qed

lemma *restrict_distr*:
assumes [*measurable*]: $f \in measurable\ M\ N$
assumes [*simp*]: $\Omega \cap space\ N \in sets\ N$ **and** *restrict*: $f \in space\ M \to \Omega$
shows $restrict\_space\ (distr\ M\ N\ f)\ \Omega = distr\ M\ (restrict\_space\ N\ \Omega)\ f$
(**is** *?l* = *?r*)

proof (*rule measure_eqI*)
fix *A* **assume** *A* ∈ *sets* (*restrict_space* (*distr M N f*) Ω)
with *restrict* **show** *emeasure ?l A* = *emeasure ?r A*
by (*subst emeasure_distr*)
(*auto simp*: *sets_restrict_space_iff emeasure_restrict_space emeasure_distr*
intro!: *measurable_restrict_space2*)
qed (*simp add*: *sets_restrict_space*)

lemma *measure_eqI_restrict_generator*:
assumes *E*: *Int_stable E E* ⊆ *Pow* Ω ⋀*X*. *X* ∈ *E* ⟹ *emeasure M X* = *emeasure N X*
assumes *sets_eq*: *sets M* = *sets N* **and** Ω: Ω ∈ *sets M*
assumes *sets* (*restrict_space M* Ω) = *sigma_sets* Ω *E*
assumes *sets* (*restrict_space N* Ω) = *sigma_sets* Ω *E*
assumes *ae*: *AE x in M*. *x* ∈ Ω *AE x in N*. *x* ∈ Ω
assumes *A*: *countable A A* ≠ {} *A* ⊆ *E* ⋃*A* = Ω ⋀*a*. *a* ∈ *A* ⟹ *emeasure M a* ≠ ∞
shows *M* = *N*
proof (*rule measure_eqI*)
fix *X* **assume** *X*: *X* ∈ *sets M*
then have *emeasure M X* = *emeasure* (*restrict_space M* Ω) (*X* ∩ Ω)
using *ae* Ω **by** (*auto simp add*: *emeasure_restrict_space intro*!: *emeasure_eq_AE*)
also have *restrict_space M* Ω = *restrict_space N* Ω
proof (*rule measure_eqI_generator_eq*)
fix *X* **assume** *X* ∈ *E*
then show *emeasure* (*restrict_space M* Ω) *X* = *emeasure* (*restrict_space N* Ω) *X*
using *E* Ω **by** (*subst* (*1 2*) *emeasure_restrict_space*) (*auto simp*: *sets_eq sets_eq*[*THEN sets_eq_imp_space_eq*])
next
show *range* (*from_nat_into A*) ⊆ *E* (⋃*i*. *from_nat_into A i*) = Ω
using *A* **by** (*auto cong del*: *SUP_cong_simp*)
next
fix *i*
have *emeasure* (*restrict_space M* Ω) (*from_nat_into A i*) = *emeasure M* (*from_nat_into A i*)
using *A* Ω **by** (*subst emeasure_restrict_space*)
(*auto simp*: *sets_eq sets_eq*[*THEN sets_eq_imp_space_eq*] *intro*: *from_nat_into*)
with *A* **show** *emeasure* (*restrict_space M* Ω) (*from_nat_into A i*) ≠ ∞
by (*auto intro*: *from_nat_into*)
qed *fact*+
also have *emeasure* (*restrict_space N* Ω) (*X* ∩ Ω) = *emeasure N X*
using *X ae* Ω **by** (*auto simp add*: *emeasure_restrict_space sets_eq intro*!: *emeasure_eq_AE*)
finally show *emeasure M X* = *emeasure N X* .
qed *fact*

7.3.15 Null measure

definition *null_measure* :: *'a measure* ⇒ *'a measure* **where**
null_measure M = *sigma* (*space M*) (*sets M*)

lemma *space_null_measure*[*simp*]: *space* (*null_measure M*) = *space M*
by (*simp add*: *null_measure_def*)

lemma *sets_null_measure*[*simp*, *measurable_cong*]: *sets* (*null_measure M*) = *sets M*
by (*simp add*: *null_measure_def*)

lemma *emeasure_null_measure*[*simp*]: *emeasure* (*null_measure M*) *X* = *0*
by (*cases X* ∈ *sets M*, *rule emeasure_measure_of*)
(*auto simp*: *positive_def countably_additive_def emeasure_notin_sets null_measure_def*
dest: *sets.sets_into_space*)

lemma *measure_null_measure*[*simp*]: *measure* (*null_measure M*) *X* = *0*
by (*intro measure_eq_emeasure_eq_ennreal*) *auto*

lemma *null_measure_idem* [*simp*]: *null_measure* (*null_measure M*) = *null_measure M*
by(*rule measure_eqI*) *simp_all*

7.3.16 Scaling a measure

definition *scale_measure* :: *ennreal* ⇒ *'a measure* ⇒ *'a measure* **where**
scale_measure r M = *measure_of* (*space M*) (*sets M*) (λ*A*. *r* ∗ *emeasure M A*)

lemma *space_scale_measure*: *space* (*scale_measure r M*) = *space M*
by (*simp add*: *scale_measure_def*)

lemma *sets_scale_measure* [*simp*, *measurable_cong*]: *sets* (*scale_measure r M*) = *sets M*
by (*simp add*: *scale_measure_def*)

lemma *emeasure_scale_measure* [*simp*]:
emeasure (*scale_measure r M*) *A* = *r* ∗ *emeasure M A*
(**is** *_* = *?μ A*)
proof(*cases A* ∈ *sets M*)
case *True*
show *?thesis* **unfolding** *scale_measure_def*
proof(*rule emeasure_measure_of_sigma*)
show *sigma_algebra* (*space M*) (*sets M*) **..**
show *positive* (*sets M*) *?μ* **by** (*simp add*: *positive_def*)
show *countably_additive* (*sets M*) *?μ*
proof (*rule countably_additiveI*)
fix *A* :: *nat* ⇒ *_* **assume** ∗: *range A* ⊆ *sets M disjoint_family A*
have (∑ *i*. *?μ* (*A i*)) = *r* ∗ (∑ *i*. *emeasure M* (*A i*))
by *simp*

also have ... = $?\mu$ ($\bigcup i.\ A\ i$) **using** $*$ **by**(*simp add*: *suminf_emeasure*)
finally show ($\sum i.\ ?\mu\ (A\ i)$) = $?\mu$ ($\bigcup i.\ A\ i$) **.**
qed
qed(*fact True*)
qed(*simp add*: *emeasure_notin_sets*)

lemma *scale_measure_1* [*simp*]: *scale_measure 1 M* = *M*
by(*rule measure_eqI*) *simp_all*

lemma *scale_measure_0*[*simp*]: *scale_measure 0 M* = *null_measure M*
by(*rule measure_eqI*) *simp_all*

lemma *measure_scale_measure* [*simp*]: *0* ≤ *r* ⟹ *measure* (*scale_measure r M*) *A* = *r* ∗ *measure M A*
using *emeasure_scale_measure*[*of r M A*]
emeasure_eq_ennreal_measure[*of M A*]
measure_eq_emeasure_eq_ennreal[*of _ scale_measure r M A*]
by (*cases emeasure* (*scale_measure r M*) *A* = *top*)
(*auto simp del*: *emeasure_scale_measure*
simp: *ennreal_top_eq_mult_iff ennreal_mult_eq_top_iff measure_zero_top ennreal_mult*[*symmetric*])

lemma *scale_scale_measure* [*simp*]:
scale_measure r (*scale_measure r′ M*) = *scale_measure* (*r* ∗ *r′*) *M*
by (*rule measure_eqI*) (*simp_all add*: *max_def mult.assoc*)

lemma *scale_null_measure* [*simp*]: *scale_measure r* (*null_measure M*) = *null_measure M*
by (*rule measure_eqI*) *simp_all*

7.3.17 Complete lattice structure on measures

lemma (**in** *finite_measure*) *finite_measure_Diff′*:
A ∈ *sets M* ⟹ *B* ∈ *sets M* ⟹ *measure M* (*A* − *B*) = *measure M A* − *measure M* (*A* ∩ *B*)
using *finite_measure_Diff*[*of A A* ∩ *B*] **by** (*auto simp*: *Diff_Int*)

lemma (**in** *finite_measure*) *finite_measure_Union′*:
A ∈ *sets M* ⟹ *B* ∈ *sets M* ⟹ *measure M* (*A* ∪ *B*) = *measure M A* + *measure M* (*B* − *A*)
using *finite_measure_Union*[*of A B* − *A*] **by** *auto*

lemma *finite_unsigned_Hahn_decomposition*:
assumes *finite_measure M finite_measure N* **and** [*simp*]: *sets N* = *sets M*
shows ∃ *Y*∈*sets M*. (∀ *X*∈*sets M*. *X* ⊆ *Y* ⟶ *N X* ≤ *M X*) ∧ (∀ *X*∈*sets M*. *X* ∩ *Y* = {} ⟶ *M X* ≤ *N X*)
proof −
interpret *M*: *finite_measure M* **by** *fact*
interpret *N*: *finite_measure N* **by** *fact*

```
  define d where d X = measure M X − measure N X for X

  have [intro]: bdd_above (d‘sets M)
    using sets.sets_into_space[of _ M]
    by (intro bdd_aboveI[where M=measure M (space M)])
     (auto simp: d_def field_simps subset_eq intro!: add_increasing M.finite_measure_mono)

  define γ where γ = (SUP X∈sets M. d X)
  have le_γ[intro]: X ∈ sets M ⟹ d X ≤ γ for X
    by (auto simp: γ_def intro!: cSUP_upper)

  have ∃f. ∀n. f n ∈ sets M ∧ d (f n) > γ − 1 / 2^n
  proof (intro choice_iff[THEN iffD1] allI)
    fix n
    have ∃X∈sets M. γ − 1 / 2^n < d X
      unfolding γ_def by (intro less_cSUP_iff[THEN iffD1]) auto
    then show ∃y. y ∈ sets M ∧ γ − 1 / 2 ^ n < d y
      by auto
  qed
  then obtain E where [measurable]: E n ∈ sets M and E: d (E n) > γ − 1 /
2^n for n
    by auto

  define F where F m n = (if m ≤ n then ⋂i∈{m..n}. E i else space M) for m
n

  have [measurable]: m ≤ n ⟹ F m n ∈ sets M for m n
    by (auto simp: F_def)

  have 1: γ − 2 / 2 ^ m + 1 / 2 ^ n ≤ d (F m n) if m ≤ n for m n
    using that
  proof (induct rule: dec_induct)
    case base with E[of m] show ?case
      by (simp add: F_def field_simps)
  next
    case (step i)
    have F_Suc: F m (Suc i) = F m i ∩ E (Suc i)
      using ‹m ≤ i› by (auto simp: F_def le_Suc_eq)

    have γ + (γ − 2 / 2^m + 1 / 2 ^ Suc i) ≤ (γ − 1 / 2^Suc i) + (γ − 2 /
2^m + 1 / 2^i)
      by (simp add: field_simps)
    also have ... ≤ d (E (Suc i)) + d (F m i)
      using E[of Suc i] by (intro add_mono step) auto
    also have ... = d (E (Suc i)) + d (F m i − E (Suc i)) + d (F m (Suc i))
    using ‹m ≤ i› by (simp add: d_def field_simps F_Suc M.finite_measure_Diff'
N.finite_measure_Diff')
    also have ... = d (E (Suc i) ∪ F m i) + d (F m (Suc i))
```

```
      using ‹m ≤ i› by (simp add: d_def field_simps M.finite_measure_Union'
N.finite_measure_Union')
    also have ... ≤ γ + d (F m (Suc i))
      using ‹m ≤ i› by auto
    finally show ?case
      by auto
  qed

  define F' where F' m = (⋂i∈{m..}. E i) for m
  have F'_eq: F' m = (⋂i. F m (i + m)) for m
    by (fastforce simp: le_iff_add[of m] F'_def F_def)

  have [measurable]: F' m ∈ sets M for m
    by (auto simp: F'_def)

  have γ_le: γ - 0 ≤ d (⋃ m. F' m)
  proof (rule LIMSEQ_le)
    show (λn. γ - 2 / 2 ^ n) ⟶ γ - 0
      by (intro tendsto_intros LIMSEQ_divide_realpow_zero) auto
    have incseq F'
      by (auto simp: incseq_def F'_def)
    then show (λm. d (F' m)) ⟶ d (⋃ m. F' m)
      unfolding d_def
    by (intro tendsto_diff M.finite_Lim_measure_incseq N.finite_Lim_measure_incseq)
auto

    have γ - 2 / 2 ^ m + 0 ≤ d (F' m) for m
    proof (rule LIMSEQ_le)
      have *: decseq (λn. F m (n + m))
        by (auto simp: decseq_def F_def)
      show (λn. d (F m n)) ⟶ d (F' m)
        unfolding d_def F'_eq
        by (rule LIMSEQ_offset[where k=m])
        (auto intro!: tendsto_diff M.finite_Lim_measure_decseq N.finite_Lim_measure_decseq
*)
      show (λn. γ - 2 / 2 ^ m + 1 / 2 ^ n) ⟶ γ - 2 / 2 ^ m + 0
        by (intro tendsto_add LIMSEQ_divide_realpow_zero tendsto_const) auto
      show ∃N. ∀n≥N. γ - 2 / 2 ^ m + 1 / 2 ^ n ≤ d (F m n)
        using 1[of m] by (intro exI[of _ m]) auto
    qed
    then show ∃N. ∀n≥N. γ - 2 / 2 ^ n ≤ d (F' n)
      by auto
  qed

  show ?thesis
  proof (safe intro!: bexI[of _ ⋃ m. F' m])
    fix X assume [measurable]: X ∈ sets M and X: X ⊆ (⋃ m. F' m)
    have d (⋃ m. F' m) - d X = d ((⋃ m. F' m) - X)
    using X by (auto simp: d_def M.finite_measure_Diff N.finite_measure_Diff)
```

also have ... $\leq \gamma$
by *auto*
finally have $0 \leq d\ X$
using γ_le **by** *auto*
then show $emeasure\ N\ X \leq emeasure\ M\ X$
by (*auto simp*: *d_def M.emeasure_eq_measure N.emeasure_eq_measure*)
next
fix X **assume** [*measurable*]: $X \in sets\ M$ **and** X: $X \cap (\bigcup m.\ F'\ m) = \{\}$
then have $d\ (\bigcup m.\ F'\ m) + d\ X = d\ (X \cup (\bigcup m.\ F'\ m))$
by (*auto simp*: *d_def M.finite_measure_Union N.finite_measure_Union*)
also have ... $\leq \gamma$
by *auto*
finally have $d\ X \leq 0$
using γ_le **by** *auto*
then show $emeasure\ M\ X \leq emeasure\ N\ X$
by (*auto simp*: *d_def M.emeasure_eq_measure N.emeasure_eq_measure*)
qed *auto*
qed

proposition *unsigned_Hahn_decomposition*:
assumes [*simp*]: $sets\ N = sets\ M$ **and** [*measurable*]: $A \in sets\ M$
and [*simp*]: $emeasure\ M\ A \neq top\ emeasure\ N\ A \neq top$
shows $\exists\ Y \in sets\ M.\ Y \subseteq A \wedge (\forall\ X \in sets\ M.\ X \subseteq Y \longrightarrow N\ X \leq M\ X) \wedge (\forall\ X \in sets\ M.\ X \subseteq A \longrightarrow X \cap Y = \{\} \longrightarrow M\ X \leq N\ X)$
proof −
have $\exists\ Y \in sets\ (restrict\_space\ M\ A).$
$(\forall\ X \in sets\ (restrict\_space\ M\ A).\ X \subseteq Y \longrightarrow (restrict\_space\ N\ A)\ X \leq (restrict\_space\ M\ A)\ X) \wedge$
$(\forall\ X \in sets\ (restrict\_space\ M\ A).\ X \cap Y = \{\} \longrightarrow (restrict\_space\ M\ A)\ X \leq (restrict\_space\ N\ A)\ X)$
proof (*rule finite_unsigned_Hahn_decomposition*)
show *finite_measure* (*restrict_space M A*) *finite_measure* (*restrict_space N A*)
by (*auto simp*: *space_restrict_space emeasure_restrict_space less_top intro*!: *finite_measureI*)
qed (*simp add*: *sets_restrict_space*)
with *assms* **show** *?thesis*
by (*metis Int_subset_iff emeasure_restrict_space sets.Int_space_eq2 sets_restrict_space_iff space_restrict_space*)
qed

Define a lexicographical order on *measure*, in the order space, sets and measure. The parts of the lexicographical order are point-wise ordered.

instantiation *measure* :: (*type*) *order_bot*
begin

inductive *less_eq_measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
$space\ M \subset space\ N \Longrightarrow less\_eq\_measure\ M\ N$
| $space\ M = space\ N \Longrightarrow sets\ M \subset sets\ N \Longrightarrow less\_eq\_measure\ M\ N$
| $space\ M = space\ N \Longrightarrow sets\ M = sets\ N \Longrightarrow emeasure\ M \leq emeasure\ N \Longrightarrow$

less_eq_measure M N

lemma *le_measure_iff*:
M ≤ *N* ⟷ (*if space M* = *space N then*
if sets M = *sets N then emeasure M* ≤ *emeasure N else sets M* ⊆ *sets N else space M* ⊆ *space N*)
by (*auto elim*: *less_eq_measure.cases intro*: *less_eq_measure.intros*)

definition *less_measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
less_measure M N ⟷ (*M* ≤ *N* ∧ ¬ *N* ≤ *M*)

definition *bot_measure* :: *'a measure* **where**
bot_measure = *sigma* {} {}

lemma
shows *space_bot*[*simp*]: *space bot* = {}
and *sets_bot*[*simp*]: *sets bot* = {{}}
and *emeasure_bot*[*simp*]: *emeasure bot X* = *0*
by (*auto simp*: *bot_measure_def sigma_sets_empty_eq emeasure_sigma*)

instance
proof *standard*
show *bot* ≤ *a* **for** *a* :: *'a measure*
by (*simp add*: *le_measure_iff bot_measure_def sigma_sets_empty_eq emeasure_sigma le_fun_def*)
qed (*auto simp*: *le_measure_iff less_measure_def split*: *if_split_asm intro*: *measure_eqI*)

end

proposition *le_measure*: *sets M* = *sets N* ⟹ *M* ≤ *N* ⟷ (∀ *A*∈*sets M*. *emeasure M A* ≤ *emeasure N A*)
by (*metis emeasure_neq_0_sets le_fun_def le_measure_iff order_class.order_eq_iff sets_eq_imp_space_eq*)

definition *sup_measure'* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
sup_measure' A B =
measure_of (*space A*) (*sets A*)
(λ*X*. *SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))

lemma assumes [*simp*]: *sets B* = *sets A*
shows *space_sup_measure'*[*simp*]: *space* (*sup_measure' A B*) = *space A*
and *sets_sup_measure'*[*simp*]: *sets* (*sup_measure' A B*) = *sets A*
using *sets_eq_imp_space_eq*[*OF assms*] **by** (*simp_all add*: *sup_measure'_def*)

lemma *emeasure_sup_measure'*:
assumes *sets_eq*[*simp*]: *sets B* = *sets A* **and** [*simp*, *intro*]: *X* ∈ *sets A*
shows *emeasure* (*sup_measure' A B*) *X* = (*SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))

```
    (is _ = ?S X)
proof −
  note sets_eq_imp_space_eq[OF sets_eq, simp]
  show ?thesis
    using sup_measure'_def
  proof (rule emeasure_measure_of)
    let ?d = λX Y. emeasure A (X ∩ Y) + emeasure B (X ∩ − Y)
    show countably_additive (sets (sup_measure' A B)) (λX. SUP Y ∈ sets A.
emeasure A (X ∩ Y) + emeasure B (X ∩ − Y))
    proof (rule countably_additiveI, goal_cases)
      case (1 X)
      then have [measurable]: ⋀i. X i ∈ sets A and disjoint_family X
        by auto
      have disjoint: disjoint_family (λi. X i ∩ Y) disjoint_family (λi. X i − Y)
for Y
        using 1(2) disjoint_family_subset by fastforce+
      have (∑ i. ?S (X i)) = (SUP Y∈sets A. ∑ i. ?d (X i) Y)
      proof (rule ennreal_suminf_SUP_eq_directed)
        fix J :: nat set and a b assume finite J and [measurable]: a ∈ sets A b ∈
sets A
        have ∃ c∈sets A. c ⊆ X i ∧ (∀ a∈sets A. ?d (X i) a ≤ ?d (X i) c) for i
        proof cases
          assume emeasure A (X i) = top ∨ emeasure B (X i) = top
          then show ?thesis
            by force
        next
          assume finite: ¬ (emeasure A (X i) = top ∨ emeasure B (X i) = top)
          then have ∃ Y∈sets A. Y ⊆ X i ∧ (∀ C∈sets A. C ⊆ Y ⟶ B C ≤ A
C) ∧ (∀ C∈sets A. C ⊆ X i ⟶ C ∩ Y = {} ⟶ A C ≤ B C)
            using unsigned_Hahn_decomposition[of B A X i] by simp
          then obtain Y where [measurable]: Y ∈ sets A and [simp]: Y ⊆ X i
            and B_le_A: ⋀C. C ∈ sets A ⟹ C ⊆ Y ⟹ B C ≤ A C
            and A_le_B: ⋀C. C ∈ sets A ⟹ C ⊆ X i ⟹ C ∩ Y = {} ⟹ A C
≤ B C
            by auto

          show ?thesis
          proof (intro bexI ballI conjI)
            fix a assume [measurable]: a ∈ sets A
            have ∗: (X i ∩ a ∩ Y ∪ (X i ∩ a − Y)) = X i ∩ a (X i − a) ∩ Y ∪
(X i − a − Y) = X i ∩ − a
              for a Y by auto
            then have ?d (X i) a =
              (A (X i ∩ a ∩ Y) + A (X i ∩ a ∩ − Y)) + (B (X i ∩ − a ∩ Y) + B
(X i ∩ − a ∩ − Y))
              by (subst (1 2) plus_emeasure) (auto simp: Diff_eq[symmetric])
            also have ... ≤ (A (X i ∩ a ∩ Y) + B (X i ∩ a ∩ − Y)) + (A (X i ∩
− a ∩ Y) + B (X i ∩ − a ∩ − Y))
                  by (intro add_mono order_refl B_le_A A_le_B) (auto simp:
```

Diff_eq[*symmetric*])
also have ... ≤ (*A* (*X i* ∩ *Y* ∩ *a*) + *A* (*X i* ∩ *Y* ∩ − *a*)) + (*B* (*X i* ∩ − *Y* ∩ *a*) + *B* (*X i* ∩ − *Y* ∩ − *a*))
by (*simp add*: *ac_simps*)
also have ... ≤ *A* (*X i* ∩ *Y*) + *B* (*X i* ∩ − *Y*)
by (*subst* (*1 2*) *plus_emeasure*) (*auto simp*: *Diff_eq*[*symmetric*] ∗)
finally show *?d* (*X i*) *a* ≤ *?d* (*X i*) *Y* **.**
qed *auto*
qed
then obtain *C* **where** [*measurable*]: *C i* ∈ *sets A* **and** *C i* ⊆ *X i*
and *C*: ⋀*a*. *a* ∈ *sets A* ⟹ *?d* (*X i*) *a* ≤ *?d* (*X i*) (*C i*) **for** *i*
by *metis*
have ∗: *X i* ∩ (⋃ *i*. *C i*) = *X i* ∩ *C i* **for** *i*
using ‹*disjoint_family X*› ‹⋀*i*. *C i* ⊆ *X i*›
by (*simp add*: *disjoint_family_on_def disjoint_iff_not_equal set_eq_iff*) (*metis subsetD*)
then have ∗∗: *X i* ∩ − (⋃ *i*. *C i*) = *X i* ∩ − *C i* **for** *i* **by** *blast*
moreover have (⋃ *i*. *C i*) ∈ *sets A*
by *fastforce*
ultimately show ∃ *c*∈*sets A*. ∀ *i*∈*J*. *?d* (*X i*) *a* ≤ *?d* (*X i*) *c* ∧ *?d* (*X i*) *b* ≤ *?d* (*X i*) *c*
by (*metis* ∗ *C* ‹*a* ∈ *sets A*› ‹*b* ∈ *sets A*›)
qed
also have ... = *?S* (⋃ *i*. *X i*)
proof −
have ⋀*Y*. *Y* ∈ *sets A* ⟹ (∑ *i*. *emeasure A* (*X i* ∩ *Y*) + *emeasure B* (*X i* ∩ − *Y*))
= *emeasure A* (⋃ *i*. *X i* ∩ *Y*) + *emeasure B* (⋃ *i*. *X i* ∩ − *Y*)
using *disjoint*
by (*auto simp flip*: *suminf_add Diff_eq simp add*: *image_subset_iff suminf_emeasure*)
then show *?thesis* **by** *force*
qed
finally show (∑ *i*. *?S* (*X i*)) = *?S* (⋃ *i*. *X i*) **.**
qed
qed (*auto dest*: *sets.sets_into_space simp*: *positive_def intro*!: *SUP_const*)
qed

lemma *le_emeasure_sup_measure'1*:
assumes *sets B* = *sets A X* ∈ *sets A* **shows** *emeasure A X* ≤ *emeasure* (*sup_measure' A B*) *X*
by (*subst emeasure_sup_measure'*[*OF assms*]) (*auto intro*!: *SUP_upper2*[*of X*] *assms*)

lemma *le_emeasure_sup_measure'2*:
assumes *sets B* = *sets A X* ∈ *sets A* **shows** *emeasure B X* ≤ *emeasure* (*sup_measure' A B*) *X*
by (*subst emeasure_sup_measure'*[*OF assms*]) (*auto intro*!: *SUP_upper2*[*of* {}]

assms)

lemma *emeasure_sup_measure'_le2*:
assumes [*simp*]: *sets B* = *sets C sets A* = *sets C* **and** [*measurable*]: *X* ∈ *sets C*
assumes *A*: ⋀*Y*. *Y* ⊆ *X* ⟹ *Y* ∈ *sets A* ⟹ *emeasure A Y* ≤ *emeasure C Y*
assumes *B*: ⋀*Y*. *Y* ⊆ *X* ⟹ *Y* ∈ *sets A* ⟹ *emeasure B Y* ≤ *emeasure C Y*
shows *emeasure* (*sup_measure' A B*) *X* ≤ *emeasure C X*
proof (*subst emeasure_sup_measure'*)
show (*SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*)) ≤ *emeasure C X*
unfolding ‹*sets A* = *sets C*›
proof (*intro SUP_least*)
fix *Y* **assume** [*measurable*]: *Y* ∈ *sets C*
have [*simp*]: *X* ∩ *Y* ∪ (*X* − *Y*) = *X*
by *auto*
have *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*) ≤ *emeasure C* (*X* ∩ *Y*) + *emeasure C* (*X* ∩ − *Y*)
by (*intro add_mono A B*) (*auto simp*: *Diff_eq*[*symmetric*])
also have ... = *emeasure C X*
by (*subst plus_emeasure*) (*auto simp*: *Diff_eq*[*symmetric*])
finally show *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*) ≤ *emeasure C X* **.**
qed
qed *simp_all*

definition *sup_lexord* :: *'a* ⇒ *'a* ⇒ (*'a* ⇒ *'b*::*order*) ⇒ *'a* ⇒ *'a* ⇒ *'a* **where**
sup_lexord A B k s c =
(*if k A* = *k B then c else*
if ¬ *k A* ≤ *k B* ∧ ¬ *k B* ≤ *k A then s else*
if k B ≤ *k A then A else B*)

lemma *sup_lexord*:
(*k A* < *k B* ⟹ *P B*) ⟹ (*k B* < *k A* ⟹ *P A*) ⟹ (*k A* = *k B* ⟹ *P c*) ⟹
(¬ *k B* ≤ *k A* ⟹ ¬ *k A* ≤ *k B* ⟹ *P s*) ⟹ *P* (*sup_lexord A B k s c*)
by (*auto simp*: *sup_lexord_def*)

lemmas *le_sup_lexord* = *sup_lexord*[**where** *P*=λ*a*. *c* ≤ *a* **for** *c*]

lemma *sup_lexord1*: *k A* = *k B* ⟹ *sup_lexord A B k s c* = *c*
by (*simp add*: *sup_lexord_def*)

lemma *sup_lexord_commute*: *sup_lexord A B k s c* = *sup_lexord B A k s c*
by (*auto simp*: *sup_lexord_def*)

lemma *sigma_sets_le_sets_iff*: (*sigma_sets* (*space x*) *A* ⊆ *sets x*) = (*A* ⊆ *sets x*)
using *sets.sigma_sets_subset*[*of A x*] **by** *auto*

lemma *sigma_le_iff*: *A* ⊆ *Pow* Ω ⟹ *sigma* Ω *A* ≤ *x* ⟷ (Ω ⊆ *space x* ∧ (*space*

```
x = Ω ⟶ 𝒜 ⊆ sets x))
  by (cases Ω = space x)
      (simp_all add: eq_commute[of _ sets x] le_measure_iff emeasure_sigma
le_fun_def
                  sigma_sets_superset_generator sigma_sets_le_sets_iff)

instantiation measure :: (type) semilattice_sup
begin

definition sup_measure :: 'a measure ⇒ 'a measure ⇒ 'a measure where
  sup_measure A B =
    sup_lexord A B space (sigma (space A ∪ space B) {})
      (sup_lexord A B sets (sigma (space A) (sets A ∪ sets B)) (sup_measure' A
B))

instance
proof
  fix x y z :: 'a measure
  show x ≤ sup x y
    unfolding sup_measure_def
  proof (intro le_sup_lexord)
    assume space x = space y
    then have ∗: sets x ∪ sets y ⊆ Pow (space x)
      using sets.space_closed by auto
    assume ¬ sets y ⊆ sets x ¬ sets x ⊆ sets y
    then have sets x ⊂ sets x ∪ sets y
      by auto
    also have ... ≤ sigma (space x) (sets x ∪ sets y)
      by (subst sets_measure_of[OF ∗]) (rule sigma_sets_superset_generator)
    finally show x ≤ sigma (space x) (sets x ∪ sets y)
      by (simp add: space_measure_of[OF ∗] less_eq_measure.intros(2))
  next
    assume ¬ space y ⊆ space x ¬ space x ⊆ space y
    then show x ≤ sigma (space x ∪ space y) {}
      by (intro less_eq_measure.intros) auto
  next
    assume sets x = sets y then show x ≤ sup_measure' x y
      by (simp add: le_measure le_emeasure_sup_measure'1)
  qed (auto intro: less_eq_measure.intros)
  show y ≤ sup x y
    unfolding sup_measure_def
  proof (intro le_sup_lexord)
    assume ∗∗: space x = space y
    then have ∗: sets x ∪ sets y ⊆ Pow (space y)
      using sets.space_closed by auto
    assume ¬ sets y ⊆ sets x ¬ sets x ⊆ sets y
    then have sets y ⊂ sets x ∪ sets y
      by auto
    also have ... ≤ sigma (space y) (sets x ∪ sets y)
```

```
    by (subst sets_measure_of[OF *]) (rule sigma_sets_superset_generator)
  finally show y ≤ sigma (space x) (sets x ∪ sets y)
    by (simp add: ** space_measure_of[OF *] less_eq_measure.intros(2))
 next
  assume ¬ space y ⊆ space x ¬ space x ⊆ space y
  then show y ≤ sigma (space x ∪ space y) {}
    by (intro less_eq_measure.intros) auto
 next
  assume sets x = sets y then show y ≤ sup_measure' x y
    by (simp add: le_measure le_emeasure_sup_measure'2)
 qed (auto intro: less_eq_measure.intros)
 show x ≤ y ⟹ z ≤ y ⟹ sup x z ≤ y
  unfolding sup_measure_def
 proof (intro sup_lexord[where P=λx. x ≤ y])
  assume x ≤ y z ≤ y and [simp]: space x = space z sets x = sets z
  from ‹x ≤ y› show sup_measure' x z ≤ y
  proof cases
    case 1 then show ?thesis
      by (intro less_eq_measure.intros(1)) simp
  next
    case 2 then show ?thesis
      by (intro less_eq_measure.intros(2)) simp_all
  next
    case 3 with ‹z ≤ y› ‹x ≤ y› show ?thesis
      by (auto simp add: le_measure intro!: emeasure_sup_measure'_le2)
  qed
 next
  assume **: x ≤ y z ≤ y space x = space z ¬ sets z ⊆ sets x ¬ sets x ⊆ sets z
  then have *: sets x ∪ sets z ⊆ Pow (space x)
    using sets.space_closed by auto
  show sigma (space x) (sets x ∪ sets z) ≤ y
   unfolding sigma_le_iff[OF *] using ** by (auto simp: le_measure_iff split:
if_split_asm)
 next
  assume x ≤ y z ≤ y ¬ space z ⊆ space x ¬ space x ⊆ space z
  then have space x ⊆ space y space z ⊆ space y
    by (auto simp: le_measure_iff split: if_split_asm)
  then show sigma (space x ∪ space z) {} ≤ y
    by (simp add: sigma_le_iff)
 qed
qed

end

lemma space_empty_eq_bot: space a = {} ⟷ a = bot
 using space_empty[of a] by (auto intro!: measure_eqI)

lemma sets_eq_iff_bounded: A ≤ B ⟹ B ≤ C ⟹ sets A = sets C ⟹ sets B
= sets A
```

by (*auto dest*: *sets_eq_imp_space_eq simp add*: *le_measure_iff split*: *if_split_asm*)

lemma *sets_sup*: *sets A* = *sets M* ⟹ *sets B* = *sets M* ⟹ *sets* (*sup A B*) = *sets M*
by (*auto simp add*: *sup_measure_def sup_lexord_def dest*: *sets_eq_imp_space_eq*)

lemma *le_measureD1*: *A* ≤ *B* ⟹ *space A* ≤ *space B*
by (*auto simp*: *le_measure_iff split*: *if_split_asm*)

lemma *le_measureD2*: *A* ≤ *B* ⟹ *space A* = *space B* ⟹ *sets A* ≤ *sets B*
by (*auto simp*: *le_measure_iff split*: *if_split_asm*)

lemma *le_measureD3*: *A* ≤ *B* ⟹ *sets A* = *sets B* ⟹ *emeasure A X* ≤ *emeasure B X*
by (*auto simp*: *le_measure_iff le_fun_def dest*: *sets_eq_imp_space_eq split*: *if_split_asm*)

lemma *UN_space_closed*: ⋃ (*sets* ‘ *S*) ⊆ *Pow* (⋃ (*space* ‘ *S*))
using *sets.space_closed* **by** *auto*

definition
Sup_lexord :: (*'a* ⇒ *'b*::*complete_lattice*) ⇒ (*'a set* ⇒ *'a*) ⇒ (*'a set* ⇒ *'a*) ⇒ *'a set* ⇒ *'a*
where
Sup_lexord k c s A =
(*let U* = (*SUP a*∈*A*. *k a*)
in if ∃ *a*∈*A*. *k a* = *U then c* {*a*∈*A*. *k a* = *U*} *else s A*)

lemma *Sup_lexord*:
(⋀*a S*. *a* ∈ *A* ⟹ *k a* = (*SUP a*∈*A*. *k a*) ⟹ *S* = {*a'*∈*A*. *k a'* = *k a*} ⟹ *P* (*c S*)) ⟹ ((⋀*a*. *a* ∈ *A* ⟹ *k a* ≠ (*SUP a*∈*A*. *k a*)) ⟹ *P* (*s A*)) ⟹
P (*Sup_lexord k c s A*)
by (*auto simp*: *Sup_lexord_def Let_def*)

lemma *Sup_lexord1*:
assumes *A*: *A* ≠ {} (⋀*a*. *a* ∈ *A* ⟹ *k a* = (⋃ *a*∈*A*. *k a*)) *P* (*c A*)
shows *P* (*Sup_lexord k c s A*)
unfolding *Sup_lexord_def Let_def*
proof (*clarsimp*, *safe*)
show ∀ *a*∈*A*. *k a* ≠ (⋃ *x*∈*A*. *k x*) ⟹ *P* (*s A*)
by (*metis assms*(*1*,*2*) *ex_in_conv*)
next
fix *a* **assume** *a* ∈ *A k a* = (⋃ *x*∈*A*. *k x*)
then have {*a* ∈ *A*. *k a* = (⋃ *x*∈*A*. *k x*)} = {*a* ∈ *A*. *k a* = *k a*}
by (*metis A*(*2*)[*symmetric*])
then show *P* (*c* {*a* ∈ *A*. *k a* = (⋃ *x*∈*A*. *k x*)})
by (*simp add*: *A*(*3*))
qed

instantiation *measure* :: (*type*) *complete_lattice*
begin

interpretation *sup_measure*: *comm_monoid_set sup bot* :: *'a measure*
by *standard* (*auto intro*!: *antisym*)

lemma *sup_measure_F_mono'*:
finite J $\Longrightarrow$ *finite I* $\Longrightarrow$ *sup_measure.F id I* $\leq$ *sup_measure.F id* (*I* $\cup$ *J*)
proof (*induction J rule*: *finite_induct*)
case *empty* **then show** *?case*
by *simp*
next
case (*insert i J*)
show *?case*
proof *cases*
assume *i* $\in$ *I* **with** *insert* **show** *?thesis*
by (*auto simp*: *insert_absorb*)
next
assume *i* $\notin$ *I*
have *sup_measure.F id I* $\leq$ *sup_measure.F id* (*I* $\cup$ *J*)
by (*intro insert*)
also have ... $\leq$ *sup_measure.F id* (*insert i* (*I* $\cup$ *J*))
using *insert* ‹*i* $\notin$ *I*› **by** (*subst sup_measure.insert*) *auto*
finally show *?thesis*
by *auto*
qed
qed

lemma *sup_measure_F_mono*: *finite I* $\Longrightarrow$ *J* $\subseteq$ *I* $\Longrightarrow$ *sup_measure.F id J* $\leq$ *sup_measure.F id I*
using *sup_measure_F_mono'*[*of I J*] **by** (*auto simp*: *finite_subset Un_absorb1*)

lemma *sets_sup_measure_F*:
finite I $\Longrightarrow$ *I* $\neq$ {} $\Longrightarrow$ ($\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ *sets i* = *sets M*) $\Longrightarrow$ *sets* (*sup_measure.F id I*) = *sets M*
by (*induction I rule*: *finite_ne_induct*) (*simp_all add*: *sets_sup*)

definition *Sup_measure'* :: *'a measure set* $\Rightarrow$ *'a measure* **where**
Sup_measure' M =
measure_of ($\bigcup$ *a*$\in$*M*. *space a*) ($\bigcup$ *a*$\in$*M*. *sets a*)
(*λX*. (*SUP P*$\in${*P*. *finite P* $\wedge$ *P* $\subseteq$ *M* }. *sup_measure.F id P X*))

lemma *space_Sup_measure'2*: *space* (*Sup_measure' M*) = ($\bigcup$ *m*$\in$*M*. *space m*)
unfolding *Sup_measure'_def* **by** (*intro space_measure_of*[*OF UN_space_closed*])

lemma *sets_Sup_measure'2*: *sets* (*Sup_measure' M*) = *sigma_sets* ($\bigcup$ *m*$\in$*M*. *space m*) ($\bigcup$ *m*$\in$*M*. *sets m*)
unfolding *Sup_measure'_def* **by** (*intro sets_measure_of*[*OF UN_space_closed*])

lemma *sets_Sup_measure'*:
assumes *sets_eq*[*simp*]: $\bigwedge m.\ m \in M \Longrightarrow sets\ m = sets\ A$ **and** $M \neq \{\}$
shows *sets* (*Sup_measure'* M) = *sets* A
using *sets_eq*[*THEN sets_eq_imp_space_eq*, *simp*] ‹$M \neq \{\}$› **by** (*simp add*: *Sup_measure'_def*)

lemma *space_Sup_measure'*:
assumes *sets_eq*[*simp*]: $\bigwedge m.\ m \in M \Longrightarrow sets\ m = sets\ A$ **and** $M \neq \{\}$
shows *space* (*Sup_measure'* M) = *space* A
using *sets_eq*[*THEN sets_eq_imp_space_eq*, *simp*] ‹$M \neq \{\}$›
by (*simp add*: *Sup_measure'_def*)

lemma *emeasure_Sup_measure'*:
assumes *sets_eq*[*simp*]: $\bigwedge m.\ m \in M \Longrightarrow sets\ m = sets\ A$ **and** $X \in sets\ A\ M \neq \{\}$
shows *emeasure* (*Sup_measure'* M) X = (*SUP* P∈{P. *finite* $P \wedge P \subseteq M$}. *sup_measure.F id* P X)
(**is** _ = *?S* X)
using *Sup_measure'_def*
proof (*rule emeasure_measure_of*)
note *sets_eq*[*THEN sets_eq_imp_space_eq*, *simp*]
have ∗: *sets* (*Sup_measure'* M) = *sets* A *space* (*Sup_measure'* M) = *space* A
using ‹$M \neq \{\}$› **by** (*simp_all add*: *Sup_measure'_def*)
let *?μ* = *sup_measure.F id*
show *countably_additive* (*sets* (*Sup_measure'* M)) *?S*
proof (*rule countably_additiveI*, *goal_cases*)
case (*1 F*)
then have ∗∗: *range* $F \subseteq$ *sets* A
by (*auto simp*: ∗)
show ($\sum i.$ *?S* ($F\ i$)) = *?S* ($\bigcup i.\ F\ i$)
proof (*subst ennreal_suminf_SUP_eq_directed*)
fix $i\ j$ **and** N :: *nat set* **assume** *ij*: $i \in \{P.\ finite\ P \wedge P \subseteq M\}$ $j \in \{P.\ finite\ P \wedge P \subseteq M\}$
have ($i \neq \{\} \longrightarrow$ *sets* (*?μ* i) = *sets* A) $\wedge$ ($j \neq \{\} \longrightarrow$ *sets* (*?μ* j) = *sets* A) $\wedge$
($i \neq \{\} \vee j \neq \{\} \longrightarrow$ *sets* (*?μ* ($i \cup j$)) = *sets* A)
using *ij* **by** (*intro impI sets_sup_measure_F conjI*) *auto*
then have *?μ* j ($F\ n$) ≤ *?μ* ($i \cup j$) ($F\ n$) $\wedge$ *?μ* i ($F\ n$) ≤ *?μ* ($i \cup j$) ($F\ n$)
for n
using *ij*
by (*cases* $i = \{\}$; *cases* $j = \{\}$)
(*auto intro*!: *le_measureD3 sup_measure_F_mono simp*: *sets_sup_measure_F*
simp del: *id_apply*)
with *ij* **show** ∃ k∈{P. *finite* $P \wedge P \subseteq M$}. ∀ n∈N. *?μ* i ($F\ n$) ≤ *?μ* k ($F\ n$) $\wedge$ *?μ* j ($F\ n$) ≤ *?μ* k ($F\ n$)
by (*safe intro*!: *bexI*[*of* _ $i \cup j$]) *auto*
next
show (*SUP* $P \in \{P.\ finite\ P \wedge P \subseteq M\}.\ \sum n.$ *?μ* P ($F\ n$)) = (*SUP* $P \in \{P.$ *finite* $P \wedge P \subseteq M\}.$ *?μ* P ($\bigcup$(F ‘ *UNIV*)))
proof (*intro arg_cong* [*of* _ _ *Sup*] *image_cong refl*)

```
        fix i assume i: i ∈ {P. finite P ∧ P ⊆ M}
        show (∑ n. ?μ i (F n)) = ?μ i (⋃ (F ‘ UNIV))
        proof cases
          assume i ≠ {} with i ** show ?thesis
          by (smt (verit, best) 1(2) Measure_Space.sets_sup_measure_F assms(1)
mem_Collect_eq subset_eq suminf_cong suminf_emeasure)
        qed simp
      qed
    qed
  qed
  show positive (sets (Sup_measure' M)) ?S
    by (auto simp: positive_def bot_ennreal[symmetric])
  show X ∈ sets (Sup_measure' M)
    using assms * by auto
qed (rule UN_space_closed)

definition Sup_measure :: 'a measure set ⇒ 'a measure where
Sup_measure =
  Sup_lexord space
    (Sup_lexord sets Sup_measure'
      (λU. sigma (⋃ u∈U. space u) (⋃ u∈U. sets u)))
    (λU. sigma (⋃ u∈U. space u) {})

definition Inf_measure :: 'a measure set ⇒ 'a measure where
  Inf_measure A = Sup {x. ∀ a∈A. x ≤ a}

definition inf_measure :: 'a measure ⇒ 'a measure ⇒ 'a measure where
  inf_measure a b = Inf {a, b}

definition top_measure :: 'a measure where
  top_measure = Inf {}

instance
proof
  note UN_space_closed [simp]
  show upper: x ≤ Sup A if x: x ∈ A for x :: 'a measure and A
    unfolding Sup_measure_def
  proof (intro Sup_lexord[where P=λy. x ≤ y])
    assume ⋀a. a ∈ A ⟹ space a ≠ (⋃ a∈A. space a)
    from this[OF ‹x ∈ A›] ‹x ∈ A› show x ≤ sigma (⋃ a∈A. space a) {}
      by (intro less_eq_measure.intros) auto
  next
    fix a S assume a ∈ A and a: space a = (⋃ a∈A. space a) and S: S = {a' ∈
A. space a' = space a}
      and neq: ⋀aa. aa ∈ S ⟹ sets aa ≠ (⋃ a∈S. sets a)
    have sp_a: space a = (⋃ (space ‘ S))
      using ‹a∈A› by (auto simp: S)
    show x ≤ sigma (⋃ (space ‘ S)) (⋃ (sets ‘ S))
    proof cases
```

```
    assume [simp]: space x = space a
    have sets x ⊂ (⋃a∈S. sets a)
      using ‹x∈A› neq[of x] by (auto simp: S)
    also have ... ⊆ sigma_sets (⋃x∈S. space x) (⋃x∈S. sets x)
      by (rule sigma_sets_superset_generator)
    finally show ?thesis
      by (intro less_eq_measure.intros(2)) (simp_all add: sp_a)
  next
    assume space x ≠ space a
    moreover have space x ≤ space a
      unfolding a using ‹x∈A› by auto
    ultimately show ?thesis
      by (intro less_eq_measure.intros) (simp add: less_le sp_a)
  qed
next
  fix a b S S' assume a ∈ A and a: space a = (⋃a∈A. space a) and S: S =
{a' ∈ A. space a' = space a}
    and b ∈ S and b: sets b = (⋃a∈S. sets a) and S': S' = {a' ∈ S. sets a' =
sets b}
  then have S' ≠ {} space b = space a
    by auto
  have sets_eq: ⋀x. x ∈ S' ⟹ sets x = sets b
    by (auto simp: S')
  note sets_eq[THEN sets_eq_imp_space_eq, simp]
  have *: sets (Sup_measure' S') = sets b space (Sup_measure' S') = space b
    using ‹S' ≠ {}› by (simp_all add: Sup_measure'_def sets_eq)
  show x ≤ Sup_measure' S'
  proof cases
    assume x ∈ S
    with ‹b ∈ S› have space x = space b
      by (simp add: S)
    show ?thesis
    proof cases
      assume x ∈ S'
      show x ≤ Sup_measure' S'
      proof (intro le_measure[THEN iffD2] ballI)
        show sets x = sets (Sup_measure' S')
          using ‹x∈S'› * by (simp add: S')
        fix X assume X ∈ sets x
        show emeasure x X ≤ emeasure (Sup_measure' S') X
        proof (subst emeasure_Sup_measure'[OF _ ‹X ∈ sets x›])
          show emeasure x X ≤ (SUP P ∈ {P. finite P ∧ P ⊆ S'}. emeasure
(sup_measure.F id P) X)
            using ‹x∈S'› by (intro SUP_upper2[where i={x}]) auto
        qed (insert ‹x∈S'› S', auto)
      qed
    next
      assume x ∉ S'
      then have sets x ≠ sets b
```

```
        using ‹x∈S› by (auto simp: S')
      moreover have sets x ≤ sets b
        using ‹x∈S› unfolding b by auto
      ultimately show ?thesis
        using ∗ ‹x ∈ S›
        by (intro less_eq_measure.intros(2))
          (simp_all add: ∗ ‹space x = space b› less_le)
    qed
  next
    assume x ∉ S
    with ‹x∈A› ‹x ∉ S› ‹space b = space a› show ?thesis
      by (intro less_eq_measure.intros)
        (simp_all add: ∗ less_le a SUP_upper S)
  qed
 qed
 show least: Sup A ≤ x if x: ⋀z. z ∈ A ⟹ z ≤ x for x :: 'a measure and A
  unfolding Sup_measure_def
 proof (intro Sup_lexord[where P=λy. y ≤ x])
  assume ⋀a. a ∈ A ⟹ space a ≠ (⋃ a∈A. space a)
  show sigma (⋃(space ‘ A)) {} ≤ x
    using x[THEN le_measureD1] by (subst sigma_le_iff) auto
 next
  fix a S assume a ∈ A space a = (⋃ a∈A. space a) and S: S = {a' ∈ A. space
a' = space a}
    ⋀a. a ∈ S ⟹ sets a ≠ (⋃ a∈S. sets a)
  have ⋃(space ‘ S) ⊆ space x
    using S le_measureD1[OF x] by auto
  moreover
  have ⋃(space ‘ S) = space a
    using ‹a∈A› S by auto
  then have space x = ⋃(space ‘ S) ⟹ ⋃(sets ‘ S) ⊆ sets x
    using ‹a ∈ A› le_measureD2[OF x] by (auto simp: S)
  ultimately show sigma (⋃(space ‘ S)) (⋃(sets ‘ S)) ≤ x
    by (subst sigma_le_iff) simp_all
 next
  fix a b S S' assume a ∈ A and a: space a = (⋃ a∈A. space a) and S: S =
{a' ∈ A. space a' = space a}
    and b ∈ S and b: sets b = (⋃ a∈S. sets a) and S': S' = {a' ∈ S. sets a' =
sets b}
  then have S' ≠ {} space b = space a
    by auto
  have sets_eq: ⋀x. x ∈ S' ⟹ sets x = sets b
    by (auto simp: S')
  note sets_eq[THEN sets_eq_imp_space_eq, simp]
  have ∗: sets (Sup_measure' S') = sets b space (Sup_measure' S') = space b
    using ‹S' ≠ {}› by (simp_all add: Sup_measure'_def sets_eq)
  show Sup_measure' S' ≤ x
  proof cases
    assume space x = space a
```

```
    show ?thesis
    proof cases
      assume **: sets x = sets b
      show ?thesis
      proof (intro le_measure[THEN iffD2] ballI)
        show ***: sets (Sup_measure' S') = sets x
          by (simp add: * **)
        fix X assume X ∈ sets (Sup_measure' S')
        show emeasure (Sup_measure' S') X ≤ emeasure x X
          unfolding ***
          proof (subst emeasure_Sup_measure'[OF _ ‹X ∈ sets (Sup_measure'
S')›])
           show (SUP P ∈ {P. finite P ∧ P ⊆ S'}. emeasure (sup_measure.F id
P) X) ≤ emeasure x X
          proof (safe intro!: SUP_least)
            fix P assume P: finite P P ⊆ S'
            show emeasure (sup_measure.F id P) X ≤ emeasure x X
            proof cases
              assume P = {} then show ?thesis
                by auto
            next
              assume P ≠ {}
              from P have finite P P ⊆ A
                unfolding S' S by (simp_all add: subset_eq)
              then have sup_measure.F id P ≤ x
                by (induction P) (auto simp: x)
              moreover have sets (sup_measure.F id P) = sets x
                using ‹finite P› ‹P ≠ {}› ‹P ⊆ S'› ‹sets x = sets b›
                by (intro sets_sup_measure_F) (auto simp: S')
             ultimately show emeasure (sup_measure.F id P) X ≤ emeasure x X
                by (rule le_measureD3)
            qed
          qed
          show m ∈ S' ⟹ sets m = sets (Sup_measure' S') for m
            unfolding * by (simp add: S')
        qed fact
      qed
    next
      assume sets x ≠ sets b
      moreover have sets b ≤ sets x
       unfolding b S using x[THEN le_measureD2] ‹space x = space a› by auto
      ultimately show Sup_measure' S' ≤ x
        using ‹space x = space a› ‹b ∈ S›
        by (intro less_eq_measure.intros(2)) (simp_all add: * S)
    qed
  next
    assume space x ≠ space a
    then have space a < space x
      using le_measureD1[OF x[OF ‹a∈A›]] by auto
```

```
    then show Sup_measure' S' ≤ x
      by (intro less_eq_measure.intros) (simp add: * ‹space b = space a›)
  qed
 qed
 show Sup {} = (bot::'a measure) Inf {} = (top::'a measure)
  by (auto intro!: antisym least simp: top_measure_def)
 show lower: x ∈ A ⟹ Inf A ≤ x for x :: 'a measure and A
  unfolding Inf_measure_def by (intro least) auto
 show greatest: (⋀z. z ∈ A ⟹ x ≤ z) ⟹ x ≤ Inf A for x :: 'a measure and A
  unfolding Inf_measure_def by (intro upper) auto
 show inf x y ≤ x inf x y ≤ y x ≤ y ⟹ x ≤ z ⟹ x ≤ inf y z for x y z :: 'a
measure
  by (auto simp: inf_measure_def intro!: lower greatest)
qed

end

lemma sets_SUP:
 assumes ⋀x. x ∈ I ⟹ sets (M x) = sets N
 shows I ≠ {} ⟹ sets (SUP i∈I. M i) = sets N
 unfolding Sup_measure_def
 using assms assms[THEN sets_eq_imp_space_eq]
  sets_Sup_measure'[where A=N and M=M‘I]
 by (intro Sup_lexord1[where P=λx. sets x = sets N]) auto

lemma emeasure_SUP:
 assumes sets: ⋀i. i ∈ I ⟹ sets (M i) = sets N X ∈ sets N I ≠ {}
 shows emeasure (SUP i∈I. M i) X = (SUP J∈{J. J ≠ {} ∧ finite J ∧ J ⊆ I}.
emeasure (SUP i∈J. M i) X)
proof −
 interpret sup_measure: comm_monoid_set sup bot :: 'b measure
  by standard (auto intro!: antisym)
 have eq: finite J ⟹ sup_measure.F id J = (SUP i∈J. i) for J :: 'b measure
set
  by (induction J rule: finite_induct) auto
 have 1: J ≠ {} ⟹ J ⊆ I ⟹ sets (SUP x∈J. M x) = sets N for J
  by (intro sets_SUP sets) (auto )
 from ‹I ≠ {}› obtain i where i∈I by auto
 have Sup_measure' (M‘I) X = (SUP P∈{P. finite P ∧ P ⊆ M‘I}. sup_measure.F
id P X)
  using sets by (intro emeasure_Sup_measure') auto
 also have Sup_measure' (M‘I) = (SUP i∈I. M i)
  unfolding Sup_measure_def using ‹I ≠ {}› sets sets(1)[THEN sets_eq_imp_space_eq]
  by (intro Sup_lexord1[where P=λx. _ = x]) auto
 also have (SUP P∈{P. finite P ∧ P ⊆ M‘I}. sup_measure.F id P X) =
  (SUP J∈{J. J ≠ {} ∧ finite J ∧ J ⊆ I}. (SUP i∈J. M i) X)
 proof (intro SUP_eq)
  fix J assume J ∈ {P. finite P ∧ P ⊆ M‘I}
  then obtain J' where J': J' ⊆ I finite J' and J: J = M‘J' and finite J
```

```
    using finite_subset_image[of J M I] by auto
  show ∃j∈{J. J ≠ {} ∧ finite J ∧ J ⊆ I}. sup_measure.F id J X ≤ (SUP i∈j.
M i) X
    proof cases
      assume J' = {} with ‹i ∈ I› show ?thesis
        by (auto simp add: J)
    next
      assume J' ≠ {} with J J' show ?thesis
        by (intro bexI[of _ J']) (auto simp add: eq simp del: id_apply)
    qed
  next
    fix J assume J: J ∈ {P. P ≠ {} ∧ finite P ∧ P ⊆ I}
    show ∃J'∈{J. finite J ∧ J ⊆ M`I}. (SUP i∈J. M i) X ≤ sup_measure.F id
J' X
      using J by (intro bexI[of _ M`J]) (auto simp add: eq simp del: id_apply)
  qed
  finally show ?thesis .
qed

lemma emeasure_SUP_chain:
  assumes sets: ⋀i. i ∈ A ⟹ sets (M i) = sets N X ∈ sets N
  assumes ch: Complete_Partial_Order.chain (≤) (M ` A) and A ≠ {}
  shows emeasure (SUP i∈A. M i) X = (SUP i∈A. emeasure (M i) X)
proof (subst emeasure_SUP[OF sets ‹A ≠ {}›])
  show (SUP J∈{J. J ≠ {} ∧ finite J ∧ J ⊆ A}. emeasure (Sup (M ` J)) X) =
(SUP i∈A. emeasure (M i) X)
  proof (rule SUP_eq)
    fix J assume J ∈ {J. J ≠ {} ∧ finite J ∧ J ⊆ A}
    then have J: Complete_Partial_Order.chain (≤) (M ` J) finite J J ≠ {} and
J ⊆ A
      using ch[THEN chain_subset, of M`J] by auto
    with in_chain_finite[OF J(1)] obtain j where j ∈ J (SUP j∈J. M j) = M j
      by auto
    with ‹J ⊆ A› show ∃j∈A. emeasure (Sup (M ` J)) X ≤ emeasure (M j) X
      by auto
  next
    fix j assume j∈A then show ∃i∈{J. J ≠ {} ∧ finite J ∧ J ⊆ A}. emeasure
(M j) X ≤ emeasure (Sup (M ` i)) X
      by (intro bexI[of _ {j}]) auto
  qed
qed
```

Supremum of a set of σ-algebras

```
lemma space_Sup_eq_UN: space (Sup M) = (⋃x∈M. space x) (is ?L=?R)
proof
  show ?L ⊆ ?R
    using Sup_lexord[where P=λx. space x = _]
    apply (clarsimp simp: Sup_measure_def)
```

by (*smt* (*verit*) *Sup_lexord_def UN_E mem_Collect_eq space_Sup_measure'2 space_measure_of_conv*)
qed (*use Sup_upper le_measureD1* **in** *fastforce*)

lemma *sets_Sup_eq*:
assumes ∗: $\bigwedge m.\ m \in M \Longrightarrow$ *space* $m = X$ **and** $M \neq \{\}$
shows *sets* (*Sup* M) = *sigma_sets* X ($\bigcup x{\in}M.$ *sets* x)
unfolding *Sup_measure_def*
proof (*rule Sup_lexord1* [*OF* ‹$M \neq \{\}$›])
show *sets* (*Sup_lexord sets Sup_measure'* ($\lambda U.$ *sigma* ($\bigcup$ (*space* ‘ U)) ($\bigcup$ (*sets* ‘ U))) M)
= *sigma_sets* X ($\bigcup$ (*sets* ‘ M))
apply (*rule Sup_lexord*)
apply (*metis* (*mono_tags*, *lifting*) ∗ *empty_iff mem_Collect_eq sets.sigma_sets_eq sets_Sup_measure'*)
by (*metis* ∗ *SUP_eq_const UN_space_closed assms*(2) *sets_measure_of*)
qed (*use* ∗ **in** *blast*)

lemma *in_sets_Sup*: ($\bigwedge m.\ m \in M \Longrightarrow$ *space* $m = X$) $\Longrightarrow$ $m \in M \Longrightarrow$ $A \in$ *sets* $m \Longrightarrow$ $A \in$ *sets* (*Sup* M)
by (*subst sets_Sup_eq*[**where** X=X]) *auto*

lemma *Sup_lexord_rel*:
assumes $\bigwedge i.\ i \in I \Longrightarrow k\ (A\ i) = k\ (B\ i)$
R (*c* (*A* ‘ {$a \in I.\ k\ (B\ a) = (SUP\ x{\in}I.\ k\ (B\ x))$})) (*c* (*B* ‘ {$a \in I.\ k\ (B\ a) = (SUP\ x{\in}I.\ k\ (B\ x))$}))
R (*s* (*A*‘*I*)) (*s* (*B*‘*I*))
shows R (*Sup_lexord k c s* (*A*‘*I*)) (*Sup_lexord k c s* (*B*‘*I*))
proof −
have *A* ‘ {$a \in I.\ k\ (B\ a) = (SUP\ x{\in}I.\ k\ (B\ x))$} = {$a \in A$ ‘ $I.\ k\ a = (SUP\ x{\in}I.\ k\ (B\ x))$}
using *assms*(1) **by** *auto*
moreover have *B* ‘ {$a \in I.\ k\ (B\ a) = (SUP\ x{\in}I.\ k\ (B\ x))$} = {$a \in B$ ‘ $I.\ k\ a = (SUP\ x{\in}I.\ k\ (B\ x))$}
by *auto*
ultimately show *?thesis*
using *assms* **by** (*auto simp*: *Sup_lexord_def Let_def image_comp*)
qed

lemma *sets_SUP_cong*:
assumes *eq*: $\bigwedge i.\ i \in I \Longrightarrow$ *sets* (*M i*) = *sets* (*N i*)
shows *sets* (*SUP* $i{\in}I.$ *M i*) = *sets* (*SUP* $i{\in}I.$ *N i*)
unfolding *Sup_measure_def*
using *eq eq*[*THEN sets_eq_imp_space_eq*]
by (*intro Sup_lexord_rel*[**where** R=$\lambda x\ y.$ *sets* x = *sets* y], *simp_all add*: *sets_Sup_measure'2*)

lemma *sets_Sup_in_sets*:

assumes $M \neq \{\}$
assumes $\bigwedge m.\ m \in M \Longrightarrow space\ m = space\ N$
assumes $\bigwedge m.\ m \in M \Longrightarrow sets\ m \subseteq sets\ N$
shows $sets\ (Sup\ M) \subseteq sets\ N$
proof −
have $*$: $\bigcup (space\ `\ M) = space\ N$
using *assms* **by** *auto*
show *?thesis*
unfolding $*$ **using** *assms* **by** (*subst sets_Sup_eq*[*of M space N*]) (*auto intro*!: *sets.sigma_sets_subset*)
qed

lemma *measurable_Sup1*:
assumes *m*: $m \in M$ **and** *f*: $f \in measurable\ m\ N$
and *const_space*: $\bigwedge m\ n.\ m \in M \Longrightarrow n \in M \Longrightarrow space\ m = space\ n$
shows $f \in measurable\ (Sup\ M)\ N$
proof −
have $space\ (Sup\ M) = space\ m$
using *m* **by** (*auto simp add*: *space_Sup_eq_UN dest*: *const_space*)
then show *?thesis*
using *m f* **unfolding** *measurable_def* **by** (*auto intro*: *in_sets_Sup*[*OF const_space*])
qed

lemma *measurable_Sup2*:
assumes *M*: $M \neq \{\}$
assumes *f*: $\bigwedge m.\ m \in M \Longrightarrow f \in measurable\ N\ m$
and *const_space*: $\bigwedge m\ n.\ m \in M \Longrightarrow n \in M \Longrightarrow space\ m = space\ n$
shows $f \in measurable\ N\ (Sup\ M)$
proof −
from *M* **obtain** *m* **where** $m \in M$ **by** *auto*
have *space_eq*: $\bigwedge n.\ n \in M \Longrightarrow space\ n = space\ m$
by (*intro const_space* ‹$m \in M$›)
have *eq*: $sets\ (sigma\ (\bigcup\ (space\ `\ M))\ (\bigcup\ (sets\ `\ M))) = sets\ (Sup\ M)$
by (*metis M SUP_eq_const UN_space_closed sets_Sup_eq sets_measure_of space_eq*)
have $f \in measurable\ N\ (sigma\ (\bigcup m{\in}M.\ space\ m)\ (\bigcup m{\in}M.\ sets\ m))$
proof (*rule measurable_measure_of*)
show $f \in space\ N \to \bigcup (space\ `\ M)$
using *measurable_space*[*OF f*] *M* **by** *auto*
qed (*auto intro*: *measurable_sets f dest*: *sets.sets_into_space*)
also have $measurable\ N\ (sigma\ (\bigcup m{\in}M.\ space\ m)\ (\bigcup m{\in}M.\ sets\ m)) = measurable\ N\ (Sup\ M)$
using *eq measurable_cong_sets* **by** *blast*
finally show *?thesis* **.**
qed

lemma *measurable_SUP2*:
$I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f \in measurable\ N\ (M\ i)) \Longrightarrow$
$(\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow space\ (M\ i) = space\ (M\ j)) \Longrightarrow f \in measurable\ N$

(*SUP i∈I. M i*)
by (*auto intro*!: *measurable_Sup2*)

lemma *sets_Sup_sigma*:
assumes [*simp*]: $M \neq \{\}$ **and** M: $\bigwedge m.\ m \in M \Longrightarrow m \subseteq Pow\ \Omega$
shows *sets* (*SUP m∈M. sigma Ω m*) = *sets* (*sigma Ω* ($\bigcup M$))
proof −
{ fix *a m* **assume** $a \in$ *sigma_sets Ω m m* $\in M$
then have $a \in$ *sigma_sets Ω* ($\bigcup M$)
by *induction* (*auto intro*: *sigma_sets.intros*(*2*−)) **}**
then have *sigma_sets Ω* ($\bigcup$ (*sigma_sets Ω* ' *M*)) = *sigma_sets Ω* ($\bigcup$ *M*)
by (*smt* (*verit*, *best*) *UN_iff Union_iff sigma_sets.Basic sigma_sets_eqI*)
then show *sets* (*SUP m∈M. sigma Ω m*) = *sets* (*sigma Ω* ($\bigcup M$))
by (*subst sets_Sup_eq*) (*fastforce simp add*: *M Union_least*)+
qed

lemma *Sup_sigma*:
assumes [*simp*]: $M \neq \{\}$ **and** M: $\bigwedge m.\ m \in M \Longrightarrow m \subseteq Pow\ \Omega$
shows (*SUP m∈M. sigma Ω m*) = (*sigma Ω* ($\bigcup M$))
proof (*intro antisym SUP_least*)
have ∗: $\bigcup M \subseteq Pow\ \Omega$
using *M* **by** *auto*
show *sigma Ω* ($\bigcup M$) ≤ (*SUP m∈M. sigma Ω m*)
proof (*intro less_eq_measure.intros*(*3*))
show *space* (*sigma Ω* ($\bigcup M$)) = *space* (*SUP m∈M. sigma Ω m*)
sets (*sigma Ω* ($\bigcup M$)) = *sets* (*SUP m∈M. sigma Ω m*)
by (*auto simp add*: *M sets_Sup_sigma sets_eq_imp_space_eq space_measure_of_conv*)
qed (*simp add*: *emeasure_sigma le_fun_def*)
fix *m* **assume** $m \in M$ **then show** *sigma Ω m* ≤ *sigma Ω* ($\bigcup M$)
by (*subst sigma_le_iff*) (*auto simp add*: *M* ∗)
qed

lemma *SUP_sigma_sigma*:
$M \neq \{\} \Longrightarrow (\bigwedge m.\ m \in M \Longrightarrow f\ m \subseteq Pow\ \Omega) \Longrightarrow$ (*SUP m∈M. sigma Ω* (*f m*))
= *sigma Ω* ($\bigcup m{\in}M.\ f\ m$)
using *Sup_sigma*[*of f*‘*M Ω*] **by** (*auto simp*: *image_comp*)

lemma *sets_vimage_Sup_eq*:
assumes ∗: $M \neq \{\}$ $f \in X \to Y$ $\bigwedge m.\ m \in M \Longrightarrow$ *space m* = *Y*
shows *sets* (*vimage_algebra X f* (*Sup M*)) = *sets* (*SUP m* ∈ *M. vimage_algebra*
X f m)
(**is** *?L* = *?R*)
proof
have $\bigwedge m.\ m \in M \Longrightarrow f \in$ *Sup* (*vimage_algebra X f* ' *M*) $\to_M$ *m*
using *assms*
by (*smt* (*verit*, *del_insts*) *Pi_iff imageE image_eqI measurable_Sup1*
measurable_vimage_algebra1 space_vimage_algebra)
then show *?L* ⊆ *?R*
by (*intro sets_image_in_sets measurable_Sup2*) (*simp_all add*: *space_Sup_eq_UN*

```
*)
  show ?R ⊆ ?L
    apply (intro sets_Sup_in_sets)
     apply (force simp add: * space_Sup_eq_UN sets_vimage_algebra2 intro:
in_sets_Sup)+
    done
qed

lemma restrict_space_eq_vimage_algebra':
  sets (restrict_space M Ω) = sets (vimage_algebra (Ω ∩ space M) (λx. x) M)
proof −
  have *: {A ∩ (Ω ∩ space M) |A. A ∈ sets M} = {A ∩ Ω |A. A ∈ sets M}
    using sets.sets_into_space[of _ M] by blast

  show ?thesis
    unfolding restrict_space_def
    by (subst sets_measure_of)
     (auto simp add: image_subset_iff sets_vimage_algebra * dest: sets.sets_into_space
intro!: arg_cong2[where f=sigma_sets])
qed

lemma sigma_le_sets:
  assumes [simp]: A ⊆ Pow X shows sets (sigma X A) ⊆ sets N ⟷ X ∈ sets
N ∧ A ⊆ sets N
proof
  have X ∈ sigma_sets X A A ⊆ sigma_sets X A
    by (auto intro: sigma_sets_top)
  moreover assume sets (sigma X A) ⊆ sets N
  ultimately show X ∈ sets N ∧ A ⊆ sets N
    by auto
next
  assume *: X ∈ sets N ∧ A ⊆ sets N
  { fix Y assume Y ∈ sigma_sets X A from this * have Y ∈ sets N
      by induction auto }
  then show sets (sigma X A) ⊆ sets N
    by auto
qed

lemma measurable_iff_sets:
  f ∈ measurable M N ⟷ (f ∈ space M → space N ∧ sets (vimage_algebra (space
M) f N) ⊆ sets M)
    unfolding measurable_def
   by (smt (verit, ccfv_threshold) mem_Collect_eq sets_vimage_algebra sigma_sets_le_sets_iff
subset_eq)

lemma sets_vimage_algebra_space: X ∈ sets (vimage_algebra X f M)
  using sets.top[of vimage_algebra X f M] by simp

lemma measurable_mono:
```

assumes N: *sets* $N' \leq$ *sets* N *space* $N =$ *space* N'
assumes M: *sets* $M \leq$ *sets* M' *space* $M =$ *space* M'
shows *measurable* M $N \subseteq$ *measurable* M' N'
unfolding *measurable_def*
proof *safe*
fix f A **assume** $f \in$ *space* $M \to$ *space* N $A \in$ *sets* N'
moreover assume $\forall y{\in}$*sets* N. $f -` y \cap$ *space* $M \in$ *sets* M **note** *this*[*THEN bspec*, *of* A]
ultimately show $f -` A \cap$ *space* $M' \in$ *sets* M'
using *assms* **by** *auto*
qed (*use* N M **in** *auto*)

lemma *measurable_Sup_measurable*:
assumes f: $f \in$ *space* $N \to A$
shows $f \in$ *measurable* N (*Sup* $\{M.$ *space* $M = A \wedge f \in$ *measurable* N $M\}$)
proof (*rule measurable_Sup2*)
show $\{M.$ *space* $M = A \wedge f \in$ *measurable* N $M\} \neq \{\}$
using f **unfolding** *ex_in_conv*[*symmetric*]
by (*intro exI*[*of* _ *sigma* A $\{\}$]) (*auto intro*!: *measurable_measure_of*)
qed *auto*

lemma (**in** *sigma_algebra*) *sigma_sets_subset'*:
assumes a: $a \subseteq M$ $\Omega' \in M$
shows *sigma_sets* Ω' $a \subseteq M$
proof
show $x \in M$ **if** x: $x \in$ *sigma_sets* Ω' a **for** x
using x **by** (*induct rule*: *sigma_sets.induct*) (*use* a **in** *auto*)
qed

lemma *in_sets_SUP*: $i \in I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow$ *space* $(M\ i) = Y) \Longrightarrow X \in$ *sets* $(M\ i) \Longrightarrow X \in$ *sets* (*SUP* $i{\in}I.$ $M\ i$)
by (*intro in_sets_Sup*[**where** $X{=}Y$]) *auto*

lemma *measurable_SUP1*:
$i \in I \Longrightarrow f \in$ *measurable* $(M\ i)$ $N \Longrightarrow (\bigwedge m\ n.\ m \in I \Longrightarrow n \in I \Longrightarrow$ *space* $(M\ m) =$ *space* $(M\ n)) \Longrightarrow$
$f \in$ *measurable* (*SUP* $i{\in}I.$ $M\ i$) N
by (*auto intro*: *measurable_Sup1*)

lemma *sets_image_in_sets'*:
assumes X: $X \in$ *sets* N
assumes f: $\bigwedge A.\ A \in$ *sets* $M \Longrightarrow f -` A \cap X \in$ *sets* N
shows *sets* (*vimage_algebra* X f M) $\subseteq$ *sets* N
unfolding *sets_vimage_algebra*
by (*rule sets.sigma_sets_subset'*) (*auto intro*!: *measurable_sets* X f)

lemma *mono_vimage_algebra*:
sets $M \leq$ *sets* $N \Longrightarrow$ *sets* (*vimage_algebra* X f M) $\subseteq$ *sets* (*vimage_algebra* X f N)

using *sets.top*[*of sigma X* {*f* −‘ *A* ∩ *X* |*A*. *A* ∈ *sets N*}]
unfolding *vimage_algebra_def*
by (*smt* (*verit*, *del_insts*) *space_measure_of sigma_le_sets Pow_iff inf_le2 mem_Collect_eq subset_eq*)

lemma *mono_restrict_space*: *sets M* ≤ *sets N* ⟹ *sets* (*restrict_space M X*) ⊆ *sets* (*restrict_space N X*)
unfolding *sets_restrict_space* **by** (*rule image_mono*)

lemma *sets_eq_bot*: *sets M* = {{}} ⟷ *M* = *bot*
by (*metis measure_eqI emeasure_empty sets_bot singletonD*)

lemma *sets_eq_bot2*: {{}} = *sets M* ⟷ *M* = *bot*
using *sets_eq_bot*[*of M*] **by** *blast*

lemma (**in** *finite_measure*) *countable_support*:
countable {*x*. *measure M* {*x*} ≠ *0*}
proof *cases*
assume *measure M* (*space M*) = *0*
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *bounded_measure measure_le_0_iff Collect_empty_eq countable_empty*)
next
let *?M* = *measure M* (*space M*) **and** *?m* = λ*x*. *measure M* {*x*}
assume *?M* ≠ *0*
then have ∗: {*x*. *?m x* ≠ *0*} = (⋃ *n*. {*x*. *?M* / *Suc n* < *?m x*})
using *reals_Archimedean*[*of ?m x* / *?M* **for** *x*]
by (*auto simp*: *field_simps not_le*[*symmetric*] *divide_le_0_iff measure_le_0_iff*)
have ∗∗: ⋀*n*. *finite* {*x*. *?M* / *Suc n* < *?m x*}
proof (*rule ccontr*)
fix *n* **assume** *infinite* {*x*. *?M* / *Suc n* < *?m x*} (**is** *infinite ?X*)
then obtain *X* **where** *finite X card X* = *Suc* (*Suc n*) *X* ⊆ *?X*
by (*metis infinite_arbitrarily_large*)
then have ∗: ⋀*x*. *x* ∈ *X* ⟹ *?M* / *Suc n* ≤ *?m x*
by *auto*
{ fix *x* **assume** *x* ∈ *X*
from ‹*?M* ≠ *0*› ∗[*OF this*] **have** *?m x* ≠ *0* **by** (*auto simp*: *field_simps measure_le_0_iff*)
then have {*x*} ∈ *sets M* **by** (*auto dest*: *measure_notin_sets*) **}**
note *singleton_sets* = *this*
have *?M* < (∑ *x*∈*X*. *?M* / *Suc n*)
using ‹*?M* ≠ *0*›
by (*simp add*: ‹*card X* = *Suc* (*Suc n*)› *field_simps less_le*)
also have ... ≤ (∑ *x*∈*X*. *?m x*)
by (*rule sum_mono*) *fact*
also have ... = *measure M* (⋃ *x*∈*X*. {*x*})
using *singleton_sets* ‹*finite X*›
by (*intro finite_measure_finite_Union*[*symmetric*]) (*auto simp*: *disjoint_family_on_def*)

finally have *?M < measure M* ($\bigcup x{\in}X.\ \{x\}$) **.**
moreover have *measure M* ($\bigcup x{\in}X.\ \{x\}$) $\leq$ *?M*
using *singleton_sets*[*THEN sets.sets_into_space*] **by** (*intro finite_measure_mono*) *auto*
ultimately show *False* **by** *simp*
qed
show *?thesis*
unfolding $*$ **by** (*intro countable_UN countableI_type countable_finite*[*OF* $**$])
qed

end

7.4 Borel Space

theory *Borel_Space*
imports
Measurable Derivative Ordered_Euclidean_Space Extended_Real_Limits
begin

lemma *is_interval_real_ereal_oo*: *is_interval* (*real_of_ereal* ‘ {*N*<..<*M*::*ereal*})
by (*auto simp*: *real_atLeastGreaterThan_eq*)

lemma *sets_Collect_eventually_sequentially*[*measurable*]:
($\bigwedge i.\ \{x{\in}space\ M.\ P\ x\ i\} \in sets\ M$) $\Longrightarrow$ {*x*∈*space M*. *eventually* (*P x*) *sequentially*} ∈ *sets M*
unfolding *eventually_sequentially* **by** *simp*

lemma *topological_basis_trivial*: *topological_basis* {*A*. *open A*}
by (*auto simp*: *topological_basis_def*)

proposition *open_prod_generated*: *open* = *generate_topology* {*A* × *B* | *A B*. *open A* ∧ *open B*}
proof −
have {*A* × *B* :: (*'a* × *'b*) *set* | *A B*. *open A* ∧ *open B*} = ((λ(*a*, *b*). *a* × *b*) ‘ ({*A*. *open A*} × {*A*. *open A*}))
by *auto*
then show *?thesis*
by (*auto intro*: *topological_basis_prod topological_basis_trivial topological_basis_imp_subbasis*)
qed

proposition *mono_on_imp_deriv_nonneg*:
assumes *mono*: *mono_on A f* **and** *deriv*: (*f has_real_derivative D*) (*at x*)
assumes *x* ∈ *interior A*
shows *D* ≥ *0*
proof (*rule tendsto_lowerbound*)
let *?A'* = (λ*y*. *y* − *x*) ‘ *interior A*
from *deriv* **show** ((λ*h*. (*f* (*x* + *h*) − *f x*) / *h*) ⟶ *D*) (*at 0*)
by (*simp add*: *field_has_derivative_at has_field_derivative_def*)
from *mono* **have** *mono'*: *mono_on* (*interior A*) *f* **by** (*rule mono_on_subset*)

(*rule interior_subset*)

show *eventually* (λ*h*. (*f* (*x* + *h*) − *f x*) / *h* ≥ *0*) (*at 0*)
proof (*subst eventually_at_topological*, *intro exI conjI ballI impI*)
have *open* (*interior A*) **by** *simp*
hence *open* ((+) (−*x*) ‘ *interior A*) **by** (*rule open_translation*)
also have ((+) (−*x*) ‘ *interior A*) = *?A′* **by** *auto*
finally show *open ?A′* **.**
next
from ‹*x* ∈ *interior A*› **show** *0* ∈ *?A′* **by** *auto*
next
fix *h* **assume** *h* ∈ *?A′*
hence *x* + *h* ∈ *interior A* **by** *auto*
with *mono′* **and** ‹*x* ∈ *interior A*› **show** (*f* (*x* + *h*) − *f x*) / *h* ≥ *0*
by (*cases h rule*: *linorder_cases*[*of _ 0*])
(*simp_all add*: *divide_nonpos_neg divide_nonneg_pos mono_onD field_simps*)
qed
qed *simp*

proposition *mono_on_ctble_discont*:
fixes *f* :: *real* ⇒ *real*
fixes *A* :: *real set*
assumes *mono_on A f*
shows *countable* {*a*∈*A*. ¬ *continuous* (*at a within A*) *f*}
proof −
have *mono*: ⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≤ *y* ⟹ *f x* ≤ *f y*
using ‹*mono_on A f*› **by** (*simp add*: *mono_on_def*)
have ∀ *a* ∈ {*a*∈*A*. ¬ *continuous* (*at a within A*) *f*}. ∃ *q* :: *nat* × *rat*.
(*fst q* = *0* ∧ *of_rat* (*snd q*) < *f a* ∧ (∀ *x* ∈ *A*. *x* < *a* ⟶ *f x* < *of_rat* (*snd q*))) ∨
(*fst q* = *1* ∧ *of_rat* (*snd q*) > *f a* ∧ (∀ *x* ∈ *A*. *x* > *a* ⟶ *f x* > *of_rat* (*snd q*)))
proof (*clarsimp simp del*: *One_nat_def*)
fix *a* **assume** *a* ∈ *A* **assume** ¬ *continuous* (*at a within A*) *f*
thus ∃ *q1 q2*.
q1 = *0* ∧ *real_of_rat q2* < *f a* ∧ (∀ *x*∈*A*. *x* < *a* ⟶ *f x* < *real_of_rat q2*) ∨
q1 = *1* ∧ *f a* < *real_of_rat q2* ∧ (∀ *x*∈*A*. *a* < *x* ⟶ *real_of_rat q2* < *f x*)
proof (*auto simp add*: *continuous_within order_tendsto_iff eventually_at*)
fix *l* **assume** *l* < *f a*
then obtain *q2* **where** *q2*: *l* < *of_rat q2 of_rat q2* < *f a*
using *of_rat_dense* **by** *blast*
assume ∗ [*rule_format*]: ∀ *d*>*0*. ∃ *x*∈*A*. *x* ≠ *a* ∧ *dist x a* < *d* ∧ ¬ *l* < *f x*
from *q2* **have** *real_of_rat q2* < *f a* ∧ (∀ *x*∈*A*. *x* < *a* ⟶ *f x* < *real_of_rat q2*)
using *q2* ∗[*of a* − _] *dist_real_def mono* **by** *fastforce*
thus *?thesis* **by** *auto*
next

```
      fix u assume u > f a
      then obtain q2 where q2: f a < of_rat q2 of_rat q2 < u
        using of_rat_dense by blast
      assume *[rule_format]: ∀ d>0. ∃ x∈A. x ≠ a ∧ dist x a < d ∧ ¬ u > f x
      from q2 have real_of_rat q2 > f a ∧ (∀ x∈A. x > a ⟶ f x > real_of_rat
q2)
        using q2 *[of _ − a] dist_real_def mono by fastforce
      thus ?thesis by auto
    qed
  qed
  then obtain g :: real ⇒ nat × rat where ∀ a ∈ {a∈A. ¬ continuous (at a within
A) f}.
      (fst (g a) = 0 ∧ of_rat (snd (g a)) < f a ∧ (∀ x ∈ A. x < a ⟶ f x < of_rat
(snd (g a)))) |
      (fst (g a) = 1 ∧ of_rat (snd (g a)) > f a ∧ (∀ x ∈ A. x > a ⟶ f x > of_rat
(snd (g a))))
    by (rule bchoice [THEN exE]) blast
  hence g: ⋀a x. a ∈ A ⟹ ¬ continuous (at a within A) f ⟹ x ∈ A ⟹
      (fst (g a) = 0 ∧ of_rat (snd (g a)) < f a ∧ (x < a ⟶ f x < of_rat (snd (g
a)))) |
      (fst (g a) = 1 ∧ of_rat (snd (g a)) > f a ∧ (x > a ⟶ f x > of_rat (snd (g
a))))
    by auto
  have inj_on g {a∈A. ¬ continuous (at a within A) f}
  proof (auto simp add: inj_on_def)
    fix w z
    assume 1: w ∈ A and 2: ¬ continuous (at w within A) f and
          3: z ∈ A and 4: ¬ continuous (at z within A) f and
          5: g w = g z
    from g [OF 1 2 3] g [OF 3 4 1] 5
    show w = z by auto
  qed
  thus ?thesis
    by (rule countableI')
qed

lemma mono_on_ctble_discont_open:
  fixes f :: real ⇒ real
  fixes A :: real set
  assumes open A mono_on A f
  shows countable {a∈A. ¬isCont f a}
  using continuous_within_open [OF _ ‹open A›] ‹mono_on A f›
  by (smt (verit, ccfv_threshold) Collect_cong mono_on_ctble_discont)

lemma mono_ctble_discont:
  fixes f :: real ⇒ real
  assumes mono f
  shows countable {a. ¬ isCont f a}
   using assms mono_on_ctble_discont [of UNIV f] unfolding mono_on_def
```

mono_def **by** *auto*

lemma *has_real_derivative_imp_continuous_on*:
assumes $\bigwedge x.\ x \in A \Longrightarrow (f \text{ has\_real\_derivative } f'\ x)\ (at\ x)$
shows *continuous_on A f*
by (*meson DERIV_isCont assms continuous_at_imp_continuous_on*)

lemma *continuous_interval_vimage_Int*:
assumes *continuous_on* $\{a::real..b\}$ *g* **and** *mono*: $\bigwedge x\ y.\ a \le x \Longrightarrow x \le y \Longrightarrow y \le b \Longrightarrow g\ x \le g\ y$
assumes $a \le b$ $(c::real) \le d$ $\{c..d\} \subseteq \{g\ a..g\ b\}$
obtains $c'\ d'$ **where** $\{a..b\} \cap g -` \{c..d\} = \{c'..d'\}$ $c' \le d'$ $g\ c' = c$ $g\ d' = d$
proof−
let *?A* $= \{a..b\} \cap g -` \{c..d\}$
from *IVT'*[*of g a c b*, *OF* _ _ ‹$a \le b$› *assms*(*1*)] *assms*(*4*,*5*)
obtain c'' **where** c'': $c'' \in$ *?A* $g\ c'' = c$ **by** *auto*
from *IVT'*[*of g a d b*, *OF* _ _ ‹$a \le b$› *assms*(*1*)] *assms*(*4*,*5*)
obtain d'' **where** d'': $d'' \in$ *?A* $g\ d'' = d$ **by** *auto*
hence [*simp*]: *?A* $\neq \{\}$ **by** *blast*

define c' **where** $c' =$ *Inf ?A*
define d' **where** $d' =$ *Sup ?A*
have *?A* $\subseteq \{c'..d'\}$ **unfolding** c'*_def* d'*_def*
by (*intro subsetI*) (*auto intro*: *cInf_lower cSup_upper*)
moreover from *assms* **have** *closed ?A*
using *continuous_on_closed_vimage*[*of* $\{a..b\}$ *g*] **by** (*subst Int_commute*) *simp*
hence $c'd'$*_in_set*: $c' \in$ *?A* $d' \in$ *?A* **unfolding** c'*_def* d'*_def*
by ((*intro closed_contains_Inf closed_contains_Sup*, *simp_all*)[])+
hence $\{c'..d'\} \subseteq$ *?A* **using** *assms*
by (*intro subsetI*)
(*auto intro*!: *order_trans*[*of c g c' g x* **for** *x*] *order_trans*[*of g x g d' d* **for** *x*]
intro!: *mono*)
moreover have $c' \le d'$ **using** $c'd'$*_in_set*(*2*) **unfolding** c'*_def* **by** (*intro cInf_lower*) *auto*
moreover have $g\ c' \le c$ $g\ d' \ge d$
using c'' d'' *calculation* **by** (*metis IntE atLeastAtMost_iff mono order_class.order_eq_iff*)+
with $c'd'$*_in_set* **have** $g\ c' = c$ $g\ d' = d$
by *auto*
ultimately show *?thesis*
using *that* **by** *blast*
qed

7.4.1 Generic Borel spaces

definition (**in** *topological_space*) *borel* :: *'a measure* **where**
borel = *sigma UNIV* $\{S.\ open\ S\}$

abbreviation *borel_measurable M* ≡ *measurable M borel*

lemma *in_borel_measurable*:
  $f \in$ *borel_measurable* $M \longleftrightarrow$
   $(\forall S \in$ *sigma_sets UNIV* $\{S.$ *open* $S\}.$ $f -` S \cap$ *space* $M \in$ *sets* $M)$
  by (*auto simp add*: *measurable_def borel_def*)

lemma *in_borel_measurable_borel*:
  $f \in$ *borel_measurable* $M \longleftrightarrow (\forall S \in$ *sets borel*. $f -` S \cap$ *space* $M \in$ *sets* $M)$
  by (*auto simp add*: *measurable_def borel_def*)

lemma *space_borel*[*simp*]: *space borel* = *UNIV*
  unfolding *borel_def* **by** *auto*

lemma *space_in_borel*[*measurable*]: *UNIV* $\in$ *sets borel*
  unfolding *borel_def* **by** *auto*

lemma *sets_borel*: *sets borel* = *sigma_sets UNIV* $\{S.$ *open* $S\}$
  unfolding *borel_def* **by** (*rule sets_measure_of*) *simp*

lemma *measurable_sets_borel*:
  $[\![f \in$ *measurable borel* M; $A \in$ *sets* $M]\!] \Longrightarrow f -` A \in$ *sets borel*
  by (*drule* (*1*) *measurable_sets*) *simp*

lemma *pred_Collect_borel*[*measurable* (*raw*)]: *Measurable.pred borel* $P \Longrightarrow \{x.$ P $x\} \in$ *sets borel*
  unfolding *borel_def pred_def* **by** *auto*

lemma *borel_open*[*measurable* (*raw generic*)]:
  assumes *open* A **shows** $A \in$ *sets borel*
  by (*simp add*: *assms sets_borel*)

lemma *borel_closed*[*measurable* (*raw generic*)]:
  assumes *closed* A **shows** $A \in$ *sets borel*
proof −
  have *space borel* − (− A) $\in$ *sets borel*
    using *assms* **unfolding** *closed_def* **by** (*blast intro*: *borel_open*)
  thus *?thesis* **by** *simp*
qed

lemma *borel_singleton*[*measurable*]:
  $A \in$ *sets borel* $\Longrightarrow$ *insert* x $A \in$ *sets* (*borel* :: *'a*::*t1_space measure*)
  unfolding *insert_def* **by** (*rule sets.Un*) *auto*

lemma *sets_borel_eq_count_space*: *sets* (*borel* :: *'a*::{*countable*, *t2_space*} *measure*) = *count_space UNIV*
  by (*simp add*: *set_eq_iff sets.countable*)

lemma *borel_comp*[*measurable*]: $A \in$ *sets borel* $\Longrightarrow -$ $A \in$ *sets borel*
  unfolding *Compl_eq_Diff_UNIV* **by** *simp*

lemma *borel_measurable_vimage*:
fixes $f :: 'a \Rightarrow 'x{::}t2\_space$
assumes *borel*[*measurable*]: $f \in borel\_measurable\ M$
shows $f -` \{x\} \cap space\ M \in sets\ M$
by *simp*

lemma *borel_measurableI*:
fixes $f :: 'a \Rightarrow 'x{::}topological\_space$
assumes $\bigwedge S.\ open\ S \Longrightarrow f -` S \cap space\ M \in sets\ M$
shows $f \in borel\_measurable\ M$
unfolding *borel_def*
proof (*rule measurable_measure_of*, *simp_all*)
fix $S :: 'x\ set$ **assume** *open S* **thus** $f -` S \cap space\ M \in sets\ M$
using *assms*[*of S*] **by** *simp*
qed

lemma *borel_measurable_const*:
$(\lambda x.\ c) \in borel\_measurable\ M$
by *auto*

lemma *borel_measurable_indicator*:
assumes *A*: $A \in sets\ M$
shows $indicator\ A \in borel\_measurable\ M$
unfolding *indicator_def* [*abs_def*] **using** *A*
by (*auto intro*!: *measurable_If_set*)

lemma *borel_measurable_count_space*[*measurable* (*raw*)]:
$f \in borel\_measurable\ (count\_space\ S)$
unfolding *measurable_def* **by** *auto*

lemma *borel_measurable_indicator'*[*measurable* (*raw*)]:
assumes [*measurable*]: $\{x \in space\ M.\ f\ x \in A\ x\} \in sets\ M$
shows $(\lambda x.\ indicator\ (A\ x)\ (f\ x)) \in borel\_measurable\ M$
unfolding *indicator_def*[*abs_def*]
by (*auto intro*!: *measurable_If*)

lemma *borel_measurable_indicator_iff*:
$(indicator\ A :: 'a \Rightarrow 'x{::}\{t1\_space, zero\_neq\_one\}) \in borel\_measurable\ M \longleftrightarrow A \cap space\ M \in sets\ M$
(**is** *?I* $\in borel\_measurable\ M \longleftrightarrow$ _)
proof
assume *?I* $\in borel\_measurable\ M$
then have *?I* $-` \{1\} \cap space\ M \in sets\ M$
unfolding *measurable_def* **by** *auto*
also have *?I* $-` \{1\} \cap space\ M = A \cap space\ M$
unfolding *indicator_def* [*abs_def*] **by** *auto*
finally show $A \cap space\ M \in sets\ M$ **.**
next
assume $A \cap space\ M \in sets\ M$

moreover have *?I ∈ borel_measurable M ⟷*
(indicator (A ∩ space M) :: 'a ⇒ 'x) ∈ borel_measurable M
by (*intro measurable_cong*) (*auto simp*: *indicator_def*)
ultimately show *?I ∈ borel_measurable M* **by** *auto*
qed

lemma *borel_measurable_subalgebra*:
assumes *sets N ⊆ sets M space N = space M f ∈ borel_measurable N*
shows *f ∈ borel_measurable M*
using *assms* **unfolding** *measurable_def* **by** *auto*

lemma *borel_measurable_restrict_space_iff_ereal*:
fixes *f :: 'a ⇒ ereal*
assumes Ω[*measurable*, *simp*]: *Ω ∩ space M ∈ sets M*
shows *f ∈ borel_measurable (restrict_space M Ω) ⟷*
*(λx. f x * indicator Ω x) ∈ borel_measurable M*
by (*subst measurable_restrict_space_iff*)
(*auto simp*: *indicator_def of_bool_def if_distrib*[**where** *f=λx. a * x* **for** *a*]
cong: *if_cong*)

lemma *borel_measurable_restrict_space_iff_ennreal*:
fixes *f :: 'a ⇒ ennreal*
assumes Ω[*measurable*, *simp*]: *Ω ∩ space M ∈ sets M*
shows *f ∈ borel_measurable (restrict_space M Ω) ⟷*
*(λx. f x * indicator Ω x) ∈ borel_measurable M*
by (*subst measurable_restrict_space_iff*)
(*auto simp*: *indicator_def of_bool_def if_distrib*[**where** *f=λx. a * x* **for** *a*]
cong: *if_cong*)

lemma *borel_measurable_restrict_space_iff*:
fixes *f :: 'a ⇒ 'b::real_normed_vector*
assumes Ω[*measurable*, *simp*]: *Ω ∩ space M ∈ sets M*
shows *f ∈ borel_measurable (restrict_space M Ω) ⟷*
*(λx. indicator Ω x *$_R$ *f x) ∈ borel_measurable M*
by (*subst measurable_restrict_space_iff*)
(*auto simp*: *indicator_def of_bool_def if_distrib*[**where** *f=λx. x *$_R$ *a* **for** *a*]
ac_simps
cong: *if_cong*)

lemma *cbox_borel*[*measurable*]: *cbox a b ∈ sets borel*
by (*auto intro*: *borel_closed*)

lemma *box_borel*[*measurable*]: *box a b ∈ sets borel*
by (*auto intro*: *borel_open*)

lemma *borel_compact*: *compact (A::'a::t2_space set) ⟹ A ∈ sets borel*
by (*simp add*: *borel_closed compact_imp_closed*)

lemma *borel_sigma_sets_subset*:

$A \subseteq$ *sets borel* $\Longrightarrow$ *sigma_sets UNIV* $A \subseteq$ *sets borel*
using *sets.sigma_sets_subset*[*of A borel*] **by** *simp*

lemma *borel_eq_sigmaI1*:
fixes $F :: 'i \Rightarrow 'a$::*topological_space set* **and** $X :: 'a$::*topological_space set set*
assumes *borel_eq*: *borel* = *sigma UNIV X*
assumes *X*: $\bigwedge x.\ x \in X \Longrightarrow x \in$ *sets* (*sigma UNIV* (*F* ‘ *A*))
assumes *F*: $\bigwedge i.\ i \in A \Longrightarrow F\ i \in$ *sets borel*
shows *borel* = *sigma UNIV* (*F* ‘ *A*)
unfolding *borel_def*
proof (*intro sigma_eqI antisym*)
have *borel_rev_eq*: *sigma_sets UNIV* {*S*::*'a set. open S*} = *sets borel*
unfolding *borel_def* **by** *simp*
also have ... = *sigma_sets UNIV X*
unfolding *borel_eq* **by** *simp*
also have ... $\subseteq$ *sigma_sets UNIV* (*F*‘*A*)
using *X* **by** (*intro sigma_algebra.sigma_sets_subset*[*OF sigma_algebra_sigma_sets*])
auto
finally show *sigma_sets UNIV* {*S. open S*} $\subseteq$ *sigma_sets UNIV* (*F*‘*A*) **.**
show *sigma_sets UNIV* (*F*‘*A*) $\subseteq$ *sigma_sets UNIV* {*S. open S*}
by (*metis F image_subset_iff sets_borel sigma_sets_mono*)
qed *auto*

lemma *borel_eq_sigmaI2*:
fixes $F :: 'i \Rightarrow 'j \Rightarrow 'a$::*topological_space set*
and $G :: 'l \Rightarrow 'k \Rightarrow 'a$::*topological_space set*
assumes *borel_eq*: *borel* = *sigma UNIV* ((λ(*i*, *j*). *G i j*)‘*B*)
assumes *X*: $\bigwedge i\ j.\ (i, j) \in B \Longrightarrow G\ i\ j \in$ *sets* (*sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘ *A*))
assumes *F*: $\bigwedge i\ j.\ (i, j) \in A \Longrightarrow F\ i\ j \in$ *sets borel*
shows *borel* = *sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘ *A*)
using *assms*
by (*smt* (*verit*, *del_insts*) *borel_eq_sigmaI1 image_iff prod.collapse split_beta*)

lemma *borel_eq_sigmaI3*:
fixes $F :: 'i \Rightarrow 'j \Rightarrow 'a$::*topological_space set* **and** $X :: 'a$::*topological_space set set*
assumes *borel_eq*: *borel* = *sigma UNIV X*
assumes *X*: $\bigwedge x.\ x \in X \Longrightarrow x \in$ *sets* (*sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘ *A*))
assumes *F*: $\bigwedge i\ j.\ (i, j) \in A \Longrightarrow F\ i\ j \in$ *sets borel*
shows *borel* = *sigma UNIV* ((λ(*i*, *j*). *F i j*) ‘ *A*)
using *assms* **by** (*intro borel_eq_sigmaI1*[**where** *X*=*X* **and** *F*=(λ(*i*, *j*). *F i j*)])
auto

lemma *borel_eq_sigmaI4*:
fixes $F :: 'i \Rightarrow 'a$::*topological_space set*
and $G :: 'l \Rightarrow 'k \Rightarrow 'a$::*topological_space set*
assumes *borel_eq*: *borel* = *sigma UNIV* ((λ(*i*, *j*). *G i j*)‘*A*)
assumes *X*: $\bigwedge i\ j.\ (i, j) \in A \Longrightarrow G\ i\ j \in$ *sets* (*sigma UNIV* (*range F*))
assumes *F*: $\bigwedge i.\ F\ i \in$ *sets borel*

shows *borel = sigma UNIV (range F)*
using *assms* **by** (*intro borel_eq_sigmaI1*[**where** *X*=(λ(*i*, *j*). *G i j*) ‘ *A* **and** *F*=*F*]) *auto*

lemma *borel_eq_sigmaI5*:
fixes *F* :: *'i* ⇒ *'j* ⇒ *'a*::*topological_space set* **and** *G* :: *'l* ⇒ *'a*::*topological_space set*
assumes *borel_eq*: *borel = sigma UNIV (range G)*
assumes *X*: ⋀*i*. *G i* ∈ *sets* (*sigma UNIV* (*range* (λ(*i*, *j*). *F i j*)))
assumes *F*: ⋀*i j*. *F i j* ∈ *sets borel*
shows *borel = sigma UNIV* (*range* (λ(*i*, *j*). *F i j*))
using *assms* **by** (*intro borel_eq_sigmaI1*[**where** *X*=*range G* **and** *F*=(λ(*i*, *j*). *F i j*)]) *auto*

theorem *second_countable_borel_measurable*:
fixes *X* :: *'a*::*second_countable_topology set set*
assumes *eq*: *open = generate_topology X*
shows *borel = sigma UNIV X*
unfolding *borel_def*
proof (*intro sigma_eqI sigma_sets_eqI*)
interpret *X*: *sigma_algebra UNIV sigma_sets UNIV X*
by (*rule sigma_algebra_sigma_sets*) *simp*

fix *S* :: *'a set* **assume** *S* ∈ *Collect open*
then have *generate_topology X S*
by (*auto simp*: *eq*)
then show *S* ∈ *sigma_sets UNIV X*
proof *induction*
case (*UN K*)
then have *K*: ⋀*k*. *k* ∈ *K* ⟹ *open k*
unfolding *eq* **by** *auto*
from *ex_countable_basis* **obtain** *B* :: *'a set set* **where**
B: ⋀*b*. *b* ∈ *B* ⟹ *open b* ⋀*X*. *open X* ⟹ ∃ *b*⊆*B*. (⋃ *b*) = *X* **and** *countable B*
by (*auto simp*: *topological_basis_def*)
from *B*(*2*)[*OF K*] **obtain** *m* **where** *m*: ⋀*k*. *k* ∈ *K* ⟹ *m k* ⊆ *B* ⋀*k*. *k* ∈ *K* ⟹ ⋃ (*m k*) = *k*
by *metis*
define *U* **where** *U* = (⋃ *k*∈*K*. *m k*)
with *m* **have** *countable U*
by (*intro countable_subset*[*OF* _ ‹*countable B*›]) *auto*
have ⋃ *U* = (⋃ *A*∈*U*. *A*) **by** *simp*
also have ... = ⋃ *K*
unfolding *U_def UN_simps* **by** (*simp add*: *m*)
finally have ⋃ *U* = ⋃ *K* **.**

have ∀ *b*∈*U*. ∃ *k*∈*K*. *b* ⊆ *k*
using *m* **by** (*auto simp*: *U_def*)
then obtain *u* **where** *u*: ⋀*b*. *b* ∈ *U* ⟹ *u b* ∈ *K* **and** ⋀*b*. *b* ∈ *U* ⟹ *b* ⊆ *u b*

```
    by metis
  then have (⋃b∈U. u b) ⊆ ⋃K ⋃U ⊆ (⋃b∈U. u b)
    by auto
  then have ⋃K = (⋃b∈U. u b)
    unfolding ‹⋃U = ⋃K› by auto
  also have ... ∈ sigma_sets UNIV X
    using u UN by (intro X.countable_UN' ‹countable U›) auto
  finally show ⋃K ∈ sigma_sets UNIV X .
 qed auto
qed (auto simp: eq intro: generate_topology.Basis)

lemma borel_eq_closed: borel = sigma UNIV (Collect closed)
proof −
 have x ∈ sigma_sets UNIV (Collect closed)
   if open x for x :: 'a set
  by (metis that Compl_eq_Diff_UNIV closed_Compl double_complement mem_Collect_eq

      sigma_sets.Basic sigma_sets.Compl)
 then show ?thesis
  unfolding borel_def
   by (metis Pow_UNIV borel_closed mem_Collect_eq sets_borel sigma_eqI
sigma_sets_eqI top_greatest)
qed

proposition borel_eq_countable_basis:
 fixes B::'a::topological_space set set
 assumes countable B
 assumes topological_basis B
 shows borel = sigma UNIV B
 unfolding borel_def
proof (intro sigma_eqI sigma_sets_eqI, safe)
 interpret countable_basis open B using assms by (rule countable_basis_openI)
 fix X::'a set assume open X
 from open_countable_basisE[OF this] obtain B' where B': B' ⊆ B X = ⋃ B'
.
 then show X ∈ sigma_sets UNIV B
  by (blast intro: sigma_sets_UNION ‹countable B› countable_subset)
next
 fix b assume b ∈ B
 hence open b by (rule topological_basis_open[OF assms(2)])
 thus b ∈ sigma_sets UNIV (Collect open) by auto
qed simp_all

lemma borel_measurable_continuous_on_restrict:
 fixes f :: 'a::topological_space ⇒ 'b::topological_space
 assumes f: continuous_on A f
 shows f ∈ borel_measurable (restrict_space borel A)
proof (rule borel_measurableI)
 fix S :: 'b set assume open S
```

with *f* **obtain** *T* **where** *f* −‘ *S* ∩ *A* = *T* ∩ *A* *open T*
by (*metis continuous_on_open_invariant*)
then show *f* −‘ *S* ∩ *space* (*restrict_space borel A*) ∈ *sets* (*restrict_space borel A*)
by (*force simp add*: *sets_restrict_space space_restrict_space*)
qed

lemma *borel_measurable_continuous_onI*: *continuous_on UNIV f* ⟹ *f* ∈ *borel_measurable borel*
by (*drule borel_measurable_continuous_on_restrict*) *simp*

lemma *borel_measurable_continuous_on_if*:
A ∈ *sets borel* ⟹ *continuous_on A f* ⟹ *continuous_on* (− *A*) *g* ⟹
(λ*x*. *if x* ∈ *A then f x else g x*) ∈ *borel_measurable borel*
by (*auto simp add*: *measurable_If_restrict_space_iff Collect_neg_eq*
intro!: *borel_measurable_continuous_on_restrict*)

lemma *borel_measurable_continuous_countable_exceptions*:
fixes *f* :: *'a*::*t1_space* ⇒ *'b*::*topological_space*
assumes *X*: *countable X*
assumes *continuous_on* (− *X*) *f*
shows *f* ∈ *borel_measurable borel*
proof (*rule measurable_discrete_difference*[*OF* _ *X*])
have *X* ∈ *sets borel*
by (*rule sets.countable*[*OF* _ *X*]) *auto*
then show (λ*x*. *if x* ∈ *X then undefined else f x*) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_on_if assms continuous_intros*)
qed *auto*

lemma *borel_measurable_continuous_on*:
assumes *f*: *continuous_on UNIV f* **and** *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *f* (*g x*)) ∈ *borel_measurable M*
using *measurable_comp*[*OF g borel_measurable_continuous_onI*[*OF f*]] **by** (*simp add*: *comp_def*)

lemma *borel_measurable_continuous_on_indicator*:
fixes *f g* :: *'a*::*topological_space* ⇒ *'b*::*real_normed_vector*
shows *A* ∈ *sets borel* ⟹ *continuous_on A f* ⟹ (λ*x*. *indicator A x* ∗$_R$ *f x*) ∈ *borel_measurable borel*
using *borel_measurable_continuous_on_restrict borel_measurable_restrict_space_iff inf_top.right_neutral* **by** *blast*

lemma *borel_measurable_Pair*[*measurable* (*raw*)]:
fixes *f* :: *'a* ⇒ *'b*::*second_countable_topology* **and** *g* :: *'a* ⇒ *'c*::*second_countable_topology*
assumes *f*[*measurable*]: *f* ∈ *borel_measurable M*
assumes *g*[*measurable*]: *g* ∈ *borel_measurable M*
shows (λ*x*. (*f x*, *g x*)) ∈ *borel_measurable M*
proof (*subst borel_eq_countable_basis*)
let *?B* = *SOME B*::*'b set set*. *countable B* ∧ *topological_basis B*

let *?C* = *SOME B::'c set set. countable B ∧ topological_basis B*
let *?P* = *(λ(b, c). b × c) ' (?B × ?C)*
show *countable ?P topological_basis ?P*
by (*auto intro*!: *countable_basis topological_basis_prod is_basis*)

show *(λx. (f x, g x)) ∈ measurable M (sigma UNIV ?P)*
proof (*rule measurable_measure_of*)
fix *S* **assume** *S ∈ ?P*
then obtain *b c* **where** *b ∈ ?B c ∈ ?C* **and** *S*: *S = b × c* **by** *auto*
then have *borel*: *open b open c*
by (*auto intro*: *is_basis topological_basis_open*)
have *(λx. (f x, g x)) -' S ∩ space M = (f -' b ∩ space M) ∩ (g -' c ∩ space M)*
unfolding *S* **by** *auto*
also have *... ∈ sets M*
using *borel* **by** *simp*
finally show *(λx. (f x, g x)) -' S ∩ space M ∈ sets M* **.**
qed *auto*
qed

lemma *borel_measurable_continuous_Pair*:
fixes *f :: 'a ⇒ 'b::second_countable_topology* **and** *g :: 'a ⇒ 'c::second_countable_topology*
assumes [*measurable*]: *f ∈ borel_measurable M*
assumes [*measurable*]: *g ∈ borel_measurable M*
assumes *H*: *continuous_on UNIV (λx. H (fst x) (snd x))*
shows *(λx. H (f x) (g x)) ∈ borel_measurable M*
proof −
have *eq*: *(λx. H (f x) (g x)) = (λx. (λx. H (fst x) (snd x)) (f x, g x))* **by** *auto*
show *?thesis*
unfolding *eq* **by** (*rule borel_measurable_continuous_on*[*OF H*]) *auto*
qed

7.4.2 Borel spaces on order topologies

lemma [*measurable*]:
fixes *a b :: 'a::linorder_topology*
shows *lessThan_borel*: *{..< a} ∈ sets borel*
and *greaterThan_borel*: *{a <..} ∈ sets borel*
and *greaterThanLessThan_borel*: *{a<..<b} ∈ sets borel*
and *atMost_borel*: *{..a} ∈ sets borel*
and *atLeast_borel*: *{a..} ∈ sets borel*
and *atLeastAtMost_borel*: *{a..b} ∈ sets borel*
and *greaterThanAtMost_borel*: *{a<..b} ∈ sets borel*
and *atLeastLessThan_borel*: *{a..<b} ∈ sets borel*
unfolding *greaterThanAtMost_def atLeastLessThan_def*
by (*blast intro*: *borel_open borel_closed open_lessThan open_greaterThan open_greaterThanLessThan closed_atMost closed_atLeast closed_atLeastAtMost*)+

lemma *borel_Iio*:

```
  borel = sigma UNIV (range lessThan :: 'a::{linorder_topology, second_countable_topology}
set set)
  unfolding second_countable_borel_measurable[OF open_generated_order]
proof (intro sigma_eqI sigma_sets_eqI)
  obtain D :: 'a set where D: countable D ⋀X. open X ⟹ X ≠ {} ⟹ ∃ d∈D.
d ∈ X
    by (rule countable_dense_setE) blast

  interpret L: sigma_algebra UNIV sigma_sets UNIV (range lessThan)
    by (rule sigma_algebra_sigma_sets) simp

  fix A :: 'a set assume A ∈ range lessThan ∪ range greaterThan
  then obtain y where A = {y <..} ∨ A = {..< y}
    by blast
  then show A ∈ sigma_sets UNIV (range lessThan)
  proof
    assume A: A = {y <..}
    show ?thesis
    proof cases
      assume ∀ x>y. ∃ d. y < d ∧ d < x
      with D(2)[of {y <..< x} for x] have ∀ x>y. ∃ d∈D. y < d ∧ d < x
        by (auto simp: set_eq_iff)
      then have A = UNIV - (⋂ d∈{d∈D. y < d}. {..< d})
        by (auto simp: A) (metis less_asym)
      also have ... ∈ sigma_sets UNIV (range lessThan)
        using D(1) by (intro L.Diff L.top L.countable_INT'') auto
      finally show ?thesis .
    next
      assume ¬ (∀ x>y. ∃ d. y < d ∧ d < x)
      then obtain x where y < x  ⋀d. y < d ⟹ ¬ d < x
        by auto
      then have A = UNIV - {..< x}
        unfolding A by (auto simp: not_less[symmetric])
      also have ... ∈ sigma_sets UNIV (range lessThan)
        by auto
      finally show ?thesis .
    qed
  qed auto
qed auto

lemma borel_Ioi:
  borel = sigma UNIV (range greaterThan :: 'a::{linorder_topology, second_countable_topology}
set set)
  unfolding second_countable_borel_measurable[OF open_generated_order]
proof (intro sigma_eqI sigma_sets_eqI)
  obtain D :: 'a set where D: countable D ⋀X. open X ⟹ X ≠ {} ⟹ ∃ d∈D.
d ∈ X
    by (rule countable_dense_setE) blast
```

```
  interpret L: sigma_algebra UNIV sigma_sets UNIV (range greaterThan)
    by (rule sigma_algebra_sigma_sets) simp

  fix A :: 'a set assume A ∈ range lessThan ∪ range greaterThan
  then obtain y where A = {y <..} ∨ A = {..< y}
    by blast
  then show A ∈ sigma_sets UNIV (range greaterThan)
  proof
    assume A: A = {..< y}
    show ?thesis
    proof cases
      assume ∀ x<y. ∃ d. x < d ∧ d < y
      with D(2)[of {x <..< y} for x] have ∀ x<y. ∃ d∈D. x < d ∧ d < y
        by (auto simp: set_eq_iff)
      then have A = UNIV - (⋂ d∈{d∈D. d < y}. {d <..})
        by (auto simp: A) (metis less_asym)
      also have ... ∈ sigma_sets UNIV (range greaterThan)
        using D(1) by (intro L.Diff L.top L.countable_INT'') auto
      finally show ?thesis .
    next
      assume ¬ (∀ x<y. ∃ d. x < d ∧ d < y)
      then obtain x where x < y  ⋀d. y > d ⟹ x ≥ d
        by (auto simp: not_less[symmetric])
      then have A = UNIV - {x <..}
        unfolding A Compl_eq_Diff_UNIV[symmetric] by auto
      also have ... ∈ sigma_sets UNIV (range greaterThan)
        by auto
      finally show ?thesis .
    qed
  qed auto
qed auto

lemma borel_measurableI_less:
  fixes f :: 'a ⇒ 'b::{linorder_topology, second_countable_topology}
  shows (⋀y. {x∈space M. f x < y} ∈ sets M) ⟹ f ∈ borel_measurable M
  unfolding borel_Iio
  by (rule measurable_measure_of) (auto simp: Int_def conj_commute)

lemma borel_measurableI_greater:
  fixes f :: 'a ⇒ 'b::{linorder_topology, second_countable_topology}
  shows (⋀y. {x∈space M. y < f x} ∈ sets M) ⟹ f ∈ borel_measurable M
  unfolding borel_Ioi
  by (rule measurable_measure_of) (auto simp: Int_def conj_commute)

lemma borel_measurableI_le:
  fixes f :: 'a ⇒ 'b::{linorder_topology, second_countable_topology}
  shows (⋀y. {x∈space M. f x ≤ y} ∈ sets M) ⟹ f ∈ borel_measurable M
  by (rule borel_measurableI_greater) (auto simp: not_le[symmetric])
```

lemma *borel_measurableI_ge*:
fixes $f :: 'a \Rightarrow 'b::\{linorder\_topology, second\_countable\_topology\}$
shows $(\bigwedge y.\ \{x \in space\ M.\ y \leq f\ x\} \in sets\ M) \Longrightarrow f \in borel\_measurable\ M$
by (*rule borel_measurableI_less*) (*auto simp*: *not_le*[*symmetric*])

lemma *borel_measurable_less*[*measurable*]:
fixes $f :: 'a \Rightarrow 'b::\{second\_countable\_topology, linorder\_topology\}$
assumes $f \in borel\_measurable\ M$
assumes $g \in borel\_measurable\ M$
shows $\{w \in space\ M.\ f\ w < g\ w\} \in sets\ M$
proof −
have $\{w \in space\ M.\ f\ w < g\ w\} = (\lambda x.\ (f\ x,\ g\ x)) -` \{x.\ fst\ x < snd\ x\} \cap space\ M$
by *auto*
also have $\ldots \in sets\ M$
by (*intro measurable_sets*[*OF borel_measurable_Pair borel_open*, *OF assms open_Collect_less*]
continuous_intros)
finally show *?thesis* **.**
qed

lemma
fixes $f :: 'a \Rightarrow 'b::\{second\_countable\_topology, linorder\_topology\}$
assumes *f*[*measurable*]: $f \in borel\_measurable\ M$
assumes *g*[*measurable*]: $g \in borel\_measurable\ M$
shows *borel_measurable_le*[*measurable*]: $\{w \in space\ M.\ f\ w \leq g\ w\} \in sets\ M$
and *borel_measurable_eq*[*measurable*]: $\{w \in space\ M.\ f\ w = g\ w\} \in sets\ M$
and *borel_measurable_neq*: $\{w \in space\ M.\ f\ w \neq g\ w\} \in sets\ M$
unfolding *eq_iff not_less*[*symmetric*]
by *measurable*

lemma *borel_measurable_SUP*[*measurable* (*raw*)]:
fixes $F :: \_ \Rightarrow \_ \Rightarrow \_::\{complete\_linorder, linorder\_topology, second\_countable\_topology\}$
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in borel\_measurable\ M$
shows $(\lambda x.\ SUP\ i{\in}I.\ F\ i\ x) \in borel\_measurable\ M$
by (*rule borel_measurableI_greater*) (*simp add*: *less_SUP_iff*)

lemma *borel_measurable_INF*[*measurable* (*raw*)]:
fixes $F :: \_ \Rightarrow \_ \Rightarrow \_::\{complete\_linorder, linorder\_topology, second\_countable\_topology\}$
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in borel\_measurable\ M$
shows $(\lambda x.\ INF\ i{\in}I.\ F\ i\ x) \in borel\_measurable\ M$
by (*rule borel_measurableI_less*) (*simp add*: *INF_less_iff*)

lemma *borel_measurable_cSUP*[*measurable* (*raw*)]:
fixes $F :: \_ \Rightarrow \_ \Rightarrow 'a::\{conditionally\_complete\_linorder, linorder\_topology, second\_countable\_topology\}$
assumes [*simp*]: *countable I*

assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in$ *borel_measurable M*
assumes *bdd*: $\bigwedge x.\ x \in$ *space M* $\Longrightarrow$ *bdd_above* $((\lambda i.\ F\ i\ x)\ `\ I)$
shows $(\lambda x.\ SUP\ i{\in}I.\ F\ i\ x) \in$ *borel_measurable M*
proof *cases*
assume $I = \{\}$ **then show** *?thesis*
by (*simp add*: *borel_measurable_const*)
next
assume $I \neq \{\}$
show *?thesis*
proof (*rule borel_measurableI_le*)
fix *y*
have $\{x \in$ *space M*$.\ \forall i{\in}I.\ F\ i\ x \leq y\} \in$ *sets M*
by *measurable*
also have $\{x \in$ *space M*$.\ \forall i{\in}I.\ F\ i\ x \leq y\} = \{x \in$ *space M*$.\ (SUP\ i{\in}I.\ F\ i\ x) \leq y\}$
by (*simp add*: *cSUP_le_iff* ‹$I \neq \{\}$› *bdd cong*: *conj_cong*)
finally show $\{x \in$ *space M*$.\ (SUP\ i{\in}I.\ F\ i\ x) \leq\ y\} \in$ *sets M* **.**
qed
qed

lemma *borel_measurable_cINF*[*measurable* (*raw*)]:
fixes *F* :: _ ⇒ _ ⇒ *'a*::{*conditionally_complete_linorder*, *linorder_topology*, *second_countable_topology*}
assumes [*simp*]: *countable I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in$ *borel_measurable M*
assumes *bdd*: $\bigwedge x.\ x \in$ *space M* $\Longrightarrow$ *bdd_below* $((\lambda i.\ F\ i\ x)\ `\ I)$
shows $(\lambda x.\ INF\ i{\in}I.\ F\ i\ x) \in$ *borel_measurable M*
proof *cases*
assume $I = \{\}$ **then show** *?thesis*
by (*simp add*: *borel_measurable_const*)
next
assume $I \neq \{\}$
show *?thesis*
proof (*rule borel_measurableI_ge*)
fix *y*
have $\{x \in$ *space M*$.\ \forall i{\in}I.\ y \leq F\ i\ x\} \in$ *sets M*
by *measurable*
also have $\{x \in$ *space M*$.\ \forall i{\in}I.\ y \leq F\ i\ x\} = \{x \in$ *space M*$.\ y \leq (INF\ i{\in}I.\ F\ i\ x)\}$
by (*simp add*: *le_cINF_iff* ‹$I \neq \{\}$› *bdd cong*: *conj_cong*)
finally show $\{x \in$ *space M*$.\ y \leq (INF\ i{\in}I.\ F\ i\ x)\} \in$ *sets M* **.**
qed
qed

lemma *borel_measurable_lfp*[*consumes 1*, *case_names continuity step*]:
fixes *F* :: (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'b*::{*complete_linorder*, *linorder_topology*, *second_countable_topology*})
assumes *sup_continuous F*
assumes ∗: $\bigwedge f.\ f \in$ *borel_measurable M* $\Longrightarrow F\ f \in$ *borel_measurable M*

shows *lfp F* ∈ *borel_measurable M*
proof −
{ fix *i* **have** ((*F* ^^ *i*) *bot*) ∈ *borel_measurable M*
by (*induct i*) (*auto intro*!: ∗) **}**
then have (λ*x*. *SUP i*. (*F* ^^ *i*) *bot x*) ∈ *borel_measurable M*
by *measurable*
also have (λ*x*. *SUP i*. (*F* ^^ *i*) *bot x*) = (*SUP i*. (*F* ^^ *i*) *bot*)
by (*auto simp add*: *image_comp*)
also have (*SUP i*. (*F* ^^ *i*) *bot*) = *lfp F*
by (*rule sup_continuous_lfp*[*symmetric*]) *fact*
finally show *?thesis* **.**
qed

lemma *borel_measurable_gfp*[*consumes 1*, *case_names continuity step*]:
fixes *F* :: (′*a* ⇒ ′*b*) ⇒ (′*a* ⇒ ′*b*::{*complete_linorder*, *linorder_topology*, *second_countable_topology*})
assumes *inf_continuous F*
assumes ∗: ⋀*f*. *f* ∈ *borel_measurable M* ⟹ *F f* ∈ *borel_measurable M*
shows *gfp F* ∈ *borel_measurable M*
proof −
{ fix *i* **have** ((*F* ^^ *i*) *top*) ∈ *borel_measurable M*
by (*induct i*) (*auto intro*!: ∗ *simp*: *bot_fun_def*) **}**
then have (λ*x*. *INF i*. (*F* ^^ *i*) *top x*) ∈ *borel_measurable M*
by *measurable*
also have (λ*x*. *INF i*. (*F* ^^ *i*) *top x*) = (*INF i*. (*F* ^^ *i*) *top*)
by (*auto simp add*: *image_comp*)
also have ... = *gfp F*
by (*rule inf_continuous_gfp*[*symmetric*]) *fact*
finally show *?thesis* **.**
qed

lemma *borel_measurable_max*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (λ*x*. *max* (*g x*) (*f x*) :: ′*b*::{*second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
by (*rule borel_measurableI_less*) *simp*

lemma *borel_measurable_min*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (λ*x*. *min* (*g x*) (*f x*) :: ′*b*::{*second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
by (*rule borel_measurableI_greater*) *simp*

lemma *borel_measurable_Min*[*measurable* (*raw*)]:
finite I ⟹ (⋀*i*. *i* ∈ *I* ⟹ *f i* ∈ *borel_measurable M*) ⟹ (λ*x*. *Min* ((λ*i*. *f i x*) ‘*I*) :: ′*b*::{*second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
proof (*induct I rule*: *finite_induct*)
case (*insert i I*) **then show** *?case*
by (*cases I* = {}) *auto*
qed *auto*

lemma *borel_measurable_Max*[*measurable* (*raw*)]:
finite I ⟹ (⋀*i*. *i* ∈ *I* ⟹ *f i* ∈ *borel_measurable M*) ⟹ (λ*x*. *Max* ((λ*i*. *f i x*)‘*I*) :: *'b*::{*second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
proof (*induct I rule*: *finite_induct*)
case (*insert i I*) **then show** *?case*
by (*cases I* = {}) *auto*
qed *auto*

lemma *borel_measurable_sup*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (λ*x*. *sup* (*g x*) (*f x*) :: *'b*::{*lattice*, *second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
unfolding *sup_max* **by** *measurable*

lemma *borel_measurable_inf*[*measurable* (*raw*)]:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (λ*x*. *inf* (*g x*) (*f x*) :: *'b*::{*lattice*, *second_countable_topology*, *linorder_topology*}) ∈ *borel_measurable M*
unfolding *inf_min* **by** *measurable*

lemma [*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ *'a* ⇒ *'b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes ⋀*i*. *f i* ∈ *borel_measurable M*
shows *borel_measurable_liminf*: (λ*x*. *liminf* (λ*i*. *f i x*)) ∈ *borel_measurable M*
and *borel_measurable_limsup*: (λ*x*. *limsup* (λ*i*. *f i x*)) ∈ *borel_measurable M*
unfolding *liminf_SUP_INF limsup_INF_SUP* **using** *assms* **by** *auto*

lemma *measurable_convergent*[*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ *'a* ⇒ *'b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows *Measurable.pred M* (λ*x*. *convergent* (λ*i*. *f i x*))
unfolding *convergent_ereal* **by** *measurable*

lemma *sets_Collect_convergent*[*measurable*]:
fixes *f* :: *nat* ⇒ *'a* ⇒ *'b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes *f*[*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows {*x*∈*space M*. *convergent* (λ*i*. *f i x*)} ∈ *sets M*
by *measurable*

lemma *borel_measurable_lim*[*measurable* (*raw*)]:
fixes *f* :: *nat* ⇒ *'a* ⇒ *'b*::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows (λ*x*. *lim* (λ*i*. *f i x*)) ∈ *borel_measurable M*
proof −
have ⋀*x*. *lim* (λ*i*. *f i x*) = (*if convergent* (λ*i*. *f i x*) *then limsup* (λ*i*. *f i x*) *else* (*THE i*. *False*))
by (*simp add*: *lim_def convergent_def convergent_limsup_cl*)
then show *?thesis*
by *simp*

qed

lemma *borel_measurable_LIMSEQ_order*:
fixes $u :: nat \Rightarrow 'a \Rightarrow 'b$::{*complete_linorder*, *second_countable_topology*, *linorder_topology*}
assumes u': $\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
and u: $\bigwedge i.\ u\ i \in borel\_measurable\ M$
shows $u' \in borel\_measurable\ M$
proof −
have $\bigwedge x.\ x \in space\ M \Longrightarrow u'\ x = liminf\ (\lambda n.\ u\ n\ x)$
using u' **by** (*simp add*: *lim_imp_Liminf*[*symmetric*])
with u **show** *?thesis* **by** (*simp cong*: *measurable_cong*)
qed

7.4.3 Borel spaces on topological monoids

lemma *borel_measurable_add*[*measurable* (*raw*)]:
fixes $f\ g :: 'a \Rightarrow 'b$::{*second_countable_topology*, *topological_monoid_add*}
assumes f: $f \in borel\_measurable\ M$
assumes g: $g \in borel\_measurable\ M$
shows $(\lambda x.\ f\ x + g\ x) \in borel\_measurable\ M$
using *f g* **by** (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

lemma *borel_measurable_sum*[*measurable* (*raw*)]:
fixes $f :: 'c \Rightarrow 'a \Rightarrow 'b$::{*second_countable_topology*, *topological_comm_monoid_add*}
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in borel\_measurable\ M$
shows $(\lambda x.\ \sum i{\in}S.\ f\ i\ x) \in borel\_measurable\ M$
proof *cases*
assume *finite S*
thus *?thesis* **using** *assms* **by** *induct auto*
qed *simp*

lemma *borel_measurable_suminf_order*[*measurable* (*raw*)]:
fixes $f :: nat \Rightarrow 'a \Rightarrow 'b$::{*complete_linorder*, *second_countable_topology*, *linorder_topology*, *topological_comm_monoid_add*}
assumes *f*[*measurable*]: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
shows $(\lambda x.\ suminf\ (\lambda i.\ f\ i\ x)) \in borel\_measurable\ M$
unfolding *suminf_def sums_def*[*abs_def*] *lim_def*[*symmetric*] **by** *simp*

7.4.4 Borel spaces on Euclidean spaces

lemma *borel_measurable_inner*[*measurable* (*raw*)]:
fixes $f\ g :: 'a \Rightarrow 'b$::{*second_countable_topology*, *real_inner*}
assumes $f \in borel\_measurable\ M$
assumes $g \in borel\_measurable\ M$
shows $(\lambda x.\ f\ x \cdot g\ x) \in borel\_measurable\ M$
using *assms*
by (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

notation
eucl_less (**infix** ‹<e› *50*)

lemma *box_oc*: $\{x.\ a <e\ x \wedge x \leq b\} = \{x.\ a <e\ x\} \cap \{..b\}$
and *box_co*: $\{x.\ a \leq x \wedge x <e\ b\} = \{a..\} \cap \{x.\ x <e\ b\}$
by *auto*

lemma *eucl_ivals*[*measurable*]:
fixes $a\ b :: 'a$::*ordered_euclidean_space*
shows $\{x.\ x <e\ a\} \in$ *sets borel*
and $\{x.\ a <e\ x\} \in$ *sets borel*
and $\{..a\} \in$ *sets borel*
and $\{a..\} \in$ *sets borel*
and $\{a..b\} \in$ *sets borel*
and $\{x.\ a <e\ x \wedge x \leq b\} \in$ *sets borel*
and $\{x.\ a \leq x \wedge\ x <e\ b\} \in$ *sets borel*
unfolding *box_oc box_co*
by (*auto intro*: *borel_open borel_closed*)

lemma
fixes $i :: 'a$::{*second_countable_topology*, *real_inner*}
shows *hafspace_less_borel*: $\{x.\ a < x \cdot i\} \in$ *sets borel*
and *hafspace_greater_borel*: $\{x.\ x \cdot i < a\} \in$ *sets borel*
and *hafspace_less_eq_borel*: $\{x.\ a \leq x \cdot i\} \in$ *sets borel*
and *hafspace_greater_eq_borel*: $\{x.\ x \cdot i \leq a\} \in$ *sets borel*
by *simp_all*

lemma *borel_eq_box*:
borel = *sigma UNIV* (*range* (λ (*a*, *b*). *box a b* :: $'a$ :: *euclidean_space set*))
(**is** _ = *?SIGMA*)
proof (*rule borel_eq_sigmaI1*[*OF borel_def*])
fix $M :: 'a$ *set* **assume** $M \in \{S.$ *open* $S\}$
then have *open M* **by** *simp*
show $M \in$ *?SIGMA*
apply (*subst open_UNION_box*[*OF* ‹*open M*›])
apply (*safe intro*!: *sets.countable_UN′ countable_PiE countable_Collect*)
apply (*auto intro*: *countable_rat*)
done
qed (*auto simp*: *box_def*)

lemma *halfspace_gt_in_halfspace*:
assumes *i*: $i \in A$
shows $\{x::'a.\ a < x \cdot i\} \in$
sigma_sets UNIV ((λ (*a*, *i*). $\{x::'a::$*euclidean_space*. $x \cdot i < a\}$) ‘ (*UNIV* × *A*))
(**is** *?set* $\in$ *?SIGMA*)
proof −
interpret *sigma_algebra UNIV ?SIGMA*
by (*intro sigma_algebra_sigma_sets*) *simp_all*
have $*$: *?set* = ($\bigcup n.$ *UNIV* − $\{x::'a.\ x \cdot i < a + 1$ / *real* (*Suc n*)$\}$)
proof (*safe*, *simp_all add*: *not_less del*: *of_nat_Suc*)

```
    fix x :: 'a assume a < x · i
    with reals_Archimedean[of x · i − a]
    obtain n where a + 1 / real (Suc n) < x · i
      by (auto simp: field_simps)
    then show ∃ n. a + 1 / real (Suc n) ≤ x · i
      by (blast intro: less_imp_le)
  next
    fix x n
    have a < a + 1 / real (Suc n) by auto
    also assume ... ≤ x
    finally show a < x .
  qed
  show ?set ∈ ?SIGMA unfolding ∗
    by (auto intro!: Diff sigma_sets_Inter i)
qed

lemma borel_eq_halfspace_less:
  borel = sigma UNIV ((λ(a, i). {x::'a::euclidean_space. x · i < a}) ' (UNIV ×
Basis))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI2[OF borel_eq_box])
  fix a b :: 'a
  have box a b = {x∈space ?SIGMA. ∀ i∈Basis. a · i < x · i ∧ x · i < b · i}
    by (auto simp: box_def)
  also have ... ∈ sets ?SIGMA
   by (intro sets.sets_Collect_conj sets.sets_Collect_finite_All sets.sets_Collect_const)
       (auto intro!: halfspace_gt_in_halfspace countable_PiE countable_rat)
  finally show box a b ∈ sets ?SIGMA .
qed auto

lemma borel_eq_halfspace_le:
  borel = sigma UNIV ((λ (a, i). {x::'a::euclidean_space. x · i ≤ a}) ' (UNIV ×
Basis))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI2[OF borel_eq_halfspace_less])
  fix a :: real and i :: 'a assume (a, i) ∈ UNIV × Basis
  then have i: i ∈ Basis by auto
  have ∗: {x::'a. x·i < a} = (⋃ n. {x. x·i ≤ a − 1/real (Suc n)})
  proof (safe, simp_all del: of_nat_Suc)
    fix x::'a assume ∗: x·i < a
    with reals_Archimedean[of a − x·i]
    obtain n where x · i < a − 1 / (real (Suc n))
      by (auto simp: field_simps)
    then show ∃ n. x · i ≤ a − 1 / (real (Suc n))
      by (blast intro: less_imp_le)
  next
    fix x::'a and n
    assume x·i ≤ a − 1 / real (Suc n)
    also have ... < a by auto
```

```
    finally show x•i < a .
  qed
  show {x. x•i < a} ∈ ?SIGMA unfolding *
    by (intro sets.countable_UN) (auto intro: i)
qed auto

lemma borel_eq_halfspace_ge:
  borel = sigma UNIV ((λ (a, i). {x::'a::euclidean_space. a ≤ x • i}) ` (UNIV ×
Basis))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI2[OF borel_eq_halfspace_less])
  fix a :: real and i :: 'a assume i: (a, i) ∈ UNIV × Basis
  have *: {x::'a. x•i < a} = space ?SIGMA − {x::'a. a ≤ x•i} by auto
  show {x. x•i < a} ∈ ?SIGMA unfolding *
    using i by (intro sets.compl_sets) auto
qed auto

lemma borel_eq_halfspace_greater:
  borel = sigma UNIV ((λ (a, i). {x::'a::euclidean_space. a < x • i}) ` (UNIV ×
Basis))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI2[OF borel_eq_halfspace_le])
  fix a :: real and i :: 'a assume (a, i) ∈ (UNIV × Basis)
  then have i: i ∈ Basis by auto
  have *: {x::'a. x•i ≤ a} = space ?SIGMA − {x::'a. a < x•i} by auto
  show {x. x•i ≤ a} ∈ ?SIGMA unfolding *
    by (intro sets.compl_sets) (auto intro: i)
qed auto

lemma borel_eq_atMost:
  borel = sigma UNIV (range (λa. {..a::'a::ordered_euclidean_space}))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI4[OF borel_eq_halfspace_le])
  fix a :: real and i :: 'a assume (a, i) ∈ UNIV × Basis
  then have i ∈ Basis by auto
  then have *: {x::'a. x•i ≤ a} = (⋃k::nat. {.. (∑ n∈Basis. (if n = i then a else
real k)*_R n)})
  proof (safe, simp_all add: eucl_le[where 'a='a] split: if_split_asm)
    fix x :: 'a
    obtain k where Max ((•) x ` Basis) ≤ real k
      using real_arch_simple by blast
    then have ⋀i. i ∈ Basis ⟹ x•i ≤ real k
      by (subst (asm) Max_le_iff) auto
    then show ∃k::nat. ∀ia∈Basis. ia ≠ i ⟶ x • ia ≤ real k
      by (auto intro!: exI[of _ k])
  qed
  show {x. x•i ≤ a} ∈ ?SIGMA unfolding *
    by (intro sets.countable_UN) auto
qed auto
```

lemma *borel_eq_greaterThan*:
borel = sigma UNIV (range (λa::'a::ordered_euclidean_space. {x. a <e x}))
(**is** *_ = ?SIGMA*)
proof (*rule borel_eq_sigmaI4*[*OF borel_eq_halfspace_le*])
fix *a* :: *real* **and** *i* :: *'a* **assume** (*a*, *i*) ∈ *UNIV* × *Basis*
then have *i*: *i* ∈ *Basis* **by** *auto*
have **: ∃ *y*. ∀ *j*∈*Basis*. *j* ≠ *i* ⟶ − *real y* < *x* · *j* **if** *a* < *x* · *i* **for** *x*
proof −
obtain *k* **where** *k*: *Max* ((·) (− *x*) ' *Basis*) < *real k*
using *reals_Archimedean2* **by** *blast*
{ fix *i* :: *'a* **assume** *i* ∈ *Basis*
then have −*x*•*i* < *real k*
using *k* **by** (*subst* (*asm*) *Max_less_iff*) *auto*
then have − *real k* < *x*•*i* **by** *simp* **}**
then show *?thesis*
by (*auto intro*!: *exI*[*of _ k*])
qed
have {*x*::*'a*. *x*•*i* ≤ *a*} = *UNIV* − {*x*::*'a*. *a* < *x*•*i*} **by** *auto*
also have *: {*x*::*'a*. *a* < *x*•*i*} = (⋃ *k*::*nat*. {*x*. (∑ *n*∈*Basis*. (*if n* = *i then a else* −*k*) *R n*) <*e x*})
using *i* ** **by** (*force simp add*: *eucl_less_def split*: *if_split_asm*)
finally have *eq*: {*x*. *x* · *i* ≤ *a*} = *UNIV* − (⋃ *x*. {*xa*. (∑ *n*∈*Basis*. (*if n* = *i then a else* − *real x*) *R n*) <*e xa*}) **.**
show {*x*. *x*•*i* ≤ *a*} ∈ *?SIGMA*
unfolding *eq* **by** (*fastforce intro*!: *sigma_sets_top sets.Diff*)
qed *auto*

lemma *borel_eq_lessThan*:
borel = sigma UNIV (range (λa::'a::ordered_euclidean_space. {x. x <e a}))
(**is** *_ = ?SIGMA*)
proof (*rule borel_eq_sigmaI4*[*OF borel_eq_halfspace_ge*])
fix *a* :: *real* **and** *i* :: *'a* **assume** (*a*, *i*) ∈ *UNIV* × *Basis*
then have *i*: *i* ∈ *Basis* **by** *auto*
have **: ∃ *y*. ∀ *j*∈*Basis*. *j* ≠ *i* ⟶ *real y* > *x* · *j* **if** *a* > *x* · *i* **for** *x*
proof −
obtain *k* **where** *k*: *Max* ((·) *x* ' *Basis*) < *real k*
using *reals_Archimedean2* **by** *blast*
{ fix *i* :: *'a* **assume** *i* ∈ *Basis*
then have *x*•*i* < *real k*
using *k* **by** (*subst* (*asm*) *Max_less_iff*) *auto*
then have *x*•*i* < *real k* **by** *simp* **}**
then show *?thesis*
by (*auto intro*!: *exI*[*of _ k*])
qed
have {*x*::*'a*. *a* ≤ *x*•*i*} = *UNIV* − {*x*::*'a*. *x*•*i* < *a*} **by** *auto*
also have *: {*x*::*'a*. *x*•*i* < *a*} = (⋃ *k*::*nat*. {*x*. *x* <*e* (∑ *n*∈*Basis*. (*if n* = *i then a else real k*) *R n*)}) **using** ‹*i*∈ *Basis*›
using *i* ** **by** (*force simp add*: *eucl_less_def split*: *if_split_asm*)

```
  finally
  have eq: {x. a ≤ x · i} =
            UNIV − (⋃k. {x. x <e (∑ n∈Basis. (if n = i then a else real k) *R
n)}) .

  show {x. a ≤ x·i} ∈ ?SIGMA
    unfolding eq by (fastforce intro!: sigma_sets_top sets.Diff)
qed auto

lemma borel_eq_atLeastAtMost:
  borel = sigma UNIV (range (λ(a,b). {a..b} ::'a::ordered_euclidean_space set))
  (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI5[OF borel_eq_atMost])
  fix a::'a
  have *: {..a} = (⋃ n::nat. {− real n *R One .. a})
  proof (safe, simp_all add: eucl_le[where 'a='a])
    fix x :: 'a
    obtain k where k: Max ((·) (− x) ` Basis) ≤ real k
      using real_arch_simple by blast
    { fix i :: 'a assume i ∈ Basis
      with k have − x·i ≤ real k
        by (subst (asm) Max_le_iff) (auto simp: field_simps)
      then have − real k ≤ x·i by simp }
    then show ∃ n::nat. ∀ i∈Basis. − real n ≤ x · i
      by (auto intro!: exI[of _ k])
  qed
  show {..a} ∈ ?SIGMA unfolding *
    by (intro sets.countable_UN)
       (auto intro!: sigma_sets_top)
qed auto

lemma borel_set_induct[consumes 1, case_names empty interval compl union]:
  assumes A ∈ sets borel
  assumes empty: P {} and int: ⋀a b. a ≤ b ⟹ P {a..b} and compl: ⋀A. A ∈
sets borel ⟹ P A ⟹ P (−A) and
          un: ⋀f. disjoint_family f ⟹ (⋀i. f i ∈ sets borel) ⟹ (⋀i. P (f i)) ⟹
P (⋃ i::nat. f i)
  shows P (A::real set)
proof −
  let ?G = range (λ(a,b). {a..b::real})
  have Int_stable ?G ?G ⊆ Pow UNIV A ∈ sigma_sets UNIV ?G
     using assms(1) by (auto simp add: borel_eq_atLeastAtMost Int_stable_def)
  thus ?thesis
  proof (induction rule: sigma_sets_induct_disjoint)
    case (union f)
    from union.hyps(2) have ⋀i. f i ∈ sets borel by (auto simp: borel_eq_atLeastAtMost)
      with union show ?case by (auto intro: un)
  next
    case (basic A)
```

```
    then obtain a b where A = {a .. b} by auto
    then show ?case
      by (cases a ≤ b) (auto intro: int empty)
  qed (auto intro: empty compl simp: Compl_eq_Diff_UNIV[symmetric] borel_eq_atLeastAtMost)
qed

lemma borel_sigma_sets_Ioc: borel = sigma UNIV (range (λ(a, b). {a <.. b::real}))
proof (rule borel_eq_sigmaI5[OF borel_eq_atMost])
  fix i :: real
  have {..i} = (⋃j::nat. {-j <.. i})
    by (auto simp: minus_less_iff reals_Archimedean2)
  also have ... ∈ sets (sigma UNIV (range (λ(i, j). {i<..j})))
    by (intro sets.countable_nat_UN) auto
  finally show {..i} ∈ sets (sigma UNIV (range (λ(i, j). {i<..j}))) .
qed simp

lemma eucl_lessThan: {x::real. x <e a} = lessThan a
  by (simp add: eucl_less_def lessThan_def)

lemma borel_eq_atLeastLessThan:
  borel = sigma UNIV (range (λ(a, b). {a ..< b :: real})) (is _ = ?SIGMA)
proof (rule borel_eq_sigmaI5[OF borel_eq_lessThan])
  have move_uminus: ⋀x y::real. -x ≤ y ⟷ -y ≤ x by auto
  fix x :: real
  have {..<x} = (⋃i::nat. {-real i ..< x})
    by (auto simp: move_uminus real_arch_simple)
  then show {y. y <e x} ∈ ?SIGMA
    by (auto intro: sigma_sets.intros(2-) simp: eucl_lessThan)
qed auto

lemma borel_measurable_halfspacesI:
  fixes f :: 'a ⇒ 'c::euclidean_space
  assumes F: borel = sigma UNIV (F ` (UNIV × Basis))
  and S_eq: ⋀a i. S a i = f -` F (a,i) ∩ space M
  shows f ∈ borel_measurable M = (∀ i∈Basis. ∀ a::real. S a i ∈ sets M)
proof safe
  fix a :: real and i :: 'b assume i: i ∈ Basis and f: f ∈ borel_measurable M
  then show S a i ∈ sets M unfolding assms
    by (auto intro!: measurable_sets simp: assms(1))
next
  assume a: ∀ i∈Basis. ∀ a. S a i ∈ sets M
  then show f ∈ borel_measurable M
    by (auto intro!: measurable_measure_of simp: S_eq F)
qed

lemma borel_measurable_iff_halfspace_le:
  fixes f :: 'a ⇒ 'c::euclidean_space
  shows f ∈ borel_measurable M = (∀ i∈Basis. ∀ a. {w ∈ space M. f w • i ≤ a}
∈ sets M)
```

by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_le*]) *auto*

lemma *borel_measurable_iff_halfspace_less*:
fixes $f :: 'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable* $M \longleftrightarrow (\forall i{\in}Basis.\ \forall a.\ \{w \in space\ M.\ f\ w \cdot i < a\} \in sets\ M)$
by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_less*]) *auto*

lemma *borel_measurable_iff_halfspace_ge*:
fixes $f :: 'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable* $M = (\forall i{\in}Basis.\ \forall a.\ \{w \in space\ M.\ a \le f\ w \cdot i\} \in sets\ M)$
by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_ge*]) *auto*

lemma *borel_measurable_iff_halfspace_greater*:
fixes $f :: 'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable* $M \longleftrightarrow (\forall i{\in}Basis.\ \forall a.\ \{w \in space\ M.\ a < f\ w \cdot i\} \in sets\ M)$
by (*rule borel_measurable_halfspacesI*[*OF borel_eq_halfspace_greater*]) *auto*

lemma *borel_measurable_iff_le*:
$(f::'a \Rightarrow real) \in$ *borel_measurable* $M = (\forall a.\ \{w \in space\ M.\ f\ w \le a\} \in sets\ M)$
using *borel_measurable_iff_halfspace_le*[**where** $'c$=*real*] **by** *simp*

lemma *borel_measurable_iff_less*:
$(f::'a \Rightarrow real) \in$ *borel_measurable* $M = (\forall a.\ \{w \in space\ M.\ f\ w < a\} \in sets\ M)$
using *borel_measurable_iff_halfspace_less*[**where** $'c$=*real*] **by** *simp*

lemma *borel_measurable_iff_ge*:
$(f::'a \Rightarrow real) \in$ *borel_measurable* $M = (\forall a.\ \{w \in space\ M.\ a \le f\ w\} \in sets\ M)$
using *borel_measurable_iff_halfspace_ge*[**where** $'c$=*real*]
by *simp*

lemma *borel_measurable_iff_greater*:
$(f::'a \Rightarrow real) \in$ *borel_measurable* $M = (\forall a.\ \{w \in space\ M.\ a < f\ w\} \in sets\ M)$
using *borel_measurable_iff_halfspace_greater*[**where** $'c$=*real*] **by** *simp*

lemma *borel_measurable_euclidean_space*:
fixes $f :: 'a \Rightarrow 'c$::*euclidean_space*
shows $f \in$ *borel_measurable* $M \longleftrightarrow (\forall i{\in}Basis.\ (\lambda x.\ f\ x \cdot i) \in borel\_measurable\ M)$
proof *safe*
assume f: $\forall i{\in}Basis.\ (\lambda x.\ f\ x \cdot i) \in$ *borel_measurable* M
then show $f \in$ *borel_measurable* M
by (*subst borel_measurable_iff_halfspace_le*) *auto*
qed *auto*

7.4.5 Borel measurable operators

lemma *borel_measurable_norm*[*measurable*]: *norm* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_sgn* [*measurable*]: (*sgn*::*'a*::*real_normed_vector* ⇒ *'a*) ∈ *borel_measurable borel*
by (*rule borel_measurable_continuous_countable_exceptions*[**where** *X*={*0*}])
(*auto intro*!: *continuous_on_sgn continuous_on_id*)

lemma *borel_measurable_uminus*[*measurable* (*raw*)]:
fixes *g* :: *'a* ⇒ *'b*::{*second_countable_topology*, *real_normed_vector*}
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. − *g x*) ∈ *borel_measurable M*
by (*rule borel_measurable_continuous_on*[*OF _ g*]) (*intro continuous_intros*)

lemma *borel_measurable_diff*[*measurable* (*raw*)]:
fixes *f* :: *'a* ⇒ *'b*::{*second_countable_topology*, *real_normed_vector*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *f x* − *g x*) ∈ *borel_measurable M*
using *borel_measurable_add* [*of f M* − *g*] *assms* **by** (*simp add*: *fun_Compl_def*)

lemma *borel_measurable_times*[*measurable* (*raw*)]:
fixes *f* :: *'a* ⇒ *'b*::{*second_countable_topology*, *real_normed_algebra*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *f x* ∗ *g x*) ∈ *borel_measurable M*
using *f g* **by** (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

lemma *borel_measurable_prod*[*measurable* (*raw*)]:
fixes *f* :: *'c* ⇒ *'a* ⇒ *'b*::{*second_countable_topology*, *real_normed_field*}
assumes ⋀*i*. *i* ∈ *S* ⟹ *f i* ∈ *borel_measurable M*
shows (λ*x*. ∏ *i*∈*S*. *f i x*) ∈ *borel_measurable M*
proof *cases*
assume *finite S*
thus *?thesis* **using** *assms* **by** *induct auto*
qed *simp*

lemma *borel_measurable_dist*[*measurable* (*raw*)]:
fixes *g f* :: *'a* ⇒ *'b*::{*second_countable_topology*, *metric_space*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*
shows (λ*x*. *dist* (*f x*) (*g x*)) ∈ *borel_measurable M*
using *f g* **by** (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

lemma *borel_measurable_scaleR*[*measurable* (*raw*)]:
fixes *g* :: *'a* ⇒ *'b*::{*second_countable_topology*, *real_normed_vector*}
assumes *f*: *f* ∈ *borel_measurable M*
assumes *g*: *g* ∈ *borel_measurable M*

shows $(\lambda x.\ f\ x *_R g\ x) \in$ *borel_measurable M*
using *f g* **by** (*rule borel_measurable_continuous_Pair*) (*intro continuous_intros*)

lemma *borel_measurable_uminus_eq* [*simp*]:
fixes $f :: 'a \Rightarrow 'b$::{*second_countable_topology*, *real_normed_vector*}
shows $(\lambda x.\ -f\ x) \in$ *borel_measurable M* $\longleftrightarrow$ $f \in$ *borel_measurable M* (**is** *?l* = *?r*)
by (*smt* (*verit*, *ccfv_SIG*) *borel_measurable_uminus equation_minus_iff measurable_cong*)

lemma *affine_borel_measurable_vector*:
fixes $f :: 'a \Rightarrow 'x$::*real_normed_vector*
assumes $f \in$ *borel_measurable M*
shows $(\lambda x.\ a + b *_R f\ x) \in$ *borel_measurable M*
proof (*rule borel_measurableI*)
fix $S :: 'x$ *set* **assume** *open S*
show $(\lambda x.\ a + b *_R f\ x) -` S \cap$ *space M* $\in$ *sets M*
proof *cases*
assume $b \neq 0$
with ‹*open S*› **have** *open* $((\lambda x.\ (-a + x) /_R b) ` S)$ (**is** *open ?S*)
using *open_affinity* [*of S inverse b − a* $/_R$ *b*]
by (*auto simp*: *algebra_simps*)
hence *?S* $\in$ *sets borel* **by** *auto*
moreover
have $\bigwedge x.\ [\![a + b *_R f\ x \in S]\!] \Longrightarrow f\ x \in (\lambda x.\ (x - a) /_R b) ` S$
using ‹$b \neq 0$› *image_iff* **by** *fastforce*
with ‹$b \neq 0$› **have** $(\lambda x.\ a + b *_R f\ x) -` S = f -` ?S$
by *auto*
ultimately show *?thesis* **using** *assms* **unfolding** *in_borel_measurable_borel*
by *auto*
qed *simp*
qed

lemma *borel_measurable_const_scaleR*[*measurable* (*raw*)]:
$f \in$ *borel_measurable M* $\Longrightarrow$ $(\lambda x.\ b *_R f\ x :: 'a$::*real_normed_vector*$) \in$ *borel_measurable M*
using *affine_borel_measurable_vector*[*of f M 0 b*] **by** *simp*

lemma *borel_measurable_const_add*[*measurable* (*raw*)]:
$f \in$ *borel_measurable M* $\Longrightarrow$ $(\lambda x.\ a + f\ x :: 'a$::*real_normed_vector*$) \in$ *borel_measurable M*
using *affine_borel_measurable_vector*[*of f M a 1*] **by** *simp*

lemma *borel_measurable_inverse*[*measurable* (*raw*)]:
fixes $f :: 'a \Rightarrow 'b$::*real_normed_div_algebra*
assumes *f*: $f \in$ *borel_measurable M*
shows $(\lambda x.\ inverse\ (f\ x)) \in$ *borel_measurable M*
proof −
have *countable* $\{0::'b\}$ *continuous_on* $(-\{0::'b\})$ *inverse*

by (*auto intro*!: *continuous_on_inverse continuous_on_id*)
then show *?thesis*
by (*metis borel_measurable_continuous_countable_exceptions f measurable_compose*)
qed

lemma *borel_measurable_divide*[*measurable* (*raw*)]:
$f \in$ *borel_measurable* $M \Longrightarrow g \in$ *borel_measurable* $M \Longrightarrow$
$(\lambda x.\ f\ x\ /\ g\ x::'b::\{second\_countable\_topology,\ real\_normed\_div\_algebra\}) \in$ *borel_measurable* M
by (*simp add*: *divide_inverse*)

lemma *borel_measurable_abs*[*measurable* (*raw*)]:
$f \in$ *borel_measurable* $M \Longrightarrow (\lambda x.\ |f\ x :: real|) \in$ *borel_measurable* M
unfolding *abs_real_def* **by** *simp*

lemma *borel_measurable_nth*[*measurable* (*raw*)]:
$(\lambda x::real^{'}n.\ x\ \$\ i) \in$ *borel_measurable borel*
by (*simp add*: *cart_eq_inner_axis*)

lemma *convex_measurable*:
fixes $A :: 'a :: euclidean\_space\ set$
shows $X \in$ *borel_measurable* $M \Longrightarrow X\ `\ space\ M \subseteq A \Longrightarrow$ *open* $A \Longrightarrow$ *convex_on* $A\ q \Longrightarrow$
$(\lambda x.\ q\ (X\ x)) \in$ *borel_measurable* M
by (*rule measurable_compose*[**where** $f{=}X$ **and** $N{=}$*restrict_space borel* A])
(*auto intro*!: *borel_measurable_continuous_on_restrict convex_on_continuous measurable_restrict_space2*)

lemma *borel_measurable_ln*[*measurable* (*raw*)]:
assumes f: $f \in$ *borel_measurable* M
shows $(\lambda x.\ ln\ (f\ x :: real)) \in$ *borel_measurable* M
using *borel_measurable_continuous_countable_exceptions*[*of* $\{0\}$] *measurable_compose*[*OF* f]
by (*auto intro*!: *continuous_on_ln continuous_on_id*)

lemma *borel_measurable_log*[*measurable* (*raw*)]:
$f \in$ *borel_measurable* $M \Longrightarrow g \in$ *borel_measurable* $M \Longrightarrow (\lambda x.\ log\ (g\ x)\ (f\ x)) \in$ *borel_measurable* M
unfolding *log_def* **by** *auto*

lemma *borel_measurable_exp*[*measurable*]:
$(exp::'a::\{real\_normed\_field,banach\} \Rightarrow 'a) \in$ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_at_imp_continuous_on ballI isCont_exp*)

lemma *measurable_real_floor*[*measurable*]:
$(floor :: real \Rightarrow int) \in$ *measurable borel* (*count_space UNIV*)
proof −
have $\bigwedge a\ x.\ \lfloor x \rfloor = a \longleftrightarrow (real\_of\_int\ a \leq x \wedge x < real\_of\_int\ (a + 1))$

by (*auto intro*: *floor_eq2*)
then show *?thesis*
by (*auto simp*: *vimage_def measurable_count_space_eq2_countable*)
qed

lemma *measurable_real_ceiling*[*measurable*]:
(*ceiling* :: *real* ⇒ *int*) ∈ *measurable borel* (*count_space UNIV*)
unfolding *ceiling_def*[*abs_def*] **by** *simp*

lemma *borel_measurable_real_floor*: (λ*x*::*real*. *real_of_int* ⌊*x*⌋) ∈ *borel_measurable borel*
by *simp*

lemma *borel_measurable_root* [*measurable*]: *root n* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_sqrt* [*measurable*]: *sqrt* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_power* [*measurable* (*raw*)]:
fixes *f* :: _ ⇒ *'b*::{*power*,*real_normed_algebra*}
assumes *f*: *f* ∈ *borel_measurable M*
shows (λ*x*. (*f x*) ^ *n*) ∈ *borel_measurable M*
by (*intro borel_measurable_continuous_on* [*OF* _ *f*] *continuous_intros*)

lemma *borel_measurable_Re* [*measurable*]: *Re* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_Im* [*measurable*]: *Im* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_of_real* [*measurable*]: (*of_real* :: _ ⇒ (_::*real_normed_algebra*)) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_sin* [*measurable*]: (*sin* :: _ ⇒ (_::{*real_normed_field*,*banach*})) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_cos* [*measurable*]: (*cos* :: _ ⇒ (_::{*real_normed_field*,*banach*})) ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_arctan* [*measurable*]: *arctan* ∈ *borel_measurable borel*
by (*intro borel_measurable_continuous_onI continuous_intros*)

lemma *borel_measurable_complex_iff*:
f ∈ *borel_measurable M* ⟷
(λ*x*. *Re* (*f x*)) ∈ *borel_measurable M* ∧ (λ*x*. *Im* (*f x*)) ∈ *borel_measurable M*

(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
using *borel_measurable_Im borel_measurable_Re measurable_compose* **by** *blast*
assume *R*: *?rhs*
then have (λx. *complex_of_real* (*Re* (*f x*)) + i $*$ *complex_of_real* (*Im* (*f x*))) $\in$ *borel_measurable M*
by (*intro borel_measurable_add*) *auto*
then show *?lhs*
using *complex_eq* **by** *force*
qed

lemma *powr_real_measurable* [*measurable*]:
assumes *f* $\in$ *measurable M borel g* $\in$ *measurable M borel*
shows (λx. *f x powr g x* :: *real*) $\in$ *measurable M borel*
using *assms* **by** (*simp_all add*: *powr_def*)

lemma *measurable_of_bool*[*measurable*]: *of_bool* $\in$ *count_space UNIV* $\rightarrow_M$ *borel*
by *simp*

7.4.6 Borel space on the extended reals

lemma *borel_measurable_ereal*[*measurable* (*raw*)]:
assumes *f*: *f* $\in$ *borel_measurable M* **shows** (λx. *ereal* (*f x*)) $\in$ *borel_measurable M*
using *continuous_on_ereal f* **by** (*rule borel_measurable_continuous_on*) (*rule continuous_on_id*)

lemma *borel_measurable_real_of_ereal*[*measurable* (*raw*)]:
fixes *f* :: *'a* $\Rightarrow$ *ereal*
assumes *f*: *f* $\in$ *borel_measurable M*
shows (λx. *real_of_ereal* (*f x*)) $\in$ *borel_measurable M*
using *measurable_compose*[*OF f*] *borel_measurable_continuous_countable_exceptions*[*of* {∞, $-\infty$ }]
by (*auto intro*: *continuous_on_real simp*: *Compl_eq_Diff_UNIV*)

lemma *borel_measurable_ereal_cases*:
fixes *f* :: *'a* $\Rightarrow$ *ereal*
assumes *f*: *f* $\in$ *borel_measurable M*
assumes *H*: (λx. *H* (*ereal* (*real_of_ereal* (*f x*)))) $\in$ *borel_measurable M*
shows (λx. *H* (*f x*)) $\in$ *borel_measurable M*
proof $-$
let *?F* = λx. *if f x* = ∞ *then H* ∞ *else if f x* = $-$ ∞ *then H* ($-\infty$) *else H* (*ereal* (*real_of_ereal* (*f x*)))
{ **fix** *x* **have** *H* (*f x*) = *?F x* **by** (*cases f x*) *auto* }
with *f H* **show** *?thesis* **by** *simp*
qed

lemma
fixes $f :: 'a \Rightarrow ereal$ **assumes** $f[measurable]$: $f \in borel\_measurable\ M$
shows $borel\_measurable\_ereal\_abs[measurable(raw)]$: $(\lambda x.\ |f\ x|) \in borel\_measurable\ M$
and $borel\_measurable\_ereal\_inverse[measurable(raw)]$: $(\lambda x.\ inverse\ (f\ x) :: ereal) \in borel\_measurable\ M$
and $borel\_measurable\_uminus\_ereal[measurable(raw)]$: $(\lambda x.\ -\ f\ x :: ereal) \in borel\_measurable\ M$
by (*auto simp del*: *abs_real_of_ereal simp*: $borel\_measurable\_ereal\_cases[OF\ f]$ *measurable_If*)

lemma $borel\_measurable\_uminus\_eq\_ereal[simp]$:
$(\lambda x.\ -\ f\ x :: ereal) \in borel\_measurable\ M \longleftrightarrow f \in borel\_measurable\ M$
by (*smt* (*verit*, *ccfv_SIG*) *borel_measurable_uminus_ereal ereal_uminus_uminus measurable_cong*)

lemma *set_Collect_ereal2*:
fixes $f\ g :: 'a \Rightarrow ereal$
assumes f: $f \in borel\_measurable\ M$
assumes g: $g \in borel\_measurable\ M$
assumes H: $\{x \in space\ M.\ H\ (ereal\ (real\_of\_ereal\ (f\ x)))\ (ereal\ (real\_of\_ereal\ (g\ x)))\} \in sets\ M$
$\{x \in space\ borel.\ H\ (-\infty)\ (ereal\ x)\} \in sets\ borel$
$\{x \in space\ borel.\ H\ (\infty)\ (ereal\ x)\} \in sets\ borel$
$\{x \in space\ borel.\ H\ (ereal\ x)\ (-\infty)\} \in sets\ borel$
$\{x \in space\ borel.\ H\ (ereal\ x)\ (\infty)\} \in sets\ borel$
shows $\{x \in space\ M.\ H\ (f\ x)\ (g\ x)\} \in sets\ M$
proof −
let $?G = \lambda y\ x.\ if\ g\ x = \infty\ then\ H\ y\ \infty\ else\ if\ g\ x = -\infty\ then\ H\ y\ (-\infty)\ else\ H\ y\ (ereal\ (real\_of\_ereal\ (g\ x)))$
let $?F = \lambda x.\ if\ f\ x = \infty\ then\ ?G\ \infty\ x\ else\ if\ f\ x = -\infty\ then\ ?G\ (-\infty)\ x\ else\ ?G\ (ereal\ (real\_of\_ereal\ (f\ x)))\ x$
{ fix x **have** $H\ (f\ x)\ (g\ x) = ?F\ x$ **by** (*cases f x g x rule*: *ereal2_cases*) *auto* **}**
note $* = this$
from *assms* **show** *?thesis*
by (*subst* $*$) (*simp del*: *space_borel split del*: *if_split*)
qed

lemma *borel_measurable_ereal_iff*:
shows $(\lambda x.\ ereal\ (f\ x)) \in borel\_measurable\ M \longleftrightarrow f \in borel\_measurable\ M$
proof
assume $(\lambda x.\ ereal\ (f\ x)) \in borel\_measurable\ M$
from $borel\_measurable\_real\_of\_ereal[OF\ this]$
show $f \in borel\_measurable\ M$ **by** *auto*
qed *auto*

lemma $borel\_measurable\_erealD[measurable\_dest]$:
$(\lambda x.\ ereal\ (f\ x)) \in borel\_measurable\ M \Longrightarrow g \in measurable\ N\ M \Longrightarrow (\lambda x.\ f\ (g\ x)) \in borel\_measurable\ N$

unfolding *borel_measurable_ereal_iff* **by** *simp*

theorem *borel_measurable_ereal_iff_real*:
fixes $f :: 'a \Rightarrow ereal$
shows $f \in borel\_measurable\ M \longleftrightarrow$
$((\lambda x.\ real\_of\_ereal\ (f\ x)) \in borel\_measurable\ M \wedge f -` \{\infty\} \cap space\ M \in sets\ M \wedge f -` \{-\infty\} \cap space\ M \in sets\ M)$
proof *safe*
assume $*$: $(\lambda x.\ real\_of\_ereal\ (f\ x)) \in borel\_measurable\ M\ f -` \{\infty\} \cap space\ M \in sets\ M\ f -` \{-\infty\} \cap space\ M \in sets\ M$
have $f -` \{\infty\} \cap space\ M = \{x \in space\ M.\ f\ x = \infty\}\ f -` \{-\infty\} \cap space\ M = \{x \in space\ M.\ f\ x = -\infty\}$ **by** *auto*
with $*$ **have** $**$: $\{x \in space\ M.\ f\ x = \infty\} \in sets\ M\ \{x \in space\ M.\ f\ x = -\infty\} \in sets\ M$ **by** *simp_all*
let *?f* $= \lambda x.\ if\ f\ x = \infty\ then\ \infty\ else\ if\ f\ x = -\infty\ then\ -\infty\ else\ ereal\ (real\_of\_ereal\ (f\ x))$
have *?f* $\in borel\_measurable\ M$ **using** $*$ $**$ **by** (*intro measurable_If*) *auto*
also have *?f* $= f$ **by** (*auto simp*: *fun_eq_iff ereal_real*)
finally show $f \in borel\_measurable\ M$ **.**
qed *simp_all*

lemma *borel_measurable_ereal_iff_Iio*:
$(f::'a \Rightarrow ereal) \in borel\_measurable\ M \longleftrightarrow (\forall a.\ f -` \{..< a\} \cap space\ M \in sets\ M)$
by (*auto simp*: *borel_Iio measurable_iff_measure_of*)

lemma *borel_measurable_ereal_iff_Ioi*:
$(f::'a \Rightarrow ereal) \in borel\_measurable\ M \longleftrightarrow (\forall a.\ f -` \{a <..\} \cap space\ M \in sets\ M)$
by (*auto simp*: *borel_Ioi measurable_iff_measure_of*)

lemma *vimage_sets_compl_iff*:
$f -` A \cap space\ M \in sets\ M \longleftrightarrow f -` (-A) \cap space\ M \in sets\ M$
by (*metis Diff_Compl Diff_Diff_Int diff_eq inf_aci(1) sets.Diff sets.top vimage_Compl*)

lemma *borel_measurable_iff_Iic_ereal*:
$(f::'a \Rightarrow ereal) \in borel\_measurable\ M \longleftrightarrow (\forall a.\ f -` \{..a\} \cap space\ M \in sets\ M)$
unfolding *borel_measurable_ereal_iff_Ioi vimage_sets_compl_iff* [**where** $A=\{a <..\}$ **for** a] **by** *simp*

lemma *borel_measurable_iff_Ici_ereal*:
$(f::'a \Rightarrow ereal) \in borel\_measurable\ M \longleftrightarrow (\forall a.\ f -` \{a..\} \cap space\ M \in sets\ M)$
unfolding *borel_measurable_ereal_iff_Iio vimage_sets_compl_iff* [**where** $A=\{..< a\}$ **for** a] **by** *simp*

lemma *borel_measurable_ereal2*:
fixes $f\ g :: 'a \Rightarrow ereal$
assumes f: $f \in borel\_measurable\ M$

assumes *g*: $g \in$ *borel_measurable M*
assumes *H*: $(\lambda x.\ H\ (ereal\ (real\_of\_ereal\ (f\ x)))\ (ereal\ (real\_of\_ereal\ (g\ x))))$
$\in$ *borel_measurable M*
$(\lambda x.\ H\ (-\infty)\ (ereal\ (real\_of\_ereal\ (g\ x)))) \in$ *borel_measurable M*
$(\lambda x.\ H\ (\infty)\ (ereal\ (real\_of\_ereal\ (g\ x)))) \in$ *borel_measurable M*
$(\lambda x.\ H\ (ereal\ (real\_of\_ereal\ (f\ x)))\ (-\infty)) \in$ *borel_measurable M*
$(\lambda x.\ H\ (ereal\ (real\_of\_ereal\ (f\ x)))\ (\infty)) \in$ *borel_measurable M*
shows $(\lambda x.\ H\ (f\ x)\ (g\ x)) \in$ *borel_measurable M*
proof −
let *?G* $= \lambda y\ x.$ *if* $g\ x = \infty$ *then* $H\ y\ \infty$ *else if* $g\ x = -\infty$ *then* $H\ y\ (-\infty)$ *else*
$H\ y\ (ereal\ (real\_of\_ereal\ (g\ x)))$
let *?F* $= \lambda x.$ *if* $f\ x = \infty$ *then* *?G* $\infty\ x$ *else if* $f\ x = -\infty$ *then* *?G* $(-\infty)\ x$ *else*
?G $(ereal\ (real\_of\_ereal\ (f\ x)))\ x$
{ fix *x* **have** $H\ (f\ x)\ (g\ x) =$ *?F* x **by** (*cases f x g x rule*: *ereal2_cases*) *auto* **}**
note $* =$ *this*
from *assms* **show** *?thesis* **unfolding** $*$ **by** *simp*
qed

lemma [*measurable*(*raw*)]:
fixes *f* :: $'a \Rightarrow$ *ereal*
assumes [*measurable*]: $f \in$ *borel_measurable M* $g \in$ *borel_measurable M*
shows *borel_measurable_ereal_add*: $(\lambda x.\ f\ x + g\ x) \in$ *borel_measurable M*
and *borel_measurable_ereal_times*: $(\lambda x.\ f\ x * g\ x) \in$ *borel_measurable M*
by (*simp_all add*: *borel_measurable_ereal2*)

lemma [*measurable*(*raw*)]:
fixes *f g* :: $'a \Rightarrow$ *ereal*
assumes $f \in$ *borel_measurable M*
assumes $g \in$ *borel_measurable M*
shows *borel_measurable_ereal_diff*: $(\lambda x.\ f\ x - g\ x) \in$ *borel_measurable M*
and *borel_measurable_ereal_divide*: $(\lambda x.\ f\ x\ /\ g\ x) \in$ *borel_measurable M*
using *assms* **by** (*simp_all add*: *minus_ereal_def divide_ereal_def*)

lemma *borel_measurable_ereal_sum*[*measurable* (*raw*)]:
fixes *f* :: $'c \Rightarrow 'a \Rightarrow$ *ereal*
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in$ *borel_measurable M*
shows $(\lambda x.\ \sum i{\in}S.\ f\ i\ x) \in$ *borel_measurable M*
using *assms* **by** (*induction S rule*: *infinite_finite_induct*) *auto*

lemma *borel_measurable_ereal_prod*[*measurable* (*raw*)]:
fixes *f* :: $'c \Rightarrow 'a \Rightarrow$ *ereal*
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in$ *borel_measurable M*
shows $(\lambda x.\ \prod i{\in}S.\ f\ i\ x) \in$ *borel_measurable M*
using *assms* **by** (*induction S rule*: *infinite_finite_induct*) *auto*

lemma *borel_measurable_extreal_suminf*[*measurable* (*raw*)]:
fixes *f* :: *nat* $\Rightarrow 'a \Rightarrow$ *ereal*
assumes [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel_measurable M*
shows $(\lambda x.\ (\sum i.\ f\ i\ x)) \in$ *borel_measurable M*

unfolding *suminf_def sums_def*[*abs_def*] *lim_def*[*symmetric*] **by** *simp*

7.4.7 Borel space on the extended non-negative reals

ennreal is a topological monoid, so no rules for plus are required, also all order statements are usually done on type classes.

lemma *measurable_enn2ereal*[*measurable*]: *enn2ereal* ∈ *borel* →$_M$ *borel*
by (*intro borel_measurable_continuous_onI continuous_on_enn2ereal*)

lemma *measurable_e2ennreal*[*measurable*]: *e2ennreal* ∈ *borel* →$_M$ *borel*
by (*intro borel_measurable_continuous_onI continuous_on_e2ennreal*)

lemma *borel_measurable_enn2real*[*measurable* (*raw*)]:
f ∈ *M* →$_M$ *borel* ⟹ (λ*x*. *enn2real* (*f x*)) ∈ *M* →$_M$ *borel*
unfolding *enn2real_def*[*abs_def*] **by** *measurable*

definition [*simp*]: *is_borel f M* ⟷ *f* ∈ *borel_measurable M*

lemma *is_borel_transfer*[*transfer_rule*]: *rel_fun* (*rel_fun* (=) *pcr_ennreal*) (=) *is_borel is_borel*
unfolding *is_borel_def*[*abs_def*]
proof (*safe intro*!: *rel_funI ext dest*!: *rel_fun_eq_pcr_ennreal*[*THEN iffD1*])
fix *f* **and** *M* :: *'a measure*
show *f* ∈ *borel_measurable M* **if** *f*: *enn2ereal* ∘ *f* ∈ *borel_measurable M*
using *measurable_compose*[*OF f measurable_e2ennreal*] **by** *simp*
qed *simp*

context
includes *ennreal.lifting*
begin

lemma *measurable_ennreal*[*measurable*]: *ennreal* ∈ *borel* →$_M$ *borel*
unfolding *is_borel_def*[*symmetric*]
by *transfer simp*

lemma *borel_measurable_ennreal_iff*[*simp*]:
assumes [*simp*]: ⋀*x*. *x* ∈ *space M* ⟹ 0 ≤ *f x*
shows (λ*x*. *ennreal* (*f x*)) ∈ *M* →$_M$ *borel* ⟷ *f* ∈ *M* →$_M$ *borel*
proof *safe*
assume (λ*x*. *ennreal* (*f x*)) ∈ *M* →$_M$ *borel*
then have (λ*x*. *enn2real* (*ennreal* (*f x*))) ∈ *M* →$_M$ *borel*
by *measurable*
then show *f* ∈ *M* →$_M$ *borel*
by (*rule measurable_cong*[*THEN iffD1*, *rotated*]) *auto*
qed *measurable*

lemma *borel_measurable_times_ennreal*[*measurable* (*raw*)]:
fixes *f g* :: *'a* ⇒ *ennreal*
shows *f* ∈ *M* →$_M$ *borel* ⟹ *g* ∈ *M* →$_M$ *borel* ⟹ (λ*x*. *f x* ∗ *g x*) ∈ *M* →$_M$

borel
unfolding *is_borel_def*[*symmetric*] **by** *transfer simp*

lemma *borel_measurable_inverse_ennreal*[*measurable* (*raw*)]:
fixes $f :: 'a \Rightarrow ennreal$
shows $f \in M \rightarrow_M borel \Longrightarrow (\lambda x.\ inverse\ (f\ x)) \in M \rightarrow_M borel$
unfolding *is_borel_def*[*symmetric*] **by** *transfer simp*

lemma *borel_measurable_divide_ennreal*[*measurable* (*raw*)]:
fixes $f :: 'a \Rightarrow ennreal$
shows $f \in M \rightarrow_M borel \Longrightarrow g \in M \rightarrow_M borel \Longrightarrow (\lambda x.\ f\ x\ /\ g\ x) \in M \rightarrow_M$ *borel*
unfolding *divide_ennreal_def* **by** *simp*

lemma *borel_measurable_minus_ennreal*[*measurable* (*raw*)]:
fixes $f :: 'a \Rightarrow ennreal$
shows $f \in M \rightarrow_M borel \Longrightarrow g \in M \rightarrow_M borel \Longrightarrow (\lambda x.\ f\ x - g\ x) \in M \rightarrow_M$ *borel*
unfolding *is_borel_def*[*symmetric*] **by** *transfer simp*

lemma *borel_measurable_power_ennreal* [*measurable* (*raw*)]:
fixes $f :: \_ \Rightarrow ennreal$
assumes *f*: $f \in borel\_measurable\ M$
shows $(\lambda x.\ (f\ x)\ \hat{}\ n) \in borel\_measurable\ M$
by (*induction n*) (*use f* **in** *auto*)

lemma *borel_measurable_prod_ennreal*[*measurable* (*raw*)]:
fixes $f :: 'c \Rightarrow 'a \Rightarrow ennreal$
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in borel\_measurable\ M$
shows $(\lambda x.\ \prod i \in S.\ f\ i\ x) \in borel\_measurable\ M$
using *assms* **by** (*induction S rule*: *infinite_finite_induct*) *auto*

end

hide_const (**open**) *is_borel*

7.4.8 LIMSEQ is borel measurable

lemma *borel_measurable_LIMSEQ_real*:
fixes $u :: nat \Rightarrow 'a \Rightarrow real$
assumes *u'*: $\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
and *u*: $\bigwedge i.\ u\ i \in borel\_measurable\ M$
shows $u' \in borel\_measurable\ M$
proof −
have $\bigwedge x.\ x \in space\ M \Longrightarrow liminf\ (\lambda n.\ ereal\ (u\ n\ x)) = ereal\ (u'\ x)$
using u' **by** (*simp add*: *lim_imp_Liminf*)
moreover from *u* **have** $(\lambda x.\ liminf\ (\lambda n.\ ereal\ (u\ n\ x))) \in borel\_measurable\ M$
by *auto*
ultimately show *?thesis* **by** (*simp cong*: *measurable_cong add*: *borel_measurable_ereal_iff*)

qed

lemma *borel_measurable_LIMSEQ_metric*:
fixes $f :: nat \Rightarrow 'a \Rightarrow 'b :: metric\_space$
assumes [*measurable*]: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
assumes *lim*: $\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ f\ i\ x) \longrightarrow g\ x$
shows $g \in borel\_measurable\ M$
unfolding *borel_eq_closed*
proof (*safe intro*!: *measurable_measure_of*)
fix $A :: 'b\ set$ **assume** *closed A*

have [*measurable*]: $(\lambda x.\ infdist\ (g\ x)\ A) \in borel\_measurable\ M$
proof (*rule borel_measurable_LIMSEQ_real*)
show $\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ infdist\ (f\ i\ x)\ A) \longrightarrow infdist\ (g\ x)\ A$
by (*intro tendsto_infdist lim*)
show $\bigwedge i.\ (\lambda x.\ infdist\ (f\ i\ x)\ A) \in borel\_measurable\ M$
by (*intro borel_measurable_continuous_on*[**where** $f{=}\lambda x.\ infdist\ x\ A$]
continuous_at_imp_continuous_on ballI continuous_infdist continuous_ident)
auto
qed

show $g -` A \cap space\ M \in sets\ M$
proof *cases*
assume $A \neq \{\}$
then have $\bigwedge x.\ infdist\ x\ A = 0 \longleftrightarrow x \in A$
using ‹*closed A*› **by** (*simp add*: *in_closed_iff_infdist_zero*)
then have $g -` A \cap space\ M = \{x{\in}space\ M.\ infdist\ (g\ x)\ A = 0\}$
by *auto*
also have $\ldots \in sets\ M$
by *measurable*
finally show *?thesis* **.**
qed *simp*
qed *auto*

lemma *sets_Collect_Cauchy*[*measurable*]:
fixes $f :: nat \Rightarrow 'a => 'b{::}\{metric\_space, second\_countable\_topology\}$
assumes f[*measurable*]: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
shows $\{x{\in}space\ M.\ Cauchy\ (\lambda i.\ f\ i\ x)\} \in sets\ M$
unfolding *metric_Cauchy_iff2* **using** *f* **by** *auto*

lemma *borel_measurable_lim_metric*[*measurable (raw)*]:
fixes $f :: nat \Rightarrow 'a \Rightarrow 'b{::}\{banach, second\_countable\_topology\}$
assumes f[*measurable*]: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
shows $(\lambda x.\ lim\ (\lambda i.\ f\ i\ x)) \in borel\_measurable\ M$
proof −
define u' **where** $u'\ x = lim\ (\lambda i.\ if\ Cauchy\ (\lambda i.\ f\ i\ x)\ then\ f\ i\ x\ else\ 0)$ **for** x
then have $*$: $\bigwedge x.\ lim\ (\lambda i.\ f\ i\ x) = (if\ Cauchy\ (\lambda i.\ f\ i\ x)\ then\ u'\ x\ else\ (THE\ x.\ False))$
by (*auto simp*: *lim_def convergent_eq_Cauchy*[*symmetric*])

```
  have u' ∈ borel_measurable M
  proof (rule borel_measurable_LIMSEQ_metric)
    fix x
    have convergent (λi. if Cauchy (λi. f i x) then f i x else 0)
      by (cases Cauchy (λi. f i x))
         (auto simp add: convergent_eq_Cauchy[symmetric] convergent_def)
    then show (λi. if Cauchy (λi. f i x) then f i x else 0) ⟶ u' x
      unfolding u'_def
      by (rule convergent_LIMSEQ_iff[THEN iffD1])
  qed measurable
  then show ?thesis
    unfolding * by measurable
qed

lemma borel_measurable_suminf[measurable (raw)]:
  fixes f :: nat ⇒ 'a ⇒ 'b::{banach, second_countable_topology}
  assumes f[measurable]: ⋀i. f i ∈ borel_measurable M
  shows (λx. suminf (λi. f i x)) ∈ borel_measurable M
  unfolding suminf_def sums_def[abs_def] lim_def[symmetric] by simp

lemma Collect_closed_imp_pred_borel: closed {x. P x} ⟹ Measurable.pred borel
P
  by (simp add: pred_def)
```

Proof by Jeremy Avigad and Luke Serafin

```
lemma isCont_borel_pred[measurable]:
  fixes f :: 'b::metric_space ⇒ 'a::metric_space
  shows Measurable.pred borel (isCont f)
proof (subst measurable_cong)
  let ?I = λj. inverse(real (Suc j))
  show isCont f x = (∀ i. ∃ j. ∀ y z. dist x y < ?I j ∧ dist x z < ?I j ⟶ dist (f y)
(f z) ≤ ?I i) for x
    unfolding continuous_at_eps_delta
  proof safe
    fix i assume ∀ e>0. ∃ d>0. ∀ y. dist y x < d ⟶ dist (f y) (f x) < e
    moreover have 0 < ?I i / 2
      by simp
    ultimately obtain d where d: 0 < d ⋀y. dist x y < d ⟹ dist (f y) (f x) <
?I i / 2
      by (metis dist_commute)
    then obtain j where j: ?I j < d
      by (metis reals_Archimedean)

    show ∃ j. ∀ y z. dist x y < ?I j ∧ dist x z < ?I j ⟶ dist (f y) (f z) ≤ ?I i
    proof (safe intro!: exI[where x=j])
      fix y z assume *: dist x y < ?I j dist x z < ?I j
      have dist (f y) (f z) ≤ dist (f y) (f x) + dist (f z) (f x)
        by (rule dist_triangle2)
      also have ... < ?I i / 2 + ?I i / 2
```

```
      by (intro add_strict_mono d less_trans[OF _ j] *)
    also have ... ≤ ?I i
      by (simp add: field_simps)
    finally show dist (f y) (f z) ≤ ?I i
      by simp
  qed
 next
  fix e::real assume 0 < e
  then obtain n where n: ?I n < e
    by (metis reals_Archimedean)
  assume ∀ i. ∃ j. ∀ y z. dist x y < ?I j ∧ dist x z < ?I j ⟶ dist (f y) (f z) ≤
?I i
  from this[THEN spec, of Suc n]
  obtain j where j: ⋀y z. dist x y < ?I j ⟹ dist x z < ?I j ⟹ dist (f y) (f
z) ≤ ?I (Suc n)
    by auto

  show ∃ d>0. ∀ y. dist y x < d ⟶ dist (f y) (f x) < e
  proof (safe intro!: exI[of _ ?I j])
    fix y assume dist y x < ?I j
    then have dist (f y) (f x) ≤ ?I (Suc n)
      by (intro j) (auto simp: dist_commute)
    also have ?I (Suc n) < ?I n
      by simp
    also note n
    finally show dist (f y) (f x) < e .
  qed simp
 qed
qed (intro pred_intros_countable closed_Collect_all closed_Collect_le open_Collect_less
       Collect_closed_imp_pred_borel closed_Collect_imp open_Collect_conj
continuous_intros)

lemma isCont_borel:
 fixes f :: 'b::metric_space ⇒ 'a::metric_space
 shows {x. isCont f x} ∈ sets borel
 by simp

lemma is_real_interval:
 assumes S: is_interval S
 shows ∃ a b::real. S = {} ∨ S = UNIV ∨ S = {..<b} ∨ S = {..b} ∨ S = {a<..}
∨ S = {a..} ∨
   S = {a<..<b} ∨ S = {a<..b} ∨ S = {a..<b} ∨ S = {a..b}
 using S unfolding is_interval_1 by (blast intro: interval_cases)

lemma real_interval_borel_measurable:
 assumes is_interval (S::real set)
 shows S ∈ sets borel
proof −
 from assms is_real_interval have ∃ a b::real. S = {} ∨ S = UNIV ∨ S = {..<b}
```

$\vee$ $S = \{..b\}$ $\vee$
$S = \{a<..\}$ $\vee$ $S = \{a..\}$ $\vee$ $S = \{a<..<b\}$ $\vee$ $S = \{a<..b\}$ $\vee$ $S = \{a..<b\}$ $\vee$ $S = \{a..b\}$ **by** *auto*
then show *?thesis*
by *auto*
qed

The next lemmas hold in any second countable linorder (including ennreal or ereal for instance), but in the current state they are restricted to reals.

lemma *borel_measurable_mono_on_fnc*:
fixes f :: *real* $\Rightarrow$ *real* **and** A :: *real set*
assumes *mono_on A f*
shows $f \in$ *borel_measurable* (*restrict_space borel A*)
proof −
have $\bigwedge x.\ x \in A \Longrightarrow \{x\} \in$ *sets* (*restrict_space borel A*)
using *sets_restrict_space* **by** *fastforce*
moreover
have *continuous_on* ($A \cap -\ \{a \in A.\ \neg$ *continuous* (*at a within A*) $f\}$) f
by (*force simp*: *continuous_on_eq_continuous_within intro*: *continuous_within_subset*)
then have $f \in$ *borel_measurable* (*restrict_space* (*restrict_space borel A*)
($-\ \{a \in A.\ \neg$ *continuous* (*at a within A*) $f\}$))
by (*smt* (*verit*, *best*) *borel_measurable_continuous_on_restrict measurable_cong_sets sets_restrict_restrict_space*)
ultimately show *?thesis*
using *measurable_restrict_countable*[*OF mono_on_ctble_discont*[*OF assms*]]
by (*smt* (*verit*, *del_insts*) *UNIV_I mem_Collect_eq space_borel*)
qed

lemma *borel_measurable_piecewise_mono*:
fixes f::*real* $\Rightarrow$ *real* **and** C::*real set set*
assumes *countable* C $\bigwedge c.\ c \in C \Longrightarrow c \in$ *sets borel* $\bigwedge c.\ c \in C \Longrightarrow$ *mono_on c f* ($\bigcup C$) = *UNIV*
shows $f \in$ *borel_measurable borel*
by (*rule measurable_piecewise_restrict*[*of C*], *auto intro*: *borel_measurable_mono_on_fnc simp*: *assms*)

lemma *borel_measurable_mono*:
fixes f :: *real* $\Rightarrow$ *real*
shows *mono* $f \Longrightarrow f \in$ *borel_measurable borel*
using *borel_measurable_mono_on_fnc*[*of UNIV f*] **by** (*simp add*: *mono_def mono_on_def*)

lemma *measurable_bdd_below_real*[*measurable* (*raw*)]:
fixes F :: $'a \Rightarrow 'i \Rightarrow$ *real*
assumes [*simp*]: *countable* I **and** [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow F\ i \in M \rightarrow_M$ *borel*
shows *Measurable.pred M* ($\lambda x.$ *bdd_below* (($\lambda i.\ F\ i\ x$) ‘I))
proof (*subst measurable_cong*)
show *bdd_below* (($\lambda i.\ F\ i\ x$) ‘I) $\longleftrightarrow$ ($\exists\ q \in \mathbb{Z}.\ \forall\ i \in I.\ q \le F\ i\ x$) **for** x
by (*auto simp*: *bdd_below_def intro*!: *bexI*[*of* _ *of_int* (*floor* _)] *intro*: *or-*

der_trans of_int_floor_le)
show *Measurable.pred M* (λ*w*. ∃ *q*∈ℤ. ∀ *i*∈*I*. *q* ≤ *F i w*)
using *countable_int* **by** *measurable*
qed

lemma *borel_measurable_cINF_real*[*measurable* (*raw*)]:
fixes *F* :: _ ⇒ _ ⇒ *real*
assumes [*simp*]: *countable I*
assumes *F*[*measurable*]: ⋀*i*. *i* ∈ *I* ⟹ *F i* ∈ *borel_measurable M*
shows (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable M*
proof (*rule measurable_piecewise_restrict*)
let *?Ω* = {*x*∈*space M*. *bdd_below* ((λ*i*. *F i x*) ‘*I*)}
show *countable* {*?Ω*, − *?Ω*} *space M* ⊆ ⋃{*?Ω*, − *?Ω*} ⋀*X*. *X* ∈ {*?Ω*, − *?Ω*}
⟹ *X* ∩ *space M* ∈ *sets M*
by *auto*
fix *X* **assume** *X* ∈ {*?Ω*, − *?Ω*} **then show** (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable*
(*restrict_space M X*)
proof *safe*
show (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable* (*restrict_space M ?Ω*)
by (*intro borel_measurable_cINF measurable_restrict_space1 F*)
(*auto simp*: *space_restrict_space*)
show (λ*x*. *INF i*∈*I*. *F i x*) ∈ *borel_measurable* (*restrict_space M* (−*?Ω*))
proof (*subst measurable_cong*)
fix *x* **assume** *x* ∈ *space* (*restrict_space M* (−*?Ω*))
then have ¬ (∀ *i*∈*I*. − *F i x* ≤ *y*) **for** *y*
by (*auto simp*: *space_restrict_space bdd_above_def bdd_above_uminus*[*symmetric*])
then show (*INF i*∈*I*. *F i x*) = − (*THE x*. *False*)
by (*auto simp*: *space_restrict_space Inf_real_def Sup_real_def Least_def*
simp del: *Set.ball_simps*(*10*))
qed *simp*
qed
qed

lemma *borel_Ici*: *borel* = *sigma UNIV* (*range* (λ*x*::*real*. {*x* ..}))
proof (*safe intro*!: *borel_eq_sigmaI1*[*OF borel_Iio*])
fix *x* :: *real*
have *eq*: {..<*x*} = *space* (*sigma UNIV* (*range atLeast*)) − {*x* ..}
by *auto*
show {..<*x*} ∈ *sets* (*sigma UNIV* (*range atLeast*))
unfolding *eq* **by** (*intro sets.compl_sets*) *auto*
qed *auto*

lemma *borel_measurable_pred_less*[*measurable* (*raw*)]:
fixes *f* :: *'a* ⇒ *'b*::{*second_countable_topology*, *linorder_topology*}
shows *f* ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ *Measurable.pred*
M (λ*w*. *f w* < *g w*)
unfolding *Measurable.pred_def* **by** (*rule borel_measurable_less*)

no_notation *eucl_less* (**infix** ‹<*e*› *50*)

lemma *borel_measurable_Max2*[*measurable* (*raw*)]:
fixes *f*::_ ⇒ _ ⇒ 'a::{*second_countable_topology*, *dense_linorder*, *linorder_topology*}
assumes *finite I*
and [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
shows (λ*x*. *Max*{*f i x* |*i*. *i* ∈ *I*}) ∈ *borel_measurable M*
by (*simp add*: *borel_measurable_Max*[*OF assms*(*1*), **where** *?f*=*f* **and** *?M*=*M*] *Setcompr_eq_image*)

lemma *measurable_compose_n* [*measurable* (*raw*)]:
assumes *T* ∈ *measurable M M*
shows (*T*^^*n*) ∈ *measurable M M*
by (*induction n*, *auto simp add*: *measurable_compose*[*OF* _ *assms*])

lemma *measurable_real_imp_nat*:
fixes *f*::'*a* ⇒ *nat*
assumes [*measurable*]: (λ*x*. *real*(*f x*)) ∈ *borel_measurable M*
shows *f* ∈ *measurable M* (*count_space UNIV*)
proof −
let *?g* = (λ*x*. *real*(*f x*))
have ⋀(*n*::*nat*). *?g*−‘({*real n*}) ∩ *space M* = *f*−‘{*n*} ∩ *space M* **by** *auto*
moreover have ⋀(*n*::*nat*). *?g*−‘({*real n*}) ∩ *space M* ∈ *sets M* **using** *assms* **by** *measurable*
ultimately have ⋀(*n*::*nat*). *f*−‘{*n*} ∩ *space M* ∈ *sets M* **by** *simp*
then show *?thesis* **using** *measurable_count_space_eq2_countable* **by** *blast*
qed

lemma *measurable_equality_set* [*measurable*]:
fixes *f g*::_⇒ 'a::{*second_countable_topology*, *t2_space*}
assumes [*measurable*]: *f* ∈ *borel_measurable M g* ∈ *borel_measurable M*
shows {*x* ∈ *space M*. *f x* = *g x*} ∈ *sets M*
proof −
define *A* **where** *A* = {*x* ∈ *space M*. *f x* = *g x*}
define *B* **where** *B* = {*y*. ∃ *x*::'*a*. *y* = (*x*,*x*)}
have *A* = (λ*x*. (*f x*, *g x*))−‘*B* ∩ *space M* **unfolding** *A_def B_def* **by** *auto*
moreover have (λ*x*. (*f x*, *g x*)) ∈ *borel_measurable M* **by** *simp*
moreover have *B* ∈ *sets borel* **unfolding** *B_def* **by** (*simp add*: *closed_diagonal*)
ultimately have *A* ∈ *sets M* **by** *simp*
then show *?thesis* **unfolding** *A_def* **by** *simp*
qed

Logically equivalent to those with the opposite orientation, still these are needed

lemma *measurable_inequality_set_flipped*:
fixes *f g*::_ ⇒ 'a::{*second_countable_topology*, *linorder_topology*}
assumes [*measurable*]: *f* ∈ *borel_measurable M g* ∈ *borel_measurable M*
shows {*x* ∈ *space M*. *f x* ≥ *g x*} ∈ *sets M*
{*x* ∈ *space M*. *f x* > *g x*} ∈ *sets M*
by *auto*

lemmas *measurable_inequality_set* [*measurable*] =
borel_measurable_le borel_measurable_less measurable_inequality_set_flipped

proposition *measurable_limit* [*measurable*]:
fixes *f*::*nat* ⇒ *'a* ⇒ *'b*::*first_countable_topology*
assumes [*measurable*]: ⋀*n*::*nat*. *f n* ∈ *borel_measurable M*
shows *Measurable.pred M* (λ*x*. (λ*n*. *f n x*) ⟶ *c*)
proof −
obtain *A* :: *nat* ⇒ *'b set* **where** *A*:
⋀*i*. *open* (*A i*)
⋀*i*. *c* ∈ *A i*
⋀*S*. *open S* ⟹ *c* ∈ *S* ⟹ *eventually* (λ*i*. *A i* ⊆ *S*) *sequentially*
by (*rule countable_basis_at_decseq*) *blast*

have [*measurable*]: ⋀*N i*. (*f N*)−‘(*A i*) ∩ *space M* ∈ *sets M* **using** *A*(*1*) **by** *auto*
then have *mes*: (⋂*i*. ⋃*n*. ⋂*N*∈{*n*..}. (*f N*)−‘(*A i*) ∩ *space M*) ∈ *sets M* **by** *blast*

have (*u* ⟶ *c*) ⟷ (∀*i*. *eventually* (λ*n*. *u n* ∈ *A i*) *sequentially*) **for** *u*::*nat* ⇒ *'b*
proof
assume *u* ⟶ *c*
then have *eventually* (λ*n*. *u n* ∈ *A i*) *sequentially* **for** *i* **using** *A*(*1*)[*of i*] *A*(*2*)[*of i*]
by (*simp add*: *topological_tendstoD*)
then show (∀*i*. *eventually* (λ*n*. *u n* ∈ *A i*) *sequentially*) **by** *auto*
next
assume *H*: (∀*i*. *eventually* (λ*n*. *u n* ∈ *A i*) *sequentially*)
show (*u* ⟶ *c*)
proof (*rule topological_tendstoI*)
fix *S* **assume** *open S c* ∈ *S*
with *A*(*3*)[*OF this*] **obtain** *i* **where** *A i* ⊆ *S*
using *eventually_False_sequentially eventually_mono* **by** *blast*
moreover have *eventually* (λ*n*. *u n* ∈ *A i*) *sequentially* **using** *H* **by** *simp*
ultimately show ∀ *F n in sequentially*. *u n* ∈ *S*
by (*simp add*: *eventually_mono subset_eq*)
qed
qed
then have {*x*. (λ*n*. *f n x*) ⟶ *c*} = (⋂*i*. ⋃*n*. ⋂*N*∈{*n*..}. (*f N*)−‘(*A i*))
by (*auto simp add*: *atLeast_def eventually_at_top_linorder*)
then have {*x* ∈ *space M*. (λ*n*. *f n x*) ⟶ *c*} = (⋂*i*. ⋃*n*. ⋂*N*∈{*n*..}. (*f N*)−‘(*A i*) ∩ *space M*)
by *auto*
then have {*x* ∈ *space M*. (λ*n*. *f n x*) ⟶ *c*} ∈ *sets M* **using** *mes* **by** *simp*
then show *?thesis* **by** *auto*
qed

lemma *measurable_limit2* [*measurable*]:

fixes u::$nat \Rightarrow 'a \Rightarrow real$
assumes [*measurable*]: $\bigwedge n.\ u\ n \in borel\_measurable\ M\ v \in borel\_measurable\ M$
shows $Measurable.pred\ M\ (\lambda x.\ (\lambda n.\ u\ n\ x) \longrightarrow v\ x)$
proof −
define w **where** $w = (\lambda n\ x.\ u\ n\ x - v\ x)$
have [*measurable*]: $w\ n \in borel\_measurable\ M$ **for** n **unfolding** w_def **by** *auto*
have $((\lambda n.\ u\ n\ x) \longrightarrow v\ x) \longleftrightarrow ((\lambda n.\ w\ n\ x) \longrightarrow 0)$ **for** x
unfolding w_def **using** Lim_null **by** *auto*
then show *?thesis* **using** $measurable\_limit$ **by** *auto*
qed

lemma $measurable\_P\_restriction$ [*measurable* (*raw*)]:
assumes [*measurable*]: $Measurable.pred\ M\ P\ A \in sets\ M$
shows $\{x \in A.\ P\ x\} \in sets\ M$
proof −
have $A \subseteq space\ M$ **using** *sets.sets_into_space*[*OF assms*(*2*)]**.**
then have $\{x \in A.\ P\ x\} = A \cap \{x \in space\ M.\ P\ x\}$ **by** *blast*
then show *?thesis* **by** *auto*
qed

lemma $measurable\_sum\_nat$ [*measurable* (*raw*)]:
fixes $f :: 'c \Rightarrow 'a \Rightarrow nat$
assumes $\bigwedge i.\ i \in S \Longrightarrow f\ i \in measurable\ M\ (count\_space\ UNIV)$
shows $(\lambda x.\ \sum i{\in}S.\ f\ i\ x) \in measurable\ M\ (count\_space\ UNIV)$
proof *cases*
assume *finite S*
then show *?thesis* **using** *assms* **by** *induct auto*
qed *simp*

lemma $measurable\_abs\_powr$ [*measurable*]:
fixes p::$real$
assumes [*measurable*]: $f \in borel\_measurable\ M$
shows $(\lambda x.\ |f\ x|\ powr\ p) \in borel\_measurable\ M$
by *simp*

The next one is a variation around $measurable\_restrict\_space$.

lemma $measurable\_restrict\_space3$:
assumes $f \in measurable\ M\ N$ **and**
$f \in A \to B$
shows $f \in measurable\ (restrict\_space\ M\ A)\ (restrict\_space\ N\ B)$
proof −
have $f \in measurable\ (restrict\_space\ M\ A)\ N$ **using** $assms(1)\ measurable\_restrict\_space1$
by *auto*
then show *?thesis* **by** (*metis Int_iff funcsetI funcset_mem*
measurable_restrict_space2[*of f*, *of restrict_space M A*, *of B*, *of N*] *assms*(*2*)
space_restrict_space)
qed

lemma *measurable_restrict_mono*:
assumes *f*: $f \in$ *restrict_space M A* $\rightarrow_M$ *N* **and** $B \subseteq A$
shows $f \in$ *restrict_space M B* $\rightarrow_M$ *N*
by (*rule measurable_compose*[*OF measurable_restrict_space3 f*]) (*use* ‹$B \subseteq A$› **in** *auto*)

The next one is a variation around *measurable_piecewise_restrict*.

lemma *measurable_piecewise_restrict2*:
assumes [*measurable*]: $\bigwedge n.\ A\ n \in$ *sets M*
and *space M* $= (\bigcup (n::nat).\ A\ n)$
$\bigwedge n.\ \exists h \in$ *measurable M N*. $(\forall x \in A\ n.\ f\ x = h\ x)$
shows $f \in$ *measurable M N*
proof (*rule measurableI*)
fix *B* **assume** [*measurable*]: $B \in$ *sets N*
{
fix *n*::*nat*
obtain *h* **where** [*measurable*]: $h \in$ *measurable M N* **and** $\forall x \in A\ n.\ f\ x = h\ x$
using *assms*(*3*) **by** *blast*
then have $*$: $f-`B \cap A\ n = h-`B \cap A\ n$ **by** *auto*
have $h-`B \cap A\ n = h-`B \cap$ *space M* $\cap A\ n$
using *assms*(*2*) *sets.sets_into_space* **by** *auto*
then have $f-`B \cap A\ n \in$ *sets M*
by (*simp add*: $*$)
}
then have $(\bigcup n.\ f-`B \cap A\ n) \in$ *sets M*
by *measurable*
moreover have $f-`B \cap$ *space M* $= (\bigcup n.\ f-`B \cap A\ n)$
using *assms*(*2*) **by** *blast*
ultimately show $f-`B \cap$ *space M* $\in$ *sets M* **by** *simp*
next
fix *x* **assume** $x \in$ *space M*
then obtain *n* **where** $x \in A\ n$
using *assms*(*2*) **by** *blast*
obtain *h* **where** [*measurable*]: $h \in$ *measurable M N* **and** $\forall x \in A\ n.\ f\ x = h\ x$
using *assms*(*3*) **by** *blast*
then show $f\ x \in$ *space N*
by (*metis* ‹$x \in A\ n$› ‹$x \in$ *space M*› *measurable_space*)
qed

end

7.5 Lebesgue Integration for Nonnegative Functions

theory *Nonnegative_Lebesgue_Integration*
imports *Measure_Space Borel_Space*
begin

7.5.1 Approximating functions

lemma *AE_upper_bound_inf_ennreal*:
fixes $F\ G::'a \Rightarrow ennreal$
assumes $\bigwedge e.\ (e::real) > 0 \Longrightarrow AE\ x\ in\ M.\ F\ x \leq G\ x + e$
shows $AE\ x\ in\ M.\ F\ x \leq G\ x$
proof −
have $AE\ x\ in\ M.\ \forall n::nat.\ F\ x \leq G\ x + ennreal\ (1\ /\ Suc\ n)$
using *assms* **by** (*auto simp*: *AE_all_countable*)
then show *?thesis*
proof (*eventually_elim*)
fix x **assume** x: $\forall n::nat.\ F\ x \leq G\ x + ennreal\ (1\ /\ Suc\ n)$
show $F\ x \leq G\ x$
proof (*rule ennreal_le_epsilon*)
fix $e :: real$ **assume** $0 < e$
then obtain n **where** n: $1\ /\ Suc\ n < e$
by (*blast elim*: *nat_approx_posE*)
have $F\ x \leq G\ x + 1\ /\ Suc\ n$
using x **by** *simp*
also have $\ldots \leq G\ x + e$
using n **by** (*intro add_mono ennreal_leI*) *auto*
finally show $F\ x \leq G\ x + ennreal\ e$ **.**
qed
qed
qed

lemma *AE_upper_bound_inf*:
fixes $F\ G::'a \Rightarrow real$
assumes $\bigwedge e.\ e > 0 \Longrightarrow AE\ x\ in\ M.\ F\ x \leq G\ x + e$
shows $AE\ x\ in\ M.\ F\ x \leq G\ x$
proof −
have $AE\ x\ in\ M.\ F\ x \leq G\ x + 1/real\ (n+1)$ **for** $n::nat$
by (*rule assms*, *auto*)
then have $AE\ x\ in\ M.\ \forall n::nat \in UNIV.\ F\ x \leq G\ x + 1/real\ (n+1)$
by (*rule AE_ball_countable'*, *auto*)
moreover
{
fix x **assume** i: $\forall n::nat \in UNIV.\ F\ x \leq G\ x + 1/real\ (n+1)$
have $(\lambda n.\ G\ x + 1/real\ (n+1)) \longrightarrow G\ x + 0$
by (*rule tendsto_add*, *simp*, *rule LIMSEQ_ignore_initial_segment*[*OF lim_1_over_n*, *of 1*])
then have $F\ x \leq G\ x$ **using** i *LIMSEQ_le_const* **by** *fastforce*
}
ultimately show *?thesis* **by** *auto*
qed

lemma *not_AE_zero_ennreal_E*:
fixes $f::'a \Rightarrow ennreal$
assumes $\neg\ (AE\ x\ in\ M.\ f\ x = 0)$ **and** [*measurable*]: $f \in borel\_measurable\ M$
shows $\exists A{\in}sets\ M.\ \exists e::real{>}0.\ emeasure\ M\ A > 0 \wedge (\forall x \in A.\ f\ x \geq e)$

proof −
{ assume ¬ (∃ *e*::*real*>*0*. {*x* ∈ *space M*. *f x* ≥ *e*} ∉ *null_sets M*)
then have *0* < *e* ⟹ *AE x in M*. *f x* ≤ *e* **for** *e* :: *real*
by (*auto simp*: *not_le less_imp_le dest*!: *AE_not_in*)
then have *AE x in M*. *f x* ≤ *0*
by (*intro AE_upper_bound_inf_ennreal*[**where** *G*=λ_. *0*]) *simp*
then have *False*
using *assms* **by** *auto* **}**
then obtain *e*::*real* **where** *e*: *e* > *0* {*x* ∈ *space M*. *f x* ≥ *e*} ∉ *null_sets M* **by** *auto*
define *A* **where** *A* = {*x* ∈ *space M*. *f x* ≥ *e*}
have *1* [*measurable*]: *A* ∈ *sets M* **unfolding** *A_def* **by** *auto*
have *2*: *emeasure M A* > *0*
using *e*(*2*) *A_def* ‹*A* ∈ *sets M*› **by** *auto*
have *3*: ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ *e* **unfolding** *A_def* **by** *auto*
show *?thesis* **using** *e*(*1*) *1 2 3* **by** *blast*
qed

lemma *not_AE_zero_E*:
fixes *f*::*'a* ⇒ *real*
assumes *AE x in M*. *f x* ≥ *0*
¬(*AE x in M*. *f x* = *0*)
and [*measurable*]: *f* ∈ *borel_measurable M*
shows ∃ *A e*. *A* ∈ *sets M* ∧ *e*>*0* ∧ *emeasure M A* > *0* ∧ (∀ *x* ∈ *A*. *f x* ≥ *e*)
proof −
have ∃ *e*. *e* > *0* ∧ {*x* ∈ *space M*. *f x* ≥ *e*} ∉ *null_sets M*
proof (*rule ccontr*)
assume ∗: ¬(∃ *e*. *e* > *0* ∧ {*x* ∈ *space M*. *f x* ≥ *e*} ∉ *null_sets M*)
{
fix *e*::*real* **assume** *e* > *0*
then have {*x* ∈ *space M*. *f x* ≥ *e*} ∈ *null_sets M* **using** ∗ **by** *blast*
then have *AE x in M*. *x* ∉ {*x* ∈ *space M*. *f x* ≥ *e*} **using** *AE_not_in* **by** *blast*
then have *AE x in M*. *f x* ≤ *e* **by** *auto*
}
then have *AE x in M*. *f x* ≤ *0* **by** (*rule AE_upper_bound_inf*, *auto*)
then have *AE x in M*. *f x* = *0* **using** *assms*(*1*) **by** *auto*
then show *False* **using** *assms*(*2*) **by** *auto*
qed
then obtain *e* **where** *e*: *e*>*0* {*x* ∈ *space M*. *f x* ≥ *e*} ∉ *null_sets M* **by** *auto*
define *A* **where** *A* = {*x* ∈ *space M*. *f x* ≥ *e*}
have *1* [*measurable*]: *A* ∈ *sets M* **unfolding** *A_def* **by** *auto*
have *2*: *emeasure M A* > *0*
using *e*(*2*) *A_def* ‹*A* ∈ *sets M*› **by** *auto*
have *3*: ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ *e* **unfolding** *A_def* **by** *auto*
show *?thesis*
using *e*(*1*) *1 2 3* **by** *blast*
qed

7.5.2 Simple function

Our simple functions are not restricted to nonnegative real numbers. Instead they are just functions with a finite range and are measurable when singleton sets are measurable.

definition *simple_function M g* ⟷
  finite (*g* ‘ *space M*) ∧
  (∀ *x* ∈ *g* ‘ *space M*. *g* −‘ {*x*} ∩ *space M* ∈ *sets M*)

lemma *simple_functionD*:
  assumes *simple_function M g*
  shows *finite* (*g* ‘ *space M*) **and** *g* −‘ *X* ∩ *space M* ∈ *sets M*
proof −
  show *finite* (*g* ‘ *space M*)
    using *assms* **unfolding** *simple_function_def* **by** *auto*
  have *g* −‘ *X* ∩ *space M* = *g* −‘ (*X* ∩ *g*‘*space M*) ∩ *space M* **by** *auto*
  also have ... = (⋃*x*∈*X* ∩ *g*‘*space M*. *g*−‘{*x*} ∩ *space M*) **by** *auto*
  finally show *g* −‘ *X* ∩ *space M* ∈ *sets M* **using** *assms*
    by (*auto simp del*: *UN_simps simp*: *simple_function_def*)
qed

lemma *measurable_simple_function*[*measurable_dest*]:
  simple_function M f ⟹ *f* ∈ *measurable M* (*count_space UNIV*)
  unfolding *simple_function_def measurable_def*
proof *safe*
  fix *A* **assume** *finite* (*f* ‘ *space M*) ∀ *x*∈*f* ‘ *space M*. *f* −‘ {*x*} ∩ *space M* ∈ *sets M*
  then have (⋃*x*∈*f* ‘ *space M*. *if x* ∈ *A then f* −‘ {*x*} ∩ *space M else* {}) ∈ *sets M*
    by (*intro sets.finite_UN*) *auto*
  also have (⋃*x*∈*f* ‘ *space M*. *if x* ∈ *A then f* −‘ {*x*} ∩ *space M else* {}) = *f* −‘ *A* ∩ *space M*
    by (*auto split*: *if_split_asm*)
  finally show *f* −‘ *A* ∩ *space M* ∈ *sets M* **.**
qed *simp*

lemma *borel_measurable_simple_function*:
  simple_function M f ⟹ *f* ∈ *borel_measurable M*
  by (*auto dest*!: *measurable_simple_function simp*: *measurable_def*)

lemma *simple_function_measurable2*[*intro*]:
  assumes *simple_function M f simple_function M g*
  shows *f* −‘ *A* ∩ *g* −‘ *B* ∩ *space M* ∈ *sets M*
proof −
  have *f* −‘ *A* ∩ *g* −‘ *B* ∩ *space M* = (*f* −‘ *A* ∩ *space M*) ∩ (*g* −‘ *B* ∩ *space M*)
    by *auto*
  then show *?thesis* **using** *assms*[*THEN simple_functionD*(*2*)] **by** *auto*
qed

lemma *simple_function_indicator_representation*:

fixes $f :: 'a \Rightarrow ennreal$
assumes f: $simple\_function\ M\ f$ **and** x: $x \in space\ M$
shows $f\ x = (\sum y \in f\ `\ space\ M.\ y * indicator\ (f\ -`\ \{y\} \cap space\ M)\ x)$
(**is** $?l = ?r$)
proof −
have $?r = (\sum y \in f\ `\ space\ M.$
$(if\ y = f\ x\ then\ y * indicator\ (f\ -`\ \{y\} \cap space\ M)\ x\ else\ 0))$
by (*auto intro*!: *sum.cong*)
also have ... = $f\ x * indicator\ (f\ -`\ \{f\ x\} \cap space\ M)\ x$
using *assms* **by** (*auto dest*: *simple_functionD*)
also have ... = $f\ x$ **using** x **by** (*auto simp*: *indicator_def*)
finally show *?thesis* **by** *auto*
qed

lemma *simple_function_notspace*:
simple_function M ($\lambda x.\ h\ x * indicator\ (-\ space\ M)\ x$::*ennreal*) (**is** *simple_function M ?h*)
proof −
have *?h ` space M* ⊆ {*0*} **unfolding** *indicator_def* **by** *auto*
hence [*simp*, *intro*]: *finite* (*?h ` space M*) **by** (*auto intro*: *finite_subset*)
have *?h* − ` {*0*} ∩ *space M* = *space M* **by** *auto*
thus *?thesis* **unfolding** *simple_function_def* **by** (*auto simp add*: *image_constant_conv*)
qed

lemma *simple_function_cong*:
assumes $\bigwedge t.\ t \in space\ M \Longrightarrow f\ t = g\ t$
shows $simple\_function\ M\ f \longleftrightarrow simple\_function\ M\ g$
proof −
have $\bigwedge x.\ f\ -`\ \{x\} \cap space\ M = g\ -`\ \{x\} \cap space\ M$
using *assms* **by** *auto*
with *assms* **show** *?thesis*
by (*simp add*: *simple_function_def cong*: *image_cong*)
qed

lemma *simple_function_cong_algebra*:
assumes *sets N* = *sets M space N* = *space M*
shows *simple_function M f* ⟷ *simple_function N f*
unfolding *simple_function_def assms* **..**

lemma *simple_function_borel_measurable*:
fixes $f :: 'a \Rightarrow 'x$::{*t2_space*}
assumes $f \in borel\_measurable\ M$ **and** *finite* (*f ` space M*)
shows *simple_function M f*
using *assms* **unfolding** *simple_function_def*
by (*auto intro*: *borel_measurable_vimage*)

lemma *simple_function_iff_borel_measurable*:
fixes $f :: 'a \Rightarrow 'x$::{*t2_space*}
shows $simple\_function\ M\ f \longleftrightarrow finite\ (f\ `\ space\ M) \wedge f \in borel\_measurable\ M$

by (*metis borel_measurable_simple_function simple_functionD*(*1*) *simple_function_borel_measurable*)

lemma *simple_function_eq_measurable*:
simple_function M f ⟷ *finite* (*f*‘*space M*) ∧ *f* ∈ *measurable M* (*count_space UNIV*)
using *measurable_simple_function*[*of M f*] **by** (*fastforce simp*: *simple_function_def*)

lemma *simple_function_const*[*intro*, *simp*]:
simple_function M (λ*x*. *c*)
by (*auto intro*: *finite_subset simp*: *simple_function_def*)
lemma *simple_function_compose*[*intro*, *simp*]:
assumes *simple_function M f*
shows *simple_function M* (*g* ∘ *f*)
unfolding *simple_function_def*
proof *safe*
show *finite* ((*g* ∘ *f*) ‘ *space M*)
using *assms* **unfolding** *simple_function_def image_comp* [*symmetric*] **by** *auto*
next
fix *x* **assume** *x* ∈ *space M*
let *?G* = *g* −‘ {*g* (*f x*)} ∩ (*f*‘*space M*)
have ∗: (*g* ∘ *f*) −‘ {(*g* ∘ *f*) *x*} ∩ *space M* =
(⋃*x*∈*?G*. *f* −‘ {*x*} ∩ *space M*) **by** *auto*
show (*g* ∘ *f*) −‘ {(*g* ∘ *f*) *x*} ∩ *space M* ∈ *sets M*
using *assms* **unfolding** *simple_function_def* ∗
by (*rule_tac sets.finite_UN*) *auto*
qed

lemma *simple_function_indicator*[*intro*, *simp*]:
assumes *A* ∈ *sets M*
shows *simple_function M* (*indicator A*)
proof −
have *indicator A* ‘ *space M* ⊆ {*0*, *1*} (**is** *?S* ⊆ _)
by (*auto simp*: *indicator_def*)
hence *finite ?S* **by** (*rule finite_subset*) *simp*
moreover have − *A* ∩ *space M* = *space M* − *A* **by** *auto*
ultimately show *?thesis* **unfolding** *simple_function_def*
using *assms* **by** (*auto simp*: *indicator_def* [*abs_def*])
qed

lemma *simple_function_Pair*[*intro*, *simp*]:
assumes *simple_function M f*
assumes *simple_function M g*
shows *simple_function M* (λ*x*. (*f x*, *g x*)) (**is** *simple_function M ?p*)
unfolding *simple_function_def*
proof *safe*
show *finite* (*?p* ‘ *space M*)
using *assms* **unfolding** *simple_function_def*
by (*rule_tac finite_subset*[*of* _ *f*‘*space M* × *g*‘*space M*]) *auto*
next

fix x **assume** $x \in$ *space* M
have $(\lambda x.\ (f\ x,\ g\ x)) -` \{(f\ x,\ g\ x)\} \cap$ *space* $M =$
$(f -` \{f\ x\} \cap$ *space* $M) \cap (g -` \{g\ x\} \cap$ *space* $M)$
by *auto*
with ‹$x \in$ *space* M› **show** $(\lambda x.\ (f\ x,\ g\ x)) -` \{(f\ x,\ g\ x)\} \cap$ *space* $M \in$ *sets* M
using *assms* **unfolding** *simple_function_def* **by** *auto*
qed

lemma *simple_function_compose1*:
assumes *simple_function* $M\ f$
shows *simple_function* $M\ (\lambda x.\ g\ (f\ x))$
using *simple_function_compose*[*OF assms*, *of g*]
by (*simp add*: *comp_def*)

lemma *simple_function_compose2*:
assumes *simple_function* $M\ f$ **and** *simple_function* $M\ g$
shows *simple_function* $M\ (\lambda x.\ h\ (f\ x)\ (g\ x))$
proof −
have *simple_function* $M\ ((\lambda(x,\ y).\ h\ x\ y) \circ (\lambda x.\ (f\ x,\ g\ x)))$
using *assms* **by** *auto*
thus *?thesis* **by** (*simp_all add*: *comp_def*)
qed

lemmas *simple_function_add*[*intro*, *simp*] = *simple_function_compose2*[**where** $h=(+)$]
and *simple_function_diff*[*intro*, *simp*] = *simple_function_compose2*[**where** $h=(-)$]
and *simple_function_uminus*[*intro*, *simp*] = *simple_function_compose*[**where** *g=uminus*]
and *simple_function_mult*[*intro*, *simp*] = *simple_function_compose2*[**where** $h=(*)$]
and *simple_function_div*[*intro*, *simp*] = *simple_function_compose2*[**where** $h=(/)$]
and *simple_function_inverse*[*intro*, *simp*] = *simple_function_compose*[**where** *g=inverse*]
and *simple_function_max*[*intro*, *simp*] = *simple_function_compose2*[**where** *h=max*]

lemma *simple_function_sum*[*intro*, *simp*]:
assumes $\bigwedge i.\ i \in P \Longrightarrow$ *simple_function* $M\ (f\ i)$
shows *simple_function* $M\ (\lambda x.\ \sum i{\in}P.\ f\ i\ x)$
proof *cases*
assume *finite* P **from** *this assms* **show** *?thesis* **by** *induct auto*
qed *auto*

lemma *simple_function_ennreal*[*intro*, *simp*]:
fixes $f\ g :: 'a \Rightarrow$ *real* **assumes** *sf*: *simple_function* $M\ f$
shows *simple_function* $M\ (\lambda x.\ ennreal\ (f\ x))$
by (*rule simple_function_compose1*[*OF sf*])

lemma *simple_function_real_of_nat*[*intro*, *simp*]:
fixes $f\ g :: 'a \Rightarrow$ *nat* **assumes** *sf*: *simple_function* $M\ f$

```
  shows simple_function M (λx. real (f x))
  by (rule simple_function_compose1[OF sf])

lemma borel_measurable_implies_simple_function_sequence:
  fixes u :: 'a ⇒ ennreal
  assumes u[measurable]: u ∈ borel_measurable M
  shows ∃f. incseq f ∧ (∀i. (∀x. f i x < top) ∧ simple_function M (f i)) ∧ u =
(SUP i. f i)
proof −
  define f where [abs_def]:
    f i x = real_of_int (floor (enn2real (min i (u x)) * 2^i)) / 2^i for i x

  have [simp]: 0 ≤ f i x for i x
     by (auto simp: f_def intro!: divide_nonneg_nonneg mult_nonneg_nonneg
enn2real_nonneg)

  have *: 2^n * real_of_int x = real_of_int (2^n * x) for n x
    by simp

  have real_of_int ⌊real i * 2 ^ i⌋ = real_of_int ⌊i * 2 ^ i⌋ for i
    by (intro arg_cong[where f=real_of_int]) simp
  then have [simp]: real_of_int ⌊real i * 2 ^ i⌋ = i * 2 ^ i for i
    unfolding floor_of_nat by simp

  have incseq f
  proof (intro monoI le_funI)
    fix m n :: nat and x assume m ≤ n
    moreover
    { fix d :: nat
      have ⌊2^d::real⌋ * ⌊2^m * enn2real (min (of_nat m) (u x))⌋ ≤
        ⌊2^d * (2^m * enn2real (min (of_nat m) (u x)))⌋
        by (rule le_mult_floor) (auto)
      also have ... ≤ ⌊2^d * (2^m * enn2real (min (of_nat d + of_nat m) (u
x)))⌋
        by (intro floor_mono mult_mono enn2real_mono min.mono)
           (auto simp: min_less_iff_disj of_nat_less_top)
      finally have f m x ≤ f (m + d) x
        unfolding f_def
        by (auto simp: field_simps power_add * simp del: of_int_mult) }
    ultimately show f m x ≤ f n x
      by (auto simp add: le_iff_add)
  qed
  then have inc_f: incseq (λi. ennreal (f i x)) for x
    by (auto simp: incseq_def le_fun_def)
  then have incseq (λi x. ennreal (f i x))
    by (auto simp: incseq_def le_fun_def)
  moreover
  have simple_function M (f i) for i
  proof (rule simple_function_borel_measurable)
```

```
    have ⌊enn2real (min (of_nat i) (u x)) * 2 ^ i⌋ ≤ ⌊int i * 2 ^ i⌋ for x
      by (cases u x rule: ennreal_cases)
         (auto split: split_min intro!: floor_mono)
    then have f i ‘ space M ⊆ (λn. real_of_int n / 2^i) ‘ {0 .. of_nat i * 2^i}
      unfolding floor_of_int by (auto simp: f_def intro!: imageI)
    then show finite (f i ‘ space M)
      by (rule finite_subset) auto
    show f i ∈ borel_measurable M
      unfolding f_def enn2real_def by measurable
  qed
  moreover
  { fix x
    have (SUP i. ennreal (f i x)) = u x
    proof (cases u x rule: ennreal_cases)
      case top then show ?thesis
      by (simp add: f_def inf_min[symmetric] ennreal_of_nat_eq_real_of_nat[symmetric]
                    ennreal_SUP_of_nat_eq_top)
    next
      case (real r)
      obtain n where r ≤ of_nat n using real_arch_simple by auto
      then have min_eq_r: ∀F x in sequentially. min (real x) r = r
        by (auto simp: eventually_sequentially intro!: exI[of _ n] split: split_min)

      have (λi. real_of_int ⌊min (real i) r * 2^i⌋ / 2^i) ⟶ r
      proof (rule tendsto_sandwich)
        show (λn. r - (1/2)^n) ⟶ r
          by (auto intro!: tendsto_eq_intros LIMSEQ_power_zero)
        show ∀F n in sequentially. real_of_int ⌊min (real n) r * 2 ^ n⌋ / 2 ^ n ≤
r
          using min_eq_r by eventually_elim (auto simp: field_simps)
        have *: r * (2 ^ n * 2 ^ n) ≤ 2^n + 2^n * real_of_int ⌊r * 2 ^ n⌋ for n
         using real_of_int_floor_ge_diff_one[of r * 2^n, THEN mult_left_mono,
of 2^n]
          by (auto simp: field_simps)
        show ∀F n in sequentially. r - (1/2)^n ≤ real_of_int ⌊min (real n) r *
2 ^ n⌋ / 2 ^ n
          using min_eq_r by eventually_elim (insert *, auto simp: field_simps)
      qed auto
      then have (λi. ennreal (f i x)) ⟶ ennreal r
        by (simp add: real f_def ennreal_of_nat_eq_real_of_nat min_ennreal)
      from LIMSEQ_unique[OF LIMSEQ_SUP[OF inc_f] this]
      show ?thesis
        by (simp add: real)
    qed }
  ultimately show ?thesis
    by (intro exI [of _ λi x. ennreal (f i x)]) (auto simp add: image_comp)
qed

lemma borel_measurable_implies_simple_function_sequence':
```

fixes $u :: 'a \Rightarrow ennreal$
assumes *u*: $u \in borel\_measurable\ M$
obtains f **where**
$\bigwedge i.\ simple\_function\ M\ (f\ i)\ incseq\ f\ \bigwedge i\ x.\ f\ i\ x < top\ \bigwedge x.\ (SUP\ i.\ f\ i\ x) = u\ x$
using *borel_measurable_implies_simple_function_sequence* [*OF u*]
by (*metis SUP_apply*)

lemma *simple_function_induct*
[*consumes 1*, *case_names cong set mult add*, *induct set*: *simple_function*]:
fixes $u :: 'a \Rightarrow ennreal$
assumes *u*: *simple_function M u*
assumes *cong*: $\bigwedge f\ g.\ simple\_function\ M\ f \Longrightarrow simple\_function\ M\ g \Longrightarrow (AE\ x\ in\ M.\ f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *set*: $\bigwedge A.\ A \in sets\ M \Longrightarrow P\ (indicator\ A)$
assumes *mult*: $\bigwedge u\ c.\ P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
assumes *add*: $\bigwedge u\ v.\ P\ u \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
shows *P u*
proof (*rule cong*)
from *AE_space* **show** $AE\ x\ in\ M.\ (\sum y \in u\ `\ space\ M.\ y * indicator\ (u -` \{y\} \cap space\ M)\ x) = u\ x$
proof *eventually_elim*
fix *x* **assume** *x*: $x \in space\ M$
from *simple_function_indicator_representation*[*OF u x*]
show $(\sum y \in u\ `\ space\ M.\ y * indicator\ (u -` \{y\} \cap space\ M)\ x) = u\ x$ **..**
qed
next
from *u* **have** *finite* $(u\ `\ space\ M)$
unfolding *simple_function_def* **by** *auto*
then show *P* $(\lambda x.\ \sum y \in u\ `\ space\ M.\ y * indicator\ (u -` \{y\} \cap space\ M)\ x)$
proof *induct*
case *empty* **show** *?case*
using *set*[*of* {}] **by** (*simp add*: *indicator_def*[*abs_def*])
qed (*auto intro*!: *add mult set simple_functionD u*)
next
show *simple_function M* $(\lambda x.\ (\sum y \in u\ `\ space\ M.\ y * indicator\ (u -` \{y\} \cap space\ M)\ x))$
apply (*subst simple_function_cong*)
apply (*rule simple_function_indicator_representation*[*symmetric*])
apply (*auto intro*: *u*)
done
qed *fact*

lemma *simple_function_induct_nn*[*consumes 1*, *case_names cong set mult add*]:
fixes $u :: 'a \Rightarrow ennreal$
assumes *u*: *simple_function M u*
assumes *cong*: $\bigwedge f\ g.\ simple\_function\ M\ f \Longrightarrow simple\_function\ M\ g \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *set*: $\bigwedge A.\ A \in sets\ M \Longrightarrow P\ (indicator\ A)$
assumes *mult*: $\bigwedge u\ c.\ simple\_function\ M\ u \Longrightarrow P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$

assumes *add*: $\bigwedge u\ v.$ *simple_function* $M\ u \Longrightarrow P\ u \Longrightarrow$ *simple_function* $M\ v \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow u\ x = 0 \vee v\ x = 0) \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
shows $P\ u$
proof −
show *?thesis*
proof (*rule cong*)
fix x **assume** x: $x \in$ *space* M
from *simple_function_indicator_representation*[*OF u x*]
show $(\sum y \in u\ `$ *space* $M.\ y *$ *indicator* $(u -` \{y\} \cap$ *space* $M)\ x) = u\ x$ **..**
next
show *simple_function* $M\ (\lambda x.\ (\sum y \in u\ `$ *space* $M.\ y *$ *indicator* $(u -` \{y\} \cap$ *space* $M)\ x))$
apply (*subst simple_function_cong*)
apply (*rule simple_function_indicator_representation*[*symmetric*])
apply (*auto intro*: *u*)
done
next
from *u* **have** *finite* $(u\ `$ *space* $M)$
unfolding *simple_function_def* **by** *auto*
then show $P\ (\lambda x.\ \sum y \in u\ `$ *space* $M.\ y *$ *indicator* $(u -` \{y\} \cap$ *space* $M)\ x)$
proof *induct*
case *empty* **show** *?case*
using *set*[*of* {}] **by** (*simp add*: *indicator_def*[*abs_def*])
next
case (*insert x S*)
{ fix z **have** $(\sum y \in S.\ y *$ *indicator* $(u -` \{y\} \cap$ *space* $M)\ z) = 0 \vee$
$x *$ *indicator* $(u -` \{x\} \cap$ *space* $M)\ z = 0$
using *insert* **by** (*subst sum_eq_0_iff*) (*auto simp*: *indicator_def*) **}**
note *disj* = *this*
from *insert* **show** *?case*
by (*auto intro*!: *add mult set simple_functionD u simple_function_sum disj*)
qed
qed *fact*
qed

lemma *borel_measurable_induct*
[*consumes 1*, *case_names cong set mult add seq*, *induct set*: *borel_measurable*]:
fixes $u :: 'a \Rightarrow$ *ennreal*
assumes u: $u \in$ *borel_measurable* M
assumes *cong*: $\bigwedge f\ g.\ f \in$ *borel_measurable* $M \Longrightarrow g \in$ *borel_measurable* $M \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ g \Longrightarrow P\ f$
assumes *set*: $\bigwedge A.\ A \in$ *sets* $M \Longrightarrow P$ (*indicator* A)
assumes *mult'*: $\bigwedge u\ c.\ c < top \Longrightarrow u \in$ *borel_measurable* $M \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow u\ x < top) \Longrightarrow P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
assumes *add*: $\bigwedge u\ v.\ u \in$ *borel_measurable* $M \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow u\ x < top) \Longrightarrow P\ u \Longrightarrow v \in$ *borel_measurable* $M \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow v\ x < top) \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow u\ x = 0 \vee v\ x = 0) \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
assumes *seq*: $\bigwedge U.\ (\bigwedge i.\ U\ i \in$ *borel_measurable* $M) \Longrightarrow (\bigwedge i\ x.\ x \in$ *space* $M \Longrightarrow U\ i\ x < top) \Longrightarrow (\bigwedge i.\ P\ (U\ i)) \Longrightarrow$ *incseq* $U \Longrightarrow u = (SUP\ i.\ U\ i) \Longrightarrow P\ (SUP$

```
i. U i)
  shows P u
  using u
proof (induct rule: borel_measurable_implies_simple_function_sequence')
  fix U assume U: ⋀i. simple_function M (U i) incseq U ⋀i x. U i x < top and
sup: ⋀x. (SUP i. U i x) = u x
  have u_eq: u = (SUP i. U i)
    using u by (auto simp add: image_comp sup)

  have not_inf: ⋀x i. x ∈ space M ⟹ U i x < top
    using U by (auto simp: image_iff eq_commute)

  from U have ⋀i. U i ∈ borel_measurable M
    by (simp add: borel_measurable_simple_function)

  show P u
    unfolding u_eq
  proof (rule seq)
    fix i show P (U i)
      using ‹simple_function M (U i)› not_inf[of _ i]
    proof (induct rule: simple_function_induct_nn)
      case (mult u c)
      show ?case
      proof cases
        assume c = 0 ∨ space M = {} ∨ (∀x∈space M. u x = 0)
        with mult(1) show ?thesis
          by (intro cong[of λx. c * u x indicator {}] set)
             (auto dest!: borel_measurable_simple_function)
      next
        assume ¬ (c = 0 ∨ space M = {} ∨ (∀x∈space M. u x = 0))
        then obtain x where space M ≠ {} and x: x ∈ space M u x ≠ 0 c ≠ 0
          by auto
        with mult(3)[of x] have c < top
          by (auto simp: ennreal_mult_less_top)
        then have u_fin: x' ∈ space M ⟹ u x' < top for x'
          using mult(3)[of x'] ‹c ≠ 0› by (auto simp: ennreal_mult_less_top)
        then have P u
          by (rule mult)
        with u_fin ‹c < top› mult(1) show ?thesis
          by (intro mult') (auto dest!: borel_measurable_simple_function)
      qed
    qed (auto intro: cong intro!: set add dest!: borel_measurable_simple_function)
  qed fact+
qed

lemma simple_function_If_set:
  assumes sf: simple_function M f simple_function M g and A: A ∩ space M ∈
sets M
  shows simple_function M (λx. if x ∈ A then f x else g x) (is simple_function
```

M ?IF)
proof −
define *F* **where** *F x* = *f* −‘ {*x*} ∩ *space M* **for** *x*
define *G* **where** *G x* = *g* −‘ {*x*} ∩ *space M* **for** *x*
show *?thesis* **unfolding** *simple_function_def*
proof *safe*
have *?IF* ‘ *space M* ⊆ *f* ‘ *space M* ∪ *g* ‘ *space M* **by** *auto*
from *finite_subset*[*OF this*] *assms*
show *finite* (*?IF* ‘ *space M*) **unfolding** *simple_function_def* **by** *auto*
next
fix *x* **assume** *x* ∈ *space M*
then have ∗: *?IF* −‘ {*?IF x*} ∩ *space M* = (*if x* ∈ *A*
then ((*F* (*f x*) ∩ (*A* ∩ *space M*)) ∪ (*G* (*f x*) − (*G* (*f x*) ∩ (*A* ∩ *space M*))))
else ((*F* (*g x*) ∩ (*A* ∩ *space M*)) ∪ (*G* (*g x*) − (*G* (*g x*) ∩ (*A* ∩ *space M*)))))
using *sets.sets_into_space*[*OF A*] **by** (*auto split*: *if_split_asm simp*: *G_def*
F_def)
have [*intro*]: ⋀*x*. *F x* ∈ *sets M* ⋀*x*. *G x* ∈ *sets M*
unfolding *F_def G_def* **using** *sf*[*THEN simple_functionD*(*2*)] **by** *auto*
show *?IF* −‘ {*?IF x*} ∩ *space M* ∈ *sets M* **unfolding** ∗ **using** *A* **by** *auto*
qed
qed

lemma *simple_function_If*:
assumes *sf*: *simple_function M f simple_function M g* **and** *P*: {*x*∈*space M*. *P*
x} ∈ *sets M*
shows *simple_function M* (λ*x*. *if P x then f x else g x*)
proof −
have {*x*∈*space M*. *P x*} = {*x*. *P x*} ∩ *space M* **by** *auto*
with *simple_function_If_set*[*OF sf*, *of* {*x*. *P x*}] *P* **show** *?thesis* **by** *simp*
qed

lemma *simple_function_subalgebra*:
assumes *simple_function N f*
and *N_subalgebra*: *sets N* ⊆ *sets M space N* = *space M*
shows *simple_function M f*
using *assms* **unfolding** *simple_function_def* **by** *auto*

lemma *simple_function_comp*:
assumes *T*: *T* ∈ *measurable M M′*
and *f*: *simple_function M′ f*
shows *simple_function M* (λ*x*. *f* (*T x*))
proof (*intro simple_function_def*[*THEN iffD2*] *conjI ballI*)
have (λ*x*. *f* (*T x*)) ‘ *space M* ⊆ *f* ‘ *space M′*
using *T* **unfolding** *measurable_def* **by** *auto*
then show *finite* ((λ*x*. *f* (*T x*)) ‘ *space M*)
using *f* **unfolding** *simple_function_def* **by** (*auto intro*: *finite_subset*)
fix *i* **assume** *i*: *i* ∈ (λ*x*. *f* (*T x*)) ‘ *space M*
then have *i* ∈ *f* ‘ *space M′*
using *T* **unfolding** *measurable_def* **by** *auto*

then have $f -` \{i\} \cap space\ M' \in sets\ M'$
using f **unfolding** *simple_function_def* **by** *auto*
then have $T -` (f -` \{i\} \cap space\ M') \cap space\ M \in sets\ M$
using T **unfolding** *measurable_def* **by** *auto*
also have $T -` (f -` \{i\} \cap space\ M') \cap space\ M = (\lambda x.\ f\ (T\ x)) -` \{i\} \cap space\ M$
using T **unfolding** *measurable_def* **by** *auto*
finally show $(\lambda x.\ f\ (T\ x)) -` \{i\} \cap space\ M \in sets\ M$ **.**
qed

7.5.3 Simple integral

definition $simple\_integral :: 'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow ennreal\ (\langle integral^S \rangle)$
where
$integral^S\ M\ f = (\sum x \in f`space\ M.\ x * emeasure\ M\ (f -` \{x\} \cap space\ M))$

syntax
$\_simple\_integral :: pttrn \Rightarrow ennreal \Rightarrow 'a\ measure \Rightarrow ennreal$
$(\langle(\langle open\_block\ notation=\langle binder\ integral\rangle\rangle \int^S \_.\ \_\ \partial\_)\rangle\ [60,61]\ 110)$
syntax_consts
$\_simple\_integral == simple\_integral$
translations
$\int^S x.\ f\ \partial M == CONST\ simple\_integral\ M\ (\%x.\ f)$

lemma *simple_integral_cong*:
assumes $\bigwedge t.\ t \in space\ M \Longrightarrow f\ t = g\ t$
shows $integral^S\ M\ f = integral^S\ M\ g$
proof −
have $f`space\ M = g`space\ M$
$\bigwedge x.\ f -` \{x\} \cap space\ M = g -` \{x\} \cap space\ M$
using *assms* **by** (*auto intro*!: *image_eqI*)
thus *?thesis* **unfolding** *simple_integral_def* **by** *simp*
qed

lemma *simple_integral_const*[*simp*]:
$(\int^S x.\ c\ \partial M) = c * (emeasure\ M)\ (space\ M)$
proof (*cases space* $M = \{\}$)
case *True* **thus** *?thesis* **unfolding** *simple_integral_def* **by** *simp*
next
case *False* **hence** $(\lambda x.\ c)`space\ M = \{c\}$ **by** *auto*
thus *?thesis* **unfolding** *simple_integral_def* **by** *simp*
qed

lemma *simple_function_partition*:
assumes *f*: *simple_function M f* **and** *g*: *simple_function M g*
assumes *sub*: $\bigwedge x\ y.\ x \in space\ M \Longrightarrow y \in space\ M \Longrightarrow g\ x = g\ y \Longrightarrow f\ x = f\ y$
assumes *v*: $\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = v\ (g\ x)$
shows $integral^S\ M\ f = (\sum y \in g`space\ M.\ v\ y * emeasure\ M\ \{x \in space\ M.\ g\ x = y\})$

```
    (is _ = ?r)
proof −
  from f g have [simp]: finite (f‘space M) finite (g‘space M)
    by (auto simp: simple_function_def)
  from f g have [measurable]: f ∈ measurable M (count_space UNIV) g ∈ mea-
surable M (count_space UNIV)
    by (auto intro: measurable_simple_function)

  { fix y assume y ∈ space M
    then have f ‘ space M ∩ {i. ∃ x∈space M. i = f x ∧ g y = g x} = {v (g y)}
      by (auto cong: sub simp: v[symmetric]) }
  note eq = this

  have integral^S M f =
    (∑ y∈f‘space M. y ∗ (∑ z∈g‘space M.
      if ∃ x∈space M. y = f x ∧ z = g x then emeasure M {x∈space M. g x = z}
else 0))
    unfolding simple_integral_def
  proof (safe intro!: sum.cong ennreal_mult_left_cong)
    fix y assume y: y ∈ space M f y ≠ 0
    have [simp]: g ‘ space M ∩ {z. ∃ x∈space M. f y = f x ∧ z = g x} =
        {z. ∃ x∈space M. f y = f x ∧ z = g x}
      by auto
    have eq:(⋃ i∈{z. ∃ x∈space M. f y = f x ∧ z = g x}. {x ∈ space M. g x = i})
=
        f −‘ {f y} ∩ space M
      by (auto simp: eq_commute cong: sub rev_conj_cong)
    have finite (g‘space M) by simp
    then have finite {z. ∃ x∈space M. f y = f x ∧ z = g x}
      by (rule rev_finite_subset) auto
    then show emeasure M (f −‘ {f y} ∩ space M) =
      (∑ z∈g ‘ space M. if ∃ x∈space M. f y = f x ∧ z = g x then emeasure M {x
∈ space M. g x = z} else 0)
      apply (simp add: sum.If_cases)
      apply (subst sum_emeasure)
      apply (auto simp: disjoint_family_on_def eq)
      done
  qed
  also have ... = (∑ y∈f‘space M. (∑ z∈g‘space M.
      if ∃ x∈space M. y = f x ∧ z = g x then y ∗ emeasure M {x∈space M. g x =
z} else 0))
    by (auto intro!: sum.cong simp: sum_distrib_left)
  also have ... = ?r
    by (subst sum.swap)
       (auto intro!: sum.cong simp: sum.If_cases scaleR_sum_right[symmetric] eq)
  finally show integral^S M f = ?r .
qed

lemma simple_integral_add[simp]:
```

assumes *f*: *simple_function M f* **and** $\bigwedge x.\ 0 \le f\ x$ **and** *g*: *simple_function M g* **and** $\bigwedge x.\ 0 \le g\ x$
shows $(\int^S x.\ f\ x + g\ x\ \partial M) = integral^S\ M\ f + integral^S\ M\ g$
proof −
have $(\int^S x.\ f\ x + g\ x\ \partial M) =$
$(\sum y \in (\lambda x.\ (f\ x,\ g\ x))\ `space\ M.\ (fst\ y + snd\ y) * emeasure\ M\ \{x \in space\ M.\ (f\ x,\ g\ x) = y\})$
by (*intro simple_function_partition*) (*auto intro*: *f g*)
also have $\ldots = (\sum y \in (\lambda x.\ (f\ x,\ g\ x))\ `space\ M.\ fst\ y * emeasure\ M\ \{x \in space\ M.\ (f\ x,\ g\ x) = y\}) +$
$(\sum y \in (\lambda x.\ (f\ x,\ g\ x))\ `space\ M.\ snd\ y * emeasure\ M\ \{x \in space\ M.\ (f\ x,\ g\ x) = y\})$
using *assms*(*2*,*4*) **by** (*auto intro*!: *sum.cong distrib_right simp*: *sum.distrib*[*symmetric*])
also have $(\sum y \in (\lambda x.\ (f\ x,\ g\ x))\ `space\ M.\ fst\ y * emeasure\ M\ \{x \in space\ M.\ (f\ x,\ g\ x) = y\}) = (\int^S x.\ f\ x\ \partial M)$
by (*intro simple_function_partition*[*symmetric*]) (*auto intro*: *f g*)
also have $(\sum y \in (\lambda x.\ (f\ x,\ g\ x))\ `space\ M.\ snd\ y * emeasure\ M\ \{x \in space\ M.\ (f\ x,\ g\ x) = y\}) = (\int^S x.\ g\ x\ \partial M)$
by (*intro simple_function_partition*[*symmetric*]) (*auto intro*: *f g*)
finally show *?thesis* **.**
qed

lemma *simple_integral_sum*[*simp*]:
assumes $\bigwedge i\ x.\ i \in P \Longrightarrow 0 \le f\ i\ x$
assumes $\bigwedge i.\ i \in P \Longrightarrow simple\_function\ M\ (f\ i)$
shows $(\int^S x.\ (\sum i \in P.\ f\ i\ x)\ \partial M) = (\sum i \in P.\ integral^S\ M\ (f\ i))$
proof *cases*
assume *finite P*
from *this assms* **show** *?thesis*
by *induct* (*auto*)
qed *auto*

lemma *simple_integral_mult*[*simp*]:
assumes *f*: *simple_function M f*
shows $(\int^S x.\ c * f\ x\ \partial M) = c * integral^S\ M\ f$
proof −
have $(\int^S x.\ c * f\ x\ \partial M) = (\sum y \in f\ `\ space\ M.\ (c * y) * emeasure\ M\ \{x \in space\ M.\ f\ x = y\})$
using *f* **by** (*intro simple_function_partition*) *auto*
also have $\ldots = c * integral^S\ M\ f$
using *f* **unfolding** *simple_integral_def*
by (*subst sum_distrib_left*) (*auto simp*: *mult.assoc Int_def conj_commute*)
finally show *?thesis* **.**
qed

lemma *simple_integral_mono_AE*:
assumes *f*[*measurable*]: *simple_function M f* **and** *g*[*measurable*]: *simple_function M g*
and *mono*: *AE x in M. f x* ≤ *g x*

```
  shows integral^S M f ≤ integral^S M g
proof −
  let ?μ = λP. emeasure M {x∈space M. P x}
  have integral^S M f = (∑ y∈(λx. (f x, g x))‘space M. fst y ∗ ?μ (λx. (f x, g x)
= y))
    using f g by (intro simple_function_partition) auto
  also have ... ≤ (∑ y∈(λx. (f x, g x))‘space M. snd y ∗ ?μ (λx. (f x, g x) = y))
  proof (clarsimp intro!: sum_mono)
    fix x assume x ∈ space M
    let ?M = ?μ (λy. f y = f x ∧ g y = g x)
    show f x ∗ ?M ≤ g x ∗ ?M
    proof cases
      assume ?M ≠ 0
      then have 0 < ?M
        by (simp add: less_le)
      also have ... ≤ ?μ (λy. f x ≤ g x)
        using mono by (force intro: emeasure_mono_AE)
      finally have ¬ ¬ f x ≤ g x
        by (intro notI) auto
      then show ?thesis
        by (intro mult_right_mono) auto
    qed simp
  qed
  also have ... = integral^S M g
    using f g by (intro simple_function_partition[symmetric]) auto
  finally show ?thesis .
qed

lemma simple_integral_mono:
  assumes simple_function M f and simple_function M g
  and mono: ⋀ x. x ∈ space M ⟹ f x ≤ g x
  shows integral^S M f ≤ integral^S M g
  using assms by (intro simple_integral_mono_AE) auto

lemma simple_integral_cong_AE:
  assumes simple_function M f and simple_function M g
  and AE x in M. f x = g x
  shows integral^S M f = integral^S M g
  using assms by (auto simp: eq_iff intro!: simple_integral_mono_AE)

lemma simple_integral_cong′:
  assumes sf: simple_function M f simple_function M g
  and mea: (emeasure M) {x∈space M. f x ≠ g x} = 0
  shows integral^S M f = integral^S M g
proof (intro simple_integral_cong_AE sf AE_I)
  show (emeasure M) {x∈space M. f x ≠ g x} = 0 by fact
  show {x ∈ space M. f x ≠ g x} ∈ sets M
    using sf[THEN borel_measurable_simple_function] by auto
qed simp
```

lemma *simple_integral_indicator*:
assumes *A*: $A \in$ *sets M*
assumes *f*: *simple_function M f*
shows $(\int^S x.\ f\ x * \mathit{indicator}\ A\ x\ \partial M) =$
$(\sum x \in f\ `\ \mathit{space}\ M.\ x * \mathit{emeasure}\ M\ (f\ -`\ \{x\} \cap \mathit{space}\ M \cap A))$
proof −
have *eq*: $(\lambda x.\ (f\ x,\ \mathit{indicator}\ A\ x))\ `\ \mathit{space}\ M \cap \{x.\ \mathit{snd}\ x = 1\} = (\lambda x.\ (f\ x,\ 1::\mathit{ennreal}))`A$
using *A*[*THEN sets.sets_into_space*] **by** (*auto simp*: *indicator_def image_iff split*: *if_split_asm*)
have *eq2*: $\bigwedge x.\ f\ x \notin f\ `\ A \Longrightarrow f\ -`\ \{f\ x\} \cap \mathit{space}\ M \cap A = \{\}$
by (*auto simp*: *image_iff*)

have $(\int^S x.\ f\ x * \mathit{indicator}\ A\ x\ \partial M) =$
$(\sum y \in (\lambda x.\ (f\ x,\ \mathit{indicator}\ A\ x))`\mathit{space}\ M.\ (\mathit{fst}\ y * \mathit{snd}\ y) * \mathit{emeasure}\ M\ \{x \in \mathit{space}\ M.\ (f\ x,\ \mathit{indicator}\ A\ x) = y\})$
using *assms* **by** (*intro simple_function_partition*) *auto*
also have $\ldots = (\sum y \in (\lambda x.\ (f\ x,\ \mathit{indicator}\ A\ x::\mathit{ennreal}))`\mathit{space}\ M.$
*if snd y = 1 then fst y * emeasure M (f -` {fst y} ∩ space M ∩ A) else 0*)
by (*auto simp*: *indicator_def split*: *if_split_asm intro*!: *arg_cong2*[**where** *f*=(∗)] *arg_cong2*[**where** *f*=*emeasure*] *sum.cong*)
also have $\ldots = (\sum y \in (\lambda x.\ (f\ x,\ 1::\mathit{ennreal}))`A.\ \mathit{fst}\ y * \mathit{emeasure}\ M\ (f\ -`\ \{\mathit{fst}\ y\} \cap \mathit{space}\ M \cap A))$
using *assms* **by** (*subst sum.If_cases*) (*auto intro*!: *simple_functionD*(*1*) *simp*: *eq*)
also have $\ldots = (\sum y \in \mathit{fst}`(\lambda x.\ (f\ x,\ 1::\mathit{ennreal}))`A.\ y * \mathit{emeasure}\ M\ (f\ -`\ \{y\} \cap \mathit{space}\ M \cap A))$
by (*subst sum.reindex* [*of fst*]) (*auto simp*: *inj_on_def*)
also have $\ldots = (\sum x \in f\ `\ \mathit{space}\ M.\ x * \mathit{emeasure}\ M\ (f\ -`\ \{x\} \cap \mathit{space}\ M \cap A))$
using *A*[*THEN sets.sets_into_space*]
by (*intro sum.mono_neutral_cong_left simple_functionD f*) (*auto simp*: *image_comp comp_def eq2*)
finally show *?thesis* **.**
qed

lemma *simple_integral_indicator_only*[*simp*]:
assumes $A \in$ *sets M*
shows $\mathit{integral}^S\ M\ (\mathit{indicator}\ A) = \mathit{emeasure}\ M\ A$
using *simple_integral_indicator*[*OF assms*, *of* $\lambda x.\ 1$] *sets.sets_into_space*[*OF assms*]
by (*simp_all add*: *image_constant_conv Int_absorb1 split*: *if_split_asm*)

lemma *simple_integral_null_set*:
assumes *simple_function M u* $\bigwedge x.\ 0 \le u\ x$ **and** $N \in$ *null_sets M*
shows $(\int^S x.\ u\ x * \mathit{indicator}\ N\ x\ \partial M) = 0$
proof −
have *AE x in M*. *indicator N x* = (*0* :: *ennreal*)

using ‹$N \in$ *null_sets* M› **by** (*auto simp*: *indicator_def intro*!: *AE_I*[*of* _ _ N])
then have $(\int^S x.\ u\ x * \mathit{indicator}\ N\ x\ \partial M) = (\int^S x.\ 0\ \partial M)$
using *assms* **apply** (*intro simple_integral_cong_AE*) **by** *auto*
then show *?thesis* **by** *simp*
qed

lemma *simple_integral_cong_AE_mult_indicator*:
assumes *sf*: *simple_function M f* **and** *eq*: *AE x in M*. $x \in S$ **and** $S \in$ *sets M*
shows $\mathit{integral}^S\ M\ f = (\int^S x.\ f\ x * \mathit{indicator}\ S\ x\ \partial M)$
using *assms* **by** (*intro simple_integral_cong_AE*) *auto*

lemma *simple_integral_cmult_indicator*:
assumes *A*: $A \in$ *sets M*
shows $(\int^S x.\ c * \mathit{indicator}\ A\ x\ \partial M) = c * \mathit{emeasure}\ M\ A$
using *simple_integral_mult*[*OF simple_function_indicator*[*OF A*]]
unfolding *simple_integral_indicator_only*[*OF A*] **by** *simp*

lemma *simple_integral_nonneg*:
assumes *f*: *simple_function M f* **and** *ae*: *AE x in M*. $0 \le f\ x$
shows $0 \le \mathit{integral}^S\ M\ f$
proof −
have $\mathit{integral}^S\ M\ (\lambda x.\ 0) \le \mathit{integral}^S\ M\ f$
using *simple_integral_mono_AE*[*OF* _ *f ae*] **by** *auto*
then show *?thesis* **by** *simp*
qed

7.5.4 Integral on nonnegative functions

definition *nn_integral* :: *'a measure* ⇒ (*'a* ⇒ *ennreal*) ⇒ *ennreal* (‹$\mathit{integral}^N$›)
where
$\mathit{integral}^N\ M\ f = (\mathit{SUP}\ g \in \{g.\ \mathit{simple\_function}\ M\ g \wedge g \le f\}.\ \mathit{integral}^S\ M\ g)$

syntax
_nn_integral :: *pttrn* ⇒ *ennreal* ⇒ *'a measure* ⇒ *ennreal* (‹(‹*indent=2 notation=‹binder integral›*›$\int^+$(*2* _./ _)/ ∂_)› [*60,61*] *110*)
syntax_consts
_nn_integral == *nn_integral*
translations
$\int^+ x.\ f\ \partial M$ == *CONST nn_integral M* $(\lambda x.\ f)$

lemma *nn_integral_def_finite*:
$\mathit{integral}^N\ M\ f = (\mathit{SUP}\ g \in \{g.\ \mathit{simple\_function}\ M\ g \wedge g \le f \wedge (\forall x.\ g\ x < \mathit{top})\}.\ \mathit{integral}^S\ M\ g)$
(**is** _ = *Sup* (*?A* ‘ *?f*))
unfolding *nn_integral_def*
proof (*safe intro*!: *antisym SUP_least*)
fix *g* **assume** *g*[*measurable*]: *simple_function M g* $g \le f$

```
  show integral^S M g ≤ Sup (?A ‘ ?f)
  proof cases
    assume ae: AE x in M. g x ≠ top
    let ?G = {x ∈ space M. g x ≠ top}
    have integral^S M g = integral^S M (λx. g x * indicator ?G x)
    proof (rule simple_integral_cong_AE)
      show AE x in M. g x = g x * indicator ?G x
        using ae AE_space by eventually_elim auto
    qed (insert g, auto)
    also have ... ≤ Sup (?A ‘ ?f)
        using g by (intro SUP_upper) (auto simp: le_fun_def less_top split:
split_indicator)
    finally show ?thesis .
  next
    assume nAE: ¬ (AE x in M. g x ≠ top)
    then have emeasure M {x∈space M. g x = top} ≠ 0 (is emeasure M ?G ≠ 0)
      by (subst (asm) AE_iff_measurable[OF _ refl]) auto
    then have top = (SUP n. (∫^S x. of_nat n * indicator ?G x ∂M))
    by (simp add: ennreal_SUP_of_nat_eq_top ennreal_top_eq_mult_iff SUP_mult_right_ennreal[symmetric])
    also have ... ≤ Sup (?A ‘ ?f)
      using g
      by (safe intro!: SUP_least SUP_upper)
           (auto simp: le_fun_def of_nat_less_top top_unique[symmetric] split:
split_indicator
                intro: order_trans[of _ g x f x for x, OF order_trans[of _ top]])
    finally show ?thesis
      by (simp add: top_unique del: SUP_eq_top_iff Sup_eq_top_iff)
  qed
qed (auto intro: SUP_upper)

lemma nn_integral_mono_AE:
  assumes ae: AE x in M. u x ≤ v x shows integral^N M u ≤ integral^N M v
  unfolding nn_integral_def
proof (safe intro!: SUP_mono)
  fix n assume n: simple_function M n n ≤ u
  from ae[THEN AE_E] obtain N
    where N: {x ∈ space M. ¬ u x ≤ v x} ⊆ N emeasure M N = 0 N ∈ sets M
    by auto
  then have ae_N: AE x in M. x ∉ N by (auto intro: AE_not_in)
  let ?n = λx. n x * indicator (space M - N) x
  have AE x in M. n x ≤ ?n x simple_function M ?n
    using n N ae_N by auto
  moreover
  { fix x have ?n x ≤ v x
    proof cases
      assume x: x ∈ space M - N
      with N have u x ≤ v x by auto
      with n(2)[THEN le_funD, of x] x show ?thesis
        by (auto simp: max_def split: if_split_asm)
```

```
    qed simp }
  then have ?n ≤ v by (auto simp: le_funI)
  moreover have integral^S M n ≤ integral^S M ?n
    using ae_N N n by (auto intro!: simple_integral_mono_AE)
  ultimately show ∃ m∈{g. simple_function M g ∧ g ≤ v}. integral^S M n ≤
integral^S M m
    by force
qed

lemma nn_integral_mono:
  (⋀x. x ∈ space M ⟹ u x ≤ v x) ⟹ integral^N M u ≤ integral^N M v
  by (auto intro: nn_integral_mono_AE)

lemma mono_nn_integral: mono F ⟹ mono (λx. integral^N M (F x))
  by (auto simp add: mono_def le_fun_def intro!: nn_integral_mono)

lemma nn_integral_cong_AE:
  AE x in M. u x = v x ⟹ integral^N M u = integral^N M v
  by (auto simp: eq_iff intro!: nn_integral_mono_AE)

lemma nn_integral_cong:
  (⋀x. x ∈ space M ⟹ u x = v x) ⟹ integral^N M u = integral^N M v
  by (auto intro: nn_integral_cong_AE)

lemma nn_integral_cong_simp:
  (⋀x. x ∈ space M =simp=> u x = v x) ⟹ integral^N M u = integral^N M v
  by (auto intro: nn_integral_cong simp: simp_implies_def)

lemma incseq_nn_integral:
  assumes incseq f shows incseq (λi. integral^N M (f i))
proof −
  have ⋀i x. f i x ≤ f (Suc i) x
    using assms by (auto dest!: incseq_SucD simp: le_fun_def)
  then show ?thesis
    by (auto intro!: incseq_SucI nn_integral_mono)
qed

lemma nn_integral_eq_simple_integral:
  assumes f: simple_function M f shows integral^N M f = integral^S M f
proof −
  let ?f = λx. f x ∗ indicator (space M) x
  have f': simple_function M ?f using f by auto
  have integral^N M ?f ≤ integral^S M ?f using f'
   by (force intro!: SUP_least simple_integral_mono simp: le_fun_def nn_integral_def)
  moreover have integral^S M ?f ≤ integral^N M ?f
    unfolding nn_integral_def
    using f' by (auto intro!: SUP_upper)
  ultimately show ?thesis
    by (simp cong: nn_integral_cong simple_integral_cong)
```

qed

Beppo-Levi monotone convergence theorem

lemma *nn_integral_monotone_convergence_SUP*:
assumes *f*: *incseq f* **and** [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel_measurable M*
shows $(\int^+ x.\ (SUP\ i.\ f\ i\ x)\ \partial M) = (SUP\ i.\ integral^N\ M\ (f\ i))$
proof (*rule antisym*)
show $(\int^+ x.\ (SUP\ i.\ f\ i\ x)\ \partial M) \leq (SUP\ i.\ (\int^+ x.\ f\ i\ x\ \partial M))$
unfolding *nn_integral_def_finite*[*of* _ $\lambda x.\ SUP\ i.\ f\ i\ x$]
proof (*safe intro*!: *SUP_least*)
fix *u* **assume** *sf_u*[*simp*]: *simple_function M u* **and**
u: $u \leq (\lambda x.\ SUP\ i.\ f\ i\ x)$ **and** *u_range*: $\forall x.\ u\ x < top$
note *sf_u*[*THEN borel_measurable_simple_function*, *measurable*]
show $integral^S\ M\ u \leq (SUP\ j.\ \int^+ x.\ f\ j\ x\ \partial M)$
proof (*rule ennreal_approx_unit*)
fix *a* :: *ennreal* **assume** $a < 1$
let *?au* $= \lambda x.\ a * u\ x$

let *?B* $= \lambda c\ i.\ \{x \in space\ M.\ ?au\ x = c \wedge c \leq f\ i\ x\}$
have $integral^S\ M\ ?au = (\sum c \in ?au`space\ M.\ c * (SUP\ i.\ emeasure\ M\ (?B\ c\ i)))$
unfolding *simple_integral_def*
proof (*intro sum.cong ennreal_mult_left_cong refl*)
fix *c* **assume** $c \in ?au\ `\ space\ M\ c \neq 0$
{ **fix** x' **assume** x': $x' \in space\ M\ ?au\ x' = c$
with ‹$c \neq 0$› *u_range* **have** $?au\ x' < 1 * u\ x'$
by (*intro ennreal_mult_strict_right_mono* ‹$a < 1$›) (*auto simp*: *less_le*)
also have $\ldots \leq (SUP\ i.\ f\ i\ x')$
using *u* **by** (*auto simp*: *le_fun_def*)
finally have $\exists i.\ ?au\ x' \leq f\ i\ x'$
by (*auto simp*: *less_SUP_iff intro*: *less_imp_le*) }
then have $*$: $?au - `\ \{c\} \cap space\ M = (\bigcup i.\ ?B\ c\ i)$
by *auto*
show $emeasure\ M\ (?au - `\ \{c\} \cap space\ M) = (SUP\ i.\ emeasure\ M\ (?B\ c\ i))$
unfolding $*$ **using** *f*
by (*intro SUP_emeasure_incseq*[*symmetric*])
(*auto simp*: *incseq_def le_fun_def intro*: *order_trans*)
qed
also have $\ldots = (SUP\ i.\ \sum c \in ?au`space\ M.\ c * emeasure\ M\ (?B\ c\ i))$
unfolding *SUP_mult_left_ennreal* **using** *f*
by (*intro ennreal_SUP_sum*[*symmetric*])
(*auto intro*!: *mult_mono emeasure_mono simp*: *incseq_def le_fun_def intro*: *order_trans*)
also have $\ldots \leq (SUP\ i.\ integral^N\ M\ (f\ i))$
proof (*intro SUP_subset_mono order_refl*)
fix *i*
have $(\sum c \in ?au`space\ M.\ c * emeasure\ M\ (?B\ c\ i)) =$
$(\int^S x.\ (a * u\ x) * indicator\ \{x \in space\ M.\ a * u\ x \leq f\ i\ x\}\ x\ \partial M)$

```
      by (subst simple_integral_indicator)
         (auto intro!: sum.cong ennreal_mult_left_cong arg_cong2[where
f=emeasure])
    also have ... = (∫⁺x. (a * u x) * indicator {x∈space M. a * u x ≤ f i x}
x ∂M)
      by (rule nn_integral_eq_simple_integral[symmetric]) simp
    also have ... ≤ (∫⁺x. f i x ∂M)
      by (intro nn_integral_mono) (auto split: split_indicator)
    finally show (∑ c∈?au`space M. c * emeasure M (?B c i)) ≤ (∫⁺x. f i x
∂M) .
  qed
  finally show a * integral^S M u ≤ (SUP i. integral^N M (f i))
    by simp
 qed
qed
qed (auto intro!: SUP_least SUP_upper nn_integral_mono)

lemma sup_continuous_nn_integral[order_continuous_intros]:
  assumes f: ⋀y. sup_continuous (f y)
  assumes [measurable]: ⋀x. (λy. f y x) ∈ borel_measurable M
  shows sup_continuous (λx. (∫⁺y. f y x ∂M))
  unfolding sup_continuous_def
proof safe
  fix C :: nat ⇒ 'b assume C: incseq C
  with sup_continuous_mono[OF f] show (∫⁺ y. f y (Sup (C ` UNIV)) ∂M) =
(SUP i. ∫⁺ y. f y (C i) ∂M)
    unfolding sup_continuousD[OF f C]
    by (subst nn_integral_monotone_convergence_SUP) (auto simp: mono_def
le_fun_def)
qed

theorem nn_integral_monotone_convergence_SUP_AE:
  assumes f: ⋀i. AE x in M. f i x ≤ f (Suc i) x ⋀i. f i ∈ borel_measurable M
  shows (∫⁺ x. (SUP i. f i x) ∂M) = (SUP i. integral^N M (f i))
proof −
  from f have AE x in M. ∀i. f i x ≤ f (Suc i) x
    by (simp add: AE_all_countable)
  from this[THEN AE_E] obtain N
    where N: {x ∈ space M. ¬ (∀i. f i x ≤ f (Suc i) x)} ⊆ N emeasure M N = 0
N ∈ sets M
    by auto
  let ?f = λi x. if x ∈ space M − N then f i x else 0
  have f_eq: AE x in M. ∀i. ?f i x = f i x using N by (auto intro!: AE_I[of _
_ N])
  then have (∫⁺ x. (SUP i. f i x) ∂M) = (∫⁺ x. (SUP i. ?f i x) ∂M)
    by (auto intro!: nn_integral_cong_AE)
  also have ... = (SUP i. (∫⁺ x. ?f i x ∂M))
  proof (rule nn_integral_monotone_convergence_SUP)
    show incseq ?f using N(1) by (force intro!: incseq_SucI le_funI)
```

```
    { fix i show (λx. if x ∈ space M − N then f i x else 0) ∈ borel_measurable M
        using f N(3) by (intro measurable_If_set) auto }
  qed
  also have ... = (SUP i. (∫⁺ x. f i x ∂M))
   using f_eq by (force intro!: arg_cong[where f = λf. Sup (range f)] nn_integral_cong_AE
ext)
  finally show ?thesis .
qed

lemma nn_integral_monotone_convergence_simple:
  incseq f ⟹ (⋀i. simple_function M (f i)) ⟹ (SUP i. ∫ˢ x. f i x ∂M) = (∫⁺x.
(SUP i. f i x) ∂M)
  using nn_integral_monotone_convergence_SUP[of f M]
 by (simp add: nn_integral_eq_simple_integral[symmetric] borel_measurable_simple_function)

lemma SUP_simple_integral_sequences:
  assumes f: incseq f ⋀i. simple_function M (f i)
  and g: incseq g ⋀i. simple_function M (g i)
  and eq: AE x in M. (SUP i. f i x) = (SUP i. g i x)
  shows (SUP i. integral^S M (f i)) = (SUP i. integral^S M (g i))
    (is Sup (?F ` _) = Sup (?G ` _))
proof −
  have (SUP i. integral^S M (f i)) = (∫⁺x. (SUP i. f i x) ∂M)
    using f by (rule nn_integral_monotone_convergence_simple)
  also have ... = (∫⁺x. (SUP i. g i x) ∂M)
    unfolding eq[THEN nn_integral_cong_AE] ..
  also have ... = (SUP i. ?G i)
    using g by (rule nn_integral_monotone_convergence_simple[symmetric])
  finally show ?thesis by simp
qed

lemma nn_integral_const[simp]: (∫⁺ x. c ∂M) = c * emeasure M (space M)
  by (subst nn_integral_eq_simple_integral) auto

lemma nn_integral_linear:
  assumes f: f ∈ borel_measurable M and g: g ∈ borel_measurable M
  shows (∫⁺ x. a * f x + g x ∂M) = a * integral^N M f + integral^N M g
    (is integral^N M ?L = _)
proof −
  obtain u
    where ⋀i. simple_function M (u i) incseq u ⋀i x. u i x < top ⋀x. (SUP i. u
i x) = f x
    using borel_measurable_implies_simple_function_sequence' f(1)
    by auto
  note u = nn_integral_monotone_convergence_simple[OF this(2,1)] this

  obtain v where
    ⋀i. simple_function M (v i) incseq v ⋀i x. v i x < top ⋀x. (SUP i. v i x) = g
x
```

```
    using borel_measurable_implies_simple_function_sequence' g(1)
    by auto
  note v = nn_integral_monotone_convergence_simple[OF this(2,1)] this

  let ?L' = λi x. a * u i x + v i x

  have ?L ∈ borel_measurable M using assms by auto
  from borel_measurable_implies_simple_function_sequence'[OF this]
  obtain l where ⋀i. simple_function M (l i) incseq l ⋀i x. l i x < top ⋀x. (SUP
i. l i x) = a * f x + g x
    by auto
  note l = nn_integral_monotone_convergence_simple[OF this(2,1)] this

  have inc: incseq (λi. a * integral^S M (u i)) incseq (λi. integral^S M (v i))
   using u v by (auto simp: incseq_Suc_iff le_fun_def intro!: add_mono mult_left_mono
simple_integral_mono)

  have l': (SUP i. integral^S M (l i)) = (SUP i. integral^S M (?L' i))
  proof (rule SUP_simple_integral_sequences[OF l(3,2)])
    show incseq ?L' ⋀i. simple_function M (?L' i)
      using u v unfolding incseq_Suc_iff le_fun_def
      by (auto intro!: add_mono mult_left_mono)
    { fix x
      have (SUP i. a * u i x + v i x) = a * (SUP i. u i x) + (SUP i. v i x)
      using u(3) v(3) u(4)[of _ x] v(4)[of _ x] unfolding SUP_mult_left_ennreal
      by (auto intro!: ennreal_SUP_add simp: incseq_Suc_iff le_fun_def add_mono
mult_left_mono) }
    then show AE x in M. (SUP i. l i x) = (SUP i. ?L' i x)
      unfolding l(5) using u(5) v(5) by (intro AE_I2) auto
  qed
  also have ... = (SUP i. a * integral^S M (u i) + integral^S M (v i))
    using u(2) v(2) by auto
  finally show ?thesis
    unfolding l(5)[symmetric] l(1)[symmetric]
   by (simp add: ennreal_SUP_add[OF inc] v u SUP_mult_left_ennreal[symmetric])
qed

lemma nn_integral_cmult: f ∈ borel_measurable M ⟹ (∫⁺ x. c * f x ∂M) = c
* integral^N M f
  using nn_integral_linear[of f M λx. 0 c] by simp

lemma nn_integral_multc: f ∈ borel_measurable M ⟹ (∫⁺ x. f x * c ∂M) =
integral^N M f * c
  unfolding mult.commute[of _ c] nn_integral_cmult by simp

lemma nn_integral_divide: f ∈ borel_measurable M ⟹ (∫⁺ x. f x / c ∂M) =
(∫⁺ x. f x ∂M) / c
   unfolding divide_ennreal_def by (rule nn_integral_multc)
```

lemma *nn_integral_indicator*[*simp*]: $A \in sets\ M \Longrightarrow (\int^+ x.\ indicator\ A\ x \partial M) = (emeasure\ M)\ A$
by (*subst nn_integral_eq_simple_integral*) (*auto simp*: *simple_integral_indicator*)

lemma *nn_integral_cmult_indicator*: $A \in sets\ M \Longrightarrow (\int^+ x.\ c * indicator\ A\ x\ \partial M) = c * emeasure\ M\ A$
by (*subst nn_integral_eq_simple_integral*) (*auto*)

lemma *nn_integral_indicator'*:
assumes [*measurable*]: $A \cap space\ M \in sets\ M$
shows $(\int^+ x.\ indicator\ A\ x\ \partial M) = emeasure\ M\ (A \cap space\ M)$
proof −
have $(\int^+ x.\ indicator\ A\ x\ \partial M) = (\int^+ x.\ indicator\ (A \cap space\ M)\ x\ \partial M)$
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also have $\ldots = emeasure\ M\ (A \cap space\ M)$
by *simp*
finally show *?thesis* **.**
qed

lemma *nn_integral_indicator_singleton*[*simp*]:
assumes [*measurable*]: $\{y\} \in sets\ M$ **shows** $(\int^+ x.\ f\ x * indicator\ \{y\}\ x\ \partial M) = f\ y * emeasure\ M\ \{y\}$
proof −
have $(\int^+ x.\ f\ x * indicator\ \{y\}\ x\ \partial M) = (\int^+ x.\ f\ y * indicator\ \{y\}\ x\ \partial M)$
by (*auto intro*!: *nn_integral_cong split*: *split_indicator*)
then show *?thesis*
by (*simp add*: *nn_integral_cmult*)
qed

lemma *nn_integral_set_ennreal*:
$(\int^+ x.\ ennreal\ (f\ x) * indicator\ A\ x\ \partial M) = (\int^+ x.\ ennreal\ (f\ x * indicator\ A\ x)\ \partial M)$
by (*rule nn_integral_cong*) (*simp split*: *split_indicator*)

lemma *nn_integral_indicator_singleton'*[*simp*]:
assumes [*measurable*]: $\{y\} \in sets\ M$
shows $(\int^+ x.\ ennreal\ (f\ x * indicator\ \{y\}\ x)\ \partial M) = f\ y * emeasure\ M\ \{y\}$
by (*subst nn_integral_set_ennreal*[*symmetric*]) (*simp*)

lemma *nn_integral_add*:
$f \in borel\_measurable\ M \Longrightarrow g \in borel\_measurable\ M \Longrightarrow (\int^+ x.\ f\ x + g\ x\ \partial M) = integral^N\ M\ f + integral^N\ M\ g$
using *nn_integral_linear*[*of f M g 1*] **by** *simp*

lemma *nn_integral_sum*:
$(\bigwedge i.\ i \in P \Longrightarrow f\ i \in borel\_measurable\ M) \Longrightarrow (\int^+ x.\ (\sum i \in P.\ f\ i\ x)\ \partial M) = (\sum i \in P.\ integral^N\ M\ (f\ i))$
by (*induction P rule*: *infinite_finite_induct*) (*auto simp*: *nn_integral_add*)

theorem *nn_integral_suminf*:
assumes *f*: $\bigwedge i.\ f\ i \in$ *borel_measurable M*
shows $(\int^+ x.\ (\sum i.\ f\ i\ x)\ \partial M) = (\sum i.\ integral^N\ M\ (f\ i))$
proof −
have *all_pos*: *AE x in M*. $\forall i.\ 0 \le f\ i\ x$
using *assms* **by** (*auto simp*: *AE_all_countable*)
have $(\sum i.\ integral^N\ M\ (f\ i)) = (SUP\ n.\ \sum i{<}n.\ integral^N\ M\ (f\ i))$
by (*rule suminf_eq_SUP*)
also have ... $= (SUP\ n.\ \int^+ x.\ (\sum i{<}n.\ f\ i\ x)\ \partial M)$
unfolding *nn_integral_sum*[*OF f*] **..**
also have ... $= \int^+ x.\ (SUP\ n.\ \sum i{<}n.\ f\ i\ x)\ \partial M$ **using** *f all_pos*
by (*intro nn_integral_monotone_convergence_SUP_AE*[*symmetric*])
(*elim AE_mp*, *auto simp*: *sum_nonneg simp del*: *sum.lessThan_Suc intro*!: *AE_I2 sum_mono2*)
also have ... $= \int^+ x.\ (\sum i.\ f\ i\ x)\ \partial M$ **using** *all_pos*
by (*intro nn_integral_cong_AE*) (*auto simp*: *suminf_eq_SUP*)
finally show *?thesis* **by** *simp*
qed

lemma *nn_integral_bound_simple_function*:
assumes *bnd*: $\bigwedge x.\ x \in space\ M \Longrightarrow f\ x < \infty$
assumes *f*[*measurable*]: *simple_function M f*
assumes *supp*: *emeasure M* $\{x{\in}space\ M.\ f\ x \neq 0\} < \infty$
shows *nn_integral M f* $< \infty$
proof *cases*
assume *space M* $= \{\}$
then have *nn_integral M f* $= (\int^+ x.\ 0\ \partial M)$
by (*intro nn_integral_cong*) *auto*
then show *?thesis* **by** *simp*
next
assume *space M* $\neq \{\}$
with *simple_functionD*(*1*)[*OF f*] *bnd* **have** *bnd*: $0 \le Max\ (f`space\ M) \wedge Max\ (f`space\ M) < \infty$
by (*subst Max_less_iff*) (*auto simp*: *Max_ge_iff*)

have *nn_integral M f* $\le (\int^+ x.\ Max\ (f`space\ M) * indicator\ \{x{\in}space\ M.\ f\ x \neq 0\}\ x\ \partial M)$
proof (*rule nn_integral_mono*)
fix *x* **assume** $x \in space\ M$
with *f* **show** $f\ x \le Max\ (f\ `\ space\ M) * indicator\ \{x \in space\ M.\ f\ x \neq 0\}\ x$
by (*auto split*: *split_indicator intro*!: *Max_ge simple_functionD*)
qed
also have ... $< \infty$
using *bnd supp* **by** (*subst nn_integral_cmult*) (*auto simp*: *ennreal_mult_less_top*)
finally show *?thesis* **.**
qed

theorem *nn_integral_Markov_inequality*:
assumes *u*: $(\lambda x.\ u\ x * indicator\ A\ x) \in$ *borel_measurable M* **and** $A \in sets\ M$

shows (*emeasure M*) ({*x*∈*A*. *1* ≤ *c* ∗ *u x*}) ≤ *c* ∗ (∫⁺ *x*. *u x* ∗ *indicator A x* ∂*M*)
(**is** (*emeasure M*) *?A* ≤ _ ∗ *?PI*)
proof −
define *u′* **where** *u′* = (λ*x*. *u x* ∗ *indicator A x*)
have [*measurable*]: *u′* ∈ *borel_measurable M*
using *u* **unfolding** *u′_def* .
have {*x*∈*space M*. *c* ∗ *u′ x* ≥ *1*} ∈ *sets M*
by *measurable*
also have {*x*∈*space M*. *c* ∗ *u′ x* ≥ *1*} = *?A*
using *sets.sets_into_space*[*OF* ‹*A* ∈ *sets M*›] **by** (*auto simp*: *u′_def indicator_def*)
finally have (*emeasure M*) *?A* = (∫⁺ *x*. *indicator ?A x* ∂*M*)
using *nn_integral_indicator* **by** *simp*
also have ... ≤ (∫⁺ *x*. *c* ∗ (*u x* ∗ *indicator A x*) ∂*M*)
using *u* **by** (*auto intro*!: *nn_integral_mono_AE simp*: *indicator_def*)
also have ... = *c* ∗ (∫⁺ *x*. *u x* ∗ *indicator A x* ∂*M*)
using *assms* **by** (*auto intro*!: *nn_integral_cmult*)
finally show *?thesis* .
qed

lemma *Chernoff_ineq_nn_integral_ge*:
assumes *s*: *s* > *0* **and** [*measurable*]: *A* ∈ *sets M*
assumes [*measurable*]: (λ*x*. *f x* ∗ *indicator A x*) ∈ *borel_measurable M*
shows *emeasure M* {*x*∈*A*. *f x* ≥ *a*} ≤
ennreal (*exp* (−*s* ∗ *a*)) ∗ *nn_integral M* (λ*x*. *ennreal* (*exp* (*s* ∗ *f x*)) ∗ *indicator A x*)
proof −
define *f′* **where** *f′* = (λ*x*. *f x* ∗ *indicator A x*)
have [*measurable*]: *f′* ∈ *borel_measurable M*
using *assms*(*3*) **unfolding** *f′_def* **by** *assumption*
have (λ*x*. *ennreal* (*exp* (*s* ∗ *f′ x*)) ∗ *indicator A x*) ∈ *borel_measurable M*
by *simp*
also have (λ*x*. *ennreal* (*exp* (*s* ∗ *f′ x*)) ∗ *indicator A x*) =
(λ*x*. *ennreal* (*exp* (*s* ∗ *f x*)) ∗ *indicator A x*)
by (*auto simp*: *f′_def indicator_def fun_eq_iff*)
finally have *meas*: ... ∈ *borel_measurable M* .

have {*x*∈*A*. *f x* ≥ *a*} = {*x*∈*A*. *ennreal* (*exp* (−*s* ∗ *a*)) ∗ *ennreal* (*exp* (*s* ∗ *f x*)) ≥ *1*}
using *s* **by** (*auto simp*: *exp_minus field_simps simp flip*: *ennreal_mult*)
also have *emeasure M* ... ≤ *ennreal* (*exp* (−*s* ∗ *a*)) ∗
(∫⁺*x*. *ennreal* (*exp* (*s* ∗ *f x*)) ∗ *indicator A x* ∂*M*)
by (*intro order.trans*[*OF nn_integral_Markov_inequality*] *meas*) *auto*
finally show *?thesis* .
qed

lemma *Chernoff_ineq_nn_integral_le*:
assumes *s*: *s* > *0* **and** [*measurable*]: *A* ∈ *sets M*

assumes [*measurable*]: $f \in$ *borel_measurable M*
shows *emeasure M* $\{x \in A.\ f\ x \leq a\} \leq$
ennreal (*exp* $(s * a)) *$ *nn_integral M* $(\lambda x.$ *ennreal* (*exp* $(-s * f\ x)) *$ *indicator A x*)
using *Chernoff_ineq_nn_integral_ge*[*of s A M* $\lambda x.\ -f\ x\ -a$] *assms* **by** *simp*

lemma *nn_integral_noteq_infinite*:
assumes *g*: $g \in$ *borel_measurable M* **and** $integral^N\ M\ g \neq \infty$
shows *AE x in M*. $g\ x \neq \infty$
proof (*rule ccontr*)
assume *c*: $\neg$ (*AE x in M*. $g\ x \neq \infty$)
have (*emeasure M*) $\{x \in$*space M*. $g\ x = \infty\} \neq 0$
using *c g* **by** (*auto simp add*: *AE_iff_null*)
then have $0 <$ (*emeasure M*) $\{x \in$*space M*. $g\ x = \infty\}$
by (*auto simp*: *zero_less_iff_neq_zero*)
then have $\infty = \infty *$ (*emeasure M*) $\{x \in$*space M*. $g\ x = \infty\}$
by (*auto simp*: *ennreal_top_eq_mult_iff*)
also have $\ldots \leq (\int^+ x.\ \infty *$ *indicator* $\{x \in$*space M*. $g\ x = \infty\}\ x\ \partial M)$
using *g* **by** (*subst nn_integral_cmult_indicator*) *auto*
also have $\ldots \leq integral^N\ M\ g$
using *assms* **by** (*auto intro*!: *nn_integral_mono_AE simp*: *indicator_def*)
finally show *False*
using ‹$integral^N\ M\ g \neq \infty$› **by** (*auto simp*: *top_unique*)
qed

lemma *nn_integral_PInf*:
assumes *f*: $f \in$ *borel_measurable M* **and** *not_Inf*: $integral^N\ M\ f \neq \infty$
shows *emeasure M* $(f -` \{\infty\} \cap$ *space M*$) = 0$
proof −
have $\infty *$ *emeasure M* $(f -` \{\infty\} \cap$ *space M*$) = (\int^+ x.\ \infty *$ *indicator* $(f -`$ $\{\infty\} \cap$ *space M*$)\ x\ \partial M)$
using *f* **by** (*subst nn_integral_cmult_indicator*) (*auto simp*: *measurable_sets*)
also have $\ldots \leq integral^N\ M\ f$
by (*auto intro*!: *nn_integral_mono simp*: *indicator_def*)
finally have $\infty *$ (*emeasure M*) $(f -` \{\infty\} \cap$ *space M*$) \leq integral^N\ M\ f$
by *simp*
then show *?thesis*
using *assms* **by** (*auto simp*: *ennreal_top_mult top_unique split*: *if_split_asm*)
qed

lemma *simple_integral_PInf*:
simple_function M f $\Longrightarrow$ $integral^S\ M\ f \neq \infty \Longrightarrow$ *emeasure M* $(f -` \{\infty\} \cap$ *space* $M) = 0$
by (*rule nn_integral_PInf*) (*auto simp*: *nn_integral_eq_simple_integral borel_measurable_simple_fun*

lemma *nn_integral_PInf_AE*:
assumes $f \in$ *borel_measurable M* $integral^N\ M\ f \neq \infty$ **shows** *AE x in M*. $f\ x \neq \infty$
proof (*rule AE_I*)

show (*emeasure M*) (*f* −‘ {∞} ∩ *space M*) = *0*
by (*rule nn_integral_PInf*[*OF assms*])
show *f* −‘ {∞} ∩ *space M* ∈ *sets M*
using *assms* **by** (*auto intro*: *borel_measurable_vimage*)
qed *auto*

lemma *nn_integral_diff*:
assumes *f*: *f* ∈ *borel_measurable M*
and *g*: *g* ∈ *borel_measurable M*
and *fin*: *integral*^{N} *M g* ≠ ∞
and *mono*: *AE x in M*. *g x* ≤ *f x*
shows (∫$^+$ *x*. *f x* − *g x* ∂*M*) = *integral*^{N} *M f* − *integral*^{N} *M g*
proof −
have *diff*: (λ*x*. *f x* − *g x*) ∈ *borel_measurable M*
using *assms* **by** *auto*
have *AE x in M*. *f x* = *f x* − *g x* + *g x*
using *diff_add_cancel_ennreal mono nn_integral_noteq_infinite*[*OF g fin*] *assms* **by** *auto*
then have ∗∗: *integral*^{N} *M f* = (∫$^+$*x*. *f x* − *g x* ∂*M*) + *integral*^{N} *M g*
unfolding *nn_integral_add*[*OF diff g*, *symmetric*]
by (*rule nn_integral_cong_AE*)
show *?thesis* **unfolding** ∗∗
using *fin*
by (*cases rule*: *ennreal2_cases*[*of* ∫$^+$ *x*. *f x* − *g x* ∂*M integral*^{N} *M g*]) *auto*
qed

lemma *nn_integral_mult_bounded_inf*:
assumes *f*: *f* ∈ *borel_measurable M* (∫$^+$*x*. *f x* ∂*M*) < ∞ **and** *c*: *c* ≠ ∞ **and** *ae*: *AE x in M*. *g x* ≤ *c* ∗ *f x*
shows (∫$^+$*x*. *g x* ∂*M*) < ∞
proof −
have (∫$^+$*x*. *g x* ∂*M*) ≤ (∫$^+$*x*. *c* ∗ *f x* ∂*M*)
by (*intro nn_integral_mono_AE ae*)
also have (∫$^+$*x*. *c* ∗ *f x* ∂*M*) < ∞
using *c f* **by** (*subst nn_integral_cmult*) (*auto simp*: *ennreal_mult_less_top top_unique not_less*)
finally show *?thesis* **.**
qed

Fatou's lemma: convergence theorem on limes inferior

lemma *nn_integral_monotone_convergence_INF_AE′*:
assumes *f*: ⋀*i*. *AE x in M*. *f* (*Suc i*) *x* ≤ *f i x* **and** [*measurable*]: ⋀*i*. *f i* ∈ *borel_measurable M*
and ∗: (∫$^+$ *x*. *f 0 x* ∂*M*) < ∞
shows (∫$^+$ *x*. (*INF i*. *f i x*) ∂*M*) = (*INF i*. *integral*^{N} *M* (*f i*))
proof (*rule ennreal_minus_cancel*)
have *integral*^{N} *M* (*f 0*) − (∫$^+$ *x*. (*INF i*. *f i x*) ∂*M*) = (∫$^+$*x*. *f 0 x* − (*INF i*. *f i x*) ∂*M*)
proof (*rule nn_integral_diff*[*symmetric*])

```
    have (∫⁺ x. (INF i. f i x) ∂M) ≤ (∫⁺ x. f 0 x ∂M)
      by (intro nn_integral_mono INF_lower) simp
    with ∗ show (∫⁺ x. (INF i. f i x) ∂M) ≠ ∞
      by simp
  qed (auto intro: INF_lower)
  also have ... = (∫⁺x. (SUP i. f 0 x − f i x) ∂M)
    by (simp add: ennreal_INF_const_minus)
  also have ... = (SUP i. (∫⁺x. f 0 x − f i x ∂M))
  proof (intro nn_integral_monotone_convergence_SUP_AE)
    show AE x in M. f 0 x − f i x ≤ f 0 x − f (Suc i) x for i
      using f[of i] by eventually_elim (auto simp: ennreal_mono_minus)
  qed simp
  also have ... = (SUP i. nn_integral M (f 0) − (∫⁺x. f i x ∂M))
  proof (subst nn_integral_diff[symmetric])
    fix i
    have dec: AE x in M. ∀ i. f (Suc i) x ≤ f i x
      unfolding AE_all_countable using f by auto
    then show AE x in M. f i x ≤ f 0 x
      using dec by eventually_elim (auto intro: lift_Suc_antimono_le[of λi. f i x
0 i for x])
    then have (∫⁺ x. f i x ∂M) ≤ (∫⁺ x. f 0 x ∂M)
      by (rule nn_integral_mono_AE)
    with ∗ show (∫⁺ x. f i x ∂M) ≠ ∞
      by simp
  qed (insert f, auto simp: decseq_def le_fun_def)
  finally show integral^N M (f 0) − (∫⁺ x. (INF i. f i x) ∂M) =
    integral^N M (f 0) − (INF i. ∫⁺ x. f i x ∂M)
    by (simp add: ennreal_INF_const_minus)
qed (insert ∗, auto intro!: nn_integral_mono intro: INF_lower)

theorem nn_integral_monotone_convergence_INF_AE:
  fixes f :: nat ⇒ 'a ⇒ ennreal
  assumes f: ⋀i. AE x in M. f (Suc i) x ≤ f i x
    and [measurable]: ⋀i. f i ∈ borel_measurable M
    and fin: (∫⁺ x. f i x ∂M) < ∞
  shows (∫⁺ x. (INF i. f i x) ∂M) = (INF i. integral^N M (f i))
proof −
  { fix f :: nat ⇒ ennreal and j assume decseq f
    then have (INF i. f i) = (INF i. f (i + j))
      apply (intro INF_eq)
      apply (rule_tac x=i in bexI)
      apply (auto simp: decseq_def le_fun_def)
      done }
  note INF_shift = this
  have mono: AE x in M. ∀ i. f (Suc i) x ≤ f i x
    using f by (auto simp: AE_all_countable)
  then have AE x in M. (INF i. f i x) = (INF n. f (n + i) x)
    by eventually_elim (auto intro!: decseq_SucI INF_shift)
  then have (∫⁺ x. (INF i. f i x) ∂M) = (∫⁺ x. (INF n. f (n + i) x) ∂M)
```

by (*rule nn_integral_cong_AE*)
also have ... = (*INF n.* (∫⁺ *x. f* (*n* + *i*) *x* ∂*M*))
by (*rule nn_integral_monotone_convergence_INF_AE'*) (*insert assms*, *auto*)
also have ... = (*INF n.* (∫⁺ *x. f n x* ∂*M*))
by (*intro INF_shift*[*symmetric*] *decseq_SucI nn_integral_mono_AE f*)
finally show *?thesis* **.**
qed

lemma *nn_integral_monotone_convergence_INF_decseq*:
assumes *f*: *decseq f* **and** ∗: ⋀*i. f i* ∈ *borel_measurable M* (∫⁺ *x. f i x* ∂*M*) < ∞
shows (∫⁺ *x.* (*INF i. f i x*) ∂*M*) = (*INF i. integral*^{N} *M* (*f i*))
using *nn_integral_monotone_convergence_INF_AE*[*of f M i*, *OF _* ∗] *f* **by** (*simp add*: *decseq_SucD le_funD*)

theorem *nn_integral_liminf*:
fixes *u* :: *nat* ⇒ *'a* ⇒ *ennreal*
assumes *u*: ⋀*i. u i* ∈ *borel_measurable M*
shows (∫⁺ *x. liminf* (λ*n. u n x*) ∂*M*) ≤ *liminf* (λ*n. integral*^{N} *M* (*u n*))
proof −
have (∫⁺ *x. liminf* (λ*n. u n x*) ∂*M*) = (*SUP n.* ∫⁺ *x.* (*INF i*∈{*n*..}. *u i x*) ∂*M*)
unfolding *liminf_SUP_INF* **using** *u*
by (*intro nn_integral_monotone_convergence_SUP_AE*)
(*auto intro*!: *AE_I2 intro*: *INF_greatest INF_superset_mono*)
also have ... ≤ *liminf* (λ*n. integral*^{N} *M* (*u n*))
by (*auto simp*: *liminf_SUP_INF intro*!: *SUP_mono INF_greatest nn_integral_mono INF_lower*)
finally show *?thesis* **.**
qed

theorem *nn_integral_limsup*:
fixes *u* :: *nat* ⇒ *'a* ⇒ *ennreal*
assumes [*measurable*]: ⋀*i. u i* ∈ *borel_measurable M w* ∈ *borel_measurable M*
assumes *bounds*: ⋀*i. AE x in M. u i x* ≤ *w x* **and** *w*: (∫⁺*x. w x* ∂*M*) < ∞
shows *limsup* (λ*n. integral*^{N} *M* (*u n*)) ≤ (∫⁺ *x. limsup* (λ*n. u n x*) ∂*M*)
proof −
have *bnd*: *AE x in M.* ∀ *i. u i x* ≤ *w x*
using *bounds* **by** (*auto simp*: *AE_all_countable*)
then have (∫⁺ *x.* (*SUP n. u n x*) ∂*M*) ≤ (∫⁺ *x. w x* ∂*M*)
by (*auto intro*!: *nn_integral_mono_AE elim*: *eventually_mono intro*: *SUP_least*)
then have (∫⁺ *x. limsup* (λ*n. u n x*) ∂*M*) = (*INF n.* ∫⁺ *x.* (*SUP i*∈{*n*..}. *u i x*) ∂*M*)
unfolding *limsup_INF_SUP* **using** *bnd w*
by (*intro nn_integral_monotone_convergence_INF_AE'*)
(*auto intro*!: *AE_I2 intro*: *SUP_least SUP_subset_mono*)
also have ... ≥ *limsup* (λ*n. integral*^{N} *M* (*u n*))
by (*auto simp*: *limsup_INF_SUP intro*!: *INF_mono SUP_least exI nn_integral_mono SUP_upper*)
finally (*xtrans*) **show** *?thesis* **.**

qed

lemma *nn_integral_LIMSEQ*:
assumes *f*: *incseq f* $\bigwedge i.$ *f i* $\in$ *borel_measurable M*
and *u*: $\bigwedge x.$ $(\lambda i.$ *f i x*$)$ $\longrightarrow$ *u x*
shows $(\lambda n.$ *integral*^{N} *M* $($*f n*$))$ $\longrightarrow$ *integral*^{N} *M u*
proof −
have $(\lambda n.$ *integral*^{N} *M* $($*f n*$))$ $\longrightarrow$ $($*SUP n. integral*^{N} *M* $($*f n*$))$
using *f* **by** (*intro LIMSEQ_SUP*[*of* $\lambda n.$ *integral*^{N} *M* $($*f n*$)$] *incseq_nn_integral*)
also have $($*SUP n. integral*^{N} *M* $($*f n*$))$ = *integral*^{N} *M* $(\lambda x.$ *SUP n. f n x*$)$
using *f* **by** (*intro nn_integral_monotone_convergence_SUP*[*symmetric*])
also have *integral*^{N} *M* $(\lambda x.$ *SUP n. f n x*$)$ = *integral*^{N} *M* $(\lambda x.$ *u x*$)$
using *f* **by** (*subst LIMSEQ_SUP*[*THEN LIMSEQ_unique*, *OF* _ *u*]) (*auto simp*: *incseq_def le_fun_def*)
finally show *?thesis* **.**
qed

theorem *nn_integral_dominated_convergence*:
assumes [*measurable*]:
$\bigwedge i.$ *u i* $\in$ *borel_measurable M u'* $\in$ *borel_measurable M w* $\in$ *borel_measurable M*
and *bound*: $\bigwedge j.$ *AE x in M. u j x* $\leq$ *w x*
and *w*: $(\int^+ x.$ *w x* $\partial M) < \infty$
and *u'*: *AE x in M.* $(\lambda i.$ *u i x*$)$ $\longrightarrow$ *u' x*
shows $(\lambda i.$ $(\int^+ x.$ *u i x* $\partial M))$ $\longrightarrow$ $(\int^+ x.$ *u' x* $\partial M)$
proof −
have *limsup* $(\lambda n.$ *integral*^{N} *M* $($*u n*$))$ $\leq$ $(\int^+ x.$ *limsup* $(\lambda n.$ *u n x*$)$ $\partial M)$
by (*intro nn_integral_limsup*[*OF* _ _ *bound w*]) *auto*
moreover have $(\int^+ x.$ *limsup* $(\lambda n.$ *u n x*$)$ $\partial M)$ = $(\int^+ x.$ *u' x* $\partial M)$
using *u'* **by** (*intro nn_integral_cong_AE*, *eventually_elim*) (*metis tendsto_iff_Liminf_eq_Limsup sequentially_bot*)
moreover have $(\int^+ x.$ *liminf* $(\lambda n.$ *u n x*$)$ $\partial M)$ = $(\int^+ x.$ *u' x* $\partial M)$
using *u'* **by** (*intro nn_integral_cong_AE*, *eventually_elim*) (*metis tendsto_iff_Liminf_eq_Limsup sequentially_bot*)
moreover have $(\int^+ x.$ *liminf* $(\lambda n.$ *u n x*$)$ $\partial M)$ $\leq$ *liminf* $(\lambda n.$ *integral*^{N} *M* $($*u n*$))$
by (*intro nn_integral_liminf*) *auto*
moreover have *liminf* $(\lambda n.$ *integral*^{N} *M* $($*u n*$))$ $\leq$ *limsup* $(\lambda n.$ *integral*^{N} *M* $($*u n*$))$
by (*intro Liminf_le_Limsup sequentially_bot*)
ultimately show *?thesis*
by (*intro Liminf_eq_Limsup*) *auto*
qed

lemma *inf_continuous_nn_integral*[*order_continuous_intros*]:
assumes *f*: $\bigwedge y.$ *inf_continuous* $($*f y*$)$
assumes [*measurable*]: $\bigwedge x.$ $(\lambda y.$ *f y x*$)$ $\in$ *borel_measurable M*
assumes *bnd*: $\bigwedge x.$ $(\int^+ y.$ *f y x* $\partial M) \neq \infty$
shows *inf_continuous* $(\lambda x.$ $(\int^+ y.$ *f y x* $\partial M))$

```
  unfolding inf_continuous_def
proof safe
  fix C :: nat ⇒ 'b assume C: decseq C
  then show (∫⁺ y. f y (Inf (C ‘ UNIV)) ∂M) = (INF i. ∫⁺ y. f y (C i) ∂M)
    using inf_continuous_mono[OF f] bnd
   by (auto simp add: inf_continuousD[OF f C] fun_eq_iff monotone_def le_fun_def
less_top
            intro!: nn_integral_monotone_convergence_INF_decseq)
qed

lemma nn_integral_null_set:
  assumes N ∈ null_sets M shows (∫⁺ x. u x * indicator N x ∂M) = 0
proof −
  have (∫⁺ x. u x * indicator N x ∂M) = (∫⁺ x. 0 ∂M)
  proof (intro nn_integral_cong_AE AE_I)
    show {x ∈ space M. u x * indicator N x ≠ 0} ⊆ N
      by (auto simp: indicator_def)
    show (emeasure M) N = 0 N ∈ sets M
      using assms by auto
  qed
  then show ?thesis by simp
qed

lemma nn_integral_0_iff:
  assumes u [measurable]: u ∈ borel_measurable M
  shows integral^N M u = 0 ⟷ emeasure M {x∈space M. u x ≠ 0} = 0
    (is _ ⟷ (emeasure M) ?A = 0)
proof −
  have u_eq: (∫⁺ x. u x * indicator ?A x ∂M) = integral^N M u
    by (auto intro!: nn_integral_cong simp: indicator_def)
  show ?thesis
  proof
    assume (emeasure M) ?A = 0
    with nn_integral_null_set[of ?A M u] u
    show integral^N M u = 0 by (simp add: u_eq null_sets_def)
  next
    assume *: integral^N M u = 0
    let ?M = λn. {x ∈ space M. 1 ≤ real (n::nat) * u x}
    have 0 = (SUP n. (emeasure M) (?M n ∩ ?A))
    proof −
      { fix n :: nat
        have emeasure M {x ∈ ?A. 1 ≤ of_nat n * u x} ≤
              of_nat n * ∫⁺ x. u x * indicator ?A x ∂M
          by (intro nn_integral_Markov_inequality) auto
        also have {x ∈ ?A. 1 ≤ of_nat n * u x} = (?M n ∩ ?A)
          by (auto simp: ennreal_of_nat_eq_real_of_nat u_eq * )
        finally have emeasure M (?M n ∩ ?A) ≤ 0
          by (simp add: ennreal_of_nat_eq_real_of_nat u_eq * )
        moreover have 0 ≤ (emeasure M) (?M n ∩ ?A) using u by auto
```

```
      ultimately have (emeasure M) (?M n ∩ ?A) = 0 by auto }
    thus ?thesis by simp
  qed
  also have ... = (emeasure M) (⋃ n. ?M n ∩ ?A)
  proof (safe intro!: SUP_emeasure_incseq)
    fix n show ?M n ∩ ?A ∈ sets M
      using u by (auto intro!: sets.Int)
  next
    show incseq (λn. {x ∈ space M. 1 ≤ real n * u x} ∩ {x ∈ space M. u x ≠
0})
    proof (safe intro!: incseq_SucI)
      fix n :: nat and x
      assume *: 1 ≤ real n * u x
      also have real n * u x ≤ real (Suc n) * u x
        by (auto intro!: mult_right_mono)
      finally show 1 ≤ real (Suc n) * u x by auto
    qed
  qed
  also have ... = (emeasure M) {x∈space M. 0 < u x}
  proof (safe intro!: arg_cong[where f=(emeasure M)])
    fix x assume 0 < u x and [simp, intro]: x ∈ space M
    show x ∈ (⋃ n. ?M n ∩ ?A)
    proof (cases u x rule: ennreal_cases)
      case (real r) with ‹0 < u x› have 0 < r by auto
      obtain j :: nat where 1 / r ≤ real j using real_arch_simple ..
      hence 1 / r * r ≤ real j * r unfolding mult_le_cancel_right using ‹0 <
r› by auto
      hence 1 ≤ real j * r using real ‹0 < r› by auto
      thus ?thesis using ‹0 < r› real
          by (auto simp: ennreal_of_nat_eq_real_of_nat ennreal_1[symmetric]
ennreal_mult[symmetric]
                simp del: ennreal_1)
    qed (insert ‹0 < u x›, auto simp: ennreal_mult_top)
  qed (auto simp: zero_less_iff_neq_zero)
  finally show emeasure M ?A = 0
    by (simp add: zero_less_iff_neq_zero)
 qed
qed

lemma nn_integral_0_iff_AE:
 assumes u: u ∈ borel_measurable M
 shows integral^N M u = 0 ⟷ (AE x in M. u x = 0)
proof -
 have sets: {x∈space M. u x ≠ 0} ∈ sets M
   using u by auto
 show integral^N M u = 0 ⟷ (AE x in M. u x = 0)
   using nn_integral_0_iff[of u] AE_iff_null[OF sets] u by auto
qed
```

lemma *AE_iff_nn_integral*:
  $\{x \in space\ M.\ P\ x\} \in sets\ M \Longrightarrow (AE\ x\ in\ M.\ P\ x) \longleftrightarrow integral^N\ M\ (indicator\ \{x.\ \neg\ P\ x\}) = 0$
  by (*subst nn_integral_0_iff_AE*) (*auto simp*: *indicator_def*[*abs_def*])

lemma *nn_integral_less*:
  assumes [*measurable*]: $f \in borel\_measurable\ M\ g \in borel\_measurable\ M$
  assumes *f*: $(\int^+ x.\ f\ x\ \partial M) \neq \infty$
  assumes *ord*: $AE\ x\ in\ M.\ f\ x \leq g\ x\ \neg\ (AE\ x\ in\ M.\ g\ x \leq f\ x)$
  shows $(\int^+ x.\ f\ x\ \partial M) < (\int^+ x.\ g\ x\ \partial M)$
proof −
  have $0 < (\int^+ x.\ g\ x - f\ x\ \partial M)$
  proof (*intro order_le_neq_trans notI*)
    assume $0 = (\int^+ x.\ g\ x - f\ x\ \partial M)$
    then have $AE\ x\ in\ M.\ g\ x - f\ x = 0$
      using *nn_integral_0_iff_AE*[*of* $\lambda x.\ g\ x - f\ x\ M$] **by** *simp*
    with *ord*(*1*) **have** $AE\ x\ in\ M.\ g\ x \leq f\ x$
      by *eventually_elim* (*auto simp*: *ennreal_minus_eq_0*)
    with *ord* **show** *False*
      by *simp*
  qed *simp*
  also have $\ldots = (\int^+ x.\ g\ x\ \partial M) - (\int^+ x.\ f\ x\ \partial M)$
    using *f* **by** (*subst nn_integral_diff*) (*auto simp*: *ord*)
  finally show *?thesis*
    using *f* **by** (*auto dest*!: *ennreal_minus_pos_iff*[*rotated*] *simp*: *less_top*)
qed

lemma *nn_integral_subalgebra*:
  assumes *f*: $f \in borel\_measurable\ N$
  and *N*: $sets\ N \subseteq sets\ M\ space\ N = space\ M\ \bigwedge A.\ A \in sets\ N \Longrightarrow emeasure\ N\ A = emeasure\ M\ A$
  shows $integral^N\ N\ f = integral^N\ M\ f$
proof −
  have [*simp*]: $\bigwedge f :: 'a \Rightarrow ennreal.\ f \in borel\_measurable\ N \Longrightarrow f \in borel\_measurable\ M$
    using *N* **by** (*auto simp*: *measurable_def*)
  have [*simp*]: $\bigwedge P.\ (AE\ x\ in\ N.\ P\ x) \Longrightarrow (AE\ x\ in\ M.\ P\ x)$
    using *N* **by** (*auto simp add*: *eventually_ae_filter null_sets_def subset_eq*)
  have [*simp*]: $\bigwedge A.\ A \in sets\ N \Longrightarrow A \in sets\ M$
    using *N* **by** *auto*
  from *f* **show** *?thesis*
    apply *induct*
   apply (*simp_all add*: *nn_integral_add nn_integral_cmult nn_integral_monotone_convergence_SUP N image_comp*)
    apply (*auto intro*!: *nn_integral_cong cong*: *nn_integral_cong simp*: *N*(*2*)[*symmetric*])
    done
qed

lemma *nn_integral_nat_function*:

fixes $f :: 'a \Rightarrow nat$
assumes $f \in measurable\ M\ (count\_space\ UNIV)$
shows $(\int^+ x.\ of\_nat\ (f\ x)\ \partial M) = (\sum t.\ emeasure\ M\ \{x \in space\ M.\ t < f\ x\})$
proof −
define F **where** $F\ i = \{x \in space\ M.\ i < f\ x\}$ **for** i
with *assms* **have** [*measurable*]: $\bigwedge i.\ F\ i \in sets\ M$
by *auto*

{ **fix** x **assume** $x \in space\ M$
have $(\lambda i.\ if\ i < f\ x\ then\ 1\ else\ 0)\ sums\ (of\_nat\ (f\ x)::real)$
using *sums_If_finite*[*of* $\lambda i.\ i < f\ x\ \lambda\_.\ 1::real$] **by** *simp*
then have $(\lambda i.\ ennreal\ (if\ i < f\ x\ then\ 1\ else\ 0))\ sums\ of\_nat(f\ x)$
unfolding *ennreal_of_nat_eq_real_of_nat*
by (*subst sums_ennreal*) *auto*
moreover have $\bigwedge i.\ ennreal\ (if\ i < f\ x\ then\ 1\ else\ 0) = indicator\ (F\ i)\ x$
using ‹$x \in space\ M$› **by** (*simp add*: *one_ennreal_def F_def*)
ultimately have $of\_nat\ (f\ x) = (\sum i.\ indicator\ (F\ i)\ x :: ennreal)$
by (*simp add*: *sums_iff*) }
then have $(\int^+ x.\ of\_nat\ (f\ x)\ \partial M) = (\int^+ x.\ (\sum i.\ indicator\ (F\ i)\ x)\ \partial M)$
by (*simp cong*: *nn_integral_cong*)
also have $\ldots = (\sum i.\ emeasure\ M\ (F\ i))$
by (*simp add*: *nn_integral_suminf*)
finally show *?thesis*
by (*simp add*: *F_def*)
qed

theorem *nn_integral_lfp*:
assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
assumes *f*: *sup_continuous f*
assumes *g*: *sup_continuous g*
assumes *meas*: $\bigwedge F.\ F \in borel\_measurable\ N \Longrightarrow f\ F \in borel\_measurable\ N$
assumes *step*: $\bigwedge F\ s.\ F \in borel\_measurable\ N \Longrightarrow integral^N\ (M\ s)\ (f\ F) = g\ (\lambda s.\ integral^N\ (M\ s)\ F)\ s$
shows $(\int^+ \omega.\ lfp\ f\ \omega\ \partial M\ s) = lfp\ g\ s$
proof (*subst lfp_transfer_bounded*[**where** $\alpha = \lambda F\ s.\ \int^+ x.\ F\ x\ \partial M\ s$ **and** $g=g$ **and** $f=f$ **and** $P=\lambda f.\ f \in borel\_measurable\ N$, *symmetric*])
fix $C :: nat \Rightarrow 'b \Rightarrow ennreal$ **assume** *incseq C* $\bigwedge i.\ C\ i \in borel\_measurable\ N$
then show $(\lambda s.\ \int^+ x.\ (SUP\ i.\ C\ i)\ x\ \partial M\ s) = (SUP\ i.\ (\lambda s.\ \int^+ x.\ C\ i\ x\ \partial M\ s))$
unfolding *SUP_apply*[*abs_def*]
by (*subst nn_integral_monotone_convergence_SUP*)
(*auto simp*: *mono_def fun_eq_iff intro*!: *arg_cong2*[**where** $f=emeasure$] *cong*: *measurable_cong_sets*)
qed (*auto simp add*: *step le_fun_def SUP_apply*[*abs_def*] *bot_fun_def bot_ennreal intro*!: *meas f g*)

theorem *nn_integral_gfp*:
assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
assumes *f*: *inf_continuous f* **and** *g*: *inf_continuous g*
assumes *meas*: $\bigwedge F.\ F \in borel\_measurable\ N \Longrightarrow f\ F \in borel\_measurable\ N$

assumes *bound*: $\bigwedge F\ s.\ F \in borel\_measurable\ N \Longrightarrow (\int^+ x.\ f\ F\ x\ \partial M\ s) < \infty$
assumes *non_zero*: $\bigwedge s.\ emeasure\ (M\ s)\ (space\ (M\ s)) \neq 0$
assumes *step*: $\bigwedge F\ s.\ F \in borel\_measurable\ N \Longrightarrow integral^N\ (M\ s)\ (f\ F) = g\ (\lambda s.\ integral^N\ (M\ s)\ F)\ s$
shows $(\int^+ \omega.\ gfp\ f\ \omega\ \partial M\ s) = gfp\ g\ s$
proof (*subst gfp_transfer_bounded*[**where** $\alpha{=}\lambda F\ s.\ \int^+ x.\ F\ x\ \partial M\ s$ **and** $g{=}g$ **and** $f{=}f$
and $P{=}\lambda F.\ F \in borel\_measurable\ N \wedge (\forall s.\ (\int^+ x.\ F\ x\ \partial M\ s) < \infty)$, *symmetric*])
fix $C :: nat \Rightarrow 'b \Rightarrow ennreal$ **assume** *decseq* C $\bigwedge i.\ C\ i \in borel\_measurable\ N \wedge (\forall s.\ integral^N\ (M\ s)\ (C\ i) < \infty)$
then show $(\lambda s.\ \int^+ x.\ (INF\ i.\ C\ i)\ x\ \partial M\ s) = (INF\ i.\ (\lambda s.\ \int^+ x.\ C\ i\ x\ \partial M\ s))$
unfolding *INF_apply*[*abs_def*]
by (*subst nn_integral_monotone_convergence_INF_decseq*)
(*auto simp*: *mono_def fun_eq_iff intro*!: *arg_cong2*[**where** $f{=}emeasure$] *cong*: *measurable_cong_sets*)
next
show $\bigwedge x.\ g\ x \leq (\lambda s.\ integral^N\ (M\ s)\ (f\ top))$
by (*subst step*)
(*auto simp add*: *top_fun_def less_le non_zero le_fun_def ennreal_top_mult*
cong del: *if_weak_cong intro*!: *monoD*[*OF inf_continuous_mono*[*OF g*], *THEN le_funD*])
next
fix C **assume** $\bigwedge i{::}nat.\ C\ i \in borel\_measurable\ N \wedge (\forall s.\ integral^N\ (M\ s)\ (C\ i) < \infty)$ *decseq* C
with *bound* **show** $Inf\ (C\ `\ UNIV) \in borel\_measurable\ N \wedge (\forall s.\ integral^N\ (M\ s)\ (Inf\ (C\ `\ UNIV)) < \infty)$
unfolding *INF_apply*[*abs_def*]
by (*subst nn_integral_monotone_convergence_INF_decseq*)
(*auto simp*: *INF_less_iff cong*: *measurable_cong_sets intro*!: *borel_measurable_INF*)
next
show $\bigwedge x.\ x \in borel\_measurable\ N \wedge (\forall s.\ integral^N\ (M\ s)\ x < \infty) \Longrightarrow (\lambda s.\ integral^N\ (M\ s)\ (f\ x)) = g\ (\lambda s.\ integral^N\ (M\ s)\ x)$
by (*subst step*) *auto*
qed (*insert bound*, *auto simp add*: *le_fun_def INF_apply*[*abs_def*] *top_fun_def intro*!: *meas f g*)

Cauchy–Schwarz inequality for $integral^N$

lemma *sum_of_squares_ge_ennreal*:
fixes $a\ b :: ennreal$
shows $2 * a * b \leq a^2 + b^2$
proof (*cases a*; *cases b*)
fix $x\ y$
assume *xy*: $x \geq 0\ y \geq 0$ **and** [*simp*]: $a = ennreal\ x\ b = ennreal\ y$
have $0 \leq (x - y)^2$
by *simp*
also have $\ldots = x^2 + y^2 - 2 * x * y$
by (*simp add*: *algebra_simps power2_eq_square*)
finally have $2 * x * y \leq x^2 + y^2$

```
    by simp
  hence ennreal (2 * x * y) ≤ ennreal (x^2 + y^2)
    by (intro ennreal_leI)
  thus ?thesis using xy
    by (simp add: ennreal_mult ennreal_power)
qed auto

lemma Cauchy_Schwarz_nn_integral:
  assumes [measurable]: f ∈ borel_measurable M g ∈ borel_measurable M
  shows (∫⁺x. f x * g x ∂M)^2 ≤ (∫⁺x. f x ^ 2 ∂M) * (∫⁺x. g x ^ 2 ∂M)
proof (cases (∫⁺x. f x * g x ∂M) = 0)
  case False
  define F where F = nn_integral M (λx. f x ^ 2)
  define G where G = nn_integral M (λx. g x ^ 2)
  from False have ¬(AE x in M. f x = 0 ∨ g x = 0)
    by (auto simp: nn_integral_0_iff_AE)
  hence ¬(AE x in M. f x = 0) and ¬(AE x in M. g x = 0)
    by (auto intro: AE_disjI1 AE_disjI2)
  hence nz: F ≠ 0 G ≠ 0
    by (auto simp: nn_integral_0_iff_AE F_def G_def)

  show ?thesis
  proof (cases F = ∞ ∨ G = ∞)
    case True
    thus ?thesis using nz
      by (auto simp: F_def G_def)
  next
    case False
    define F' where F' = ennreal (sqrt (enn2real F))
    define G' where G' = ennreal (sqrt (enn2real G))
    from False have fin: F < top G < top
      by (simp_all add: top.not_eq_extremum)
    have F'_sqr: F'^2 = F
      using False by (cases F) (auto simp: F'_def ennreal_power)
    have G'_sqr: G'^2 = G
      using False by (cases G) (auto simp: G'_def ennreal_power)
    have nz': F' ≠ 0 G' ≠ 0 and fin': F' ≠ ∞ G' ≠ ∞
      using F'_sqr G'_sqr nz fin by auto
    from fin' have fin'': F' < top G' < top
      by (auto simp: top.not_eq_extremum)

    have 2 * (F' / F') * (G' / G') * (∫⁺x. f x * g x ∂M) =
          F' * G' * (∫⁺x. 2 * (f x / F') * (g x / G') ∂M)
      using nz' fin''
      by (simp add: divide_ennreal_def algebra_simps ennreal_inverse_mult flip:
nn_integral_cmult)
    also have F'/ F' = 1
      using nz' fin'' by simp
    also have G'/ G' = 1
```

```
    using nz' fin'' by simp
  also have 2 * 1 * 1 = (2 :: ennreal) by simp
  also have F' * G' * (∫⁺ x. 2 * (f x / F') * (g x / G') ∂M) ≤
            F' * G' * (∫⁺x. (f x / F')² + (g x / G')² ∂M)
    by (intro mult_left_mono nn_integral_mono sum_of_squares_ge_ennreal)
auto
  also have ... = F' * G' * (F / F'² + G / G'²) using nz
    by (auto simp: nn_integral_add algebra_simps nn_integral_divide F_def
G_def)
  also have F / F'² = 1
    using nz F'_sqr fin by simp
  also have G / G'² = 1
    using nz G'_sqr fin by simp
  also have F' * G' * (1 + 1) = 2 * (F' * G')
    by (simp add: mult_ac)
  finally have (∫⁺x. f x * g x ∂M) ≤ F' * G'
    by (subst (asm) ennreal_mult_le_mult_iff) auto
  hence (∫⁺x. f x * g x ∂M)² ≤ (F' * G')²
    by (intro power_mono_ennreal)
  also have ... = F * G
    by (simp add: algebra_simps F'_sqr G'_sqr)
  finally show ?thesis
    by (simp add: F_def G_def)
 qed
qed auto
```

7.5.5 Integral under concrete measures

```
lemma nn_integral_mono_measure:
 assumes sets M = sets N M ≤ N shows nn_integral M f ≤ nn_integral N f
 unfolding nn_integral_def
proof (intro SUP_subset_mono)
 note ‹sets M = sets N›[simp] ‹sets M = sets N›[THEN sets_eq_imp_space_eq,
simp]
 show {g. simple_function M g ∧ g ≤ f} ⊆ {g. simple_function N g ∧ g ≤ f}
   by (simp add: simple_function_def)
 show integral^S M x ≤ integral^S N x for x
   using le_measureD3[OF ‹M ≤ N›]
   by (auto simp add: simple_integral_def intro!: sum_mono mult_mono)
qed

lemma nn_integral_empty:
 assumes space M = {}
 shows nn_integral M f = 0
proof −
 have (∫⁺ x. f x ∂M) = (∫⁺ x. 0 ∂M)
   by(rule nn_integral_cong)(simp add: assms)
 thus ?thesis by simp
qed
```

lemma *nn_integral_bot*[*simp*]: *nn_integral bot f = 0*
by (*simp add*: *nn_integral_empty*)

Distributions

lemma *nn_integral_distr*:
assumes *T*: *T ∈ measurable M M'* **and** *f*: *f ∈ borel_measurable* (*distr M M' T*)
shows *integral*^{N} (*distr M M' T*) *f* = (∫$^+$ *x*. *f* (*T x*) ∂*M*)
using *f*
proof *induct*
case (*cong f g*)
with *T* **show** *?case*
apply (*subst nn_integral_cong*[*of _ f g*])
apply *simp*
apply (*subst nn_integral_cong*[*of _ λx. f* (*T x*) *λx. g* (*T x*)])
apply (*simp add*: *measurable_def Pi_iff*)
apply *simp*
done
next
case (*set A*)
then have *eq*: ⋀*x*. *x ∈ space M* ⟹ *indicator A* (*T x*) = *indicator* (*T* −‘ *A* ∩ *space M*) *x*
by (*auto simp*: *indicator_def*)
from *set T* **show** *?case*
by (*subst nn_integral_cong*[*OF eq*])
(*auto simp add*: *emeasure_distr intro*!: *nn_integral_indicator*[*symmetric*] *measurable_sets*)
qed (*simp_all add*: *measurable_compose*[*OF T*] *T nn_integral_cmult nn_integral_add*
nn_integral_monotone_convergence_SUP le_fun_def incseq_def image_comp)

Counting space

lemma *simple_function_count_space*[*simp*]:
simple_function (*count_space A*) *f* ⟷ *finite* (*f* ‘ *A*)
unfolding *simple_function_def* **by** *simp*

lemma *nn_integral_count_space*:
assumes *A*: *finite* {*a*∈*A*. *0* < *f a*}
shows *integral*^{N} (*count_space A*) *f* = (∑ *a*|*a*∈*A* ∧ *0* < *f a*. *f a*)
proof −
have ∗: (∫$^+$*x*. *max 0* (*f x*) ∂*count_space A*) =
(∫$^+$ *x*. (∑ *a*|*a*∈*A* ∧ *0* < *f a*. *f a* ∗ *indicator* {*a*} *x*) ∂*count_space A*)
by (*auto intro*!: *nn_integral_cong*
simp add: *indicator_def of_bool_def if_distrib sum.If_cases*[*OF A*] *max_def le_less*)
also have ... = (∑ *a*|*a*∈*A* ∧ *0* < *f a*. ∫$^+$ *x*. *f a* ∗ *indicator* {*a*} *x* ∂*count_space A*)

by (*subst nn_integral_sum*) (*simp_all add*: *AE_count_space less_imp_le*)
also have ... = ($\sum$ *a*|*a*∈*A* ∧ *0* < *f a*. *f a*)
by (*auto intro*!: *sum.cong simp*: *one_ennreal_def*[*symmetric*] *max_def*)
finally show *?thesis* **by** (*simp add*: *max.absorb2*)
qed

lemma *nn_integral_count_space_finite*:
finite A ⟹ ($\int^+$*x*. *f x* ∂*count_space A*) = ($\sum$ *a*∈*A*. *f a*)
by (*auto intro*!: *sum.mono_neutral_left simp*: *nn_integral_count_space less_le*)

lemma *nn_integral_count_space'*:
assumes *finite A* ⋀*x*. *x* ∈ *B* ⟹ *x* ∉ *A* ⟹ *f x* = *0* *A* ⊆ *B*
shows ($\int^+$*x*. *f x* ∂*count_space B*) = ($\sum$ *x*∈*A*. *f x*)
proof −
have ($\int^+$*x*. *f x* ∂*count_space B*) = ($\sum$ *a* | *a* ∈ *B* ∧ *0* < *f a*. *f a*)
using *assms*(*2*,*3*)
by (*intro nn_integral_count_space finite_subset*[*OF* _ ‹*finite A*›]) (*auto simp*: *less_le*)
also have ... = ($\sum$ *a*∈*A*. *f a*)
using *assms* **by** (*intro sum.mono_neutral_cong_left*) (*auto simp*: *less_le*)
finally show *?thesis* **.**
qed

lemma *nn_integral_bij_count_space*:
assumes *g*: *bij_betw g A B*
shows ($\int^+$*x*. *f* (*g x*) ∂*count_space A*) = ($\int^+$*x*. *f x* ∂*count_space B*)
using *g*[*THEN bij_betw_imp_funcset*]
by (*subst distr_bij_count_space*[*OF g*, *symmetric*])
(*auto intro*!: *nn_integral_distr*[*symmetric*])

lemma *nn_integral_indicator_finite*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite A* **and** [*measurable*]: ⋀*a*. *a* ∈ *A* ⟹ {*a*} ∈ *sets M*
shows ($\int^+$*x*. *f x* ∗ *indicator A x* ∂*M*) = ($\sum$ *x*∈*A*. *f x* ∗ *emeasure M* {*x*})
proof −
from *f* **have** ($\int^+$*x*. *f x* ∗ *indicator A x* ∂*M*) = ($\int^+$*x*. ($\sum$ *a*∈*A*. *f a* ∗ *indicator* {*a*} *x*) ∂*M*)
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def if_distrib*[**where** *f*=λ*a*. *x* ∗ *a* **for** *x*] *sum.If_cases*)
also have ... = ($\sum$ *a*∈*A*. *f a* ∗ *emeasure M* {*a*})
by (*subst nn_integral_sum*) *auto*
finally show *?thesis* **.**
qed

lemma *nn_integral_count_space_nat*:
fixes *f* :: *nat* ⇒ *ennreal*
shows ($\int^+$*i*. *f i* ∂*count_space UNIV*) = ($\sum$ *i*. *f i*)
proof −
have ($\int^+$*i*. *f i* ∂*count_space UNIV*) =

```
    (∫⁺i. (∑j. f j * indicator {j} i) ∂count_space UNIV)
  proof (intro nn_integral_cong)
    fix i
    have f i = (∑j∈{i}. f j * indicator {j} i)
      by simp
    also have ... = (∑j. f j * indicator {j} i)
      by (rule suminf_finite[symmetric]) auto
    finally show f i = (∑j. f j * indicator {j} i) .
  qed
  also have ... = (∑j. (∫⁺i. f j * indicator {j} i ∂count_space UNIV))
    by (rule nn_integral_suminf) auto
  finally show ?thesis
    by simp
qed

lemma nn_integral_enat_function:
  assumes f: f ∈ measurable M (count_space UNIV)
  shows (∫⁺ x. ennreal_of_enat (f x) ∂M) = (∑ t. emeasure M {x ∈ space M.
t < f x})
proof −
  define F where F i = {x∈space M. i < f x} for i :: nat
  with assms have [measurable]: ⋀i. F i ∈ sets M
    by auto

  { fix x assume x ∈ space M
    have (λi::nat. if i < f x then 1 else 0) sums ennreal_of_enat (f x)
      using sums_If_finite[of λr. r < f x λ_. 1 :: ennreal]
      by (cases f x) (simp_all add: sums_def of_nat_tendsto_top_ennreal)
    also have (λi. (if i < f x then 1 else 0)) = (λi. indicator (F i) x)
      using ‹x ∈ space M› by (simp add: one_ennreal_def F_def fun_eq_iff)
    finally have ennreal_of_enat (f x) = (∑ i. indicator (F i) x)
      by (simp add: sums_iff) }
  then have (∫⁺x. ennreal_of_enat (f x) ∂M) = (∫⁺x. (∑ i. indicator (F i) x)
∂M)
    by (simp cong: nn_integral_cong)
  also have ... = (∑ i. emeasure M (F i))
    by (simp add: nn_integral_suminf)
  finally show ?thesis
    by (simp add: F_def)
qed

lemma nn_integral_count_space_nn_integral:
  fixes f :: 'i ⇒ 'a ⇒ ennreal
  assumes countable I and [measurable]: ⋀i. i ∈ I ⟹ f i ∈ borel_measurable M
 shows (∫⁺x. ∫⁺i. f i x ∂count_space I ∂M) = (∫⁺i. ∫⁺x. f i x ∂M ∂count_space
I)
proof cases
  assume finite I then show ?thesis
    by (simp add: nn_integral_count_space_finite nn_integral_sum)
```

next
assume *infinite I*
then have [*simp*]: $I \neq \{\}$
by *auto*
note $* = bij\_betw\_from\_nat\_into[OF \langle countable\ I\rangle \langle infinite\ I\rangle]$
have $**$: $\bigwedge f.\ (\bigwedge i.\ 0 \le f\ i) \Longrightarrow (\int^+ i.\ f\ i\ \partial count\_space\ I) = (\sum n.\ f\ (from\_nat\_into\ I\ n))$
by (*simp add*: *nn_integral_bij_count_space*[*symmetric*, *OF* $*$] *nn_integral_count_space_nat*)
show *?thesis*
by (*simp add*: $**$ *nn_integral_suminf from_nat_into*)
qed

lemma *of_bool_Bex_eq_nn_integral*:
assumes *unique*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in X \Longrightarrow P\ x \Longrightarrow P\ y \Longrightarrow x = y$
shows *of_bool* $(\exists\, y{\in}X.\ P\ y) = (\int^+ y.\ of\_bool\ (P\ y)\ \partial count\_space\ X)$
proof *cases*
assume $\exists\, y{\in}X.\ P\ y$
then obtain y **where** $P\ y\ y \in X$ **by** *auto*
then show *?thesis*
by (*subst nn_integral_count_space'*[**where** $A=\{y\}$]) (*auto dest*: *unique*)
qed (*auto cong*: *nn_integral_cong_simp*)

lemma *emeasure_UN_countable*:
assumes *sets*[*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in sets\ M$ **and** I[*simp*]: *countable I*
assumes *disj*: *disjoint_family_on X I*
shows *emeasure M* $(\bigcup (X\ `\ I)) = (\int^+ i.\ emeasure\ M\ (X\ i)\ \partial count\_space\ I)$
proof −
have *eq*: $\bigwedge x.\ indicator\ (\bigcup (X\ `\ I))\ x = \int^+ i.\ indicator\ (X\ i)\ x\ \partial count\_space\ I$
proof *cases*
fix x **assume** x: $x \in \bigcup (X\ `\ I)$
then obtain j **where** j: $x \in X\ j\ j \in I$
by *auto*
with *disj* **have** $\bigwedge i.\ i \in I \Longrightarrow indicator\ (X\ i)\ x = (indicator\ \{j\}\ i::ennreal)$
by (*auto simp*: *disjoint_family_on_def split*: *split_indicator*)
with $x\ j$ **show** *?thesis x*
by (*simp cong*: *nn_integral_cong_simp*)
qed (*auto simp*: *nn_integral_0_iff_AE*)

note *sets.countable_UN'*[*unfolded subset_eq*, *measurable*]
have *emeasure M* $(\bigcup (X\ `\ I)) = (\int^+ x.\ indicator\ (\bigcup (X\ `\ I))\ x\ \partial M)$
by *simp*
also have $\ldots = (\int^+ i.\ \int^+ x.\ indicator\ (X\ i)\ x\ \partial M\ \partial count\_space\ I)$
by (*simp add*: *eq nn_integral_count_space_nn_integral*)
finally show *?thesis*
by (*simp cong*: *nn_integral_cong_simp*)
qed

lemma *emeasure_countable_singleton*:
assumes *sets*: $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in sets\ M$ **and** X: *countable X*

shows *emeasure M X* = (∫⁺*x*. *emeasure M* {*x*} *∂count_space X*)
proof −
have *emeasure M* (⋃*i*∈*X*. {*i*}) = (∫⁺*x*. *emeasure M* {*x*} *∂count_space X*)
using *assms* **by** (*intro emeasure_UN_countable*) (*auto simp*: *disjoint_family_on_def*)
also have (⋃*i*∈*X*. {*i*}) = *X* **by** *auto*
finally show *?thesis* **.**
qed

lemma *measure_eqI_countable*:
assumes [*simp*]: *sets M* = *Pow A sets N* = *Pow A* **and** *A*: *countable A*
assumes *eq*: ⋀*a*. *a* ∈ *A* ⟹ *emeasure M* {*a*} = *emeasure N* {*a*}
shows *M* = *N*
proof (*rule measure_eqI*)
fix *X* **assume** *X* ∈ *sets M*
then have *X*: *X* ⊆ *A* **by** *auto*
moreover from *A X* **have** *countable X* **by** (*auto dest*: *countable_subset*)
ultimately have
emeasure M X = (∫⁺*a*. *emeasure M* {*a*} *∂count_space X*)
emeasure N X = (∫⁺*a*. *emeasure N* {*a*} *∂count_space X*)
by (*auto intro*!: *emeasure_countable_singleton*)
moreover have (∫⁺*a*. *emeasure M* {*a*} *∂count_space X*) = (∫⁺*a*. *emeasure N* {*a*} *∂count_space X*)
using *X* **by** (*intro nn_integral_cong eq*) *auto*
ultimately show *emeasure M X* = *emeasure N X*
by *simp*
qed *simp*

lemma *measure_eqI_countable_AE*:
assumes [*simp*]: *sets M* = *UNIV sets N* = *UNIV*
assumes *ae*: *AE x in M*. *x* ∈ Ω *AE x in N*. *x* ∈ Ω **and** [*simp*]: *countable* Ω
assumes *eq*: ⋀*x*. *x* ∈ Ω ⟹ *emeasure M* {*x*} = *emeasure N* {*x*}
shows *M* = *N*
proof (*rule measure_eqI*)
fix *A*
have *emeasure N A* = *emeasure N* {*x*∈Ω. *x* ∈ *A*}
using *ae* **by** (*intro emeasure_eq_AE*) *auto*
also have ... = (∫⁺*x*. *emeasure N* {*x*} *∂count_space* {*x*∈Ω. *x* ∈ *A*})
by (*intro emeasure_countable_singleton*) *auto*
also have ... = (∫⁺*x*. *emeasure M* {*x*} *∂count_space* {*x*∈Ω. *x* ∈ *A*})
by (*intro nn_integral_cong eq*[*symmetric*]) *auto*
also have ... = *emeasure M* {*x*∈Ω. *x* ∈ *A*}
by (*intro emeasure_countable_singleton*[*symmetric*]) *auto*
also have ... = *emeasure M A*
using *ae* **by** (*intro emeasure_eq_AE*) *auto*
finally show *emeasure M A* = *emeasure N A* **..**
qed *simp*

lemma *nn_integral_monotone_convergence_SUP_nat*:
fixes *f* :: *'a* ⇒ *nat* ⇒ *ennreal*

assumes *chain*: *Complete_Partial_Order.chain* (≤) (*f* ‘ *Y*)
and *nonempty*: *Y* ≠ {}
shows (∫⁺ *x*. (*SUP i*∈*Y*. *f i x*) ∂*count_space UNIV*) = (*SUP i*∈*Y*. (∫⁺ *x*. *f i x* ∂*count_space UNIV*))
(**is** *?lhs* = *?rhs* **is** *integral*^{N} *?M* _ = _)
proof (*rule order_class.order.antisym*)
show *?rhs* ≤ *?lhs*
by (*auto intro*!: *SUP_least SUP_upper nn_integral_mono*)
next
have ∃ *g*. *incseq g* ∧ *range g* ⊆ (λ*i*. *f i x*) ‘ *Y* ∧ (*SUP i*∈*Y*. *f i x*) = (*SUP i*. *g i*) **for** *x*
by (*rule ennreal_Sup_countable_SUP*) (*simp add*: *nonempty*)
then obtain *g* **where** *incseq*: ⋀*x*. *incseq* (*g x*)
and *range*: ⋀*x*. *range* (*g x*) ⊆ (λ*i*. *f i x*) ‘ *Y*
and *sup*: ⋀*x*. (*SUP i*∈*Y*. *f i x*) = (*SUP i*. *g x i*) **by** *moura*
from *incseq* **have** *incseq'*: *incseq* (λ*i x*. *g x i*)
by(*blast intro*: *incseq_SucI le_funI dest*: *incseq_SucD*)

have *?lhs* = ∫⁺ *x*. (*SUP i*. *g x i*) ∂*?M* **by**(*simp add*: *sup*)
also have ... = (*SUP i*. ∫⁺ *x*. *g x i* ∂*?M*) **using** *incseq'*
by(*rule nn_integral_monotone_convergence_SUP*) *simp*
also have ... ≤ (*SUP i*∈*Y*. ∫⁺ *x*. *f i x* ∂*?M*)
proof(*rule SUP_least*)
fix *n*
have ⋀*x*. ∃ *i*. *g x n* = *f i x* ∧ *i* ∈ *Y* **using** *range* **by** *blast*
then obtain *I* **where** *I*: ⋀*x*. *g x n* = *f* (*I x*) *x* ⋀*x*. *I x* ∈ *Y* **by** *moura*

have (∫⁺ *x*. *g x n* ∂*count_space UNIV*) = (∑ *x*. *g x n*)
by(*rule nn_integral_count_space_nat*)
also have ... = (*SUP m*. ∑ *x*<*m*. *g x n*)
by(*rule suminf_eq_SUP*)
also have ... ≤ (*SUP i*∈*Y*. ∫⁺ *x*. *f i x* ∂*?M*)
proof(*rule SUP_mono*)
fix *m*
show ∃ *m'*∈*Y*. (∑ *x*<*m*. *g x n*) ≤ (∫⁺ *x*. *f m' x* ∂*?M*)
proof(*cases m* > *0*)
case *False*
thus *?thesis* **using** *nonempty* **by** *auto*
next
case *True*
let *?Y* = *I* ‘ {..<*m*}
have *f* ‘ *?Y* ⊆ *f* ‘ *Y* **using** *I* **by** *auto*
with *chain* **have** *chain'*: *Complete_Partial_Order.chain* (≤) (*f* ‘ *?Y*) **by**(*rule chain_subset*)
hence *Sup* (*f* ‘ *?Y*) ∈ *f* ‘ *?Y*
by(*rule ccpo_class.in_chain_finite*)(*auto simp add*: *True lessThan_empty_iff*)
then obtain *m'* **where** *m'* < *m* **and** *m'*: (*SUP i*∈*?Y*. *f i*) = *f* (*I m'*) **by** *auto*
have *I m'* ∈ *Y* **using** *I* **by** *blast*

```
      have (∑x<m. g x n) ≤ (∑x<m. f (I m') x)
      proof(rule sum_mono)
        fix x
        assume x ∈ {..<m}
        hence x < m by simp
        have g x n = f (I x) x by(simp add: I)
       also have ... ≤ (SUP i∈?Y. f i) x unfolding Sup_fun_def image_image
          using ‹x ∈ {..<m}› by (rule Sup_upper [OF imageI])
        also have ... = f (I m') x unfolding m' by simp
        finally show g x n ≤ f (I m') x .
      qed
      also have ... ≤ (SUP m. (∑x<m. f (I m') x))
        by(rule SUP_upper) simp
      also have ... = (∑x. f (I m') x)
        by(rule suminf_eq_SUP[symmetric])
      also have ... = (∫⁺ x. f (I m') x ∂?M)
        by(rule nn_integral_count_space_nat[symmetric])
      finally show ?thesis using ‹I m' ∈ Y› by blast
    qed
   qed
   finally show (∫⁺ x. g x n ∂count_space UNIV) ≤ ... .
  qed
  finally show ?lhs ≤ ?rhs .
qed

lemma power_series_tendsto_at_left:
  assumes nonneg: ⋀i. 0 ≤ f i and summable: ⋀z. 0 ≤ z ⟹ z < 1 ⟹ summable
(λn. f n * z^n)
   shows ((λz. ennreal (∑n. f n * z^n)) ⟶ (∑n. ennreal (f n))) (at_left
(1::real))
proof (intro tendsto_at_left_sequentially)
  show 0 < (1::real) by simp
  fix S :: nat ⇒ real assume S: ⋀n. S n < 1 ⋀n. 0 < S n S ⟶ 1 incseq S
  then have S_nonneg: ⋀i. 0 ≤ S i by (auto intro: less_imp_le)

  have (λi. (∫⁺n. f n * S i^n ∂count_space UNIV)) ⟶ (∫⁺n. ennreal (f n)
∂count_space UNIV)
  proof (rule nn_integral_LIMSEQ)
    show incseq (λi n. ennreal (f n * S i^n))
      using S by (auto intro!: mult_mono power_mono nonneg ennreal_leI
                    simp: incseq_def le_fun_def less_imp_le)
    fix n have (λi. ennreal (f n * S i^n)) ⟶ ennreal (f n * 1^n)
      by (intro tendsto_intros tendsto_ennrealI S)
    then show (λi. ennreal (f n * S i^n)) ⟶ ennreal (f n)
      by simp
  qed (auto simp: S_nonneg intro!: mult_nonneg_nonneg nonneg)
  also have (λi. (∫⁺n. f n * S i^n ∂count_space UNIV)) = (λi. ∑n. f n * S
i^n)
    by (subst nn_integral_count_space_nat)
```

(*intro ext suminf_ennreal2 mult_nonneg_nonneg nonneg S_nonneg*
zero_le_power summable S)+
also have $(\int^+ n.\ ennreal\ (f\ n)\ \partial count\_space\ UNIV) = (\sum n.\ ennreal\ (f\ n))$
by (*simp add*: *nn_integral_count_space_nat nonneg*)
finally show $(\lambda n.\ ennreal\ (\sum na.\ f\ na * S\ n\ \widehat{}\ na)) \longrightarrow (\sum n.\ ennreal\ (f\ n))$
.
qed

Measures with Restricted Space

lemma *simple_function_restrict_space_ennreal*:
fixes $f :: 'a \Rightarrow ennreal$
assumes $\Omega \cap space\ M \in sets\ M$
shows *simple_function* (*restrict_space M* Ω) $f \longleftrightarrow$ *simple_function M* $(\lambda x.\ f\ x * indicator\ \Omega\ x)$
proof −
{ **assume** *finite* $(f\ `\ space\ (restrict\_space\ M\ \Omega))$
then have *finite* $(f\ `\ space\ (restrict\_space\ M\ \Omega) \cup \{0\})$ **by** *simp*
then have *finite* $((\lambda x.\ f\ x * indicator\ \Omega\ x)\ `\ space\ M)$
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}
moreover
{ **assume** *finite* $((\lambda x.\ f\ x * indicator\ \Omega\ x)\ `\ space\ M)$
then have *finite* $(f\ `\ space\ (restrict\_space\ M\ \Omega))$
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}
ultimately show *?thesis*
unfolding
simple_function_iff_borel_measurable borel_measurable_restrict_space_iff_ennreal[*OF assms*]
by *auto*
qed

lemma *simple_function_restrict_space*:
fixes $f :: 'a \Rightarrow 'b{::}real\_normed\_vector$
assumes $\Omega \cap space\ M \in sets\ M$
shows *simple_function* (*restrict_space M* Ω) $f \longleftrightarrow$ *simple_function M* $(\lambda x.\ indicator\ \Omega\ x *_R f\ x)$
proof −
{ **assume** *finite* $(f\ `\ space\ (restrict\_space\ M\ \Omega))$
then have *finite* $(f\ `\ space\ (restrict\_space\ M\ \Omega) \cup \{0\})$ **by** *simp*
then have *finite* $((\lambda x.\ indicator\ \Omega\ x *_R f\ x)\ `\ space\ M)$
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}
moreover
{ **assume** *finite* $((\lambda x.\ indicator\ \Omega\ x *_R f\ x)\ `\ space\ M)$
then have *finite* $(f\ `\ space\ (restrict\_space\ M\ \Omega))$
by (*rule rev_finite_subset*) (*auto split*: *split_indicator simp*: *space_restrict_space*)
}

ultimately show *?thesis*
unfolding *simple_function_iff_borel_measurable*
borel_measurable_restrict_space_iff[*OF assms*]
by *auto*
qed

lemma *simple_integral_restrict_space*:
assumes Ω: Ω ∩ *space M* ∈ *sets M simple_function* (*restrict_space M* Ω) *f*
shows *simple_integral* (*restrict_space M* Ω) *f* = *simple_integral M* (λ*x*. *f x* ∗ *indicator* Ω *x*)
using *simple_function_restrict_space_ennreal*[*THEN iffD1*, *OF* Ω, *THEN simple_functionD*(*1*)]
by (*auto simp add*: *space_restrict_space emeasure_restrict_space*[*OF* Ω(*1*)] *le_infI2 simple_integral_def*
split: *split_indicator split_indicator_asm*
intro!: *sum.mono_neutral_cong_left ennreal_mult_left_cong arg_cong2*[**where** *f*=*emeasure*])

lemma *nn_integral_restrict_space*:
assumes Ω[*simp*]: Ω ∩ *space M* ∈ *sets M*
shows *nn_integral* (*restrict_space M* Ω) *f* = *nn_integral M* (λ*x*. *f x* ∗ *indicator* Ω *x*)
proof −
let *?R* = *restrict_space M* Ω **and** *?X* = λ*M f*. {*s*. *simple_function M s* ∧ *s* ≤ *f* ∧ (∀ *x*. *s x* < *top*)}
have *integral*^{S} *?R* ‘ *?X ?R f* = *integral*^{S} *M* ‘ *?X M* (λ*x*. *f x* ∗ *indicator* Ω *x*)
proof (*safe intro*!: *image_eqI*)
fix *s* **assume** *s*: *simple_function ?R s s* ≤ *f* ∀ *x*. *s x* < *top*
from *s* **show** *integral*^{S} (*restrict_space M* Ω) *s* = *integral*^{S} *M* (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*intro simple_integral_restrict_space*) *auto*
from *s* **show** *simple_function M* (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*simp add*: *simple_function_restrict_space_ennreal*)
from *s* **show** (λ*x*. *s x* ∗ *indicator* Ω *x*) ≤ (λ*x*. *f x* ∗ *indicator* Ω *x*)
⋀*x*. *s x* ∗ *indicator* Ω *x* < *top*
by (*auto split*: *split_indicator simp*: *le_fun_def image_subset_iff*)
next
fix *s* **assume** *s*: *simple_function M s s* ≤ (λ*x*. *f x* ∗ *indicator* Ω *x*) ∀ *x*. *s x* < *top*
then have *simple_function M* (λ*x*. *s x* ∗ *indicator* (Ω ∩ *space M*) *x*) (**is** *?s′*)
by (*intro simple_function_mult simple_function_indicator*) *auto*
also have *?s′* ⟷ *simple_function M* (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*rule simple_function_cong*) (*auto split*: *split_indicator*)
finally show *sf*: *simple_function* (*restrict_space M* Ω) *s*
by (*simp add*: *simple_function_restrict_space_ennreal*)

from *s* **have** *s_eq*: *s* = (λ*x*. *s x* ∗ *indicator* Ω *x*)
by (*auto simp add*: *fun_eq_iff le_fun_def image_subset_iff*
split: *split_indicator split_indicator_asm*

intro: *antisym*)

show $integral^S\ M\ s = integral^S\ (restrict\_space\ M\ \Omega)\ s$
by (*subst s_eq*) (*rule simple_integral_restrict_space*[*symmetric*, *OF* Ω *sf*])
show $\bigwedge x.\ s\ x < top$
using *s* **by** (*auto simp*: *image_subset_iff*)
from *s* **show** $s \le f$
by (*subst s_eq*) (*auto simp*: *image_subset_iff le_fun_def split*: *split_indicator split_indicator_asm*)
qed
then show *?thesis*
unfolding *nn_integral_def_finite* **by** (*simp cong del*: *SUP_cong_simp*)
qed

lemma *nn_integral_count_space_indicator*:
assumes *NO_MATCH* (*UNIV*::'*a set*) (*X*::'*a set*)
shows $(\int^+ x.\ f\ x\ \partial count\_space\ X) = (\int^+ x.\ f\ x * indicator\ X\ x\ \partial count\_space\ UNIV)$
by (*simp add*: *nn_integral_restrict_space*[*symmetric*] *restrict_count_space*)

lemma *nn_integral_count_space_eq*:
$(\bigwedge x.\ x \in A - B \Longrightarrow f\ x = 0) \Longrightarrow (\bigwedge x.\ x \in B - A \Longrightarrow f\ x = 0) \Longrightarrow$
$(\int^+ x.\ f\ x\ \partial count\_space\ A) = (\int^+ x.\ f\ x\ \partial count\_space\ B)$
by (*auto simp*: *nn_integral_count_space_indicator intro*!: *nn_integral_cong split*: *split_indicator*)

lemma *nn_integral_ge_point*:
assumes $x \in A$
shows $p\ x \le \int^+ x.\ p\ x\ \partial count\_space\ A$
proof −
from *assms* **have** $p\ x \le \int^+ x.\ p\ x\ \partial count\_space\ \{x\}$
by(*auto simp add*: *nn_integral_count_space_finite max_def*)
also have $\ldots = \int^+ x'.\ p\ x' * indicator\ \{x\}\ x'\ \partial count\_space\ A$
using *assms* **by**(*auto simp add*: *nn_integral_count_space_indicator indicator_def intro*!: *nn_integral_cong*)
also have $\ldots \le \int^+ x.\ p\ x\ \partial count\_space\ A$
by(*rule nn_integral_mono*)(*simp add*: *indicator_def*)
finally show *?thesis* **.**
qed

Measure spaces with an associated density

definition *density* :: '*a measure* ⇒ ('*a* ⇒ *ennreal*) ⇒ '*a measure* **where**
density M f = *measure_of* (*space M*) (*sets M*) ($\lambda A.\ \int^+ x.\ f\ x * indicator\ A\ x\ \partial M$)

lemma
shows *sets_density*[*simp*, *measurable_cong*]: *sets* (*density M f*) = *sets M*
and *space_density*[*simp*]: *space* (*density M f*) = *space M*

by (*auto simp*: *density_def*)

lemma *space_density_imp*[*measurable_dest*]:
⋀*x M f*. *x* ∈ *space* (*density M f*) ⟹ *x* ∈ *space M* **by** *auto*

lemma
shows *measurable_density_eq1*[*simp*]: *g* ∈ *measurable* (*density Mg f*) *Mg′* ⟷ *g* ∈ *measurable Mg Mg′*
and *measurable_density_eq2*[*simp*]: *h* ∈ *measurable Mh* (*density Mh′ f*) ⟷ *h* ∈ *measurable Mh Mh′*
and *simple_function_density_eq*[*simp*]: *simple_function* (*density Mu f*) *u* ⟷ *simple_function Mu u*
unfolding *measurable_def simple_function_def* **by** *simp_all*

lemma *density_cong*: *f* ∈ *borel_measurable M* ⟹ *f′* ∈ *borel_measurable M* ⟹
(*AE x in M*. *f x* = *f′ x*) ⟹ *density M f* = *density M f′*
unfolding *density_def* **by** (*auto intro*!: *measure_of_eq nn_integral_cong_AE sets.space_closed*)

lemma *emeasure_density*:
assumes *f*[*measurable*]: *f* ∈ *borel_measurable M* **and** *A*[*measurable*]: *A* ∈ *sets M*
shows *emeasure* (*density M f*) *A* = (∫⁺ *x*. *f x* ∗ *indicator A x* ∂*M*)
(**is** *_* = *?μ A*)
unfolding *density_def*
proof (*rule emeasure_measure_of_sigma*)
show *sigma_algebra* (*space M*) (*sets M*) **..**
show *positive* (*sets M*) *?μ*
using *f* **by** (*auto simp*: *positive_def*)
show *countably_additive* (*sets M*) *?μ*
proof (*intro countably_additiveI*)
fix *A* :: *nat* ⇒ *'a set* **assume** *range A* ⊆ *sets M*
then have ⋀*i*. *A i* ∈ *sets M* **by** *auto*
then have ∗: ⋀*i*. (λ*x*. *f x* ∗ *indicator* (*A i*) *x*) ∈ *borel_measurable M*
by *auto*
assume *disj*: *disjoint_family A*
then have (∑ *n*. *?μ* (*A n*)) = (∫⁺ *x*. (∑ *n*. *f x* ∗ *indicator* (*A n*) *x*) ∂*M*)
using *f* ∗ **by** (*subst nn_integral_suminf*) *auto*
also have (∫⁺ *x*. (∑ *n*. *f x* ∗ *indicator* (*A n*) *x*) ∂*M*) = (∫⁺ *x*. *f x* ∗ (∑ *n*. *indicator* (*A n*) *x*) ∂*M*)
using *f* **by** (*auto intro*!: *ennreal_suminf_cmult nn_integral_cong_AE*)
also have ... = (∫⁺ *x*. *f x* ∗ *indicator* (⋃ *n*. *A n*) *x* ∂*M*)
unfolding *suminf_indicator*[*OF disj*] **..**
finally show (∑ *i*. ∫⁺ *x*. *f x* ∗ *indicator* (*A i*) *x* ∂*M*) = ∫⁺ *x*. *f x* ∗ *indicator* (⋃ *i*. *A i*) *x* ∂*M* **.**
qed
qed *fact*

lemma *null_sets_density_iff*:

assumes *f*: $f \in$ *borel_measurable M*
shows $A \in$ *null_sets* (*density M f*) $\longleftrightarrow A \in$ *sets M* $\wedge$ (*AE x in M*. $x \in A \longrightarrow f\ x = 0$)
proof −
{ assume $A \in$ *sets M*
have $(\int^+ x.\ f\ x * \mathit{indicator}\ A\ x\ \partial M) = 0 \longleftrightarrow \mathit{emeasure}\ M\ \{x \in \mathit{space}\ M.\ f\ x * \mathit{indicator}\ A\ x \neq 0\} = 0$
using *f* ‹$A \in$ *sets M*› **by** (*intro nn_integral_0_iff*) *auto*
also have ... $\longleftrightarrow$ (*AE x in M*. $f\ x * \mathit{indicator}\ A\ x = 0$)
using *f* ‹$A \in$ *sets M*› **by** (*intro AE_iff_measurable*[*OF _ refl*, *symmetric*]) *auto*
also have (*AE x in M*. $f\ x * \mathit{indicator}\ A\ x = 0$) $\longleftrightarrow$ (*AE x in M*. $x \in A \longrightarrow f\ x \leq 0$)
by (*auto simp add*: *indicator_def max_def split*: *if_split_asm*)
finally have $(\int^+ x.\ f\ x * \mathit{indicator}\ A\ x\ \partial M) = 0 \longleftrightarrow$ (*AE x in M*. $x \in A \longrightarrow f\ x \leq 0$) **. }**
with *f* **show** *?thesis*
by (*simp add*: *null_sets_def emeasure_density cong*: *conj_cong*)
qed

lemma *AE_density*:
assumes *f*: $f \in$ *borel_measurable M*
shows (*AE x in density M f*. *P x*) $\longleftrightarrow$ (*AE x in M*. $0 < f\ x \longrightarrow P\ x$)
proof
assume *AE x in density M f*. *P x*
with *f* **obtain** *N* **where** $\{x \in \mathit{space}\ M.\ \neg\ P\ x\} \subseteq N\ N \in \mathit{sets}\ M$ **and** *ae*: *AE x in M*. $x \in N \longrightarrow f\ x = 0$
by (*auto simp*: *eventually_ae_filter null_sets_density_iff*)
then have *AE x in M*. $x \notin N \longrightarrow P\ x$ **by** *auto*
with *ae* **show** *AE x in M*. $0 < f\ x \longrightarrow P\ x$
by (*rule eventually_elim2*) *auto*
next
fix *N* **assume** *ae*: *AE x in M*. $0 < f\ x \longrightarrow P\ x$
then obtain *N* **where** $\{x \in \mathit{space}\ M.\ \neg\ (0 < f\ x \longrightarrow P\ x)\} \subseteq N\ N \in \mathit{null\_sets}\ M$
by (*auto simp*: *eventually_ae_filter*)
then have $*$: $\{x \in \mathit{space}\ (\mathit{density}\ M\ f).\ \neg\ P\ x\} \subseteq N \cup \{x \in \mathit{space}\ M.\ f\ x = 0\}$ $N \cup \{x \in \mathit{space}\ M.\ f\ x = 0\} \in \mathit{sets}\ M$ **and** *ae2*: *AE x in M*. $x \notin N$
using *f* **by** (*auto simp*: *subset_eq zero_less_iff_neq_zero intro*!: *AE_not_in*)
show *AE x in density M f*. *P x*
using *ae2*
unfolding *eventually_ae_filter*[*of _ density M f*] *Bex_def null_sets_density_iff*[*OF f*]
by (*intro exI*[*of _* $N \cup \{x \in \mathit{space}\ M.\ f\ x = 0\}$] *conjI* $*$) (*auto elim*: *eventually_elim2*)
qed

lemma *nn_integral_density*:
assumes *f*: $f \in$ *borel_measurable M*

```
  assumes g: g ∈ borel_measurable M
  shows integral^N (density M f) g = (∫⁺ x. f x * g x ∂M)
using g proof induct
  case (cong u v)
  then show ?case
    apply (subst nn_integral_cong[OF cong(3)])
    apply (simp_all cong: nn_integral_cong)
    done
next
  case (set A) then show ?case
    by (simp add: emeasure_density f)
next
  case (mult u c)
  moreover have ⋀x. f x * (c * u x) = c * (f x * u x) by (simp add: field_simps)
  ultimately show ?case
    using f by (simp add: nn_integral_cmult)
next
  case (add u v)
  then have ⋀x. f x * (v x + u x) = f x * v x + f x * u x
    by (simp add: distrib_left)
  with add f show ?case
    by (auto simp add: nn_integral_add intro!: nn_integral_add[symmetric])
next
  case (seq U)
  have eq: AE x in M. f x * (SUP i. U i x) = (SUP i. f x * U i x)
    by eventually_elim (simp add: SUP_mult_left_ennreal seq)
  from seq f show ?case
    apply (simp add: nn_integral_monotone_convergence_SUP image_comp)
    apply (subst nn_integral_cong_AE[OF eq])
    apply (subst nn_integral_monotone_convergence_SUP_AE)
    apply (auto simp: incseq_def le_fun_def intro!: mult_left_mono)
    done
qed

lemma density_distr:
  assumes [measurable]: f ∈ borel_measurable N X ∈ measurable M N
  shows density (distr M N X) f = distr (density M (λx. f (X x))) N X
  by (intro measure_eqI)
     (auto simp add: emeasure_density nn_integral_distr emeasure_distr
           split: split_indicator intro!: nn_integral_cong)

lemma emeasure_restricted:
  assumes S: S ∈ sets M and X: X ∈ sets M
  shows emeasure (density M (indicator S)) X = emeasure M (S ∩ X)
proof −
  have emeasure (density M (indicator S)) X = (∫⁺x. indicator S x * indicator
X x ∂M)
    using S X by (simp add: emeasure_density)
  also have ... = (∫⁺x. indicator (S ∩ X) x ∂M)
```

by (*auto intro*!: *nn_integral_cong simp*: *indicator_def*)
also have ... = *emeasure M* (*S* ∩ *X*)
using *S X* **by** (*simp add*: *sets.Int*)
finally show *?thesis* **.**
qed

lemma *measure_restricted*:
S ∈ *sets M* ⟹ *X* ∈ *sets M* ⟹ *measure* (*density M* (*indicator S*)) *X* = *measure M* (*S* ∩ *X*)
by (*simp add*: *emeasure_restricted measure_def*)

lemma (**in** *finite_measure*) *finite_measure_restricted*:
S ∈ *sets M* ⟹ *finite_measure* (*density M* (*indicator S*))
by *standard* (*simp add*: *emeasure_restricted*)

lemma *emeasure_density_const*:
A ∈ *sets M* ⟹ *emeasure* (*density M* (λ_. *c*)) *A* = *c* ∗ *emeasure M A*
by (*auto simp*: *nn_integral_cmult_indicator emeasure_density*)

lemma *measure_density_const*:
A ∈ *sets M* ⟹ *c* ≠ ∞ ⟹ *measure* (*density M* (λ_. *c*)) *A* = *enn2real c* ∗ *measure M A*
by (*auto simp*: *emeasure_density_const measure_def enn2real_mult*)

lemma *density_density_eq*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹
density (*density M f*) *g* = *density M* (λ*x*. *f x* ∗ *g x*)
by (*auto intro*!: *measure_eqI simp*: *emeasure_density nn_integral_density ac_simps*)

lemma *distr_density_distr*:
assumes *T*: *T* ∈ *measurable M M'* **and** *T'*: *T'* ∈ *measurable M' M*
and *inv*: ∀ *x*∈*space M*. *T'* (*T x*) = *x*
assumes *f*: *f* ∈ *borel_measurable M'*
shows *distr* (*density* (*distr M M' T*) *f*) *M T'* = *density M* (*f* ∘ *T*) (**is** *?R* = *?L*)
proof (*rule measure_eqI*)
fix *A* **assume** *A*: *A* ∈ *sets ?R*
{ **fix** *x* **assume** *x* ∈ *space M*
with *sets.sets_into_space*[*OF A*]
have *indicator* (*T'* −‘ *A* ∩ *space M'*) (*T x*) = (*indicator A x* :: *ennreal*)
using *T inv* **by** (*auto simp*: *indicator_def measurable_space*) }
with *A T T' f* **show** *emeasure ?R A* = *emeasure ?L A*
by (*simp add*: *measurable_comp emeasure_density emeasure_distr*
nn_integral_distr measurable_sets cong: *nn_integral_cong*)
qed *simp*

lemma *density_density_divide*:
fixes *f g* :: *'a* ⇒ *real*
assumes *f*: *f* ∈ *borel_measurable M AE x in M*. *0* ≤ *f x*

assumes *g*: $g \in$ *borel_measurable M AE x in M*. $0 \leq g\ x$
assumes *ac*: *AE x in M*. $f\ x = 0 \longrightarrow g\ x = 0$
shows *density* (*density M f*) (λx. $g\ x$ / $f\ x$) = *density M g*
proof −
have *density M g* = *density M* (λx. *ennreal* ($f\ x$) ∗ *ennreal* ($g\ x$ / $f\ x$))
using *f g ac* **by** (*auto intro*!: *density_cong measurable_If simp*: *ennreal_mult*[*symmetric*])
then show *?thesis*
using *f g* **by** (*subst density_density_eq*) *auto*
qed

lemma *density_1*: *density M* (λ_. *1*) = *M*
by (*intro measure_eqI*) (*auto simp*: *emeasure_density*)

lemma *emeasure_density_add*:
assumes *X*: $X \in$ *sets M*
assumes *Mf*[*measurable*]: $f \in$ *borel_measurable M*
assumes *Mg*[*measurable*]: $g \in$ *borel_measurable M*
shows *emeasure* (*density M f*) *X* + *emeasure* (*density M g*) *X* =
emeasure (*density M* (λx. $f\ x + g\ x$)) *X*
using *assms*
apply (*subst* (*1 2 3*) *emeasure_density*, *simp_all*) []
apply (*subst nn_integral_add*[*symmetric*], *simp_all*) []
apply (*intro nn_integral_cong*, *simp split*: *split_indicator*)
done

Point measure

definition *point_measure* :: *'a set* ⇒ (*'a* ⇒ *ennreal*) ⇒ *'a measure* **where**
point_measure A f = *density* (*count_space A*) *f*

lemma
shows *space_point_measure*: *space* (*point_measure A f*) = *A*
and *sets_point_measure*: *sets* (*point_measure A f*) = *Pow A*
by (*auto simp*: *point_measure_def*)

lemma *sets_point_measure_count_space*[*measurable_cong*]: *sets* (*point_measure A f*) = *sets* (*count_space A*)
by (*simp add*: *sets_point_measure*)

lemma *measurable_point_measure_eq1*[*simp*]:
$g \in$ *measurable* (*point_measure A f*) *M* ⟷ $g \in A \rightarrow$ *space M*
unfolding *point_measure_def* **by** *simp*

lemma *measurable_point_measure_eq2_finite*[*simp*]:
finite A ⟹
$g \in$ *measurable M* (*point_measure A f*) ⟷
($g \in$ *space M* → *A* ∧ (∀ *a*∈*A*. g −‘ {*a*} ∩ *space M* ∈ *sets M*))
unfolding *point_measure_def* **by** (*simp add*: *measurable_count_space_eq2*)

lemma *simple_function_point_measure*[*simp*]:
simple_function (*point_measure A f*) *g* ⟷ *finite* (*g* ‘ *A*)
by (*simp add*: *point_measure_def*)

lemma *emeasure_point_measure*:
assumes *A*: *finite* {*a*∈*X*. *0* < *f a*} *X* ⊆ *A*
shows *emeasure* (*point_measure A f*) *X* = (∑ *a*|*a*∈*X* ∧ *0* < *f a*. *f a*)
proof −
have {*a*. (*a* ∈ *X* ⟶ *a* ∈ *A* ∧ *0* < *f a*) ∧ *a* ∈ *X*} = {*a*∈*X*. *0* < *f a*}
using ‹*X* ⊆ *A*› **by** *auto*
with *A* **show** *?thesis*
by (*simp add*: *emeasure_density nn_integral_count_space point_measure_def indicator_def of_bool_def*)
qed

lemma *emeasure_point_measure_finite*:
finite A ⟹ *X* ⊆ *A* ⟹ *emeasure* (*point_measure A f*) *X* = (∑ *a*∈*X*. *f a*)
by (*subst emeasure_point_measure*) (*auto dest*: *finite_subset intro*!: *sum.mono_neutral_left simp*: *less_le*)

lemma *emeasure_point_measure_finite_if*:
finite A ⟹ *emeasure* (*point_measure A f*) *X* = (*if X* ⊆ *A then* ∑ *a*∈*X*. *f a else 0*)
by (*simp add*: *emeasure_point_measure_finite emeasure_notin_sets sets_point_measure*)

lemma *measure_point_measure_finite_if*:
assumes *finite A* **and** ⋀*x*. *x* ∈ *A* ⟹ *f x* ≥ *0*
shows *measure* (*point_measure A f*) *X* = (*if X* ⊆ *A then* ∑ *a*∈*X*. *f a else 0*)
by (*simp add*: *Sigma_Algebra.measure_def assms emeasure_point_measure_finite_if subset_eq sum_nonneg*)

lemma *emeasure_point_measure_finite2*:
X ⊆ *A* ⟹ *finite X* ⟹ *emeasure* (*point_measure A f*) *X* = (∑ *a*∈*X*. *f a*)
by (*subst emeasure_point_measure*)
(*auto dest*: *finite_subset intro*!: *sum.mono_neutral_left simp*: *less_le*)

lemma *null_sets_point_measure_iff*:
X ∈ *null_sets* (*point_measure A f*) ⟷ *X* ⊆ *A* ∧ (∀ *x*∈*X*. *f x* = *0*)
by (*auto simp*: *AE_count_space null_sets_density_iff point_measure_def*)

lemma *AE_point_measure*:
(*AE x in point_measure A f*. *P x*) ⟷ (∀ *x*∈*A*. *0* < *f x* ⟶ *P x*)
unfolding *point_measure_def*
by (*subst AE_density*) (*auto simp*: *AE_density AE_count_space point_measure_def*)

lemma *nn_integral_point_measure*:
finite {*a*∈*A*. *0* < *f a* ∧ *0* < *g a*} ⟹
integral^N (*point_measure A f*) *g* = (∑ *a*|*a*∈*A* ∧ *0* < *f a* ∧ *0* < *g a*. *f a* ∗ *g a*)
unfolding *point_measure_def*

by (*subst nn_integral_density*)
(*simp_all add*: *nn_integral_density nn_integral_count_space ennreal_zero_less_mult_iff*)

lemma *nn_integral_point_measure_finite*:
finite A $\Longrightarrow$ *integral*^{N} (*point_measure A f*) *g* = ($\sum$ *a*∈*A*. *f a* ∗ *g a*)
by (*subst nn_integral_point_measure*) (*auto intro*!: *sum.mono_neutral_left simp*: *less_le*)

Uniform measure

definition *uniform_measure M A* = *density M* (λ*x*. *indicator A x* / *emeasure M A*)

lemma
shows *sets_uniform_measure*[*simp*, *measurable_cong*]: *sets* (*uniform_measure M A*) = *sets M*
and *space_uniform_measure*[*simp*]: *space* (*uniform_measure M A*) = *space M*
by (*auto simp*: *uniform_measure_def*)

lemma *emeasure_uniform_measure*[*simp*]:
assumes *A*: *A* ∈ *sets M* **and** *B*: *B* ∈ *sets M*
shows *emeasure* (*uniform_measure M A*) *B* = *emeasure M* (*A* ∩ *B*) / *emeasure M A*
proof −
from *A B* **have** *emeasure* (*uniform_measure M A*) *B* = ($\int^+$*x*. (*1* / *emeasure M A*) ∗ *indicator* (*A* ∩ *B*) *x* ∂*M*)
by (*auto simp add*: *uniform_measure_def emeasure_density divide_ennreal_def split*: *split_indicator*
intro!: *nn_integral_cong*)
also have ... = *emeasure M* (*A* ∩ *B*) / *emeasure M A*
using *A B*
by (*subst nn_integral_cmult_indicator*) (*simp_all add*: *sets.Int divide_ennreal_def mult.commute*)
finally show *?thesis* **.**
qed

lemma *measure_uniform_measure*[*simp*]:
assumes *A*: *emeasure M A* ≠ *0 emeasure M A* ≠ ∞ **and** *B*: *B* ∈ *sets M*
shows *measure* (*uniform_measure M A*) *B* = *measure M* (*A* ∩ *B*) / *measure M A*
using *emeasure_uniform_measure*[*OF emeasure_neq_0_sets*[*OF A*(*1*)] *B*] *A*
by (*cases emeasure M A emeasure M* (*A* ∩ *B*) *rule*: *ennreal2_cases*)
(*simp_all add*: *measure_def divide_ennreal top_ennreal.rep_eq top_ereal_def ennreal_top_divide*)

lemma *AE_uniform_measureI*:
A ∈ *sets M* $\Longrightarrow$ (*AE x in M*. *x* ∈ *A* ⟶ *P x*) $\Longrightarrow$ (*AE x in uniform_measure M A*. *P x*)
unfolding *uniform_measure_def* **by** (*auto simp*: *AE_density divide_ennreal_def*)

lemma *emeasure_uniform_measure_1*:
emeasure M S ≠ 0 ⟹ emeasure M S ≠ ∞ ⟹ emeasure (uniform_measure M S) S = 1
by (*subst emeasure_uniform_measure*)
(*simp_all add*: *emeasure_neq_0_sets emeasure_eq_ennreal_measure divide_ennreal zero_less_iff_neq_zero*[*symmetric*])

lemma *nn_integral_uniform_measure*:
assumes *f*[*measurable*]: *f ∈ borel_measurable M* **and** *S*[*measurable*]: *S ∈ sets M*
shows *(∫⁺x. f x ∂uniform_measure M S) = (∫⁺x. f x * indicator S x ∂M) / emeasure M S*
proof −
{ assume *emeasure M S = ∞*
then have *?thesis*
by (*simp add*: *uniform_measure_def nn_integral_density f*) **}**
moreover
{ assume [*simp*]: *emeasure M S = 0*
then have *ae*: *AE x in M. x ∉ S*
using *sets.sets_into_space*[*OF S*]
by (*subst AE_iff_measurable*[*OF _ refl*]) (*simp_all add*: *subset_eq cong*: *rev_conj_cong*)
from *ae* **have** *(∫⁺ x. indicator S x / 0 * f x ∂M) = 0*
by (*subst nn_integral_0_iff_AE*) *auto*
moreover from *ae* **have** *(∫⁺ x. f x * indicator S x ∂M) = 0*
by (*subst nn_integral_0_iff_AE*) *auto*
ultimately have *?thesis*
by (*simp add*: *uniform_measure_def nn_integral_density f*) **}**
moreover have *emeasure M S ≠ 0 ⟹ emeasure M S ≠ ∞ ⟹ ?thesis*
unfolding *uniform_measure_def*
by (*subst nn_integral_density*)
(*auto simp*: *ennreal_times_divide f nn_integral_divide*[*symmetric*] *mult.commute*)
ultimately show *?thesis* **by** *blast*
qed

lemma *AE_uniform_measure*:
assumes *emeasure M A ≠ 0 emeasure M A < ∞*
shows *(AE x in uniform_measure M A. P x) ⟷ (AE x in M. x ∈ A ⟶ P x)*
proof −
have *A ∈ sets M*
using ‹*emeasure M A ≠ 0*› **by** (*metis emeasure_notin_sets*)
moreover have *⋀x. 0 < indicator A x / emeasure M A ⟷ x ∈ A*
using *assms*
by (*cases emeasure M A*) (*auto split*: *split_indicator simp*: *ennreal_zero_less_divide*)
ultimately show *?thesis*
unfolding *uniform_measure_def* **by** (*simp add*: *AE_density*)
qed

Null measure

lemma *null_measure_eq_density*: *null_measure M = density M* (λ_. *0*)
by (*intro measure_eqI*) (*simp_all add*: *emeasure_density*)

lemma *nn_integral_null_measure*[*simp*]: (∫⁺*x*. *f x* ∂*null_measure M*) = *0*
by (*auto simp add*: *nn_integral_def simple_integral_def SUP_constant bot_ennreal_def le_fun_def*
intro!: *exI*[*of* _ λ*x*. *0*])

lemma *density_null_measure*[*simp*]: *density* (*null_measure M*) *f* = *null_measure M*
proof (*intro measure_eqI*)
fix *A* **show** *emeasure* (*density* (*null_measure M*) *f*) *A* = *emeasure* (*null_measure M*) *A*
by (*simp add*: *density_def*) (*simp only*: *null_measure_def*[*symmetric*] *emeasure_null_measure*)
qed *simp*

Uniform count measure

definition *uniform_count_measure A* = *point_measure A* (λ*x*. *1* / *card A*)

lemma
shows *space_uniform_count_measure*: *space* (*uniform_count_measure A*) = *A*
and *sets_uniform_count_measure*: *sets* (*uniform_count_measure A*) = *Pow A*
unfolding *uniform_count_measure_def* **by** (*auto simp*: *space_point_measure sets_point_measure*)

lemma *sets_uniform_count_measure_count_space*[*measurable_cong*]:
sets (*uniform_count_measure A*) = *sets* (*count_space A*)
by (*simp add*: *sets_uniform_count_measure*)

lemma *emeasure_uniform_count_measure*:
finite A ⟹ *X* ⊆ *A* ⟹ *emeasure* (*uniform_count_measure A*) *X* = *card X* / *card A*
by (*simp add*: *emeasure_point_measure_finite uniform_count_measure_def divide_inverse ennreal_mult*
ennreal_of_nat_eq_real_of_nat)

lemma *emeasure_uniform_count_measure_if*:
finite A ⟹ *emeasure* (*uniform_count_measure A*) *X* = (*if X* ⊆ *A then card X* / *card A else 0*)
by (*simp add*: *emeasure_notin_sets emeasure_uniform_count_measure sets_uniform_count_measure*)

lemma *measure_uniform_count_measure*:
finite A ⟹ *X* ⊆ *A* ⟹ *measure* (*uniform_count_measure A*) *X* = *card X* / *card A*
by (*simp add*: *emeasure_point_measure_finite uniform_count_measure_def mea-*

sure_def enn2real_mult)

lemma *measure_uniform_count_measure_if*:
finite A $\Longrightarrow$ *measure* (*uniform_count_measure A*) *X* = (*if X* $\subseteq$ *A then card X*
/ card A else 0)
by (*simp add*: *measure_uniform_count_measure measure_notin_sets sets_uniform_count_measure*)

lemma *space_uniform_count_measure_empty_iff* [*simp*]:
space (*uniform_count_measure X*) = {} $\longleftrightarrow$ *X* = {}
by(*simp add*: *space_uniform_count_measure*)

lemma *sets_uniform_count_measure_eq_UNIV* [*simp*]:
sets (*uniform_count_measure UNIV*) = *UNIV* $\longleftrightarrow$ *True*
UNIV = *sets* (*uniform_count_measure UNIV*) $\longleftrightarrow$ *True*
by(*simp_all add*: *sets_uniform_count_measure*)

Scaled measure

lemma *nn_integral_scale_measure*:
assumes *f*: *f* $\in$ *borel_measurable M*
shows *nn_integral* (*scale_measure r M*) *f* = *r* $*$ *nn_integral M f*
using *f*
proof *induction*
case (*cong f g*)
thus *?case*
by(*simp add*: *cong.hyps space_scale_measure cong*: *nn_integral_cong_simp*)
next
case (*mult f c*)
thus *?case*
by(*simp add*: *nn_integral_cmult max_def mult.assoc mult.left_commute*)
next
case (*add f g*)
thus *?case*
by(*simp add*: *nn_integral_add distrib_left*)
next
case (*seq U*)
thus *?case*
by(*simp add*: *nn_integral_monotone_convergence_SUP SUP_mult_left_ennreal*
image_comp)
qed *simp*

end

7.6 Binary Product Measure

theory *Binary_Product_Measure*
imports *Nonnegative_Lebesgue_Integration*
begin

lemma *Pair_vimage_times*[*simp*]: *Pair x −‘ (A × B) = (if x ∈ A then B else {})*
by *auto*

lemma *rev_Pair_vimage_times*[*simp*]: *(λx. (x, y)) −‘ (A × B) = (if y ∈ B then A else {})*
by *auto*

7.6.1 Binary products

definition *pair_measure* (**infixr** ‹⨂$_M$› *80*) **where**
A ⨂$_M$ B = measure_of (space A × space B)
{a × b | a b. a ∈ sets A ∧ b ∈ sets B}
(λX. ∫⁺x. (∫⁺y. indicator X (x,y) ∂B) ∂A)

lemma *pair_measure_closed*: *{a × b | a b. a ∈ sets A ∧ b ∈ sets B} ⊆ Pow (space A × space B)*
using *sets.space_closed*[*of A*] *sets.space_closed*[*of B*] **by** *auto*

lemma *space_pair_measure*:
space (A ⨂$_M$ B) = space A × space B
unfolding *pair_measure_def* **using** *pair_measure_closed*[*of A B*]
by (*rule space_measure_of*)

lemma *SIGMA_Collect_eq*: *(SIGMA x:space M. {y∈space N. P x y}) = {x∈space (M ⨂$_M$ N). P (fst x) (snd x)}*
by (*auto simp*: *space_pair_measure*)

lemma *sets_pair_measure*:
sets (A ⨂$_M$ B) = sigma_sets (space A × space B) {a × b | a b. a ∈ sets A ∧ b ∈ sets B}
unfolding *pair_measure_def* **using** *pair_measure_closed*[*of A B*]
by (*rule sets_measure_of*)

lemma *sets_pair_measure_cong*[*measurable_cong*, *cong*]:
sets M1 = sets M1′ ⟹ sets M2 = sets M2′ ⟹ sets (M1 ⨂$_M$ M2) = sets (M1′ ⨂$_M$ M2′)
unfolding *sets_pair_measure* **by** (*simp cong*: *sets_eq_imp_space_eq*)

lemma *pair_measureI*[*intro*, *simp*, *measurable*]:
x ∈ sets A ⟹ y ∈ sets B ⟹ x × y ∈ sets (A ⨂$_M$ B)
by (*auto simp*: *sets_pair_measure*)

lemma *sets_Pair*: *{x} ∈ sets M1 ⟹ {y} ∈ sets M2 ⟹ {(x, y)} ∈ sets (M1 ⨂$_M$ M2)*
using *pair_measureI*[*of {x} M1 {y} M2*] **by** *simp*

lemma *measurable_pair_measureI*:
assumes *1*: *f ∈ space M → space M1 × space M2*
assumes *2*: *⋀A B. A ∈ sets M1 ⟹ B ∈ sets M2 ⟹ f −‘ (A × B) ∩ space*

$M \in sets\ M$
shows $f \in measurable\ M\ (M1 \bigotimes_M M2)$
unfolding *pair_measure_def* **using** *1 2*
by (*intro measurable_measure_of*) (*auto dest*: *sets.sets_into_space*)

lemma *measurable_split_replace*[*measurable* (*raw*)]:
$(\lambda x.\ f\ x\ (fst\ (g\ x))\ (snd\ (g\ x))) \in measurable\ M\ N \Longrightarrow (\lambda x.\ case\_prod\ (f\ x)\ (g\ x)) \in measurable\ M\ N$
unfolding *split_beta'* **.**

lemma *measurable_Pair*[*measurable* (*raw*)]:
assumes *f*: $f \in measurable\ M\ M1$ **and** *g*: $g \in measurable\ M\ M2$
shows $(\lambda x.\ (f\ x,\ g\ x)) \in measurable\ M\ (M1 \bigotimes_M M2)$
proof (*rule measurable_pair_measureI*)
show $(\lambda x.\ (f\ x,\ g\ x)) \in space\ M \to space\ M1 \times space\ M2$
using *f g* **by** (*auto simp*: *measurable_def*)
fix *A B* **assume** $*$: $A \in sets\ M1\ B \in sets\ M2$
have $(\lambda x.\ (f\ x,\ g\ x)) -` (A \times B) \cap space\ M = (f -` A \cap space\ M) \cap (g -` B \cap space\ M)$
by *auto*
also have $\ldots \in sets\ M$
by (*rule sets.Int*) (*auto intro*!: *measurable_sets* $*$ *f g*)
finally show $(\lambda x.\ (f\ x,\ g\ x)) -` (A \times B) \cap space\ M \in sets\ M$ **.**
qed

lemma *measurable_fst*[*intro*!, *simp*, *measurable*]: $fst \in measurable\ (M1 \bigotimes_M M2)\ M1$
by (*auto simp*: *fst_vimage_eq_Times space_pair_measure sets.sets_into_space Times_Int_Times*
measurable_def)

lemma *measurable_snd*[*intro*!, *simp*, *measurable*]: $snd \in measurable\ (M1 \bigotimes_M M2)\ M2$
by (*auto simp*: *snd_vimage_eq_Times space_pair_measure sets.sets_into_space Times_Int_Times*
measurable_def)

lemma *measurable_Pair_compose_split*[*measurable_dest*]:
assumes *f*: $case\_prod\ f \in measurable\ (M1 \bigotimes_M M2)\ N$
assumes *g*: $g \in measurable\ M\ M1$ **and** *h*: $h \in measurable\ M\ M2$
shows $(\lambda x.\ f\ (g\ x)\ (h\ x)) \in measurable\ M\ N$
using *measurable_compose*[*OF measurable_Pair f*, *OF g h*] **by** *simp*

lemma *measurable_Pair1_compose*[*measurable_dest*]:
assumes *f*: $(\lambda x.\ (f\ x,\ g\ x)) \in measurable\ M\ (M1 \bigotimes_M M2)$
assumes [*measurable*]: $h \in measurable\ N\ M$
shows $(\lambda x.\ f\ (h\ x)) \in measurable\ N\ M1$
using *measurable_compose*[*OF f measurable_fst*] **by** *simp*

lemma *measurable_Pair2_compose*[*measurable_dest*]:
assumes *f*: $(\lambda x.\ (f\ x,\ g\ x)) \in$ *measurable M* $(M1 \bigotimes_M M2)$
assumes [*measurable*]: $h \in$ *measurable N M*
shows $(\lambda x.\ g\ (h\ x)) \in$ *measurable N M2*
using *measurable_compose*[*OF f measurable_snd*] **by** *simp*

lemma *measurable_pair*:
assumes $(fst \circ f) \in$ *measurable M M1* $(snd \circ f) \in$ *measurable M M2*
shows $f \in$ *measurable M* $(M1 \bigotimes_M M2)$
using *measurable_Pair*[*OF assms*] **by** *simp*

lemma
assumes *f*[*measurable*]: $f \in$ *measurable M* $(N \bigotimes_M P)$
shows *measurable_fst'*: $(\lambda x.\ fst\ (f\ x)) \in$ *measurable M N*
and *measurable_snd'*: $(\lambda x.\ snd\ (f\ x)) \in$ *measurable M P*
by *simp_all*

lemma
assumes *f*[*measurable*]: $f \in$ *measurable M N*
shows *measurable_fst''*: $(\lambda x.\ f\ (fst\ x)) \in$ *measurable* $(M \bigotimes_M P)$ *N*
and *measurable_snd''*: $(\lambda x.\ f\ (snd\ x)) \in$ *measurable* $(P \bigotimes_M M)$ *N*
by *simp_all*

lemma *sets_pair_in_sets*:
assumes $\bigwedge a\ b.\ a \in sets\ A \Longrightarrow b \in sets\ B \Longrightarrow a \times b \in sets\ N$
shows *sets* $(A \bigotimes_M B) \subseteq sets\ N$
unfolding *sets_pair_measure*
by (*intro sets.sigma_sets_subset'*) (*auto intro*!: *assms*)

lemma *sets_pair_eq_sets_fst_snd*:
sets $(A \bigotimes_M B) =$ *sets* (*Sup* {*vimage_algebra* (*space A* × *space B*) *fst A*, *vimage_algebra* (*space A* × *space B*) *snd B*})
(**is** *?P* = *sets* (*Sup* {*?fst*, *?snd*}))
proof −
{ fix *a b* **assume** *ab*: *a* ∈ *sets A b* ∈ *sets B*
then have *a* × *b* = (*fst* −‘ *a* ∩ (*space A* × *space B*)) ∩ (*snd* −‘ *b* ∩ (*space A* × *space B*))
by (*auto dest*: *sets.sets_into_space*)
also have ... ∈ *sets* (*Sup* {*?fst*, *?snd*})
apply (*rule sets.Int*)
apply (*rule in_sets_Sup*)
apply *auto* []
apply (*rule insertI1*)
apply (*auto intro*: *ab in_vimage_algebra*) []
apply (*rule in_sets_Sup*)
apply *auto* []
apply (*rule insertI2*)
apply (*auto intro*: *ab in_vimage_algebra*)
done

finally have $a \times b \in sets\ (Sup\ \{?fst, ?snd\})$ **. }**
moreover have $sets\ ?fst \subseteq sets\ (A \bigotimes_M B)$
by (*rule sets_image_in_sets*) (*auto simp*: *space_pair_measure*[*symmetric*])
moreover have $sets\ ?snd \subseteq sets\ (A \bigotimes_M B)$
by (*rule sets_image_in_sets*) (*auto simp*: *space_pair_measure*)
ultimately show *?thesis*
apply (*intro antisym*[*of sets A* **for** *A*] *sets_Sup_in_sets sets_pair_in_sets*)
apply *simp*
apply *simp*
apply *simp*
apply (*elim disjE*)
apply (*simp add*: *space_pair_measure*)
apply (*simp add*: *space_pair_measure*)
apply (*auto simp add*: *space_pair_measure*)
done
qed

lemma *measurable_pair_iff*:
$f \in measurable\ M\ (M1 \bigotimes_M M2) \longleftrightarrow (fst \circ f) \in measurable\ M\ M1 \wedge (snd \circ f) \in measurable\ M\ M2$
by (*auto intro*: *measurable_pair*[*of f M M1 M2*])

lemma *measurable_split_conv*:
$(\lambda(x, y).\ f\ x\ y) \in measurable\ A\ B \longleftrightarrow (\lambda x.\ f\ (fst\ x)\ (snd\ x)) \in measurable\ A\ B$
by (*intro arg_cong2*[**where** $f=(\in)$]) *auto*

lemma *measurable_pair_swap'*: $(\lambda(x,y).\ (y, x)) \in measurable\ (M1 \bigotimes_M M2)\ (M2 \bigotimes_M M1)$
by (*auto intro*!: *measurable_Pair simp*: *measurable_split_conv*)

lemma *measurable_pair_swap*:
assumes *f*: $f \in measurable\ (M1 \bigotimes_M M2)\ M$ **shows** $(\lambda(x,y).\ f\ (y, x)) \in measurable\ (M2 \bigotimes_M M1)\ M$
using *measurable_comp*[*OF measurable_Pair f*] **by** (*auto simp*: *measurable_split_conv comp_def*)

lemma *measurable_pair_swap_iff*:
$f \in measurable\ (M2 \bigotimes_M M1)\ M \longleftrightarrow (\lambda(x,y).\ f\ (y,x)) \in measurable\ (M1 \bigotimes_M M2)\ M$
by (*auto dest*: *measurable_pair_swap*)

lemma *measurable_Pair1'*: $x \in space\ M1 \Longrightarrow Pair\ x \in measurable\ M2\ (M1 \bigotimes_M M2)$
by *simp*

lemma *sets_Pair1*[*measurable* (*raw*)]:
assumes *A*: $A \in sets\ (M1 \bigotimes_M M2)$ **shows** $Pair\ x -` A \in sets\ M2$
proof −
have $Pair\ x -` A = (if\ x \in space\ M1\ then\ Pair\ x -` A \cap space\ M2\ else\ \{\})$

using *A*[*THEN sets.sets_into_space*] **by** (*auto simp*: *space_pair_measure*)
also have ... ∈ *sets M2*
using *A* **by** (*auto simp add*: *measurable_Pair1′ intro*!: *measurable_sets split*: *if_split_asm*)
finally show *?thesis* **.**
qed

lemma *measurable_Pair2′*: *y* ∈ *space M2* ⟹ (λ*x*. (*x*, *y*)) ∈ *measurable M1* (*M1* ⨂$_M$ *M2*)
by (*auto intro*!: *measurable_Pair*)

lemma *sets_Pair2*: **assumes** *A*: *A* ∈ *sets* (*M1* ⨂$_M$ *M2*) **shows** (λ*x*. (*x*, *y*)) −‘ *A* ∈ *sets M1*
proof −
have (λ*x*. (*x*, *y*)) −‘ *A* = (*if y* ∈ *space M2 then* (λ*x*. (*x*, *y*)) −‘ *A* ∩ *space M1 else* {})
using *A*[*THEN sets.sets_into_space*] **by** (*auto simp*: *space_pair_measure*)
also have ... ∈ *sets M1*
using *A* **by** (*auto simp add*: *measurable_Pair2′ intro*!: *measurable_sets split*: *if_split_asm*)
finally show *?thesis* **.**
qed

lemma *measurable_Pair2*:
assumes *f*: *f* ∈ *measurable* (*M1* ⨂$_M$ *M2*) *M* **and** *x*: *x* ∈ *space M1*
shows (λ*y*. *f* (*x*, *y*)) ∈ *measurable M2 M*
using *measurable_comp*[*OF measurable_Pair1′ f*, *OF x*]
by (*simp add*: *comp_def*)

lemma *measurable_Pair1*:
assumes *f*: *f* ∈ *measurable* (*M1* ⨂$_M$ *M2*) *M* **and** *y*: *y* ∈ *space M2*
shows (λ*x*. *f* (*x*, *y*)) ∈ *measurable M1 M*
using *measurable_comp*[*OF measurable_Pair2′ f*, *OF y*]
by (*simp add*: *comp_def*)

lemma *Int_stable_pair_measure_generator*: *Int_stable* {*a* × *b* | *a b*. *a* ∈ *sets A* ∧ *b* ∈ *sets B*}
unfolding *Int_stable_def*
by *safe* (*auto simp add*: *Times_Int_Times*)

lemma (**in** *finite_measure*) *finite_measure_cut_measurable*:
assumes [*measurable*]: *Q* ∈ *sets* (*N* ⨂$_M$ *M*)
shows (λ*x*. *emeasure M* (*Pair x* −‘ *Q*)) ∈ *borel_measurable N*
(**is** *?s Q* ∈ _)
using *Int_stable_pair_measure_generator pair_measure_closed assms*
unfolding *sets_pair_measure*
proof (*induct rule*: *sigma_sets_induct_disjoint*)
case (*compl A*)
with *sets.sets_into_space* **have** ⋀*x*. *emeasure M* (*Pair x* −‘ ((*space N* × *space*

```
M) - A)) =
      (if x ∈ space N then emeasure M (space M) - ?s A x else 0)
    unfolding sets_pair_measure[symmetric]
    by (auto intro!: emeasure_compl simp: vimage_Diff sets_Pair1)
  with compl sets.top show ?case
    by (auto intro!: measurable_If simp: space_pair_measure)
next
  case (union F)
  then have ⋀x. emeasure M (Pair x -` (⋃ i. F i)) = (∑ i. ?s (F i) x)
    by (simp add: suminf_emeasure disjoint_family_on_vimageI subset_eq vim-
age_UN sets_pair_measure[symmetric])
  with union show ?case
    unfolding sets_pair_measure[symmetric] by simp
qed (auto simp add: if_distrib Int_def[symmetric] intro!: measurable_If)

lemma (in sigma_finite_measure) measurable_emeasure_Pair:
  assumes Q: Q ∈ sets (N ⨂M M) shows (λx. emeasure M (Pair x -` Q)) ∈
borel_measurable N (is ?s Q ∈ _)
proof -
  obtain F :: nat ⇒ 'a set where F:
    range F ⊆ sets M
    ⋃ (range F) = space M
    ⋀i. emeasure M (F i) ≠ ∞
    disjoint_family F by (blast intro: sigma_finite_disjoint)
  then have F_sets: ⋀i. F i ∈ sets M by auto
  let ?C = λx i. F i ∩ Pair x -` Q
  { fix i
    have [simp]: space N × F i ∩ space N × space M = space N × F i
      using F sets.sets_into_space by auto
    let ?R = density M (indicator (F i))
    have finite_measure ?R
      using F by (intro finite_measureI) (auto simp: emeasure_restricted sub-
set_eq)
    then have (λx. emeasure ?R (Pair x -` (space N × space ?R ∩ Q))) ∈
borel_measurable N
      by (rule finite_measure.finite_measure_cut_measurable) (auto intro: Q)
    moreover have ⋀x. emeasure ?R (Pair x -` (space N × space ?R ∩ Q))
      = emeasure M (F i ∩ Pair x -` (space N × space ?R ∩ Q))
      using Q F_sets by (intro emeasure_restricted) (auto intro: sets_Pair1)
    moreover have ⋀x. F i ∩ Pair x -` (space N × space ?R ∩ Q) = ?C x i
      using sets.sets_into_space[OF Q] by (auto simp: space_pair_measure)
    ultimately have (λx. emeasure M (?C x i)) ∈ borel_measurable N
      by simp }
  moreover
  { fix x
    have (∑ i. emeasure M (?C x i)) = emeasure M (⋃ i. ?C x i)
    proof (intro suminf_emeasure)
      show range (?C x) ⊆ sets M
        using F ‹Q ∈ sets (N ⨂M M)› by (auto intro!: sets_Pair1)
```

```
    have disjoint_family F using F by auto
    show disjoint_family (?C x)
      by (rule disjoint_family_on_bisimulation[OF ‹disjoint_family F›]) auto
  qed
  also have (⋃ i. ?C x i) = Pair x -‘ Q
    using F sets.sets_into_space[OF ‹Q ∈ sets (N ⨂M M)›]
    by (auto simp: space_pair_measure)
  finally have emeasure M (Pair x -‘ Q) = (∑ i. emeasure M (?C x i))
    by simp }
 ultimately show ?thesis using ‹Q ∈ sets (N ⨂M M)› F_sets
  by auto
qed

lemma (in sigma_finite_measure) measurable_emeasure[measurable (raw)]:
 assumes space: ⋀x. x ∈ space N ⟹ A x ⊆ space M
 assumes A: {x∈space (N ⨂M M). snd x ∈ A (fst x)} ∈ sets (N ⨂M M)
 shows (λx. emeasure M (A x)) ∈ borel_measurable N
proof -
 from space have ⋀x. x ∈ space N ⟹ Pair x -‘ {x ∈ space (N ⨂M M). snd
x ∈ A (fst x)} = A x
   by (auto simp: space_pair_measure)
 with measurable_emeasure_Pair[OF A] show ?thesis
   by (auto cong: measurable_cong)
qed

lemma (in sigma_finite_measure) emeasure_pair_measure:
 assumes X ∈ sets (N ⨂M M)
 shows emeasure (N ⨂M M) X = (∫+ x. ∫+ y. indicator X (x, y) ∂M ∂N)
(is _ = ?μ X)
proof (rule emeasure_measure_of[OF pair_measure_def])
 show positive (sets (N ⨂M M)) ?μ
   by (auto simp: positive_def)
 have eq[simp]: ⋀A x y. indicator A (x, y) = indicator (Pair x -‘ A) y
   by (auto simp: indicator_def)
 show countably_additive (sets (N ⨂M M)) ?μ
 proof (rule countably_additiveI)
   fix F :: nat ⇒ ('b × 'a) set assume F: range F ⊆ sets (N ⨂M M) dis-
joint_family F
   from F have ∗: ⋀i. F i ∈ sets (N ⨂M M) by auto
   moreover have ⋀x. disjoint_family (λi. Pair x -‘ F i)
     by (intro disjoint_family_on_bisimulation[OF F(2)]) auto
   moreover have ⋀x. range (λi. Pair x -‘ F i) ⊆ sets M
     using F by (auto simp: sets_Pair1)
   ultimately show (∑ n. ?μ (F n)) = ?μ (⋃ i. F i)
   by (auto simp add: nn_integral_suminf[symmetric] vimage_UN suminf_emeasure
            intro!: nn_integral_cong nn_integral_indicator[symmetric])
 qed
 show {a × b |a b. a ∈ sets N ∧ b ∈ sets M} ⊆ Pow (space N × space M)
   using sets.space_closed[of N] sets.space_closed[of M] by auto
```

qed *fact*

lemma (**in** *sigma_finite_measure*) *emeasure_pair_measure_alt*:
assumes *X*: $X \in$ *sets* ($N \bigotimes_M M$)
shows *emeasure* ($N \bigotimes_M M$) $X = (\int^+ x.$ *emeasure* M (*Pair* x −‘ X) ∂N)
proof −
have [*simp*]: $\bigwedge x\ y.$ *indicator* X (x, y) = *indicator* (*Pair* x −‘ X) y
by (*auto simp*: *indicator_def*)
show *?thesis*
using X **by** (*auto intro*!: *nn_integral_cong simp*: *emeasure_pair_measure sets_Pair1*)
qed

proposition (**in** *sigma_finite_measure*) *emeasure_pair_measure_Times*:
assumes *A*: $A \in$ *sets* N **and** *B*: $B \in$ *sets* M
shows *emeasure* ($N \bigotimes_M M$) ($A \times B$) = *emeasure* N A ∗ *emeasure* M B
proof −
have *emeasure* ($N \bigotimes_M M$) ($A \times B$) = ($\int^+ x.$ *emeasure* M B ∗ *indicator* A x ∂N)
using A B **by** (*auto intro*!: *nn_integral_cong simp*: *emeasure_pair_measure_alt*)
also have ... = *emeasure* M B ∗ *emeasure* N A
using A **by** (*simp add*: *nn_integral_cmult_indicator*)
finally show *?thesis*
by (*simp add*: *ac_simps*)
qed

lemma (**in** *sigma_finite_measure*) *times_in_null_sets1* [*intro*]:
assumes $A \in$ *null_sets* N $B \in$ *sets* M
shows $A \times B \in$ *null_sets* ($N \bigotimes_M M$)
using *assms* **by** (*simp add*: *null_sets_def emeasure_pair_measure_Times*)

lemma (**in** *sigma_finite_measure*) *times_in_null_sets2* [*intro*]:
assumes $A \in$ *sets* N $B \in$ *null_sets* M
shows $A \times B \in$ *null_sets* ($N \bigotimes_M M$)
using *assms* **by** (*simp add*: *null_sets_def emeasure_pair_measure_Times*)

7.6.2 Binary products of σ-finite emeasure spaces

locale *pair_sigma_finite* = *M1?*: *sigma_finite_measure M1* + *M2?*: *sigma_finite_measure M2*
for *M1* :: *'a measure* **and** *M2* :: *'b measure*

lemma (**in** *pair_sigma_finite*) *measurable_emeasure_Pair1*:
$Q \in$ *sets* (*M1* $\bigotimes_M$ *M2*) ⟹ ($\lambda x.$ *emeasure M2* (*Pair* x −‘ Q)) ∈ *borel_measurable M1*
using *M2.measurable_emeasure_Pair* .

lemma (**in** *pair_sigma_finite*) *measurable_emeasure_Pair2*:
assumes *Q*: $Q \in$ *sets* (*M1* $\bigotimes_M$ *M2*) **shows** ($\lambda y.$ *emeasure M1* (($\lambda x.$ (x, y)) −‘

```
Q)) ∈ borel_measurable M2
proof −
  have (λ(x, y). (y, x)) −‘ Q ∩ space (M2 ⨂M M1) ∈ sets (M2 ⨂M M1)
    using Q measurable_pair_swap′ by (auto intro: measurable_sets)
  note M1.measurable_emeasure_Pair[OF this]
  moreover have ⋀y. Pair y −‘ ((λ(x, y). (y, x)) −‘ Q ∩ space (M2 ⨂M M1))
= (λx. (x, y)) −‘ Q
    using Q[THEN sets.sets_into_space] by (auto simp: space_pair_measure)
  ultimately show ?thesis by simp
qed

proposition (in pair_sigma_finite) sigma_finite_up_in_pair_measure_generator:
  defines E ≡ {A × B | A B. A ∈ sets M1 ∧ B ∈ sets M2}
  shows ∃F::nat ⇒ ('a × 'b) set. range F ⊆ E ∧ incseq F ∧ (⋃i. F i) = space
M1 × space M2 ∧
    (∀i. emeasure (M1 ⨂M M2) (F i) ≠ ∞)
proof −
  obtain F1 where F1: range F1 ⊆ sets M1
    ⋃ (range F1) = space M1
    ⋀i. emeasure M1 (F1 i) ≠ ∞
    incseq F1
    by (rule M1.sigma_finite_incseq) blast
  obtain F2 where F2: range F2 ⊆ sets M2
    ⋃ (range F2) = space M2
    ⋀i. emeasure M2 (F2 i) ≠ ∞
    incseq F2
    by (rule M2.sigma_finite_incseq) blast
  from F1 F2 have space: space M1 = (⋃i. F1 i) space M2 = (⋃i. F2 i) by
auto
  let ?F = λi. F1 i × F2 i
  show ?thesis
  proof (intro exI[of _ ?F] conjI allI)
   show range ?F ⊆ E using F1 F2 by (auto simp: E_def) (metis range_subsetD)
  next
    have space M1 × space M2 ⊆ (⋃i. ?F i)
    proof (intro subsetI)
      fix x assume x ∈ space M1 × space M2
      then obtain i j where fst x ∈ F1 i snd x ∈ F2 j
        by (auto simp: space)
      then have fst x ∈ F1 (max i j) snd x ∈ F2 (max j i)
        using ‹incseq F1› ‹incseq F2› unfolding incseq_def
        by (force split: split_max)+
      then have (fst x, snd x) ∈ F1 (max i j) × F2 (max i j)
        by (intro SigmaI) (auto simp add: max.commute)
      then show x ∈ (⋃i. ?F i) by auto
    qed
    then show (⋃i. ?F i) = space M1 × space M2
      using space by (auto simp: space)
  next
```

fix *i* **show** *incseq* (λ*i*. *F1 i* × *F2 i*)
using ‹*incseq F1*› ‹*incseq F2*› **unfolding** *incseq_Suc_iff* **by** *auto*
next
fix *i*
from *F1 F2* **have** *F1 i* ∈ *sets M1 F2 i* ∈ *sets M2* **by** *auto*
with *F1 F2* **show** *emeasure* (*M1* ⨂$_M$ *M2*) (*F1 i* × *F2 i*) ≠ ∞
by (*auto simp add*: *emeasure_pair_measure_Times ennreal_mult_eq_top_iff*)
qed
qed

sublocale *pair_sigma_finite* ⊆ *P?*: *sigma_finite_measure M1* ⨂$_M$ *M2*
proof
obtain *F1* :: *'a set set* **and** *F2* :: *'b set set* **where**
countable F1 ∧ *F1* ⊆ *sets M1* ∧ ⋃ *F1* = *space M1* ∧ (∀ *a*∈*F1*. *emeasure M1 a* ≠ ∞)
countable F2 ∧ *F2* ⊆ *sets M2* ∧ ⋃ *F2* = *space M2* ∧ (∀ *a*∈*F2*. *emeasure M2 a* ≠ ∞)
using *M1.sigma_finite_countable M2.sigma_finite_countable* **by** *auto*
then show
∃ *A*. *countable A* ∧ *A* ⊆ *sets* (*M1* ⨂$_M$ *M2*) ∧ ⋃ *A* = *space* (*M1* ⨂$_M$ *M2*) ∧ (∀ *a*∈*A*. *emeasure* (*M1* ⨂$_M$ *M2*) *a* ≠ ∞)
by (*intro exI*[*of* _ (λ(*a*, *b*). *a* × *b*) ‘ (*F1* × *F2*)] *conjI*)
(*auto simp*: *M2.emeasure_pair_measure_Times space_pair_measure set_eq_iff subset_eq ennreal_mult_eq_top_iff*)
qed

lemma *sigma_finite_pair_measure*:
assumes *A*: *sigma_finite_measure A* **and** *B*: *sigma_finite_measure B*
shows *sigma_finite_measure* (*A* ⨂$_M$ *B*)
proof −
interpret *A*: *sigma_finite_measure A* **by** *fact*
interpret *B*: *sigma_finite_measure B* **by** *fact*
interpret *AB*: *pair_sigma_finite A B* **..**
show *?thesis* **..**
qed

lemma *sets_pair_swap*:
assumes *A* ∈ *sets* (*M1* ⨂$_M$ *M2*)
shows (λ(*x*, *y*). (*y*, *x*)) −‘ *A* ∩ *space* (*M2* ⨂$_M$ *M1*) ∈ *sets* (*M2* ⨂$_M$ *M1*)
using *measurable_pair_swap'* *assms* **by** (*rule measurable_sets*)

lemma (**in** *pair_sigma_finite*) *distr_pair_swap*:
M1 ⨂$_M$ *M2* = *distr* (*M2* ⨂$_M$ *M1*) (*M1* ⨂$_M$ *M2*) (λ(*x*, *y*). (*y*, *x*)) (**is** *?P* = *?D*)
proof −
let *?E* = {*a* × *b* | *a b*. *a* ∈ *sets M1* ∧ *b* ∈ *sets M2*}
obtain *F* :: *nat* ⇒ (*'a* × *'b*) *set* **where** *F*: *range F* ⊆ *?E*
incseq F ⋃ (*range F*) = *space M1* × *space M2* ∀ *i*. *emeasure* (*M1* ⨂$_M$ *M2*) (*F i*) ≠ ∞

```
    using sigma_finite_up_in_pair_measure_generator by auto
  show ?thesis
  proof (rule measure_eqI_generator_eq[OF Int_stable_pair_measure_generator[of
M1 M2]])
    show ?E ⊆ Pow (space ?P)
        using sets.space_closed[of M1] sets.space_closed[of M2] by (auto simp:
space_pair_measure)
    show sets ?P = sigma_sets (space ?P) ?E
      by (simp add: sets_pair_measure space_pair_measure)
    then show sets ?D = sigma_sets (space ?P) ?E
      by simp
    from F show range F ⊆ ?E (⋃i. F i) = space ?P ⋀i. emeasure ?P (F i) ≠
∞
      by (auto simp: space_pair_measure)
  next
    fix X assume X ∈ ?E
    then obtain A B where X[simp]: X = A × B and A: A ∈ sets M1 and B:
B ∈ sets M2 by auto
    have (λ(y, x). (x, y)) -` X ∩ space (M2 ⨂M M1) = B × A
      using sets.sets_into_space[OF A] sets.sets_into_space[OF B] by (auto simp:
space_pair_measure)
    with A B show emeasure (M1 ⨂M M2) X = emeasure ?D X
    by (simp add: M2.emeasure_pair_measure_Times M1.emeasure_pair_measure_Times
emeasure_distr
                  measurable_pair_swap' ac_simps)
  qed
qed

lemma (in pair_sigma_finite) emeasure_pair_measure_alt2:
  assumes A: A ∈ sets (M1 ⨂M M2)
  shows emeasure (M1 ⨂M M2) A = (∫+y. emeasure M1 ((λx. (x, y)) -` A)
∂M2)
    (is _ = ?ν A)
proof -
  have [simp]: ⋀y. (Pair y -` ((λ(x, y). (y, x)) -` A ∩ space (M2 ⨂M M1))) =
(λx. (x, y)) -` A
    using sets.sets_into_space[OF A] by (auto simp: space_pair_measure)
  show ?thesis using A
    by (subst distr_pair_swap)
      (simp_all del: vimage_Int add: measurable_sets[OF measurable_pair_swap']
                    M1.emeasure_pair_measure_alt emeasure_distr[OF measur-
able_pair_swap' A])
qed

lemma (in pair_sigma_finite) AE_pair:
  assumes AE x in (M1 ⨂M M2). Q x
  shows AE x in M1. (AE y in M2. Q (x, y))
proof -
  obtain N where N: N ∈ sets (M1 ⨂M M2) emeasure (M1 ⨂M M2) N = 0
```

$\{x{\in}space\ (M1 \bigotimes_M M2).\ \neg\ Q\ x\} \subseteq N$
using *assms* **unfolding** *eventually_ae_filter* **by** *auto*
show *?thesis*
proof (*rule AE_I*)
from *N measurable_emeasure_Pair1*[*OF* ‹$N \in sets\ (M1 \bigotimes_M M2)$›]
show *emeasure M1* $\{x{\in}space\ M1.\ emeasure\ M2\ (Pair\ x\ -`\ N) \neq 0\} = 0$
by (*auto simp*: *M2.emeasure_pair_measure_alt nn_integral_0_iff*)
show $\{x \in space\ M1.\ emeasure\ M2\ (Pair\ x\ -`\ N) \neq 0\} \in sets\ M1$
by (*intro borel_measurable_eq measurable_emeasure_Pair1 N sets.sets_Collect_neg N*) *simp*
{ fix *x* **assume** $x \in space\ M1\ emeasure\ M2\ (Pair\ x\ -`\ N) = 0$
have *AE y in M2. Q* (*x*, *y*)
proof (*rule AE_I*)
show *emeasure M2* $(Pair\ x\ -`\ N) = 0$ **by** *fact*
show $Pair\ x\ -`\ N \in sets\ M2$ **using** *N*(*1*) **by** (*rule sets_Pair1*)
show $\{y \in space\ M2.\ \neg\ Q\ (x, y)\} \subseteq Pair\ x\ -`\ N$
using *N* ‹$x \in space\ M1$› **unfolding** *space_pair_measure* **by** *auto*
qed }
then show $\{x \in space\ M1.\ \neg\ (AE\ y\ in\ M2.\ Q\ (x, y))\} \subseteq \{x \in space\ M1.\ emeasure\ M2\ (Pair\ x\ -`\ N) \neq 0\}$
by *auto*
qed
qed

lemma (**in** *pair_sigma_finite*) *AE_pair_measure*:
assumes $\{x{\in}space\ (M1 \bigotimes_M M2).\ P\ x\} \in sets\ (M1 \bigotimes_M M2)$
assumes *ae*: *AE x in M1. AE y in M2. P* (*x*, *y*)
shows *AE x in M1* $\bigotimes_M$ *M2. P x*
proof (*subst AE_iff_measurable*[*OF _ refl*])
show $\{x{\in}space\ (M1 \bigotimes_M M2).\ \neg\ P\ x\} \in sets\ (M1 \bigotimes_M M2)$
by (*rule sets.sets_Collect*) *fact*
then have $emeasure\ (M1 \bigotimes_M M2)\ \{x \in space\ (M1 \bigotimes_M M2).\ \neg\ P\ x\} =$
$(\int^+ x.\ \int^+ y.\ indicator\ \{x \in space\ (M1 \bigotimes_M M2).\ \neg\ P\ x\}\ (x, y)\ \partial M2\ \partial M1)$
by (*simp add*: *M2.emeasure_pair_measure*)
also have $\ldots = (\int^+ x.\ \int^+ y.\ 0\ \partial M2\ \partial M1)$
using *ae*
apply (*safe intro*!: *nn_integral_cong_AE*)
apply (*intro AE_I2*)
apply (*safe intro*!: *nn_integral_cong_AE*)
apply *auto*
done
finally show $emeasure\ (M1 \bigotimes_M M2)\ \{x \in space\ (M1 \bigotimes_M M2).\ \neg\ P\ x\} = 0$
by *simp*
qed

lemma (**in** *pair_sigma_finite*) *AE_pair_iff*:
$\{x{\in}space\ (M1 \bigotimes_M M2).\ P\ (fst\ x)\ (snd\ x)\} \in sets\ (M1 \bigotimes_M M2) \Longrightarrow$
$(AE\ x\ in\ M1.\ AE\ y\ in\ M2.\ P\ x\ y) \longleftrightarrow (AE\ x\ in\ (M1 \bigotimes_M M2).\ P\ (fst\ x)\ (snd\ x))$

using *AE_pair*[*of* $\lambda x.\ P\ (fst\ x)\ (snd\ x)$] *AE_pair_measure*[*of* $\lambda x.\ P\ (fst\ x)\ (snd\ x)$] **by** *auto*

lemma (**in** *pair_sigma_finite*) *AE_commute*:
assumes *P*: $\{x \in space\ (M1 \bigotimes_M M2).\ P\ (fst\ x)\ (snd\ x)\} \in sets\ (M1 \bigotimes_M M2)$
shows $(AE\ x\ in\ M1.\ AE\ y\ in\ M2.\ P\ x\ y) \longleftrightarrow (AE\ y\ in\ M2.\ AE\ x\ in\ M1.\ P\ x\ y)$
proof −
interpret *Q*: *pair_sigma_finite M2 M1* **..**
have [*simp*]: $\bigwedge x.\ (fst\ (case\ x\ of\ (x,\ y) \Rightarrow (y,\ x))) = snd\ x\ \bigwedge x.\ (snd\ (case\ x\ of\ (x,\ y) \Rightarrow (y,\ x))) = fst\ x$
by *auto*
have $\{x \in space\ (M2 \bigotimes_M M1).\ P\ (snd\ x)\ (fst\ x)\} =$
$(\lambda(x,\ y).\ (y,\ x)) -` \{x \in space\ (M1 \bigotimes_M M2).\ P\ (fst\ x)\ (snd\ x)\} \cap space\ (M2 \bigotimes_M M1)$
by (*auto simp*: *space_pair_measure*)
also have $\ldots \in sets\ (M2 \bigotimes_M M1)$
by (*intro sets_pair_swap P*)
finally show *?thesis*
apply (*subst AE_pair_iff*[*OF P*])
apply (*subst distr_pair_swap*)
apply (*subst AE_distr_iff*[*OF measurable_pair_swap′ P*])
apply (*subst Q.AE_pair_iff*)
apply *simp_all*
done
qed

7.6.3 Fubinis theorem

lemma *measurable_compose_Pair1*:
$x \in space\ M1 \Longrightarrow g \in measurable\ (M1 \bigotimes_M M2)\ L \Longrightarrow (\lambda y.\ g\ (x,\ y)) \in measurable\ M2\ L$
by *simp*

lemma (**in** *sigma_finite_measure*) *borel_measurable_nn_integral_fst*:
assumes *f*: $f \in borel\_measurable\ (M1 \bigotimes_M M)$
shows $(\lambda x.\ \int^+ y.\ f\ (x,\ y)\ \partial M) \in borel\_measurable\ M1$
using *f* **proof** *induct*
case (*cong u v*)
then have $\bigwedge w\ x.\ w \in space\ M1 \Longrightarrow x \in space\ M \Longrightarrow u\ (w,\ x) = v\ (w,\ x)$
by (*auto simp*: *space_pair_measure*)
show *?case*
apply (*subst measurable_cong*)
apply (*rule nn_integral_cong*)
apply *fact+*
done
next
case (*set Q*)
have [*simp*]: $\bigwedge x\ y.\ indicator\ Q\ (x,\ y) = indicator\ (Pair\ x -` Q)\ y$

by (*auto simp*: *indicator_def*)
have $\bigwedge x.\ x \in space\ M1 \Longrightarrow emeasure\ M\ (Pair\ x -` Q) = \int^+ y.\ indicator\ Q\ (x,\ y)\ \partial M$
by (*simp add*: *sets_Pair1*[*OF set*])
from *this measurable_emeasure_Pair*[*OF set*] **show** *?case*
by (*rule measurable_cong*[*THEN iffD1*])
qed (*simp_all add*: *nn_integral_add nn_integral_cmult measurable_compose_Pair1*
nn_integral_monotone_convergence_SUP incseq_def le_fun_def image_comp
cong: *measurable_cong*)

lemma (**in** *sigma_finite_measure*) *nn_integral_fst*:
assumes *f*: $f \in borel\_measurable\ (M1 \bigotimes_M M)$
shows $(\int^+ x.\ \int^+ y.\ f\ (x,\ y)\ \partial M\ \partial M1) = integral^N\ (M1 \bigotimes_M M)\ f$ (**is** *?I f* = _)
using *f* **proof** *induct*
case (*cong u v*)
then have *?I u* = *?I v*
by (*intro nn_integral_cong*) (*auto simp*: *space_pair_measure*)
with *cong* **show** *?case*
by (*simp cong*: *nn_integral_cong*)
qed (*simp_all add*: *emeasure_pair_measure nn_integral_cmult nn_integral_add*
nn_integral_monotone_convergence_SUP measurable_compose_Pair1
borel_measurable_nn_integral_fst nn_integral_mono incseq_def le_fun_def image_comp
cong: *nn_integral_cong*)

lemma (**in** *sigma_finite_measure*) *borel_measurable_nn_integral*[*measurable* (*raw*)]:
case_prod f $\in$ *borel_measurable* $(N \bigotimes_M M) \Longrightarrow (\lambda x.\ \int^+ y.\ f\ x\ y\ \partial M) \in$ *borel_measurable N*
using *borel_measurable_nn_integral_fst*[*of case_prod f N*] **by** *simp*

proposition (**in** *pair_sigma_finite*) *nn_integral_snd*:
assumes *f*[*measurable*]: $f \in borel\_measurable\ (M1 \bigotimes_M M2)$
shows $(\int^+ y.\ (\int^+ x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = integral^N\ (M1 \bigotimes_M M2)\ f$
proof −
note *measurable_pair_swap*[*OF f*]
from *M1.nn_integral_fst*[*OF this*]
have $(\int^+ y.\ (\int^+ x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = (\int^+ (x,\ y).\ f\ (y,\ x)\ \partial(M2 \bigotimes_M M1))$
by *simp*
also have $(\int^+ (x,\ y).\ f\ (y,\ x)\ \partial(M2 \bigotimes_M M1)) = integral^N\ (M1 \bigotimes_M M2)\ f$
by (*subst distr_pair_swap*) (*auto simp add*: *nn_integral_distr intro*!: *nn_integral_cong*)
finally show *?thesis* **.**
qed

theorem (**in** *pair_sigma_finite*) *Fubini*:
assumes *f*: $f \in borel\_measurable\ (M1 \bigotimes_M M2)$
shows $(\int^+ y.\ (\int^+ x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = (\int^+ x.\ (\int^+ y.\ f\ (x,\ y)\ \partial M2)$

∂M1)
unfolding *nn_integral_snd*[*OF assms*] *M2.nn_integral_fst*[*OF assms*] **..**

theorem (**in** *pair_sigma_finite*) *Fubini'*:
assumes *f*: *case_prod f* ∈ *borel_measurable* (*M1* ⨂$_M$ *M2*)
shows (∫$^+$ *y*. (∫$^+$ *x*. *f x y ∂M1*) *∂M2*) = (∫$^+$ *x*. (∫$^+$ *y*. *f x y ∂M2*) *∂M1*)
using *Fubini*[*OF f*] **by** *simp*

7.6.4 Products on counting spaces, densities and distributions

proposition *sigma_prod*:
assumes *X_cover*: ∃ *E*⊆*A*. *countable E* ∧ *X* = ⋃ *E* **and** *A*: *A* ⊆ *Pow X*
assumes *Y_cover*: ∃ *E*⊆*B*. *countable E* ∧ *Y* = ⋃ *E* **and** *B*: *B* ⊆ *Pow Y*
shows *sigma X A* ⨂$_M$ *sigma Y B* = *sigma* (*X* × *Y*) {*a* × *b* | *a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
(**is** *?P* = *?S*)
proof (*rule measure_eqI*)
have [*simp*]: *snd* ∈ *X* × *Y* → *Y fst* ∈ *X* × *Y* → *X*
by *auto*
let *?XY* = {{*fst* −‘ *a* ∩ *X* × *Y* | *a*. *a* ∈ *A*}, {*snd* −‘ *b* ∩ *X* × *Y* | *b*. *b* ∈ *B*}}
have *sets ?P* = *sets* (*SUP xy*∈*?XY*. *sigma* (*X* × *Y*) *xy*)
by (*simp add*: *vimage_algebra_sigma sets_pair_eq_sets_fst_snd A B*)
also have ... = *sets* (*sigma* (*X* × *Y*) (⋃ *?XY*))
by (*intro Sup_sigma arg_cong*[**where** *f*=*sets*]) *auto*
also have ... = *sets ?S*
proof (*intro arg_cong*[**where** *f*=*sets*] *sigma_eqI sigma_sets_eqI*)
show ⋃ *?XY* ⊆ *Pow* (*X* × *Y*) {*a* × *b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*} ⊆ *Pow* (*X* × *Y*)
using *A B* **by** *auto*
next
interpret *XY*: *sigma_algebra X* × *Y sigma_sets* (*X* × *Y*) {*a* × *b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
using *A B* **by** (*intro sigma_algebra_sigma_sets*) *auto*
fix *Z* **assume** *Z* ∈ ⋃ *?XY*
then show *Z* ∈ *sigma_sets* (*X* × *Y*) {*a* × *b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
proof *safe*
fix *a* **assume** *a* ∈ *A*
from *Y_cover* **obtain** *E* **where** *E*: *E* ⊆ *B countable E* **and** *Y* = ⋃ *E*
by *auto*
with ‹*a* ∈ *A*› *A* **have** *eq*: *fst* −‘ *a* ∩ *X* × *Y* = (⋃ *e*∈*E*. *a* × *e*)
by *auto*
show *fst* −‘ *a* ∩ *X* × *Y* ∈ *sigma_sets* (*X* × *Y*) {*a* × *b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
using ‹*a* ∈ *A*› *E* **unfolding** *eq* **by** (*auto intro*!: *XY.countable_UN'*)
next
fix *b* **assume** *b* ∈ *B*
from *X_cover* **obtain** *E* **where** *E*: *E* ⊆ *A countable E* **and** *X* = ⋃ *E*
by *auto*
with ‹*b* ∈ *B*› *B* **have** *eq*: *snd* −‘ *b* ∩ *X* × *Y* = (⋃ *e*∈*E*. *e* × *b*)
by *auto*

```
    show snd −‘ b ∩ X × Y ∈ sigma_sets (X × Y) {a × b |a b. a ∈ A ∧ b ∈
B}
      using ‹b ∈ B› E unfolding eq by (auto intro!: XY.countable_UN′)
  qed
 next
  fix Z assume Z ∈ {a × b |a b. a ∈ A ∧ b ∈ B}
  then obtain a b where Z = a × b and ab: a ∈ A b ∈ B
    by auto
  then have Z: Z = (fst −‘ a ∩ X × Y) ∩ (snd −‘ b ∩ X × Y)
    using A B by auto
  interpret XY: sigma_algebra X × Y sigma_sets (X × Y) (⋃ ?XY)
    by (intro sigma_algebra_sigma_sets) auto
  show Z ∈ sigma_sets (X × Y) (⋃ ?XY)
    unfolding Z by (rule XY.Int) (blast intro: ab)+
 qed
 finally show sets ?P = sets ?S .
next
 interpret finite_measure sigma X A for X A
  proof qed (simp add: emeasure_sigma)
 fix A assume A ∈ sets ?P then show emeasure ?P A = emeasure ?S A
  by (simp add: emeasure_pair_measure_alt emeasure_sigma)
qed

lemma sigma_sets_pair_measure_generator_finite:
 assumes finite A and finite B
 shows sigma_sets (A × B) { a × b | a b. a ⊆ A ∧ b ⊆ B} = Pow (A × B)
 (is sigma_sets ?prod ?sets = _)
proof safe
 have fin: finite (A × B) using assms by (rule finite_cartesian_product)
 fix x assume subset: x ⊆ A × B
 hence finite x using fin by (rule finite_subset)
 from this subset show x ∈ sigma_sets ?prod ?sets
 proof (induct x)
  case empty show ?case by (rule sigma_sets.Empty)
 next
  case (insert a x)
  hence {a} ∈ sigma_sets ?prod ?sets by auto
  moreover have x ∈ sigma_sets ?prod ?sets using insert by auto
 ultimately show ?case unfolding insert_is_Un[of a x] by (rule sigma_sets_Un)
 qed
next
 fix x a b
 assume x ∈ sigma_sets ?prod ?sets and (a, b) ∈ x
 from sigma_sets_into_sp[OF _ this(1)] this(2)
 show a ∈ A and b ∈ B by auto
qed

proposition sets_pair_eq:
 assumes Ea: Ea ⊆ Pow (space A) sets A = sigma_sets (space A) Ea
```

and *Ca*: *countable Ca Ca* $\subseteq$ *Ea* $\bigcup$ *Ca* = *space A*
and *Eb*: *Eb* $\subseteq$ *Pow* (*space B*) *sets B* = *sigma_sets* (*space B*) *Eb*
and *Cb*: *countable Cb Cb* $\subseteq$ *Eb* $\bigcup$ *Cb* = *space B*
shows *sets* (*A* $\bigotimes_M$ *B*) = *sets* (*sigma* (*space A* $\times$ *space B*) { *a* $\times$ *b* | *a b*. *a* $\in$ *Ea* $\wedge$ *b* $\in$ *Eb* })
(**is** _ = *sets* (*sigma ?Ω ?E*))
proof
show *sets* (*sigma ?Ω ?E*) $\subseteq$ *sets* (*A* $\bigotimes_M$ *B*)
using *Ea*(*1*) *Eb*(*1*) **by** (*subst sigma_le_sets*) (*auto simp*: *Ea*(*2*) *Eb*(*2*))
have *?E* $\subseteq$ *Pow ?Ω*
using *Ea*(*1*) *Eb*(*1*) **by** *auto*
then have *E*: *a* $\in$ *Ea* $\Longrightarrow$ *b* $\in$ *Eb* $\Longrightarrow$ *a* $\times$ *b* $\in$ *sets* (*sigma ?Ω ?E*) **for** *a b*
by *auto*
have *sets* (*A* $\bigotimes_M$ *B*) $\subseteq$ *sets* (*Sup* {*vimage_algebra ?Ω fst A*, *vimage_algebra ?Ω snd B*})
unfolding *sets_pair_eq_sets_fst_snd* **..**
also have *vimage_algebra ?Ω fst A* = *vimage_algebra ?Ω fst* (*sigma* (*space A*) *Ea*)
by (*intro vimage_algebra_cong*[*OF refl refl*]) (*simp add*: *Ea*)
also have ... = *sigma ?Ω* {*fst* $-$‘ *A* $\cap$ *?Ω* |*A*. *A* $\in$ *Ea*}
by (*intro Ea vimage_algebra_sigma*) *auto*
also have *vimage_algebra ?Ω snd B* = *vimage_algebra ?Ω snd* (*sigma* (*space B*) *Eb*)
by (*intro vimage_algebra_cong*[*OF refl refl*]) (*simp add*: *Eb*)
also have ... = *sigma ?Ω* {*snd* $-$‘ *A* $\cap$ *?Ω* |*A*. *A* $\in$ *Eb*}
by (*intro Eb vimage_algebra_sigma*) *auto*
also have {*sigma ?Ω* {*fst* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Ea*}, *sigma ?Ω* {*snd* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Eb*}} =
sigma ?Ω ‘ {{*fst* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Ea*}, {*snd* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Eb*}}
by *auto*
also have *sets* (*SUP S*$\in${{*fst* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Ea*}, {*snd* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Eb*}}. *sigma ?Ω S*) =
sets (*sigma ?Ω* ($\bigcup$ {{*fst* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Ea*}, {*snd* $-$‘ *Aa* $\cap$ *?Ω* |*Aa*. *Aa* $\in$ *Eb*}}))
using *Ea*(*1*) *Eb*(*1*) **by** (*intro sets_Sup_sigma*) *auto*
also have ... $\subseteq$ *sets* (*sigma ?Ω ?E*)
proof (*subst sigma_le_sets*, *safe intro*!: *space_in_measure_of*)
fix *a* **assume** *a* $\in$ *Ea*
then have *fst* $-$‘ *a* $\cap$ *?Ω* = ($\bigcup$ *b*$\in$*Cb*. *a* $\times$ *b*)
using *Cb*(*3*)[*symmetric*] *Ea*(*1*) **by** *auto*
then show *fst* $-$‘ *a* $\cap$ *?Ω* $\in$ *sets* (*sigma ?Ω ?E*)
using *Cb* ‹*a* $\in$ *Ea*› **by** (*auto intro*!: *sets.countable_UN′ E*)
next
fix *b* **assume** *b* $\in$ *Eb*
then have *snd* $-$‘ *b* $\cap$ *?Ω* = ($\bigcup$ *a*$\in$*Ca*. *a* $\times$ *b*)
using *Ca*(*3*)[*symmetric*] *Eb*(*1*) **by** *auto*
then show *snd* $-$‘ *b* $\cap$ *?Ω* $\in$ *sets* (*sigma ?Ω ?E*)
using *Ca* ‹*b* $\in$ *Eb*› **by** (*auto intro*!: *sets.countable_UN′ E*)
qed

finally show *sets* $(A \bigotimes_M B) \subseteq$ *sets* (*sigma* ?Ω ?*E*) **.**
qed

proposition *borel_prod*:
(*borel* $\bigotimes_M$ *borel*) = (*borel* :: ('*a*::*second_countable_topology* × '*b*::*second_countable_topology*) *measure*)
(**is** ?*P* = ?*B*)
proof −
have ?*B* = *sigma UNIV* {*A* × *B* | *A B*. *open A* ∧ *open B*}
by (*rule second_countable_borel_measurable*[*OF open_prod_generated*])
also have ... = ?*P*
unfolding *borel_def*
by (*subst sigma_prod*) (*auto intro*!: *exI*[*of* _ {*UNIV*}])
finally show ?*thesis* **..**
qed

proposition *pair_measure_count_space*:
assumes *A*: *finite A* **and** *B*: *finite B*
shows *count_space A* $\bigotimes_M$ *count_space B* = *count_space* (*A* × *B*) (**is** ?*P* = ?*C*)
proof (*rule measure_eqI*)
interpret *A*: *finite_measure count_space A* **by** (*rule finite_measure_count_space*) *fact*
interpret *B*: *finite_measure count_space B* **by** (*rule finite_measure_count_space*) *fact*
interpret *P*: *pair_sigma_finite count_space A count_space B* **..**
show *eq*: *sets* ?*P* = *sets* ?*C*
by (*simp add*: *sets_pair_measure sigma_sets_pair_measure_generator_finite A B*)
fix *X* **assume** *X*: *X* ∈ *sets* ?*P*
with *eq* **have** *X_subset*: *X* ⊆ *A* × *B* **by** *simp*
with *A B* **have** *fin_Pair*: ⋀*x*. *finite* (*Pair x* −‘ *X*)
by (*intro finite_subset*[*OF* _ *B*]) *auto*
have *fin_X*: *finite X* **using** *X_subset* **by** (*rule finite_subset*) (*auto simp*: *A B*)
have *card*: *0* < *card* (*Pair a* −‘ *X*) **if** (*a*, *b*) ∈ *X* **for** *a b*
using *card_gt_0_iff fin_Pair that* **by** *auto*
then have *emeasure* ?*P X* = ∫⁺ *x*. *emeasure* (*count_space B*) (*Pair x* −‘ *X*)
∂*count_space A*
by (*simp add*: *B.emeasure_pair_measure_alt X*)
also have ... = *emeasure* ?*C X*
apply (*subst emeasure_count_space*)
using *card X_subset A fin_Pair fin_X*
apply (*auto simp add*: *nn_integral_count_space*
of_nat_sum[*symmetric*] *card_SigmaI*[*symmetric*]
simp del: *card_SigmaI*
intro!: *arg_cong*[**where** *f*=*card*])
done
finally show *emeasure* ?*P X* = *emeasure* ?*C X* **.**
qed

lemma *emeasure_prod_count_space*:
assumes *A*: $A \in sets\ (count\_space\ UNIV \bigotimes_M M)$ (**is** $A \in sets\ (?A \bigotimes_M ?B)$)
shows $emeasure\ (?A \bigotimes_M ?B)\ A = (\int^+ x.\ \int^+ y.\ indicator\ A\ (x, y)\ \partial ?B\ \partial ?A)$
by (*rule emeasure_measure_of*[*OF pair_measure_def*])
(*auto simp*: *countably_additive_def positive_def suminf_indicator A*
nn_integral_suminf[*symmetric*] *dest*: *sets.sets_into_space*)

lemma *emeasure_prod_count_space_single*[*simp*]: $emeasure\ (count\_space\ UNIV \bigotimes_M count\_space\ UNIV)\ \{x\} = 1$
proof −
have [*simp*]: $\bigwedge a\ b\ x\ y.\ indicator\ \{(a, b)\}\ (x, y) = (indicator\ \{a\}\ x * indicator\ \{b\}\ y :: ennreal)$
by (*auto split*: *split_indicator*)
show *?thesis*
by (*cases x*) (*auto simp*: *emeasure_prod_count_space nn_integral_cmult sets_Pair*)
qed

lemma *emeasure_count_space_prod_eq*:
fixes $A :: ('a \times 'b)\ set$
assumes *A*: $A \in sets\ (count\_space\ UNIV \bigotimes_M count\_space\ UNIV)$ (**is** $A \in sets\ (?A \bigotimes_M ?B)$)
shows $emeasure\ (?A \bigotimes_M ?B)\ A = emeasure\ (count\_space\ UNIV)\ A$
proof −
{ fix $A :: ('a \times 'b)\ set$ **assume** *countable A*
then have $emeasure\ (?A \bigotimes_M ?B)\ (\bigcup a \in A.\ \{a\}) = (\int^+ a.\ emeasure\ (?A \bigotimes_M ?B)\ \{a\}\ \partial count\_space\ A)$
by (*intro emeasure_UN_countable*) (*auto simp*: *sets_Pair disjoint_family_on_def*)
also have $\ldots = (\int^+ a.\ indicator\ A\ a\ \partial count\_space\ UNIV)$
by (*subst nn_integral_count_space_indicator*) *auto*
finally have $emeasure\ (?A \bigotimes_M ?B)\ A = emeasure\ (count\_space\ UNIV)\ A$
by *simp* **}**
note $* = this$

show *?thesis*
proof *cases*
assume *finite A* **then show** *?thesis*
by (*intro* $*$ *countable_finite*)
next
assume *infinite A*
then obtain *C* **where** *countable C* **and** *infinite C* **and** $C \subseteq A$
by (*auto dest*: *infinite_countable_subset'*)
with *A* **have** $emeasure\ (?A \bigotimes_M ?B)\ C \leq emeasure\ (?A \bigotimes_M ?B)\ A$
by (*intro emeasure_mono*) *auto*
also have $emeasure\ (?A \bigotimes_M ?B)\ C = emeasure\ (count\_space\ UNIV)\ C$
using ‹*countable C*› **by** (*rule* $*$)
finally show *?thesis*
using ‹*infinite C*› ‹*infinite A*› **by** (*simp add*: *top_unique*)

```
  qed
qed

lemma nn_integral_count_space_prod_eq:
   nn_integral (count_space UNIV ⨂M count_space UNIV) f = nn_integral
(count_space UNIV) f
    (is nn_integral ?P f = _)
proof cases
  assume cntbl: countable {x. f x ≠ 0}
  have [simp]: ⋀x. card ({x} ∩ {x. f x ≠ 0}) = (indicator {x. f x ≠ 0} x::ennreal)
    by (auto split: split_indicator)
  have [measurable]: ⋀y. (λx. indicator {y} x) ∈ borel_measurable ?P
    by (rule measurable_discrete_difference[of λx. 0 _ borel {y} λx. indicator {y}
x for y])
       (auto intro: sets_Pair)

  have (∫⁺x. f x ∂?P) = (∫⁺x. ∫⁺x'. f x * indicator {x} x' ∂count_space {x. f
x ≠ 0} ∂?P)
    by (auto simp add: nn_integral_cmult nn_integral_indicator' intro!: nn_integral_cong
split: split_indicator)
  also have ... = (∫⁺x. ∫⁺x'. f x' * indicator {x'} x ∂count_space {x. f x ≠ 0}
∂?P)
    by (auto intro!: nn_integral_cong split: split_indicator)
  also have ... = (∫⁺x'. ∫⁺x. f x' * indicator {x'} x ∂?P ∂count_space {x. f x
≠ 0})
    by (intro nn_integral_count_space_nn_integral cntbl) auto
  also have ... = (∫⁺x'. f x' ∂count_space {x. f x ≠ 0})
    by (intro nn_integral_cong) (auto simp: nn_integral_cmult sets_Pair)
  finally show ?thesis
   by (auto simp add: nn_integral_count_space_indicator intro!: nn_integral_cong
split: split_indicator)
next
  { fix x assume f x ≠ 0
    then have (∃r≥0. 0 < r ∧ f x = ennreal r) ∨ f x = ∞
      by (cases f x rule: ennreal_cases) (auto simp: less_le)
    then have ∃n. ennreal (1 / real (Suc n)) ≤ f x
      by (auto elim!: nat_approx_posE intro!: less_imp_le) }
  note ∗ = this

  assume cntbl: uncountable {x. f x ≠ 0}
  also have {x. f x ≠ 0} = (⋃n. {x. 1/Suc n ≤ f x})
    using ∗ by auto
  finally obtain n where infinite {x. 1/Suc n ≤ f x}
    by (meson countableI_type countable_UN uncountable_infinite)
  then obtain C where C: C ⊆ {x. 1/Suc n ≤ f x} and countable C infinite C
    by (metis infinite_countable_subset')

  have [measurable]: C ∈ sets ?P
    using sets.countable[OF _ ‹countable C›, of ?P] by (auto simp: sets_Pair)
```

have $(\int^+ x.\ ennreal\ (1/Suc\ n) * indicator\ C\ x\ \partial ?P) \leq nn\_integral\ ?P\ f$
using C **by** (*intro nn_integral_mono*) (*auto split*: *split_indicator simp*: *zero_ereal_def*[*symmetric*])
moreover have $(\int^+ x.\ ennreal\ (1/Suc\ n) * indicator\ C\ x\ \partial ?P) = \infty$
using ‹*infinite C*› **by** (*simp add*: *nn_integral_cmult emeasure_count_space_prod_eq ennreal_mult_top*)
moreover have $(\int^+ x.\ ennreal\ (1/Suc\ n) * indicator\ C\ x\ \partial count\_space\ UNIV) \leq nn\_integral\ (count\_space\ UNIV)\ f$
using C **by** (*intro nn_integral_mono*) (*auto split*: *split_indicator simp*: *zero_ereal_def*[*symmetric*])
moreover have $(\int^+ x.\ ennreal\ (1/Suc\ n) * indicator\ C\ x\ \partial count\_space\ UNIV) = \infty$
using ‹*infinite C*› **by** (*simp add*: *nn_integral_cmult ennreal_mult_top*)
ultimately show *?thesis*
by (*simp add*: *top_unique*)
qed

theorem *pair_measure_density*:
assumes *f*: $f \in borel\_measurable\ M1$
assumes *g*: $g \in borel\_measurable\ M2$
assumes $sigma\_finite\_measure\ M2\ sigma\_finite\_measure\ (density\ M2\ g)$
shows $density\ M1\ f \bigotimes_M density\ M2\ g = density\ (M1 \bigotimes_M M2)\ (\lambda(x,y).\ f\ x * g\ y)$ (**is** *?L* = *?R*)
proof (*rule measure_eqI*)
interpret *M2*: *sigma_finite_measure M2* **by** *fact*
interpret *D2*: $sigma\_finite\_measure\ density\ M2\ g$ **by** *fact*

fix *A* **assume** *A*: $A \in sets\ ?L$
with *f g* **have** $(\int^+ x.\ f\ x * \int^+ y.\ g\ y * indicator\ A\ (x, y)\ \partial M2\ \partial M1) = (\int^+ x.\ \int^+ y.\ f\ x * g\ y * indicator\ A\ (x, y)\ \partial M2\ \partial M1)$
by (*intro nn_integral_cong_AE*)
(*auto simp add*: *nn_integral_cmult*[*symmetric*] *ac_simps*)
with *A f g* **show** *emeasure ?L A* = *emeasure ?R A*
by (*simp add*: *D2.emeasure_pair_measure emeasure_density nn_integral_density*
M2.nn_integral_fst[*symmetric*]
cong: *nn_integral_cong*)
qed *simp*

lemma *sigma_finite_measure_distr*:
assumes $sigma\_finite\_measure\ (distr\ M\ N\ f)$ **and** *f*: $f \in measurable\ M\ N$
shows *sigma_finite_measure M*
proof −
interpret *sigma_finite_measure distr M N f* **by** *fact*
obtain *A* **where** *A*: *countable A* $A \subseteq sets\ (distr\ M\ N\ f)$ $\bigcup A = space\ (distr\ M\ N\ f)$ $\forall a{\in}A.\ emeasure\ (distr\ M\ N\ f)\ a \neq \infty$
using *sigma_finite_countable* **by** *auto*
show *?thesis*
proof
show $\exists A.\ countable\ A \wedge A \subseteq sets\ M \wedge \bigcup A = space\ M \wedge (\forall a{\in}A.\ emeasure\ M\ a \neq \infty)$

```
    using A f
    by (intro exI[of _ (λa. f -` a ∩ space M) ` A])
       (auto simp: emeasure_distr set_eq_iff subset_eq intro: measurable_space)
  qed
qed

lemma pair_measure_distr:
  assumes f: f ∈ measurable M S and g: g ∈ measurable N T
  assumes sigma_finite_measure (distr N T g)
  shows distr M S f ⨂M distr N T g = distr (M ⨂M N) (S ⨂M T) (λ(x, y).
(f x, g y)) (is ?P = ?D)
proof (rule measure_eqI)
  interpret T: sigma_finite_measure distr N T g by fact
  interpret N: sigma_finite_measure N by (rule sigma_finite_measure_distr)
fact+

  fix A assume A: A ∈ sets ?P
  with f g show emeasure ?P A = emeasure ?D A
    by (auto simp add: N.emeasure_pair_measure_alt space_pair_measure emea-
sure_distr
                       T.emeasure_pair_measure_alt nn_integral_distr
             intro!: nn_integral_cong arg_cong[where f=emeasure N])
qed simp

lemma pair_measure_eqI:
  assumes sigma_finite_measure M1 sigma_finite_measure M2
  assumes sets: sets (M1 ⨂M M2) = sets M
  assumes emeasure: ⋀A B. A ∈ sets M1 ⟹ B ∈ sets M2 ⟹ emeasure M1 A
* emeasure M2 B = emeasure M (A × B)
  shows M1 ⨂M M2 = M
proof -
  interpret M1: sigma_finite_measure M1 by fact
  interpret M2: sigma_finite_measure M2 by fact
  interpret pair_sigma_finite M1 M2 ..
  let ?E = {a × b |a b. a ∈ sets M1 ∧ b ∈ sets M2}
  let ?P = M1 ⨂M M2
  obtain F :: nat ⇒ ('a × 'b) set where F:
    range F ⊆ ?E incseq F ⋃ (range F) = space M1 × space M2 ∀i. emeasure ?P
(F i) ≠ ∞
    using sigma_finite_up_in_pair_measure_generator
    by blast
  show ?thesis
  proof (rule measure_eqI_generator_eq[OF Int_stable_pair_measure_generator[of
M1 M2]])
    show ?E ⊆ Pow (space ?P)
      using sets.space_closed[of M1] sets.space_closed[of M2] by (auto simp:
space_pair_measure)
    show sets ?P = sigma_sets (space ?P) ?E
      by (simp add: sets_pair_measure space_pair_measure)
```

```
    then show sets M = sigma_sets (space ?P) ?E
      using sets[symmetric] by simp
  next
    show range F ⊆ ?E (⋃i. F i) = space ?P ⋀i. emeasure ?P (F i) ≠ ∞
      using F by (auto simp: space_pair_measure)
  next
    fix X assume X ∈ ?E
    then obtain A B where X[simp]: X = A × B and A: A ∈ sets M1 and B:
B ∈ sets M2 by auto
    then have emeasure ?P X = emeasure M1 A * emeasure M2 B
      by (simp add: M2.emeasure_pair_measure_Times)
    also have ... = emeasure M (A × B)
      using A B emeasure by auto
    finally show emeasure ?P X = emeasure M X
      by simp
  qed
qed

lemma sets_pair_countable:
  assumes countable S1 countable S2
  assumes M: sets M = Pow S1 and N: sets N = Pow S2
  shows sets (M ⨂M N) = Pow (S1 × S2)
proof auto
  fix x a b assume x: x ∈ sets (M ⨂M N) (a, b) ∈ x
  from sets.sets_into_space[OF x(1)] x(2)
      sets_eq_imp_space_eq[of N count_space S2] sets_eq_imp_space_eq[of M
count_space S1] M N
  show a ∈ S1 b ∈ S2
    by (auto simp: space_pair_measure)
next
  fix X assume X: X ⊆ S1 × S2
  then have countable X
    by (metis countable_subset ‹countable S1› ‹countable S2› countable_SIGMA)
  have X = (⋃(a, b)∈X. {a} × {b}) by auto
  also have ... ∈ sets (M ⨂M N)
    using X
   by (safe intro!: sets.countable_UN' ‹countable X› subsetI pair_measureI) (auto
simp: M N)
  finally show X ∈ sets (M ⨂M N) .
qed

lemma pair_measure_countable:
  assumes countable S1 countable S2
  shows count_space S1 ⨂M count_space S2 = count_space (S1 × S2)
proof (rule pair_measure_eqI)
 show sigma_finite_measure (count_space S1) sigma_finite_measure (count_space
S2)
    using assms by (auto intro!: sigma_finite_measure_count_space_countable)
  show sets (count_space S1 ⨂M count_space S2) = sets (count_space (S1 ×
```

S2))
by (*subst sets_pair_countable*[*OF assms*]) *auto*
next
fix *A B* **assume** *A* ∈ *sets* (*count_space S1*) *B* ∈ *sets* (*count_space S2*)
then show *emeasure* (*count_space S1*) *A* ∗ *emeasure* (*count_space S2*) *B* =
emeasure (*count_space* (*S1* × *S2*)) (*A* × *B*)
by (*subst* (*1 2 3*) *emeasure_count_space*) (*auto simp*: *finite_cartesian_product_iff ennreal_mult_top ennreal_top_mult*)
qed

proposition *nn_integral_fst_count_space*:
(∫⁺ *x*. ∫⁺ *y*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = *integral*^{N} (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)
proof(*cases*)
assume ∗: *countable* {*xy*. *f xy* ≠ *0*}
let *?A* = *fst* ‘ {*xy*. *f xy* ≠ *0*}
let *?B* = *snd* ‘ {*xy*. *f xy* ≠ *0*}
from ∗ **have** [*simp*]: *countable ?A countable ?B* **by**(*rule countable_image*)+
have *?lhs* = (∫⁺ *x*. ∫⁺ *y*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space ?A*)
by(*rule nn_integral_count_space_eq*)
(*auto simp add*: *nn_integral_0_iff_AE AE_count_space not_le intro*: *rev_image_eqI*)
also have ... = (∫⁺ *x*. ∫⁺ *y*. *f* (*x*, *y*) ∂*count_space ?B* ∂*count_space ?A*)
by(*intro nn_integral_count_space_eq nn_integral_cong*)(*auto intro*: *rev_image_eqI*)
also have ... = (∫⁺ *xy*. *f xy* ∂*count_space* (*?A* × *?B*))
by(*subst sigma_finite_measure.nn_integral_fst*)
(*simp_all add*: *sigma_finite_measure_count_space_countable pair_measure_countable*)
also have ... = *?rhs*
by(*rule nn_integral_count_space_eq*)(*auto intro*: *rev_image_eqI*)
finally show *?thesis* **.**
next
{ fix *xy* **assume** *f xy* ≠ *0*
then have (∃ *r*≥*0*. *0* < *r* ∧ *f xy* = *ennreal r*) ∨ *f xy* = ∞
by (*cases f xy rule*: *ennreal_cases*) (*auto simp*: *less_le*)
then have ∃ *n*. *ennreal* (*1* / *real* (*Suc n*)) ≤ *f xy*
by (*auto elim*!: *nat_approx_posE intro*!: *less_imp_le*) **}**
note ∗ = *this*

assume *cntbl*: *uncountable* {*xy*. *f xy* ≠ *0*}
also have {*xy*. *f xy* ≠ *0*} = (⋃ *n*. {*xy*. *1*/*Suc n* ≤ *f xy*})
using ∗ **by** *auto*
finally obtain *n* **where** *infinite* {*xy*. *1*/*Suc n* ≤ *f xy*}
by (*meson countableI_type countable_UN uncountable_infinite*)
then obtain *C* **where** *C*: *C* ⊆ {*xy*. *1*/*Suc n* ≤ *f xy*} **and** *countable C infinite C*
by (*metis infinite_countable_subset'*)

have ∞ = (∫⁺ *xy*. *ennreal* (*1* / *Suc n*) ∗ *indicator C xy* ∂*count_space UNIV*)
using ‹*infinite C*› **by**(*simp add*: *nn_integral_cmult ennreal_mult_top*)

also have ... $\leq$ *?rhs* **using** *C*
by(*intro nn_integral_mono*)(*auto split*: *split_indicator*)
finally have *?rhs* $= \infty$ **by** (*simp add*: *top_unique*)
moreover have *?lhs* $= \infty$
proof(*cases finite* (*fst* ‘ *C*))
case *True*
then obtain *x C′* **where** *x*: $x \in$ *fst* ‘ *C*
and *C′*: $C' =$ *fst* $-$‘ $\{x\} \cap C$
and *infinite C′*
using ‹*infinite C*› **by**(*auto elim*!: *inf_img_fin_domE′*)
from *x C C′* **have** $**$: $C' \subseteq \{xy.\ 1\ /\ Suc\ n \leq f\ xy\}$ **by** *auto*

from *C′* ‹*infinite C′*› **have** *infinite* (*snd* ‘ *C′*)
by(*auto dest*!: *finite_imageD simp add*: *inj_on_def*)
then have $\infty = (\int^+ y.$ *ennreal* (*1 / Suc n*) $*$ *indicator* (*snd* ‘ *C′*) *y* ∂*count_space UNIV*)
by(*simp add*: *nn_integral_cmult ennreal_mult_top*)
also have ... $= (\int^+ y.$ *ennreal* (*1 / Suc n*) $*$ *indicator C′* (*x*, *y*) ∂*count_space UNIV*)
by(*rule nn_integral_cong*)(*force split*: *split_indicator intro*: *rev_image_eqI simp add*: *C′*)
also have ... $= (\int^+ x'.\ (\int^+ y.$ *ennreal* (*1 / Suc n*) $*$ *indicator C′* (*x*, *y*) ∂*count_space UNIV*) $*$ *indicator* $\{x\}$ *x′* ∂*count_space UNIV*)
by(*simp add*: *one_ereal_def*[*symmetric*])
also have ... $\leq (\int^+ x.\ \int^+ y.$ *ennreal* (*1 / Suc n*) $*$ *indicator C′* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*)
by(*rule nn_integral_mono*)(*simp split*: *split_indicator*)
also have ... $\leq$ *?lhs* **using** $**$
by(*intro nn_integral_mono*)(*auto split*: *split_indicator*)
finally show *?thesis* **by** (*simp add*: *top_unique*)
next
case *False*
define *C′* **where** *C′* = *fst* ‘ *C*
have $\infty = \int^+ x.$ *ennreal* (*1 / Suc n*) $*$ *indicator C′ x* ∂*count_space UNIV*
using *C′_def False* **by**(*simp add*: *nn_integral_cmult ennreal_mult_top*)
also have ... $= \int^+ x.\ \int^+ y.$ *ennreal* (*1 / Suc n*) $*$ *indicator C′ x* $*$ *indicator* $\{$*SOME y*. (*x*, *y*) $\in$ *C*$\}$ *y* ∂*count_space UNIV* ∂*count_space UNIV*
by(*auto simp add*: *one_ereal_def*[*symmetric*] *max_def intro*: *nn_integral_cong*)
also have ... $\leq \int^+ x.\ \int^+ y.$ *ennreal* (*1 / Suc n*) $*$ *indicator C* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*
by(*intro nn_integral_mono*)(*auto simp add*: *C′_def split*: *split_indicator intro*: *someI*)
also have ... $\leq$ *?lhs* **using** *C*
by(*intro nn_integral_mono*)(*auto split*: *split_indicator*)
finally show *?thesis* **by** (*simp add*: *top_unique*)
qed
ultimately show *?thesis* **by** *simp*
qed

proposition *nn_integral_snd_count_space*:
(∫⁺ *y*. ∫⁺ *x*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = *integral*^{N} (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = (∫⁺ *y*. ∫⁺ *x*. (λ(*y*, *x*). *f* (*x*, *y*)) (*y*, *x*) ∂*count_space UNIV* ∂*count_space UNIV*)
by(*simp*)
also have ... = ∫⁺ *yx*. (λ(*y*, *x*). *f* (*x*, *y*)) *yx* ∂*count_space UNIV*
by(*rule nn_integral_fst_count_space*)
also have ... = ∫⁺ *xy*. *f xy* ∂*count_space* ((λ(*x*, *y*). (*y*, *x*)) ‘ *UNIV*)
by(*subst nn_integral_bij_count_space*[*OF inj_on_imp_bij_betw*, *symmetric*])
(*simp_all add*: *inj_on_def split_def*)
also have ... = *?rhs* **by**(*rule nn_integral_count_space_eq*) *auto*
finally show *?thesis* **.**
qed

lemma *measurable_pair_measure_countable1*:
assumes *countable A*
and [*measurable*]: ⋀*x*. *x* ∈ *A* ⟹ (λ*y*. *f* (*x*, *y*)) ∈ *measurable N K*
shows *f* ∈ *measurable* (*count_space A* ⨂$_M$ *N*) *K*
using _ _ *assms*(*1*)
by(*rule measurable_compose_countable'*[**where** *f*=λ*a b*. *f* (*a*, *snd b*) **and** *g*=*fst* **and** *I*=*A*, *simplified*])*simp_all*

7.6.5 Product of Borel spaces

theorem *borel_Times*:
fixes *A* :: *'a*::*topological_space set* **and** *B* :: *'b*::*topological_space set*
assumes *A*: *A* ∈ *sets borel* **and** *B*: *B* ∈ *sets borel*
shows *A* × *B* ∈ *sets borel*
proof −
have *A* × *B* = (*A*×*UNIV*) ∩ (*UNIV* × *B*)
by *auto*
moreover
{ **have** *A* ∈ *sigma_sets UNIV* {*S*. *open S*} **using** *A* **by** (*simp add*: *sets_borel*)
then have *A*×*UNIV* ∈ *sets borel*
proof (*induct A*)
case (*Basic S*) **then show** *?case*
by (*auto intro*!: *borel_open open_Times*)
next
case (*Compl A*)
moreover have ∗: (*UNIV* − *A*) × *UNIV* = *UNIV* − (*A* × *UNIV*)
by *auto*
ultimately show *?case*
unfolding ∗ **by** *auto*
next
case (*Union A*)
moreover have ∗: (⋃(*A* ‘ *UNIV*)) × *UNIV* = ⋃((λ*i*. *A i* × *UNIV*) ‘ *UNIV*)

```
      by auto
    ultimately show ?case
      unfolding ∗ by auto
  qed simp }
 moreover
 { have B ∈ sigma_sets UNIV {S. open S} using B by (simp add: sets_borel)
  then have UNIV×B ∈ sets borel
  proof (induct B)
    case (Basic S) then show ?case
      by (auto intro!: borel_open open_Times)
  next
    case (Compl B)
    moreover have ∗: UNIV × (UNIV − B) = UNIV − (UNIV × B)
      by auto
    ultimately show ?case
      unfolding ∗ by auto
  next
    case (Union B)
   moreover have ∗: UNIV × (⋃(B ‘ UNIV)) = ⋃((λi. UNIV × B i) ‘ UNIV)
      by auto
    ultimately show ?case
      unfolding ∗ by auto
  qed simp }
 ultimately show ?thesis
  by auto
qed

lemma finite_measure_pair_measure:
 assumes finite_measure M finite_measure N
 shows finite_measure (N ⨂M M)
proof (rule finite_measureI)
 interpret M: finite_measure M by fact
 interpret N: finite_measure N by fact
 show emeasure (N ⨂M M) (space (N ⨂M M)) ≠ ∞
   by (auto simp: space_pair_measure M.emeasure_pair_measure_Times en-
nreal_mult_eq_top_iff)
qed

end
```

7.7 Finite Product Measure

```
theory Finite_Product_Measure
imports Binary_Product_Measure Function_Topology
begin

lemma Pi_choice: (∀ i∈I. ∃ x∈F i. P i x) ⟷ (∃ f∈Pi I F. ∀ i∈I. P i (f i))
 by (metis Pi_iff)
```

lemma *PiE_choice*: $(\forall\, i{\in}I.\ \exists\, x{\in}F\ i.\ P\ i\ x) \longleftrightarrow (\exists\, f{\in}Pi_E\ I\ F.\ \forall\, i{\in}I.\ P\ i\ (f\ i))$
unfolding *Pi_choice* **by** (*metis Int_iff PiE_def restrict_PiE restrict_apply*)

lemma *case_prod_const*: $(\lambda(i,\ j).\ c) = (\lambda\_.\ c)$
by *auto*

7.7.1 More about Function restricted by *extensional*

definition
merge I J $= (\lambda(x,\ y)\ i.$ *if* $i \in I$ *then* $x\ i$ *else if* $i \in J$ *then* $y\ i$ *else undefined*)

lemma *merge_apply*[*simp*]:
$I \cap J = \{\} \Longrightarrow i \in I \Longrightarrow$ *merge I J* $(x,\ y)\ i = x\ i$
$I \cap J = \{\} \Longrightarrow i \in J \Longrightarrow$ *merge I J* $(x,\ y)\ i = y\ i$
$J \cap I = \{\} \Longrightarrow i \in I \Longrightarrow$ *merge I J* $(x,\ y)\ i = x\ i$
$J \cap I = \{\} \Longrightarrow i \in J \Longrightarrow$ *merge I J* $(x,\ y)\ i = y\ i$
$i \notin I \Longrightarrow i \notin J \Longrightarrow$ *merge I J* $(x,\ y)\ i =$ *undefined*
unfolding *merge_def* **by** *auto*

lemma *merge_commute*:
$I \cap J = \{\} \Longrightarrow$ *merge I J* $(x,\ y) =$ *merge J I* $(y,\ x)$
by (*force simp*: *merge_def*)

lemma *Pi_cancel_merge_range*[*simp*]:
$I \cap J = \{\} \Longrightarrow x \in Pi\ I\ (merge\ I\ J\ (A,\ B)) \longleftrightarrow x \in Pi\ I\ A$
$I \cap J = \{\} \Longrightarrow x \in Pi\ I\ (merge\ J\ I\ (B,\ A)) \longleftrightarrow x \in Pi\ I\ A$
$J \cap I = \{\} \Longrightarrow x \in Pi\ I\ (merge\ I\ J\ (A,\ B)) \longleftrightarrow x \in Pi\ I\ A$
$J \cap I = \{\} \Longrightarrow x \in Pi\ I\ (merge\ J\ I\ (B,\ A)) \longleftrightarrow x \in Pi\ I\ A$
by (*auto simp*: *Pi_def*)

lemma *Pi_cancel_merge*[*simp*]:
$I \cap J = \{\} \Longrightarrow$ *merge I J* $(x,\ y) \in Pi\ I\ B \longleftrightarrow x \in Pi\ I\ B$
$J \cap I = \{\} \Longrightarrow$ *merge I J* $(x,\ y) \in Pi\ I\ B \longleftrightarrow x \in Pi\ I\ B$
$I \cap J = \{\} \Longrightarrow$ *merge I J* $(x,\ y) \in Pi\ J\ B \longleftrightarrow y \in Pi\ J\ B$
$J \cap I = \{\} \Longrightarrow$ *merge I J* $(x,\ y) \in Pi\ J\ B \longleftrightarrow y \in Pi\ J\ B$
by (*auto simp*: *Pi_def*)

lemma *extensional_merge*[*simp*]: *merge I J* $(x,\ y) \in$ *extensional* $(I \cup J)$
by (*auto simp*: *extensional_def*)

lemma *restrict_merge*[*simp*]:
$I \cap J = \{\} \Longrightarrow$ *restrict* $(merge\ I\ J\ (x,\ y))\ I =$ *restrict x I*
$I \cap J = \{\} \Longrightarrow$ *restrict* $(merge\ I\ J\ (x,\ y))\ J =$ *restrict y J*
$J \cap I = \{\} \Longrightarrow$ *restrict* $(merge\ I\ J\ (x,\ y))\ I =$ *restrict x I*
$J \cap I = \{\} \Longrightarrow$ *restrict* $(merge\ I\ J\ (x,\ y))\ J =$ *restrict y J*
by (*auto simp*: *restrict_def*)

lemma *split_merge*: $P\ (merge\ I\ J\ (x,y)\ i) \longleftrightarrow (i \in I \longrightarrow P\ (x\ i)) \wedge (i \in J - I \longrightarrow P\ (y\ i)) \wedge (i \notin I \cup J \longrightarrow P$ *undefined*)

unfolding *merge_def* **by** *auto*

lemma *PiE_cancel_merge*[*simp*]:
I ∩ *J* = {} ⟹
merge I J (*x*, *y*) ∈ *Pi*$_E$ (*I* ∪ *J*) *B* ⟷ *x* ∈ *Pi I B* ∧ *y* ∈ *Pi J B*
by (*auto simp*: *PiE_def restrict_Pi_cancel*)

lemma *merge_singleton*[*simp*]: *i* ∉ *I* ⟹ *merge I* {*i*} (*x*,*y*) = *restrict* (*x*(*i* := *y i*)) (*insert i I*)
unfolding *merge_def* **by** (*auto simp*: *fun_eq_iff*)

lemma *extensional_merge_sub*: *I* ∪ *J* ⊆ *K* ⟹ *merge I J* (*x*, *y*) ∈ *extensional K*
unfolding *merge_def extensional_def* **by** *auto*

lemma *merge_restrict*[*simp*]:
merge I J (*restrict x I*, *y*) = *merge I J* (*x*, *y*)
merge I J (*x*, *restrict y J*) = *merge I J* (*x*, *y*)
unfolding *merge_def* **by** *auto*

lemma *merge_x_x_eq_restrict*[*simp*]:
merge I J (*x*, *x*) = *restrict x* (*I* ∪ *J*)
unfolding *merge_def* **by** *auto*

lemma *injective_vimage_restrict*:
assumes *J*: *J* ⊆ *I*
and *sets*: *A* ⊆ (Π$_E$ *i*∈*J*. *S i*) *B* ⊆ (Π$_E$ *i*∈*J*. *S i*) **and** *ne*: (Π$_E$ *i*∈*I*. *S i*) ≠ {}
and *eq*: (λ*x*. *restrict x J*) −‘ *A* ∩ (Π$_E$ *i*∈*I*. *S i*) = (λ*x*. *restrict x J*) −‘ *B* ∩ (Π$_E$ *i*∈*I*. *S i*)
shows *A* = *B*
proof (*intro set_eqI*)
fix *x*
from *ne* **obtain** *y* **where** *y*: ⋀*i*. *i* ∈ *I* ⟹ *y i* ∈ *S i* **by** *auto*
have *J* ∩ (*I* − *J*) = {} **by** *auto*
show *x* ∈ *A* ⟷ *x* ∈ *B*
proof *cases*
assume *x*: *x* ∈ (Π$_E$ *i*∈*J*. *S i*)
have *x* ∈ *A* ⟷ *merge J* (*I* − *J*) (*x*,*y*) ∈ (λ*x*. *restrict x J*) −‘ *A* ∩ (Π$_E$ *i*∈*I*. *S i*)
using *y x* ‹*J* ⊆ *I*› *PiE_cancel_merge*[*of J I* − *J x y S*]
by (*auto simp del*: *PiE_cancel_merge simp add*: *Un_absorb1*)
then show *x* ∈ *A* ⟷ *x* ∈ *B*
using *y x* ‹*J* ⊆ *I*› *PiE_cancel_merge*[*of J I* − *J x y S*]
by (*auto simp del*: *PiE_cancel_merge simp add*: *Un_absorb1 eq*)
qed (*use sets* **in** *auto*)
qed

lemma *restrict_vimage*:
I ∩ *J* = {} ⟹
(λ*x*. (*restrict x I*, *restrict x J*)) −‘ (*Pi*$_E$ *I E* × *Pi*$_E$ *J F*) = *Pi* (*I* ∪ *J*) (*merge*

I J (*E*, *F*))
by (*auto simp*: *restrict_Pi_cancel PiE_def*)

lemma *merge_vimage*:
I ∩ *J* = {} ⟹ *merge I J* −‘ *Pi*$_E$ (*I* ∪ *J*) *E* = *Pi I E* × *Pi J E*
by (*auto simp*: *restrict_Pi_cancel PiE_def*)

7.7.2 Finite product spaces

definition *prod_emb* **where**
prod_emb I M K X = (λ*x*. *restrict x K*) −‘ *X* ∩ (Π$_E$ *i*∈*I*. *space* (*M i*))

lemma *prod_emb_iff*:
f ∈ *prod_emb I M K X* ⟷ *f* ∈ *extensional I* ∧ (*restrict f K* ∈ *X*) ∧ (∀ *i*∈*I*. *f i* ∈ *space* (*M i*))
unfolding *prod_emb_def PiE_def* **by** *auto*

lemma
shows *prod_emb_empty*[*simp*]: *prod_emb M L K* {} = {}
and *prod_emb_Un*[*simp*]: *prod_emb M L K* (*A* ∪ *B*) = *prod_emb M L K A* ∪ *prod_emb M L K B*
and *prod_emb_Int*: *prod_emb M L K* (*A* ∩ *B*) = *prod_emb M L K A* ∩ *prod_emb M L K B*
and *prod_emb_UN*[*simp*]: *prod_emb M L K* (⋃ *i*∈*I*. *F i*) = (⋃ *i*∈*I*. *prod_emb M L K* (*F i*))
and *prod_emb_INT*[*simp*]: *I* ≠ {} ⟹ *prod_emb M L K* (⋂ *i*∈*I*. *F i*) = (⋂ *i*∈*I*. *prod_emb M L K* (*F i*))
and *prod_emb_Diff*[*simp*]: *prod_emb M L K* (*A* − *B*) = *prod_emb M L K A* − *prod_emb M L K B*
by (*auto simp*: *prod_emb_def*)

lemma *prod_emb_PiE*: *J* ⊆ *I* ⟹ (⋀*i*. *i* ∈ *J* ⟹ *E i* ⊆ *space* (*M i*)) ⟹
prod_emb I M J (Π$_E$ *i*∈*J*. *E i*) = (Π$_E$ *i*∈*I*. *if i* ∈ *J then E i else space* (*M i*))
by (*force simp*: *prod_emb_def PiE_iff if_split_mem2*)

lemma *prod_emb_PiE_same_index*[*simp*]:
(⋀*i*. *i* ∈ *I* ⟹ *E i* ⊆ *space* (*M i*)) ⟹ *prod_emb I M I* (*Pi*$_E$ *I E*) = *Pi*$_E$ *I E*
by (*auto simp*: *prod_emb_def PiE_iff*)

lemma *prod_emb_trans*[*simp*]:
J ⊆ *K* ⟹ *K* ⊆ *L* ⟹ *prod_emb L M K* (*prod_emb K M J X*) = *prod_emb L M J X*
by (*auto simp add*: *Int_absorb1 prod_emb_def PiE_def*)

lemma *prod_emb_Pi*:
assumes *X* ∈ (Π *j*∈*J*. *sets* (*M j*)) *J* ⊆ *K*
shows *prod_emb K M J* (*Pi*$_E$ *J X*) = (Π$_E$ *i*∈*K*. *if i* ∈ *J then X i else space* (*M i*))
using *assms sets.space_closed*

by (*auto simp*: *prod_emb_def PiE_iff split*: *if_split_asm*) *blast+*

lemma *prod_emb_id*:
B ⊆ (Π$_E$ *i*∈*L*. *space* (*M i*)) ⟹ *prod_emb L M L B* = *B*
by (*auto simp*: *prod_emb_def subset_eq extensional_restrict*)

lemma *prod_emb_mono*:
F ⊆ *G* ⟹ *prod_emb A M B F* ⊆ *prod_emb A M B G*
by (*auto simp*: *prod_emb_def*)

definition *PiM* :: *'i set* ⇒ (*'i* ⇒ *'a measure*) ⇒ (*'i* ⇒ *'a*) *measure* **where**
PiM I M = *extend_measure* (Π$_E$ *i*∈*I*. *space* (*M i*))
{(*J*, *X*). (*J* ≠ {} ∨ *I* = {}) ∧ *finite J* ∧ *J* ⊆ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}
(λ(*J*, *X*). *prod_emb I M J* (Π$_E$ *j*∈*J*. *X j*))
(λ(*J*, *X*). ∏*j*∈*J* ∪ {*i*∈*I*. *emeasure* (*M i*) (*space* (*M i*)) ≠ *1*}. *if j* ∈ *J then emeasure* (*M j*) (*X j*) *else emeasure* (*M j*) (*space* (*M j*)))

definition *prod_algebra* :: *'i set* ⇒ (*'i* ⇒ *'a measure*) ⇒ (*'i* ⇒ *'a*) *set set* **where**
prod_algebra I M = (λ(*J*, *X*). *prod_emb I M J* (Π$_E$ *j*∈*J*. *X j*)) ‘
{(*J*, *X*). (*J* ≠ {} ∨ *I* = {}) ∧ *finite J* ∧ *J* ⊆ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}

abbreviation
Pi$_M$ I M ≡ *PiM I M*

syntax
_PiM :: *pttrn* ⇒ *'i set* ⇒ *'a measure* ⇒ (*'i* => *'a*) *measure*
(‹(‹*indent*=*3 notation*=‹*binder* Π$_M$››Π$_M$ *_*∈*_*./ *_*)› *10*)
syntax_consts
_PiM == *PiM*
translations
Π$_M$ *x*∈*I*. *M* == *CONST PiM I* (%*x*. *M*)

lemma *extend_measure_cong*:
assumes Ω = Ω' *I* = *I'* *G* = *G'* ⋀*i*. *i* ∈ *I'* ⟹ μ *i* = μ' *i*
shows *extend_measure* Ω *I G* μ = *extend_measure* Ω' *I' G'* μ'
unfolding *extend_measure_def* **by** (*auto simp add*: *assms*)

lemma *Pi_cong_sets*:
⟦*I* = *J*; ⋀*x*. *x* ∈ *I* ⟹ *M x* = *N x*⟧ ⟹ *Pi I M* = *Pi J N*
by *auto*

lemma *PiM_cong*:
assumes *I* = *J* ⋀*x*. *x* ∈ *I* ⟹ *M x* = *N x*
shows *PiM I M* = *PiM J N*
unfolding *PiM_def*
proof (*rule extend_measure_cong*, *goal_cases*)
case *1*
show *?case* **using** *assms*
by (*subst assms*(*1*), *intro PiE_cong*[*of J* λ*i*. *space* (*M i*) λ*i*. *space* (*N i*)])

simp_all
next
case *2*
have $\bigwedge K.\ K \subseteq J \Longrightarrow (\Pi\ j{\in}K.\ sets\ (M\ j)) = (\Pi\ j{\in}K.\ sets\ (N\ j))$
using *assms* **by** (*intro Pi_cong_sets*) *auto*
thus *?case* **by** (*auto simp*: *assms*)
next
case *3*
show *?case* **using** *assms*
by (*intro ext*) (*auto simp*: *prod_emb_def dest*: *PiE_mem*)
next
case (*4 x*)
thus *?case* **using** *assms*
by (*auto intro*!: *prod.cong split*: *if_split_asm*)
qed

lemma *prod_algebra_sets_into_space*:
prod_algebra I M $\subseteq$ *Pow* $(\Pi_E\ i{\in}I.\ space\ (M\ i))$
by (*auto simp*: *prod_emb_def prod_algebra_def*)

lemma *prod_algebra_eq_finite*:
assumes *I*: *finite I*
shows *prod_algebra I M* $= \{(\Pi_E\ i{\in}I.\ X\ i)\ |X.\ X \in (\Pi\ j{\in}I.\ sets\ (M\ j))\}$ (**is** *?L* = *?R*)
proof (*intro iffI set_eqI*)
fix *A* **assume** *A* $\in$ *?L*
then obtain *J E* **where** *J*: $J \neq \{\} \vee I = \{\}$ *finite J* $J \subseteq I$ $\forall i{\in}J.\ E\ i \in sets\ (M\ i)$
and *A*: $A = prod\_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ E\ j)$
by (*auto simp*: *prod_algebra_def*)
let *?A* $= \Pi_E\ i{\in}I.$ *if i* $\in$ *J then E i else space* $(M\ i)$
have *A*: $A = ?A$
unfolding *A* **using** *J* **by** (*intro prod_emb_PiE sets.sets_into_space*) *auto*
show $A \in ?R$ **unfolding** *A* **using** *J sets.top*
by (*intro CollectI exI*[*of* _ $\lambda i.$ *if i* $\in$ *J then E i else space* $(M\ i)$]) *simp*
next
fix *A* **assume** $A \in ?R$
then obtain *X* **where** *A*: $A = (\Pi_E\ i{\in}I.\ X\ i)$ **and** *X*: $X \in (\Pi\ j{\in}I.\ sets\ (M\ j))$
by *auto*
then have *A*: $A = prod\_emb\ I\ M\ I\ (\Pi_E\ i{\in}I.\ X\ i)$
by (*simp add*: *prod_emb_PiE_same_index*[*OF sets.sets_into_space*] *Pi_iff*)
from *X I* **show** $A \in ?L$ **unfolding** *A*
by (*auto simp*: *prod_algebra_def*)
qed

lemma *prod_algebraI*:
finite J $\Longrightarrow (J \neq \{\} \vee I = \{\}) \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow E\ i \in sets\ (M\ i))$
$\Longrightarrow prod\_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ E\ j) \in prod\_algebra\ I\ M$

by (*auto simp*: *prod_algebra_def*)

lemma *prod_algebraI_finite*:
finite I $\Longrightarrow$ $(\forall\, i{\in}I.\ E\ i \in sets\ (M\ i)) \Longrightarrow (Pi_E\ I\ E) \in prod\_algebra\ I\ M$
using *prod_algebraI*[*of I I E M*] *prod_emb_PiE_same_index*[*of I E M*, *OF sets.sets_into_space*] **by** *simp*

lemma *Int_stable_PiE*: *Int_stable* $\{Pi_E\ J\ E \mid E.\ \forall\, i{\in}I.\ E\ i \in sets\ (M\ i)\}$
proof (*safe intro*!: *Int_stableI*)
fix *E F* **assume** $\forall\, i{\in}I.\ E\ i \in sets\ (M\ i)$ $\forall\, i{\in}I.\ F\ i \in sets\ (M\ i)$
then show $\exists\, G.\ Pi_E\ J\ E \cap Pi_E\ J\ F = Pi_E\ J\ G \wedge (\forall\, i{\in}I.\ G\ i \in sets\ (M\ i))$
by (*auto intro*!: *exI*[*of _* $\lambda i.\ E\ i \cap F\ i$] *simp*: *PiE_Int*)
qed

lemma *prod_algebraE*:
assumes *A*: $A \in prod\_algebra\ I\ M$
obtains *J E* **where** $A = prod\_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ E\ j)$
finite J $J \neq \{\} \vee I = \{\}$ $J \subseteq I$ $\bigwedge i.\ i \in J \Longrightarrow E\ i \in sets\ (M\ i)$
using *A* **by** (*auto simp*: *prod_algebra_def*)

lemma *prod_algebraE_all*:
assumes *A*: $A \in prod\_algebra\ I\ M$
obtains *E* **where** $A = Pi_E\ I\ E\ E \in (\Pi\ i{\in}I.\ sets\ (M\ i))$
proof −
from *A* **obtain** *E J* **where** *A*: $A = prod\_emb\ I\ M\ J\ (Pi_E\ J\ E)$
and *J*: $J \subseteq I$ **and** *E*: $E \in (\Pi\ i{\in}J.\ sets\ (M\ i))$
by (*auto simp*: *prod_algebra_def*)
from *E* **have** $\bigwedge i.\ i \in J \Longrightarrow E\ i \subseteq space\ (M\ i)$
using *sets.sets_into_space* **by** *auto*
then have $A = (\Pi_E\ i{\in}I.\ \textit{if } i{\in}J \textit{ then } E\ i \textit{ else } space\ (M\ i))$
using *A J* **by** (*auto simp*: *prod_emb_PiE*)
moreover have $(\lambda i.\ \textit{if } i{\in}J \textit{ then } E\ i \textit{ else } space\ (M\ i)) \in (\Pi\ i{\in}I.\ sets\ (M\ i))$
using *sets.top E* **by** *auto*
ultimately show *?thesis* **using** *that* **by** *auto*
qed

lemma *Int_stable_prod_algebra*: *Int_stable* (*prod_algebra I M*)
unfolding *Int_stable_def*
proof *safe*
fix *A* **assume** $A \in prod\_algebra\ I\ M$
from *prod_algebraE*[*OF this*] **obtain** *J E* **where** *A*:
$A = prod\_emb\ I\ M\ J\ (Pi_E\ J\ E)$
finite J
$J \neq \{\} \vee I = \{\}$
$J \subseteq I$
$\bigwedge i.\ i \in J \Longrightarrow E\ i \in sets\ (M\ i)$
by *auto*
fix *B* **assume** $B \in prod\_algebra\ I\ M$
from *prod_algebraE*[*OF this*] **obtain** *K F* **where** *B*:

$B = prod\_emb\ I\ M\ K\ (Pi_E\ K\ F)$
$finite\ K$
$K \neq \{\} \vee I = \{\}$
$K \subseteq I$
$\bigwedge i.\ i \in K \Longrightarrow F\ i \in sets\ (M\ i)$
by *auto*
have $A \cap B = prod\_emb\ I\ M\ (J \cup K)\ (\Pi_E\ i{\in}J \cup K.\ (if\ i \in J\ then\ E\ i\ else\ space\ (M\ i)) \cap$
$(if\ i \in K\ then\ F\ i\ else\ space\ (M\ i)))$
unfolding *A B* **using** *A*(*2*,*3*,*4*) *A*(*5*)[*THEN sets.sets_into_space*] *B*(*2*,*3*,*4*)
B(*5*)[*THEN sets.sets_into_space*]
apply (*subst* (*1 2 3*) *prod_emb_PiE*)
apply (*simp_all add*: *subset_eq PiE_Int*)
apply *blast*
apply (*intro PiE_cong*)
apply *auto*
done
also have $\ldots \in prod\_algebra\ I\ M$
using *A B* **by** (*auto intro*!: *prod_algebraI*)
finally show $A \cap B \in prod\_algebra\ I\ M$ **.**
qed

proposition *prod_algebra_mono*:
assumes *space*: $\bigwedge i.\ i \in I \Longrightarrow space\ (E\ i) = space\ (F\ i)$
assumes *sets*: $\bigwedge i.\ i \in I \Longrightarrow sets\ (E\ i) \subseteq sets\ (F\ i)$
shows $prod\_algebra\ I\ E \subseteq prod\_algebra\ I\ F$
proof
fix *A* **assume** $A \in prod\_algebra\ I\ E$
then obtain *J G* **where** *J*: $J \neq \{\} \vee I = \{\}$ *finite J* $J \subseteq I$
and *A*: $A = prod\_emb\ I\ E\ J\ (\Pi_E\ i{\in}J.\ G\ i)$
and *G*: $\bigwedge i.\ i \in J \Longrightarrow G\ i \in sets\ (E\ i)$
by (*auto simp*: *prod_algebra_def*)
moreover
from *space* **have** $(\Pi_E\ i{\in}I.\ space\ (E\ i)) = (\Pi_E\ i{\in}I.\ space\ (F\ i))$
by (*rule PiE_cong*)
with *A* **have** $A = prod\_emb\ I\ F\ J\ (\Pi_E\ i{\in}J.\ G\ i)$
by (*simp add*: *prod_emb_def*)
moreover
from *sets G J* **have** $\bigwedge i.\ i \in J \Longrightarrow G\ i \in sets\ (F\ i)$
by *auto*
ultimately show $A \in prod\_algebra\ I\ F$
apply (*simp add*: *prod_algebra_def image_iff*)
apply (*intro exI*[*of* _ *J*] *exI*[*of* _ *G*] *conjI*)
apply *auto*
done
qed
proposition *prod_algebra_cong*:
assumes $I = J$ **and** $(\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sets\ (N\ i))$
shows $prod\_algebra\ I\ M = prod\_algebra\ J\ N$

by (*metis assms prod_algebra_mono sets_eq_imp_space_eq subsetI subset_antisym*)

lemma *space_in_prod_algebra*: $(\Pi_E\ i{\in}I.\ space\ (M\ i)) \in prod\_algebra\ I\ M$
proof *cases*
assume $I = \{\}$ **then show** *?thesis*
by (*auto simp add*: *prod_algebra_def image_iff prod_emb_def*)
next
assume $I \neq \{\}$
then obtain i **where** $i \in I$ **by** *auto*
then have $(\Pi_E\ i{\in}I.\ space\ (M\ i)) = prod\_emb\ I\ M\ \{i\}\ (\Pi_E\ i{\in}\{i\}.\ space\ (M\ i))$
by (*auto simp*: *prod_emb_def*)
then show *?thesis*
by (*simp add*: ‹$i \in I$› *prod_algebraI*)
qed

lemma *space_PiM*: $space\ (\Pi_M\ i{\in}I.\ M\ i) = (\Pi_E\ i{\in}I.\ space\ (M\ i))$
using *prod_algebra_sets_into_space* **unfolding** *PiM_def prod_algebra_def* **by** (*intro space_extend_measure*) *simp*

lemma *prod_emb_subset_PiM*[*simp*]: $prod\_emb\ I\ M\ K\ X \subseteq space\ (PiM\ I\ M)$
by (*auto simp*: *prod_emb_def space_PiM*)

lemma *space_PiM_empty_iff*[*simp*]: $space\ (PiM\ I\ M) = \{\} \longleftrightarrow\ (\exists\, i{\in}I.\ space\ (M\ i) = \{\})$
by (*auto simp*: *space_PiM PiE_eq_empty_iff*)

lemma *undefined_in_PiM_empty*[*simp*]: $(\lambda x.\ undefined) \in space\ (PiM\ \{\}\ M)$
by (*auto simp*: *space_PiM*)

lemma *sets_PiM*: $sets\ (\Pi_M\ i{\in}I.\ M\ i) = sigma\_sets\ (\Pi_E\ i{\in}I.\ space\ (M\ i))\ (prod\_algebra\ I\ M)$
using *prod_algebra_sets_into_space* **unfolding** *PiM_def prod_algebra_def* **by** (*intro sets_extend_measure*) *simp*

proposition *sets_PiM_single*: $sets\ (PiM\ I\ M) =$
$sigma\_sets\ (\Pi_E\ i{\in}I.\ space\ (M\ i))\ \{\{f{\in}\Pi_E\ i{\in}I.\ space\ (M\ i).\ f\ i \in A\} \mid i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
(**is** $\_ = sigma\_sets\ ?\Omega\ ?R$)
unfolding *sets_PiM*
proof (*rule sigma_sets_eqI*)
interpret *R*: *sigma_algebra* $?\Omega$ *sigma_sets* $?\Omega\ ?R$ **by** (*rule sigma_algebra_sigma_sets*) *auto*
fix A **assume** $A \in prod\_algebra\ I\ M$
from *prod_algebraE*[*OF this*] **obtain** $J\ X$ **where** X:
$A = prod\_emb\ I\ M\ J\ (Pi_E\ J\ X)$
finite J
$J \neq \{\} \vee I = \{\}$
$J \subseteq I$
$\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)$

```
    by auto
  show A ∈ sigma_sets ?Ω ?R
  proof cases
    assume I = {}
    with X show ?thesis
        by (metis (no_types, lifting) PiE_cong R.top empty_iff prod_emb_PiE
subset_eq)
  next
    assume I ≠ {}
    with X have A = (⋂j∈J. {f∈(Π_E i∈I. space (M i)). f j ∈ X j})
      by (auto simp: prod_emb_def)
    also have ... ∈ sigma_sets ?Ω ?R
      using X ‹I ≠ {}› by (intro R.finite_INT sigma_sets.Basic) auto
    finally show A ∈ sigma_sets ?Ω ?R .
  qed
next
  fix A assume A ∈ ?R
  then obtain i B where A: A = {f∈Π_E i∈I. space (M i). f i ∈ B} i ∈ I B ∈
sets (M i)
    by auto
  then have A = prod_emb I M {i} (Π_E i∈{i}. B)
     by (auto simp: prod_emb_def)
  also have ... ∈ sigma_sets ?Ω (prod_algebra I M)
    using A by (intro sigma_sets.Basic prod_algebraI) auto
  finally show A ∈ sigma_sets ?Ω (prod_algebra I M) .
qed

lemma sets_PiM_eq_proj:
  assumes I ≠ {}
  shows sets (PiM I M) = sets (SUP i∈I. vimage_algebra (Π_E i∈I. space (M i))
(λx. x i) (M i))
        (is ?lhs = ?rhs)
proof −
  have ?lhs =
        sigma_sets (Π_E i∈I. space (M i)) {{f ∈ Π_E i∈I. space (M i). f i ∈ A} |i
A. i ∈ I ∧ A ∈ sets (M i)}
    by (simp add: sets_PiM_single)
  also have ... = sigma_sets (Π_E i∈I. space (M i))
                (⋃x∈I. sets (vimage_algebra (Π_E i∈I. space (M i)) (λxa. xa x) (M
x)))
    apply (subst arg_cong [of _ _ Sup, OF image_cong, OF refl])
     apply (rule sets_vimage_algebra2)
    by (auto intro!: arg_cong2[where f=sigma_sets])
  also have ... = sigma_sets (Π_E i∈I. space (M i))
     (⋃ (sets ‘ (λi. vimage_algebra (Π_E i∈I. space (M i)) (λx. x i) (M i)) ‘ I))
    by simp
  also have ... = ?rhs
    by (subst sets_Sup_eq[where X=Π_E i∈I. space (M i)]) (use assms in auto)
  finally show ?thesis .
```

qed

lemma
shows *space_PiM_empty*: *space* (Pi_M {} M) = {$\lambda k.$ *undefined*}
and *sets_PiM_empty*: *sets* (Pi_M {} M) = { {}, {$\lambda k.$ *undefined*} }
by (*simp_all add*: *space_PiM sets_PiM_single image_constant sigma_sets_empty_eq*)

proposition *sets_PiM_sigma*:
assumes Ω_*cover*: $\bigwedge i.\ i \in I \Longrightarrow \exists S{\subseteq}E\ i.\ countable\ S \wedge \Omega\ i = \bigcup S$
assumes *E*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq Pow\ (\Omega\ i)$
assumes *J*: $\bigwedge j.\ j \in J \Longrightarrow$ *finite* $j \bigcup J = I$
defines $P \equiv \{\{f{\in}(\Pi_E\ i{\in}I.\ \Omega\ i).\ \forall i{\in}j.\ f\ i \in A\ i\} \mid A\ j.\ j \in J \wedge A \in Pi\ j\ E\}$
shows *sets* ($\Pi_M\ i{\in}I.$ *sigma* ($\Omega\ i$) ($E\ i$)) = *sets* (*sigma* ($\Pi_E\ i{\in}I.\ \Omega\ i$) P)
proof *cases*
assume $I = \{\}$
with ‹$\bigcup J = I$› **have** $P = \{\{\lambda\_.\ undefined\}\} \vee P = \{\}$
by (*auto simp*: *P_def*)
with ‹$I = \{\}$› **show** *?thesis*
by (*auto simp add*: *sets_PiM_empty sigma_sets_empty_eq*)
next
let *?F* = $\lambda i.\ \{(\lambda x.\ x\ i) -` A \cap Pi_E\ I\ \Omega\ |A.\ A \in E\ i\}$
assume $I \neq \{\}$
then have *sets* ($Pi_M\ I$ ($\lambda i.$ *sigma* ($\Omega\ i$) ($E\ i$))) =
sets (*SUP* $i{\in}I.$ *vimage_algebra* ($\Pi_E\ i{\in}I.\ \Omega\ i$) ($\lambda x.\ x\ i$) (*sigma* ($\Omega\ i$) ($E\ i$)))
by (*subst sets_PiM_eq_proj*) (*auto simp*: *space_measure_of_conv*)
also have ... = *sets* (*SUP* $i{\in}I.$ *sigma* ($Pi_E\ I\ \Omega$) (*?F* i))
using E **by** (*intro sets_SUP_cong arg_cong*[**where** f=*sets*] *vimage_algebra_sigma*)
auto
also have ... = *sets* (*sigma* ($Pi_E\ I\ \Omega$) ($\bigcup i{\in}I.$ *?F* i))
using ‹$I \neq \{\}$› **by** (*intro arg_cong*[**where** f=*sets*] *SUP_sigma_sigma*) *auto*
also have ... = *sets* (*sigma* ($Pi_E\ I\ \Omega$) P)
proof (*intro arg_cong*[**where** f=*sets*] *sigma_eqI sigma_sets_eqI*)
show ($\bigcup i{\in}I.$ *?F* i) $\subseteq$ $Pow\ (Pi_E\ I\ \Omega)$ $P \subseteq Pow\ (Pi_E\ I\ \Omega)$
by (*auto simp*: *P_def*)
next
interpret P: *sigma_algebra* $\Pi_E\ i{\in}I.\ \Omega\ i$ *sigma_sets* ($\Pi_E\ i{\in}I.\ \Omega\ i$) P
by (*auto intro*!: *sigma_algebra_sigma_sets simp*: *P_def*)

fix Z **assume** $Z \in (\bigcup i{\in}I.$ *?F* i)
then obtain i A **where** i: $i \in I\ A \in E\ i$ **and** *Z_def*: $Z = (\lambda x.\ x\ i) -` A \cap Pi_E\ I\ \Omega$
by *auto*
from ‹$i \in I$› J **obtain** j **where** j: $i \in j\ j \in J\ j \subseteq I$ *finite* j
by *auto*
obtain S **where** S: $\bigwedge i.\ i \in j \Longrightarrow S\ i \subseteq E\ i\ \bigwedge i.\ i \in j \Longrightarrow countable\ (S\ i)$
$\bigwedge i.\ i \in j \Longrightarrow \Omega\ i = \bigcup (S\ i)$
by (*metis subset_eq* Ω_*cover* ‹$j \subseteq I$›)
define A' **where** $A'\ n = n(i := A)$ **for** n
then have A'_i: $\bigwedge n.\ A'\ n\ i = A$

```
    by simp
  { fix n assume n ∈ Pi_E (j − {i}) S
    then have A' n ∈ Pi j E
      unfolding PiE_def Pi_def using S(1) by (auto simp: A'_def ‹A ∈ E i› )
    with ‹j ∈ J› have {f ∈ Pi_E I Ω. ∀ i∈j. f i ∈ A' n i} ∈ P
      by (auto simp: P_def) }
  note A'_in_P = this

  { fix x assume x i ∈ A x ∈ Pi_E I Ω
    with S(3) ‹j ⊆ I› have ∀ i∈j. ∃ s∈S i. x i ∈ s
      by (auto simp: PiE_def Pi_def)
    then obtain s where s: ⋀i. i ∈ j ⟹ s i ∈ S i ⋀i. i ∈ j ⟹ x i ∈ s i
      by metis
    with ‹x i ∈ A› have ∃ n∈Pi_E (j−{i}) S. ∀ i∈j. x i ∈ A' n i
      by (intro bexI[of _ restrict (s(i := A)) (j−{i})]) (auto simp: A'_def split:
if_splits) }
  then have Z = (⋃ n∈Pi_E (j−{i}) S. {f∈(Π_E i∈I. Ω i). ∀ i∈j. f i ∈ A' n i})
    unfolding Z_def
    by (auto simp add: set_eq_iff ball_conj_distrib ‹i∈j› A'_i dest: bspec[OF _
‹i∈j›]
          cong: conj_cong)
  also have ... ∈ sigma_sets (Π_E i∈I. Ω i) P
    using ‹finite j› S(2)
    by (intro P.countable_UN' countable_PiE) (simp_all add: image_subset_iff
A'_in_P)
  finally show Z ∈ sigma_sets (Π_E i∈I. Ω i) P .
next
  interpret F: sigma_algebra Π_E i∈I. Ω i sigma_sets (Π_E i∈I. Ω i) (⋃i∈I.
?F i)
    by (auto intro!: sigma_algebra_sigma_sets)

  fix b assume b ∈ P
  then obtain A j where b: b = {f∈(Π_E i∈I. Ω i). ∀ i∈j. f i ∈ A i} j ∈ J A ∈
Pi j E
    by (auto simp: P_def)
  show b ∈ sigma_sets (Pi_E I Ω) (⋃i∈I. ?F i)
  proof cases
    assume j = {}
    with b have b = (Π_E i∈I. Ω i)
      by auto
    then show ?thesis
      by blast
  next
    assume j ≠ {}
    with J b(2,3) have eq: b = (⋂ i∈j. ((λx. x i) −‘ A i ∩ Pi_E I Ω))
      unfolding b(1)
      by (auto simp: PiE_def Pi_def)
    show ?thesis
      unfolding eq using ‹A ∈ Pi j E› ‹j ∈ J› J(2)
```

```
      by (intro F.finite_INT J ‹j ∈ J› ‹j ≠ {}› sigma_sets.Basic) blast
    qed
  qed
  finally show ?thesis .
qed

lemma sets_PiM_in_sets:
  assumes space: space N = (Π_E i∈I. space (M i))
  assumes sets: ⋀i A. i ∈ I ⟹ A ∈ sets (M i) ⟹ {x∈space N. x i ∈ A} ∈ sets
N
  shows sets (Π_M i ∈ I. M i) ⊆ sets N
  unfolding sets_PiM_single space[symmetric]
  by (intro sets.sigma_sets_subset subsetI) (auto intro: sets)

lemma sets_PiM_cong[measurable_cong]:
  assumes I = J ⋀i. i ∈ J ⟹ sets (M i) = sets (N i) shows sets (PiM I M) =
sets (PiM J N)
  using assms sets_eq_imp_space_eq[OF assms(2)] by (simp add: sets_PiM_single
cong: PiE_cong conj_cong)

lemma sets_PiM_I:
  assumes finite J J ⊆ I ∀i∈J. E i ∈ sets (M i)
  shows prod_emb I M J (Π_E j∈J. E j) ∈ sets (Π_M i∈I. M i)
proof cases
  assume J = {}
  then have prod_emb I M J (Π_E j∈J. E j) = (Π_E j∈I. space (M j))
    by (auto simp: prod_emb_def)
  then show ?thesis
    by (auto simp add: sets_PiM intro!: sigma_sets_top)
next
  assume J ≠ {} with assms show ?thesis
    by (force simp add: sets_PiM prod_algebra_def)
qed

proposition measurable_PiM:
  assumes space: f ∈ space N → (Π_E i∈I. space (M i))
  assumes sets: ⋀X J. J ≠ {} ∨ I = {} ⟹ finite J ⟹ J ⊆ I ⟹ (⋀i. i ∈ J
⟹ X i ∈ sets (M i)) ⟹
    f −‘ prod_emb I M J (Pi_E J X) ∩ space N ∈ sets N
  shows f ∈ measurable N (PiM I M)
  using sets_PiM prod_algebra_sets_into_space space
proof (rule measurable_sigma_sets)
  fix A assume A ∈ prod_algebra I M
  from prod_algebraE[OF this] obtain J X where
    A = prod_emb I M J (Pi_E J X)
    finite J
    J ≠ {} ∨ I = {}
    J ⊆ I
    ⋀i. i ∈ J ⟹ X i ∈ sets (M i)
```

by *auto*
with *sets*[*of J X*] **show** $f -` A \cap space\ N \in sets\ N$ **by** *auto*
qed

lemma *measurable_PiM_Collect*:
assumes *space*: $f \in space\ N \to (\Pi_E\ i \in I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge X\ J.\ J \neq \{\} \vee I = \{\} \Longrightarrow finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)) \Longrightarrow \{\omega \in space\ N.\ \forall i \in J.\ f\ \omega\ i \in X\ i\} \in sets\ N$
shows $f \in measurable\ N\ (PiM\ I\ M)$
using *sets_PiM prod_algebra_sets_into_space space*
proof (*rule measurable_sigma_sets*)
fix *A* **assume** $A \in prod\_algebra\ I\ M$
from *prod_algebraE*[*OF this*] **obtain** *J X*
where *X*:
$A = prod\_emb\ I\ M\ J\ (Pi_E\ J\ X)$
$finite\ J$
$J \neq \{\} \vee I = \{\}$
$J \subseteq I$
$\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)$
by *auto*
then have $f -` A \cap space\ N = \{\omega \in space\ N.\ \forall i \in J.\ f\ \omega\ i \in X\ i\}$
using *space* **by** (*auto simp*: *prod_emb_def del*: *PiE_I*)
also have $\ldots \in sets\ N$ **using** *X*(*3*,*2*,*4*,*5*) **by** (*rule sets*)
finally show $f -` A \cap space\ N \in sets\ N$ **.**
qed

lemma *measurable_PiM_single*:
assumes *space*: $f \in space\ N \to (\Pi_E\ i \in I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge A\ i.\ i \in I \Longrightarrow A \in sets\ (M\ i) \Longrightarrow \{\omega \in space\ N.\ f\ \omega\ i \in A\} \in sets\ N$
shows $f \in measurable\ N\ (PiM\ I\ M)$
using *sets_PiM_single*
proof (*rule measurable_sigma_sets*)
fix *A* **assume** $A \in \{\{f \in \Pi_E\ i \in I.\ space\ (M\ i).\ f\ i \in A\}\ |i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
then obtain *B i* **where** $A = \{f \in \Pi_E\ i \in I.\ space\ (M\ i).\ f\ i \in B\}$ **and** *B*: $i \in I\ B \in sets\ (M\ i)$
by *auto*
with *space* **have** $f -` A \cap space\ N = \{\omega \in space\ N.\ f\ \omega\ i \in B\}$ **by** *auto*
also have $\ldots \in sets\ N$ **using** *B* **by** (*rule sets*)
finally show $f -` A \cap space\ N \in sets\ N$ **.**
qed (*auto simp*: *space*)

lemma *measurable_PiM_single'*:
assumes *f*: $\bigwedge i.\ i \in I \Longrightarrow f\ i \in measurable\ N\ (M\ i)$
and $(\lambda\omega\ i.\ f\ i\ \omega) \in space\ N \to (\Pi_E\ i \in I.\ space\ (M\ i))$
shows $(\lambda\omega\ i.\ f\ i\ \omega) \in measurable\ N\ (Pi_M\ I\ M)$
proof (*rule measurable_PiM_single*)

fix A i **assume** A: $i \in I$ $A \in sets\ (M\ i)$
then have $\{\omega \in space\ N.\ f\ i\ \omega \in A\} = f\ i\ -`\ A \cap space\ N$
by *auto*
then show $\{\omega \in space\ N.\ f\ i\ \omega \in A\} \in sets\ N$
using A f **by** (*auto intro*!: *measurable_sets*)
qed *fact*

lemma *sets_PiM_I_finite*[*measurable*]:
assumes *finite* I **and** *sets*: $(\bigwedge i.\ i \in I \Longrightarrow E\ i \in sets\ (M\ i))$
shows $(\Pi_E\ j{\in}I.\ E\ j) \in sets\ (\Pi_M\ i{\in}I.\ M\ i)$
using *sets_PiM_I*[*of I I E M*] *sets.sets_into_space*[*OF sets*] ‹*finite I*› *sets* **by** *auto*

lemma *measurable_component_singleton*[*measurable* (*raw*)]:
assumes $i \in I$ **shows** $(\lambda x.\ x\ i) \in measurable\ (Pi_M\ I\ M)\ (M\ i)$
proof (*unfold measurable_def*, *intro CollectI conjI ballI*)
fix A **assume** $A \in sets\ (M\ i)$
then have $(\lambda x.\ x\ i)\ -`\ A \cap space\ (Pi_M\ I\ M) = prod\_emb\ I\ M\ \{i\}\ (\Pi_E\ j{\in}\{i\}.\ A)$
using *sets.sets_into_space* ‹$i \in I$›
by (*fastforce dest*: *Pi_mem simp*: *prod_emb_def space_PiM split*: *if_split_asm*)
then show $(\lambda x.\ x\ i)\ -`\ A \cap space\ (Pi_M\ I\ M) \in sets\ (Pi_M\ I\ M)$
using ‹$A \in sets\ (M\ i)$› ‹$i \in I$› **by** (*auto intro*!: *sets_PiM_I*)
qed (*use* ‹$i \in I$› **in** ‹*auto simp*: *space_PiM*›)

lemma *measurable_component_singleton'*[*measurable_dest*]:
assumes f: $f \in measurable\ N\ (Pi_M\ I\ M)$
assumes g: $g \in measurable\ L\ N$
assumes i: $i \in I$
shows $(\lambda x.\ (f\ (g\ x))\ i) \in measurable\ L\ (M\ i)$
using *measurable_compose*[*OF measurable_compose*[*OF g f*] *measurable_component_singleton*, *OF i*] **.**

lemma *measurable_PiM_component_rev*:
$i \in I \Longrightarrow f \in measurable\ (M\ i)\ N \Longrightarrow (\lambda x.\ f\ (x\ i)) \in measurable\ (PiM\ I\ M)\ N$
by *simp*

lemma *measurable_case_nat*[*measurable* (*raw*)]:
assumes [*measurable* (*raw*)]: $i = 0 \Longrightarrow f \in measurable\ M\ N$
$\bigwedge j.\ i = Suc\ j \Longrightarrow (\lambda x.\ g\ x\ j) \in measurable\ M\ N$
shows $(\lambda x.\ case\_nat\ (f\ x)\ (g\ x)\ i) \in measurable\ M\ N$
by (*cases i*) *simp_all*

lemma *measurable_case_nat'*[*measurable* (*raw*)]:
assumes *fg*[*measurable*]: $f \in measurable\ N\ M\ g \in measurable\ N\ (\Pi_M\ i{\in}UNIV.\ M)$
shows $(\lambda x.\ case\_nat\ (f\ x)\ (g\ x)) \in measurable\ N\ (\Pi_M\ i{\in}UNIV.\ M)$
using *fg*[*THEN measurable_space*]
by (*auto intro*!: *measurable_PiM_single' simp add*: *space_PiM PiE_iff split*:

nat.split)

lemma *measurable_add_dim*[*measurable*]:
(λ(*f*, *y*). *f*(*i* := *y*)) ∈ *measurable* (*Pi*$_M$ *I M* ⨂$_M$ *M i*) (*Pi*$_M$ (*insert i I*) *M*)
(**is** *?f* ∈ *measurable ?P ?I*)
proof (*rule measurable_PiM_single*)
fix *j A* **assume** *j*: *j* ∈ *insert i I* **and** *A*: *A* ∈ *sets* (*M j*)
have {ω ∈ *space ?P*. (λ(*f*, *x*). *fun_upd f i x*) ω *j* ∈ *A*} =
(*if j* = *i then space* (*Pi*$_M$ *I M*) × *A else* ((λ*x*. *x j*) ∘ *fst*) −‘ *A* ∩ *space ?P*)
using *sets.sets_into_space*[*OF A*] **by** (*auto simp add*: *space_pair_measure space_PiM*)
also have ... ∈ *sets ?P*
using *A j*
by (*auto intro*!: *measurable_sets*[*OF measurable_comp*, *OF _ measurable_component_singleton*])
finally show {ω ∈ *space ?P*. *case_prod* (λ*f*. *fun_upd f i*) ω *j* ∈ *A*} ∈ *sets ?P* **.**
qed (*auto simp*: *space_pair_measure space_PiM PiE_def*)

proposition *measurable_fun_upd*:
assumes *I*: *I* = *J* ∪ {*i*}
assumes *f*[*measurable*]: *f* ∈ *measurable N* (*PiM J M*)
assumes *h*[*measurable*]: *h* ∈ *measurable N* (*M i*)
shows (λ*x*. (*f x*) (*i* := *h x*)) ∈ *measurable N* (*PiM I M*)
proof (*intro measurable_PiM_single'*)
fix *j* **assume** *j* ∈ *I* **then show** (λω. ((*f* ω)(*i* := *h* ω)) *j*) ∈ *measurable N* (*M j*)
unfolding *I* **by** (*cases j* = *i*) *auto*
next
show (λ*x*. (*f x*)(*i* := *h x*)) ∈ *space N* → (Π$_E$ *i*∈*I*. *space* (*M i*))
using *I f*[*THEN measurable_space*] *h*[*THEN measurable_space*]
by (*auto simp*: *space_PiM PiE_iff extensional_def*)
qed

lemma *measurable_component_update*:
x ∈ *space* (*Pi*$_M$ *I M*) ⟹ *i* ∉ *I* ⟹ (λ*v*. *x*(*i* := *v*)) ∈ *measurable* (*M i*) (*Pi*$_M$ (*insert i I*) *M*)
by *simp*

lemma *measurable_merge*[*measurable*]:
merge I J ∈ *measurable* (*Pi*$_M$ *I M* ⨂$_M$ *Pi*$_M$ *J M*) (*Pi*$_M$ (*I* ∪ *J*) *M*)
(**is** *?f* ∈ *measurable ?P ?U*)
proof (*rule measurable_PiM_single*)
fix *i A* **assume** *A*: *A* ∈ *sets* (*M i*) *i* ∈ *I* ∪ *J*
then have {ω ∈ *space ?P*. *merge I J* ω *i* ∈ *A*} =
(*if i* ∈ *I then* ((λ*x*. *x i*) ∘ *fst*) −‘ *A* ∩ *space ?P else* ((λ*x*. *x i*) ∘ *snd*) −‘ *A* ∩ *space ?P*)
by (*auto simp*: *merge_def*)
also have ... ∈ *sets ?P*
using *A*
by (*auto intro*!: *measurable_sets*[*OF measurable_comp*, *OF _ measurable_component_singleton*])
finally show {ω ∈ *space ?P*. *merge I J* ω *i* ∈ *A*} ∈ *sets ?P* **.**

qed (*auto simp*: *space_pair_measure space_PiM PiE_iff merge_def extensional_def*)

lemma *measurable_restrict*[*measurable* (*raw*)]:
assumes X: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in measurable\ N\ (M\ i)$
shows $(\lambda x.\ \lambda i{\in}I.\ X\ i\ x) \in measurable\ N\ (Pi_M\ I\ M)$
proof (*rule measurable_PiM_single*)
fix A i **assume** A: $i \in I\ A \in sets\ (M\ i)$
then have $\{\omega \in space\ N.\ (\lambda i{\in}I.\ X\ i\ \omega)\ i \in A\} = X\ i\ -`\ A \cap space\ N$
by *auto*
then show $\{\omega \in space\ N.\ (\lambda i{\in}I.\ X\ i\ \omega)\ i \in A\} \in sets\ N$
using A X **by** (*auto intro*!: *measurable_sets*)
next
show $(\lambda x.\ \lambda i{\in}I.\ X\ i\ x) \in space\ N \to (\Pi_E\ i{\in}I.\ space\ (M\ i))$
using X **by** (*auto simp add*: *PiE_def dest*: *measurable_space*)
qed

lemma *measurable_abs_UNIV*:
$(\bigwedge n.\ (\lambda\omega.\ f\ n\ \omega) \in measurable\ M\ (N\ n)) \Longrightarrow (\lambda\omega\ n.\ f\ n\ \omega) \in measurable\ M\ (PiM\ UNIV\ N)$
by (*intro measurable_PiM_single*) (*auto dest*: *measurable_space*)

lemma *measurable_restrict_subset*: $J \subseteq L \Longrightarrow (\lambda f.\ restrict\ f\ J) \in measurable\ (Pi_M\ L\ M)\ (Pi_M\ J\ M)$
by (*intro measurable_restrict measurable_component_singleton*) *auto*

lemma *measurable_restrict_subset'*:
assumes $J \subseteq L\ \bigwedge x.\ x \in J \Longrightarrow sets\ (M\ x) = sets\ (N\ x)$
shows $(\lambda f.\ restrict\ f\ J) \in measurable\ (Pi_M\ L\ M)\ (Pi_M\ J\ N)$
by (*metis* (*no_types*) *assms measurable_cong_sets measurable_restrict_subset sets_PiM_cong*)

lemma *measurable_prod_emb*[*intro*, *simp*]:
$J \subseteq L \Longrightarrow X \in sets\ (Pi_M\ J\ M) \Longrightarrow prod\_emb\ L\ M\ J\ X \in sets\ (Pi_M\ L\ M)$
unfolding *prod_emb_def space_PiM*[*symmetric*]
by (*auto intro*!: *measurable_sets measurable_restrict measurable_component_singleton*)

lemma *merge_in_prod_emb*:
assumes $y \in space\ (PiM\ I\ M)$ $x \in X$ **and** X: $X \in sets\ (Pi_M\ J\ M)$ **and** $J \subseteq I$
shows $merge\ J\ I\ (x, y) \in prod\_emb\ I\ M\ J\ X$
using *assms sets.sets_into_space*[*OF X*]
by (*simp add*: *merge_def prod_emb_def subset_eq space_PiM PiE_def extensional_restrict Pi_iff*
cong: *if_cong restrict_cong*)
(*simp add*: *extensional_def*)

lemma *prod_emb_eq_emptyD*:
assumes J: $J \subseteq I$ **and** *ne*: $space\ (PiM\ I\ M) \neq \{\}$ **and** X: $X \in sets\ (Pi_M\ J\ M)$
and $*$: $prod\_emb\ I\ M\ J\ X = \{\}$
shows $X = \{\}$

proof *safe*
fix x **assume** $x \in X$
obtain ω **where** $\omega \in space\ (PiM\ I\ M)$
using *ne* **by** *blast*
from *merge_in_prod_emb*[*OF this* ‹$x \in X$› $X\ J$] $*$ **show** $x \in \{\}$ **by** *auto*
qed

lemma *sets_in_Pi_aux*:
$finite\ I \Longrightarrow (\bigwedge j.\ j \in I \Longrightarrow \{x \in space\ (M\ j).\ x \in F\ j\} \in sets\ (M\ j)) \Longrightarrow$
$\{x \in space\ (PiM\ I\ M).\ x \in Pi\ I\ F\} \in sets\ (PiM\ I\ M)$
by (*simp add*: *subset_eq Pi_iff*)

lemma *sets_in_Pi*[*measurable* (*raw*)]:
$finite\ I \Longrightarrow f \in measurable\ N\ (PiM\ I\ M) \Longrightarrow$
$(\bigwedge j.\ j \in I \Longrightarrow \{x \in space\ (M\ j).\ x \in F\ j\} \in sets\ (M\ j)) \Longrightarrow$
$Measurable.pred\ N\ (\lambda x.\ f\ x \in Pi\ I\ F)$
unfolding *pred_def*
by (*rule measurable_sets_Collect*[*of f N PiM I M*, *OF* _ *sets_in_Pi_aux*]) *auto*

lemma *sets_in_extensional_aux*:
$\{x \in space\ (PiM\ I\ M).\ x \in extensional\ I\} \in sets\ (PiM\ I\ M)$
by (*smt* (*verit*) *PiE_iff mem_Collect_eq sets.top space_PiM subsetI subset_antisym*)

lemma *sets_in_extensional*[*measurable* (*raw*)]:
$f \in measurable\ N\ (PiM\ I\ M) \Longrightarrow Measurable.pred\ N\ (\lambda x.\ f\ x \in extensional\ I)$
unfolding *pred_def*
by (*rule measurable_sets_Collect*[*of f N PiM I M*, *OF* _ *sets_in_extensional_aux*])
auto

lemma *sets_PiM_I_countable*:
assumes *I*: *countable I* **and** *E*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \in sets\ (M\ i)$ **shows** $Pi_E\ I\ E$
$\in sets\ (Pi_M\ I\ M)$
proof *cases*
assume $I \neq \{\}$
then have $Pi_E\ I\ E = (\bigcap i \in I.\ prod\_emb\ I\ M\ \{i\}\ (Pi_E\ \{i\}\ E))$
using *E*[*THEN sets.sets_into_space*] **by** (*auto simp*: *PiE_iff prod_emb_def*
fun_eq_iff)
also have $\ldots \in sets\ (PiM\ I\ M)$
using *I* ‹$I \neq \{\}$› **by** (*simp add*: *E sets.countable_INT' sets_PiM_I subset_eq*)
finally show *?thesis* **.**
qed (*simp add*: *sets_PiM_empty*)

lemma *sets_PiM_D_countable*:
assumes *A*: $A \in PiM\ I\ M$
shows $\exists J \subseteq I.\ \exists X \in PiM\ J\ M.\ countable\ J \wedge A = prod\_emb\ I\ M\ J\ X$
using *A*[*unfolded sets_PiM_single*]
proof *induction*
case (*Basic A*)
then obtain *i X* **where** $*$: $i \in I\ X \in sets\ (M\ i)$ **and** $A = \{f \in \Pi_E\ i \in I.\ space$

```
(M i). f i ∈ X}
    by auto
  then have A: A = prod_emb I M {i} (Π_E _∈{i}. X)
    by (auto simp: prod_emb_def)
  then show ?case
    by (intro exI[of _ {i}] conjI bexI[of _ Π_E _∈{i}. X])
       (auto intro: countable_finite * sets_PiM_I_finite)
next
  case Empty then show ?case
    by (intro exI[of _ {}] conjI bexI[of _ {}]) auto
next
  case (Compl A)
  then obtain J X where J ⊆ I X ∈ sets (Pi_M J M) countable J A = prod_emb
I M J X
    by auto
  then show ?case
    by (intro exI[of _ J] bexI[of _ space (PiM J M) - X] conjI)
       (auto simp add: space_PiM prod_emb_PiE intro!: sets_PiM_I_countable)
next
  case (Union K)
  obtain J X where J: ⋀i. J i ⊆ I ⋀i. countable (J i) and X: ⋀i. X i ∈ sets
(Pi_M (J i) M)
    and K: ⋀i. K i = prod_emb I M (J i) (X i)
    by (metis Union.IH)
  show ?case
  proof (intro exI bexI conjI)
    show (⋃i. J i) ⊆ I countable (⋃i. J i)
      using J by auto
    with J show ⋃(K ` UNIV) = prod_emb I M (⋃i. J i) (⋃i. prod_emb (⋃i.
J i) M (J i) (X i))
      by (simp add: K[abs_def] SUP_upper)
  qed(auto intro: X)
qed

proposition measure_eqI_PiM_finite:
  assumes [simp]: finite I sets P = PiM I M sets Q = PiM I M
  assumes eq: ⋀A. (⋀i. i ∈ I ⟹ A i ∈ sets (M i)) ⟹ P (Pi_E I A) = Q (Pi_E
I A)
  assumes A: range A ⊆ prod_algebra I M (⋃i. A i) = space (PiM I M) ⋀i::nat.
P (A i) ≠ ∞
  shows P = Q
proof (rule measure_eqI_generator_eq[OF Int_stable_prod_algebra prod_algebra_sets_into_space])
  show range A ⊆ prod_algebra I M (⋃i. A i) = (Π_E i∈I. space (M i)) ⋀i. P
(A i) ≠ ∞
    unfolding space_PiM[symmetric] by fact+
  fix X assume X ∈ prod_algebra I M
  then obtain J E where X: X = prod_emb I M J (Π_E j∈J. E j)
    and J: finite J J ⊆ I ⋀j. j ∈ J ⟹ E j ∈ sets (M j)
    by (force elim!: prod_algebraE)
```

then show *emeasure P X = emeasure Q X*
unfolding *X* **by** (*subst* (*1 2*) *prod_emb_Pi*) (*auto simp*: *eq*)
qed (*simp_all add*: *sets_PiM*)

proposition *measure_eqI_PiM_infinite*:
assumes [*simp*]: *sets P = PiM I M sets Q = PiM I M*
assumes *eq*: $\bigwedge A\ J.$ *finite J* $\Longrightarrow$ *J* $\subseteq$ *I* $\Longrightarrow$ ($\bigwedge i.$ *i* $\in$ *J* $\Longrightarrow$ *A i* $\in$ *sets* (*M i*)) $\Longrightarrow$ *P* (*prod_emb I M J* (*Pi*$_E$ *J A*)) = *Q* (*prod_emb I M J* (*Pi*$_E$ *J A*))
assumes *A*: *finite_measure P*
shows *P = Q*
proof (*rule measure_eqI_generator_eq*[*OF Int_stable_prod_algebra prod_algebra_sets_into_space*])
interpret *finite_measure P* **by** *fact*
define *i* **where** *i* = (*SOME i. i* $\in$ *I*)
have *i*: *I* $\neq$ {} $\Longrightarrow$ *i* $\in$ *I*
unfolding *i_def* **by** (*rule someI_ex*) *auto*
define *A* **where** *A n* =
(*if I* = {} *then prod_emb I M* {} (Π_E *i*$\in${}. {}) *else prod_emb I M* {*i*} (Π_E *i*$\in${*i*}. *space* (*M i*)))
for *n* :: *nat*
then show *range A* $\subseteq$ *prod_algebra I M*
using *prod_algebraI*[*of* {} *I* $\lambda i.$ *space* (*M i*) *M*] **by** (*auto intro*!: *prod_algebraI i*)
have $\bigwedge i.$ *A i = space* (*PiM I M*)
by (*auto simp*: *prod_emb_def space_PiM PiE_iff A_def i ex_in_conv*[*symmetric*] *exI*)
then show ($\bigcup i.$ *A i*) = (Π_E *i*$\in$*I*. *space* (*M i*)) $\bigwedge i.$ *emeasure P* (*A i*) $\neq \infty$
by (*auto simp*: *space_PiM*)
next
fix *X* **assume** *X*: *X* $\in$ *prod_algebra I M*
then obtain *J E* **where** *X*: *X = prod_emb I M J* (Π_E *j*$\in$*J*. *E j*)
and *J*: *finite J J* $\subseteq$ *I* $\bigwedge j.$ *j* $\in$ *J* $\Longrightarrow$ *E j* $\in$ *sets* (*M j*)
by (*force elim*!: *prod_algebraE*)
then show *emeasure P X = emeasure Q X*
by (*auto intro*!: *eq*)
qed (*auto simp*: *sets_PiM*)

locale *product_sigma_finite* =
fixes *M* :: *'i* $\Rightarrow$ *'a measure*
assumes *sigma_finite_measures*: $\bigwedge i.$ *sigma_finite_measure* (*M i*)

sublocale *product_sigma_finite* $\subseteq$ *M*?: *sigma_finite_measure M i* **for** *i*
by (*rule sigma_finite_measures*)

locale *finite_product_sigma_finite* = *product_sigma_finite M* **for** *M* :: *'i* $\Rightarrow$ *'a measure* +
fixes *I* :: *'i set*
assumes *finite_index*: *finite I*

proposition (**in** *finite_product_sigma_finite*) *sigma_finite_pairs*:
  $\exists F{::}'i \Rightarrow nat \Rightarrow 'a\ set.$
    $(\forall i{\in}I.\ range\ (F\ i) \subseteq sets\ (M\ i)) \wedge$
    $(\forall k.\ \forall i{\in}I.\ emeasure\ (M\ i)\ (F\ i\ k) \neq \infty) \wedge incseq\ (\lambda k.\ \Pi_E\ i{\in}I.\ F\ i\ k) \wedge$
    $(\bigcup k.\ \Pi_E\ i{\in}I.\ F\ i\ k) = space\ (PiM\ I\ M)$
proof −
  have $\forall i{::}'i.\ \exists F{::}nat \Rightarrow 'a\ set.\ range\ F \subseteq sets\ (M\ i) \wedge incseq\ F \wedge (\bigcup i.\ F\ i) = space\ (M\ i) \wedge (\forall k.\ emeasure\ (M\ i)\ (F\ k) \neq \infty)$
    using *M.sigma_finite_incseq* **by** *metis*
  then obtain $F :: 'i \Rightarrow nat \Rightarrow 'a\ set$
    where $\forall x.\ range\ (F\ x) \subseteq sets\ (M\ x) \wedge incseq\ (F\ x) \wedge \bigcup\ (range\ (F\ x)) = space\ (M\ x) \wedge (\forall k.\ emeasure\ (M\ x)\ (F\ x\ k) \neq \infty)$
    by *metis*
  then have *F*: $\bigwedge i.\ range\ (F\ i) \subseteq sets\ (M\ i)\ \bigwedge i.\ incseq\ (F\ i)\ \bigwedge i.\ (\bigcup j.\ F\ i\ j) = space\ (M\ i)\ \bigwedge i\ k.\ emeasure\ (M\ i)\ (F\ i\ k) \neq \infty$
    by *auto*
  let *?F* = $\lambda k.\ \Pi_E\ i{\in}I.\ F\ i\ k$
  note *space_PiM*[*simp*]
  show *?thesis*
  proof (*intro exI*[*of* _ *F*] *conjI allI incseq_SucI set_eqI iffI ballI*)
    fix *i* **show** *range* (*F i*) ⊆ *sets* (*M i*) **by** *fact*
  next
    fix *i k* **show** *emeasure* (*M i*) (*F i k*) ≠ ∞ **by** *fact*
  next
    fix *x* **assume** *x* ∈ (⋃ *i*. *?F i*) **with** *F*(*1*) **show** *x* ∈ *space* (*PiM I M*)
      by (*auto simp*: *PiE_def dest*!: *sets.sets_into_space*)
  next
    fix *f* **assume** *f* ∈ *space* (*PiM I M*)
    with *Pi_UN*[*OF finite_index*, *of* λ*k i*. *F i k*] *F*
    show *f* ∈ (⋃ *i*. *?F i*) **by** (*auto simp*: *incseq_def PiE_def*)
  next
    fix *i* **show** *?F i* ⊆ *?F* (*Suc i*)
      using ‹⋀*i*. *incseq* (*F i*)›[*THEN incseq_SucD*] **by** *auto*
  qed
qed

lemma *emeasure_PiM_empty*[*simp*]: *emeasure* (*PiM* {} *M*) {λ_. *undefined*} = *1*
proof −
  let *?μ* = λ*A*. *if A* = {} *then 0 else* (*1*::*ennreal*)
  have *emeasure* (*Pi*$_M$ {} *M*) (*prod_emb* {} *M* {} (Π$_E$ *i*∈{}. {})) = *1*
  proof (*subst emeasure_extend_measure_Pair*[*OF PiM_def*])
    show *positive* (*PiM* {} *M*) *?μ*
      by (*auto simp*: *positive_def*)
    show *countably_additive* (*PiM* {} *M*) *?μ*
      by (*rule sets.countably_additiveI_finite*)
        (*auto simp*: *additive_def positive_def sets_PiM_empty space_PiM_empty intro*!:)
  qed (*auto simp*: *prod_emb_def*)
  also have (*prod_emb* {} *M* {} (Π$_E$ *i*∈{}. {})) = {λ_. *undefined*}

```
    by (auto simp: prod_emb_def)
  finally show ?thesis
    by simp
qed

lemma PiM_empty: PiM {} M = count_space {λ_. undefined}
  by (rule measure_eqI) (auto simp add: sets_PiM_empty)

lemma (in product_sigma_finite) emeasure_PiM:
  finite I ⟹ (⋀i. i∈I ⟹ A i ∈ sets (M i)) ⟹ emeasure (PiM I M) (Pi_E I A)
= (∏ i∈I. emeasure (M i) (A i))
proof (induct I arbitrary: A rule: finite_induct)
  case (insert i I)
  interpret finite_product_sigma_finite M I by standard fact
  have finite (insert i I) using ‹finite I› by auto
  interpret I': finite_product_sigma_finite M insert i I by standard fact
  let ?h = (λ(f, y). f(i := y))

  let ?P = distr (Pi_M I M ⨂_M M i) (Pi_M (insert i I) M) ?h
  let ?μ = emeasure ?P
  let ?I = {j ∈ insert i I. emeasure (M j) (space (M j)) ≠ 1}
  let ?f = λJ E j. if j ∈ J then emeasure (M j) (E j) else emeasure (M j) (space
(M j))

  have emeasure (Pi_M (insert i I) M) (prod_emb (insert i I) M (insert i I) (Pi_E
(insert i I) A)) =
    (∏ i∈insert i I. emeasure (M i) (A i))
  proof (subst emeasure_extend_measure_Pair[OF PiM_def])
    fix J E assume (J ≠ {} ∨ insert i I = {}) ∧ finite J ∧ J ⊆ insert i I ∧ E ∈
(Π j∈J. sets (M j))
    then have J: J ≠ {} finite J J ⊆ insert i I and E: ∀j∈J. E j ∈ sets (M j)
by auto
    let ?p = prod_emb (insert i I) M J (Pi_E J E)
    let ?p' = prod_emb I M (J - {i}) (Π_E j∈J-{i}. E j)
    have ?μ ?p =
      emeasure (Pi_M I M ⨂_M (M i)) (?h -` ?p ∩ space (Pi_M I M ⨂_M M i))
      by (intro emeasure_distr measurable_add_dim sets_PiM_I) fact+
    also have ?h -` ?p ∩ space (Pi_M I M ⨂_M M i) = ?p' × (if i ∈ J then E i
else space (M i))
      using J E[rule_format, THEN sets.sets_into_space]
     by (force simp: space_pair_measure space_PiM prod_emb_iff PiE_def Pi_iff
split: if_split_asm)
    also have emeasure (Pi_M I M ⨂_M (M i)) (?p' × (if i ∈ J then E i else space
(M i))) =
      emeasure (Pi_M I M) ?p' * emeasure (M i) (if i ∈ J then (E i) else space (M
i))
      using J E by (intro M.emeasure_pair_measure_Times sets_PiM_I) auto
    also have ?p' = (Π_E j∈I. if j ∈ J-{i} then E j else space (M j))
      using J E[rule_format, THEN sets.sets_into_space]
```

```
    by (auto simp: prod_emb_iff PiE_def Pi_iff split: if_split_asm) blast+
  also have emeasure (Pi_M I M) (Π_E j∈I. if j ∈ J−{i} then E j else space (M
j)) =
    (∏ j∈I. if j ∈ J−{i} then emeasure (M j) (E j) else emeasure (M j) (space
(M j)))
    using E by (subst insert) (auto intro!: prod.cong)
  also have (∏j∈I. if j ∈ J − {i} then emeasure (M j) (E j) else emeasure (M
j) (space (M j))) *
     emeasure (M i) (if i ∈ J then E i else space (M i)) = (∏j∈insert i I. ?f J E
j)
    using insert by (auto simp: mult.commute intro!: arg_cong2[where f=(*)]
prod.cong)
  also have ... = (∏j∈J ∪ ?I. ?f J E j)
    using insert(1,2) J E by (intro prod.mono_neutral_right) auto
  finally show ?μ ?p = ... .

  show prod_emb (insert i I) M J (Pi_E J E) ∈ Pow (Π_E i∈insert i I. space (M
i))
  using J E[rule_format, THEN sets.sets_into_space] by (auto simp: prod_emb_iff
PiE_def)
next
  show positive (sets (Pi_M (insert i I) M)) ?μ countably_additive (sets (Pi_M
(insert i I) M)) ?μ
    using emeasure_positive[of ?P] emeasure_countably_additive[of ?P] by
simp_all
next
  show (insert i I ≠ {} ∨ insert i I = {}) ∧ finite (insert i I) ∧
    insert i I ⊆ insert i I ∧ A ∈ (Π j∈insert i I. sets (M j))
    using insert by auto
qed (auto intro!: prod.cong)
with insert show ?case
 by (subst (asm) prod_emb_PiE_same_index) (auto intro!: sets.sets_into_space)
qed simp

lemma (in product_sigma_finite) PiM_eqI:
assumes I[simp]: finite I and P: sets P = PiM I M
assumes eq: ⋀A. (⋀i. i ∈ I ⟹ A i ∈ sets (M i)) ⟹ P (Pi_E I A) = (∏ i∈I.
emeasure (M i) (A i))
shows P = PiM I M
proof −
interpret finite_product_sigma_finite M I
  by standard fact
from sigma_finite_pairs obtain C where C:
  ∀ i∈I. range (C i) ⊆ sets (M i) ∀ k. ∀ i∈I. emeasure (M i) (C i k) ≠ ∞
  incseq (λk. Π_E i∈I. C i k) (⋃ k. Π_E i∈I. C i k) = space (Pi_M I M)
  by auto
show ?thesis
proof (rule measure_eqI_PiM_finite[OF I refl P, symmetric])
  show (⋀i. i ∈ I ⟹ A i ∈ sets (M i)) ⟹ (Pi_M I M) (Pi_E I A) = P (Pi_E I
```

A) **for** *A*
by (*simp add*: *eq emeasure_PiM*)
define *A* **where** *A n* = (Π$_E$ *i*∈*I*. *C i n*) **for** *n*
with *C* **show** *range A* ⊆ *prod_algebra I M* ⋀*i*. *emeasure* (*Pi*$_M$ *I M*) (*A i*) ≠ ∞ (⋃ *i*. *A i*) = *space* (*PiM I M*)
by (*auto intro*!: *prod_algebraI_finite simp*: *emeasure_PiM subset_eq ennreal_prod_eq_top*)
qed
qed

lemma (**in** *product_sigma_finite*) *sigma_finite*:
assumes *finite I*
shows *sigma_finite_measure* (*PiM I M*)
proof
interpret *finite_product_sigma_finite M I* **by** *standard fact*

obtain *F* **where** *F*: ⋀*j*. *countable* (*F j*) ⋀*j f*. *f* ∈ *F j* ⟹ *f* ∈ *sets* (*M j*) ⋀*j f*. *f* ∈ *F j* ⟹ *emeasure* (*M j*) *f* ≠ ∞ **and**
in_space: ⋀*j*. *space* (*M j*) = ⋃(*F j*)
using *sigma_finite_countable* **by** (*metis subset_eq*)
moreover have (⋃(*Pi*$_E$ *I* ‘ *Pi*$_E$ *I F*)) = *space* (*Pi*$_M$ *I M*)
using *in_space* **by** (*auto simp*: *space_PiM PiE_iff intro*!: *PiE_choice*[*THEN iffD1*])
ultimately show ∃ *A*. *countable A* ∧ *A* ⊆ *sets* (*Pi*$_M$ *I M*) ∧ ⋃ *A* = *space* (*Pi*$_M$ *I M*) ∧ (∀ *a*∈*A*. *emeasure* (*Pi*$_M$ *I M*) *a* ≠ ∞)
by (*intro exI*[*of* _ *Pi*$_E$ *I* ‘ *Pi*$_E$ *I F*])
(*auto intro*!: *countable_PiE sets_PiM_I_finite*
simp: *PiE_iff emeasure_PiM finite_index ennreal_prod_eq_top*)
qed

sublocale *finite_product_sigma_finite* ⊆ *sigma_finite_measure Pi*$_M$ *I M*
using *sigma_finite*[*OF finite_index*] **.**

lemma (**in** *finite_product_sigma_finite*) *measure_times*:
(⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *sets* (*M i*)) ⟹ *emeasure* (*Pi*$_M$ *I M*) (*Pi*$_E$ *I A*) = (∏ *i*∈*I*. *emeasure* (*M i*) (*A i*))
using *emeasure_PiM*[*OF finite_index*] **by** *auto*

lemma (**in** *product_sigma_finite*) *nn_integral_empty*:
0 ≤ *f* (λ*k*. *undefined*) ⟹ *integral*^{N} (*Pi*$_M$ {} *M*) *f* = *f* (λ*k*. *undefined*)
by (*simp add*: *PiM_empty nn_integral_count_space_finite max.absorb2*)

lemma (**in** *product_sigma_finite*) *distr_merge*:
assumes *IJ*[*simp*]: *I* ∩ *J* = {} **and** *fin*: *finite I finite J*
shows *distr* (*Pi*$_M$ *I M* ⨂$_M$ *Pi*$_M$ *J M*) (*Pi*$_M$ (*I* ∪ *J*) *M*) (*merge I J*) = *Pi*$_M$ (*I* ∪ *J*) *M*
(**is** *?D* = *?P*)
proof (*rule PiM_eqI*)
interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*

interpret J: *finite_product_sigma_finite M J* **by** *standard fact*
fix A **assume** A: $\bigwedge i.\ i \in I \cup J \Longrightarrow A\ i \in sets\ (M\ i)$
have $*$: $(merge\ I\ J\ -`\ Pi_E\ (I \cup J)\ A \cap space\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)) = Pi_E\ I\ A \times Pi_E\ J\ A$
using A[*THEN sets.sets_into_space*] **by** (*auto simp*: *space_PiM space_pair_measure*)
from A *fin* **show** $emeasure\ (distr\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)\ (merge\ I\ J))\ (Pi_E\ (I \cup J)\ A) =$
$(\prod i \in I \cup J.\ emeasure\ (M\ i)\ (A\ i))$
by (*subst emeasure_distr*)
(*auto simp*: $*$ *J.emeasure_pair_measure_Times I.measure_times J.measure_times prod.union_disjoint*)
qed (*use fin* **in** *simp_all*)

proposition (**in** *product_sigma_finite*) *product_nn_integral_fold*:
assumes *IJ*: $I \cap J = \{\}$ *finite I finite J*
and f[*measurable*]: $f \in borel\_measurable\ (Pi_M\ (I \cup J)\ M)$
shows $integral^N\ (Pi_M\ (I \cup J)\ M)\ f = (\int^+ x.\ (\int^+ y.\ f\ (merge\ I\ J\ (x, y))\ \partial(Pi_M\ J\ M))\ \partial(Pi_M\ I\ M))$
(**is** *?lhs* = *?rhs*)
proof −
interpret I: *finite_product_sigma_finite M I* **by** *standard fact*
interpret J: *finite_product_sigma_finite M J* **by** *standard fact*
interpret P: *pair_sigma_finite* $Pi_M\ I\ M\ Pi_M\ J\ M$ **by** *standard*
have *P_borel*: $(\lambda x.\ f\ (merge\ I\ J\ x)) \in borel\_measurable\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)$
using *measurable_comp*[*OF measurable_merge f*] **by** (*simp add*: *comp_def*)
have *?lhs* = $integral^N\ (distr\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)\ (merge\ I\ J))\ f$
by (*simp add*: *I.finite_index J.finite_index assms*(1) *distr_merge*)
also have ... = $\int^+ x.\ f\ (merge\ I\ J\ x)\ \partial(Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)$
by (*simp add*: *nn_integral_distr*)
also have ... = *?rhs*
using *P.Fubini P.nn_integral_snd* **by** *force*
finally show *?thesis* **.**
qed

lemma (**in** *product_sigma_finite*) *distr_singleton*:
$distr\ (Pi_M\ \{i\}\ M)\ (M\ i)\ (\lambda x.\ x\ i) = M\ i$ (**is** *?D* = _)
proof (*intro measure_eqI*[*symmetric*])
interpret I: *finite_product_sigma_finite M* $\{i\}$ **by** *standard simp*
fix A **assume** A: $A \in sets\ (M\ i)$
then have $(\lambda x.\ x\ i)\ -`\ A \cap space\ (Pi_M\ \{i\}\ M) = (\Pi_E\ i \in \{i\}.\ A)$
using *sets.sets_into_space* **by** (*auto simp*: *space_PiM*)
then show $emeasure\ (M\ i)\ A = emeasure$ *?D A*
using *A I.measure_times*[*of* $\lambda\_.\ A$]
by (*simp add*: *emeasure_distr measurable_component_singleton*)
qed *simp*

lemma (**in** *product_sigma_finite*) *product_nn_integral_singleton*:

assumes f: $f \in$ *borel_measurable* $(M\ i)$
shows $integral^N\ (Pi_M\ \{i\}\ M)\ (\lambda x.\ f\ (x\ i)) = integral^N\ (M\ i)\ f$
proof −
interpret I: *finite_product_sigma_finite* $M\ \{i\}$ **by** *standard simp*
from f **show** *?thesis*
by (*metis distr_singleton insert_iff measurable_component_singleton nn_integral_distr*)
qed

proposition (**in** *product_sigma_finite*) *product_nn_integral_insert*:
assumes I[*simp*]: *finite* $I\ i \notin I$
and f: $f \in$ *borel_measurable* $(Pi_M\ (insert\ i\ I)\ M)$
shows $integral^N\ (Pi_M\ (insert\ i\ I)\ M)\ f = (\int^+ x.\ (\int^+ y.\ f\ (x(i := y))\ \partial(M\ i))$
$\partial(Pi_M\ I\ M))$
proof −
interpret I: *finite_product_sigma_finite* $M\ I$ **by** *standard auto*
interpret i: *finite_product_sigma_finite* $M\ \{i\}$ **by** *standard auto*
have IJ: $I \cap \{i\} = \{\}$ **and** *insert*: $I \cup \{i\} = insert\ i\ I$
using f **by** *auto*
show *?thesis*
unfolding *product_nn_integral_fold*[*OF IJ*, *unfolded insert*, *OF I*(*1*) *i.finite_index*
f]
proof (*rule nn_integral_cong*, *subst product_nn_integral_singleton*[*symmetric*])
fix x **assume** x: $x \in space\ (Pi_M\ I\ M)$
let *?f* $= \lambda y.\ f\ (x(i := y))$
show *?f* $\in$ *borel_measurable* $(M\ i)$
using *measurable_comp*[*OF measurable_component_update f*, *OF x* ‹$i \notin I$›]
unfolding *comp_def* **.**
show $(\int^+ y.\ f\ (merge\ I\ \{i\}\ (x,\ y))\ \partial Pi_M\ \{i\}\ M) = (\int^+ y.\ f\ (x(i := y\ i))$
$\partial Pi_M\ \{i\}\ M)$
using x
by (*auto intro*!: *nn_integral_cong arg_cong*[**where** f=f]
simp add: *space_PiM extensional_def PiE_def*)
qed
qed

lemma (**in** *product_sigma_finite*) *product_nn_integral_insert_rev*:
assumes I[*simp*]: *finite* $I\ i \notin I$
and [*measurable*]: $f \in$ *borel_measurable* $(Pi_M\ (insert\ i\ I)\ M)$
shows $integral^N\ (Pi_M\ (insert\ i\ I)\ M)\ f = (\int^+ y.\ (\int^+ x.\ f\ (x(i := y))\ \partial(Pi_M$
$I\ M))\ \partial(M\ i))$
apply (*subst product_nn_integral_insert*[*OF assms*])
apply (*rule pair_sigma_finite.Fubini'*)
apply (*simp add*: *local.sigma_finite pair_sigma_finite.intro sigma_finite_measures*)
apply *measurable*
done

lemma (**in** *product_sigma_finite*) *product_nn_integral_prod*:
assumes *finite* I $\bigwedge i.\ i \in I \Longrightarrow f\ i \in$ *borel_measurable* $(M\ i)$
shows $(\int^+ x.\ (\prod i{\in}I.\ f\ i\ (x\ i))\ \partial Pi_M\ I\ M) = (\prod i{\in}I.\ integral^N\ (M\ i)\ (f\ i))$

using *assms* **proof** (*induction I*)
case (*insert i I*)
note *insert.prems*[*measurable*]
note ‹*finite I*›[*intro*, *simp*]
interpret *I*: *finite_product_sigma_finite M I* **by** *standard auto*
have ∗: ⋀*x y*. (∏ *j*∈*I*. *f j* (*if j* = *i then y else x j*)) = (∏ *j*∈*I*. *f j* (*x j*))
using *insert* **by** (*auto intro*!: *prod.cong*)
have *prod*: ⋀*J*. *J* ⊆ *insert i I* ⟹ (λ*x*. (∏ *i*∈*J*. *f i* (*x i*))) ∈ *borel_measurable* (*Pi*$_M$ *J M*)
using *sets.sets_into_space insert*
by (*intro borel_measurable_prod_ennreal*
measurable_comp[*OF measurable_component_singleton*, *unfolded comp_def*])
auto
then show *?case*
using *product_nn_integral_insert*[*OF insert*(*1*,*2*)]
by (*simp add*: *insert*(*2*−) ∗ *nn_integral_multc nn_integral_cmult*)
qed (*simp add*: *space_PiM*)

proposition (**in** *product_sigma_finite*) *product_nn_integral_pair*:
assumes [*measurable*]: *case_prod f* ∈ *borel_measurable* (*M x* ⨂$_M$ *M y*)
assumes *xy*: *x* ≠ *y*
shows (∫$^+$σ. *f* (σ *x*) (σ *y*) ∂*PiM* {*x*, *y*} *M*) = (∫$^+$*z*. *f* (*fst z*) (*snd z*) ∂(*M x* ⨂$_M$ *M y*))
proof −
interpret *psm*: *pair_sigma_finite M x M y*
unfolding *pair_sigma_finite_def* **using** *sigma_finite_measures* **by** *simp_all*
have {*x*, *y*} = {*y*, *x*} **by** *auto*
also have (∫$^+$σ. *f* (σ *x*) (σ *y*) ∂*PiM* {*y*, *x*} *M*) = (∫$^+$*y*. ∫$^+$σ. *f* (σ *x*) *y* ∂*PiM* {*x*} *M* ∂*M y*)
using *xy* **by** (*subst product_nn_integral_insert_rev*) *simp_all*
also have ... = (∫$^+$*y*. ∫$^+$*x*. *f x y* ∂*M x* ∂*M y*)
by (*intro nn_integral_cong*, *subst product_nn_integral_singleton*) *simp_all*
also have ... = (∫$^+$*z*. *f* (*fst z*) (*snd z*) ∂(*M x* ⨂$_M$ *M y*))
by (*subst psm.nn_integral_snd*[*symmetric*]) *simp_all*
finally show *?thesis* **.**
qed

lemma (**in** *product_sigma_finite*) *distr_component*:
distr (*M i*) (*Pi*$_M$ {*i*} *M*) (λ*x*. λ*i*∈{*i*}. *x*) = *Pi*$_M$ {*i*} *M* (**is** *?D* = *?P*)
proof (*intro PiM_eqI*)
fix *A* **assume** *A*: ⋀*ia*. *ia* ∈ {*i*} ⟹ *A ia* ∈ *sets* (*M ia*)
then have (λ*x*. λ*i*∈{*i*}. *x*) −‘ *Pi*$_E$ {*i*} *A* ∩ *space* (*M i*) = *A i*
by (*fastforce dest*: *sets.sets_into_space*)
with *A* **show** *emeasure* (*distr* (*M i*) (*Pi*$_M$ {*i*} *M*) (λ*x*. λ*i*∈{*i*}. *x*)) (*Pi*$_E$ {*i*} *A*) = (∏ *i*∈{*i*}. *emeasure* (*M i*) (*A i*))
by (*subst emeasure_distr*) (*auto intro*!: *sets_PiM_I_finite measurable_restrict*)
qed *simp_all*

lemma (**in** *product_sigma_finite*)
assumes *IJ*: $I \cap J = \{\}$ *finite I finite J* **and** *A*: $A \in sets\ (Pi_M\ (I \cup J)\ M)$
shows *emeasure_fold_integral*:
$emeasure\ (Pi_M\ (I \cup J)\ M)\ A = (\int^+ x.\ emeasure\ (Pi_M\ J\ M)\ ((\lambda y.\ merge\ I\ J\ (x, y)) -` A \cap space\ (Pi_M\ J\ M))\ \partial Pi_M\ I\ M)$
(**is** *?lhs* = *?rhs*)
and *emeasure_fold_measurable*:
$(\lambda x.\ emeasure\ (Pi_M\ J\ M)\ ((\lambda y.\ merge\ I\ J\ (x, y)) -` A \cap space\ (Pi_M\ J\ M))) \in borel\_measurable\ (Pi_M\ I\ M)$ (**is** *?B*)
proof −
interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*
interpret *J*: *finite_product_sigma_finite M J* **by** *standard fact*
interpret *IJ*: *pair_sigma_finite* $Pi_M\ I\ M\ Pi_M\ J\ M$ **..**
have *merge*: $merge\ I\ J -` A \cap space\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M) \in sets\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)$
by (*intro measurable_sets*[*OF* _ *A*] *measurable_merge assms*)
have $?lhs = emeasure\ (distr\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)\ (merge\ I\ J))\ A$
by (*simp add*: *I.finite_index J.finite_index assms*(*1*) *distr_merge*)
also have ... = $emeasure\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (merge\ I\ J -` A \cap space\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M))$
by (*meson A emeasure_distr measurable_merge*)
also have ... = *?rhs*
apply (*subst J.emeasure_pair_measure_alt*[*OF merge*])
by (*auto intro*!: *nn_integral_cong arg_cong2*[**where** *f*=*emeasure*] *simp*: *space_pair_measure*)
finally show *?lhs* = *?rhs* **.**

show *?B*
using *IJ.measurable_emeasure_Pair1*[*OF merge*]
by (*simp add*: *vimage_comp comp_def space_pair_measure cong*: *measurable_cong*)
qed

lemma *sets_Collect_single*:
$i \in I \Longrightarrow A \in sets\ (M\ i) \Longrightarrow \{\ x \in space\ (Pi_M\ I\ M).\ x\ i \in A\ \} \in sets\ (Pi_M\ I\ M)$
by *simp*
lemma *pair_measure_eq_distr_PiM*:
fixes *M1* :: *'a measure* **and** *M2* :: *'a measure*
assumes *sigma_finite_measure M1 sigma_finite_measure M2*
shows $(M1 \bigotimes_M M2) = distr\ (Pi_M\ UNIV\ (case\_bool\ M1\ M2))\ (M1 \bigotimes_M M2)\ (\lambda x.\ (x\ True, x\ False))$
(**is** *?P* = *?D*)
proof (*rule pair_measure_eqI*[*OF assms*])
interpret *B*: *product_sigma_finite case_bool M1 M2*
unfolding *product_sigma_finite_def* **using** *assms* **by** (*auto split*: *bool.split*)
let *?B* = $Pi_M\ UNIV\ (case\_bool\ M1\ M2)$

have [*simp*]: $fst \circ (\lambda x.\ (x\ True, x\ False)) = (\lambda x.\ x\ True)\ snd \circ (\lambda x.\ (x\ True, x$

False)) = (λ*x*. *x False*)
by *auto*
fix *A B* **assume** *A*: *A* ∈ *sets M1* **and** *B*: *B* ∈ *sets M2*
have *emeasure M1 A* ∗ *emeasure M2 B* = (∏ *i*∈*UNIV*. *emeasure* (*case_bool M1 M2 i*) (*case_bool A B i*))
by (*simp add*: *UNIV_bool ac_simps*)
also have ... = *emeasure ?B* (*Pi*$_E$ *UNIV* (*case_bool A B*))
using *A B* **by** (*subst B.emeasure_PiM*) (*auto split*: *bool.split*)
also have *Pi*$_E$ *UNIV* (*case_bool A B*) = (λ*x*. (*x True*, *x False*)) −‘ (*A* × *B*) ∩ *space ?B*
using *A*[*THEN sets.sets_into_space*] *B*[*THEN sets.sets_into_space*]
by (*auto simp*: *PiE_iff all_bool_eq space_PiM split*: *bool.split*)
finally show *emeasure M1 A* ∗ *emeasure M2 B* = *emeasure ?D* (*A* × *B*)
using *A B*
measurable_component_singleton[*of True UNIV case_bool M1 M2*]
measurable_component_singleton[*of False UNIV case_bool M1 M2*]
by (*subst emeasure_distr*) (*auto simp*: *measurable_pair_iff*)
qed *simp*

lemma *infprod_in_sets*[*intro*]:
fixes *E* :: *nat* ⇒ *'a set* **assumes** *E*: ⋀*i*. *E i* ∈ *sets* (*M i*)
shows *Pi UNIV E* ∈ *sets* (Π$_M$ *i*∈*UNIV*::*nat set*. *M i*)
proof −
have *Pi UNIV E* = (⋂ *i*. *prod_emb UNIV M* {..*i*} (Π$_E$ *j*∈{..*i*}. *E j*))
using *E E*[*THEN sets.sets_into_space*]
by (*auto simp*: *prod_emb_def Pi_iff extensional_def*)
with *E* **show** *?thesis* **by** *auto*
qed

7.7.3 Measurability

There are two natural sigma-algebras on a product space: the borel sigma algebra, generated by open sets in the product, and the product sigma algebra, countably generated by products of measurable sets along finitely many coordinates. The second one is defined and studied in `Finite_Product_Measure.thy`.

These sigma-algebra share a lot of natural properties (measurability of coordinates, for instance), but there is a fundamental difference: open sets are generated by arbitrary unions, not only countable ones, so typically many open sets will not be measurable with respect to the product sigma algebra (while all sets in the product sigma algebra are borel). The two sigma algebras coincide only when everything is countable (i.e., the product is countable, and the borel sigma algebra in the factor is countably generated).

In this paragraph, we develop basic measurability properties for the borel sigma algebra, and compare it with the product sigma algebra as explained above.

lemma *measurable_product_coordinates* [*measurable* (*raw*)]:
(λ*x*. *x i*) ∈ *measurable borel borel*

by (*rule borel_measurable_continuous_onI*[*OF continuous_on_product_coordinates*])

lemma *measurable_product_then_coordinatewise*:
fixes f::$'a \Rightarrow 'b \Rightarrow$ ($'c$::*topological_space*)
assumes [*measurable*]: $f \in$ *borel_measurable M*
shows $(\lambda x.\ f\ x\ i) \in$ *borel_measurable M*
proof −
have $(\lambda x.\ f\ x\ i) = (\lambda y.\ y\ i)\ o\ f$
unfolding *comp_def* **by** *auto*
then show *?thesis* **by** *simp*
qed

To compare the Borel sigma algebra with the product sigma algebra, we give a presentation of the product sigma algebra that is more similar to the one we used above for the product topology.

lemma *sets_PiM_finite*:
sets $(Pi_M\ I\ M)$ = *sigma_sets* $(\Pi_E\ i{\in}I.\ space\ (M\ i))$
$\{(\Pi_E\ i{\in}I.\ X\ i)\ |X.\ (\forall i.\ X\ i \in sets\ (M\ i)) \wedge finite\ \{i.\ X\ i \neq space\ (M\ i)\}\}$
proof
have $\{(\Pi_E\ i{\in}I.\ X\ i)\ |X.\ (\forall i.\ X\ i \in sets\ (M\ i)) \wedge finite\ \{i.\ X\ i \neq space\ (M\ i)\}\} \subseteq sets\ (Pi_M\ I\ M)$
proof *clarify*
fix X **assume** H: $\forall i.\ X\ i \in sets\ (M\ i)$ *finite* $\{i.\ X\ i \neq space\ (M\ i)\}$
then have $*$: $X\ i \in sets\ (M\ i)$ **for** i **by** *simp*
define J **where** $J = \{i \in I.\ X\ i \neq space\ (M\ i)\}$
have *finite* J $J \subseteq I$ **unfolding** *J_def* **using** H **by** *auto*
define Y **where** $Y = (\Pi_E\ j{\in}J.\ X\ j)$
have *prod_emb I M J Y* $\in sets\ (Pi_M\ I\ M)$
unfolding *Y_def* **apply** (*rule sets_PiM_I*) **using** ‹*finite J*› ‹$J \subseteq I$› $*$ **by** *auto*
moreover have *prod_emb I M J Y* = $(\Pi_E\ i{\in}I.\ X\ i)$
unfolding *prod_emb_def Y_def J_def* **using** H *sets.sets_into_space*[*OF* $*$]
by (*auto simp add*: *PiE_iff*, *blast*)
ultimately show $Pi_E\ I\ X \in sets\ (Pi_M\ I\ M)$ **by** *simp*
qed
then show *sigma_sets* $(\Pi_E\ i{\in}I.\ space\ (M\ i))$ $\{(\Pi_E\ i{\in}I.\ X\ i)\ |X.\ (\forall i.\ X\ i \in sets\ (M\ i)) \wedge finite\ \{i.\ X\ i \neq space\ (M\ i)\}\}$
$\subseteq sets\ (Pi_M\ I\ M)$
by (*metis* (*mono_tags*, *lifting*) *sets.sigma_sets_subset'* *sets.top space_PiM*)

have $*$: $\exists X.\ \{f.\ (\forall i{\in}I.\ f\ i \in space\ (M\ i)) \wedge f \in extensional\ I \wedge f\ i \in A\} = Pi_E\ I\ X \wedge$
$(\forall i.\ X\ i \in sets\ (M\ i)) \wedge finite\ \{i.\ X\ i \neq space\ (M\ i)\}$
if $i \in I$ $A \in sets\ (M\ i)$ **for** i A
proof −
define X **where** $X = (\lambda j.\ if\ j = i\ then\ A\ else\ space\ (M\ j))$
have $\{f.\ (\forall i{\in}I.\ f\ i \in space\ (M\ i)) \wedge f \in extensional\ I \wedge f\ i \in A\} = Pi_E\ I\ X$
unfolding *X_def* **using** *sets.sets_into_space*[*OF* ‹$A \in sets\ (M\ i)$›] ‹$i \in I$›
by (*auto simp add*: *PiE_iff extensional_def*, *metis subsetCE*, *metis*)

```
    moreover have X j ∈ sets (M j) for j
      unfolding X_def using ‹A ∈ sets (M i)› by auto
    moreover have finite {j. X j ≠ space (M j)}
      unfolding X_def by simp
    ultimately show ?thesis by auto
  qed
  show sets (Pi_M I M) ⊆ sigma_sets (Π_E i∈I. space (M i)) {(Π_E i∈I. X i) |X.
(∀ i. X i ∈ sets (M i)) ∧ finite {i. X i ≠ space (M i)}}
    unfolding sets_PiM_single
    by (intro sigma_sets_mono') (auto simp add: PiE_iff ∗)
qed

lemma sets_PiM_subset_borel:
  sets (Pi_M UNIV (λ_. borel)) ⊆ sets borel
proof −
  have ∗: Pi_E UNIV X ∈ sets borel if [measurable]: ⋀i. X i ∈ sets borel finite {i.
X i ≠ UNIV} for X::'a ⇒ 'b set
  proof −
    define I where I = {i. X i ≠ UNIV}
    have finite I unfolding I_def using that by simp
    have Pi_E UNIV X = (⋂ i∈I. (λx. x i)−‘(X i) ∩ space borel) ∩ space borel
      unfolding I_def by auto
    also have ... ∈ sets borel
      using that ‹finite I› by measurable
    finally show ?thesis by simp
  qed
  then have {(Π_E i∈UNIV. X i) |X::('a ⇒ 'b set). (∀ i. X i ∈ sets borel) ∧ finite
{i. X i ≠ space borel}} ⊆ sets borel
    by auto
  then show ?thesis unfolding sets_PiM_finite space_borel
    by (simp add: ∗ sets.sigma_sets_subset')
qed

proposition sets_PiM_equal_borel:
  sets (Pi_M UNIV (λi::('a::countable). borel::('b::second_countable_topology mea-
sure))) = sets borel
proof
  obtain K::('a ⇒ 'b) set set where K: topological_basis K countable K
          ⋀k. k ∈ K ⟹ ∃ X. (k = Pi_E UNIV X) ∧ (∀ i. open (X i)) ∧ finite {i.
X i ≠ UNIV}
    using product_topology_countable_basis by fast
  have ∗: k ∈ sets (Pi_M UNIV (λ_. borel)) if k ∈ K for k
  proof −
    obtain X where H: k = Pi_E UNIV X ⋀i. open (X i) finite {i. X i ≠ UNIV}
      using K(3)[OF ‹k ∈ K›] by blast
   show ?thesis unfolding H(1) sets_PiM_finite space_borel using borel_open[OF
H(2)] H(3) by auto
  qed
  have ∗∗: U ∈ sets (Pi_M UNIV (λ_. borel)) if open U for U::('a ⇒ 'b) set
```

proof −
obtain B **where** $B \subseteq K$ $U = (\bigcup B)$
using ‹*open U*› ‹*topological_basis K*› **by** (*metis topological_basis_def*)
have *countable B* **using** ‹$B \subseteq K$› ‹*countable K*› *countable_subset* **by** *blast*
moreover have $k \in$ *sets* (*Pi*$_M$ *UNIV* (λ_. *borel*)) **if** $k \in B$ **for** k
using ‹$B \subseteq K$› ∗ *that* **by** *auto*
ultimately show *?thesis* **unfolding** ‹$U = (\bigcup B)$› **by** *auto*
qed
have *sigma_sets UNIV* (*Collect open*) ⊆ *sets* (*Pi*$_M$ *UNIV* (λ*i*::′*a*. (*borel*::(′*b measure*))))
by (*metis ∗∗ mem_Collect_eq open_UNIV sets.sigma_sets_subset′ subsetI*)
then show *sets* (*borel*::(′*a* ⇒ ′*b*) *measure*) ⊆ *sets* (*Pi*$_M$ *UNIV* (λ_. *borel*))
unfolding *borel_def* **by** *auto*
qed (*simp add*: *sets_PiM_subset_borel*)

lemma *measurable_coordinatewise_then_product*:
fixes f::′*a* ⇒ (′*b*::*countable*) ⇒ (′*c*::*second_countable_topology*)
assumes [*measurable*]: ⋀*i*. (λ*x*. *f x i*) ∈ *borel_measurable M*
shows $f \in$ *borel_measurable M*
proof −
have $f \in$ *measurable M* (*Pi*$_M$ *UNIV* (λ_. *borel*))
by (*rule measurable_PiM_single′*, *auto simp add*: *assms*)
then show *?thesis* **using** *sets_PiM_equal_borel measurable_cong_sets* **by** *blast*
qed

end

7.8 Caratheodory Extension Theorem

theory *Caratheodory*
imports *Measure_Space*
begin

Originally from the Hurd/Coble measure theory development, translated by Lawrence Paulson.

lemma *suminf_ennreal_2dimen*:
fixes f:: *nat* × *nat* ⇒ *ennreal*
assumes ⋀*m*. *g m* = (∑ *n*. *f* (*m*,*n*))
shows (∑ *i*. *f* (*prod_decode i*)) = *suminf g*
proof −
have *g_def*: *g* = (λ*m*. (∑ *n*. *f* (*m*,*n*)))
using *assms* **by** (*simp add*: *fun_eq_iff*)
have *reindex*: ⋀*B*. (∑ *x*∈*B*. *f* (*prod_decode x*)) = *sum f* (*prod_decode* ‘ *B*)
by (*simp add*: *sum.reindex*[*OF inj_prod_decode*] *comp_def*)
have (*SUP n*. ∑ *i*<*n*. *f* (*prod_decode i*)) = (*SUP p* ∈ *UNIV* × *UNIV*. ∑ *i*<*fst p*. ∑ *n*<*snd p*. *f* (*i*, *n*))
proof (*intro SUP_eq*; *clarsimp simp*: *sum.cartesian_product reindex*)
fix *n*

let *?M* = λf. *Suc* (*Max* (*f* ‘ *prod_decode* ‘ {..<*n*}))
{ fix *a b x* **assume** *x* < *n* **and** [*symmetric*]: (*a*, *b*) = *prod_decode x*
then have *a* < *?M fst b* < *?M snd*
by (*auto intro*!: *Max_ge le_imp_less_Suc image_eqI*) **}**
then have *sum f* (*prod_decode* ‘ {..<*n*}) ≤ *sum f* ({..<*?M fst*} × {..<*?M snd*})
by (*auto intro*!: *sum_mono2*)
then show ∃ *a b*. *sum f* (*prod_decode* ‘ {..<*n*}) ≤ *sum f* ({..<*a*} × {..<*b*}) **by** *auto*
next
fix *a b*
let *?M* = *prod_decode* ‘ {..<*Suc* (*Max* (*prod_encode* ‘ ({..<*a*} × {..<*b*})))}
{ fix *a′ b′* **assume** *a′* < *a b′* < *b* **then have** (*a′*, *b′*) ∈ *?M*
by (*auto intro*!: *Max_ge le_imp_less_Suc image_eqI*[**where** *x*=*prod_encode* (*a′*, *b′*)]) **}**
then have *sum f* ({..<*a*} × {..<*b*}) ≤ *sum f ?M*
by (*auto intro*!: *sum_mono2*)
then show ∃ *n*. *sum f* ({..<*a*} × {..<*b*}) ≤ *sum f* (*prod_decode* ‘ {..<*n*})
by *auto*
qed
also have ... = (*SUP p*. ∑*i*<*p*. ∑*n*. *f* (*i*, *n*))
unfolding *suminf_sum*[*OF summableI*, *symmetric*]
by (*simp add*: *suminf_eq_SUP SUP_pair sum.swap*[*of* _ {..< *fst* _}])
finally show *?thesis* **unfolding** *g_def*
by (*simp add*: *suminf_eq_SUP*)
qed

7.8.1 Characterizations of Measures

definition *outer_measure_space* **where**
outer_measure_space M f ⟷ *positive M f* ∧ *increasing M f* ∧ *countably_subadditive M f*

Lambda Systems

definition *lambda_system* :: *'a set* ⇒ *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *'a set set* **where**
lambda_system Ω *M f* = {*l* ∈ *M*. ∀ *x* ∈ *M*. *f* (*l* ∩ *x*) + *f* ((Ω − *l*) ∩ *x*) = *f x*}

lemma (**in** *algebra*) *lambda_system_eq*:
lambda_system Ω *M f* = {*l* ∈ *M*. ∀ *x* ∈ *M*. *f* (*x* ∩ *l*) + *f* (*x* − *l*) = *f x*}
proof −
have [*simp*]: ⋀*l x*. *l* ∈ *M* ⟹ *x* ∈ *M* ⟹ (Ω − *l*) ∩ *x* = *x* − *l*
by (*metis Int_Diff Int_absorb1 Int_commute sets_into_space*)
show *?thesis*
by (*auto simp add*: *lambda_system_def*) (*metis Int_commute*)+
qed

lemma (**in** *algebra*) *lambda_system_empty*: *positive M f* ⟹ {} ∈ *lambda_system* Ω *M f*

by (*auto simp add*: *positive_def lambda_system_eq*)

lemma *lambda_system_sets*: $x \in$ *lambda_system* Ω M $f \Longrightarrow x \in M$
by (*simp add*: *lambda_system_def*)

lemma (**in** *algebra*) *lambda_system_Compl*:
fixes f:: $'a$ *set* $\Rightarrow$ *ennreal*
assumes x: $x \in$ *lambda_system* Ω M f
shows $\Omega - x \in$ *lambda_system* Ω M f
proof −
have $x \subseteq \Omega$
by (*metis sets_into_space lambda_system_sets x*)
hence $\Omega - (\Omega - x) = x$
by (*metis double_diff equalityE*)
with x **show** *?thesis*
by (*force simp add*: *lambda_system_def ac_simps*)
qed

lemma (**in** *algebra*) *lambda_system_Int*:
fixes f:: $'a$ *set* $\Rightarrow$ *ennreal*
assumes xl: $x \in$ *lambda_system* Ω M f **and** yl: $y \in$ *lambda_system* Ω M f
shows $x \cap y \in$ *lambda_system* Ω M f
proof −
from xl yl **show** *?thesis*
proof (*auto simp add*: *positive_def lambda_system_eq Int*)
fix u
assume x: $x \in M$ **and** y: $y \in M$ **and** u: $u \in M$
and fx: $\forall z{\in}M.\ f\ (z \cap x) + f\ (z - x) = f\ z$
and fy: $\forall z{\in}M.\ f\ (z \cap y) + f\ (z - y) = f\ z$
have $u - x \cap y \in M$
by (*metis Diff Diff_Int Un u x y*)
moreover
have $(u - (x \cap y)) \cap y = u \cap y - x$ **by** *blast*
moreover
have $u - x \cap y - y = u - y$ **by** *blast*
ultimately
have ey: $f\ (u - x \cap y) = f\ (u \cap y - x) + f\ (u - y)$ **using** fy
by *force*
have $f\ (u \cap (x \cap y)) + f\ (u - x \cap y)$
$= (f\ (u \cap (x \cap y)) + f\ (u \cap y - x)) + f\ (u - y)$
by (*simp add*: *ey ac_simps*)
also have ... = $(f\ ((u \cap y) \cap x) + f\ (u \cap y - x)) + f\ (u - y)$
by (*simp add*: *Int_ac*)
also have ... = $f\ (u \cap y) + f\ (u - y)$
using fx [*THEN bspec*, *of* $u \cap y$] *Int* y u
by *force*
also have ... = $f\ u$
by (*metis fy u*)
finally show $f\ (u \cap (x \cap y)) + f\ (u - x \cap y) = f\ u$ **.**

```
  qed
qed

lemma (in algebra) lambda_system_Un:
  fixes f:: 'a set ⇒ ennreal
  assumes xl: x ∈ lambda_system Ω M f and yl: y ∈ lambda_system Ω M f
  shows x ∪ y ∈ lambda_system Ω M f
proof −
  have (Ω − x) ∩ (Ω − y) ∈ M
    by (metis Diff_Un Un compl_sets lambda_system_sets xl yl)
  moreover
  have x ∪ y = Ω − ((Ω − x) ∩ (Ω − y))
    by auto (metis subsetD lambda_system_sets sets_into_space xl yl)+
  ultimately show ?thesis
    by (metis lambda_system_Compl lambda_system_Int xl yl)
qed

lemma (in algebra) lambda_system_algebra:
  positive M f ⟹ algebra Ω (lambda_system Ω M f)
  apply (auto simp add: algebra_iff_Un)
  apply (metis lambda_system_sets subsetD sets_into_space)
  apply (metis lambda_system_empty)
  apply (metis lambda_system_Compl)
  apply (metis lambda_system_Un)
  done

lemma (in algebra) lambda_system_strong_additive:
  assumes z: z ∈ M and disj: x ∩ y = {}
     and xl: x ∈ lambda_system Ω M f and yl: y ∈ lambda_system Ω M f
  shows f (z ∩ (x ∪ y)) = f (z ∩ x) + f (z ∩ y)
proof −
  have z ∩ x = (z ∩ (x ∪ y)) ∩ x using disj by blast
  moreover
  have z ∩ y = (z ∩ (x ∪ y)) − x using disj by blast
  moreover
  have (z ∩ (x ∪ y)) ∈ M
    by (metis Int Un lambda_system_sets xl yl z)
  ultimately show ?thesis using xl yl
    by (simp add: lambda_system_eq)
qed

lemma (in algebra) lambda_system_additive: additive (lambda_system Ω M f) f
proof (auto simp add: additive_def)
  fix x and y
  assume disj: x ∩ y = {}
     and xl: x ∈ lambda_system Ω M f and yl: y ∈ lambda_system Ω M f
  hence  x ∈ M y ∈ M by (blast intro: lambda_system_sets)+
  thus f (x ∪ y) = f x + f y
    using lambda_system_strong_additive [OF top disj xl yl]
```

by (*simp add*: *Un*)
qed

lemma *lambda_system_increasing*: *increasing M f* $\Longrightarrow$ *increasing* (*lambda_system* Ω *M f*) *f*
by (*simp add*: *increasing_def lambda_system_def*)

lemma *lambda_system_positive*: *positive M f* $\Longrightarrow$ *positive* (*lambda_system* Ω *M f*) *f*
by (*simp add*: *positive_def lambda_system_def*)

lemma (**in** *algebra*) *lambda_system_strong_sum*:
fixes *A*:: *nat* $\Rightarrow$ *'a set* **and** *f* :: *'a set* $\Rightarrow$ *ennreal*
assumes *f*: *positive M f* **and** *a*: $a \in M$
and *A*: *range A* $\subseteq$ *lambda_system* Ω *M f*
and *disj*: *disjoint_family A*
shows $(\sum i = 0..<n.\ f\ (a \cap A\ i)) = f\ (a \cap (\bigcup i \in \{0..<n\}.\ A\ i))$
proof (*induct n*)
case *0* **show** *?case* **using** *f* **by** (*simp add*: *positive_def*)
next
case (*Suc n*)
have *2*: $A\ n \cap \bigcup\ (A\ `\ \{0..<n\}) = \{\}$ **using** *disj*
by (*force simp add*: *disjoint_family_on_def neq_iff*)
have *3*: $A\ n \in$ *lambda_system* Ω *M f* **using** *A*
by *blast*
interpret *l*: *algebra* Ω *lambda_system* Ω *M f*
using *f* **by** (*rule lambda_system_algebra*)
have *4*: $\bigcup\ (A\ `\ \{0..<n\}) \in$ *lambda_system* Ω *M f*
using *A l.UNION_in_sets* **by** *simp*
from *Suc.hyps* **show** *?case*
by (*simp add*: *atLeastLessThanSuc lambda_system_strong_additive* [*OF a 2 3 4*])
qed

proposition (**in** *sigma_algebra*) *lambda_system_caratheodory*:
assumes *oms*: *outer_measure_space M f*
and *A*: *range A* $\subseteq$ *lambda_system* Ω *M f*
and *disj*: *disjoint_family A*
shows $(\bigcup i.\ A\ i) \in$ *lambda_system* Ω *M f* $\wedge$ $(\sum i.\ f\ (A\ i)) = f\ (\bigcup i.\ A\ i)$
proof −
have *pos*: *positive M f* **and** *inc*: *increasing M f*
and *csa*: *countably_subadditive M f*
by (*metis oms outer_measure_space_def*)+
have *sa*: *subadditive M f*
by (*metis countably_subadditive_subadditive csa pos*)
have *A'*: $\bigwedge S.\ A\ `S \subseteq$ (*lambda_system* Ω *M f*) **using** *A*
by *auto*
interpret *ls*: *algebra* Ω *lambda_system* Ω *M f*
using *pos* **by** (*rule lambda_system_algebra*)

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  have A'': range A ⊆ M
    by (metis A image_subset_iff lambda_system_sets)

  have U_in: (⋃ i. A i) ∈ M
    by (metis A'' countable_UN)
  have U_eq: f (⋃ i. A i) = (∑ i. f (A i))
  proof (rule antisym)
    show f (⋃ i. A i) ≤ (∑ i. f (A i))
      using csa[unfolded countably_subadditive_def] A'' disj U_in by auto
   have dis: ⋀N. disjoint_family_on A {..<N} by (intro disjoint_family_on_mono[OF
_ disj]) auto
    show (∑ i. f (A i)) ≤ f (⋃ i. A i)
     using ls.additive_sum [OF lambda_system_positive[OF pos] lambda_system_additive
_ A' dis] A''
      by (intro suminf_le_const[OF summableI]) (auto intro!: increasingD[OF inc]
countable_UN)
  qed
  have f (a ∩ (⋃ i. A i)) + f (a − (⋃ i. A i)) = f a
    if a [iff]: a ∈ M for a
  proof (rule antisym)
    have range (λi. a ∩ A i) ⊆ M using A''
      by blast
    moreover
    have disjoint_family (λi. a ∩ A i) using disj
      by (auto simp add: disjoint_family_on_def)
    moreover
    have a ∩ (⋃ i. A i) ∈ M
      by (metis Int U_in a)
    ultimately
    have f (a ∩ (⋃ i. A i)) ≤ (∑ i. f (a ∩ A i))
     using csa[unfolded countably_subadditive_def, rule_format, of (λi. a ∩ A i)]
      by (simp add: o_def)
    hence f (a ∩ (⋃ i. A i)) + f (a − (⋃ i. A i)) ≤ (∑ i. f (a ∩ A i)) + f (a −
(⋃ i. A i))
      by (rule add_right_mono)
    also have ... ≤ f a
    proof (intro ennreal_suminf_bound_add)
      fix n
      have UNION_in: (⋃ i∈{0..<n}. A i) ∈ M
        by (metis A'' UNION_in_sets)
      have le_fa: f (⋃ (A ` {0..<n}) ∩ a) ≤ f a using A''
        by (blast intro: increasingD [OF inc] A'' UNION_in_sets)
      have ls: (⋃ i∈{0..<n}. A i) ∈ lambda_system Ω M f
        using ls.UNION_in_sets by (simp add: A)
       hence eq_fa: f a = f (a ∩ (⋃ i∈{0..<n}. A i)) + f (a − (⋃ i∈{0..<n}. A
i))
        by (simp add: lambda_system_eq UNION_in)
      have f (a − (⋃ i. A i)) ≤ f (a − (⋃ i∈{0..<n}. A i))
        by (blast intro: increasingD [OF inc] UNION_in U_in)
```

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      thus (∑ i<n. f (a ∩ A i)) + f (a − (⋃ i. A i)) ≤ f a
       by (simp add: lambda_system_strong_sum pos A disj eq_fa add_left_mono
atLeast0LessThan[symmetric])
    qed
    finally show f (a ∩ (⋃ i. A i)) + f (a − (⋃ i. A i)) ≤ f a
      by simp
  next
    have f a ≤ f (a ∩ (⋃ i. A i) ∪ (a − (⋃ i. A i)))
      by (blast intro: increasingD [OF inc] U_in)
    also have ... ≤ f (a ∩ (⋃ i. A i)) + f (a − (⋃ i. A i))
      by (blast intro: subadditiveD [OF sa] U_in)
    finally show f a ≤ f (a ∩ (⋃ i. A i)) + f (a − (⋃ i. A i)) .
  qed
  thus ?thesis
    by (simp add: lambda_system_eq sums_iff U_eq U_in)
qed

proposition (in sigma_algebra) caratheodory_lemma:
  assumes oms: outer_measure_space M f
  defines L ≡ lambda_system Ω M f
  shows measure_space Ω L f
proof −
  have pos: positive M f
    by (metis oms outer_measure_space_def)
  have alg: algebra Ω L
    using lambda_system_algebra [of f, OF pos]
    by (simp add: algebra_iff_Un L_def)
  then
  have sigma_algebra Ω L
    using lambda_system_caratheodory [OF oms]
    by (simp add: sigma_algebra_disjoint_iff L_def)
  moreover
  have countably_additive L f positive L f
    using pos lambda_system_caratheodory [OF oms]
     by (auto simp add: lambda_system_sets L_def countably_additive_def posi-
tive_def)
  ultimately
  show ?thesis
    using pos by (simp add: measure_space_def)
qed

definition outer_measure :: 'a set set ⇒ ('a set ⇒ ennreal) ⇒ 'a set ⇒ ennreal
where
   outer_measure M f X =
     (INF A∈{A. range A ⊆ M ∧ disjoint_family A ∧ X ⊆ (⋃ i. A i)}. ∑ i. f (A
i))

lemma (in ring_of_sets) outer_measure_agrees:
  assumes posf: positive M f and ca: countably_additive M f and s: s ∈ M
```

shows *outer_measure M f s = f s*
unfolding *outer_measure_def*
proof (*safe intro*!: *antisym INF_greatest*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *range A* ⊆ *M* **and** *dA*: *disjoint_family A* **and** *sA*: *s* ⊆ (⋃ *x*. *A x*)
have *inc*: *increasing M f*
by (*metis additive_increasing ca countably_additive_additive posf*)
have *f s* = *f* (⋃ *i*. *A i* ∩ *s*)
using *sA* **by** (*auto simp*: *Int_absorb1*)
also have ... = (∑ *i*. *f* (*A i* ∩ *s*))
using *sA dA A s*
by (*intro ca*[*unfolded countably_additive_def*, *rule_format*, *symmetric*])
(*auto simp*: *Int_absorb1 disjoint_family_on_def*)
also have ... ≤ (∑ *i*. *f* (*A i*))
using *A s* **by** (*auto intro*!: *suminf_le increasingD*[*OF inc*])
finally show *f s* ≤ (∑ *i*. *f* (*A i*)) **.**
next
have (∑ *i*. *f* (*if i* = *0 then s else* {})) ≤ *f s*
using *positiveD1*[*OF posf*] **by** (*subst suminf_finite*[*of* {*0*}]) *auto*
with *s* **show** (*INF A*∈{*A*. *range A* ⊆ *M* ∧ *disjoint_family A* ∧ *s* ⊆ ⋃(*A* ‘ *UNIV*)}. ∑ *i*. *f* (*A i*)) ≤ *f s*
by (*intro INF_lower2*[*of* λ*i*. *if i* = *0 then s else* {}])
(*auto simp*: *disjoint_family_on_def*)
qed

lemma *outer_measure_empty*:
positive M f ⟹ {} ∈ *M* ⟹ *outer_measure M f* {} = *0*
unfolding *outer_measure_def*
by (*intro antisym INF_lower2*[*of* λ_. {}]) (*auto simp*: *disjoint_family_on_def positive_def*)

lemma (**in** *ring_of_sets*) *positive_outer_measure*:
assumes *positive M f* **shows** *positive* (*Pow* Ω) (*outer_measure M f*)
unfolding *positive_def* **by** (*auto simp*: *assms outer_measure_empty*)

lemma (**in** *ring_of_sets*) *increasing_outer_measure*: *increasing* (*Pow* Ω) (*outer_measure M f*)
by (*force simp*: *increasing_def outer_measure_def intro*!: *INF_greatest intro*: *INF_lower*)

lemma (**in** *ring_of_sets*) *outer_measure_le*:
assumes *pos*: *positive M f* **and** *inc*: *increasing M f* **and** *A*: *range A* ⊆ *M* **and** *X*: *X* ⊆ (⋃ *i*. *A i*)
shows *outer_measure M f X* ≤ (∑ *i*. *f* (*A i*))
unfolding *outer_measure_def*
proof (*safe intro*!: *INF_lower2*[*of disjointed A*] *del*: *subsetI*)
show *dA*: *range* (*disjointed A*) ⊆ *M*
by (*auto intro*!: *A range_disjointed_sets*)
have ∀ *n*. *f* (*disjointed A n*) ≤ *f* (*A n*)

by (*metis increasingD* [*OF inc*] *UNIV_I dA image_subset_iff disjointed_subset A*)
then show $(\sum i.\ f\ (disjointed\ A\ i)) \leq (\sum i.\ f\ (A\ i))$
by (*blast intro*!: *suminf_le*)
qed (*auto simp*: *X UN_disjointed_eq disjoint_family_disjointed*)

lemma (**in** *ring_of_sets*) *outer_measure_close*:
outer_measure M f X $< e \Longrightarrow \exists A.$ *range A* $\subseteq M \wedge$ *disjoint_family A* $\wedge X \subseteq (\bigcup i.\ A\ i) \wedge (\sum i.\ f\ (A\ i)) < e$
unfolding *outer_measure_def INF_less_iff* **by** *auto*

lemma (**in** *ring_of_sets*) *countably_subadditive_outer_measure*:
assumes *posf*: *positive M f* **and** *inc*: *increasing M f*
shows *countably_subadditive* (*Pow* Ω) (*outer_measure M f*)
proof (*simp add*: *countably_subadditive_def*, *safe*)
fix *A* :: *nat* $\Rightarrow$ _ **assume** *A*: *range A* $\subseteq$ *Pow* (Ω) **and** *sb*: $(\bigcup i.\ A\ i) \subseteq \Omega$
let *?O* = *outer_measure M f*
show *?O* $(\bigcup i.\ A\ i) \leq (\sum n.$ *?O* $(A\ n))$
proof (*rule ennreal_le_epsilon*)
fix *b* **and** *e* :: *real* **assume** $0 < e$ $(\sum n.$ *outer_measure M f* $(A\ n)) <$ *top*
then have $*$: $\bigwedge n.$ *outer_measure M f* $(A\ n) <$ *outer_measure M f* $(A\ n) + e * (1/2)\hat{}Suc\ n$
by (*auto simp add*: *less_top dest*!: *ennreal_suminf_lessD*)
obtain *B*
where *B*: $\bigwedge n.$ *range* $(B\ n) \subseteq M$
and *sbB*: $\bigwedge n.\ A\ n \subseteq (\bigcup i.\ B\ n\ i)$
and *Ble*: $\bigwedge n.\ (\sum i.\ f\ (B\ n\ i)) \leq$ *?O* $(A\ n) + e * (1/2)\hat{}(Suc\ n)$
by (*metis less_imp_le outer_measure_close*[*OF* $*$])

define *C* **where** *C* = *case_prod B o prod_decode*
from *B* **have** *B_in_M*: $\bigwedge i\ j.\ B\ i\ j \in M$
by (*rule range_subsetD*)
then have *C*: *range C* $\subseteq M$
by (*auto simp add*: *C_def split_def*)
have *A_C*: $(\bigcup i.\ A\ i) \subseteq (\bigcup i.\ C\ i)$
using *sbB* **by** (*auto simp add*: *C_def subset_eq*) (*metis prod.case prod_encode_inverse*)

have *?O* $(\bigcup i.\ A\ i) \leq$ *?O* $(\bigcup i.\ C\ i)$
using *A_C A C* **by** (*intro increasing_outer_measure*[*THEN increasingD*]) (*auto dest*!: *sets_into_space*)
also have ... $\leq (\sum i.\ f\ (C\ i))$
using *C* **by** (*intro outer_measure_le*[*OF posf inc*]) *auto*
also have ... $= (\sum n.\ \sum i.\ f\ (B\ n\ i))$
using *B_in_M* **unfolding** *C_def comp_def* **by** (*intro suminf_ennreal_2dimen*) *auto*
also have ... $\leq (\sum n.$ *?O* $(A\ n) + e * (1/2)\ \hat{}\ Suc\ n)$
using *B_in_M* **by** (*intro suminf_le suminf_nonneg allI Ble*) *auto*
also have ... $= (\sum n.$ *?O* $(A\ n)) + (\sum n.$ *ennreal e* $*$ *ennreal* $((1/2)\ \hat{}\ Suc\ n))$
using ‹$0 < e$› **by** (*subst suminf_add*[*symmetric*])

(*auto simp del*: *ennreal_suminf_cmult simp add*: *ennreal_mult*[*symmetric*])
also have ... = ($\sum$ *n*. *?O* (*A n*)) + *e*
unfolding *ennreal_suminf_cmult*
by (*subst suminf_ennreal_eq*[*OF zero_le_power power_half_series*]) *auto*
finally show *?O* ($\bigcup$ *i*. *A i*) ≤ ($\sum$ *n*. *?O* (*A n*)) + *e* **.**
qed
qed

lemma (**in** *ring_of_sets*) *outer_measure_space_outer_measure*:
positive M f ⟹ *increasing M f* ⟹ *outer_measure_space* (*Pow* Ω) (*outer_measure M f*)
by (*simp add*: *outer_measure_space_def*
positive_outer_measure increasing_outer_measure countably_subadditive_outer_measure)

lemma (**in** *ring_of_sets*) *algebra_subset_lambda_system*:
assumes *posf*: *positive M f* **and** *inc*: *increasing M f*
and *add*: *additive M f*
shows *M* ⊆ *lambda_system* Ω (*Pow* Ω) (*outer_measure M f*)
proof (*auto dest*: *sets_into_space*
simp add: *algebra.lambda_system_eq* [*OF algebra_Pow*])
fix *x s* **assume** *x*: *x* ∈ *M* **and** *s*: *s* ⊆ Ω
have [*simp*]: ⋀*x*. *x* ∈ *M* ⟹ *s* ∩ (Ω − *x*) = *s* − *x* **using** *s*
by *blast*
have *outer_measure M f* (*s* ∩ *x*) + *outer_measure M f* (*s* − *x*) ≤ *outer_measure M f s*
unfolding *outer_measure_def*[*of M f s*]
proof (*safe intro*!: *INF_greatest*)
fix *A* :: *nat* ⇒ *'a set* **assume** *A*: *disjoint_family A range A* ⊆ *M s* ⊆ ($\bigcup$ *i*. *A i*)
have *outer_measure M f* (*s* ∩ *x*) ≤ ($\sum$ *i*. *f* (*A i* ∩ *x*))
unfolding *outer_measure_def*
proof (*safe intro*!: *INF_lower2*[*of* λ*i*. *A i* ∩ *x*])
from *A*(*1*) **show** *disjoint_family* (λ*i*. *A i* ∩ *x*)
by (*rule disjoint_family_on_bisimulation*) *auto*
qed (*insert x A*, *auto*)
moreover
have *outer_measure M f* (*s* − *x*) ≤ ($\sum$ *i*. *f* (*A i* − *x*))
unfolding *outer_measure_def*
proof (*safe intro*!: *INF_lower2*[*of* λ*i*. *A i* − *x*])
from *A*(*1*) **show** *disjoint_family* (λ*i*. *A i* − *x*)
by (*rule disjoint_family_on_bisimulation*) *auto*
qed (*insert x A*, *auto*)
ultimately have *outer_measure M f* (*s* ∩ *x*) + *outer_measure M f* (*s* − *x*) ≤
($\sum$ *i*. *f* (*A i* ∩ *x*)) + ($\sum$ *i*. *f* (*A i* − *x*)) **by** (*rule add_mono*)
also have ... = ($\sum$ *i*. *f* (*A i* ∩ *x*) + *f* (*A i* − *x*))
using *A*(*2*) *x posf* **by** (*subst suminf_add*) (*auto simp*: *positive_def*)
also have ... = ($\sum$ *i*. *f* (*A i*))
using *A x*

by (*subst add*[*THEN additiveD*, *symmetric*])
(*auto intro*!: *arg_cong*[**where** *f*=*suminf*] *arg_cong*[**where** *f*=*f*])
finally show *outer_measure M f* (*s* ∩ *x*) + *outer_measure M f* (*s* − *x*) ≤ (∑ *i*. *f* (*A i*)) **.**
qed
moreover
have *outer_measure M f s* ≤ *outer_measure M f* (*s* ∩ *x*) + *outer_measure M f* (*s* − *x*)
proof −
have *outer_measure M f s* = *outer_measure M f* ((*s* ∩ *x*) ∪ (*s* − *x*))
by (*metis Un_Diff_Int Un_commute*)
also have ... ≤ *outer_measure M f* (*s* ∩ *x*) + *outer_measure M f* (*s* − *x*)
apply (*rule subadditiveD*)
apply (*rule ring_of_sets.countably_subadditive_subadditive* [*OF ring_of_sets_Pow*])
apply (*simp add*: *positive_def outer_measure_empty*[*OF posf*])
apply (*rule countably_subadditive_outer_measure*)
using *s* **by** (*auto intro*!: *posf inc*)
finally show *?thesis* **.**
qed
ultimately
show *outer_measure M f* (*s* ∩ *x*) + *outer_measure M f* (*s* − *x*) = *outer_measure M f s*
by (*rule order_antisym*)
qed

lemma *measure_down*: *measure_space* Ω *N* μ ⟹ *sigma_algebra* Ω *M* ⟹ *M* ⊆ *N* ⟹ *measure_space* Ω *M* μ
by (*auto simp add*: *measure_space_def positive_def countably_additive_def subset_eq*)

7.8.2 Caratheodory's theorem

theorem (**in** *ring_of_sets*) *caratheodory'*:
assumes *posf*: *positive M f* **and** *ca*: *countably_additive M f*
shows ∃μ :: *'a set* ⇒ *ennreal*. (∀ *s* ∈ *M*. μ *s* = *f s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) μ
proof −
have *inc*: *increasing M f*
by (*metis additive_increasing ca countably_additive_additive posf*)
let *?O* = *outer_measure M f*
define *ls* **where** *ls* = *lambda_system* Ω (*Pow* Ω) *?O*
have *mls*: *measure_space* Ω *ls ?O*
using *sigma_algebra.caratheodory_lemma*
[*OF sigma_algebra_Pow outer_measure_space_outer_measure* [*OF posf inc*]]
by (*simp add*: *ls_def*)
hence *sls*: *sigma_algebra* Ω *ls*
by (*simp add*: *measure_space_def*)
have *M* ⊆ *ls*

by (*simp add*: *ls_def*)
(*metis ca posf inc countably_additive_additive algebra_subset_lambda_system*)
hence *sgs_sb*: *sigma_sets* (Ω) (*M*) ⊆ *ls*
using *sigma_algebra.sigma_sets_subset* [*OF sls*, *of M*]
by *simp*
have *measure_space* Ω (*sigma_sets* Ω *M*) *?O*
by (*rule measure_down* [*OF mls*], *rule sigma_algebra_sigma_sets*)
(*simp_all add*: *sgs_sb space_closed*)
thus *?thesis* **using** *outer_measure_agrees* [*OF posf ca*]
by (*intro exI*[*of* _ *?O*]) *auto*
qed

lemma (**in** *ring_of_sets*) *caratheodory_empty_continuous*:
assumes *f*: *positive M f additive M f* **and** *fin*: ⋀*A*. *A* ∈ *M* ⟹ *f A* ≠ ∞
assumes *cont*: ⋀*A*. *range A* ⊆ *M* ⟹ *decseq A* ⟹ (⋂ *i*. *A i*) = {} ⟹ (λ*i*. *f* (*A i*)) ⟶ *0*
shows ∃ μ :: *'a set* ⇒ *ennreal*. (∀ *s* ∈ *M*. μ *s* = *f s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) μ
proof (*intro caratheodory' empty_continuous_imp_countably_additive f*)
show ∀ *A*∈*M*. *f A* ≠ ∞ **using** *fin* **by** *auto*
qed (*rule cont*)

7.8.3 Volumes

definition *volume* :: *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *bool* **where**
volume M f ⟷
(*f* {} = *0*) ∧ (∀ *a*∈*M*. *0* ≤ *f a*) ∧
(∀ *C*⊆*M*. *disjoint C* ⟶ *finite C* ⟶ ⋃ *C* ∈ *M* ⟶ *f* (⋃ *C*) = (∑ *c*∈*C*. *f c*))

lemma *volumeI*:
assumes *f* {} = *0*
assumes ⋀*a*. *a* ∈ *M* ⟹ *0* ≤ *f a*
assumes ⋀*C*. *C* ⊆ *M* ⟹ *disjoint C* ⟹ *finite C* ⟹ ⋃ *C* ∈ *M* ⟹ *f* (⋃ *C*) = (∑ *c*∈*C*. *f c*)
shows *volume M f*
using *assms* **by** (*auto simp*: *volume_def*)

lemma *volume_positive*:
volume M f ⟹ *a* ∈ *M* ⟹ *0* ≤ *f a*
by (*auto simp*: *volume_def*)

lemma *volume_empty*:
volume M f ⟹ *f* {} = *0*
by (*auto simp*: *volume_def*)

proposition *volume_finite_additive*:
assumes *volume M f*
assumes *A*: ⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *M disjoint_family_on A I finite I* ⋃ (*A* ‘ *I*) ∈ *M*

shows $f\ (\bigcup (A\ `\ I)) = (\sum i{\in}I.\ f\ (A\ i))$
proof −
have $A`I \subseteq M$ *disjoint* $(A`I)$ *finite* $(A`I)$ $\bigcup (A`I) \in M$
using A **by** (*auto simp*: *disjoint_family_on_disjoint_image*)
with ‹*volume* $M\ f$› **have** $f\ (\bigcup (A`I)) = (\sum a{\in}A`I.\ f\ a)$
unfolding *volume_def* **by** *blast*
also have $\ldots = (\sum i{\in}I.\ f\ (A\ i))$
proof (*subst sum.reindex_nontrivial*)
fix $i\ j$ **assume** $i \in I\ j \in I\ i \neq j\ A\ i = A\ j$
with ‹*disjoint_family_on* $A\ I$› **have** $A\ i = \{\}$
by (*auto simp*: *disjoint_family_on_def*)
then show $f\ (A\ i) = 0$
using *volume_empty*[*OF* ‹*volume* $M\ f$›] **by** *simp*
qed (*auto intro*: ‹*finite* I›)
finally show $f\ (\bigcup (A\ `\ I)) = (\sum i{\in}I.\ f\ (A\ i))$
by *simp*
qed

lemma (**in** *ring_of_sets*) *volume_additiveI*:
assumes *pos*: $\bigwedge a.\ a \in M \Longrightarrow 0 \leq \mu\ a$
assumes [*simp*]: $\mu\ \{\} = 0$
assumes *add*: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cap b = \{\} \Longrightarrow \mu\ (a \cup b) = \mu\ a + \mu\ b$
shows *volume* $M\ \mu$
proof (*unfold volume_def*, *safe*)
fix C **assume** *finite* $C\ C \subseteq M$ *disjoint* C
then show $\mu\ (\bigcup C) = sum\ \mu\ C$
proof (*induct* C)
case (*insert* $c\ C$)
from *insert*(*1*,*2*,*4*,*5*) **have** $\mu\ (\bigcup (insert\ c\ C)) = \mu\ c + \mu\ (\bigcup C)$
by (*auto intro*!: *add simp*: *disjoint_def*)
with *insert* **show** *?case*
by (*simp add*: *disjoint_def*)
qed *simp*
qed *fact*+

proposition (**in** *semiring_of_sets*) *extend_volume*:
assumes *volume* $M\ \mu$
shows $\exists \mu'.$ *volume generated_ring* $\mu' \wedge (\forall a{\in}M.\ \mu'\ a = \mu\ a)$
proof −
let *?R* = *generated_ring*
have $\forall a{\in}?R.\ \exists m.\ \exists C{\subseteq}M.\ a = \bigcup C \wedge$ *finite* $C \wedge$ *disjoint* $C \wedge m = (\sum c{\in}C.\ \mu\ c)$
by (*auto simp*: *generated_ring_def*)
from *bchoice*[*OF this*] **obtain** μ'
where μ'*_spec*: $\forall x{\in}$*generated_ring*. $\exists C{\subseteq}M.\ x = \bigcup\ C \wedge$ *finite* $C \wedge$ *disjoint* $C \wedge \mu'\ x = sum\ \mu\ C$
by *blast*
{ fix C **assume** C: $C \subseteq M$ *finite* C *disjoint* C

```
  fix D assume D: D ⊆ M finite D disjoint D
  assume ⋃ C = ⋃ D
  have (∑ d∈D. μ d) = (∑ d∈D. ∑ c∈C. μ (c ∩ d))
  proof (intro sum.cong refl)
    fix d assume d ∈ D
    have Un_eq_d: (⋃ c∈C. c ∩ d) = d
      using ‹d ∈ D› ‹⋃ C = ⋃ D› by auto
    moreover have μ (⋃ c∈C. c ∩ d) = (∑ c∈C. μ (c ∩ d))
    proof (rule volume_finite_additive)
      { fix c assume c ∈ C then show c ∩ d ∈ M
          using C D ‹d ∈ D› by auto }
      show (⋃ a∈C. a ∩ d) ∈ M
        unfolding Un_eq_d using ‹d ∈ D› D by auto
      show disjoint_family_on (λa. a ∩ d) C
        using ‹disjoint C› by (auto simp: disjoint_family_on_def disjoint_def)
    qed fact+
    ultimately show μ d = (∑ c∈C. μ (c ∩ d)) by simp
  qed }
note split_sum = this

{ fix C assume C: C ⊆ M finite C disjoint C
  fix D assume D: D ⊆ M finite D disjoint D
  assume ⋃ C = ⋃ D
  with split_sum[OF C D] split_sum[OF D C]
  have (∑ d∈D. μ d) = (∑ c∈C. μ c)
    by (simp, subst sum.swap, simp add: ac_simps) }
note sum_eq = this

{ fix C assume C: C ⊆ M finite C disjoint C
  then have ⋃ C ∈ ?R by (auto simp: generated_ring_def)
  with μ'_spec[THEN bspec, of ⋃ C]
  obtain D where
    D: D ⊆ M finite D disjoint D ⋃ C = ⋃ D and μ' (⋃ C) = (∑ d∈D. μ d)
    by auto
  with sum_eq[OF C D] have μ' (⋃ C) = (∑ c∈C. μ c) by simp }
note μ' = this

show ?thesis
proof (intro exI conjI ring_of_sets.volume_additiveI[OF generating_ring] ballI)
  fix a assume a ∈ M with μ'[of {a}] show μ' a = μ a
    by (simp add: disjoint_def)
next
  fix a assume a ∈ ?R
  then obtain Ca where Ca: finite Ca disjoint Ca Ca ⊆ M a = ⋃ Ca ..
  with μ'[of Ca] ‹volume M μ›[THEN volume_positive]
  show 0 ≤ μ' a
    by (auto intro!: sum_nonneg)
next
  show μ' {} = 0 using μ'[of {}] by auto
```

```
  next
    fix a b assume a ∈ ?R b ∈ ?R
    then obtain Ca Cb
      where Ca: finite Ca disjoint Ca Ca ⊆ M a = ⋃ Ca
        and Cb: finite Cb disjoint Cb Cb ⊆ M b = ⋃ Cb
      by (meson generated_ringE)
    assume a ∩ b = {}
    with Ca Cb have Ca ∩ Cb ⊆ {{}} by auto
    then have C_Int_cases: Ca ∩ Cb = {{}} ∨ Ca ∩ Cb = {} by auto

    from ‹a ∩ b = {}› have μ' (⋃(Ca ∪ Cb)) = (∑ c∈Ca ∪ Cb. μ c)
      using Ca Cb by (intro μ') (auto intro!: disjoint_union)
    also have ... = (∑ c∈Ca ∪ Cb. μ c) + (∑ c∈Ca ∩ Cb. μ c)
        using C_Int_cases volume_empty[OF ‹volume M μ›] by (elim disjE)
simp_all
    also have ... = (∑ c∈Ca. μ c) + (∑ c∈Cb. μ c)
      using Ca Cb by (simp add: sum.union_inter)
    also have ... = μ' a + μ' b
      using Ca Cb by (simp add: μ')
    finally show μ' (a ∪ b) = μ' a + μ' b
      using Ca Cb by simp
  qed
qed
```

Caratheodory on semirings

```
theorem (in semiring_of_sets) caratheodory:
  assumes pos: positive M μ and ca: countably_additive M μ
   shows ∃μ' :: 'a set ⇒ ennreal. (∀ s ∈ M. μ' s = μ s) ∧ measure_space Ω
(sigma_sets Ω M) μ'
proof −
  have volume M μ
  proof (rule volumeI)
    { fix a assume a ∈ M then show 0 ≤ μ a
       using pos unfolding positive_def by auto }
    note p = this

    fix C assume sets_C: C ⊆ M ⋃ C ∈ M and disjoint C finite C
    have ∃ F'. bij_betw F' {..<card C} C
      by (rule finite_same_card_bij[OF _ ‹finite C›]) auto
    then obtain F' where bij_betw F' {..<card C} C ..
    then have F': C = F' ‘ {..< card C} inj_on F' {..< card C}
      by (auto simp: bij_betw_def)
    { fix i j assume ∗: i < card C j < card C i ≠ j
      with F' have F' i ∈ C F' j ∈ C F' i ≠ F' j
        unfolding inj_on_def by auto
      with ‹disjoint C›[THEN disjointD]
      have F' i ∩ F' j = {}
        by auto }
```

```
    note F'_disj = this
    define F where F i = (if i < card C then F' i else {}) for i
    then have disjoint_family F
      using F'_disj by (auto simp: disjoint_family_on_def)
    moreover from F' have (⋃ i. F i) = ⋃ C
      by (auto simp add: F_def split: if_split_asm cong del: SUP_cong)
    moreover have sets_F: ⋀i. F i ∈ M
      using F' sets_C by (auto simp: F_def)
    moreover note sets_C
    ultimately have μ (⋃ C) = (∑ i. μ (F i))
      using ca[unfolded countably_additive_def, THEN spec, of F] by auto
    also have ... = (∑ i<card C. μ (F' i))
    proof −
      have (λi. if i ∈ {..< card C} then μ (F' i) else 0) sums (∑ i<card C. μ (F' i))
        by (rule sums_If_finite_set) auto
      also have (λi. if i ∈ {..< card C} then μ (F' i) else 0) = (λi. μ (F i))
        using pos by (auto simp: positive_def F_def)
      finally show (∑ i. μ (F i)) = (∑ i<card C. μ (F' i))
        by (simp add: sums_iff)
    qed
    also have ... = (∑ c∈C. μ c)
      using F'(2) by (subst (2) F') (simp add: sum.reindex)
    finally show μ (⋃ C) = (∑ c∈C. μ c) .
  next
    show μ {} = 0
      using ‹positive M μ› by (rule positiveD1)
  qed
  from extend_volume[OF this] obtain μ_r where
    V: volume generated_ring μ_r ⋀a. a ∈ M ⟹ μ a = μ_r a
    by auto

  interpret G: ring_of_sets Ω generated_ring
    by (rule generating_ring)

  have pos: positive generated_ring μ_r
      using V unfolding positive_def by (auto simp: positive_def intro!: volume_positive volume_empty)

  have countably_additive generated_ring μ_r
  proof (rule countably_additiveI)
    fix A' :: nat ⇒ 'a set
    assume A': range A' ⊆ generated_ring disjoint_family A'
      and Un_A: (⋃ i. A' i) ∈ generated_ring

    obtain C' where C': finite C' disjoint C' C' ⊆ M ⋃ (range A') = ⋃ C'
      using generated_ringE[OF Un_A] by auto

    { fix c assume c ∈ C'
```

moreover define A **where** [*abs_def*]: $A\ i = A'\ i \cap c$ **for** i
ultimately have A: *range* $A \subseteq$ *generated_ring disjoint_family* A
and *Un_A*: $(\bigcup i.\ A\ i) \in$ *generated_ring*
using $A'\ C'$
by (*auto intro*!: *G.Int G.finite_Union intro*: *generated_ringI_Basic simp*: *disjoint_family_on_def*)
from $A\ C'$ ‹$c \in C'$› **have** *UN_eq*: $(\bigcup i.\ A\ i) = c$
by (*auto simp*: *A_def*)

have $\forall i{::}nat.\ \exists f{::}nat \Rightarrow 'a\ set.\ \mu\_r\ (A\ i) = (\sum j.\ \mu\_r\ (f\ j)) \wedge$ *disjoint_family* $f \wedge \bigcup(range\ f) = A\ i \wedge (\forall j.\ f\ j \in M)$
(**is** $\forall i.\ ?P\ i$)
proof
fix i
from A **have** *Ai*: $A\ i \in$ *generated_ring* **by** *auto*
from *generated_ringE*[*OF this*] **obtain** C
where C: *finite* C *disjoint* C $C \subseteq M$ $A\ i = \bigcup\ C$ **by** *auto*
have $\exists F'.$ *bij_betw* F' $\{..<card\ C\}$ C
by (*rule finite_same_card_bij*[*OF* _ ‹*finite* C›]) *auto*
then obtain F **where** F: *bij_betw* F $\{..<card\ C\}$ C **..**
define f **where** [*abs_def*]: $f\ i = (if\ i < card\ C\ then\ F\ i\ else\ \{\})$ **for** i
then have f: *bij_betw* f $\{..<card\ C\}$ C
by (*intro bij_betw_cong*[*THEN iffD1*, *OF* _ F]) *auto*
with C **have** $\forall j.\ f\ j \in M$
by (*auto simp*: *Pi_iff f_def dest*!: *bij_betw_imp_funcset*)
moreover
from f C **have** *d_f*: *disjoint_family_on* f $\{..<card\ C\}$
by (*intro disjoint_image_disjoint_family_on*) (*auto simp*: *bij_betw_def*)
then have *disjoint_family* f
by (*auto simp*: *disjoint_family_on_def f_def*)
moreover
have *Ai_eq*: $A\ i = (\bigcup x{<}card\ C.\ f\ x)$
using f C *Ai* **unfolding** *bij_betw_def* **by** *auto*
then have $\bigcup(range\ f) = A\ i$
using f **by** (*auto simp add*: *f_def*)
moreover
{ have $(\sum j.\ \mu\_r\ (f\ j)) = (\sum j.\ if\ j \in \{..<\ card\ C\}\ then\ \mu\_r\ (f\ j)\ else\ 0)$
using *volume_empty*[*OF V(1)*] **by** (*auto intro*!: *arg_cong*[**where** *f=suminf*] *simp*: *f_def*)
also have $\ldots = (\sum j{<}card\ C.\ \mu\_r\ (f\ j))$
by (*rule sums_If_finite_set*[*THEN sums_unique*, *symmetric*]) *simp*
also have $\ldots = \mu\_r\ (A\ i)$
using C f[*THEN bij_betw_imp_funcset*] **unfolding** *Ai_eq*
by (*intro volume_finite_additive*[*OF V(1)* _ *d_f*, *symmetric*])
(*auto simp*: *Pi_iff Ai_eq intro*: *generated_ringI_Basic*)
finally have $\mu\_r\ (A\ i) = (\sum j.\ \mu\_r\ (f\ j))$ **.. }**
ultimately show $?P\ i$
by *blast*
qed

```
    from choice[OF this] obtain f
      where f: ∀x. μ_r (A x) = (∑j. μ_r (f x j)) ∧ disjoint_family (f x) ∧ ⋃
(range (f x)) = A x ∧ (∀j. f x j ∈ M)
      ..
    then have UN_f_eq: (⋃i. case_prod f (prod_decode i)) = (⋃i. A i)
     unfolding UN_extend_simps surj_prod_decode by (auto simp: set_eq_iff)

    have d: disjoint_family (λi. case_prod f (prod_decode i))
      unfolding disjoint_family_on_def
    proof (intro ballI impI)
      fix m n :: nat assume m ≠ n
      then have neq: prod_decode m ≠ prod_decode n
        using inj_prod_decode[of UNIV] by (auto simp: inj_on_def)
      show case_prod f (prod_decode m) ∩ case_prod f (prod_decode n) = {}
      proof cases
        assume fst (prod_decode m) = fst (prod_decode n)
        then show ?thesis
          using neq f by (fastforce simp: disjoint_family_on_def)
      next
        assume neq: fst (prod_decode m) ≠ fst (prod_decode n)
        have case_prod f (prod_decode m) ⊆ A (fst (prod_decode m))
          case_prod f (prod_decode n) ⊆ A (fst (prod_decode n))
          using f[THEN spec, of fst (prod_decode m)]
          using f[THEN spec, of fst (prod_decode n)]
          by (auto simp: set_eq_iff)
        with f A neq show ?thesis
          by (fastforce simp: disjoint_family_on_def subset_eq set_eq_iff)
      qed
    qed
   from f have (∑n. μ_r (A n)) = (∑n. μ_r (case_prod f (prod_decode n)))
      by (intro suminf_ennreal_2dimen[symmetric] generated_ringI_Basic)
       (auto split: prod.split)
    also have ... = (∑n. μ (case_prod f (prod_decode n)))
      using f V(2) by (auto intro!: arg_cong[where f=suminf] split: prod.split)
    also have ... = μ (⋃i. case_prod f (prod_decode i))
      using f ‹c ∈ C'› C'
      by (intro ca[unfolded countably_additive_def, rule_format])
        (auto split: prod.split simp: UN_f_eq d UN_eq)
    finally have (∑n. μ_r (A' n ∩ c)) = μ c
      using UN_f_eq UN_eq by (simp add: A_def) }
  note eq = this

  have (∑n. μ_r (A' n)) = (∑n. ∑c∈C'. μ_r (A' n ∩ c))
    using C' A'
    by (subst volume_finite_additive[symmetric, OF V(1)])
      (auto simp: disjoint_def disjoint_family_on_def
           intro!: G.Int G.finite_Union arg_cong[where f=λX. suminf (λi. μ_r
(X i))] ext
            intro: generated_ringI_Basic)
```

```
    also have ... = (∑ c∈C'. ∑ n. μ_r (A' n ∩ c))
      using C' A'
     by (intro suminf_sum G.Int G.finite_Union) (auto intro: generated_ringI_Basic)
    also have ... = (∑ c∈C'. μ_r c)
      using eq V C' by (auto intro!: sum.cong)
    also have ... = μ_r (⋃ C')
      using C' Un_A
      by (subst volume_finite_additive[symmetric, OF V(1)])
         (auto simp: disjoint_family_on_def disjoint_def
               intro: generated_ringI_Basic)
    finally show (∑ n. μ_r (A' n)) = μ_r (⋃ i. A' i)
      using C' by simp
  qed
   obtain μ' where (∀ s∈generated_ring. μ' s = μ_r s) ∧ measure_space Ω
(sigma_sets Ω generated_ring) μ'
     using G.caratheodory'[OF pos ‹countably_additive generated_ring μ_r›] by
auto
  with V show ?thesis
    unfolding sigma_sets_generated_ring_eq
    by (intro exI[of _ μ']) (auto intro: generated_ringI_Basic)
qed

lemma extend_measure_caratheodory:
  fixes G :: 'i ⇒ 'a set
  assumes M: M = extend_measure Ω I G μ
  assumes i ∈ I
  assumes semiring_of_sets Ω (G ‘ I)
  assumes empty: ⋀i. i ∈ I ⟹ G i = {} ⟹ μ i = 0
  assumes inj: ⋀i j. i ∈ I ⟹ j ∈ I ⟹ G i = G j ⟹ μ i = μ j
  assumes nonneg: ⋀i. i ∈ I ⟹ 0 ≤ μ i
  assumes add: ⋀A::nat ⇒ 'i. ⋀j. A ∈ UNIV → I ⟹ j ∈ I ⟹ disjoint_family
(G ∘ A) ⟹
    (⋃ i. G (A i)) = G j ⟹ (∑ n. μ (A n)) = μ j
  shows emeasure M (G i) = μ i

proof −
  interpret semiring_of_sets Ω G ‘ I
    by fact
  have ∀ g∈G‘I. ∃ i∈I. g = G i
    by auto
  then obtain sel where sel: ⋀g. g ∈ G ‘ I ⟹ sel g ∈ I ⋀g. g ∈ G ‘ I ⟹ G
(sel g) = g
    by metis

  have ∃ μ'. (∀ s∈G ‘ I. μ' s = μ (sel s)) ∧ measure_space Ω (sigma_sets Ω (G ‘
I)) μ'
  proof (rule caratheodory)
    show positive (G ‘ I) (λs. μ (sel s))
      by (auto simp: positive_def intro!: empty sel nonneg)
```

```
    show countably_additive (G ‘ I) (λs. μ (sel s))
    proof (rule countably_additiveI)
      fix A :: nat ⇒ 'a set assume range A ⊆ G ‘ I disjoint_family A (⋃i. A i)
∈ G ‘ I
      then show (∑i. μ (sel (A i))) = μ (sel (⋃i. A i))
        by (intro add) (auto simp: sel image_subset_iff_funcset comp_def Pi_iff
intro!: sel)
    qed
  qed
  then obtain μ' where μ': ∀ s∈G ‘ I. μ' s = μ (sel s) measure_space Ω (sigma_sets
Ω (G ‘ I)) μ'
    by metis

  show ?thesis
  proof (rule emeasure_extend_measure[OF M])
    { fix i assume i ∈ I then show μ' (G i) = μ i
      using μ' by (auto intro!: inj sel) }
    show G ‘ I ⊆ Pow Ω
      by (rule space_closed)
    then show positive (sets M) μ' countably_additive (sets M) μ'
      using μ' by (simp_all add: M sets_extend_measure measure_space_def)
  qed fact
qed

proposition extend_measure_caratheodory_pair:
  fixes G :: 'i ⇒ 'j ⇒ 'a set
  assumes M: M = extend_measure Ω {(a, b). P a b} (λ(a, b). G a b) (λ(a, b).
μ a b)
  assumes P i j
  assumes semiring: semiring_of_sets Ω {G a b | a b. P a b}
  assumes empty: ⋀i j. P i j ⟹ G i j = {} ⟹ μ i j = 0
  assumes inj: ⋀i j k l. P i j ⟹ P k l ⟹ G i j = G k l ⟹ μ i j = μ k l
  assumes nonneg: ⋀i j. P i j ⟹ 0 ≤ μ i j
  assumes add: ⋀A::nat ⇒ 'i. ⋀B::nat ⇒ 'j. ⋀j k.
    (⋀n. P (A n) (B n)) ⟹ P j k ⟹ disjoint_family (λn. G (A n) (B n)) ⟹
    (⋃i. G (A i) (B i)) = G j k ⟹ (∑n. μ (A n) (B n)) = μ j k
  shows emeasure M (G i j) = μ i j
proof −
  have emeasure M ((λ(a, b). G a b) (i, j)) = (λ(a, b). μ a b) (i, j)
  proof (rule extend_measure_caratheodory[OF M])
    show semiring_of_sets Ω ((λ(a, b). G a b) ‘ {(a, b). P a b})
      using semiring by (simp add: image_def conj_commute)
  next
    fix A :: nat ⇒ ('i × 'j) and j assume A ∈ UNIV → {(a, b). P a b} j ∈ {(a,
b). P a b}
      disjoint_family ((λ(a, b). G a b) ∘ A)
      (⋃i. case A i of (a, b) ⇒ G a b) = (case j of (a, b) ⇒ G a b)
    then show (∑n. case A n of (a, b) ⇒ μ a b) = (case j of (a, b) ⇒ μ a b)
      using add[of λi. fst (A i) λi. snd (A i) fst j snd j]
```

by (*simp add*: *split_beta'* *comp_def* *Pi_iff*)
qed (*auto split*: *prod.splits intro*: *assms*)
then show *?thesis* **by** *simp*
qed

end

7.9 Bochner Integration for Vector-Valued Functions

theory *Bochner_Integration*
imports *Finite_Product_Measure*
begin

In the following development of the Bochner integral we use second countable topologies instead of separable spaces. A second countable topology is also separable.

proposition *borel_measurable_implies_sequence_metric*:
fixes $f :: 'a \Rightarrow 'b :: \{metric\_space, second\_countable\_topology\}$
assumes [*measurable*]: $f \in borel\_measurable\ M$
shows $\exists F.\ (\forall i.\ simple\_function\ M\ (F\ i)) \wedge (\forall x \in space\ M.\ (\lambda i.\ F\ i\ x) \longrightarrow f\ x) \wedge$
$(\forall i.\ \forall x \in space\ M.\ dist\ (F\ i\ x)\ z \leq 2 * dist\ (f\ x)\ z)$
proof −
obtain $D :: 'b\ set$ **where** *countable* D **and** D: $\bigwedge X.\ open\ X \Longrightarrow X \neq \{\} \Longrightarrow \exists d \in D.\ d \in X$
by (*erule countable_dense_setE*)

define e **where** $e = from\_nat\_into\ D$
{ fix $n\ x$
obtain d **where** $d \in D$ **and** d: $d \in ball\ x\ (1\ /\ Suc\ n)$
using D[*of ball x (1 / Suc n)*] **by** *auto*
from ‹$d \in D$› D[*of UNIV*] ‹*countable D*› **obtain** i **where** $d = e\ i$
unfolding *e_def* **by** (*auto dest*: *from_nat_into_surj*)
with d **have** $\exists i.\ dist\ x\ (e\ i) < 1\ /\ Suc\ n$
by *auto* **}**
note $e = this$

define A **where** [*abs_def*]: $A\ m\ n =$
$\{x \in space\ M.\ dist\ (f\ x)\ (e\ n) < 1\ /\ (Suc\ m) \wedge 1\ /\ (Suc\ m) \leq dist\ (f\ x)\ z\}$ **for** $m\ n$
define B **where** [*abs_def*]: $B\ m = disjointed\ (A\ m)$ **for** m

define m **where** [*abs_def*]: $m\ N\ x = Max\ \{m.\ m \leq N \wedge x \in (\bigcup n \leq N.\ B\ m\ n)\}$ **for** $N\ x$
define F **where** [*abs_def*]: $F\ N\ x =$
$(if\ (\exists m \leq N.\ x \in (\bigcup n \leq N.\ B\ m\ n)) \wedge (\exists n \leq N.\ x \in B\ (m\ N\ x)\ n)$
$then\ e\ (LEAST\ n.\ x \in B\ (m\ N\ x)\ n)\ else\ z)$ **for** $N\ x$

```
  have B_imp_A[intro, simp]: ⋀x m n. x ∈ B m n ⟹ x ∈ A m n
    using disjointed_subset[of A m for m] unfolding B_def by auto

  { fix m
    have ⋀n. A m n ∈ sets M
      by (auto simp: A_def)
    then have ⋀n. B m n ∈ sets M
      using sets.range_disjointed_sets[of A m M] by (auto simp: B_def) }
  note this[measurable]

  { fix N i x assume ∃m≤N. x ∈ (⋃n≤N. B m n)
    then have m N x ∈ {m::nat. m ≤ N ∧ x ∈ (⋃n≤N. B m n)}
      unfolding m_def by (intro Max_in) auto
    then have m N x ≤ N ∃n≤N. x ∈ B (m N x) n
      by auto }
  note m = this

  { fix j N i x assume j ≤ N i ≤ N x ∈ B j i
    then have j ≤ m N x
      unfolding m_def by (intro Max_ge) auto }
  note m_upper = this

  show ?thesis
    unfolding simple_function_def
  proof (safe intro!: exI[of _ F])
    have [measurable]: ⋀i. F i ∈ borel_measurable M
      unfolding F_def m_def by measurable
    show ⋀x i. F i -` {x} ∩ space M ∈ sets M
      by measurable

    { fix i
      { fix n x assume x ∈ B (m i x) n
        then have (LEAST n. x ∈ B (m i x) n) ≤ n
          by (intro Least_le)
        also assume n ≤ i
        finally have (LEAST n. x ∈ B (m i x) n) ≤ i . }
      then have F i ` space M ⊆ {z} ∪ e ` {.. i}
        by (auto simp: F_def)
      then show finite (F i ` space M)
        by (rule finite_subset) auto }

    { fix N i n x assume i ≤ N n ≤ N x ∈ B i n
      then have 1: ∃m≤N. x ∈ (⋃n≤N. B m n) by auto
      from m[OF this] obtain n where n: m N x ≤ N n ≤ N x ∈ B (m N x) n
by auto
      moreover
      define L where L = (LEAST n. x ∈ B (m N x) n)
      have dist (f x) (e L) < 1 / Suc (m N x)
```

```
    proof −
      have x ∈ B (m N x) L
        using n(3) unfolding L_def by (rule LeastI)
      then have x ∈ A (m N x) L
        by auto
      then show ?thesis
        unfolding A_def by simp
    qed
    ultimately have dist (f x) (F N x) < 1 / Suc (m N x)
      by (auto simp: F_def L_def) }
  note ∗ = this

  fix x assume x ∈ space M
  show (λi. F i x) ⟶ f x
  proof (cases f x = z)
    case True
    then have ⋀i n. x ∉ A i n
      unfolding A_def by auto
    then show ?thesis
      by (metis B_imp_A F_def LIMSEQ_def True dist_self)
  next
    case False
    show ?thesis
    proof (rule tendstoI)
      fix e :: real assume 0 < e
      with ‹f x ≠ z› obtain n where 1 / Suc n < e 1 / Suc n < dist (f x) z
        by (metis dist_nz order_less_trans neq_iff nat_approx_posE)
      with ‹x∈space M› ‹f x ≠ z› have x ∈ (⋃i. B n i)
        unfolding A_def B_def UN_disjointed_eq using e by auto
      then obtain i where i: x ∈ B n i by auto

      show eventually (λi. dist (F i x) (f x) < e) sequentially
        using eventually_ge_at_top[of max n i]
      proof eventually_elim
        fix j assume j: max n i ≤ j
        with i have dist (f x) (F j x) < 1 / Suc (m j x)
          by (intro ∗[OF _ _ i]) auto
        also have ... ≤ 1 / Suc n
          using j m_upper[OF _ _ i]
          by (auto simp: field_simps)
        also note ‹1 / Suc n < e›
        finally show dist (F j x) (f x) < e
          by (simp add: less_imp_le dist_commute)
      qed
    qed
  qed
  fix i
  { fix n m assume x ∈ A n m
    then have dist (e m) (f x) + dist (f x) z ≤ 2 ∗ dist (f x) z
```

```
      unfolding A_def by (auto simp: dist_commute)
    also have dist (e m) z ≤ dist (e m) (f x) + dist (f x) z
      by (rule dist_triangle)
    finally (xtrans) have dist (e m) z ≤ 2 * dist (f x) z . }
  then show dist (F i x) z ≤ 2 * dist (f x) z
    unfolding F_def
  by (smt (verit, ccfv_threshold) LeastI2 B_imp_A dist_eq_0_iff zero_le_dist)
 qed
qed

lemma
 fixes f :: 'a ⇒ 'b::semiring_1 assumes finite A
 shows sum_mult_indicator[simp]: (∑ x ∈ A. f x * indicator (B x) (g x)) =
(∑ x∈{x∈A. g x ∈ B x}. f x)
  and sum_indicator_mult[simp]: (∑ x ∈ A. indicator (B x) (g x) * f x) =
(∑ x∈{x∈A. g x ∈ B x}. f x)
 unfolding indicator_def
 using assms by (auto intro!: sum.mono_neutral_cong_right split: if_split_asm)

lemma borel_measurable_induct_real[consumes 2, case_names set mult add seq]:
 fixes P :: ('a ⇒ real) ⇒ bool
 assumes u: u ∈ borel_measurable M ⋀x. 0 ≤ u x
 assumes set: ⋀A. A ∈ sets M ⟹ P (indicator A)
 assumes mult: ⋀u c. 0 ≤ c ⟹ u ∈ borel_measurable M ⟹ (⋀x. 0 ≤ u x)
⟹ P u ⟹ P (λx. c * u x)
 assumes add: ⋀u v. u ∈ borel_measurable M ⟹ (⋀x. 0 ≤ u x) ⟹ P u ⟹
v ∈ borel_measurable M ⟹ (⋀x. 0 ≤ v x) ⟹ (⋀x. x ∈ space M ⟹ u x = 0
∨ v x = 0) ⟹ P v ⟹ P (λx. v x + u x)
 assumes seq: ⋀U. (⋀i. U i ∈ borel_measurable M) ⟹ (⋀i x. 0 ≤ U i x) ⟹
(⋀i. P (U i)) ⟹ incseq U ⟹ (⋀x. x ∈ space M ⟹ (λi. U i x) ⟶ u x)
⟹ P u
 shows P u
proof −
 have (λx. ennreal (u x)) ∈ borel_measurable M using u by auto
 from borel_measurable_implies_simple_function_sequence'[OF this]
 obtain U where U: ⋀i. simple_function M (U i) incseq U ⋀i x. U i x < top
and
   sup: ⋀x. (SUP i. U i x) = ennreal (u x)
   by blast

 define U' where [abs_def]: U' i x = indicator (space M) x * enn2real (U i x)
for i x
 then have U'_sf[measurable]: ⋀i. simple_function M (U' i)
   using U by (auto intro!: simple_function_compose1[where g=enn2real])

 show P u
 proof (rule seq)
   show U': U' i ∈ borel_measurable M ⋀x. 0 ≤ U' i x for i
     using U'_sf by (auto simp: U'_def borel_measurable_simple_function)
```

```
    show incseq U'
      using U(2,3)
      by (auto simp: incseq_def le_fun_def image_iff eq_commute U'_def indicator_def ennreal_mono)

    fix x assume x: x ∈ space M
    have (λi. U i x) ⟶ (SUP i. U i x)
      using U(2) by (intro LIMSEQ_SUP) (auto simp: incseq_def le_fun_def)
    moreover have (λi. U i x) = (λi. ennreal (U' i x))
      using x U(3) by (auto simp: fun_eq_iff U'_def image_iff eq_commute)
    moreover have (SUP i. U i x) = ennreal (u x)
      using sup u(2) by (simp add: max_def)
    ultimately show (λi. U' i x) ⟶ u x
      using u U' by simp
  next
    fix i
    have U' i ‘ space M ⊆ ennreal ‘ (U i ‘ space M) finite (U i ‘ space M)
      unfolding U'_def using U(1) by (auto dest: simple_functionD)
    then have fin: finite (U' i ‘ space M)
      by (metis finite_subset finite_imageI)
    moreover have ⋀z. {y. U' i z = y ∧ y ∈ U' i ‘ space M ∧ z ∈ space M} =
(if z ∈ space M then {U' i z} else {})
      by auto
    ultimately have U': (λz. ∑ y∈U' i‘space M. y * indicator {x∈space M. U' i x = y} z) = U' i
      by (simp add: U'_def fun_eq_iff)
    have ⋀x. x ∈ U' i ‘ space M ⟹ 0 ≤ x
      by (auto simp: U'_def)
    with fin have P (λz. ∑ y∈U' i‘space M. y * indicator {x∈space M. U' i x = y} z)
    proof induct
      case empty from set[of {}] show ?case
        by (simp add: indicator_def[abs_def])
    next
      case (insert x F)
      from insert.prems have nonneg: x ≥ 0 ⋀y. y ∈ F ⟹ y ≥ 0
        by simp_all
      hence *: P (λxa. x * indicat_real {x' ∈ space M. U' i x' = x} xa)
        by (intro mult set) auto
      have P (λz. x * indicat_real {x' ∈ space M. U' i x' = x} z +
                (∑ y∈F. y * indicat_real {x ∈ space M. U' i x = y} z))
        using insert(1-3)
        by (intro add * sum_nonneg mult_nonneg_nonneg)
           (auto simp: nonneg indicator_def of_bool_def sum_nonneg_eq_0_iff)
      thus ?case
        using insert.hyps by (subst sum.insert) auto
    qed
    with U' show P (U' i) by simp
  qed
```

qed

lemma *scaleR_cong_right*:
fixes $x :: 'a :: real\_vector$
shows $(x \neq 0 \Longrightarrow r = p) \Longrightarrow r *_R x = p *_R x$
by *auto*

inductive *simple_bochner_integrable* :: $'a$ *measure* $\Rightarrow$ ($'a \Rightarrow 'b$::*real_vector*) $\Rightarrow$ *bool* **for** *M f* **where**
simple_function M f $\Longrightarrow$ *emeasure M* $\{y \in space\ M.\ f\ y \neq 0\} \neq \infty \Longrightarrow$
simple_bochner_integrable M f

lemma *simple_bochner_integrable_compose2*:
assumes *p_0*: *p 0 0* $= 0$
shows *simple_bochner_integrable M f* $\Longrightarrow$ *simple_bochner_integrable M g* $\Longrightarrow$
simple_bochner_integrable M $(\lambda x.\ p\ (f\ x)\ (g\ x))$
proof (*safe intro*!: *simple_bochner_integrable.intros elim*!: *simple_bochner_integrable.cases del*: *notI*)
assume *sf*: *simple_function M f simple_function M g*
then show *simple_function M* $(\lambda x.\ p\ (f\ x)\ (g\ x))$
by (*rule simple_function_compose2*)

from *sf* **have** [*measurable*]:
$f \in measurable\ M\ (count\_space\ UNIV)$
$g \in measurable\ M\ (count\_space\ UNIV)$
by (*auto intro*: *measurable_simple_function*)

assume *fin*: *emeasure M* $\{y \in space\ M.\ f\ y \neq 0\} \neq \infty$ *emeasure M* $\{y \in space\ M.\ g\ y \neq 0\} \neq \infty$

have *emeasure M* $\{x \in space\ M.\ p\ (f\ x)\ (g\ x) \neq 0\} \leq$
emeasure M $(\{x \in space\ M.\ f\ x \neq 0\} \cup \{x \in space\ M.\ g\ x \neq 0\})$
by (*intro emeasure_mono*) (*auto simp*: *p_0*)
also have $\ldots \leq$ *emeasure M* $\{x \in space\ M.\ f\ x \neq 0\} +$ *emeasure M* $\{x \in space\ M.\ g\ x \neq 0\}$
by (*intro emeasure_subadditive*) *auto*
finally show *emeasure M* $\{y \in space\ M.\ p\ (f\ y)\ (g\ y) \neq 0\} \neq \infty$
using *fin* **by** (*auto simp*: *top_unique*)
qed

lemma *simple_function_finite_support*:
assumes *f*: *simple_function M f* **and** *fin*: $(\int^+ x.\ f\ x\ \partial M) < \infty$ **and** *nn*: $\bigwedge x.\ 0 \leq f\ x$
shows *emeasure M* $\{x \in space\ M.\ f\ x \neq 0\} \neq \infty$
proof *cases*
from *f* **have** *meas*[*measurable*]: $f \in borel\_measurable\ M$
by (*rule borel_measurable_simple_function*)

assume *non_empty*: $\exists x \in space\ M.\ f\ x \neq 0$

```
  define m where m = Min (f`space M - {0})
  have m ∈ f`space M - {0}
    unfolding m_def using f non_empty by (intro Min_in) (auto simp: simple_function_def)
  then have m: 0 < m
    using nn by (auto simp: less_le)

  from m have m * emeasure M {x∈space M. 0 ≠ f x} =
    (∫⁺x. m * indicator {x∈space M. 0 ≠ f x} x ∂M)
    using f by (intro nn_integral_cmult_indicator[symmetric]) auto
  also have ... ≤ (∫⁺x. f x ∂M)
    using AE_space
  proof (intro nn_integral_mono_AE, eventually_elim)
    fix x assume x ∈ space M
    with nn show m * indicator {x ∈ space M. 0 ≠ f x} x ≤ f x
      using f by (auto split: split_indicator simp: simple_function_def m_def)
  qed
  also note ‹... < ∞›
  finally show ?thesis
    using m by (auto simp: ennreal_mult_less_top)
next
  assume ¬ (∃x∈space M. f x ≠ 0)
  with nn have *: {x∈space M. f x ≠ 0} = {}
    by auto
  show ?thesis unfolding * by simp
qed

lemma simple_bochner_integrableI_bounded:
  assumes f: simple_function M f and fin: (∫⁺x. norm (f x) ∂M) < ∞
  shows simple_bochner_integrable M f
proof
  have emeasure M {y ∈ space M. ennreal (norm (f y)) ≠ 0} ≠ ∞
   using simple_function_finite_support simple_function_compose1 f fin by force
  then show emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞ by simp
qed fact

definition simple_bochner_integral :: 'a measure ⇒ ('a ⇒ 'b::real_vector) ⇒ 'b
where
  simple_bochner_integral M f = (∑ y∈f`space M. measure M {x∈space M. f x = y} *R y)

proposition simple_bochner_integral_partition:
  assumes f: simple_bochner_integrable M f and g: simple_function M g
  assumes sub: ⋀x y. x ∈ space M ⟹ y ∈ space M ⟹ g x = g y ⟹ f x = f y
  assumes v: ⋀x. x ∈ space M ⟹ f x = v (g x)
  shows simple_bochner_integral M f = (∑ y∈g ` space M. measure M {x∈space M. g x = y} *R v y)
    (is _ = ?r)
```

```
proof −
  from f g have [simp]: finite (f‘space M) finite (g‘space M)
    by (auto simp: simple_function_def elim: simple_bochner_integrable.cases)

  from f have [measurable]: f ∈ measurable M (count_space UNIV)
   by (auto intro: measurable_simple_function elim: simple_bochner_integrable.cases)

  from g have [measurable]: g ∈ measurable M (count_space UNIV)
   by (auto intro: measurable_simple_function elim: simple_bochner_integrable.cases)

  { fix y assume y ∈ space M
    then have f ‘ space M ∩ {i. ∃x∈space M. i = f x ∧ g y = g x} = {v (g y)}
      by (auto cong: sub simp: v[symmetric]) }
  note eq = this

  have simple_bochner_integral M f =
    (∑ y∈f‘space M. (∑ z∈g‘space M.
      if ∃x∈space M. y = f x ∧ z = g x then measure M {x∈space M. g x = z}
else 0) ∗R y)
    unfolding simple_bochner_integral_def
  proof (safe intro!: sum.cong scaleR_cong_right)
    fix y assume y: y ∈ space M f y ≠ 0
    have [simp]: g ‘ space M ∩ {z. ∃x∈space M. f y = f x ∧ z = g x} =
        {z. ∃x∈space M. f y = f x ∧ z = g x}
      by auto
    have eq:{x ∈ space M. f x = f y} =
        (⋃ i∈{z. ∃x∈space M. f y = f x ∧ z = g x}. {x ∈ space M. g x = i})
      by (auto simp: eq_commute cong: sub rev_conj_cong)
    have finite (g‘space M) by simp
    then have finite {z. ∃x∈space M. f y = f x ∧ z = g x}
      by (rule rev_finite_subset) auto
    moreover
    { fix x assume x ∈ space M f x = f y
      then have x ∈ space M f x ≠ 0
        using y by auto
      then have emeasure M {y ∈ space M. g y = g x} ≤ emeasure M {y ∈ space
M. f y ≠ 0}
        by (auto intro!: emeasure_mono cong: sub)
      then have emeasure M {xa ∈ space M. g xa = g x} < ∞
        using f by (auto simp: simple_bochner_integrable.simps less_top) }
    ultimately
    show measure M {x ∈ space M. f x = f y} =
      (∑ z∈g ‘ space M. if ∃x∈space M. f y = f x ∧ z = g x then measure M {x
∈ space M. g x = z} else 0)
      apply (simp add: sum.If_cases eq)
      apply (subst measure_finite_Union[symmetric])
      apply (auto simp: disjoint_family_on_def less_top)
      done
  qed
```

```
  also have ... = (∑ y∈f`space M. (∑ z∈g`space M.
      if ∃ x∈space M. y = f x ∧ z = g x then measure M {x∈space M. g x = z} *R
y else 0))
    by (auto intro!: sum.cong simp: scaleR_sum_left)
  also have ... = ?r
    by (subst sum.swap)
       (auto intro!: sum.cong simp: sum.If_cases scaleR_sum_right[symmetric] eq)
  finally show simple_bochner_integral M f = ?r .
qed

lemma simple_bochner_integral_add:
  assumes f: simple_bochner_integrable M f and g: simple_bochner_integrable
M g
  shows simple_bochner_integral M (λx. f x + g x) =
    simple_bochner_integral M f + simple_bochner_integral M g
proof −
  from f g have simple_bochner_integral M (λx. f x + g x) =
    (∑ y∈(λx. (f x, g x)) ` space M. measure M {x ∈ space M. (f x, g x) = y} *R
(fst y + snd y))
    by (intro simple_bochner_integral_partition)
       (auto simp: simple_bochner_integrable_compose2 elim: simple_bochner_integrable.cases)
  moreover from f g have simple_bochner_integral M f =
    (∑ y∈(λx. (f x, g x)) ` space M. measure M {x ∈ space M. (f x, g x) = y} *R
fst y)
    by (intro simple_bochner_integral_partition)
       (auto simp: simple_bochner_integrable_compose2 elim: simple_bochner_integrable.cases)
  moreover from f g have simple_bochner_integral M g =
    (∑ y∈(λx. (f x, g x)) ` space M. measure M {x ∈ space M. (f x, g x) = y} *R
snd y)
    by (intro simple_bochner_integral_partition)
       (auto simp: simple_bochner_integrable_compose2 elim: simple_bochner_integrable.cases)
  ultimately show ?thesis
    by (simp add: sum.distrib[symmetric] scaleR_add_right)
qed

lemma simple_bochner_integral_linear:
  assumes linear f
  assumes g: simple_bochner_integrable M g
  shows simple_bochner_integral M (λx. f (g x)) = f (simple_bochner_integral
M g)
proof −
  interpret linear f by fact
  from g have simple_bochner_integral M (λx. f (g x)) =
    (∑ y∈g ` space M. measure M {x ∈ space M. g x = y} *R f y)
    by (intro simple_bochner_integral_partition)
       (auto simp: simple_bochner_integrable_compose2[where p=λx y. f x]
             elim: simple_bochner_integrable.cases)
  also have ... = f (simple_bochner_integral M g)
    by (simp add: simple_bochner_integral_def sum scale)
```

finally show *?thesis* **.**
qed

lemma *simple_bochner_integral_minus*:
assumes *f*: *simple_bochner_integrable M f*
shows *simple_bochner_integral M* (λ*x*. − *f x*) = − *simple_bochner_integral M f*
proof −
from *linear_uminus f* **show** *?thesis*
by (*rule simple_bochner_integral_linear*)
qed

lemma *simple_bochner_integral_diff*:
assumes *f*: *simple_bochner_integrable M f* **and** *g*: *simple_bochner_integrable M g*
shows *simple_bochner_integral M* (λ*x*. *f x* − *g x*) =
simple_bochner_integral M f − *simple_bochner_integral M g*
unfolding *diff_conv_add_uminus* **using** *f g*
by (*subst simple_bochner_integral_add*)
(*auto simp*: *simple_bochner_integral_minus simple_bochner_integrable_compose2*[**where** *p*=λ*x y*. − *y*])

lemma *simple_bochner_integral_norm_bound*:
assumes *f*: *simple_bochner_integrable M f*
shows *norm* (*simple_bochner_integral M f*) ≤ *simple_bochner_integral M* (λ*x*. *norm* (*f x*))
proof −
have *norm* (*simple_bochner_integral M f*) ≤
(∑ *y*∈*f* ‘ *space M*. *norm* (*measure M* {*x* ∈ *space M*. *f x* = *y*} ∗$_R$ *y*))
unfolding *simple_bochner_integral_def* **by** (*rule norm_sum*)
also have ... = (∑ *y*∈*f* ‘ *space M*. *measure M* {*x* ∈ *space M*. *f x* = *y*} ∗$_R$ *norm y*)
by *simp*
also have ... = *simple_bochner_integral M* (λ*x*. *norm* (*f x*))
using *f*
by (*intro simple_bochner_integral_partition*[*symmetric*])
(*auto intro*: *f simple_bochner_integrable_compose2 elim*: *simple_bochner_integrable.cases*)
finally show *?thesis* **.**
qed

lemma *simple_bochner_integral_nonneg*[*simp*]:
fixes *f* :: *'a* ⇒ *real*
shows (⋀*x*. *0* ≤ *f x*) ⟹ *0* ≤ *simple_bochner_integral M f*
by (*force simp*: *simple_bochner_integral_def intro*: *sum_nonneg*)

lemma *simple_bochner_integral_eq_nn_integral*:
assumes *f*: *simple_bochner_integrable M f* ⋀*x*. *0* ≤ *f x*
shows *simple_bochner_integral M f* = (∫$^+$*x*. *f x* ∂*M*)
proof −

```
  have ennreal_cong_mult: (x ≠ 0 ⟹ y = z) ⟹ ennreal x * y = ennreal x * z
for x y z
    by fastforce

  have [measurable]: f ∈ borel_measurable M
   by (meson borel_measurable_simple_function f(1) simple_bochner_integrable.cases)

  { fix y assume y: y ∈ space M f y ≠ 0
    have ennreal (measure M {x ∈ space M. f x = f y}) = emeasure M {x ∈ space
M. f x = f y}
    proof (rule emeasure_eq_ennreal_measure[symmetric])
      have emeasure M {x ∈ space M. f x = f y} ≤ emeasure M {x ∈ space M. f
x ≠ 0}
        using y by (intro emeasure_mono) auto
      with f show emeasure M {x ∈ space M. f x = f y} ≠ top
        by (auto simp: simple_bochner_integrable.simps top_unique)
    qed
    moreover have {x ∈ space M. f x = f y} = (λx. ennreal (f x)) -` {ennreal (f
y)} ∩ space M
      using f by auto
    ultimately have ennreal (measure M {x ∈ space M. f x = f y}) =
        emeasure M ((λx. ennreal (f x)) -` {ennreal (f y)} ∩ space M) by simp }
  with f have simple_bochner_integral M f = (∫S x. f x ∂M)
    unfolding simple_integral_def
   by (subst simple_bochner_integral_partition[OF f(1), where g=λx. ennreal (f
x) and v=enn2real])
     (auto intro: f simple_function_compose1 elim: simple_bochner_integrable.cases
           intro!: sum.cong ennreal_cong_mult
           simp: ac_simps ennreal_mult
           simp flip: sum_ennreal)
  also have ... = (∫+x. f x ∂M)
    using f
      by (metis nn_integral_eq_simple_integral simple_bochner_integrable.cases
simple_function_compose1)
  finally show ?thesis .
qed

lemma simple_bochner_integral_bounded:
  fixes f :: 'a ⇒ 'b::{real_normed_vector, second_countable_topology}
  assumes f[measurable]: f ∈ borel_measurable M
  assumes s: simple_bochner_integrable M s and t: simple_bochner_integrable M
t
  shows ennreal (norm (simple_bochner_integral M s - simple_bochner_integral
M t)) ≤
    (∫+ x. norm (f x - s x) ∂M) + (∫+ x. norm (f x - t x) ∂M)
    (is ennreal (norm (?s - ?t)) ≤ ?S + ?T)
proof -
  have [measurable]: s ∈ borel_measurable M t ∈ borel_measurable M
   using s t by (auto intro: borel_measurable_simple_function elim: simple_bochner_integrable.cases)
```

```
  have ennreal (norm (?s - ?t)) = norm (simple_bochner_integral M (λx. s x -
t x))
    using s t by (subst simple_bochner_integral_diff) auto
  also have ... ≤ simple_bochner_integral M (λx. norm (s x - t x))
    using simple_bochner_integrable_compose2[of (-) M s t] s t
    by (auto intro!: simple_bochner_integral_norm_bound)
  also have ... = (∫⁺x. norm (s x - t x) ∂M)
    using simple_bochner_integrable_compose2[of λx y. norm (x - y) M s t] s t
    by (auto intro!: simple_bochner_integral_eq_nn_integral)
  also have ... ≤ (∫⁺x. ennreal (norm (f x - s x)) + ennreal (norm (f x - t x))
∂M)
  proof -
    have ⋀x. x ∈ space M ⟹
        norm (s x - t x) ≤ norm (f x - s x) + norm (f x - t x)
      by (metis dual_order.refl norm_diff_triangle_le norm_minus_commute)
    then show ?thesis
      by (metis (mono_tags, lifting) ennreal_leI ennreal_plus nn_integral_mono
norm_ge_zero)
  qed
  also have ... = ?S + ?T
   by (rule nn_integral_add) auto
  finally show ?thesis .
qed

inductive has_bochner_integral :: 'a measure ⇒ ('a ⇒ 'b) ⇒ 'b::{real_normed_vector,
second_countable_topology} ⇒ bool
  for M f x where
  f ∈ borel_measurable M ⟹
    (⋀i. simple_bochner_integrable M (s i)) ⟹
    (λi. ∫⁺x. norm (f x - s i x) ∂M) ⟶ 0 ⟹
    (λi. simple_bochner_integral M (s i)) ⟶ x ⟹
    has_bochner_integral M f x

lemma has_bochner_integral_cong:
  assumes M = N ⋀x. x ∈ space N ⟹ f x = g x x = y
  shows has_bochner_integral M f x ⟷ has_bochner_integral N g y
  unfolding has_bochner_integral.simps assms(1,3)
  using assms(2) by (simp cong: measurable_cong_simp nn_integral_cong_simp)

lemma has_bochner_integral_cong_AE:
  f ∈ borel_measurable M ⟹ g ∈ borel_measurable M ⟹ (AE x in M. f x = g
x) ⟹
    has_bochner_integral M f x ⟷ has_bochner_integral M g x
  unfolding has_bochner_integral.simps
  by (intro arg_cong[where f=Ex] ext conj_cong rev_conj_cong refl arg_cong[where
f=λx. x ⟶ 0]
            nn_integral_cong_AE)
     auto
```

lemma *borel_measurable_has_bochner_integral*:
has_bochner_integral M f x $\Longrightarrow$ *f* $\in$ *borel_measurable M*
by (*rule has_bochner_integral.cases*)

lemma *borel_measurable_has_bochner_integral'*[*measurable_dest*]:
has_bochner_integral M f x $\Longrightarrow$ *g* $\in$ *measurable N M* $\Longrightarrow$ (λx. *f* (*g x*)) $\in$ *borel_measurable N*
using *borel_measurable_has_bochner_integral*[*measurable*] **by** *measurable*

lemma *has_bochner_integral_simple_bochner_integrable*:
simple_bochner_integrable M f $\Longrightarrow$ *has_bochner_integral M f* (*simple_bochner_integral M f*)
by (*rule has_bochner_integral.intros*[**where** *s*=λ_. *f*])
(*auto intro*: *borel_measurable_simple_function*
elim: *simple_bochner_integrable.cases*
simp flip: *zero_ennreal_def*)

lemma *has_bochner_integral_real_indicator*:
assumes [*measurable*]: *A* $\in$ *sets M* **and** *A*: *emeasure M A* $< \infty$
shows *has_bochner_integral M* (*indicator A*) (*measure M A*)
proof −
have *sbi*: *simple_bochner_integrable M* (*indicator A*::*'a* $\Rightarrow$ *real*)
proof
have {*y* $\in$ *space M*. (*indicator A y*::*real*) $\neq$ *0*} = *A*
using *sets.sets_into_space*[*OF* ‹*A*$\in$*sets M*›] **by** (*auto split*: *split_indicator*)
then show *emeasure M* {*y* $\in$ *space M*. (*indicator A y*::*real*) $\neq$ *0*} $\neq \infty$
using *A* **by** *auto*
qed (*rule simple_function_indicator assms*)+
moreover have *simple_bochner_integral M* (*indicator A*) = *measure M A*
using *simple_bochner_integral_eq_nn_integral*[*OF sbi*] *A*
by (*simp add*: *ennreal_indicator emeasure_eq_ennreal_measure*)
ultimately show *?thesis*
by (*metis has_bochner_integral_simple_bochner_integrable*)
qed

lemma *has_bochner_integral_add*[*intro*]:
has_bochner_integral M f x $\Longrightarrow$ *has_bochner_integral M g y* $\Longrightarrow$
has_bochner_integral M (λx. *f x* + *g x*) (*x* + *y*)
proof (*safe intro*!: *has_bochner_integral.intros elim*!: *has_bochner_integral.cases*)
fix *sf sg*
assume *f_sf*: (λi. $\int^+$ *x*. *norm* (*f x* − *sf i x*) ∂M) $\longrightarrow$ *0*
assume *g_sg*: (λi. $\int^+$ *x*. *norm* (*g x* − *sg i x*) ∂M) $\longrightarrow$ *0*

assume *sf*: $\forall$ *i*. *simple_bochner_integrable M* (*sf i*)
and *sg*: $\forall$ *i*. *simple_bochner_integrable M* (*sg i*)
then have [*measurable*]: $\bigwedge$*i*. *sf i* $\in$ *borel_measurable M* $\bigwedge$*i*. *sg i* $\in$ *borel_measurable M*
by (*auto intro*: *borel_measurable_simple_function elim*: *simple_bochner_integrable.cases*)

```
  assume [measurable]: f ∈ borel_measurable M g ∈ borel_measurable M

  show ⋀i. simple_bochner_integrable M (λx. sf i x + sg i x)
    using sf sg by (simp add: simple_bochner_integrable_compose2)

  show (λi. ∫⁺ x. (norm (f x + g x − (sf i x + sg i x))) ∂M) ⟶ 0
    (is ?f ⟶ 0)
  proof (rule tendsto_sandwich)
    show eventually (λn. 0 ≤ ?f n) sequentially (λ_. 0) ⟶ 0
      by auto
    show eventually (λi. ?f i ≤ (∫⁺ x. (norm (f x − sf i x)) ∂M) + ∫⁺ x. (norm
(g x − sg i x)) ∂M) sequentially
      (is eventually (λi. ?f i ≤ ?g i) sequentially)
    proof (intro always_eventually allI)
      fix i have ?f i ≤ (∫⁺ x. (norm (f x − sf i x)) + ennreal (norm (g x − sg i
x)) ∂M)
        by (auto intro!: nn_integral_mono norm_diff_triangle_ineq
              simp flip: ennreal_plus)
      also have ... = ?g i
        by (intro nn_integral_add) auto
      finally show ?f i ≤ ?g i .
    qed
    show ?g ⟶ 0
      using tendsto_add[OF f_sf g_sg] by simp
  qed
qed (auto simp: simple_bochner_integral_add tendsto_add)

lemma has_bochner_integral_bounded_linear:
  assumes bounded_linear T
  shows has_bochner_integral M f x ⟹ has_bochner_integral M (λx. T (f x))
(T x)
proof (safe intro!: has_bochner_integral.intros elim!: has_bochner_integral.cases)
  interpret T: bounded_linear T by fact
  have [measurable]: T ∈ borel_measurable borel
   by (intro borel_measurable_continuous_onI T.continuous_on continuous_on_id)
  assume [measurable]: f ∈ borel_measurable M
  then show (λx. T (f x)) ∈ borel_measurable M
    by auto

  fix s assume f_s: (λi. ∫⁺ x. norm (f x − s i x) ∂M) ⟶ 0
  assume s: ∀ i. simple_bochner_integrable M (s i)
  then show ⋀i. simple_bochner_integrable M (λx. T (s i x))
    by (auto intro: simple_bochner_integrable_compose2 T.zero)

  have [measurable]: ⋀i. s i ∈ borel_measurable M
   using s by (auto intro: borel_measurable_simple_function elim: simple_bochner_integrable.cases)

  obtain K where K: K > 0 ⋀x i. norm (T (f x) − T (s i x)) ≤ norm (f x − s
i x) * K
```

```
    using T.pos_bounded by (auto simp: T.diff[symmetric])

  show (λi. ∫⁺ x. norm (T (f x) − T (s i x)) ∂M) ⟶ 0
    (is ?f ⟶ 0)
  proof (rule tendsto_sandwich)
    show eventually (λn. 0 ≤ ?f n) sequentially (λ_. 0) ⟶ 0
      by auto

    show eventually (λi. ?f i ≤ K * (∫⁺ x. norm (f x − s i x) ∂M)) sequentially
      (is eventually (λi. ?f i ≤ ?g i) sequentially)
    proof (intro always_eventually allI)
      fix i have ?f i ≤ (∫⁺ x. ennreal K * norm (f x − s i x) ∂M)
      using K by (intro nn_integral_mono) (auto simp: ac_simps ennreal_mult[symmetric])
      also have ... = ?g i
        using K by (intro nn_integral_cmult) auto
      finally show ?f i ≤ ?g i .
    qed
    show ?g ⟶ 0
      using ennreal_tendsto_cmult[OF _ f_s] by simp
  qed

  assume (λi. simple_bochner_integral M (s i)) ⟶ x
  with s show (λi. simple_bochner_integral M (λx. T (s i x))) ⟶ T x
   by (auto intro!: T.tendsto simp: simple_bochner_integral_linear T.linear_axioms)
qed

lemma has_bochner_integral_zero[intro]: has_bochner_integral M (λx. 0) 0
  by (auto intro!: has_bochner_integral.intros[where s=λ_ _. 0]
          simp: zero_ennreal_def[symmetric] simple_bochner_integrable.simps
               simple_bochner_integral_def image_constant_conv)

lemma has_bochner_integral_scaleR_left[intro]:
  (c ≠ 0 ⟹ has_bochner_integral M f x) ⟹ has_bochner_integral M (λx. f x
*R c) (x *R c)
  by (cases c = 0) (auto simp: has_bochner_integral_bounded_linear[OF bounded_linear_scaleR_left])

lemma has_bochner_integral_scaleR_right[intro]:
  (c ≠ 0 ⟹ has_bochner_integral M f x) ⟹ has_bochner_integral M (λx. c *R
f x) (c *R x)
  by (cases c = 0) (auto simp: has_bochner_integral_bounded_linear[OF bounded_linear_scaleR_right])

lemma has_bochner_integral_mult_left[intro]:
  fixes c :: _::{real_normed_algebra,second_countable_topology}
  shows (c ≠ 0 ⟹ has_bochner_integral M f x) ⟹ has_bochner_integral M
(λx. f x * c) (x * c)
  by (cases c = 0) (auto simp: has_bochner_integral_bounded_linear[OF bounded_linear_mult_left])

lemma has_bochner_integral_mult_right[intro]:
  fixes c :: _::{real_normed_algebra,second_countable_topology}
```

shows $(c \neq 0 \Longrightarrow \mathit{has\_bochner\_integral}\ M\ f\ x) \Longrightarrow \mathit{has\_bochner\_integral}\ M$
$(\lambda x.\ c * f\ x)\ (c * x)$
by $(\mathit{cases}\ c = 0)\ (\mathit{auto\ simp}\colon \mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_mult\_right}])$

lemmas $\mathit{has\_bochner\_integral\_divide} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_divide}]$

lemma $\mathit{has\_bochner\_integral\_divide\_zero}[\mathit{intro}]$:
fixes $c :: \_::\{\mathit{real\_normed\_field}, \mathit{field}, \mathit{second\_countable\_topology}\}$
shows $(c \neq 0 \Longrightarrow \mathit{has\_bochner\_integral}\ M\ f\ x) \Longrightarrow \mathit{has\_bochner\_integral}\ M$
$(\lambda x.\ f\ x\ /\ c)\ (x\ /\ c)$
using $\mathit{has\_bochner\_integral\_divide}$ **by** $(\mathit{cases}\ c = 0)\ \mathit{auto}$

lemma $\mathit{has\_bochner\_integral\_inner\_left}[\mathit{intro}]$:
$(c \neq 0 \Longrightarrow \mathit{has\_bochner\_integral}\ M\ f\ x) \Longrightarrow \mathit{has\_bochner\_integral}\ M\ (\lambda x.\ f\ x \cdot$
$c)\ (x \cdot c)$
by $(\mathit{cases}\ c = 0)\ (\mathit{auto\ simp}\colon \mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_inner\_left}])$

lemma $\mathit{has\_bochner\_integral\_inner\_right}[\mathit{intro}]$:
$(c \neq 0 \Longrightarrow \mathit{has\_bochner\_integral}\ M\ f\ x) \Longrightarrow \mathit{has\_bochner\_integral}\ M\ (\lambda x.\ c \cdot f$
$x)\ (c \cdot x)$
by $(\mathit{cases}\ c = 0)\ (\mathit{auto\ simp}\colon \mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_inner\_right}])$

lemmas $\mathit{has\_bochner\_integral\_minus} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_minus}[\mathit{OF}\ \mathit{bounded\_linear\_ident}]]$
lemmas $\mathit{has\_bochner\_integral\_Re} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_Re}]$
lemmas $\mathit{has\_bochner\_integral\_Im} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_Im}]$
lemmas $\mathit{has\_bochner\_integral\_cnj} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_cnj}]$
lemmas $\mathit{has\_bochner\_integral\_of\_real} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_of\_real}]$
lemmas $\mathit{has\_bochner\_integral\_fst} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_fst}]$
lemmas $\mathit{has\_bochner\_integral\_snd} =$
$\mathit{has\_bochner\_integral\_bounded\_linear}[\mathit{OF}\ \mathit{bounded\_linear\_snd}]$

lemma $\mathit{has\_bochner\_integral\_indicator}$:
$A \in \mathit{sets}\ M \Longrightarrow \mathit{emeasure}\ M\ A < \infty \Longrightarrow$
$\mathit{has\_bochner\_integral}\ M\ (\lambda x.\ \mathit{indicator}\ A\ x *_R c)\ (\mathit{measure}\ M\ A *_R c)$
by $(\mathit{intro}\ \mathit{has\_bochner\_integral\_scaleR\_left}\ \mathit{has\_bochner\_integral\_real\_indicator})$

lemma $\mathit{has\_bochner\_integral\_diff}$:
$\mathit{has\_bochner\_integral}\ M\ f\ x \Longrightarrow \mathit{has\_bochner\_integral}\ M\ g\ y \Longrightarrow$
$\mathit{has\_bochner\_integral}\ M\ (\lambda x.\ f\ x - g\ x)\ (x - y)$
unfolding $\mathit{diff\_conv\_add\_uminus}$
by $(\mathit{intro}\ \mathit{has\_bochner\_integral\_add}\ \mathit{has\_bochner\_integral\_minus})$

lemma *has_bochner_integral_sum*:
(⋀*i*. *i* ∈ *I* ⟹ *has_bochner_integral M* (*f i*) (*x i*)) ⟹
has_bochner_integral M (λ*x*. ∑ *i*∈*I*. *f i x*) (∑ *i*∈*I*. *x i*)
by (*induct I rule*: *infinite_finite_induct*) *auto*

proposition *has_bochner_integral_implies_finite_norm*:
has_bochner_integral M f x ⟹ (∫⁺*x*. *norm* (*f x*) ∂*M*) < ∞
proof (*elim has_bochner_integral.cases*)
fix *s v*
assume [*measurable*]: *f* ∈ *borel_measurable M* **and** *s*: ⋀*i*. *simple_bochner_integrable M* (*s i*) **and**
lim_0: (λ*i*. ∫⁺ *x*. *ennreal* (*norm* (*f x* − *s i x*)) ∂*M*) ⟶ *0*
from *order_tendstoD*[*OF lim_0*, *of* ∞]
obtain *i* **where** *f_s_fin*: (∫⁺ *x*. *ennreal* (*norm* (*f x* − *s i x*)) ∂*M*) < ∞
by (*auto simp*: *eventually_sequentially*)

have [*measurable*]: ⋀*i*. *s i* ∈ *borel_measurable M*
using *s* **by** (*auto intro*: *borel_measurable_simple_function elim*: *simple_bochner_integrable.cases*)

define *m* **where** *m* = (*if space M* = {} *then 0 else Max* ((λ*x*. *norm* (*s i x*))‘*space M*))
have *finite* (*s i* ‘ *space M*)
using *s* **by** (*auto simp*: *simple_function_def simple_bochner_integrable.simps*)
then have *finite* (*norm* ‘ *s i* ‘ *space M*)
by (*rule finite_imageI*)
then have ⋀*x*. *x* ∈ *space M* ⟹ *norm* (*s i x*) ≤ *m* *0* ≤ *m*
by (*auto simp*: *m_def image_comp comp_def Max_ge_iff*)
then have (∫⁺*x*. *norm* (*s i x*) ∂*M*) ≤ (∫⁺*x*. *ennreal m* ∗ *indicator* {*x*∈*space M*. *s i x* ≠ *0*} *x* ∂*M*)
by (*auto split*: *split_indicator intro*!: *Max_ge nn_integral_mono simp*:)
also have ... < ∞
using *s* **by** (*subst nn_integral_cmult_indicator*) (*auto simp*: ‹*0* ≤ *m*› *simple_bochner_integrable.simps ennreal_mult_less_top less_top*)
finally have *s_fin*: (∫⁺*x*. *norm* (*s i x*) ∂*M*) < ∞ **.**

have (∫⁺ *x*. *norm* (*f x*) ∂*M*) ≤ (∫⁺ *x*. *ennreal* (*norm* (*f x* − *s i x*)) + *ennreal* (*norm* (*s i x*)) ∂*M*)
by (*auto intro*!: *nn_integral_mono simp flip*: *ennreal_plus*)
(*metis add.commute norm_triangle_sub*)
also have ... = (∫⁺*x*. *norm* (*f x* − *s i x*) ∂*M*) + (∫⁺*x*. *norm* (*s i x*) ∂*M*)
by (*rule nn_integral_add*) *auto*
also have ... < ∞
using *s_fin f_s_fin* **by** *auto*
finally show (∫⁺ *x*. *ennreal* (*norm* (*f x*)) ∂*M*) < ∞ **.**
qed

proposition *has_bochner_integral_norm_bound*:
assumes *i*: *has_bochner_integral M f x*
shows *norm x* ≤ (∫⁺*x*. *norm* (*f x*) ∂*M*)

```
using assms proof
  fix s assume
    x: (λi. simple_bochner_integral M (s i)) ⟶ x (is ?s ⟶ x) and
    s[simp]: ⋀i. simple_bochner_integrable M (s i) and
    lim: (λi. ∫⁺ x. ennreal (norm (f x − s i x)) ∂M) ⟶ 0 and
    f[measurable]: f ∈ borel_measurable M

  have [measurable]: ⋀i. s i ∈ borel_measurable M
   using s by (auto simp: simple_bochner_integrable.simps intro: borel_measurable_simple_function)

  show norm x ≤ (∫⁺x. norm (f x) ∂M)
  proof (rule LIMSEQ_le)
    show (λi. ennreal (norm (?s i))) ⟶ norm x
      using x by (auto simp: tendsto_ennreal_iff intro: tendsto_intros)
    show ∃N. ∀n≥N. norm (?s n) ≤ (∫⁺x. norm (f x − s n x) ∂M) + (∫⁺x.
norm (f x) ∂M)
      (is ∃N. ∀n≥N. _ ≤ ?t n)
    proof (intro exI allI impI)
      fix n
     have ennreal (norm (?s n)) ≤ simple_bochner_integral M (λx. norm (s n x))
        by (auto intro!: simple_bochner_integral_norm_bound)
      also have ... = (∫⁺x. norm (s n x) ∂M)
        by (intro simple_bochner_integral_eq_nn_integral)
           (auto intro: s simple_bochner_integrable_compose2)
      also have ... ≤ (∫⁺x. ennreal (norm (f x − s n x)) + norm (f x) ∂M)
        by (auto intro!: nn_integral_mono simp flip: ennreal_plus)
           (metis add.commute norm_minus_commute norm_triangle_sub)
      also have ... = ?t n
        by (rule nn_integral_add) auto
      finally show norm (?s n) ≤ ?t n .
    qed
    have ?t ⟶ 0 + (∫⁺ x. ennreal (norm (f x)) ∂M)
      using has_bochner_integral_implies_finite_norm[OF i]
      by (intro tendsto_add tendsto_const lim)
    then show ?t ⟶ ∫⁺ x. ennreal (norm (f x)) ∂M
      by simp
  qed
qed

lemma has_bochner_integral_eq:
  has_bochner_integral M f x ⟹ has_bochner_integral M f y ⟹ x = y
proof (elim has_bochner_integral.cases)
  assume f[measurable]: f ∈ borel_measurable M

  fix s t
  assume (λi. ∫⁺ x. norm (f x − s i x) ∂M) ⟶ 0 (is ?S ⟶ 0)
  assume (λi. ∫⁺ x. norm (f x − t i x) ∂M) ⟶ 0 (is ?T ⟶ 0)
  assume s: ⋀i. simple_bochner_integrable M (s i)
  assume t: ⋀i. simple_bochner_integrable M (t i)
```

```
  have [measurable]: ⋀i. s i ∈ borel_measurable M ⋀i. t i ∈ borel_measurable M
   using s t by (auto intro: borel_measurable_simple_function elim: simple_bochner_integrable.cases)

  let ?s = λi. simple_bochner_integral M (s i)
  let ?t = λi. simple_bochner_integral M (t i)
  assume ?s ⟶ x ?t ⟶ y
  then have (λi. norm (?s i − ?t i)) ⟶ norm (x − y)
    by (intro tendsto_intros)
  moreover
  have (λi. ennreal (norm (?s i − ?t i))) ⟶ ennreal 0
  proof (rule tendsto_sandwich)
    show eventually (λi. 0 ≤ ennreal (norm (?s i − ?t i))) sequentially (λ_. 0)
⟶ ennreal 0
      by auto

    show eventually (λi. norm (?s i − ?t i) ≤ ?S i + ?T i) sequentially
      by (intro always_eventually allI simple_bochner_integral_bounded s t f)
    show (λi. ?S i + ?T i) ⟶ ennreal 0
      using tendsto_add[OF ‹?S ⟶ 0› ‹?T ⟶ 0›] by simp
  qed
  then have (λi. norm (?s i − ?t i)) ⟶ 0
    by (simp flip: ennreal_0)
  ultimately have norm (x − y) = 0
    by (rule LIMSEQ_unique)
  then show x = y by simp
qed

lemma has_bochner_integralI_AE:
  assumes f: has_bochner_integral M f x
    and g: g ∈ borel_measurable M
    and ae: AE x in M. f x = g x
  shows has_bochner_integral M g x
  using f
proof (safe intro!: has_bochner_integral.intros elim!: has_bochner_integral.cases)
  fix s assume (λi. ∫⁺ x. ennreal (norm (f x − s i x)) ∂M) ⟶ 0
  also have (λi. ∫⁺ x. ennreal (norm (f x − s i x)) ∂M) = (λi. ∫⁺ x. ennreal
(norm (g x − s i x)) ∂M)
    using ae
    by (intro ext nn_integral_cong_AE, eventually_elim) simp
  finally show (λi. ∫⁺ x. ennreal (norm (g x − s i x)) ∂M) ⟶ 0 .
qed (auto intro: g)

lemma has_bochner_integral_eq_AE:
  assumes has_bochner_integral M f x and has_bochner_integral M g y
    and AE x in M. f x = g x
  shows x = y
  by (meson assms has_bochner_integral.simps has_bochner_integral_cong_AE
has_bochner_integral_eq)
```

lemma *simple_bochner_integrable_restrict_space*:
fixes $f :: \_ \Rightarrow 'b$::*real_normed_vector*
assumes Ω: $\Omega \cap$ *space* $M \in$ *sets* M
shows *simple_bochner_integrable* (*restrict_space* $M\ \Omega$) $f \longleftrightarrow$
simple_bochner_integrable M (λx. *indicator* $\Omega\ x *_R f\ x$)
by (*simp add*: *simple_bochner_integrable.simps space_restrict_space*
simple_function_restrict_space[*OF* Ω] *emeasure_restrict_space*[*OF* Ω] *Collect_restrict*
indicator_eq_0_iff conj_left_commute)

lemma *simple_bochner_integral_restrict_space*:
fixes $f :: \_ \Rightarrow 'b$::*real_normed_vector*
assumes Ω: $\Omega \cap$ *space* $M \in$ *sets* M
assumes f: *simple_bochner_integrable* (*restrict_space* $M\ \Omega$) f
shows *simple_bochner_integral* (*restrict_space* $M\ \Omega$) $f =$
simple_bochner_integral M (λx. *indicator* $\Omega\ x *_R f\ x$)
proof −
have *finite* ((λx. *indicator* $\Omega\ x *_R f\ x$)‘*space* M)
using f *simple_bochner_integrable_restrict_space*[*OF* Ω, *of* f]
by (*simp add*: *simple_bochner_integrable.simps simple_function_def*)
then show *?thesis*
by (*auto simp*: *space_restrict_space measure_restrict_space*[*OF* $\Omega(1)$] *le_infI2*
simple_bochner_integral_def Collect_restrict
split: *split_indicator split_indicator_asm*
intro!: *sum.mono_neutral_cong_left arg_cong2*[**where** f=*measure*])
qed

context
notes [[*inductive_internals*]]
begin

inductive *integrable* **for** $M\ f$ **where**
has_bochner_integral $M\ f\ x \Longrightarrow$ *integrable* $M\ f$

end

definition *lebesgue_integral* (‹*integral*^{L}›) **where**
integral^L $M\ f$ = (*if* $\exists x$. *has_bochner_integral* $M\ f\ x$ *then THE* x. *has_bochner_integral* $M\ f\ x$ *else* 0)

syntax
_lebesgue_integral :: *pttrn* ⇒ *real* ⇒ $'a$ *measure* ⇒ *real*
(‹(‹*indent*=1 *notation*=‹*binder integral*››$\int$ (2 _./ _)/ ∂_)› [60,61] 110)
syntax_consts
_lebesgue_integral == *lebesgue_integral*
translations
$\int$ x. f ∂M == *CONST lebesgue_integral* M (λx. f)

syntax
_ascii_lebesgue_integral :: *pttrn* ⇒ *'a measure* ⇒ *real* ⇒ *real*
(‹(‹*indent=3 notation=‹mixfix LINT›*›*LINT* (*1_*)/|(*_*)./ *_*)› [*0*,*110*,*60*] *60*)
syntax_consts
_ascii_lebesgue_integral == *lebesgue_integral*
translations
LINT x|M. f == *CONST lebesgue_integral M* (λ*x. f*)

lemma *has_bochner_integral_integral_eq*: *has_bochner_integral M f x* ⟹ *integral*^{L} *M f* = *x*
by (*metis the_equality has_bochner_integral_eq lebesgue_integral_def*)

lemma *has_bochner_integral_integrable*:
integrable M f ⟹ *has_bochner_integral M f* (*integral*^{L} *M f*)
by (*auto simp*: *has_bochner_integral_integral_eq integrable.simps*)

lemma *has_bochner_integral_iff*:
has_bochner_integral M f x ⟷ *integrable M f* ∧ *integral*^{L} *M f* = *x*
by (*metis has_bochner_integral_integrable has_bochner_integral_integral_eq integrable.intros*)

lemma *simple_bochner_integrable_eq_integral*:
simple_bochner_integrable M f ⟹ *simple_bochner_integral M f* = *integral*^{L} *M f*
using *has_bochner_integral_simple_bochner_integrable*[*of M f*]
by (*simp add*: *has_bochner_integral_integral_eq*)

lemma *not_integrable_integral_eq*: ¬ *integrable M f* ⟹ *integral*^{L} *M f* = *0*
unfolding *integrable.simps lebesgue_integral_def* **by** (*auto intro*!: *arg_cong*[**where** *f*=*The*])

lemma *integral_eq_cases*:
integrable M f ⟷ *integrable N g* ⟹
(*integrable M f* ⟹ *integrable N g* ⟹ *integral*^{L} *M f* = *integral*^{L} *N g*) ⟹
integral^L *M f* = *integral*^{L} *N g*
by (*metis not_integrable_integral_eq*)

lemma *borel_measurable_integrable*[*measurable_dest*]: *integrable M f* ⟹ *f* ∈ *borel_measurable M*
by (*auto elim*: *integrable.cases has_bochner_integral.cases*)

lemma *borel_measurable_integrable'*[*measurable_dest*]:
integrable M f ⟹ *g* ∈ *measurable N M* ⟹ (λ*x. f* (*g x*)) ∈ *borel_measurable N*
using *borel_measurable_integrable*[*measurable*] **by** *measurable*

lemma *integrable_cong*:
M = *N* ⟹ (⋀*x. x* ∈ *space N* ⟹ *f x* = *g x*) ⟹ *integrable M f* ⟷ *integrable N g*
by (*simp cong*: *has_bochner_integral_cong add*: *integrable.simps*)

lemma *integrable_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ *AE x in M*. *f x* = *g x* ⟹
integrable M f ⟷ *integrable M g*
unfolding *integrable.simps*
by (*intro has_bochner_integral_cong_AE arg_cong*[**where** *f*=*Ex*] *ext*)

lemma *integrable_cong_AE_imp*:
integrable M g ⟹ *f* ∈ *borel_measurable M* ⟹ (*AE x in M*. *g x* = *f x*) ⟹ *integrable M f*
using *integrable_cong_AE*[*of f M g*] **by** (*auto simp*: *eq_commute*)

lemma *integral_cong*:
M = *N* ⟹ (⋀*x*. *x* ∈ *space N* ⟹ *f x* = *g x*) ⟹ *integral*^{L} *M f* = *integral*^{L} *N g*
by (*simp cong*: *has_bochner_integral_cong cong del*: *if_weak_cong add*: *lebesgue_integral_def*)

lemma *integral_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ *AE x in M*. *f x* = *g x* ⟹
integral^L *M f* = *integral*^{L} *M g*
unfolding *lebesgue_integral_def*
by (*rule arg_cong*[**where** *x*=*has_bochner_integral M f*]) (*intro has_bochner_integral_cong_AE ext*)

lemma *integrable_add*[*simp*, *intro*]: *integrable M f* ⟹ *integrable M g* ⟹ *integrable M* (λ*x*. *f x* + *g x*)
by (*auto simp*: *integrable.simps*)

lemma *integrable_zero*[*simp*, *intro*]: *integrable M* (λ*x*. *0*)
by (*metis has_bochner_integral_zero integrable.simps*)

lemma *integrable_sum*[*simp*, *intro*]: (⋀*i*. *i* ∈ *I* ⟹ *integrable M* (*f i*)) ⟹ *integrable M* (λ*x*. ∑ *i*∈*I*. *f i x*)
by (*metis has_bochner_integral_sum integrable.simps*)

lemma *integrable_indicator*[*simp*, *intro*]: *A* ∈ *sets M* ⟹ *emeasure M A* < ∞ ⟹
integrable M (λ*x*. *indicator A x* ∗$_R$ *c*)
by (*metis has_bochner_integral_indicator integrable.simps*)

lemma *integrable_real_indicator*[*simp*, *intro*]: *A* ∈ *sets M* ⟹ *emeasure M A* < ∞ ⟹
integrable M (*indicator A* :: *'a* ⇒ *real*)
by (*metis has_bochner_integral_real_indicator integrable.simps*)

lemma *integrable_diff*[*simp*, *intro*]: *integrable M f* ⟹ *integrable M g* ⟹ *integrable M* (λ*x*. *f x* − *g x*)
by (*auto simp*: *integrable.simps intro*: *has_bochner_integral_diff*)

lemma *integrable_bounded_linear*: *bounded_linear T* $\Longrightarrow$ *integrable M f* $\Longrightarrow$ *integrable M* ($\lambda x.$ *T* (*f x*))
by (*auto simp*: *integrable.simps intro*: *has_bochner_integral_bounded_linear*)

lemma *integrable_scaleR_left*[*simp*, *intro*]: (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* ($\lambda x.$ *f x* $*_R$ *c*)
unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_scaleR_right*[*simp*, *intro*]: (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* ($\lambda x.$ *c* $*_R$ *f x*)
unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_mult_left*[*simp*, *intro*]:
fixes *c* :: _::{*real_normed_algebra*,*second_countable_topology*}
shows (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* ($\lambda x.$ *f x* $*$ *c*)
unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_mult_right*[*simp*, *intro*]:
fixes *c* :: _::{*real_normed_algebra*,*second_countable_topology*}
shows (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* ($\lambda x.$ *c* $*$ *f x*)
unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_divide_zero*[*simp*, *intro*]:
fixes *c* :: _::{*real_normed_field*, *field*, *second_countable_topology*}
shows (*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* ($\lambda x.$ *f x* / *c*)
unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_inner_left*[*simp*, *intro*]:
(*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* ($\lambda x.$ *f x* $\cdot$ *c*)
unfolding *integrable.simps* **by** *fastforce*

lemma *integrable_inner_right*[*simp*, *intro*]:
(*c* $\neq$ *0* $\Longrightarrow$ *integrable M f*) $\Longrightarrow$ *integrable M* ($\lambda x.$ *c* $\cdot$ *f x*)
unfolding *integrable.simps* **by** *fastforce*

lemmas *integrable_minus*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_minus*[*OF bounded_linear_ident*]]
lemmas *integrable_divide*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_divide*]
lemmas *integrable_Re*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_Re*]
lemmas *integrable_Im*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_Im*]
lemmas *integrable_cnj*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_cnj*]
lemmas *integrable_of_real*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_of_real*]
lemmas *integrable_fst*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_fst*]

lemmas *integrable_snd*[*simp*, *intro*] =
integrable_bounded_linear[*OF bounded_linear_snd*]

lemma *integral_zero*[*simp*]: $integral^L\ M\ (\lambda x.\ 0) = 0$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_zero*)

lemma *integral_add*[*simp*]: *integrable M f* $\Longrightarrow$ *integrable M g* $\Longrightarrow$
$integral^L\ M\ (\lambda x.\ f\ x + g\ x) = integral^L\ M\ f + integral^L\ M\ g$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_add has_bochner_integral_integrab*

lemma *integral_diff*[*simp*]: *integrable M f* $\Longrightarrow$ *integrable M g* $\Longrightarrow$
$integral^L\ M\ (\lambda x.\ f\ x - g\ x) = integral^L\ M\ f - integral^L\ M\ g$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_diff has_bochner_integral_integrabl*

lemma *integral_sum*: $(\bigwedge i.\ i \in I \Longrightarrow integrable\ M\ (f\ i)) \Longrightarrow$
$integral^L\ M\ (\lambda x.\ \sum i{\in}I.\ f\ i\ x) = (\sum i{\in}I.\ integral^L\ M\ (f\ i))$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_sum has_bochner_integral_integral*

lemma *integral_sum'*[*simp*]: $(\bigwedge i.\ i \in I =simp=> integrable\ M\ (f\ i)) \Longrightarrow$
$integral^L\ M\ (\lambda x.\ \sum i{\in}I.\ f\ i\ x) = (\sum i{\in}I.\ integral^L\ M\ (f\ i))$
unfolding *simp_implies_def* **by** (*rule integral_sum*)

lemma *integral_bounded_linear*: *bounded_linear T* $\Longrightarrow$ *integrable M f* $\Longrightarrow$
$integral^L\ M\ (\lambda x.\ T\ (f\ x)) = T\ (integral^L\ M\ f)$
by (*metis has_bochner_integral_bounded_linear has_bochner_integral_integrable*
has_bochner_integral_integral_eq)

lemma *integral_bounded_linear'*:
assumes *T*: *bounded_linear T* **and** *T'*: *bounded_linear T'*
assumes $*$: $\neg\ (\forall x.\ T\ x = 0) \Longrightarrow (\forall x.\ T'\ (T\ x) = x)$
shows $integral^L\ M\ (\lambda x.\ T\ (f\ x)) = T\ (integral^L\ M\ f)$
proof *cases*
assume $(\forall x.\ T\ x = 0)$ **then show** *?thesis*
by *simp*
next
assume $**$: $\neg\ (\forall x.\ T\ x = 0)$
show *?thesis*
proof *cases*
assume *integrable M f* **with** *T* **show** *?thesis*
by (*rule integral_bounded_linear*)
next
assume *not*: $\neg$ *integrable M f*
moreover have $\neg$ *integrable M* $(\lambda x.\ T\ (f\ x))$
by (*metis* (*full_types*) $*$ $**$ *T' integrable_bounded_linear integrable_cong not*)
ultimately show *?thesis*
using *T* **by** (*simp add*: *not_integrable_integral_eq linear_simps*)
qed
qed

lemma *integral_scaleR_left*[*simp*]: $(c \neq 0 \Longrightarrow \textit{integrable}\ M\ f) \Longrightarrow (\int\ x.\ f\ x *_R c\ \partial M) = \textit{integral}^L\ M\ f *_R c$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_scaleR_left*)

lemma *integral_scaleR_right*[*simp*]: $(\int\ x.\ c *_R f\ x\ \partial M) = c *_R \textit{integral}^L\ M\ f$
by (*rule integral_bounded_linear'*[*OF bounded_linear_scaleR_right bounded_linear_scaleR_right*[*of 1 / c*]]) *simp*

lemma *integral_mult_left*[*simp*]:
fixes $c :: \_ :: \{\textit{real\_normed\_algebra}, \textit{second\_countable\_topology}\}$
shows $(c \neq 0 \Longrightarrow \textit{integrable}\ M\ f) \Longrightarrow (\int\ x.\ f\ x * c\ \partial M) = \textit{integral}^L\ M\ f * c$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_mult_left*)

lemma *integral_mult_right*[*simp*]:
fixes $c :: \_ :: \{\textit{real\_normed\_algebra}, \textit{second\_countable\_topology}\}$
shows $(c \neq 0 \Longrightarrow \textit{integrable}\ M\ f) \Longrightarrow (\int\ x.\ c * f\ x\ \partial M) = c * \textit{integral}^L\ M\ f$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_mult_right*)

lemma *integral_mult_left_zero*[*simp*]:
fixes $c :: \_ :: \{\textit{real\_normed\_field}, \textit{second\_countable\_topology}\}$
shows $(\int\ x.\ f\ x * c\ \partial M) = \textit{integral}^L\ M\ f * c$
by (*rule integral_bounded_linear'*[*OF bounded_linear_mult_left bounded_linear_mult_left*[*of 1 / c*]]) *simp*

lemma *integral_mult_right_zero*[*simp*]:
fixes $c :: \_ :: \{\textit{real\_normed\_field}, \textit{second\_countable\_topology}\}$
shows $(\int\ x.\ c * f\ x\ \partial M) = c * \textit{integral}^L\ M\ f$
by (*rule integral_bounded_linear'*[*OF bounded_linear_mult_right bounded_linear_mult_right*[*of 1 / c*]]) *simp*

lemma *integral_inner_left*[*simp*]: $(c \neq 0 \Longrightarrow \textit{integrable}\ M\ f) \Longrightarrow (\int\ x.\ f\ x \cdot c\ \partial M) = \textit{integral}^L\ M\ f \cdot c$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_inner_left*)

lemma *integral_inner_right*[*simp*]: $(c \neq 0 \Longrightarrow \textit{integrable}\ M\ f) \Longrightarrow (\int\ x.\ c \cdot f\ x\ \partial M) = c \cdot \textit{integral}^L\ M\ f$
by (*intro has_bochner_integral_integral_eq has_bochner_integral_integrable has_bochner_integral_inner_right*)

lemma *integral_divide_zero*[*simp*]:
fixes $c :: \_ :: \{\textit{real\_normed\_field}, \textit{field}, \textit{second\_countable\_topology}\}$
shows $\textit{integral}^L\ M\ (\lambda x.\ f\ x\ /\ c) = \textit{integral}^L\ M\ f\ /\ c$
by (*rule integral_bounded_linear'*[*OF bounded_linear_divide bounded_linear_mult_left*[*of c*]]) *simp*

lemma *integral_minus*[*simp*]: $\textit{integral}^L\ M\ (\lambda x.\ -\ f\ x) = -\ \textit{integral}^L\ M\ f$
by (*rule integral_bounded_linear'*[*OF bounded_linear_minus*[*OF bounded_linear_ident*] *bounded_linear_minus*[*OF bounded_linear_ident*]]) *simp*

lemma *integral_complex_of_real*[*simp*]: $\textit{integral}^L\ M\ (\lambda x.\ \textit{complex\_of\_real}\ (f\ x))$

$= of\_real\ (integral^L\ M\ f)$
by (*rule integral_bounded_linear'*[*OF bounded_linear_of_real bounded_linear_Re*])
simp

lemma *integral_cnj*[*simp*]: $integral^L\ M\ (\lambda x.\ cnj\ (f\ x)) = cnj\ (integral^L\ M\ f)$
by (*rule integral_bounded_linear'*[*OF bounded_linear_cnj bounded_linear_cnj*])
simp

lemmas *integral_divide*[*simp*] =
integral_bounded_linear[*OF bounded_linear_divide*]
lemmas *integral_Re*[*simp*] =
integral_bounded_linear[*OF bounded_linear_Re*]
lemmas *integral_Im*[*simp*] =
integral_bounded_linear[*OF bounded_linear_Im*]
lemmas *integral_of_real*[*simp*] =
integral_bounded_linear[*OF bounded_linear_of_real*]
lemmas *integral_fst*[*simp*] =
integral_bounded_linear[*OF bounded_linear_fst*]
lemmas *integral_snd*[*simp*] =
integral_bounded_linear[*OF bounded_linear_snd*]

lemma *integral_norm_bound_ennreal*:
$integrable\ M\ f \Longrightarrow norm\ (integral^L\ M\ f) \leq (\int^+ x.\ norm\ (f\ x)\ \partial M)$
by (*metis has_bochner_integral_integrable has_bochner_integral_norm_bound*)

lemma *integrableI_sequence*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: $f \in borel\_measurable\ M$
assumes *s*: $\bigwedge i.\ simple\_bochner\_integrable\ M\ (s\ i)$
assumes *lim*: $(\lambda i.\ \int^+ x.\ norm\ (f\ x - s\ i\ x)\ \partial M) \longrightarrow 0$ (**is** $?S \longrightarrow 0$)
shows $integrable\ M\ f$
proof −
let $?s = \lambda n.\ simple\_bochner\_integral\ M\ (s\ n)$

have $\exists x.\ ?s \longrightarrow x$
unfolding *convergent_eq_Cauchy*
proof (*rule metric_CauchyI*)
fix *e* :: *real* **assume** $0 < e$
then have $0 < ennreal\ (e\ /\ 2)$ **by** *auto*
from *order_tendstoD*(2)[*OF lim this*]
obtain *M* **where** *M*: $\bigwedge n.\ M \leq n \Longrightarrow ?S\ n < e\ /\ 2$
by (*auto simp*: *eventually_sequentially*)
show $\exists M.\ \forall m{\geq}M.\ \forall n{\geq}M.\ dist\ (?s\ m)\ (?s\ n) < e$
proof (*intro exI allI impI*)
fix *m n* **assume** *m*: $M \leq m$ **and** *n*: $M \leq n$
have $?S\ n \neq \infty$
using *M*[*OF n*] **by** *auto*
have $norm\ (?s\ n - ?s\ m) \leq ?S\ n + ?S\ m$
by (*intro simple_bochner_integral_bounded s f*)

```
    also have ... < ennreal (e / 2) + e / 2
      by (intro add_strict_mono M n m)
    also have ... = e using ‹0<e› by (simp flip: ennreal_plus)
    finally show dist (?s n) (?s m) < e
      using ‹0<e› by (simp add: dist_norm ennreal_less_iff)
  qed
 qed
 then obtain x where ?s ⟶ x ..
 show ?thesis
  by (rule, rule) fact+
qed

proposition nn_integral_dominated_convergence_norm:
 fixes u' :: _ ⇒ _::{real_normed_vector, second_countable_topology}
 assumes [measurable]:
    ⋀i. u i ∈ borel_measurable M u' ∈ borel_measurable M w ∈ borel_measurable
M
  and bound: ⋀j. AE x in M. norm (u j x) ≤ w x
  and w: (∫⁺x. w x ∂M) < ∞
  and u': AE x in M. (λi. u i x) ⟶ u' x
 shows (λi. (∫⁺x. norm (u' x − u i x) ∂M)) ⟶ 0
proof −
 have AE x in M. ∀j. norm (u j x) ≤ w x
  unfolding AE_all_countable by rule fact
 with u' have bnd: AE x in M. ∀j. norm (u' x − u j x) ≤ 2 ∗ w x
 proof (eventually_elim, intro allI)
  fix i x assume (λi. u i x) ⟶ u' x ∀j. norm (u j x) ≤ w x ∀j. norm (u j
x) ≤ w x
  then have norm (u' x) ≤ w x norm (u i x) ≤ w x
   by (auto intro: LIMSEQ_le_const2 tendsto_norm)
  then have norm (u' x) + norm (u i x) ≤ 2 ∗ w x
   by simp
  also have norm (u' x − u i x) ≤ norm (u' x) + norm (u i x)
   by (rule norm_triangle_ineq4)
  finally (xtrans) show norm (u' x − u i x) ≤ 2 ∗ w x .
 qed
 have w_nonneg: AE x in M. 0 ≤ w x
  using bound[of 0] by (auto intro: order_trans[OF norm_ge_zero])

 have (λi. (∫⁺x. norm (u' x − u i x) ∂M)) ⟶ (∫⁺x. 0 ∂M)
 proof (rule nn_integral_dominated_convergence)
  show (∫⁺x. 2 ∗ w x ∂M) < ∞
   by (rule nn_integral_mult_bounded_inf[OF _ w, of 2]) (insert w_nonneg,
auto simp: ennreal_mult )
  show AE x in M. (λi. ennreal (norm (u' x − u i x))) ⟶ 0
   using u'
  proof eventually_elim
   fix x assume (λi. u i x) ⟶ u' x
   from tendsto_diff[OF tendsto_const[of u' x] this]
```

```
    show (λi. ennreal (norm (u' x − u i x))) ⟶ 0
      by (simp add: tendsto_norm_zero_iff flip: ennreal_0)
  qed
 qed (use bnd w_nonneg in auto)
 then show ?thesis by simp
qed

proposition integrableI_bounded:
 fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
 assumes f[measurable]: f ∈ borel_measurable M and fin: (∫⁺x. norm (f x) ∂M)
< ∞
 shows integrable M f
proof −
 from borel_measurable_implies_sequence_metric[OF f, of 0] obtain s where
   s: ⋀i. simple_function M (s i) and
   pointwise: ⋀x. x ∈ space M ⟹ (λi. s i x) ⟶ f x and
   bound: ⋀i x. x ∈ space M ⟹ norm (s i x) ≤ 2 * norm (f x)
   by simp metis

 show ?thesis
 proof (rule integrableI_sequence)
   { fix i
     have (∫⁺x. norm (s i x) ∂M) ≤ (∫⁺x. ennreal (2 * norm (f x)) ∂M)
       by (intro nn_integral_mono) (simp add: bound)
     also have ... = 2 * (∫⁺x. ennreal (norm (f x)) ∂M)
       by (simp add: ennreal_mult nn_integral_cmult)
     also have ... < top
       using fin by (simp add: ennreal_mult_less_top)
     finally have (∫⁺x. norm (s i x) ∂M) < ∞
       by simp }
   note fin_s = this

   show ⋀i. simple_bochner_integrable M (s i)
     by (rule simple_bochner_integrableI_bounded) fact+

   show (λi. ∫⁺ x. ennreal (norm (f x − s i x)) ∂M) ⟶ 0
   proof (rule nn_integral_dominated_convergence_norm)
     show ⋀j. AE x in M. norm (s j x) ≤ 2 * norm (f x)
       using bound by auto
    show ⋀i. s i ∈ borel_measurable M (λx. 2 * norm (f x)) ∈ borel_measurable
M
       using s by (auto intro: borel_measurable_simple_function)
     show (∫⁺ x. ennreal (2 * norm (f x)) ∂M) < ∞
     using fin by (simp add: nn_integral_cmult ennreal_mult ennreal_mult_less_top)
     show AE x in M. (λi. s i x) ⟶ f x
       using pointwise by auto
   qed fact
 qed fact
qed
```

lemma *integrableI_bounded_set*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
assumes [*measurable*]: $A \in sets\ M\ f \in borel\_measurable\ M$
assumes *finite*: $emeasure\ M\ A < \infty$
and *bnd*: $AE\ x\ in\ M.\ x \in A \longrightarrow norm\ (f\ x) \leq B$
and *null*: $AE\ x\ in\ M.\ x \notin A \longrightarrow f\ x = 0$
shows *integrable M f*
proof (*rule integrableI_bounded*)
{ fix $x :: 'b$ **have** $norm\ x \leq B \Longrightarrow 0 \leq B$
using *norm_ge_zero*[*of x*] **by** *arith* **}**
with *bnd null* **have** $(\int^+ x.\ ennreal\ (norm\ (f\ x))\ \partial M) \leq (\int^+ x.\ ennreal\ (max\ 0\ B) * indicator\ A\ x\ \partial M)$
by (*intro nn_integral_mono_AE*) (*auto split*: *split_indicator split_max*)
also have $\ldots < \infty$
using *finite* **by** (*subst nn_integral_cmult_indicator*) (*auto simp*: *ennreal_mult_less_top*)
finally show $(\int^+ x.\ ennreal\ (norm\ (f\ x))\ \partial M) < \infty$ **.**
qed *simp*

lemma *integrableI_bounded_set_indicator*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
shows $A \in sets\ M \Longrightarrow f \in borel\_measurable\ M \Longrightarrow$
$emeasure\ M\ A < \infty \Longrightarrow (AE\ x\ in\ M.\ x \in A \longrightarrow norm\ (f\ x) \leq B) \Longrightarrow$
$integrable\ M\ (\lambda x.\ indicator\ A\ x *_R f\ x)$
by (*rule integrableI_bounded_set*[**where** *A*=*A*]) *auto*

lemma *integrableI_nonneg*:
fixes $f :: 'a \Rightarrow real$
assumes $f \in borel\_measurable\ M\ AE\ x\ in\ M.\ 0 \leq f\ x\ (\int^+ x.\ f\ x\ \partial M) < \infty$
shows *integrable M f*
proof −
have $(\int^+ x.\ norm\ (f\ x)\ \partial M) = (\int^+ x.\ f\ x\ \partial M)$
using *assms* **by** (*intro nn_integral_cong_AE*) *auto*
then show *?thesis*
using *assms* **by** (*intro integrableI_bounded*) *auto*
qed

lemma *integrable_iff_bounded*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
shows $integrable\ M\ f \longleftrightarrow f \in borel\_measurable\ M \wedge (\int^+ x.\ norm\ (f\ x)\ \partial M) < \infty$
using *integrableI_bounded*[*of f M*] *has_bochner_integral_implies_finite_norm*[*of M f*]
unfolding *integrable.simps has_bochner_integral.simps*[*abs_def*] **by** *auto*

lemma (**in** *finite_measure*) *square_integrable_imp_integrable*:
fixes $f :: 'a \Rightarrow 'b :: \{second\_countable\_topology, banach, real\_normed\_div\_algebra\}$
assumes [*measurable*]: $f \in borel\_measurable\ M$
assumes $integrable\ M\ (\lambda x.\ f\ x\ \widehat{}\ 2)$

```
  shows   integrable M f
proof −
  have less_top: emeasure M (space M) < top
    using finite_emeasure_space by (meson top.not_eq_extremum)
  have nn_integral M (λx. norm (f x)) ^ 2 ≤
        nn_integral M (λx. norm (f x) ^ 2) * emeasure M (space M)
    using Cauchy_Schwarz_nn_integral[of λx. norm (f x) M λ_. 1]
    by (simp add: ennreal_power)
  also have ... < ∞
    using assms(2) less_top
   by (subst (asm) integrable_iff_bounded) (auto simp: norm_power ennreal_mult_less_top)
  finally have nn_integral M (λx. norm (f x)) < ∞
    by (simp add: power_less_top_ennreal)
  thus ?thesis
    by (simp add: integrable_iff_bounded)
qed

lemma integrable_bound:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
    and g :: 'a ⇒ 'c::{banach, second_countable_topology}
  shows integrable M f ⟹ g ∈ borel_measurable M ⟹ (AE x in M. norm (g x)
≤ norm (f x)) ⟹
    integrable M g
  unfolding integrable_iff_bounded
  by (smt (verit) AE_cong ennreal_le_iff nn_integral_mono_AE norm_ge_zero
order_less_subst2 linorder_not_le order_less_le_trans)


lemma integrable_mult_indicator:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  shows A ∈ sets M ⟹ integrable M f ⟹ integrable M (λx. indicator A x *R f
x)
  by (rule integrable_bound[of M f]) (auto split: split_indicator)

lemma integrable_real_mult_indicator:
  fixes f :: 'a ⇒ real
  shows A ∈ sets M ⟹ integrable M f ⟹ integrable M (λx. f x * indicator A x)
  using integrable_mult_indicator[of A M f] by (simp add: mult_ac)

lemma integrable_abs[simp, intro]:
  fixes f :: 'a ⇒ real
  assumes [measurable]: integrable M f shows integrable M (λx. |f x|)
  using assms by (rule integrable_bound) auto

lemma integrable_norm[simp, intro]:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes [measurable]: integrable M f shows integrable M (λx. norm (f x))
  using assms by (rule integrable_bound) auto
```

lemma *integrable_norm_cancel*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
assumes [*measurable*]: *integrable M* $(\lambda x.\ norm\ (f\ x))$ $f \in$ *borel_measurable M*
shows *integrable M f*
using *assms* **by** (*rule integrable_bound*) *auto*

lemma *integrable_norm_iff*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
shows $f \in$ *borel_measurable M* $\Longrightarrow$ *integrable M* $(\lambda x.\ norm\ (f\ x)) \longleftrightarrow$ *integrable M f*
by (*auto intro*: *integrable_norm_cancel*)

lemma *integrable_abs_cancel*:
fixes $f :: 'a \Rightarrow real$
assumes [*measurable*]: *integrable M* $(\lambda x.\ |f\ x|)$ $f \in$ *borel_measurable M* **shows** *integrable M f*
using *assms* **by** (*rule integrable_bound*) *auto*

lemma *integrable_abs_iff*:
fixes $f :: 'a \Rightarrow real$
shows $f \in$ *borel_measurable M* $\Longrightarrow$ *integrable M* $(\lambda x.\ |f\ x|) \longleftrightarrow$ *integrable M f*
by (*auto intro*: *integrable_abs_cancel*)

lemma *integrable_max*[*simp*, *intro*]:
fixes $f :: 'a \Rightarrow real$
assumes *fg*[*measurable*]: *integrable M f integrable M g*
shows *integrable M* $(\lambda x.\ max\ (f\ x)\ (g\ x))$
using *integrable_add*[*OF integrable_norm*[*OF fg*(*1*)] *integrable_norm*[*OF fg*(*2*)]]
by (*rule integrable_bound*) *auto*

lemma *integrable_min*[*simp*, *intro*]:
fixes $f :: 'a \Rightarrow real$
assumes *fg*[*measurable*]: *integrable M f integrable M g*
shows *integrable M* $(\lambda x.\ min\ (f\ x)\ (g\ x))$
using *integrable_add*[*OF integrable_norm*[*OF fg*(*1*)] *integrable_norm*[*OF fg*(*2*)]]
by (*rule integrable_bound*) *auto*

lemma *integral_minus_iff*[*simp*]:
integrable M $(\lambda x.\ -f\ x ::'a::\{banach, second\_countable\_topology\})$ $\longleftrightarrow$ *integrable M f*
unfolding *integrable_iff_bounded*
by (*auto*)

lemma *integrable_indicator_iff*:
integrable M (*indicator* $A::\_ \Rightarrow real$) $\longleftrightarrow$ $A \cap$ *space M* $\in$ *sets M* $\wedge$ *emeasure M* $(A \cap$ *space M*$) < \infty$
by (*simp add*: *integrable_iff_bounded borel_measurable_indicator_iff ennreal_indicator nn_integral_indicator'*
cong: *conj_cong*)

lemma *integral_indicator*[*simp*]: $integral^L\ M\ (indicator\ A) = measure\ M\ (A \cap space\ M)$
proof *cases*
assume $*$: $A \cap space\ M \in sets\ M \wedge emeasure\ M\ (A \cap space\ M) < \infty$
have $integral^L\ M\ (indicator\ A) = integral^L\ M\ (indicator\ (A \cap space\ M))$
by (*metis* (*no_types*, *lifting*) *Int_iff indicator_simps integral_cong*)
also have $\ldots = measure\ M\ (A \cap space\ M)$
using $*$ **by** (*intro has_bochner_integral_integral_eq has_bochner_integral_real_indicator*) *auto*
finally show *?thesis* **.**
next
assume $*$: $\neg\ (A \cap space\ M \in sets\ M \wedge emeasure\ M\ (A \cap space\ M) < \infty)$
have $integral^L\ M\ (indicator\ A) = integral^L\ M\ (indicator\ (A \cap space\ M) :: \_ \Rightarrow real)$
by (*intro integral_cong*) (*auto split*: *split_indicator*)
also have $\ldots = 0$
using $*$ **by** (*subst not_integrable_integral_eq*) (*auto simp*: *integrable_indicator_iff*)
also have $\ldots = measure\ M\ (A \cap space\ M)$
using $*$ **by** (*auto simp*: *measure_def emeasure_notin_sets not_less top_unique*)
finally show *?thesis* **.**
qed

lemma *integrable_discrete_difference_aux*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
assumes *f*: *integrable M f* **and** *X*: *countable X*
assumes *null*: $\bigwedge x.\ x \in X \Longrightarrow emeasure\ M\ \{x\} = 0$
assumes *sets*: $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in sets\ M$
assumes *eq*: $\bigwedge x.\ x \in space\ M \Longrightarrow x \notin X \Longrightarrow f\ x = g\ x$
shows *integrable M g*
unfolding *integrable_iff_bounded*
proof
have *fmeas*: $f \in borel\_measurable\ M\ (\int^+ x.\ ennreal\ (norm\ (f\ x))\ \partial M) < \infty$
using *f integrable_iff_bounded* **by** *auto*
then show $g \in borel\_measurable\ M$
using *measurable_discrete_difference*[**where** *X*=*X*]
by (*smt* (*verit*) *UNIV_I X eq sets space_borel*)
have *AE x in M*. $x \notin X$
using *AE_discrete_difference X null sets* **by** *blast*
with *fmeas* **show** $(\int^+ x.\ ennreal\ (norm\ (g\ x))\ \partial M) < \infty$
by (*metis* (*mono_tags*, *lifting*) *AE_I2 AE_mp eq nn_integral_cong_AE*)
qed

lemma *integrable_discrete_difference*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
assumes *X*: *countable X*
assumes *null*: $\bigwedge x.\ x \in X \Longrightarrow emeasure\ M\ \{x\} = 0$
assumes *sets*: $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in sets\ M$
assumes *eq*: $\bigwedge x.\ x \in space\ M \Longrightarrow x \notin X \Longrightarrow f\ x = g\ x$

shows *integrable M f* $\longleftrightarrow$ *integrable M g*
by (*metis X eq integrable_discrete_difference_aux null sets*)

lemma *integral_discrete_difference*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*banach*, *second_countable_topology*}
assumes *X*: *countable X*
assumes *null*: $\bigwedge x.\ x \in X \Longrightarrow$ *emeasure M* {*x*} = *0*
assumes *sets*: $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in$ *sets M*
assumes *eq*: $\bigwedge x.\ x \in$ *space M* $\Longrightarrow x \notin X \Longrightarrow f\ x = g\ x$
shows $integral^L\ M\ f = integral^L\ M\ g$
proof (*rule integral_eq_cases*)
show *eq*: *integrable M f* $\longleftrightarrow$ *integrable M g*
by (*rule integrable_discrete_difference*[**where** *X*=*X*]) *fact+*

assume *f*: *integrable M f*
show $integral^L\ M\ f = integral^L\ M\ g$
proof (*rule integral_cong_AE*)
show *f* $\in$ *borel_measurable M g* $\in$ *borel_measurable M*
using *f eq* **by** (*auto intro*: *borel_measurable_integrable*)
have *AE x in M*. $x \notin X$
by (*rule AE_discrete_difference*) *fact+*
with *AE_space* **show** *AE x in M*. $f\ x = g\ x$
by *eventually_elim fact*
qed
qed

lemma *has_bochner_integral_discrete_difference*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*banach*, *second_countable_topology*}
assumes *countable X*
assumes $\bigwedge x.\ x \in X \Longrightarrow$ *emeasure M* {*x*} = *0*
assumes $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in$ *sets M*
assumes $\bigwedge x.\ x \in$ *space M* $\Longrightarrow x \notin X \Longrightarrow f\ x = g\ x$
shows *has_bochner_integral M f x* $\longleftrightarrow$ *has_bochner_integral M g x*
by (*metis assms has_bochner_integral_iff integrable_discrete_difference integral_discrete_difference*)

lemma
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*banach*, *second_countable_topology*} **and** *w* :: *'a* $\Rightarrow$ *real*
assumes *f* $\in$ *borel_measurable M* $\bigwedge i.\ s\ i \in$ *borel_measurable M integrable M w*
assumes *lim*: *AE x in M*. $(\lambda i.\ s\ i\ x) \longrightarrow f\ x$
assumes *bound*: $\bigwedge i.$ *AE x in M*. *norm* $(s\ i\ x) \leq w\ x$
shows *integrable_dominated_convergence*: *integrable M f*
and *integrable_dominated_convergence2*: $\bigwedge i.$ *integrable M* $(s\ i)$
and *integral_dominated_convergence*: $(\lambda i.\ integral^L\ M\ (s\ i)) \longrightarrow integral^L\ M\ f$
proof −
have *w_nonneg*: *AE x in M*. $0 \leq w\ x$
using *bound*[*of 0*] **by** *eventually_elim* (*auto intro*: *norm_ge_zero order_trans*)
then have $(\int^+ x.\ w\ x\ \partial M) = (\int^+ x.\ norm\ (w\ x)\ \partial M)$

```
    by (intro nn_integral_cong_AE) auto
  with ‹integrable M w› have w: w ∈ borel_measurable M (∫⁺x. w x ∂M) < ∞
    unfolding integrable_iff_bounded by auto

  show int_s: ⋀i. integrable M (s i)
    unfolding integrable_iff_bounded
  proof
    fix i
    have (∫⁺ x. ennreal (norm (s i x)) ∂M) ≤ (∫⁺x. w x ∂M)
      using bound[of i] w_nonneg by (intro nn_integral_mono_AE) auto
    with w show (∫⁺ x. ennreal (norm (s i x)) ∂M) < ∞ by auto
  qed fact

  have all_bound: AE x in M. ∀i. norm (s i x) ≤ w x
    using bound unfolding AE_all_countable by auto

  show int_f: integrable M f
    unfolding integrable_iff_bounded
  proof
    have (∫⁺ x. ennreal (norm (f x)) ∂M) ≤ (∫⁺x. w x ∂M)
      using all_bound lim w_nonneg
    proof (intro nn_integral_mono_AE, eventually_elim)
      fix x assume ∀i. norm (s i x) ≤ w x (λi. s i x) ⟶ f x 0 ≤ w x
      then show ennreal (norm (f x)) ≤ ennreal (w x)
        by (metis LIMSEQ_le_const2 ennreal_leI tendsto_norm)
    qed
    with w show (∫⁺ x. ennreal (norm (f x)) ∂M) < ∞ by auto
  qed fact

  have (λn. ennreal (norm (integral^L M (s n) - integral^L M f))) ⟶ ennreal
0 (is ?d ⟶ ennreal 0)
  proof (rule tendsto_sandwich)
    show eventually (λn. ennreal 0 ≤ ?d n) sequentially (λ_. ennreal 0) ⟶
ennreal 0 by auto
    show eventually (λn. ?d n ≤ (∫⁺x. norm (s n x - f x) ∂M)) sequentially
    proof (intro always_eventually allI)
      fix n
      have ?d n = norm (integral^L M (λx. s n x - f x))
        using int_f int_s by simp
      also have ... ≤ (∫⁺x. norm (s n x - f x) ∂M)
        by (intro int_f int_s integrable_diff integral_norm_bound_ennreal)
      finally show ?d n ≤ (∫⁺x. norm (s n x - f x) ∂M) .
    qed
    show (λn. ∫⁺x. norm (s n x - f x) ∂M) ⟶ ennreal 0
      unfolding ennreal_0
      apply (subst norm_minus_commute)
    proof (rule nn_integral_dominated_convergence_norm[where w=w])
      show ⋀n. s n ∈ borel_measurable M
        using int_s unfolding integrable_iff_bounded by auto
```

```
    qed fact+
  qed
  then have (λn. integral^L M (s n) − integral^L M f) ⟶ 0
    by (simp add: tendsto_norm_zero_iff del: ennreal_0)
  from tendsto_add[OF this tendsto_const[of integral^L M f]]
  show (λi. integral^L M (s i)) ⟶ integral^L M f by simp
qed

context
  fixes s :: real ⇒ 'a ⇒ 'b::{banach, second_countable_topology} and w :: 'a ⇒
real
    and f :: 'a ⇒ 'b and M
  assumes f ∈ borel_measurable M ⋀t. s t ∈ borel_measurable M integrable M w
  assumes lim: AE x in M. ((λi. s i x) ⟶ f x) at_top
  assumes bound: ∀_F i in at_top. AE x in M. norm (s i x) ≤ w x
begin

lemma integral_dominated_convergence_at_top: ((λt. integral^L M (s t)) ⟶
integral^L M f) at_top
proof (rule tendsto_at_topI_sequentially)
  fix X :: nat ⇒ real assume X: filterlim X at_top sequentially
  from filterlim_iff[THEN iffD1, OF this, rule_format, OF bound]
  obtain N where w: ⋀n. N ≤ n ⟹ AE x in M. norm (s (X n) x) ≤ w x
    by (auto simp: eventually_sequentially)

  show (λn. integral^L M (s (X n))) ⟶ integral^L M f
  proof (rule LIMSEQ_offset, rule integral_dominated_convergence)
    show AE x in M. norm (s (X (n + N)) x) ≤ w x for n
      by (rule w) auto
    show AE x in M. (λn. s (X (n + N)) x) ⟶ f x
      using lim
    proof eventually_elim
      fix x assume ((λi. s i x) ⟶ f x) at_top
      then show (λn. s (X (n + N)) x) ⟶ f x
        by (intro LIMSEQ_ignore_initial_segment filterlim_compose[OF _ X])
    qed
  qed fact+
qed

lemma integrable_dominated_convergence_at_top: integrable M f
proof −
  from bound obtain N where w: ⋀n. N ≤ n ⟹ AE x in M. norm (s n x) ≤
w x
    by (auto simp: eventually_at_top_linorder)
  show ?thesis
  proof (rule integrable_dominated_convergence)
    show AE x in M. norm (s (N + i) x) ≤ w x for i :: nat
      by (intro w) auto
    show AE x in M. (λi. s (N + real i) x) ⟶ f x
```

```
      using lim
    proof eventually_elim
      fix x assume ((λi. s i x) ⟶ f x) at_top
      then show (λn. s (N + n) x) ⟶ f x
        by (rule filterlim_compose)
           (auto intro!: filterlim_tendsto_add_at_top filterlim_real_sequentially)
    qed
  qed fact+
qed

end

lemma integrable_mult_left_iff [simp]:
  fixes f :: 'a ⇒ real
  shows integrable M (λx. c * f x) ⟷ c = 0 ∨ integrable M f
  using integrable_mult_left[of c M f] integrable_mult_left[of 1 / c M λx. c * f
x]
  by (cases c = 0) auto

lemma integrable_mult_right_iff [simp]:
  fixes f :: 'a ⇒ real
  shows integrable M (λx. f x * c) ⟷ c = 0 ∨ integrable M f
  using integrable_mult_left_iff [of M c f] by (simp add: mult.commute)

lemma integrableI_nn_integral_finite:
  assumes [measurable]: f ∈ borel_measurable M
    and nonneg: AE x in M. 0 ≤ f x
    and finite: (∫⁺x. f x ∂M) = ennreal x
  shows integrable M f
proof (rule integrableI_bounded)
  have (∫⁺ x. ennreal (norm (f x)) ∂M) = (∫⁺ x. ennreal (f x) ∂M)
    using nonneg by (intro nn_integral_cong_AE) auto
  with finite show (∫⁺ x. ennreal (norm (f x)) ∂M) < ∞
    by auto
qed simp

lemma integral_nonneg_AE:
  fixes f :: 'a ⇒ real
  assumes nonneg: AE x in M. 0 ≤ f x
  shows 0 ≤ integral^L M f
proof cases
  assume f: integrable M f
  then have [measurable]: f ∈ M →_M borel
    by auto
  have (λx. max 0 (f x)) ∈ M →_M borel ⋀x. 0 ≤ max 0 (f x) integrable M (λx.
max 0 (f x))
    using f by auto
  from this have 0 ≤ integral^L M (λx. max 0 (f x))
  proof (induction rule: borel_measurable_induct_real)
```

```
    case (add f g)
    then have integrable M f integrable M g
      by (auto intro!: integrable_bound[OF add.prems])
    with add show ?case
      by (simp add: nn_integral_add)
  next
    case (seq U)
    show ?case
    proof (rule LIMSEQ_le_const)
      have U_le: x ∈ space M ⟹ U i x ≤ max 0 (f x) for x i
        using seq by (intro incseq_le) (auto simp: incseq_def le_fun_def)
      with seq nonneg show (λi. integral^L M (U i)) ⟶ LINT x|M. max 0 (f
x)
        by (intro integral_dominated_convergence)
          (simp_all, fastforce)
      have integrable M (U i) for i
        using seq.prems by (rule integrable_bound) (insert U_le seq, auto)
      with seq show ∃N. ∀n≥N. 0 ≤ integral^L M (U n)
        by auto
    qed
  qed (auto)
  also have ... = integral^L M f
    using nonneg by (auto intro!: integral_cong_AE)
  finally show ?thesis .
qed (simp add: not_integrable_integral_eq)

lemma integral_nonneg[simp]:
  fixes f :: 'a ⇒ real
  shows (⋀x. x ∈ space M ⟹ 0 ≤ f x) ⟹ 0 ≤ integral^L M f
  by (intro integral_nonneg_AE) auto

proposition nn_integral_eq_integral:
  assumes f: integrable M f
  assumes nonneg: AE x in M. 0 ≤ f x
  shows (∫⁺ x. f x ∂M) = integral^L M f
proof −
  { fix f :: 'a ⇒ real assume f: f ∈ borel_measurable M ⋀x. 0 ≤ f x integrable
M f
    then have (∫⁺ x. f x ∂M) = integral^L M f
    proof (induct rule: borel_measurable_induct_real)
      case (set A) then show ?case
      by (simp add: integrable_indicator_iff ennreal_indicator emeasure_eq_ennreal_measure)
    next
      case (mult f c) then show ?case
        by (auto simp: nn_integral_cmult ennreal_mult integral_nonneg_AE)
    next
      case (add g f)
      then have integrable M f integrable M g
        by (auto intro!: integrable_bound[OF add.prems])
```

```
    with add show ?case
      by (simp add: nn_integral_add integral_nonneg_AE)
  next
    case (seq U)
    show ?case
    proof (rule LIMSEQ_unique)
      have U_le_f: x ∈ space M ⟹ U i x ≤ f x for x i
        using seq by (intro incseq_le) (auto simp: incseq_def le_fun_def)
      have int_U: ⋀i. integrable M (U i)
        using seq f U_le_f by (intro integrable_bound[OF f(3)]) auto
      from U_le_f seq have (λi. integral^L M (U i)) ⟶ integral^L M f
        by (intro integral_dominated_convergence) auto
      then show (λi. ennreal (integral^L M (U i))) ⟶ ennreal (integral^L M
f)
        using seq f int_U by (simp add: f integral_nonneg_AE)
      have (λi. ∫⁺ x. U i x ∂M) ⟶ ∫⁺ x. f x ∂M
        using seq U_le_f f
        by (intro nn_integral_dominated_convergence[where w=f]) (auto simp:
integrable_iff_bounded)
      then show (λi. ∫ x. U i x ∂M) ⟶ ∫⁺x. f x ∂M
        using seq int_U by simp
    qed
  qed }
 from this[of λx. max 0 (f x)] assms have (∫⁺ x. max 0 (f x) ∂M) = integral^L
M (λx. max 0 (f x))
   by simp
 also have ... = integral^L M f
   using assms by (auto intro!: integral_cong_AE simp: integral_nonneg_AE)
 also have (∫⁺ x. max 0 (f x) ∂M) = (∫⁺ x. f x ∂M)
   using assms by (auto intro!: nn_integral_cong_AE simp: max_def)
 finally show ?thesis .
qed

lemma nn_integral_eq_integrable:
 assumes f ∈ M →_M borel AE x in M. 0 ≤ f x and 0 ≤ x
 shows (∫⁺x. f x ∂M) = ennreal x ⟷ (integrable M f ∧ integral^L M f = x)
 by (metis assms ennreal_inj integrableI_nn_integral_finite integral_nonneg_AE
nn_integral_eq_integral)

lemma
 fixes f :: _ ⇒ _ ⇒ 'a :: {banach, second_countable_topology}
 assumes integrable[measurable]: ⋀i. integrable M (f i)
 and summable: AE x in M. summable (λi. norm (f i x))
 and sums: summable (λi. (∫ x. norm (f i x) ∂M))
 shows integrable_suminf: integrable M (λx. (∑ i. f i x)) (is integrable M ?S)
   and sums_integral: (λi. integral^L M (f i)) sums (∫ x. (∑ i. f i x) ∂M) (is ?f
sums ?x)
   and integral_suminf: (∫ x. (∑ i. f i x) ∂M) = (∑ i. integral^L M (f i))
   and summable_integral: summable (λi. integral^L M (f i))
```

proof −
have *1*: *integrable M* (λ*x*. ∑ *i*. *norm* (*f i x*))
proof (*rule integrableI_bounded*)
have ⋀*x*. *summable* (λ*i*. *norm* (*f i x*)) ⟹
ennreal (*norm* (∑ *i*. *norm* (*f i x*))) = (∑ *i*. *ennreal* (*norm* (*f i x*)))
by (*simp add*: *suminf_nonneg ennreal_suminf_neq_top*)
then have (∫⁺ *x*. *ennreal* (*norm* (∑ *i*. *norm* (*f i x*))) ∂*M*) = (∫⁺ *x*. (∑ *i*. *ennreal* (*norm* (*f i x*))) ∂*M*)
using *local.summable* **by** (*intro nn_integral_cong_AE*) *auto*
also have ... = (∑ *i*. ∫⁺ *x*. *norm* (*f i x*) ∂*M*)
by (*intro nn_integral_suminf*) *auto*
also have ... = (∑ *i*. *ennreal* (∫ *x*. *norm* (*f i x*) ∂*M*))
by (*intro arg_cong*[**where** *f*=*suminf*] *ext nn_integral_eq_integral integrable_norm integrable*) *auto*
finally show (∫⁺ *x*. *ennreal* (*norm* (∑ *i*. *norm* (*f i x*))) ∂*M*) < ∞
by (*simp add*: *sums ennreal_suminf_neq_top less_top*[*symmetric*] *integral_nonneg_AE*)
qed *simp*

have *2*: *AE x in M*. (λ*n*. ∑ *i*<*n*. *f i x*) ⟶ (∑ *i*. *f i x*)
using *summable* **by** *eventually_elim* (*auto intro*: *summable_LIMSEQ summable_norm_cancel*)

have *3*: ⋀*j*. *AE x in M*. *norm* (∑ *i*<*j*. *f i x*) ≤ (∑ *i*. *norm* (*f i x*))
using *summable*
proof *eventually_elim*
fix *j x* **assume** [*simp*]: *summable* (λ*i*. *norm* (*f i x*))
have *norm* (∑ *i*<*j*. *f i x*) ≤ (∑ *i*<*j*. *norm* (*f i x*)) **by** (*rule norm_sum*)
also have ... ≤ (∑ *i*. *norm* (*f i x*))
using *sum_le_suminf*[*of* λ*i*. *norm* (*f i x*)] **unfolding** *sums_iff* **by** *auto*
finally show *norm* (∑ *i*<*j*. *f i x*) ≤ (∑ *i*. *norm* (*f i x*)) **by** *simp*
qed

note *ibl* = *integrable_dominated_convergence*[*OF _ _ 1 2 3*]
note *int* = *integral_dominated_convergence*[*OF _ _ 1 2 3*]

show *integrable M ?S*
by (*rule ibl*) *measurable*

show *?f sums ?x* **unfolding** *sums_def*
using *int* **by** (*simp add*: *integrable*)
then show *?x* = *suminf ?f summable ?f*
unfolding *sums_iff* **by** *auto*
qed

proposition *integral_norm_bound* [*simp*]:
fixes *f* :: _ ⇒ *'a* :: {*banach*, *second_countable_topology*}
shows *norm* (*integral*^{L} *M f*) ≤ (∫ *x*. *norm* (*f x*) ∂*M*)
proof (*cases integrable M f*)
case *True* **then show** *?thesis*

using *nn_integral_eq_integral*[*of M* λ*x*. *norm* (*f x*)] *integral_norm_bound_ennreal*[*of M f*]
by (*simp add*: *integral_nonneg_AE*)
next
case *False*
then have *norm* (*integral*^{L} *M f*) = *0* **by** (*simp add*: *not_integrable_integral_eq*)
moreover have (∫ *x*. *norm* (*f x*) ∂*M*) ≥ *0* **by** *auto*
ultimately show *?thesis* **by** *simp*
qed

proposition *integral_abs_bound* [*simp*]:
fixes *f* :: *'a* ⇒ *real* **shows** *abs* (∫ *x*. *f x* ∂*M*) ≤ (∫ *x*. |*f x*| ∂*M*)
using *integral_norm_bound*[*of M f*] **by** *auto*

lemma *integral_eq_nn_integral*:
assumes *f* ∈ *borel_measurable M*
assumes *AE x in M*. *0* ≤ *f x*
shows *integral*^{L} *M f* = *enn2real* (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*)
by (*metis assms enn2real_ennreal enn2real_top infinity_ennreal_def integrableI_nonneg integral_nonneg_AE*
less_top nn_integral_eq_integral not_integrable_integral_eq)

lemma *enn2real_nn_integral_eq_integral*:
assumes *AE x in M*. *f x* = *ennreal* (*g x*) **and** *nn*: *AE x in M*. *0* ≤ *g x*
and *g* ∈ *M* →$_M$ *borel*
shows *enn2real* (∫$^+$*x*. *f x* ∂*M*) = (∫ *x*. *g x* ∂*M*)
by (*metis assms integral_eq_nn_integral nn nn_integral_cong_AE*)

lemma *has_bochner_integral_nn_integral*:
assumes *f* ∈ *borel_measurable M AE x in M*. *0* ≤ *f x 0* ≤ *x*
assumes (∫$^+$*x*. *f x* ∂*M*) = *ennreal x*
shows *has_bochner_integral M f x*
by (*metis assms has_bochner_integral_iff nn_integral_eq_integrable*)

lemma *integrableI_simple_bochner_integrable*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
shows *simple_bochner_integrable M f* ⟹ *integrable M f*
using *has_bochner_integral_simple_bochner_integrable integrable.intros* **by** *blast*

proposition *integrable_induct*[*consumes 1*, *case_names base add lim*, *induct pred*: *integrable*]:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *integrable M f*
assumes *base*: ⋀*A c*. *A* ∈ *sets M* ⟹ *emeasure M A* < ∞ ⟹ *P* (λ*x*. *indicator A x* ∗$_R$ *c*)
assumes *add*: ⋀*f g*. *integrable M f* ⟹ *P f* ⟹ *integrable M g* ⟹ *P g* ⟹ *P* (λ*x*. *f x* + *g x*)
assumes *lim*: ⋀*f s*. (⋀*i*. *integrable M* (*s i*)) ⟹ (⋀*i*. *P* (*s i*)) ⟹
(⋀*x*. *x* ∈ *space M* ⟹ (λ*i*. *s i x*) ⟶ *f x*) ⟹

```
  (⋀i x. x ∈ space M ⟹ norm (s i x) ≤ 2 * norm (f x)) ⟹ integrable M f ⟹
P f
  shows P f
proof −
  from ‹integrable M f› have f: f ∈ borel_measurable M (∫⁺x. norm (f x) ∂M)
< ∞
    unfolding integrable_iff_bounded by auto
  from borel_measurable_implies_sequence_metric[OF f(1)]
  obtain s where s: ⋀i. simple_function M (s i) ⋀x. x ∈ space M ⟹ (λi. s i
x) ⟶ f x
    ⋀i x. x ∈ space M ⟹ norm (s i x) ≤ 2 * norm (f x)
    unfolding norm_conv_dist by metis

  { fix f A
    have [simp]: P (λx. 0)
      using base[of {} undefined] by simp
    have (⋀i::'b. i ∈ A ⟹ integrable M (f i::'a ⇒ 'b)) ⟹
    (⋀i. i ∈ A ⟹ P (f i)) ⟹ P (λx. ∑ i∈A. f i x)
    by (induct A rule: infinite_finite_induct) (auto intro!: add) }
  note sum = this

  define s' where [abs_def]: s' i z = indicator (space M) z *R s i z for i z
  then have s'_eq_s: ⋀i x. x ∈ space M ⟹ s' i x = s i x
    by simp

  have sf[measurable]: ⋀i. simple_function M (s' i)
    unfolding s'_def using s(1)
   by (intro simple_function_compose2[where h=(*R)] simple_function_indicator)
auto

  { fix i
    have ⋀z. {y. s' i z = y ∧ y ∈ s' i ‘ space M ∧ y ≠ 0 ∧ z ∈ space M} =
        (if z ∈ space M ∧ s' i z ≠ 0 then {s' i z} else {})
      by (auto simp: s'_def split: split_indicator)
    then have ⋀z. s' i = (λz. ∑ y∈s' i‘space M − {0}. indicator {x∈space M. s'
i x = y} z *R y)
      using sf by (auto simp: fun_eq_iff simple_function_def s'_def) }
  note s'_eq = this

  show P f
  proof (rule lim)
    fix i

    have (∫⁺x. norm (s' i x) ∂M) ≤ (∫⁺x. ennreal (2 * norm (f x)) ∂M)
      using s by (intro nn_integral_mono) (auto simp: s'_eq_s)
    also have ... < ∞
    using f by (simp add: nn_integral_cmult ennreal_mult_less_top ennreal_mult)
    finally have sbi: simple_bochner_integrable M (s' i)
      using sf by (intro simple_bochner_integrableI_bounded) auto
```

```
    then show integrable M (s' i)
      by (rule integrableI_simple_bochner_integrable)

    { fix x assumex ∈ space M s' i x ≠ 0
      then have emeasure M {y ∈ space M. s' i y = s' i x} ≤ emeasure M {y ∈
space M. s' i y ≠ 0}
        by (intro emeasure_mono) auto
      also have ... < ∞
        using sbi by (auto elim: simple_bochner_integrable.cases simp: less_top)
      finally have emeasure M {y ∈ space M. s' i y = s' i x} ≠ ∞ by simp }
    then show P (s' i)
      by (subst s'_eq) (auto intro!: sum base simp: less_top)

    fix x assume x ∈ space M with s show (λi. s' i x) ⟶ f x
      by (simp add: s'_eq_s)
    show norm (s' i x) ≤ 2 * norm (f x)
      using ‹x ∈ space M› s by (simp add: s'_eq_s)
  qed fact
qed

lemma integral_eq_zero_AE:
  (AE x in M. f x = 0) ⟹ integral^L M f = 0
  using integral_cong_AE[of f M λ_. 0]
  by (cases integrable M f) (simp_all add: not_integrable_integral_eq)

lemma integral_nonneg_eq_0_iff_AE:
  fixes f :: _ ⇒ real
  assumes f[measurable]: integrable M f and nonneg: AE x in M. 0 ≤ f x
  shows integral^L M f = 0 ⟷ (AE x in M. f x = 0)
proof
  assume integral^L M f = 0
  then have integral^N M f = 0
    using nn_integral_eq_integral[OF f nonneg] by simp
  then have AE x in M. ennreal (f x) ≤ 0
    by (simp add: nn_integral_0_iff_AE)
  with nonneg show AE x in M. f x = 0
    by auto
qed (auto simp: integral_eq_zero_AE)

lemma integral_mono_AE:
  fixes f :: 'a ⇒ real
  assumes integrable M f integrable M g AE x in M. f x ≤ g x
  shows integral^L M f ≤ integral^L M g
proof −
  have 0 ≤ integral^L M (λx. g x − f x)
    using assms by (intro integral_nonneg_AE integrable_diff assms) auto
  also have ... = integral^L M g − integral^L M f
    by (intro integral_diff assms)
  finally show ?thesis by simp
```

qed

lemma *integral_mono*:
fixes $f :: 'a \Rightarrow real$
shows *integrable M f* $\Longrightarrow$ *integrable M g* $\Longrightarrow$ $(\bigwedge x.\ x \in space\ M \Longrightarrow f\ x \leq g\ x) \Longrightarrow$ $integral^L\ M\ f \leq integral^L\ M\ g$
by (*intro integral_mono_AE*) *auto*

lemma *integral_norm_bound_integral*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::\{banach, second\_countable\_topology\}$
assumes *integrable M f integrable M g* $\bigwedge x.\ x \in space\ M \Longrightarrow norm(f\ x) \leq g\ x$
shows $norm\ (\int x.\ f\ x\ \partial M) \leq (\int x.\ g\ x\ \partial M)$
by (*smt* (*verit*, *best*) *assms integrable_norm integral_mono integral_norm_bound*)

lemma *integral_abs_bound_integral*:
fixes $f :: 'a::euclidean\_space \Rightarrow real$
assumes *integrable M f integrable M g* $\bigwedge x.\ x \in space\ M \Longrightarrow |f\ x| \leq g\ x$
shows $|\int x.\ f\ x\ \partial M| \leq (\int x.\ g\ x\ \partial M)$
by (*metis integral_norm_bound_integral assms real_norm_def*)

The next two statements are useful to bound Lebesgue integrals, as they avoid one integrability assumption. The price to pay is that the upper function has to be nonnegative, but this is often true and easy to check in computations.

lemma *integral_mono_AE'*:
fixes $f::\_ \Rightarrow real$
assumes *integrable M f AE x in M. g x* $\leq$ *f x AE x in M.* $0 \leq f\ x$
shows $(\int x.\ g\ x\ \partial M) \leq (\int x.\ f\ x\ \partial M)$
by (*metis assms integral_mono_AE integral_nonneg_AE not_integrable_integral_eq*)

lemma *integral_mono'*:
fixes $f::\_ \Rightarrow real$
assumes *integrable M f* $\bigwedge x.\ x \in space\ M \Longrightarrow g\ x \leq f\ x\ \bigwedge x.\ x \in space\ M \Longrightarrow 0 \leq f\ x$
shows $(\int x.\ g\ x\ \partial M) \leq (\int x.\ f\ x\ \partial M)$
by (*rule integral_mono_AE'*, *insert assms*, *auto*)

lemma (**in** *finite_measure*) *integrable_measure*:
assumes *I*: *disjoint_family_on X I countable I*
shows *integrable* (*count_space I*) ($\lambda i.$ *measure M* (*X i*))
proof −
have $(\int^+ i.$ *measure M* (*X i*) $\partial count\_space\ I) = (\int^+ i.$ *measure M* (*if X i* $\in$ *sets M then X i else* {}) $\partial count\_space\ I)$
by (*auto intro*!: *nn_integral_cong measure_notin_sets*)
also have ... = *measure M* $(\bigcup i \in I.$ *if X i* $\in$ *sets M then X i else* {})
using *I* **unfolding** *emeasure_eq_measure*[*symmetric*]
by (*subst emeasure_UN_countable*) (*auto simp*: *disjoint_family_on_def*)
finally show *?thesis*

by (*auto intro*!: *integrableI_bounded*)
qed

lemma *nn_integral_nonneg_infinite*:
fixes $f::'a \Rightarrow real$
assumes $f \in borel\_measurable\ M$ $\neg$ *integrable M f AE x in M*. $f\ x \geq 0$
shows $(\int^+ x.\ f\ x\ \partial M) = \infty$
using *assms integrableI_nonneg less_top* **by** *auto*

lemma *integral_real_bounded*:
assumes $0 \leq r$ $integral^N\ M\ f \leq ennreal\ r$
shows $integral^L\ M\ f \leq r$
proof *cases*
assume [*simp*]: *integrable M f*

have $integral^L\ M\ (\lambda x.\ max\ 0\ (f\ x)) = integral^N\ M\ (\lambda x.\ max\ 0\ (f\ x))$
by (*intro nn_integral_eq_integral*[*symmetric*]) *auto*
also have $\ldots = integral^N\ M\ f$
by (*intro nn_integral_cong*) (*simp add*: *max_def ennreal_neg*)
also have $\ldots \leq r$
by *fact*
finally have $integral^L\ M\ (\lambda x.\ max\ 0\ (f\ x)) \leq r$
using ‹$0 \leq r$› **by** *simp*

moreover have $integral^L\ M\ f \leq integral^L\ M\ (\lambda x.\ max\ 0\ (f\ x))$
by (*rule integral_mono_AE*) *auto*
ultimately show *?thesis*
by *simp*
next
assume $\neg$ *integrable M f* **then show** *?thesis*
using ‹$0 \leq r$› **by** (*simp add*: *not_integrable_integral_eq*)
qed

lemma *integrable_MIN*:
fixes $f::\_ \Rightarrow \_ \Rightarrow real$
shows $[\![\ finite\ I;\ I \neq \{\};\ \bigwedge i.\ i \in I \Longrightarrow integrable\ M\ (f\ i)\ ]\!]$
$\Longrightarrow integrable\ M\ (\lambda x.\ MIN\ i{\in}I.\ f\ i\ x)$
by (*induct rule*: *finite_ne_induct*) *simp*+

lemma *integrable_MAX*:
fixes $f::\_ \Rightarrow \_ \Rightarrow real$
shows $[\![\ finite\ I;\ I \neq \{\};\ \bigwedge i.\ i \in I \Longrightarrow integrable\ M\ (f\ i)\ ]\!]$
$\Longrightarrow integrable\ M\ (\lambda x.\ MAX\ i{\in}I.\ f\ i\ x)$
by (*induct rule*: *finite_ne_induct*) *simp*+

theorem *integral_Markov_inequality*:
assumes [*measurable*]: *integrable M u* **and** *AE x in M*. $0 \leq u\ x\ 0 < (c::real)$
shows $(emeasure\ M)\ \{x{\in}space\ M.\ u\ x \geq c\} \leq (1/c) * (\int x.\ u\ x\ \partial M)$
proof −

have $(\int^+ x.\ ennreal(u\ x) * indicator\ (space\ M)\ x\ \partial M) \leq (\int^+ x.\ u\ x\ \partial M)$
by (*rule nn_integral_mono_AE*, *auto simp*: ‹*c*>*0*› *less_eq_real_def*)
also have $\ldots = (\int x.\ u\ x\ \partial M)$
by (*rule nn_integral_eq_integral*, *auto simp*: *assms*)
finally have $*$: $(\int^+ x.\ ennreal(u\ x) * indicator\ (space\ M)\ x\ \partial M) \leq (\int x.\ u\ x\ \partial M)$
by *simp*

have $\{x \in space\ M.\ u\ x \geq c\} = \{x \in space\ M.\ ennreal(1/c) * u\ x \geq 1\} \cap (space\ M)$
using ‹*c*>*0*› **by** (*auto simp*: *ennreal_mult'*[*symmetric*])
then have $emeasure\ M\ \{x \in space\ M.\ u\ x \geq c\} = emeasure\ M\ (\{x \in space\ M.\ ennreal(1/c) * u\ x \geq 1\})$
by *simp*
also have $\ldots \leq ennreal(1/c) * (\int^+ x.\ ennreal(u\ x) * indicator\ (space\ M)\ x\ \partial M)$
by (*rule nn_integral_Markov_inequality*) (*auto simp*: *assms*)
also have $\ldots \leq ennreal(1/c) * (\int x.\ u\ x\ \partial M)$
by (*rule mult_left_mono*) (*use* $*$ ‹*c* > *0*› **in** *auto*)
finally show *?thesis*
using ‹*0*<*c*› **by** (*simp add*: *ennreal_mult'*[*symmetric*])
qed

theorem *integral_Markov_inequality_measure*:
assumes [*measurable*]: *integrable M u* **and** $A \in sets\ M$ **and** *AE x in M*. $0 \leq u\ x$ $0 < (c::real)$
shows $measure\ M\ \{x \in space\ M.\ u\ x \geq c\} \leq (\int x.\ u\ x\ \partial M)\ /\ c$
proof −
have *le*: $emeasure\ M\ \{x \in space\ M.\ u\ x \geq c\} \leq ennreal\ ((1/c) * (\int x.\ u\ x\ \partial M))$
by (*rule integral_Markov_inequality*) (*use assms* **in** *auto*)
also have $\ldots < top$
by *auto*
finally have $ennreal\ (measure\ M\ \{x \in space\ M.\ u\ x \geq c\}) = emeasure\ M\ \{x \in space\ M.\ u\ x \geq c\}$
by (*intro emeasure_eq_ennreal_measure* [*symmetric*]) *auto*
also note *le*
finally show *?thesis*
by (*simp add*: *assms divide_nonneg_pos integral_nonneg_AE*)
qed

theorem (**in** *finite_measure*) *second_moment_method*:
assumes [*measurable*]: $f \in M \to_M borel$
assumes $integrable\ M\ (\lambda x.\ f\ x\ \hat{}\ 2)$
defines $\mu \equiv lebesgue\_integral\ M\ f$
assumes $a > 0$
shows $measure\ M\ \{x \in space\ M.\ |f\ x| \geq a\} \leq lebesgue\_integral\ M\ (\lambda x.\ f\ x\ \hat{}\ 2)\ /\ a^2$
proof −
have *integrable*: *integrable M f*

```
    using assms by (blast dest: square_integrable_imp_integrable)
  have {x∈space M. |f x| ≥ a} = {x∈space M. f x ^ 2 ≥ a²}
    using ‹a > 0› by (simp flip: abs_le_square_iff)
  hence measure M {x∈space M. |f x| ≥ a} = measure M {x∈space M. f x ^ 2 ≥
a²}
    by simp
  also have ... ≤ lebesgue_integral M (λx. f x ^ 2) / a²
    using assms by (intro integral_Markov_inequality_measure) auto
  finally show ?thesis .
qed

lemma integral_ineq_eq_0_then_AE:
  fixes f::_ ⇒ real
  assumes AE x in M. f x ≤ g x integrable M f integrable M g
          (∫ x. f x ∂M) = (∫ x. g x ∂M)
  shows AE x in M. f x = g x
proof −
  define h where h = (λx. g x − f x)
  have AE x in M. h x = 0
    apply (subst integral_nonneg_eq_0_iff_AE[symmetric])
    unfolding h_def using assms by auto
  then show ?thesis unfolding h_def by auto
qed

lemma not_AE_zero_int_E:
  fixes f::'a ⇒ real
  assumes AE x in M. f x ≥ 0 (∫ x. f x ∂M) > 0
      and [measurable]: f ∈ borel_measurable M
  shows ∃ A e. A ∈ sets M ∧ e>0 ∧ emeasure M A > 0 ∧ (∀ x ∈ A. f x ≥ e)
proof (rule not_AE_zero_E, auto simp: assms)
  assume ∗: AE x in M. f x = 0
  have (∫ x. f x ∂M) = (∫ x. 0 ∂M)
    by (rule integral_cong_AE, auto simp: ∗)
  then have (∫ x. f x ∂M) = 0 by simp
  then show False using assms(2) by simp
qed

proposition tendsto_L1_int:
  fixes u :: _ ⇒ _ ⇒ 'b::{banach, second_countable_topology}
  assumes [measurable]: ⋀n. integrable M (u n) integrable M f
          and ((λn. (∫⁺x. norm(u n x − f x) ∂M)) ⟶ 0) F
  shows ((λn. (∫ x. u n x ∂M)) ⟶ (∫ x. f x ∂M)) F
proof −
  have ((λn. norm((∫ x. u n x ∂M) − (∫ x. f x ∂M))) ⟶ (0::ennreal)) F
  proof (rule tendsto_sandwich[of λ_. 0, where ?h = λn. (∫⁺x. norm(u n x −
f x) ∂M)], auto simp: assms)
    {
      fix n
      have (∫ x. u n x ∂M) − (∫ x. f x ∂M) = (∫ x. u n x − f x ∂M)
```

```
      by (simp add: assms)
    then have norm((∫ x. u n x ∂M) − (∫ x. f x ∂M)) = norm (∫ x. u n x − f
x ∂M)
      by auto
    also have ... ≤ (∫ x. norm(u n x − f x) ∂M)
      by (rule integral_norm_bound)
   finally have ennreal(norm((∫ x. u n x ∂M) − (∫ x. f x ∂M))) ≤ (∫ x. norm(u
n x − f x) ∂M)
      by simp
    also have ... = (∫⁺x. norm(u n x − f x) ∂M)
      by (simp add: assms nn_integral_eq_integral)
    finally have norm((∫ x. u n x ∂M) − (∫ x. f x ∂M)) ≤ (∫⁺x. norm(u n x
− f x) ∂M)
      by simp
  }
  then show eventually (λn. norm((∫ x. u n x ∂M) − (∫ x. f x ∂M)) ≤ (∫⁺x.
norm(u n x − f x) ∂M)) F
    by auto
 qed
 then have ((λn. norm((∫ x. u n x ∂M) − (∫ x. f x ∂M))) ⟶ 0) F
  by (simp flip: ennreal_0)
 then have ((λn. ((∫ x. u n x ∂M) − (∫ x. f x ∂M))) ⟶ 0) F
  using tendsto_norm_zero_iff by blast
 then show ?thesis
  using Lim_null by auto
qed
```

The next lemma asserts that, if a sequence of functions converges in L^1, then it admits a subsequence that converges almost everywhere.

```
proposition tendsto_L1_AE_subseq:
 fixes u :: nat ⇒ 'a ⇒ 'b::{banach, second_countable_topology}
 assumes [measurable]: ⋀n. integrable M (u n)
    and (λn. (∫ x. norm(u n x) ∂M)) ⟶ 0
 shows ∃ r::nat⇒nat. strict_mono r ∧ (AE x in M. (λn. u (r n) x) ⟶ 0)
proof −
 {
  fix k
  have eventually (λn. (∫ x. norm(u n x) ∂M) < (1/2)^k) sequentially
   using order_tendstoD(2)[OF assms(2)] by auto
  with eventually_elim2[OF eventually_gt_at_top[of k] this]
  have ∃ n>k. (∫ x. norm(u n x) ∂M) < (1/2)^k
   by (metis eventually_False_sequentially)
 }
 then have ∃ r. ∀ n. True ∧ (r (Suc n) > r n ∧ (∫ x. norm(u (r (Suc n)) x) ∂M)
< (1/2)^(r n))
  by (intro dependent_nat_choice, auto)
 then obtain r0 where r0: strict_mono r0 ⋀n. (∫ x. norm(u (r0 (Suc n)) x)
∂M) < (1/2)^(r0 n)
  by (auto simp: strict_mono_Suc_iff)
```

```
  define r where r = (λn. r0(n+1))
 have strict_mono r unfolding r_def using r0(1) by (simp add: strict_mono_Suc_iff)
  have I: (∫⁺x. norm(u (r n) x) ∂M) < ennreal((1/2)^n) for n
  proof −
    have r0 n ≥ n using ‹strict_mono r0› by (simp add: seq_suble)
     have (1/2::real)^(r0 n) ≤ (1/2)^n by (rule power_decreasing[OF ‹r0 n ≥
n›], auto)
    then have (∫ x. norm(u (r0 (Suc n)) x) ∂M) < (1/2)^n
      using r0(2) less_le_trans by blast
    then have (∫ x. norm(u (r n) x) ∂M) < (1/2)^n
      unfolding r_def by auto
    moreover have (∫⁺x. norm(u (r n) x) ∂M) = (∫ x. norm(u (r n) x) ∂M)
     by (rule nn_integral_eq_integral, auto simp: integrable_norm[OF assms(1)[of
r n]])
    ultimately show ?thesis by (auto intro: ennreal_lessI)
  qed

  have AE x in M. limsup (λn. ennreal (norm(u (r n) x))) ≤ 0
  proof (rule AE_upper_bound_inf_ennreal)
    fix e::real assume e > 0
    define A where A = (λn. {x ∈ space M. norm(u (r n) x) ≥ e})
    have A_meas [measurable]: ⋀n. A n ∈ sets M unfolding A_def by auto
    have A_bound: emeasure M (A n) < (1/e) * ennreal((1/2)^n) for n
    proof −
      have ∗: indicator (A n) x ≤ (1/e) * ennreal(norm(u (r n) x)) for x
      using ‹0 < e› by (cases x ∈ A n) (auto simp: A_def ennreal_mult[symmetric])
      have emeasure M (A n) = (∫⁺x. indicator (A n) x ∂M) by auto
      also have ... ≤ (∫⁺x. (1/e) * ennreal(norm(u (r n) x)) ∂M)
        by (meson ∗ nn_integral_mono)
      also have ... = (1/e) * (∫⁺x. norm(u (r n) x) ∂M)
        using ‹e > 0› by (force simp add: intro: nn_integral_cmult)
      also have ... < (1/e) * ennreal((1/2)^n)
        using I[of n] ‹e > 0› by (intro ennreal_mult_strict_left_mono) auto
      finally show ?thesis by simp
    qed
    have A_fin: emeasure M (A n) < ∞ for n
      using ‹e > 0› A_bound[of n]
      by (auto simp: ennreal_mult less_top[symmetric])

    have A_sum: summable (λn. measure M (A n))
    proof (rule summable_comparison_test′)
      have summable (λn. (1/(2::real))^n)
        by (simp add: summable_geometric)
     then show summable (λn. (1/e) * (1/2)^n) using summable_mult by blast
      fix n::nat assume n ≥ 0
      have norm(measure M (A n)) = measure M (A n) by simp
       also have ... = enn2real(emeasure M (A n)) unfolding measure_def by
simp
      also have ... < enn2real((1/e) * (1/2)^n)
```

```
      using A_bound[of n] ‹emeasure M (A n) < ∞› ‹0 < e›
        by (auto simp: emeasure_eq_ennreal_measure ennreal_mult[symmetric]
ennreal_less_iff)
    also have ... = (1/e) * (1/2)^n
      using ‹0<e› by auto
    finally show norm(measure M (A n)) ≤ (1/e) * (1/2)^n by simp
  qed

  have AE x in M. eventually (λn. x ∈ space M − A n) sequentially
    by (rule borel_cantelli_AE1[OF A_meas A_fin A_sum])
  moreover
  {
    fix x assume eventually (λn. x ∈ space M − A n) sequentially
    moreover have norm(u (r n) x) ≤ ennreal e if x ∈ space M − A n for n
      using that unfolding A_def by (auto intro: ennreal_leI)
    ultimately have eventually (λn. norm(u (r n) x) ≤ ennreal e) sequentially
      by (simp add: eventually_mono)
    then have limsup (λn. ennreal (norm(u (r n) x))) ≤ e
      by (simp add: Limsup_bounded)
  }
  ultimately show AE x in M. limsup (λn. ennreal (norm(u (r n) x))) ≤ 0 +
ennreal e
    by auto
 qed
 moreover
 {
  fix x assume x: limsup (λn. ennreal (norm(u (r n) x))) ≤ 0
  moreover have liminf (λn. ennreal (norm(u (r n) x))) ≤ 0
    by (metis x Liminf_le_Limsup le_zero_eq sequentially_bot)
  ultimately have (λn. ennreal (norm(u (r n) x))) ⟶ 0
   using tendsto_Limsup[of sequentially λn. ennreal (norm(u (r n) x))] by auto
  then have (λn. norm(u (r n) x)) ⟶ 0
    by (simp flip: ennreal_0)
  then have (λn. u (r n) x) ⟶ 0
    by (simp add: tendsto_norm_zero_iff)
 }
 ultimately have AE x in M. (λn. u (r n) x) ⟶ 0 by auto
 then show ?thesis using ‹strict_mono r› by auto
qed
```

7.9.1 Restricted measure spaces

```
lemma integrable_restrict_space:
 fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
 assumes Ω[simp]: Ω ∩ space M ∈ sets M
 shows integrable (restrict_space M Ω) f ⟷ integrable M (λx. indicator Ω x
*R f x)
 unfolding integrable_iff_bounded
   borel_measurable_restrict_space_iff[OF Ω]
```

nn_integral_restrict_space[*OF* Ω]
by (*simp add*: *ac_simps ennreal_indicator ennreal_mult*)

lemma *integral_restrict_space*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes Ω[*simp*]: Ω ∩ *space M* ∈ *sets M*
shows *integral*^{L} (*restrict_space M* Ω) *f* = *integral*^{L} *M* (λ*x*. *indicator* Ω *x* ∗$_R$ *f x*)
proof (*rule integral_eq_cases*)
assume *integrable* (*restrict_space M* Ω) *f*
then show *?thesis*
proof *induct*
case (*base A c*) **then show** *?case*
by (*simp add*: *indicator_inter_arith*[*symmetric*] *sets_restrict_space_iff*
emeasure_restrict_space Int_absorb1 measure_restrict_space)
next
case (*add g f*) **then show** *?case*
by (*simp add*: *scaleR_add_right integrable_restrict_space*)
next
case (*lim f s*)
show *?case*
proof (*rule LIMSEQ_unique*)
show (λ*i*. *integral*^{L} (*restrict_space M* Ω) (*s i*)) ⟶ *integral*^{L} (*restrict_space M* Ω) *f*
using *lim* **by** (*intro integral_dominated_convergence*[**where** *w*=λ*x*. *2* ∗ *norm* (*f x*)]) *simp_all*

show (λ*i*. *integral*^{L} (*restrict_space M* Ω) (*s i*)) ⟶ (∫ *x*. *indicator* Ω *x* ∗$_R$ *f x* ∂*M*)
unfolding *lim*
using *lim*
by (*intro integral_dominated_convergence*[**where** *w*=λ*x*. *2* ∗ *norm* (*indicator* Ω *x* ∗$_R$ *f x*)])
(*auto simp*: *space_restrict_space integrable_restrict_space simp del*: *norm_scaleR*
split: *split_indicator*)
qed
qed
qed (*simp add*: *integrable_restrict_space*)

lemma *integral_empty*:
assumes *space M* = {}
shows *integral*^{L} *M f* = *0*
by (*metis AE_I2 assms empty_iff integral_eq_zero_AE*)

7.9.2 Measure spaces with an associated density

lemma *integrable_density*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*} **and** *g* :: *'a* ⇒ *real*

assumes [*measurable*]: $f \in$ *borel_measurable* M $g \in$ *borel_measurable* M
and *nn*: *AE* x *in* M. $0 \leq g\ x$
shows *integrable* (*density* M g) $f \longleftrightarrow$ *integrable* M ($\lambda x.\ g\ x *_R f\ x$)
unfolding *integrable_iff_bounded* **using** *nn*
apply (*simp add*: *nn_integral_density less_top*[*symmetric*])
apply (*intro arg_cong2*[**where** f=(=)] *refl nn_integral_cong_AE*)
apply (*auto simp*: *ennreal_mult*)
done

lemma *integral_density*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*} **and** $g :: 'a \Rightarrow$ *real*
assumes f: $f \in$ *borel_measurable* M
and g[*measurable*]: $g \in$ *borel_measurable* M *AE* x *in* M. $0 \leq g\ x$
shows *integral*^{L} (*density* M g) f = *integral*^{L} M ($\lambda x.\ g\ x *_R f\ x$)
proof (*rule integral_eq_cases*)
assume *integrable* (*density* M g) f
then show *?thesis*
proof *induct*
case (*base* A c)
then have [*measurable*]: $A \in$ *sets* M **by** *auto*

have *int*: *integrable* M ($\lambda x.\ g\ x * $ *indicator* $A\ x$)
using g *base integrable_density*[*of indicator* $A :: 'a \Rightarrow$ *real* $M\ g$] **by** *simp*
then have *integral*^{L} M ($\lambda x.\ g\ x * $ *indicator* $A\ x$) = ($\int^+ x.$ *ennreal* ($g\ x *$ *indicator* $A\ x$) ∂M)
using g **by** (*subst nn_integral_eq_integral*) *auto*
also have ... = ($\int^+ x.$ *ennreal* ($g\ x$) $*$ *indicator* $A\ x\ \partial M$)
by (*intro nn_integral_cong*) (*auto split*: *split_indicator*)
also have ... = *emeasure* (*density* M g) A
by (*rule emeasure_density*[*symmetric*]) *auto*
also have ... = *ennreal* (*measure* (*density* M g) A)
using *base* **by** (*auto intro*: *emeasure_eq_ennreal_measure*)
also have ... = *integral*^{L} (*density* M g) (*indicator* A)
using *base* **by** *simp*
finally show *?case*
using *base* g
apply (*simp add*: *int integral_nonneg_AE*)
apply (*subst* (*asm*) *ennreal_inj*)
apply (*auto intro*!: *integral_nonneg_AE*)
done
next
case (*add* f h)
then have [*measurable*]: $f \in$ *borel_measurable* M $h \in$ *borel_measurable* M
by (*auto dest*!: *borel_measurable_integrable*)
from *add* g **show** *?case*
by (*simp add*: *scaleR_add_right integrable_density*)
next
case (*lim* f s)
have [*measurable*]: $f \in$ *borel_measurable* M $\bigwedge i.\ s\ i \in$ *borel_measurable* M

using *lim*(*1*,*5*)[*THEN borel_measurable_integrable*] **by** *auto*

show *?case*
proof (*rule LIMSEQ_unique*)
show (λ*i*. *integral*^{L} *M* (λ*x*. *g x* ∗$_R$ *s i x*)) ⟶ *integral*^{L} *M* (λ*x*. *g x* ∗$_R$ *f x*)
proof (*rule integral_dominated_convergence*)
show *integrable M* (λ*x*. *2* ∗ *norm* (*g x* ∗$_R$ *f x*))
by (*intro integrable_mult_right integrable_norm integrable_density*[*THEN iffD1*] *lim g*) *auto*
show *AE x in M*. (λ*i*. *g x* ∗$_R$ *s i x*) ⟶ *g x* ∗$_R$ *f x*
using *lim*(*3*) **by** (*auto intro*!: *tendsto_scaleR AE_I2*[*of M*])
show ⋀*i*. *AE x in M*. *norm* (*g x* ∗$_R$ *s i x*) ≤ *2* ∗ *norm* (*g x* ∗$_R$ *f x*)
using *lim*(*4*) *g* **by** (*auto intro*!: *AE_I2*[*of M*] *mult_left_mono simp*: *field_simps*)
qed *auto*
show (λ*i*. *integral*^{L} *M* (λ*x*. *g x* ∗$_R$ *s i x*)) ⟶ *integral*^{L} (*density M g*) *f*
unfolding *lim*(*2*)[*symmetric*]
by (*rule integral_dominated_convergence*[**where** *w*=λ*x*. *2* ∗ *norm* (*f x*)])
(*use lim* **in** *auto*)
qed
qed
qed (*simp add*: *f g integrable_density*)

lemma
fixes *g* :: *'a* ⇒ *real*
assumes *f* ∈ *borel_measurable M AE x in M*. *0* ≤ *f x g* ∈ *borel_measurable M*
shows *integral_real_density*: *integral*^{L} (*density M f*) *g* = (∫ *x*. *f x* ∗ *g x* ∂*M*)
and *integrable_real_density*: *integrable* (*density M f*) *g* ⟷ *integrable M* (λ*x*. *f x* ∗ *g x*)
using *assms integral_density*[*of g M f*] *integrable_density*[*of g M f*] **by** *auto*

lemma *has_bochner_integral_density*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*} **and** *g* :: *'a* ⇒ *real*
shows *f* ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹ (*AE x in M*. *0* ≤ *g x*) ⟹
has_bochner_integral M (λ*x*. *g x* ∗$_R$ *f x*) *x* ⟹ *has_bochner_integral* (*density M g*) *f x*
by (*simp add*: *has_bochner_integral_iff integrable_density integral_density*)

7.9.3 Distributions

lemma *integrable_distr_eq*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes [*measurable*]: *g* ∈ *measurable M N f* ∈ *borel_measurable N*
shows *integrable* (*distr M N g*) *f* ⟷ *integrable M* (λ*x*. *f* (*g x*))
unfolding *integrable_iff_bounded* **by** (*simp_all add*: *nn_integral_distr*)

lemma *integrable_distr*:

fixes $f :: 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
shows $T \in measurable\ M\ M' \Longrightarrow integrable\ (distr\ M\ M'\ T)\ f \Longrightarrow integrable\ M\ (\lambda x.\ f\ (T\ x))$
by (*metis integrable_distr_eq integrable_iff_bounded measurable_distr_eq1*)

lemma *integral_distr*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
assumes *g*[*measurable*]: $g \in measurable\ M\ N$ **and** *f*: $f \in borel\_measurable\ N$
shows $integral^L\ (distr\ M\ N\ g)\ f = integral^L\ M\ (\lambda x.\ f\ (g\ x))$
proof (*rule integral_eq_cases*)
assume *integrable* (*distr M N g*) *f*
then show *?thesis*
proof *induct*
case (*base A c*)
then have [*measurable*]: $A \in sets\ N$ **by** *auto*
from *base* **have** *int*: *integrable* (*distr M N g*) ($\lambda a.$ *indicator A a* $*_R$ *c*)
by (*intro integrable_indicator*)

have $integral^L\ (distr\ M\ N\ g)\ (\lambda a.\ indicator\ A\ a\ *_R\ c) = measure\ (distr\ M\ N\ g)\ A\ *_R\ c$
using *base* **by** *auto*
also have ... = *measure M* (*g* −‘ *A* ∩ *space M*) $*_R$ *c*
by (*subst measure_distr*) *auto*
also have ... = $integral^L\ M\ (\lambda a.\ indicator\ (g\ -`\ A \cap space\ M)\ a\ *_R\ c)$
using *base* **by** (*auto simp*: *emeasure_distr*)
also have ... = $integral^L\ M\ (\lambda a.\ indicator\ A\ (g\ a)\ *_R\ c)$
using *int base* **by** (*intro integral_cong_AE*) (*auto simp*: *emeasure_distr split*: *split_indicator*)
finally show *?case* **.**
next
case (*add f h*)
then have [*measurable*]: $f \in borel\_measurable\ N\ h \in borel\_measurable\ N$
by (*auto dest*!: *borel_measurable_integrable*)
from *add g* **show** *?case*
by (*simp add*: *scaleR_add_right integrable_distr_eq*)
next
case (*lim f s*)
have [*measurable*]: $f \in borel\_measurable\ N\ \bigwedge i.\ s\ i \in borel\_measurable\ N$
using *lim*(*1*,*5*)[*THEN borel_measurable_integrable*] **by** *auto*

show *?case*
proof (*rule LIMSEQ_unique*)
show $(\lambda i.\ integral^L\ M\ (\lambda x.\ s\ i\ (g\ x))) \longrightarrow integral^L\ M\ (\lambda x.\ f\ (g\ x))$
proof (*rule integral_dominated_convergence*)
show *integrable M* ($\lambda x.$ *2* $*$ *norm* (*f* (*g x*)))
using *lim* **by** (*auto simp*: *integrable_distr_eq*)
show *AE x in M*. ($\lambda i.$ *s i* (*g x*)) $\longrightarrow$ *f* (*g x*)
using *lim*(*3*) *g*[*THEN measurable_space*] **by** *auto*
show $\bigwedge i.$ *AE x in M*. *norm* (*s i* (*g x*)) $\leq$ *2* $*$ *norm* (*f* (*g x*))

```
        using lim(4) g[THEN measurable_space] by auto
    qed auto
    show (λi. integral^L M (λx. s i (g x))) ⟶ integral^L (distr M N g) f
      unfolding lim(2)[symmetric]
      by (rule integral_dominated_convergence[where w=λx. 2 * norm (f x)])
        (use lim in auto)
  qed
 qed
qed (simp add: f g integrable_distr_eq)

lemma has_bochner_integral_distr:
 fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
 shows f ∈ borel_measurable N ⟹ g ∈ measurable M N ⟹
   has_bochner_integral M (λx. f (g x)) x ⟹ has_bochner_integral (distr M N
g) f x
 by (simp add: has_bochner_integral_iff integrable_distr_eq integral_distr)
```

7.9.4 Lebesgue integration on *count_space*

```
lemma integrable_count_space:
 fixes f :: 'a ⇒ 'b::{banach,second_countable_topology}
 shows finite X ⟹ integrable (count_space X) f
 by (auto simp: nn_integral_count_space integrable_iff_bounded)

lemma measure_count_space[simp]:
 B ⊆ A ⟹ finite B ⟹ measure (count_space A) B = card B
 unfolding measure_def by (subst emeasure_count_space ) auto

lemma lebesgue_integral_count_space_finite_support:
 assumes f: finite {a∈A. f a ≠ 0}
 shows (∫ x. f x ∂count_space A) = (∑ a | a ∈ A ∧ f a ≠ 0. f a)
proof −
 have eq: ⋀x. x ∈ A ⟹ (∑ a | x = a ∧ a ∈ A ∧ f a ≠ 0. f a) = (∑ x∈{x}. f x)
   by (intro sum.mono_neutral_cong_left) auto

 have (∫ x. f x ∂count_space A) = (∫ x. (∑ a | a ∈ A ∧ f a ≠ 0. indicator {a}
x *_R f a) ∂count_space A)
   by (intro integral_cong refl) (simp add: f eq)
 also have ... = (∑ a | a ∈ A ∧ f a ≠ 0. measure (count_space A) {a} *_R f a)
   by (subst integral_sum) (auto intro!: sum.cong)
 finally show ?thesis
   by auto
qed

lemma lebesgue_integral_count_space_finite: finite A ⟹ (∫ x. f x ∂count_space
A) = (∑ a∈A. f a)
 by (subst lebesgue_integral_count_space_finite_support)
   (auto intro!: sum.mono_neutral_cong_left)
```

lemma *integrable_count_space_nat_iff*:
fixes $f :: nat \Rightarrow \_::\{banach,second\_countable\_topology\}$
shows *integrable* (*count_space UNIV*) $f \longleftrightarrow$ *summable* ($\lambda x.$ *norm* ($f\ x$))
by (*auto simp*: *integrable_iff_bounded nn_integral_count_space_nat ennreal_suminf_neq_top*
intro: *summable_suminf_not_top*)

lemma *sums_integral_count_space_nat*:
fixes $f :: nat \Rightarrow \_::\{banach,second\_countable\_topology\}$
assumes $*$: *integrable* (*count_space UNIV*) f
shows f *sums* (*integral*^{L} (*count_space UNIV*) f)
proof −
let *?f* $= \lambda n\ i.$ *indicator* $\{n\}\ i *_R f\ i$
have f': $\bigwedge n\ i.$ *?f* $n\ i =$ *indicator* $\{n\}\ i *_R f\ n$
by (*auto simp*: *fun_eq_iff split*: *split_indicator*)

have ($\lambda i. \int n.$ *?f* $i\ n\ \partial$*count_space UNIV*) *sums* $\int n. (\sum i.$ *?f* $i\ n)\ \partial$*count_space UNIV*
proof (*rule sums_integral*)
show $\bigwedge i.$ *integrable* (*count_space UNIV*) (*?f* i)
using $*$ **by** (*intro integrable_mult_indicator*) *auto*
show *AE* n *in count_space UNIV*. *summable* ($\lambda i.$ *norm* (*?f* $i\ n$))
using *summable_finite*[*of* $\{n\}$ $\lambda i.$ *norm* (*?f* $i\ n$) **for** n] **by** *simp*
show *summable* ($\lambda i. \int n.$ *norm* (*?f* $i\ n$) ∂*count_space UNIV*)
using $*$ **by** (*subst* f') (*simp add*: *integrable_count_space_nat_iff*)
qed
also have ($\int n. (\sum i.$ *?f* $i\ n)\ \partial$*count_space UNIV*) = ($\int n.\ f\ n\ \partial$*count_space UNIV*)
using *suminf_finite*[*of* $\{n\}$ $\lambda i.$ *?f* $i\ n$ **for** n] **by** (*auto intro*!: *integral_cong*)
also have ($\lambda i. \int n.$ *?f* $i\ n\ \partial$*count_space UNIV*) = f
by (*subst* f') *simp*
finally show *?thesis* **.**
qed

lemma *integral_count_space_nat*:
fixes $f :: nat \Rightarrow \_::\{banach,second\_countable\_topology\}$
shows *integrable* (*count_space UNIV*) $f \Longrightarrow$ *integral*^{L} (*count_space UNIV*) f = ($\sum x.\ f\ x$)
using *sums_integral_count_space_nat sums_unique* **by** *auto*

lemma *integrable_bij_count_space*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\_countable\_topology\}$
assumes g: *bij_betw* $g\ A\ B$
shows *integrable* (*count_space* A) ($\lambda x.\ f\ (g\ x)$) $\longleftrightarrow$ *integrable* (*count_space* B) f
unfolding *integrable_iff_bounded* **by** (*subst nn_integral_bij_count_space*[*OF* g]) *auto*

lemma *integral_bij_count_space*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\_countable\_topology\}$

assumes *g*: *bij_betw g A B*
shows *integral*^{L} (*count_space A*) (λ*x*. *f* (*g x*)) = *integral*^{L} (*count_space B*) *f*
using *g*[*THEN bij_betw_imp_funcset*] *distr_bij_count_space* [*OF g*]
by (*metis borel_measurable_count_space integral_distr measurable_count_space_eq1 space_count_space*)

lemma *has_bochner_integral_count_space_nat*:
fixes *f* :: *nat* ⇒ _::{*banach*,*second_countable_topology*}
shows *has_bochner_integral* (*count_space UNIV*) *f x* ⟹ *f sums x*
unfolding *has_bochner_integral_iff* **by** (*auto intro*!: *sums_integral_count_space_nat*)

7.9.5 Point measure

lemma *lebesgue_integral_point_measure_finite*:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
shows *finite A* ⟹ (⋀*a*. *a* ∈ *A* ⟹ 0 ≤ *f a*) ⟹
integral^L (*point_measure A f*) *g* = (∑ *a*∈*A*. *f a* ∗$_R$ *g a*)
by (*simp add*: *lebesgue_integral_count_space_finite AE_count_space integral_density point_measure_def*)

proposition *integrable_point_measure_finite*:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*} **and** *f* :: *'a* ⇒ *real*
assumes *finite A*
shows *integrable* (*point_measure A f*) *g*
proof −
have *integrable* (*density* (*count_space A*) (λ*x*. *ennreal* (*max 0* (*f x*)))) *g*
using *assms*
by (*simp add*: *integrable_count_space integrable_density*)
then show *?thesis*
by (*smt* (*verit*) *AE_I2 borel_measurable_count_space density_cong ennreal_neg point_measure_def*)
qed

lemma *integral_uniform_count_measure*:
assumes *finite A*
shows *integral*^{L} (*uniform_count_measure A*) *f* = *sum f A* / (*card A*)
proof −
have *integral*^{L} (*uniform_count_measure A*) *f* = (∑ *x*∈*A*. *f x* / *card A*)
using *assms* **by** (*simp add*: *uniform_count_measure_def lebesgue_integral_point_measure_finite*)
with *assms* **show** *?thesis*
by (*simp add*: *sum_divide_distrib nn_integral_count_space_finite*)
qed

7.9.6 Lebesgue integration on *null_measure*

lemma *has_bochner_integral_null_measure_iff*[*iff*]:
has_bochner_integral (*null_measure M*) *f* 0 ⟷ *f* ∈ *borel_measurable M*
by (*auto simp*: *has_bochner_integral.simps simple_bochner_integral_def*[*abs_def*]
intro!: *exI*[*of* _ λ*n x*. 0] *simple_bochner_integrable.intros*)

lemma *integrable_null_measure_iff*[*iff*]: *integrable* (*null_measure M*) *f* ⟷ *f* ∈ *borel_measurable M*
by (*auto simp*: *integrable.simps*)

lemma *integral_null_measure*[*simp*]: *integral*^{L} (*null_measure M*) *f* = *0*
using *integral_norm_bound_ennreal not_integrable_integral_eq* **by** *fastforce*

7.9.7 Legacy lemmas for the real-valued Lebesgue integral

theorem *real_lebesgue_integral_def*:
assumes *f*[*measurable*]: *integrable M f*
shows *integral*^{L} *M f* = *enn2real* (∫$^+$*x*. *f x* ∂*M*) − *enn2real* (∫$^+$*x*. *ennreal* (− *f x*) ∂*M*)
proof −
have *integral*^{L} *M f* = *integral*^{L} *M* (λ*x*. *max 0* (*f x*) − *max 0* (− *f x*))
by (*auto intro*!: *arg_cong*[**where** *f*=*integral*^{L} *M*])
also have ... = *integral*^{L} *M* (λ*x*. *max 0* (*f x*)) − *integral*^{L} *M* (λ*x*. *max 0* (− *f x*))
by (*intro integral_diff integrable_max integrable_minus integrable_zero f*)
also have *integral*^{L} *M* (λ*x*. *max 0* (*f x*)) = *enn2real* (∫$^+$*x*. *ennreal* (*f x*) ∂*M*)
by (*subst integral_eq_nn_integral*) (*auto intro*!: *arg_cong*[**where** *f*=*enn2real*] *nn_integral_cong simp*: *max_def ennreal_neg*)
also have *integral*^{L} *M* (λ*x*. *max 0* (− *f x*)) = *enn2real* (∫$^+$*x*. *ennreal* (− *f x*) ∂*M*)
by (*subst integral_eq_nn_integral*) (*auto intro*!: *arg_cong*[**where** *f*=*enn2real*] *nn_integral_cong simp*: *max_def ennreal_neg*)
finally show *?thesis* **.**
qed

theorem *real_integrable_def*:
integrable M f ⟷ *f* ∈ *borel_measurable M* ∧
(∫$^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞ ∧ (∫$^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≠ ∞
unfolding *integrable_iff_bounded*
proof (*safe del*: *notI*)
assume ∗: (∫$^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*M*) < ∞
have (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*) ≤ (∫$^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*M*)
by (*intro nn_integral_mono*) *auto*
also note ∗
finally show (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞
by *simp*
have (∫$^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≤ (∫$^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*M*)
by (*intro nn_integral_mono*) *auto*
also note ∗
finally show (∫$^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≠ ∞
by *simp*
next
assume [*measurable*]: *f* ∈ *borel_measurable M*
assume *fin*: (∫$^+$ *x*. *ennreal* (*f x*) ∂*M*) ≠ ∞ (∫$^+$ *x*. *ennreal* (− *f x*) ∂*M*) ≠ ∞
have (∫$^+$ *x*. *norm* (*f x*) ∂*M*) = (∫$^+$ *x*. *ennreal* (*f x*) + *ennreal* (− *f x*) ∂*M*)

by (*intro nn_integral_cong*) (*auto simp*: *abs_real_def ennreal_neg*)
also have... = $(\int^+ x.\ ennreal\ (f\ x)\ \partial M) + (\int^+ x.\ ennreal\ (-f\ x)\ \partial M)$
by (*intro nn_integral_add*) *auto*
also have ... $< \infty$
using *fin* **by** (*auto simp*: *less_top*)
finally show $(\int^+ x.\ norm\ (f\ x)\ \partial M) < \infty$ **.**
qed

lemma *integrableD*[*dest*]:
assumes *integrable M f*
shows $f \in borel\_measurable\ M$ $(\int^+ x.\ ennreal\ (f\ x)\ \partial M) \neq \infty$ $(\int^+ x.\ ennreal\ (-f\ x)\ \partial M) \neq \infty$
using *assms* **unfolding** *real_integrable_def* **by** *auto*

lemma *integrableE*:
assumes *integrable M f*
obtains *r q* **where** $0 \leq r$ $0 \leq q$
$(\int^+x.\ ennreal\ (f\ x)\partial M) = ennreal\ r$
$(\int^+x.\ ennreal\ (-f\ x)\partial M) = ennreal\ q$
$f \in borel\_measurable\ M$ $integral^L\ M\ f = r - q$
using *assms* **unfolding** *real_integrable_def real_lebesgue_integral_def*[*OF assms*]
by (*cases rule*: *ennreal2_cases*[*of* $(\int^+x.\ ennreal\ (-f\ x)\partial M)$ $(\int^+x.\ ennreal\ (f\ x)\partial M)$]) *auto*

lemma *integral_monotone_convergence_nonneg*:
fixes *f* :: $nat \Rightarrow 'a \Rightarrow real$
assumes *i*: $\bigwedge i.\ integrable\ M\ (f\ i)$ **and** *mono*: *AE x in M*. *mono* $(\lambda n.\ f\ n\ x)$
and *pos*: $\bigwedge i.$ *AE x in M*. $0 \leq f\ i\ x$
and *lim*: *AE x in M*. $(\lambda i.\ f\ i\ x) \longrightarrow u\ x$
and *ilim*: $(\lambda i.\ integral^L\ M\ (f\ i)) \longrightarrow x$
and *u*: $u \in borel\_measurable\ M$
shows *integrable M u*
and $integral^L\ M\ u = x$
proof −
have *nn*: *AE x in M*. $\forall i.\ 0 \leq f\ i\ x$
using *pos* **unfolding** *AE_all_countable* **by** *auto*
with *lim* **have** *u_nn*: *AE x in M*. $0 \leq u\ x$
by *eventually_elim* (*auto intro*: *LIMSEQ_le_const*)
have [*simp*]: $0 \leq x$
by (*intro LIMSEQ_le_const*[*OF ilim*] *allI exI impI integral_nonneg_AE pos*)
have $(\int^+ x.\ ennreal\ (u\ x)\ \partial M) = (SUP\ n.\ (\int^+ x.\ ennreal\ (f\ n\ x)\ \partial M))$
proof (*subst nn_integral_monotone_convergence_SUP_AE*[*symmetric*])
fix *i*
from *mono nn* **show** *AE x in M*. $ennreal\ (f\ i\ x) \leq ennreal\ (f\ (Suc\ i)\ x)$
by *eventually_elim* (*auto simp*: *mono_def*)
show $(\lambda x.\ ennreal\ (f\ i\ x)) \in borel\_measurable\ M$
using *i* **by** *auto*
next
show $(\int^+ x.\ ennreal\ (u\ x)\ \partial M) = \int^+ x.\ (SUP\ i.\ ennreal\ (f\ i\ x))\ \partial M$

proof (*rule nn_integral_cong_AE*)
show *AE x in M. ennreal (u x) = (SUP i. ennreal (f i x))*
using *lim mono nn u_nn*
by *eventually_elim* (*simp add*: *LIMSEQ_unique*[*OF _ LIMSEQ_SUP*] *incseq_def*)
qed
qed
also have *... = ennreal x*
using *mono i nn* **unfolding** *nn_integral_eq_integral*[*OF i pos*]
by (*subst LIMSEQ_unique*[*OF LIMSEQ_SUP*]) (*auto simp*: *mono_def integral_nonneg_AE pos intro*!: *integral_mono_AE ilim*)
finally have $(\int^+ x.\ ennreal\ (u\ x)\ \partial M) = ennreal\ x$ **.**
moreover have $(\int^+ x.\ ennreal\ (-\ u\ x)\ \partial M) = 0$
using *u u_nn* **by** (*subst nn_integral_0_iff_AE*) (*auto simp*: *ennreal_neg*)
ultimately show *integrable M u integral*^{L} *M u = x*
by (*auto simp*: *real_integrable_def real_lebesgue_integral_def u*)
qed

lemma
fixes *f* :: *nat* ⇒ *'a* ⇒ *real*
assumes *f*: $\bigwedge i.$ *integrable M* (*f i*) **and** *mono*: *AE x in M. mono* (λ*n. f n x*)
and *lim*: *AE x in M.* (λ*i. f i x*) ⟶ *u x*
and *ilim*: (λ*i. integral*^{L} *M* (*f i*)) ⟶ *x*
and *u*: *u* ∈ *borel_measurable M*
shows *integrable_monotone_convergence*: *integrable M u*
and *integral_monotone_convergence*: *integral*^{L} *M u = x*
and *has_bochner_integral_monotone_convergence*: *has_bochner_integral M u x*
proof −
have *1*: $\bigwedge i.$ *integrable M* (λ*x. f i x − f 0 x*)
using *f* **by** *auto*
have *2*: *AE x in M. mono* (λ*n. f n x − f 0 x*)
using *mono* **by** (*auto simp*: *mono_def le_fun_def*)
have *3*: $\bigwedge n.$ *AE x in M. 0 ≤ f n x − f 0 x*
using *mono* **by** (*auto simp*: *field_simps mono_def le_fun_def*)
have *4*: *AE x in M.* (λ*i. f i x − f 0 x*) ⟶ *u x − f 0 x*
using *lim* **by** (*auto intro*!: *tendsto_diff*)
have *5*: (λ*i.* (∫ *x. f i x − f 0 x* ∂*M*)) ⟶ *x − integral*^{L} *M* (*f 0*)
using *f ilim* **by** (*auto intro*!: *tendsto_diff*)
have *6*: (λ*x. u x − f 0 x*) ∈ *borel_measurable M*
using *f*[*of 0*] *u* **by** *auto*
note *diff* = *integral_monotone_convergence_nonneg*[*OF 1 2 3 4 5 6*]
have *integrable M* (λ*x.* (*u x − f 0 x*) + *f 0 x*)
using *diff*(*1*) *f* **by** (*rule integrable_add*)
with *diff*(*2*) *f* **show** *integrable M u integral*^{L} *M u = x*
by *auto*
then show *has_bochner_integral M u x*
by (*metis has_bochner_integral_integrable*)
qed

lemma *integral_norm_eq_0_iff*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
assumes $f[measurable]$: *integrable M f*
shows $(\int x.\ norm\ (f\ x)\ \partial M) = 0 \longleftrightarrow emeasure\ M\ \{x \in space\ M.\ f\ x \neq 0\} = 0$
proof −
have $(\int^+ x.\ norm\ (f\ x)\ \partial M) = (\int x.\ norm\ (f\ x)\ \partial M)$
using *f* **by** (*intro nn_integral_eq_integral integrable_norm*) *auto*
then have $(\int x.\ norm\ (f\ x)\ \partial M) = 0 \longleftrightarrow (\int^+ x.\ norm\ (f\ x)\ \partial M) = 0$
by *simp*
also have $\ldots \longleftrightarrow emeasure\ M\ \{x \in space\ M.\ ennreal\ (norm\ (f\ x)) \neq 0\} = 0$
by (*intro nn_integral_0_iff*) *auto*
finally show *?thesis*
by *simp*
qed

lemma *integral_0_iff*:
fixes $f :: 'a \Rightarrow real$
shows *integrable M f* $\Longrightarrow (\int x.\ |f\ x|\ \partial M) = 0 \longleftrightarrow emeasure\ M\ \{x \in space\ M.\ f\ x \neq 0\} = 0$
using *integral_norm_eq_0_iff*[*of M f*] **by** *simp*

lemma (**in** *finite_measure*) *integrable_const*[*intro*!, *simp*]: *integrable M* $(\lambda x.\ a)$
using *integrable_indicator*[*of space M M a*] **by** (*simp cong*: *integrable_cong add*: *less_top*[*symmetric*])

lemma *lebesgue_integral_const*[*simp*]:
fixes $a :: 'a :: \{banach, second\_countable\_topology\}$
shows $(\int x.\ a\ \partial M) = measure\ M\ (space\ M) *_R a$
proof −
{ assume $emeasure\ M\ (space\ M) = \infty\ a \neq 0$
then have *?thesis*
by (*auto simp*: *not_integrable_integral_eq ennreal_mult_less_top measure_def integrable_iff_bounded*) **}**
moreover
{ assume $a = 0$ **then have** *?thesis* **by** *simp* **}**
moreover
{ assume $emeasure\ M\ (space\ M) \neq \infty$
interpret *finite_measure M*
proof qed *fact*
have $(\int x.\ a\ \partial M) = (\int x.\ indicator\ (space\ M)\ x *_R a\ \partial M)$
by (*intro integral_cong*) *auto*
also have $\ldots = measure\ M\ (space\ M) *_R a$
by (*simp add*: *less_top*[*symmetric*])
finally have *?thesis* **. }**
ultimately show *?thesis* **by** *blast*
qed

lemma (**in** *finite_measure*) *integrable_const_bound*:

fixes $f :: 'a \Rightarrow 'b::\{banach,second\_countable\_topology\}$
shows $AE\ x\ in\ M.\ norm\ (f\ x) \leq B \Longrightarrow f \in borel\_measurable\ M \Longrightarrow integrable\ M\ f$
using $integrable\_bound[OF\ integrable\_const[of\ B],\ of\ f]$
by $(smt\ (verit,\ ccfv\_SIG)\ eventually\_elim2\ real\_norm\_def)$

lemma (**in** *finite_measure*) *integral_bounded_eq_bound_then_AE*:
assumes $AE\ x\ in\ M.\ f\ x \leq (c::real)$
$integrable\ M\ f\ (\int x.\ f\ x\ \partial M) = c * measure\ M\ (space\ M)$
shows $AE\ x\ in\ M.\ f\ x = c$
using *assms* **by** $(intro\ integral\_ineq\_eq\_0\_then\_AE)\ auto$

lemma *integral_indicator_finite_real*:
fixes $f :: 'a \Rightarrow real$
assumes [*simp*]: *finite A*
assumes [*measurable*]: $\bigwedge a.\ a \in A \Longrightarrow \{a\} \in sets\ M$
assumes *finite*: $\bigwedge a.\ a \in A \Longrightarrow emeasure\ M\ \{a\} < \infty$
shows $(\int x.\ f\ x * indicator\ A\ x\ \partial M) = (\sum a{\in}A.\ f\ a * measure\ M\ \{a\})$
proof −
have $(\int x.\ f\ x * indicator\ A\ x\ \partial M) = (\int x.\ (\sum a{\in}A.\ f\ a * indicator\ \{a\}\ x)\ \partial M)$
proof $(intro\ integral\_cong\ refl)$
fix x **show** $f\ x * indicator\ A\ x = (\sum a{\in}A.\ f\ a * indicator\ \{a\}\ x)$
by $(auto\ split: split\_indicator\ simp: eq\_commute[of\ x]\ cong: conj\_cong)$
qed
also have $\ldots = (\sum a{\in}A.\ f\ a * measure\ M\ \{a\})$
using *finite* **by** $(subst\ integral\_sum)\ (auto)$
finally show *?thesis* **.**
qed

lemma (**in** *finite_measure*) *ennreal_integral_real*:
assumes [*measurable*]: $f \in borel\_measurable\ M$
assumes *ae*: $AE\ x\ in\ M.\ f\ x \leq ennreal\ B\ 0 \leq B$
shows $ennreal\ (\int x.\ enn2real\ (f\ x)\ \partial M) = (\int^+ x.\ f\ x\ \partial M)$
proof $(subst\ nn\_integral\_eq\_integral[symmetric])$
show $integrable\ M\ (\lambda x.\ enn2real\ (f\ x))$
using *ae* **by** $(intro\ integrable\_const\_bound[\textbf{where}\ B{=}B])\ (auto\ simp: enn2real\_leI)$
show $(\int^+ x.\ ennreal\ (enn2real\ (f\ x))\ \partial M) = integral^N\ M\ f$
using *ae* **by** $(intro\ nn\_integral\_cong\_AE)\ (auto\ simp: le\_less\_trans[OF\ \_\ ennreal\_less\_top])$
qed *auto*

lemma (**in** *finite_measure*) *integral_less_AE*:
fixes $X\ Y :: 'a \Rightarrow real$
assumes *int*: $integrable\ M\ X\ integrable\ M\ Y$
assumes *A*: $(emeasure\ M)\ A \neq 0\ A \in sets\ M\ AE\ x\ in\ M.\ x \in A \longrightarrow X\ x \neq Y\ x$
assumes *gt*: $AE\ x\ in\ M.\ X\ x \leq Y\ x$
shows $integral^L\ M\ X < integral^L\ M\ Y$
proof −
have $integral^L\ M\ X \leq integral^L\ M\ Y$

```
    using gt int by (intro integral_mono_AE) auto
  moreover
  have integral^L M X ≠ integral^L M Y
  proof
    assume eq: integral^L M X = integral^L M Y
    have integral^L M (λx. |Y x − X x|) = integral^L M (λx. Y x − X x)
      using gt int by (intro integral_cong_AE) auto
    also have ... = 0
      using eq int by simp
    finally have (emeasure M) {x ∈ space M. Y x − X x ≠ 0} = 0
      using int by (simp add: integral_0_iff)
    moreover
    have (∫⁺x. indicator A x ∂M) ≤ (∫⁺x. indicator {x ∈ space M. Y x − X x
≠ 0} x ∂M)
      using A by (intro nn_integral_mono_AE) auto
    then have (emeasure M) A ≤ (emeasure M) {x ∈ space M. Y x − X x ≠ 0}
      using int A by (simp add: integrable_def)
    ultimately have emeasure M A = 0
      by simp
    with ‹(emeasure M) A ≠ 0› show False by auto
  qed
  ultimately show ?thesis by auto
qed

lemma (in finite_measure) integral_less_AE_space:
  fixes X Y :: 'a ⇒ real
  assumes int: integrable M X integrable M Y
  assumes gt: AE x in M. X x < Y x emeasure M (space M) ≠ 0
  shows integral^L M X < integral^L M Y
  using gt by (intro integral_less_AE[OF int, where A=space M]) auto

lemma tendsto_integral_at_top:
  fixes f :: real ⇒ 'a::{banach, second_countable_topology}
  assumes [measurable_cong]: sets M = sets borel and f[measurable]: integrable
M f
  shows ((λy. ∫ x. indicator {.. y} x *_R f x ∂M) ⟶ ∫ x. f x ∂M) at_top
proof (rule tendsto_at_topI_sequentially)
  fix X :: nat ⇒ real assume filterlim X at_top sequentially
  show (λn. ∫ x. indicator {..X n} x *_R f x ∂M) ⟶ integral^L M f
  proof (rule integral_dominated_convergence)
    show integrable M (λx. norm (f x))
      by (rule integrable_norm) fact
    show AE x in M. (λn. indicator {..X n} x *_R f x) ⟶ f x
    proof
      fix x
      from ‹filterlim X at_top sequentially›
      have eventually (λn. x ≤ X n) sequentially
        unfolding filterlim_at_top_ge[where c=x] by auto
      then show (λn. indicator {..X n} x *_R f x) ⟶ f x
```

```
    by (intro tendsto_eventually) (auto simp: frequently_def split: split_indicator)
  qed
  fix n show AE x in M. norm (indicator {..X n} x *R f x) ≤ norm (f x)
    by (auto split: split_indicator)
 qed auto
qed

lemma
 fixes f :: real ⇒ real
 assumes M: sets M = sets borel
 assumes nonneg: AE x in M. 0 ≤ f x
 assumes borel: f ∈ borel_measurable borel
 assumes int: ⋀y. integrable M (λx. f x * indicator {.. y} x)
 assumes conv: ((λy. ∫ x. f x * indicator {.. y} x ∂M) ⟶ x) at_top
 shows has_bochner_integral_monotone_convergence_at_top: has_bochner_integral
M f x
   and integrable_monotone_convergence_at_top: integrable M f
   and integral_monotone_convergence_at_top:integral^L M f = x
proof −
 from nonneg have AE x in M. mono (λn::nat. f x * indicator {..real n} x)
   by (auto split: split_indicator intro!: monoI)
 { fix x have eventually (λn. f x * indicator {..real n} x = f x) sequentially
     by (rule eventually_sequentiallyI[of nat ⌈x⌉])
       (auto split: split_indicator simp: nat_le_iff ceiling_le_iff) }
 from filterlim_cong[OF refl refl this]
 have AE x in M. (λi. f x * indicator {..real i} x) ⟶ f x
   by simp
 have (λi. ∫ x. f x * indicator {..real i} x ∂M) ⟶ x
   using conv filterlim_real_sequentially by (rule filterlim_compose)
 have M_measure[simp]: borel_measurable M = borel_measurable borel
   using M by (simp add: sets_eq_imp_space_eq measurable_def)
 have f ∈ borel_measurable M
   using borel by simp
 show has_bochner_integral M f x
   by (rule has_bochner_integral_monotone_convergence) fact+
 then show integrable M f integral^L M f = x
   by (auto simp: _has_bochner_integral_iff)
qed
```

7.9.8 Product measure

```
lemma (in sigma_finite_measure) borel_measurable_lebesgue_integrable[measurable
(raw)]:
 fixes f :: _ ⇒ _ ⇒ _::{banach, second_countable_topology}
 assumes [measurable]: case_prod f ∈ borel_measurable (N ⨂M M)
 shows Measurable.pred N (λx. integrable M (f x))
proof −
 have [simp]: ⋀x. x ∈ space N ⟹ integrable M (f x) ⟷ (∫+y. norm (f x y)
∂M) < ∞
```

```
    unfolding integrable_iff_bounded by simp
  show ?thesis
    by (simp cong: measurable_cong)
qed

lemma (in sigma_finite_measure) measurable_measure[measurable (raw)]:
  (⋀x. x ∈ space N ⟹ A x ⊆ space M) ⟹
    {x ∈ space (N ⨂M M). snd x ∈ A (fst x)} ∈ sets (N ⨂M M) ⟹
    (λx. measure M (A x)) ∈ borel_measurable N
  unfolding measure_def by (intro measurable_emeasure borel_measurable_enn2real)
auto

proposition (in sigma_finite_measure) borel_measurable_lebesgue_integral[measurable
(raw)]:
  fixes f :: _ ⇒ _ ⇒ _::{banach, second_countable_topology}
  assumes f[measurable]: case_prod f ∈ borel_measurable (N ⨂M M)
  shows (λx. ∫ y. f x y ∂M) ∈ borel_measurable N
proof −
  from borel_measurable_implies_sequence_metric[OF f, of 0]
  obtain s where s: ⋀i. simple_function (N ⨂M M) (s i)
    and ∀x∈space (N ⨂M M). (λi. s i x) ⟶ (case x of (x, y) ⇒ f x y)
    and ∀ i. ∀x∈space (N ⨂M M). dist (s i x) 0 ≤ 2 ∗ dist (case x of (x, xa) ⇒
f x xa) 0
    by auto
  then have ∗:
    ⋀x y. x ∈ space N ⟹ y ∈ space M ⟹ (λi. s i (x, y)) ⟶ f x y
    ⋀i x y. x ∈ space N ⟹ y ∈ space M ⟹ norm (s i (x, y)) ≤ 2 ∗ norm (f x y)
    by (auto simp: space_pair_measure)

  have [measurable]: ⋀i. s i ∈ borel_measurable (N ⨂M M)
    by (rule borel_measurable_simple_function) fact

  have ⋀i. s i ∈ measurable (N ⨂M M) (count_space UNIV)
    by (rule measurable_simple_function) fact

  define f′ where [abs_def]: f′ i x =
    (if integrable M (f x) then simple_bochner_integral M (λy. s i (x, y)) else 0)
for i x

  { fix i x assume x ∈ space N
    then have simple_bochner_integral M (λy. s i (x, y)) =
      (∑z∈s i ‘ (space N × space M). measure M {y ∈ space M. s i (x, y) = z}
∗R z)
      using s[THEN simple_functionD(1)]
      unfolding simple_bochner_integral_def
      by (intro sum.mono_neutral_cong_left)
         (auto simp: eq_commute space_pair_measure image_iff cong: conj_cong)
}
  note eq = this
```

```
  show ?thesis
  proof (rule borel_measurable_LIMSEQ_metric)
    fix i show f' i ∈ borel_measurable N
      unfolding f'_def by (simp_all add: eq cong: measurable_cong if_cong)
  next
    fix x assume x: x ∈ space N
    { assume int_f: integrable M (f x)
      have int_2f: integrable M (λy. 2 * norm (f x y))
        by (intro integrable_norm integrable_mult_right int_f)
      have (λi. integral^L M (λy. s i (x, y))) ⟶ integral^L M (f x)
      proof (rule integral_dominated_convergence)
        from int_f show f x ∈ borel_measurable M by auto
        show ⋀i. (λy. s i (x, y)) ∈ borel_measurable M
          using x by simp
        show AE xa in M. (λi. s i (x, xa)) ⟶ f x xa
          using x * by auto
        show ⋀i. AE xa in M. norm (s i (x, xa)) ≤ 2 * norm (f x xa)
          using x * by auto
      qed fact
      moreover
      { fix i
        have simple_bochner_integrable M (λy. s i (x, y))
        proof (rule simple_bochner_integrableI_bounded)
          have (λy. s i (x, y)) ' space M ⊆ s i ' (space N × space M)
            using x by auto
          then show simple_function M (λy. s i (x, y))
            using simple_functionD(1)[OF s(1), of i] x
            by (intro simple_function_borel_measurable)
               (auto simp: space_pair_measure dest: finite_subset)
          have (∫⁺ y. ennreal (norm (s i (x, y))) ∂M) ≤ (∫⁺ y. 2 * norm (f x y)
∂M)
            using x * by (intro nn_integral_mono) auto
          also have (∫⁺ y. 2 * norm (f x y) ∂M) < ∞
            using int_2f unfolding integrable_iff_bounded by simp
          finally show (∫⁺ xa. ennreal (norm (s i (x, xa))) ∂M) < ∞ .
        qed
        then have integral^L M (λy. s i (x, y)) = simple_bochner_integral M (λy.
s i (x, y))
          by (rule simple_bochner_integrable_eq_integral[symmetric]) }
       ultimately have (λi. simple_bochner_integral M (λy. s i (x, y))) ⟶
integral^L M (f x)
        by simp }
    then
    show (λi. f' i x) ⟶ integral^L M (f x)
      unfolding f'_def
      by (cases integrable M (f x)) (simp_all add: not_integrable_integral_eq)
  qed
qed
```

lemma (**in** *pair_sigma_finite*) *integrable_product_swap*:
fixes $f :: \_ \Rightarrow \_::\{banach, second\_countable\_topology\}$
assumes *integrable* $(M1 \bigotimes_M M2)$ f
shows *integrable* $(M2 \bigotimes_M M1)$ $(\lambda(x,y).\ f\ (y,x))$
by (*smt* (*verit*) *assms distr_pair_swap integrable_cong integrable_distr measurable_pair_swap'* *prod.case_distrib split_cong*)

lemma (**in** *pair_sigma_finite*) *integrable_product_swap_iff*:
fixes $f :: \_ \Rightarrow \_::\{banach, second\_countable\_topology\}$
shows *integrable* $(M2 \bigotimes_M M1)$ $(\lambda(x,y).\ f\ (y,x)) \longleftrightarrow$ *integrable* $(M1 \bigotimes_M M2)$ f
proof −
interpret *Q*: *pair_sigma_finite M2 M1* **..**
from *Q.integrable_product_swap*[*of* $\lambda(x,y).\ f\ (y,x)$] *integrable_product_swap*[*of* f]
show *?thesis* **by** *auto*
qed

lemma (**in** *pair_sigma_finite*) *integral_product_swap*:
fixes $f :: \_ \Rightarrow \_::\{banach, second\_countable\_topology\}$
assumes f: $f \in$ *borel_measurable* $(M1 \bigotimes_M M2)$
shows $(\int (x,y).\ f\ (y,x)\ \partial(M2 \bigotimes_M M1)) =$ *integral*^{L} $(M1 \bigotimes_M M2)$ f
by (*smt* (*verit*) *distr_pair_swap f integral_cong integral_distr measurable_pair_swap'* *prod.case_distrib split_cong*)

theorem (**in** *pair_sigma_finite*) *Fubini_integrable*:
fixes $f :: \_ \Rightarrow \_::\{banach, second\_countable\_topology\}$
assumes f[*measurable*]: $f \in$ *borel_measurable* $(M1 \bigotimes_M M2)$
and *integ1*: *integrable M1* $(\lambda x.\ \int\ y.\ norm\ (f\ (x,\ y))\ \partial M2)$
and *integ2*: *AE x in M1*. *integrable M2* $(\lambda y.\ f\ (x,\ y))$
shows *integrable* $(M1 \bigotimes_M M2)$ f
proof (*rule integrableI_bounded*)
have $(\int^+ p.\ norm\ (f\ p)\ \partial(M1 \bigotimes_M M2)) = (\int^+ x.\ (\int^+ y.\ norm\ (f\ (x,\ y))\ \partial M2)\ \partial M1)$
by (*simp add*: *M2.nn_integral_fst* [*symmetric*])
also have $\ldots = (\int^+ x.\ |\int y.\ norm\ (f\ (x,\ y))\ \partial M2|\ \partial M1)$
apply (*intro nn_integral_cong_AE*)
using *integ2*
proof *eventually_elim*
fix x **assume** *integrable M2* $(\lambda y.\ f\ (x,\ y))$
then have f: *integrable M2* $(\lambda y.\ norm\ (f\ (x,\ y)))$
by *simp*
then have $(\int^+ y.\ ennreal\ (norm\ (f\ (x,\ y)))\ \partial M2) = ennreal\ (LINT\ y|M2.\ norm\ (f\ (x,\ y)))$
by (*rule nn_integral_eq_integral*) *simp*
also have $\ldots = ennreal\ |LINT\ y|M2.\ norm\ (f\ (x,\ y))|$
using f **by** *simp*
finally show $(\int^+ y.\ ennreal\ (norm\ (f\ (x,\ y)))\ \partial M2) = ennreal\ |LINT\ y|M2.$

norm (*f* (*x*, *y*))| .
qed
also have ... < ∞
using *integ1* **by** (*simp add*: *integrable_iff_bounded integral_nonneg_AE*)
finally show (∫⁺ *p*. *norm* (*f p*) ∂(*M1* ⨂_M *M2*)) < ∞ .
qed *fact*

lemma (**in** *pair_sigma_finite*) *emeasure_pair_measure_finite*:
assumes *A*: *A* ∈ *sets* (*M1* ⨂_M *M2*) **and** *finite*: *emeasure* (*M1* ⨂_M *M2*) *A* < ∞
shows *AE x in M1*. *emeasure M2* {*y*∈*space M2*. (*x*, *y*) ∈ *A*} < ∞
proof −
from *M2.emeasure_pair_measure_alt*[*OF A*] *finite*
have (∫⁺ *x*. *emeasure M2* (*Pair x* −‘ *A*) ∂*M1*) ≠ ∞
by *simp*
then have *AE x in M1*. *emeasure M2* (*Pair x* −‘ *A*) ≠ ∞
by (*rule nn_integral_PInf_AE*[*rotated*]) (*intro M2.measurable_emeasure_Pair A*)
moreover have ⋀*x*. *x* ∈ *space M1* ⟹ *Pair x* −‘ *A* = {*y*∈*space M2*. (*x*, *y*) ∈ *A*}
using *sets.sets_into_space*[*OF A*] **by** (*auto simp*: *space_pair_measure*)
ultimately show *?thesis* **by** (*auto simp*: *less_top*)
qed

lemma (**in** *pair_sigma_finite*) *AE_integrable_fst'*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *integrable* (*M1* ⨂_M *M2*) *f*
shows *AE x in M1*. *integrable M2* (λ*y*. *f* (*x*, *y*))
proof −
have (∫⁺*x*. (∫⁺*y*. *norm* (*f* (*x*, *y*)) ∂*M2*) ∂*M1*) = (∫⁺*x*. *norm* (*f x*) ∂(*M1* ⨂_M *M2*))
by (*rule M2.nn_integral_fst*) *simp*
also have (∫⁺*x*. *norm* (*f x*) ∂(*M1* ⨂_M *M2*)) ≠ ∞
using *f* **unfolding** *integrable_iff_bounded* **by** *simp*
finally have *AE x in M1*. (∫⁺*y*. *norm* (*f* (*x*, *y*)) ∂*M2*) ≠ ∞
by (*intro nn_integral_PInf_AE M2.borel_measurable_nn_integral*)
(*auto simp*: *measurable_split_conv*)
with *AE_space* **show** *?thesis*
by *eventually_elim*
(*auto simp*: *integrable_iff_bounded measurable_compose*[*OF _ borel_measurable_integrable*[*OF f*]] *less_top*)
qed

lemma (**in** *pair_sigma_finite*) *integrable_fst'*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *integrable* (*M1* ⨂_M *M2*) *f*
shows *integrable M1* (λ*x*. ∫ *y*. *f* (*x*, *y*) ∂*M2*)
unfolding *integrable_iff_bounded*
proof

show ($\lambda x.\ \int\ y.\ f\ (x,\ y)\ \partial M2) \in$ *borel_measurable M1*
by (*rule M2.borel_measurable_lebesgue_integral*) *simp*
have ($\int^+\ x.$ *ennreal* (*norm* ($\int\ y.\ f\ (x,\ y)\ \partial M2$)) $\partial M1$) $\leq$ ($\int^+ x.$ ($\int^+ y.$ *norm* ($f\ (x,\ y)$) $\partial M2$) $\partial M1$)
using *AE_integrable_fst'*[*OF f*] **by** (*auto intro*!: *nn_integral_mono_AE integral_norm_bound_ennreal*)
also have ($\int^+ x.$ ($\int^+ y.$ *norm* ($f\ (x,\ y)$) $\partial M2$) $\partial M1$) = ($\int^+ x.$ *norm* ($f\ x$) ∂(*M1* $\bigotimes_M$ *M2*))
by (*rule M2.nn_integral_fst*) *simp*
also have ($\int^+ x.$ *norm* ($f\ x$) ∂(*M1* $\bigotimes_M$ *M2*)) $< \infty$
using *f* **unfolding** *integrable_iff_bounded* **by** *simp*
finally show ($\int^+\ x.$ *ennreal* (*norm* ($\int\ y.\ f\ (x,\ y)\ \partial M2$)) $\partial M1$) $< \infty$.
qed

proposition (**in** *pair_sigma_finite*) *integral_fst'*:
fixes *f* :: _ $\Rightarrow$ _::{*banach*, *second_countable_topology*}
assumes *f*: *integrable* (*M1* $\bigotimes_M$ *M2*) *f*
shows ($\int x.$ ($\int y.\ f\ (x,\ y)\ \partial M2$) $\partial M1$) = *integral*^{L} (*M1* $\bigotimes_M$ *M2*) *f*
using *f* **proof** *induct*
case (*base A c*)
have *A*[*measurable*]: *A* $\in$ *sets* (*M1* $\bigotimes_M$ *M2*) **by** *fact*

have *eq*: $\bigwedge x\ y.\ x \in$ *space M1* $\Longrightarrow$ *indicator A* ($x,\ y$) = *indicator* {*y*∈*space M2*. ($x,\ y$) $\in A$} *y*
using *sets.sets_into_space*[*OF A*] **by** (*auto split*: *split_indicator simp*: *space_pair_measure*)

have *int_A*: *integrable* (*M1* $\bigotimes_M$ *M2*) (*indicator A* :: _ $\Rightarrow$ *real*)
using *base* **by** (*rule integrable_real_indicator*)

have ($\int\ x.\ \int\ y.$ *indicator A* ($x,\ y$) $*_R$ *c* $\partial M2\ \partial M1$) = ($\int x.$ *measure M2* {*y*∈*space M2*. ($x,\ y$) $\in A$} $*_R$ *c* $\partial M1$)
proof (*intro integral_cong_AE*, *simp*, *simp*)
from *AE_integrable_fst'*[*OF int_A*] *AE_space*
show *AE x in M1*. ($\int y.$ *indicator A* ($x,\ y$) $*_R$ *c* $\partial M2$) = *measure M2* {*y*∈*space M2*. ($x,\ y$) $\in A$} $*_R$ *c*
by *eventually_elim* (*simp add*: *eq integrable_indicator_iff*)
qed
also have ... = *measure* (*M1* $\bigotimes_M$ *M2*) *A* $*_R$ *c*
proof (*subst integral_scaleR_left*)
have ($\int^+ x.$ *ennreal* (*measure M2* {$y \in$ *space M2*. ($x,\ y$) $\in A$}) $\partial M1$) =
($\int^+ x.$ *emeasure M2* {$y \in$ *space M2*. ($x,\ y$) $\in A$} $\partial M1$)
using *emeasure_pair_measure_finite*[*OF base*]
by (*intro nn_integral_cong_AE*, *eventually_elim*) (*simp add*: *emeasure_eq_ennreal_measure*)
also have ... = *emeasure* (*M1* $\bigotimes_M$ *M2*) *A*
using *sets.sets_into_space*[*OF A*]
by (*subst M2.emeasure_pair_measure_alt*)
(*auto intro*!: *nn_integral_cong arg_cong*[**where** *f*=*emeasure M2*] *simp*: *space_pair_measure*)
finally have $*$: ($\int^+ x.$ *ennreal* (*measure M2* {$y \in$ *space M2*. ($x,\ y$) $\in A$}) $\partial M1$)

```
= emeasure (M1 ⨂M M2) A .

    from base ∗ show integrable M1 (λx. measure M2 {y ∈ space M2. (x, y) ∈
A})
      by (simp add: integrable_iff_bounded)
    then have (∫ x. measure M2 {y ∈ space M2. (x, y) ∈ A} ∂M1) =
      (∫+x. ennreal (measure M2 {y ∈ space M2. (x, y) ∈ A}) ∂M1)
      by (rule nn_integral_eq_integral[symmetric]) simp
    also note ∗
    finally show (∫ x. measure M2 {y ∈ space M2. (x, y) ∈ A} ∂M1) ∗R c =
measure (M1 ⨂M M2) A ∗R c
      using base by (simp add: emeasure_eq_ennreal_measure)
  qed
  also have ... = (∫ a. indicator A a ∗R c ∂(M1 ⨂M M2))
    using base by simp
  finally show ?case .
next
  case (add f g)
  then have [measurable]: f ∈ borel_measurable (M1 ⨂M M2) g ∈ borel_measurable
(M1 ⨂M M2)
    by auto
  have AE x in M1. LINT y|M2. f (x, y) + g (x, y) =
                (LINT y|M2. f (x, y)) + (LINT y|M2. g (x, y))
    using AE_integrable_fst'[OF add(1)] AE_integrable_fst'[OF add(3)]
    by eventually_elim simp
  then have (∫ x. ∫ y. f (x, y) + g (x, y) ∂M2 ∂M1) =
    (∫ x. (∫ y. f (x, y) ∂M2) + (∫ y. g (x, y) ∂M2) ∂M1)
    by (intro integral_cong_AE) auto
  also have ... = (∫ x. f x ∂(M1 ⨂M M2)) + (∫ x. g x ∂(M1 ⨂M M2))
    using integrable_fst'[OF add(1)] integrable_fst'[OF add(3)] add(2,4) by simp
  finally show ?case
    using add by simp
next
  case (lim f s)
  then have [measurable]: f ∈ borel_measurable (M1 ⨂M M2) ⋀i. s i ∈ borel_measurable
(M1 ⨂M M2)
    by auto

  show ?case
  proof (rule LIMSEQ_unique)
    show (λi. integral^L (M1 ⨂M M2) (s i)) ⟶ integral^L (M1 ⨂M M2) f
    proof (rule integral_dominated_convergence)
      show integrable (M1 ⨂M M2) (λx. 2 ∗ norm (f x))
        using lim(5) by auto
    qed (insert lim, auto)
    have (λi. ∫ x. ∫ y. s i (x, y) ∂M2 ∂M1) ⟶ ∫ x. ∫ y. f (x, y) ∂M2 ∂M1
    proof (rule integral_dominated_convergence)
      have AE x in M1. ∀i. integrable M2 (λy. s i (x, y))
        unfolding AE_all_countable using AE_integrable_fst'[OF lim(1)] ..
```

```
    with AE_space AE_integrable_fst'[OF lim(5)]
    show AE x in M1. (λi. ∫ y. s i (x, y) ∂M2) ⟶ ∫ y. f (x, y) ∂M2
    proof eventually_elim
      fix x assume x: x ∈ space M1 and
        s: ∀ i. integrable M2 (λy. s i (x, y)) and f: integrable M2 (λy. f (x, y))
      show (λi. ∫ y. s i (x, y) ∂M2) ⟶ ∫ y. f (x, y) ∂M2
      proof (rule integral_dominated_convergence)
        show integrable M2 (λy. 2 * norm (f (x, y)))
          using f by auto
        show AE xa in M2. (λi. s i (x, xa)) ⟶ f (x, xa)
          using x lim(3) by (auto simp: space_pair_measure)
        show ⋀i. AE xa in M2. norm (s i (x, xa)) ≤ 2 * norm (f (x, xa))
          using x lim(4) by (auto simp: space_pair_measure)
      qed (insert x, measurable)
    qed
    show integrable M1 (λx. (∫ y. 2 * norm (f (x, y)) ∂M2))
      by (intro integrable_mult_right integrable_norm integrable_fst' lim)
    fix i show AE x in M1. norm (∫ y. s i (x, y) ∂M2) ≤ (∫ y. 2 * norm (f
(x, y)) ∂M2)
     using AE_space AE_integrable_fst'[OF lim(1), of i] AE_integrable_fst'[OF
lim(5)]
    proof eventually_elim
      fix x assume x: x ∈ space M1
        and s: integrable M2 (λy. s i (x, y)) and f: integrable M2 (λy. f (x, y))
      from s have norm (∫ y. s i (x, y) ∂M2) ≤ (∫⁺y. norm (s i (x, y)) ∂M2)
        by (rule integral_norm_bound_ennreal)
      also have ... ≤ (∫⁺y. 2 * norm (f (x, y)) ∂M2)
       using x lim by (auto intro!: nn_integral_mono simp: space_pair_measure)
      also have ... = (∫ y. 2 * norm (f (x, y)) ∂M2)
        using f by (intro nn_integral_eq_integral) auto
       finally show norm (∫ y. s i (x, y) ∂M2) ≤ (∫ y. 2 * norm (f (x, y))
∂M2)
        by simp
    qed
   qed simp_all
   then show (λi. integral^L (M1 ⨂M M2) (s i)) ⟶ ∫ x. ∫ y. f (x, y) ∂M2
∂M1
    using lim by simp
  qed
qed

lemma (in pair_sigma_finite)
  fixes f :: _ ⇒ _ ⇒ _::{banach, second_countable_topology}
  assumes f: integrable (M1 ⨂M M2) (case_prod f)
  shows AE_integrable_fst: AE x in M1. integrable M2 (λy. f x y) (is ?AE)
    and integrable_fst: integrable M1 (λx. ∫ y. f x y ∂M2) (is ?INT)
   and integral_fst: (∫ x. (∫ y. f x y ∂M2) ∂M1) = integral^L (M1 ⨂M M2) (λ(x,
y). f x y) (is ?EQ)
  using AE_integrable_fst'[OF f] integrable_fst'[OF f] integral_fst'[OF f] by auto
```

lemma (**in** *pair_sigma_finite*)
fixes $f :: \_ \Rightarrow \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
assumes $f[measurable]$: *integrable* $(M1 \bigotimes_M M2)$ (*case_prod f*)
shows *AE_integrable_snd*: *AE y in M2. integrable M1* $(\lambda x.\ f\ x\ y)$ (**is** *?AE*)
and *integrable_snd*: *integrable M2* $(\lambda y.\ \int x.\ f\ x\ y\ \partial M1)$ (**is** *?INT*)
and *integral_snd*: $(\int y.\ (\int x.\ f\ x\ y\ \partial M1)\ \partial M2) = integral^L\ (M1 \bigotimes_M M2)$ (*case_prod f*) (**is** *?EQ*)
proof −
interpret *Q*: *pair_sigma_finite M2 M1* **..**
have *Q_int*: *integrable* $(M2 \bigotimes_M M1)\ (\lambda(x, y).\ f\ y\ x)$
using *f* **unfolding** *integrable_product_swap_iff*[*symmetric*] **by** *simp*
show *?AE* **using** *Q.AE_integrable_fst'*[*OF Q_int*] **by** *simp*
show *?INT* **using** *Q.integrable_fst'*[*OF Q_int*] **by** *simp*
show *?EQ* **using** *Q.integral_fst'*[*OF Q_int*]
using *integral_product_swap*[*of case_prod f*] **by** *simp*
qed

proposition (**in** *pair_sigma_finite*) *Fubini_integral*:
fixes $f :: \_ \Rightarrow \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
assumes *f*: *integrable* $(M1 \bigotimes_M M2)$ (*case_prod f*)
shows $(\int y.\ (\int x.\ f\ x\ y\ \partial M1)\ \partial M2) = (\int x.\ (\int y.\ f\ x\ y\ \partial M2)\ \partial M1)$
unfolding *integral_snd*[*OF assms*] *integral_fst*[*OF assms*] **..**

lemma (**in** *product_sigma_finite*) *product_integral_singleton*:
fixes $f :: \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
shows $f \in borel\_measurable\ (M\ i) \Longrightarrow (\int x.\ f\ (x\ i)\ \partial Pi_M\ \{i\}\ M) = integral^L\ (M\ i)\ f$
by (*metis* (*no_types*) *distr_singleton insert_iff integral_distr measurable_component_singleton*)

lemma (**in** *product_sigma_finite*) *product_integral_fold*:
fixes $f :: \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
assumes *IJ*[*simp*]: $I \cap J = \{\}$ **and** *fin*: *finite I finite J*
and *f*: *integrable* $(Pi_M\ (I \cup J)\ M)\ f$
shows $integral^L\ (Pi_M\ (I \cup J)\ M)\ f = (\int x.\ (\int y.\ f\ (merge\ I\ J\ (x, y))\ \partial Pi_M\ J\ M)\ \partial Pi_M\ I\ M)$
proof −
interpret *I*: *finite_product_sigma_finite M I* **by** *standard fact*
interpret *J*: *finite_product_sigma_finite M J* **by** *standard fact*
have *finite* $(I \cup J)$ **using** *fin* **by** *auto*
interpret *IJ*: *finite_product_sigma_finite M* $I \cup J$ **by** *standard fact*
interpret *P*: *pair_sigma_finite* $Pi_M\ I\ M\ Pi_M\ J\ M$ **..**
let $?M = merge\ I\ J$
let $?f = \lambda x.\ f\ (?M\ x)$
from *f* **have** *f_borel*: $f \in borel\_measurable\ (Pi_M\ (I \cup J)\ M)$
by *auto*
have *P_borel*: $(\lambda x.\ f\ (merge\ I\ J\ x)) \in borel\_measurable\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)$
using *measurable_comp*[*OF measurable_merge f_borel*] **by** (*simp add*: *comp_def*)

```
  have f_int: integrable (Pi_M I M ⨂_M Pi_M J M) ?f
    by (rule integrable_distr[OF measurable_merge]) (simp add: distr_merge[OF
IJ fin] f)
  have LINT x|(Pi_M I M ⨂_M Pi_M J M). f (merge I J x) =
        LINT x|Pi_M I M. LINT y|Pi_M J M. f (merge I J (x, y))
    by (simp add: P.integral_fst'[symmetric, OF f_int])
  then show ?thesis
    apply (subst distr_merge[symmetric, OF IJ fin])
    by (simp add: integral_distr[OF measurable_merge f_borel])
qed

lemma (in product_sigma_finite) product_integral_insert:
  fixes f :: _ ⇒ _::{banach, second_countable_topology}
  assumes I: finite I i ∉ I
    and f: integrable (Pi_M (insert i I) M) f
  shows integral^L (Pi_M (insert i I) M) f = (∫ x. (∫ y. f (x(i:=y)) ∂M i) ∂Pi_M I
M)
proof −
  have integral^L (Pi_M (insert i I) M) f = integral^L (Pi_M (I ∪ {i}) M) f
    by simp
  also have ... = (∫ x. (∫ y. f (merge I {i} (x,y)) ∂Pi_M {i} M) ∂Pi_M I M)
    using f I by (intro product_integral_fold) auto
  also have ... = (∫ x. (∫ y. f (x(i := y)) ∂M i) ∂Pi_M I M)
  proof (rule integral_cong[OF refl], subst product_integral_singleton[symmetric])
    fix x assume x: x ∈ space (Pi_M I M)
    have f_borel: f ∈ borel_measurable (Pi_M (insert i I) M)
      using f by auto
    show (λy. f (x(i := y))) ∈ borel_measurable (M i)
      using measurable_comp[OF measurable_component_update f_borel, OF x ‹i
∉ I›]
      unfolding comp_def .
    from x I show (∫ y. f (merge I {i} (x,y)) ∂Pi_M {i} M) = (∫ xa. f (x(i :=
xa i)) ∂Pi_M {i} M)
    by (auto intro!: integral_cong arg_cong[where f=f] simp: merge_def space_PiM
extensional_def PiE_def)
  qed
  finally show ?thesis .
qed

lemma (in product_sigma_finite) product_integrable_prod:
  fixes f :: 'i ⇒ 'a ⇒ _::{real_normed_field,banach,second_countable_topology}
  assumes [simp]: finite I and integrable: ⋀i. i ∈ I ⟹ integrable (M i) (f i)
  shows integrable (Pi_M I M) (λx. (∏ i∈I. f i (x i))) (is integrable _ ?f)
proof (unfold integrable_iff_bounded, intro conjI)
  interpret finite_product_sigma_finite M I by standard fact

  show ?f ∈ borel_measurable (Pi_M I M)
    using assms by simp
  have (∫^+ x. ennreal (norm (∏ i∈I. f i (x i))) ∂Pi_M I M) =
```

```
    (∫⁺ x. (∏ i∈I. ennreal (norm (f i (x i)))) ∂Pi_M I M)
    by (simp add: prod_norm prod_ennreal)
  also have ... = (∏ i∈I. ∫⁺ x. ennreal (norm (f i x)) ∂M i)
    using assms by (intro product_nn_integral_prod) auto
  also have ... < ∞
    using integrable by (simp add: less_top[symmetric] ennreal_prod_eq_top integrable_iff_bounded)
  finally show (∫⁺ x. ennreal (norm (∏ i∈I. f i (x i))) ∂Pi_M I M) < ∞ .
qed

lemma (in product_sigma_finite) product_integral_prod:
  fixes f :: 'i ⇒ 'a ⇒ _::{real_normed_field,banach,second_countable_topology}
  assumes finite I and integrable: ⋀i. i ∈ I ⟹ integrable (M i) (f i)
  shows (∫ x. (∏ i∈I. f i (x i)) ∂Pi_M I M) = (∏ i∈I. integral^L (M i) (f i))
using assms proof induct
  case empty
  interpret finite_measure Pi_M {} M
    by rule (simp add: space_PiM)
  show ?case by (simp add: space_PiM measure_def)
next
  case (insert i I)
  then have iI: finite (insert i I) by auto
  then have prod: ⋀J. J ⊆ insert i I ⟹
    integrable (Pi_M J M) (λx. (∏ i∈J. f i (x i)))
    by (intro product_integrable_prod insert(4)) (auto intro: finite_subset)
  interpret I: finite_product_sigma_finite M I by standard fact
  have *: ⋀x y. (∏ j∈I. f j (if j = i then y else x j)) = (∏ j∈I. f j (x j))
    using ‹i ∉ I› by (auto intro!: prod.cong)
  show ?case
    unfolding product_integral_insert[OF insert(1,2) prod[OF subset_refl]]
    by (simp add: * insert prod subset_insertI)
qed

lemma integrable_subalgebra:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes borel: f ∈ borel_measurable N
  and N: sets N ⊆ sets M space N = space M ⋀A. A ∈ sets N ⟹ emeasure N A = emeasure M A
  shows integrable N f ⟷ integrable M f (is ?P)
proof −
  have f ∈ borel_measurable M
    using assms by (auto simp: measurable_def)
  with assms show ?thesis
    using assms by (auto simp: integrable_iff_bounded nn_integral_subalgebra)
qed

lemma integral_subalgebra:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes borel: f ∈ borel_measurable N
```

and N: *sets* $N \subseteq$ *sets* M *space* $N =$ *space* M $\bigwedge A$. $A \in$ *sets* $N \Longrightarrow$ *emeasure* N $A =$ *emeasure* M A
shows $integral^L\ N\ f = integral^L\ M\ f$
proof *cases*
assume *integrable* N f
then show *?thesis*
proof *induct*
case *base* **with** *assms* **show** *?case* **by** (*auto simp*: *subset_eq measure_def*)
next
case (*add* f g)
then have $(\int\ a.\ f\ a + g\ a\ \partial N) = integral^L\ M\ f + integral^L\ M\ g$
by *simp*
also have $\ldots = (\int\ a.\ f\ a + g\ a\ \partial M)$
using *add integrable_subalgebra*[*OF* _ N, *of* f] *integrable_subalgebra*[*OF* _ N, *of* g] **by** *simp*
finally show *?case* **.**
next
case (*lim* f s)
then have M: $\bigwedge i.$ *integrable* M (s i) *integrable* M f
using *integrable_subalgebra*[*OF* _ N, *of* f] *integrable_subalgebra*[*OF* _ N, *of* s i **for** i] **by** *simp_all*
show *?case*
proof (*intro LIMSEQ_unique*)
show $(\lambda i.\ integral^L\ N\ (s\ i)) \longrightarrow integral^L\ N\ f$
apply (*rule integral_dominated_convergence*[**where** w=$\lambda x.\ 2 * norm\ (f\ x)$])
using *lim* **by** *auto*
show $(\lambda i.\ integral^L\ N\ (s\ i)) \longrightarrow integral^L\ M\ f$
unfolding *lim*
apply (*rule integral_dominated_convergence*[**where** w=$\lambda x.\ 2 * norm\ (f\ x)$])
using *lim* M N **by** *auto*
qed
qed
qed (*simp add*: *not_integrable_integral_eq integrable_subalgebra*[*OF assms*])

hide_const (**open**) *simple_bochner_integral*
hide_const (**open**) *simple_bochner_integrable*

end

7.10 Complete Measures

theory *Complete_Measure*
imports *Bochner_Integration*
begin

locale *complete_measure* =
fixes M :: $'a$ *measure*

assumes *complete*: $\bigwedge A\ B.\ B \subseteq A \Longrightarrow A \in null\_sets\ M \Longrightarrow B \in sets\ M$

definition
split_completion M A p = (*if A* ∈ *sets M then p* = (*A*, {}) *else*
∃ *N'*. *A* = *fst p* ∪ *snd p* ∧ *fst p* ∩ *snd p* = {} ∧ *fst p* ∈ *sets M* ∧ *snd p* ⊆ *N'* ∧ *N'* ∈ *null_sets M*)

definition
main_part M A = *fst* (*Eps* (*split_completion M A*))

definition
null_part M A = *snd* (*Eps* (*split_completion M A*))

definition *completion* :: *'a measure* ⇒ *'a measure* **where**
completion M = *measure_of* (*space M*) { *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* }
(*emeasure M* ∘ *main_part M*)

lemma *completion_into_space*:
{ *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* } ⊆ *Pow* (*space M*)
using *sets.sets_into_space* **by** *auto*

lemma *space_completion*[*simp*]: *space* (*completion M*) = *space M*
unfolding *completion_def* **using** *space_measure_of*[*OF completion_into_space*]
by *simp*

lemma *completionI*:
assumes *A* = *S* ∪ *N N* ⊆ *N' N'* ∈ *null_sets M S* ∈ *sets M*
shows *A* ∈ { *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* }
using *assms* **by** *auto*

lemma *completionE*:
assumes *A* ∈ { *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* }
obtains *S N N'* **where** *A* = *S* ∪ *N N* ⊆ *N' N'* ∈ *null_sets M S* ∈ *sets M*
using *assms* **by** *auto*

lemma *sigma_algebra_completion*:
sigma_algebra (*space M*) { *S* ∪ *N* |*S N N'*. *S* ∈ *sets M* ∧ *N'* ∈ *null_sets M* ∧ *N* ⊆ *N'* }
(**is** *sigma_algebra _ ?A*)
unfolding *sigma_algebra_iff2*
proof (*intro conjI ballI allI impI*)
show *?A* ⊆ *Pow* (*space M*)
using *sets.sets_into_space* **by** *auto*
next
show {} ∈ *?A* **by** *auto*
next
let *?C* = *space M*

fix A **assume** $A \in ?A$
then obtain $S\ N\ N'$
where $A = S \cup N\ N \subseteq N'\ N' \in null\_sets\ M\ S \in sets\ M$
by (*rule completionE*)
then show *space* $M - A \in ?A$
by (*intro completionI*[*of* _ $(?C - S) \cap (?C - N')$ $(?C - S) \cap N' \cap (?C - N)$]) *auto*
next
fix $A :: nat \Rightarrow 'a\ set$ **assume** A: *range* $A \subseteq ?A$
then have $\forall n.\ \exists S\ N\ N'.\ A\ n = S \cup N \wedge S \in sets\ M \wedge N' \in null\_sets\ M \wedge N \subseteq N'$
by (*auto simp*: *image_subset_iff*)
then obtain $S\ N\ N'$ **where** $\forall x.\ A\ x = S\ x \cup N\ x \wedge S\ x \in sets\ M \wedge N'\ x \in null\_sets\ M \wedge N\ x \subseteq N'\ x$
by *metis*
then show $\bigcup (A\ `\ UNIV) \in ?A$
using *null_sets_UN*[*of N'*]
by (*intro completionI*[*of* _ $\bigcup (S\ `\ UNIV) \bigcup (N\ `\ UNIV) \bigcup (N'\ `\ UNIV)$]) *auto*
qed

lemma *sets_completion*:
sets (*completion* M) = { $S \cup N$ |$S\ N\ N'.\ S \in sets\ M \wedge N' \in null\_sets\ M \wedge N \subseteq N'$ }
using *sigma_algebra.sets_measure_of_eq*[*OF sigma_algebra_completion*]
by (*simp add*: *completion_def*)

lemma *sets_completionE*:
assumes $A \in sets$ (*completion* M)
obtains $S\ N\ N'$ **where** $A = S \cup N\ N \subseteq N'\ N' \in null\_sets\ M\ S \in sets\ M$
using *assms* **unfolding** *sets_completion* **by** *auto*

lemma *sets_completionI*:
assumes $A = S \cup N\ N \subseteq N'\ N' \in null\_sets\ M\ S \in sets\ M$
shows $A \in sets$ (*completion* M)
using *assms* **unfolding** *sets_completion* **by** *auto*

lemma *sets_completionI_sets*[*intro*, *simp*]:
$A \in sets\ M \Longrightarrow A \in sets$ (*completion* M)
unfolding *sets_completion* **by** *force*

lemma *measurable_completion*: $f \in M \rightarrow_M N \Longrightarrow f \in completion\ M \rightarrow_M N$
by (*auto simp*: *measurable_def*)

lemma *null_sets_completion*:
assumes $N' \in null\_sets\ M\ N \subseteq N'$ **shows** $N \in sets$ (*completion* M)
using *assms* **by** (*intro sets_completionI*[*of* N {} $N\ N'$]) *auto*

lemma *split_completion*:
assumes $A \in sets$ (*completion* M)

shows *split_completion M A* (*main_part M A*, *null_part M A*)
proof *cases*
assume *A* ∈ *sets M* **then show** *?thesis*
by (*simp add*: *split_completion_def*[*abs_def*] *main_part_def null_part_def*)
next
assume *nA*: *A* ∉ *sets M*
show *?thesis*
unfolding *main_part_def null_part_def if_not_P*[*OF nA*]
proof (*rule someI2_ex*)
from *assms* **obtain** *S N N'* **where** *A*: *A* = *S* ∪ *N N* ⊆ *N' N'* ∈ *null_sets M* *S* ∈ *sets M*
by (*blast intro*: *sets_completionE*)
let *?P* = (*S*, *N* − *S*)
show ∃ *p*. *split_completion M A p*
unfolding *split_completion_def if_not_P*[*OF nA*] **using** *A*
proof (*intro exI conjI*)
show *A* = *fst ?P* ∪ *snd ?P* **using** *A* **by** *auto*
show *snd ?P* ⊆ *N'* **using** *A* **by** *auto*
qed *auto*
qed *auto*
qed

lemma *sets_restrict_space_subset*:
assumes *S*: *S* ∈ *sets* (*completion M*)
shows *sets* (*restrict_space* (*completion M*) *S*) ⊆ *sets* (*completion M*)
by (*metis assms sets.Int_space_eq2 sets_restrict_space_iff subsetI*)

lemma
assumes *S* ∈ *sets* (*completion M*)
shows *main_part_sets*[*intro*, *simp*]: *main_part M S* ∈ *sets M*
and *main_part_null_part_Un*[*simp*]: *main_part M S* ∪ *null_part M S* = *S*
and *main_part_null_part_Int*[*simp*]: *main_part M S* ∩ *null_part M S* = {}
using *split_completion*[*OF assms*]
by (*auto simp*: *split_completion_def split*: *if_split_asm*)

lemma *main_part*[*simp*]: *S* ∈ *sets M* ⟹ *main_part M S* = *S*
using *split_completion*[*of S M*]
by (*auto simp*: *split_completion_def split*: *if_split_asm*)

lemma *null_part*:
assumes *S* ∈ *sets* (*completion M*) **shows** ∃ *N*. *N*∈*null_sets M* ∧ *null_part M* *S* ⊆ *N*
using *split_completion*[*OF assms*] **by** (*auto simp*: *split_completion_def split*: *if_split_asm*)

lemma *null_part_sets*[*intro*, *simp*]:
assumes *S* ∈ *sets M* **shows** *null_part M S* ∈ *sets M emeasure M* (*null_part M* *S*) = *0*
proof −

have S: $S \in sets\ (completion\ M)$ **using** *assms* **by** *auto*
have $*$: $S - main\_part\ M\ S \in sets\ M$ **using** *assms* **by** *auto*
from *main_part_null_part_Un*[*OF S*] *main_part_null_part_Int*[*OF S*]
have $S - main\_part\ M\ S = null\_part\ M\ S$ **by** *auto*
with $*$ **show** *sets*: $null\_part\ M\ S \in sets\ M$ **by** *auto*
from *null_part*[*OF S*] **obtain** N **where** $N \in null\_sets\ M \wedge null\_part\ M\ S \subseteq N$ **..**
with *emeasure_eq_0*[*of N _ null_part M S*] *sets*
show $emeasure\ M\ (null\_part\ M\ S) = 0$ **by** *auto*
qed

lemma *emeasure_main_part_UN*:
fixes $S :: nat \Rightarrow 'a\ set$
assumes $range\ S \subseteq sets\ (completion\ M)$
shows $emeasure\ M\ (main\_part\ M\ (\bigcup i.\ (S\ i))) = emeasure\ M\ (\bigcup i.\ main\_part\ M\ (S\ i))$
proof −
have S: $\bigwedge i.\ S\ i \in sets\ (completion\ M)$ **using** *assms* **by** *auto*
then have *UN*: $(\bigcup i.\ S\ i) \in sets\ (completion\ M)$ **by** *auto*
have $\forall i.\ \exists N.\ N \in null\_sets\ M \wedge null\_part\ M\ (S\ i) \subseteq N$
using *null_part*[*OF S*] **by** *auto*
then obtain N **where** N: $\forall x.\ N\ x \in null\_sets\ M \wedge null\_part\ M\ (S\ x) \subseteq N\ x$ **by** *metis*
then have *UN_N*: $(\bigcup i.\ N\ i) \in null\_sets\ M$ **by** (*intro null_sets_UN*) *auto*
from S **have** $(\bigcup i.\ S\ i) \in sets\ (completion\ M)$ **by** *auto*
from *null_part*[*OF this*] **obtain** N' **where** N': $N' \in null\_sets\ M \wedge null\_part\ M\ (\bigcup\ (range\ S)) \subseteq N'$ **..**
let $?N = (\bigcup i.\ N\ i) \cup N'$
have *null_set*: $?N \in null\_sets\ M$ **using** N' *UN_N* **by** (*intro null_sets.Un*) *auto*
have $main\_part\ M\ (\bigcup i.\ S\ i) \cup ?N = (main\_part\ M\ (\bigcup i.\ S\ i) \cup null\_part\ M\ (\bigcup i.\ S\ i)) \cup ?N$
using N' **by** *auto*
also have $\ldots = (\bigcup i.\ main\_part\ M\ (S\ i) \cup null\_part\ M\ (S\ i)) \cup ?N$
unfolding *main_part_null_part_Un*[*OF S*] *main_part_null_part_Un*[*OF UN*] **by** *auto*
also have $\ldots = (\bigcup i.\ main\_part\ M\ (S\ i)) \cup ?N$
using N **by** *auto*
finally have $*$: $main\_part\ M\ (\bigcup i.\ S\ i) \cup ?N = (\bigcup i.\ main\_part\ M\ (S\ i)) \cup ?N$ **.**
have $emeasure\ M\ (main\_part\ M\ (\bigcup i.\ S\ i)) = emeasure\ M\ (main\_part\ M\ (\bigcup i.\ S\ i) \cup ?N)$
using *null_set UN* **by** (*intro emeasure_Un_null_set*[*symmetric*]) *auto*
also have $\ldots = emeasure\ M\ ((\bigcup i.\ main\_part\ M\ (S\ i)) \cup ?N)$
unfolding $*$ **..**
also have $\ldots = emeasure\ M\ (\bigcup i.\ main\_part\ M\ (S\ i))$
using *null_set S* **by** (*intro emeasure_Un_null_set*) *auto*
finally show *?thesis* **.**
qed

lemma *emeasure_completion*[*simp*]:
assumes *S*: $S \in sets\ (completion\ M)$
shows $emeasure\ (completion\ M)\ S = emeasure\ M\ (main\_part\ M\ S)$
proof (*subst emeasure_measure_of*[*OF completion_def completion_into_space*])
let $?\mu = emeasure\ M \circ main\_part\ M$
show $S \in sets\ (completion\ M)\ ?\mu\ S = emeasure\ M\ (main\_part\ M\ S)$ **using** *S*
by *simp_all*
show $positive\ (sets\ (completion\ M))\ ?\mu$
by (*simp add*: *positive_def*)
show $countably\_additive\ (sets\ (completion\ M))\ ?\mu$
proof (*intro countably_additiveI*)
fix $A :: nat \Rightarrow 'a\ set$ **assume** *A*: $range\ A \subseteq sets\ (completion\ M)\ disjoint\_family\ A$
have $disjoint\_family\ (\lambda i.\ main\_part\ M\ (A\ i))$
proof (*intro disjoint_family_on_bisimulation*[*OF A*(*2*)])
fix *n m* **assume** $A\ n \cap A\ m = \{\}$
then have $(main\_part\ M\ (A\ n) \cup null\_part\ M\ (A\ n)) \cap (main\_part\ M\ (A\ m) \cup null\_part\ M\ (A\ m)) = \{\}$
using *A* **by** (*subst* (*1 2*) *main_part_null_part_Un*) *auto*
then show $main\_part\ M\ (A\ n) \cap main\_part\ M\ (A\ m) = \{\}$ **by** *auto*
qed
then have $(\sum n.\ emeasure\ M\ (main\_part\ M\ (A\ n))) = emeasure\ M\ (\bigcup i.\ main\_part\ M\ (A\ i))$
using *A* **by** (*auto intro*!: *suminf_emeasure*)
then show $(\sum n.\ ?\mu\ (A\ n)) = ?\mu\ (\bigcup (A\ `\ UNIV))$
by (*simp add*: *completion_def emeasure_main_part_UN*[*OF A*(*1*)])
qed
qed

lemma *measure_completion*[*simp*]: $S \in sets\ M \Longrightarrow measure\ (completion\ M)\ S = measure\ M\ S$
unfolding *measure_def* **by** *auto*

lemma *emeasure_completion_UN*:
$range\ S \subseteq sets\ (completion\ M) \Longrightarrow$
$emeasure\ (completion\ M)\ (\bigcup i{::}nat.\ (S\ i)) = emeasure\ M\ (\bigcup i.\ main\_part\ M\ (S\ i))$
by (*subst emeasure_completion*) (*auto simp add*: *emeasure_main_part_UN*)

lemma *emeasure_completion_Un*:
assumes *S*: $S \in sets\ (completion\ M)$ **and** *T*: $T \in sets\ (completion\ M)$
shows $emeasure\ (completion\ M)\ (S \cup T) = emeasure\ M\ (main\_part\ M\ S \cup main\_part\ M\ T)$
proof (*subst emeasure_completion*)
have *UN*: $(\bigcup i.\ binary\ (main\_part\ M\ S)\ (main\_part\ M\ T)\ i) = (\bigcup i.\ main\_part\ M\ (binary\ S\ T\ i))$
unfolding *binary_def* **by** (*auto split*: *if_split_asm*)
show $emeasure\ M\ (main\_part\ M\ (S \cup T)) = emeasure\ M\ (main\_part\ M\ S \cup$

```
main_part M T)
    using emeasure_main_part_UN[of binary S T M] assms
    by (simp add: range_binary_eq, simp add: Un_range_binary UN)
qed (auto intro: S T)

lemma sets_completionI_sub:
  assumes N: N' ∈ null_sets M N ⊆ N'
  shows N ∈ sets (completion M)
  using assms by (intro sets_completionI[of _ {} N N'']) auto

lemma completion_ex_simple_function:
  assumes f: simple_function (completion M) f
  shows ∃f'. simple_function M f' ∧ (AE x in M. f x = f' x)
proof −
  let ?F = λx. f −‘ {x} ∩ space M
  have F: ⋀x. ?F x ∈ sets (completion M) and fin: finite (f‘space M)
    using simple_functionD[OF f] simple_functionD[OF f] by simp_all
  have ∀x. ∃N. N ∈ null_sets M ∧ null_part M (?F x) ⊆ N
    using F null_part by auto
  from choice[OF this] obtain N where
    N: ⋀x. null_part M (?F x) ⊆ N x ⋀x. N x ∈ null_sets M by auto
  let ?N = ⋃x∈f‘space M. N x
  let ?f' = λx. if x ∈ ?N then undefined else f x
  have sets: ?N ∈ null_sets M using N fin by (intro null_sets.finite_UN) auto
  show ?thesis unfolding simple_function_def
  proof (safe intro!: exI[of _ ?f'])
    have ?f' ‘ space M ⊆ f‘space M ∪ {undefined} by auto
    from finite_subset[OF this] simple_functionD(1)[OF f]
    show finite (?f' ‘ space M) by auto
  next
    fix x assume x ∈ space M
    have ?f' −‘ {?f' x} ∩ space M =
      (if x ∈ ?N then ?F undefined ∪ ?N
       else if f x = undefined then ?F (f x) ∪ ?N
       else ?F (f x) − ?N)
     using N(2) sets.sets_into_space by (auto split: if_split_asm simp: null_sets_def)
    moreover { fix y have ?F y ∪ ?N ∈ sets M
      proof cases
        assume y: y ∈ f‘space M
        have ?F y ∪ ?N = (main_part M (?F y) ∪ null_part M (?F y)) ∪ ?N
          using main_part_null_part_Un[OF F] by auto
        also have ... = main_part M (?F y) ∪ ?N
          using y N by auto
        finally show ?thesis
          using F sets by auto
      next
        assume y ∉ f‘space M then have ?F y = {} by auto
        then show ?thesis using sets by auto
      qed }
```

```
    moreover {
      have ?F (f x) - ?N = main_part M (?F (f x)) ∪ null_part M (?F (f x)) -
?N
        using main_part_null_part_Un[OF F] by auto
      also have ... = main_part M (?F (f x)) - ?N
        using N ‹x ∈ space M› by auto
      finally have ?F (f x) - ?N ∈ sets M
        using F sets by auto }
    ultimately show ?f' -` {?f' x} ∩ space M ∈ sets M by auto
  next
    show AE x in M. f x = ?f' x
      by (rule AE_I', rule sets) auto
  qed
qed

lemma completion_ex_borel_measurable:
  fixes g :: 'a ⇒ ennreal
  assumes g: g ∈ borel_measurable (completion M)
  shows ∃ g'∈borel_measurable M. (AE x in M. g x = g' x)
proof -
  obtain f :: nat ⇒ 'a ⇒ ennreal
    where *: ⋀i. simple_function (completion M) (f i)
      and **: ⋀x. (SUP i. f i x) = g x
   using g[THEN borel_measurable_implies_simple_function_sequence'] by metis
  from *[THEN completion_ex_simple_function]
  have ∀ i. ∃ f'. simple_function M f' ∧ (AE x in M. f i x = f' x) ..
  then obtain f'
    where sf: ⋀i. simple_function M (f' i)
      and AE: ∀ i. AE x in M. f i x = f' i x
    by metis
  show ?thesis
  proof (intro bexI)
    from AE[unfolded AE_all_countable[symmetric]]
    show AE x in M. g x = (SUP i. f' i x) (is AE x in M. g x = ?f x)
    proof (elim AE_mp, safe intro!: AE_I2)
      fix x assume eq: ∀ i. f i x = f' i x
      have g x = (SUP i. f i x) by (auto simp: ** split: split_max)
      with eq show g x = ?f x by auto
    qed
    show ?f ∈ borel_measurable M
      using sf[THEN borel_measurable_simple_function] by auto
  qed
qed

lemma null_sets_completionI: N ∈ null_sets M ⟹ N ∈ null_sets (completion
M)
  by (auto simp: null_sets_def)

lemma AE_completion: (AE x in M. P x) ⟹ (AE x in completion M. P x)
```

unfolding *eventually_ae_filter* **by** (*auto intro*: *null_sets_completionI*)

lemma *null_sets_completion_iff*: $N \in sets\ M \Longrightarrow N \in null\_sets\ (completion\ M) \longleftrightarrow N \in null\_sets\ M$
by (*auto simp*: *null_sets_def*)

lemma *sets_completion_AE*: $(AE\ x\ in\ M.\ \neg\ P\ x) \Longrightarrow Measurable.pred\ (completion\ M)\ P$
unfolding *pred_def sets_completion eventually_ae_filter*
by *auto*

lemma *null_sets_completion_iff2*:
$A \in null\_sets\ (completion\ M) \longleftrightarrow (\exists\ N' \in null\_sets\ M.\ A \subseteq N')$
proof *safe*
assume $A \in null\_sets\ (completion\ M)$
then have *A*: $A \in sets\ (completion\ M)$ **and** $main\_part\ M\ A \in null\_sets\ M$
by (*auto simp*: *null_sets_def*)
moreover obtain *N* **where** $N \in null\_sets\ M\ null\_part\ M\ A \subseteq N$
using *null_part*[*OF A*] **by** *auto*
ultimately show $\exists\ N' \in null\_sets\ M.\ A \subseteq N'$
proof (*intro bexI*)
show $A \subseteq N \cup main\_part\ M\ A$
using ‹*null_part M A* ⊆ *N*› **by** (*subst main_part_null_part_Un*[*OF A*, *symmetric*]) *auto*
qed *auto*
next
fix *N* **assume** $N \in null\_sets\ M\ A \subseteq N$
then have $A \in sets\ (completion\ M)$ **and** *N*: $N \in sets\ M\ A \subseteq N\ emeasure\ M\ N = 0$
by (*auto intro*: *null_sets_completion*)
moreover have $emeasure\ (completion\ M)\ A = 0$
using *N* **by** (*intro emeasure_eq_0*[*of N _ A*]) *auto*
ultimately show $A \in null\_sets\ (completion\ M)$
by *auto*
qed

lemma *null_sets_completion_subset*:
$B \subseteq A \Longrightarrow A \in null\_sets\ (completion\ M) \Longrightarrow B \in null\_sets\ (completion\ M)$
unfolding *null_sets_completion_iff2* **by** *auto*

interpretation *completion*: *complete_measure completion M* **for** *M*
proof
show $B \subseteq A \Longrightarrow A \in null\_sets\ (completion\ M) \Longrightarrow B \in sets\ (completion\ M)$
for *B A*
using *null_sets_completion_subset*[*of B A M*] **by** (*simp add*: *null_sets_def*)
qed

lemma *null_sets_restrict_space*:
$\Omega \in sets\ M \Longrightarrow A \in null\_sets\ (restrict\_space\ M\ \Omega) \longleftrightarrow A \subseteq \Omega \wedge A \in null\_sets$

M
by (*auto simp*: *null_sets_def emeasure_restrict_space sets_restrict_space*)

lemma *completion_ex_borel_measurable_real*:
fixes *g* :: *'a* ⇒ *real*
assumes *g*: *g* ∈ *borel_measurable* (*completion M*)
shows ∃ *g'*∈*borel_measurable M*. (*AE x in M*. *g x* = *g' x*)
proof −
have (λ*x*. *ennreal* (*g x*)) ∈ *completion M* →$_M$ *borel* (λ*x*. *ennreal* (− *g x*)) ∈ *completion M* →$_M$ *borel*
using *g* **by** *auto*
from *this*[*THEN completion_ex_borel_measurable*]
obtain *pf nf* :: *'a* ⇒ *ennreal*
where [*measurable*]: *nf* ∈ *M* →$_M$ *borel pf* ∈ *M* →$_M$ *borel*
and *ae*: *AE x in M*. *pf x* = *ennreal* (*g x*) *AE x in M*. *nf x* = *ennreal* (− *g x*)
by (*auto simp*: *eq_commute*)
then have *AE x in M*. *pf x* = *ennreal* (*g x*) ∧ *nf x* = *ennreal* (− *g x*)
by *auto*
then obtain *N* **where** *N* ∈ *null_sets M* {*x*∈*space M*. *pf x* ≠ *ennreal* (*g x*) ∧ *nf x* ≠ *ennreal* (− *g x*)} ⊆ *N*
by (*auto elim*!: *AE_E*)
show *?thesis*
proof
let *?F* = λ*x*. *indicator* (*space M* − *N*) *x* ∗ (*enn2real* (*pf x*) − *enn2real* (*nf x*))
show *?F* ∈ *M* →$_M$ *borel*
using ‹*N* ∈ *null_sets M*› **by** *auto*
show *AE x in M*. *g x* = *?F x*
using ‹*N* ∈ *null_sets M*›[*THEN AE_not_in*] *ae AE_space*
apply *eventually_elim*
subgoal for *x*
by (*cases 0*::*real g x rule*: *linorder_le_cases*) (*auto simp*: *ennreal_neg*)
done
qed
qed

lemma *simple_function_completion*: *simple_function M f* ⟹ *simple_function* (*completion M*) *f*
by (*simp add*: *simple_function_def*)

lemma *simple_integral_completion*:
simple_function M f ⟹ *simple_integral* (*completion M*) *f* = *simple_integral M f*
unfolding *simple_integral_def* **by** *simp*

lemma *nn_integral_completion*: *nn_integral* (*completion M*) *f* = *nn_integral M f*
unfolding *nn_integral_def*
proof (*safe intro*!: *SUP_eq*)
fix *s* **assume** *s*: *simple_function* (*completion M*) *s* **and** *s* ≤ *f*

```
  then obtain s' where s': simple_function M s' AE x in M. s x = s' x
    by (auto dest: completion_ex_simple_function)
  then obtain N where N: N ∈ null_sets M {x∈space M. s x ≠ s' x} ⊆ N
    by (auto elim!: AE_E)
  then have ae_N: AE x in M. (s x ≠ s' x ⟶ x ∈ N) ∧ x ∉ N
    by (auto dest: AE_not_in)
  define s'' where s'' x = (if x ∈ N then 0 else s x) for x
  then have ae_s_eq_s'': AE x in completion M. s x = s'' x
    using s' ae_N by (intro AE_completion) auto
  have s'': simple_function M s''
  proof (subst simple_function_cong)
    show t ∈ space M ⟹ s'' t = (if t ∈ N then 0 else s' t) for t
      using N by (auto simp: s''_def dest: sets.sets_into_space)
    show simple_function M (λt. if t ∈ N then 0 else s' t)
      unfolding s''_def[abs_def] using N by (auto intro!: simple_function_If s')
  qed

  show ∃j∈{g. simple_function M g ∧ g ≤ f}. integral^S (completion M) s ≤
integral^S M j
  proof (safe intro!: bexI[of _ s''])
    have integral^S (completion M) s = integral^S (completion M) s''
    by (intro simple_integral_cong_AE s simple_function_completion s'' ae_s_eq_s'')
    then show integral^S (completion M) s ≤ integral^S M s''
      using s'' by (simp add: simple_integral_completion)
    from ‹s ≤ f› show s'' ≤ f
      unfolding s''_def le_fun_def by auto
  qed fact
next
  fix s assume simple_function M s s ≤ f
  then show ∃j∈{g. simple_function (completion M) g ∧ g ≤ f}. integral^S M s
≤ integral^S (completion M) j
   by (intro bexI[of _ s]) (auto simp: simple_integral_completion simple_function_completion)
qed

lemma integrable_completion:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  shows f ∈ M →_M borel ⟹ integrable (completion M) f ⟷ integrable M f
  by (rule integrable_subalgebra[symmetric]) auto

lemma integral_completion:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes f: f ∈ M →_M borel shows integral^L (completion M) f = integral^L M
f
  by (rule integral_subalgebra[symmetric]) (auto intro: f)

locale semifinite_measure =
  fixes M :: 'a measure
  assumes semifinite:
    ⋀A. A ∈ sets M ⟹ emeasure M A = ∞ ⟹ ∃B∈sets M. B ⊆ A ∧ emeasure
```

$M\ B < \infty$

locale *locally_determined_measure* = *semifinite_measure* +
  assumes *locally_determined*:
    $\bigwedge A.\ A \subseteq space\ M \Longrightarrow (\bigwedge B.\ B \in sets\ M \Longrightarrow emeasure\ M\ B < \infty \Longrightarrow A \cap B \in sets\ M) \Longrightarrow A \in sets\ M$

locale *cld_measure* =
  complete_measure M + *locally_determined_measure M* **for** *M* :: *'a measure*

definition *outer_measure_of* :: *'a measure* ⇒ *'a set* ⇒ *ennreal*
  where *outer_measure_of M A* = (*INF B* ∈ {*B*∈*sets M*. *A* ⊆ *B*}. *emeasure M B*)

lemma *outer_measure_of_eq*[*simp*]: *A* ∈ *sets M* ⟹ *outer_measure_of M A* = *emeasure M A*
  by (*auto simp*: *outer_measure_of_def intro*!: *INF_eqI emeasure_mono*)

lemma *outer_measure_of_mono*: *A* ⊆ *B* ⟹ *outer_measure_of M A* ≤ *outer_measure_of M B*
  unfolding *outer_measure_of_def* **by** (*intro INF_superset_mono*) *auto*

lemma *outer_measure_of_attain*:
  assumes *A* ⊆ *space M*
  shows ∃ *E*∈*sets M*. *A* ⊆ *E* ∧ *outer_measure_of M A* = *emeasure M E*
proof −
  have *emeasure M* ‘ {*B* ∈ *sets M*. *A* ⊆ *B*} ≠ {}
    using ‹*A* ⊆ *space M*› **by** *auto*
  from *ennreal_Inf_countable_INF*[*OF this*]
  obtain *f*
    where *f*: *range f* ⊆ *emeasure M* ‘ {*B* ∈ *sets M*. *A* ⊆ *B*} *decseq f*
      and *outer_measure_of M A* = (*INF i*. *f i*)
    unfolding *outer_measure_of_def* **by** *auto*
  have ∃ *E*. ∀ *n*. (*E n* ∈ *sets M* ∧ *A* ⊆ *E n* ∧ *emeasure M* (*E n*) ≤ *f n*) ∧ *E* (*Suc n*) ⊆ *E n*
  proof (*rule dependent_nat_choice*)
    show ∃ *x*. *x* ∈ *sets M* ∧ *A* ⊆ *x* ∧ *emeasure M x* ≤ *f 0*
      using *f*(*1*) **by** (*fastforce simp*: *image_subset_iff image_iff intro*: *eq_refl*[*OF sym*])
  next
    fix *E n* **assume** *E* ∈ *sets M* ∧ *A* ⊆ *E* ∧ *emeasure M E* ≤ *f n*
    moreover obtain *F* **where** *F* ∈ *sets M A* ⊆ *F f* (*Suc n*) = *emeasure M F*
      using *f*(*1*) **by** (*auto simp*: *image_subset_iff image_iff*)
    ultimately show ∃ *y*. (*y* ∈ *sets M* ∧ *A* ⊆ *y* ∧ *emeasure M y* ≤ *f* (*Suc n*)) ∧ *y* ⊆ *E*
      by (*auto intro*!: *exI*[*of* _ *F* ∩ *E*] *emeasure_mono*)
  qed
  then obtain *E*
    where [*simp*]: ⋀*n*. *E n* ∈ *sets M*

```
    and ⋀n. A ⊆ E n
    and le_f: ⋀n. emeasure M (E n) ≤ f n
    and decseq E
   by (auto simp: decseq_Suc_iff)
  show ?thesis
  proof cases
   assume fin: ∃ i. emeasure M (E i) < ∞
   show ?thesis
   proof (intro bexI[of _ ⋂ i. E i] conjI)
    show A ⊆ (⋂ i. E i) (⋂ i. E i) ∈ sets M
     using ‹⋀n. A ⊆ E n› by auto

    have  (INF i. emeasure M (E i)) ≤ outer_measure_of M A
     unfolding ‹outer_measure_of M A = (INF n. f n)›
     by (intro INF_superset_mono le_f) auto
    moreover have outer_measure_of M A ≤ (INF i. outer_measure_of M (E
i))
     by (intro INF_greatest outer_measure_of_mono ‹⋀n. A ⊆ E n›)
    ultimately have outer_measure_of M A = (INF i. emeasure M (E i))
     by auto
    also have ... = emeasure M (⋂ i. E i)
     using fin by (intro INF_emeasure_decseq' ‹decseq E›) (auto simp: less_top)
    finally show outer_measure_of M A = emeasure M (⋂ i. E i) .
   qed
  next
   assume ∄ i. emeasure M (E i) < ∞
   then have f n = ∞ for n
    using le_f by (auto simp: not_less top_unique)
   moreover have ∃ E∈sets M. A ⊆ E ∧ f 0 = emeasure M E
    using f by auto
   ultimately show ?thesis
    unfolding ‹outer_measure_of M A = (INF n. f n)› by simp
  qed
qed

lemma SUP_outer_measure_of_incseq:
  assumes A: ⋀n. A n ⊆ space M and incseq A
  shows (SUP n. outer_measure_of M (A n)) = outer_measure_of M (⋃ i. A i)
proof (rule antisym)
  obtain E
   where E: ⋀n. E n ∈ sets M ⋀n. A n ⊆ E n ⋀n. outer_measure_of M (A n)
= emeasure M (E n)
   using outer_measure_of_attain[OF A] by metis

  define F where F n = (⋂ i∈{n ..}. E i) for n
  with E have F: incseq F ⋀n. F n ∈ sets M
   by (auto simp: incseq_def)
  have A n ⊆ F n for n
   using incseqD[OF ‹incseq A›, of n] ‹⋀n. A n ⊆ E n› by (auto simp: F_def)
```

```
  have eq: outer_measure_of M (A n) = outer_measure_of M (F n) for n
  proof (intro antisym)
    have outer_measure_of M (F n) ≤ outer_measure_of M (E n)
      by (intro outer_measure_of_mono) (auto simp add: F_def)
    with E show outer_measure_of M (F n) ≤ outer_measure_of M (A n)
      by auto
    show outer_measure_of M (A n) ≤ outer_measure_of M (F n)
      by (intro outer_measure_of_mono ‹A n ⊆ F n›)
  qed

  have outer_measure_of M (⋃ n. A n) ≤ outer_measure_of M (⋃ n. F n)
    using ‹⋀n. A n ⊆ F n› by (intro outer_measure_of_mono) auto
  also have ... = (SUP n. emeasure M (F n))
    using F by (simp add: SUP_emeasure_incseq subset_eq)
  finally show outer_measure_of M (⋃ n. A n) ≤ (SUP n. outer_measure_of M
(A n))
    by (simp add: eq F)
qed (auto intro: SUP_least outer_measure_of_mono)

definition measurable_envelope :: 'a measure ⇒ 'a set ⇒ 'a set ⇒ bool
  where measurable_envelope M A E ⟷
   (A ⊆ E ∧ E ∈ sets M ∧ (∀ F∈sets M. emeasure M (F ∩ E) = outer_measure_of
M (F ∩ A)))

lemma measurable_envelopeD:
  assumes measurable_envelope M A E
  shows A ⊆ E
    and E ∈ sets M
    and ⋀F. F ∈ sets M ⟹ emeasure M (F ∩ E) = outer_measure_of M (F ∩
A)
    and A ⊆ space M
 using assms sets.sets_into_space[of E] by (auto simp: measurable_envelope_def)

lemma measurable_envelopeD1:
  assumes E: measurable_envelope M A E and F: F ∈ sets M F ⊆ E − A
  shows emeasure M F = 0
proof −
  have emeasure M F = emeasure M (F ∩ E)
    using F by (intro arg_cong2[where f=emeasure]) auto
  also have ... = outer_measure_of M (F ∩ A)
     using measurable_envelopeD[OF E] ‹F ∈ sets M› by (auto simp: measur-
able_envelope_def)
  also have ... = outer_measure_of M {}
    using ‹F ⊆ E − A› by (intro arg_cong2[where f=outer_measure_of]) auto
  finally show emeasure M F = 0
    by simp
qed
```

lemma *measurable_envelope_eq1*:
assumes $A \subseteq E$ $E \in sets\ M$
shows *measurable_envelope M A E* $\longleftrightarrow$ ($\forall F{\in}sets\ M.\ F \subseteq E - A \longrightarrow emeasure\ M\ F = 0$)
proof *safe*
assume $*$: $\forall F{\in}sets\ M.\ F \subseteq E - A \longrightarrow emeasure\ M\ F = 0$
show *measurable_envelope M A E*
unfolding *measurable_envelope_def*
proof (*rule ccontr*, *auto simp add*: ‹$E \in sets\ M$› ‹$A \subseteq E$›)
fix F **assume** $F \in sets\ M$ *emeasure M* $(F \cap E) \neq$ *outer_measure_of M* $(F \cap A)$
then have *outer_measure_of M* $(F \cap A) <$ *emeasure M* $(F \cap E)$
using *outer_measure_of_mono*[*of* $F \cap A$ $F \cap E$ *M*] ‹$A \subseteq E$› ‹$E \in sets\ M$›
by (*auto simp*: *less_le*)
then obtain G **where** G: $G \in sets\ M$ $F \cap A \subseteq G$ **and** *less*: *emeasure M G* $<$ *emeasure M* $(E \cap F)$
unfolding *outer_measure_of_def INF_less_iff* **by** (*auto simp*: *ac_simps*)
have *le*: *emeasure M* $(G \cap E \cap F) \leq$ *emeasure M G*
using ‹$E \in sets\ M$› ‹$G \in sets\ M$› ‹$F \in sets\ M$› **by** (*auto intro*!: *emeasure_mono*)

from G **have** $E \cap F - G \in sets\ M$ $E \cap F - G \subseteq E - A$
using ‹$F \in sets\ M$› ‹$E \in sets\ M$› **by** *auto*
with $*$ **have** $0 =$ *emeasure M* $(E \cap F - G)$
by *auto*
also have $E \cap F - G = E \cap F - (G \cap E \cap F)$
by *auto*
also have *emeasure M* $(E \cap F - (G \cap E \cap F)) =$ *emeasure M* $(E \cap F) -$ *emeasure M* $(G \cap E \cap F)$
using ‹$E \in sets\ M$› ‹$F \in sets\ M$› *le less G* **by** (*intro emeasure_Diff*) (*auto simp*: *top_unique*)
also have $\ldots > 0$
using *le less* **by** (*intro diff_gr0_ennreal*) *auto*
finally show *False* **by** *auto*
qed
qed (*rule measurable_envelopeD1*)

lemma *measurable_envelopeD2*:
assumes *E*: *measurable_envelope M A E* **shows** *emeasure M E* = *outer_measure_of M A*
proof −
from ‹*measurable_envelope M A E*› **have** *emeasure M* $(E \cap E) =$ *outer_measure_of M* $(E \cap A)$
by (*auto simp*: *measurable_envelope_def*)
with *measurable_envelopeD*[*OF E*] **show** *emeasure M E* = *outer_measure_of M A*
by (*auto simp*: *Int_absorb1*)
qed

lemma *measurable_envelope_eq2*:
assumes $A \subseteq E$ $E \in$ *sets* M *emeasure* M $E < \infty$
shows *measurable_envelope* M A $E \longleftrightarrow$ (*emeasure* M E = *outer_measure_of* M A)
proof *safe*
assume $*$: *emeasure* M E = *outer_measure_of* M A
show *measurable_envelope* M A E
unfolding *measurable_envelope_eq1*[*OF* ‹$A \subseteq E$› ‹$E \in$ *sets* M›]
proof (*intro conjI ballI impI assms*)
fix F **assume** F: $F \in$ *sets* M $F \subseteq E - A$
with ‹$E \in$ *sets* M› **have** *le*: *emeasure* M $F \leq$ *emeasure* M E
by (*intro emeasure_mono*) *auto*
from F ‹$A \subseteq E$› **have** *outer_measure_of* M $A \leq$ *outer_measure_of* M ($E - F$)
by (*intro outer_measure_of_mono*) *auto*
then have *emeasure* M $E - 0 \leq$ *emeasure* M ($E - F$)
using $*$ ‹$E \in$ *sets* M› ‹$F \in$ *sets* M› **by** *simp*
also have ... = *emeasure* M E − *emeasure* M F
using ‹$E \in$ *sets* M› ‹*emeasure* M $E < \infty$› F *le* **by** (*intro emeasure_Diff*) (*auto simp*: *top_unique*)
finally show *emeasure* M F = 0
using *ennreal_mono_minus_cancel*[*of emeasure* M E 0 *emeasure* M F] *le assms* **by** *auto*
qed
qed (*auto intro*: *measurable_envelopeD2*)

lemma *measurable_envelopeI_countable*:
fixes A :: *nat* ⇒ 'a *set*
assumes E: $\bigwedge n$. *measurable_envelope* M (A n) (E n)
shows *measurable_envelope* M ($\bigcup n$. A n) ($\bigcup n$. E n)
proof (*subst measurable_envelope_eq1*)
show ($\bigcup n$. A n) ⊆ ($\bigcup n$. E n) ($\bigcup n$. E n) ∈ *sets* M
using *measurable_envelopeD*(1,2)[*OF* E] **by** *auto*
show ∀ F∈*sets* M. $F \subseteq$ ($\bigcup n$. E n) − ($\bigcup n$. A n) ⟶ *emeasure* M F = 0
proof *safe*
fix F **assume** F: $F \in$ *sets* M $F \subseteq$ ($\bigcup n$. E n) − ($\bigcup n$. A n)
then have $F \cap E$ $n \in$ *sets* M $F \cap E$ $n \subseteq E$ $n - A$ n $F \subseteq$ ($\bigcup n$. E n) **for** n
using *measurable_envelopeD*(1,2)[*OF* E] **by** *auto*
then have *emeasure* M ($\bigcup n$. $F \cap E$ n) = 0
by (*intro emeasure_UN_eq_0 measurable_envelopeD1*[*OF* E]) *auto*
then show *emeasure* M F = 0
using ‹$F \subseteq$ ($\bigcup n$. E n)› **by** (*auto simp*: *Int_absorb2*)
qed
qed

lemma *measurable_envelopeI_countable_cover*:
fixes A **and** C :: *nat* ⇒ 'a *set*
assumes C: $A \subseteq$ ($\bigcup n$. C n) $\bigwedge n$. C $n \in$ *sets* M $\bigwedge n$. *emeasure* M (C n) $< \infty$
shows ∃ $E \subseteq$ ($\bigcup n$. C n). *measurable_envelope* M A E

proof −
have $A \cap C\ n \subseteq$ *space* M **for** n
using ‹$C\ n \in$ *sets* M› **by** (*auto dest*: *sets.sets_into_space*)
then have $\forall n.\ \exists E{\in}$*sets* $M.\ A \cap C\ n \subseteq E \wedge$ *outer_measure_of* $M\ (A \cap C\ n) =$ *emeasure* $M\ E$
using *outer_measure_of_attain*[*of* $A \cap C\ n\ M$ **for** n] **by** *auto*
then obtain E
where E: $\bigwedge n.\ E\ n \in$ *sets* $M\ \bigwedge n.\ A \cap C\ n \subseteq E\ n$
and *eq*: $\bigwedge n.$ *outer_measure_of* $M\ (A \cap C\ n) =$ *emeasure* $M\ (E\ n)$
by *metis*

have *outer_measure_of* $M\ (A \cap C\ n) \leq$ *outer_measure_of* $M\ (E\ n \cap C\ n)$ **for** n
using E **by** (*intro outer_measure_of_mono*) *auto*
moreover have *outer_measure_of* $M\ (E\ n \cap C\ n) \leq$ *outer_measure_of* $M\ (E\ n)$ **for** n
by (*intro outer_measure_of_mono*) *auto*
ultimately have *eq*: *outer_measure_of* $M\ (A \cap C\ n) =$ *emeasure* $M\ (E\ n \cap C\ n)$ **for** n
using E C **by** (*intro antisym*) (*auto simp*: *eq*)

{ fix n
have *outer_measure_of* $M\ (A \cap C\ n) \leq$ *outer_measure_of* $M\ (C\ n)$
by (*intro outer_measure_of_mono*) *simp*
also have $\ldots < \infty$
using *assms* **by** *auto*
finally have *emeasure* $M\ (E\ n \cap C\ n) < \infty$
using *eq* **by** *simp* **}**
then have *measurable_envelope* $M\ (\bigcup n.\ A \cap C\ n)\ (\bigcup n.\ E\ n \cap C\ n)$
using E C **by** (*intro measurable_envelopeI_countable measurable_envelope_eq2*[*THEN iffD2*]) (*auto simp*: *eq*)
with ‹$A \subseteq (\bigcup n.\ C\ n)$› **show** *?thesis*
by (*intro exI*[*of* _ $(\bigcup n.\ E\ n \cap C\ n)$]) (*auto simp add*: *Int_absorb2*)
qed

lemma (**in** *complete_measure*) *complete_sets_sandwich*:
assumes [*measurable*]: $A \in$ *sets* M $C \in$ *sets* M **and** *subset*: $A \subseteq B$ $B \subseteq C$
and *measure*: *emeasure* $M\ A =$ *emeasure* $M\ C$ *emeasure* $M\ A < \infty$
shows $B \in$ *sets* M
proof −
have $B - A \in$ *sets* M
proof (*rule complete*)
show $B - A \subseteq C - A$
using *subset* **by** *auto*
show $C - A \in$ *null_sets* M
using *measure subset* **by**(*simp add*: *emeasure_Diff null_setsI*)
qed
then have $A \cup (B - A) \in$ *sets* M
by *measurable*

also have $A \cup (B - A) = B$
using ‹$A \subseteq B$› **by** *auto*
finally show *?thesis* **.**
qed

lemma (**in** *complete_measure*) *complete_sets_sandwich_fmeasurable*:
assumes [*measurable*]: $A \in$ *fmeasurable* M $C \in$ *fmeasurable* M **and** *subset*: $A \subseteq B$ $B \subseteq C$
and *measure*: *measure* M A = *measure* M C
shows $B \in$ *fmeasurable* M
proof (*rule fmeasurableI2*)
show $B \subseteq C$ $C \in$ *fmeasurable* M **by** *fact+*
show $B \in$ *sets* M
proof (*rule complete_sets_sandwich*)
show $A \in$ *sets* M $C \in$ *sets* M $A \subseteq B$ $B \subseteq C$
using *assms* **by** *auto*
show *emeasure* M $A < \infty$
using ‹$A \in$ *fmeasurable* M› **by** (*auto simp*: *fmeasurable_def*)
show *emeasure* M A = *emeasure* M C
using *assms* **by** (*simp add*: *emeasure_eq_measure2*)
qed
qed

lemma *AE_completion_iff*: (*AE* x *in completion* M. P x) $\longleftrightarrow$ (*AE* x *in* M. P x)
proof
assume *AE* x *in completion* M. P x
then obtain N **where** $N \in$ *null_sets* (*completion* M) **and** P: {x∈*space* M. ¬ P x} $\subseteq N$
unfolding *eventually_ae_filter* **by** *auto*
then obtain N' **where** N': $N' \in$ *null_sets* M **and** $N \subseteq N'$
unfolding *null_sets_completion_iff2* **by** *auto*
from P ‹$N \subseteq N'$› **have** {x∈*space* M. ¬ P x} $\subseteq N'$
by *auto*
with N' **show** *AE* x *in* M. P x
unfolding *eventually_ae_filter* **by** *auto*
qed (*rule AE_completion*)

lemma *null_part_null_sets*: $S \in$ *completion* $M \Longrightarrow$ *null_part* M $S \in$ *null_sets* (*completion* M)
by (*auto dest*!: *null_part intro*: *null_sets_completionI null_sets_completion_subset*)

lemma *AE_notin_null_part*: $S \in$ *completion* $M \Longrightarrow$ (*AE* x *in* M. $x \notin$ *null_part* M S)
by (*auto dest*!: *null_part_null_sets AE_not_in simp*: *AE_completion_iff*)

lemma *completion_upper*:
assumes A: $A \in$ *sets* (*completion* M)
shows $\exists A' \in$*sets* M. $A \subseteq A' \wedge$ *emeasure* (*completion* M) A = *emeasure* M A'
proof −

```
 from AE_notin_null_part[OF A] obtain N where N: N ∈ null_sets M null_part
M A ⊆ N
  unfolding eventually_ae_filter using null_part_null_sets[OF A, THEN null_setsD2,
THEN sets.sets_into_space] by auto
 show ?thesis
 proof (intro bexI conjI)
  show A ⊆ main_part M A ∪ N
   using ‹null_part M A ⊆ N› by (subst main_part_null_part_Un[symmetric,
OF A]) auto
  show emeasure (completion M) A = emeasure M (main_part M A ∪ N)
   using A ‹N ∈ null_sets M› by (simp add: emeasure_Un_null_set)
 qed (use A N in auto)
qed

lemma AE_in_main_part:
 assumes A: A ∈ completion M shows AE x in M. x ∈ main_part M A ⟷ x
∈ A
 using AE_notin_null_part[OF A]
 by (subst (2) main_part_null_part_Un[symmetric, OF A]) auto

lemma completion_density_eq:
 assumes ae: AE x in M. 0 < f x and [measurable]: f ∈ M →M borel
 shows completion (density M f) = density (completion M) f
proof (intro measure_eqI)
 have N' ∈ sets M ∧ (AE x∈N' in M. f x = 0) ⟷ N' ∈ null_sets M for N'
 proof safe
  assume N': N' ∈ sets M and ae_N': AE x∈N' in M. f x = 0
  from ae_N' ae have AE x in M. x ∉ N'
   by eventually_elim auto
  then show N' ∈ null_sets M
   using N' by (simp add: AE_iff_null_sets)
 next
  assume N': N' ∈ null_sets M then show N' ∈ sets M AE x∈N' in M. f x =
0
   using ae AE_not_in[OF N'] by (auto simp: less_le)
 qed
 then show sets_eq: sets (completion (density M f)) = sets (density (completion
M) f)
  by (simp add: sets_completion null_sets_density_iff)

 fix A assume A: ‹A ∈ completion (density M f)›
 moreover
 have A ∈ completion M
  using A unfolding sets_eq by simp
 moreover
 have main_part (density M f) A ∈ M
  using A main_part_sets[of A density M f] unfolding sets_density sets_eq by
simp
 moreover have AE x in M. x ∈ main_part (density M f) A ⟷ x ∈ A
```

using *AE_in_main_part*[*OF* ‹*A* ∈ *completion* (*density M f*)›] *ae* **by** (*auto simp add*: *AE_density*)
ultimately show *emeasure* (*completion* (*density M f*)) *A* = *emeasure* (*density* (*completion M*) *f*) *A*
by (*auto simp add*: *emeasure_density measurable_completion nn_integral_completion intro*!: *nn_integral_cong_AE*)
qed

lemma *null_sets_subset*: *B* ∈ *null_sets M* ⟹ *A* ∈ *sets M* ⟹ *A* ⊆ *B* ⟹ *A* ∈ *null_sets M*
using *emeasure_mono*[*of A B M*] **by** (*simp add*: *null_sets_def*)

lemma (**in** *complete_measure*) *complete2*: *A* ⊆ *B* ⟹ *B* ∈ *null_sets M* ⟹ *A* ∈ *null_sets M*
using *complete*[*of A B*] *null_sets_subset*[*of B M A*] **by** *simp*

lemma (**in** *complete_measure*) *AE_iff_null_sets*: (*AE x in M*. *P x*) ⟷ {*x*∈*space M*. ¬ *P x*} ∈ *null_sets M*
unfolding *eventually_ae_filter* **by** (*auto intro*: *complete2*)

lemma (**in** *complete_measure*) *null_sets_iff_AE*: *A* ∈ *null_sets M* ⟷ ((*AE x in M*. *x* ∉ *A*) ∧ *A* ⊆ *space M*)
unfolding *AE_iff_null_sets* **by** (*auto cong*: *rev_conj_cong dest*: *sets.sets_into_space simp*: *subset_eq*)

lemma (**in** *complete_measure*) *in_sets_AE*:
assumes *ae*: *AE x in M*. *x* ∈ *A* ⟷ *x* ∈ *B* **and** *A*: *A* ∈ *sets M* **and** *B*: *B* ⊆ *space M*
shows *B* ∈ *sets M*
proof −
have (*AE x in M*. *x* ∉ *B* − *A* ∧ *x* ∉ *A* − *B*)
using *ae* **by** *eventually_elim auto*
then have *B* − *A* ∈ *null_sets M A* − *B* ∈ *null_sets M*
using *A B* **unfolding** *null_sets_iff_AE* **by** (*auto dest*: *sets.sets_into_space*)
then have *A* ∪ (*B* − *A*) − (*A* − *B*) ∈ *sets M*
using *A* **by** *blast*
also have *A* ∪ (*B* − *A*) − (*A* − *B*) = *B*
by *auto*
finally show *B* ∈ *sets M* **.**
qed

lemma (**in** *complete_measure*) *vimage_null_part_null_sets*:
assumes *f*: *f* ∈ *M* →$_M$ *N* **and** *eq*: *null_sets N* ⊆ *null_sets* (*distr M N f*)
and *A*: *A* ∈ *completion N*
shows *f* −‘ *null_part N A* ∩ *space M* ∈ *null_sets M*
proof −
obtain *N*′ **where** *N*′ ∈ *null_sets N null_part N A* ⊆ *N*′
using *null_part*[*OF A*] **by** *auto*
then have *N*′: *N*′ ∈ *null_sets* (*distr M N f*)

```
    using eq by auto
  show ?thesis
  proof (rule complete2)
    show f −‘ null_part N A ∩ space M ⊆ f −‘ N′ ∩ space M
      using ‹null_part N A ⊆ N′› by auto
    show f −‘ N′ ∩ space M ∈ null_sets M
      using f N′ by (auto simp: null_sets_def emeasure_distr)
  qed
qed

lemma (in complete_measure) vimage_null_part_sets:
  f ∈ M →M N ⟹ null_sets N ⊆ null_sets (distr M N f) ⟹ A ∈ completion
N ⟹
  f −‘ null_part N A ∩ space M ∈ sets M
  using vimage_null_part_null_sets[of f N A] by auto

lemma (in complete_measure) measurable_completion2:
  assumes f: f ∈ M →M N and null_sets: null_sets N ⊆ null_sets (distr M N
f)
  shows f ∈ M →M completion N
proof (rule measurableI)
  show x ∈ space M ⟹ f x ∈ space (completion N) for x
    using f[THEN measurable_space] by auto
  fix A assume A: A ∈ sets (completion N)
  have f −‘ A ∩ space M = (f −‘ main_part N A ∩ space M) ∪ (f −‘ null_part
N A ∩ space M)
    using main_part_null_part_Un[OF A] by auto
  then show f −‘ A ∩ space M ∈ sets M
    using f A null_sets by (auto intro: vimage_null_part_sets measurable_sets)
qed

lemma (in complete_measure) completion_distr_eq:
  assumes X: X ∈ M →M N and null_sets: null_sets (distr M N X) = null_sets
N
  shows completion (distr M N X) = distr M (completion N) X
proof (rule measure_eqI)
  show eq: sets (completion (distr M N X)) = sets (distr M (completion N) X)
    by (simp add: sets_completion null_sets)

  fix A assume A: A ∈ completion (distr M N X)
  moreover have A′: A ∈ completion N
    using A by (simp add: eq)
  moreover have main_part (distr M N X) A ∈ sets N
    using main_part_sets[OF A] by simp
  moreover have X −‘ A ∩ space M = (X −‘ main_part (distr M N X) A ∩
space M) ∪ (X −‘ null_part (distr M N X) A ∩ space M)
    using main_part_null_part_Un[OF A] by auto
  moreover have X −‘ null_part (distr M N X) A ∩ space M ∈ null_sets M
    using X A by (intro vimage_null_part_null_sets) (auto cong: distr_cong)
```

ultimately show *emeasure* (*completion* (*distr M N X*)) *A* = *emeasure* (*distr M* (*completion N*) *X*) *A*
using *X* **by** (*auto simp*: *emeasure_distr measurable_completion null_sets measurable_completion2*
intro!: *emeasure_Un_null_set*[*symmetric*])
qed

lemma *distr_completion*: *X* ∈ *M* →$_M$ *N* ⟹ *distr* (*completion M*) *N X* = *distr M N X*
by (*intro measure_eqI*) (*auto simp*: *emeasure_distr measurable_completion*)

proposition (**in** *complete_measure*) *fmeasurable_inner_outer*:
S ∈ *fmeasurable M* ⟷
(∀ *e*>*0*. ∃ *T*∈*fmeasurable M*. ∃ *U*∈*fmeasurable M*. *T* ⊆ *S* ∧ *S* ⊆ *U* ∧ |*measure M T* − *measure M U*| < *e*)
(**is** _ ⟷ *?approx*)
proof *safe*
let *?μ* = *measure M* **let** *?D* = λ*T U* . |*?μ T* − *?μ U*|
assume *?approx*
have ∃ *A*. ∀ *n*. (*fst* (*A n*) ∈ *fmeasurable M* ∧ *snd* (*A n*) ∈ *fmeasurable M* ∧ *fst* (*A n*) ⊆ *S* ∧ *S* ⊆ *snd* (*A n*) ∧
?D (*fst* (*A n*)) (*snd* (*A n*)) < *1*/*Suc n*) ∧ (*fst* (*A n*) ⊆ *fst* (*A* (*Suc n*)) ∧ *snd* (*A* (*Suc n*)) ⊆ *snd* (*A n*))
(**is** ∃ *A*. ∀ *n*. *?P n* (*A n*) ∧ *?Q* (*A n*) (*A* (*Suc n*)))
proof (*intro dependent_nat_choice*)
show ∃ *A*. *?P 0 A*
using ‹*?approx*›[*THEN spec*, *of 1*] **by** *auto*
next
fix *A n* **assume** *?P n A*
moreover
from ‹*?approx*›[*THEN spec*, *of 1*/*Suc* (*Suc n*)]
obtain *T U* **where** ∗: *T* ∈ *fmeasurable M U* ∈ *fmeasurable M T* ⊆ *S S* ⊆ *U ?D T U* < *1* / *Suc* (*Suc n*)
by *auto*
ultimately have *?μ T* ≤ *?μ* (*T* ∪ *fst A*) *?μ* (*U* ∩ *snd A*) ≤ *?μ U*
?μ T ≤ *?μ U ?μ* (*T* ∪ *fst A*) ≤ *?μ* (*U* ∩ *snd A*)
by (*auto intro*!: *measure_mono_fmeasurable*)
then have *?D* (*T* ∪ *fst A*) (*U* ∩ *snd A*) ≤ *?D T U*
by *auto*
also have *?D T U* < *1*/*Suc* (*Suc n*) **by** *fact*
finally show ∃ *B*. *?P* (*Suc n*) *B* ∧ *?Q A B*
using ‹*?P n A*› ∗
by (*intro exI*[*of* _ (*T* ∪ *fst A*, *U* ∩ *snd A*)] *conjI*) *auto*
qed
then obtain *A*
where *lm*: ⋀*n*. *fst* (*A n*) ∈ *fmeasurable M* ⋀*n*. *snd* (*A n*) ∈ *fmeasurable M*
and *set_bound*: ⋀*n*. *fst* (*A n*) ⊆ *S* ⋀*n*. *S* ⊆ *snd* (*A n*)
and *mono*: ⋀*n*. *fst* (*A n*) ⊆ *fst* (*A* (*Suc n*)) ⋀*n*. *snd* (*A* (*Suc n*)) ⊆ *snd* (*A n*)

```
    and bound: ⋀n. ?D (fst (A n)) (snd (A n)) < 1/Suc n
   by metis

  have INT_sA: (⋂ n. snd (A n)) ∈ fmeasurable M
   using lm by (intro fmeasurable_INT[of _ 0]) auto
  have UN_fA: (⋃ n. fst (A n)) ∈ fmeasurable M
   using lm order_trans[OF set_bound(1) set_bound(2)[of 0]] by (intro fmeasurable_UN[of _ _ snd (A 0)]) auto

  have (λn. ?μ (fst (A n)) - ?μ (snd (A n))) ⟶ 0
   using bound
   by (subst tendsto_rabs_zero_iff[symmetric])
    (intro tendsto_sandwich[OF _ _ tendsto_const LIMSEQ_inverse_real_of_nat];
      auto intro!: always_eventually less_imp_le simp: divide_inverse)
  moreover
  have (λn. ?μ (fst (A n)) - ?μ (snd (A n))) ⟶ ?μ (⋃ n. fst (A n)) - ?μ (⋂ n. snd (A n))
  proof (intro tendsto_diff Lim_measure_incseq Lim_measure_decseq)
   show range (λi. fst (A i)) ⊆ sets M range (λi. snd (A i)) ⊆ sets M
    incseq (λi. fst (A i)) decseq (λn. snd (A n))
     using mono lm by (auto simp: incseq_Suc_iff decseq_Suc_iff intro!: measure_mono_fmeasurable)
   show emeasure M (⋃ x. fst (A x)) ≠ ∞ emeasure M (snd (A n)) ≠ ∞ for n
    using lm(2)[of n] UN_fA by (auto simp: fmeasurable_def)
  qed
  ultimately have eq: 0 = ?μ (⋃ n. fst (A n)) - ?μ (⋂ n. snd (A n))
   by (rule LIMSEQ_unique)

  show S ∈ fmeasurable M
   using UN_fA INT_sA
  proof (rule complete_sets_sandwich_fmeasurable)
   show (⋃ n. fst (A n)) ⊆ S S ⊆ (⋂ n. snd (A n))
    using set_bound by auto
   show ?μ (⋃ n. fst (A n)) = ?μ (⋂ n. snd (A n))
    using eq by auto
  qed
qed (auto intro!: bexI[of _ S])

lemma (in complete_measure) fmeasurable_measure_inner_outer:
  (S ∈ fmeasurable M ∧ m = measure M S) ⟷
    (∀ e>0. ∃ T∈fmeasurable M. T ⊆ S ∧ m - e < measure M T) ∧
    (∀ e>0. ∃ U∈fmeasurable M. S ⊆ U ∧ measure M U < m + e)
  (is ?lhs = ?rhs)
proof
  assume RHS: ?rhs
  then have T: ⋀e. 0 < e ⟶ (∃ T∈fmeasurable M. T ⊆ S ∧ m - e < measure M T)
      and U: ⋀e. 0 < e ⟶ (∃ U∈fmeasurable M. S ⊆ U ∧ measure M U < m + e)
```

```
    by auto
  have S ∈ fmeasurable M
  proof (subst fmeasurable_inner_outer, safe)
    fix e::real assume 0 < e
     with RHS obtain T U where T: T ∈ fmeasurable M T ⊆ S m − e/2 <
measure M T
                          and U: U ∈ fmeasurable M S ⊆ U measure M U < m + e/2
      by (meson half_gt_zero)+
    moreover have measure M U − measure M T < (m + e/2) − (m − e/2)
      by (intro diff_strict_mono) fact+
    moreover have measure M T ≤ measure M U
      using T U by (intro measure_mono_fmeasurable) auto
    ultimately show ∃ T∈fmeasurable M. ∃ U∈fmeasurable M. T ⊆ S ∧ S ⊆ U
∧ |measure M T − measure M U| < e
      apply (rule_tac bexI[OF _ ‹T ∈ fmeasurable M›])
      apply (rule_tac bexI[OF _ ‹U ∈ fmeasurable M›])
      by auto
  qed
  moreover have m = measure M S
    using ‹S ∈ fmeasurable M› U[of measure M S − m] T[of m − measure M S]
    by (cases m measure M S rule: linorder_cases)
       (auto simp: not_le[symmetric] measure_mono_fmeasurable)
  ultimately show ?lhs
    by simp
qed (auto intro!: bexI[of _ S])

lemma (in complete_measure) null_sets_outer:
  S ∈ null_sets M ⟷ (∀ e>0. ∃ T∈fmeasurable M. S ⊆ T ∧ measure M T < e)
proof −
  have S ∈ null_sets M ⟷ (S ∈ fmeasurable M ∧ 0 = measure M S)
   by (auto simp: null_sets_def emeasure_eq_measure2 intro: fmeasurableI) (simp
add: measure_def)
  also have ... = (∀ e>0. ∃ T∈fmeasurable M. S ⊆ T ∧ measure M T < e)
    unfolding fmeasurable_measure_inner_outer by auto
  finally show ?thesis .
qed

lemma (in complete_measure) fmeasurable_measure_inner_outer_le:
    (S ∈ fmeasurable M ∧ m = measure M S) ⟷
      (∀ e>0. ∃ T∈fmeasurable M. T ⊆ S ∧ m − e ≤ measure M T) ∧
      (∀ e>0. ∃ U∈fmeasurable M. S ⊆ U ∧ measure M U ≤ m + e) (is ?T1)
  and null_sets_outer_le:
    S ∈ null_sets M ⟷ (∀ e>0. ∃ T∈fmeasurable M. S ⊆ T ∧ measure M T ≤
e) (is ?T2)
proof −
  have e > 0 ∧ m − e/2 ≤ t ⟹ m − e < t
       e > 0 ∧ t ≤ m + e/2 ⟹ t < m + e
       e > 0 ⟷ e/2 > 0
       for e t
```

```
    by auto
  then show ?T1 ?T2
    unfolding fmeasurable_measure_inner_outer null_sets_outer
    by (meson dense le_less_trans less_imp_le)+
qed

lemma (in cld_measure) notin_sets_outer_measure_of_cover:
  assumes E: E ⊆ space M E ∉ sets M
  shows ∃ B∈sets M. 0 < emeasure M B ∧ emeasure M B < ∞ ∧
    outer_measure_of M (B ∩ E) = emeasure M B ∧ outer_measure_of M (B −
E) = emeasure M B
proof −
  from locally_determined[OF ‹E ⊆ space M›] ‹E ∉ sets M›
  obtain F
    where [measurable]: F ∈ sets M and emeasure M F < ∞ E ∩ F ∉ sets M
    by blast
  then obtain H H'
    where H: measurable_envelope M (F ∩ E) H and H': measurable_envelope
M (F − E) H'
    using measurable_envelopeI_countable_cover[of F ∩ E λ_. F M]
      measurable_envelopeI_countable_cover[of F − E λ_. F M]
    by auto
  note measurable_envelopeD(2)[OF H', measurable] measurable_envelopeD(2)[OF
H, measurable]

  from measurable_envelopeD(1)[OF H'] measurable_envelopeD(1)[OF H]
  have subset: F − H' ⊆ F ∩ E F ∩ E ⊆ F ∩ H
    by auto
  moreover define G where G = (F ∩ H) − (F − H')
  ultimately have G: G = F ∩ H ∩ H'
    by auto
  have emeasure M (F ∩ H) ≠ 0
  proof
    assume emeasure M (F ∩ H) = 0
    then have F ∩ H ∈ null_sets M
      by auto
    with ‹E ∩ F ∉ sets M› show False
      using complete[OF ‹F ∩ E ⊆ F ∩ H›] by (auto simp: Int_commute)
  qed
  moreover
  have emeasure M (F − H') ≠ emeasure M (F ∩ H)
  proof
    assume emeasure M (F − H') = emeasure M (F ∩ H)
    with ‹E ∩ F ∉ sets M› emeasure_mono[of F ∩ H F M] ‹emeasure M F < ∞›
    have F ∩ E ∈ sets M
      by (intro complete_sets_sandwich[OF _ _ subset]) auto
    with ‹E ∩ F ∉ sets M› show False
      by (simp add: Int_commute)
  qed
```

```
  moreover have emeasure M (F − H') ≤ emeasure M (F ∩ H)
    using subset by (intro emeasure_mono) auto
  ultimately have emeasure M G ≠ 0
    unfolding G_def using subset
    by (subst emeasure_Diff) (auto simp: top_unique diff_eq_0_iff_ennreal)
  show ?thesis
  proof (intro bexI conjI)
    have emeasure M G ≤ emeasure M F
      unfolding G by (auto intro!: emeasure_mono)
    with ‹emeasure M F < ∞› show 0 < emeasure M G emeasure M G < ∞
      using ‹emeasure M G ≠ 0› by (auto simp: zero_less_iff_neq_zero)
    show [measurable]: G ∈ sets M
      unfolding G by auto

    have emeasure M G = outer_measure_of M (F ∩ H' ∩ (F ∩ E))
      using measurable_envelopeD(3)[OF H, of F ∩ H'] unfolding G by (simp
add: ac_simps)
    also have ... ≤ outer_measure_of M (G ∩ E)
      using measurable_envelopeD(1)[OF H] by (intro outer_measure_of_mono)
(auto simp: G)
    finally show outer_measure_of M (G ∩ E) = emeasure M G
      using outer_measure_of_mono[of G ∩ E G M] by auto

    have emeasure M G = outer_measure_of M (F ∩ H ∩ (F − E))
      using measurable_envelopeD(3)[OF H', of F ∩ H] unfolding G by (simp
add: ac_simps)
    also have ... ≤ outer_measure_of M (G − E)
     using measurable_envelopeD(1)[OF H'] by (intro outer_measure_of_mono)
(auto simp: G)
    finally show outer_measure_of M (G − E) = emeasure M G
      using outer_measure_of_mono[of G − E G M] by auto
  qed
qed
```

The following theorem is a specialization of D.H. Fremlin, Measure Theory vol 4I (413G). We only show one direction and do not use a inner regular family K.

```
lemma (in cld_measure) borel_measurable_cld:
  fixes f :: 'a ⇒ real
  assumes ⋀A a b. A ∈ sets M ⟹ 0 < emeasure M A ⟹ emeasure M A < ∞
⟹ a < b ⟹
      min (outer_measure_of M {x∈A. f x ≤ a}) (outer_measure_of M {x∈A. b
≤ f x}) < emeasure M A
  shows f ∈ M →M borel
proof (rule ccontr)
  let ?E = λa. {x∈space M. f x ≤ a} and ?F = λa. {x∈space M. a ≤ f x}

  assume f ∉ M →M borel
  then obtain a where ?E a ∉ sets M
```

```
    unfolding borel_measurable_iff_le by blast
  from notin_sets_outer_measure_of_cover[OF _ this]
  obtain K
    where K: K ∈ sets M 0 < emeasure M K emeasure M K < ∞
      and eq1: outer_measure_of M (K ∩ ?E a) = emeasure M K
      and eq2: outer_measure_of M (K - ?E a) = emeasure M K
    by auto
  then have me_K: measurable_envelope M (K ∩ ?E a) K
    by (subst measurable_envelope_eq2) auto

  define b where b n = a + inverse (real (Suc n)) for n
  have (SUP n. outer_measure_of M (K ∩ ?F (b n))) = outer_measure_of M
(⋃n. K ∩ ?F (b n))
  proof (intro SUP_outer_measure_of_incseq)
    have x ≤ y ⟹ b y ≤ b x for x y
      by (auto simp: b_def field_simps)
    then show incseq (λn. K ∩ {x ∈ space M. b n ≤ f x})
      by (auto simp: incseq_def intro: order_trans)
  qed auto
  also have (⋃n. K ∩ ?F (b n)) = K - ?E a
  proof -
    have b ⟶ a
      unfolding b_def by (rule LIMSEQ_inverse_real_of_nat_add)
    then have ∀n. ¬ b n ≤ f x ⟹ f x ≤ a for x
      by (rule LIMSEQ_le_const) (auto intro: less_imp_le simp: not_le)
    moreover have ¬ b n ≤ a for n
      by (auto simp: b_def)
    ultimately show ?thesis
      using ‹K ∈ sets M› [THEN sets.sets_into_space] by (auto simp: subset_eq
intro: order_trans)
  qed
  finally have 0 < (SUP n. outer_measure_of M (K ∩ ?F (b n)))
    using K by (simp add: eq2)
  then obtain n where pos_b: 0 < outer_measure_of M (K ∩ ?F (b n)) and a
< b n
    unfolding less_SUP_iff by (auto simp: b_def)
  from measurable_envelopeI_countable_cover[of K ∩ ?F (b n) λ_. K M] K
  obtain K' where K' ⊆ K and me_K': measurable_envelope M (K ∩ ?F (b
n)) K'
    by auto
  then have K'_le_K: emeasure M K' ≤ emeasure M K
    by (intro emeasure_mono K)
  have K' ∈ sets M
    using me_K' by (rule measurable_envelopeD)

  have min (outer_measure_of M {x∈K'. f x ≤ a}) (outer_measure_of M {x∈K'.
b n ≤ f x}) < emeasure M K'
  proof (rule assms)
    show 0 < emeasure M K' emeasure M K' < ∞
```

using *measurable_envelopeD2*[*OF me_K'*] *pos_b K K'_le_K* **by** *auto*
qed *fact*+
also have $\{x{\in}K'.\ f\ x \le a\} = K' \cap (K \cap ?E\ a)$
using ‹$K' \in$ *sets M*›[*THEN sets.sets_into_space*] ‹$K' \subseteq K$› **by** *auto*
also have $\{x{\in}K'.\ b\ n \le f\ x\} = K' \cap (K \cap ?F\ (b\ n))$
using ‹$K' \in$ *sets M*›[*THEN sets.sets_into_space*] ‹$K' \subseteq K$› **by** *auto*
finally have *min* (*emeasure M K*) (*emeasure M K'*) < *emeasure M K'*
unfolding
measurable_envelopeD(*3*)[*OF me_K* ‹$K' \in$ *sets M*›, *symmetric*]
measurable_envelopeD(*3*)[*OF me_K'* ‹$K' \in$ *sets M*›, *symmetric*]
using ‹$K' \subseteq K$› **by** (*simp add*: *Int_absorb1 Int_absorb2*)
with *K'_le_K* **show** *False*
by (*auto simp*: *min_def split*: *if_split_asm*)
qed

end

7.11 Regularity of Measures

theory *Regularity*
imports *Measure_Space Borel_Space*
begin

theorem
fixes *M*::'*a*::{*second_countable_topology*, *complete_space*} *measure*
assumes *sb*: *sets M* = *sets borel*
assumes *emeasure M* (*space M*) ≠ ∞
assumes *B* ∈ *sets borel*
shows *inner_regular*: *emeasure M B* =
(*SUP K* ∈ {*K*. *K* ⊆ *B* ∧ *compact K*}. *emeasure M K*) (**is** *?inner B*)
and *outer_regular*: *emeasure M B* =
(*INF U* ∈ {*U*. *B* ⊆ *U* ∧ *open U*}. *emeasure M U*) (**is** *?outer B*)
proof −
have *Us*: *UNIV* = *space M* **by** (*metis assms*(*1*) *sets_eq_imp_space_eq space_borel*)
hence *sU*: *space M* = *UNIV* **by** *simp*
interpret *finite_measure M* **by** *rule fact*
have *approx_inner*: ⋀*A*. *A* ∈ *sets M* ⟹
(⋀*e*. *e* > *0* ⟹ ∃ *K*. *K* ⊆ *A* ∧ *compact K* ∧ *emeasure M A* ≤ *emeasure M K*
+ *ennreal e*) ⟹ *?inner A*
by (*rule ennreal_approx_SUP*)
(*force intro*!: *emeasure_mono simp*: *compact_imp_closed emeasure_eq_measure*)+
have *approx_outer*: ⋀*A*. *A* ∈ *sets M* ⟹
(⋀*e*. *e* > *0* ⟹ ∃ *B*. *A* ⊆ *B* ∧ *open B* ∧ *emeasure M B* ≤ *emeasure M A* +
ennreal e) ⟹ *?outer A*
by (*rule ennreal_approx_INF*)
(*force intro*!: *emeasure_mono simp*: *emeasure_eq_measure sb*)+
from *countable_dense_setE* **obtain** *X* :: '*a set*
where *X*: *countable X* ⋀*Y* :: '*a set*. *open Y* ⟹ *Y* ≠ {} ⟹ ∃ *d*∈*X*. *d* ∈ *Y*
by *auto*

```
 {
  fix r::real assume r > 0 hence ⋀y. open (ball y r) ⋀y. ball y r ≠ {} by auto
  with X(2)[OF this]
  have x: space M = (⋃x∈X. cball x r)
    by (auto simp add: sU) (metis dist_commute order_less_imp_le)
  let ?U = ⋃k. (⋃n∈{0..k}. cball (from_nat_into X n) r)
  have (λk. emeasure M (⋃n∈{0..k}. cball (from_nat_into X n) r)) ⟶ M
?U
     by (rule Lim_emeasure_incseq) (auto intro!: borel_closed bexI simp: inc-
seq_def Us sb)
  also have ?U = space M
  proof safe
    fix x from X(2)[OF open_ball[of x r]] ‹r > 0› obtain d where d: d∈X d
∈ ball x r by auto
    show x ∈ ?U
     using X(1) d
     by simp (auto intro!: exI [where x = to_nat_on X d] simp: dist_commute
Bex_def)
  qed (simp add: sU)
  finally have (λk. M (⋃n∈{0..k}. cball (from_nat_into X n) r)) ⟶ M
(space M) .
 } note M_space = this
 {
  fix e ::real and n :: nat assume e > 0 n > 0
  hence 1/n > 0 e * 2 powr - n > 0 by (auto)
  from M_space[OF ‹1/n>0›]
   have (λk. measure M (⋃i∈{0..k}. cball (from_nat_into X i) (1/real n)))
⟶ measure M (space M)
    unfolding emeasure_eq_measure by (auto)
  from metric_LIMSEQ_D[OF this ‹0 < e * 2 powr -n›]
   obtain k where dist (measure M (⋃i∈{0..k}. cball (from_nat_into X i)
(1/real n))) (measure M (space M)) <
    e * 2 powr -n
    by auto
  hence measure M (⋃i∈{0..k}. cball (from_nat_into X i) (1/real n)) ≥
    measure M (space M) - e * 2 powr -real n
    by (auto simp: dist_real_def)
  hence ∃k. measure M (⋃i∈{0..k}. cball (from_nat_into X i) (1/real n)) ≥
    measure M (space M) - e * 2 powr - real n ..
 } note k=this
 hence ∀e∈{0<..}. ∀(n::nat)∈{0<..}. ∃k.
   measure M (⋃i∈{0..k}. cball (from_nat_into X i) (1/real n)) ≥ measure M
(space M) - e * 2 powr - real n
  by blast
 then obtain k where k: ∀e∈{0<..}. ∀n∈{0<..}. measure M (space M) - e *
2 powr - real (n::nat)
  ≤ measure M (⋃i∈{0..k e n}. cball (from_nat_into X i) (1 / n))
  by metis
 hence k: ⋀e n. e > 0 ⟹ n > 0 ⟹ measure M (space M) - e * 2 powr - n
```

```
    ≤ measure M (⋃ i∈{0..k e n}. cball (from_nat_into X i) (1 / n))
    unfolding Ball_def by blast
  have approx_space:
    ∃ K ∈ {K. K ⊆ space M ∧ compact K}. emeasure M (space M) ≤ emeasure
M K + ennreal e
    (is ?thesis e) if 0 < e for e :: real
  proof −
    define B where [abs_def]:
      B n = (⋃ i∈{0..k e (Suc n)}. cball (from_nat_into X i) (1 / Suc n)) for n
    have ⋀n. closed (B n) by (auto simp: B_def)
    hence [simp]: ⋀n. B n ∈ sets M by (simp add: sb)
    from k[OF ‹e > 0› zero_less_Suc]
    have ⋀n. measure M (space M) − measure M (B n) ≤ e * 2 powr − real (Suc
n)
      by (simp add: algebra_simps B_def finite_measure_compl)
    hence B_compl_le: ⋀n::nat. measure M (space M − B n) ≤ e * 2 powr −
real (Suc n)
      by (simp add: finite_measure_compl)
    define K where K = (⋂ n. B n)
    from ‹closed (B _)› have closed K by (auto simp: K_def)
    hence [simp]: K ∈ sets M by (simp add: sb)
    have measure M (space M) − measure M K = measure M (space M − K)
      by (simp add: finite_measure_compl)
    also have ... = emeasure M (⋃ n. space M − B n) by (auto simp: K_def
emeasure_eq_measure)
    also have ... ≤ (∑ n. emeasure M (space M − B n))
      by (rule emeasure_subadditive_countably) (auto simp: summable_def)
    also have ... ≤ (∑ n. ennreal (e*2 powr − real (Suc n)))
    using B_compl_le by (intro suminf_le) (simp_all add: emeasure_eq_measure
ennreal_leI)
    also have ... ≤ (∑ n. ennreal (e * (1 / 2) ^ Suc n))
      by (simp add: powr_minus powr_realpow field_simps del: of_nat_Suc)
    also have ... = ennreal e * (∑ n. ennreal ((1 / 2) ^ Suc n))
      unfolding ennreal_power[symmetric]
      using ‹0 < e›
    by (simp add: ac_simps ennreal_mult' divide_ennreal[symmetric] divide_ennreal_def
                  ennreal_power[symmetric])
    also have ... = e
      by (subst suminf_ennreal_eq[OF zero_le_power power_half_series]) auto
    finally have measure M (space M) ≤ measure M K + e
      using ‹0 < e› by simp
    hence emeasure M (space M) ≤ emeasure M K + e
      using ‹0 < e› by (simp add: emeasure_eq_measure flip: ennreal_plus)
    moreover have compact K
      unfolding compact_eq_totally_bounded
    proof safe
      show complete K using ‹closed K› by (simp add: complete_eq_closed)
      fix e'::real assume 0 < e'
      then obtain n where n: 1 / real (Suc n) < e' by (rule nat_approx_posE)
```

```
    let ?k = from_nat_into X ` {0..k e (Suc n)}
    have finite ?k by simp
    moreover have K ⊆ (⋃x∈?k. ball x e') unfolding K_def B_def using n
by force
    ultimately show ∃k. finite k ∧ K ⊆ (⋃x∈k. ball x e') by blast
  qed
  ultimately
  show ?thesis by (auto simp: sU)
 qed
 { fix A::'a set assume closed A hence A ∈ sets borel by (simp add: com-
pact_imp_closed)
  hence [simp]: A ∈ sets M by (simp add: sb)
  have ?inner A
  proof (rule approx_inner)
   fix e::real assume e > 0
   from approx_space[OF this] obtain K where
     K: K ⊆ space M compact K emeasure M (space M) ≤ emeasure M K + e
     by (auto simp: emeasure_eq_measure)
   hence [simp]: K ∈ sets M by (simp add: sb compact_imp_closed)
   have measure M A − measure M (A ∩ K) = measure M (A − A ∩ K)
     by (subst finite_measure_Diff) auto
   also have A − A ∩ K = A ∪ K − K by auto
   also have measure M ... = measure M (A ∪ K) − measure M K
     by (subst finite_measure_Diff) auto
   also have ... ≤ measure M (space M) − measure M K
     by (simp add: emeasure_eq_measure sU sb finite_measure_mono)
   also have ... ≤ e
     using K ‹0 < e› by (simp add: emeasure_eq_measure flip: ennreal_plus)
   finally have emeasure M A ≤ emeasure M (A ∩ K) + ennreal e
       using ‹0<e› by (simp add: emeasure_eq_measure algebra_simps flip:
ennreal_plus)
    moreover have A ∩ K ⊆ A compact (A ∩ K) using ‹closed A› ‹compact
K› by auto
   ultimately show ∃K ⊆ A. compact K ∧ emeasure M A ≤ emeasure M K +
ennreal e
     by blast
  qed simp
  have ?outer A
  proof cases
   assume A ≠ {}
   let ?G = λd. {x. infdist x A < d}
   {
    fix d
    have ?G d = (λx. infdist x A) −` {..<d} by auto
    also have open ...
         by (intro continuous_open_vimage) (auto intro!: continuous_infdist
continuous_ident)
    finally have open (?G d) .
   } note open_G = this
```

```
    from in_closed_iff_infdist_zero[OF ‹closed A› ‹A ≠ {}›]
    have A = {x. infdist x A = 0} by auto
    also have ... = (⋂i. ?G (1/real (Suc i)))
    proof (auto simp del: of_nat_Suc, rule ccontr)
      fix x
      assume infdist x A ≠ 0
      then have pos: infdist x A > 0 using infdist_nonneg[of x A] by simp
        then obtain n where n: 1 / real (Suc n) < infdist x A by (rule
nat_approx_posE)
      assume ∀i. infdist x A < 1 / real (Suc i)
      then have infdist x A < 1 / real (Suc n) by auto
      with n show False by simp
    qed
    also have M ... = (INF n. emeasure M (?G (1 / real (Suc n))))
    proof (rule INF_emeasure_decseq[symmetric], safe)
      fix i::nat
      from open_G[of 1 / real (Suc i)]
      show ?G (1 / real (Suc i)) ∈ sets M by (simp add: sb borel_open)
    next
      show decseq (λi. {x. infdist x A < 1 / real (Suc i)})
        by (auto intro: less_trans intro!: divide_strict_left_mono
          simp: decseq_def le_eq_less_or_eq)
    qed simp
    finally
    have emeasure M A = (INF n. emeasure M {x. infdist x A < 1 / real (Suc
n)}) .
    moreover
    have ... ≥ (INF U∈{U. A ⊆ U ∧ open U}. emeasure M U)
    proof (intro INF_mono)
      fix m
      have ?G (1 / real (Suc m)) ∈ {U. A ⊆ U ∧ open U} using open_G by
auto
      moreover have M (?G (1 / real (Suc m))) ≤ M (?G (1 / real (Suc m)))
by simp
      ultimately show ∃U∈{U. A ⊆ U ∧ open U}.
        emeasure M U ≤ emeasure M {x. infdist x A < 1 / real (Suc m)}
        by blast
    qed
    moreover
    have emeasure M A ≤ (INF U∈{U. A ⊆ U ∧ open U}. emeasure M U)
      by (rule INF_greatest) (auto intro!: emeasure_mono simp: sb)
    ultimately show ?thesis by simp
  qed (auto intro!: INF_eqI)
  note ‹?inner A› ‹?outer A› }
 note closed_in_D = this
 from ‹B ∈ sets borel›
 have Int_stable (Collect closed) Collect closed ⊆ Pow UNIV B ∈ sigma_sets
UNIV (Collect closed)
  by (auto simp: Int_stable_def borel_eq_closed)
```

```
  then show ?inner B ?outer B
  proof (induct B rule: sigma_sets_induct_disjoint)
    case empty
    { case 1 show ?case by (intro SUP_eqI[symmetric]) auto }
    { case 2 show ?case by (intro INF_eqI[symmetric]) (auto elim!: meta_allE[of
_ {}]) }
  next
    case (basic B)
    { case 1 from basic closed_in_D show ?case by auto }
    { case 2 from basic closed_in_D show ?case by auto }
  next
    case (compl B)
    note inner = compl(2) and outer = compl(3)
    from compl have [simp]: B ∈ sets M by (auto simp: sb borel_eq_closed)
    case 2
     have M (space M − B) = M (space M) − emeasure M B by (auto simp:
emeasure_compl)
    also have ... = (INF K∈{K. K ⊆ B ∧ compact K}. M (space M) − M K)
     by (subst ennreal_SUP_const_minus) (auto simp: less_top[symmetric] inner)
    also have ... = (INF U∈{U. U ⊆ B ∧ compact U}. M (space M − U))
      by (auto simp add: emeasure_compl sb compact_imp_closed)
    also have ... ≥ (INF U∈{U. U ⊆ B ∧ closed U}. M (space M − U))
      by (rule INF_superset_mono) (auto simp add: compact_imp_closed)
    also have (INF U∈{U. U ⊆ B ∧ closed U}. M (space M − U)) =
        (INF U∈{U. space M − B ⊆ U ∧ open U}. emeasure M U)
      apply (rule arg_cong [of _ _ Inf])
      using sU
      apply (auto simp add: image_iff)
      apply (rule exI [of _ UNIV − y for y])
      apply safe
        apply (auto simp add: double_diff)
      done
    finally have
      (INF U∈{U. space M − B ⊆ U ∧ open U}. emeasure M U) ≤ emeasure M
(space M − B) .
    moreover have
      (INF U∈{U. space M − B ⊆ U ∧ open U}. emeasure M U) ≥ emeasure M
(space M − B)
      by (auto simp: sb sU intro!: INF_greatest emeasure_mono)
   ultimately show ?case by (auto intro!: antisym simp: sets_eq_imp_space_eq[OF
sb])

    case 1
     have M (space M − B) = M (space M) − emeasure M B by (auto simp:
emeasure_compl)
    also have ... = (SUP U∈ {U. B ⊆ U ∧ open U}. M (space M) − M U)
      unfolding outer by (subst ennreal_INF_const_minus) auto
    also have ... = (SUP U∈{U. B ⊆ U ∧ open U}. M (space M − U))
      by (auto simp add: emeasure_compl sb compact_imp_closed)
```

also have ... = (*SUP* K∈{K. $K \subseteq$ *space* $M - B \wedge$ *closed* K}. *emeasure* M K)
unfolding *SUP_image* [*of* _ λu. *space* $M - u$ _, *symmetric*, *unfolded comp_def*]
apply (*rule arg_cong* [*of* _ _ *Sup*])
using *sU* **apply** (*auto intro*!: *imageI*)
done
also have ... = (*SUP* K∈{K. $K \subseteq$ *space* $M - B \wedge$ *compact* K}. *emeasure* M K)
proof (*safe intro*!: *antisym SUP_least*)
fix K **assume** *closed* K $K \subseteq$ *space* $M - B$
from *closed_in_D*[*OF* ‹*closed* K›]
have *K_inner*: *emeasure* M K = (*SUP* K∈{*Ka*. *Ka* $\subseteq$ $K \wedge$ *compact Ka*}. *emeasure* M K) **by** *simp*
show *emeasure* M $K \leq$ (*SUP* K∈{K. $K \subseteq$ *space* $M - B \wedge$ *compact* K}. *emeasure* M K)
unfolding *K_inner* **using** ‹$K \subseteq$ *space* $M - B$›
by (*auto intro*!: *SUP_upper SUP_least*)
qed (*fastforce intro*!: *SUP_least SUP_upper simp*: *compact_imp_closed*)
finally show *?case* **by** (*auto intro*!: *antisym simp*: *sets_eq_imp_space_eq*[*OF sb*])
next
case (*union D*)
then have *range D* $\subseteq$ *sets M* **by** (*auto simp*: *sb borel_eq_closed*)
with *union* **have** *M*[*symmetric*]: ($\sum i.$ *M* (*D i*)) = *M* ($\bigcup i.$ *D i*) **by** (*intro suminf_emeasure*)
also have ($\lambda n.$ $\sum i{<}n.$ *M* (*D i*)) ⟶ ($\sum i.$ *M* (*D i*))
by (*intro summable_LIMSEQ*) *auto*
finally have *measure_LIMSEQ*: ($\lambda n.$ $\sum i{<}n.$ *measure M* (*D i*)) ⟶ *measure M* ($\bigcup i.$ *D i*)
by (*simp add*: *emeasure_eq_measure sum_nonneg*)
have ($\bigcup i.$ *D i*) ∈ *sets M* **using** ‹*range D* $\subseteq$ *sets M*› **by** *auto*

case *1*
show *?case*
proof (*rule approx_inner*)
fix *e*::*real* **assume** *e* > *0*
with *measure_LIMSEQ*
have $\exists$ *no*. $\forall$ *n*≥*no*. |($\sum i{<}n.$ *measure M* (*D i*)) −*measure M* ($\bigcup x.$ *D x*)| < *e*/*2*
by (*auto simp*: *lim_sequentially dist_real_def simp del*: *less_divide_eq_numeral1*)
hence $\exists$ *n0*. |($\sum i{<}n0.$ *measure M* (*D i*)) − *measure M* ($\bigcup x.$ *D x*)| < *e*/*2* **by** *auto*
then obtain *n0* **where** *n0*: |($\sum i{<}n0.$ *measure M* (*D i*)) − *measure M* ($\bigcup i.$ *D i*)| < *e*/*2*
unfolding *choice_iff* **by** *blast*
have *ennreal* ($\sum i{<}n0.$ *measure M* (*D i*)) = ($\sum i{<}n0.$ *M* (*D i*))
by (*auto simp add*: *emeasure_eq_measure*)
also have ... ≤ ($\sum i.$ *M* (*D i*)) **by** (*rule sum_le_suminf*) *auto*

also have ... = M ($\bigcup i.\ D\ i$) **by** (*simp add*: M)
also have ... = *measure* M ($\bigcup i.\ D\ i$) **by** (*simp add*: *emeasure_eq_measure*)
finally have *n0*: *measure* M ($\bigcup i.\ D\ i$) − ($\sum i{<}n0.$ *measure* M ($D\ i$)) < $e/2$
using *n0* **by** (*auto simp*: *sum_nonneg*)
have $\forall i.\ \exists K.\ K \subseteq D\ i \wedge$ *compact* $K \wedge$ *emeasure* M ($D\ i$) ≤ *emeasure* $M\ K$ + $e/(2{*}Suc\ n0)$
proof
fix i
from ‹0 < *e*› **have** $0 < e/(2{*}Suc\ n0)$ **by** *simp*
have *emeasure* M ($D\ i$) = ($SUP\ K{\in}\{K.\ K \subseteq (D\ i) \wedge$ *compact* $K\}.$ *emeasure* $M\ K$)
using *union* **by** *blast*
from *SUP_approx_ennreal*[*OF* ‹0 < $e/(2{*}Suc\ n0)$› _ *this*]
show $\exists K.\ K \subseteq D\ i \wedge$ *compact* $K \wedge$ *emeasure* M ($D\ i$) ≤ *emeasure* $M\ K$ + $e/(2{*}Suc\ n0)$
by (*auto simp*: *emeasure_eq_measure intro*: *less_imp_le compact_empty*)
qed
then obtain K **where** K: $\bigwedge i.\ K\ i \subseteq D\ i\ \bigwedge i.$ *compact* ($K\ i$) $\bigwedge i.$ *emeasure* M ($D\ i$) ≤ *emeasure* M ($K\ i$) + $e/(2{*}Suc\ n0)$
unfolding *choice_iff* **by** *blast*
let *?K* = $\bigcup i{\in}\{..{<}n0\}.\ K\ i$
have *disjoint_family_on* K $\{..{<}n0\}$ **using** K ‹*disjoint_family D*›
unfolding *disjoint_family_on_def* **by** *blast*
hence *mK*: *measure* M *?K* = ($\sum i{<}n0.$ *measure* M ($K\ i$)) **using** K
by (*intro finite_measure_finite_Union*) (*auto simp*: *sb compact_imp_closed*)
have *measure* M ($\bigcup i.\ D\ i$) < ($\sum i{<}n0.$ *measure* M ($D\ i$)) + $e/2$ **using** *n0* **by** *simp*
also have ($\sum i{<}n0.$ *measure* M ($D\ i$)) ≤ ($\sum i{<}n0.$ *measure* M ($K\ i$) + $e/(2{*}Suc\ n0)$)
using K ‹0 < *e*›
by (*auto intro*: *sum_mono simp*: *emeasure_eq_measure simp flip*: *ennreal_plus*)
also have ... = ($\sum i{<}n0.$ *measure* M ($K\ i$)) + ($\sum i{<}n0.$ $e/(2{*}Suc\ n0)$)
by (*simp add*: *sum.distrib*)
also have ... ≤ ($\sum i{<}n0.$ *measure* M ($K\ i$)) + $e\ /\ 2$ **using** ‹0 < *e*›
by (*auto simp*: *field_simps intro*!: *mult_left_mono*)
finally
have *measure* M ($\bigcup i.\ D\ i$) < ($\sum i{<}n0.$ *measure* M ($K\ i$)) + $e\ /\ 2 + e\ /\ 2$
by *auto*
hence M ($\bigcup i.\ D\ i$) < M *?K* + e
using ‹0<*e*› **by** (*auto simp*: *mK emeasure_eq_measure sum_nonneg ennreal_less_iff simp flip*: *ennreal_plus*)
moreover
have *?K* ⊆ ($\bigcup i.\ D\ i$) **using** K **by** *auto*
moreover
have *compact ?K* **using** K **by** *auto*
ultimately
have *?K*⊆($\bigcup i.\ D\ i$) ∧ *compact ?K* ∧ *emeasure* M ($\bigcup i.\ D\ i$) ≤ *emeasure* M *?K* + *ennreal* e **by** *simp*

```
    thus ∃K⊆⋃i. D i. compact K ∧ emeasure M (⋃i. D i) ≤ emeasure M K +
ennreal e ..
  qed fact
  case 2
  show ?case
  proof (rule approx_outer[OF ‹(⋃i. D i) ∈ sets M›])
    fix e::real assume e > 0
    have ∀i::nat. ∃U. D i ⊆ U ∧ open U ∧ e/(2 powr Suc i) > emeasure M U
− emeasure M (D i)
    proof
      fix i::nat
      from ‹0 < e› have 0 < e/(2 powr Suc i) by simp
       have emeasure M (D i) = (INF U∈{U. (D i) ⊆ U ∧ open U}. emeasure
M U)
        using union by blast
      from INF_approx_ennreal[OF ‹0 < e/(2 powr Suc i)› this]
       show ∃U. D i ⊆ U ∧ open U ∧ e/(2 powr Suc i) > emeasure M U −
emeasure M (D i)
        using ‹0<e›
          by (auto simp: emeasure_eq_measure sum_nonneg ennreal_less_iff
ennreal_minus
                    finite_measure_mono sb
              simp flip: ennreal_plus)
    qed
    then obtain U where U: ⋀i. D i ⊆ U i ⋀i. open (U i)
     ⋀i. e/(2 powr Suc i) > emeasure M (U i) − emeasure M (D i)
     unfolding choice_iff by blast
    let ?U = ⋃i. U i
    have ennreal (measure M ?U − measure M (⋃i. D i)) = M ?U − M (⋃i.
D i)
      using U(1,2)
      by (subst ennreal_minus[symmetric])
        (auto intro!: finite_measure_mono simp: sb emeasure_eq_measure)
    also have ... = M (?U − (⋃i. D i)) using U ‹(⋃i. D i) ∈ sets M›
      by (subst emeasure_Diff) (auto simp: sb)
    also have ... ≤ M (⋃i. U i − D i) using U ‹range D ⊆ sets M›
      by (intro emeasure_mono) (auto simp: sb intro!: sets.countable_nat_UN
sets.Diff)
    also have ... ≤ (∑i. M (U i − D i)) using U ‹range D ⊆ sets M›
      by (intro emeasure_subadditive_countably) (auto intro!: sets.Diff simp: sb)
    also have ... ≤ (∑i. ennreal e/(2 powr Suc i)) using U ‹range D ⊆ sets
M›
      using ‹0<e›
      by (intro suminf_le, subst emeasure_Diff)
        (auto simp: emeasure_Diff emeasure_eq_measure sb ennreal_minus
                  finite_measure_mono divide_ennreal ennreal_less_iff
            intro: less_imp_le)
    also have ... ≤ (∑n. ennreal (e * (1 / 2) ^ Suc n))
      using ‹0<e›
```

by (*simp add*: *powr_minus powr_realpow field_simps divide_ennreal del*: *of_nat_Suc*)
also have ... = *ennreal e* * (∑ *n*. *ennreal* ((*1* / *2*) ^ *Suc n*))
unfolding *ennreal_power*[*symmetric*]
using ‹*0* < *e*›
by (*simp add*: *ac_simps ennreal_mult'* *divide_ennreal*[*symmetric*] *divide_ennreal_def*
ennreal_power[*symmetric*])
also have ... = *ennreal e*
by (*subst suminf_ennreal_eq*[*OF zero_le_power power_half_series*]) *auto*
finally have *emeasure M ?U* ≤ *emeasure M* (⋃ *i*. *D i*) + *ennreal e*
using ‹*0*<*e*› **by** (*simp add*: *emeasure_eq_measure flip*: *ennreal_plus*)
moreover
have (⋃ *i*. *D i*) ⊆ *?U* **using** *U* **by** *auto*
moreover
have *open ?U* **using** *U* **by** *auto*
ultimately
have (⋃ *i*. *D i*) ⊆ *?U* ∧ *open ?U* ∧ *emeasure M ?U* ≤ *emeasure M* (⋃ *i*. *D i*) + *ennreal e* **by** *simp*
thus ∃ *B*. (⋃ *i*. *D i*) ⊆ *B* ∧ *open B* ∧ *emeasure M B* ≤ *emeasure M* (⋃ *i*. *D i*) + *ennreal e* **..**
qed
qed
qed

end

7.12 Lebesgue Measure

theory *Lebesgue_Measure*
imports
Finite_Product_Measure
Caratheodory
Complete_Measure
Summation_Tests
Regularity
begin

lemma *measure_eqI_lessThan*:
fixes *M N* :: *real measure*
assumes *sets*: *sets M* = *sets borel sets N* = *sets borel*
assumes *fin*: ⋀*x*. *emeasure M* {*x* <..} < ∞
assumes ⋀*x*. *emeasure M* {*x* <..} = *emeasure N* {*x* <..}
shows *M* = *N*
proof (*rule measure_eqI_generator_eq_countable*)
let *?LT* = λ*a*::*real*. {*a* <..} **let** *?E* = *range ?LT*
show *Int_stable ?E*
by (*auto simp*: *Int_stable_def lessThan_Int_lessThan*)

show *?E* ⊆ *Pow UNIV* **sets** *M* = *sigma_sets UNIV ?E* **sets** *N* = *sigma_sets UNIV ?E*
unfolding *sets borel_Ioi* **by** *auto*

show *?LT'Rats* ⊆ *?E* (⋃ *i*∈*Rats*. *?LT i*) = *UNIV* ⋀*a*. *a* ∈ *?LT'Rats* ⟹ *emeasure M a* ≠ ∞
using *fin* **by** (*auto intro*: *Rats_no_bot_less simp*: *less_top*)
qed (*auto intro*: *assms countable_rat*)

7.12.1 Measures defined by monotonous functions

Every right-continuous and nondecreasing function gives rise to a measure on the reals:

definition *interval_measure* :: (*real* ⇒ *real*) ⇒ *real measure* **where**
interval_measure F =
extend_measure UNIV {(*a*, *b*). *a* ≤ *b*} (λ(*a*, *b*). {*a*<..*b*}) (λ(*a*, *b*). *ennreal* (*F b* − *F a*))

lemma *emeasure_interval_measure_Ioc*:
assumes *a* ≤ *b*
assumes *mono_F*: ⋀*x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y*
assumes *right_cont_F* : ⋀*a*. *continuous* (*at_right a*) *F*
shows *emeasure* (*interval_measure F*) {*a*<..*b*} = *F b* − *F a*
proof (*rule extend_measure_caratheodory_pair*[*OF interval_measure_def* ‹*a* ≤ *b*›])
show *semiring_of_sets UNIV* {{*a*<..*b*} |*a b* :: *real*. *a* ≤ *b*}
proof (*unfold_locales*, *safe*)
fix *a b c d* :: *real* **assume** ∗: *a* ≤ *b c* ≤ *d*
then show ∃ *C*⊆{{*a*<..*b*} |*a b*. *a* ≤ *b*}. *finite C* ∧ *disjoint C* ∧ {*a*<..*b*} − {*c*<..*d*} = ⋃ *C*
proof *cases*
let *?C* = {{*a*<..*b*}}
assume *b* < *c* ∨ *d* ≤ *a* ∨ *d* ≤ *c*
with ∗ **have** *?C* ⊆ {{*a*<..*b*} |*a b*. *a* ≤ *b*} ∧ *finite ?C* ∧ *disjoint ?C* ∧ {*a*<..*b*} − {*c*<..*d*} = ⋃ *?C*
by (*auto simp add*: *disjoint_def*)
thus *?thesis* **..**
next
let *?C* = {{*a*<..*c*}, {*d*<..*b*}}
assume ¬ (*b* < *c* ∨ *d* ≤ *a* ∨ *d* ≤ *c*)
with ∗ **have** *?C* ⊆ {{*a*<..*b*} |*a b*. *a* ≤ *b*} ∧ *finite ?C* ∧ *disjoint ?C* ∧ {*a*<..*b*} − {*c*<..*d*} = ⋃ *?C*
by (*auto simp add*: *disjoint_def Ioc_inj*) (*metis linear*)+
thus *?thesis* **..**
qed
qed (*auto simp*: *Ioc_inj*, *metis linear*)
next
fix *l r* :: *nat* ⇒ *real* **and** *a b* :: *real*
assume *l_r*[*simp*]: ⋀*n*. *l n* ≤ *r n* **and** *a* ≤ *b* **and** *disj*: *disjoint_family* (λ*n*. {*l*

```
n<..r n})
  assume lr_eq_ab: (⋃ i. {l i<..r i}) = {a<..b}

  have [intro, simp]: ⋀a b. a ≤ b ⟹ F a ≤ F b
    by (auto intro!: l_r mono_F)

  { fix S :: nat set assume finite S
    moreover note ‹a ≤ b›
    moreover have ⋀i. i ∈ S ⟹ {l i <.. r i} ⊆ {a <.. b}
      unfolding lr_eq_ab[symmetric] by auto
    ultimately have (∑ i∈S. F (r i) − F (l i)) ≤ F b − F a
    proof (induction S arbitrary: a rule: finite_psubset_induct)
      case (psubset S)
      show ?case
      proof cases
        assume ∃ i∈S. l i < r i
        with ‹finite S› have Min (l ‘ {i∈S. l i < r i}) ∈ l ‘ {i∈S. l i < r i}
          by (intro Min_in) auto
        then obtain m where m: m ∈ S l m < r m l m = Min (l ‘ {i∈S. l i < r
i})
          by fastforce

        have (∑ i∈S. F (r i) − F (l i)) = (F (r m) − F (l m)) + (∑ i∈S − {m}.
F (r i) − F (l i))
          using m psubset by (intro sum.remove) auto
        also have (∑ i∈S − {m}. F (r i) − F (l i)) ≤ F b − F (r m)
        proof (intro psubset.IH)
          show S − {m} ⊂ S
            using ‹m∈S› by auto
          show r m ≤ b
            using psubset.prems(2)[OF ‹m∈S›] ‹l m < r m› by auto
        next
          fix i assume i ∈ S − {m}
          then have i: i ∈ S i ≠ m by auto
          { assume i': l i < r i l i < r m
            with ‹finite S› i m have l m ≤ l i
              by auto
            with i' have {l i <.. r i} ∩ {l m <.. r m} ≠ {}
              by auto
            then have False
              using disjoint_family_onD[OF disj, of i m] i by auto }
          then have l i ≠ r i ⟹ r m ≤ l i
            unfolding not_less[symmetric] using l_r[of i] by auto
          then show {l i <.. r i} ⊆ {r m <.. b}
            using psubset.prems(2)[OF ‹i∈S›] by auto
        qed
        also have F (r m) − F (l m) ≤ F (r m) − F a
          using psubset.prems(2)[OF ‹m ∈ S›] ‹l m < r m›
          by (auto simp add: Ioc_subset_iff intro!: mono_F)
```

```
      finally show ?case
        by (auto intro: add_mono)
    qed (auto simp add: ‹a ≤ b› less_le)
  qed }
 note claim1 = this


 { fix S u v and l r :: nat ⇒ real
  assume finite S ⋀i. i∈S ⟹ l i < r i {u..v} ⊆ (⋃i∈S. {l i<..< r i})
  then have F v − F u ≤ (∑ i∈S. F (r i) − F (l i))
  proof (induction arbitrary: v u rule: finite_psubset_induct)
    case (psubset S)
    show ?case
    proof cases
      assume S = {} then show ?case
        using psubset by (simp add: mono_F)
    next
      assume S ≠ {}
      then obtain j where j ∈ S
        by auto

      let ?R = r j < u ∨ l j > v ∨ (∃ i∈S−{j}. l i ≤ l j ∧ r j ≤ r i)
      show ?case
      proof cases
        assume ?R
        with ‹j ∈ S› psubset.prems have {u..v} ⊆ (⋃i∈S−{j}. {l i<..< r i})
          apply (simp add: subset_eq Ball_def Bex_def)
        by (metis order_le_less_trans order_less_le_trans order_less_not_sym)
        with ‹j ∈ S› have F v − F u ≤ (∑ i∈S − {j}. F (r i) − F (l i))
          by (intro psubset) auto
        also have ... ≤ (∑ i∈S. F (r i) − F (l i))
          using psubset.prems
          by (intro sum_mono2 psubset) (auto intro: less_imp_le)
        finally show ?thesis .
      next
        assume ¬ ?R
        then have j: u ≤ r j l j ≤ v ⋀i. i ∈ S − {j} ⟹ r i < r j ∨ l i > l j
          by (auto simp: not_less)
        let ?S1 = {i ∈ S. l i < l j}
        let ?S2 = {i ∈ S. r i > r j}
        have ∗: ?S1 ∩ ?S2 = {}
          using j by (fastforce simp add: disjoint_iff)
        have (∑ i∈S. F (r i) − F (l i)) ≥ (∑ i∈?S1 ∪ ?S2 ∪ {j}. F (r i) − F
(l i))
          using ‹j ∈ S› ‹finite S› psubset.prems j
          by (intro sum_mono2) (auto intro: less_imp_le)
        also have (∑ i∈?S1 ∪ ?S2 ∪ {j}. F (r i) − F (l i)) =
          (∑ i∈?S1. F (r i) − F (l i)) + (∑ i∈?S2 . F (r i) − F (l i)) + (F (r
```

```
j) − F (l j))
        using psubset(1) by (simp add: ∗ sum.union_disjoint disjoint_iff_not_equal)
         also (xtrans) have (∑ i∈?S1. F (r i) − F (l i)) ≥ F (l j) − F u
           using ‹j ∈ S› ‹finite S› psubset.prems j
           apply (intro psubset.IH psubset)
           apply (auto simp: subset_eq Ball_def)
           apply (metis less_le_trans not_le)
           done
         also (xtrans) have (∑ i∈?S2. F (r i) − F (l i)) ≥ F v − F (r j)
           using ‹j ∈ S› ‹finite S› psubset.prems j
           apply (intro psubset.IH psubset)
           apply (auto simp: subset_eq Ball_def)
           apply (metis le_less_trans not_le)
           done
         finally (xtrans) show ?case
           by (auto simp: add_mono)
       qed
     qed
   qed }
  note claim2 = this


  have ennreal (F b − F a) ≤ (∑ i. ennreal (F (r i) − F (l i)))
  proof (rule ennreal_le_epsilon)
    fix epsilon :: real assume egt0: epsilon > 0
    have ∀ i. ∃ d>0. F (r i + d) < F (r i) + epsilon / 2^(i+2)
    proof
      fix i
      note right_cont_F [of r i]
      then have ∃ d>0. F (r i + d) − F (r i) < epsilon / 2 ^(i + 2)
        by (simp add: continuous_at_right_real_increasing egt0)
      thus ∃ d>0. F (r i + d) < F (r i) + epsilon / 2^(i+2)
        by force
    qed
    then obtain delta where
       deltai_gt0: ⋀i. delta i > 0 and
       deltai_prop: ⋀i. F (r i + delta i) < F (r i) + epsilon / 2^(i+2)
      by metis
    have ∃ a' > a. F a' − F a < epsilon / 2
      using right_cont_F [of a]
    by (metis continuous_at_right_real_increasing egt0 half_gt_zero less_add_same_cancel1
mono_F)
    then obtain a' where a'lea [arith]: a' > a and
      a_prop: F a' − F a < epsilon / 2
      by auto
    define S' where S' = {i. l i < r i}
    obtain S :: nat set where S ⊆ S' and finS: finite S
      and Sprop: {a'..b} ⊆ (⋃ i ∈ S. {l i<..<r i + delta i})
    proof (rule compactE_image)
```

show *compact* $\{a'..b\}$
by (*rule compact_Icc*)
show $\bigwedge i.\ i \in S' \Longrightarrow$ *open* $(\{l\ i<..<r\ i + delta\ i\})$ **by** *auto*
have $\{a'..b\} \subseteq \{a <.. b\}$
by *auto*
also have $\{a <.. b\} = (\bigcup i{\in}S'.\ \{l\ i<..r\ i\})$
unfolding *lr_eq_ab*[*symmetric*] **by** (*fastforce simp add*: *S'_def intro*: *less_le_trans*)
also have $\ldots \subseteq (\bigcup i \in S'.\ \{l\ i<..<r\ i + delta\ i\})$
by (*intro UN_mono*; *simp add*: *add.commute add_strict_increasing deltai_gt0 subset_iff*)
finally show $\{a'..b\} \subseteq (\bigcup i \in S'.\ \{l\ i<..<r\ i + delta\ i\})$ **.**
qed
with *S'_def* **have** *Sprop2*: $\bigwedge i.\ i \in S \Longrightarrow l\ i < r\ i$ **by** *auto*
obtain *n* **where** *Sbound*: $\bigwedge i.\ i \in S \Longrightarrow i \leq n$
using *Max_ge finS* **by** *blast*
have $F\ b - F\ a = (F\ b - F\ a') + (F\ a' - F\ a)$
by *auto*
also have $\ldots \leq (F\ b - F\ a') + epsilon\ /\ 2$
using *a_prop* **by** (*intro add_left_mono*) *simp*
also have $\ldots \leq (\sum i{\in}S.\ F\ (r\ i) - F\ (l\ i)) + epsilon\ /\ 2 + epsilon\ /\ 2$
proof −
have $F\ b - F\ a' \leq (\sum i{\in}S.\ F\ (r\ i + delta\ i) - F\ (l\ i))$
using *claim2 l_r Sprop* **by** (*simp add*: *deltai_gt0 finS add.commute add_strict_increasing*)
also have $\ldots \leq (\sum i \in S.\ F(r\ i) - F(l\ i) + epsilon\ /\ 2\hat{}(i+2))$
by (*smt* (*verit*) *sum_mono deltai_prop*)
also have $\ldots = (\sum i \in S.\ F(r\ i) - F(l\ i)) +$
$(epsilon\ /\ 4) * (\sum i \in S.\ (1\ /\ 2)\hat{}i)$ (**is** $\_ = ?t + \_$)
by (*subst sum.distrib*) (*simp add*: *field_simps sum_distrib_left*)
also have $\ldots \leq ?t + (epsilon\ /\ 4) * (\sum\ i < Suc\ n.\ (1\ /\ 2)\hat{}i)$
using *egt0 Sbound* **by** (*intro add_left_mono mult_left_mono sum_mono2*) *force*+
also have $\ldots \leq ?t + (epsilon\ /\ 2)$
using *egt0* **by** (*simp add*: *geometric_sum add_left_mono mult_left_mono*)
finally have $F\ b - F\ a' \leq (\sum i{\in}S.\ F\ (r\ i) - F\ (l\ i)) + epsilon\ /\ 2$
by *simp*
then show *?thesis*
by *linarith*
qed
also have $\ldots = (\sum i{\in}S.\ F\ (r\ i) - F\ (l\ i)) + epsilon$
by *auto*
also have $\ldots \leq (\sum i{\leq}n.\ F\ (r\ i) - F\ (l\ i)) + epsilon$
using *finS Sbound Sprop* **by** (*auto intro*!: *add_right_mono sum_mono2*)
finally have $ennreal\ (F\ b - F\ a) \leq (\sum i{\leq}n.\ ennreal\ (F\ (r\ i) - F\ (l\ i))) + epsilon$
using *egt0* **by** (*simp add*: *sum_nonneg flip*: *ennreal_plus*)
then show $ennreal\ (F\ b - F\ a) \leq (\sum i.\ ennreal\ (F\ (r\ i) - F\ (l\ i))) + (epsilon :: real)$

```
      by (rule order_trans) (auto intro!: add_mono sum_le_suminf simp del:
sum_ennreal)
  qed
  moreover have (∑ i. ennreal (F (r i) − F (l i))) ≤ ennreal (F b − F a)
    using ‹a ≤ b› by (auto intro!: suminf_le_const ennreal_le_iff[THEN iffD2]
claim1)
  ultimately show (∑ n. ennreal (F (r n) − F (l n))) = ennreal (F b − F a)
    by (rule antisym[rotated])
qed (auto simp: Ioc_inj mono_F)

lemma measure_interval_measure_Ioc:
  assumes a ≤ b and ⋀x y. x ≤ y ⟹ F x ≤ F y and ⋀a. continuous (at_right
a) F
  shows measure (interval_measure F) {a <.. b} = F b − F a
  unfolding measure_def
  by (simp add: assms emeasure_interval_measure_Ioc)

lemma emeasure_interval_measure_Ioc_eq:
  (⋀x y. x ≤ y ⟹ F x ≤ F y) ⟹ (⋀a. continuous (at_right a) F) ⟹
    emeasure (interval_measure F) {a <.. b} = (if a ≤ b then F b − F a else 0)
  using emeasure_interval_measure_Ioc[of a b F] by auto

lemma sets_interval_measure [simp, measurable_cong]:
    sets (interval_measure F) = sets borel
 apply (simp add: sets_extend_measure interval_measure_def borel_sigma_sets_Ioc
image_def split: prod.split)
  by (metis greaterThanAtMost_empty nle_le)

lemma space_interval_measure [simp]: space (interval_measure F) = UNIV
  by (simp add: interval_measure_def space_extend_measure)

lemma emeasure_interval_measure_Icc:
  assumes a ≤ b
  assumes mono_F: ⋀x y. x ≤ y ⟹ F x ≤ F y
  assumes cont_F : continuous_on UNIV F
  shows emeasure (interval_measure F) {a .. b} = F b − F a
proof (rule tendsto_unique)
  { fix a b :: real assume a ≤ b then have emeasure (interval_measure F) {a
<.. b} = F b − F a
      using cont_F
      by (subst emeasure_interval_measure_Ioc)
      (auto intro: mono_F continuous_within_subset simp: continuous_on_eq_continuous_within)
}
  note ∗ = this

  let ?F = interval_measure F
  show ((λa. F b − F a) ⟶ emeasure ?F {a..b}) (at_left a)
  proof (rule tendsto_at_left_sequentially)
    show a − 1 < a by simp
```

fix X **assume** X: $\bigwedge n.\ X\ n < a$ *incseq* X $X \longrightarrow a$
then have *emeasure* (*interval_measure* F) $\{X\ n<..b\} \neq \infty$ **for** n
by (*smt* (*verit*) $*$ ‹$a \leq b$› *ennreal_neq_top infinity_ennreal_def*)
with X **have** ($\lambda n.$ *emeasure* *?F* $\{X\ n<..b\}$) $\longrightarrow$ *emeasure* *?F* ($\bigcap n.$ $\{X\ n <..b\}$)
by (*intro Lim_emeasure_decseq*; *force simp*: *decseq_def incseq_def emeasure_interval_measure_Ioc* $*$)
also have ($\bigcap n.$ $\{X\ n <..b\}$) = $\{a..b\}$
apply *auto*
apply (*rule LIMSEQ_le_const2*[*OF* ‹$X \longrightarrow a$›])
using *less_eq_real_def* **apply** *presburger*
using $X(1)$ *order_less_le_trans* **by** *blast*
also have ($\lambda n.$ *emeasure* *?F* $\{X\ n<..b\}$) = ($\lambda n.$ $F\ b - F\ (X\ n)$)
using ‹$\bigwedge n.\ X\ n < a$› ‹$a \leq b$› **by** (*subst* $*$) (*auto intro*: *less_imp_le less_le_trans*)
finally show ($\lambda n.$ $F\ b - F\ (X\ n)$) $\longrightarrow$ *emeasure* *?F* $\{a..b\}$ **.**
qed
show (($\lambda a.$ *ennreal* ($F\ b - F\ a$)) $\longrightarrow$ $F\ b - F\ a$) (*at_left* a)
by (*rule continuous_on_tendsto_compose*[**where** g=$\lambda x.\ x$ **and** s=*UNIV*])
(*auto simp*: *continuous_on_ennreal continuous_on_diff cont_F*)
qed (*rule trivial_limit_at_left_real*)

lemma *sigma_finite_interval_measure*:
assumes *mono_F*: $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$
assumes *right_cont_F* : $\bigwedge a.$ *continuous* (*at_right* a) F
shows *sigma_finite_measure* (*interval_measure* F)
apply *unfold_locales*
apply (*intro exI*[*of* _ ($\lambda(a,\ b).$ $\{a <..\ b\}$) ‘ ($\mathbb{Q} \times \mathbb{Q}$)])
apply (*auto intro*!: *Rats_no_top_le Rats_no_bot_less countable_rat simp*: *emeasure_interval_measure_Ioc_eq*[*OF assms*])
done

7.12.2 Lebesgue-Borel measure

definition *lborel* :: ('*a* :: *euclidean_space*) *measure* **where**
lborel = *distr* (Π_M b∈*Basis*. *interval_measure* ($\lambda x.\ x$)) *borel* ($\lambda f.$ $\sum b$∈*Basis*. $f\ b *_R b$)

abbreviation *lebesgue* :: '*a*::*euclidean_space measure*
where *lebesgue* ≡ *completion lborel*

abbreviation *lebesgue_on* :: '*a set* ⇒ '*a*::*euclidean_space measure*
where *lebesgue_on* Ω ≡ *restrict_space* (*completion lborel*) Ω

lemma *lebesgue_on_mono*:
assumes *major*: *AE x in lebesgue_on S*. *P x* **and** *minor*: $\bigwedge x.$⟦*P x*; $x \in S$⟧ ⟹ *Q x*
shows *AE x in lebesgue_on S*. *Q x*
proof −

have *AE a in lebesgue_on S. P a ⟶ Q a*
using *minor space_restrict_space* **by** *fastforce*
then show *?thesis*
using *major* **by** *auto*
qed

lemma *integral_eq_zero_null_sets*:
assumes *S ∈ null_sets lebesgue*
shows *integral^L (lebesgue_on S) f = 0*
proof (*rule integral_eq_zero_AE*)
show *AE x in lebesgue_on S. f x = 0*
by (*metis* (*no_types*, *lifting*) *assms AE_not_in lebesgue_on_mono null_setsD2 null_sets_restrict_space order_refl*)
qed

lemma
shows *sets_lborel*[*simp*, *measurable_cong*]: *sets lborel = sets borel*
and *space_lborel*[*simp*]: *space lborel = space borel*
and *measurable_lborel1*[*simp*]: *measurable M lborel = measurable M borel*
and *measurable_lborel2*[*simp*]: *measurable lborel M = measurable borel M*
by (*simp_all add*: *lborel_def*)

lemma *space_lebesgue_on* [*simp*]: *space (lebesgue_on S) = S*
by (*simp add*: *space_restrict_space*)

lemma *sets_lebesgue_on_refl* [*iff*]: *S ∈ sets (lebesgue_on S)*
by (*metis inf_top.right_neutral sets.top space_borel space_completion space_lborel space_restrict_space*)

lemma *Compl_in_sets_lebesgue*: *−A ∈ sets lebesgue ⟷ A ∈ sets lebesgue*
by (*metis Compl_eq_Diff_UNIV double_compl space_borel space_completion space_lborel Sigma_Algebra.sets.compl_sets*)

lemma *measurable_lebesgue_cong*:
assumes *⋀x. x ∈ S ⟹ f x = g x*
shows *f ∈ measurable (lebesgue_on S) M ⟷ g ∈ measurable (lebesgue_on S) M*
by (*metis* (*mono_tags*, *lifting*) *IntD1 assms measurable_cong_simp space_restrict_space*)

lemma *lebesgue_on_UNIV_eq*: *lebesgue_on UNIV = lebesgue*
by (*simp add*: *emeasure_restrict_space measure_eqI*)

lemma *integral_restrict_Int*:
fixes *f :: 'a::euclidean_space ⇒ 'b::euclidean_space*
assumes *S ∈ sets lebesgue T ∈ sets lebesgue*
shows *integral^L (lebesgue_on T) (λx. if x ∈ S then f x else 0) = integral^L (lebesgue_on (S ∩ T)) f*
proof −
have *(λx. indicat_real T x *_R (if x ∈ S then f x else 0)) = (λx. indicat_real (S*

$\cap\ T)\ x *_R f x)$
by (*force simp*: *indicator_def*)
then show *?thesis*
by (*simp add*: *assms sets.Int Bochner_Integration.integral_restrict_space*)
qed

lemma *integral_restrict*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $S \subseteq T\ S \in sets\ lebesgue\ T \in sets\ lebesgue$
shows $integral^L\ (lebesgue\_on\ T)\ (\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0) = integral^L\ (lebesgue\_on\ S)\ f$
using *integral_restrict_Int* [*of S T f*] *assms*
by (*simp add*: *Int_absorb2*)

lemma *integral_restrict_UNIV*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $S \in sets\ lebesgue$
shows $integral^L\ lebesgue\ (\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0) = integral^L\ (lebesgue\_on\ S)\ f$
using *integral_restrict_Int* [*of S UNIV f*] *assms*
by (*simp add*: *lebesgue_on_UNIV_eq*)

lemma *integrable_lebesgue_on_empty* [*iff*]:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::\{second\_countable\_topology,banach\}$
shows *integrable* (*lebesgue_on* {}) *f*
by (*simp add*: *integrable_restrict_space*)

lemma *integral_lebesgue_on_empty* [*simp*]:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::\{second\_countable\_topology,banach\}$
shows $integral^L\ (lebesgue\_on\ \{\})\ f = 0$
by (*simp add*: *Bochner_Integration.integral_empty*)
lemma *has_bochner_integral_restrict_space*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\_countable\_topology\}$
assumes Ω: $\Omega \cap space\ M \in sets\ M$
shows *has_bochner_integral* (*restrict_space M* Ω) *f i*
$\longleftrightarrow$ *has_bochner_integral M* ($\lambda x.$ *indicator* Ω *x* $*_R$ *f x*) *i*
by (*simp add*: *integrable_restrict_space* [*OF assms*] *integral_restrict_space* [*OF assms*] *has_bochner_integral_iff*)

lemma *integrable_restrict_UNIV*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::\{banach,\ second\_countable\_topology\}$
assumes S: $S \in sets\ lebesgue$
shows *integrable lebesgue* ($\lambda x.$ *if x* $\in$ *S then f x else 0*) $\longleftrightarrow$ *integrable* (*lebesgue_on S*) *f*
using *has_bochner_integral_restrict_space* [*of S lebesgue f*] *assms*
by (*simp add*: *integrable.simps indicator_scaleR_eq_if*)

lemma *integral_mono_lebesgue_on_AE*:
fixes $f::\_ \Rightarrow real$

assumes f: *integrable* (*lebesgue_on* T) f
and gf: *AE* x *in* (*lebesgue_on* S). $g\ x \leq f\ x$
and $f0$: *AE* x *in* (*lebesgue_on* T). $0 \leq f\ x$
and $S \subseteq T$ **and** S: $S \in$ *sets lebesgue* **and** T: $T \in$ *sets lebesgue*
shows $(\int x.\ g\ x\ \partial(\text{lebesgue\_on}\ S)) \leq (\int x.\ f\ x\ \partial(\text{lebesgue\_on}\ T))$
proof −
have $(\int x.\ g\ x\ \partial(\text{lebesgue\_on}\ S)) = (\int x.\ (\text{if}\ x \in S\ \text{then}\ g\ x\ \text{else}\ 0)\ \partial\text{lebesgue})$
by (*simp add*: *Lebesgue_Measure.integral_restrict_UNIV* S)
also have $\ldots \leq (\int x.\ (\text{if}\ x \in T\ \text{then}\ f\ x\ \text{else}\ 0)\ \partial\text{lebesgue})$
proof (*rule Bochner_Integration.integral_mono_AE'*)
show *integrable lebesgue* $(\lambda x.\ \text{if}\ x \in T\ \text{then}\ f\ x\ \text{else}\ 0)$
by (*simp add*: *integrable_restrict_UNIV* T f)
show *AE* x *in lebesgue*. $(\text{if}\ x \in S\ \text{then}\ g\ x\ \text{else}\ 0) \leq (\text{if}\ x \in T\ \text{then}\ f\ x\ \text{else}\ 0)$
using *assms* **by** (*auto simp*: *AE_restrict_space_iff*)
show *AE* x *in lebesgue*. $0 \leq (\text{if}\ x \in T\ \text{then}\ f\ x\ \text{else}\ 0)$
using $f0$ **by** (*simp add*: *AE_restrict_space_iff* T)
qed
also have $\ldots = (\int x.\ f\ x\ \partial(\text{lebesgue\_on}\ T))$
using *Lebesgue_Measure.integral_restrict_UNIV* T **by** *blast*
finally show *?thesis* **.**
qed

7.12.3 Borel measurability

lemma *borel_measurable_if_I*:
fixes $f :: 'a{::}\text{euclidean\_space} \Rightarrow 'b{::}\text{euclidean\_space}$
assumes f: $f \in$ *borel_measurable* (*lebesgue_on* S) **and** S: $S \in$ *sets lebesgue*
shows $(\lambda x.\ \text{if}\ x \in S\ \text{then}\ f\ x\ \text{else}\ 0) \in$ *borel_measurable lebesgue*
proof −
have *eq*: $\{x.\ x \notin S\} \cup \{x.\ f\ x \in Y\} = \{x.\ x \notin S\} \cup \{x.\ f\ x \in Y\} \cap S$ **for** Y
by *blast*
show *?thesis*
using f S
apply (*simp add*: *vimage_def in_borel_measurable_borel Ball_def*)
apply (*elim all_forward imp_forward asm_rl*)
apply (*simp only*: *Collect_conj_eq Collect_disj_eq imp_conv_disj eq*)
apply (*auto simp*: *Compl_eq* [*symmetric*] *Compl_in_sets_lebesgue sets_restrict_space_iff*)
done
qed

lemma *borel_measurable_if_D*:
fixes $f :: 'a{::}\text{euclidean\_space} \Rightarrow 'b{::}\text{euclidean\_space}$
assumes $(\lambda x.\ \text{if}\ x \in S\ \text{then}\ f\ x\ \text{else}\ 0) \in$ *borel_measurable lebesgue*
shows $f \in$ *borel_measurable* (*lebesgue_on* S)
using *assms* **by** (*smt* (*verit*) *measurable_lebesgue_cong measurable_restrict_space1*)

lemma *borel_measurable_if*:
fixes $f :: 'a{::}\text{euclidean\_space} \Rightarrow 'b{::}\text{euclidean\_space}$
assumes $S \in$ *sets lebesgue*

shows $(\lambda x.\ \mathit{if}\ x \in S\ \mathit{then}\ f\ x\ \mathit{else}\ 0) \in \mathit{borel\_measurable}\ \mathit{lebesgue} \longleftrightarrow f \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ S)$
using *assms borel_measurable_if_D borel_measurable_if_I* **by** *blast*

lemma *borel_measurable_if_lebesgue_on*:
fixes $f :: 'a::\mathit{euclidean\_space} \Rightarrow 'b::\mathit{euclidean\_space}$
assumes $S \in \mathit{sets}\ \mathit{lebesgue}\ T \in \mathit{sets}\ \mathit{lebesgue}\ S \subseteq T$
shows $(\lambda x.\ \mathit{if}\ x \in S\ \mathit{then}\ f\ x\ \mathit{else}\ 0) \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ T) \longleftrightarrow f \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ S)$
(**is** *?lhs = ?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *measurable_restrict_mono* [*OF _* ‹$S \subseteq T$›]
by (*subst measurable_lebesgue_cong* [**where** $g = (\lambda x.\ \mathit{if}\ x \in S\ \mathit{then}\ f\ x\ \mathit{else}\ 0)$]) *auto*
next
assume *?rhs* **then show** *?lhs*
by (*simp add*: ‹$S \in \mathit{sets}\ \mathit{lebesgue}$› *borel_measurable_if_I measurable_restrict_space1*)
qed

lemma *borel_measurable_vimage_halfspace_component_lt*:
$f \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ S) \longleftrightarrow$
$(\forall a\ i.\ i \in \mathit{Basis} \longrightarrow \{x \in S.\ f\ x \cdot i < a\} \in \mathit{sets}\ (\mathit{lebesgue\_on}\ S))$
by (*force simp add*: *space_restrict_space trans* [*OF borel_measurable_iff_halfspace_less*])

lemma *borel_measurable_vimage_halfspace_component_ge*:
fixes $f :: 'a::\mathit{euclidean\_space} \Rightarrow 'b::\mathit{euclidean\_space}$
shows $f \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ S) \longleftrightarrow$
$(\forall a\ i.\ i \in \mathit{Basis} \longrightarrow \{x \in S.\ f\ x \cdot i \geq a\} \in \mathit{sets}\ (\mathit{lebesgue\_on}\ S))$
by (*force simp add*: *space_restrict_space trans* [*OF borel_measurable_iff_halfspace_ge*])

lemma *borel_measurable_vimage_halfspace_component_gt*:
fixes $f :: 'a::\mathit{euclidean\_space} \Rightarrow 'b::\mathit{euclidean\_space}$
shows $f \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ S) \longleftrightarrow$
$(\forall a\ i.\ i \in \mathit{Basis} \longrightarrow \{x \in S.\ f\ x \cdot i > a\} \in \mathit{sets}\ (\mathit{lebesgue\_on}\ S))$
by (*force simp add*: *space_restrict_space trans* [*OF borel_measurable_iff_halfspace_greater*])

lemma *borel_measurable_vimage_halfspace_component_le*:
fixes $f :: 'a::\mathit{euclidean\_space} \Rightarrow 'b::\mathit{euclidean\_space}$
shows $f \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ S) \longleftrightarrow$
$(\forall a\ i.\ i \in \mathit{Basis} \longrightarrow \{x \in S.\ f\ x \cdot i \leq a\} \in \mathit{sets}\ (\mathit{lebesgue\_on}\ S))$
by (*force simp add*: *space_restrict_space trans* [*OF borel_measurable_iff_halfspace_le*])

lemma
fixes $f :: 'a::\mathit{euclidean\_space} \Rightarrow 'b::\mathit{euclidean\_space}$
shows *borel_measurable_vimage_open_interval*:
$f \in \mathit{borel\_measurable}\ (\mathit{lebesgue\_on}\ S) \longleftrightarrow$
$(\forall a\ b.\ \{x \in S.\ f\ x \in \mathit{box}\ a\ b\} \in \mathit{sets}\ (\mathit{lebesgue\_on}\ S))$ (**is** *?thesis1*)
and *borel_measurable_vimage_open*:

```
        f ∈ borel_measurable (lebesgue_on S) ⟷
        (∀ T. open T ⟶ {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S)) (is ?thesis2)
proof −
  have {x ∈ S. f x ∈ box a b} ∈ sets (lebesgue_on S) if f ∈ borel_measurable
(lebesgue_on S) for a b
  proof −
    have S = S ∩ space lebesgue
      by simp
    then have S ∩ (f −‘ box a b) ∈ sets (lebesgue_on S)
      by (metis (no_types) box_borel in_borel_measurable_borel inf_sup_aci(1)
space_restrict_space that)
    then show ?thesis
      by (simp add: Collect_conj_eq vimage_def)
  qed
  moreover
  have {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S)
      if T: ⋀a b. {x ∈ S. f x ∈ box a b} ∈ sets (lebesgue_on S) open T for T
  proof −
    obtain 𝒟 where countable 𝒟 and 𝒟: ⋀X. X ∈ 𝒟 ⟹ ∃ a b. X = box a b ⋃𝒟
= T
      using open_countable_Union_open_box that ‹open T› by metis
    then have eq: {x ∈ S. f x ∈ T} = (⋃ U ∈ 𝒟. {x ∈ S. f x ∈ U})
      by blast
    have {x ∈ S. f x ∈ U} ∈ sets (lebesgue_on S) if U ∈ 𝒟 for U
      using that T 𝒟 by blast
    then show ?thesis
      by (auto simp: eq intro: Sigma_Algebra.sets.countable_UN′ [OF ‹countable
𝒟›])
  qed
  moreover
  have eq: {x ∈ S. f x · i < a} = {x ∈ S. f x ∈ {y. y · i < a}} for i a
    by auto
  have f ∈ borel_measurable (lebesgue_on S)
    if ⋀T. open T ⟹ {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S)
    by (metis (no_types) eq borel_measurable_vimage_halfspace_component_lt
open_halfspace_component_lt that)
  ultimately show ?thesis1 ?thesis2
    by blast+
qed

lemma borel_measurable_vimage_closed:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  shows f ∈ borel_measurable (lebesgue_on S) ⟷
        (∀ T. closed T ⟶ {x ∈ S. f x ∈ T} ∈ sets (lebesgue_on S))
proof −
  have eq: {x ∈ S. f x ∈ T} = S − (S ∩ f −‘ (− T)) for T
    by auto
  show ?thesis
    unfolding borel_measurable_vimage_open eq
```

by (*metis Diff_Diff_Int closed_Compl diff_eq open_Compl sets.Diff sets_lebesgue_on_refl vimage_Compl*)
qed

lemma *borel_measurable_vimage_closed_interval*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
shows $f \in borel\_measurable\ (lebesgue\_on\ S) \longleftrightarrow$
$(\forall a\ b.\ \{x \in S.\ f\ x \in cbox\ a\ b\} \in sets\ (lebesgue\_on\ S))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *borel_measurable_vimage_closed* **by** *blast*
next
assume *RHS*: *?rhs*
have $\{x \in S.\ f\ x \in T\} \in sets\ (lebesgue\_on\ S)$ **if** *open T* **for** *T*
proof −
obtain $\mathcal{D}$ **where** *countable* $\mathcal{D}$ **and** $\mathcal{D}$: $\mathcal{D} \subseteq Pow\ T\ \bigwedge X.\ X \in \mathcal{D} \Longrightarrow \exists a\ b.\ X = cbox\ a\ b\ \bigcup \mathcal{D} = T$
using *open_countable_Union_open_cbox that* ‹*open T*› **by** *metis*
then have *eq*: $\{x \in S.\ f\ x \in T\} = (\bigcup U \in \mathcal{D}.\ \{x \in S.\ f\ x \in U\})$
by *blast*
have $\{x \in S.\ f\ x \in U\} \in sets\ (lebesgue\_on\ S)$ **if** $U \in \mathcal{D}$ **for** *U*
using *that* $\mathcal{D}$ **by** (*metis RHS*)
then show *?thesis*
by (*auto simp*: *eq intro*: *Sigma_Algebra.sets.countable_UN'* [*OF* ‹*countable* $\mathcal{D}$›])
qed
then show *?lhs*
by (*simp add*: *borel_measurable_vimage_open*)
qed

lemma *borel_measurable_vimage_borel*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
shows $f \in borel\_measurable\ (lebesgue\_on\ S) \longleftrightarrow$
$(\forall T.\ T \in sets\ borel \longrightarrow \{x \in S.\ f\ x \in T\} \in sets\ (lebesgue\_on\ S))$
(**is** *?lhs* = *?rhs*)
proof
assume *f*: *?lhs*
then show *?rhs*
using *measurable_sets* [*OF f*]
by (*simp add*: *Collect_conj_eq inf_sup_aci*(1) *space_restrict_space vimage_def*)
qed (*simp add*: *borel_measurable_vimage_open_interval*)

lemma *lebesgue_measurable_vimage_borel*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $f \in borel\_measurable\ lebesgue\ T \in sets\ borel$
shows $\{x.\ f\ x \in T\} \in sets\ lebesgue$
using *assms borel_measurable_vimage_borel* [*of f UNIV*] **by** *auto*

lemma *borel_measurable_lebesgue_preimage_borel*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
shows $f \in borel\_measurable\ lebesgue \longleftrightarrow$
$(\forall T.\ T \in sets\ borel \longrightarrow \{x.\ f\ x \in T\} \in sets\ lebesgue)$
by (*smt* (*verit*, *best*) *Collect_cong UNIV_I borel_measurable_vimage_borel lebesgue_on_UNIV_eq*)

7.12.4 Measurability of continuous functions

lemma *continuous_imp_measurable_on_sets_lebesgue*:
assumes *f*: *continuous_on S f* **and** *S*: $S \in sets\ lebesgue$
shows $f \in borel\_measurable\ (lebesgue\_on\ S)$
by (*metis borel_measurable_continuous_on_restrict borel_measurable_subalgebra f*
lebesgue_on_UNIV_eq mono_restrict_space sets_completionI_sets sets_lborel space_borel
space_lebesgue_on space_restrict_space subsetI)

lemma *id_borel_measurable_lebesgue* [*iff*]: $id \in borel\_measurable\ lebesgue$
by (*simp add*: *measurable_completion*)

lemma *id_borel_measurable_lebesgue_on* [*iff*]: $id \in borel\_measurable\ (lebesgue\_on\ S)$
by (*simp add*: *measurable_completion measurable_restrict_space1*)

context
begin

interpretation *sigma_finite_measure interval_measure* $(\lambda x.\ x)$
by (*rule sigma_finite_interval_measure*) *auto*
interpretation *finite_product_sigma_finite* $\lambda\_.$ *interval_measure* $(\lambda x.\ x)$ *Basis*
proof qed *simp*

lemma *lborel_eq_real*: $lborel = interval\_measure\ (\lambda x.\ x)$
unfolding *lborel_def Basis_real_def*
using *distr_id*[*of interval_measure* $(\lambda x.\ x)$]
by (*subst distr_component*[*symmetric*])
(*simp_all add*: *distr_distr comp_def del*: *distr_id cong*: *distr_cong*)

lemma *lborel_eq*: $lborel = distr\ (\Pi_M\ b{\in}Basis.\ lborel)\ borel\ (\lambda f.\ \sum b{\in}Basis.\ f\ b *_R\ b)$
by (*subst lborel_def*) (*simp add*: *lborel_eq_real*)

lemma *nn_integral_lborel_prod*:
assumes [*measurable*]: $\bigwedge b.\ b \in Basis \Longrightarrow f\ b \in borel\_measurable\ borel$
assumes *nn*[*simp*]: $\bigwedge b\ x.\ b \in Basis \Longrightarrow 0 \le f\ b\ x$
shows $(\int^+ x.\ (\prod b{\in}Basis.\ f\ b\ (x \cdot b))\ \partial lborel) = (\prod b{\in}Basis.\ (\int^+ x.\ f\ b\ x\ \partial lborel))$
by (*simp add*: *lborel_def nn_integral_distr product_nn_integral_prod*
product_nn_integral_singleton)

lemma *emeasure_lborel_Icc*[*simp*]:
fixes $l\ u :: real$
assumes [*simp*]: $l \leq u$
shows $emeasure\ lborel\ \{l\ ..\ u\} = u - l$
by (*simp add*: *emeasure_interval_measure_Icc lborel_eq_real*)

lemma *emeasure_lborel_Icc_eq*: $emeasure\ lborel\ \{l\ ..\ u\} = ennreal\ (if\ l \leq u\ then\ u - l\ else\ 0)$
by *simp*

lemma *emeasure_lborel_cbox*[*simp*]:
assumes [*simp*]: $\bigwedge b.\ b \in Basis \Longrightarrow l \cdot b \leq u \cdot b$
shows $emeasure\ lborel\ (cbox\ l\ u) = (\prod b{\in}Basis.\ (u - l) \cdot b)$
proof −
have $(\lambda x.\ \prod b{\in}Basis.\ indicator\ \{l{\cdot}b\ ..\ u{\cdot}b\}\ (x \cdot b) :: ennreal) = indicator\ (cbox\ l\ u)$
by (*auto simp*: *fun_eq_iff cbox_def split*: *split_indicator*)
then have $emeasure\ lborel\ (cbox\ l\ u) = (\int^+ x.\ (\prod b{\in}Basis.\ indicator\ \{l{\cdot}b\ ..\ u{\cdot}b\}\ (x \cdot b))\ \partial lborel)$
by *simp*
also have $\ldots = (\prod b{\in}Basis.\ (u - l) \cdot b)$
by (*subst nn_integral_lborel_prod*) (*simp_all add*: *prod_ennreal inner_diff_left*)
finally show *?thesis* **.**
qed

lemma *AE_lborel_singleton*: $AE\ x\ in\ lborel::'a::euclidean\_space\ measure.\ x \neq c$
using *SOME_Basis AE_discrete_difference* [*of* $\{c\}$ *lborel*] *emeasure_lborel_cbox* [*of c c*]
by (*auto simp add*: *power_0_left*)

lemma *emeasure_lborel_Ioo*[*simp*]:
assumes [*simp*]: $l \leq u$
shows $emeasure\ lborel\ \{l\ <..<\ u\} = ennreal\ (u - l)$
proof −
have $emeasure\ lborel\ \{l\ <..<\ u\} = emeasure\ lborel\ \{l\ ..\ u\}$
using *AE_lborel_singleton*[*of u*] *AE_lborel_singleton*[*of l*] **by** (*intro emeasure_eq_AE*) *auto*
then show *?thesis*
by *simp*
qed

lemma *emeasure_lborel_Ioc*[*simp*]:
assumes [*simp*]: $l \leq u$
shows $emeasure\ lborel\ \{l\ <..\ u\} = ennreal\ (u - l)$
by (*simp add*: *emeasure_interval_measure_Ioc lborel_eq_real*)

lemma *emeasure_lborel_Ico*[*simp*]:
assumes [*simp*]: $l \leq u$

shows *emeasure lborel* $\{l ..< u\} = ennreal\ (u - l)$
proof −
have *emeasure lborel* $\{l ..< u\} =$ *emeasure lborel* $\{l .. u\}$
using *AE_lborel_singleton*[*of u*] *AE_lborel_singleton*[*of l*] **by** (*intro emeasure_eq_AE*) *auto*
then show *?thesis*
by *simp*
qed

lemma *emeasure_lborel_box*[*simp*]:
assumes [*simp*]: $\bigwedge b.\ b \in Basis \Longrightarrow l \cdot b \leq u \cdot b$
shows *emeasure lborel* (*box l u*) = $(\prod b{\in}Basis.\ (u - l) \cdot b)$
proof −
have $(\lambda x.\ \prod b{\in}Basis.\ indicator\ \{l{\cdot}b <..< u{\cdot}b\}\ (x \cdot b) :: ennreal) = indicator\ (box\ l\ u)$
by (*auto simp*: *fun_eq_iff box_def split*: *split_indicator*)
then have *emeasure lborel* (*box l u*) = $(\int^+ x.\ (\prod b{\in}Basis.\ indicator\ \{l{\cdot}b <..< u{\cdot}b\}\ (x \cdot b))\ \partial lborel)$
by *simp*
also have ... = $(\prod b{\in}Basis.\ (u - l) \cdot b)$
by (*subst nn_integral_lborel_prod*) (*simp_all add*: *prod_ennreal inner_diff_left*)
finally show *?thesis* **.**
qed

lemma *emeasure_lborel_cbox_eq*:
emeasure lborel (*cbox l u*) = (*if* $\forall b{\in}Basis.\ l \cdot b \leq u \cdot b$ *then* $\prod b{\in}Basis.\ (u - l) \cdot b$ *else 0*)
using *box_eq_empty*(*2*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le*)

lemma *emeasure_lborel_box_eq*:
emeasure lborel (*box l u*) = (*if* $\forall b{\in}Basis.\ l \cdot b \leq u \cdot b$ *then* $\prod b{\in}Basis.\ (u - l) \cdot b$ *else 0*)
using *box_eq_empty*(*1*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le dest*!: *less_imp_le*) *force*

lemma *emeasure_lborel_singleton*[*simp*]: *emeasure lborel* $\{x\} = 0$
using *emeasure_lborel_cbox*[*of x x*] *nonempty_Basis*
by (*auto simp del*: *emeasure_lborel_cbox nonempty_Basis*)

lemma *emeasure_lborel_cbox_finite*: *emeasure lborel* (*cbox a b*) $< \infty$
by (*auto simp*: *emeasure_lborel_cbox_eq*)

lemma *emeasure_lborel_box_finite*: *emeasure lborel* (*box a b*) $< \infty$
by (*auto simp*: *emeasure_lborel_box_eq*)

lemma *emeasure_lborel_ball_finite*: *emeasure lborel* (*ball c r*) $< \infty$
by (*metis bounded_ball bounded_subset_cbox_symmetric cbox_borel emeasure_lborel_cbox_finite emeasure_mono order_le_less_trans sets_lborel*)

lemma *emeasure_lborel_cball_finite*: *emeasure lborel* (*cball c r*) $< \infty$
by (*metis bounded_cball bounded_subset_cbox_symmetric cbox_borel emeasure_lborel_cbox_finite*
emeasure_mono order_le_less_trans sets_lborel)

lemma *fmeasurable_cbox* [*iff*]: *cbox a b* $\in$ *fmeasurable lborel*
and *fmeasurable_box* [*iff*]: *box a b* $\in$ *fmeasurable lborel*
by (*auto simp*: *fmeasurable_def emeasure_lborel_box_eq emeasure_lborel_cbox_eq*)

lemma
fixes *l u* :: *real*
assumes [*simp*]: $l \le u$
shows *measure_lborel_Icc*[*simp*]: *measure lborel* $\{l .. u\} = u - l$
and *measure_lborel_Ico*[*simp*]: *measure lborel* $\{l ..< u\} = u - l$
and *measure_lborel_Ioc*[*simp*]: *measure lborel* $\{l <.. u\} = u - l$
and *measure_lborel_Ioo*[*simp*]: *measure lborel* $\{l <..< u\} = u - l$
by (*simp_all add*: *measure_def*)

lemma
assumes [*simp*]: $\bigwedge b.\ b \in Basis \Longrightarrow l \cdot b \le u \cdot b$
shows *measure_lborel_box*[*simp*]: *measure lborel* (*box l u*) = ($\prod b{\in}Basis.\ (u - l) \cdot b$)
and *measure_lborel_cbox*[*simp*]: *measure lborel* (*cbox l u*) = ($\prod b{\in}Basis.\ (u - l) \cdot b$)
by (*simp_all add*: *measure_def inner_diff_left prod_nonneg*)

lemma *measure_lborel_cbox_eq*:
measure lborel (*cbox l u*) = (*if* $\forall\, b{\in}Basis.\ l \cdot b \le u \cdot b$ *then* $\prod b{\in}Basis.\ (u - l) \cdot b$ *else 0*)
using *box_eq_empty*(*2*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le*)

lemma *measure_lborel_box_eq*:
measure lborel (*box l u*) = (*if* $\forall\, b{\in}Basis.\ l \cdot b \le u \cdot b$ *then* $\prod b{\in}Basis.\ (u - l) \cdot b$ *else 0*)
using *box_eq_empty*(*1*)[*THEN iffD2*, *of u l*] **by** (*auto simp*: *not_le dest*!: *less_imp_le*) *force*

lemma *measure_lborel_singleton*[*simp*]: *measure lborel* $\{x\} = 0$
by (*simp add*: *measure_def*)

lemma *sigma_finite_lborel*: *sigma_finite_measure lborel*
proof
show $\exists A{::}'a\ set\ set.\ countable\ A \wedge A \subseteq sets\ lborel \wedge \bigcup A = space\ lborel \wedge (\forall\, a{\in}A.\ emeasure\ lborel\ a \neq \infty)$
by (*intro exI*[*of* _ *range* ($\lambda n{::}nat.\ box\ (-\ real\ n *_R One)\ (real\ n *_R One)$)])
(*auto simp*: *emeasure_lborel_cbox_eq UN_box_eq_UNIV*)
qed

end

lemma *emeasure_lborel_UNIV* [*simp*]: *emeasure lborel* (*UNIV*::*'a*::*euclidean_space set*) = ∞
proof −
  { fix *n*::*nat*
    let *?Ba* = *Basis* :: *'a set*
    have *real n* ≤ (*2*::*real*) ^ *card ?Ba* ∗ *real n*
      by (*simp add*: *mult_le_cancel_right1*)
    also
    have ... ≤ (*2*::*real*) ^ *card ?Ba* ∗ *real* (*Suc n*) ^ *card ?Ba*
      apply (*rule mult_left_mono*)
     apply (*metis DIM_positive One_nat_def less_eq_Suc_le less_imp_le of_nat_le_iff of_nat_power self_le_power zero_less_Suc*)
      apply (*simp*)
      done
    finally have *real n* ≤ (*2*::*real*) ^ *card ?Ba* ∗ *real* (*Suc n*) ^ *card ?Ba* **.**
  } note [*intro*!] = *this*
  show *?thesis*
    unfolding *UN_box_eq_UNIV*[*symmetric*]
    apply (*subst SUP_emeasure_incseq*[*symmetric*])
    apply (*auto simp*: *incseq_def subset_box inner_add_left*
      simp del: *Sup_eq_top_iff SUP_eq_top_iff*
      intro!: *ennreal_SUP_eq_top*)
    done
qed

lemma *emeasure_lborel_countable*:
  fixes *A* :: *'a*::*euclidean_space set*
  assumes *countable A*
  shows *emeasure lborel A* = *0*
proof −
   have *A* ⊆ (⋃ *i*. {*from_nat_into A i*}) **using** *from_nat_into_surj assms* **by** *force*
  then have *emeasure lborel A* ≤ *emeasure lborel* (⋃ *i*. {*from_nat_into A i*})
    by (*intro emeasure_mono*) *auto*
  also have *emeasure lborel* (⋃ *i*. {*from_nat_into A i*}) = *0*
    by (*rule emeasure_UN_eq_0*) *auto*
  finally show *?thesis*
    by *simp*
qed

lemma *countable_imp_null_set_lborel*: *countable A* ⟹ *A* ∈ *null_sets lborel*
  by (*simp add*: *null_sets_def emeasure_lborel_countable sets.countable*)

lemma *finite_imp_null_set_lborel*: *finite A* ⟹ *A* ∈ *null_sets lborel*
  by (*intro countable_imp_null_set_lborel countable_finite*)

lemma *insert_null_sets_iff* [*simp*]: *insert a N* ∈ *null_sets lebesgue* ⟷ *N* ∈ *null_sets lebesgue*

by (*meson completion.complete2 finite.simps finite_imp_null_set_lborel null_sets.insert_in_sets*

null_sets_completionI subset_insertI)

lemma *insert_null_sets_lebesgue_on_iff* [*simp*]:
assumes $a \in S$ $S \in$ *sets lebesgue*
shows *insert a N* $\in$ *null_sets* (*lebesgue_on S*) $\longleftrightarrow$ $N \in$ *null_sets* (*lebesgue_on S*)
by (*simp add*: *assms null_sets_restrict_space*)

lemma *lborel_neq_count_space*[*simp*]:
fixes A :: ('*a*::*ordered_euclidean_space*) *set*
shows *lborel* $\neq$ *count_space A*
by (*metis finite.simps finite_imp_null_set_lborel insert_not_empty null_sets_count_space singleton_iff*)

lemma *mem_closed_if_AE_lebesgue_open*:
assumes *open S closed C*
assumes *AE x* $\in$ *S in lebesgue.* $x \in C$
assumes $x \in S$
shows $x \in C$
proof (*rule ccontr*)
assume *xC*: $x \notin C$
with *openE*[*of S* $-$ *C*] *assms*
obtain *e* **where** *e*: $0 < e$ *ball x e* $\subseteq S - C$
by *blast*
then obtain *a b* **where** *box*: $x \in$ *box a b box a b* $\subseteq S - C$
by (*metis rational_boxes order_trans*)
then have $0 <$ *emeasure lebesgue* (*box a b*)
by (*auto simp*: *emeasure_lborel_box_eq mem_box algebra_simps intro*!: *prod_pos*)
also have $\ldots \leq$ *emeasure lebesgue* ($S - C$)
using *assms box*
by (*auto intro*!: *emeasure_mono*)
also have $\ldots = 0$
using *assms*
by (*auto simp*: *eventually_ae_filter completion.complete2 set_diff_eq null_setsD1*)
finally show *False* **by** *simp*
qed

lemma *mem_closed_if_AE_lebesgue*: *closed C* $\Longrightarrow$ (*AE x in lebesgue.* $x \in C$) $\Longrightarrow x \in C$
using *mem_closed_if_AE_lebesgue_open*[*OF open_UNIV*] **by** *simp*

7.12.5 Affine transformation on the Lebesgue-Borel

lemma *lborel_eqI*:
fixes M :: '*a*::*euclidean_space measure*
assumes *emeasure_eq*: $\bigwedge l\ u.\ (\bigwedge b.\ b \in Basis \Longrightarrow l \cdot b \leq u \cdot b) \Longrightarrow$ *emeasure M* (*box l u*) $= (\prod b \in Basis.\ (u - l) \cdot b)$

assumes *sets_eq*: *sets M = sets borel*
shows *lborel = M*
proof (*rule measure_eqI_generator_eq*)
let *?E = range* (λ(*a*, *b*). *box a b*::′*a set*)
show *Int_stable ?E*
by (*auto simp*: *Int_stable_def box_Int_box*)

show *?E* ⊆ *Pow UNIV sets lborel = sigma_sets UNIV ?E sets M = sigma_sets UNIV ?E*
by (*simp_all add*: *borel_eq_box sets_eq*)

let *?A* = λ*n*::*nat*. *box* (− (*real n* ∗$_R$ *One*)) (*real n* ∗$_R$ *One*) :: ′*a set*
show *range ?A* ⊆ *?E* (⋃ *i*. *?A i*) = *UNIV*
unfolding *UN_box_eq_UNIV* **by** *auto*
show *emeasure lborel* (*?A i*) ≠ ∞ **for** *i* **by** *auto*
show *emeasure lborel X = emeasure M X* **if** *X* ∈ *?E* **for** *X*
using *that box_eq_empty*(*1*) **by** (*fastforce simp*: *emeasure_eq emeasure_lborel_box_eq*)
qed

lemma *lborel_affine_euclidean*:
fixes *c* :: ′*a*::*euclidean_space* ⇒ *real* **and** *t*
defines *T x* ≡ *t* + (∑ *j*∈*Basis*. (*c j* ∗ (*x* ∙ *j*)) ∗$_R$ *j*)
assumes *c*: ⋀*j*. *j* ∈ *Basis* ⟹ *c j* ≠ *0*
shows *lborel = density* (*distr lborel borel T*) (λ_. (∏ *j*∈*Basis*. |*c j*|)) (**is** _ = *?D*)
proof (*rule lborel_eqI*)
let *?B = Basis* :: ′*a set*
fix *l u* **assume** *le*: ⋀*b*. *b* ∈ *?B* ⟹ *l* ∙ *b* ≤ *u* ∙ *b*
have [*measurable*]: *T* ∈ *borel* →$_M$ *borel*
by (*simp add*: *T_def*[*abs_def*])
have *eq*: *T* −‘ *box l u = box*
(∑ *j*∈*Basis*. (((*if 0 < c j then l − t else u − t*) ∙ *j*) / *c j*) ∗$_R$ *j*)
(∑ *j*∈*Basis*. (((*if 0 < c j then u − t else l − t*) ∙ *j*) / *c j*) ∗$_R$ *j*)
using *c* **by** (*auto simp*: *box_def T_def field_simps inner_simps divide_less_eq*)
with *le c* **show** *emeasure ?D* (*box l u*) = (∏ *b*∈*?B*. (*u* − *l*) ∙ *b*)
by (*auto simp*: *emeasure_density emeasure_distr nn_integral_multc emeasure_lborel_box_eq inner_simps*
field_split_simps ennreal_mult′[*symmetric*] *prod_nonneg prod.distrib*[*symmetric*]
intro!: *prod.cong*)
qed *simp*

lemma *lborel_affine*:
fixes *t* :: ′*a*::*euclidean_space*
shows *c* ≠ *0* ⟹ *lborel = density* (*distr lborel borel* (λ*x*. *t* + *c* ∗$_R$ *x*)) (λ_. |*c*|^*DIM*(′*a*))
using *lborel_affine_euclidean*[**where** *c*=λ_::′*a*. *c* **and** *t*=*t*]
unfolding *scaleR_scaleR*[*symmetric*] *scaleR_sum_right*[*symmetric*] *euclidean_representation prod_constant* **by** *simp*

lemma *lborel_real_affine*:
  $c \neq 0 \Longrightarrow$ *lborel* = *density* (*distr lborel borel* ($\lambda x.\ t + c * x$)) ($\lambda\_.$ *ennreal* (*abs c*))
  using *lborel_affine*[*of c t*] **by** *simp*

lemma *AE_borel_affine*:
  fixes *P* :: *real* ⇒ *bool*
  shows $c \neq 0 \Longrightarrow$ *Measurable.pred borel P* $\Longrightarrow$ *AE x in lborel. P x* $\Longrightarrow$ *AE x in lborel. P* ($t + c * x$)
  by (*subst lborel_real_affine*[**where** *t*=− *t* / *c* **and** *c*=*1* / *c*])
    (*simp_all add*: *AE_density AE_distr_iff field_simps*)

lemma *nn_integral_real_affine*:
  fixes *c* :: *real* **assumes** [*measurable*]: $f \in$ *borel_measurable borel* **and** *c*: $c \neq 0$
  shows ($\int^+ x.\ f\ x\ \partial lborel$) = $|c| *$ ($\int^+ x.\ f\ (t + c * x)\ \partial lborel$)
  by (*subst lborel_real_affine*[*OF c, of t*])
    (*simp add*: *nn_integral_density nn_integral_distr nn_integral_cmult*)

lemma *lborel_integrable_real_affine*:
  fixes *f* :: *real* ⇒ *'a* :: {*banach, second_countable_topology*}
  assumes *f*: *integrable lborel f*
  shows $c \neq 0 \Longrightarrow$ *integrable lborel* ($\lambda x.\ f\ (t + c * x)$)
  using *f f*[*THEN borel_measurable_integrable*] **unfolding** *integrable_iff_bounded*
  by (*subst* (*asm*) *nn_integral_real_affine*[**where** *c*=*c* **and** *t*=*t*]) (*auto simp*: *ennreal_mult_less_top*)

lemma *lborel_integrable_real_affine_iff*:
  fixes *f* :: *real* ⇒ *'a* :: {*banach, second_countable_topology*}
  shows $c \neq 0 \Longrightarrow$ *integrable lborel* ($\lambda x.\ f\ (t + c * x)$) $\longleftrightarrow$ *integrable lborel f*
  using
    lborel_integrable_real_affine[*of f c t*]
    lborel_integrable_real_affine[*of* $\lambda x.\ f\ (t + c * x)$ *1*/*c* −*t*/*c*]
  by (*auto simp add*: *field_simps*)

lemma *lborel_integral_real_affine*:
  fixes *f* :: *real* ⇒ *'a* :: {*banach, second_countable_topology*} **and** *c* :: *real*
  assumes *c*: $c \neq 0$ **shows** ($\int x.\ f\ x\ \partial$ *lborel*) = $|c| *_R$ ($\int x.\ f\ (t + c * x)\ \partial lborel$)
proof *cases*
  assume *f*[*measurable*]: *integrable lborel f* **then show** *?thesis*
   using *c f f*[*THEN borel_measurable_integrable*] *f*[*THEN lborel_integrable_real_affine, of c t*]
    by (*subst lborel_real_affine*[*OF c, of t*])
      (*simp add*: *integral_density integral_distr*)
next
  assume ¬ *integrable lborel f* **with** *c* **show** *?thesis*
    by (*simp add*: *lborel_integrable_real_affine_iff not_integrable_integral_eq*)
qed

lemma

fixes $c :: 'a::euclidean\_space \Rightarrow real$ **and** t
assumes c: $\bigwedge j.\ j \in Basis \Longrightarrow c\ j \neq 0$
defines $T == (\lambda x.\ t + (\sum j{\in}Basis.\ (c\ j * (x \cdot j))\ *_R\ j))$
shows *lebesgue_affine_euclidean*: $lebesgue = density\ (distr\ lebesgue\ lebesgue\ T)\ (\lambda\_.\ (\prod j{\in}Basis.\ |c\ j|))$ (**is** $\_ = ?D$)
and *lebesgue_affine_measurable*: $T \in lebesgue \rightarrow_M lebesgue$
proof −
have *T_borel*[*measurable*]: $T \in borel \rightarrow_M borel$
by (*auto simp*: *T_def*[*abs_def*])
{ fix $A :: 'a\ set$ **assume** A: $A \in sets\ borel$
then have $emeasure\ lborel\ A = 0 \longleftrightarrow emeasure\ (density\ (distr\ lborel\ borel\ T)\ (\lambda\_.\ (\prod j{\in}Basis.\ |c\ j|)))\ A = 0$
unfolding *T_def* **using** c **by** (*subst lborel_affine_euclidean*[*symmetric*]) *auto*
also have $\ldots \longleftrightarrow emeasure\ (distr\ lebesgue\ lborel\ T)\ A = 0$
using A c **by** (*simp add*: *distr_completion emeasure_density nn_integral_cmult prod_nonneg cong*: *distr_cong*)
finally have $emeasure\ lborel\ A = 0 \longleftrightarrow emeasure\ (distr\ lebesgue\ lborel\ T)\ A = 0$ **. }**
then have *eq*: $null\_sets\ lborel = null\_sets\ (distr\ lebesgue\ lborel\ T)$
by (*auto simp*: *null_sets_def*)

show $T \in lebesgue \rightarrow_M lebesgue$
by (*simp add*: *completion.measurable_completion2 eq measurable_completion*)

have $lebesgue = completion\ (density\ (distr\ lborel\ borel\ T)\ (\lambda\_.\ (\prod j{\in}Basis.\ |c\ j|)))$
using c **by** (*subst lborel_affine_euclidean*[*of c t*]) (*simp_all add*: *T_def*[*abs_def*])
also have $\ldots = density\ (completion\ (distr\ lebesgue\ lborel\ T))\ (\lambda\_.\ (\prod j{\in}Basis.\ |c\ j|))$
using c **by** (*auto intro*!: *always_eventually prod_pos completion_density_eq simp*: *distr_completion cong*: *distr_cong*)
also have $\ldots = density\ (distr\ lebesgue\ lebesgue\ T)\ (\lambda\_.\ (\prod j{\in}Basis.\ |c\ j|))$
by (*subst completion.completion_distr_eq*) (*auto simp*: *eq measurable_completion*)
finally show $lebesgue = density\ (distr\ lebesgue\ lebesgue\ T)\ (\lambda\_.\ (\prod j{\in}Basis.\ |c\ j|))$ **.**
qed

corollary *lebesgue_real_affine*:
$c \neq 0 \Longrightarrow lebesgue = density\ (distr\ lebesgue\ lebesgue\ (\lambda x.\ t + c * x))\ (\lambda\_.\ ennreal\ (abs\ c))$
using *lebesgue_affine_euclidean* [**where** $c{=}\ \lambda x{::}real.\ c$] **by** *simp*

lemma *nn_integral_real_affine_lebesgue*:
fixes $c :: real$ **assumes** f[*measurable*]: $f \in borel\_measurable\ lebesgue$ **and** c: $c \neq 0$
shows $(\int^+ x.\ f\ x\ \partial lebesgue) = ennreal|c| * (\int^+ x.\ f(t + c * x)\ \partial lebesgue)$
proof −
have $(\int^+ x.\ f\ x\ \partial lebesgue) = (\int^+ x.\ f\ x\ \partial density\ (distr\ lebesgue\ lebesgue\ (\lambda x.\ t + c * x))\ (\lambda x.\ ennreal\ |c|))$

```
    using lebesgue_real_affine c by auto
  also have ... = ∫⁺ x. ennreal |c| * f x ∂distr lebesgue lebesgue (λx. t + c * x)
    by (subst nn_integral_density) auto
  also have ... = ennreal |c| * integral^N (distr lebesgue lebesgue (λx. t + c * x))
f
    using f measurable_distr_eq1 nn_integral_cmult by blast
  also have ... = |c| * (∫⁺x. f(t + c * x) ∂lebesgue)
    using lebesgue_affine_measurable[where c= λx::real. c]
    by (subst nn_integral_distr) (force+)
  finally show ?thesis .
qed

lemma lebesgue_measurable_scaling[measurable]: (*R) x ∈ lebesgue →M lebesgue
proof cases
  assume x = 0
  then have (*R) x = (λx. 0::'a)
    by (auto simp: fun_eq_iff)
  then show ?thesis by auto
next
  assume x ≠ 0 then show ?thesis
    using lebesgue_affine_measurable[of λ_. x 0]
   unfolding scaleR_scaleR[symmetric] scaleR_sum_right[symmetric] euclidean_representation
    by (auto simp add: ac_simps)
qed

lemma
  fixes m :: real and δ :: 'a::euclidean_space
  defines T r d x ≡ r *R x + d
  shows emeasure_lebesgue_affine: emeasure lebesgue (T m δ ' S) = |m| ^ DIM('a)
* emeasure lebesgue S (is ?e)
    and measure_lebesgue_affine: measure lebesgue (T m δ ' S) = |m| ^ DIM('a)
* measure lebesgue S (is ?m)
proof −
  show ?e
  proof cases
    assume m = 0 then show ?thesis
      by (simp add: image_constant_conv T_def[abs_def])
  next
    let ?T = T m δ and ?T' = T (1 / m) (− ((1/m) *R δ))
    assume m ≠ 0
    then have s_comp_s: ?T' ∘ ?T = id ?T ∘ ?T' = id
    by (auto simp: T_def[abs_def] fun_eq_iff scaleR_add_right scaleR_diff_right)
    then have inv ?T' = ?T bij ?T'
      by (auto intro: inv_unique_comp o_bij)
    then have eq: T m δ ' S = T (1 / m) ((−1/m) *R δ) −' S ∩ space lebesgue
      using bij_vimage_eq_inv_image[OF ‹bij ?T'›, of S] by auto

    have trans_eq_T: (λx. δ + (∑j∈Basis. (m * (x • j)) *R j)) = T m δ for m δ
     unfolding T_def[abs_def] scaleR_scaleR[symmetric] scaleR_sum_right[symmetric]
```

```
    by (auto simp add: euclidean_representation ac_simps)

  have T[measurable]: T r d ∈ lebesgue →M lebesgue for r d
    using lebesgue_affine_measurable[of λ_. r d]
    by (cases r = 0) (auto simp: trans_eq_T T_def[abs_def])

  show ?thesis
  proof cases
    assume S ∈ sets lebesgue with ‹m ≠ 0› show ?thesis
      unfolding eq
      apply (subst lebesgue_affine_euclidean[of λ_. m δ])
       apply (simp_all add: emeasure_density trans_eq_T nn_integral_cmult
emeasure_distr
                    del: space_completion emeasure_completion)
      apply (simp add: vimage_comp s_comp_s)
      done
  next
    assume S ∉ sets lebesgue
    moreover have ?T ‘ S ∉ sets lebesgue
    proof
      assume ?T ‘ S ∈ sets lebesgue
      then have ?T −‘ (?T ‘ S) ∩ space lebesgue ∈ sets lebesgue
        by (rule measurable_sets[OF T])
      also have ?T −‘ (?T ‘ S) ∩ space lebesgue = S
        by (simp add: vimage_comp s_comp_s eq)
      finally show False using ‹S ∉ sets lebesgue› by auto
    qed
    ultimately show ?thesis
      by (simp add: emeasure_notin_sets)
  qed
 qed
 show ?m
  unfolding measure_def ‹?e› by (simp add: enn2real_mult prod_nonneg)
qed

lemma lebesgue_real_scale:
 assumes c ≠ 0
 shows   lebesgue = density (distr lebesgue lebesgue (λx. c * x)) (λx. ennreal |c|)
 using assms by (subst lebesgue_affine_euclidean[of λ_. c 0]) simp_all

lemma lborel_has_bochner_integral_real_affine_iff:
 fixes x :: 'a :: {banach, second_countable_topology}
 shows c ≠ 0 ⟹
   has_bochner_integral lborel f x ⟷
   has_bochner_integral lborel (λx. f (t + c * x)) (x /R |c|)
 unfolding has_bochner_integral_iff lborel_integrable_real_affine_iff
  by (simp_all add: lborel_integral_real_affine[symmetric] divideR_right cong:
conj_cong)
```

lemma *lborel_distr_uminus*: *distr lborel borel uminus* = (*lborel* :: *real measure*)
by (*subst lborel_real_affine*[*of* −*1 0*])
(*auto simp*: *density_1 one_ennreal_def*[*symmetric*])

lemma *lborel_distr_mult*:
assumes (*c*::*real*) ≠ *0*
shows *distr lborel borel* ((∗) *c*) = *density lborel* (λ_. *inverse* |*c*|)
proof−
have *distr lborel borel* ((∗) *c*) = *distr lborel lborel* ((∗) *c*) **by** (*simp cong*: *distr_cong*)
also from *assms* **have** ... = *density lborel* (λ_. *inverse* |*c*|)
by (*subst lborel_real_affine*[*of inverse c 0*]) (*auto simp*: *o_def distr_density_distr*)
finally show *?thesis* **.**
qed

lemma *lborel_distr_mult′*:
assumes (*c*::*real*) ≠ *0*
shows *lborel* = *density* (*distr lborel borel* ((∗) *c*)) (λ_. |*c*|)
proof−
have *lborel* = *density lborel* (λ_. *1*) **by** (*rule density_1*[*symmetric*])
also from *assms* **have** (λ_. *1* :: *ennreal*) = (λ_. *inverse* |*c*| ∗ |*c*|) **by** (*intro ext*)
simp
also have *density lborel* ... = *density* (*density lborel* (λ_. *inverse* |*c*|)) (λ_. |*c*|)
by (*subst density_density_eq*) (*auto simp*: *ennreal_mult*)
also from *assms* **have** *density lborel* (λ_. *inverse* |*c*|) = *distr lborel borel* ((∗) *c*)
by (*rule lborel_distr_mult*[*symmetric*])
finally show *?thesis* **.**
qed

lemma *lborel_distr_plus*:
fixes *c* :: *′a*::*euclidean_space*
shows *distr lborel borel* ((+) *c*) = *lborel*
by (*subst lborel_affine*[*of 1 c*], *auto simp*: *density_1*)

interpretation *lborel*: *sigma_finite_measure lborel*
by (*rule sigma_finite_lborel*)

interpretation *lborel_pair*: *pair_sigma_finite lborel lborel* **..**

lemma *lborel_prod*:
lborel ⨂$_M$ *lborel* = (*lborel* :: (*′a*::*euclidean_space* × *′b*::*euclidean_space*) *measure*)
proof (*rule lborel_eqI*[*symmetric*], *clarify*)
fix *la ua* :: *′a* **and** *lb ub* :: *′b*
assume *lu*: ⋀*a b*. (*a*, *b*) ∈ *Basis* ⟹ (*la*, *lb*) · (*a*, *b*) ≤ (*ua*, *ub*) · (*a*, *b*)
have [*simp*]:
⋀*b*. *b* ∈ *Basis* ⟹ *la* · *b* ≤ *ua* · *b*
⋀*b*. *b* ∈ *Basis* ⟹ *lb* · *b* ≤ *ub* · *b*
inj_on (λ*u*. (*u*, *0*)) *Basis inj_on* (λ*u*. (*0*, *u*)) *Basis*
(λ*u*. (*u*, *0*)) ‘ *Basis* ∩ (λ*u*. (*0*, *u*)) ‘ *Basis* = {}

box (*la*, *lb*) (*ua*, *ub*) = *box la ua* × *box lb ub*
using *lu*[*of* _ *0*] *lu*[*of 0*] **by** (*auto intro*!: *inj_onI simp add*: *Basis_prod_def ball_Un box_def*)
show *emeasure* (*lborel* ⨂$_M$ *lborel*) (*box* (*la*, *lb*) (*ua*, *ub*)) =
ennreal (*prod* ((·) ((*ua*, *ub*) − (*la*, *lb*))) *Basis*)
by (*simp add*: *lborel.emeasure_pair_measure_Times Basis_prod_def prod.union_disjoint prod.reindex ennreal_mult inner_diff_left prod_nonneg*)
qed (*simp add*: *borel_prod*[*symmetric*])

lemma *lborelD_Collect*[*measurable* (*raw*)]: {*x*∈*space borel*. *P x*} ∈ *sets borel* ⟹ {*x*∈*space lborel*. *P x*} ∈ *sets lborel*
by *simp*

lemma *lborelD*[*measurable* (*raw*)]: *A* ∈ *sets borel* ⟹ *A* ∈ *sets lborel*
by *simp*

lemma *emeasure_bounded_finite*:
assumes *bounded A* **shows** *emeasure lborel A* < ∞
proof −
obtain *a b* **where** *A* ⊆ *cbox a b*
by (*meson bounded_subset_cbox_symmetric* ‹*bounded A*›)
then have *emeasure lborel A* ≤ *emeasure lborel* (*cbox a b*)
by (*intro emeasure_mono*) *auto*
then show *?thesis*
by (*auto simp*: *emeasure_lborel_cbox_eq prod_nonneg less_top*[*symmetric*] *top_unique split*: *if_split_asm*)
qed

lemma *emeasure_compact_finite*: *compact A* ⟹ *emeasure lborel A* < ∞
using *emeasure_bounded_finite*[*of A*] **by** (*auto intro*: *compact_imp_bounded*)

lemma *borel_integrable_compact*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *compact S continuous_on S f*
shows *integrable lborel* (λ*x*. *indicator S x* ∗$_R$ *f x*)
proof *cases*
assume *S* ≠ {}
have *continuous_on S* (λ*x*. *norm* (*f x*))
using *assms* **by** (*intro continuous_intros*)
from *continuous_attains_sup*[*OF* ‹*compact S*› ‹*S* ≠ {}› *this*]
obtain *M* **where** *M*: ⋀*x*. *x* ∈ *S* ⟹ *norm* (*f x*) ≤ *M*
by *auto*
show *?thesis*
proof (*rule integrable_bound*)
show *integrable lborel* (λ*x*. *indicator S x* ∗ *M*)
using *assms* **by** (*auto intro*!: *emeasure_compact_finite borel_compact integrable_mult_left*)
show (λ*x*. *indicator S x* ∗$_R$ *f x*) ∈ *borel_measurable lborel*

```
      using assms by (auto intro!: borel_measurable_continuous_on_indicator
borel_compact)
    show AE x in lborel. norm (indicator S x *R f x) ≤ norm (indicator S x * M)
      by (auto split: split_indicator simp: abs_real_def dest!: M)
  qed
qed simp

lemma borel_integrable_atLeastAtMost:
  fixes f :: real ⇒ real
  assumes f: ⋀x. a ≤ x ⟹ x ≤ b ⟹ isCont f x
  shows integrable lborel (λx. f x * indicator {a .. b} x) (is integrable _ ?f)
proof −
  have integrable lborel (λx. indicator {a .. b} x *R f x)
  proof (rule borel_integrable_compact)
    from f show continuous_on {a..b} f
      by (auto intro: continuous_at_imp_continuous_on)
  qed simp
  then show ?thesis
    by (auto simp: mult.commute)
qed
```

7.12.6 Lebesgue measurable sets

```
abbreviation lmeasurable :: 'a::euclidean_space set set
where
  lmeasurable ≡ fmeasurable lebesgue

lemma not_measurable_UNIV [simp]: UNIV ∉ lmeasurable
  by (simp add: fmeasurable_def)

lemma lmeasurable_iff_integrable:
  S ∈ lmeasurable ⟷ integrable lebesgue (indicator S :: 'a::euclidean_space ⇒
real)
 by (auto simp: fmeasurable_def integrable_iff_bounded borel_measurable_indicator_iff
ennreal_indicator)

lemma lmeasurable_cbox [iff]: cbox a b ∈ lmeasurable
  and lmeasurable_box [iff]: box a b ∈ lmeasurable
 by (auto simp: fmeasurable_def emeasure_lborel_box_eq emeasure_lborel_cbox_eq)

lemma
  fixes a::real
  shows lmeasurable_interval [iff]: {a..b} ∈ lmeasurable {a<..<b} ∈ lmeasurable
  by (metis box_real lmeasurable_box lmeasurable_cbox)+

lemma fmeasurable_compact: compact S ⟹ S ∈ fmeasurable lborel
 using emeasure_compact_finite[of S] by (intro fmeasurableI) (auto simp: borel_compact)

lemma lmeasurable_compact: compact S ⟹ S ∈ lmeasurable
```

using *fmeasurable_compact* **by** (*force simp*: *fmeasurable_def*)

lemma *measure_frontier*:
bounded S $\Longrightarrow$ *measure lebesgue* (*frontier S*) = *measure lebesgue* (*closure S*) − *measure lebesgue* (*interior S*)
using *closure_subset interior_subset*
by (*auto simp*: *frontier_def fmeasurable_compact intro*!: *measurable_measure_Diff*)

lemma *lmeasurable_closure*:
bounded S $\Longrightarrow$ *closure S* $\in$ *lmeasurable*
by (*simp add*: *lmeasurable_compact*)

lemma *lmeasurable_frontier*:
bounded S $\Longrightarrow$ *frontier S* $\in$ *lmeasurable*
by (*simp add*: *compact_frontier_bounded lmeasurable_compact*)

lemma *lmeasurable_open*: *bounded S* $\Longrightarrow$ *open S* $\Longrightarrow$ *S* $\in$ *lmeasurable*
using *emeasure_bounded_finite*[*of S*] **by** (*intro fmeasurableI*) (*auto simp*: *borel_open*)

lemma *lmeasurable_ball* [*simp*]: *ball a r* $\in$ *lmeasurable*
by (*simp add*: *lmeasurable_open*)

lemma *lmeasurable_cball* [*simp*]: *cball a r* $\in$ *lmeasurable*
by (*simp add*: *lmeasurable_compact*)

lemma *lmeasurable_interior*: *bounded S* $\Longrightarrow$ *interior S* $\in$ *lmeasurable*
by (*simp add*: *bounded_interior lmeasurable_open*)

lemma *null_sets_cbox_Diff_box*: *cbox a b* − *box a b* $\in$ *null_sets lborel*
by (*simp add*: *emeasure_Diff emeasure_lborel_box_eq emeasure_lborel_cbox_eq null_setsI subset_box*)

lemma *bounded_set_imp_lmeasurable*:
assumes *bounded S S* $\in$ *sets lebesgue* **shows** *S* $\in$ *lmeasurable*
by (*metis assms bounded_Un emeasure_bounded_finite emeasure_completion fmeasurableI main_part_null_part_Un*)

lemma *finite_measure_lebesgue_on*: *S* $\in$ *lmeasurable* $\Longrightarrow$ *finite_measure* (*lebesgue_on S*)
by (*auto simp*: *finite_measureI fmeasurable_def emeasure_restrict_space*)

lemma *integrable_const_ivl* [*iff*]:
fixes *a*::′*a*::*ordered_euclidean_space*
shows *integrable* (*lebesgue_on* {*a*..*b*}) (λ*x*. *c*)
by (*metis cbox_interval finite_measure.integrable_const finite_measure_lebesgue_on lmeasurable_cbox*)

7.12.7 Translation preserves Lebesgue measure

lemma *sigma_sets_image*:
assumes *S*: $S \in$ *sigma_sets* Ω M **and** $M \subseteq$ *Pow* Ω f ‘ $\Omega = \Omega$ *inj_on* f Ω
and *M*: $\bigwedge y.\ y \in M \Longrightarrow f$ ‘ $y \in M$
shows (f ‘ S) $\in$ *sigma_sets* Ω M
using *S*
proof (*induct S rule*: *sigma_sets.induct*)
case (*Basic a*) **then show** *?case*
by (*simp add*: *M*)
next
case *Empty* **then show** *?case*
by (*simp add*: *sigma_sets.Empty*)
next
case (*Compl a*)
with *assms* **show** *?case*
by (*metis inj_on_image_set_diff sigma_sets.Compl sigma_sets_into_sp*)
next
case (*Union a*) **then show** *?case*
by (*metis image_UN sigma_sets.simps*)
qed

lemma *null_sets_translation*:
assumes $N \in$ *null_sets lborel* **shows** $\{x.\ x - a \in N\} \in$ *null_sets lborel*
proof −
have [*simp*]: $(\lambda x.\ x + a)$ ‘ $N = \{x.\ x - a \in N\}$
by *force*
show *?thesis*
using *assms emeasure_lebesgue_affine* [*of 1 a N*] **by** (*auto simp*: *null_sets_def*)
qed

lemma *lebesgue_sets_translation*:
fixes $f :: 'a \Rightarrow 'a$::*euclidean_space*
assumes *S*: $S \in$ *sets lebesgue*
shows $((\lambda x.\ a + x)$ ‘ $S) \in$ *sets lebesgue*
proof −
have *im_eq*: $(+)\ a$ ‘ $A = \{x.\ x - a \in A\}$ **for** A
by *force*
have $((\lambda x.\ a + x)$ ‘ $S) = ((\lambda x.\ -a + x)$ −‘ $S) \cap$ (*space lebesgue*)
using *image_iff* **by** *fastforce*
also have $\ldots \in$ *sets lebesgue*
proof (*rule measurable_sets* [*OF measurableI assms*])
fix $A :: 'b$ *set*
assume *A*: $A \in$ *sets lebesgue*
have *vim_eq*: $(\lambda x.\ x - a)$ −‘ $A = (+)\ a$ ‘ A **for** A
by *force*
have $\exists s\ n\ N'.\ (+)\ a$ ‘ $(S \cup N) = s \cup n \wedge s \in$ *sets borel* $\wedge\ N' \in$ *null_sets lborel* $\wedge\ n \subseteq N'$
if $S \in$ *sets borel* **and** $N' \in$ *null_sets lborel* **and** $N \subseteq N'$ **for** $S\ N\ N'$
proof (*intro exI conjI*)

```
    show (+) a ‘ (S ∪ N) = (λx. a + x) ‘ S ∪ (λx. a + x) ‘ N
      by auto
    show (λx. a + x) ‘ N' ∈ null_sets lborel
      using that by (auto simp: null_sets_translation im_eq)
  qed (use that im_eq in auto)
  with A have (λx. x − a) −‘ A ∈ sets lebesgue
    by (force simp: vim_eq completion_def intro!: sigma_sets_image)
  then show (+) (− a) −‘ A ∩ space lebesgue ∈ sets lebesgue
    by (auto simp: vimage_def im_eq)
 qed auto
 finally show ?thesis .
qed

lemma measurable_translation:
  S ∈ lmeasurable ⟹ ((+) a ‘ S) ∈ lmeasurable
 using emeasure_lebesgue_affine [of 1 a S]
 by (smt (verit, best) add.commute ennreal_1 fmeasurable_def image_cong lambda_one

       lebesgue_sets_translation mem_Collect_eq power_one scaleR_one)

lemma measurable_translation_subtract:
  S ∈ lmeasurable ⟹ ((λx. x − a) ‘ S) ∈ lmeasurable
 using measurable_translation [of S − a] by (simp cong: image_cong_simp)

lemma measure_translation:
 measure lebesgue ((+) a ‘ S) = measure lebesgue S
  using measure_lebesgue_affine [of 1 a S] by (simp add: ac_simps cong: im-
age_cong_simp)

lemma measure_translation_subtract:
 measure lebesgue ((λx. x − a) ‘ S) = measure lebesgue S
 using measure_translation [of − a] by (simp cong: image_cong_simp)
```

7.12.8 A nice lemma for negligibility proofs

```
lemma summable_iff_suminf_neq_top: (⋀n. f n ≥ 0) ⟹ ¬ summable f ⟹
(∑ i. ennreal (f i)) = top
 by (metis summable_suminf_not_top)

proposition starlike_negligible_bounded_gmeasurable:
 fixes S :: 'a :: euclidean_space set
 assumes S: S ∈ sets lebesgue and bounded S
    and eq1: ⋀c x. ⟦(c *R x) ∈ S; 0 ≤ c; x ∈ S⟧ ⟹ c = 1
  shows S ∈ null_sets lebesgue
proof −
 obtain M where 0 < M S ⊆ ball 0 M
  using ‹bounded S› by (auto dest: bounded_subset_ballD)

 let ?f = λn. root DIM('a) (Suc n)
```

```
  have ?f n *R x ∈ S ⟹ x ∈ (*R) (1 / ?f n) ‘ S for x n
    by (rule image_eqI[of _ _ ?f n *R x]) auto
  then have vimage_eq_image: (*R) (?f n) −‘ S = (*R) (1 / ?f n) ‘ S for n
    by auto

  have eq: (1 / ?f n) ^ DIM('a) = 1 / Suc n for n
    by (simp add: field_simps)

  { fix n x assume x: root DIM('a) (1 + real n) *R x ∈ S
    have 1 * norm x ≤ root DIM('a) (1 + real n) * norm x
      by (rule mult_mono) auto
    also have ... < M
      using x ‹S ⊆ ball 0 M› by auto
    finally have norm x < M by simp }
  note less_M = this

  have (∑ n. ennreal (1 / Suc n)) = top
      using not_summable_harmonic[where 'a=real] summable_Suc_iff[where
f=λn. 1 / (real n)]
    by (intro summable_iff_suminf_neq_top) (auto simp add: inverse_eq_divide)
  then have top * emeasure lebesgue S = (∑ n. (1 / ?f n)^DIM('a) * emeasure
lebesgue S)
    unfolding ennreal_suminf_multc eq by simp
  also have ... = (∑ n. emeasure lebesgue ((*R) (?f n) −‘ S))
    unfolding vimage_eq_image using emeasure_lebesgue_affine[of 1 / ?f n 0 S
for n] by simp
  also have ... = emeasure lebesgue (⋃ n. (*R) (?f n) −‘ S)
  proof (intro suminf_emeasure)
    show disjoint_family (λn. (*R) (?f n) −‘ S)
      unfolding disjoint_family_on_def
    proof safe
      fix m n :: nat and x assume m ≠ n ?f m *R x ∈ S ?f n *R x ∈ S
      with eq1[of ?f m / ?f n ?f n *R x] show x ∈ {}
        by auto
    qed
    have (*R) (?f i) −‘ S ∈ sets lebesgue for i
      using measurable_sets[OF lebesgue_measurable_scaling[of ?f i] S] by auto
    then show range (λi. (*R) (?f i) −‘ S) ⊆ sets lebesgue
      by auto
  qed
  also have ... ≤ emeasure lebesgue (ball 0 M :: 'a set)
    using less_M by (intro emeasure_mono) auto
  also have ... < top
    using lmeasurable_ball by (auto simp: fmeasurable_def)
  finally have emeasure lebesgue S = 0
    by (simp add: ennreal_top_mult split: if_split_asm)
  then show S ∈ null_sets lebesgue
    unfolding null_sets_def using ‹S ∈ sets lebesgue› by auto
```

qed

corollary *starlike_negligible_compact*:
compact S $\Longrightarrow$ ($\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \le c;\ x \in S]\!] \Longrightarrow c = 1$) $\Longrightarrow$ *S* $\in$ *null_sets lebesgue*
using *starlike_negligible_bounded_gmeasurable*[*of S*] **by** (*auto simp*: *compact_eq_bounded_closed*)

proposition *outer_regular_lborel_le*:
assumes *B*[*measurable*]: *B* $\in$ *sets borel* **and** *0* $<$ (*e*::*real*)
obtains *U* **where** *open U B* $\subseteq$ *U* **and** *emeasure lborel* (*U* $-$ *B*) $\le$ *e*
proof $-$
let *?μ* = *emeasure lborel*
let *?B* = λ*n*::*nat*. *ball 0 n* :: *'a set*
let *?e* = λ*n*. *e*$*$((*1*/*2*)^Suc *n*)
have $\forall$ *n*. $\exists$ *U*. *open U* $\wedge$ *?B n* $\cap$ *B* $\subseteq$ *U* $\wedge$ *?μ* (*U* $-$ *B*) $<$ *?e n*
proof
fix *n* :: *nat*
let *?A* = *density lborel* (*indicator* (*?B n*))
have *emeasure_A*: *X* $\in$ *sets borel* $\Longrightarrow$ *emeasure ?A X* = *?μ* (*?B n* $\cap$ *X*) **for** *X*
by (*auto simp*: *emeasure_density borel_measurable_indicator indicator_inter_arith*[*symmetric*])

have *finite_A*: *emeasure ?A* (*space ?A*) $\neq \infty$
using *emeasure_bounded_finite*[*of ?B n*] **by** (*auto simp*: *emeasure_A*)
interpret *A*: *finite_measure ?A*
by *rule fact*
have *emeasure ?A B* + *?e n* $>$ (*INF U*$\in${*U*. *B* $\subseteq$ *U* $\wedge$ *open U*}. *emeasure ?A U*)
using ‹*0*<*e*› **by** (*auto simp*: *outer_regular*[*OF _ finite_A B*, *symmetric*])
then obtain *U* **where** *U*: *B* $\subseteq$ *U open U* **and** *muU*: *?μ* (*?B n* $\cap$ *B*) + *?e n* $>$ *?μ* (*?B n* $\cap$ *U*)
unfolding *INF_less_iff* **by** (*auto simp*: *emeasure_A*)
moreover
{ have *?μ* ((*?B n* $\cap$ *U*) $-$ *B*) = *?μ* ((*?B n* $\cap$ *U*) $-$ (*?B n* $\cap$ *B*))
using *U* **by** (*intro arg_cong*[**where** *f*=*?μ*]) *auto*
also have ... = *?μ* (*?B n* $\cap$ *U*) $-$ *?μ* (*?B n* $\cap$ *B*)
using *U A.emeasure_finite*[*of B*]
by (*intro emeasure_Diff*) (*auto simp del*: *A.emeasure_finite simp*: *emeasure_A*)
also have ... $<$ *?e n*
using *U muU A.emeasure_finite*[*of B*]
by (*subst minus_less_iff_ennreal*)
(*auto simp del*: *A.emeasure_finite simp*: *emeasure_A less_top ac_simps intro*!: *emeasure_mono*)
finally have *?μ* ((*?B n* $\cap$ *U*) $-$ *B*) $<$ *?e n* **. }**
ultimately show $\exists$ *U*. *open U* $\wedge$ *?B n* $\cap$ *B* $\subseteq$ *U* $\wedge$ *?μ* (*U* $-$ *B*) $<$ *?e n*
by (*intro exI*[*of _ ?B n* $\cap$ *U*]) *auto*
qed
then obtain *U*
where *U*: $\bigwedge$*n*. *open* (*U n*) $\bigwedge$*n*. *?B n* $\cap$ *B* $\subseteq$ *U n* $\bigwedge$*n*. *?μ* (*U n* $-$ *B*) $<$ *?e n*

```
    by metis
  show ?thesis
  proof
    { fix x assume x ∈ B
      moreover
      obtain n where norm x < real n
        using reals_Archimedean2 by blast
      ultimately have x ∈ (⋃ n. U n)
        using U(2)[of n] by auto }
    note * = this
    then show open (⋃ n. U n) B ⊆ (⋃ n. U n)
      using U by auto
    have ?μ (⋃ n. U n − B) ≤ (∑ n. ?μ (U n − B))
      using U(1) by (intro emeasure_subadditive_countably) auto
    also have ... ≤ (∑ n. ennreal (?e n))
      using U(3) by (intro suminf_le) (auto intro: less_imp_le)
    also have ... = ennreal (e * 1)
      using ‹0<e› by (intro suminf_ennreal_eq sums_mult power_half_series)
auto
    finally show emeasure lborel ((⋃ n. U n) − B) ≤ ennreal e
      by simp
  qed
qed

lemma outer_regular_lborel:
  assumes B: B ∈ sets borel and 0 < (e::real)
  obtains U where open U B ⊆ U emeasure lborel (U − B) < e
proof −
  obtain U where U: open U B ⊆ U and emeasure lborel (U−B) ≤ e/2
    using outer_regular_lborel_le [OF B, of e/2] ‹e > 0›
    by force
  moreover have ennreal (e/2) < ennreal e
    using ‹e > 0› by (simp add: ennreal_lessI)
  ultimately have emeasure lborel (U−B) < e
    by auto
  with U show ?thesis
    using that by auto
qed

lemma completion_upper:
  assumes A: A ∈ sets (completion M)
  obtains A' where A ⊆ A' A' ∈ sets M A' − A ∈ null_sets (completion M)
                   emeasure (completion M) A = emeasure M A'
proof −
 from AE_notin_null_part[OF A] obtain N where N: N ∈ null_sets M null_part
M A ⊆ N
    by (meson assms null_part)
  let ?A' = main_part M A ∪ N
  show ?thesis
```

```
  proof
    show A ⊆ ?A'
      using ‹null_part M A ⊆ N› assms main_part_null_part_Un by blast
    have main_part M A ⊆ A
      using assms main_part_null_part_Un by auto
    then have ?A' − A ⊆ N
      by blast
    with N show ?A' − A ∈ null_sets (completion M)
      by (blast intro: null_sets_completionI completion.complete_measure_axioms
complete_measure.complete2)
    show emeasure (completion M) A = emeasure M (main_part M A ∪ N)
      using A ‹N ∈ null_sets M› by (simp add: emeasure_Un_null_set)
  qed (use A N in auto)
qed

lemma sets_lebesgue_outer_open:
  fixes e::real
  assumes S: S ∈ sets lebesgue and e > 0
  obtains T where open T S ⊆ T (T − S) ∈ lmeasurable emeasure lebesgue (T
− S) < ennreal e
proof −
  obtain S' where S': S ⊆ S' S' ∈ sets borel
            and null: S' − S ∈ null_sets lebesgue
            and em: emeasure lebesgue S = emeasure lborel S'
    using completion_upper[of S lborel] S by auto
  then have f_S': S' ∈ sets borel
    using S by (auto simp: fmeasurable_def)
  with outer_regular_lborel[OF _ ‹0<e›]
  obtain U where U: open U S' ⊆ U emeasure lborel (U − S') < e
    by blast
  show thesis
  proof
    show open U S ⊆ U
      using f_S' U S' by auto
  have (U − S) = (U − S') ∪ (S' − S)
    using S' U by auto
  then have eq: emeasure lebesgue (U − S) = emeasure lborel (U − S')
    using null  by (simp add: U(1) emeasure_Un_null_set f_S' sets.Diff)
  have (U − S) ∈ sets lebesgue
    by (simp add: S U(1) sets.Diff)
  then show (U − S) ∈ lmeasurable
    unfolding fmeasurable_def using U(3) eq less_le_trans by fastforce
  with eq U show emeasure lebesgue (U − S) < e
    by (simp add: eq)
  qed
qed

lemma sets_lebesgue_inner_closed:
  fixes e::real
```

assumes $S \in$ *sets lebesgue* $e > 0$
obtains T **where** *closed* T $T \subseteq S$ $S-T \in$ *lmeasurable* *emeasure lebesgue* $(S - T) <$ *ennreal* e
proof −
have $-S \in$ *sets lebesgue*
using *assms* **by** (*simp add*: *Compl_in_sets_lebesgue*)
then obtain T **where** *open* T $-S \subseteq T$
and T: $(T - -S) \in$ *lmeasurable* *emeasure lebesgue* $(T - -S) < e$
using *sets_lebesgue_outer_open assms* **by** *blast*
show *thesis*
proof
show *closed* $(-T)$
using ‹*open* T› **by** *blast*
show $-T \subseteq S$
using ‹$- S \subseteq T$› **by** *auto*
show $S - (-T) \in$ *lmeasurable* *emeasure lebesgue* $(S - (- T)) < e$
using T **by** (*auto simp*: *Int_commute*)
qed
qed

lemma *lebesgue_openin*:
⟦*openin* (*top_of_set* S) T; $S \in$ *sets lebesgue*⟧ ⟹ $T \in$ *sets lebesgue*
by (*metis borel_open openin_open sets.Int sets_completionI_sets sets_lborel*)

lemma *lebesgue_closedin*:
⟦*closedin* (*top_of_set* S) T; $S \in$ *sets lebesgue*⟧ ⟹ $T \in$ *sets lebesgue*
by (*metis borel_closed closedin_closed sets.Int sets_completionI_sets sets_lborel*)

7.12.9 *F_sigma* and *G_delta* sets.

inductive *fsigma* :: *'a*::*topological_space set* ⇒ *bool* **where**
(⋀*n*::*nat*. *closed* (*F n*)) ⟹ *fsigma* (⋃(*F* ‘ *UNIV*))

inductive *gdelta* :: *'a*::*topological_space set* ⇒ *bool* **where**
(⋀*n*::*nat*. *open* (*F n*)) ⟹ *gdelta* (⋂(*F* ‘ *UNIV*))

lemma *fsigma_UNIV* [*iff*]: *fsigma* (*UNIV* :: *'a*::*real_inner set*)
proof −
have (*UNIV* ::*'a set*) = (⋃ *i*. *cball 0* (*of_nat i*))
by (*auto simp*: *real_arch_simple*)
then show *?thesis*
by (*metis closed_cball fsigma.intros*)
qed

lemma *fsigma_Union_compact*:
fixes S :: *'a*::{*real_normed_vector*,*heine_borel*} *set*
shows *fsigma* $S \longleftrightarrow$ (∃ *F*::*nat* ⇒ *'a set*. *range* $F \subseteq$ *Collect compact* ∧ $S = \bigcup$(*F* ‘ *UNIV*))
proof *safe*

```
  assume fsigma S
  then obtain F :: nat ⇒ 'a set where F: range F ⊆ Collect closed S = ⋃(F ‘ UNIV)
    by (meson fsigma.cases image_subsetI mem_Collect_eq)
  then have ∃ D::nat ⇒ 'a set. range D ⊆ Collect compact ∧ ⋃(D ‘ UNIV) = F i for i
    using closed_Union_compact_subsets [of F i]
    by (metis image_subsetI mem_Collect_eq range_subsetD)
  then obtain D :: nat ⇒ nat ⇒ 'a set
    where D: ⋀i. range (D i) ⊆ Collect compact ∧ ⋃((D i) ‘ UNIV) = F i
    by metis
  let ?DD = λn. (λ(i,j). D i j) (prod_decode n)
  show ∃ F::nat ⇒ 'a set. range F ⊆ Collect compact ∧ S = ⋃(F ‘ UNIV)
  proof (intro exI conjI)
    show range ?DD ⊆ Collect compact
      using D by clarsimp (metis mem_Collect_eq rangeI split_conv subsetCE surj_pair)
    show S = ⋃ (range ?DD)
    proof
      show S ⊆ ⋃ (range ?DD)
        using D F
      by clarsimp (metis UN_iff old.prod.case prod_decode_inverse prod_encode_eq)
      show ⋃ (range ?DD) ⊆ S
        using D F  by fastforce
    qed
  qed
next
  fix F :: nat ⇒ 'a set
  assume range F ⊆ Collect compact and S = ⋃(F ‘ UNIV)
  then show fsigma (⋃(F ‘ UNIV))
    by (simp add: compact_imp_closed fsigma.intros image_subset_iff)
qed

lemma gdelta_imp_fsigma: gdelta S ⟹ fsigma (− S)
proof (induction rule: gdelta.induct)
  case (1 F)
  have − ⋂(F ‘ UNIV) = (⋃ i. −(F i))
    by auto
  then show ?case
    by (simp add: fsigma.intros closed_Compl 1)
qed

lemma fsigma_imp_gdelta: fsigma S ⟹ gdelta (− S)
proof (induction rule: fsigma.induct)
  case (1 F)
  have − ⋃(F ‘ UNIV) = (⋂ i. −(F i))
    by auto
  then show ?case
    by (simp add: 1 gdelta.intros open_closed)
```

qed

lemma *gdelta_complement*: *gdelta*(− *S*) ⟷ *fsigma S*
using *fsigma_imp_gdelta gdelta_imp_fsigma* **by** *force*

lemma *lebesgue_set_almost_fsigma*:
assumes *S* ∈ *sets lebesgue*
obtains *C T* **where** *fsigma C T* ∈ *null_sets lebesgue C* ∪ *T* = *S disjnt C T*
proof −
{ fix *n*::*nat*
obtain *T* **where** *closed T T* ⊆ *S S*−*T* ∈ *lmeasurable emeasure lebesgue* (*S* − *T*) < *ennreal* (*1* / *Suc n*)
using *sets_lebesgue_inner_closed* [*OF assms*]
by (*metis of_nat_0_less_iff zero_less_Suc zero_less_divide_1_iff*)
then have ∃ *T*. *closed T* ∧ *T* ⊆ *S* ∧ *S* − *T* ∈ *lmeasurable* ∧ *measure lebesgue* (*S*−*T*) < *1* / *Suc n*
by (*metis emeasure_eq_measure2 ennreal_leI not_le*)
}
then obtain *F* **where** *F*: ⋀*n*::*nat*. *closed* (*F n*) ∧ *F n* ⊆ *S* ∧ *S* − *F n* ∈ *lmeasurable* ∧ *measure lebesgue* (*S* − *F n*) < *1* / *Suc n*
by *metis*
let *?C* = ⋃(*F* ‘ *UNIV*)
show *thesis*
proof
show *fsigma ?C*
using *F* **by** (*simp add*: *fsigma.intros*)
show (*S* − *?C*) ∈ *null_sets lebesgue*
proof (*clarsimp simp add*: *completion.null_sets_outer_le*)
fix *e* :: *real*
assume *0* < *e*
then obtain *n* **where** *n*: *1* / *Suc n* < *e*
using *nat_approx_posE* **by** *metis*
show ∃ *T* ∈ *lmeasurable*. *S* − (⋃*x*. *F x*) ⊆ *T* ∧ *measure lebesgue T* ≤ *e*
proof (*intro bexI conjI*)
show *measure lebesgue* (*S* − *F n*) ≤ *e*
by (*meson F n less_trans not_le order.asym*)
qed (*use F* **in** *auto*)
qed
show *?C* ∪ (*S* − *?C*) = *S*
using *F* **by** *blast*
show *disjnt ?C* (*S* − *?C*)
by (*auto simp*: *disjnt_def*)
qed
qed

lemma *lebesgue_set_almost_gdelta*:
assumes *S* ∈ *sets lebesgue*
obtains *C T* **where** *gdelta C T* ∈ *null_sets lebesgue S* ∪ *T* = *C disjnt S T*
proof −

have $-S \in$ *sets lebesgue*
using *assms Compl_in_sets_lebesgue* **by** *blast*
then obtain *C T* **where** *C*: *fsigma C T* $\in$ *null_sets lebesgue C* $\cup$ *T* $= -S$ *disjnt C T*
using *lebesgue_set_almost_fsigma* **by** *metis*
show *thesis*
proof
show *gdelta* $(-C)$
by (*simp add*: ‹*fsigma C*› *fsigma_imp_gdelta*)
show $S \cup T = -C$ *disjnt S T*
using *C* **by** (*auto simp*: *disjnt_def*)
qed (*use C* **in** *auto*)
qed

end

7.13 Tagged Divisions for Henstock-Kurzweil Integration

theory *Tagged_Division*
imports *Topology_Euclidean_Space*
begin

lemma *sum_Sigma_product*:
assumes *finite S*
and $\bigwedge i.\ i \in S \Longrightarrow$ *finite* $(T\ i)$
shows $(\sum i{\in}S.\ sum\ (x\ i)\ (T\ i)) = (\sum (i,\ j){\in}Sigma\ S\ T.\ x\ i\ j)$
using *assms*
by *induction* (*auto simp*: *sum.Sigma*)

lemmas *scaleR_simps* = *scaleR_zero_left scaleR_minus_left scaleR_left_diff_distrib scaleR_zero_right scaleR_minus_right scaleR_right_diff_distrib scaleR_eq_0_iff scaleR_cancel_left scaleR_cancel_right scaleR_add_right scaleR_add_left real_vector_class.scaleR_*

7.13.1 Some useful lemmas about intervals

lemma *interior_subset_union_intervals*:
fixes *a b c d*
defines *i* $\equiv$ *cbox a b*
defines *j* $\equiv$ *cbox c d*
assumes *interior j* $\neq \{\}$
and $i \subseteq j \cup S$
and *interior i* $\cap$ *interior j* $= \{\}$
shows *interior i* $\subseteq$ *interior S*
by (*smt* (*verit*, *del_insts*) *IntI Int_interval_mixed_eq_empty UnE assms empty_iff interior_cbox interior_maximal interior_subset open_interior subset_eq*)

lemma *interior_Union_subset_cbox*:

assumes *finite* $\mathcal{F}$
assumes $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \exists\, a\ b.\ S = cbox\ a\ b\ \bigwedge S.\ S \in \mathcal{F} \Longrightarrow interior\ S \subseteq T$
and T: *closed* T
shows *interior* $(\bigcup \mathcal{F}) \subseteq T$
proof −
have *clo*[*simp*]: $S \in \mathcal{F} \Longrightarrow$ *closed* S **for** S
using $\mathcal{F}$ **by** *auto*
define E **where** $E = \{s{\in}\mathcal{F}.\ interior\ s = \{\}\}$
then have *finite* E $E \subseteq \{s{\in}\mathcal{F}.\ interior\ s = \{\}\}$
using ‹*finite* $\mathcal{F}$› **by** *auto*
then have *interior* $(\bigcup \mathcal{F}) =$ *interior* $(\bigcup (\mathcal{F} - E))$
proof (*induction* E *rule*: *finite_subset_induct'*)
case (*insert* s f')
have *interior* $(\bigcup (\mathcal{F} -$ *insert* s $f') \cup s) =$ *interior* $(\bigcup (\mathcal{F} -$ *insert* s $f'))$
using *insert.hyps* ‹*finite* $\mathcal{F}$› **by** (*intro interior_closed_Un_empty_interior*) *auto*
also have $\bigcup (\mathcal{F} -$ *insert* s $f') \cup s = \bigcup (\mathcal{F} - f')$
using *insert.hyps* **by** *auto*
finally show *?case*
by (*simp add*: *insert.IH*)
qed *simp*
also have $\ldots \subseteq \bigcup (\mathcal{F} - E)$
by (*rule interior_subset*)
also have $\ldots \subseteq T$
proof (*rule Union_least*)
fix s **assume** $s \in \mathcal{F} - E$
with $\mathcal{F}$[*of* s] **obtain** a b **where** s: $s \in \mathcal{F}$ $s = cbox\ a\ b$ $box\ a\ b \neq \{\}$
by (*fastforce simp*: *E_def*)
have *closure* (*interior* s) $\subseteq$ *closure* T
by (*intro closure_mono* $\mathcal{F}$ ‹$s \in \mathcal{F}$›)
with s ‹*closed* T› **show** $s \subseteq T$
by *simp*
qed
finally show *?thesis* **.**
qed

lemma *Int_interior_Union_intervals*:
⟦*finite* $\mathcal{F}$; *open* S; $\bigwedge T.\ T{\in}\mathcal{F} \Longrightarrow \exists\, a\ b.\ T = cbox\ a\ b$; $\bigwedge T.\ T{\in}\mathcal{F} \Longrightarrow S \cap (interior\ T) = \{\}$⟧
$\Longrightarrow S \cap interior\ (\bigcup \mathcal{F}) = \{\}$
using *interior_Union_subset_cbox*[*of* $\mathcal{F}$ *UNIV* − S] **by** *auto*

lemma *interval_split*:
fixes a :: *'a*::*euclidean_space*
assumes $k \in Basis$
shows
$cbox\ a\ b \cap \{x.\ x{\cdot}k \leq c\} = cbox\ a\ (\sum i{\in}Basis.\ (if\ i = k\ then\ min\ (b{\cdot}k)\ c\ else\ b{\cdot}i) *_R i)$
$cbox\ a\ b \cap \{x.\ x{\cdot}k \geq c\} = cbox\ (\sum i{\in}Basis.\ (if\ i = k\ then\ max\ (a{\cdot}k)\ c\ else$

$a \cdot i) *_R i)\ b$
using *assms* **by** (*rule_tac set_eqI*; *auto simp*: *mem_box*)+

lemma *interval_not_empty*: $\forall i{\in}Basis.\ a \cdot i \leq b \cdot i \Longrightarrow cbox\ a\ b \neq \{\}$
by (*simp add*: *box_ne_empty*)

7.13.2 Bounds on intervals where they exist

definition *interval_upperbound* :: ('*a*::*euclidean_space*) *set* ⇒ '*a*
where *interval_upperbound s* = $(\sum i{\in}Basis.\ (SUP\ x{\in}s.\ x \cdot i) *_R i)$

definition *interval_lowerbound* :: ('*a*::*euclidean_space*) *set* ⇒ '*a*
where *interval_lowerbound s* = $(\sum i{\in}Basis.\ (INF\ x{\in}s.\ x \cdot i) *_R i)$

lemma *interval_upperbound*[*simp*]:
$\forall i{\in}Basis.\ a \cdot i \leq b \cdot i \Longrightarrow$
interval_upperbound (*cbox a b*) = (*b*::'*a*::*euclidean_space*)
unfolding *interval_upperbound_def euclidean_representation_sum cbox_def*
by (*safe intro*!: *cSup_eq*) *auto*

lemma *interval_lowerbound*[*simp*]:
$\forall i{\in}Basis.\ a \cdot i \leq b \cdot i \Longrightarrow$
interval_lowerbound (*cbox a b*) = (*a*::'*a*::*euclidean_space*)
unfolding *interval_lowerbound_def euclidean_representation_sum cbox_def*
by (*safe intro*!: *cInf_eq*) *auto*

lemmas *interval_bounds* = *interval_upperbound interval_lowerbound*

lemma
fixes *X*::*real set*
shows *interval_upperbound_real*[*simp*]: *interval_upperbound X* = *Sup X*
and *interval_lowerbound_real*[*simp*]: *interval_lowerbound X* = *Inf X*
by (*auto simp*: *interval_upperbound_def interval_lowerbound_def*)

lemma *interval_bounds'*[*simp*]:
assumes *cbox a b* ≠ {}
shows *interval_upperbound* (*cbox a b*) = *b*
and *interval_lowerbound* (*cbox a b*) = *a*
using *assms* **unfolding** *box_ne_empty* **by** *auto*

lemma *interval_upperbound_Times*:
assumes *A* ≠ {} **and** *B* ≠ {}
shows *interval_upperbound* (*A* × *B*) = (*interval_upperbound A*, *interval_upperbound B*)
proof−
from *assms* **have** *fst_image_times'*: *A* = *fst* ‘ (*A* × *B*) **by** *simp*
have $(\sum i{\in}Basis.\ (SUP\ x{\in}A \times B.\ x \cdot (i, 0)) *_R i) = (\sum i{\in}Basis.\ (SUP\ x{\in}A.\ x \cdot i) *_R i)$
by (*subst* (*2*) *fst_image_times'*) (*simp del*: *fst_image_times add*: *image_comp*

inner_Pair_0)
moreover from *assms* **have** *snd_image_times'*: $B = snd\ `\ (A \times B)$ **by** *simp*
have $(\sum i{\in}Basis.\ (SUP\ x{\in}A \times B.\ x \cdot (0, i)) *_R i) = (\sum i{\in}Basis.\ (SUP\ x{\in}B.\ x \cdot i) *_R i)$
by (*subst* (2) *snd_image_times'*) (*simp del*: *snd_image_times add*: *image_comp inner_Pair_0*)
ultimately show *?thesis* **unfolding** *interval_upperbound_def*
by (*subst sum_Basis_prod_eq*) (*auto simp add*: *sum_prod*)
qed

lemma *interval_lowerbound_Times*:
assumes $A \neq \{\}$ **and** $B \neq \{\}$
shows *interval_lowerbound* $(A \times B) = (interval\_lowerbound\ A, interval\_lowerbound\ B)$
proof−
from *assms* **have** *fst_image_times'*: $A = fst\ `\ (A \times B)$ **by** *simp*
have $(\sum i{\in}Basis.\ (INF\ x{\in}A \times B.\ x \cdot (i, 0)) *_R i) = (\sum i{\in}Basis.\ (INF\ x{\in}A.\ x \cdot i) *_R i)$
by (*subst* (2) *fst_image_times'*) (*simp del*: *fst_image_times add*: *image_comp inner_Pair_0*)
moreover from *assms* **have** *snd_image_times'*: $B = snd\ `\ (A \times B)$ **by** *simp*
have $(\sum i{\in}Basis.\ (INF\ x{\in}A \times B.\ x \cdot (0, i)) *_R i) = (\sum i{\in}Basis.\ (INF\ x{\in}B.\ x \cdot i) *_R i)$
by (*subst* (2) *snd_image_times'*) (*simp del*: *snd_image_times add*: *image_comp inner_Pair_0*)
ultimately show *?thesis* **unfolding** *interval_lowerbound_def*
by (*subst sum_Basis_prod_eq*) (*auto simp add*: *sum_prod*)
qed

7.13.3 The notion of a gauge — simply an open set containing the point

definition $gauge\ \gamma \longleftrightarrow (\forall x.\ x \in \gamma\ x \wedge open\ (\gamma\ x))$

lemma *gaugeI*:
assumes $\bigwedge x.\ x \in \gamma\ x$ **and** $\bigwedge x.\ open\ (\gamma\ x)$
shows $gauge\ \gamma$
using *assms* **unfolding** *gauge_def* **by** *auto*

lemma *gaugeD*[*dest*]:
assumes $gauge\ \gamma$ **shows** $x \in \gamma\ x$
and $open\ (\gamma\ x)$
using *assms* **unfolding** *gauge_def* **by** *auto*

lemma *gauge_ball_dependent*: $\forall x.\ 0 < e\ x \Longrightarrow gauge\ (\lambda x.\ ball\ x\ (e\ x))$
unfolding *gauge_def* **by** *auto*

lemma *gauge_ball*[*intro*]: $0 < e \Longrightarrow gauge\ (\lambda x.\ ball\ x\ e)$
unfolding *gauge_def* **by** *auto*

lemma *gauge_trivial*[*intro*!]: *gauge* (λ*x*. *ball x 1*)
by *auto*

lemma *gauge_Int*[*intro*]: *gauge γ1* ⟹ *gauge γ2* ⟹ *gauge* (λ*x*. *γ1 x* ∩ *γ2 x*)
unfolding *gauge_def* **by** *auto*

lemma *gauge_reflect*:
fixes *γ* :: *'a*::*euclidean_space* ⇒ *'a set*
shows *gauge γ* ⟹ *gauge* (λ*x*. *uminus* ‘ *γ* (− *x*))
by (*metis* (*mono_tags*, *lifting*) *gauge_def imageI open_negations minus_minus*)

lemma *gauge_Inter*:
assumes *finite S*
and ⋀*u*. *u*∈*S* ⟹ *gauge* (*f u*)
shows *gauge* (λ*x*. ⋂{*f u x* | *u*. *u* ∈ *S*})
proof −
have ∗: ⋀*x*. {*f u x* |*u*. *u* ∈ *S*} = (λ*u*. *f u x*) ‘ *S*
by *auto*
show *?thesis*
unfolding *gauge_def* **unfolding** ∗
by (*simp add*: *assms gaugeD open_INT*)
qed

lemma *gauge_existence_lemma*:
(∀ *x*. ∃ *d* :: *real*. *p x* ⟶ *0* < *d* ∧ *q d x*) ⟷ (∀ *x*. ∃ *d*>*0*. *p x* ⟶ *q d x*)
by (*metis zero_less_one*)

7.13.4 Attempt a systematic general set of "offset" results for components

lemma *gauge_modify*:
assumes (∀ *S*. *open S* ⟶ *open* {*x*. *f*(*x*) ∈ *S*}) *gauge d*
shows *gauge* (λ*x*. {*y*. *f y* ∈ *d* (*f x*)})
using *assms* **unfolding** *gauge_def* **by** *force*

7.13.5 Divisions

definition *division_of* (**infixl** ‹*division'_of*› *40*)
where
s division_of i ⟷
finite s ∧
(∀ *K*∈*s*. *K* ⊆ *i* ∧ *K* ≠ {} ∧ (∃ *a b*. *K* = *cbox a b*)) ∧
(∀ *K1*∈*s*. ∀ *K2*∈*s*. *K1* ≠ *K2* ⟶ *interior*(*K1*) ∩ *interior*(*K2*) = {}) ∧
(⋃ *s* = *i*)

lemma *division_ofD*[*dest*]:
assumes *s division_of i*
shows *finite s*

and $\bigwedge K.\ K \in s \Longrightarrow K \subseteq i$
and $\bigwedge K.\ K \in s \Longrightarrow K \neq \{\}$
and $\bigwedge K.\ K \in s \Longrightarrow \exists a\ b.\ K = cbox\ a\ b$
and $\bigwedge K1\ K2.\ K1 \in s \Longrightarrow K2 \in s \Longrightarrow K1 \neq K2 \Longrightarrow interior(K1) \cap interior(K2) = \{\}$
and $\bigcup s = i$
using *assms* **unfolding** *division_of_def* **by** *auto*

lemma *division_ofI*:
assumes *finite s*
and $\bigwedge K.\ K \in s \Longrightarrow K \subseteq i$
and $\bigwedge K.\ K \in s \Longrightarrow K \neq \{\}$
and $\bigwedge K.\ K \in s \Longrightarrow \exists a\ b.\ K = cbox\ a\ b$
and $\bigwedge K1\ K2.\ K1 \in s \Longrightarrow K2 \in s \Longrightarrow K1 \neq K2 \Longrightarrow interior\ K1 \cap interior\ K2 = \{\}$
and $\bigcup s = i$
shows *s division_of i*
using *assms* **unfolding** *division_of_def* **by** *auto*

lemma *division_of_finite*: *s division_of i* $\Longrightarrow$ *finite s*
by *auto*

lemma *division_of_self*[*intro*]: *cbox a b* $\neq \{\} \Longrightarrow \{cbox\ a\ b\}$ *division_of* (*cbox a b*)
unfolding *division_of_def* **by** *auto*

lemma *division_of_trivial*[*simp*]: *s division_of* $\{\} \longleftrightarrow s = \{\}$
unfolding *division_of_def* **by** *auto*

lemma *division_of_sing*[*simp*]:
s division_of cbox a (*a*::'*a*::*euclidean_space*) $\longleftrightarrow s = \{cbox\ a\ a\}$
(**is** *?l* = *?r*)
proof
assume *?l*
have $x = \{a\}$ **if** $x \in s$ **for** *x*
by (*metis* ‹*s division_of cbox a a*› *cbox_idem division_ofD*(2) *division_ofD*(3) *subset_singletonD that*)
moreover have $s \neq \{\}$
using ‹*s division_of cbox a a*› **by** *auto*
ultimately show *?r*
unfolding *cbox_idem* **by** *auto*
qed (*use cbox_idem* **in** *blast*)

lemma *elementary_empty*: **obtains** *p* **where** *p division_of* $\{\}$
by *simp*

lemma *elementary_interval*: **obtains** *p* **where** *p division_of* (*cbox a b*)
by (*metis division_of_trivial division_of_self*)

lemma *division_contains*: *s division_of i* $\Longrightarrow \forall x{\in}i.\ \exists k{\in}s.\ x \in k$
unfolding *division_of_def* **by** *auto*

lemma *forall_in_division*:
d division_of i $\Longrightarrow (\forall x{\in}d.\ P\ x) \longleftrightarrow (\forall a\ b.\ cbox\ a\ b \in d \longrightarrow P\ (cbox\ a\ b))$
unfolding *division_of_def* **by** *fastforce*

lemma *cbox_division_memE*:
assumes $\mathcal{D}$: $K \in \mathcal{D}$ $\mathcal{D}$ *division_of S*
obtains *a b* **where** $K = cbox\ a\ b$ $K \neq \{\}$ $K \subseteq S$
using *assms* **unfolding** *division_of_def* **by** *metis*

lemma *division_of_subset*:
assumes *p division_of* $(\bigcup p)$
and $q \subseteq p$
shows *q division_of* $(\bigcup q)$
proof (*rule division_ofI*)
show *finite q*
using *assms finite_subset* **by** *blast*
next
fix *k*
assume $k \in q$
show $k \subseteq \bigcup q$
using ‹$k \in q$› **by** *auto*
show $\exists a\ b.\ k = cbox\ a\ b\ k \neq \{\}$
using *assms* ‹$k \in q$› **by** *blast+*
next
fix *k1 k2*
assume $k1 \in q$ $k2 \in q$ $k1 \neq k2$
then show *interior k1* $\cap$ *interior k2* $= \{\}$
using *assms* **by** *blast*
qed *auto*

lemma *division_of_union_self*[*intro*]: *p division_of s* $\Longrightarrow$ *p division_of* $(\bigcup p)$
by *blast*

lemma *division_inter*:
fixes *s1 s2* :: *'a*::*euclidean_space set*
assumes *p1 division_of s1*
and *p2 division_of s2*
shows $\{k1 \cap k2 \mid k1\ k2.\ k1 \in p1 \wedge k2 \in p2 \wedge k1 \cap k2 \neq \{\}\}$ *division_of* (*s1* $\cap$ *s2*)
(**is** *?A' division_of _*)
proof −
let *?A* $= \{s.\ s \in\ (\lambda(k1,k2).\ k1 \cap k2)\ `\ (p1 \times p2) \wedge s \neq \{\}\}$
have $*$: *?A'* $=$ *?A* **by** *auto*
show *?thesis*
unfolding $*$
proof (*rule division_ofI*)

```
    have ?A ⊆ (λ(x, y). x ∩ y) ‘ (p1 × p2)
      by auto
    moreover have finite (p1 × p2)
      using assms unfolding division_of_def by auto
    ultimately show finite ?A by auto
    have ∗: ⋀s. ⋃{x∈s. x ≠ {}} = ⋃s
      by auto
    show UA: ⋃ ?A = s1 ∩ s2
      unfolding ∗
     using division_ofD(6)[OF assms(1)] and division_ofD(6)[OF assms(2)] by
auto
    {
      fix k
      assume kA: k ∈ ?A
      then obtain k1 k2 where k: k = k1 ∩ k2 k1 ∈ p1 k2 ∈ p2 k ≠ {}
        by auto
      then show k ≠ {}
        by auto
      show k ⊆ s1 ∩ s2
        using UA kA by blast
      show ∃ a b. k = cbox a b
        using k by (metis (no_types, lifting) Int_interval assms division_ofD(4))
    }
    fix k1 k2
    assume k1 ∈ ?A
    then obtain x1 y1 where k1: k1 = x1 ∩ y1 x1 ∈ p1 y1 ∈ p2 k1 ≠ {}
      by auto
    assume k2 ∈ ?A
    then obtain x2 y2 where k2: k2 = x2 ∩ y2 x2 ∈ p1 y2 ∈ p2 k2 ≠ {}
      by auto
    assume k1 ≠ k2
    then show interior k1 ∩ interior k2 = {}
      unfolding k1 k2
      using assms division_ofD(5) k1 k2 by auto
  qed
qed

lemma division_inter_1:
  assumes d division_of i
    and cbox a (b::'a::euclidean_space) ⊆ i
  shows {cbox a b ∩ k | k. k ∈ d ∧ cbox a b ∩ k ≠ {}} division_of (cbox a b)
proof (cases cbox a b = {})
  case True
  show ?thesis
    unfolding True and division_of_trivial by auto
next
  case False
  have ∗: cbox a b ∩ i = cbox a b using assms(2) by auto
  show ?thesis
```

using *division_inter*[*OF division_of_self*[*OF False*] *assms*(*1*)]
unfolding $*$ **by** *auto*
qed

lemma *elementary_Int*:
fixes *s t* :: *'a*::*euclidean_space set*
assumes *p1 division_of s* **and** *p2 division_of t*
shows $\exists$ *p*. *p division_of* (*s* $\cap$ *t*)
using *assms division_inter* **by** *blast*

lemma *elementary_Inter*:
assumes *finite* $\mathcal{F}$
and $\mathcal{F} \neq \{\}$
and $\forall$ *s*$\in\mathcal{F}$. $\exists$ *p*. *p division_of* (*s*::(*'a*::*euclidean_space*) *set*)
shows $\exists$ *p*. *p division_of* ($\bigcap \mathcal{F}$)
using *assms*
proof (*induct* $\mathcal{F}$ *rule*: *finite_induct*)
case (*insert x* $\mathcal{F}$)
then show *?case*
by (*metis cInf_singleton complete_lattice_class.Inf_insert elementary_Int insert_iff*)
qed *auto*

lemma *division_disjoint_union*:
assumes *p1 division_of s1*
and *p2 division_of s2*
and *interior s1* $\cap$ *interior s2* = {}
shows (*p1* $\cup$ *p2*) *division_of* (*s1* $\cup$ *s2*)
proof (*rule division_ofI*)
note *d1* = *division_ofD*[*OF assms*(*1*)]
note *d2* = *division_ofD*[*OF assms*(*2*)]
fix *k1 k2*
assume *k1* $\in$ *p1* $\cup$ *p2 k2* $\in$ *p1* $\cup$ *p2 k1* $\neq$ *k2*
with *assms* **show** *interior k1* $\cap$ *interior k2* = {}
by (*smt* (*verit*, *best*) *IntE Un_iff disjoint_iff_not_equal division_ofD*(*2*) *division_ofD*(*5*) *inf.orderE interior_Int*)
qed (*use division_ofD assms* **in** *auto*)

lemma *partial_division_extend_1*:
fixes *a b c d* :: *'a*::*euclidean_space*
assumes *incl*: *cbox c d* $\subseteq$ *cbox a b*
and *nonempty*: *cbox c d* $\neq$ {}
obtains *p* **where** *p division_of* (*cbox a b*) *cbox c d* $\in$ *p*
proof
let *?B* = λ*f*::*'a*$\Rightarrow$*'a* $\times$ *'a*.
cbox ($\sum$ *i*$\in$*Basis*. (*fst* (*f i*) $\cdot$ *i*) $*_R$ *i*) ($\sum$ *i*$\in$*Basis*. (*snd* (*f i*) $\cdot$ *i*) $*_R$ *i*)
define *p* **where** *p* = *?B* ' (*Basis* $\rightarrow_E$ {(*a*, *c*), (*c*, *d*), (*d*, *b*)})

show *cbox c d* $\in$ *p*

```
    unfolding p_def
   by (auto simp add: box_eq_empty cbox_def intro!: image_eqI[where x=λ(i::'a)∈Basis.
(c, d)])
  have ord: a · i ≤ c · i c · i ≤ d · i d · i ≤ b · i if i ∈ Basis for i
    using incl nonempty that
    unfolding box_eq_empty subset_box by (auto simp: not_le)

  show p division_of (cbox a b)
  proof (rule division_ofI)
    show finite p
      unfolding p_def by (auto intro!: finite_PiE)
    {
      fix K
      assume K ∈ p
      then obtain f where f: f ∈ Basis →E {(a, c), (c, d), (d, b)} and k: K =
?B f
        by (auto simp: p_def)
      then show ∃ a b. K = cbox a b
        by auto
      {
        fix l
        assume l ∈ p
        then obtain g where g: g ∈ Basis →E {(a, c), (c, d), (d, b)} and l: l =
?B g
          by (auto simp: p_def)
        assume l ≠ K
        have ∃ i∈Basis. f i ≠ g i
          using ‹l ≠ K› l k f g by auto
        then obtain i where ∗: i ∈ Basis f i ≠ g i ..
        then have f i = (a, c) ∨ f i = (c, d) ∨ f i = (d, b)
          g i = (a, c) ∨ g i = (c, d) ∨ g i = (d, b)
          using f g by (auto simp: PiE_iff)
        with ∗ ord[of i] show interior l ∩ interior K = {}
          by (auto simp add: l k disjoint_interval intro!: bexI[of _ i])
      }
      have a · i ≤ fst (f i) · i snd (f i) · i ≤ b · i fst (f i) · i ≤ snd (f i) · i
        if i ∈ Basis for i
      proof −
        have f i = (a, c) ∨ f i = (c, d) ∨ f i = (d, b)
          using that f by (auto simp: PiE_iff)
        with that ord[of i]
        show a · i ≤ fst (f i) · i snd (f i) · i ≤ b · i fst (f i) · i ≤ snd (f i) · i
          by auto
      qed
      then show K ≠ {} K ⊆ cbox a b
        by (auto simp add: k box_eq_empty subset_box not_less)
    }
    moreover
    have ∃ k∈p. x ∈ k if x: x ∈ cbox a b for x
```

```
    proof −
      have ∃ l. x · i ∈ {fst l · i .. snd l · i} ∧ l ∈ {(a, c), (c, d), (d, b)} if i ∈
Basis for i
      proof −
        have (a · i ≤ x · i ∧ x · i ≤ c · i) ∨ (c · i ≤ x · i ∧ x · i ≤ d · i) ∨
            (d · i ≤ x · i ∧ x · i ≤ b · i)
          using that x ord[of i]
          by (auto simp: cbox_def)
        then show ∃ l. x · i ∈ {fst l · i .. snd l · i} ∧ l ∈ {(a, c), (c, d), (d, b)}
          by auto
      qed
      then obtain f where
       f: ∀ i∈Basis. x · i ∈ {fst (f i) · i..snd (f i) · i} ∧ f i ∈ {(a, c), (c, d), (d, b)}
        by metis
     moreover from f have x ∈ ?B (restrict f Basis) restrict f Basis ∈ Basis →E
{(a,c), (c,d), (d,b)}
       by (auto simp: mem_box)
      ultimately show ?thesis
        unfolding p_def by blast
    qed
    ultimately show ⋃ p = cbox a b
      by auto
  qed
qed

proposition partial_division_extend_interval:
  assumes p division_of (⋃ p) (⋃ p) ⊆ cbox a b
  obtains q where p ⊆ q q division_of cbox a (b::'a::euclidean_space)
proof (cases p = {})
  case True
  then show ?thesis
    using elementary_interval that by auto
next
  case False
  note p = division_ofD[OF assms(1)]
  have ∀ k∈p. ∃ q. q division_of cbox a b ∧ k ∈ q
  proof
    fix k
    assume kp: k ∈ p
    obtain c d where k: k = cbox c d
      using p(4)[OF kp] by blast
    have ∗: cbox c d ⊆ cbox a b cbox c d ≠ {}
      using p(2,3)[OF kp, unfolded k] using assms(2)
      by (blast intro: order.trans)+
    obtain q where q division_of cbox a b cbox c d ∈ q
      by (rule partial_division_extend_1[OF ∗])
    then show ∃ q. q division_of cbox a b ∧ k ∈ q
      unfolding k by auto
  qed
```

then obtain q **where** q: $\bigwedge x.\ x \in p \Longrightarrow q\ x\ division\_of\ cbox\ a\ b\ \bigwedge x.\ x \in p \Longrightarrow x \in q\ x$
by *metis*
have $q\ x\ division\_of \bigcup (q\ x)$ **if** x: $x \in p$ **for** x
using $q(1)\ x$ **by** *blast*
then have di: $\bigwedge x.\ x \in p \Longrightarrow \exists d.\ d\ division\_of \bigcup (q\ x - \{x\})$
by (*meson Diff_subset division_of_subset*)
have $\forall s \in (\lambda i.\ \bigcup (q\ i - \{i\}))\ `\ p.\ \exists d.\ d\ division\_of\ s$
using di **by** *blast*
then obtain d **where** d: $d\ division\_of \bigcap ((\lambda i.\ \bigcup (q\ i - \{i\}))\ `\ p)$
by (*meson False elementary_Inter finite_imageI image_is_empty p(1)*)
have $d \cup p\ division\_of\ cbox\ a\ b$
proof −
have *te*: $\bigwedge S\ f\ T.\ S \neq \{\} \Longrightarrow \forall i \in S.\ f\ i \cup i = T \Longrightarrow T = \bigcap (f\ `\ S) \cup \bigcup S$ **by** *auto*
have *cbox_eq*: $cbox\ a\ b = \bigcap ((\lambda i.\ \bigcup (q\ i - \{i\}))\ `\ p) \cup \bigcup p$
proof (*rule te*[*OF False*], *clarify*)
fix i
assume $i \in p$
then show $\bigcup (q\ i - \{i\}) \cup i = cbox\ a\ b$
by (*metis Un_commute complete_lattice_class.Sup_insert division_ofD(6) insert_Diff q*)
qed
have [*simp*]: $interior\ (\bigcap i \in p.\ \bigcup (q\ i - \{i\})) \cap interior\ K = \{\}$ **if** K: $K \in p$ **for** K
proof −
note $qk = division\_ofD[OF\ q(1)[OF\ K]]$
have $*$: $\bigwedge U\ T\ S.\ T \cap S = \{\} \Longrightarrow U \subseteq S \Longrightarrow U \cap T = \{\}$
by *auto*
show *?thesis*
proof (*rule* $*$[*OF Int_interior_Union_intervals*])
show $\bigwedge T.\ T \in q\ K - \{K\} \Longrightarrow interior\ K \cap interior\ T = \{\}$
using $K\ q(2)\ qk(5)$ **by** *auto*
show $interior\ (\bigcap i \in p.\ \bigcup (q\ i - \{i\})) \subseteq interior\ (\bigcup (q\ K - \{K\}))$
by (*meson INF_lower K interior_mono*)
qed (*use qk* **in** *auto*)
qed
show $d \cup p\ division\_of\ (cbox\ a\ b)$
by (*simp add*: *Int_interior_Union_intervals assms(1) cbox_eq d division_disjoint_union p(1) p(4)*)
qed
then show *?thesis*
by (*meson Un_upper2 that*)
qed

lemma *elementary_bounded*[*dest*]:
shows $p\ division\_of\ S \Longrightarrow bounded\ S$
unfolding *division_of_def* **by** (*metis bounded_Union bounded_cbox*)

lemma *elementary_subset_cbox*:
p division_of S $\Longrightarrow$ $\exists$ *a b*. *S* $\subseteq$ *cbox a b*
by (*meson bounded_subset_cbox_symmetric elementary_bounded*)

proposition *division_union_intervals_exists*:
assumes *cbox a b* $\neq$ {}
obtains *p* **where** (*insert* (*cbox a b*) *p*) *division_of* (*cbox a b* $\cup$ *cbox c d*)
proof (*cases cbox c d* = {})
case *True*
with *assms that* **show** *?thesis* **by** *force*
next
case *False*
show *?thesis*
proof (*cases cbox a b* $\cap$ *cbox c d* = {})
case *True*
then show *?thesis*
by (*metis that False assms division_disjoint_union division_of_self insert_is_Un interior_Int interior_empty*)
next
case *False*
obtain *u v* **where** *uv*: *cbox a b* $\cap$ *cbox c d* = *cbox u v*
unfolding *Int_interval* **by** *auto*
have *uv_sub*: *cbox u v* $\subseteq$ *cbox c d* **using** *uv* **by** *auto*
obtain *p* **where** *pd*: *p division_of cbox c d* **and** *cbox u v* $\in$ *p*
by (*rule partial_division_extend_1*[*OF uv_sub False*[*unfolded uv*]])
note *p* = *this division_ofD*[*OF pd*]
have *interior* (*cbox a b* $\cap$ $\bigcup$(*p* $-$ {*cbox u v*})) = *interior*(*cbox u v*) $\cap$ *interior* ($\bigcup$(*p* $-$ {*cbox u v*}))
by (*metis Diff_subset Int_absorb1 Int_assoc Sup_subset_mono interior_Int p*(*8*) *uv*)
also have ... = {}
using *p*(*6*) *p*(*7*)[*OF p*(*2*)] ‹*finite p*›
by (*intro Int_interior_Union_intervals*) *auto*
finally have *disj*: *interior* (*cbox a b*) $\cap$ *interior* ($\bigcup$(*p* $-$ {*cbox u v*})) = {}
by *simp*
have *cbe*: *cbox a b* $\cup$ *cbox c d* = *cbox a b* $\cup$ $\bigcup$(*p* $-$ {*cbox u v*})
using *p*(*8*) **unfolding** *uv*[*symmetric*] **by** *auto*
have *insert* (*cbox a b*) (*p* $-$ {*cbox u v*}) *division_of cbox a b* $\cup$ $\bigcup$(*p* $-$ {*cbox u v*})
by (*metis Diff_subset assms disj division_disjoint_union division_of_self division_of_subset insert_is_Un p*(*8*) *pd*)
with *that*[*of p* $-$ {*cbox u v*}] **show** *?thesis*
by (*simp add*: *cbe*)
qed
qed

lemma *division_of_Union*:
assumes *finite f*
and $\bigwedge$*p*. *p* $\in$ *f* $\Longrightarrow$ *p division_of* ($\bigcup$*p*)

and $\bigwedge k1\ k2.\ k1 \in \bigcup f \Longrightarrow k2 \in \bigcup f \Longrightarrow k1 \neq k2 \Longrightarrow interior\ k1 \cap interior\ k2 = \{\}$
shows $\bigcup f\ division\_of\ \bigcup(\bigcup f)$
using *assms* **by** (*auto intro*!: *division_ofI*)

lemma *elementary_union_interval*:
fixes $a\ b :: 'a::euclidean\_space$
assumes $p\ division\_of\ \bigcup p$
obtains q **where** $q\ division\_of\ (cbox\ a\ b \cup \bigcup p)$
proof (*cases* $p = \{\} \vee cbox\ a\ b = \{\}$)
case *True*
obtain p **where** $p\ division\_of\ (cbox\ a\ b)$
by (*rule elementary_interval*)
then show *thesis*
using *True assms that* **by** *auto*
next
case *False*
then have $p \neq \{\}\ cbox\ a\ b \neq \{\}$
by *auto*
note $pdiv = division\_ofD[OF\ assms]$
show *?thesis*
proof (*cases interior* $(cbox\ a\ b) = \{\}$)
case *True*
show *?thesis*
by (*metis True assms division_disjoint_union elementary_interval inf_bot_left that*)
next
case *nonempty*: *False*
have $\forall K{\in}p.\ \exists q.\ (insert\ (cbox\ a\ b)\ q)\ division\_of\ (cbox\ a\ b \cup K)$
by (*metis* ‹$cbox\ a\ b \neq \{\}$› *division_union_intervals_exists pdiv*(*4*))
then obtain q **where** $\forall x{\in}p.\ insert\ (cbox\ a\ b)\ (q\ x)\ division\_of\ (cbox\ a\ b) \cup x$
by *metis*
note $q = division\_ofD[OF\ this[rule\_format]]$
let $?D = \bigcup\{insert\ (cbox\ a\ b)\ (q\ K) \mid K.\ K \in p\}$
show *thesis*
proof (*rule that*[*OF division_ofI*])
have $*$: $\{insert\ (cbox\ a\ b)\ (q\ K)\ |K.\ K \in p\} = (\lambda K.\ insert\ (cbox\ a\ b)\ (q\ K))\ `\ p$
by *auto*
show *finite ?D*
using $*\ pdiv(1)\ q(1)$ **by** *auto*
have $\bigcup ?D = (\bigcup x{\in}p.\ \bigcup(insert\ (cbox\ a\ b)\ (q\ x)))$
by *auto*
also have $... = \bigcup\{cbox\ a\ b \cup t\ |t.\ t \in p\}$
using $q(6)$ **by** *auto*
also have $... = cbox\ a\ b \cup \bigcup p$
using ‹$p \neq \{\}$› **by** *auto*
finally show $\bigcup ?D = cbox\ a\ b \cup \bigcup p$ **.**

```
      show K ⊆ cbox a b ∪ ⋃p K ≠ {} if K ∈ ?D for K
        using q that by blast+
      show ∃ a b. K = cbox a b if K ∈ ?D for K
        using q(4) that by auto
    next
      fix K' K
      assume K: K ∈ ?D and K': K' ∈ ?D K ≠ K'
      obtain x where x: K ∈ insert (cbox a b) (q x) x∈p
        using K by auto
      obtain x' where x': K'∈insert (cbox a b) (q x') x'∈p
        using K' by auto
      show interior K ∩ interior K' = {}
      proof (cases x = x' ∨ K = cbox a b ∨ K' = cbox a b)
        case True
        then show ?thesis
          using True K' q(5) x' x by auto
      next
        case False
        then have as': K ≠ cbox a b K' ≠ cbox a b
          by auto
        obtain c d where K: K = cbox c d
          using q(4) x by blast
        have interior K ∩ interior (cbox a b) = {}
          using as' K'(2) q(5) x by blast
        then have interior K ⊆ interior x
       by (metis ‹interior (cbox a b) ≠ {}› K q(2) x interior_subset_union_intervals)
        moreover
        obtain c d where c_d: K' = cbox c d
          using q(4) x'(1) x'(2) by presburger
        have interior K' ∩ interior (cbox a b) = {}
          using as'(2) q(5) x' by blast
        then have interior K' ⊆ interior x'
          by (metis nonempty c_d interior_subset_union_intervals q(2) x')
        moreover have interior x ∩ interior x' = {}
          by (meson False assms division_ofD(5) x'(2) x(2))
        ultimately show ?thesis
          using ‹interior K ⊆ interior x› ‹interior K' ⊆ interior x'› by auto
      qed
    qed
  qed
qed



lemma elementary_unions_intervals:
  assumes fin: finite ℱ
    and ⋀s. s ∈ ℱ ⟹ ∃ a b. s = cbox a (b::'a::euclidean_space)
  obtains p where p division_of (⋃ℱ)
proof −
```

```
  have ∃p. p division_of (⋃ℱ)
  proof (induct_tac ℱ rule:finite_subset_induct)
    show ∃p. p division_of ⋃{} using elementary_empty by auto
  next
    fix x F
    assume as: finite F x ∉ F ∃p. p division_of ⋃F x∈ℱ
    then obtain a b where x: x = cbox a b
      by (meson assms(2))
    then show ∃p. p division_of ⋃(insert x F)
    using elementary_union_interval by (metis Union_insert as(3) division_ofD(6))
  qed (use assms in auto)
  then show ?thesis
    using that by auto
qed

lemma elementary_union:
  assumes ps division_of S pt division_of T
  obtains p where p division_of (S ∪ T)
proof −
  have S ∪ T = ⋃ps ∪ ⋃pt
    using assms unfolding division_of_def by auto
  with elementary_unions_intervals[of ps ∪ pt] assms
  show ?thesis
    by (metis Un_iff Union_Un_distrib division_of_def finite_Un that)
qed

lemma partial_division_extend:
  fixes T :: 'a::euclidean_space set
  assumes p division_of S
    and q division_of T
    and S ⊆ T
  obtains r where p ⊆ r and r division_of T
proof −
  note divp = division_ofD[OF assms(1)] and divq = division_ofD[OF assms(2)]
  obtain a b where ab: T ⊆ cbox a b
    using elementary_subset_cbox[OF assms(2)] by auto
  obtain r1 where p ⊆ r1 r1 division_of (cbox a b)
    using assms
    by (metis ab dual_order.trans partial_division_extend_interval divp(6))
  note r1 = this division_ofD[OF this(2)]
  obtain p' where p' division_of ⋃(r1 − p)
    by (metis Diff_subset division_of_subset r1(2) r1(8))
  then obtain r2 where r2: r2 division_of (⋃(r1 − p)) ∩ (⋃q)
    by (metis assms(2) divq(6) elementary_Int)
  {
    fix x
    assume x: x ∈ T x ∉ S
    then obtain R where r: R ∈ r1 x ∈ R
      unfolding r1 using ab
```

```
    by (meson division_contains r1(2) subsetCE)
   moreover have R ∉ p
    using divp(6) r(2) x(2) by blast
   ultimately have x∈⋃(r1 − p) by auto
  }
  then have Teq: T = ⋃p ∪ (⋃(r1 − p) ∩ ⋃q)
   unfolding divp divq using assms(3) by auto
  have interior S ∩ interior (⋃(r1−p)) = {}
  proof (rule Int_interior_Union_intervals)
   have ∗: ⋀S. (⋀x. x ∈ S ⟹ False) ⟹ S = {}
    by auto
   show interior S ∩ interior m = {} if m ∈ r1 − p for m
   proof −
    have interior m ∩ interior (⋃p) = {}
    proof (rule Int_interior_Union_intervals)
     show ⋀T. T∈p ⟹ interior m ∩ interior T = {}
      by (metis DiffD1 DiffD2 that r1(1) r1(7) rev_subsetD)
    qed (use divp in auto)
    then show interior S ∩ interior m = {}
     unfolding divp by auto
   qed
  qed (use r1 in auto)
  then have interior S ∩ interior (⋃(r1−p) ∩ (⋃q)) = {}
   using interior_subset by auto
  then have div: p ∪ r2 division_of ⋃p ∪ ⋃(r1 − p) ∩ ⋃q
   by (simp add: assms(1) division_disjoint_union divp(6) r2)
  show ?thesis
   by (metis Teq div sup_ge1 that)
qed


lemma division_split:
  assumes p division_of (cbox a b)
   and k: k∈Basis
  shows {l ∩ {x. x•k ≤ c} | l. l ∈ p ∧ l ∩ {x. x•k ≤ c} ≠ {}} division_of(cbox a
b ∩ {x. x•k ≤ c})
     (is ?p1 division_of ?I1)
   and {l ∩ {x. x•k ≥ c} | l. l ∈ p ∧ l ∩ {x. x•k ≥ c} ≠ {}} division_of (cbox a
b ∩ {x. x•k ≥ c})
     (is ?p2 division_of ?I2)
proof (rule_tac[!] division_ofI)
  note p = division_ofD[OF assms(1)]
  show finite ?p1 finite ?p2
   using p(1) by auto
  show ⋃?p1 = ?I1 ⋃?p2 = ?I2
   unfolding p(6)[symmetric] by auto
  {
   fix K
   assume K ∈ ?p1
```

```
    then obtain l where l: K = l ∩ {x. x · k ≤ c} l ∈ p l ∩ {x. x · k ≤ c} ≠ {}
      by blast
    obtain u v where uv: l = cbox u v
      using assms(1) l(2) by blast
    show K ⊆ ?I1
      using l p(2) uv by force
    show  K ≠ {}
      by (simp add: l)
    have ∃ a b. cbox u v ∩ {x. x · k ≤ c} = cbox a b
      unfolding interval_split[OF k] by (auto intro: order.trans)
    then show  ∃ a b. K = cbox a b
      by (simp add: l(1) uv)
    fix K'
    assume K' ∈ ?p1
    then obtain l' where l': K' = l' ∩ {x. x · k ≤ c} l' ∈ p l' ∩ {x. x · k ≤ c}
≠ {}
      by blast
    assume K ≠ K'
    then show interior K ∩ interior K' = {}
      unfolding l l' using p(5)[OF l(2) l'(2)] by auto
  }
  {
    fix K
    assume K ∈ ?p2
    then obtain l where l: K = l ∩ {x. c ≤ x · k} l ∈ p l ∩ {x. c ≤ x · k} ≠ {}
      by blast
    obtain u v where uv: l = cbox u v
      using l(2) p(4) by blast
    show K ⊆ ?I2
      using l p(2) uv by force
    show  K ≠ {}
      by (simp add: l)
    have ∃ a b. cbox u v ∩ {x. c ≤ x · k} = cbox a b
      unfolding interval_split[OF k] by (auto intro: order.trans)
    then show  ∃ a b. K = cbox a b
      by (simp add: l(1) uv)
    fix K'
    assume K' ∈ ?p2
    then obtain l' where l': K' = l' ∩ {x. c ≤ x · k} l' ∈ p l' ∩ {x. c ≤ x · k}
≠ {}
      by blast
    assume K ≠ K'
    then show interior K ∩ interior K' = {}
      unfolding l l' using p(5)[OF l(2) l'(2)] by auto
  }
qed
```

7.13.6 Tagged (partial) divisions

definition *tagged_partial_division_of* (**infixr** ‹*tagged'_partial'_division'_of*› *40*)
where *s tagged_partial_division_of i* ⟷
finite s ∧
(∀ *x K*. (*x*, *K*) ∈ *s* ⟶ *x* ∈ *K* ∧ *K* ⊆ *i* ∧ (∃ *a b*. *K* = *cbox a b*)) ∧
(∀ *x1 K1 x2 K2*. (*x1*, *K1*) ∈ *s* ∧ (*x2*, *K2*) ∈ *s* ∧ (*x1*, *K1*) ≠ (*x2*, *K2*) ⟶
interior K1 ∩ *interior K2* = {})

lemma *tagged_partial_division_ofD*:
assumes *s tagged_partial_division_of i*
shows *finite s*
and ⋀*x K*. (*x*,*K*) ∈ *s* ⟹ *x* ∈ *K*
and ⋀*x K*. (*x*,*K*) ∈ *s* ⟹ *K* ⊆ *i*
and ⋀*x K*. (*x*,*K*) ∈ *s* ⟹ ∃ *a b*. *K* = *cbox a b*
and ⋀*x1 K1 x2 K2*. (*x1*,*K1*) ∈ *s* ⟹
(*x2*, *K2*) ∈ *s* ⟹ (*x1*, *K1*) ≠ (*x2*, *K2*) ⟹ *interior K1* ∩ *interior K2* = {}
using *assms* **unfolding** *tagged_partial_division_of_def* **by** *blast+*

definition *tagged_division_of* (**infixr** ‹*tagged'_division'_of*› *40*)
where *s tagged_division_of i* ⟷ *s tagged_partial_division_of i* ∧ (⋃{*K*. ∃ *x*. (*x*,*K*) ∈ *s*} = *i*)

lemma *tagged_division_of_finite*: *s tagged_division_of i* ⟹ *finite s*
unfolding *tagged_division_of_def tagged_partial_division_of_def* **by** *auto*

lemma *tagged_division_of*:
s tagged_division_of i ⟷
finite s ∧
(∀ *x K*. (*x*, *K*) ∈ *s* ⟶ *x* ∈ *K* ∧ *K* ⊆ *i* ∧ (∃ *a b*. *K* = *cbox a b*)) ∧
(∀ *x1 K1 x2 K2*. (*x1*, *K1*) ∈ *s* ∧ (*x2*, *K2*) ∈ *s* ∧ (*x1*, *K1*) ≠ (*x2*, *K2*) ⟶
interior K1 ∩ *interior K2* = {}) ∧
(⋃{*K*. ∃ *x*. (*x*,*K*) ∈ *s*} = *i*)
unfolding *tagged_division_of_def tagged_partial_division_of_def* **by** *auto*

lemma *tagged_division_ofI*:
assumes *finite s*
and ⋀*x K*. (*x*,*K*) ∈ *s* ⟹ *x* ∈ *K*
and ⋀*x K*. (*x*,*K*) ∈ *s* ⟹ *K* ⊆ *i*
and ⋀*x K*. (*x*,*K*) ∈ *s* ⟹ ∃ *a b*. *K* = *cbox a b*
and ⋀*x1 K1 x2 K2*. (*x1*,*K1*) ∈ *s* ⟹ (*x2*, *K2*) ∈ *s* ⟹ (*x1*, *K1*) ≠ (*x2*, *K2*)
⟹
interior K1 ∩ *interior K2* = {}
and (⋃{*K*. ∃ *x*. (*x*,*K*) ∈ *s*} = *i*)
shows *s tagged_division_of i*
unfolding *tagged_division_of*
using *assms* **by** *fastforce*

lemma *tagged_division_ofD*[*dest*]:
assumes *s tagged_division_of i*

shows *finite s*
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow x \in K$
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow K \subseteq i$
and $\bigwedge x\ K.\ (x,K) \in s \Longrightarrow \exists\, a\ b.\ K = \mathit{cbox}\ a\ b$
and $\bigwedge x1\ K1\ x2\ K2.\ (x1,\ K1) \in s \Longrightarrow (x2,\ K2) \in s \Longrightarrow (x1,\ K1) \neq (x2,\ K2)$
$\Longrightarrow$
$\mathit{interior}\ K1 \cap \mathit{interior}\ K2 = \{\}$
and $(\bigcup\{K.\ \exists\, x.\ (x,K) \in s\} = i)$
using *assms* **unfolding** *tagged_division_of* **by** *blast+*

lemma *division_of_tagged_division*:
assumes *s tagged_division_of i*
shows (*snd ‘ s*) *division_of i*
proof (*rule division_ofI*)
note *assm* = *tagged_division_ofD*[*OF assms*]
show $\bigcup(\mathit{snd}\ `\ s) = i$ *finite* (*snd ‘ s*)
using *assm* **by** *auto*
fix *K*
assume *k*: $K \in \mathit{snd}\ `\ s$
then show $K \subseteq i\ K \neq \{\}\ \exists\, a\ b.\ K = \mathit{cbox}\ a\ b$
using *assm* **by** *fastforce+*
fix K'
assume k': $K' \in \mathit{snd}\ `\ s\ K \neq K'$
then show $\mathit{interior}\ K \cap \mathit{interior}\ K' = \{\}$
by (*metis* (*no_types*, *lifting*) *assms imageE k prod.collapse tagged_division_ofD*(*5*))
qed

lemma *partial_division_of_tagged_division*:
assumes *s tagged_partial_division_of i*
shows (*snd ‘ s*) *division_of* $\bigcup(\mathit{snd}\ `\ s)$
proof (*rule division_ofI*)
note *assm* = *tagged_partial_division_ofD*[*OF assms*]
show *finite* (*snd ‘ s*) $\bigcup(\mathit{snd}\ `\ s) = \bigcup(\mathit{snd}\ `\ s)$
using *assm* **by** *auto*
fix *K*
assume *K*: $K \in \mathit{snd}\ `\ s$
then show $K \neq \{\}\ \exists\, a\ b.\ K = \mathit{cbox}\ a\ b\ K \subseteq \bigcup(\mathit{snd}\ `\ s)$
using *assm* **by** *fastforce+*
fix K'
assume $K' \in \mathit{snd}\ `\ s\ K \neq K'$
then show $\mathit{interior}\ K \cap \mathit{interior}\ K' = \{\}$
using *assm*(*5*) *K* **by** *force*
qed

lemma *tagged_partial_division_subset*:
assumes *s tagged_partial_division_of i* **and** $t \subseteq s$
shows *t tagged_partial_division_of i*
using *assms finite_subset*[*OF assms*(*2*)]
unfolding *tagged_partial_division_of_def*

by *blast*

lemma *tag_in_interval*: *p tagged_division_of i* $\Longrightarrow$ $(x, k) \in p \Longrightarrow x \in i$
by *auto*

lemma *tagged_division_of_empty*: $\{\}$ *tagged_division_of* $\{\}$
unfolding *tagged_division_of* **by** *auto*

lemma *tagged_partial_division_of_trivial*[*simp*]: *p tagged_partial_division_of* $\{\}$
$\longleftrightarrow$ $p = \{\}$
unfolding *tagged_partial_division_of_def* **by** *auto*

lemma *tagged_division_of_trivial*[*simp*]: *p tagged_division_of* $\{\} \longleftrightarrow p = \{\}$
unfolding *tagged_division_of* **by** *auto*

lemma *tagged_division_of_self*: $x \in$ *cbox a b* $\Longrightarrow$ $\{(x,$*cbox a b*$)\}$ *tagged_division_of*
(*cbox a b*)
by (*rule tagged_division_ofI*) *auto*

lemma *tagged_division_of_self_real*: $x \in \{a$ *..* b::*real*$\}$ $\Longrightarrow$ $\{(x,\{a$ *..* $b\})\}$ *tagged_division_of*
$\{a$ *..* $b\}$
by (*metis box_real*(*2*) *tagged_division_of_self*)

lemma *tagged_division_Un*:
assumes *p1 tagged_division_of s1*
and *p2 tagged_division_of s2*
and *interior s1* $\cap$ *interior s2* $= \{\}$
shows (*p1* $\cup$ *p2*) *tagged_division_of* (*s1* $\cup$ *s2*)
proof (*rule tagged_division_ofI*)
note *p1* $=$ *tagged_division_ofD*[*OF assms*(*1*)]
note *p2* $=$ *tagged_division_ofD*[*OF assms*(*2*)]
show *finite* (*p1* $\cup$ *p2*)
using *p1*(*1*) *p2*(*1*) **by** *auto*
show $\bigcup\{k.\ \exists x.\ (x, k) \in p1 \cup p2\} = s1 \cup s2$
using *p1*(*6*) *p2*(*6*) **by** *blast*
fix *x K*
assume *xK*: $(x, K) \in p1 \cup p2$
show $x \in K$ $\exists a\ b.\ K =$ *cbox a b* $K \subseteq s1 \cup s2$
using *xK p1 p2* **by** *auto*
fix x' K'
assume *xk'*: $(x', K') \in p1 \cup p2$ $(x, K) \neq (x', K')$
have $*$: $\bigwedge a\ b.\ a \subseteq s1 \Longrightarrow b \subseteq s2 \Longrightarrow$ *interior a* $\cap$ *interior b* $= \{\}$
using *assms*(*3*) *interior_mono* **by** *blast*
with *assms* **show** *interior K* $\cap$ *interior* $K' = \{\}$
by (*metis Un_iff inf_commute p1*(*3*) *p2*(*3*) *tagged_division_ofD*(*5*) *xK xk'*)
qed

lemma *tagged_division_Union*:
assumes *finite I*

and *tag*: $\bigwedge i.\ i{\in}I \Longrightarrow$ *pfn i tagged_division_of i*
and *disj*: $\bigwedge i1\ i2.\ [\![i1 \in I;\ i2 \in I;\ i1 \neq i2]\!] \Longrightarrow$ *interior*(*i1*) $\cap$ *interior*(*i2*) = {}
shows $\bigcup$(*pfn* ‘ *I*) *tagged_division_of* ($\bigcup$ *I*)
proof (*rule tagged_division_ofI*)
note *assm* = *tagged_division_ofD*[*OF tag*]
show *finite* ($\bigcup$(*pfn* ‘ *I*))
using *assms* **by** *auto*
have $\bigcup\{k.\ \exists x.\ (x,\ k) \in \bigcup(pfn\ `\ I)\} = \bigcup((\lambda i.\ \bigcup\{k.\ \exists x.\ (x,\ k) \in pfn\ i\})\ `\ I)$
by *blast*
also have ... = $\bigcup$ *I*
using *assm*(*6*) **by** *auto*
finally show $\bigcup\{k.\ \exists x.\ (x,\ k) \in \bigcup(pfn\ `\ I)\} = \bigcup I$.
fix *x k*
assume *xk*: (*x*, *k*) $\in \bigcup$(*pfn* ‘ *I*)
then obtain *i* **where** *i*: *i* $\in$ *I* (*x*, *k*) $\in$ *pfn i*
by *auto*
show *x* $\in$ *k* $\exists a\ b.\ k = cbox\ a\ b \subseteq \bigcup I$
using *assm*(*2*−*4*)[*OF i*] **using** *i*(*1*) **by** *auto*
fix *x′ k′*
assume *xk′*: (*x′*, *k′*) $\in \bigcup$(*pfn* ‘ *I*) (*x*, *k*) $\neq$ (*x′*, *k′*)
then obtain *i′* **where** *i′*: *i′* $\in$ *I* (*x′*, *k′*) $\in$ *pfn i′*
by *auto*
have $*$: $\bigwedge a\ b.\ i \neq i' \Longrightarrow a \subseteq i \Longrightarrow b \subseteq i' \Longrightarrow$ *interior a* $\cap$ *interior b* = {}
using *i*(*1*) *i′*(*1*) *disj interior_mono* **by** *blast*
show *interior k* $\cap$ *interior k′* = {}
proof (*cases i* = *i′*)
case *True* **then show** *?thesis*
using *assm*(*5*) *i′ i xk′*(*2*) **by** *blast*
next
case *False* **then show** *?thesis*
using $*$ *assm*(*3*) *i′ i* **by** *auto*
qed
qed

lemma *tagged_partial_division_of_Union_self*:
assumes *p tagged_partial_division_of s*
shows *p tagged_division_of* ($\bigcup$(*snd* ‘ *p*))
using *tagged_partial_division_ofD*[*OF assms*]
by (*intro tagged_division_ofI*) *auto*

lemma *tagged_division_of_union_self*:
assumes *p tagged_division_of s*
shows *p tagged_division_of* ($\bigcup$(*snd* ‘ *p*))
using *tagged_division_ofD*[*OF assms*]
by (*intro tagged_division_ofI*) *auto*

lemma *tagged_division_Un_interval*:
assumes *p1 tagged_division_of* (*cbox a b* $\cap$ {*x*. *x*$\cdot$*k* $\leq$ (*c*::*real*)})

```
    and p2 tagged_division_of (cbox a b ∩ {x. x•k ≥ c})
    and k: k ∈ Basis
  shows (p1 ∪ p2) tagged_division_of (cbox a b)
proof −
  have ∗: cbox a b = (cbox a b ∩ {x. x•k ≤ c}) ∪ (cbox a b ∩ {x. x•k ≥ c})
    by auto
  have interior (cbox a b ∩ {x. x • k ≤ c}) ∩ interior (cbox a b ∩ {x. c ≤ x • k})
= {}
    using k by (force simp: interval_split[OF k] box_def)
  with assms show ?thesis
    by (metis ∗ tagged_division_Un)
qed

lemma tagged_division_Un_interval_real:
  fixes a :: real
  assumes p1 tagged_division_of ({a .. b} ∩ {x. x•k ≤ (c::real)})
    and p2 tagged_division_of ({a .. b} ∩ {x. x•k ≥ c})
    and k: k ∈ Basis
  shows (p1 ∪ p2) tagged_division_of {a .. b}
  using assms by (metis box_real(2) tagged_division_Un_interval)

lemma tagged_division_split_left_inj:
  assumes d: d tagged_division_of i
    and tags: (x1, K1) ∈ d (x2, K2) ∈ d
    and K1 ≠ K2
    and eq: K1 ∩ {x. x • k ≤ c} = K2 ∩ {x. x • k ≤ c}
  shows interior (K1 ∩ {x. x•k ≤ c}) = {}
 by (smt (verit) Int_Un_eq(1) Un_Int_distrib interior_Int prod.inject sup_bot.right_neutral
tagged_division_ofD(5) assms)

lemma tagged_division_split_right_inj:
  assumes d: d tagged_division_of i
    and tags: (x1, K1) ∈ d (x2, K2) ∈ d
    and K1 ≠ K2
    and eq: K1 ∩ {x. x•k ≥ c} = K2 ∩ {x. x•k ≥ c}
  shows interior (K1 ∩ {x. x•k ≥ c}) = {}
 by (smt (verit) Int_Un_eq(1) Un_Int_distrib interior_Int prod.inject sup_bot.right_neutral
tagged_division_ofD(5) assms)

lemma (in comm_monoid_set) over_tagged_division_lemma:
  assumes p tagged_division_of i
    and ⋀u v. box u v = {} ⟹ d (cbox u v) = 1
  shows F (λ(_, k). d k) p = F d (snd ‘ p)
proof −
  have ∗: (λ(_ ,k). d k) = d ∘ snd
    by (simp add: fun_eq_iff)
  note assm = tagged_division_ofD[OF assms(1)]
  show ?thesis
    unfolding ∗
```

```
  proof (rule reindex_nontrivial[symmetric])
    show finite p
      using assm by auto
    fix x y
    assume x∈p y∈p x≠y snd x = snd y
    obtain a b where ab: snd x = cbox a b
      using assm(4)[of fst x snd x] ‹x∈p› by auto
    have (fst x, snd y) ∈ p (fst x, snd y) ≠ y
      using ‹x ∈ p› ‹x ≠ y› ‹snd x = snd y› [symmetric] by auto
    with ‹x∈p› ‹y∈p› have box a b = {}
      by (metis ‹snd x = snd y› ab assm(5) inf.idem interior_cbox prod.collapse)
    then show d (snd x) = 1
      by (simp add: ab assms(2))
  qed
qed
```

7.13.7 Functions closed on boxes: morphisms from boxes to monoids

This auxiliary structure is used to sum up over the elements of a division. Main theorem is *operative_division*. Instances for the monoid are *'a option*, *real*, and *bool*.

Using additivity of lifted function to encode definedness. **definition** *lift_option* :: ('a ⇒ 'b ⇒ 'c) ⇒ 'a option ⇒ 'b option ⇒ 'c option
where
  lift_option f a' b' = Option.bind a' (λa. Option.bind b' (λb. Some (f a b)))

lemma *lift_option_simps*[*simp*]:
  lift_option f (Some a) (Some b) = Some (f a b)
  lift_option f None b' = None
  lift_option f a' None = None
  by (*auto simp*: *lift_option_def*)

lemma *comm_monoid_lift_option*:
  assumes *comm_monoid f z*
  shows *comm_monoid (lift_option f) (Some z)*
proof −
  from *assms* **interpret** *comm_monoid f z* **.**
  show *?thesis*
    by *standard* (*auto simp*: *lift_option_def ac_simps split*: *bind_split*)
qed

lemma *comm_monoid_set_and*: *comm_monoid_set HOL.conj True*
  by (*simp add*: *comm_monoid_set.intro conj.comm_monoid_axioms*)

Misc **lemma** *interval_real_split*:
  {a .. b::real} ∩ {x. x ≤ c} = {a .. min b c}

$\{a\ ..\ b\} \cap \{x.\ c \le x\} = \{max\ a\ c\ ..\ b\}$
by *auto*

lemma *bgauge_existence_lemma*: $(\forall x{\in}s.\ \exists d{::}real.\ 0 < d \wedge q\ d\ x) \longleftrightarrow (\forall x.\ \exists d{>}0.\ x{\in}s \longrightarrow q\ d\ x)$
by (*meson zero_less_one*)

Division points **definition** *division_points* $(k{::}('a{::}euclidean\_space)\ set)\ d =$
$\{(j,x).\ j \in Basis \wedge (interval\_lowerbound\ k){\cdot}j < x \wedge x < (interval\_upperbound\ k){\cdot}j \wedge$
$(\exists i{\in}d.\ (interval\_lowerbound\ i){\cdot}j = x \vee (interval\_upperbound\ i){\cdot}j = x)\}$

lemma *division_points_finite*:
fixes $i :: 'a{::}euclidean\_space\ set$
assumes $d\ division\_of\ i$
shows *finite* $(division\_points\ i\ d)$
proof −
note $assm = division\_ofD[OF\ assms]$
let $?M = \lambda j.\ \{(j,x)|x.\ (interval\_lowerbound\ i){\cdot}j < x \wedge x < (interval\_upperbound\ i){\cdot}j \wedge$
$(\exists i{\in}d.\ (interval\_lowerbound\ i){\cdot}j = x \vee (interval\_upperbound\ i){\cdot}j = x)\}$
have $*$: $division\_points\ i\ d = \bigcup (?M\ `\ Basis)$
unfolding *division_points_def* **by** *auto*
show *?thesis*
unfolding $*$ **using** *assm* **by** *auto*
qed

lemma *division_points_subset*:
fixes $a :: 'a{::}euclidean\_space$
assumes $d\ division\_of\ (cbox\ a\ b)$
and $\forall i{\in}Basis.\ a{\cdot}i < b{\cdot}i\ \ a{\cdot}k < c\ c < b{\cdot}k$
and k: $k \in Basis$
shows $division\_points\ (cbox\ a\ b \cap \{x.\ x{\cdot}k \le c\})\ \{l \cap \{x.\ x{\cdot}k \le c\} \mid l\ .\ l \in d \wedge l \cap \{x.\ x{\cdot}k \le c\} \neq \{\}\} \subseteq$
$division\_points\ (cbox\ a\ b)\ d$ (**is** *?t1*)
and $division\_points\ (cbox\ a\ b \cap \{x.\ x{\cdot}k \ge c\})\ \{l \cap \{x.\ x{\cdot}k \ge c\} \mid l\ .\ l \in d \wedge \neg(l \cap \{x.\ x{\cdot}k \ge c\} = \{\})\} \subseteq$
$division\_points\ (cbox\ a\ b)\ d$ (**is** *?t2*)
proof −
note $assm = division\_ofD[OF\ assms(1)]$
have $*$: $\forall i{\in}Basis.\ a{\cdot}i \le b{\cdot}i$
$\forall i{\in}Basis.\ a{\cdot}i \le (\sum i{\in}Basis.\ (if\ i = k\ then\ min\ (b \cdot k)\ c\ else\ \ b \cdot i) *_R i) \cdot i$
$\forall i{\in}Basis.\ (\sum i{\in}Basis.\ (if\ i = k\ then\ max\ (a \cdot k)\ c\ else\ a \cdot i) *_R i) \cdot i \le b{\cdot}i$
$min\ (b \cdot k)\ c = c\ max\ (a \cdot k)\ c = c$
using *assms* **using** *less_imp_le* **by** *auto*
have $\exists i{\in}d.\ interval\_lowerbound\ i \cdot x = y \vee interval\_upperbound\ i \cdot x = y$
if $a \cdot x < y\ y < (if\ x = k\ then\ c\ else\ b \cdot x)$
$interval\_lowerbound\ i \cdot x = y \vee interval\_upperbound\ i \cdot x = y$
$i = l \cap \{x.\ x \cdot k \le c\}\ l \in d\ l \cap \{x.\ x \cdot k \le c\} \neq \{\}$

```
      x ∈ Basis for i l x y
  proof −
    obtain u v where l: l = cbox u v
      using ‹l ∈ d› assms(1) by blast
    have ∀ i∈Basis. u•i ≤ v•i
      using l using that(6) unfolding box_ne_empty[symmetric] by auto
    then show ?thesis
      using that ‹x ∈ Basis› unfolding l interval_split[OF k]
      by (force split: if_split_asm)
  qed
  moreover have ⋀x y. ⟦y < (if x = k then c else b • x)⟧ ⟹ y < b • x
    using ‹c < b•k› by (auto split: if_split_asm)
  ultimately show ?t1
    unfolding division_points_def interval_split[OF k, of a b]
   unfolding interval_bounds[OF ∗(1)] interval_bounds[OF ∗(2)] interval_bounds[OF
∗(3)]
    unfolding ∗ by force
  have ⋀x y i l. (if x = k then c else a • x) < y ⟹ a • x < y
    using ‹a•k < c› by (auto split: if_split_asm)
  moreover have ∃ i∈d. interval_lowerbound i • x = y ∨
                       interval_upperbound i • x = y
    if (if x = k then c else a • x) < y y < b • x
      interval_lowerbound i • x = y ∨ interval_upperbound i • x = y
      i = l ∩ {x. c ≤ x • k} l ∈ d l ∩ {x. c ≤ x • k} ≠ {}
      x ∈ Basis for x y i l
  proof −
    obtain u v where l: l = cbox u v
      using ‹l ∈ d› assm(4) by blast
    have ∀ i∈Basis. u•i ≤ v•i
      using l using that(6) unfolding box_ne_empty[symmetric] by auto
    then show ∃ i∈d. interval_lowerbound i • x = y ∨ interval_upperbound i • x
= y
      using that ‹x ∈ Basis› unfolding l interval_split[OF k]
      by (force split: if_split_asm)
  qed
  ultimately show ?t2
    unfolding division_points_def interval_split[OF k, of a b]
   unfolding interval_bounds[OF ∗(1)] interval_bounds[OF ∗(2)] interval_bounds[OF
∗(3)]
    unfolding ∗
    by force
qed

lemma division_points_psubset:
  fixes a :: 'a::euclidean_space
  assumes d: d division_of (cbox a b)
      and altb: ∀ i∈Basis. a•i < b•i  a•k < c c < b•k
      and l ∈ d
      and disj: interval_lowerbound l•k = c ∨ interval_upperbound l•k = c
```

```
    and k: k ∈ Basis
  shows division_points (cbox a b ∩ {x. x•k ≤ c}) {l ∩ {x. x•k ≤ c} | l. l∈d ∧ l
∩ {x. x•k ≤ c} ≠ {}} ⊂
        division_points (cbox a b) d (is ?D1 ⊂ ?D)
    and division_points (cbox a b ∩ {x. x•k ≥ c}) {l ∩ {x. x•k ≥ c} | l. l∈d ∧ l ∩
{x. x•k ≥ c} ≠ {}} ⊂
        division_points (cbox a b) d (is ?D2 ⊂ ?D)
proof −
  have ab: ∀ i∈Basis. a•i ≤ b•i
    using altb by (auto intro!:less_imp_le)
  obtain u v where l: l = cbox u v
    using d ‹l ∈ d› by blast
  have uv: ∀ i∈Basis. u•i ≤ v•i ∀ i∈Basis. a•i ≤ u•i ∧ v•i ≤ b•i
    apply (metis assms(5) box_ne_empty(1) cbox_division_memE d l)
   by (metis assms(5) box_ne_empty(1) cbox_division_memE d l subset_box(1))
  have ∗: interval_upperbound (cbox a b ∩ {x. x • k ≤ interval_upperbound l • k})
• k = interval_upperbound l • k
       interval_upperbound (cbox a b ∩ {x. x • k ≤ interval_lowerbound l • k}) •
k = interval_lowerbound l • k
    unfolding l interval_split[OF k] interval_bounds[OF uv(1)]
    using uv[rule_format, of k] ab k
    by auto
  have ∃ x. x ∈ ?D − ?D1
    using assms(3−)
    unfolding division_points_def interval_bounds[OF ab]
    by (force simp add: ∗)
  moreover have ?D1 ⊆ ?D
    by (auto simp add: assms division_points_subset)
  ultimately show ?D1 ⊂ ?D
    by blast
  have ∗: interval_lowerbound (cbox a b ∩ {x. x • k ≥ interval_lowerbound l • k})
• k = interval_lowerbound l • k
   interval_lowerbound (cbox a b ∩ {x. x • k ≥ interval_upperbound l • k}) • k =
interval_upperbound l • k
    unfolding l interval_split[OF k] interval_bounds[OF uv(1)]
    using uv[rule_format, of k] ab k
    by auto
  have ∃ x. x ∈ ?D − ?D2
    using assms(3−)
    unfolding division_points_def interval_bounds[OF ab]
    by (force simp add: ∗)
  moreover have ?D2 ⊆ ?D
    by (auto simp add: assms division_points_subset)
  ultimately show ?D2 ⊂ ?D
    by blast
qed

lemma division_split_left_inj:
  fixes S :: 'a::euclidean_space set
```

assumes *div*: $\mathcal{D}$ *division_of* S
and *eq*: $K1 \cap \{x::'a.\ x{\cdot}k \le c\} = K2 \cap \{x.\ x{\cdot}k \le c\}$
and $K1 \in \mathcal{D}$ $K2 \in \mathcal{D}$ $K1 \neq K2$
shows *interior* $(K1 \cap \{x.\ x{\cdot}k \le c\}) = \{\}$
proof −
have *interior* $K2 \cap$ *interior* $\{a.\ a \cdot k \le c\} =$ *interior* $K1 \cap$ *interior* $\{a.\ a \cdot k \le c\}$
by (*metis* (*no_types*) *eq interior_Int*)
moreover have $\bigwedge A.$ *interior* $A \cap$ *interior* $K2 = \{\} \vee A = K2 \vee A \notin \mathcal{D}$
by (*meson div* ‹$K2 \in \mathcal{D}$› *division_of_def*)
ultimately show *?thesis*
using ‹$K1 \in \mathcal{D}$› ‹$K1 \neq K2$› **by** *auto*
qed

lemma *division_split_right_inj*:
fixes S :: $'a$::*euclidean_space set*
assumes *div*: $\mathcal{D}$ *division_of* S
and *eq*: $K1 \cap \{x::'a.\ x{\cdot}k \ge c\} = K2 \cap \{x.\ x{\cdot}k \ge c\}$
and $K1 \in \mathcal{D}$ $K2 \in \mathcal{D}$ $K1 \neq K2$
shows *interior* $(K1 \cap \{x.\ x{\cdot}k \ge c\}) = \{\}$
proof −
have *interior* $K2 \cap$ *interior* $\{a.\ a \cdot k \ge c\} =$ *interior* $K1 \cap$ *interior* $\{a.\ a \cdot k \ge c\}$
by (*metis* (*no_types*) *eq interior_Int*)
moreover have $\bigwedge A.$ *interior* $A \cap$ *interior* $K2 = \{\} \vee A = K2 \vee A \notin \mathcal{D}$
by (*meson div* ‹$K2 \in \mathcal{D}$› *division_of_def*)
ultimately show *?thesis*
using ‹$K1 \in \mathcal{D}$› ‹$K1 \neq K2$› **by** *auto*
qed

lemma *interval_doublesplit*:
fixes a :: $'a$::*euclidean_space*
assumes $k \in$ *Basis*
shows *cbox* $a\ b \cap \{x\ .\ |x{\cdot}k - c| \le (e::real)\} =$
cbox $(\sum i{\in}Basis.\ (if\ i = k\ then\ max\ (a{\cdot}k)\ (c - e)\ else\ a{\cdot}i) *_R i)$
$(\sum i{\in}Basis.\ (if\ i = k\ then\ min\ (b{\cdot}k)\ (c + e)\ else\ b{\cdot}i) *_R i)$
proof −
have $*$: $\bigwedge x\ c\ e::real.\ |x - c| \le e \longleftrightarrow x \ge c - e \wedge x \le c + e$
by *auto*
have $**$: $\bigwedge s\ P\ Q.\ s \cap \{x.\ P\ x \wedge Q\ x\} = (s \cap \{x.\ Q\ x\}) \cap \{x.\ P\ x\}$
by *blast*
show *?thesis*
unfolding $*$ $**$ *interval_split*[*OF assms*] **by** (*rule refl*)
qed

lemma *division_doublesplit*:
fixes a :: $'a$::*euclidean_space*
assumes p *division_of* (*cbox* $a\ b$)
and k: $k \in$ *Basis*

shows $(\lambda l.\ l \cap \{x.\ |x \cdot k - c| \leq e\})$ ` $\{l \in p.\ l \cap \{x.\ |x \cdot k - c| \leq e\} \neq \{\}\}$
division_of $(cbox\ a\ b \cap \{x.\ |x \cdot k - c| \leq e\})$
proof −
have $**$: $\bigwedge p\ q\ p'\ q'.\ p$ *division_of* $q \Longrightarrow p = p' \Longrightarrow q = q' \Longrightarrow p'$ *division_of* q'
by *auto*
note *division_split*(*1*)[*OF assms*, **where** *c*=*c*+*e*,*unfolded interval_split*[*OF k*]]
note *division_split*(*2*)[*OF this*, **where** *c*=*c*−*e* **and** *k*=*k*,*OF k*]
then show *?thesis*
apply (*rule* $**$)
subgoal
apply (*simp add*: *abs_diff_le_iff field_simps Collect_conj_eq setcompr_eq_image* [*symmetric*] *cong*: *image_cong_simp*)
apply (*rule equalityI*)
apply *blast*
apply *clarsimp*
apply (*rename_tac S*)
apply (*rule_tac x*=*S* $\cap \{x.\ c + e \geq x \cdot k\}$ **in** *exI*)
apply *auto*
done
by (*simp add*: *interval_split k interval_doublesplit*)
qed

Operative **locale** *operative* = *comm_monoid_set* +
fixes *g* :: *'b*::*euclidean_space set* ⇒ *'a*
assumes *box_empty_imp*: $\bigwedge a\ b.$ *box a b* = {} $\Longrightarrow g\ (cbox\ a\ b) = \mathbf{1}$
and *Basis_imp*: $\bigwedge a\ b\ c\ k.\ k \in Basis \Longrightarrow g\ (cbox\ a\ b) = g\ (cbox\ a\ b \cap \{x.\ x \cdot k \leq c\}) * g\ (cbox\ a\ b \cap \{x.\ x \cdot k \geq c\})$
begin

lemma *empty* [*simp*]: $g\ \{\} = \mathbf{1}$
by (*metis box_empty_imp box_subset_cbox empty_as_interval subset_empty*)

lemma *division*:
$F\ g\ d = g\ (cbox\ a\ b)$ **if** *d division_of* $(cbox\ a\ b)$
proof −
define *C* **where** [*abs_def*]: $C = card\ (division\_points\ (cbox\ a\ b)\ d)$
then show *?thesis*
using *that* **proof** (*induction C arbitrary*: *a b d rule*: *less_induct*)
case (*less a b d*)
show *?case*
proof (*cases box a b* = {})
case *True*
{ fix *k* **assume** *k*∈*d*
then obtain *a' b'* **where** *k*: $k = cbox\ a'\ b'$
using *division_ofD*(*4*)[*OF less.prems*] **by** *blast*
then have $cbox\ a'\ b' \subseteq cbox\ a\ b$
using ‹$k \in d$› *less.prems* **by** *blast*
then have $box\ a'\ b' \subseteq box\ a\ b$
unfolding *subset_box* **by** *auto*

```
      then have g k = 1
        using box_empty_imp [of a' b'] k by (simp add: True)
    }
    with True show F g d = g (cbox a b)
      by (auto intro!: neutral simp: box_empty_imp)
  next
    case False
    then have ab: ∀ i∈Basis. a•i < b•i and ab': ∀ i∈Basis. a•i ≤ b•i
      by (auto simp: box_ne_empty)
    show F g d = g (cbox a b)
    proof (cases division_points (cbox a b) d = {})
      case True
      { fix u v and j :: 'b
        assume j: j ∈ Basis and as: cbox u v ∈ d
        then have cbox u v ≠ {}
          using less.prems by blast
        then have uv: ∀ i∈Basis. u•i ≤ v•i u•j ≤ v•j
          using j unfolding box_ne_empty by auto
        have *: ⋀p r Q. ¬ j∈Basis ∨ p ∨ r ∨ (∀ x∈d. Q x) ⟹ p ∨ r ∨ Q (cbox
u v)
          using as j by auto
        have (j, u•j) ∉ division_points (cbox a b) d
             (j, v•j) ∉ division_points (cbox a b) d using True by auto
       note this[unfolded de_Morgan_conj division_points_def mem_Collect_eq
split_conv interval_bounds[OF ab'] bex_simps]
        note *[OF this(1)] *[OF this(2)] note this[unfolded interval_bounds[OF
uv(1)]]
        moreover
        have a•j ≤ u•j v•j ≤ b•j
          by (meson as division_ofD(2) j less.prems subset_box(1) uv(1))+
        ultimately have u•j = a•j ∧ v•j = a•j ∨ u•j = b•j ∧ v•j = b•j ∨ u•j =
a•j ∧ v•j = b•j
          using uv(2) by force
      }
      then have d': ∀ i∈d. ∃ u v. i = cbox u v ∧
        (∀ j∈Basis. u•j = a•j ∧ v•j = a•j ∨ u•j = b•j ∧ v•j = b•j ∨ u•j = a•j ∧
v•j = b•j)
        unfolding forall_in_division[OF less.prems] by blast
      have (1/2) *R (a+b) ∈ cbox a b
        unfolding mem_box using ab by (auto simp: inner_simps)
      note this[unfolded division_ofD(6)[OF ‹d division_of cbox a b›,symmetric]
Union_iff]
      then obtain i where i: i ∈ d (1 / 2) *R (a + b) ∈ i ..
      obtain u v where uv: i = cbox u v
                       ∀ j∈Basis. u • j = a • j ∧ v • j = a • j ∨
                                  u • j = b • j ∧ v • j = b • j ∨
                                  u • j = a • j ∧ v • j = b • j
        using d' i(1) by auto
      have cbox a b ∈ d
```

```
      proof −
        have u = a v = b
          unfolding euclidean_eq_iff[where 'a='b]
        proof safe
          fix j :: 'b
          assume j: j ∈ Basis
          note i(2)[unfolded uv mem_box]
          then show u · j = a · j and v · j = b · j
            by (smt (verit) False box_midpoint j mem_box(1) uv(2))+
        qed
        then have i = cbox a b using uv by auto
        then show ?thesis using i by auto
      qed
      then have deq: d = insert (cbox a b) (d − {cbox a b})
        by auto
      have F g (d − {cbox a b}) = 1
      proof (intro neutral ballI)
        fix x
        assume x: x ∈ d − {cbox a b}
        then have x∈d
          by auto note d'[rule_format,OF this]
        then obtain u v where uv: x = cbox u v
                  ∀ j∈Basis. u · j = a · j ∧ v · j = a · j ∨
                             u · j = b · j ∧ v · j = b · j ∨
                             u · j = a · j ∧ v · j = b · j
          by blast
        have u ≠ a ∨ v ≠ b
          using x[unfolded uv] by auto
        then obtain j where u·j ≠ a·j ∨ v·j ≠ b·j and j: j ∈ Basis
          unfolding euclidean_eq_iff[where 'a='b] by auto
        then have u·j = v·j
          using uv(2)[rule_format,OF j] by auto
        then show g x = 1
          by (smt (verit) box_empty_imp box_eq_empty(1) j uv(1))
      qed
      then show F g d = g (cbox a b)
    by (metis deq division_of_finite insert_Diff_single insert_remove less.prems
right_neutral)
  next
    case False
    then have ∃ x. x ∈ division_points (cbox a b) d
      by auto
    then obtain k c
      where k ∈ Basis interval_lowerbound (cbox a b) · k < c
            c < interval_upperbound (cbox a b) · k
            ∃ i∈d. interval_lowerbound i · k = c ∨ interval_upperbound i · k = c
      unfolding division_points_def by auto
    then obtain j where a · k < c c < b · k
        and j ∈ d and j: interval_lowerbound j · k = c ∨ interval_upperbound
```

```
j · k = c
        by (metis division_of_trivial empty_iff interval_bounds' less.prems)
      let ?lec = {x. x·k ≤ c} let ?gec = {x. x·k ≥ c}
      define d1 where d1 = {l ∩ ?lec | l. l ∈ d ∧ l ∩ ?lec ≠ {}}
      define d2 where d2 = {l ∩ ?gec | l. l ∈ d ∧ l ∩ ?gec ≠ {}}
      define cb where cb = (∑ i∈Basis. (if i = k then c else b·i) *R i)
      define ca where ca = (∑ i∈Basis. (if i = k then c else a·i) *R i)
      have division_points (cbox a b ∩ ?lec) {l ∩ ?lec |l. l ∈ d ∧ l ∩ ?lec ≠ {}}
            ⊂ division_points (cbox a b) d
        by (rule division_points_psubset[OF ‹d division_of cbox a b› ab ‹a · k <
c› ‹c < b · k› ‹j ∈ d› j ‹k ∈ Basis›])
      with division_points_finite[OF ‹d division_of cbox a b›]
      have card
       (division_points (cbox a b ∩ ?lec) {l ∩ ?lec |l. l ∈ d ∧ l ∩ ?lec ≠ {}})
         < card (division_points (cbox a b) d)
        by (rule psubset_card_mono)
      moreover have division_points (cbox a b ∩ {x. c ≤ x · k}) {l ∩ {x. c ≤ x
· k} |l. l ∈ d ∧ l ∩ {x. c ≤ x · k} ≠ {}}
            ⊂ division_points (cbox a b) d
        by (rule division_points_psubset[OF ‹d division_of cbox a b› ab ‹a · k <
c› ‹c < b · k› ‹j ∈ d› j ‹k ∈ Basis›])
      with division_points_finite[OF ‹d division_of cbox a b›]
      have card (division_points (cbox a b ∩ ?gec) {l ∩ ?gec |l. l ∈ d ∧ l ∩ ?gec
≠ {}})
            < card (division_points (cbox a b) d)
        by (rule psubset_card_mono)
      ultimately have *: F g d1 = g (cbox a b ∩ ?lec) F g d2 = g (cbox a b ∩
?gec)
        unfolding interval_split[OF ‹k ∈ Basis›]
        apply (rule_tac[!] less.hyps)
        using division_split[OF ‹d division_of cbox a b›, where k=k and c=c]
‹k ∈ Basis›
        by (simp_all add: interval_split d1_def d2_def division_points_finite[OF
‹d division_of cbox a b›])
      have fxk_le: g (l ∩ ?lec) = 1
        if l ∈ d y ∈ d l ∩ ?lec = y ∩ ?lec l ≠ y for l y
      proof −
        obtain u v where leq: l = cbox u v
          using ‹l ∈ d› less.prems by auto
        have interior (cbox u v ∩ ?lec) = {}
          using that division_split_left_inj leq less.prems by blast
        then show ?thesis
          unfolding leq interval_split [OF ‹k ∈ Basis›]
          by (auto intro: box_empty_imp)
      qed
      have fxk_ge: g (l ∩ {x. x · k ≥ c}) = 1
        if l ∈ d y ∈ d l ∩ ?gec = y ∩ ?gec l ≠ y for l y
      proof −
        obtain u v where leq: l = cbox u v
```

```
        using ‹l ∈ d› less.prems by auto
      have interior (cbox u v ∩ ?gec) = {}
        using that division_split_right_inj leq less.prems by blast
      then show ?thesis
        unfolding leq interval_split[OF ‹k ∈ Basis›]
        by (auto intro: box_empty_imp)
    qed
    have d1_alt: d1 = (λl. l ∩ ?lec) ` {l ∈ d. l ∩ ?lec ≠ {}}
      using d1_def by auto
    have d2_alt: d2 = (λl. l ∩ ?gec) ` {l ∈ d. l ∩ ?gec ≠ {}}
      using d2_def by auto
    have g (cbox a b) = F g d1 ∗ F g d2 (is _ = ?prev)
      unfolding ∗ using ‹k ∈ Basis›
      by (auto dest: Basis_imp)
    also have F g d1 = F (λl. g (l ∩ ?lec)) d
      unfolding d1_alt using division_of_finite[OF less.prems] fxk_le
      by (subst reindex_nontrivial) (auto intro!: mono_neutral_cong_left)
    also have F g d2 = F (λl. g (l ∩ ?gec)) d
      unfolding d2_alt using division_of_finite[OF less.prems] fxk_ge
      by (subst reindex_nontrivial) (auto intro!: mono_neutral_cong_left)
    also have ∗: ∀x∈d. g x = g (x ∩ ?lec) ∗ g (x ∩ ?gec)
      unfolding forall_in_division[OF ‹d division_of cbox a b›]
      using ‹k ∈ Basis›
      by (auto dest: Basis_imp)
    have F (λl. g (l ∩ ?lec)) d ∗ F (λl. g (l ∩ ?gec)) d = F g d
      using ∗ by (simp add: distrib)
    finally show ?thesis by auto
  qed
 qed
qed

proposition tagged_division:
 assumes d tagged_division_of (cbox a b)
 shows F (λ(_, l). g l) d = g (cbox a b)
 by (metis assms box_empty_imp division division_of_tagged_division over_tagged_division_lemma)

end

locale operative_real = comm_monoid_set +
 fixes g :: real set ⇒ 'a
 assumes neutral: b ≤ a ⟹ g {a..b} = 1
 assumes coalesce_less: a < c ⟹ c < b ⟹ g {a..c} ∗ g {c..b} = g {a..b}
begin

sublocale operative where g = g
 rewrites box = (greaterThanLessThan :: real ⇒ _)
  and cbox = (atLeastAtMost :: real ⇒ _)
  and ⋀x::real. x ∈ Basis ⟷ x = 1
```

```
proof −
  show operative f z g
  proof
    show g (cbox a b) = 1 if box a b = {} for a b
      using that by (simp add: neutral)
    show g (cbox a b) = g (cbox a b ∩ {x. x · k ≤ c}) ∗ g (cbox a b ∩ {x. c ≤ x ·
k})
      if k ∈ Basis for a b c k
    proof −
      from that have [simp]: k = 1
        by simp
      from neutral [of 0 1] neutral [of a a for a] coalesce_less
      have [simp]: g {} = 1 ⋀a. g {a} = 1
        ⋀a b c. a < c ⟹ c < b ⟹ g {a..c} ∗ g {c..b} = g {a..b}
        by auto
      have g {a..b} = g {a..min b c} ∗ g {max a c..b}
        by (auto simp: min_def max_def le_less)
      then show g (cbox a b) = g (cbox a b ∩ {x. x · k ≤ c}) ∗ g (cbox a b ∩ {x.
c ≤ x · k})
        by (simp add: atMost_def [symmetric] atLeast_def [symmetric])
    qed
  qed
  show box = (greaterThanLessThan :: real ⇒ _)
    and cbox = (atLeastAtMost :: real ⇒ _)
    and ⋀x::real. x ∈ Basis ⟷ x = 1
    by (simp_all add: fun_eq_iff)
qed

lemma coalesce_less_eq:
  g {a..c} ∗ g {c..b} = g {a..b} if a ≤ c c ≤ b
  by (metis coalesce_less commute left_neutral less_eq_real_def neutral that)

end

lemma operative_realI:
  operative_real f z g if operative f z g
proof −
  interpret operative f z g
    using that .
  show ?thesis
  proof
    show g {a..b} = z if b ≤ a for a b
      using that box_empty_imp by simp
    show f (g {a..c}) (g {c..b}) = g {a..b} if a < c c < b for a b c
      using that Basis_imp [of 1 a b c]
     by (simp_all add: atMost_def [symmetric] atLeast_def [symmetric] max_def
min_def)
  qed
qed
```

7.13.8 Special case of additivity we need for the FTC

lemma *additive_tagged_division_1*:
fixes f :: *real* $\Rightarrow$ *'a*::*real_normed_vector*
assumes $a \leq b$
and p *tagged_division_of* $\{a..b\}$
shows *sum* $(\lambda(x,K).\ f(Sup\ K) - f(Inf\ K))\ p = f\ b - f\ a$
proof −
let *?f* = $(\lambda K$::*real set*. *if* $K = \{\}$ *then 0 else* f(*interval_upperbound* K) − f(*interval_lowerbound* K))
interpret *operative_real plus 0 ?f*
rewrites *comm_monoid_set.F* $(+)$ *0* = *sum*
by *standard*[*1*] (*auto simp add*: *sum_def*)
have *p_td*: p *tagged_division_of cbox a b*
using *assms*(*2*) *box_real*(*2*) **by** *presburger*
have $**$: *cbox a b* $\neq \{\}$
using *assms*(*1*) **by** *auto*
then have $f\ b - f\ a = (\sum (x,\ l) \in p.$ *if* $l = \{\}$ *then 0 else* f (*interval_upperbound* l) − f (*interval_lowerbound* l))
using *assms*(*2*) *tagged_division* **by** *force*
then show *?thesis*
using *assms* **by** (*auto intro*!: *sum.cong*)
qed

7.13.9 Fine-ness of a partition w.r.t. a gauge

definition *fine* (**infixr** ‹*fine*› *46*)
where d *fine* $s \longleftrightarrow (\forall (x,k) \in s.\ k \subseteq d\ x)$

lemma *fineI*:
assumes $\bigwedge x\ k.\ (x,\ k) \in s \Longrightarrow k \subseteq d\ x$
shows d *fine* s
using *assms* **unfolding** *fine_def* **by** *auto*

lemma *fineD*[*dest*]:
assumes d *fine* s
shows $\bigwedge x\ k.\ (x,k) \in s \Longrightarrow k \subseteq d\ x$
using *assms* **unfolding** *fine_def* **by** *auto*

lemma *fine_Int*: $(\lambda x.\ d1\ x \cap d2\ x)$ *fine* $p \longleftrightarrow$ *d1 fine p* $\wedge$ *d2 fine p*
unfolding *fine_def* **by** *auto*

lemma *fine_Inter*:
$(\lambda x.\ \bigcap \{f\ d\ x \mid d.\ \ d \in s\})$ *fine* $p \longleftrightarrow (\forall d \in s.\ (f\ d)$ *fine* $p)$
unfolding *fine_def* **by** *blast*

lemma *fine_Un*: d *fine p1* $\Longrightarrow$ d *fine p2* $\Longrightarrow$ d *fine* $(p1 \cup p2)$
unfolding *fine_def* **by** *blast*

lemma *fine_Union*: $(\bigwedge p.\ p \in ps \Longrightarrow d$ *fine* $p) \Longrightarrow d$ *fine* $(\bigcup ps)$

unfolding *fine_def* **by** *auto*

lemma *fine_subset*: $p \subseteq q \Longrightarrow d$ *fine* $q \Longrightarrow d$ *fine* p
unfolding *fine_def* **by** *blast*

7.13.10 Some basic combining lemmas

lemma *tagged_division_Union_exists*:
assumes *finite* I
and $\forall i{\in}I.\ \exists p.\ p$ *tagged_division_of* $i \wedge d$ *fine* p
and $\forall i1{\in}I.\ \forall i2{\in}I.\ i1 \neq i2 \longrightarrow$ *interior* $i1 \cap$ *interior* $i2 = \{\}$
and $\bigcup I = i$
obtains p **where** p *tagged_division_of* i **and** d *fine* p
proof −
obtain *pfn* **where** *pfn*:
$\bigwedge x.\ x \in I \Longrightarrow$ *pfn* x *tagged_division_of* x
$\bigwedge x.\ x \in I \Longrightarrow d$ *fine* *pfn* x
using *assms* **by** *metis*
show *thesis*
proof
show $\bigcup$ (*pfn* ‘ I) *tagged_division_of* i
using *assms pfn*(1) *tagged_division_Union* **by** *force*
show d *fine* $\bigcup$ (*pfn* ‘ I)
by (*metis* (*mono_tags*, *lifting*) *fine_Union imageE pfn*(2))
qed
qed

7.13.11 The set we’re concerned with must be closed

lemma *division_of_closed*:
fixes $i :: 'n{::}$*euclidean_space set*
shows s *division_of* $i \Longrightarrow$ *closed* i
by *blast*

7.13.12 General bisection principle for intervals; might be useful elsewhere

lemma *interval_bisection_step*:
fixes *type* :: $'a{::}$*euclidean_space*
assumes *emp*: $P\ \{\}$
and *Un*: $\bigwedge S\ T.\ [\![P\ S;\ P\ T;\ \mathit{interior}(S) \cap \mathit{interior}(T) = \{\}]\!] \Longrightarrow P\ (S \cup T)$
and *non*: $\neg\ P\ (\mathit{cbox}\ a\ (b{::}'a))$
obtains $c\ d$ **where** $\neg\ P\ (\mathit{cbox}\ c\ d)$
and $\bigwedge i.\ i \in \mathit{Basis} \Longrightarrow a{\cdot}i \leq c{\cdot}i \wedge c{\cdot}i \leq d{\cdot}i \wedge d{\cdot}i \leq b{\cdot}i \wedge 2 * (d{\cdot}i - c{\cdot}i) \leq b{\cdot}i - a{\cdot}i$
proof −
have *cbox a b* $\neq \{\}$
using *emp non* **by** *metis*
then have *ab*: $\bigwedge i.\ i{\in}\mathit{Basis} \Longrightarrow a \cdot i \leq b \cdot i$
by (*force simp*: *mem_box*)

have *UN_cases*: ⟦*finite* $\mathcal{F}$;
$\bigwedge S.\ S{\in}\mathcal{F} \Longrightarrow P\ S$;
$\bigwedge S.\ S{\in}\mathcal{F} \Longrightarrow \exists\, a\ b.\ S = cbox\ a\ b$;
$\bigwedge S\ T.\ S{\in}\mathcal{F} \Longrightarrow T{\in}\mathcal{F} \Longrightarrow S \neq T \Longrightarrow interior\ S \cap interior\ T = \{\}$⟧ $\Longrightarrow$
$P\ (\bigcup \mathcal{F})$ **for** $\mathcal{F}$
proof (*induct* $\mathcal{F}$ *rule*: *finite_induct*)
case *empty* **show** *?case*
using *emp* **by** *auto*
next
case (*insert x f*)
then show *?case*
unfolding *Union_insert* **by** (*metis Int_interior_Union_intervals Un insert_iff open_interior*)
qed
let *?ab* = $\lambda i.\ (a{\cdot}i + b{\cdot}i)\ /\ 2$
let *?A* = $\{cbox\ c\ d \mid c\ d{::}'a.\ \forall i{\in}Basis.\ (c{\cdot}i = a{\cdot}i) \wedge (d{\cdot}i = ?ab\ i) \vee (c{\cdot}i = ?ab\ i) \wedge (d{\cdot}i = b{\cdot}i)\}$
have *P* $(\bigcup ?A)$
if $\bigwedge c\ d.\ \forall i{\in}Basis.\ a{\cdot}i \le c{\cdot}i \wedge c{\cdot}i \le d{\cdot}i \wedge d{\cdot}i \le b{\cdot}i \wedge 2 * (d{\cdot}i - c{\cdot}i) \le b{\cdot}i - a{\cdot}i \Longrightarrow P\ (cbox\ c\ d)$
proof (*rule UN_cases*)
let *?B* = $(\lambda S.\ cbox\ (\sum i{\in}Basis.\ (if\ i \in S\ then\ a{\cdot}i\ else\ ?ab\ i) *_R i{::}'a)$
$(\sum i{\in}Basis.\ (if\ i \in S\ then\ ?ab\ i\ else\ b{\cdot}i) *_R i))\ `\ \{s.\ s \subseteq Basis\}$
have *?A* $\subseteq$ *?B*
proof
fix *x*
assume $x \in ?A$
then obtain *c d*
where *x*: $x = cbox\ c\ d$
$\bigwedge i.\ i \in Basis \Longrightarrow c \cdot i = a \cdot i \wedge d \cdot i = ?ab\ i \vee c \cdot i = ?ab\ i \wedge d \cdot i = b \cdot i$
by *blast*
have $c = (\sum i{\in}Basis.\ (if\ c \cdot i = a \cdot i\ then\ a \cdot i\ else\ ?ab\ i) *_R i)$
$d = (\sum i{\in}Basis.\ (if\ c \cdot i = a \cdot i\ then\ ?ab\ i\ else\ b \cdot i) *_R i)$
using *x*(2) *ab* **by** (*fastforce simp add*: *euclidean_eq_iff*[**where** *'a*=*'a*])+
then show $x \in ?B$
unfolding *x* **by** (*rule_tac x*=$\{i.\ i{\in}Basis \wedge c{\cdot}i = a{\cdot}i\}$ **in** *image_eqI*) *auto*
qed
then show *finite ?A*
by (*rule finite_subset*) *auto*
next
fix *S*
assume $S \in ?A$
then obtain *c d*
where *s*: $S = cbox\ c\ d$
$\bigwedge i.\ i \in Basis \Longrightarrow c \cdot i = a \cdot i \wedge d \cdot i = ?ab\ i \vee c \cdot i = ?ab\ i \wedge d \cdot i = b \cdot i$
by *blast*
show *P S*
unfolding *s* **using** *ab s*(2) **by** (*fastforce intro*!: *that*)

show $\exists a\ b.\ S = cbox\ a\ b$
unfolding s **by** *auto*
fix T
assume $T \in ?A$
then obtain $e\ f$ **where** t:
$T = cbox\ e\ f$
$\bigwedge i.\ i \in Basis \Longrightarrow e \cdot i = a \cdot i \wedge f \cdot i = ?ab\ i \vee e \cdot i = ?ab\ i \wedge f \cdot i = b \cdot i$
by *blast*
assume $S \neq T$
then have $\neg\ (c = e \wedge d = f)$
unfolding $s\ t$ **by** *auto*
then obtain i **where** $c{\cdot}i \neq e{\cdot}i \vee d{\cdot}i \neq f{\cdot}i$ **and** i': $i \in Basis$
unfolding *euclidean_eq_iff*[**where** $'a='a$] **by** *auto*
then have i: $c{\cdot}i \neq e{\cdot}i\ d{\cdot}i \neq f{\cdot}i$
using $s(2)\ t(2)$ **by** *fastforce+*
have $*$: $\bigwedge s\ t.\ (\bigwedge a.\ a \in s \Longrightarrow a \in t \Longrightarrow False) \Longrightarrow s \cap t = \{\}$
by *auto*
show $interior\ S \cap interior\ T = \{\}$
unfolding $s\ t$ *interior_cbox*
proof (*rule* $*$)
fix x
assume $x \in box\ c\ d\ x \in box\ e\ f$
then have $c{\cdot}i < f{\cdot}i\ e{\cdot}i < d{\cdot}i$
unfolding *mem_box* **using** i' **by** *force+*
with $i\ i'$ **show** *False*
using $s(2)\ t(2)$ **by** *fastforce*
qed
qed
also have $\bigcup ?A = cbox\ a\ b$
proof (*rule set_eqI*,*rule*)
fix x
assume $x \in \bigcup ?A$
then obtain $c\ d$ **where** x:
$x \in cbox\ c\ d$
$\bigwedge i.\ i \in Basis \Longrightarrow c \cdot i = a \cdot i \wedge d \cdot i = ?ab\ i \vee c \cdot i = ?ab\ i \wedge d \cdot i = b \cdot i$
by *blast*
then show $x \in cbox\ a\ b$
unfolding *mem_box* **by** *force*
next
fix x
assume x: $x \in cbox\ a\ b$
then have $\forall i \in Basis.\ \exists c\ d.\ (c = a{\cdot}i \wedge d = ?ab\ i \vee c = ?ab\ i \wedge d = b{\cdot}i) \wedge c \leq x{\cdot}i \wedge x{\cdot}i \leq d$
(**is** $\forall i \in Basis.\ \exists c\ d.\ ?P\ i\ c\ d$)
unfolding *mem_box* **by** (*metis linear*)
then obtain $\alpha\ \beta$ **where** $\forall i \in Basis.\ (\alpha \cdot i = a \cdot i \wedge \beta \cdot i = ?ab\ i \vee \alpha \cdot i = ?ab\ i \wedge \beta \cdot i = b \cdot i) \wedge \alpha \cdot i \leq x \cdot i \wedge x \cdot i \leq \beta \cdot i$
by (*auto simp*: *choice_Basis_iff*)
then show $x \in \bigcup ?A$

```
    by (force simp add: mem_box)
  qed
  finally show thesis
    by (metis (no_types, lifting) assms(3) that)
qed

lemma interval_bisection:
  fixes type :: 'a::euclidean_space
  assumes P {}
    and Un: ⋀S T. ⟦P S; P T; interior(S) ∩ interior(T) = {}⟧ ⟹ P (S ∪ T)
    and ¬ P (cbox a (b::'a))
  obtains x where x ∈ cbox a b
    and ∀ e>0. ∃ c d. x ∈ cbox c d ∧ cbox c d ⊆ ball x e ∧ cbox c d ⊆ cbox a b ∧
¬ P (cbox c d)
proof −
  have ∀ x. ∃ y. ¬ P (cbox (fst x) (snd x)) ⟶ (¬ P (cbox (fst y) (snd y)) ∧
    (∀ i∈Basis. fst x•i ≤ fst y•i ∧ fst y•i ≤ snd y•i ∧ snd y•i ≤ snd x•i ∧
      2 ∗ (snd y•i − fst y•i) ≤ snd x•i − fst x•i)) (is ∀ x. ?P x)
  proof
    show ?P x for x
    proof (cases P (cbox (fst x) (snd x)))
      case True
      then show ?thesis by auto
    next
      case False
      obtain c d where ¬ P (cbox c d)
        ⋀i. i ∈ Basis ⟹
          fst x • i ≤ c • i ∧
          c • i ≤ d • i ∧
          d • i ≤ snd x • i ∧
          2 ∗ (d • i − c • i) ≤ snd x • i − fst x • i
        by (blast intro: interval_bisection_step[of P, OF assms(1−2) False])
      then show ?thesis
        by (rule_tac x=(c,d) in exI) auto
    qed
  qed
  then obtain f where f:
    ∀ x.
      ¬ P (cbox (fst x) (snd x)) ⟶
      ¬ P (cbox (fst (f x)) (snd (f x))) ∧
        (∀ i∈Basis.
            fst x • i ≤ fst (f x) • i ∧
            fst (f x) • i ≤ snd (f x) • i ∧
            snd (f x) • i ≤ snd x • i ∧
            2 ∗ (snd (f x) • i − fst (f x) • i) ≤ snd x • i − fst x • i) by metis
  define AB A B where ab_def: AB n = (f ^^ n) (a,b) A n = fst(AB n) B n =
snd(AB n) for n
  have [simp]: A 0 = a B 0 = b
    and ABRAW: ⋀n. ¬ P (cbox (A(Suc n)) (B(Suc n))) ∧
```

```
            (∀ i∈Basis. A(n)•i ≤ A(Suc n)•i ∧ A(Suc n)•i ≤ B(Suc n)•i ∧ B(Suc
n)•i ≤ B(n)•i ∧
              2 * (B(Suc n)•i − A(Suc n)•i) ≤ B(n)•i − A(n)•i) (is ⋀n. ?P n)
  proof −
    show A 0 = a B 0 = b
      unfolding ab_def by auto
    note S = ab_def funpow.simps o_def id_apply
    show ?P n for n
    proof (induct n)
      case 0
      then show ?case
        unfolding S using ‹¬ P (cbox a b)› f by auto
    next
      case (Suc n)
      then show ?case
        unfolding S
        by (force intro!: f[rule_format])
    qed
  qed
  then have AB: A(n)•i ≤ A(Suc n)•i A(Suc n)•i ≤ B(Suc n)•i
                B(Suc n)•i ≤ B(n)•i 2 * (B(Suc n)•i − A(Suc n)•i) ≤ B(n)•i −
A(n)•i
              if i∈Basis for i n
    using that by blast+
  have notPAB: ¬ P (cbox (A(Suc n)) (B(Suc n))) for n
    using ABRAW by blast
  have interv: ∃ n. ∀ x∈cbox (A n) (B n). ∀ y∈cbox (A n) (B n). dist x y < e
    if e: 0 < e for e
  proof −
    obtain n where n: (∑ i∈Basis. b • i − a • i) / e < 2 ^ n
      using real_arch_pow[of 2 (sum (λi. b•i − a•i) Basis) / e] by auto
    show ?thesis
    proof (rule exI [where x=n], clarify)
      fix x y
      assume xy: x∈cbox (A n) (B n) y∈cbox (A n) (B n)
      have dist x y ≤ sum (λi. |(x − y)•i|) Basis
        unfolding dist_norm by(rule norm_le_l1)
      also have ... ≤ sum (λi. B n•i − A n•i) Basis
      proof (rule sum_mono)
        fix i :: 'a
        assume i ∈ Basis
        with xy show |(x − y) • i| ≤ B n • i − A n • i
          by (smt (verit, best) inner_diff_left mem_box(2))
      qed
      also have ... ≤ sum (λi. b•i − a•i) Basis / 2^n
        unfolding sum_divide_distrib
      proof (rule sum_mono)
        show B n • i − A n • i ≤ (b • i − a • i) / 2 ^ n if i: i ∈ Basis for i
        proof (induct n)
```

```
      case 0
      then show ?case
        unfolding AB by auto
    next
      case (Suc n)
      have B (Suc n) · i − A (Suc n) · i ≤ (B n · i − A n · i) / 2
        using AB(3) that AB(4)[of i n] using i by auto
      also have ... ≤ (b · i − a · i) / 2 ^ Suc n
        using Suc by (auto simp add: field_simps)
      finally show ?case .
    qed
  qed
  also have ... < e
    using n using e by (auto simp add: field_simps)
  finally show dist x y < e .
 qed
qed
have ABsubset: cbox (A n) (B n) ⊆ cbox (A m) (B m) if m ≤ n for m n
 using that
proof (induction rule: inc_induct)
 case (step i) show ?case
  by (smt (verit, ccfv_SIG) ABRAW in_mono mem_box(2) step.IH subsetI)
qed simp
have ⋀n. cbox (A n) (B n) ≠ {}
 by (meson AB dual_order.trans interval_not_empty)
then obtain x0 where x0: ⋀n. x0 ∈ cbox (A n) (B n)
 using decreasing_closed_nest [OF closed_cbox] ABsubset interv by blast
show thesis
proof (intro that strip)
 show x0 ∈ cbox a b
  using ‹A 0 = a› ‹B 0 = b› x0 by blast
next
 fix e :: real
 assume e > 0
 from interv[OF this] obtain n
  where n: ∀x∈cbox (A n) (B n). ∀y∈cbox (A n) (B n). dist x y < e ..
 have ¬ P (cbox (A n) (B n))
 proof (cases 0 < n)
  case True then show ?thesis
   by (metis Suc_pred' notPAB)
 next
  case False then show ?thesis
   using ‹A 0 = a› ‹B 0 = b› ‹¬ P (cbox a b)› by blast
 qed
 moreover have cbox (A n) (B n) ⊆ ball x0 e
  using n using x0[of n] by auto
 moreover have cbox (A n) (B n) ⊆ cbox a b
  using ABsubset ‹A 0 = a› ‹B 0 = b› by blast
 ultimately show ∃c d. x0 ∈ cbox c d ∧ cbox c d ⊆ ball x0 e ∧ cbox c d ⊆
```

```
cbox a b ∧ ¬ P (cbox c d)
      by (meson x0)
  qed
qed
```

7.13.13 Cousin's lemma

```
lemma fine_division_exists:
  fixes a b :: 'a::euclidean_space
  assumes gauge g
  obtains p where p tagged_division_of (cbox a b) g fine p
proof (cases ∃p. p tagged_division_of (cbox a b) ∧ g fine p)
  case True
  then show ?thesis
    using that by auto
next
  case False
  assume ¬ (∃p. p tagged_division_of (cbox a b) ∧ g fine p)
  obtain x where x:
      x ∈ (cbox a b)
      ⋀e. 0 < e ⟹
        ∃c d.
          x ∈ cbox c d ∧
          cbox c d ⊆ ball x e ∧
          cbox c d ⊆ (cbox a b) ∧
          ¬ (∃p. p tagged_division_of cbox c d ∧ g fine p)
    proof (rule interval_bisection[of λs. ∃p. p tagged_division_of s ∧ _ p, OF _
_ False])
      show ∃p. p tagged_division_of {} ∧ g fine p
        by (simp add: fine_def)
    qed (meson tagged_division_Un fine_Un)+
  obtain e where e: e > 0 ball x e ⊆ g x
    by (meson assms gauge_def openE)
  then obtain c d where c_d: x ∈ cbox c d
                           cbox c d ⊆ ball x e
                           cbox c d ⊆ cbox a b
                           ¬ (∃p. p tagged_division_of cbox c d ∧ g fine p)
    by (meson x(2))
  have g fine {(x, cbox c d)}
    unfolding fine_def using e using c_d(2) by auto
  then show ?thesis
    using tagged_division_of_self[OF c_d(1)] using c_d by simp
qed

lemma fine_division_exists_real:
  fixes a b :: real
  assumes gauge g
  obtains p where p tagged_division_of {a .. b} g fine p
  by (metis assms box_real(2) fine_division_exists)
```

7.13.14 A technical lemma about "refinement" of division

lemma *tagged_division_finer*:
fixes p :: ('*a*::*euclidean_space* × ('*a*::*euclidean_space set*)) *set*
assumes *ptag*: p *tagged_division_of* (*cbox a b*)
and *gauge d*
obtains q **where** q *tagged_division_of* (*cbox a b*)
and d *fine* q
and $\forall (x,K) \in p.\ K \subseteq d(x) \longrightarrow (x,K) \in q$
proof −
have *p*: *finite p p tagged_partial_division_of* (*cbox a b*)
using *ptag tagged_division_of_def* **by** *blast+*
have ($\exists q.\ q$ *tagged_division_of* ($\bigcup \{k.\ \exists x.\ (x,k) \in p\}$) ∧ d *fine* q ∧ ($\forall (x,k) \in p.\ k \subseteq d(x) \longrightarrow (x,k) \in q$))
if *finite p p tagged_partial_division_of* (*cbox a b*) *gauge d* **for** p
using *that*
proof (*induct p*)
case *empty*
show *?case*
by (*force simp add*: *fine_def*)
next
case (*insert xk p*)
obtain x k **where** *xk*: $xk = (x, k)$
using *surj_pair*[*of xk*] **by** *metis*
obtain *q1* **where** *q1*: *q1 tagged_division_of* $\bigcup \{k.\ \exists x.\ (x, k) \in p\}$
and d *fine q1*
and *q1I*: $\bigwedge x\ k.\ [\![(x, k) \in p;\ \ k \subseteq d\ x]\!] \Longrightarrow (x, k) \in q1$
using *insert*
by (*metis* (*mono_tags*, *lifting*) *case_prodD subset_insertI tagged_partial_division_subset*)
have ∗: $\bigcup \{l.\ \exists y.\ (y,l) \in$ *insert xk p*$\} = k \cup \bigcup \{l.\ \exists y.\ (y,l) \in p\}$
unfolding *xk* **by** *auto*
note p = *tagged_partial_division_ofD*[*OF insert*(*4*)]
obtain u v **where** *uv*: $k =$ *cbox u v*
using $p(4)$ *xk* **by** *blast*
have $\{K.\ \exists x.\ (x, K) \in p\} \subseteq$ *snd* ‘ p
by *force*
then have *finite* $\{K.\ \exists x.\ (x, K) \in p\}$
using *finite_surj insert.hyps*(*1*) **by** *blast*
then have *int*: *interior* (*cbox u v*) ∩ *interior* ($\bigcup \{k.\ \exists x.\ (x, k) \in p\}$) = {}
proof (*rule Int_interior_Union_intervals*)
show *open* (*interior* (*cbox u v*))
by *auto*
show $\bigwedge T.\ T \in \{K.\ \exists x.\ (x, K) \in p\} \Longrightarrow \exists a\ b.\ T =$ *cbox a b*
using $p(4)$ **by** *auto*
show $\bigwedge T.\ T \in \{K.\ \exists x.\ (x, K) \in p\} \Longrightarrow$ *interior* (*cbox u v*) ∩ *interior* T = {}
using *insert.hyps*(*2*) $p(5)$ *uv xk* **by** *blast*
qed
show *?case*
proof (*cases cbox u v* ⊆ d x)

case *True*
have {(*x*, *cbox u v*)} *tagged_division_of cbox u v*
by (*simp add*: *p*(2) *uv xk tagged_division_of_self*)
then have {(*x*, *cbox u v*)} ∪ *q1 tagged_division_of* ⋃{*K*. ∃*x*. (*x*, *K*) ∈ *insert xk p*}
by (*smt* (*verit*, *best*) ∗ *int q1 tagged_division_Un uv*)
moreover have *d fine* ({(*x*,*cbox u v*)} ∪ *q1*)
using *True* ‹*d fine q1*› *fine_def* **by** *fastforce*
ultimately show *?thesis*
by (*metis* (*no_types*, *lifting*) *case_prodI2 insert_iff insert_is_Un q1I uv xk*)
next
case *False*
obtain *q2* **where** *q2*: *q2 tagged_division_of cbox u v d fine q2*
using *fine_division_exists*[*OF assms*(2)] **by** *blast*
show *?thesis*
proof (*intro exI conjI*)
show *q2* ∪ *q1 tagged_division_of* ⋃ {*k*. ∃*x*. (*x*, *k*) ∈ *insert xk p*}
by (*smt* (*verit*, *best*) ∗ *int q1 q2*(1) *tagged_division_Un uv*)
show *d fine q2* ∪ *q1*
by (*simp add*: ‹*d fine q1*› *fine_Un q2*(2))
qed (*use False uv xk q1I* **in** *auto*)
qed
qed
with *p* **obtain** *q* **where** *q*: *q tagged_division_of* ⋃{*k*. ∃*x*. (*x*, *k*) ∈ *p*} *d fine q* ∀(*x*, *k*)∈*p*. *k* ⊆ *d x* ⟶ (*x*, *k*) ∈ *q*
by (*meson* ‹*gauge d*›)
with *ptag that* **show** *?thesis* **by** *auto*
qed

Covering lemma

Some technical lemmas used in the approximation results that follow. Proof of the covering lemma is an obvious multidimensional generalization of Lemma 3, p65 of Swartz's "Introduction to Gauge Integrals".

proposition *covering_lemma*:
assumes *S* ⊆ *cbox a b box a b* ≠ {} *gauge g*
obtains 𝒟 **where**
countable 𝒟 ⋃𝒟 ⊆ *cbox a b*
⋀*K*. *K* ∈ 𝒟 ⟹ *interior K* ≠ {} ∧ (∃*c d*. *K* = *cbox c d*)
pairwise (λ*A B*. *interior A* ∩ *interior B* = {}) 𝒟
⋀*K*. *K* ∈ 𝒟 ⟹ ∃*x* ∈ *S* ∩ *K*. *K* ⊆ *g x*
⋀*u v*. *cbox u v* ∈ 𝒟 ⟹ ∃*n*. ∀*i* ∈ *Basis*. *v* ∙ *i* − *u* ∙ *i* = (*b* ∙ *i* − *a* ∙ *i*) / 2^*n*
S ⊆ ⋃𝒟
proof −
have *aibi*: ⋀*i*. *i* ∈ *Basis* ⟹ *a* ∙ *i* ≤ *b* ∙ *i* **and** *normab*: 0 < *norm*(*b* − *a*)
using ‹*box a b* ≠ {}› *box_eq_empty box_idem* **by** *fastforce*+
let *?K0* = λ(*n*, *f*::'*a*⇒*nat*).
cbox (∑*i* ∈ *Basis*. (*a* ∙ *i* + (*f i* / 2^*n*) ∗ (*b* ∙ *i* − *a* ∙ *i*)) ∗$_R$ *i*)

$(\sum i \in Basis.\ (a \cdot i + ((f\ i + 1)\ /\ 2\hat{}n) * (b \cdot i - a \cdot i)) *_R i)$
let *?D0* = *?K0* ‘ (*SIGMA n:UNIV. Pi*$_E$ *Basis* (λ*i*::*'a. lessThan* (*2*^*n*)))
obtain $\mathcal{D}0$ **where** *count*: *countable* $\mathcal{D}0$
and *sub*: $\bigcup \mathcal{D}0 \subseteq$ *cbox a b*
and *int*: $\bigwedge K.\ K \in \mathcal{D}0 \Longrightarrow$ (*interior K* $\neq \{\}$) $\wedge$ ($\exists\, c\ d.\ K =$ *cbox c d*)
and *intdj*: $\bigwedge A\ B.\ [\![A \in \mathcal{D}0;\ B \in \mathcal{D}0]\!] \Longrightarrow A \subseteq B \vee B \subseteq A \vee$ *interior* $A \cap$ *interior* $B = \{\}$
and *SK*: $\bigwedge x.\ x \in S \Longrightarrow \exists\, K \in \mathcal{D}0.\ x \in K \wedge K \subseteq g\ x$
and *cbox*: $\bigwedge u\ v.$ *cbox u v* $\in \mathcal{D}0 \Longrightarrow \exists\, n.\ \forall\, i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)\ /\ 2\hat{}n$
and *fin*: $\bigwedge K.\ K \in \mathcal{D}0 \Longrightarrow$ *finite* $\{L \in \mathcal{D}0.\ K \subseteq L\}$
proof
show *countable ?D0*
by (*simp add*: *countable_PiE*)
next
show $\bigcup$ *?D0* $\subseteq$ *cbox a b*
apply (*simp add*: *UN_subset_iff*)
apply (*intro conjI allI ballI subset_box_imp*)
apply (*simp add*: *field_simps aibi mult_right_mono*)
apply (*force simp*: *aibi scaling_mono nat_less_real_le dest*: *PiE_mem intro*: *mult_right_mono*)
done
next
show $\bigwedge K.\ K \in$ *?D0* $\Longrightarrow$ *interior K* $\neq \{\} \wedge (\exists\, c\ d.\ K =$ *cbox c d*)
using ‹*box a b* $\neq \{\}$›
by (*clarsimp simp*: *box_eq_empty*) (*fastforce simp add*: *field_split_simps dest*: *PiE_mem*)
next
have *realff*: (*real w*) $*$ *2*^*m* $<$ (*real v*) $*$ *2*^*n* $\longleftrightarrow$ *w* $*$ *2*^*m* $<$ *v* $*$ *2*^*n* **for** *m n v w*
using *of_nat_less_iff less_imp_of_nat_less* **by** *fastforce*
have $*$: $\forall\, v\ w.$ *?K0*(*m*,*v*) $\subseteq$ *?K0*(*n*,*w*) $\vee$ *?K0*(*n*,*w*) $\subseteq$ *?K0*(*m*,*v*) $\vee$ *interior*(*?K0*(*m*,*v*)) $\cap$ *interior*(*?K0*(*n*,*w*)) $= \{\}$
for *m n* — The symmetry argument requires a single HOL formula
proof (*rule linorder_wlog* [**where** *a*=*m* **and** *b*=*n*], *intro allI impI*)
fix *v w m* **and** *n*::*nat*
assume $m \leq n$ — WLOG we can assume $m \leq n$, when the first disjunct becomes impossible
have *?K0*(*n*,*w*) $\subseteq$ *?K0*(*m*,*v*) $\vee$ *interior*(*?K0*(*m*,*v*)) $\cap$ *interior*(*?K0*(*n*,*w*)) $= \{\}$
apply (*rule ccontr*)
apply (*simp add*: *subset_box disjoint_interval*)
apply (*clarsimp simp add*: *aibi mult_le_cancel_right divide_le_cancel not_less not_le*)
apply (*drule_tac x*=*i* **in** *bspec*, *assumption*)
using ‹*m*≤*n*› *realff* [*of* _ _ *1*+_] *realff* [*of 1*+_ _ *1*+_]
apply (*auto simp*: *divide_simps add.commute not_le nat_le_iff_add realff*)
apply (*metis* (*no_types*, *opaque_lifting*) *power_add mult_Suc mult_less_cancel2 not_less_eq mult.assoc*)+

```
      done
    then show ?K0(m,v) ⊆ ?K0(n,w) ∨ ?K0(n,w) ⊆ ?K0(m,v) ∨ interior(?K0(m,v))
∩ interior(?K0(n,w)) = {}
      by meson
    qed auto
  show ⋀A B. ⟦A ∈ ?D0; B ∈ ?D0⟧ ⟹ A ⊆ B ∨ B ⊆ A ∨ interior A ∩ interior
B = {}
    using ∗ by fastforce
 next
  show ∃K ∈ ?D0. x ∈ K ∧ K ⊆ g x if x ∈ S for x
  proof (simp only: bex_simps split_paired_Bex_Sigma)
   show ∃n. ∃f ∈ Basis →E {..<2 ^ n}. x ∈ ?K0(n,f) ∧ ?K0(n,f) ⊆ g x
   proof −
    obtain e where 0 < e
              and e: ⋀y. (⋀i. i ∈ Basis ⟹ |x · i − y · i| ≤ e) ⟹ y ∈ g x
    proof −
     have x ∈ g x open (g x)
      using ‹gauge g› by (auto simp: gauge_def)
     then obtain ε where 0 < ε and ε: ball x ε ⊆ g x
      using openE by blast
     have norm (x − y) < ε
        if (⋀i. i ∈ Basis ⟹ |x · i − y · i| ≤ ε / (2 ∗ real DIM('a))) for y
     proof −
      have norm (x − y) ≤ (∑ i∈Basis. |x · i − y · i|)
       by (metis (no_types, lifting) inner_diff_left norm_le_l1 sum.cong)
      also have ... ≤ DIM('a) ∗ (ε / (2 ∗ real DIM('a)))
       by (meson sum_bounded_above that)
      also have ... = ε / 2
       by (simp add: field_split_simps)
      finally show ?thesis
       using ‹0 < ε› by linarith
     qed
     then show ?thesis
       by (rule_tac e = ε / 2 / DIM('a) in that) (simp_all add: ‹0 < ε›
dist_norm subsetD [OF ε])
    qed
    have xab: x ∈ cbox a b
     using ‹x ∈ S› ‹S ⊆ cbox a b› by blast
    obtain n where n: norm (b − a) / 2^n < e
     using real_arch_pow_inv [of e / norm(b − a) 1/2] normab ‹0 < e›
     by (auto simp: field_split_simps)
    then have norm (b − a) < e ∗ 2^n
     by (auto simp: field_split_simps)
    then have bai: b · i − a · i < e ∗ 2 ^ n if i ∈ Basis for i
     by (smt (verit, del_insts) Basis_le_norm diff_add_cancel inner_simps(1)
that)
     have D: (a + n ≤ x ∧ x ≤ a + m) ⟹ (a + n ≤ y ∧ y ≤ a + m) ⟹
abs(x − y) ≤ m − n
          for a m n x and y::real
```

```
      by auto
    have ∀ i∈Basis. ∃ k<2 ^ n. (a · i + real k * (b · i − a · i) / 2 ^ n ≤ x · i ∧
          x · i ≤ a · i + (real k + 1) * (b · i − a · i) / 2 ^ n)
     proof
      fix i::'a assume i ∈ Basis
      consider x · i = b · i | x · i < b · i
        using ‹i ∈ Basis› mem_box(2) xab by force
      then show ∃ k<2 ^ n. (a · i + real k * (b · i − a · i) / 2 ^ n ≤ x · i ∧
                  x · i ≤ a · i + (real k + 1) * (b · i − a · i) / 2 ^ n)
      proof cases
        case 1 then show ?thesis
            by (rule_tac x = 2^n − 1 in exI) (auto simp: algebra_simps
field_split_simps of_nat_diff ‹i ∈ Basis› aibi)
      next
        case 2
        then have abi_less: a · i < b · i
          using ‹i ∈ Basis› xab by (auto simp: mem_box)
        let ?k = nat ⌊2 ^ n * (x · i − a · i) / (b · i − a · i)⌋
        show ?thesis
        proof (intro exI conjI)
          show ?k < 2 ^ n
            using aibi xab ‹i ∈ Basis›
              by (force simp: nat_less_iff floor_less_iff field_split_simps 2
mem_box)
        next
          have a · i + real ?k * (b · i − a · i) / 2 ^ n ≤
              a · i + (2 ^ n * (x · i − a · i) / (b · i − a · i)) * (b · i − a · i) /
2 ^ n
            using aibi [OF ‹i ∈ Basis›] xab 2
             apply (intro add_left_mono mult_right_mono divide_right_mono
of_nat_floor)
              apply (auto simp: ‹i ∈ Basis› mem_box field_split_simps)
            done
          also have ... = x · i
            using abi_less by (simp add: field_split_simps)
          finally show a · i + real ?k * (b · i − a · i) / 2 ^ n ≤ x · i .
        next
          have x · i ≤ a · i + (2 ^ n * (x · i − a · i) / (b · i − a · i)) * (b · i
− a · i) / 2 ^ n
            using abi_less by (simp add: field_split_simps)
          also have ... ≤ a · i + (real ?k + 1) * (b · i − a · i) / 2 ^ n
            using aibi [OF ‹i ∈ Basis›] xab
             apply (intro add_left_mono mult_right_mono divide_right_mono
of_nat_floor)
              apply (auto simp: ‹i ∈ Basis› mem_box divide_simps)
            done
          finally show x · i ≤ a · i + (real ?k + 1) * (b · i − a · i) / 2 ^ n .
        qed
      qed
```

```
      qed
      then have ∃f∈Basis →E {..<2 ^ n}. x ∈ ?K0(n,f)
        apply (simp add: mem_box Bex_def)
        apply (clarify dest!: bchoice)
        apply (rule_tac x=restrict f Basis in exI, simp)
        done
      moreover have ⋀f. x ∈ ?K0(n,f) ⟹ ?K0(n,f) ⊆ g x
        apply (clarsimp simp add: mem_box)
        apply (rule e)
        apply (drule bspec D, assumption)+
        apply (erule order_trans)
        apply (simp add: divide_simps)
        using bai apply (force simp add: algebra_simps)
        done
      ultimately show ?thesis by auto
    qed
  qed
 next
  show ⋀u v. cbox u v ∈ ?D0 ⟹ ∃n. ∀i∈Basis. v · i − u · i = (b · i − a ·
i) / 2^n
    by (force simp: eq_cbox box_eq_empty field_simps dest!: aibi)
 next
  obtain j::'a where j ∈ Basis
    using nonempty_Basis by blast
  have finite {L ∈ ?D0. ?K0(n,f) ⊆ L} if f ∈ Basis →E {..<2 ^ n} for n f
  proof (rule finite_subset)
    let ?B = (λ(n, f::'a⇒nat). cbox (∑i∈Basis. (a · i + (f i) / 2^n * (b · i −
a · i)) *R i)
                                    (∑i∈Basis. (a · i + ((f i) + 1) / 2^n * (b · i −
a · i)) *R i))
            ' (SIGMA m:atMost n. Pi_E Basis (λi::'a. lessThan (2^m)))
     have ?K0(m,g) ∈ ?B if g ∈ Basis →E {..<2 ^ m} ?K0(n,f) ⊆ ?K0(m,g)
for m g
     proof −
       have dd: w / m ≤ v / n ∧ (v+1) / n ≤ (w+1) / m
             ⟹ inverse n ≤ inverse m for w m v n::real
         by (auto simp: field_split_simps)
       have bjaj: b · j − a · j > 0
         using ‹j ∈ Basis› ‹box a b ≠ {}› box_eq_empty(1) by fastforce
       have ((g j) / 2 ^ m) * (b · j − a · j) ≤ ((f j) / 2 ^ n) * (b · j − a · j) ∧
            (((f j) + 1) / 2 ^ n) * (b · j − a · j) ≤ (((g j) + 1) / 2 ^ m) * (b · j
− a · j)
         using that ‹j ∈ Basis› by (simp add: subset_box field_split_simps aibi)
       then have ((g j) / 2 ^ m) ≤ ((f j) / 2 ^ n) ∧
                 ((real(f j) + 1) / 2 ^ n) ≤ ((real(g j) + 1) / 2 ^ m)
      by (metis bjaj mult.commute of_nat_1 of_nat_add mult_le_cancel_left_pos)
       then have inverse (2^n) ≤ (inverse (2^m) :: real)
         by (rule dd)
       then have m ≤ n
```

```
        by auto
      show ?thesis
        by (rule imageI) (simp add: ‹m ≤ n› that)
    qed
    then show {L ∈ ?D0. ?K0(n,f) ⊆ L} ⊆ ?B
      by auto
    show finite ?B
        by (intro finite_imageI finite_SigmaI finite_atMost finite_lessThan finite_PiE finite_Basis)
  qed
  then show finite {L ∈ ?D0. K ⊆ L} if K ∈ ?D0 for K
    using that by auto
 qed
 let ?D1 = {K ∈ 𝒟0. ∃x ∈ S ∩ K. K ⊆ g x}
 obtain 𝒟 where count: countable 𝒟
          and sub: ⋃𝒟 ⊆ cbox a b  S ⊆ ⋃𝒟
          and int:  ⋀K. K ∈ 𝒟 ⟹ (interior K ≠ {}) ∧ (∃c d. K = cbox c d)
          and intdj: ⋀A B. ⟦A ∈ 𝒟; B ∈ 𝒟⟧ ⟹ A ⊆ B ∨ B ⊆ A ∨ interior A ∩ interior B = {}
          and SK: ⋀K. K ∈ 𝒟 ⟹ ∃x. x ∈ S ∩ K ∧ K ⊆ g x
          and cbox: ⋀u v. cbox u v ∈ 𝒟 ⟹ ∃n. ∀i ∈ Basis. v ∙ i − u ∙ i = (b ∙ i − a ∙ i) / 2^n
          and fin: ⋀K. K ∈ 𝒟 ⟹ finite {L. L ∈ 𝒟 ∧ K ⊆ L}
 proof
   show countable ?D1 using count countable_subset
     by (simp add: count countable_subset)
   show ⋃?D1 ⊆ cbox a b
     using sub by blast
   show S ⊆ ⋃?D1
     using SK by (force simp:)
   show ⋀K. K ∈ ?D1 ⟹ (interior K ≠ {}) ∧ (∃c d. K = cbox c d)
     using int by blast
  show ⋀A B. ⟦A ∈ ?D1; B ∈ ?D1⟧ ⟹ A ⊆ B ∨ B ⊆ A ∨ interior A ∩ interior B = {}
     using intdj by blast
   show ⋀K. K ∈ ?D1 ⟹ ∃x. x ∈ S ∩ K ∧ K ⊆ g x
     by auto
   show ⋀u v. cbox u v ∈ ?D1 ⟹ ∃n. ∀i ∈ Basis. v ∙ i − u ∙ i = (b ∙ i − a ∙ i) / 2^n
     using cbox by blast
   show ⋀K. K ∈ ?D1 ⟹ finite {L. L ∈ ?D1 ∧ K ⊆ L}
    using fin by simp (metis (mono_tags, lifting) Collect_mono rev_finite_subset)
 qed
 let ?𝒟 = {K ∈ 𝒟. ∀K'. K' ∈ 𝒟 ∧ K ≠ K' ⟶ ¬(K ⊆ K')}
 show ?thesis
 proof
   show countable ?𝒟
     by (blast intro: countable_subset [OF _ count])
   show ⋃?𝒟 ⊆ cbox a b
```

```
    using sub by blast
  show S ⊆ ⋃ ?D
  proof clarsimp
    fix x
    assume x ∈ S
    then obtain X where x ∈ X X ∈ D using ‹S ⊆ ⋃ D› by blast
    let ?R = {(K,L). K ∈ D ∧ L ∈ D ∧ L ⊂ K}
    have irrR: irrefl ?R by (force simp: irrefl_def)
    have traR: trans ?R by (force simp: trans_def)
    have finR: ⋀x. finite {y. (y, x) ∈ ?R}
    by simp (metis (mono_tags, lifting) fin ‹X ∈ D› finite_subset mem_Collect_eq
psubset_imp_subset subsetI)
    have {X ∈ D. x ∈ X} ≠ {}
      using ‹X ∈ D› ‹x ∈ X› by blast
    then obtain Y where Y ∈ {X ∈ D. x ∈ X} ⋀Y'. (Y', Y) ∈ ?R ⟹ Y' ∉
{X ∈ D. x ∈ X}
      by (rule wfE_min' [OF wf_finite_segments [OF irrR traR finR]]) blast
    then show ∃ Y. Y ∈ D ∧ (∀ K'. K' ∈ D ∧ Y ≠ K' ⟶ ¬ Y ⊆ K') ∧ x ∈ Y
      by blast
  qed
  show ⋀K. K ∈ ?D ⟹ interior K ≠ {} ∧ (∃ c d. K = cbox c d)
    by (blast intro: dest: int)
  show pairwise (λA B. interior A ∩ interior B = {}) ?D
    using intdj by (simp add: pairwise_def) metis
  show ⋀K. K ∈ ?D ⟹ ∃ x ∈ S ∩ K. K ⊆ g x
    using SK by force
  show ⋀u v. cbox u v ∈ ?D ⟹ ∃ n. ∀ i∈Basis. v · i − u · i = (b · i − a · i) /
2^n
    using cbox by force
  qed
qed
```

7.13.15 Division filter

Divisions over all gauges towards finer divisions.

```
definition division_filter :: 'a::euclidean_space set ⇒ ('a × 'a set) set filter
 where division_filter s = (INF g∈{g. gauge g}. principal {p. p tagged_division_of
s ∧ g fine p})

proposition eventually_division_filter:
  (∀F p in division_filter s. P p) ⟷
    (∃ g. gauge g ∧ (∀ p. p tagged_division_of s ∧ g fine p ⟶ P p))
  unfolding division_filter_def
proof (subst eventually_INF_base; clarsimp)
  fix g1 g2 :: 'a ⇒ 'a set show gauge g1 ⟹ gauge g2 ⟹ ∃ x. gauge x ∧
    {p. p tagged_division_of s ∧ x fine p} ⊆ {p. p tagged_division_of s ∧ g1 fine
p} ∧
    {p. p tagged_division_of s ∧ x fine p} ⊆ {p. p tagged_division_of s ∧ g2 fine
p}
```

by (*intro exI*[*of* _ $\lambda x.$ *g1 x* $\cap$ *g2 x*]) (*auto simp*: *fine_Int*)
qed (*auto simp*: *eventually_principal*)

lemma *division_filter_not_empty*: *division_filter* (*cbox a b*) $\neq$ *bot*
unfolding *trivial_limit_def eventually_division_filter*
by (*auto elim*: *fine_division_exists*)

lemma *eventually_division_filter_tagged_division*:
eventually ($\lambda p.$ *p tagged_division_of s*) (*division_filter s*)
using *eventually_division_filter* **by** *auto*

end

7.14 Henstock-Kurzweil Gauge Integration in Many Dimensions

theory *Henstock_Kurzweil_Integration*
imports
Lebesgue_Measure Tagged_Division
begin

lemma *norm_diff2*: $[\![y = y1 + y2;\ x = x1 + x2;\ e = e1 + e2;\ norm(y1 - x1) \leq e1;\ norm(y2 - x2) \leq e2]\!]$
$\Longrightarrow$ $norm(y-x) \leq e$
by (*smt* (*verit*, *ccfv_SIG*) *norm_diff_triangle_ineq*)

lemma *setcomp_dot1*: $\{z.\ P\ (z \cdot (i,0))\} = \{(x,y).\ P(x \cdot i)\}$
by *auto*

lemma *setcomp_dot2*: $\{z.\ P\ (z \cdot (0,i))\} = \{(x,y).\ P(y \cdot i)\}$
by *auto*

lemma *Sigma_Int_Paircomp1*: (*Sigma A B*) $\cap$ $\{(x,\ y).\ P\ x\}$ = *Sigma* ($A \cap \{x.\ P\ x\}$) *B*
by *blast*

lemma *Sigma_Int_Paircomp2*: (*Sigma A B*) $\cap$ $\{(x,\ y).\ P\ y\}$ = *Sigma A* ($\lambda z.$ *B z* $\cap$ $\{y.\ P\ y\}$)
by *blast*

7.14.1 Content (length, area, volume...) of an interval

abbreviation *content* :: *'a*::*euclidean_space set* $\Rightarrow$ *real*
where *content s* $\equiv$ *measure lborel s*

lemma *content_cbox_cases*:
content (*cbox a b*) = (*if* $\forall$ *i*$\in$*Basis*. *a*$\cdot$*i* $\leq$ *b*$\cdot$*i then prod* ($\lambda i.$ *b*$\cdot$*i* $-$ *a*$\cdot$*i*) *Basis else 0*)

by (*simp add*: *measure_lborel_cbox_eq inner_diff*)

lemma *content_cbox*: $\forall i{\in}Basis.\ a{\cdot}i \leq b{\cdot}i \Longrightarrow content\ (cbox\ a\ b) = (\prod i{\in}Basis.\ b{\cdot}i - a{\cdot}i)$
unfolding *content_cbox_cases* **by** *simp*

lemma *content_cbox'*: $cbox\ a\ b \neq \{\} \Longrightarrow content\ (cbox\ a\ b) = (\prod i{\in}Basis.\ b{\cdot}i - a{\cdot}i)$
by (*simp add*: *box_ne_empty inner_diff*)

lemma *content_cbox_if*: $content\ (cbox\ a\ b) = (if\ cbox\ a\ b = \{\}\ then\ 0\ else\ \prod i{\in}Basis.\ b{\cdot}i - a{\cdot}i)$
by (*simp add*: *content_cbox'*)

lemma *content_cbox_cart*:
$cbox\ a\ b \neq \{\} \Longrightarrow content(cbox\ a\ b) = prod\ (\lambda i.\ b\$i - a\$i)\ UNIV$
by (*simp add*: *content_cbox_if Basis_vec_def cart_eq_inner_axis axis_eq_axis prod.UNION_disjoint*)

lemma *content_cbox_if_cart*:
$content(cbox\ a\ b) = (if\ cbox\ a\ b = \{\}\ then\ 0\ else\ prod\ (\lambda i.\ b\$i - a\$i)\ UNIV)$
by (*simp add*: *content_cbox_cart*)

lemma *content_division_of*:
assumes $K \in \mathcal{D}\ \mathcal{D}\ division\_of\ S$
shows $content\ K = (\prod i \in Basis.\ interval\_upperbound\ K \cdot i - interval\_lowerbound\ K \cdot i)$
proof −
obtain *a b* **where** $K = cbox\ a\ b$
using *cbox_division_memE assms* **by** *metis*
then show *?thesis*
using *assms* **by** (*force simp*: *division_of_def content_cbox'*)
qed

lemma *content_real*: $a \leq b \Longrightarrow content\ \{a..b\} = b - a$
by *simp*

lemma *abs_eq_content*: $|y - x| = (if\ x{\leq}y\ then\ content\ \{x..y\}\ else\ content\ \{y..x\})$
by (*auto simp*: *content_real*)

lemma *content_singleton*: $content\ \{a\} = 0$
by *simp*

lemma *content_unit*[*iff*]: $content\ (cbox\ 0\ (One{::}'a{::}euclidean\_space)) = 1$
by *simp*

lemma *content_pos_le* [*iff*]: $0 \leq content\ X$
by *simp*

corollary *content_nonneg* [*simp*]: ¬ *content* (*cbox a b*) < *0*
using *not_le* **by** *blast*

lemma *content_pos_lt*: ∀ *i*∈*Basis*. *a*•*i* < *b*•*i* ⟹ *0* < *content* (*cbox a b*)
by (*auto simp*: *less_imp_le inner_diff box_eq_empty intro*!: *prod_pos*)

lemma *content_eq_0*: *content* (*cbox a b*) = *0* ⟷ (∃ *i*∈*Basis*. *b*•*i* ≤ *a*•*i*)
by (*auto simp*: *content_cbox_cases not_le intro*: *less_imp_le antisym eq_refl*)

lemma *content_eq_0_interior*: *content* (*cbox a b*) = *0* ⟷ *interior*(*cbox a b*) = {}
unfolding *content_eq_0 interior_cbox box_eq_empty* **by** *auto*

lemma *content_pos_lt_eq*: *0* < *content* (*cbox a* (*b*::′*a*::*euclidean_space*)) ⟷ (∀ *i*∈*Basis*. *a*•*i* < *b*•*i*)
by (*auto simp add*: *content_cbox_cases less_le prod_nonneg*)

lemma *content_empty* [*simp*]: *content* {} = *0*
by *simp*

lemma *content_real_if* [*simp*]: *content* {*a*..*b*} = (*if a* ≤ *b then b* − *a else 0*)
by (*simp add*: *content_real*)

lemma *content_subset*: *cbox a b* ⊆ *cbox c d* ⟹ *content* (*cbox a b*) ≤ *content* (*cbox c d*)
unfolding *measure_def*
by (*intro enn2real_mono emeasure_mono*) (*auto simp*: *emeasure_lborel_cbox_eq*)

lemma *content_lt_nz*: *0* < *content* (*cbox a b*) ⟷ *content* (*cbox a b*) ≠ *0*
by (*fact zero_less_measure_iff*)

lemma *content_Pair*: *content* (*cbox* (*a*,*c*) (*b*,*d*)) = *content* (*cbox a b*) ∗ *content* (*cbox c d*)
unfolding *measure_lborel_cbox_eq Basis_prod_def*
apply (*subst prod.union_disjoint*)
apply (*auto simp*: *bex_Un ball_Un*)
apply (*subst* (*1 2*) *prod.reindex_nontrivial*)
apply *auto*
done

lemma *content_cbox_pair_eq0_D*:
content (*cbox* (*a*,*c*) (*b*,*d*)) = *0* ⟹ *content* (*cbox a b*) = *0* ∨ *content* (*cbox c d*) = *0*
by (*simp add*: *content_Pair*)

lemma *content_cbox_plus*:
fixes *x* :: ′*a*::*euclidean_space*
shows *content*(*cbox x* (*x* + *h* ∗$_R$ *One*)) = (*if h* ≥ *0 then h* ^ *DIM*(′*a*) *else 0*)
by (*simp add*: *algebra_simps content_cbox_if box_eq_empty*)

lemma *content_0_subset*: *content*(*cbox a b*) = *0* ⟹ *s* ⊆ *cbox a b* ⟹ *content s* = *0*
using *emeasure_mono*[*of s cbox a b lborel*]
by (*auto simp*: *measure_def enn2real_eq_0_iff emeasure_lborel_cbox_eq*)

lemma *content_ball_pos*:
assumes *r* > *0*
shows *content* (*ball c r*) > *0*
proof −
from *rational_boxes*[*OF assms*, *of c*] **obtain** *a b* **where** *ab*: *c* ∈ *box a b box a b* ⊆ *ball c r*
by *auto*
then have *0* < *content* (*box a b*)
by (*subst measure_lborel_box_eq*) (*auto intro*!: *prod_pos simp*: *algebra_simps box_def*)
have *emeasure lborel* (*box a b*) ≤ *emeasure lborel* (*ball c r*)
using *ab* **by** (*intro emeasure_mono*) *auto*
then show *?thesis*
using ‹*content* (*box a b*) > *0*›
by (*smt* (*verit*, *best*) *Sigma_Algebra.measure_def emeasure_lborel_ball_finite enn2real_mono infinity_ennreal_def*)
qed

lemma *content_cball_pos*:
assumes *r* > *0*
shows *content* (*cball c r*) > *0*
proof −
from *rational_boxes*[*OF assms*, *of c*] **obtain** *a b* **where** *ab*: *c* ∈ *box a b box a b* ⊆ *ball c r*
by *auto*
then have *0* < *content* (*box a b*)
by (*subst measure_lborel_box_eq*) (*auto intro*!: *prod_pos simp*: *algebra_simps box_def*)
have *emeasure lborel* (*box a b*) ≤ *emeasure lborel* (*cball c r*)
using *ab* **by** (*intro emeasure_mono*) *auto*
also have *emeasure lborel* (*box a b*) = *ennreal* (*content* (*box a b*))
using *emeasure_lborel_box_finite*[*of a b*] **by** (*intro emeasure_eq_ennreal_measure*) *auto*
also have *emeasure lborel* (*cball c r*) = *ennreal* (*content* (*cball c r*))
using *emeasure_lborel_cball_finite*[*of c r*] **by** (*intro emeasure_eq_ennreal_measure*) *auto*
finally show *?thesis*
using ‹*content* (*box a b*) > *0*› **by** *simp*
qed

lemma *content_split*:
fixes *a* :: *'a*::*euclidean_space*
assumes *k* ∈ *Basis*

shows *content* (*cbox a b*) = *content*(*cbox a b* ∩ {*x*. *x*·*k* ≤ *c*}) + *content*(*cbox a b* ∩ {*x*. *x*·*k* ≥ *c*})
— Prove using measure theory
proof (*cases* ∀ *i*∈*Basis*. *a* · *i* ≤ *b* · *i*)
case *True*
have *1*: ⋀*X Y Z*. (∏ *i*∈*Basis*. *Z i* (*if i* = *k then X else Y i*)) = *Z k X* ∗ (∏ *i*∈*Basis*−{*k*}. *Z i* (*Y i*))
by (*simp add*: *if_distrib prod.delta_remove assms*)
note *simps* = *interval_split*[*OF assms*] *content_cbox_cases*
have *2*: (∏ *i*∈*Basis*. *b*·*i* − *a*·*i*) = (∏ *i*∈*Basis*−{*k*}. *b*·*i* − *a*·*i*) ∗ (*b*·*k* − *a*·*k*)
by (*metis* (*no_types*, *lifting*) *assms finite_Basis mult.commute prod.remove*)
have ⋀*x*. *min* (*b* · *k*) *c* = *max* (*a* · *k*) *c* ⟹
x ∗ (*b*·*k* − *a*·*k*) = *x* ∗ (*max* (*a* · *k*) *c* − *a* · *k*) + *x* ∗ (*b* · *k* − *max* (*a* · *k*) *c*)
by (*auto simp add*: *field_simps*)
moreover
have ∗∗: (∏ *i*∈*Basis*. ((∑ *i*∈*Basis*. (*if i* = *k then min* (*b* · *k*) *c else b* · *i*) ∗*R i*) · *i* − *a* · *i*)) =
(∏ *i*∈*Basis*. (*if i* = *k then min* (*b* · *k*) *c else b* · *i*) − *a* · *i*)
(∏ *i*∈*Basis*. *b* · *i* − ((∑ *i*∈*Basis*. (*if i* = *k then max* (*a* · *k*) *c else a* · *i*) ∗*R i*) · *i*)) =
(∏ *i*∈*Basis*. *b* · *i* − (*if i* = *k then max* (*a* · *k*) *c else a* · *i*))
by (*auto intro*!: *prod.cong*)
have ¬ *a* · *k* ≤ *c* ⟹ ¬ *c* ≤ *b* · *k* ⟹ *False*
unfolding *not_le* **using** *True assms* **by** *auto*
ultimately show *?thesis*
using *assms* **unfolding** *simps* ∗∗ *1*[*of* λ*i x*. *b*·*i* − *x*] *1*[*of* λ*i x*. *x* − *a*·*i*] *2*
by *auto*
next
case *False*
then have *cbox a b* = {}
unfolding *box_eq_empty* **by** (*auto simp*: *not_le*)
then show *?thesis*
by (*auto simp*: *not_le*)
qed

lemma *division_of_content_0*:
assumes *content* (*cbox a b*) = *0 d division_of* (*cbox a b*) *K* ∈ *d*
shows *content K* = *0*
unfolding *forall_in_division*[*OF assms*(*2*)]
by (*meson assms content_0_subset division_of_def*)

lemma *sum_content_null*:
assumes *content* (*cbox a b*) = *0*
and *p tagged_division_of* (*cbox a b*)
shows (∑ (*x*,*K*)∈*p*. *content K* ∗*R f x*) = (*0*::′*a*::*real_normed_vector*)
proof (*intro sum.neutral strip*)
fix *y*
assume *y*: *y* ∈ *p*
obtain *x K* **where** *xk*: *y* = (*x*, *K*)

using *surj_pair*[*of y*] **by** *blast*
then obtain *c d* **where** *k*: $K = cbox\ c\ d\ K \subseteq cbox\ a\ b$
by (*metis assms*(2) *tagged_division_ofD*(3) *tagged_division_ofD*(4) *y*)
have (λ(*x'*,*K'*). *content K'* $*_R$ *f x'*) *y* = *content K* $*_R$ *f x*
unfolding *xk* **by** *auto*
also have ... = *0*
using *assms*(1) *content_0_subset k*(2) **by** *auto*
finally show (λ(*x*, *k*). *content k* $*_R$ *f x*) *y* = *0* **.**
qed

global_interpretation *sum_content*: *operative plus 0 content*
rewrites *comm_monoid_set.F plus 0* = *sum*
proof −
interpret *operative plus 0 content*
by *standard* (*auto simp add*: *content_split* [*symmetric*] *content_eq_0_interior*)
show *operative plus 0 content*
by *standard*
show *comm_monoid_set.F plus 0* = *sum*
by (*simp add*: *sum_def*)
qed

lemma *additive_content_division*: *d division_of* (*cbox a b*) $\Longrightarrow$ *sum content d* = *content* (*cbox a b*)
by (*fact sum_content.division*)

lemma *additive_content_tagged_division*:
d tagged_division_of (*cbox a b*) $\Longrightarrow$ *sum* (λ(*x*,*l*). *content l*) *d* = *content* (*cbox a b*)
by (*fact sum_content.tagged_division*)

lemma *subadditive_content_division*:
assumes $\mathcal{D}$ *division_of S S* $\subseteq$ *cbox a b*
shows *sum content* $\mathcal{D} \leq$ *content*(*cbox a b*)
proof −
have $\mathcal{D}$ *division_of* $\bigcup\mathcal{D}$ $\bigcup\mathcal{D}$ $\subseteq$ *cbox a b*
using *assms* **by** *auto*
then obtain $\mathcal{D}'$ **where** $\mathcal{D} \subseteq \mathcal{D}'$ $\mathcal{D}'$ *division_of cbox a b*
using *partial_division_extend_interval* **by** *metis*
then have *sum content* $\mathcal{D} \leq$ *sum content* $\mathcal{D}'$
using *sum_mono2* **by** *blast*
also have ... $\leq$ *content*(*cbox a b*)
by (*simp add*: ‹$\mathcal{D}'$ *division_of cbox a b*› *additive_content_division less_eq_real_def*)
finally show *?thesis* **.**
qed

lemma *content_real_eq_0*: *content* {*a*..*b*::*real*} = *0* $\longleftrightarrow$ *a* $\geq$ *b*
by *simp*

lemma *property_empty_interval*: $\forall$ *a b*. *content* (*cbox a b*) = *0* $\longrightarrow$ *P* (*cbox a b*)

$\Longrightarrow P\ \{\}$
using *content_empty* **unfolding** *empty_as_interval* **by** *auto*

lemma *interval_bounds_nz_content* [*simp*]:
assumes *content* (*cbox a b*) $\neq$ *0*
shows *interval_upperbound* (*cbox a b*) = *b*
and *interval_lowerbound* (*cbox a b*) = *a*
by (*metis assms content_empty interval_bounds'*)+

7.14.2 Gauge integral

Case distinction to define it first on compact intervals first, then use a limit. This is only much later unified. In Fremlin: Measure Theory, Volume 4I this is generalized using residual sets.

definition *has_integral* :: ('*n*::*euclidean_space* $\Rightarrow$ '*b*::*real_normed_vector*) $\Rightarrow$ '*b* $\Rightarrow$ '*n set* $\Rightarrow$ *bool*
(**infixr** ‹*has'_integral*› *46*)
where (*f has_integral I*) *s* $\longleftrightarrow$
(*if* $\exists$ *a b*. *s* = *cbox a b*
then (λ*p*. $\sum$ (*x*,*k*)$\in$*p*. *content k* $*_R$ *f x*) $\longrightarrow$ *I*) (*division_filter s*)
else ($\forall$ *e*>*0*. $\exists$ *B*>*0*. $\forall$ *a b*. *ball 0 B* $\subseteq$ *cbox a b* $\longrightarrow$
($\exists$ *z*. (λ*p*. $\sum$ (*x*,*k*)$\in$*p*. *content k* $*_R$ (*if x* $\in$ *s then f x else 0*)) $\longrightarrow$ *z*) (*division_filter* (*cbox a b*)) $\wedge$
norm (*z* $-$ *I*) < *e*)))

lemma *has_integral_cbox*:
(*f has_integral I*) (*cbox a b*) $\longleftrightarrow$ ((λ*p*. $\sum$ (*x*,*k*)$\in$*p*. *content k* $*_R$ *f x*) $\longrightarrow$ *I*) (*division_filter* (*cbox a b*))
by (*auto simp add*: *has_integral_def*)

lemma *has_integral*:
(*f has_integral y*) (*cbox a b*) $\longleftrightarrow$
($\forall$ *e*>*0*. $\exists\gamma$. *gauge* γ $\wedge$
($\forall\mathcal{D}$. $\mathcal{D}$ *tagged_division_of* (*cbox a b*) $\wedge$ γ *fine* $\mathcal{D}$ $\longrightarrow$
norm (*sum* (λ(*x*,*k*). *content*(*k*) $*_R$ *f x*) $\mathcal{D}$ $-$ *y*) < *e*))
by (*auto simp*: *dist_norm eventually_division_filter has_integral_def tendsto_iff*)

lemma *has_integral_real*:
(*f has_integral y*) {*a*..*b*::*real*} $\longleftrightarrow$
($\forall$ *e*>*0*. $\exists\gamma$. *gauge* γ $\wedge$
($\forall\mathcal{D}$. $\mathcal{D}$ *tagged_division_of* {*a*..*b*} $\wedge$ γ *fine* $\mathcal{D}$ $\longrightarrow$
norm (*sum* (λ(*x*,*k*). *content*(*k*) $*_R$ *f x*) $\mathcal{D}$ $-$ *y*) < *e*))
unfolding *box_real*[*symmetric*] **by** (*rule has_integral*)

lemma *has_integralD*[*dest*]:
assumes (*f has_integral y*) (*cbox a b*)
and *e* > *0*
obtains γ
where *gauge* γ

and $\bigwedge\mathcal{D}$. $\mathcal{D}$ *tagged_division_of* (*cbox a b*) $\Longrightarrow$ γ *fine* $\mathcal{D}$ $\Longrightarrow$
norm (($\sum$ (*x*,*k*)$\in\mathcal{D}$. *content k* $*_R$ *f x*) − *y*) < *e*
using *assms* **unfolding** *has_integral* **by** *auto*

lemma *has_integral_alt*:
(*f has_integral y*) *i* $\longleftrightarrow$
(*if* $\exists$ *a b*. *i* = *cbox a b*
then (*f has_integral y*) *i*
else ($\forall$ *e*>0. $\exists$ *B*>0. $\forall$ *a b*. *ball 0 B* $\subseteq$ *cbox a b* $\longrightarrow$
($\exists$ *z*. ((*λx*. *if x* $\in$ *i then f x else 0*) *has_integral z*) (*cbox a b*) $\wedge$ *norm* (*z* − *y*) < *e*)))
by (*subst has_integral_def*) (*auto simp add*: *has_integral_cbox*)

lemma *has_integral_altD*:
assumes (*f has_integral y*) *i*
and ¬ ($\exists$ *a b*. *i* = *cbox a b*)
and *e*>0
obtains *B* **where** *B* > 0
and $\forall$ *a b*. *ball 0 B* $\subseteq$ *cbox a b* $\longrightarrow$
($\exists$ *z*. ((*λx*. *if x* $\in$ *i then f*(*x*) *else 0*) *has_integral z*) (*cbox a b*) $\wedge$ *norm*(*z* − *y*) < *e*)
using *assms has_integral_alt*[*of f y i*] **by** *auto*

definition *integrable_on* (**infixr** ‹*integrable'_on*› *46*)
where *f integrable_on i* $\longleftrightarrow$ ($\exists$ *y*. (*f has_integral y*) *i*)

definition *integral i f* = (*SOME y*. (*f has_integral y*) *i* $\vee$ ¬ *f integrable_on i* $\wedge$ *y*=*0*)

lemma *integrable_integral*[*intro*]: *f integrable_on i* $\Longrightarrow$ (*f has_integral* (*integral i f*)) *i*
unfolding *integrable_on_def integral_def* **by** (*metis* (*mono_tags*, *lifting*))

lemma *not_integrable_integral*: ¬ *f integrable_on i* $\Longrightarrow$ *integral i f* = *0*
unfolding *integrable_on_def integral_def* **by** *blast*

lemma *has_integral_integrable*[*dest*]: (*f has_integral i*) *s* $\Longrightarrow$ *f integrable_on s*
unfolding *integrable_on_def* **by** *auto*

lemma *has_integral_integral*: *f integrable_on s* $\longleftrightarrow$ (*f has_integral* (*integral s f*)) *s*
by *auto*

7.14.3 Basic theorems about integrals

lemma *has_integral_eq_rhs*: (*f has_integral j*) *S* $\Longrightarrow$ *i* = *j* $\Longrightarrow$ (*f has_integral i*) *S*
by (*rule forw_subst*)

lemma *has_integral_unique_cbox*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
shows (*f has_integral k1*) (*cbox a b*) ⟹ (*f has_integral k2*) (*cbox a b*) ⟹ *k1* = *k2*
by (*meson division_filter_not_empty has_integral_cbox tendsto_unique*)

lemma *has_integral_unique*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
assumes (*f has_integral k1*) *i* (*f has_integral k2*) *i*
shows *k1* = *k2*
proof (*rule ccontr*)
let *?e* = *norm* (*k1* − *k2*)/*2*
let *?F* = (λ*x*. *if x* ∈ *i then f x else 0*)
assume *k1* ≠ *k2*
then have *e*: *?e* > *0*
by *auto*
have *nonbox*: ¬ (∃ *a b*. *i* = *cbox a b*)
using ‹*k1* ≠ *k2*› *assms has_integral_unique_cbox* **by** *blast*
obtain *B1* **where** *B1*:
0 < *B1*
⋀*a b*. *ball 0 B1* ⊆ *cbox a b* ⟹
∃ *z*. (*?F has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *k1*) < *norm* (*k1* − *k2*)/*2*
by (*rule has_integral_altD*[*OF assms*(*1*) *nonbox*,*OF e*]) *blast*
obtain *B2* **where** *B2*:
0 < *B2*
⋀*a b*. *ball 0 B2* ⊆ *cbox a b* ⟹
∃ *z*. (*?F has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *k2*) < *norm* (*k1* − *k2*)/*2*
by (*rule has_integral_altD*[*OF assms*(*2*) *nonbox*,*OF e*]) *blast*
obtain *a b* :: *'n* **where** *ab*: *ball 0 B1* ⊆ *cbox a b ball 0 B2* ⊆ *cbox a b*
by (*metis Un_subset_iff bounded_Un bounded_ball bounded_subset_cbox_symmetric*)
obtain *w* **where** *w*: (*?F has_integral w*) (*cbox a b*) *norm* (*w* − *k1*) < *norm* (*k1* − *k2*)/*2*
using *B1*(*2*)[*OF ab*(*1*)] **by** *blast*
obtain *z* **where** *z*: (*?F has_integral z*) (*cbox a b*) *norm* (*z* − *k2*) < *norm* (*k1* − *k2*)/*2*
using *B2*(*2*)[*OF ab*(*2*)] **by** *blast*
have *z* = *w*
using *has_integral_unique_cbox*[*OF w*(*1*) *z*(*1*)] **by** *auto*
then have *norm* (*k1* − *k2*) ≤ *norm* (*z* − *k2*) + *norm* (*w* − *k1*)
using *norm_triangle_ineq4* [*of k1* − *w k2* − *z*]
by (*auto simp add*: *norm_minus_commute*)
also have ... < *norm* (*k1* − *k2*)/*2* + *norm* (*k1* − *k2*)/*2*
by (*metis add_strict_mono z*(*2*) *w*(*2*))
finally show *False* **by** *auto*
qed

lemma *integral_unique* [*intro*]: (*f has_integral y*) *k* ⟹ *integral k f* = *y*
unfolding *integral_def*
by (*rule some_equality*) (*auto intro*: *has_integral_unique*)

lemma *has_integral_iff*: (*f has_integral i*) *S* ⟷ (*f integrable_on S* ∧ *integral S f* = *i*)
by *blast*

lemma *eq_integralD*: *integral k f* = *y* ⟹ (*f has_integral y*) *k* ∨ ¬ *f integrable_on k* ∧ *y*=*0*
unfolding *integral_def integrable_on_def*
by (*metis* (*mono_tags*, *lifting*))

lemma *has_integral_const* [*intro*]:
fixes *a b* :: *'a*::*euclidean_space*
shows ((λ*x*. *c*) *has_integral* (*content* (*cbox a b*) ∗$_R$ *c*)) (*cbox a b*)
using *eventually_division_filter_tagged_division*[*of cbox a b*]
additive_content_tagged_division[*of* _ *a b*]
by (*auto simp*: *has_integral_cbox split_beta' scaleR_sum_left*[*symmetric*]
elim!: *eventually_mono intro*!: *tendsto_cong*[*THEN iffD1*, *OF* _ *tendsto_const*])

lemma *has_integral_const_real* [*intro*]:
fixes *a b* :: *real*
shows ((λ*x*. *c*) *has_integral* (*content* {*a*..*b*} ∗$_R$ *c*)) {*a*..*b*}
by (*metis box_real*(*2*) *has_integral_const*)

lemma *has_integral_integrable_integral*: (*f has_integral i*) *s* ⟷ *f integrable_on s* ∧ *integral s f* = *i*
by *blast*

lemma *integral_const* [*simp*]:
fixes *a b* :: *'a*::*euclidean_space*
shows *integral* (*cbox a b*) (λ*x*. *c*) = *content* (*cbox a b*) ∗$_R$ *c*
by *blast*

lemma *integral_const_real* [*simp*]:
fixes *a b* :: *real*
shows *integral* {*a*..*b*} (λ*x*. *c*) = *content* {*a*..*b*} ∗$_R$ *c*
by *blast*

lemma *has_integral_is_0_cbox*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
assumes ⋀*x*. *x* ∈ *cbox a b* ⟹ *f x* = *0*
shows (*f has_integral 0*) (*cbox a b*)
unfolding *has_integral_cbox*
using *eventually_division_filter_tagged_division*[*of cbox a b*] *assms*
by (*subst tendsto_cong*[**where** *g*=λ_. *0*])
(*auto elim*!: *eventually_mono intro*!: *sum.neutral simp*: *tag_in_interval*)

lemma *has_integral_is_0*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*real_normed_vector*

```
  assumes ⋀x. x ∈ S ⟹ f x = 0
  shows (f has_integral 0) S
proof (cases (∃ a b. S = cbox a b))
  case True with assms has_integral_is_0_cbox show ?thesis
    by blast
next
  case False
  have ∗: (λx. if x ∈ S then f x else 0) = (λx. 0)
    by (auto simp add: assms)
  show ?thesis
    using has_integral_is_0_cbox False
    by (subst has_integral_alt) (force simp add: ∗)
qed

lemma has_integral_0[simp]: ((λx::'n::euclidean_space. 0) has_integral 0) S
  by (rule has_integral_is_0) auto

lemma has_integral_0_eq[simp]: ((λx. 0) has_integral i) S ⟷ i = 0
  using has_integral_unique[OF has_integral_0] by auto

lemma has_integral_linear_cbox:
  fixes f :: 'n::euclidean_space ⇒ 'a::real_normed_vector
  assumes f: (f has_integral y) (cbox a b)
    and h: bounded_linear h
  shows ((h ∘ f) has_integral (h y)) (cbox a b)
proof −
  interpret bounded_linear h using h .
  show ?thesis
   unfolding has_integral_cbox using tendsto [OF f [unfolded has_integral_cbox]]
    by (simp add: sum scaleR split_beta')
qed

lemma has_integral_linear:
  fixes f :: 'n::euclidean_space ⇒ 'a::real_normed_vector
  assumes f: (f has_integral y) S
    and h: bounded_linear h
  shows ((h ∘ f) has_integral (h y)) S
proof (cases (∃ a b. S = cbox a b))
  case True with f h has_integral_linear_cbox show ?thesis
    by blast
next
  case False
  interpret bounded_linear h using h .
  from pos_bounded obtain B where B: 0 < B ⋀x. norm (h x) ≤ norm x ∗ B
    by blast
  let ?S = λf x. if x ∈ S then f x else 0
  show ?thesis
  proof (subst has_integral_alt, clarsimp simp: False)
    fix e :: real
```

```
    assume e: e > 0
    have *: 0 < e/B using e B(1) by simp
    obtain M where M:
      M > 0
      ⋀a b. ball 0 M ⊆ cbox a b ⟹
        ∃z. (?S f has_integral z) (cbox a b) ∧ norm (z − y) < e/B
      using has_integral_altD[OF f False *] by blast
    show ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
      (∃z. (?S(h ∘ f) has_integral z) (cbox a b) ∧ norm (z − h y) < e)
    proof (rule exI, intro allI conjI impI)
      show M > 0 using M by metis
    next
      fix a b::'n
      assume sb: ball 0 M ⊆ cbox a b
      obtain z where z: (?S f has_integral z) (cbox a b) norm (z − y) < e/B
        using M(2)[OF sb] by blast
      have *: ?S(h ∘ f) = h ∘ ?S f
        using zero by auto
      show ∃z. (?S(h ∘ f) has_integral z) (cbox a b) ∧ norm (z − h y) < e
      proof (intro exI conjI)
        show (?S(h ∘ f) has_integral h z) (cbox a b)
          by (simp add: * has_integral_linear_cbox[OF z(1) h])
        show norm (h z − h y) < e
          by (metis B diff le_less_trans pos_less_divide_eq z(2))
      qed
    qed
  qed
qed

lemma has_integral_scaleR_left:
  (f has_integral y) S ⟹ ((λx. f x *R c) has_integral (y *R c)) S
   using has_integral_linear[OF _ bounded_linear_scaleR_left] by (simp add:
comp_def)

lemma integrable_on_scaleR_left:
  assumes f integrable_on A
  shows (λx. f x *R y) integrable_on A
  using assms has_integral_scaleR_left unfolding integrable_on_def by blast

lemma has_integral_mult_left:
  fixes c :: _ :: real_normed_algebra
  shows (f has_integral y) S ⟹ ((λx. f x * c) has_integral (y * c)) S
 using has_integral_linear[OF _ bounded_linear_mult_left] by (simp add: comp_def)

lemma integrable_on_mult_left:
  fixes c :: 'a :: real_normed_algebra
  assumes f integrable_on A
  shows   (λx. f x * c) integrable_on A
  using assms has_integral_mult_left by blast
```

lemma *has_integral_divide*:
fixes *c* :: _ :: *real_normed_div_algebra*
shows (*f has_integral y*) *S* ⟹ ((λ*x*. *f x* / *c*) *has_integral* (*y* / *c*)) *S*
unfolding *divide_inverse* **by** (*simp add*: *has_integral_mult_left*)

lemma *integrable_on_divide*:
fixes *c* :: *'a* :: *real_normed_div_algebra*
assumes *f integrable_on A*
shows (λ*x*. *f x* / *c*) *integrable_on A*
using *assms has_integral_divide* **by** *blast*

The case analysis eliminates the condition *f integrable_on S* at the cost of the type class constraint *division_ring*

corollary *integral_mult_left* [*simp*]:
fixes *c*:: *'a*::{*real_normed_algebra*,*division_ring*}
shows *integral S* (λ*x*. *f x* ∗ *c*) = *integral S f* ∗ *c*
proof (*cases f integrable_on S* ∨ *c* = *0*)
case *True* **then show** *?thesis*
by (*force intro*: *has_integral_mult_left*)
next
case *False* **then have** ¬ (λ*x*. *f x* ∗ *c*) *integrable_on S*
using *has_integral_mult_left* [*of* (λ*x*. *f x* ∗ *c*) _ *S inverse c*]
by (*auto simp add*: *mult.assoc*)
with *False* **show** *?thesis* **by** (*simp add*: *not_integrable_integral*)
qed

corollary *integral_mult_right* [*simp*]:
fixes *c*:: *'a*::{*real_normed_field*}
shows *integral S* (λ*x*. *c* ∗ *f x*) = *c* ∗ *integral S f*
by (*simp add*: *mult.commute* [*of c*])

corollary *integral_divide* [*simp*]:
fixes *z* :: *'a*::*real_normed_field*
shows *integral S* (λ*x*. *f x* / *z*) = *integral S* (λ*x*. *f x*) / *z*
using *integral_mult_left* [*of S f inverse z*]
by (*simp add*: *divide_inverse_commute*)

lemma *has_integral_mult_right*:
fixes *c* :: *'a* :: *real_normed_algebra*
shows (*f has_integral y*) *A* ⟹ ((λ*x*. *c* ∗ *f x*) *has_integral* (*c* ∗ *y*)) *A*
using *has_integral_linear*[*OF* _ *bounded_linear_mult_right*] **by** (*simp add*: *comp_def*)

lemma *integrable_on_mult_right*:
fixes *c* :: *'a* :: *real_normed_algebra*
assumes *f integrable_on A*
shows (λ*x*. *c* ∗ *f x*) *integrable_on A*
using *assms has_integral_mult_right* **by** *blast*

lemma *integrable_on_mult_right_iff* [*simp*]:
fixes $c :: 'a :: real\_normed\_field$
assumes $c \neq 0$
shows $(\lambda x.\ c * f\ x)$ *integrable_on* $A \longleftrightarrow f$ *integrable_on* A
using *integrable_on_mult_right*[*of* $f\ A\ c$]
integrable_on_mult_right[*of* $\lambda x.\ c * f\ x\ A$ *inverse* c] *assms*
by (*auto simp*: *field_simps*)

lemma *integrable_on_mult_left_iff* [*simp*]:
fixes $c :: 'a :: real\_normed\_field$
assumes $c \neq 0$
shows $(\lambda x.\ f\ x * c)$ *integrable_on* $A \longleftrightarrow f$ *integrable_on* A
using *integrable_on_mult_right_iff*[*OF assms*, *of f A*] **by** (*simp add*: *mult.commute*)

lemma *integrable_on_div_iff* [*simp*]:
fixes $c :: 'a :: real\_normed\_field$
assumes $c \neq 0$
shows $(\lambda x.\ f\ x\ /\ c)$ *integrable_on* $A \longleftrightarrow f$ *integrable_on* A
using *integrable_on_mult_right_iff*[*of inverse c f A*] *assms* **by** (*simp add*: *field_simps*)

lemma *has_integral_cmul*: $(f$ *has_integral* $k)\ S \Longrightarrow ((\lambda x.\ c *_R f\ x)$ *has_integral* $(c *_R k))\ S$
unfolding *o_def*[*symmetric*]
by (*metis has_integral_linear bounded_linear_scaleR_right*)

lemma *has_integral_cmult_real*:
fixes $c :: real$
assumes $c \neq 0 \Longrightarrow (f$ *has_integral* $x)\ A$
shows $((\lambda x.\ c * f\ x)$ *has_integral* $c * x)\ A$
by (*metis assms has_integral_is_0 has_integral_mult_right lambda_zero*)

lemma *has_integral_neg*: $(f$ *has_integral* $k)\ S \Longrightarrow ((\lambda x.\ -(f\ x))$ *has_integral* $-k)\ S$
by (*drule_tac c*=-1 **in** *has_integral_cmul*) *auto*

lemma *has_integral_neg_iff*: $((\lambda x.\ -f\ x)$ *has_integral* $k)\ S \longleftrightarrow (f$ *has_integral* $-k)\ S$
using *has_integral_neg*[*of f* $-k$] *has_integral_neg*[*of* $\lambda x.\ -f\ x\ k$] **by** *auto*

lemma *has_integral_add_cbox*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::real\_normed\_vector$
assumes $(f$ *has_integral* $k)$ $(cbox\ a\ b)$ $(g$ *has_integral* $l)$ $(cbox\ a\ b)$
shows $((\lambda x.\ f\ x + g\ x)$ *has_integral* $(k + l))$ $(cbox\ a\ b)$
using *assms*
unfolding *has_integral_cbox*
by (*simp add*: *split_beta' scaleR_add_right sum.distrib*[*abs_def*] *tendsto_add*)

lemma *has_integral_add*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
assumes *f*: (*f has_integral k*) *S* **and** *g*: (*g has_integral l*) *S*
shows ((λ*x*. *f x* + *g x*) *has_integral* (*k* + *l*)) *S*
proof (*cases* ∃ *a b*. *S* = *cbox a b*)
case *True* **with** *has_integral_add_cbox assms* **show** *?thesis*
by *blast*
next
let *?S* = λ*f x*. *if x* ∈ *S then f x else 0*
case *False*
then show *?thesis*
proof (*subst has_integral_alt*, *clarsimp*, *goal_cases*)
case (*1 e*)
then have *e2*: *e*/*2* > *0*
by *auto*
obtain *Bf* **where** *0* < *Bf*
and *Bf*: ⋀*a b*. *ball 0 Bf* ⊆ *cbox a b* ⟹
∃ *z*. (*?S f has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *k*) < *e*/*2*
using *has_integral_altD*[*OF f False e2*] **by** *blast*
obtain *Bg* **where** *0* < *Bg*
and *Bg*: ⋀*a b*. *ball 0 Bg* ⊆ (*cbox a b*) ⟹
∃ *z*. (*?S g has_integral z*) (*cbox a b*) ∧ *norm* (*z* − *l*) < *e*/*2*
using *has_integral_altD*[*OF g False e2*] **by** *blast*
show *?case*
proof (*rule_tac x*=*max Bf Bg* **in** *exI*, *clarsimp simp add*: *max.strict_coboundedI1*
‹*0* < *Bf*›)
fix *a b*
assume *ball 0* (*max Bf Bg*) ⊆ *cbox a* (*b*::*'n*)
then have *fs*: *ball 0 Bf* ⊆ *cbox a* (*b*::*'n*) **and** *gs*: *ball 0 Bg* ⊆ *cbox a* (*b*::*'n*)
by *auto*
obtain *w* **where** *w*: (*?S f has_integral w*) (*cbox a b*) *norm* (*w* − *k*) < *e*/*2*
using *Bf*[*OF fs*] **by** *blast*
obtain *z* **where** *z*: (*?S g has_integral z*) (*cbox a b*) *norm* (*z* − *l*) < *e*/*2*
using *Bg*[*OF gs*] **by** *blast*
have ∗: ⋀*x*. (*if x* ∈ *S then f x* + *g x else 0*) = (*?S f x*) + (*?S g x*)
by *auto*
show ∃ *z*. (*?S*(λ*x*. *f x* + *g x*) *has_integral z*) (*cbox a b*) ∧ *norm* (*z* − (*k* +
l)) < *e*
proof (*intro exI conjI*)
show (*?S*(λ*x*. *f x* + *g x*) *has_integral* (*w* + *z*)) (*cbox a b*)
by (*simp add*: *has_integral_add_cbox*[*OF w*(*1*) *z*(*1*), *unfolded* ∗[*symmetric*]])
show *norm* (*w* + *z* − (*k* + *l*)) < *e*
by (*metis dist_norm dist_triangle_add_half w*(*2*) *z*(*2*))
qed
qed
qed
qed

lemma *has_integral_diff*:

(*f has_integral k*) *S* ⟹ (*g has_integral l*) *S* ⟹
((λ*x*. *f x* − *g x*) *has_integral* (*k* − *l*)) *S*
using *has_integral_add*[*OF* _ *has_integral_neg*, *of f k S g l*]
by (*auto simp*: *algebra_simps*)

lemma *integral_0* [*simp*]:
integral S (λ*x*::′*n*::*euclidean_space*. 0::′*m*::*real_normed_vector*) = 0
by *auto*

lemma *integral_add*: *f integrable_on S* ⟹ *g integrable_on S* ⟹
integral S (λ*x*. *f x* + *g x*) = *integral S f* + *integral S g*
by (*rule integral_unique*) (*metis integrable_integral has_integral_add*)

lemma *integral_cmul* [*simp*]: *integral S* (λ*x*. *c* ∗*R f x*) = *c* ∗*R integral S f*
proof (*cases f integrable_on S* ∨ *c* = 0)
case *True* **with** *has_integral_cmul integrable_integral* **show** *?thesis*
by *fastforce*
next
case *False* **then have** ¬ (λ*x*. *c* ∗*R f x*) *integrable_on S*
using *has_integral_cmul* [*of* (λ*x*. *c* ∗*R f x*) _ *S inverse c*] **by** *auto*
with *False* **show** *?thesis* **by** (*simp add*: *not_integrable_integral*)
qed

lemma *integral_mult*:
fixes *K*::*real*
shows *f integrable_on X* ⟹ *K* ∗ *integral X f* = *integral X* (λ*x*. *K* ∗ *f x*)
by *simp*

lemma *integral_neg* [*simp*]: *integral S* (λ*x*. − *f x*) = − *integral S f*
by (*metis eq_integralD equation_minus_iff has_integral_iff has_integral_neg_iff neg_equal_0_iff_equal*)

lemma *integral_diff*: *f integrable_on S* ⟹ *g integrable_on S* ⟹
integral S (λ*x*. *f x* − *g x*) = *integral S f* − *integral S g*
by (*rule integral_unique*) (*metis integrable_integral has_integral_diff*)

lemma *integrable_0*: (λ*x*. 0) *integrable_on S*
unfolding *integrable_on_def* **using** *has_integral_0* **by** *auto*

lemma *integrable_add*: *f integrable_on S* ⟹ *g integrable_on S* ⟹ (λ*x*. *f x* + *g x*) *integrable_on S*
unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_add*)

lemma *integrable_cmul*: *f integrable_on S* ⟹ (λ*x*. *c* ∗*R f*(*x*)) *integrable_on S*
unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_cmul*)

lemma *integrable_on_scaleR_iff* [*simp*]:
fixes *c* :: *real*
assumes *c* ≠ 0

shows $(\lambda x.\ c *_R f\ x)$ *integrable_on* $S \longleftrightarrow f$ *integrable_on* S
using *integrable_cmul*[*of* $\lambda x.\ c *_R f\ x\ S\ 1\ /\ c$] *integrable_cmul*[*of* $f\ S\ c$] $\langle c \neq 0 \rangle$
by *auto*

lemma *integrable_on_cmult_iff* [*simp*]:
fixes $c :: real$
assumes $c \neq 0$
shows $(\lambda x.\ c * f\ x)$ *integrable_on* $S \longleftrightarrow f$ *integrable_on* S
using *integrable_on_scaleR_iff* [*of* $c\ f$] *assms* **by** *simp*

lemma *integrable_on_cmult_left*:
assumes f *integrable_on* S
shows $(\lambda x.\ of\_real\ c * f\ x)$ *integrable_on* S
using *integrable_cmul*[*of* $f\ S$ *of_real* c] *assms*
by (*simp add*: *scaleR_conv_of_real*)

lemma *integrable_neg*: f *integrable_on* $S \Longrightarrow (\lambda x.\ -f(x))$ *integrable_on* S
unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_neg*)

lemma *integrable_neg_iff*: $(\lambda x.\ -f(x))$ *integrable_on* $S \longleftrightarrow f$ *integrable_on* S
using *integrable_neg* **by** *fastforce*

lemma *integrable_diff*:
f *integrable_on* $S \Longrightarrow g$ *integrable_on* $S \Longrightarrow (\lambda x.\ f\ x - g\ x)$ *integrable_on* S
unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_diff*)

lemma *integrable_linear*:
f *integrable_on* $S \Longrightarrow$ *bounded_linear* $h \Longrightarrow (h \circ f)$ *integrable_on* S
unfolding *integrable_on_def* **by**(*auto intro*: *has_integral_linear*)

lemma *integral_linear*:
f *integrable_on* $S \Longrightarrow$ *bounded_linear* $h \Longrightarrow$ *integral* $S\ (h \circ f) = h\ (integral\ S\ f)$
by (*meson has_integral_iff has_integral_linear*)

lemma *integrable_on_cnj_iff*:
$(\lambda x.\ cnj\ (f\ x))$ *integrable_on* $A \longleftrightarrow f$ *integrable_on* A
using *integrable_linear*[*OF* _ *bounded_linear_cnj*, *of* $f\ A$]
integrable_linear[*OF* _ *bounded_linear_cnj*, *of* $cnj \circ f\ A$]
by (*auto simp*: *o_def*)

lemma *integral_cnj*: $cnj\ (integral\ A\ f) = integral\ A\ (\lambda x.\ cnj\ (f\ x))$
by (*cases f integrable_on A*)
(*simp_all add*: *integral_linear*[*OF* _ *bounded_linear_cnj*, *symmetric*]
o_def integrable_on_cnj_iff not_integrable_integral)

lemma *has_integral_cnj*: $(cnj \circ f\ has\_integral\ (cnj\ I))\ A\ = (f\ has\_integral\ I)\ A$
unfolding *has_integral_iff comp_def*
by (*metis integral_cnj complex_cnj_cancel_iff integrable_on_cnj_iff*)

lemma *integral_component_eq*[*simp*]:
fixes $f :: 'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
assumes *f integrable_on S*
shows *integral S* $(\lambda x.\ f\ x \cdot k) =$ *integral S f* $\cdot\ k$
unfolding *integral_linear*[*OF assms*(*1*) *bounded_linear_inner_left*,*unfolded o_def*]
..

lemma *has_integral_sum*:
assumes *finite T*
and $\bigwedge a.\ a \in T \Longrightarrow ((f\ a)\ has\_integral\ (i\ a))\ S$
shows $((\lambda x.\ sum\ (\lambda a.\ f\ a\ x)\ T)\ has\_integral\ (sum\ i\ T))\ S$
using ‹*finite T*› *subset_refl*[*of T*]
by (*induct rule*: *finite_subset_induct*) (*use assms* **in** ‹*auto simp*: *has_integral_add*›)

lemma *integral_sum*:
$[\![$*finite I*; $\bigwedge a.\ a \in I \Longrightarrow f\ a$ *integrable_on S*$]\!] \Longrightarrow$
integral S $(\lambda x.\ \sum a{\in}I.\ f\ a\ x) = (\sum a{\in}I.\ integral\ S\ (f\ a))$
by (*simp add*: *has_integral_sum integrable_integral integral_unique*)

lemma *integrable_sum*:
$[\![$*finite I*; $\bigwedge a.\ a \in I \Longrightarrow f\ a$ *integrable_on S*$]\!] \Longrightarrow (\lambda x.\ \sum a{\in}I.\ f\ a\ x)$ *integrable_on S*
unfolding *integrable_on_def* **using** *has_integral_sum*[*of I*] **by** *metis*

lemma *has_integral_eq*:
assumes $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
and *f*: (*f has_integral k*) *s*
shows (*g has_integral k*) *s*
using *has_integral_diff*[*OF f*, *of* $\lambda x.\ f\ x - g\ x\ 0$]
using *has_integral_is_0*[*of s* $\lambda x.\ f\ x - g\ x$]
using *assms*
by *auto*

lemma *integrable_eq*: $[\![$*f integrable_on s*; $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x]\!] \Longrightarrow$ *g integrable_on s*
unfolding *integrable_on_def*
using *has_integral_eq*[*of s f g*] *has_integral_eq* **by** *blast*

lemma *has_integral_cong*:
assumes $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
shows (*f has_integral i*) *s* = (*g has_integral i*) *s*
by (*metis assms has_integral_eq*)

lemma *integrable_cong*:
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x$
shows *f integrable_on A* $\longleftrightarrow$ *g integrable_on A*
using *has_integral_cong* [*OF assms*] **by** *fast*

lemma *integral_cong*:

assumes $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$
shows *integral s f = integral s g*
unfolding *integral_def*
by (*metis* (*full_types*, *opaque_lifting*) *assms has_integral_cong integrable_eq*)

lemma *integrable_on_cmult_left_iff* [*simp*]:
assumes $c \neq 0$
shows ($\lambda x.$ *of_real c * f x*) *integrable_on s* $\longleftrightarrow$ *f integrable_on s*
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then have ($\lambda x.$ *of_real (1 / c) * (of_real c * f x)*) *integrable_on s*
using *integrable_cmul*[*of* $\lambda x.$ *of_real c * f x s 1 / of_real c*]
by (*simp add*: *scaleR_conv_of_real*)
then have ($\lambda x.$ (*of_real (1 / c) * of_real c * f x*)) *integrable_on s*
by (*simp add*: *algebra_simps*)
with ‹$c \neq 0$› **show** *?rhs*
by (*metis* (*no_types*, *lifting*) *integrable_eq mult.left_neutral nonzero_divide_eq_eq of_real_1 of_real_mult*)
qed (*blast intro*: *integrable_on_cmult_left*)

lemma *integrable_on_cmult_right*:
fixes *f* :: _ ⇒ *'b* :: {*comm_ring*,*real_algebra_1*,*real_normed_vector*}
assumes *f integrable_on s*
shows ($\lambda x.$ *f x * of_real c*) *integrable_on s*
using *integrable_on_cmult_left* [*OF assms*] **by** (*simp add*: *mult.commute*)

lemma *integrable_on_cmult_right_iff* [*simp*]:
fixes *f* :: _ ⇒ *'b* :: {*comm_ring*,*real_algebra_1*,*real_normed_vector*}
assumes $c \neq 0$
shows ($\lambda x.$ *f x * of_real c*) *integrable_on s* $\longleftrightarrow$ *f integrable_on s*
using *integrable_on_cmult_left_iff* [*OF assms*] **by** (*simp add*: *mult.commute*)

lemma *integrable_on_cdivide_iff* [*simp*]:
fixes *f* :: _ ⇒ *'b* :: *real_normed_field*
assumes $c \neq 0$
shows ($\lambda x.$ *f x / of_real c*) *integrable_on s* $\longleftrightarrow$ *f integrable_on s*
by (*simp add*: *divide_inverse assms flip*: *of_real_inverse*)

lemma *has_integral_null* [*intro*]: *content*(*cbox a b*) = *0* $\Longrightarrow$ (*f has_integral 0*) (*cbox a b*)
unfolding *has_integral_cbox*
using *eventually_division_filter_tagged_division*[*of cbox a b*]
by (*subst tendsto_cong*[**where** *g*=λ_. *0*]) (*auto elim*: *eventually_mono intro*: *sum_content_null*)

lemma *has_integral_null_real* [*intro*]: *content* {*a*..*b*::*real*} = *0* $\Longrightarrow$ (*f has_integral 0*) {*a*..*b*}
by (*metis box_real*(*2*) *has_integral_null*)

lemma *has_integral_null_eq*[*simp*]: *content* (*cbox a b*) = *0* ⟹ (*f has_integral i*) (*cbox a b*) ⟷ *i* = *0*
by (*auto simp add*: *has_integral_null dest*!: *integral_unique*)

lemma *integral_null* [*simp*]: *content* (*cbox a b*) = *0* ⟹ *integral* (*cbox a b*) *f* = *0*
by (*metis has_integral_null integral_unique*)

lemma *integrable_on_null* [*intro*]: *content* (*cbox a b*) = *0* ⟹ *f integrable_on* (*cbox a b*)
by (*simp add*: *has_integral_integrable*)

lemma *has_integral_empty*[*intro*]: (*f has_integral 0*) {}
by (*meson ex_in_conv has_integral_is_0*)

lemma *has_integral_empty_eq*[*simp*]: (*f has_integral i*) {} ⟷ *i* = *0*
by (*auto simp add*: *has_integral_empty has_integral_unique*)

lemma *integrable_on_empty*[*intro*]: *f integrable_on* {}
unfolding *integrable_on_def* **by** *auto*

lemma *integral_empty*[*simp*]: *integral* {} *f* = *0*
by *blast*

lemma *has_integral_refl*[*intro*]:
fixes *a* :: *'a*::*euclidean_space*
shows (*f has_integral 0*) (*cbox a a*)
and (*f has_integral 0*) {*a*}
proof −
show (*f has_integral 0*) (*cbox a a*)
by (*rule has_integral_null*) *simp*
then show (*f has_integral 0*) {*a*}
by *simp*
qed

lemma *integrable_on_refl*[*intro*]: *f integrable_on cbox a a*
unfolding *integrable_on_def* **by** *auto*

lemma *integral_refl* [*simp*]: *integral* (*cbox a a*) *f* = *0*
by *auto*

lemma *integral_singleton* [*simp*]: *integral* {*a*} *f* = *0*
by *auto*

lemma *integral_blinfun_apply*:
assumes *f integrable_on s*
shows *integral s* (λ*x*. *blinfun_apply h* (*f x*)) = *blinfun_apply h* (*integral s f*)
by (*subst integral_linear*[*symmetric*, *OF assms blinfun.bounded_linear_right*])
(*simp add*: *o_def*)

lemma *blinfun_apply_integral*:
assumes *f integrable_on s*
shows *blinfun_apply (integral s f) x = integral s (λy. blinfun_apply (f y) x)*
by (*metis* (*no_types*, *lifting*) *assms blinfun.prod_left.rep_eq integral_blinfun_apply integral_cong*)

lemma *has_integral_componentwise_iff*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
shows (*f has_integral y*) *A* ⟷ (∀ *b*∈*Basis*. ((λ*x*. *f x* ∙ *b*) *has_integral* (*y* ∙ *b*)) *A*)
proof *safe*
fix *b* :: *'b* **assume** (*f has_integral y*) *A*
from *has_integral_linear*[*OF this*(*1*) *bounded_linear_inner_left*, *of b*]
show ((λ*x*. *f x* ∙ *b*) *has_integral* (*y* ∙ *b*)) *A* **by** (*simp add*: *o_def*)
next
assume (∀ *b*∈*Basis*. ((λ*x*. *f x* ∙ *b*) *has_integral* (*y* ∙ *b*)) *A*)
hence ∀ *b*∈*Basis*. (((λ*x*. *x* ∗*R b*) ∘ (λ*x*. *f x* ∙ *b*)) *has_integral* ((*y* ∙ *b*) ∗*R b*)) *A*
by (*intro ballI has_integral_linear*) (*simp_all add*: *bounded_linear_scaleR_left*)
hence ((λ*x*. ∑ *b*∈*Basis*. (*f x* ∙ *b*) ∗*R b*) *has_integral* (∑ *b*∈*Basis*. (*y* ∙ *b*) ∗*R b*)) *A*
by (*intro has_integral_sum*) (*simp_all add*: *o_def*)
thus (*f has_integral y*) *A* **by** (*simp add*: *euclidean_representation*)
qed

lemma *has_integral_componentwise*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
shows (⋀*b*. *b* ∈ *Basis* ⟹ ((λ*x*. *f x* ∙ *b*) *has_integral* (*y* ∙ *b*)) *A*) ⟹ (*f has_integral y*) *A*
by (*subst has_integral_componentwise_iff*) *blast*

lemma *integrable_componentwise_iff*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
shows *f integrable_on A* ⟷ (∀ *b*∈*Basis*. (λ*x*. *f x* ∙ *b*) *integrable_on A*)
proof
assume *f integrable_on A*
then obtain *y* **where** (*f has_integral y*) *A* **by** (*auto simp*: *integrable_on_def*)
hence (∀ *b*∈*Basis*. ((λ*x*. *f x* ∙ *b*) *has_integral* (*y* ∙ *b*)) *A*)
by (*subst* (*asm*) *has_integral_componentwise_iff*)
thus (∀ *b*∈*Basis*. (λ*x*. *f x* ∙ *b*) *integrable_on A*) **by** (*auto simp*: *integrable_on_def*)
next
assume (∀ *b*∈*Basis*. (λ*x*. *f x* ∙ *b*) *integrable_on A*)
then obtain *y* **where** ∀ *b*∈*Basis*. ((λ*x*. *f x* ∙ *b*) *has_integral y b*) *A*
unfolding *integrable_on_def* **by** (*subst* (*asm*) *bchoice_iff*) *blast*
hence ∀ *b*∈*Basis*. (((λ*x*. *x* ∗*R b*) ∘ (λ*x*. *f x* ∙ *b*)) *has_integral* (*y b* ∗*R b*)) *A*
by (*intro ballI has_integral_linear*) (*simp_all add*: *bounded_linear_scaleR_left*)
hence ((λ*x*. ∑ *b*∈*Basis*. (*f x* ∙ *b*) ∗*R b*) *has_integral* (∑ *b*∈*Basis*. *y b* ∗*R b*)) *A*
by (*intro has_integral_sum*) (*simp_all add*: *o_def*)
thus *f integrable_on A* **by** (*auto simp*: *integrable_on_def o_def euclidean_representation*)

qed

lemma *integrable_componentwise*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: euclidean\_space$
shows $(\bigwedge b.\ b \in Basis \Longrightarrow (\lambda x.\ f\ x \cdot b)\ integrable\_on\ A) \Longrightarrow f\ integrable\_on\ A$
by (*subst integrable_componentwise_iff*) *blast*

lemma *integral_componentwise*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: euclidean\_space$
assumes *f integrable_on A*
shows $integral\ A\ f = (\sum b{\in}Basis.\ integral\ A\ (\lambda x.\ (f\ x \cdot b) *_R b))$
proof −
from *assms* **have** *integrable*: $\forall\, b{\in}Basis.\ (\lambda x.\ x *_R b) \circ (\lambda x.\ (f\ x \cdot b))\ integrable\_on\ A$
by (*subst* (*asm*) *integrable_componentwise_iff*, *intro integrable_linear ballI*) (*simp_all add*: *bounded_linear_scaleR_left*)
have $integral\ A\ f = integral\ A\ (\lambda x.\ \sum b{\in}Basis.\ (f\ x \cdot b) *_R b)$
by (*simp add*: *euclidean_representation*)
also from *integrable* **have** $\ldots = (\sum a{\in}Basis.\ integral\ A\ (\lambda x.\ (f\ x \cdot a) *_R a))$
by (*subst integral_sum*) (*simp_all add*: *o_def*)
finally show *?thesis* **.**
qed

lemma *integrable_component*:
f integrable_on A $\Longrightarrow (\lambda x.\ f\ x \cdot (y :: 'b :: euclidean\_space))\ integrable\_on\ A$
by (*drule integrable_linear*[*OF* _ *bounded_linear_inner_left*[*of y*]]) (*simp add*: *o_def*)

7.14.4 Cauchy-type criterion for integrability

proposition *integrable_Cauchy*:
fixes $f :: 'n{::}euclidean\_space \Rightarrow 'a{::}\{real\_normed\_vector, complete\_space\}$
shows *f integrable_on cbox a b* $\longleftrightarrow$
$(\forall\, e{>}0.\ \exists\gamma.\ gauge\ \gamma \wedge$
$(\forall\, \mathcal{D}1\ \mathcal{D}2.\ \mathcal{D}1\ tagged\_division\_of\ (cbox\ a\ b) \wedge \gamma\ fine\ \mathcal{D}1 \wedge$
$\mathcal{D}2\ tagged\_division\_of\ (cbox\ a\ b) \wedge \gamma\ fine\ \mathcal{D}2 \longrightarrow$
$norm\ ((\sum (x,K){\in}\mathcal{D}1.\ content\ K *_R f\ x) - (\sum (x,K){\in}\mathcal{D}2.\ content\ K *_R f\ x)) < e))$
(**is** $?l = (\forall\, e{>}0.\ \exists\gamma.\ ?P\ e\ \gamma)$)
proof (*intro iffI allI impI*)
assume *?l*
then obtain *y*
where y: $\bigwedge e.\ e > 0 \Longrightarrow$
$\exists\gamma.\ gauge\ \gamma \wedge$
$(\forall\, \mathcal{D}.\ \mathcal{D}\ tagged\_division\_of\ cbox\ a\ b \wedge \gamma\ fine\ \mathcal{D} \longrightarrow$
$norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ x) - y) < e)$
by (*auto simp*: *integrable_on_def has_integral*)
show $\exists\gamma.\ ?P\ e\ \gamma$ **if** $e > 0$ **for** *e*
proof −

```
    have e/2 > 0 using that by auto
    with y obtain γ where gauge γ
      and γ: ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ γ fine 𝒟 ⟹
                norm ((∑(x,K)∈𝒟. content K *R f x) − y) < e/2
      by meson
    show ?thesis
    apply (rule_tac x=γ in exI, clarsimp simp: ‹gauge γ›)
       by (blast intro!: γ dist_triangle_half_l[where y=y,unfolded dist_norm])
    qed
next
  assume ∀e>0. ∃γ. ?P e γ
  then have ∀n::nat. ∃γ. ?P (1 / (n + 1)) γ
    by auto
  then obtain γ :: nat ⇒ 'n ⇒ 'n set where γ:
    ⋀m. gauge (γ m)
    ⋀m 𝒟1 𝒟2. ⟦𝒟1 tagged_division_of cbox a b;
            γ m fine 𝒟1; 𝒟2 tagged_division_of cbox a b; γ m fine 𝒟2⟧
              ⟹ norm ((∑(x,K) ∈ 𝒟1. content K *R f x) − (∑(x,K) ∈ 𝒟2.
content K *R f x))
                 < 1 / (m + 1)
    by metis
  have gauge (λx. ⋂{γ i x |i. i ∈ {0..n}}) for n
    using γ by (intro gauge_Inter) auto
   then have ∀n. ∃p. p tagged_division_of (cbox a b) ∧ (λx. ⋂{γ i x |i. i ∈
{0..n}}) fine p
    by (meson fine_division_exists)
  then obtain p where p: ⋀z. p z tagged_division_of cbox a b
                         ⋀z. (λx. ⋂{γ i x |i. i ∈ {0..z}}) fine p z
    by meson
  have dp: ⋀i n. i≤n ⟹ γ i fine p n
    using p unfolding fine_Inter
    using atLeastAtMost_iff by blast
  have Cauchy (λn. sum (λ(x,K). content K *R (f x)) (p n))
  proof (rule CauchyI)
    fix e::real
    assume 0 < e
    then obtain N where N ≠ 0 and N: inverse (real N) < e
      using real_arch_inverse[of e] by blast
     show ∃M. ∀m≥M. ∀n≥M. norm ((∑(x,K) ∈ p m. content K *R f x) −
(∑(x,K) ∈ p n. content K *R f x)) < e
    proof (intro exI allI impI)
      fix m n
      assume mn: N ≤ m N ≤ n
      have norm ((∑(x,K) ∈ p m. content K *R f x) − (∑(x,K) ∈ p n. content
K *R f x)) < 1 / (real N + 1)
        by (simp add: p(1) dp mn γ)
      also have ... < e
        using  N ‹N ≠ 0› ‹0 < e› by (auto simp: field_simps)
      finally show norm ((∑(x,K) ∈ p m. content K *R f x) − (∑(x,K) ∈ p n.
```

content K $*_R$ *f x*)) < *e* .
qed
qed
then obtain *y* **where** *y*: ∃*no*. ∀*n*≥*no*. *norm* ((∑(*x*,*K*) ∈ *p n*. *content K* $*_R$ *f x*) − *y*) < *r* **if** *r* > *0* **for** *r*
by (*auto simp*: *convergent_eq_Cauchy*[*symmetric*] *dest*: *LIMSEQ_D*)
show *?l*
unfolding *integrable_on_def has_integral*
proof (*rule_tac x*=*y* **in** *exI*, *clarify*)
fix *e* :: *real*
assume *e*>*0*
then have *e2*: *e*/*2* > *0* **by** *auto*
then obtain *N1*::*nat* **where** *N1*: *N1* ≠ *0 inverse* (*real N1*) < *e*/*2*
using *real_arch_inverse* **by** *blast*
obtain *N2*::*nat* **where** *N2*: ⋀*n*. *n* ≥ *N2* ⟹ *norm* ((∑(*x*,*K*) ∈ *p n*. *content K* $*_R$ *f x*) − *y*) < *e*/*2*
using *y*[*OF e2*] **by** *metis*
show ∃γ. *gauge* γ ∧
(∀*D*. *D tagged_division_of* (*cbox a b*) ∧ γ *fine D* ⟶
norm ((∑(*x*,*K*) ∈ *D*. *content K* $*_R$ *f x*) − *y*) < *e*)
proof (*intro exI conjI allI impI*)
show *gauge* (γ (*N1*+*N2*))
using γ **by** *auto*
show *norm* ((∑(*x*,*K*) ∈ *q*. *content K* $*_R$ *f x*) − *y*) < *e*
if *q tagged_division_of cbox a b* ∧ γ (*N1*+*N2*) *fine q* **for** *q*
proof (*rule norm_triangle_half_r*)
have *norm* ((∑(*x*,*K*) ∈ *p* (*N1*+*N2*). *content K* $*_R$ *f x*) − (∑(*x*,*K*) ∈ *q*. *content K* $*_R$ *f x*))
< *1* / (*real* (*N1*+*N2*) + *1*)
by (*rule* γ; *simp add*: *dp p that*)
also have ... < *e*/*2*
using *N1* ‹*0* < *e*› **by** (*auto simp*: *field_simps intro*: *less_le_trans*)
finally show *norm* ((∑(*x*,*K*) ∈ *p* (*N1*+*N2*). *content K* $*_R$ *f x*) − (∑(*x*,*K*) ∈ *q*. *content K* $*_R$ *f x*)) < *e*/*2* .
show *norm* ((∑(*x*,*K*) ∈ *p* (*N1*+*N2*). *content K* $*_R$ *f x*) − *y*) < *e*/*2*
using *N2 le_add_same_cancel2* **by** *blast*
qed
qed
qed
qed

7.14.5 Additivity of integral on abutting intervals

lemma *tagged_division_split_left_inj_content*:
assumes *D*: *D tagged_division_of S*
and (*x1*, *K1*) ∈ *D* (*x2*, *K2*) ∈ *D K1* ≠ *K2 K1* ∩ {*x*. *x*•*k* ≤ *c*} = *K2* ∩ {*x*. *x*•*k* ≤ *c*} *k* ∈ *Basis*
shows *content* (*K1* ∩ {*x*. *x*•*k* ≤ *c*}) = *0*
proof −

```
  from tagged_division_ofD(4)[OF 𝒟 ‹(x1, K1) ∈ 𝒟›] obtain a b where K1:
K1 = cbox a b
    by auto
  then have interior (K1 ∩ {x. x · k ≤ c}) = {}
    by (metis tagged_division_split_left_inj assms)
  then show ?thesis
   unfolding K1 interval_split[OF ‹k ∈ Basis›] by (auto simp: content_eq_0_interior)
qed

lemma tagged_division_split_right_inj_content:
  assumes 𝒟: 𝒟 tagged_division_of S
    and (x1, K1) ∈ 𝒟 (x2, K2) ∈ 𝒟 K1 ≠ K2 K1 ∩ {x. x•k ≥ c} = K2 ∩ {x.
x•k ≥ c} k ∈ Basis
  shows content (K1 ∩ {x. x•k ≥ c}) = 0
proof −
  from tagged_division_ofD(4)[OF 𝒟 ‹(x1, K1) ∈ 𝒟›] obtain a b where K1:
K1 = cbox a b
    by auto
  then have interior (K1 ∩ {x. c ≤ x · k}) = {}
    by (metis tagged_division_split_right_inj assms)
  then show ?thesis
    unfolding K1 interval_split[OF ‹k ∈ Basis›]
    by (auto simp: content_eq_0_interior)
qed


proposition has_integral_split:
  fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
  assumes fi: (f has_integral i) (cbox a b ∩ {x. x•k ≤ c})
      and fj: (f has_integral j) (cbox a b ∩ {x. x•k ≥ c})
      and k: k ∈ Basis
shows (f has_integral (i + j)) (cbox a b)
  unfolding has_integral
proof clarify
  fix e::real
  assume 0 < e
  then have e: e/2 > 0
    by auto
    obtain γ1 where γ1: gauge γ1
      and γ1norm:
        ⋀𝒟. ⟦𝒟 tagged_division_of cbox a b ∩ {x. x · k ≤ c}; γ1 fine 𝒟⟧
            ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − i) < e/2
      apply (rule has_integralD[OF fi[unfolded interval_split[OF k]] e])
      apply (simp add: interval_split[symmetric] k)
      done
    obtain γ2 where γ2: gauge γ2
      and γ2norm:
        ⋀𝒟. ⟦𝒟 tagged_division_of cbox a b ∩ {x. c ≤ x · k}; γ2 fine 𝒟⟧
            ⟹ norm ((∑(x, k) ∈ 𝒟. content k *R f x) − j) < e/2
```

```
      apply (rule has_integralD[OF fj[unfolded interval_split[OF k]] e])
      apply (simp add: interval_split[symmetric] k)
      done
  let ?γ = λx. if x·k = c then (γ1 x ∩ γ2 x) else ball x |x·k − c| ∩ γ1 x ∩ γ2 x
  have gauge ?γ
    using γ1 γ2 unfolding gauge_def by auto
  then show ∃γ. gauge γ ∧
              (∀ D. D tagged_division_of cbox a b ∧ γ fine D ⟶
                  norm ((∑(x, k)∈D. content k *R f x) − (i + j)) < e)
  proof (rule_tac x=?γ in exI, safe)
    fix p
    assume p: p tagged_division_of (cbox a b) and ?γ fine p
    have ab_eqp: cbox a b = ⋃{K. ∃x. (x, K) ∈ p}
      using p by blast
    have xk_le_c: x·k ≤ c if as: (x,K) ∈ p and K: K ∩ {x. x·k ≤ c} ≠ {} for x
K
    proof (rule ccontr)
      assume **: ¬ x · k ≤ c
      then have K ⊆ ball x |x · k − c|
        using ‹?γ fine p› as by (fastforce simp: not_le algebra_simps)
      with K obtain y where y: y ∈ ball x |x · k − c| y·k ≤ c
        by blast
      then have |x · k − y · k| < |x · k − c|
        using Basis_le_norm[OF k, of x − y]
        by (auto simp add: dist_norm inner_diff_left intro: le_less_trans)
      with y show False
        using ** by (auto simp add: field_simps)
    qed
    have xk_ge_c: x·k ≥ c if as: (x,K) ∈ p and K: K ∩ {x. x·k ≥ c} ≠ {} for x
K
    proof (rule ccontr)
      assume **: ¬ x · k ≥ c
      then have K ⊆ ball x |x · k − c|
        using ‹?γ fine p› as by (fastforce simp: not_le algebra_simps)
      with K obtain y where y: y ∈ ball x |x · k − c| y·k ≥ c
        by blast
      then have |x · k − y · k| < |x · k − c|
        using Basis_le_norm[OF k, of x − y]
        by (auto simp add: dist_norm inner_diff_left intro: le_less_trans)
      with y show False
        using ** by (auto simp add: field_simps)
    qed
    have fin_finite: finite {(x,f K) | x K. (x,K) ∈ s ∧ P x K}
      if finite s for s and f :: 'a set ⇒ 'a set and P :: 'a ⇒ 'a set ⇒ bool
    proof −
      from that have finite ((λ(x,K). (x, f K)) ` s)
        by auto
      then show ?thesis
        by (rule rev_finite_subset) auto
```

```
  qed
  { fix 𝒢 :: 'a set ⇒ 'a set
    fix i :: 'a × 'a set
    assume i ∈ (λ(x, k). (x, 𝒢 k)) ` p - {(x, 𝒢 k) |x k. (x, k) ∈ p ∧ 𝒢 k ≠ {}}
    then obtain x K where xk: i = (x, 𝒢 K)  (x,K) ∈ p
                                (x, 𝒢 K) ∉ {(x, 𝒢 K) |x K. (x,K) ∈ p ∧ 𝒢 K ≠ {}}
      by auto
    have content (𝒢 K) = 0
      using xk using content_empty by auto
    then have (λ(x,K). content K *R f x) i = 0
      unfolding xk split_conv by auto
  } note [simp] = this
  have finite p
    using p by blast
  let ?M1 = {(x, K ∩ {x. x•k ≤ c}) |x K. (x,K) ∈ p ∧ K ∩ {x. x•k ≤ c} ≠ {}}
  have γ1_fine: γ1 fine ?M1
    using ‹?γ fine p› by (fastforce simp: fine_def split: if_split_asm)
  have norm ((∑(x, k)∈?M1. content k *R f x) - i) < e/2
  proof (rule γ1norm [OF tagged_division_ofI γ1_fine])
    show finite ?M1
      by (rule fin_finite) (use p in blast)
    show ⋃{k. ∃x. (x, k) ∈ ?M1} = cbox a b ∩ {x. x•k ≤ c}
      by (auto simp: ab_eqp)

    fix x L
    assume xL: (x, L) ∈ ?M1
    then obtain x' L' where xL': x = x' L = L' ∩ {x. x • k ≤ c}
                                (x', L') ∈ p L' ∩ {x. x • k ≤ c} ≠ {}
      by blast
    then obtain a' b' where ab': L' = cbox a' b'
      using p by blast
    show x ∈ L L ⊆ cbox a b ∩ {x. x • k ≤ c}
      using p xk_le_c xL' by auto
    show ∃a b. L = cbox a b
      using p xL' ab' by (auto simp add: interval_split[OF k,where c=c])

    fix y R
    assume yR: (y, R) ∈ ?M1
    then obtain y' R' where yR': y = y' R = R' ∩ {x. x • k ≤ c}
                                (y', R') ∈ p R' ∩ {x. x • k ≤ c} ≠ {}
      by blast
    assume as: (x, L) ≠ (y, R)
    show interior L ∩ interior R = {}
    proof (cases L' = R' ⟶ x' = y')
      case False
      have interior R' = {}
        by (metis (no_types) False Pair_inject inf.idem tagged_division_ofD(5)
[OF p] xL'(3) yR'(3))
      then show ?thesis
```

```
        using yR' by simp
      next
        case True
        then have L' ≠ R'
          using as unfolding xL' yR' by auto
        have interior L' ∩ interior R' = {}
          by (metis (no_types) Pair_inject ‹L' ≠ R'› p tagged_division_ofD(5)
xL'(3) yR'(3))
        then show ?thesis
          using xL'(2) yR'(2) by auto
      qed
    qed
    moreover
    let ?M2 = {(x,K ∩ {x. x•k ≥ c}) |x K. (x,K) ∈ p ∧ K ∩ {x. x•k ≥ c} ≠ {}}
    have γ2_fine: γ2 fine ?M2
      using ‹?γ fine p› by (fastforce simp: fine_def split: if_split_asm)
    have norm ((∑(x, k)∈?M2. content k *R f x) - j) < e/2
    proof (rule γ2norm [OF tagged_division_ofI γ2_fine])
      show finite ?M2
        by (rule fin_finite) (use p in blast)
      show ⋃{k. ∃x. (x, k) ∈ ?M2} = cbox a b ∩ {x. x•k ≥ c}
        by (auto simp: ab_eqp)

      fix x L
      assume xL: (x, L) ∈ ?M2
      then obtain x' L' where xL': x = x' L = L' ∩ {x. x • k ≥ c}
                               (x', L') ∈ p L' ∩ {x. x • k ≥ c} ≠ {}
        by blast
      then obtain a' b' where ab': L' = cbox a' b'
        using p by blast
      show x ∈ L L ⊆ cbox a b ∩ {x. x • k ≥ c}
        using p xk_ge_c xL' by auto
      show ∃a b. L = cbox a b
        using p xL' ab' by (auto simp add: interval_split[OF k,where c=c])

      fix y R
      assume yR: (y, R) ∈ ?M2
      then obtain y' R' where yR': y = y' R = R' ∩ {x. x • k ≥ c}
                               (y', R') ∈ p R' ∩ {x. x • k ≥ c} ≠ {}
        by blast
      assume as: (x, L) ≠ (y, R)
      show interior L ∩ interior R = {}
      proof (cases L' = R' ⟶ x' = y')
        case False
        have interior R' = {}
          by (metis (no_types) False Pair_inject inf.idem tagged_division_ofD(5)
[OF p] xL'(3) yR'(3))
        then show ?thesis
          using yR' by simp
```

```
    next
      case True
      then have L' ≠ R'
        using as unfolding xL' yR' by auto
      have interior L' ∩ interior R' = {}
        by (metis (no_types) Pair_inject ‹L' ≠ R'› p tagged_division_ofD(5)
xL'(3) yR'(3))
      then show ?thesis
        using xL'(2) yR'(2) by auto
    qed
  qed
  ultimately
  have norm (((∑(x,K) ∈ ?M1. content K *R f x) − i) + ((∑(x,K) ∈ ?M2.
content K *R f x) − j)) < e/2 + e/2
    using norm_add_less by blast
  moreover have ((∑(x,K) ∈ ?M1. content K *R f x) − i) +
                ((∑(x,K) ∈ ?M2. content K *R f x) − j) =
                (∑(x, ka)∈p. content ka *R f x) − (i + j)
  proof −
    have eq0: ⋀x y. x = (0::real) ⟹ x *R (y::'b) = 0
      by auto
    have cont_eq: ⋀g. (λ(x,l). content l *R f x) ∘ (λ(x,l). (x,g l)) = (λ(x,l).
content (g l) *R f x)
      by auto
    have *: ⋀G :: 'a set ⇒ 'a set.
              (∑(x,K)∈{(x, G K) |x K. (x,K) ∈ p ∧ G K ≠ {}}. content K *R f
x) =
              (∑(x,K)∈(λ(x,K). (x, G K)) ‘ p. content K *R f x)
      by (rule sum.mono_neutral_left) (auto simp: ‹finite p›)
    have ((∑(x, k)∈?M1. content k *R f x) − i) + ((∑(x, k)∈?M2. content k
*R f x) − j) =
      (∑(x, k)∈?M1. content k *R f x) + (∑(x, k)∈?M2. content k *R f x) −
(i + j)
      by auto
    moreover have ... = (∑(x,K) ∈ p. content (K ∩ {x. x • k ≤ c}) *R f x) +
      (∑(x,K) ∈ p. content (K ∩ {x. c ≤ x • k}) *R f x) − (i + j)
      unfolding *
      apply (subst (1 2) sum.reindex_nontrivial)
          apply (auto intro!: k p eq0 tagged_division_split_left_inj_content
tagged_division_split_right_inj_content
                  simp: cont_eq ‹finite p›)
      done
    moreover have ⋀x. x ∈ p ⟹ (λ(a,B). content (B ∩ {a. a • k ≤ c}) *R f
a) x +
                          (λ(a,B). content (B ∩ {a. c ≤ a • k}) *R f a) x =
                          (λ(a,B). content B *R f a) x
    proof clarify
      fix a B
      assume (a, B) ∈ p
```

```
      with p obtain u v where uv: B = cbox u v by blast
      then show content (B ∩ {x. x · k ≤ c}) *R f a + content (B ∩ {x. c ≤ x
· k}) *R f a = content B *R f a
        by (auto simp: scaleR_left_distrib uv content_split[OF k,of u v c])
    qed
    ultimately show ?thesis
      by (auto simp: sum.distrib[symmetric])
    qed
  ultimately show norm ((∑(x, k)∈p. content k *R f x) − (i + j)) < e
    by auto
 qed
qed
```

7.14.6 A sort of converse, integrability on subintervals

```
lemma has_integral_separate_sides:
 fixes f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
 assumes f: (f has_integral i) (cbox a b)
   and e > 0
   and k: k ∈ Basis
 obtains d where gauge d
   ∀p1 p2. p1 tagged_division_of (cbox a b ∩ {x. x•k ≤ c}) ∧ d fine p1 ∧
     p2 tagged_division_of (cbox a b ∩ {x. x•k ≥ c}) ∧ d fine p2 ⟶
     norm ((sum (λ(x,k). content k *R f x) p1 + sum (λ(x,k). content k *R f x)
p2) − i) < e
proof −
 obtain γ where d: gauge γ
     ⋀p. ⟦p tagged_division_of cbox a b; γ fine p⟧
         ⟹ norm ((∑(x, k)∈p. content k *R f x) − i) < e
   using has_integralD[OF f ‹e > 0›] by metis
 { fix p1 p2
   assume tdiv1: p1 tagged_division_of (cbox a b) ∩ {x. x · k ≤ c} and γ fine
p1
   note p1=tagged_division_ofD[OF this(1)]
   assume tdiv2: p2 tagged_division_of (cbox a b) ∩ {x. c ≤ x · k} and γ fine
p2
   note p2=tagged_division_ofD[OF this(1)]
   note tagged_division_Un_interval[OF tdiv1 tdiv2]
   note p12 = tagged_division_ofD[OF this] this
   { fix a b
     assume ab: (a, b) ∈ p1 ∩ p2
     have (a, b) ∈ p1
       using ab by auto
     obtain u v where uv: b = cbox u v
       using ‹(a, b) ∈ p1› p1(4) by moura
     have b ⊆ {x. x•k = c}
       using ab p1(3)[of a b] p2(3)[of a b] by fastforce
     moreover
     have interior {x::'a. x · k = c} = {}
```

```
    proof (rule ccontr)
      assume ¬ ?thesis
      then obtain x where x: x ∈ interior {x::'a. x•k = c}
        by auto
      then obtain ε where 0 < ε and ε: ball x ε ⊆ {x. x • k = c}
        using mem_interior by metis
      have x: x•k = c
        using x interior_subset by fastforce
      have *: ⋀i. i ∈ Basis ⟹ |(x - (x + (ε/2) *R k)) • i| = (if i = k then ε/2
else 0)
        using ‹0 < ε› k by (auto simp: inner_simps inner_not_same_Basis)
      have (∑ i∈Basis. |(x - (x + (ε/2 ) *R k)) • i|) =
            (∑ i∈Basis. (if i = k then ε/2 else 0))
        using * by (blast intro: sum.cong)
      also have ... < ε
        by (subst sum.delta) (use ‹0 < ε› in auto)
      finally have x + (ε/2) *R k ∈ ball x ε
        unfolding mem_ball dist_norm by(rule le_less_trans[OF norm_le_l1])
      then have x + (ε/2) *R k ∈ {x. x•k = c}
        using ε by auto
      then show False
        using ‹0 < ε› x k by (auto simp: inner_simps)
    qed
    ultimately have content b = 0
      unfolding uv content_eq_0_interior
      using interior_mono by blast
    then have content b *R f a = 0
      by auto
  }
  then have norm ((∑(x, k)∈p1. content k *R f x) + (∑(x, k)∈p2. content k
*R f x) - i) =
            norm ((∑(x, k)∈p1 ∪ p2. content k *R f x) - i)
    by (subst sum.union_inter_neutral) (auto simp: p1 p2)
  also have ... < e
    using d(2) p12 by (simp add: fine_Un k ‹γ fine p1› ‹γ fine p2›)
  finally have norm ((∑(x, k)∈p1. content k *R f x) + (∑(x, k)∈p2. content
k *R f x) - i) < e .
 }
 then show ?thesis
  using d(1) that by auto
qed

lemma integrable_split [intro]:
 fixes f :: 'a::euclidean_space ⇒ 'b::{real_normed_vector,complete_space}
 assumes f: f integrable_on cbox a b
    and k: k ∈ Basis
  shows f integrable_on (cbox a b ∩ {x. x•k ≤ c})   (is ?thesis1)
  and   f integrable_on (cbox a b ∩ {x. x•k ≥ c})   (is ?thesis2)
proof -
```

```
  obtain y where y: (f has_integral y) (cbox a b)
    using f by blast
  define a' where a' = (∑ i∈Basis. (if i = k then max (a•k) c else a•i)*R i)
  define b' where b' = (∑ i∈Basis. (if i = k then min (b•k) c else b•i)*R i)
  have ∃ d. gauge d ∧
          (∀ p1 p2. p1 tagged_division_of cbox a b ∩ {x. x • k ≤ c} ∧ d fine p1 ∧
                    p2 tagged_division_of cbox a b ∩ {x. x • k ≤ c} ∧ d fine p2 ⟶
                      norm ((∑ (x,K) ∈ p1. content K *R f x) − (∑ (x,K) ∈ p2.
content K *R f x)) < e)
    if e > 0 for e
  proof −
    have e/2 > 0 using that by auto
  with has_integral_separate_sides[OF y this k, of c]
  obtain d
    where gauge d
        and d: ⋀p1 p2. ⟦p1 tagged_division_of cbox a b ∩ {x. x • k ≤ c}; d fine
p1;
                        p2 tagged_division_of cbox a b ∩ {x. c ≤ x • k}; d fine p2⟧
                  ⟹ norm ((∑ (x,K)∈p1. content K *R f x) + (∑ (x,K)∈p2. content
K *R f x) − y) < e/2
    by metis
  show ?thesis
    proof (rule_tac x=d in exI, clarsimp simp add: ‹gauge d›)
      fix p1 p2
      assume as: p1 tagged_division_of (cbox a b) ∩ {x. x • k ≤ c} d fine p1
                 p2 tagged_division_of (cbox a b) ∩ {x. x • k ≤ c} d fine p2
      show norm ((∑ (x, k)∈p1. content k *R f x) − (∑ (x, k)∈p2. content k *R
f x)) < e
      proof (rule fine_division_exists[OF ‹gauge d›, of a' b])
        fix p
        assume p tagged_division_of cbox a' b d fine p
        then show ?thesis
          using as norm_triangle_half_l[OF d[of p1 p] d[of p2 p]]
          unfolding interval_split[OF k] b'_def[symmetric] a'_def[symmetric]
          by (auto simp add: algebra_simps)
      qed
    qed
  qed
  with f show ?thesis1
    by (simp add: interval_split[OF k] integrable_Cauchy)
  have ∃ d. gauge d ∧
          (∀ p1 p2. p1 tagged_division_of cbox a b ∩ {x. x • k ≥ c} ∧ d fine p1 ∧
                    p2 tagged_division_of cbox a b ∩ {x. x • k ≥ c} ∧ d fine p2 ⟶
                      norm ((∑ (x,K) ∈ p1. content K *R f x) − (∑ (x,K) ∈ p2.
content K *R f x)) < e)
    if e > 0 for e
  proof −
    have e/2 > 0 using that by auto
  with has_integral_separate_sides[OF y this k, of c]
```

```
  obtain d
    where gauge d
        and d: ⋀p1 p2. ⟦p1 tagged_division_of cbox a b ∩ {x. x • k ≤ c}; d fine
p1;
                       p2 tagged_division_of cbox a b ∩ {x. c ≤ x • k}; d fine p2⟧
                  ⟹ norm ((∑(x,K)∈p1. content K *R f x) + (∑(x,K)∈p2. content
K *R f x) − y) < e/2
    by metis
  show ?thesis
    proof (rule_tac x=d in exI, clarsimp simp add: ‹gauge d›)
      fix p1 p2
      assume as: p1 tagged_division_of (cbox a b) ∩ {x. x • k ≥ c} d fine p1
                 p2 tagged_division_of (cbox a b) ∩ {x. x • k ≥ c} d fine p2
      show norm ((∑(x, k)∈p1. content k *R f x) − (∑(x, k)∈p2. content k *R
f x)) < e
      proof (rule fine_division_exists[OF ‹gauge d›, of a b'])
        fix p
        assume p tagged_division_of cbox a b' d fine p
        then show ?thesis
          using as norm_triangle_half_l[OF d[of p p1] d[of p p2]]
          unfolding interval_split[OF k] b'_def[symmetric] a'_def[symmetric]
          by (auto simp add: algebra_simps)
      qed
    qed
  qed
  with f show ?thesis2
    by (simp add: interval_split[OF k] integrable_Cauchy)
qed

lemma operative_integralI:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  shows operative (lift_option (+)) (Some 0)
    (λi. if f integrable_on i then Some (integral i f) else None)
proof −
  interpret comm_monoid lift_option plus Some (0::'b)
    by (rule comm_monoid_lift_option)
      (rule add.comm_monoid_axioms)
  show ?thesis
  proof
    fix a b c
    fix k :: 'a
    assume k: k ∈ Basis
    show (if f integrable_on cbox a b then Some (integral (cbox a b) f) else None)
=
          lift_option (+) (if f integrable_on cbox a b ∩ {x. x • k ≤ c} then Some
(integral (cbox a b ∩ {x. x • k ≤ c}) f) else None)
         (if f integrable_on cbox a b ∩ {x. c ≤ x • k} then Some (integral (cbox a b
∩ {x. c ≤ x • k}) f) else None)
    proof (cases f integrable_on cbox a b)
```

```
      case True
      with k show ?thesis
      by (auto simp: integrable_split intro: integral_unique [OF has_integral_split[OF
_ _ k]])
    next
    case False
     have ¬ (f integrable_on cbox a b ∩ {x. x · k ≤ c}) ∨ ¬ ( f integrable_on cbox
a b ∩ {x. c ≤ x · k})
      proof (rule ccontr)
        assume ¬ ?thesis
        then have f integrable_on cbox a b
          unfolding integrable_on_def
          apply (rule_tac x=integral (cbox a b ∩ {x. x · k ≤ c}) f + integral (cbox
a b ∩ {x. x · k ≥ c}) f in exI)
          apply (auto intro: has_integral_split[OF _ _ k])
          done
        then show False
          using False by auto
      qed
      then show ?thesis
        using False by auto
    qed
  next
    fix a b :: 'a
    assume box a b = {}
    then show (if f integrable_on cbox a b then Some (integral (cbox a b) f) else
None) = Some 0
      using has_integral_null_eq
      by (auto simp: integrable_on_null content_eq_0_interior)
  qed
qed
```

7.14.7 Bounds on the norm of Riemann sums and the integral itself

```
lemma dsum_bound:
  assumes p: p division_of (cbox a b)
    and norm c ≤ e
  shows norm (sum (λl. content l *R c) p) ≤ e * content(cbox a b)
proof −
  have sumeq: (∑ i∈p. |content i|) = sum content p
    by simp
  have e: 0 ≤ e
    using assms(2) norm_ge_zero order_trans by blast
  have norm (sum (λl. content l *R c) p) ≤ (∑ i∈p. norm (content i *R c))
    using norm_sum by blast
  also have ... ≤ e * (∑ i∈p. |content i|)
   by (simp add: sum_distrib_left[symmetric] mult.commute assms(2) mult_right_mono
sum_nonneg)
```

```
  also have ... ≤ e * content (cbox a b)
    by (metis additive_content_division p eq_iff sumeq)
  finally show ?thesis .
qed

lemma rsum_bound:
  assumes p: p tagged_division_of (cbox a b)
      and ∀x∈cbox a b. norm (f x) ≤ e
    shows norm (sum (λ(x,k). content k *R f x) p) ≤ e * content (cbox a b)
proof (cases cbox a b = {})
  case True show ?thesis
    using p unfolding True tagged_division_of_trivial by auto
next
  case False
  then have e: e ≥ 0
    by (meson ex_in_conv assms(2) norm_ge_zero order_trans)
  have sum_le: sum (content ∘ snd) p ≤ content (cbox a b)
    unfolding additive_content_tagged_division[OF p, symmetric] split_def
    by (auto intro: eq_refl)
  have con: ⋀xk. xk ∈ p ⟹ 0 ≤ content (snd xk)
    using tagged_division_ofD(4) [OF p] content_pos_le
    by force
  have norm (sum (λ(x,k). content k *R f x) p) ≤ (∑i∈p. norm (case i of (x, k)
⇒ content k *R f x))
    by (rule norm_sum)
  also have ... ≤ e * content (cbox a b)
  proof −
    have ⋀xk. xk ∈ p ⟹ norm (f (fst xk)) ≤ e
      using assms(2) p tag_in_interval by force
    moreover have (∑i∈p. |content (snd i)| * e) ≤ e * content (cbox a b)
      unfolding sum_distrib_right[symmetric]
      using con sum_le by (auto simp: mult.commute intro: mult_left_mono [OF
_ e])
    ultimately show ?thesis
      unfolding split_def norm_scaleR
        by (metis (no_types, lifting) mult_left_mono[OF _ abs_ge_zero]    or-
der_trans[OF sum_mono])
  qed
  finally show ?thesis .
qed

lemma rsum_diff_bound:
  assumes p tagged_division_of (cbox a b)
    and ∀x∈cbox a b. norm (f x − g x) ≤ e
  shows norm (sum (λ(x,k). content k *R f x) p − sum (λ(x,k). content k *R g x)
p) ≤
        e * content (cbox a b)
  using order_trans[OF _ rsum_bound[OF assms]]
  by (simp add: split_def scaleR_diff_right sum_subtractf eq_refl)
```

lemma *has_integral_bound*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes *0* ≤ *B*
and *f*: (*f has_integral i*) (*cbox a b*)
and ⋀*x*. *x*∈*cbox a b* ⟹ *norm* (*f x*) ≤ *B*
shows *norm i* ≤ *B* ∗ *content* (*cbox a b*)
proof (*rule ccontr*)
assume ¬ *?thesis*
then have *norm i* − *B* ∗ *content* (*cbox a b*) > *0*
by *auto*
with *f*[*unfolded has_integral*]
obtain *γ* **where** *gauge γ* **and** *γ*:
⋀*p*. ⟦*p tagged_division_of cbox a b*; *γ fine p*⟧
⟹ *norm* ((∑ (*x*, *K*)∈*p*. *content K* ∗*R f x*) − *i*) < *norm i* − *B* ∗ *content* (*cbox a b*)
by *metis*
then obtain *p* **where** *p*: *p tagged_division_of cbox a b* **and** *γ fine p*
using *fine_division_exists* **by** *blast*
have ⋀*s B*. *norm s* ≤ *B* ⟹ ¬ *norm* (*s* − *i*) < *norm i* − *B*
unfolding *not_less*
by (*metis diff_left_mono dist_commute dist_norm norm_triangle_ineq2 order_trans*)
then show *False*
using *γ* [*OF p* ‹*γ fine p*›] *rsum_bound*[*OF p*] *assms* **by** *metis*
qed

corollary *integrable_bound*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*real_normed_vector*
assumes *0* ≤ *B*
and *f integrable_on* (*cbox a b*)
and ⋀*x*. *x*∈*cbox a b* ⟹ *norm* (*f x*) ≤ *B*
shows *norm* (*integral* (*cbox a b*) *f*) ≤ *B* ∗ *content* (*cbox a b*)
by (*metis integrable_integral has_integral_bound assms*)

7.14.8 Similar theorems about relationship among components

lemma *rsum_component_le*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *p*: *p tagged_division_of* (*cbox a b*)
and ⋀*x*. *x* ∈ *cbox a b* ⟹ (*f x*)∙*i* ≤ (*g x*)∙*i*
shows (∑ (*x*, *K*)∈*p*. *content K* ∗*R f x*) ∙ *i* ≤ (∑ (*x*, *K*)∈*p*. *content K* ∗*R g x*) ∙ *i*
unfolding *inner_sum_left*
proof (*rule sum_mono*, *clarify*)
fix *x K*
assume *ab*: (*x*, *K*) ∈ *p*
with *p* **obtain** *u v* **where** *K*: *K* = *cbox u v*

```
    by blast
  then show (content K *R f x) · i ≤ (content K *R g x) · i
   by (metis ab assms inner_scaleR_left measure_nonneg mult_left_mono tag_in_interval)
qed

lemma has_integral_component_le:
  fixes f g :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes k: k ∈ Basis
  assumes (f has_integral i) S (g has_integral j) S
    and f_le_g: ⋀x. x ∈ S ⟹ (f x)•k ≤ (g x)•k
  shows i•k ≤ j•k
proof −
  have ik_le_jk: i•k ≤ j•k
    if f_i: (f has_integral i) (cbox a b)
    and g_j: (g has_integral j) (cbox a b)
    and le: ∀x∈cbox a b. (f x)•k ≤ (g x)•k
    for a b i and j :: 'b and f g :: 'a ⇒ 'b
  proof (rule ccontr)
    assume ¬ ?thesis
    then have *: 0 < (i•k − j•k) / 3
      by auto
    obtain γ1 where gauge γ1
      and γ1: ⋀p. ⟦p tagged_division_of cbox a b; γ1 fine p⟧
            ⟹ norm ((∑(x, k)∈p. content k *R f x) − i) < (i · k − j · k) / 3
      using f_i[unfolded has_integral,rule_format,OF *] by fastforce
    obtain γ2 where gauge γ2
      and γ2: ⋀p. ⟦p tagged_division_of cbox a b; γ2 fine p⟧
            ⟹ norm ((∑(x, k)∈p. content k *R g x) − j) < (i · k − j · k) / 3
      using g_j[unfolded has_integral,rule_format,OF *] by fastforce
    obtain p where p: p tagged_division_of cbox a b and γ1 fine p γ2 fine p
      using fine_division_exists[OF gauge_Int[OF ‹gauge γ1› ‹gauge γ2›], of a
b] unfolding fine_Int
      by metis
    then have |((∑(x, k)∈p. content k *R f x) − i) · k| < (i · k − j · k) / 3
        |((∑(x, k)∈p. content k *R g x) − j) · k| < (i · k − j · k) / 3
      using le_less_trans[OF Basis_le_norm[OF k]] k γ1 γ2 by metis+
    then show False
      unfolding inner_simps
      using rsum_component_le[OF p] le
      by (fastforce simp add: abs_real_def split: if_split_asm)
  qed
  show ?thesis
  proof (cases ∃a b. S = cbox a b)
    case True
    with ik_le_jk assms show ?thesis
      by auto
  next
    case False
    show ?thesis
```

```
    proof (rule ccontr)
      assume ¬ i·k ≤ j·k
      then have ij: (i·k − j·k) / 3 > 0
        by auto
      obtain B1 where 0 < B1
          and B1: ⋀a b. ball 0 B1 ⊆ cbox a b ⟹
                  ∃z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b) ∧
                    norm (z − i) < (i · k − j · k) / 3
        using has_integral_altD[OF _ False ij] assms by blast
      obtain B2 where 0 < B2
          and B2: ⋀a b. ball 0 B2 ⊆ cbox a b ⟹
                  ∃z. ((λx. if x ∈ S then g x else 0) has_integral z) (cbox a b) ∧
                    norm (z − j) < (i · k − j · k) / 3
        using has_integral_altD[OF _ False ij] assms by blast
      have bounded (ball 0 B1 ∪ ball (0::'a) B2)
        unfolding bounded_Un by(rule conjI bounded_ball)+
      from bounded_subset_cbox_symmetric[OF this]
      obtain a b::'a where ab: ball 0 B1 ⊆ cbox a b ball 0 B2 ⊆ cbox a b
        by (meson Un_subset_iff)
     then obtain w1 w2 where int_w1: ((λx. if x ∈ S then f x else 0) has_integral
w1) (cbox a b)
                         and norm_w1:  norm (w1 − i) < (i · k − j · k) / 3
                          and int_w2: ((λx. if x ∈ S then g x else 0) has_integral w2)
(cbox a b)
                         and norm_w2: norm (w2 − j) < (i · k − j · k) / 3
        using B1 B2 by blast
      have ∗: ⋀w1 w2 j i::real .|w1 − i| < (i − j) / 3 ⟹ |w2 − j| < (i − j) / 3
⟹ w1 ≤ w2 ⟹ False
        by (simp add: abs_real_def split: if_split_asm)
      have |(w1 − i) · k| < (i · k − j · k) / 3
           |(w2 − j) · k| < (i · k − j · k) / 3
        using Basis_le_norm k le_less_trans norm_w1 norm_w2 by blast+
      moreover
      have w1·k ≤ w2·k
        using ik_le_jk int_w1 int_w2 f_le_g by auto
      ultimately show False
        unfolding inner_simps by(rule ∗)
    qed
  qed
qed

lemma integral_component_le:
  fixes g f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes k ∈ Basis
    and f integrable_on S g integrable_on S
    and ⋀x. x ∈ S ⟹ (f x)·k ≤ (g x)·k
  shows (integral S f)·k ≤ (integral S g)·k
  using has_integral_component_le assms by blast
```

lemma *has_integral_component_nonneg*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $k \in Basis$
and $(f\ has\_integral\ i)\ S$
and $\bigwedge x.\ x \in S \Longrightarrow 0 \le (f\ x) \cdot k$
shows $0 \le i \cdot k$
by (*metis* (*no_types*, *lifting*) *assms euclidean_all_zero_iff has_integral_0 has_integral_component_le*)

lemma *integral_component_nonneg*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $k \in Basis$
and $\bigwedge x.\ x \in S \Longrightarrow 0 \le (f\ x) \cdot k$
shows $0 \le (integral\ S\ f) \cdot k$
by (*smt* (*verit*, *ccfv_threshold*) *assms eq_integralD euclidean_all_zero_iff has_integral_component_no*

lemma *has_integral_component_neg*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $k \in Basis$
and $(f\ has\_integral\ i)\ S$
and $\bigwedge x.\ x \in S \Longrightarrow (f\ x) \cdot k \le 0$
shows $i \cdot k \le 0$
by (*metis* (*no_types*, *lifting*) *assms has_integral_0 has_integral_component_le inner_zero_left*)

lemma *has_integral_component_lbound*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $(f\ has\_integral\ i)\ (cbox\ a\ b)$
and $\forall x \in cbox\ a\ b.\ B \le f(x) \cdot k$
and $k \in Basis$
shows $B * content\ (cbox\ a\ b) \le i \cdot k$
using *has_integral_component_le*[*OF assms*(*3*) *has_integral_const assms*(*1*),*of* $(\sum i \in Basis.\ B *_R\ i) :: 'b$] *assms*(*2*−)
by (*auto simp add*: *field_simps*)

lemma *has_integral_component_ubound*:
fixes $f :: 'a::euclidean\_space => 'b::euclidean\_space$
assumes $(f\ has\_integral\ i)\ (cbox\ a\ b)$
and $\forall x \in cbox\ a\ b.\ f\ x \cdot k \le B$
and $k \in Basis$
shows $i \cdot k \le B * content\ (cbox\ a\ b)$
using *has_integral_component_le*[*OF assms*(*3*,*1*) *has_integral_const*, *of* $\sum i \in Basis.\ B *_R\ i$] *assms*(*2*−)
by (*auto simp add*: *field_simps*)

lemma *integral_component_lbound*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *f integrable_on cbox a b*
and $\forall x \in cbox\ a\ b.\ B \le f(x) \cdot k$
and $k \in Basis$

shows $B * content\ (cbox\ a\ b) \leq (integral(cbox\ a\ b)\ f)\cdot k$
using *assms has_integral_component_lbound* **by** *blast*

lemma *integral_component_lbound_real*:
assumes *f integrable_on* $\{a ::real..b\}$
and $\forall x{\in}\{a..b\}.\ B \leq f(x)\cdot k$
and $k \in Basis$
shows $B * content\ \{a..b\} \leq (integral\ \{a..b\}\ f)\cdot k$
using *assms* **by** (*metis box_real*(2) *integral_component_lbound*)

lemma *integral_component_ubound*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *f integrable_on cbox a b*
and $\forall x{\in}cbox\ a\ b.\ f\ x\cdot k \leq B$
and $k \in Basis$
shows $(integral\ (cbox\ a\ b)\ f)\cdot k \leq B * content\ (cbox\ a\ b)$
using *assms has_integral_component_ubound* **by** *blast*

lemma *integral_component_ubound_real*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
assumes *f integrable_on* $\{a..b\}$
and $\forall x{\in}\{a..b\}.\ f\ x\cdot k \leq B$
and $k \in Basis$
shows $(integral\ \{a..b\}\ f)\cdot k \leq B * content\ \{a..b\}$
using *assms* **by** (*metis box_real*(2) *integral_component_ubound*)

7.14.9 Uniform limit of integrable functions is integrable

lemma *real_arch_invD*:
$0 < (e::real) \Longrightarrow (\exists n::nat.\ n \neq 0 \wedge 0 < inverse\ (real\ n) \wedge inverse\ (real\ n) < e)$
by (*subst*(*asm*) *real_arch_inverse*)

lemma *integrable_uniform_limit*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::banach$
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists g.\ (\forall x{\in}cbox\ a\ b.\ norm\ (f\ x - g\ x) \leq e) \wedge g$ *integrable_on cbox a b*
shows *f integrable_on cbox a b*
proof (*cases content* $(cbox\ a\ b) > 0$)
case *False* **then show** *?thesis*
using *has_integral_null* **by** (*simp add*: *content_lt_nz integrable_on_def*)
next
case *True*
have $1\ /\ (real\ n + 1) > 0$ **for** *n*
by *auto*
then have $\exists g.\ (\forall x{\in}cbox\ a\ b.\ norm\ (f\ x - g\ x) \leq 1\ /\ (real\ n + 1)) \wedge g$ *integrable_on cbox a b* **for** *n*
using *assms* **by** *blast*
then obtain *g* **where** *g_near_f*: $\bigwedge n\ x.\ x \in cbox\ a\ b \Longrightarrow norm\ (f\ x - g\ n\ x) \leq$

```
1 / (real n + 1)
                and int_g: ⋀n. g n integrable_on cbox a b
    by metis
  then obtain h where h: ⋀n. (g n has_integral h n) (cbox a b)
    unfolding integrable_on_def by metis
  have Cauchy h
    unfolding Cauchy_def
  proof clarify
    fix e :: real
    assume e>0
    then have e/4 / content (cbox a b) > 0
      using True by (auto simp: field_simps)
    then obtain M where M ≠ 0 and M: 1 / (real M) < e/4 / content (cbox a
b)
      by (metis inverse_eq_divide real_arch_inverse)
    show ∃M. ∀m≥M. ∀n≥M. dist (h m) (h n) < e
    proof (intro exI strip)
      fix m n
      assume m: M ≤ m and n: M ≤ n
      have e/4>0 using ‹e>0› by auto
      then obtain gm gn where gauge gm gauge gn
            and gm: ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ gm fine 𝒟
                        ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R g m x) − h m) <
e/4
            and gn: ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ gn fine 𝒟 ⟹
                  norm ((∑(x,K) ∈ 𝒟. content K *R g n x) − h n) < e/4
        using h[unfolded has_integral] by meson
      then obtain 𝒟 where 𝒟: 𝒟 tagged_division_of cbox a b (λx. gm x ∩ gn x)
fine 𝒟
        by (metis (full_types) fine_division_exists gauge_Int)
      have triangle3: norm (i1 − i2) < e
        if no: norm(s2 − s1) ≤ e/2 norm (s1 − i1) < e/4 norm (s2 − i2) < e/4
for s1 s2 i1 and i2::'b
      proof −
        have norm (i1 − i2) ≤ norm (i1 − s1) + norm (s1 − s2) + norm (s2 −
i2)
          using norm_triangle_ineq[of i1 − s1 s1 − i2]
          using norm_triangle_ineq[of s1 − s2 s2 − i2]
          by (auto simp: algebra_simps)
        also have ... < e
          using no by (auto simp: algebra_simps norm_minus_commute)
        finally show ?thesis .
      qed
      have finep: gm fine 𝒟 gn fine 𝒟
        using fine_Int 𝒟 by auto
      have norm_le: norm (g n x − g m x) ≤ 2 / real M if x: x ∈ cbox a b for x
      proof −
        have norm (f x − g n x) + norm (f x − g m x) ≤ 1 / (real n + 1) + 1 /
(real m + 1)
```

```
      using g_near_f[OF x, of n] g_near_f[OF x, of m] by simp
    also have ... ≤ 1 / (real M) + 1 / (real M)
      using ‹M ≠ 0› m n by (intro add_mono; force simp: field_split_simps)
    also have ... = 2 / real M
      by auto
    finally show norm (g n x − g m x) ≤ 2 / real M
      using norm_triangle_le[of g n x − f x f x − g m x 2 / real M]
      by (auto simp: algebra_simps simp add: norm_minus_commute)
  qed
  have norm ((∑(x,K) ∈ 𝒟. content K *R g n x) − (∑(x,K) ∈ 𝒟. content K
*R g m x)) ≤ 2 / real M * content (cbox a b)
   by (blast intro: norm_le rsum_diff_bound[OF 𝒟(1), where e=2 / real M])
  also have ... ≤ e/2
    using M True
    by (auto simp: field_simps)
  finally have le_e2: norm ((∑(x,K) ∈ 𝒟. content K *R g n x) − (∑(x,K)
∈ 𝒟. content K *R g m x)) ≤ e/2 .
  then show dist (h m) (h n) < e
    unfolding dist_norm using gm gn 𝒟 finep by (auto intro!: triangle3)
 qed
qed
then obtain s where s: h ⟶ s
 using convergent_eq_Cauchy[symmetric] by blast
show ?thesis
 unfolding integrable_on_def has_integral
proof (rule_tac x=s in exI, clarify)
 fix e::real
 assume e: 0 < e
 then have e/3 > 0 by auto
 then obtain N1 where N1: ∀n≥N1. norm (h n − s) < e/3
  using LIMSEQ_D [OF s] by metis
 from e True have e/3 / content (cbox a b) > 0
  by (auto simp: field_simps)
 then obtain N2 :: nat
     where N2 ≠ 0 and N2: 1 / (real N2) < e/3 / content (cbox a b)
  by (metis inverse_eq_divide real_arch_inverse)
 obtain g' where gauge g'
       and g': ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ g' fine 𝒟 ⟹
            norm ((∑(x,K) ∈ 𝒟. content K *R g (N1 + N2) x) − h (N1 +
N2)) < e/3
  by (metis h has_integral ‹e/3 > 0›)
 have *: norm (sf − s) < e
   if no: norm (sf − sg) ≤ e/3 norm(h − s) < e/3 norm (sg − h) < e/3 for
sf sg h
 proof −
  have norm (sf − s) ≤ norm (sf − sg) + norm (sg − h) + norm (h − s)
   using norm_triangle_ineq[of sf − sg sg − s]
   using norm_triangle_ineq[of sg −  h  h − s]
   by (auto simp: algebra_simps)
```

```
    also have ... < e
      using no by (auto simp: algebra_simps norm_minus_commute)
    finally show ?thesis .
  qed
  { fix 𝒟
    assume ptag: 𝒟 tagged_division_of (cbox a b) and g' fine 𝒟
    then have norm_less: norm ((∑(x,K) ∈ 𝒟. content K *R g (N1 + N2) x)
− h (N1 + N2)) < e/3
      using g' by blast
    have content (cbox a b) < e/3 * (of_nat N2)
      using ‹N2 ≠ 0› N2 using True by (auto simp: field_split_simps)
    moreover have e/3 * of_nat N2 ≤ e/3 * (of_nat (N1 + N2) + 1)
      using ‹e>0› by auto
    ultimately have content (cbox a b) < e/3 * (of_nat (N1 + N2) + 1)
      by linarith
    then have le_e3: 1 / (real (N1 + N2) + 1) * content (cbox a b) ≤ e/3
      unfolding inverse_eq_divide
      by (auto simp: field_simps)
    have ne3: norm (h (N1 + N2) − s) < e/3
      using N1 by auto
    have norm ((∑(x,K) ∈ 𝒟. content K *R f x) − (∑(x,K) ∈ 𝒟. content K
*R g (N1 + N2) x))
          ≤ 1 / (real (N1 + N2) + 1) * content (cbox a b)
      by (blast intro: g_near_f rsum_diff_bound[OF ptag])
    then have norm ((∑(x,K) ∈ 𝒟. content K *R f x) − s) < e
      by (rule *[OF order_trans [OF _ le_e3] ne3 norm_less])
  }
  then show ∃d. gauge d ∧
        (∀𝒟. 𝒟 tagged_division_of cbox a b ∧ d fine 𝒟 ⟶ norm ((∑(x,K) ∈
𝒟. content K *R f x) − s) < e)
    by (blast intro: g' ‹gauge g'›)
 qed
qed
```

lemmas *integrable_uniform_limit_real* = *integrable_uniform_limit* [**where** *'a=real*, *simplified*]

7.14.10 Negligible sets

definition *negligible* (*s*:: *'a*::*euclidean_space set*) ⟷
  (∀ *a b*. ((*indicator s* :: *'a*⇒*real*) *has_integral 0*) (*cbox a b*))

Negligibility of hyperplane

lemma *content_doublesplit*:
  fixes *a* :: *'a*::*euclidean_space*
  assumes *0 < e*
    and *k*: *k* ∈ *Basis*
  obtains *d* **where** *0 < d* **and** *content* (*cbox a b* ∩ {*x*. |*x*•*k* − *c*| ≤ *d*}) < *e*
proof *cases*
```
```

assume $*$: $a \cdot k \le c \wedge c \le b \cdot k \wedge (\forall j{\in}Basis.\ a \cdot j \le b \cdot j)$
define a' **where** $a'\ d = (\sum j{\in}Basis.\ (if\ j = k\ then\ max\ (a{\cdot}j)\ (c - d)\ else\ a{\cdot}j) *_R j)$ **for** d
define b' **where** $b'\ d = (\sum j{\in}Basis.\ (if\ j = k\ then\ min\ (b{\cdot}j)\ (c + d)\ else\ b{\cdot}j) *_R j)$ **for** d

have $((\lambda d.\ \prod j{\in}Basis.\ (b'\ d - a'\ d) \cdot j) \longrightarrow (\prod j{\in}Basis.\ (b'\ 0 - a'\ 0) \cdot j))\ (at\_right\ 0)$
by (*auto simp*: *b'_def a'_def intro*!: *tendsto_min tendsto_max tendsto_eq_intros*)
also have $(\prod j{\in}Basis.\ (b'\ 0 - a'\ 0) \cdot j) = 0$
using k $*$
by (*intro prod_zero bexI*[*OF* _ *k*])
(*auto simp*: *b'_def a'_def inner_diff inner_sum_left inner_not_same_Basis intro*!: *sum.cong*)
also have $((\lambda d.\ \prod j{\in}Basis.\ (b'\ d - a'\ d) \cdot j) \longrightarrow 0)\ (at\_right\ 0) =$
$((\lambda d.\ content\ (cbox\ a\ b \cap \{x.\ |x{\cdot}k - c| \le d\})) \longrightarrow 0)\ (at\_right\ 0)$
proof (*intro tendsto_cong eventually_at_rightI*)
fix $d :: real$ **assume** d: $d \in \{0<..<1\}$
have $cbox\ a\ b \cap \{x.\ |x{\cdot}k - c| \le d\} = cbox\ (a'\ d)\ (b'\ d)$ **for** d
using $*$ d k **by** (*auto simp add*: *cbox_def set_eq_iff Int_def ball_conj_distrib abs_diff_le_iff a'_def b'_def*)
moreover have $j \in Basis \Longrightarrow a'\ d \cdot j \le b'\ d \cdot j$ **for** j
using $*$ d k **by** (*auto simp*: *a'_def b'_def*)
ultimately show $(\prod j{\in}Basis.\ (b'\ d - a'\ d) \cdot j) = content\ (cbox\ a\ b \cap \{x.\ |x{\cdot}k - c| \le d\})$
by *simp*
qed *simp*
finally have $((\lambda d.\ content\ (cbox\ a\ b \cap \{x.\ |x \cdot k - c| \le d\})) \longrightarrow 0)\ (at\_right\ 0)$ **.**
from *order_tendstoD*(*2*)[*OF this* ‹*0*<*e*›]
obtain d' **where** $0 < d'$ **and** d': $\bigwedge y.\ y > 0 \Longrightarrow y < d' \Longrightarrow content\ (cbox\ a\ b \cap \{x.\ |x \cdot k - c| \le y\}) < e$
by (*subst* (*asm*) *eventually_at_right*[*of* _ *1*]) *auto*
show *?thesis*
by (*rule that*[*of* $d'/2$], *insert* ‹*0*<d'› d'[*of* $d'/2$], *auto*)
next
assume $*$: $\neg\ (a \cdot k \le c \wedge c \le b \cdot k \wedge (\forall j{\in}Basis.\ a \cdot j \le b \cdot j))$
then have $(\exists j{\in}Basis.\ b \cdot j < a \cdot j) \vee (c < a \cdot k \vee b \cdot k < c)$
by (*auto simp*: *not_le*)
show *thesis*
proof *cases*
assume $\exists j{\in}Basis.\ b \cdot j < a \cdot j$
then have [*simp*]: $cbox\ a\ b = \{\}$
using *box_ne_empty*(*1*)[*of a b*] **by** *auto*
show *?thesis*
by (*rule that*[*of 1*]) (*simp_all add*: ‹*0*<*e*›)
next
assume $\neg\ (\exists j{\in}Basis.\ b \cdot j < a \cdot j)$
with $*$ **have** $c < a \cdot k \vee b \cdot k < c$

```
    by auto
  then show thesis
  proof
    assume c: c < a · k
    moreover have x ∈ cbox a b ⟹ c ≤ x · k for x
      using k c by (auto simp: cbox_def)
    ultimately have cbox a b ∩ {x. |x · k − c| ≤ (a · k − c)/2} = {}
      using k by (auto simp: cbox_def)
    with ‹0<e› c that[of (a · k − c)/2] show ?thesis
      by auto
  next
    assume c: b · k < c
    moreover have x ∈ cbox a b ⟹ x · k ≤ c for x
      using k c by (auto simp: cbox_def)
    ultimately have cbox a b ∩ {x. |x · k − c| ≤ (c − b · k)/2} = {}
      using k by (auto simp: cbox_def)
    with ‹0<e› c that[of (c − b · k)/2] show ?thesis
      by auto
  qed
 qed
qed


proposition negligible_standard_hyperplane[intro]:
 fixes k :: 'a::euclidean_space
 assumes k: k ∈ Basis
 shows negligible {x. x•k = c}
 unfolding negligible_def has_integral
proof clarsimp
 fix a b and e::real assume e > 0
 with k obtain d where 0 < d and d: content (cbox a b ∩ {x. |x · k − c| ≤ d})
< e
   by (metis content_doublesplit)
 let ?i = indicator {x::'a. x•k = c} :: 'a⇒real
 show ∃γ. gauge γ ∧
         (∀D. D tagged_division_of cbox a b ∧ γ fine D ⟶
              |∑(x,K) ∈ D. content K * ?i x| < e)
 proof (intro exI, safe)
   show gauge (λx. ball x d)
     using ‹0 < d› by blast
 next
   fix D
   assume p: D tagged_division_of (cbox a b) (λx. ball x d) fine D
   have content L = content (L ∩ {x. |x · k − c| ≤ d})
     if (x, L) ∈ D ?i x ≠ 0 for x L
   proof −
     have xk: x•k = c
       using that by (simp add: indicator_def split: if_split_asm)
     have L ⊆ {x. |x · k − c| ≤ d}
```

```
      proof
        fix y
        assume y: y ∈ L
        have L ⊆ ball x d
          using p(2) that(1) by auto
        then have norm (x − y) < d
          by (simp add: dist_norm subset_iff y)
        then have |(x − y) · k| < d
          using k norm_bound_Basis_lt by blast
        then show y ∈ {x. |x · k − c| ≤ d}
          unfolding inner_simps xk by auto
      qed
      then show content L = content (L ∩ {x. |x · k − c| ≤ d})
        by (metis inf.orderE)
    qed
    then have ∗: (∑(x,K)∈𝒟. content K ∗ ?i x) = (∑(x,K)∈𝒟. content (K ∩
{x. |x·k − c| ≤ d}) ∗R ?i x)
      by (force simp add: split_paired_all intro!: sum.cong [OF refl])
  note p'= tagged_division_ofD[OF p(1)] and p''=division_of_tagged_division[OF
p(1)]
    have (∑(x,K)∈𝒟. content (K ∩ {x. |x · k − c| ≤ d}) ∗ indicator {x. x · k =
c} x) < e
    proof −
      have (∑(x,K)∈𝒟. content (K ∩ {x. |x · k − c| ≤ d}) ∗ ?i x) ≤ (∑(x,K)∈𝒟.
content (K ∩ {x. |x · k − c| ≤ d}))
        by (force simp add: indicator_def intro!: sum_mono)
      also have ... < e
      proof (subst sum.over_tagged_division_lemma[OF p(1)])
        fix u v::'a
        assume box u v = {}
        then have ∗: content (cbox u v) = 0
          unfolding content_eq_0_interior by simp
        have cbox u v ∩ {x. |x · k − c| ≤ d} ⊆ cbox u v
          by auto
        then have content (cbox u v ∩ {x. |x · k − c| ≤ d}) ≤ content (cbox u v)
          unfolding interval_doublesplit[OF k] by (rule content_subset)
        then show content (cbox u v ∩ {x. |x · k − c| ≤ d}) = 0
          unfolding ∗ interval_doublesplit[OF k]
          by (blast intro: antisym)
      next
        have (∑l∈snd ‘ 𝒟. content (l ∩ {x. |x · k − c| ≤ d})) =
          sum content ((λl. l ∩ {x. |x · k − c| ≤ d}) ‘ {l∈snd ‘ 𝒟. l ∩ {x. |x · k − c|
≤ d} ≠ {}})
        proof (subst (2) sum.reindex_nontrivial)
          fix x y assume x ∈ {l ∈ snd ‘ 𝒟. l ∩ {x. |x · k − c| ≤ d} ≠ {}} y ∈ {l
∈ snd ‘ 𝒟. l ∩ {x. |x · k − c| ≤ d} ≠ {}}
            x ≠ y and eq: x ∩ {x. |x · k − c| ≤ d} = y ∩ {x. |x · k − c| ≤ d}
          then obtain x' y' where (x', x) ∈ 𝒟 x ∩ {x. |x · k − c| ≤ d} ≠ {} (y',
y) ∈ 𝒟 y ∩ {x. |x · k − c| ≤ d} ≠ {}
```

```
          by (auto)
        from p'(5)[OF ‹(x', x) ∈ 𝒟› ‹(y', y) ∈ 𝒟›] ‹x ≠ y› have interior (x ∩ y)
= {}
          by auto
         moreover have interior ((x ∩ {x. |x · k − c| ≤ d}) ∩ (y ∩ {x. |x · k −
c| ≤ d})) ⊆ interior (x ∩ y)
          by (auto intro: interior_mono)
         ultimately have interior (x ∩ {x. |x · k − c| ≤ d}) = {}
          by (auto simp: eq)
         then show content (x ∩ {x. |x · k − c| ≤ d}) = 0
          using p'(4)[OF ‹(x', x) ∈ 𝒟›] by (auto simp: interval_doublesplit[OF k]
content_eq_0_interior simp del: interior_Int)
       qed (insert p'(1), auto intro!: sum.mono_neutral_right)
       also have ... ≤ norm (∑ l∈(λl. l ∩ {x. |x · k − c| ≤ d}) ‘ {l∈snd ‘ 𝒟. l ∩
{x. |x · k − c| ≤ d} ≠ {}}. content l *R 1::real)
         by simp
       also have ... ≤ 1 * content (cbox a b ∩ {x. |x · k − c| ≤ d})
         using division_doublesplit[OF p'' k, unfolded interval_doublesplit[OF k]]
         unfolding interval_doublesplit[OF k] by (intro dsum_bound) auto
       also have ... < e
         using d by simp
       finally show (∑ K∈snd ‘ 𝒟. content (K ∩ {x. |x · k − c| ≤ d})) < e .
     qed
     finally show (∑ (x, K)∈𝒟. content (K ∩ {x. |x · k − c| ≤ d}) * ?i x) < e .
   qed
   then show |∑ (x, K)∈𝒟. content K * ?i x| < e
     unfolding *  by (simp add: sum_nonneg split: prod.split)
  qed
qed

corollary negligible_standard_hyperplane_cart:
  fixes k :: 'a::finite
  shows negligible {x. x$k = (0::real)}
  by (simp add: cart_eq_inner_axis negligible_standard_hyperplane)
```

Hence the main theorem about negligible sets

```
lemma has_integral_negligible_cbox:
  fixes f :: 'b::euclidean_space ⇒ 'a::real_normed_vector
  assumes negs: negligible S
    and 0: ⋀x. x ∉ S ⟹ f x = 0
  shows (f has_integral 0) (cbox a b)
  unfolding has_integral
proof clarify
  fix e::real
  assume e > 0
  then have nn_gt0: e/2 / ((real n+1) * (2 ^ n)) > 0 for n
    by simp
  then have ∃γ. gauge γ ∧
```

```
                (∀ 𝒟. 𝒟 tagged_division_of cbox a b ∧ γ fine 𝒟 ⟶
                    |∑(x,K) ∈ 𝒟. content K *R indicator S x|
                    < e/2 / ((real n + 1) * 2 ^ n)) for n
    using negs [unfolded negligible_def has_integral] by auto
  then obtain γ where
    gd: ⋀n. gauge (γ n)
    and γ: ⋀n 𝒟. ⟦𝒟 tagged_division_of cbox a b; γ n fine 𝒟⟧
              ⟹ |∑(x,K) ∈ 𝒟. content K *R indicator S x| < e/2 / ((real n +
1) * 2 ^ n)
    by metis
  show ∃γ. gauge γ ∧
          (∀ 𝒟. 𝒟 tagged_division_of cbox a b ∧ γ fine 𝒟 ⟶
              norm ((∑(x,K) ∈ 𝒟. content K *R f x) − 0) < e)
  proof (intro exI, safe)
    show gauge (λx. γ (nat ⌊norm (f x)⌋) x)
      using gd by (auto simp: gauge_def)

    show norm ((∑(x,K) ∈ 𝒟. content K *R f x) − 0) < e
      if 𝒟 tagged_division_of (cbox a b) (λx. γ (nat ⌊norm (f x)⌋) x) fine 𝒟 for 𝒟
    proof (cases 𝒟 = {})
      case True with ‹0 < e› show ?thesis by simp
    next
      case False
      obtain N where Max ((λ(x, K). norm (f x)) ` 𝒟) ≤ real N
        using real_arch_simple by blast
      then have N: ⋀x. x ∈ (λ(x, K). norm (f x)) ` 𝒟 ⟹ x ≤ real N
      by (meson Max_ge that(1) dual_order.trans finite_imageI tagged_division_of_finite)
      have ∀ i. ∃ q. q tagged_division_of (cbox a b) ∧ (γ i) fine q ∧ (∀(x,K) ∈ 𝒟.
K ⊆ (γ i) x ⟶ (x, K) ∈ q)
        by (auto intro: tagged_division_finer[OF that(1) gd])
      from choice[OF this]
      obtain q where q: ⋀n. q n tagged_division_of cbox a b
                        ⋀n. γ n fine q n
                        ⋀n x K. ⟦(x, K) ∈ 𝒟; K ⊆ γ n x⟧ ⟹ (x, K) ∈ q n
        by fastforce
      have finite 𝒟
        using that(1) by blast
      then have sum_le_inc: ⟦finite T; ⋀x y. (x,y) ∈ T ⟹ (0::real) ≤ g(x,y);
                        ⋀y. y∈𝒟 ⟹ ∃ x. (x,y) ∈ T ∧ f(y) ≤ g(x,y)⟧ ⟹ sum f 𝒟 ≤
sum g T for f g T
        by (rule sum_le_included[of 𝒟 T g snd f]; force)
      have norm (∑(x,K) ∈ 𝒟. content K *R f x) ≤ (∑(x,K) ∈ 𝒟. norm (content
K *R f x))
        unfolding split_def by (rule norm_sum)
      also have ... ≤ (∑(i, j) ∈ Sigma {..N + 1} q.
                          (real i + 1) * (case j of (x, K) ⇒ content K *R indicator S
x))
      proof (rule sum_le_inc, safe)
        show finite (Sigma {..N+1} q)
```

```
      by (meson finite_SigmaI finite_atMost tagged_division_of_finite q(1))
    next
      fix x K
      assume xk: (x, K) ∈ 𝒟
      define n where n = nat ⌊norm (f x)⌋
      have ∗: norm (f x) ∈ (λ(x, K). norm (f x)) ` 𝒟
        using xk by auto
      have nfx: real n ≤ norm (f x) norm (f x) ≤ real n + 1
        unfolding n_def by auto
      then have n ∈ {0..N + 1}
        using N[OF ∗] by auto
      moreover have K ⊆ γ (nat ⌊norm (f x)⌋) x
        using that(2) xk by auto
      moreover then have (x, K) ∈ q (nat ⌊norm (f x)⌋)
        by (simp add: q(3) xk)
      moreover then have (x, K) ∈ q n
        using n_def by blast
      moreover
      have norm (content K ∗R f x) ≤ (real n + 1) ∗ (content K ∗ indicator S x)
      proof (cases x ∈ S)
        case False
        then show ?thesis by (simp add: 0)
      next
        case True
        have ∗: content K ≥ 0
          using tagged_division_ofD(4)[OF that(1) xk] by auto
        moreover have content K ∗ norm (f x) ≤ content K ∗ (real n + 1)
          by (simp add: mult_left_mono nfx(2))
        ultimately show ?thesis
          using nfx True by (auto simp: field_simps)
      qed
      ultimately show ∃ y. (y, x, K) ∈ (Sigma {..N + 1} q) ∧ norm (content
K ∗R f x) ≤
        (real y + 1) ∗ (content K ∗R indicator S x)
        by force
    qed auto
    also have ... = (∑ i≤N + 1. ∑ j∈q i. (real i + 1) ∗ (case j of (x, K) ⇒
content K ∗R indicator S x))
      using q(1) by (intro sum_Sigma_product [symmetric]) auto
    also have ... ≤ (∑ i≤N + 1. (real i + 1) ∗ |∑ (x,K) ∈ q i. content K ∗R
indicator S x|)
      by (rule sum_mono) (simp add: sum_distrib_left [symmetric])
    also have ... ≤ (∑ i≤N + 1. e/2/2 ^ i)
    proof (rule sum_mono)
      show (real i + 1) ∗ |∑ (x,K) ∈ q i. content K ∗R indicator S x| ≤ e/2/2
^ i
        if i ∈ {..N + 1} for i
        using γ[of q i i] q by (simp add: divide_simps mult.left_commute)
    qed
```

```
    also have ... = e/2 * (∑i≤N + 1. (1/2) ^ i)
      unfolding sum_distrib_left by (metis divide_inverse inverse_eq_divide
power_one_over)
    also have ... < e/2 * 2
    proof (rule mult_strict_left_mono)
      have sum (power (1/2)) {..N + 1} = sum (power (1/2::real)) {..<N +
2}
        using lessThan_Suc_atMost by auto
      also have ... < 2
        by (auto simp: geometric_sum)
      finally show sum (power (1/2::real)) {..N + 1} < 2 .
    qed (use ‹0 < e› in auto)
    finally  show ?thesis by auto
  qed
 qed
qed


proposition has_integral_negligible:
 fixes f :: 'b::euclidean_space ⇒ 'a::real_normed_vector
 assumes negs: negligible S
   and ⋀x. x ∈ (T - S) ⟹ f x = 0
 shows (f has_integral 0) T
proof (cases ∃ a b. T = cbox a b)
 case True
 then have ((λx. if x ∈ T then f x else 0) has_integral 0) T
   using assms by (auto intro!: has_integral_negligible_cbox)
 then show ?thesis
   by (rule has_integral_eq [rotated]) auto
next
 case False
 let ?f = (λx. if x ∈ T then f x else 0)
 have ((λx. if x ∈ T then f x else 0) has_integral 0) T
   apply (auto simp: False has_integral_alt [of ?f])
   apply (rule_tac x=1 in exI, auto)
  apply (rule_tac x=0 in exI, simp add: has_integral_negligible_cbox [OF negs]
assms)
   done
 then show ?thesis
   by (rule_tac f=?f in has_integral_eq) auto
qed

lemma
 assumes negligible S
 shows integrable_negligible: f integrable_on S and integral_negligible: integral S
f = 0
 using has_integral_negligible [OF assms]
 by (auto simp: has_integral_iff)
```

lemma *has_integral_spike*:
fixes *f* :: *'b*::*euclidean_space* ⇒ *'a*::*real_normed_vector*
assumes *negligible S*
and *gf*: ⋀*x*. *x* ∈ *T* − *S* ⟹ *g x* = *f x*
and *fint*: (*f has_integral y*) *T*
shows (*g has_integral y*) *T*
proof −
have ∗: (*g has_integral y*) (*cbox a b*)
if (*f has_integral y*) (*cbox a b*) ∀*x* ∈ *cbox a b* − *S*. *g x* = *f x* **for** *a b f* **and** *g*:: *'b* ⇒ *'a* **and** *y*
proof −
have ((λ*x*. *f x* + (*g x* − *f x*)) *has_integral* (*y* + *0*)) (*cbox a b*)
using *that* **by** (*intro has_integral_add has_integral_negligible*) (*auto intro*!: ‹*negligible S*›)
then show *?thesis*
by *auto*
qed
have §: ⋀*a b z*. ⟦⋀*x*. *x* ∈ *T* ∧ *x* ∉ *S* ⟹ *g x* = *f x*;
((λ*x*. *if x* ∈ *T then f x else 0*) *has_integral z*) (*cbox a b*)⟧
⟹ ((λ*x*. *if x* ∈ *T then g x else 0*) *has_integral z*) (*cbox a b*)
by (*auto dest*!: ∗[**where** *f*=λ*x*. *if x*∈*T then f x else 0* **and** *g*=λ*x*. *if x* ∈ *T then g x else 0*])
show *?thesis*
using *fint gf*
apply (*subst has_integral_alt*)
apply (*subst* (*asm*) *has_integral_alt*)
apply (*auto split*: *if_split_asm*)
apply (*blast dest*: ∗)
using § **by** *meson*
qed

lemma *has_integral_spike_eq*:
assumes *negligible S* **and** ⋀*x*. *x* ∈ *T* − *S* ⟹ *g x* = *f x*
shows (*f has_integral y*) *T* ⟷ (*g has_integral y*) *T*
by (*metis assms has_integral_spike*)

lemma *integrable_spike*:
assumes *f integrable_on T negligible S* ⋀*x*. *x* ∈ *T* − *S* ⟹ *g x* = *f x*
shows *g integrable_on T*
using *assms* **unfolding** *integrable_on_def* **by** (*blast intro*: *has_integral_spike*)

lemma *integral_spike*:
assumes *negligible S*
and ⋀*x*. *x* ∈ *T* − *S* ⟹ *g x* = *f x*
shows *integral T f* = *integral T g*
using *has_integral_spike_eq*[*OF assms*]
by (*auto simp*: *integral_def integrable_on_def*)

7.14.11 Some other trivialities about negligible sets

lemma *negligible_subset*:
assumes *negligible s* $t \subseteq s$
shows *negligible t*
unfolding *negligible_def*
by (*metis* (*no_types*) *Diff_iff assms contra_subsetD has_integral_negligible indicator_simps*(*2*))

lemma *negligible_diff*[*intro?*]:
assumes *negligible s*
shows *negligible* $(s - t)$
using *assms* **by** (*meson Diff_subset negligible_subset*)

lemma *negligible_Int*:
assumes *negligible s* $\vee$ *negligible t*
shows *negligible* $(s \cap t)$
using *assms negligible_subset* **by** *force*

lemma *negligible_Un*:
assumes *negligible S* **and** *T*: *negligible T*
shows *negligible* $(S \cup T)$
proof −
have (*indicat_real* $(S \cup T)$ *has_integral 0*) (*cbox a b*)
if *S0*: (*indicat_real S has_integral 0*) (*cbox a b*)
and (*indicat_real T has_integral 0*) (*cbox a b*) **for** *a b*
proof (*subst has_integral_spike_eq*[*OF T*])
show *indicat_real S x* = *indicat_real* $(S \cup T)$ *x* **if** $x \in$ *cbox a b* − *T* **for** *x*
using *that* **by** (*simp add*: *indicator_def*)
show (*indicat_real S has_integral 0*) (*cbox a b*)
by (*simp add*: *S0*)
qed
with *assms* **show** *?thesis*
unfolding *negligible_def* **by** *blast*
qed

lemma *negligible_Un_eq*[*simp*]: *negligible* $(s \cup t) \longleftrightarrow$ *negligible s* $\wedge$ *negligible t*
using *negligible_Un negligible_subset* **by** *blast*

lemma *negligible_sing*[*intro*]: *negligible* {*a*::′*a*::*euclidean_space*}
using *negligible_standard_hyperplane*[*OF SOME_Basis*, *of a* · (*SOME i*. *i* ∈ *Basis*)] *negligible_subset* **by** *blast*

lemma *negligible_insert*[*simp*]: *negligible* (*insert a s*) $\longleftrightarrow$ *negligible s*
by (*metis insert_is_Un negligible_Un_eq negligible_sing*)

lemma *negligible_empty*[*iff*]: *negligible* {}
using *negligible_insert* **by** *blast*

Useful in this form for backchaining

lemma *empty_imp_negligible*: $S = \{\} \Longrightarrow$ *negligible* S
by *simp*

lemma *negligible_finite*[*intro*]:
assumes *finite* s
shows *negligible* s
using *assms* **by** (*induct* s) *auto*

lemma *negligible_Union*[*intro*]:
assumes *finite* $\mathcal{T}$
and $\bigwedge t.\ t \in \mathcal{T} \Longrightarrow$ *negligible* t
shows *negligible*$(\bigcup \mathcal{T})$
using *assms* **by** *induct auto*

lemma *negligible*: *negligible* $S \longleftrightarrow (\forall\, T.\ (\mathit{indicat\_real}\ S\ \mathit{has\_integral}\ 0)\ T)$
proof (*intro iffI allI*)
fix T
assume *negligible* S
then show (*indicator* S *has_integral* 0) T
by (*meson Diff_iff has_integral_negligible indicator_simps*(*2*))
qed (*simp add*: *negligible_def*)

7.14.12 Finite case of the spike theorem is quite commonly needed

lemma *has_integral_spike_finite*:
assumes *finite* S
and $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
and (f *has_integral* y) T
shows (g *has_integral* y) T
using *assms has_integral_spike negligible_finite* **by** *blast*

lemma *has_integral_spike_finite_eq*:
assumes *finite* S
and $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
shows ((f *has_integral* y) $T \longleftrightarrow$ (g *has_integral* y) T)
by (*metis assms has_integral_spike_finite*)

lemma *integrable_spike_finite*:
assumes *finite* S
and $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
and f *integrable_on* T
shows g *integrable_on* T
using *assms has_integral_spike_finite* **by** *blast*

lemma *integrable_spike_finite_eq*:
assumes *finite* S
and $\bigwedge x.\ x \in T - S \Longrightarrow f\ x = g\ x$
shows f *integrable_on* $T \longleftrightarrow g$ *integrable_on* T

by (*metis assms integrable_spike_finite*)

lemma *has_integral_bound_spike_finite*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::real\_normed\_vector$
assumes $0 \le B$ *finite* S
and f: $(f\ has\_integral\ i)\ (cbox\ a\ b)$
and *leB*: $\bigwedge x.\ x \in cbox\ a\ b - S \Longrightarrow norm\ (f\ x) \le B$
shows $norm\ i \le B * content\ (cbox\ a\ b)$
proof −
define g **where** $g \equiv (\lambda x.\ if\ x \in S\ then\ 0\ else\ f\ x)$
then have $\bigwedge x.\ x \in cbox\ a\ b - S \Longrightarrow norm\ (g\ x) \le B$
using *leB* **by** *simp*
moreover have $(g\ has\_integral\ i)\ (cbox\ a\ b)$
using *has_integral_spike_finite* [*OF* ‹*finite S*› _ f]
by (*simp add*: *g_def*)
ultimately show *?thesis*
by (*simp add*: ‹$0 \le B$› *g_def has_integral_bound*)
qed

corollary *has_integral_bound_real*:
fixes $f :: real \Rightarrow 'b::real\_normed\_vector$
assumes $0 \le B$ *finite* S
and $(f\ has\_integral\ i)\ \{a..b\}$
and $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow norm\ (f\ x) \le B$
shows $norm\ i \le B * content\ \{a..b\}$
by (*metis assms box_real*(2) *has_integral_bound_spike_finite*)

7.14.13 In particular, the boundary of an interval is negligible

lemma *negligible_frontier_interval*: $negligible(cbox\ (a::'a::euclidean\_space)\ b - box\ a\ b)$
proof −
let $?A = \bigcup((\lambda k.\ \{x.\ x \cdot k = a \cdot k\} \cup \{x::'a.\ x \cdot k = b \cdot k\})\ `\ Basis)$
have *negligible ?A*
by (*force simp add*: *negligible_Union*[*OF finite_imageI*])
moreover have $cbox\ a\ b - box\ a\ b \subseteq ?A$
by (*force simp add*: *mem_box*)
ultimately show *?thesis*
by (*rule negligible_subset*)
qed

lemma *has_integral_spike_interior*:
assumes f: $(f\ has\_integral\ y)\ (cbox\ a\ b)$ **and** *gf*: $\bigwedge x.\ x \in box\ a\ b \Longrightarrow g\ x = f\ x$
shows $(g\ has\_integral\ y)\ (cbox\ a\ b)$
by (*meson Diff_iff gf has_integral_spike*[*OF negligible_frontier_interval* _ f])

lemma *has_integral_spike_interior_eq*:
assumes $\bigwedge x.\ x \in box\ a\ b \Longrightarrow g\ x = f\ x$

shows (*f has_integral y*) (*cbox a b*) ⟷ (*g has_integral y*) (*cbox a b*)
by (*metis assms has_integral_spike_interior*)

lemma *integrable_spike_interior*:
assumes ⋀*x*. *x* ∈ *box a b* ⟹ *g x* = *f x*
and *f integrable_on cbox a b*
shows *g integrable_on cbox a b*
using *assms has_integral_spike_interior_eq* **by** *blast*

7.14.14 Integrability of continuous functions

lemma *operative_approximableI*:
fixes *f* :: *'b*::*euclidean_space* ⇒ *'a*::*banach*
assumes *0* ≤ *e*
shows *operative conj True* (λ*i*. ∃ *g*. (∀ *x*∈*i*. *norm* (*f x* − *g* (*x*::*'b*)) ≤ *e*) ∧ *g integrable_on i*)
proof −
interpret *comm_monoid conj True*
by *standard auto*
show *?thesis*
proof (*standard*, *safe*)
fix *a b* :: *'b*
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b*
if *box a b* = {} **for** *a b*
using *assms that*
by (*metis content_eq_0_interior integrable_on_null interior_cbox norm_zero right_minus_eq*)
{
fix *c g* **and** *k* :: *'b*
assume *fg*: ∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e* **and** *g*: *g integrable_on cbox a b*
assume *k*: *k* ∈ *Basis*
show ∃ *g*. (∀ *x*∈*cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}
∃ *g*. (∀ *x*∈*cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}
using *fg g k* **by** *auto*
}
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b*
if *fg1*: ∀ *x*∈*cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}. *norm* (*f x* − *g1 x*) ≤ *e*
and *g1*: *g1 integrable_on cbox a b* ∩ {*x*. *x* · *k* ≤ *c*}
and *fg2*: ∀ *x*∈*cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}. *norm* (*f x* − *g2 x*) ≤ *e*
and *g2*: *g2 integrable_on cbox a b* ∩ {*x*. *c* ≤ *x* · *k*}
and *k*: *k* ∈ *Basis*
for *c k g1 g2*
proof −
let *?g* = λ*x*. *if x*·*k* = *c then f x else if x*·*k* ≤ *c then g1 x else g2 x*
show ∃ *g*. (∀ *x*∈*cbox a b*. *norm* (*f x* − *g x*) ≤ *e*) ∧ *g integrable_on cbox a b*
proof (*intro exI conjI ballI*)

```
      show norm (f x − ?g x) ≤ e if x ∈ cbox a b for x
        by (auto simp: that assms fg1 fg2)
      show ?g integrable_on cbox a b
      proof −
        have ?g integrable_on cbox a b ∩ {x. x • k ≤ c} ?g integrable_on cbox a
b ∩ {x. x • k ≥ c}
          by(rule integrable_spike[OF _ negligible_standard_hyperplane[of k c]],
use k g1 g2 in auto)+
        with has_integral_split[OF _ _ k] show ?thesis
          unfolding integrable_on_def by blast
      qed
    qed
  qed
qed

lemma comm_monoid_set_F_and: comm_monoid_set.F (∧) True f s ⟷ (finite
s ⟶ (∀ x∈s. f x))
proof −
  interpret bool: comm_monoid_set ‹(∧)› True ..
  show ?thesis
    by (induction s rule: infinite_finite_induct) auto
qed

lemma approximable_on_division:
  fixes f :: 'b::euclidean_space ⇒ 'a::banach
  assumes 0 ≤ e
    and d: d division_of (cbox a b)
    and f: ∀ i∈d. ∃ g. (∀ x∈i. norm (f x − g x) ≤ e) ∧ g integrable_on i
  obtains g where ∀ x∈cbox a b. norm (f x − g x) ≤ e g integrable_on cbox a b
proof −
  interpret operative conj True λi. ∃ g. (∀ x∈i. norm (f x − g (x::'b)) ≤ e) ∧ g
integrable_on i
    using ‹0 ≤ e› by (rule operative_approximableI)
  from f local.division [OF d] that show thesis
    by auto
qed

lemma integrable_continuous:
  fixes f :: 'b::euclidean_space ⇒ 'a::banach
  assumes continuous_on (cbox a b) f
  shows f integrable_on cbox a b
proof (rule integrable_uniform_limit)
  fix e :: real
  assume e: e > 0
  then obtain d where 0 < d and d: ⋀x x'. ⟦x ∈ cbox a b; x' ∈ cbox a b; dist x'
x < d⟧ ⟹ dist (f x') (f x) < e
    using compact_uniformly_continuous[OF assms compact_cbox] unfolding
uniformly_continuous_on_def by metis
```

```
  obtain p where ptag: p tagged_division_of cbox a b and finep: (λx. ball x d) fine p
    using fine_division_exists[OF gauge_ball[OF ‹0 < d›], of a b] .
  have ∗: ∀ i∈snd ‘ p. ∃ g. (∀ x∈i. norm (f x − g x) ≤ e) ∧ g integrable_on i
  proof (safe, unfold snd_conv)
    fix x l
    assume as: (x, l) ∈ p
    obtain a b where l: l = cbox a b
      using as ptag by blast
    then have x: x ∈ cbox a b
      using as ptag by auto
    show ∃ g. (∀ x∈l. norm (f x − g x) ≤ e) ∧ g integrable_on l
    proof (intro exI conjI strip)
      show (λy. f x) integrable_on l
        unfolding integrable_on_def l by blast
    next
      fix y
      assume y: y ∈ l
      then have y ∈ ball x d
        using as finep by fastforce
      then show norm (f y − f x) ≤ e
        using d x y as l
        by (metis dist_commute dist_norm less_imp_le mem_ball ptag subsetCE
tagged_division_ofD(3))
    qed
  qed
  from e have e ≥ 0
    by auto
  from approximable_on_division[OF this division_of_tagged_division[OF ptag]
∗]
  show ∃ g. (∀ x∈cbox a b. norm (f x − g x) ≤ e) ∧ g integrable_on cbox a b
    by metis
qed

lemma integrable_continuous_interval:
  fixes f :: 'b::ordered_euclidean_space ⇒ 'a::banach
  assumes continuous_on {a..b} f
  shows f integrable_on {a..b}
  by (metis assms integrable_continuous interval_cbox)

lemmas integrable_continuous_real = integrable_continuous_interval[where 'b=real]

lemma integrable_continuous_closed_segment:
  fixes f :: real ⇒ 'a::banach
  assumes continuous_on (closed_segment a b) f
  shows f integrable_on (closed_segment a b)
  by (metis assms closed_segment_eq_real_ivl integrable_continuous_interval)
```

7.14.15 Specialization of additivity to one dimension

7.14.16 A useful lemma allowing us to factor out the content size

lemma *has_integral_factor_content*:
(*f has_integral i*) (*cbox a b*) ⟷
(∀ *e*>*0*. ∃ *d*. *gauge d* ∧ (∀ *p*. *p tagged_division_of* (*cbox a b*) ∧ *d fine p* ⟶
norm (*sum* (λ(*x*,*k*). *content k* ∗$_R$ *f x*) *p* − *i*) ≤ *e* ∗ *content* (*cbox a b*)))
proof (*cases content* (*cbox a b*) = *0*)
case *True*
have ⋀*e p*. *p tagged_division_of cbox a b* ⟹ *norm* ((∑ (*x*, *k*)∈*p*. *content k* ∗$_R$ *f x*)) ≤ *e* ∗ *content* (*cbox a b*)
unfolding *sum_content_null*[*OF True*] *True* **by** *force*
moreover have *i* = *0*
if ⋀*e*. *e* > *0* ⟹ ∃ *d*. *gauge d* ∧
(∀ *p*. *p tagged_division_of cbox a b* ∧
d fine p ⟶
norm ((∑ (*x*, *k*)∈*p*. *content k* ∗$_R$ *f x*) − *i*) ≤ *e* ∗ *content* (*cbox a b*))
using *that* [*of 1*]
by (*force simp add*: *True sum_content_null*[*OF True*] *intro*: *fine_division_exists*[*of* _ *a b*])
ultimately show *?thesis*
unfolding *has_integral_null_eq*[*OF True*]
by (*force simp add*:)
next
case *False*
then have *F*: *0* < *content* (*cbox a b*)
using *zero_less_measure_iff* **by** *blast*
let *?P* = λ*e opp*. ∃ *d*. *gauge d* ∧
(∀ *p*. *p tagged_division_of* (*cbox a b*) ∧ *d fine p* ⟶ *opp* (*norm* ((∑ (*x*, *k*)∈*p*. *content k* ∗$_R$ *f x*) − *i*)) *e*)
show *?thesis*
proof (*subst has_integral*, *safe*)
fix *e* :: *real*
assume *e*: *e* > *0*
show *?P* (*e* ∗ *content* (*cbox a b*)) (≤) **if** §[*rule_format*]: ∀ε>*0*. *?P* ε (<)
using § [*of e* ∗ *content* (*cbox a b*)]
by (*meson F e less_imp_le mult_pos_pos*)
show *?P e* (<) **if** §[*rule_format*]: ∀ε>*0*. *?P* (ε ∗ *content* (*cbox a b*)) (≤)
using § [*of e*/*2* / *content* (*cbox a b*)]
using *F e* **by** (*force simp add*: *algebra_simps*)
qed
qed

lemma *has_integral_factor_content_real*:
(*f has_integral i*) {*a*..*b*::*real*} ⟷
(∀ *e*>*0*. ∃ *d*. *gauge d* ∧ (∀ *p*. *p tagged_division_of* {*a*..*b*} ∧ *d fine p* ⟶
norm (*sum* (λ(*x*,*k*). *content k* ∗$_R$ *f x*) *p* − *i*) ≤ *e* ∗ *content* {*a*..*b*}))
unfolding *box_real*[*symmetric*]

by (*rule has_integral_factor_content*)

7.14.17 Fundamental theorem of calculus

lemma *interval_bounds_real*:
fixes $q\ b :: real$
assumes $a \le b$
shows $Sup\ \{a..b\} = b$
and $Inf\ \{a..b\} = a$
using *assms* **by** *auto*

theorem *fundamental_theorem_of_calculus*:
fixes $f :: real \Rightarrow 'a::banach$
assumes $a \le b$
and *vecd*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x\ within\ \{a..b\})$
shows $(f'\ has\_integral\ (f\ b - f\ a))\ \{a..b\}$
unfolding *has_integral_factor_content box_real*[*symmetric*]
proof *safe*
fix $e :: real$
assume $e > 0$
then have $\forall x.\ \exists d{>}0.\ x \in \{a..b\} \longrightarrow (\forall y{\in}\{a..b\}.\ norm\ (y{-}x) < d \longrightarrow norm\ (f\ y - f\ x - (y{-}x) *_R f'\ x) \le e * norm\ (y{-}x))$
using *vecd*[*unfolded has_vector_derivative_def has_derivative_within_alt*] **by** *blast*
then obtain d **where** d: $\bigwedge x.\ 0 < d\ x$
$\bigwedge x\ y.\ [\![x \in \{a..b\};\ y \in \{a..b\};\ norm\ (y{-}x) < d\ x]\!] \Longrightarrow norm\ (f\ y - f\ x - (y{-}x) *_R f'\ x) \le e * norm\ (y{-}x)$
by *metis*
show $\exists d.\ gauge\ d \wedge (\forall p.\ p\ tagged\_division\_of\ (cbox\ a\ b) \wedge d\ fine\ p \longrightarrow norm\ ((\sum (x,\ k){\in}p.\ content\ k *_R f'\ x) - (f\ b - f\ a)) \le e * content\ (cbox\ a\ b))$
proof (*rule exI*, *safe*)
show *gauge* $(\lambda x.\ ball\ x\ (d\ x))$
using *d*(*1*) *gauge_ball_dependent* **by** *blast*
next
fix p
assume *ptag*: $p\ tagged\_division\_of\ cbox\ a\ b$ **and** *finep*: $(\lambda x.\ ball\ x\ (d\ x))\ fine\ p$
have *ba*: $b - a = (\sum (x,K){\in}p.\ Sup\ K - Inf\ K)\ f\ b - f\ a = (\sum (x,K){\in}p.\ f(Sup\ K) - f(Inf\ K))$
using *additive_tagged_division_1*[**where** $f{=}\lambda x.\ x$] *additive_tagged_division_1*[**where** $f{=}f$]
$\langle a \le b\rangle$ *ptag* **by** *auto*
have $norm\ (\sum (x,\ K) \in p.\ (content\ K *_R f'\ x) - (f\ (Sup\ K) - f\ (Inf\ K))) \le (\sum n{\in}p.\ e * (case\ n\ of\ (x,\ k) \Rightarrow Sup\ k - Inf\ k))$
proof (*rule sum_norm_le*,*safe*)
fix $x\ K$
assume $(x,\ K) \in p$
then have $x \in K$ **and** *kab*: $K \subseteq cbox\ a\ b$

```
      using ptag by blast+
    then obtain u v where k: K = cbox u v and x ∈ K and kab: K ⊆ cbox a b
      using ptag ‹(x, K) ∈ p› by auto
    have u ≤ v
      using ‹x ∈ K› unfolding k by auto
    have ball: ∀ y∈K. y ∈ ball x (d x)
      using finep ‹(x, K) ∈ p› by blast
    have u ∈ K v ∈ K
      by (simp_all add: ‹u ≤ v› k)
    have norm ((v − u) *R f′ x − (f v − f u)) = norm (f u − f x − (u − x) *R
f′ x − (f v − f x − (v − x) *R f′ x))
      by (auto simp add: algebra_simps)
    also have ... ≤ norm (f u − f x − (u − x) *R f′ x) + norm (f v − f x − (v
− x) *R f′ x)
      by (rule norm_triangle_ineq4)
    finally have norm ((v − u) *R f′ x − (f v − f u)) ≤
      norm (f u − f x − (u − x) *R f′ x) + norm (f v − f x − (v − x) *R f′ x) .
    also have ... ≤ e * norm (u − x) + e * norm (v − x)
    proof (rule add_mono)
      show norm (f u − f x − (u − x) *R f′ x) ≤ e * norm (u − x)
      proof (rule d)
        show norm (u − x) < d x
          using ‹u ∈ K› ball by (auto simp add: dist_real_def)
      qed (use ‹x ∈ K› ‹u ∈ K› kab in auto)
      show norm (f v − f x − (v − x) *R f′ x) ≤ e * norm (v − x)
      proof (rule d)
        show norm (v − x) < d x
          using ‹v ∈ K› ball by (auto simp add: dist_real_def)
      qed (use ‹x ∈ K› ‹v ∈ K› kab in auto)
    qed
    also have ... ≤ e * (Sup K − Inf K)
        using ‹x ∈ K› by (auto simp: k interval_bounds_real[OF ‹u ≤ v›]
field_simps)
    finally show norm (content K *R f′ x − (f (Sup K) − f (Inf K))) ≤ e *
(Sup K − Inf K)
      using ‹u ≤ v› by (simp add: k)
  qed
  with ‹a ≤ b› show norm ((∑(x, K)∈p. content K *R f′ x) − (f b − f a)) ≤
e * content (cbox a b)
    by (auto simp: ba split_def sum_subtractf [symmetric] sum_distrib_left)
 qed
qed

lemma has_complex_derivative_imp_has_vector_derivative:
 fixes f :: complex ⇒ complex
  assumes (f has_field_derivative f′) (at (of_real a) within (cbox (of_real x)
(of_real y)))
 shows ((f o of_real) has_vector_derivative f′) (at a within {x..y})
 using has_derivative_in_compose[of of_real of_real a {x..y} f (*) f′] assms
```

by (*simp add*: *scaleR_conv_of_real ac_simps has_vector_derivative_def has_field_derivative_def o_def cbox_complex_of_real*)

lemma *ident_has_integral*:
fixes *a*::*real*
assumes $a \leq b$
shows $((\lambda x.\ x)$ *has_integral* $(b^2 - a^2)/2)$ $\{a..b\}$
proof −
have $((\lambda x.\ x)$ *has_integral inverse 2* $*$ $b^2 -$ *inverse 2* $*$ $a^2)$ $\{a..b\}$
unfolding *power2_eq_square*
by (*rule fundamental_theorem_of_calculus* [*OF assms*] *derivative_eq_intros* | *simp*)+
then show *?thesis*
by (*simp add*: *field_simps*)
qed

lemma *integral_ident* [*simp*]:
fixes *a*::*real*
assumes $a \leq b$
shows *integral* $\{a..b\}$ $(\lambda x.\ x) =$ (*if* $a \leq b$ *then* $(b^2 - a^2)/2$ *else* 0)
by (*metis assms ident_has_integral integral_unique*)

lemma *ident_integrable_on*:
fixes *a*::*real*
shows $(\lambda x.\ x)$ *integrable_on* $\{a..b\}$
using *continuous_on_id integrable_continuous_real* **by** *blast*

lemma *integral_sin* [*simp*]:
fixes *a*::*real*
assumes $a \leq b$ **shows** *integral* $\{a..b\}$ $sin = cos\ a - cos\ b$
proof −
have (*sin has_integral* $(-\ cos\ b - -\ cos\ a))$ $\{a..b\}$
proof (*rule fundamental_theorem_of_calculus*)
show $((\lambda a.\ -\ cos\ a)$ *has_vector_derivative sin x*) (*at x within* $\{a..b\}$) **for** *x*
unfolding *has_real_derivative_iff_has_vector_derivative* [*symmetric*]
by (*rule derivative_eq_intros* | *force*)+
qed (*use assms* **in** *auto*)
then show *?thesis*
by (*simp add*: *integral_unique*)
qed

lemma *integral_cos* [*simp*]:
fixes *a*::*real*
assumes $a \leq b$ **shows** *integral* $\{a..b\}$ $cos = sin\ b - sin\ a$
proof −
have (*cos has_integral* $(sin\ b - sin\ a))$ $\{a..b\}$
proof (*rule fundamental_theorem_of_calculus*)
show (*sin has_vector_derivative cos x*) (*at x within* $\{a..b\}$) **for** *x*
unfolding *has_real_derivative_iff_has_vector_derivative* [*symmetric*]

```
    by (rule derivative_eq_intros | force)+
  qed (use assms in auto)
  then show ?thesis
    by (simp add: integral_unique)
qed

lemma integral_exp [simp]:
  fixes a::real
  assumes a ≤ b shows integral {a..b} exp = exp b − exp a
 by (meson DERIV_exp assms fundamental_theorem_of_calculus has_real_derivative_iff_has_vector_derivative
      has_vector_derivative_at_within integral_unique)


lemma has_integral_sin_nx: ((λx. sin(real_of_int n ∗ x)) has_integral 0) {−pi..pi}
proof (cases n = 0)
  case False
  have ((λx. sin (n ∗ x)) has_integral (− cos (n ∗ pi)/n − − cos (n ∗ − pi)/n))
{−pi..pi}
  proof (rule fundamental_theorem_of_calculus)
    show ((λx. − cos (n ∗ x) / n) has_vector_derivative sin (n ∗ a)) (at a within
{−pi..pi})
      if a ∈ {−pi..pi} for a :: real
      using that False
      unfolding has_vector_derivative_def
      by (intro derivative_eq_intros | force)+
  qed auto
  then show ?thesis
    by simp
qed auto

lemma integral_sin_nx:
   integral {−pi..pi} (λx. sin(x ∗ real_of_int n)) = 0
  using has_integral_sin_nx [of n] by (force simp: mult.commute)

lemma has_integral_cos_nx:
 ((λx. cos(real_of_int n ∗ x)) has_integral (if n = 0 then 2 ∗ pi else 0)) {−pi..pi}
proof (cases n = 0)
  case True
  then show ?thesis
    using has_integral_const_real [of 1::real −pi pi] by auto
next
  case False
   have ((λx. cos (n ∗ x)) has_integral (sin (n ∗ pi)/n − sin (n ∗ − pi)/n))
{−pi..pi}
  proof (rule fundamental_theorem_of_calculus)
     show ((λx. sin (n ∗ x) / n) has_vector_derivative cos (n ∗ x)) (at x within
{−pi..pi})
      if x ∈ {−pi..pi}
      for x :: real
```

using *that False*
unfolding *has_vector_derivative_def*
by (*intro derivative_eq_intros* | *force*)+
qed *auto*
with *False* **show** *?thesis*
by (*simp add*: *mult.commute*)
qed

lemma *integral_cos_nx*:
integral {−*pi*..*pi*} (λ*x*. *cos*(*x* ∗ *real_of_int n*)) = (*if n* = *0 then 2* ∗ *pi else 0*)
using *has_integral_cos_nx* [*of n*] **by** (*force simp*: *mult.commute*)

7.14.18 Taylor series expansion

lemma *mvt_integral*:
fixes *f*::′*a*::*real_normed_vector*⇒′*b*::*banach*
assumes *f*′[*derivative_intros*]:
⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f*′ *x*) (*at x within S*)
assumes *line_in*: ⋀*t*. *t* ∈ {*0*..*1*} ⟹ *x* + *t* ∗*R y* ∈ *S*
shows *f* (*x* + *y*) − *f x* = *integral* {*0*..*1*} (λ*t*. *f*′ (*x* + *t* ∗*R y*) *y*)
proof −
from *assms* **have** *subset*: (λ*xa*. *x* + *xa* ∗*R y*) ‘ {*0*..*1*} ⊆ *S* **by** *auto*
note [*derivative_intros*] =
has_derivative_subset[*OF* _ *subset*]
has_derivative_in_compose[**where** *f*=(λ*xa*. *x* + *xa* ∗*R y*) **and** *g* = *f*]
note [*continuous_intros*] =
continuous_on_compose2[**where** *f*=(λ*xa*. *x* + *xa* ∗*R y*)]
continuous_on_subset[*OF* _ *subset*]
have ⋀*t*. *t* ∈ {*0*..*1*} ⟹
((λ*t*. *f* (*x* + *t* ∗*R y*)) *has_vector_derivative f*′ (*x* + *t* ∗*R y*) *y*)
(*at t within* {*0*..*1*})
using *assms*
by (*auto simp*: *has_vector_derivative_def*
linear_cmul[*OF has_derivative_linear*[*OF f*′], *symmetric*]
intro!: *derivative_eq_intros*)
from *fundamental_theorem_of_calculus*[*rule_format*, *OF* _ *this*]
show *?thesis*
by (*auto intro*!: *integral_unique*[*symmetric*])
qed

lemma (**in** *bounded_bilinear*) *sum_prod_derivatives_has_vector_derivative*:
assumes *p*>*0*
and *f0*: *Df 0* = *f*
and *Df*: ⋀*m t*. *m* < *p* ⟹ *a* ≤ *t* ⟹ *t* ≤ *b* ⟹
(*Df m has_vector_derivative Df* (*Suc m*) *t*) (*at t within* {*a*..*b*})
and *g0*: *Dg 0* = *g*
and *Dg*: ⋀*m t*. *m* < *p* ⟹ *a* ≤ *t* ⟹ *t* ≤ *b* ⟹
(*Dg m has_vector_derivative Dg* (*Suc m*) *t*) (*at t within* {*a*..*b*})
and *ivl*: *a* ≤ *t t* ≤ *b*

shows $((\lambda t.\ \sum i{<}p.\ (-1)\hat{}i *_R prod\ (Df\ i\ t)\ (Dg\ (p - Suc\ i)\ t))$
has_vector_derivative
$prod\ (f\ t)\ (Dg\ p\ t) - (-1)\hat{}p *_R prod\ (Df\ p\ t)\ (g\ t))$
$(at\ t\ within\ \{a..b\})$
using *assms*
proof *cases*
assume p: $p \neq 1$
define p' **where** $p' = p - 2$
from *assms* p **have** p': $\{..{<}p\} = \{..Suc\ p'\}$ $p = Suc\ (Suc\ p')$
by (*auto simp*: p'_*def*)
have $*$: $\bigwedge i.\ i \leq p' \Longrightarrow Suc\ (Suc\ p' - i) = (Suc\ (Suc\ p') - i)$
by *auto*
let *?f* $= \lambda i.\ (-1)\ \hat{}\ i *_R (prod\ (Df\ i\ t)\ (Dg\ ((p - i))\ t))$
have $(\sum i{<}p.\ (-1)\ \hat{}\ i *_R (prod\ (Df\ i\ t)\ (Dg\ (Suc\ (p - Suc\ i))\ t) +$
$prod\ (Df\ (Suc\ i)\ t)\ (Dg\ (p - Suc\ i)\ t))) =$
$(\sum i{\leq}(Suc\ p').\ ?f\ i - ?f\ (Suc\ i))$
by (*auto simp*: *algebra_simps* $p'(2)$ *numeral_2_eq_2* $*$ *lessThan_Suc_atMost*)
also note *sum_telescope*
finally
have $(\sum i{<}p.\ (-1)\ \hat{}\ i *_R (prod\ (Df\ i\ t)\ (Dg\ (Suc\ (p - Suc\ i))\ t) +$
$prod\ (Df\ (Suc\ i)\ t)\ (Dg\ (p - Suc\ i)\ t)))$
$= prod\ (f\ t)\ (Dg\ p\ t) - (-\ 1)\ \hat{}\ p *_R prod\ (Df\ p\ t)\ (g\ t)$
unfolding $p'[symmetric]$
by (*simp add*: *assms*)
thus *?thesis*
using *assms*
by (*auto intro*!: *derivative_eq_intros has_vector_derivative*)
qed (*auto intro*!: *derivative_eq_intros has_vector_derivative*)

lemma
fixes *f*::*real*⇒'*a*::*banach*
assumes *p*>*0*
and *f0*: *Df 0* $= f$
and *Df*: $\bigwedge m\ t.\ m < p \Longrightarrow a \leq t \Longrightarrow t \leq b \Longrightarrow$
$(Df\ m\ has\_vector\_derivative\ Df\ (Suc\ m)\ t)\ (at\ t\ within\ \{a..b\})$
and *ivl*: $a \leq b$
defines $i \equiv \lambda x.\ ((b - x)\ \hat{}\ (p - 1)\ /\ fact\ (p - 1)) *_R Df\ p\ x$
shows *Taylor_has_integral*:
$(i\ has\_integral\ f\ b - (\sum i{<}p.\ ((b{-}a)\ \hat{}\ i\ /\ fact\ i) *_R Df\ i\ a))\ \{a..b\}$
and *Taylor_integral*:
$f\ b = (\sum i{<}p.\ ((b{-}a)\ \hat{}\ i\ /\ fact\ i) *_R Df\ i\ a) + integral\ \{a..b\}\ i$
and *Taylor_integrable*:
$i\ integrable\_on\ \{a..b\}$
proof *goal_cases*
case *1*
interpret *bounded_bilinear scaleR*::*real*⇒'*a*⇒'*a*
by (*rule bounded_bilinear_scaleR*)
define g **where** $g\ s = (b - s)\hat{}(p - 1)/fact\ (p - 1)$ **for** s
define *Dg* **where** [*abs_def*]:

```
    Dg n s = (if n < p then (−1)^n * (b − s)^(p − 1 − n) / fact (p − 1 − n)
else 0) for n s
  have g0: Dg 0 = g
    using ‹p > 0›
    by (auto simp add: Dg_def field_split_simps g_def split: if_split_asm)
  have fact_eq: real (p − Suc m) * fact (p − Suc (Suc m)) = fact (p − Suc m)
    if p > Suc m for m
    by (metis Suc_diff_Suc fact_Suc that)
  have Dg: ⋀m t. m < p ⟹ a ≤ t ⟹ t ≤ b ⟹
    (Dg m has_vector_derivative Dg (Suc m) t) (at t within {a..b})
    unfolding Dg_def has_vector_derivative_def
     by (auto intro!: derivative_eq_intros simp: fact_eq divide_simps simp del:
of_nat_diff)
  let ?sum = λt. ∑ i<p. (− 1) ^ i *R Dg i t *R Df (p − Suc i) t
  from sum_prod_derivatives_has_vector_derivative[of _ Dg _ _ _ Df,
      OF ‹p > 0› g0 Dg f0 Df]
  have deriv: ⋀t. a ≤ t ⟹ t ≤ b ⟹
    (?sum has_vector_derivative
      g t *R Df p t − (− 1) ^ p *R Dg p t *R f t) (at t within {a..b})
    by auto
  from fundamental_theorem_of_calculus[rule_format, OF ‹a ≤ b› deriv]
  have (i has_integral ?sum b − ?sum a) {a..b}
    using atLeastatMost_empty'[simp del]
    by (simp add: i_def g_def Dg_def)
  also
  have one: (− 1) ^ p' * (− 1) ^ p' = (1::real) {..<p} ∩ {i. p = Suc i} = {p −
1} for p'
    using ‹p > 0› by (auto simp: power_mult_distrib)
  then have ?sum b = f b
    using Suc_pred'[OF ‹p > 0›]
    by (simp add: diff_eq_eq Dg_def power_0_left le_Suc_eq if_distrib
       if_distribR sum.If_cases f0)
  also
  have ?sum a = (∑ i<p. ((b−a) ^ i / fact i) *R Df i a)
  proof (rule sum.reindex_cong)
    have ⋀i. i < p ⟹ ∃j<p. i = p − Suc j
    by (metis Suc_diff_Suc ‹p>0› diff_Suc_less diff_diff_cancel less_or_eq_imp_le)
    then show {..<p} = (λx. p − x − 1) ‘ {..<p}
      by force
  qed (auto simp add: inj_on_def Dg_def one)
  finally show c: ?case .
  case 2 show ?case using c integral_unique
    by (metis (lifting) add.commute diff_eq_eq integral_unique)
  case 3 show ?case using c by force
qed
```

7.14.19 Only need trivial subintervals if the interval itself is trivial

proposition *division_of_nontrivial*:
fixes $\mathcal{D}$ :: *'a*::*euclidean_space set set*
assumes *sdiv*: $\mathcal{D}$ *division_of* (*cbox a b*)
and *cont0*: *content* (*cbox a b*) $\neq$ *0*
shows $\{k.\ k \in \mathcal{D} \wedge$ *content* $k \neq 0\}$ *division_of* (*cbox a b*)
using *sdiv*
proof (*induction card* $\mathcal{D}$ *arbitrary*: $\mathcal{D}$ *rule*: *less_induct*)
case *less*
note $\mathcal{D}$ = *division_ofD*[*OF less.prems*]
{
presume $*$: $\{k \in \mathcal{D}.$ *content* $k \neq 0\} \neq \mathcal{D} \Longrightarrow$ *?case*
then show *?case*
using *less.prems* **by** *fastforce*
}
assume *noteq*: $\{k \in \mathcal{D}.$ *content* $k \neq 0\} \neq \mathcal{D}$
then obtain *K c d* **where** $K \in \mathcal{D}$ **and** *contk*: *content K* = *0* **and** *keq*: *K* = *cbox c d*
using $\mathcal{D}(4)$ **by** *blast*
then have *card* $\mathcal{D}$ > *0*
unfolding *card_gt_0_iff* **using** *less* **by** *auto*
then have *card*: *card* ($\mathcal{D} - \{K\}$) < *card* $\mathcal{D}$
using *less* ‹$K \in \mathcal{D}$› **by** (*simp add*: $\mathcal{D}(1)$)
have *closed*: *closed* ($\bigcup(\mathcal{D} - \{K\})$)
using *less.prems* **by** *auto*
have *x islimpt* $\bigcup(\mathcal{D} - \{K\})$ **if** $x \in K$ **for** *x*
unfolding *islimpt_approachable*
proof (*intro allI impI*)
fix *e*::*real*
assume *e* > *0*
obtain *i* **where** *i*: *c*·*i* = *d*·*i* *i*∈*Basis*
using *contk* $\mathcal{D}(3)$ [*OF* ‹$K \in \mathcal{D}$›] **unfolding** *box_ne_empty keq*
by (*meson content_eq_0 dual_order.antisym*)
then have *xi*: *x*·*i* = *d*·*i*
using ‹$x \in K$› **unfolding** *keq mem_box* **by** (*metis antisym*)
define *y* **where** *y* = ($\sum j$∈*Basis*. (*if j* = *i then if c*·*i* ≤ (*a*·*i* + *b*·*i*)/*2 then c*·*i* +
min e (*b*·*i* − *c*·*i*)/*2 else c*·*i* − *min e* (*c*·*i* − *a*·*i*)/*2 else x*·*j*) $*_R$ *j*)
show $\exists x'$∈$\bigcup(\mathcal{D} - \{K\})$. $x' \neq x \wedge$ *dist x' x* < *e*
proof (*intro bexI conjI*)
have *d* ∈ *cbox c d*
using $\mathcal{D}(3)$[*OF* ‹$K \in \mathcal{D}$›] **by** (*simp add*: *box_ne_empty*(*1*) *keq mem_box*(*2*))
then have *d* ∈ *cbox a b*
using $\mathcal{D}(2)$[*OF* ‹$K \in \mathcal{D}$›] **by** (*auto simp*: *keq*)
then have *di*: $a \cdot i \leq d \cdot i \wedge d \cdot i \leq b \cdot i$
using ‹*i* ∈ *Basis*› *mem_box*(*2*) **by** *blast*
then have *xyi*: *y*·*i* ≠ *x*·*i*
unfolding *y_def i xi* **using** ‹*e* > *0*› *cont0* ‹*i* ∈ *Basis*›

```
      by (auto simp: content_eq_0 elim!: ballE[of _ _ i])
    then show y ≠ x
      unfolding euclidean_eq_iff[where 'a='a] using i by auto
    have norm (y−x) ≤ (∑ b∈Basis. |(y − x) · b|)
      by (rule norm_le_l1)
    also have ... = |(y − x) · i| + (∑ b ∈ Basis − {i}. |(y − x) · b|)
      by (meson finite_Basis i(2) sum.remove)
    also have ... <  e + sum (λi. 0) Basis
    proof (rule add_less_le_mono)
      show |(y−x) · i| < e
        using di ‹e > 0› y_def i xi by (auto simp: inner_simps)
      show (∑ i∈Basis − {i}. |(y−x) · i|) ≤ (∑ i∈Basis. 0)
        unfolding y_def by (auto simp: inner_simps)
    qed
    finally have norm (y−x) < e + sum (λi. 0) Basis .
    then show dist y x < e
      unfolding dist_norm by auto
    have y ∉ K
      unfolding keq mem_box using i(1) i(2) xi xyi by fastforce
    moreover have y ∈ ⋃ 𝒟
      using subsetD[OF 𝒟(2)[OF ‹K ∈ 𝒟›] ‹x ∈ K›] ‹e > 0› di i
      by (auto simp: 𝒟 mem_box y_def field_simps elim!: ballE[of _ _ i])
    ultimately show y ∈ ⋃ (𝒟 − {K}) by auto
  qed
 qed
 then have K ⊆ ⋃ (𝒟 − {K})
  using closed closed_limpt by blast
 then have ⋃ (𝒟 − {K}) = cbox a b
  unfolding 𝒟(6)[symmetric] by auto
 then have 𝒟 − {K} division_of cbox a b
  by (metis Diff_subset less.prems division_of_subset 𝒟(6))
 then have {ka ∈ 𝒟 − {K}. content ka ≠ 0} division_of (cbox a b)
  using card less.hyps by blast
 moreover have {ka ∈ 𝒟 − {K}. content ka ≠ 0} = {K ∈ 𝒟. content K ≠ 0}
  using contk by auto
 ultimately show ?case by auto
qed
```

7.14.20 Integrability on subintervals

```
lemma operative_integrableI:
 fixes f :: 'b::euclidean_space ⇒ 'a::banach
 assumes 0 ≤ e
 shows operative conj True (λi. f integrable_on i)
proof −
 interpret comm_monoid conj True
 proof qed
 show ?thesis
 proof
```

```
    show ⋀a b. box a b = {} ⟹ (f integrable_on cbox a b) = True
      by (simp add: content_eq_0_interior integrable_on_null)
    show ⋀a b c k.
           k ∈ Basis ⟹
           (f integrable_on cbox a b) ⟷
           (f integrable_on cbox a b ∩ {x. x · k ≤ c} ∧ f integrable_on cbox a b ∩
{x. c ≤ x · k})
      unfolding integrable_on_def by (auto intro!: has_integral_split)
  qed
qed

lemma integrable_subinterval:
  fixes f :: 'b::euclidean_space ⇒ 'a::banach
  assumes f: f integrable_on cbox a b
    and cd: cbox c d ⊆ cbox a b
  shows f integrable_on cbox c d
proof −
  interpret operative conj True λi. f integrable_on i
    using order_refl by (rule operative_integrableI)
  show ?thesis
    by (metis cd division division_of_finite empty f partial_division_extend_1
remove)
qed

lemma integrable_subinterval_real:
  fixes f :: real ⇒ 'a::banach
  assumes f integrable_on {a..b}
    and {c..d} ⊆ {a..b}
  shows f integrable_on {c..d}
  by (metis assms box_real(2) integrable_subinterval)
```

7.14.21 Combining adjacent intervals in 1 dimension

```
lemma has_integral_combine:
  fixes a b c :: real and j :: 'a::banach
  assumes a ≤ c
    and c ≤ b
    and ac: (f has_integral i) {a..c}
    and cb: (f has_integral j) {c..b}
  shows (f has_integral (i + j)) {a..b}
proof −
  interpret operative_real lift_option plus Some 0
    λi. if f integrable_on i then Some (integral i f) else None
    using operative_integralI by (rule operative_realI)
  from ‹a ≤ c› ‹c ≤ b› ac cb coalesce_less_eq
  have ∗: lift_option (+)
           (if f integrable_on {a..c} then Some (integral {a..c} f) else None)
           (if f integrable_on {c..b} then Some (integral {c..b} f) else None) =
          (if f integrable_on {a..b} then Some (integral {a..b} f) else None)
```

```
    by (auto simp: split: if_split_asm)
  then have f integrable_on cbox a b
    using ac cb by (auto split: if_split_asm)
  with * show ?thesis
     using ac cb by (auto simp add: integrable_on_def integral_unique split:
if_split_asm)
qed

lemma integral_combine:
  fixes f :: real ⇒ 'a::banach
  assumes a ≤ c
    and c ≤ b
    and ab: f integrable_on {a..b}
  shows integral {a..c} f + integral {c..b} f = integral {a..b} f
proof −
  have (f has_integral integral {a..c} f) {a..c}
    using ab ‹c ≤ b› integrable_subinterval_real by fastforce
  moreover
  have (f has_integral integral {c..b} f) {c..b}
    using ab ‹a ≤ c› integrable_subinterval_real by fastforce
  ultimately show ?thesis
    by (smt (verit, best) assms has_integral_combine integral_unique)
qed

lemma integrable_combine:
  fixes f :: real ⇒ 'a::banach
  assumes a ≤ c
    and c ≤ b
    and f integrable_on {a..c}
    and f integrable_on {c..b}
  shows f integrable_on {a..b}
  using assms has_integral_combine by blast

lemma integral_minus_sets:
  fixes f::real ⇒ 'a::banach
  shows c ≤ a ⟹ c ≤ b ⟹ f integrable_on {c .. max a b} ⟹
    integral {c .. a} f − integral {c .. b} f =
    (if a ≤ b then − integral {a .. b} f else integral {b .. a} f)
  using integral_combine[of c a b f]  integral_combine[of c b a f]
  by (auto simp: algebra_simps max_def)

lemma integral_minus_sets':
  fixes f::real ⇒ 'a::banach
  shows c ≥ a ⟹ c ≥ b ⟹ f integrable_on {min a b .. c} ⟹
    integral {a .. c} f − integral {b .. c} f =
    (if a ≤ b then integral {a .. b} f else − integral {b .. a} f)
  using integral_combine[of b a c f] integral_combine[of a b c f]
  by (auto simp: algebra_simps min_def)
```

7.14.22 Reduce integrability to "local" integrability

lemma *integrable_on_little_subintervals*:
fixes *f* :: *'b::euclidean_space* ⇒ *'a::banach*
assumes ∀ *x*∈*cbox a b*. ∃ *d*>*0*. ∀ *u v*. *x* ∈ *cbox u v* ∧ *cbox u v* ⊆ *ball x d* ∧ *cbox u v* ⊆ *cbox a b* ⟶
f integrable_on cbox u v
shows *f integrable_on cbox a b*
proof −
interpret *operative conj True* λ*i*. *f integrable_on i*
using *order_refl* **by** (*rule operative_integrableI*)
have ∀ *x*. ∃ *d*>*0*. *x*∈*cbox a b* ⟶ (∀ *u v*. *x* ∈ *cbox u v* ∧ *cbox u v* ⊆ *ball x d* ∧ *cbox u v* ⊆ *cbox a b* ⟶
f integrable_on cbox u v)
using *assms* **by** (*metis zero_less_one*)
then obtain *d* **where** *d*: ⋀*x*. *0* < *d x*
⋀*x u v*. ⟦*x* ∈ *cbox a b*; *x* ∈ *cbox u v*; *cbox u v* ⊆ *ball x* (*d x*); *cbox u v* ⊆ *cbox a b*⟧
⟹ *f integrable_on cbox u v*
by *metis*
obtain *p* **where** *p*: *p tagged_division_of cbox a b* (λ*x*. *ball x* (*d x*)) *fine p*
using *fine_division_exists*[*OF gauge_ball_dependent*,*of d a b*] *d*(*1*) **by** *blast*
then have *sndp*: *snd* ‘ *p division_of cbox a b*
by (*metis division_of_tagged_division*)
have *f integrable_on k* **if** (*x*, *k*) ∈ *p* **for** *x k*
using *tagged_division_ofD*(*2*−*4*)[*OF p*(*1*) *that*] *fineD*[*OF p*(*2*) *that*] *d*[*of x*]
by *auto*
then show *?thesis*
unfolding *division* [*symmetric*, *OF sndp*] *comm_monoid_set_F_and*
by *auto*
qed

7.14.23 Second FTC or existence of antiderivative

lemma *integrable_const*[*intro*]: (λ*x*. *c*) *integrable_on cbox a b*
unfolding *integrable_on_def* **by** *blast*

lemma *integral_has_vector_derivative_continuous_at*:
fixes *f* :: *real* ⇒ *'a::banach*
assumes *f*: *f integrable_on* {*a*..*b*}
and *x*: *x* ∈ {*a*..*b*} − *S*
and *finite S*
and *fx*: *continuous* (*at x within* ({*a*..*b*} − *S*)) *f*
shows ((λ*u*. *integral* {*a*..*u*} *f*) *has_vector_derivative f x*) (*at x within* ({*a*..*b*} − *S*))
proof −
let *?I* = λ*a b*. *integral* {*a*..*b*} *f*
{ fix *e*::*real*
assume *e* > *0*
obtain *d* **where** *d*>*0* **and** *d*: ⋀*x'*. ⟦*x'* ∈ {*a*..*b*} − *S*; |*x'* − *x*| < *d*⟧ ⟹ *norm*(*f*

$x' - f\ x) \leq e$
using ‹*e>0*› *fx* **by** (*auto simp*: *continuous_within_eps_delta dist_norm less_imp_le*)
have *norm* (*integral* {*a..y*} *f* − *integral* {*a..x*} *f* − (*y*−*x*) ∗$_R$ *f x*) ≤ *e* ∗ |*y* − *x*| (**is** *?lhs* ≤ *?rhs*)
if *y*: *y* ∈ {*a..b*} − *S* **and** *yx*: |*y* − *x*| < *d* **for** *y*
proof (*cases y* < *x*)
case *False*
have *f integrable_on* {*a..y*}
using *f y* **by** (*simp add*: *integrable_subinterval_real*)
then have *Idiff*: *?I a y* − *?I a x* = *?I x y*
using *False x* **by** (*simp add*: *algebra_simps integral_combine*)
have *fux_int*: ((λ*u*. *f u* − *f x*) *has_integral integral* {*x..y*} *f* − (*y*−*x*) ∗$_R$ *f x*) {*x..y*}
proof (*rule has_integral_diff*)
show (*f has_integral integral* {*x..y*} *f*) {*x..y*}
using *x y* **by** (*auto intro*: *integrable_integral* [*OF integrable_subinterval_real* [*OF f*]])
show ((λ*u*. *f x*) *has_integral* (*y* − *x*) ∗$_R$ *f x*) {*x..y*}
using *has_integral_const_real* [*of f x x y*] *False* **by** *simp*
qed
have *?lhs* ≤ *e* ∗ *content* {*x..y*}
using *yx False d x y* ‹*e>0*› *assms*
by (*intro has_integral_bound_real*[**where** *f*=(λ*u*. *f u* − *f x*)]) (*auto simp*: *Idiff fux_int*)
also have ... ≤ *?rhs*
using *False* **by** *auto*
finally show *?thesis* **.**
next
case *True*
have *f integrable_on* {*a..x*}
using *f x* **by** (*simp add*: *integrable_subinterval_real*)
then have *Idiff*: *?I a x* − *?I a y* = *?I y x*
using *True x y* **by** (*simp add*: *algebra_simps integral_combine*)
have *fux_int*: ((λ*u*. *f u* − *f x*) *has_integral integral* {*y..x*} *f* − (*x* − *y*) ∗$_R$ *f x*) {*y..x*}
proof (*rule has_integral_diff*)
show (*f has_integral integral* {*y..x*} *f*) {*y..x*}
using *x y* **by** (*auto intro*: *integrable_integral* [*OF integrable_subinterval_real* [*OF f*]])
show ((λ*u*. *f x*) *has_integral* (*x* − *y*) ∗$_R$ *f x*) {*y..x*}
using *has_integral_const_real* [*of f x y x*] *True* **by** *simp*
qed
have *norm* (*integral* {*a..x*} *f* − *integral* {*a..y*} *f* − (*x* − *y*) ∗$_R$ *f x*) ≤ *e* ∗ *content* {*y..x*}
using *yx True d x y* ‹*e>0*› *assms*
by (*intro has_integral_bound_real*[**where** *f*=(λ*u*. *f u* − *f x*)]) (*auto simp*: *Idiff fux_int*)
also have ... ≤ *e* ∗ |*y* − *x*|

using *True* **by** *auto*
finally have *norm* (*integral* {*a*..*x*} *f* − *integral* {*a*..*y*} *f* − (*x* − *y*) ∗$_R$ *f x*) ≤ *e* ∗ |*y* − *x*| **.**
then show *?thesis*
by (*simp add*: *algebra_simps norm_minus_commute*)
qed
then have ∃ *d*>*0*. ∀ *y*∈{*a*..*b*} − *S*. |*y* − *x*| < *d* ⟶ *norm* (*integral* {*a*..*y*} *f* − *integral* {*a*..*x*} *f* − (*y*−*x*) ∗$_R$ *f x*) ≤ *e* ∗ |*y* − *x*|
using ‹*d*>*0*› **by** *blast*
}
then show *?thesis*
by (*simp add*: *has_vector_derivative_def has_derivative_within_alt bounded_linear_scaleR_left*)
qed

lemma *integral_has_vector_derivative*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *continuous_on* {*a*..*b*} *f*
and *x* ∈ {*a*..*b*}
shows ((λ*u*. *integral* {*a*..*u*} *f*) *has_vector_derivative f*(*x*)) (*at x within* {*a*..*b*})
using *assms integral_has_vector_derivative_continuous_at* [*OF integrable_continuous_real*]
by (*fastforce simp*: *continuous_on_eq_continuous_within*)

lemma *integral_has_real_derivative*:
assumes *continuous_on* {*a*..*b*} *g*
assumes *t* ∈ {*a*..*b*}
shows ((λ*x*. *integral* {*a*..*x*} *g*) *has_real_derivative g t*) (*at t within* {*a*..*b*})
using *integral_has_vector_derivative*[*of a b g t*] *assms*
by (*auto simp*: *has_real_derivative_iff_has_vector_derivative*)

lemma *antiderivative_continuous*:
fixes *q b* :: *real*
assumes *continuous_on* {*a*..*b*} *f*
obtains *g* **where** ⋀*x*. *x* ∈ {*a*..*b*} ⟹ (*g has_vector_derivative* (*f x*::_::*banach*)) (*at x within* {*a*..*b*})
using *integral_has_vector_derivative*[*OF assms*] **by** *auto*

7.14.24 Combined fundamental theorem of calculus

lemma *antiderivative_integral_continuous*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *continuous_on* {*a*..*b*} *f*
obtains *g* **where** ∀ *u*∈{*a*..*b*}. ∀ *v* ∈ {*a*..*b*}. *u* ≤ *v* ⟶ (*f has_integral* (*g v* − *g u*)) {*u*..*v*}
proof −
obtain *g*
where *g*: ⋀*x*. *x* ∈ {*a*..*b*} ⟹ (*g has_vector_derivative f x*) (*at x within* {*a*..*b*})
using *antiderivative_continuous*[*OF assms*] **by** *metis*

```
  have (f has_integral g v − g u) {u..v} if u ∈ {a..b} v ∈ {a..b} u ≤ v for u v
  proof −
    have ⋀x. x ∈ cbox u v ⟹ (g has_vector_derivative f x) (at x within cbox u v)
      by (metis atLeastAtMost_iff atLeastatMost_subset_iff box_real(2) g
          has_vector_derivative_within_subset subsetCE that(1,2))
    then show ?thesis
      by (metis box_real(2) that(3) fundamental_theorem_of_calculus)
  qed
  then show ?thesis
    using that by blast
qed
```

7.14.25 General "twiddling" for interval-to-interval function image

```
lemma has_integral_twiddle:
  assumes 0 < r
    and hg: ⋀x. h(g x) = x
    and gh: ⋀x. g(h x) = x
    and contg: ⋀x. continuous (at x) g
    and g: ⋀u v. ∃w z. g ‘ cbox u v = cbox w z
    and h: ⋀u v. ∃w z. h ‘ cbox u v = cbox w z
    and r: ⋀u v. content(g ‘ cbox u v) = r * content (cbox u v)
    and intfi: (f has_integral i) (cbox a b)
  shows ((λx. f(g x)) has_integral (1 / r) *R i) (h ‘ cbox a b)
proof (cases cbox a b = {})
  case True
  then show ?thesis
    using intfi by auto
next
  case False
  obtain w z where wz: h ‘ cbox a b = cbox w z
    using h by blast
  have inj: inj g inj h
    using hg gh injI by metis+
  from h obtain ha hb where h_eq: h ‘ cbox a b = cbox ha hb by blast
  have ∃d. gauge d ∧ (∀p. p tagged_division_of h ‘ cbox a b ∧ d fine p
            ⟶ norm ((∑(x, k)∈p. content k *R f (g x)) − (1 / r) *R i) < e)
    if e > 0 for e
  proof −
    have e * r > 0 using that ‹0 < r› by simp
    with intfi[unfolded has_integral]
    obtain d where gauge d
              and d: ⋀p. p tagged_division_of cbox a b ∧ d fine p
                     ⟹ norm ((∑(x, k)∈p. content k *R f x) − i) < e * r
      by metis
    define d′ where d′ x = g −‘ d (g x) for x
    show ?thesis
    proof (rule_tac x=d′ in exI, safe)
```

```
    show gauge d'
      using ‹gauge d› continuous_open_vimage[OF _ contg] by (auto simp:
gauge_def d'_def)
  next
    fix p
    assume ptag: p tagged_division_of h ‘ cbox a b and finep: d' fine p
    note p = tagged_division_ofD[OF ptag]
    have gab: g y ∈ cbox a b if y ∈ K (x, K) ∈ p for x y K
      by (metis hg inj(2) inj_image_mem_iff p(3) subsetCE that that)
    have gimp: (λ(x,K). (g x, g ‘ K)) ‘ p tagged_division_of (cbox a b) ∧
          d fine (λ(x, k). (g x, g ‘ k)) ‘ p
      unfolding tagged_division_of
    proof safe
      show finite ((λ(x, k). (g x, g ‘ k)) ‘ p)
        using ptag by auto
      show d fine (λ(x, k). (g x, g ‘ k)) ‘ p
        using finep unfolding fine_def d'_def by auto
    next
      fix x K
      assume xk: (x, K) ∈ p
      show g x ∈ g ‘ K
        using p(2)[OF xk] by auto
      show ∃ u v. g ‘ K = cbox u v
        using p(4)[OF xk] using assms(5−6) by auto
      fix x' K' u
      assume xk': (x', K') ∈ p and u: u ∈ interior (g ‘ K) u ∈ interior (g ‘ K')
      have interior K ∩ interior K' ≠ {}
      proof
        assume interior K ∩ interior K' = {}
        moreover have u ∈ g ‘ (interior K ∩ interior K')
          using interior_image_subset[OF ‹inj g› contg] u
          unfolding image_Int[OF inj(1)] by blast
        ultimately show False by blast
      qed
      then have same: (x, K) = (x', K')
        using ptag xk' xk by blast
      then show g x = g x'
        by auto
      show g u ∈ g ‘ K'if u ∈ K for u
        using that same by auto
      show g u ∈ g ‘ Kif u ∈ K' for u
        using that same by auto
    next
      fix x
      assume x ∈ cbox a b
      then have h x ∈ ⋃{k. ∃ x. (x, k) ∈ p}
        using p(6) by auto
      then obtain X y where h x ∈ X (y, X) ∈ p by blast
      then show x ∈ ⋃{k. ∃ x. (x, k) ∈ (λ(x, k). (g x, g ‘ k)) ‘ p}
```

```
        by clarsimp (metis (no_types, lifting) gh image_eqI pair_imageI)
      qed (use gab in auto)
      have ∗: inj_on (λ(x, k). (g x, g ‘ k)) p
        using inj(1) unfolding inj_on_def by fastforce
      have (∑(x,K)∈(λ(y,L). (g y, g ‘ L)) ‘ p. content K ∗R f x)
          = (∑u∈p. case case u of (x,K) ⇒ (g x, g ‘ K) of (y,L) ⇒ content L ∗R
f y)
        by (metis (mono_tags, lifting) ∗ sum.reindex_cong)
      also have ... = (∑(x,K)∈p. r ∗R content K ∗R f (g x))
        using r by (auto intro!: ∗ sum.cong simp: bij_betw_def dest!: p(4))
      finally
       have (∑(x, K)∈(λ(x,K). (g x, g ‘ K)) ‘ p. content K ∗R f x) − i = r ∗R
(∑(x,K)∈p. content K ∗R f (g x)) − i
        by (simp add: scaleR_right.sum split_def)
      also have ... = r ∗R ((∑(x,K)∈p. content K ∗R f (g x)) − (1 / r) ∗R i)
        using ‹0 < r› by (auto simp: scaleR_diff_right)
      finally show norm ((∑(x,K)∈p. content K ∗R f (g x)) − (1 / r) ∗R i) < e
        using d[OF gimp] ‹0 < r› by auto
    qed
  qed
  then show ?thesis
    by (auto simp: h_eq has_integral)
qed
```

7.14.26 Special case of a basic affine transformation

```
lemma AE_lborel_inner_neq:
  assumes k: k ∈ Basis
  shows AE x in lborel. x • k ≠ c
proof −
  interpret finite_product_sigma_finite λ_. lborel Basis
    proof qed simp
    have emeasure lborel {x∈space lborel. x • k = c}
        = emeasure (ΠM j::'a∈Basis. lborel) (ΠE j∈Basis. if j = k then {c} else
UNIV)
    using k
   by (auto simp add: lborel_eq[where 'a='a] emeasure_distr intro!: arg_cong2[where
f=emeasure])
       (auto simp: space_PiM PiE_iff extensional_def split: if_split_asm)
  also have ... = (∏j∈Basis. emeasure lborel (if j = k then {c} else UNIV))
    by (intro measure_times) auto
  also have ... = 0
    by (intro prod_zero bexI[OF _ k]) auto
  finally show ?thesis
    by (subst AE_iff_measurable[OF _ refl]) auto
qed

lemma content_image_stretch_interval:
  fixes m :: 'a::euclidean_space ⇒ real
```

```
  defines s f x ≡ (∑ k::'a∈Basis. (f k * (x•k)) *R k)
  shows content (s m ‘ cbox a b) = |∏ k∈Basis. m k| * content (cbox a b)
proof cases
  have s[measurable]: s f ∈ borel →M borel for f
    by (auto simp: s_def[abs_def])
  assume m: ∀ k∈Basis. m k ≠ 0
  then have s_comp_s: s (λk. 1 / m k) ∘ s m = id s m ∘ s (λk. 1 / m k) = id
    by (auto simp: s_def[abs_def] fun_eq_iff euclidean_representation)
  then have inv (s (λk. 1 / m k)) = s m bij (s (λk. 1 / m k))
    by (auto intro: inv_unique_comp o_bij)
  then have eq: s m ‘ cbox a b = s (λk. 1 / m k) −‘ cbox a b
    using bij_vimage_eq_inv_image[OF ‹bij (s (λk. 1 / m k))›, of cbox a b] by
auto
  show ?thesis
    using m unfolding eq measure_def
    by (subst lborel_affine_euclidean[where c=m and t=0])
       (simp_all add: emeasure_density measurable_sets_borel[OF s] abs_prod
nn_integral_cmult
                      s_def[symmetric] emeasure_distr vimage_comp s_comp_s
enn2real_mult prod_nonneg)
next
  assume ¬ (∀ k∈Basis. m k ≠ 0)
  then obtain k where k: k ∈ Basis m k = 0 by auto
  then have [simp]: (∏ k∈Basis. m k) = 0
    by (intro prod_zero) auto
  have emeasure lborel {x∈space lborel. x ∈ s m ‘ cbox a b} = 0
  proof (rule emeasure_eq_0_AE)
    show AE x in lborel. x ∉ s m ‘ cbox a b
      using AE_lborel_inner_neq[OF ‹k∈Basis›]
    proof eventually_elim
      show x • k ≠ 0 ⟹ x ∉ s m ‘ cbox a b for x
        using k by (auto simp: s_def[abs_def] cbox_def)
    qed
  qed
  then show ?thesis
    by (simp add: measure_def)
qed

lemma interval_image_affinity_interval:
  ∃ u v. (λx. m *R (x::'a::euclidean_space) + c) ‘ cbox a b = cbox u v
  unfolding image_affinity_cbox
  by auto

lemma content_image_affinity_cbox:
  content((λx::'a::euclidean_space. m *R x + c) ‘ cbox a b) =
    |m| ^ DIM('a) * content (cbox a b) (is ?l = ?r)
proof (cases cbox a b = {})
  case True then show ?thesis by simp
next
```

```
  case False
  show ?thesis
  proof (cases m ≥ 0)
    case True
    with ‹cbox a b ≠ {}› have cbox (m *R a + c) (m *R b + c) ≠ {}
      by (simp add: box_ne_empty inner_left_distrib mult_left_mono)
    moreover from True have *: ⋀i. (m *R b + c) · i − (m *R a + c) · i = m
*R (b−a) · i
      by (simp add: inner_simps field_simps)
    ultimately show ?thesis
        by (simp add: image_affinity_cbox True content_cbox' prod.distrib in-
ner_diff_left)
  next
    case False
    with ‹cbox a b ≠ {}› have cbox (m *R b + c) (m *R a + c) ≠ {}
      by (simp add: box_ne_empty inner_left_distrib mult_left_mono)
    moreover from False have *: ⋀i. (m *R a + c) · i − (m *R b + c) · i =
(−m) *R (b−a) · i
      by (simp add: inner_simps field_simps)
    ultimately show ?thesis using False
      by (simp add: image_affinity_cbox content_cbox'
        prod.distrib[symmetric] inner_diff_left flip: prod_constant)
  qed
qed

lemma has_integral_affinity:
  fixes a :: 'a::euclidean_space
  assumes (f has_integral i) (cbox a b)
      and m ≠ 0
  shows ((λx. f(m *R x + c)) has_integral (1 / (|m| ^ DIM('a))) *R i) ((λx. (1
/ m) *R x + −((1 / m) *R c)) ‘ cbox a b)
proof (rule has_integral_twiddle)
  show ∃w z. (λx. (1 / m) *R x + − ((1 / m) *R c)) ‘ cbox u v = cbox w z
       ∃w z. (λx. m *R x + c) ‘ cbox u v = cbox w z for u v
    using interval_image_affinity_interval by blast+
  show content ((λx. m *R x + c) ‘ cbox u v) = |m| ^ DIM('a) * content (cbox u
v) for u v
    using content_image_affinity_cbox by blast
qed (use assms zero_less_power in ‹auto simp: field_simps›)

lemma integrable_affinity:
  assumes f integrable_on cbox a b
    and m ≠ 0
  shows (λx. f(m *R x + c)) integrable_on ((λx. (1 / m) *R x + −((1/m) *R
c)) ‘ cbox a b)
  using has_integral_affinity assms
  unfolding integrable_on_def by blast

lemmas has_integral_affinity01 = has_integral_affinity [of _ _ 0 1::real, sim-
```

plified]

lemma *integrable_on_affinity*:
assumes $m \neq 0$ f *integrable_on* (*cbox* a b)
shows $(\lambda x.\ f\ (m *_R x + c))$ *integrable_on* $((\lambda x.\ (1\ /\ m) *_R x - ((1\ /\ m) *_R c))$ ‘ *cbox* a b)
proof −
from *assms* **obtain** I **where** (f *has_integral* I) (*cbox* a b)
by (*auto simp*: *integrable_on_def*)
from *has_integral_affinity*[*OF this assms*(1), *of c*] **show** *?thesis*
by (*auto simp*: *integrable_on_def*)
qed

lemma *has_integral_cmul_iff*:
assumes $c \neq 0$
shows $((\lambda x.\ c *_R f\ x)$ *has_integral* $(c *_R I))\ A \longleftrightarrow (f$ *has_integral* $I)\ A$
using *assms has_integral_cmul*[*of f I A c*]
has_integral_cmul[*of* $\lambda x.\ c *_R f\ x\ c *_R I\ A$ *inverse c*]
by (*auto simp*: *field_simps*)

lemma *has_integral_cmul_iff'*:
assumes $c \neq 0$
shows $((\lambda x.\ c *_R f\ x)$ *has_integral* $I)\ A \longleftrightarrow (f$ *has_integral* $I\ /_R\ c)\ A$
using *assms* **by** (*metis divideR_right has_integral_cmul_iff*)

lemma *has_integral_affinity'*:
fixes $a :: 'a$::*euclidean_space*
assumes (f *has_integral* i) (*cbox* a b) **and** $m > 0$
shows $((\lambda x.\ f(m *_R x + c))$ *has_integral* $(i /_R m\ \hat{}\ DIM('a)))$
$(cbox\ ((a - c) /_R m)\ ((b - c) /_R m))$
proof (*cases cbox a b* = {})
case *True*
hence (*cbox* $((a - c) /_R m)\ ((b - c) /_R m)) = \{\}$
using ‹$m > 0$› **unfolding** *box_eq_empty* **by** (*auto simp*: *algebra_simps*)
with *True* **and** *assms* **show** *?thesis* **by** *simp*
next
case *False*
have $((\lambda x.\ f\ (m *_R x + c))$ *has_integral* $(1\ /\ |m|\ \hat{}\ DIM('a)) *_R i)$
$((\lambda x.\ (1\ /\ m) *_R x + - ((1\ /\ m) *_R c))$ ‘ *cbox* a b)
using *assms* **by** (*intro has_integral_affinity*) *auto*
also have $((\lambda x.\ (1\ /\ m) *_R x + - ((1\ /\ m) *_R c))$ ‘ *cbox* a $b) =$
$((\lambda x.\ - ((1\ /\ m) *_R c) + x)$ ‘ $(\lambda x.\ (1\ /\ m) *_R x)$ ‘ *cbox* a $b)$
by (*simp add*: *image_image algebra_simps*)
also have $(\lambda x.\ (1\ /\ m) *_R x)$ ‘ *cbox* a $b =$ *cbox* $((1\ /\ m) *_R a)\ ((1\ /\ m) *_R b)$
using ‹$m > 0$› *False*
by (*subst image_smult_cbox*) *simp_all*
also have $(\lambda x.\ - ((1\ /\ m) *_R c) + x)$ ‘ ... = *cbox* $((a - c) /_R m)\ ((b - c) /_R m)$
by (*subst cbox_translation* [*symmetric*]) (*simp add*: *field_simps vector_add_divide_simps*)

finally show *?thesis* **using** ‹$m > 0$›
by (*simp add*: *field_simps*)
qed

lemma *has_integral_affinity_iff*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: real\_normed\_vector$
assumes $m > 0$
shows $((\lambda x.\ f\ (m *_R x + c))\ has\_integral\ (I\ /_R\ m\ \widehat{}\ DIM('a)))$
$(cbox\ ((a - c)\ /_R\ m)\ ((b - c)\ /_R\ m)) \longleftrightarrow$
$(f\ has\_integral\ I)\ (cbox\ a\ b)$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
from *has_integral_affinity'*[*OF this*, *of* $1\ /\ m\ -c\ /_R\ m$] **and** ‹$m > 0$›
show *?rhs* **by** (*simp add*: *vector_add_divide_simps*) (*simp add*: *field_simps*)
next
assume *?rhs*
from *has_integral_affinity'*[*OF this*, *of* $m\ c$] **and** ‹$m > 0$›
show *?lhs* **by** *simp*
qed

7.14.27 Special case of stretching coordinate axes separately

lemma *has_integral_stretch*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::real\_normed\_vector$
assumes $(f\ has\_integral\ i)\ (cbox\ a\ b)$
and $\forall k{\in}Basis.\ m\ k \neq 0$
shows $((\lambda x.\ f\ (\sum k{\in}Basis.\ (m\ k * (x{\cdot}k)) *_R\ k))\ has\_integral$
$((1/\ |prod\ m\ Basis|) *_R\ i))\ ((\lambda x.\ (\sum k{\in}Basis.\ (1\ /\ m\ k * (x{\cdot}k)) *_R\ k))\ `\ cbox\ a\ b)$
apply (*rule has_integral_twiddle*[**where** $f{=}f$])
unfolding *zero_less_abs_iff content_image_stretch_interval*
unfolding *image_stretch_interval empty_as_interval euclidean_eq_iff*[**where** $'a{=}'a$]
using *assms*
by *auto*

lemma *has_integral_stretch_real*:
fixes $f :: real \Rightarrow 'b::real\_normed\_vector$
assumes $(f\ has\_integral\ i)\ \{a..b\}$ **and** $m \neq 0$
shows $((\lambda x.\ f\ (m * x))\ has\_integral\ (1\ /\ |m|) *_R\ i)\ ((\lambda x.\ x\ /\ m)\ `\ \{a..b\})$
using *has_integral_stretch* [*of f i a b* $\lambda b.\ m$] *assms* **by** *simp*

lemma *integrable_stretch*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::real\_normed\_vector$
assumes *f integrable_on cbox a b*
and $\forall k{\in}Basis.\ m\ k \neq 0$
shows $(\lambda x::'a.\ f\ (\sum k{\in}Basis.\ (m\ k * (x{\cdot}k)) *_R\ k))\ integrable\_on$
$((\lambda x.\ \sum k{\in}Basis.\ (1\ /\ m\ k * (x{\cdot}k)) *_R\ k)\ `\ cbox\ a\ b)$
using *assms* **unfolding** *integrable_on_def*

by (*force dest*: *has_integral_stretch*)

lemma *vec_lambda_eq_sum*:
(χ *k*. *f k* (*x* $ *k*)) = (∑ *k*∈*Basis*. (*f* (*axis_index k*) (*x* ⋅ *k*)) ∗$_R$ *k*) (**is** *?lhs* = *?rhs*)
proof −
have *?lhs* = (χ *k*. *f k* (*x* ⋅ *axis k 1*))
by (*simp add*: *cart_eq_inner_axis*)
also have ... = (∑ *u*∈*UNIV*. *f u* (*x* ⋅ *axis u 1*) ∗$_R$ *axis u 1*)
by (*simp add*: *vec_eq_iff axis_def if_distrib cong*: *if_cong*)
also have ... = *?rhs*
by (*simp add*: *Basis_vec_def UNION_singleton_eq_range sum.reindex axis_eq_axis inj_on_def*)
finally show *?thesis* .
qed

lemma *has_integral_stretch_cart*:
fixes *m* :: *'n*::*finite* ⇒ *real*
assumes *f*: (*f has_integral i*) (*cbox a b*) **and** *m*: ⋀*k*. *m k* ≠ *0*
shows ((λ*x*. *f*(χ *k*. *m k* ∗ *x*$*k*)) *has_integral i* /$_R$ |*prod m UNIV*|)
((λ*x*. χ *k*. *x*$*k* / *m k*) ‘ (*cbox a b*))
proof −
have ∗: ∀ *k*:: *real*^*'n* ∈ *Basis*. *m* (*axis_index k*) ≠ *0*
using *axis_index* **by** (*simp add*: *m*)
have *eqp*: (∏ *k*:: *real*^*'n* ∈ *Basis*. *m* (*axis_index k*)) = *prod m UNIV*
by (*simp add*: *Basis_vec_def UNION_singleton_eq_range prod.reindex axis_eq_axis inj_on_def*)
show *?thesis*
using *has_integral_stretch* [*OF f* ∗] *vec_lambda_eq_sum* [**where** *f*=λ*i x*. *m i* ∗ *x*] *vec_lambda_eq_sum* [**where** *f*=λ*i x*. *x* / *m i*]
by (*simp add*: *field_simps eqp*)
qed

lemma *image_stretch_interval_cart*:
fixes *m* :: *'n*::*finite* ⇒ *real*
shows (λ*x*. χ *k*. *m k* ∗ *x*$*k*) ‘ *cbox a b* =
(*if cbox a b* = {} *then* {}
else cbox (χ *k*. *min* (*m k* ∗ *a*$*k*) (*m k* ∗ *b*$*k*)) (χ *k*. *max* (*m k* ∗ *a*$*k*) (*m k* ∗ *b*$*k*)))
proof −
have ∗: (∑ *k*∈*Basis*. *min* (*m* (*axis_index k*) ∗ (*a* ⋅ *k*)) (*m* (*axis_index k*) ∗ (*b* ⋅ *k*)) ∗$_R$ *k*)
= (χ *k*. *min* (*m k* ∗ *a* $ *k*) (*m k* ∗ *b* $ *k*))
(∑ *k*∈*Basis*. *max* (*m* (*axis_index k*) ∗ (*a* ⋅ *k*)) (*m* (*axis_index k*) ∗ (*b* ⋅ *k*)) ∗$_R$ *k*)
= (χ *k*. *max* (*m k* ∗ *a* $ *k*) (*m k* ∗ *b* $ *k*))
apply (*simp_all add*: *Basis_vec_def cart_eq_inner_axis UNION_singleton_eq_range sum.reindex axis_eq_axis inj_on_def*)
apply (*simp_all add*: *vec_eq_iff axis_def if_distrib cong*: *if_cong*)

done
show *?thesis*
by (*simp add*: *vec_lambda_eq_sum* [**where** *f*=λ*i x*. *m i* ∗ *x*] *image_stretch_interval eq_cbox* ∗)
qed

7.14.28 even more special cases

lemma *uminus_interval_vector*[*simp*]:
fixes *a b* :: *'a*::*euclidean_space*
shows *uminus* ‘ *cbox a b* = *cbox* (−*b*) (−*a*)
proof −
have *x* ∈ *uminus* ‘ *cbox a b* **if** *x* ∈ *cbox* (− *b*) (− *a*) **for** *x*
by (*smt* (*verit*) *add.inverse_inverse image_iff inner_minus_left mem_box*(2) *that*)
then show *?thesis*
by (*auto simp*: *mem_box*)
qed

lemma *has_integral_reflect_lemma*[*intro*]:
assumes (*f has_integral i*) (*cbox a b*)
shows ((λ*x*. *f*(−*x*)) *has_integral i*) (*cbox* (−*b*) (−*a*))
using *has_integral_affinity*[*OF assms*, *of* −*1 0*]
by *auto*

lemma *has_integral_reflect_lemma_real*[*intro*]:
assumes (*f has_integral i*) {*a*..*b*::*real*}
shows ((λ*x*. *f*(−*x*)) *has_integral i*) {−*b* .. −*a*}
by (*metis has_integral_reflect_lemma interval_cbox assms*)

lemma *has_integral_reflect*[*simp*]:
((λ*x*. *f* (−*x*)) *has_integral i*) (*cbox* (−*b*) (−*a*)) ⟷ (*f has_integral i*) (*cbox a b*)
by (*auto dest*: *has_integral_reflect_lemma*)

lemma *has_integral_reflect_real*[*simp*]:
fixes *a b*::*real*
shows ((λ*x*. *f* (−*x*)) *has_integral i*) {−*b*..−*a*} ⟷ (*f has_integral i*) {*a*..*b*}
by (*metis has_integral_reflect interval_cbox*)

lemma *integrable_reflect*[*simp*]: (λ*x*. *f*(−*x*)) *integrable_on cbox* (−*b*) (−*a*) ⟷ *f integrable_on cbox a b*
unfolding *integrable_on_def* **by** *auto*

lemma *integrable_reflect_real*[*simp*]: (λ*x*. *f*(−*x*)) *integrable_on* {−*b* .. −*a*} ⟷ *f integrable_on* {*a*..*b*::*real*}
unfolding *box_real*[*symmetric*]
by (*rule integrable_reflect*)

lemma *integral_reflect*[*simp*]: *integral* (*cbox* (−*b*) (−*a*)) (λ*x*. *f* (−*x*)) = *integral*

(*cbox a b*) *f*
unfolding *integral_def* **by** *auto*

lemma *integral_reflect_real*[*simp*]: *integral* {−*b* .. −*a*} (λ*x*. *f* (−*x*)) = *integral* {*a*..*b*::*real*} *f*
unfolding *box_real*[*symmetric*]
by (*rule integral_reflect*)

7.14.29 Stronger form of FCT; quite a tedious proof

lemma *split_minus*[*simp*]: (λ(*x*, *k*). *f x k*) *x* − (λ(*x*, *k*). *g x k*) *x* = (λ(*x*, *k*). *f x k* − *g x k*) *x*
by (*simp add*: *split_def*)

theorem *fundamental_theorem_of_calculus_interior*:
fixes *f* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *a* ≤ *b*
and *contf*: *continuous_on* {*a*..*b*} *f*
and *derf*: ⋀*x*. *x* ∈ {*a* <..< *b*} ⟹ (*f has_vector_derivative f' x*) (*at x*)
shows (*f' has_integral* (*f b* − *f a*)) {*a*..*b*}
proof (*cases a* = *b*)
case *True*
then have ∗: *cbox a b* = {*b*} *f b* − *f a* = *0*
by (*auto simp add*: *order_antisym*)
with *True* **show** *?thesis* **by** *auto*
next
case *False*
with ‹*a* ≤ *b*› **have** *ab*: *a* < *b* **by** *arith*
show *?thesis*
unfolding *has_integral_factor_content_real*
proof (*intro allI impI*)
fix *e* :: *real*
assume *e*: *e* > *0*
then have *eba8*: (*e* ∗ (*b*−*a*)) / *8* > *0*
using *ab* **by** (*auto simp add*: *field_simps*)
note *derf_exp* = *derf*[*unfolded has_vector_derivative_def has_derivative_at_alt*, *THEN conjunct2*, *rule_format*]
have *bounded*: ⋀*x*. *x* ∈ {*a*<..<*b*} ⟹ *bounded_linear* (λ*u*. *u* ∗*R* *f' x*)
by (*simp add*: *bounded_linear_scaleR_left*)
have ∀ *x* ∈ *box a b*. ∃ *d*>*0*. ∀ *y*. *norm* (*y*−*x*) < *d* ⟶ *norm* (*f y* − *f x* − (*y*−*x*) ∗*R* *f' x*) ≤ *e*/*2* ∗ *norm* (*y*−*x*)
(**is** ∀ *x* ∈ *box a b*. *?Q x*) — The explicit quantifier is required by the following step
proof
fix *x* **assume** *x* ∈ *box a b*
with *e* **show** *?Q x*
using *derf_exp* [*of x e*/*2*] **by** *auto*
qed
then obtain *d* **where** *d*: ⋀*x*. *0* < *d x*

```
    ⋀x y. ⟦x ∈ box a b; norm (y−x) < d x⟧ ⟹ norm (f y − f x − (y−x) *R f′
x) ≤ e/2 * norm (y−x)
     unfolding bgauge_existence_lemma by metis
   have bounded (f ‘ cbox a b)
     using compact_cbox assms by (auto simp: compact_imp_bounded com-
pact_continuous_image)
   then obtain B
     where 0 < B and B: ⋀x. x ∈ f ‘ cbox a b ⟹ norm x ≤ B
     unfolding bounded_pos by metis
   obtain da where 0 < da
     and da: ⋀c. ⟦a ≤ c; {a..c} ⊆ {a..b}; {a..c} ⊆ ball a da⟧
                  ⟹ norm (content {a..c} *R f′ a − (f c − f a)) ≤ (e * (b−a))
/ 4
   proof −
     have continuous (at a within {a..b}) f
       using contf continuous_on_eq_continuous_within by force
     with eba8 obtain k where 0 < k
      and k: ⋀x. ⟦x ∈ {a..b}; 0 < norm (x−a); norm (x−a) < k⟧ ⟹ norm (f x
− f a) < e * (b−a) / 8
       unfolding continuous_within Lim_within dist_norm by metis
     obtain l where l: 0 < l norm (l *R f′ a) ≤ e * (b−a) / 8
     proof (cases f′ a = 0)
       case True with ab e that show ?thesis by auto
     next
       case False
       show ?thesis
       proof
         show norm ((e * (b − a) / 8 / norm (f′ a)) *R f′ a) ≤ e * (b − a) / 8
              0 < e * (b − a) / 8 / norm (f′ a)
           using False ab e by (auto simp add: field_simps)
       qed
     qed
     have norm (content {a..c} *R f′ a − (f c − f a)) ≤ e * (b−a) / 4
       if a ≤ c {a..c} ⊆ {a..b} and bmin: {a..c} ⊆ ball a (min k l) for c
     proof −
       have minkl: |a − x| < min k l if x ∈ {a..c} for x
         using bmin dist_real_def that by auto
       then have lel: |c − a| ≤ |l|
         using that by force
        have norm ((c − a) *R f′ a − (f c − f a)) ≤ norm ((c − a) *R f′ a) +
norm (f c − f a)
         by (rule norm_triangle_ineq4)
       also have ... ≤ e * (b−a) / 8 + e * (b−a) / 8
       proof (rule add_mono)
         have norm ((c − a) *R f′ a) ≤ norm (l *R f′ a)
           by (auto intro: mult_right_mono [OF lel])
         with l show norm ((c − a) *R f′ a) ≤ e * (b−a) / 8
           by linarith
       next
```

```
        have norm (f c − f a) < e * (b−a) / 8
        proof (cases a = c)
          case True then show ?thesis
            using eba8 by auto
        next
          case False show ?thesis
            by (rule k) (use minkl ‹a ≤ c› that False in auto)
        qed
        then show norm (f c − f a) ≤ e * (b−a) / 8 by simp
      qed
      finally show norm (content {a..c} *R f' a − (f c − f a)) ≤ e * (b−a) / 4
        unfolding content_real[OF ‹a ≤ c›] by auto
    qed
    then show ?thesis
      by (rule_tac da=min k l in that) (auto simp: l ‹0 < k›)
  qed
  obtain db where 0 < db
    and db: ⋀c. ⟦c ≤ b; {c..b} ⊆ {a..b}; {c..b} ⊆ ball b db⟧
                ⟹ norm (content {c..b} *R f' b − (f b − f c)) ≤ (e * (b−a))
/ 4
  proof −
    have continuous (at b within {a..b}) f
      using contf continuous_on_eq_continuous_within by force
    with eba8 obtain k
      where 0 < k
        and k: ⋀x. ⟦x ∈ {a..b}; 0 < norm(x−b); norm(x−b) < k⟧
               ⟹ norm (f b − f x) < e * (b−a) / 8
    unfolding continuous_within Lim_within dist_norm norm_minus_commute
by metis
    obtain l where l: 0 < l norm (l *R f' b) ≤ (e * (b−a)) / 8
    proof (cases f' b = 0)
      case True thus ?thesis
        using ab e that by auto
    next
      case False show ?thesis
      proof
        show norm ((e * (b − a) / 8 / norm (f' b)) *R f' b) ≤ e * (b − a) / 8
             0 < e * (b − a) / 8 / norm (f' b)
          using False ab e by (auto simp add: field_simps)
      qed
    qed
    have norm (content {c..b} *R f' b − (f b − f c)) ≤ e * (b−a) / 4
      if c ≤ b {c..b} ⊆ {a..b} and bmin: {c..b} ⊆ ball b (min k l) for c
    proof −
      have minkl: |b − x| < min k l if x ∈ {c..b} for x
        using bmin dist_real_def that by auto
      then have lel: |b − c| ≤ |l|
        using that by force
       have norm ((b − c) *R f' b − (f b − f c)) ≤ norm ((b − c) *R f' b) +
```

```
norm (f b − f c)
        by (rule norm_triangle_ineq4)
      also have ... ≤ e * (b−a) / 8 + e * (b−a) / 8
      proof (rule add_mono)
        have norm ((b − c) *R f' b) ≤ norm (l *R f' b)
          by (auto intro: mult_right_mono [OF lel])
        also have ... ≤ e * (b−a) / 8
          by (rule l)
        finally show norm ((b − c) *R f' b) ≤ e * (b−a) / 8 .
      next
        have norm (f b − f c) < e * (b−a) / 8
        proof (cases b = c)
          case True with eba8 show ?thesis
            by auto
        next
          case False show ?thesis
            by (rule k) (use minkl ‹c ≤ b› that False in auto)
        qed
        then show norm (f b − f c) ≤ e * (b−a) / 8 by simp
      qed
      finally show norm (content {c..b} *R f' b − (f b − f c)) ≤ e * (b−a) / 4
        unfolding content_real[OF ‹c ≤ b›] by auto
    qed
    then show ?thesis
      by (rule_tac db=min k l in that) (auto simp: l ‹0 < k›)
  qed
  let ?d = (λx. ball x (if x=a then da else if x=b then db else d x))
  show ∃ d. gauge d ∧ (∀ p. p tagged_division_of {a..b} ∧ d fine p ⟶
          norm ((∑ (x,K)∈p. content K *R f' x) − (f b − f a)) ≤ e * content
{a..b})
  proof (rule exI, safe)
    show gauge ?d
      using ab ‹db > 0› ‹da > 0› d(1) by (auto intro: gauge_ball_dependent)
  next
    fix p
    assume ptag: p tagged_division_of {a..b} and fine: ?d fine p
    let ?A = {t. fst t ∈ {a, b}}
    note p = tagged_division_ofD[OF ptag]
    have pA: p = (p ∩ ?A) ∪ (p − ?A) finite (p ∩ ?A) finite (p − ?A) (p ∩ ?A)
∩ (p − ?A) = {}
      using ptag fine by auto
    have le_xz: ⋀w x y z::real. y ≤ z/2 ⟹ w − x ≤ z/2 ⟹ w + y ≤ x + z
      by arith
    have non: False if xk: (x,K) ∈ p and x ≠ a x ≠ b
     and less: e * (Sup K − Inf K)/2 < norm (content K *R f' x − (f (Sup K)
− f (Inf K)))
    for x K
    proof −
      obtain u v where k: K = cbox u v
```

```
      using p(4) xk by blast
    then have u ≤ v and uv: {u, v} ⊆ cbox u v
      using p(2)[OF xk] by auto
    then have result: e * (v − u)/2 < norm ((v − u) *R f′ x − (f v − f u))
     using less[unfolded k box_real interval_bounds_real content_real] by auto
    then have x ∈ box a b
      using p(2) p(3) ‹x ≠ a› ‹x ≠ b› xk by fastforce
    with d have *: ⋀y. norm (y−x) < d x
          ⟹ norm (f y − f x − (y−x) *R f′ x) ≤ e/2 * norm (y−x)
      by metis
    have xd: norm (u − x) < d x norm (v − x) < d x
      using fineD[OF fine xk] ‹x ≠ a› ‹x ≠ b› uv
      by (auto simp add: k subset_eq dist_commute dist_real_def)
    have norm ((v − u) *R f′ x − (f v − f u)) =
        norm ((f u − f x − (u − x) *R f′ x) − (f v − f x − (v − x) *R f′ x))
      by (rule arg_cong[where f=norm]) (auto simp: scaleR_left.diff)
    also have ... ≤ e/2 * norm (u − x) + e/2 * norm (v − x)
      by (metis norm_triangle_le_diff add_mono * xd)
    also have ... ≤ e/2 * norm (v − u)
      using p(2)[OF xk] by (auto simp add: field_simps k)
    also have ... < norm ((v − u) *R f′ x − (f v − f u))
      using result by (simp add: ‹u ≤ v›)
    finally have e * (v − u)/2 < e * (v − u)/2
      using uv by auto
    then show False by auto
  qed
   have norm (∑(x, K)∈p − ?A. content K *R f′ x − (f (Sup K) − f (Inf
K)))
        ≤ (∑(x, K)∈p − ?A. norm (content K *R f′ x − (f (Sup K) − f (Inf
K))))
    by (auto intro: sum_norm_le)
  also have ... ≤ (∑n∈p − ?A. e * (case n of (x, k) ⇒ Sup k − Inf k)/2)
    using non by (fastforce intro: sum_mono)
  finally have I: norm (∑(x, k)∈p − ?A. content k *R f′ x − (f (Sup k) − f
(Inf k)))
                ≤ (∑n∈p − ?A. e * (case n of (x, k) ⇒ Sup k − Inf k))/2
    by (simp add: sum_divide_distrib)
   have II: norm (∑(x, k)∈p ∩ ?A. content k *R f′ x − (f (Sup k) − f (Inf
k))) −
        (∑n∈p ∩ ?A. e * (case n of (x, k) ⇒ Sup k − Inf k))
        ≤ (∑n∈p − ?A. e * (case n of (x, k) ⇒ Sup k − Inf k))/2
  proof −
    have ge0: 0 ≤ e * (Sup k − Inf k) if xkp: (x, k) ∈ p ∩ ?A for x k
    proof −
      obtain u v where uv: k = cbox u v
        by (meson Int_iff xkp p(4))
      with p that have cbox u v ≠ {}
        by blast
      then show 0 ≤ e * ((Sup k) − (Inf k))
```

```
        unfolding uv using e by (auto simp add: field_simps)
      qed
      let ?B = λx. {t ∈ p. fst t = x ∧ content (snd t) ≠ 0}
      let ?C = {t ∈ p. fst t ∈ {a, b} ∧ content (snd t) ≠ 0}
      have norm (∑(x, k)∈p ∩ {t. fst t ∈ {a, b}}. content k *R f' x − (f (Sup
k) − f (Inf k))) ≤ e * (b−a)/2
      proof −
        have *: ⋀S f e. sum f S = sum f (p ∩ ?C) ⟹ norm (sum f (p ∩ ?C))
≤ e ⟹ norm (sum f S) ≤ e
          by auto
        have 1: content K *R (f' x) − (f ((Sup K)) − f ((Inf K))) = 0
          if (x,K) ∈ p ∩ {t. fst t ∈ {a, b}} − p ∩ ?C for x K
        proof −
          have xk: (x,K) ∈ p and k0: content K = 0
            using that by auto
          then obtain u v where uv: K = cbox u v u = v
            using xk k0 p by fastforce
          then show content K *R (f' x) − (f ((Sup K)) − f ((Inf K))) = 0
            using xk unfolding uv by auto
        qed
        have 2: norm(∑(x,K)∈p ∩ ?C. content K *R f' x − (f (Sup K) − f (Inf
K))) ≤ e * (b−a)/2
        proof −
          have norm_le: norm (sum f S) ≤ e
           if ⋀x y. ⟦x ∈ S; y ∈ S⟧ ⟹ x = y ⋀x. x ∈ S ⟹ norm (f x) ≤ e e > 0
            for S f and e :: real
          proof (cases S = {})
            case True
            with that show ?thesis by auto
          next
            case False then obtain x where x ∈ S
              by auto
            then have S = {x}
              using that(1) by auto
            then show ?thesis
              using ‹x ∈ S› that(2) by auto
          qed
          have *: p ∩ ?C = ?B a ∪ ?B b
            by blast
         then have norm (∑(x,K)∈p ∩ ?C. content K *R f' x − (f (Sup K) −
f (Inf K))) =
                    norm (∑(x,K) ∈ ?B a ∪ ?B b. content K *R f' x − (f (Sup
K) − f (Inf K)))
            by simp
          also have ... = norm ((∑(x,K) ∈ ?B a. content K *R f' x − (f (Sup
K) − f (Inf K))) +
                                (∑(x,K) ∈ ?B b. content K *R f' x − (f (Sup K) − f
(Inf K))))
            using p(1) ab e by (subst sum.union_disjoint) auto
```

```
      also have ... ≤ e * (b − a) / 4 + e * (b − a) / 4
      proof (rule norm_triangle_le [OF add_mono])
        have pa: ∃ v. k = cbox a v ∧ a ≤ v if (a, k) ∈ p for k
          using p that by fastforce
        show norm (∑(x,K) ∈ ?B a. content K *R f' x − (f (Sup K) − f
(Inf K))) ≤ e * (b − a) / 4
        proof (intro norm_le; clarsimp)
          fix K K'
          assume K: (a, K) ∈ p (a, K') ∈ p and ne0: content K ≠ 0 content
K' ≠ 0
            with pa obtain v v' where v: K = cbox a v a ≤ v and v': K' =
cbox a v' a ≤ v'
              by blast
          let ?v = min v v'
          have box a ?v ⊆ K ∩ K'
            unfolding v v' by (auto simp add: mem_box)
          then have interior (box a (min v v')) ⊆ interior K ∩ interior K'
            using interior_Int interior_mono by blast
          moreover have (a + ?v)/2 ∈ box a ?v
            using ne0  unfolding v v' content_eq_0 not_le
            by (auto simp add: mem_box)
          ultimately have (a + ?v)/2 ∈ interior K ∩ interior K'
            unfolding interior_open[OF open_box] by auto
          then show K = K'
            using p(5)[OF K] by auto
        next
          fix K
          assume K: (a, K) ∈ p and ne0: content K ≠ 0
          show norm (content c *R f' a − (f (Sup c) − f (Inf c))) * 4 ≤ e *
(b−a)
            if (a, c) ∈ p and ne0: content c ≠ 0 for c
          proof −
            obtain v where v: c = cbox a v and a ≤ v
              using pa[OF ‹(a, c) ∈ p›] by metis
            then have a ∈ {a..v} v ≤ b
              using p(3)[OF ‹(a, c) ∈ p›] by auto
            moreover have {a..v} ⊆ ball a da
              using fineD[OF ‹?d fine p› ‹(a, c) ∈ p›] by (simp add: v split:
if_split_asm)
            ultimately show ?thesis
              unfolding v interval_bounds_real[OF ‹a ≤ v›] box_real
              using da ‹a ≤ v› by auto
          qed
        qed (use ab e in auto)
      next
        have pb: ∃ v. k = cbox v b ∧ b ≥ v if (b, k) ∈ p for k
          using p that by fastforce
        show norm (∑(x,K) ∈ ?B b. content K *R f' x − (f (Sup K) − f
(Inf K))) ≤ e * (b − a) / 4
```

```
          proof (intro norm_le; clarsimp)
            fix K K'
            assume K: (b, K) ∈ p (b, K') ∈ p and ne0: content K ≠ 0 content
K' ≠ 0
              with pb obtain v v' where v: K = cbox v b v ≤ b and v': K' =
cbox v' b v' ≤ b
              by blast
            let ?v = max v v'
            have box ?v b ⊆ K ∩ K'
              unfolding v v' by (auto simp: mem_box)
            then have interior (box (max v v') b) ⊆ interior K ∩ interior K'
              using interior_Int interior_mono by blast
            moreover have ((b + ?v)/2) ∈ box ?v b
                using ne0 unfolding v v' content_eq_0 not_le by (auto simp:
mem_box)
            ultimately have ((b + ?v)/2) ∈ interior K ∩ interior K'
              unfolding interior_open[OF open_box] by auto
            then show K = K'
              using p(5)[OF K] by auto
          next
            fix K
            assume K: (b, K) ∈ p and ne0: content K ≠ 0
            show norm (content c *R f' b − (f (Sup c) − f (Inf c))) * 4 ≤ e *
(b−a)
              if (b, c) ∈ p and ne0: content c ≠ 0 for c
            proof −
            obtain v where v: c = cbox v b and v ≤ b
              using ‹(b, c) ∈ p› pb by blast
            then have v ≥ ab ∈ {v.. b}
              using p(3)[OF ‹(b, c) ∈ p›] by auto
            moreover have {v..b} ⊆ ball b db
               using fineD[OF ‹?d fine p› ‹(b, c) ∈ p›] box_real(2) v False by
force
            ultimately show ?thesis
              using db v by auto
            qed
          qed (use ab e in auto)
        qed
        also have ... = e * (b−a)/2
          by simp
        finally show norm (∑(x,k)∈p ∩ ?C.
                  content k *R f' x − (f (Sup k) − f (Inf k))) ≤ e * (b−a)/2 .
      qed
      show norm (∑(x, k)∈p ∩ ?A. content k *R f' x − (f ((Sup k)) − f ((Inf
k)))) ≤ e * (b−a)/2
        apply (rule * [OF sum.mono_neutral_right[OF pA(2)]])
        using 1 2 by (auto simp: split_paired_all)
    qed
    also have ... = (∑ n∈p. e * (case n of (x, k) ⇒ Sup k − Inf k))/2
```

```
        unfolding sum_distrib_left[symmetric]
        by (subst additive_tagged_division_1[OF ‹a ≤ b› ptag]) auto
      finally have norm_le: norm (∑(x,K)∈p ∩ {t. fst t ∈ {a, b}}. content K
*R f' x − (f (Sup K) − f (Inf K)))
            ≤ (∑n∈p. e * (case n of (x, K) ⇒ Sup K − Inf K))/2 .
      have le2: ⋀x s1 s2::real. 0 ≤ s1 ⟹ x ≤ (s1 + s2)/2 ⟹ x − s1 ≤ s2/2
        by auto
      show ?thesis
        apply (rule le2 [OF sum_nonneg])
        using ge0 apply force
          by (metis (no_types, lifting) Diff_Diff_Int Diff_subset norm_le p(1)
sum.subset_diff)
    qed
    note * = additive_tagged_division_1[OF assms(1) ptag, symmetric]
    have norm (∑(x,K)∈p ∩ ?A ∪ (p − ?A). content K *R f' x − (f (Sup K)
− f (Inf K)))
          ≤ e * (∑(x,K)∈p ∩ ?A ∪ (p − ?A). Sup K − Inf K)
      unfolding sum_distrib_left
      unfolding sum.union_disjoint[OF pA(2−)]
      using le_xz norm_triangle_le I II by blast
    then
    show norm ((∑(x,K)∈p. content K *R f' x) − (f b − f a)) ≤ e * content
{a..b}
    by (simp only: content_real[OF ‹a ≤ b›] *[of λx. x] *[of f] sum_subtractf[symmetric]
split_minus pA(1) [symmetric])
  qed
 qed
qed
```

7.14.30 Stronger form with finite number of exceptional points

```
lemma fundamental_theorem_of_calculus_interior_strong:
 fixes f :: real ⇒ 'a::banach
 assumes finite S
   and a ≤ b ⋀x. x ∈ {a <..< b} − S ⟹ (f has_vector_derivative f'(x)) (at x)
   and continuous_on {a .. b} f
 shows (f' has_integral (f b − f a)) {a .. b}
  using assms
proof (induction arbitrary: a b)
case empty
  then show ?case
    using fundamental_theorem_of_calculus_interior by force
next
case (insert x S)
  show ?case
  proof (cases x ∈ {a<..<b})
    case False then show ?thesis
      using insert by blast
  next
```

```
    case True then have a < x x < b
      by auto
    have (f' has_integral f x − f a) {a..x} (f' has_integral f b − f x) {x..b}
      using ‹continuous_on {a..b} f› ‹a < x› ‹x < b› continuous_on_subset by
(force simp: intro!: insert)+
    then have (f' has_integral f x − f a + (f b − f x)) {a..b}
      using ‹a < x› ‹x < b› has_integral_combine less_imp_le by blast
    then show ?thesis
      by simp
  qed
qed

corollary fundamental_theorem_of_calculus_strong:
  fixes f :: real ⇒ 'a::banach
  assumes finite S
    and a ≤ b
    and vec: ⋀x. x ∈ {a..b} − S ⟹ (f has_vector_derivative f'(x)) (at x)
    and continuous_on {a..b} f
  shows (f' has_integral (f b − f a)) {a..b}
 by (rule fundamental_theorem_of_calculus_interior_strong [OF ‹finite S›]) (force
simp: assms)+

proposition indefinite_integral_continuous_left:
  fixes f:: real ⇒ 'a::banach
  assumes intf: f integrable_on {a..b} and a < c c ≤ b e > 0
  obtains d where d > 0
    and ∀ t. c − d < t ∧ t ≤ c ⟶ norm (integral {a..c} f − integral {a..t} f) <
e
proof −
  obtain w where w > 0 and w: ⋀t. ⟦c − w < t; t < c⟧ ⟹ norm (f c) * norm(c
− t) < e/3
  proof (cases f c = 0)
    case False
    hence e3: 0 < e/3 / norm (f c) using ‹e>0› by simp
    moreover have norm (f c) * norm (c − t) < e/3
      if t < c and c − e/3 / norm (f c) < t for t
      unfolding real_norm_def
        by (smt (verit) False divide_right_mono nonzero_mult_div_cancel_left
norm_eq_zero norm_ge_zero that)
    ultimately show ?thesis
      using that by auto
  qed (use ‹e > 0› in auto)

  let ?SUM = λp. (∑(x,K) ∈ p. content K *R f x)
  have e3: e/3 > 0
    using ‹e > 0› by auto
  have f integrable_on {a..c}
    using ‹a < c› ‹c ≤ b› by (auto intro: integrable_subinterval_real[OF intf])
  then obtain d1 where gauge d1 and d1:
```

⋀*p*. ⟦*p tagged_division_of* {*a*..*c*}; *d1 fine p*⟧ ⟹ *norm* (*?SUM p* − *integral* {*a*..*c*} *f*) < *e*/3
using *integrable_integral has_integral_real e3* **by** *metis*
define *d* **where** [*abs_def*]: *d x* = *ball x w* ∩ *d1 x* **for** *x*
have *gauge d*
unfolding *d_def* **using** ‹*w* > 0› ‹*gauge d1*› **by** *auto*
then obtain *k* **where** 0 < *k* **and** *k*: *ball c k* ⊆ *d c*
by (*meson gauge_def open_contains_ball*)

let *?d* = *min k* (*c* − *a*)/2
show *thesis*
proof (*intro that*[*of ?d*] *allI impI*, *safe*)
show *?d* > 0
using ‹0 < *k*› ‹*a* < *c*› **by** *auto*
next
fix *t*
assume *t*: *c* − *?d* < *t t* ≤ *c*
show *norm* (*integral* ({*a*..*c*}) *f* − *integral* ({*a*..*t*}) *f*) < *e*
proof (*cases t* < *c*)
case *False* **with** ‹*t* ≤ *c*› **show** *?thesis*
by (*simp add*: ‹*e* > 0›)
next
case *True*
have *f integrable_on* {*a*..*t*}
using ‹*t* < *c*› ‹*c* ≤ *b*› **by** (*auto intro*: *integrable_subinterval_real*[*OF intf*])
then obtain *d2* **where** *d2*: *gauge d2*
⋀*p*. *p tagged_division_of* {*a*..*t*} ∧ *d2 fine p* ⟹ *norm* (*?SUM p* − *integral* {*a*..*t*} *f*) < *e*/3
using *integrable_integral has_integral_real e3* **by** *metis*
define *d3* **where** *d3 x* = (*if x* ≤ *t then d1 x* ∩ *d2 x else d1 x*) **for** *x*
have *gauge d3*
using ‹*gauge d1*› ‹*gauge d2*› **unfolding** *d3_def gauge_def* **by** *auto*
then obtain *p* **where** *ptag*: *p tagged_division_of* {*a*..*t*} **and** *pfine*: *d3 fine p*
by (*metis box_real*(2) *fine_division_exists*)
note *p'* = *tagged_division_ofD*[*OF ptag*]
have *pt*: (*x*,*K*)∈*p* ⟹ *x* ≤ *t* **for** *x K*
by (*meson atLeastAtMost_iff p'*(2) *p'*(3) *subsetCE*)
with *pfine* **have** *d2 fine p*
unfolding *fine_def d3_def* **by** *fastforce*
then have *d2_fin*: *norm* (*?SUM p* − *integral* {*a*..*t*} *f*) < *e*/3
using *d2*(2) *ptag* **by** *auto*
have *eqs*: {*a*..*c*} ∩ {*x*. *x* ≤ *t*} = {*a*..*t*} {*a*..*c*} ∩ {*x*. *x* ≥ *t*} = {*t*..*c*}
using *t* **by** (*auto simp add*: *field_simps*)
have *p* ∪ {(*c*, {*t*..*c*})} *tagged_division_of* {*a*..*c*}
proof (*intro tagged_division_Un_interval_real*)
show {(*c*, {*t*..*c*})} *tagged_division_of* {*a*..*c*} ∩ {*x*. *t* ≤ *x* ⋅ 1}
using ‹*t* ≤ *c*› **by** (*auto simp*: *eqs tagged_division_of_self_real*)
qed (*auto simp*: *eqs ptag*)
moreover

```
    have d1 fine p ∪ {(c, {t..c})}
      unfolding fine_def
    proof safe
      fix x K y
      assume (x,K) ∈ p and y ∈ K then show y ∈ d1 x
        by (metis Int_iff d3_def subsetD fineD pfine)
    next
      fix x assume x ∈ {t..c}
      then have dist c x < k
        using t(1) by (auto simp add: field_simps dist_real_def)
      with k show x ∈ d1 c
        unfolding d_def by auto
    qed
    ultimately have d1_fin: norm (?SUM(p ∪ {(c, {t..c})}) − integral {a..c}
f) < e/3
      using d1 by metis
    have SUMEQ: ?SUM(p ∪ {(c, {t..c})}) = (c − t) *R f c + ?SUM p
    proof −
      have ?SUM(p ∪ {(c, {t..c})}) = (content{t..c} *R f c) + ?SUM p
      proof (subst sum.union_disjoint)
        show p ∩ {(c, {t..c})} = {}
          using ‹t < c› pt by force
      qed (use p'(1) in auto)
      also have ... = (c − t) *R f c + ?SUM p
        using ‹t ≤ c› by auto
      finally show ?thesis .
    qed
    have c − k < t
      using ‹k>0› t(1) by (auto simp add: field_simps)
    moreover have k ≤ w
    proof (rule ccontr)
      assume ¬ k ≤ w
      then have c + (k + w) / 2 ∉ d c
        by (auto simp add: field_simps not_le not_less dist_real_def d_def)
      then have c + (k + w) / 2 ∉ ball c k
        using k by blast
      then show False
        using ‹0 < w› ‹¬ k ≤ w› dist_real_def by auto
    qed
    ultimately have cwt: c − w < t
      by (auto simp add: field_simps)
    have eq: integral {a..c} f − integral {a..t} f = −(((c − t) *R f c + ?SUM
p) −
            integral {a..c} f) + (?SUM p − integral {a..t} f) + (c − t) *R f c
      by auto
    have norm (integral {a..c} f − integral {a..t} f) < e/3 + e/3 + e/3
      unfolding eq
    proof (intro norm_triangle_lt add_strict_mono)
      show norm (− ((c − t) *R f c + ?SUM p − integral {a..c} f)) < e/3
```

```
        by (metis SUMEQ d1_fin norm_minus_cancel)
      show norm (?SUM p − integral {a..t} f) < e/3
        using d2_fin by blast
      show norm ((c − t) *R f c) < e/3
        using w cwt ‹t < c› by simp (simp add: field_simps)
    qed
    then show ?thesis by simp
  qed
 qed
qed

lemma indefinite_integral_continuous_right:
 fixes f :: real ⇒ 'a::banach
 assumes f integrable_on {a..b}
  and a ≤ c
  and c < b
  and e > 0
 obtains d where 0 < d
  and ∀ t. c ≤ t ∧ t < c + d ⟶ norm (integral {a..c} f − integral {a..t} f) <
e
proof −
 have intm: (λx. f (− x)) integrable_on {−b .. −a} − b < − c − c ≤ − a
  using assms by auto
 from indefinite_integral_continuous_left[OF intm ‹e>0›]
 obtain d where 0 < d
  and d: ⋀t. ⟦− c − d < t; t ≤ −c⟧
         ⟹ norm (integral {− b..− c} (λx. f (−x)) − integral {− b..t} (λx. f
(−x))) < e
  by metis
 let ?d = min d (b − c)
 show ?thesis
 proof (intro that[of ?d] allI impI)
  show 0 < ?d
   using ‹0 < d› ‹c < b› by auto
  fix t :: real
  assume t: c ≤ t ∧ t < c + ?d
  have ∗: integral {a..c} f = integral {a..b} f − integral {c..b} f
       integral {a..t} f = integral {a..b} f − integral {t..b} f
   using assms t by (auto simp: algebra_simps integral_combine)
  have (− c) − d < (− t) − t ≤ − c
   using t by auto
  from d[OF this] show norm (integral {a..c} f − integral {a..t} f) < e
   by (auto simp add: algebra_simps norm_minus_commute ∗)
 qed
qed

lemma indefinite_integral_continuous_1:
 fixes f :: real ⇒ 'a::banach
 assumes f integrable_on {a..b}
```

```
  shows continuous_on {a..b} (λx. integral {a..x} f)
proof −
  have ∃ d>0. ∀ x'∈{a..b}. dist x' x < d ⟶ dist (integral {a..x'} f) (integral
{a..x} f) < e
      if x: x ∈ {a..b} and e > 0 for x e :: real
  proof (cases a = b)
    case True
    with that show ?thesis by force
  next
    case False
    with x have a < b by force
    with x consider x = a | x = b | a < x x < b
      by force
    then show ?thesis
    proof cases
      case 1 then show ?thesis
      by (force simp: dist_norm algebra_simps intro: indefinite_integral_continuous_right
[OF assms _ ‹a < b› ‹e > 0›])
    next
      case 2 then show ?thesis
        by (force simp: dist_norm norm_minus_commute algebra_simps intro:
indefinite_integral_continuous_left [OF assms ‹a < b› _ ‹e > 0›])
    next
      case 3
      obtain d1 where 0 < d1
        and d1: ⋀t. ⟦x − d1 < t; t ≤ x⟧ ⟹ norm (integral {a..x} f − integral
{a..t} f) < e
        using 3 by (auto intro: indefinite_integral_continuous_left [OF assms ‹a
< x› _ ‹e > 0›])
      obtain d2 where 0 < d2
        and d2: ⋀t. ⟦x ≤ t; t < x + d2⟧ ⟹ norm (integral {a..x} f − integral
{a..t} f) < e
       using 3 by (auto intro: indefinite_integral_continuous_right [OF assms _
‹x < b› ‹e > 0›])
      show ?thesis
      proof (intro exI ballI conjI impI)
        show 0 < min d1 d2
          using ‹0 < d1› ‹0 < d2› by simp
        show dist (integral {a..y} f) (integral {a..x} f) < e
          if dist y x < min d1 d2 for y
          by (smt (verit) d1 d2 dist_norm dist_real_def norm_minus_commute
that)
      qed
    qed
  qed
  then show ?thesis
    by (auto simp: continuous_on_iff)
qed
```

lemma *indefinite_integral_continuous_1'*:
fixes *f*::*real* ⇒ *'a*::*banach*
assumes *f integrable_on* {*a*..*b*}
shows *continuous_on* {*a*..*b*} (λ*x*. *integral* {*x*..*b*} *f*)
proof −
have *integral* {*a*..*b*} *f* − *integral* {*a*..*x*} *f* = *integral* {*x*..*b*} *f* **if** *x* ∈ {*a*..*b*} **for** *x*
using *integral_combine*[*OF* _ _ *assms*, *of x*] *that*
by (*auto simp*: *algebra_simps*)
with _ **show** *?thesis*
by (*rule continuous_on_eq*) (*auto intro*!: *continuous_intros indefinite_integral_continuous_1 assms*)
qed

theorem *integral_has_vector_derivative'*:
fixes *f* :: *real* ⇒ *'b*::*banach*
assumes *continuous_on* {*a*..*b*} *f*
and *x* ∈ {*a*..*b*}
shows ((λ*u*. *integral* {*u*..*b*} *f*) *has_vector_derivative* − *f x*) (*at x within* {*a*..*b*})
proof −
have ∗: *integral* {*x*..*b*} *f* = *integral* {*a* .. *b*} *f* − *integral* {*a* .. *x*} *f* **if** *a* ≤ *x x* ≤ *b* **for** *x*
using *integral_combine*[*of a x b* **for** *x*, *OF that integrable_continuous_real*[*OF assms*(*1*)]]
by (*simp add*: *algebra_simps*)
show *?thesis*
using ‹*x* ∈ _› ∗
by (*rule has_vector_derivative_transform*)
(*auto intro*!: *derivative_eq_intros assms integral_has_vector_derivative*)
qed

lemma *integral_has_real_derivative'*:
assumes *continuous_on* {*a*..*b*} *g*
assumes *t* ∈ {*a*..*b*}
shows ((λ*x*. *integral* {*x*..*b*} *g*) *has_real_derivative* −*g t*) (*at t within* {*a*..*b*})
using *integral_has_vector_derivative'*[*OF assms*]
by (*auto simp*: *has_real_derivative_iff_has_vector_derivative*)

7.14.31 This doesn't directly involve integration, but that gives an easy proof

lemma *has_derivative_zero_unique_strong_interval*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes *finite k*
and *contf*: *continuous_on* {*a*..*b*} *f*
and *f a* = *y*
and *fder*: ⋀*x*. *x* ∈ {*a*..*b*} − *k* ⟹ (*f has_derivative* (λ*h*. *0*)) (*at x within* {*a*..*b*})
and *x*: *x* ∈ {*a*..*b*}
shows *f x* = *y*
proof −

```
  have a ≤ b a ≤ x
    using assms by auto
  have ((λx. 0::'a) has_integral f x − f a) {a..x}
  proof (rule fundamental_theorem_of_calculus_interior_strong[OF ‹finite k› ‹a
≤ x›]; clarify?)
    have {a..x} ⊆ {a..b}
      using x by auto
    then show continuous_on {a..x} f
      by (rule continuous_on_subset[OF contf])
    show (f has_vector_derivative 0) (at z) if z: z ∈ {a<..<x} and notin: z ∉ k
for z
      unfolding has_vector_derivative_def
    proof (simp add: at_within_open[OF z, symmetric])
      show (f has_derivative (λx. 0)) (at z within {a<..<x})
        by (rule has_derivative_subset [OF fder]) (use x z notin in auto)
    qed
  qed
  from has_integral_unique[OF has_integral_0 this]
  show ?thesis
    unfolding assms by auto
qed
```

7.14.32 Generalize a bit to any convex set

```
lemma has_derivative_zero_unique_strong_convex:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  assumes convex S finite K
    and contf: continuous_on S f
    and c ∈ S f c = y
    and derf: ⋀x. x ∈ S − K ⟹ (f has_derivative (λh. 0)) (at x within S)
    and x ∈ S
  shows f x = y
proof (cases x = c)
  case True with ‹f c = y› show ?thesis
    by blast
next
  case False
  let ?φ = λu. (1 − u) *R c + u *R x
  have contf': continuous_on {0 ..1} (f ∘ ?φ)
  proof (rule continuous_intros continuous_on_subset[OF contf])+
    show (λu. (1 − u) *R c + u *R x) ` {0..1} ⊆ S
      using ‹convex S› ‹x ∈ S› ‹c ∈ S› by (auto simp add: convex_alt alge-
bra_simps)
  qed
  have t = u if ?φ t = ?φ u for t u
  proof −
    from that have (t − u) *R x = (t − u) *R c
      by (auto simp add: algebra_simps)
    then show ?thesis
```

```
    using ‹x ≠ c› by auto
  qed
  then have eq: (SOME t. ?φ t = ?φ u) = u for u
    by blast
  then have (?φ −‘ K) ⊆ (λz. SOME t. ?φ t = z) ‘ K
    by (clarsimp simp: image_iff) (metis (no_types) eq)
  then have fin: finite (?φ −‘ K)
    by (rule finite_surj[OF ‹finite K›])

  have derf': ((λu. f (?φ u)) has_derivative (λh. 0)) (at t within {0..1})
            if t ∈ {0..1} − {t. ?φ t ∈ K} for t
  proof −
    have df: (f has_derivative (λh. 0)) (at (?φ t) within ?φ ‘ {0..1})
      using ‹convex S› ‹x ∈ S› ‹c ∈ S› that
      by (auto simp add: convex_alt algebra_simps intro: has_derivative_subset
[OF derf])
    have (f ∘ ?φ has_derivative (λx. 0) ∘ (λz. (0 − z *R c) + z *R x)) (at t within
{0..1})
      by (rule derivative_eq_intros df | simp)+
    then show ?thesis
      unfolding o_def .
  qed
  have (f ∘ ?φ) 1 = y
    apply (rule has_derivative_zero_unique_strong_interval[OF fin contf'])
    unfolding o_def using ‹f c = y› derf' by auto
  then show ?thesis
    by auto
qed
```

Also to any open connected set with finite set of exceptions. Could generalize to locally convex set with limpt-free set of exceptions.

```
lemma has_derivative_zero_unique_strong_connected:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  assumes connected S
    and open S
    and finite K
    and contf: continuous_on S f
    and c ∈ S
    and f c = y
    and derf: ⋀x. x ∈ S − K ⟹ (f has_derivative (λh. 0)) (at x within S)
    and x ∈ S
  shows f x = y
proof −
  have ∃e>0. ball x e ⊆ (S ∩ f −‘ {f x}) if x ∈ S for x
  proof −
    obtain e where 0 < e and e: ball x e ⊆ S
      using ‹x ∈ S› ‹open S› open_contains_ball by blast
    have ball x e ⊆ {u ∈ S. f u ∈ {f x}}
    proof safe
```

```
      fix y
      assume y: y ∈ ball x e
      then show y ∈ S
        using e by auto
      show f y = f x
      proof (rule has_derivative_zero_unique_strong_convex[OF convex_ball ‹finite K›])
        show continuous_on (ball x e) f
          using contf continuous_on_subset e by blast
        show (f has_derivative (λh. 0)) (at u within ball x e)
            if u ∈ ball x e − K for u
          by (metis Diff_iff contra_subsetD derf e has_derivative_subset that)
      qed (use y e ‹0 < e› in auto)
    qed
    then show ∃e>0. ball x e ⊆ (S ∩ f −‘ {f x})
      using ‹0 < e› by blast
  qed
  then have openin (top_of_set S) (S ∩ f −‘ {y})
    by (auto intro!: open_openin_trans[OF ‹open S›] simp: open_contains_ball)
  moreover have closedin (top_of_set S) (S ∩ f −‘ {y})
    by (force intro!: continuous_closedin_preimage [OF contf])
  ultimately have (S ∩ f −‘ {y}) = {} ∨ (S ∩ f −‘ {y}) = S
    using ‹connected S› by (simp add: connected_clopen)
  then show ?thesis
    using ‹x ∈ S› ‹f c = y› ‹c ∈ S› by auto
qed

lemma has_derivative_zero_connected_constant_on:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  assumes connected S open S finite K continuous_on S f
      and ∀x∈S−K. (f has_derivative (λh. 0)) (at x within S)
  shows   f constant_on S
 by (smt (verit, best) assms constant_on_def has_derivative_zero_unique_strong_connected)

lemma DERIV_zero_connected_constant_on:
  fixes f :: 'a::{real_normed_field,euclidean_space} ⇒ 'a
  assumes ∗: connected S open S finite K continuous_on S f
      and 0: ∀x∈S−K. DERIV f x :> 0
  shows   f constant_on S
  using has_derivative_zero_connected_constant_on [OF ∗] 0
  by (metis has_derivative_at_withinI has_field_derivative_def lambda_zero)

lemma DERIV_zero_connected_constant:
  fixes f :: 'a::{real_normed_field,euclidean_space} ⇒ 'a
  assumes connected S and open S and finite K and continuous_on S f
      and ∀x∈S−K. DERIV f x :> 0
    obtains c where ⋀x. x ∈ S ⟹ f(x) = c
  by (metis DERIV_zero_connected_constant_on [OF assms] constant_on_def)
```

lemma *has_field_derivative_0_imp_constant_on*:
fixes *f* :: *'a*::{*real_normed_field*,*euclidean_space*} ⇒ *'a*
assumes ⋀*z*. *z* ∈ *S* ⟹ (*f has_field_derivative 0*) (*at z*) **and** *S*: *connected S open S*
shows *f constant_on S*
using *DERIV_zero_connected_constant_on* [**where** *K*=*Basis*]
by (*metis DERIV_isCont Diff_iff assms continuous_at_imp_continuous_on eucl.finite_Basis*)

7.14.33 Integrating characteristic function of an interval

lemma *has_integral_restrict_open_subinterval*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
assumes *intf*: (*f has_integral i*) (*cbox c d*)
and *cb*: *cbox c d* ⊆ *cbox a b*
shows ((λ*x*. *if x* ∈ *box c d then f x else 0*) *has_integral i*) (*cbox a b*)
proof (*cases cbox c d* = {})
case *True*
then have *box c d* = {}
by (*metis bot.extremum_uniqueI box_subset_cbox*)
then show *?thesis*
using *True intf* **by** *auto*
next
case *False*
then obtain *p* **where** *pdiv*: *p division_of cbox a b* **and** *inp*: *cbox c d* ∈ *p*
using *cb partial_division_extend_1* **by** *blast*
define *g* **where** [*abs_def*]: *g x* = (*if x* ∈*box c d then f x else 0*) **for** *x*
interpret *operative lift_option plus Some* (*0* :: *'b*)
λ*i*. *if g integrable_on i then Some* (*integral i g*) *else None*
by (*fact operative_integralI*)
note *operat* = *division* [*OF pdiv*, *symmetric*]
show *?thesis*
proof (*cases* (*g has_integral i*) (*cbox a b*))
case *True* **then show** *?thesis*
by (*simp add*: *g_def*)
next
case *False*
have *iterate*:*F* (λ*i*. *if g integrable_on i then Some* (*integral i g*) *else None*) (*p* − {*cbox c d*}) = *Some 0*
proof (*intro neutral ballI*)
fix *x*
assume *x*: *x* ∈ *p* − {*cbox c d*}
then have *x* ∈ *p*
by *auto*
then obtain *u v* **where** *uv*: *x* = *cbox u v*
using *pdiv* **by** *blast*
have *interior x* ∩ *interior* (*cbox c d*) = {}
using *pdiv inp x* **by** *blast*
then have (*g has_integral 0*) *x*

```
      unfolding uv using has_integral_spike_interior[where f=λx. 0]
    by (metis (no_types, lifting) disjoint_iff_not_equal g_def has_integral_0_eq
interior_cbox)
    then show (if g integrable_on x then Some (integral x g) else None) = Some
0
      by auto
  qed
  interpret comm_monoid_set lift_option plus Some (0 :: 'b)
  by (intro comm_monoid_set.intro comm_monoid_lift_option add.comm_monoid_axioms)
  have intg: g integrable_on cbox c d
    using integrable_spike_interior[where f=f]
    by (meson g_def has_integral_integrable intf)
  moreover have integral (cbox c d) g = i
    by (meson g_def has_integral_iff has_integral_spike_interior intf)
  ultimately have F (λA. if g integrable_on A then Some (integral A g) else
None) p = Some i
      by (metis (full_types, lifting) division_of_finite inp iterate pdiv remove
right_neutral)
  with False show ?thesis
    by (metis integrable_integral not_None_eq operat option.inject)
 qed
qed


lemma has_integral_restrict_closed_subinterval:
 fixes f :: 'a::euclidean_space ⇒ 'b::banach
 assumes (f has_integral i) (cbox c d)
   and cbox c d ⊆ cbox a b
 shows ((λx. if x ∈ cbox c d then f x else 0) has_integral i) (cbox a b)
proof −
 note has_integral_restrict_open_subinterval[OF assms]
 note ∗ = has_integral_spike[OF negligible_frontier_interval _ this]
 show ?thesis
   by (rule ∗[of c d]) (use box_subset_cbox[of c d] in auto)
qed

lemma has_integral_restrict_closed_subintervals_eq:
 fixes f :: 'a::euclidean_space ⇒ 'b::banach
 assumes cbox c d ⊆ cbox a b
 shows ((λx. if x ∈ cbox c d then f x else 0) has_integral i) (cbox a b) ⟷ (f
has_integral i) (cbox c d)
 (is ?l = ?r)
proof (cases cbox c d = {})
 case False
 let ?g = λx. if x ∈ cbox c d then f x else 0
 show ?thesis
 proof
   assume ?l
   then have ?g integrable_on cbox c d
```

```
    using assms has_integral_integrable integrable_subinterval by blast
  then have f integrable_on cbox c d
    by (rule integrable_eq) auto
  moreover then have i = integral (cbox c d) f
    by (meson ‹((λx. if x ∈ cbox c d then f x else 0) has_integral i) (cbox
a b)› assms has_integral_restrict_closed_subinterval has_integral_unique inte-
grable_integral)
  ultimately show ?r by auto
 next
  assume ?r then show ?l
    by (rule has_integral_restrict_closed_subinterval[OF _ assms])
 qed
qed auto
```

Hence we can apply the limit process uniformly to all integrals.

```
lemma has_integral':
 fixes f :: 'n::euclidean_space ⇒ 'a::banach
 shows (f has_integral i) S ⟷
   (∀e>0. ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
     (∃z. ((λx. if x ∈ S then f(x) else 0) has_integral z) (cbox a b) ∧ norm(z −
i) < e))
 (is ?l ⟷ (∀e>0. ?r e))
proof (cases ∃a b. S = cbox a b)
 case False then show ?thesis
   by (simp add: has_integral_alt)
next
 case True
 then obtain a b where S: S = cbox a b
   by blast
 obtain B where  0 < B and B: ⋀x. x ∈ cbox a b ⟹ norm x ≤ B
   using bounded_cbox[unfolded bounded_pos] by blast
 show ?thesis
 proof safe
   fix e :: real
   assume ?l and e > 0
   have ((λx. if x ∈ S then f x else 0) has_integral i) (cbox c d)
     if ball 0 (B+1) ⊆ cbox c d for c d
     unfolding S using B that
     by (force intro: ‹?l›[unfolded S] has_integral_restrict_closed_subinterval)
   then show ?r e
     by (meson ‹0 < B› ‹0 < e› add_pos_pos le_less_trans zero_less_one
norm_pths(2))
 next
   assume as: ∀e>0. ?r e
   then obtain C
     where C: ⋀a b. ball 0 C ⊆ cbox a b ⟹
           ∃z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b)
     by (meson zero_less_one)
   define c :: 'n where c = (∑i∈Basis. (− max B C) *R i)
```

```
    define d :: 'n where d = (∑ i∈Basis. max B C *R i)
    have c • i ≤ x • i ∧ x • i ≤ d • i if norm x ≤ B i ∈ Basis for x i
      using that and Basis_le_norm[OF ‹i∈Basis›, of x]
      by (auto simp add: field_simps sum_negf c_def d_def)
    then have c_d: cbox a b ⊆ cbox c d
      by (meson B mem_box(2) subsetI)
    have c • i ≤ x • i ∧ x • i ≤ d • i
      if x: norm (0 − x) < C and i: i ∈ Basis for x i
       using Basis_le_norm[OF i, of x] x i by (auto simp: sum_negf c_def d_def)
      then have ball 0 C ⊆ cbox c d
        by (auto simp: mem_box dist_norm)
    with C obtain y where y: (f has_integral y) (cbox a b)
      using c_d has_integral_restrict_closed_subintervals_eq S by blast
    have y = i
    proof (rule ccontr)
      assume y ≠ i
      then have 0 < norm (y − i)
        by auto
      from as[rule_format,OF this]
      obtain C where C: ⋀a b. ball 0 C ⊆ cbox a b ⟹
          ∃ z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b) ∧ norm (z−i)
< norm (y−i)
        by auto
      define c :: 'n where c = (∑ i∈Basis. (− max B C) *R i)
      define d :: 'n where d = (∑ i∈Basis. max B C *R i)
      have c • i ≤ x • i ∧ x • i ≤ d • i
        if norm x ≤ B and i ∈ Basis for x i
        using that Basis_le_norm[of i x] by (auto simp add: field_simps sum_negf
c_def d_def)
        then have c_d: cbox a b ⊆ cbox c d
        by (simp add: B mem_box(2) subset_eq)
      have c • i ≤ x • i ∧ x • i ≤ d • i if norm (0 − x) < C and i ∈ Basis for x i
        using Basis_le_norm[of i x] that by (auto simp: sum_negf c_def d_def)
      then have ball 0 C ⊆ cbox c d
        by (auto simp: mem_box dist_norm)
      with C obtain z where z: (f has_integral z) (cbox a b) norm (z−i) < norm
(y−i)
        using has_integral_restrict_closed_subintervals_eq[OF c_d] S by blast
      moreover then have z = y
        by (blast intro: has_integral_unique[OF _ y])
      ultimately show False
        by auto
    qed
    then show ?l
      using y by (auto simp: S)
  qed
qed

lemma has_integral_le:
```

fixes $f :: 'n::euclidean\_space \Rightarrow real$
assumes *fg*: $(f\ has\_integral\ i)\ S\ (g\ has\_integral\ j)\ S$
and *le*: $\bigwedge x.\ x \in S \Longrightarrow f\ x \leq g\ x$
shows $i \leq j$
using *has_integral_component_le*[*OF _ fg*, *of 1*] *le* **by** *auto*

lemma *integral_le*:
fixes $f :: 'n::euclidean\_space \Rightarrow real$
assumes *f integrable_on S*
and *g integrable_on S*
and $\bigwedge x.\ x \in S \Longrightarrow f\ x \leq g\ x$
shows *integral S f* $\leq$ *integral S g*
by (*meson assms has_integral_le integrable_integral*)

lemma *has_integral_nonneg*:
fixes $f :: 'n::euclidean\_space \Rightarrow real$
assumes $(f\ has\_integral\ i)\ S$ **and** $\bigwedge x.\ x \in S \Longrightarrow 0 \leq f\ x$
shows $0 \leq i$
using *assms has_integral_0 has_integral_le* **by** *blast*

lemma *integral_nonneg*:
fixes $f :: 'n::euclidean\_space \Rightarrow real$
assumes *f*: *f integrable_on S* **and** *0*: $\bigwedge x.\ x \in S \Longrightarrow 0 \leq f\ x$
shows *0* $\leq$ *integral S f*
by (*rule has_integral_nonneg*[*OF f*[*unfolded has_integral_integral*] *0*])

Hence a general restriction property.

lemma *has_integral_restrict* [*simp*]:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
assumes $S \subseteq T$
shows $((\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ T \longleftrightarrow (f\ has\_integral\ i)\ S$
proof −
have $*$: $\bigwedge x.\ (if\ x \in T\ then\ if\ x \in S\ then\ f\ x\ else\ 0\ else\ 0) = (if\ x \in S\ then\ f\ x\ else\ 0)$
using *assms* **by** *auto*
show *?thesis*
apply (*subst*(*2*) *has_integral'*)
apply (*subst has_integral'*)
apply (*simp add*: $*$)
done
qed

corollary *has_integral_restrict_UNIV*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
shows $((\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ UNIV \longleftrightarrow (f\ has\_integral\ i)\ s$
by *auto*

lemma *has_integral_restrict_Int*:

fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
shows $((\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ T \longleftrightarrow (f\ has\_integral\ i)\ (S \cap T)$
proof −
have $((\lambda x.\ if\ x \in T\ then\ if\ x \in S\ then\ f\ x\ else\ 0\ else\ 0)\ has\_integral\ i)\ UNIV =$
$((\lambda x.\ if\ x \in S \cap T\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ UNIV$
by (*rule has_integral_cong*) *auto*
then show *?thesis*
using *has_integral_restrict_UNIV* **by** *fastforce*
qed

lemma *integral_restrict_Int*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
shows $integral\ T\ (\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0) = integral\ (S \cap T)\ f$
by (*metis* (*no_types*, *lifting*) *has_integral_cong has_integral_restrict_Int integrable_integral integral_unique not_integrable_integral*)

lemma *integrable_restrict_Int*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
shows $(\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ integrable\_on\ T \longleftrightarrow f\ integrable\_on\ (S \cap T)$
using *has_integral_restrict_Int* **by** *fastforce*

lemma *has_integral_on_superset*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes $(f\ has\_integral\ i)\ S$
and $\bigwedge x.\ x \notin S \Longrightarrow f\ x = 0$
and $S \subseteq T$
shows $(f\ has\_integral\ i)\ T$
by (*smt* (*verit*, *ccfv_SIG*) *assms has_integral_cong has_integral_restrict*)

lemma *integrable_on_superset*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes $f\ integrable\_on\ S$ **and** $\bigwedge x.\ x \notin S \Longrightarrow f\ x = 0$ **and** $S \subseteq t$
shows $f\ integrable\_on\ t$
by (*meson assms has_integral_on_superset integrable_integral integrable_on_def*)

lemma *integral_subset_negligible*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
assumes $S \subseteq T\ negligible\ (T - S)$
shows $integral\ S\ f = integral\ T\ f$
proof −
have $integral\ T\ f = integral\ T\ (\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)$
by (*rule integral_spike*[*of* $T - S$]) (*use assms* **in** *auto*)
also have $\ldots = integral\ (S \cap T)\ f$
by (*subst integral_restrict_Int*) *auto*
also have $S \cap T = S$ **using** *assms* **by** *auto*
finally show *?thesis* **..**
qed

lemma *integral_restrict_UNIV*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
shows $integral\ UNIV\ (\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0) = integral\ S\ f$
by (*simp add*: *integral_restrict_Int*)

lemma *integrable_restrict_UNIV*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
shows $(\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)\ integrable\_on\ UNIV \longleftrightarrow f\ integrable\_on\ s$
unfolding *integrable_on_def*
by *auto*

lemma *has_integral_subset_component_le*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
assumes k: $k \in Basis$
and $S \subseteq T\ (f\ has\_integral\ i)\ S\ (f\ has\_integral\ j)\ T\ \bigwedge x.\ x \in T \Longrightarrow 0 \le f(x) \cdot k$
shows $i \cdot k \le j \cdot k$
proof −
have §: $((\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ UNIV$
$((\lambda x.\ if\ x \in T\ then\ f\ x\ else\ 0)\ has\_integral\ j)\ UNIV$
by (*simp_all add*: *assms*)
show *?thesis*
using *assms* **by** (*force intro*!: *has_integral_component_le*[*OF k* §])
qed

7.14.34 Integrals on set differences

lemma *has_integral_setdiff*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::banach$
assumes S: $(f\ has\_integral\ i)\ S$ **and** T: $(f\ has\_integral\ j)\ T$
and *neg*: $negligible\ (T - S)$
shows $(f\ has\_integral\ (i - j))\ (S - T)$
proof −
show *?thesis*
unfolding *has_integral_restrict_UNIV* [*symmetric*, *of f*]
proof (*rule has_integral_spike* [*OF neg*])
have *eq*: $(\lambda x.\ (if\ x \in S\ then\ f\ x\ else\ 0) - (if\ x \in T\ then\ f\ x\ else\ 0)) =$
$(\lambda x.\ if\ x \in T - S\ then - f\ x\ else\ if\ x \in S - T\ then\ f\ x\ else\ 0)$
by (*force simp add*:)
have $((\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ has\_integral\ i)\ UNIV$
$((\lambda x.\ if\ x \in T\ then\ f\ x\ else\ 0)\ has\_integral\ j)\ UNIV$
using *S T has_integral_restrict_UNIV* **by** *auto*
from *has_integral_diff* [*OF this*]
show $((\lambda x.\ if\ x \in T - S\ then - f\ x\ else\ if\ x \in S - T\ then\ f\ x\ else\ 0)$
$has\_integral\ i - j)\ UNIV$
by (*simp add*: *eq*)
qed *force*
qed

lemma *integral_setdiff*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
assumes *f integrable_on S f integrable_on T negligible*(*T* − *S*)
shows *integral* (*S* − *T*) *f* = *integral S f* − *integral T f*
by (*rule integral_unique*) (*simp add*: *assms has_integral_setdiff integrable_integral*)

lemma *integrable_setdiff*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*banach*
assumes (*f has_integral i*) *S* (*f has_integral j*) *T negligible* (*T* − *S*)
shows *f integrable_on* (*S* − *T*)
using *has_integral_setdiff* [*OF assms*]
by (*simp add*: *has_integral_iff*)

lemma *negligible_setdiff* [*simp*]: *T* ⊆ *S* ⟹ *negligible* (*T* − *S*)
by (*metis Diff_eq_empty_iff negligible_empty*)

lemma *negligible_on_intervals*: *negligible s* ⟷ (∀ *a b*. *negligible*(*s* ∩ *cbox a b*))
(**is** *?l* ⟷ *?r*)
proof
assume *R*: *?r*
show *?l*
unfolding *negligible_def*
by (*metis Diff_iff Int_iff R has_integral_negligible indicator_simps*(*2*))
qed (*simp add*: *negligible_Int*)

lemma *negligible_translation*:
assumes *negligible S*
shows *negligible* ((+) *c* ‘ *S*)
proof −
have *inj*: *inj* ((+) *c*)
by *simp*
show *?thesis*
using *assms*
proof (*clarsimp simp*: *negligible_def*)
fix *a b*
assume ∀ *x y*. (*indicator S has_integral 0*) (*cbox x y*)
then have ∗: (*indicator S has_integral 0*) (*cbox* (*a*−*c*) (*b*−*c*))
by (*meson Diff_iff assms has_integral_negligible indicator_simps*(*2*))
have *eq*: *indicator* ((+) *c* ‘ *S*) = (λ*x*. *indicator S* (*x* − *c*))
by (*force simp add*: *indicator_def*)
show (*indicator* ((+) *c* ‘ *S*) *has_integral 0*) (*cbox a b*)
using *has_integral_affinity* [*OF* ∗, *of 1* −*c*]
cbox_translation [*of c* −*c*+*a* −*c*+*b*]
by (*simp add*: *eq*) (*simp add*: *ac_simps*)
qed
qed

lemma *negligible_translation_rev*:
assumes *negligible* ((+) *c* ‘ *S*)

shows *negligible S*
by (*metis negligible_translation* [*OF assms*, *of* $-c$] *translation_galois*)

lemma *negligible_atLeastAtMostI*: $b \leq a \Longrightarrow$ *negligible* $\{a..(b::real)\}$
using *negligible_insert* **by** *fastforce*

lemma *has_integral_spike_set_eq*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes *negligible* $\{x \in S - T.\ f\ x \neq 0\}$ *negligible* $\{x \in T - S.\ f\ x \neq 0\}$
shows $(f\ has\_integral\ y)\ S \longleftrightarrow (f\ has\_integral\ y)\ T$
proof −
have $((\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)\ has\_integral\ y)\ UNIV =$
$((\lambda x.\ if\ x \in T\ then\ f\ x\ else\ 0)\ has\_integral\ y)\ UNIV$
proof (*rule has_integral_spike_eq*)
show *negligible* $(\{x \in S - T.\ f\ x \neq 0\} \cup \{x \in T - S.\ f\ x \neq 0\})$
by (*rule negligible_Un* [*OF assms*])
qed *auto*
then show *?thesis*
by (*simp add*: *has_integral_restrict_UNIV*)
qed

corollary *integral_spike_set*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes *negligible* $\{x \in S - T.\ f\ x \neq 0\}$ *negligible* $\{x \in T - S.\ f\ x \neq 0\}$
shows *integral S f* = *integral T f*
using *has_integral_spike_set_eq* [*OF assms*]
by (*metis eq_integralD integral_unique*)

lemma *integrable_spike_set*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes *f*: *f integrable_on S* **and** *neg*: *negligible* $\{x \in S - T.\ f\ x \neq 0\}$ *negligible* $\{x \in T - S.\ f\ x \neq 0\}$
shows *f integrable_on T*
using *has_integral_spike_set_eq* [*OF neg*] *f* **by** *blast*

lemma *integrable_spike_set_eq*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes *negligible* $((S - T) \cup (T - S))$
shows *f integrable_on S* $\longleftrightarrow$ *f integrable_on T*
by (*blast intro*: *integrable_spike_set assms negligible_subset*)

lemma *integrable_on_insert_iff*: *f integrable_on* (*insert x X*) $\longleftrightarrow$ *f integrable_on X*
for $f::\_ \Rightarrow 'a::banach$
by (*rule integrable_spike_set_eq*) (*auto simp*: *insert_Diff_if*)

lemma *has_integral_interior*:
fixes $f :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
shows *negligible*(*frontier S*) $\Longrightarrow$ (*f has_integral y*) (*interior S*) $\longleftrightarrow$ (*f has_integral*

y) *S*
by (*rule has_integral_spike_set_eq* [*OF empty_imp_negligible negligible_subset*])
(*use interior_subset* **in** ‹*auto simp*: *frontier_def closure_def*›)

lemma *has_integral_closure*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows *negligible*(*frontier S*) ⟹ (*f has_integral y*) (*closure S*) ⟷ (*f has_integral*
y) *S*
by (*rule has_integral_spike_set_eq* [*OF negligible_subset empty_imp_negligible*])
(*auto simp*: *closure_Un_frontier*)

lemma *has_integral_open_interval*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows (*f has_integral y*) (*box a b*) ⟷ (*f has_integral y*) (*cbox a b*)
unfolding *interior_cbox* [*symmetric*]
by (*metis frontier_cbox has_integral_interior negligible_frontier_interval*)

lemma *integrable_on_open_interval*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows *f integrable_on box a b* ⟷ *f integrable_on cbox a b*
by (*simp add*: *has_integral_open_interval integrable_on_def*)

lemma *integral_open_interval*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
shows *integral*(*box a b*) *f* = *integral*(*cbox a b*) *f*
by (*metis has_integral_integrable_integral has_integral_open_interval not_integrable_integral*)

lemma *integrable_on_open_interval_real*:
fixes *f* :: *real* ⇒ *'b* :: *banach*
shows *f integrable_on* {*a*<..<*b*} ⟷ *f integrable_on* {*a*..*b*}
using *integrable_on_open_interval*[*of f a b*] **by** *simp*

lemma *integral_open_interval_real*:
integral {*a*..*b*} (*f* :: *real* ⇒ *'a* :: *banach*) = *integral* {*a*<..<(*b*::*real*)} *f*
using *integral_open_interval*[*of a b f*] **by** *simp*

lemma *has_integral_Icc_iff_Ioo*:
fixes *f* :: *real* ⇒ *'a* :: *banach*
shows (*f has_integral I*) {*a*..*b*} ⟷ (*f has_integral I*) {*a*<..<*b*}
by (*metis box_real*(*1*) *cbox_interval has_integral_open_interval*)

lemma *integrable_on_Icc_iff_Ioo*:
fixes *f* :: *real* ⇒ *'a* :: *banach*
shows *f integrable_on* {*a*..*b*} ⟷ *f integrable_on* {*a*<..<*b*}
using *has_integral_Icc_iff_Ioo* **by** *blast*

7.14.35 More lemmas that are useful later

lemma *has_integral_subset_le*:

fixes $f :: 'n::euclidean\_space \Rightarrow real$
assumes $s \subseteq t$
and $(f\ has\_integral\ i)\ s$
and $(f\ has\_integral\ j)\ t$
and $\forall x{\in}t.\ 0 \le f\ x$
shows $i \le j$
using *assms has_integral_subset_component_le*[*OF _ assms*(1), *of 1 f i j*]
by *auto*

lemma *integral_subset_component_le*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
assumes $k \in Basis$
and $s \subseteq t$
and $f\ integrable\_on\ s$
and $f\ integrable\_on\ t$
and $\forall x \in t.\ 0 \le f\ x \cdot k$
shows $(integral\ s\ f){\cdot}k \le (integral\ t\ f){\cdot}k$
by (*meson assms has_integral_subset_component_le integrable_integral*)

lemma *integral_subset_le*:
fixes $f :: 'n::euclidean\_space \Rightarrow real$
assumes $s \subseteq t$
and $f\ integrable\_on\ s$
and $f\ integrable\_on\ t$
and $\forall x \in t.\ 0 \le f\ x$
shows $integral\ s\ f \le integral\ t\ f$
using *assms has_integral_subset_le* **by** *blast*

lemma *has_integral_alt'*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
shows $(f\ has\_integral\ i)\ s \longleftrightarrow$
$(\forall a\ b.\ (\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)\ integrable\_on\ cbox\ a\ b) \wedge$
$(\forall e{>}0.\ \exists B{>}0.\ \forall a\ b.\ ball\ 0\ B \subseteq cbox\ a\ b \longrightarrow$
$norm\ (integral\ (cbox\ a\ b)\ (\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0) - i) < e)$
(**is** $?l = ?r$)
proof
assume *rhs*: $?r$
show $?l$
proof (*subst has_integral'*, *intro allI impI*)
fix $e::real$
assume $e > 0$
from *rhs*[*THEN conjunct2*,*rule_format*,*OF this*]
show $\exists B{>}0.\ \forall a\ b.\ ball\ 0\ B \subseteq cbox\ a\ b \longrightarrow$
$(\exists z.\ ((\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)\ has\_integral\ z)$
$(cbox\ a\ b) \wedge norm\ (z - i) < e)$
by (*simp add*: *has_integral_iff rhs*)
qed
next
let $?\Phi = \lambda e\ a\ b.\ \exists z.\ ((\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)\ has\_integral\ z)\ (cbox\ a\ b) \wedge$

norm (*z* − *i*) < *e*
assume *?l*
then have *lhs*: ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶ *?Φ e a b* **if** *e* > *0* **for** *e*
using *that has_integral'*[*of f*] **by** *auto*
let *?f* = λ*x*. *if x* ∈ *s then f x else 0*
show *?r*
proof (*intro conjI allI impI*)
fix *a b* :: *'n*
from *lhs*[*OF zero_less_one*]
obtain *B* **where** *0* < *B* **and** *B*: ⋀*a b*. *ball 0 B* ⊆ *cbox a b* ⟹ *?Φ 1 a b*
by *blast*
let *?a* = ∑ *i*∈*Basis*. *min* (*a*•*i*) (−*B*) ∗$_R$ *i*::*'n*
let *?b* = ∑ *i*∈*Basis*. *max* (*b*•*i*) *B* ∗$_R$ *i*::*'n*
show *?f integrable_on cbox a b*
proof (*rule integrable_subinterval*[*of _ ?a ?b*])
have *?a* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *?b* • *i* **if** *norm* (*0* − *x*) < *B i* ∈ *Basis* **for** *x i*
using *Basis_le_norm*[*of i x*] *that* **by** (*auto simp add:field_simps*)
then have *ball 0 B* ⊆ *cbox ?a ?b*
by (*auto simp*: *mem_box dist_norm*)
then show *?f integrable_on cbox ?a ?b*
unfolding *integrable_on_def* **using** *B* **by** *blast*
show *cbox a b* ⊆ *cbox ?a ?b*
by (*force simp*: *mem_box*)
qed
fix *e* :: *real*
assume *e* > *0*
with *lhs* **show** ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
norm (*integral* (*cbox a b*) (λ*x*. *if x* ∈ *s then f x else 0*) − *i*) < *e*
by (*metis* (*no_types*, *lifting*) *has_integral_integrable_integral*)
qed
qed

7.14.36 Continuity of the integral (for a 1-dimensional interval)

lemma *integrable_alt*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
shows *f integrable_on s* ⟷
(∀ *a b*. (λ*x*. *if x* ∈ *s then f x else 0*) *integrable_on cbox a b*) ∧
(∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b c d*. *ball 0 B* ⊆ *cbox a b* ∧ *ball 0 B* ⊆ *cbox c d* ⟶
norm (*integral* (*cbox a b*) (λ*x*. *if x* ∈ *s then f x else 0*) −
integral (*cbox c d*) (λ*x*. *if x* ∈ *s then f x else 0*)) < *e*)
(**is** *?l* = *?r*)
proof
let *?F* = λ*x*. *if x* ∈ *s then f x else 0*
assume *?l*
then obtain *y* **where** *intF*: ⋀*a b*. *?F integrable_on cbox a b*
and *y*: ⋀*e*. *0* < *e* ⟹
∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶ *norm* (*integral* (*cbox a b*) *?F* −

```
y) < e
    unfolding integrable_on_def has_integral_alt'[of f] by auto
  show ?r
  proof (intro conjI allI impI intF)
    fix e::real
    assume e > 0
    then have e/2 > 0
      by auto
    obtain B where 0 < B
      and B: ⋀a b. ball 0 B ⊆ cbox a b ⟹ norm (integral (cbox a b) ?F − y) <
e/2
      using ‹0 < e/2› y by blast
    show ∃B>0. ∀a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d ⟶
                norm (integral (cbox a b) ?F − integral (cbox c d) ?F) < e
    proof (intro conjI exI impI allI, rule ‹0 < B›)
      fix a b c d::'n
      assume sub: ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d
      show norm (integral (cbox a b) ?F − integral (cbox c d) ?F) < e
        using sub by (auto intro: norm_triangle_half_l dest: B)
    qed
  qed
next
  let ?F = λx. if x ∈ s then f x else 0
  assume rhs: ?r
  let ?cube = λn. cbox (∑ i∈Basis. − real n *R i::'n) (∑ i∈Basis. real n *R i)
  have Cauchy (λn. integral (?cube n) ?F)
    unfolding Cauchy_def
  proof (intro allI impI)
    fix e::real
    assume e > 0
    with rhs obtain B where 0 < B
      and B: ⋀a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d
                  ⟹ norm (integral (cbox a b) ?F − integral (cbox c d) ?F) < e
      by blast
    obtain N where N: B ≤ real N
      using real_arch_simple by blast
    have ball 0 B ⊆ ?cube n if n: n ≥ N for n
    proof −
      have sum ((*R) (− real n)) Basis · i ≤ x · i ∧
            x · i ≤ sum ((*R) (real n)) Basis · i
        if norm x < B i ∈ Basis for x i::'n
          using Basis_le_norm[of i x] n N that by (auto simp add: field_simps
sum_negf)
      then show ?thesis
        by (auto simp: mem_box dist_norm)
    qed
   then show ∃M. ∀m≥M. ∀n≥M. dist (integral (?cube m) ?F) (integral (?cube
n) ?F) < e
      by (fastforce simp add: dist_norm intro!: B)
```

```
  qed
  then obtain i where i: (λn. integral (?cube n) ?F) ⟶ i
    using convergent_eq_Cauchy by blast
  have ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶ norm (integral (cbox a b) ?F − i)
< e
    if e > 0 for e
  proof −
    have ∗: e/2 > 0 using that by auto
    then obtain N where N: ⋀n. N ≤ n ⟹ norm (i − integral (?cube n) ?F)
< e/2
      using i[THEN LIMSEQ_D, simplified norm_minus_commute] by meson
    obtain B where 0 < B
      and B: ⋀a b c d. ⟦ball 0 B ⊆ cbox a b; ball 0 B ⊆ cbox c d⟧ ⟹
                norm (integral (cbox a b) ?F − integral (cbox c d) ?F) < e/2
      using rhs ∗ by meson
    let ?B = max (real N) B
    show ?thesis
    proof (intro exI conjI allI impI)
      show 0 < ?B
        using ‹B > 0› by auto
      fix a b :: 'n
      assume ball 0 ?B ⊆ cbox a b
      moreover obtain n where n: max (real N) B ≤ real n
        using real_arch_simple by blast
      moreover have ball 0 B ⊆ ?cube n
      proof
        fix x :: 'n
        assume x: x ∈ ball 0 B
        have ⟦norm (0 − x) < B; i ∈ Basis⟧
             ⟹ sum ((∗R) (−n)) Basis · i≤ x · i ∧ x · i ≤ sum ((∗R) n) Basis · i
for i
          using Basis_le_norm[of i x] n by (auto simp add: field_simps sum_negf)
        then show x ∈ ?cube n
          using x by (auto simp: mem_box dist_norm)
      qed
      ultimately show norm (integral (cbox a b) ?F − i) < e
        using norm_triangle_half_l [OF B N] by force
    qed
  qed
  then show ?l unfolding integrable_on_def has_integral_alt'[of f]
    using rhs by blast
qed

lemma integrable_altD:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes f integrable_on s
  shows ⋀a b. (λx. if x ∈ s then f x else 0) integrable_on cbox a b
    and ⋀e. e > 0 ⟹ ∃B>0. ∀a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c
d ⟶
```

```
    norm (integral (cbox a b) (λx. if x ∈ s then f x else 0) − integral (cbox c d)
(λx. if x ∈ s then f x else 0)) < e
  using assms[unfolded integrable_alt[of f]] by auto

lemma integrable_alt_subset:
  fixes f :: 'a::euclidean_space ⇒ 'b::banach
  shows
    f integrable_on S ⟷
     (∀ a b. (λx. if x ∈ S then f x else 0) integrable_on cbox a b) ∧
     (∀ e>0. ∃ B>0. ∀ a b c d.
                  ball 0 B ⊆ cbox a b ∧ cbox a b ⊆ cbox c d
                  ⟶ norm(integral (cbox a b) (λx. if x ∈ S then f x else 0) −
                        integral (cbox c d) (λx. if x ∈ S then f x else 0)) < e)
     (is _ = ?rhs)
proof −
  let ?g = λx. if x ∈ S then f x else 0
  have f integrable_on S ⟷
       (∀ a b. ?g integrable_on cbox a b) ∧
       (∀ e>0. ∃ B>0. ∀ a b c d. ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d ⟶
          norm (integral (cbox a b) ?g − integral (cbox c d)  ?g) < e)
    by (rule integrable_alt)
  also have ... = ?rhs
  proof −
    { fix e :: real
      assume e: ⋀e. e>0 ⟹ ∃ B>0. ∀ a b c d. ball 0 B ⊆ cbox a b ∧ cbox a b ⊆
cbox c d ⟶
                                  norm (integral (cbox a b) ?g − integral (cbox c d) ?g)
< e
       and e > 0
      obtain B where B > 0
        and B: ⋀a b c d. ⟦ball 0 B ⊆ cbox a b; cbox a b ⊆ cbox c d⟧ ⟹
                      norm (integral (cbox a b) ?g − integral (cbox c d) ?g) < e/2
        using ‹e > 0› e [of e/2] by force
      have ∃ B>0. ∀ a b c d.
              ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d ⟶
              norm (integral (cbox a b) ?g − integral (cbox c d) ?g) < e
      proof (intro exI allI conjI impI)
        fix a b c d :: 'a
        let ?α = ∑ i∈Basis. max (a • i) (c • i) *R i
        let ?β = ∑ i∈Basis. min (b • i) (d • i) *R i
        show norm (integral (cbox a b) ?g − integral (cbox c d) ?g) < e
          if ball: ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d
        proof −
          have B': norm (integral (cbox a b ∩ cbox c d) ?g − integral (cbox x y) ?g)
< e/2
            if cbox a b ∩ cbox c d ⊆ cbox x y for x y
            using B [of ?α ?β x y] ball that by (simp add: Int_interval [symmetric])
          show ?thesis
            using B' [of a b] B' [of c d] norm_triangle_half_r by blast
```

```
      qed
    qed (use ‹B > 0› in auto)}
  then show ?thesis
    by force
 qed
 finally show ?thesis .
qed

lemma integrable_on_subcbox:
 fixes f :: 'n::euclidean_space ⇒ 'a::banach
 assumes intf: f integrable_on S
   and sub: cbox a b ⊆ S
 shows f integrable_on cbox a b
proof −
 have (λx. if x ∈ S then f x else 0) integrable_on cbox a b
   by (simp add: intf integrable_altD(1))
then show ?thesis
  by (metis (mono_tags) sub integrable_restrict_Int le_inf_iff order_refl subset_antisym)
qed
```

7.14.37 A straddling criterion for integrability

```
lemma integrable_straddle_interval:
 fixes f :: 'n::euclidean_space ⇒ real
 assumes ⋀e. e>0 ⟹ ∃g h i j. (g has_integral i) (cbox a b) ∧ (h has_integral j) (cbox a b) ∧
                              |i − j| < e ∧ (∀x∈cbox a b. (g x) ≤ f x ∧ f x ≤ h x)
 shows f integrable_on cbox a b
proof −
 have ∃d. gauge d ∧
         (∀p1 p2. p1 tagged_division_of cbox a b ∧ d fine p1 ∧
                  p2 tagged_division_of cbox a b ∧ d fine p2 ⟶
                  |(∑(x,K)∈p1. content K *R f x) − (∑(x,K)∈p2. content K *R f x)| < e)
   if e > 0 for e
 proof −
   have e: e/3 > 0
     using that by auto
   then obtain g h i j where ij: |i − j| < e/3
         and (g has_integral i) (cbox a b)
         and (h has_integral j) (cbox a b)
         and fgh: ⋀x. x ∈ cbox a b ⟹ g x ≤ f x ∧ f x ≤ h x
     using assms real_norm_def by metis
   then obtain d1 d2 where gauge d1 gauge d2
         and d1: ⋀p. ⟦p tagged_division_of cbox a b; d1 fine p⟧ ⟹
                     |(∑(x,K)∈p. content K *R g x) − i| < e/3
         and d2: ⋀p. ⟦p tagged_division_of cbox a b; d2 fine p⟧ ⟹
                     |(∑(x,K) ∈ p. content K *R h x) − j| < e/3
```

```
      by (metis e has_integral real_norm_def)
    have |(∑(x,K) ∈ p1. content K *R f x) − (∑(x,K) ∈ p2. content K *R f x)|
< e
      if p1: p1 tagged_division_of cbox a b and 11: d1 fine p1 and 21: d2 fine p1
       and p2: p2 tagged_division_of cbox a b and 12: d1 fine p2 and 22: d2 fine
p2 for p1 p2
    proof −
      have *: ⋀g1 g2 h1 h2 f1 f2.
                ⟦|g2 − i| < e/3; |g1 − i| < e/3; |h2 − j| < e/3; |h1 − j| < e/3;
                 g1 − h2 ≤ f1 − f2; f1 − f2 ≤ h1 − g2⟧
                ⟹ |f1 − f2| < e
        using ‹e > 0› ij by arith
      have 0: (∑(x, k)∈p1. content k *R f x) − (∑(x, k)∈p1. content k *R g x)
≥ 0
            0 ≤ (∑(x, k)∈p2. content k *R h x) − (∑(x, k)∈p2. content k *R f x)
            (∑(x, k)∈p2. content k *R f x) − (∑(x, k)∈p2. content k *R g x) ≥ 0
            0 ≤ (∑(x, k)∈p1. content k *R h x) − (∑(x, k)∈p1. content k *R f x)
        unfolding sum_subtractf[symmetric]
          apply (auto intro!: sum_nonneg)
         apply (meson fgh measure_nonneg mult_left_mono tag_in_interval that
sum_nonneg)+
        done
      show ?thesis
      proof (rule *)
        show |(∑(x,K) ∈ p2. content K *R g x) − i| < e/3
          by (rule d1[OF p2 12])
        show |(∑(x,K) ∈ p1. content K *R g x) − i| < e/3
          by (rule d1[OF p1 11])
        show |(∑(x,K) ∈ p2. content K *R h x) − j| < e/3
          by (rule d2[OF p2 22])
        show |(∑(x,K) ∈ p1. content K *R h x) − j| < e/3
          by (rule d2[OF p1 21])
      qed (use 0 in auto)
    qed
    then show ?thesis
      by (rule_tac x=λx. d1 x ∩ d2 x in exI)
        (auto simp: fine_Int intro: ‹gauge d1› ‹gauge d2› d1 d2)
  qed
  then show ?thesis
    by (simp add: integrable_Cauchy)
qed

lemma integrable_straddle:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes ⋀e. e>0 ⟹ ∃g h i j. (g has_integral i) s ∧ (h has_integral j) s ∧
                        |i − j| < e ∧ (∀x∈s. g x ≤ f x ∧ f x ≤ h x)
  shows f integrable_on s
proof −
  let ?fs = (λx. if x ∈ s then f x else 0)
```

have *?fs integrable_on cbox a b* **for** *a b*
proof (*rule integrable_straddle_interval*)
fix *e*::*real*
assume *e* > *0*
then have ∗: *e*/*4* > *0*
by *auto*
with *assms* **obtain** *g h i j* **where** *g*: (*g has_integral i*) *s* **and** *h*: (*h has_integral j*) *s*
and *ij*: |*i* − *j*| < *e*/*4*
and *fgh*: ⋀*x*. *x* ∈ *s* ⟹ *g x* ≤ *f x* ∧ *f x* ≤ *h x*
by *metis*
let *?gs* = (λ*x*. *if x* ∈ *s then g x else 0*)
let *?hs* = (λ*x*. *if x* ∈ *s then h x else 0*)
obtain *Bg* **where** *Bg*: ⋀*a b*. *ball 0 Bg* ⊆ *cbox a b* ⟹ |*integral* (*cbox a b*) *?gs* − *i*| < *e*/*4*
and *int_g*: ⋀*a b*. *?gs integrable_on cbox a b*
using *g* ∗ **unfolding** *has_integral_alt*′ *real_norm_def* **by** *meson*
obtain *Bh* **where**
Bh: ⋀*a b*. *ball 0 Bh* ⊆ *cbox a b* ⟹ |*integral* (*cbox a b*) *?hs* − *j*| < *e*/*4*
and *int_h*: ⋀*a b*. *?hs integrable_on cbox a b*
using *h* ∗ **unfolding** *has_integral_alt*′ *real_norm_def* **by** *meson*
define *c* **where** *c* = (∑ *i*∈*Basis*. *min* (*a*•*i*) (− (*max Bg Bh*)) ∗$_R$ *i*)
define *d* **where** *d* = (∑ *i*∈*Basis*. *max* (*b*•*i*) (*max Bg Bh*) ∗$_R$ *i*)
have ⟦*norm* (*0* − *x*) < *Bg*; *i* ∈ *Basis*⟧ ⟹ *c* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *d* • *i* **for** *x i*
using *Basis_le_norm*[*of i x*] **unfolding** *c_def d_def* **by** *auto*
then have *ballBg*: *ball 0 Bg* ⊆ *cbox c d*
by (*auto simp*: *mem_box dist_norm*)
have ⟦*norm* (*0* − *x*) < *Bh*; *i* ∈ *Basis*⟧ ⟹ *c* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *d* • *i* **for** *x i*
using *Basis_le_norm*[*of i x*] **unfolding** *c_def d_def* **by** *auto*
then have *ballBh*: *ball 0 Bh* ⊆ *cbox c d*
by (*auto simp*: *mem_box dist_norm*)
have *ab_cd*: *cbox a b* ⊆ *cbox c d*
by (*auto simp*: *c_def d_def subset_box_imp*)
have ∗∗: ⋀*ch cg ag ah*::*real*. ⟦|*ah* − *ag*| ≤ |*ch* − *cg*|; |*cg* − *i*| < *e*/*4*; |*ch* − *j*| < *e*/*4*⟧
⟹ |*ag* − *ah*| < *e*
using *ij* **by** *arith*
show ∃ *g h i j*. (*g has_integral i*) (*cbox a b*) ∧ (*h has_integral j*) (*cbox a b*) ∧ |*i* − *j*| < *e* ∧
(∀ *x*∈*cbox a b*. *g x* ≤ (*if x* ∈ *s then f x else 0*) ∧
(*if x* ∈ *s then f x else 0*) ≤ *h x*)
proof (*intro exI ballI conjI*)
have *eq*: ⋀*x f g*. (*if x* ∈ *s then f x else 0*) − (*if x* ∈ *s then g x else 0*) =
(*if x* ∈ *s then f x* − *g x else* (*0*::*real*))
by *auto*
have *int_hg*: (λ*x*. *if x* ∈ *s then h x* − *g x else 0*) *integrable_on cbox a b*
(λ*x*. *if x* ∈ *s then h x* − *g x else 0*) *integrable_on cbox c d*
by (*metis* (*no_types*) *integrable_diff g h has_integral_integrable integrable_altD*(*1*))+

```
    show (?gs has_integral integral (cbox a b) ?gs) (cbox a b)
        (?hs has_integral integral (cbox a b) ?hs) (cbox a b)
      by (intro integrable_integral int_g int_h)+
    then have integral (cbox a b) ?gs ≤ integral (cbox a b) ?hs
      using fgh by (force intro: has_integral_le)
    then have 0 ≤ integral (cbox a b) ?hs − integral (cbox a b) ?gs
      by simp
    then have |integral (cbox a b) ?hs − integral (cbox a b) ?gs|
          ≤ |integral (cbox c d) ?hs − integral (cbox c d) ?gs|
      apply (simp add: integral_diff [symmetric] int_g int_h)
       apply (subst abs_of_nonneg[OF integral_nonneg[OF integrable_diff, OF
int_h int_g]])
         using fgh apply (force simp: eq intro!: integral_subset_le [OF ab_cd
int_hg])+
      done
    then show |integral (cbox a b) ?gs − integral (cbox a b) ?hs| < e
      using ∗∗ Bg ballBg Bh ballBh by blast
    show ⋀x. x ∈ cbox a b ⟹ ?gs x ≤ ?fs x ⋀x. x ∈ cbox a b ⟹ ?fs x ≤ ?hs x
      using fgh by auto
  qed
 qed
 then have int_f: ?fs integrable_on cbox a b for a b
  by simp
 have ∃B>0. ∀a b c d.
            ball 0 B ⊆ cbox a b ∧ ball 0 B ⊆ cbox c d ⟶
            abs (integral (cbox a b) ?fs − integral (cbox c d) ?fs) < e
    if 0 < e for e
 proof −
  have ∗: e/3 > 0
    using that by auto
  with assms obtain g h i j where g: (g has_integral i) s and h: (h has_integral
j) s
              and ij: |i − j| < e/3
              and fgh: ⋀x. x ∈ s ⟹ g x ≤ f x ∧ f x ≤ h x
    by metis
  let ?gs = (λx. if x ∈ s then g x else 0)
  let ?hs = (λx. if x ∈ s then h x else 0)
  obtain Bg where Bg > 0
           and Bg: ⋀a b. ball 0 Bg ⊆ cbox a b ⟹ |integral (cbox a b) ?gs − i| <
e/3
           and int_g: ⋀a b. ?gs integrable_on cbox a b
    using g ∗ unfolding has_integral_alt' real_norm_def by meson
  obtain Bh where Bh > 0
           and Bh: ⋀a b. ball 0 Bh ⊆ cbox a b ⟹ |integral (cbox a b) ?hs − j|
< e/3
           and int_h: ⋀a b. ?hs integrable_on cbox a b
    using h ∗ unfolding has_integral_alt' real_norm_def by meson
  { fix a b c d :: 'n
    assume as: ball 0 (max Bg Bh) ⊆ cbox a b ball 0 (max Bg Bh) ⊆ cbox c d
```

have $**$: *ball 0 Bg* $\subseteq$ *ball* (*0*::*'n*) (*max Bg Bh*) *ball 0 Bh* $\subseteq$ *ball* (*0*::*'n*) (*max Bg Bh*)
by *auto*
have $*$: $\bigwedge$*ga gc ha hc fa fc*. $[\![|ga - i| < e/3; |gc - i| < e/3; |ha - j| < e/3; |hc - j| < e/3; ga \leq fa; fa \leq ha; gc \leq fc; fc \leq hc]\!] \Longrightarrow$ $|fa - fc| < e$
using *ij* **by** *arith*
have *abs* (*integral* (*cbox a b*) (λ*x*. *if x* $\in$ *s then f x else 0*) $-$ *integral* (*cbox c d*) (λ*x*. *if x* $\in$ *s then f x else 0*)) $<$ *e*
proof (*rule* $*$)
show $|$*integral* (*cbox a b*) *?gs* $-$ *i*$|$ $<$ *e*/*3*
using $**$ *Bg as* **by** *blast*
show $|$*integral* (*cbox c d*) *?gs* $-$ *i*$|$ $<$ *e*/*3*
using $**$ *Bg as* **by** *blast*
show $|$*integral* (*cbox a b*) *?hs* $-$ *j*$|$ $<$ *e*/*3*
using $**$ *Bh as* **by** *blast*
show $|$*integral* (*cbox c d*) *?hs* $-$ *j*$|$ $<$ *e*/*3*
using $**$ *Bh as* **by** *blast*
qed (*use int_f int_g int_h fgh* **in** ‹*simp_all add*: *integral_le*›)
}
then show *?thesis*
apply (*rule_tac x*=*max Bg Bh* **in** *exI*)
using ‹*Bg* $>$ *0*› **by** *auto*
qed
then show *?thesis*
unfolding *integrable_alt*[*of f*] *real_norm_def* **by** (*blast intro*: *int_f*)
qed

7.14.38 Adding integrals over several sets

lemma *has_integral_Un*:
fixes *f* :: *'n*::*euclidean_space* $\Rightarrow$ *'a*::*banach*
assumes *f*: (*f has_integral i*) *S* (*f has_integral j*) *T*
and *neg*: *negligible* (*S* $\cap$ *T*)
shows (*f has_integral* (*i* + *j*)) (*S* $\cup$ *T*)
unfolding *has_integral_restrict_UNIV*[*symmetric*, *of f*]
proof (*rule has_integral_spike*[*OF neg*])
let *?f* = λ*x*. (*if x* $\in$ *S then f x else 0*) + (*if x* $\in$ *T then f x else 0*)
show (*?f has_integral i* + *j*) *UNIV*
by (*simp add*: *f has_integral_add*)
qed *auto*

lemma *integral_Un* [*simp*]:
fixes *f* :: *'n*::*euclidean_space* $\Rightarrow$ *'a*::*banach*
assumes *f integrable_on S f integrable_on T negligible* (*S* $\cap$ *T*)
shows *integral* (*S* $\cup$ *T*) *f* = *integral S f* + *integral T f*
by (*simp add*: *has_integral_Un assms integrable_integral integral_unique*)

lemma *integrable_Un*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b :: banach$
assumes *negligible* $(A \cap B)$ *f integrable_on A f integrable_on B*
shows *f integrable_on* $(A \cup B)$
proof −
from *assms* **obtain** *y z* **where** $(f\ has\_integral\ y)\ A\ (f\ has\_integral\ z)\ B$
by (*auto simp*: *integrable_on_def*)
from *has_integral_Un*[*OF this assms*(*1*)] **show** *?thesis* **by** (*auto simp*: *integrable_on_def*)
qed

lemma *integrable_Un'*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b :: banach$
assumes *f integrable_on A f integrable_on B negligible* $(A \cap B)$ $C = A \cup B$
shows *f integrable_on C*
using *integrable_Un*[*of A B f*] *assms* **by** *simp*

lemma *has_integral_UN*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes *finite I*
and *int*: $\bigwedge i.\ i \in I \Longrightarrow (f\ has\_integral\ (g\ i))\ (\mathcal{T}\ i)$
and *neg*: *pairwise* $(\lambda i\ i'.\ negligible\ (\mathcal{T}\ i \cap \mathcal{T}\ i'))\ I$
shows $(f\ has\_integral\ (sum\ g\ I))\ (\bigcup i{\in}I.\ \mathcal{T}\ i)$
proof −
let $?\mathcal{U} = ((\lambda(a,b).\ \mathcal{T}\ a \cap \mathcal{T}\ b)\ `\ \{(a,b).\ a \in I \wedge b \in I - \{a\}\})$
have $((\lambda x.\ if\ x \in (\bigcup i{\in}I.\ \mathcal{T}\ i)\ then\ f\ x\ else\ 0)\ has\_integral\ sum\ g\ I)\ UNIV$
proof (*rule has_integral_spike*)
show *negligible* $(\bigcup ?\mathcal{U})$
proof (*rule negligible_Union*)
have *finite* $(I \times I)$
by (*simp add*: ‹*finite I*›)
moreover have $\{(a,b).\ a \in I \wedge b \in I - \{a\}\} \subseteq I \times I$
by *auto*
ultimately show *finite ?U*
by (*simp add*: *finite_subset*)
show $\bigwedge t.\ t \in ?\mathcal{U} \Longrightarrow negligible\ t$
using *neg* **unfolding** *pairwise_def* **by** *auto*
qed
next
show $(if\ x \in (\bigcup i{\in}I.\ \mathcal{T}\ i)\ then\ f\ x\ else\ 0) = (\sum i{\in}I.\ if\ x \in \mathcal{T}\ i\ then\ f\ x\ else\ 0)$
if $x \in UNIV - (\bigcup ?\mathcal{U})$ **for** *x*
proof *clarsimp*
fix *i* **assume** *i*: $i \in I\ x \in \mathcal{T}\ i$
then have $\forall j{\in}I.\ x \in \mathcal{T}\ j \longleftrightarrow j = i$
using *that* **by** *blast*
with *i* **show** $f\ x = (\sum i{\in}I.\ if\ x \in \mathcal{T}\ i\ then\ f\ x\ else\ 0)$
by (*simp add*: *sum.delta*[*OF* ‹*finite I*›])
qed
next

```
    show ((λx. (∑ i∈I. if x ∈ 𝒯 i then f x else 0)) has_integral sum g I) UNIV
      using int by (simp add: has_integral_restrict_UNIV has_integral_sum [OF ‹finite I›])
  qed
  then show ?thesis
    using has_integral_restrict_UNIV by blast
qed

lemma has_integral_Union:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes finite 𝒯
    and ⋀S. S ∈ 𝒯 ⟹ (f has_integral (i S)) S
    and pairwise (λS S'. negligible (S ∩ S')) 𝒯
  shows (f has_integral (sum i 𝒯)) (⋃𝒯)
proof −
  have (f has_integral (sum i 𝒯)) (⋃ S∈𝒯. S)
    by (intro has_integral_UN assms)
  then show ?thesis
    by force
qed
```

In particular adding integrals over a division, maybe not of an interval.

```
lemma has_integral_combine_division:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes 𝒟 division_of S
    and ⋀k. k ∈ 𝒟 ⟹ (f has_integral (i k)) k
  shows (f has_integral (sum i 𝒟)) S
proof −
  note 𝒟 = division_ofD[OF assms(1)]
  have neg: negligible (S ∩ s') if S ∈ 𝒟 s' ∈ 𝒟 S ≠ s' for S s'
  proof −
    obtain a c b 𝒟 where obt: S = cbox a b s' = cbox c 𝒟
      by (meson ‹S ∈ 𝒟› ‹s' ∈ 𝒟› 𝒟(4))
    from 𝒟(5)[OF that] show ?thesis
      unfolding obt interior_cbox
      by (metis (no_types, lifting) Diff_empty Int_interval box_Int_box negligible_frontier_interval)
  qed
  show ?thesis
    unfolding 𝒟(6)[symmetric]
    by (auto intro: 𝒟 neg assms has_integral_Union pairwiseI)
qed

lemma integral_combine_division_bottomup:
  fixes f :: 'n::euclidean_space ⇒ 'a::banach
  assumes 𝒟 division_of S ⋀k. k ∈ 𝒟 ⟹ f integrable_on k
  shows integral S f = sum (λi. integral i f) 𝒟
 by (meson assms integral_unique has_integral_combine_division has_integral_integrable_integral)
```

lemma *has_integral_combine_division_topdown*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes f: f *integrable_on* S
and $\mathcal{D}$: $\mathcal{D}$ *division_of* K
and $K \subseteq S$
shows (f *has_integral* (*sum* ($\lambda i.$ *integral* i f) $\mathcal{D}$)) K
proof −
have f *integrable_on* L **if** $L \in \mathcal{D}$ **for** L
by (*smt* (*verit*, *best*) *assms cbox_division_memE f integrable_on_subcbox subset_trans that*)
then show *?thesis*
by (*meson* $\mathcal{D}$ *has_integral_combine_division has_integral_integrable_integral*)
qed

lemma *integral_combine_division_topdown*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes f *integrable_on* S
and $\mathcal{D}$ *division_of* S
shows *integral* S f = *sum* ($\lambda i.$ *integral* i f) $\mathcal{D}$
using *assms has_integral_combine_division_topdown* **by** *blast*

lemma *integrable_combine_division*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes $\mathcal{D}$: $\mathcal{D}$ *division_of* S
and f: $\bigwedge i.\ i \in \mathcal{D} \Longrightarrow f$ *integrable_on* i
shows f *integrable_on* S
using f **unfolding** *integrable_on_def* **by** (*metis has_integral_combine_division*[*OF* $\mathcal{D}$])

lemma *integrable_on_subdivision*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes $\mathcal{D}$: $\mathcal{D}$ *division_of* i
and f: f *integrable_on* S
and $i \subseteq S$
shows f *integrable_on* i
proof −
have f *integrable_on* i **if** $i \in \mathcal{D}$ **for** i
by (*smt* (*verit*, *best*) *assms cbox_division_memE f integrable_on_subcbox order_trans that*)
then show *?thesis*
using $\mathcal{D}$ *integrable_combine_division* **by** *blast*
qed

7.14.39 Also tagged divisions

lemma *has_integral_combine_tagged_division*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes p *tagged_division_of* S
and $\bigwedge x\ k.\ (x,k) \in p \Longrightarrow (f$ *has_integral* $(i\ k))\ k$

shows (*f has_integral* ($\sum$ (*x*,*k*)∈*p*. *i k*)) *S*
proof −
have ∗: (*f has_integral* ($\sum$ *k*∈*snd*'*p*. *integral k f*)) *S*
by (*smt* (*verit*, *del_insts*) *assms division_of_tagged_division has_integral_combine_division has_integral_iff imageE prod.collapse*)
also have ($\sum$ *k*∈*snd*'*p*. *integral k f*) = ($\sum$ (*x*, *k*)∈*p*. *integral k f*)
by (*intro sum.over_tagged_division_lemma*[*OF assms*(*1*), *symmetric*] *integral_null*)
(*simp add*: *content_eq_0_interior*)
finally show *?thesis*
using *assms* **by** (*auto simp add*: *has_integral_iff intro*!: *sum.cong*)
qed

lemma *integral_combine_tagged_division_bottomup*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *p*: *p tagged_division_of* (*cbox a b*)
and *f*: ⋀*x k*. (*x*,*k*)∈*p* ⟹ *f integrable_on k*
shows *integral* (*cbox a b*) *f* = *sum* (λ(*x*,*k*). *integral k f*) *p*
by (*simp add*: *has_integral_combine_tagged_division*[*OF p*] *integral_unique f integrable_integral*)

lemma *has_integral_combine_tagged_division_topdown*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *f*: *f integrable_on cbox a b*
and *p*: *p tagged_division_of* (*cbox a b*)
shows (*f has_integral* (*sum* (λ(*x*,*K*). *integral K f*) *p*)) (*cbox a b*)
proof −
have (*f has_integral integral K f*) *K* **if** (*x*,*K*) ∈ *p* **for** *x K*
by (*metis assms integrable_integral integrable_on_subcbox tagged_division_ofD*(*3*,*4*) *that*)
then show *?thesis*
by (*simp add*: *has_integral_combine_tagged_division p*)
qed

lemma *integral_combine_tagged_division_topdown*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *f integrable_on cbox a b*
and *p tagged_division_of* (*cbox a b*)
shows *integral* (*cbox a b*) *f* = *sum* (λ(*x*,*k*). *integral k f*) *p*
using *assms* **by** (*auto intro*: *integral_unique* [*OF has_integral_combine_tagged_division_topdown*])

7.14.40 Henstock's lemma

lemma *Henstock_lemma_part1*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *intf*: *f integrable_on cbox a b*
and *e* > *0*
and *gauge d*
and *less_e*: ⋀*p*. ⟦*p tagged_division_of* (*cbox a b*); *d fine p*⟧ ⟹

```
                norm (sum (λ(x,K). content K *R f x) p − integral(cbox a b) f)
< e
    and p: p tagged_partial_division_of (cbox a b) d fine p
  shows norm (sum (λ(x,K). content K *R f x − integral K f) p) ≤ e (is ?lhs ≤
e)
proof (rule field_le_epsilon)
  fix k :: real
  assume k > 0
  let ?SUM = λp. (∑(x,K) ∈ p. content K *R f x)
  note p' = tagged_partial_division_ofD[OF p(1)]
  have ⋃(snd ` p) ⊆ cbox a b
    using p'(3) by fastforce
  then obtain q where q: snd ` p ⊆ q and qdiv: q division_of cbox a b
   by (meson p(1) partial_division_extend_interval partial_division_of_tagged_division)
  note q' = division_ofD[OF qdiv]
  define r where r = q − snd ` p
  have snd ` p ∩ r = {}
    unfolding r_def by auto
  have finite r
    using q' unfolding r_def by auto
  have ∃p. p tagged_division_of i ∧ d fine p ∧
        norm (?SUM p − integral i f) < k / (real (card r) + 1)
    if i∈r for i
  proof −
    have gt0: k / (real (card r) + 1) > 0 using ‹k > 0› by simp
    have i: i ∈ q
      using that unfolding r_def by auto
    then obtain u v where uv: i = cbox u v
      using q'(4) by blast
    then have cbox u v ⊆ cbox a b
      using i q'(2) by auto
    then have f integrable_on cbox u v
      by (rule integrable_subinterval[OF intf])
    with integrable_integral[OF this, unfolded has_integral[of f]]
    obtain dd where gauge dd and dd:
      ⋀D. ⟦D tagged_division_of cbox u v; dd fine D⟧ ⟹
     norm (?SUM D − integral (cbox u v) f) < k / (real (card r) + 1)
      using gt0 by auto
    with gauge_Int[OF ‹gauge d› ‹gauge dd›]
    obtain qq where qq: qq tagged_division_of cbox u v (λx. d x ∩ dd x) fine qq
      using fine_division_exists by blast
    with dd[of qq]  show ?thesis
      by (auto simp: fine_Int uv)
  qed
  then obtain qq where qq: ⋀i. i ∈ r ⟹ qq i tagged_division_of i ∧
      d fine qq i ∧ norm (?SUM (qq i) − integral i f) < k / (real (card r) + 1)
    by metis

  let ?p = p ∪ ⋃(qq ` r)
```

```
  have norm (?SUM ?p − integral (cbox a b) f) < e
  proof (rule less_e)
    show d fine ?p
      by (metis (mono_tags, opaque_lifting) qq fine_Un fine_Union imageE p(2))
    note ptag = tagged_partial_division_of_Union_self[OF p(1)]
    have p ∪ ⋃(qq ‘ r) tagged_division_of ⋃(snd ‘ p) ∪ ⋃r
     proof (rule tagged_division_Un[OF ptag tagged_division_Union [OF ‹finite
r›]])
      show ⋀i. i ∈ r ⟹ qq i tagged_division_of i
        using qq by auto
      show ⋀i1 i2. ⟦i1 ∈ r; i2 ∈ r; i1 ≠ i2⟧ ⟹ interior i1 ∩ interior i2 = {}
        by (simp add: q′(5) r_def)
      show interior (⋃(snd ‘ p)) ∩ interior (⋃r) = {}
      proof (rule Int_interior_Union_intervals [OF ‹finite r›])
        show open (interior (⋃(snd ‘ p)))
          by blast
        show ⋀T. T ∈ r ⟹ ∃a b. T = cbox a b
          by (simp add: q′(4) r_def)
        have interior T ∩ interior (⋃(snd ‘ p)) = {} if T ∈ r for T
        proof (rule Int_interior_Union_intervals)
          show ⋀U. U ∈ snd ‘ p ⟹ ∃a b. U = cbox a b
            using q q′(4) by blast
          show ⋀U. U ∈ snd ‘ p ⟹ interior T ∩ interior U = {}
            by (metis DiffE q q′(5) r_def subsetD that)
        qed (use p′ in auto)
        then show ⋀T. T ∈ r ⟹ interior (⋃(snd ‘ p)) ∩ interior T = {}
          by (metis Int_commute)
      qed
    qed
    moreover have ⋃(snd ‘ p) ∪ ⋃r = cbox a b and {qq i |i. i ∈ r} = qq ‘ r
      using qdiv q unfolding Union_Un_distrib[symmetric] r_def by auto
    ultimately show ?p tagged_division_of (cbox a b)
      by fastforce
  qed
  then have norm (?SUM p + (?SUM (⋃(qq ‘ r))) − integral (cbox a b) f) < e
  proof (subst sum.union_inter_neutral[symmetric, OF ‹finite p›], safe)
    show content L *R f x = 0 if (x, L) ∈ p (x, L) ∈ qq K K ∈ r for x K L
    proof −
      obtain u v where uv: L = cbox u v
        using ‹(x,L) ∈ p› p′(4) by blast
      have L ⊆ K
        using  qq[OF that(3)] tagged_division_ofD(3) ‹(x,L) ∈ qq K› by metis
      have L ∈ snd ‘ p
        using ‹(x,L) ∈ p› image_iff by fastforce
      then have L ∈ q K ∈ q L ≠ K
        using that q(1) unfolding r_def by auto
      with q′(5) show content L *R f x = 0
      by (metis ‹L ⊆ K› content_eq_0_interior inf.orderE interior_Int scaleR_eq_0_iff
uv)
```

```
    qed
    show finite (⋃(qq ' r))
      by (meson finite_UN qq ‹finite r› tagged_division_of_finite)
  qed
  moreover have content M *R f x = 0
      if x: (x,M) ∈ qq K (x,M) ∈ qq L and KL: qq K ≠ qq L and r: K ∈ r L ∈ r
    for x M K L
  proof −
    note kl = tagged_division_ofD(3,4)[OF qq[THEN conjunct1]]
    obtain u v where uv: M = cbox u v
      using ‹(x, M) ∈ qq L› ‹L ∈ r› kl(2) by blast
    have empty: interior (K ∩ L) = {}
      by (metis DiffD1 interior_Int q'(5) r_def KL r)
    with that kl show content M *R f x = 0
      by (metis content_eq_0_interior dual_order.refl inf.orderE inf_mono interior_mono
                scaleR_eq_0_iff subset_empty uv x)
  qed
  ultimately have norm (?SUM p + sum ?SUM (qq ' r) − integral (cbox a b) f)
< e
    apply (subst (asm) sum.Union_comp)
    using qq by (force simp: split_paired_all)+
  moreover have content M *R f x = 0
      if K ∈ r L ∈ r K ≠ L qq K = qq L (x, M) ∈ qq K for K L x M
    using tagged_division_ofD(6) qq that by (metis (no_types, lifting))
  ultimately have less_e: norm (?SUM p + sum (?SUM ∘ qq) r − integral (cbox
a b) f) < e
  proof (subst (asm) sum.reindex_nontrivial [OF ‹finite r›])
    qed (auto simp: split_paired_all sum.neutral)
  have norm_le: norm (cp − ip) ≤ e + k
              if norm ((cp + cr) − i) < e norm (cr − ir) < k ip + ir = i
              for ir ip i cr cp::'a
    using norm_triangle_le[of cp + cr − i − (cr − ir)] that
    unfolding that(3)[symmetric] norm_minus_cancel
    by (auto simp add: algebra_simps)

  have ?lhs =  norm (?SUM p − (∑(x, k)∈p. integral k f))
    unfolding split_def sum_subtractf ..
  also have ... ≤ e + k
  proof (rule norm_le[OF less_e])
    have lessk: k * real (card r) / (1 + real (card r)) < k
      using ‹k>0› by (auto simp add: field_simps)
    have norm (sum (?SUM ∘ qq) r − (∑k∈r. integral k f)) ≤ (∑x∈r. k / (real
(card r) + 1))
     unfolding sum_subtractf[symmetric] by (force dest: qq intro!: sum_norm_le)
    also have ... < k
      by (simp add: lessk add.commute mult.commute)
    finally show norm (sum (?SUM ∘ qq) r − (∑k∈r. integral k f)) < k .
  next
```

```
    from q(1) have [simp]: snd ` p ∪ q = q by auto
    have integral l f = 0
      if inp: (x, l) ∈ p (y, m) ∈ p and ne: (x, l) ≠ (y, m) and l = m for x l y m
    proof −
      obtain u v where uv: l = cbox u v
        using inp p'(4) by blast
      then show ?thesis
        using uv that p
        by (metis content_eq_0_interior dual_order.refl inf.orderE integral_null
ne tagged_partial_division_ofD(5))
    qed
    then have (∑(x, K)∈p. integral K f) = (∑K∈snd ` p. integral K f)
      apply (subst sum.reindex_nontrivial [OF ‹finite p›])
      unfolding split_paired_all split_def by auto
    then show (∑(x, k)∈p. integral k f) + (∑k∈r. integral k f) = integral (cbox
a b) f
      unfolding integral_combine_division_topdown[OF intf qdiv] r_def
      using q'(1) p'(1) sum.union_disjoint [of snd ` p q − snd ` p, symmetric]
        by simp
  qed
  finally show ?lhs ≤ e + k .
qed

lemma Henstock_lemma_part2:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes fed: f integrable_on cbox a b e > 0 gauge d
    and less_e: ⋀D. ⟦D tagged_division_of (cbox a b); d fine D⟧ ⟹
                  norm (sum (λ(x,k). content k *R f x) D − integral (cbox a b) f)
< e
    and tag: p tagged_partial_division_of (cbox a b)
    and d fine p
  shows sum (λ(x,k). norm (content k *R f x − integral k f)) p ≤ 2 * real
(DIM('n)) * e
proof −
  have finite p
    using tag tagged_partial_division_ofD by blast
  then show ?thesis
    unfolding split_def
  proof (rule sum_norm_allsubsets_bound)
    fix Q
    assume Q: Q ⊆ p
    then have fine: d fine Q
      by (simp add: ‹d fine p› fine_subset)
    show norm (∑x∈Q. content (snd x) *R f (fst x) − integral (snd x) f) ≤ e
      apply (rule Henstock_lemma_part1[OF fed less_e, unfolded split_def])
      using Q tag tagged_partial_division_subset by (force simp add: fine)+
  qed
qed
```

lemma *Henstock_lemma*:
fixes $f :: 'm::euclidean\_space \Rightarrow 'n::euclidean\_space$
assumes *intf*: f *integrable_on cbox a b*
and $e > 0$
obtains γ **where** *gauge* γ
and $\bigwedge p.\ [\![p\ tagged\_partial\_division\_of\ (cbox\ a\ b);\ \gamma\ fine\ p]\!] \Longrightarrow$
$sum\ (\lambda(x,k).\ norm(content\ k *_R f\ x - integral\ k\ f))\ p < e$
proof −
have $*$: $e/(2 * (real\ DIM('n) + 1)) > 0$ **using** ‹$e > 0$› **by** *simp*
with *integrable_integral*[*OF intf*, *unfolded has_integral*]
obtain γ **where** *gauge* γ
and γ: $\bigwedge \mathcal{D}.\ [\![\mathcal{D}\ tagged\_division\_of\ cbox\ a\ b;\ \gamma\ fine\ \mathcal{D}]\!] \Longrightarrow$
$norm\ ((\sum (x,K)\in\mathcal{D}.\ content\ K *_R f\ x) - integral\ (cbox\ a\ b)\ f)$
$< e/(2 * (real\ DIM('n) + 1))$
by *metis*
show *thesis*
proof (*rule that* [*OF* ‹*gauge* γ›])
fix p
assume p: $p\ tagged\_partial\_division\_of\ cbox\ a\ b\ \gamma\ fine\ p$
have $(\sum (x,K)\in p.\ norm\ (content\ K *_R f\ x - integral\ K\ f))$
$\leq 2 * real\ DIM('n) * (e/(2 * (real\ DIM('n) + 1)))$
using *Henstock_lemma_part2*[*OF intf* $*$ ‹*gauge* γ› γ p] **by** *metis*
also have $... < e$
using ‹$e > 0$› **by** (*auto simp add*: *field_simps*)
finally
show $(\sum (x,K)\in p.\ norm\ (content\ K *_R f\ x - integral\ K\ f)) < e$ **.**
qed
qed

7.14.41 Monotone convergence (bounded interval first)

lemma *bounded_increasing_convergent*:
fixes $f :: nat \Rightarrow real$
shows $[\![bounded\ (range\ f);\ \bigwedge n.\ f\ n \leq f\ (Suc\ n)]\!] \Longrightarrow \exists l.\ f \longrightarrow l$
using *Bseq_mono_convergent*[*of f*] *incseq_Suc_iff*[*of f*]
by (*auto simp*: *image_def Bseq_eq_bounded convergent_def incseq_def*)

lemma *monotone_convergence_interval*:
fixes $f :: nat \Rightarrow 'n::euclidean\_space \Rightarrow real$
assumes *intf*: $\bigwedge k.\ (f\ k)\ integrable\_on\ cbox\ a\ b$
and *le*: $\bigwedge k\ x.\ x \in cbox\ a\ b \Longrightarrow (f\ k\ x) \leq f\ (Suc\ k)\ x$
and *fg*: $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow ((\lambda k.\ f\ k\ x) \longrightarrow g\ x)\ sequentially$
and *bou*: *bounded* (*range* ($\lambda k.$ *integral* (*cbox a b*) (*f k*)))
shows *g integrable_on cbox a b* $\wedge$ (($\lambda k.$ *integral* (*cbox a b*) (*f k*)) $\longrightarrow$ *integral* (*cbox a b*) *g*) *sequentially*
proof (*cases content* (*cbox a b*) = *0*)
case *True* **then show** *?thesis*
by *auto*
next

```
  case False
  have fg1: (f k x) ≤ (g x) if x: x ∈ cbox a b for x k
  proof −
    have ∀F j in sequentially. f k x ≤ f j x
    proof (rule eventually_sequentiallyI [of k])
      show ⋀j. k ≤ j ⟹ f k x ≤ f j x
        using le x by (force intro: transitive_stepwise_le)
    qed
    then show f k x ≤ g x
      using tendsto_lowerbound [OF fg] x trivial_limit_sequentially by blast
  qed
  have int_inc: ⋀n. integral (cbox a b) (f n) ≤ integral (cbox a b) (f (Suc n))
    by (metis integral_le intf le)
  then obtain i where i: (λk. integral (cbox a b) (f k)) ⟶ i
    using bounded_increasing_convergent bou by blast
  have ⋀k. ∀F x in sequentially. integral (cbox a b) (f k) ≤ integral (cbox a b) (f
x)
    unfolding eventually_sequentially
    by (force intro: transitive_stepwise_le int_inc)
  then have i': ⋀k. (integral(cbox a b) (f k)) ≤ i
    using tendsto_le [OF trivial_limit_sequentially i] by blast
  have (g has_integral i) (cbox a b)
    unfolding has_integral real_norm_def
  proof clarify
    fix e::real
    assume e: e > 0
    have ⋀k. (∃γ. gauge γ ∧ (∀D. D tagged_division_of (cbox a b) ∧ γ fine D
⟶
      abs ((∑(x,K)∈D. content K *R f k x) − integral (cbox a b) (f k)) < e/2 ^(k
+ 2)))
      using intf e by (auto simp: has_integral_integral has_integral)
    then obtain c where c: ⋀x. gauge (c x)
          ⋀x D. ⟦D tagged_division_of cbox a b; c x fine D⟧ ⟹
            abs ((∑(u,K)∈D. content K *R f x u) − integral (cbox a b) (f x))
            < e/2 ^ (x + 2)
      by metis

    have ∃r. ∀k≥r. 0 ≤ i − (integral (cbox a b) (f k)) ∧ i − (integral (cbox a b)
(f k)) < e/4
    proof −
      have e/4 > 0
        using e by auto
      show ?thesis
        using LIMSEQ_D [OF i ‹e/4 > 0›] i' by auto
    qed
    then obtain r where r: ⋀k. r ≤ k ⟹ 0 ≤ i − integral (cbox a b) (f k)
                           ⋀k. r ≤ k ⟹ i − integral (cbox a b) (f k) < e/4
      by metis
   have ∃n≥r. ∀k≥n. 0 ≤ (g x) − (f k x) ∧ (g x) − (f k x) < e/(4 * content(cbox
```

```
a b))
    if x ∈ cbox a b for x
  proof −
    have e/(4 * content (cbox a b)) > 0
      by (simp add: False content_lt_nz e)
    with fg that LIMSEQ_D
    obtain N where ∀n≥N. norm (f n x − g x) < e/(4 * content (cbox a b))
      by metis
    then show ∃n≥r. ∀k≥n. 0 ≤ g x − f k x ∧ g x − f k x < e/(4 * content
(cbox a b))
      apply (rule_tac x=N + r in exI)
      using fg1[OF that] by (auto simp add: field_simps)
  qed
  then obtain m where r_le_m: ⋀x. x ∈ cbox a b ⟹ r ≤ m x
    and m: ⋀x k. ⟦x ∈ cbox a b; m x ≤ k⟧
                ⟹ 0 ≤ g x − f k x ∧ g x − f k x < e/(4 * content (cbox a b))
    by metis
  define d where d x = c (m x) x for x
  show ∃γ. gauge γ ∧
        (∀D. D tagged_division_of cbox a b ∧
            γ fine D ⟶ abs ((∑(x,K)∈D. content K *R g x) − i) < e)
  proof (rule exI, safe)
    show gauge d
      using c(1) unfolding gauge_def d_def by auto
  next
    fix D
    assume ptag: D tagged_division_of (cbox a b) and d fine D
    note p'=tagged_division_ofD[OF ptag]
    obtain s where s: ⋀x. x ∈ D ⟹ m (fst x) ≤ s
    by (metis finite_imageI finite_nat_set_iff_bounded_le p'(1) rev_image_eqI)
    have *: |a − d| < e if |a − b| ≤ e/4 |b − c| < e/2 |c − d| < e/4 for a b c d
      using that norm_triangle_lt[of a − b b − c 3* e/4]
        norm_triangle_lt[of a − b + (b − c) c − d e]
      by (auto simp add: algebra_simps)
    show |(∑(x, k)∈D. content k *R g x) − i| < e
    proof (rule *)
      have |(∑(x,K)∈D. content K *R g x) − (∑(x,K)∈D. content K *R f (m
x) x)|
          ≤ (∑i∈D. |(case i of (x, K) ⇒ content K *R g x) − (case i of (x, K)
⇒ content K *R f (m x) x)|)
        by (metis (mono_tags) sum_subtractf sum_abs)
      also have ... ≤ (∑(x, k)∈D. content k * (e/(4 * content (cbox a b))))
      proof (rule sum_mono, simp add: split_paired_all)
        fix x K
        assume xk: (x,K) ∈ D
        with ptag have x: x ∈ cbox a b
          by blast
        then have abs (content K * (g x − f (m x) x)) ≤ content K * (e/(4 *
content (cbox a b)))
```

by (*metis m*[*OF x*] *mult_nonneg_nonneg abs_of_nonneg less_eq_real_def measure_nonneg mult_left_mono order_refl*)
then show |*content K* ∗ *g x* − *content K* ∗ *f* (*m x*) *x*| ≤ *content K* ∗ *e*/(4 ∗ *content* (*cbox a b*))
by (*simp add*: *algebra_simps*)
qed
also have ... = (*e*/(4 ∗ *content* (*cbox a b*))) ∗ (∑(*x*, *k*)∈$\mathcal{D}$. *content k*)
by (*simp add*: *sum_distrib_left sum_divide_distrib split_def mult.commute*)
also have ... ≤ *e*/4
by (*metis False additive_content_tagged_division* [*OF ptag*] *nonzero_mult_divide_mult_cancel_r order_refl times_divide_eq_left*)
finally show |(∑(*x*,*K*)∈$\mathcal{D}$. *content K* ∗$_R$ *g x*) − (∑(*x*,*K*)∈$\mathcal{D}$. *content K* ∗$_R$ *f* (*m x*) *x*)| ≤ *e*/4 **.**

next
have *norm* ((∑(*x*,*K*)∈$\mathcal{D}$. *content K* ∗$_R$ *f* (*m x*) *x*) − (∑(*x*,*K*)∈$\mathcal{D}$. *integral K* (*f* (*m x*))))
≤ *norm* (∑*j* = 0..*s*. ∑(*x*,*K*)∈{*xk* ∈ $\mathcal{D}$. *m* (*fst xk*) = *j*}. *content K* ∗$_R$ *f* (*m x*) *x* − *integral K* (*f* (*m x*)))
apply (*subst sum.group*)
using *s* **by** (*auto simp*: *sum_subtractf split_def p'*(1))
also have ... < *e*/2
proof −
have *norm* (∑*j* = 0..*s*. ∑(*x*, *k*)∈{*xk* ∈ $\mathcal{D}$. *m* (*fst xk*) = *j*}. *content k* ∗$_R$ *f* (*m x*) *x* − *integral k* (*f* (*m x*)))
≤ (∑*i* = 0..*s*. *e*/2 ^ (*i* + 2))
proof (*rule sum_norm_le*)
fix *t*
assume *t* ∈ {0..*s*}
have *norm* (∑(*x*,*k*)∈{*xk* ∈ $\mathcal{D}$. *m* (*fst xk*) = *t*}. *content k* ∗$_R$ *f* (*m x*) *x* − *integral k* (*f* (*m x*))) =
norm (∑(*x*,*k*)∈{*xk* ∈ $\mathcal{D}$. *m* (*fst xk*) = *t*}. *content k* ∗$_R$ *f t x* − *integral k* (*f t*))
by (*force intro*!: *sum.cong arg_cong*[**where** *f*=*norm*])
also have ... ≤ *e*/2 ^ (*t* + 2)
proof (*rule Henstock_lemma_part1* [*OF intf*])
show {*xk* ∈ $\mathcal{D}$. *m* (*fst xk*) = *t*} *tagged_partial_division_of cbox a b*
proof (*rule tagged_partial_division_subset*[*of* $\mathcal{D}$])
show $\mathcal{D}$ *tagged_partial_division_of cbox a b*
using *ptag tagged_division_of_def* **by** *blast*
qed *auto*
show *c t fine* {*xk* ∈ $\mathcal{D}$. *m* (*fst xk*) = *t*}
using ‹*d fine* $\mathcal{D}$› **by** (*auto simp*: *fine_def d_def*)
qed (*use c e* **in** *auto*)
finally show *norm* (∑(*x*,*K*)∈{*xk* ∈ $\mathcal{D}$. *m* (*fst xk*) = *t*}. *content K* ∗$_R$ *f* (*m x*) *x* −
integral K (*f* (*m x*))) ≤ *e*/2 ^ (*t* + 2) **.**
qed
also have ... = (*e*/2/2) ∗ (∑*i* = 0..*s*. (1/2) ^ *i*)

```
        by (simp add: sum_distrib_left field_simps)
      also have ... < e/2
        by (simp add: sum_gp mult_strict_left_mono[OF _ e])
      finally show norm (∑j = 0..s. ∑(x, k)∈{xk ∈ 𝒟.
        m (fst xk) = j}. content k *R f (m x) x − integral k (f (m x))) < e/2 .
    qed
  finally show |(∑(x,K)∈𝒟. content K *R f (m x) x) − (∑(x,K)∈𝒟. integral
K (f (m x)))| < e/2
    by simp
  next
    have comb: integral (cbox a b) (f y) = (∑(x, k)∈𝒟. integral k (f y)) for y
      using integral_combine_tagged_division_topdown[OF intf ptag] by metis
    have f_le: ⋀y m n. ⟦y ∈ cbox a b; n≥m⟧ ⟹ f m y ≤ f n y
      using le by (auto intro: transitive_stepwise_le)
    have (∑(x, k)∈𝒟. integral k (f r)) ≤ (∑(x, K)∈𝒟. integral K (f (m x)))
    proof (rule sum_mono, simp add: split_paired_all)
      fix x K
      assume xK: (x, K) ∈ 𝒟
      show integral K (f r) ≤ integral K (f (m x))
      proof (rule integral_le)
        show f r integrable_on K
          by (metis integrable_on_subcbox intf p'(3) p'(4) xK)
        show f (m x) integrable_on K
            by (metis elementary_interval integrable_on_subdivision intf p'(3)
p'(4) xK)
        show f r y ≤ f (m x) y if y ∈ K for y
          using that r_le_m[of x] p'(2−3)[OF xK] f_le by auto
      qed
    qed
  moreover have (∑(x, K)∈𝒟. integral K (f (m x))) ≤ (∑(x, k)∈𝒟. integral
k (f s))
    proof (rule sum_mono, simp add: split_paired_all)
      fix x K
      assume xK: (x, K) ∈ 𝒟
      show integral K (f (m x)) ≤ integral K (f s)
      proof (rule integral_le)
        show f (m x) integrable_on K
            by (metis elementary_interval integrable_on_subdivision intf p'(3)
p'(4) xK)
        show f s integrable_on K
          by (metis integrable_on_subcbox intf p'(3) p'(4) xK)
        show f (m x) y ≤ f s y if y ∈ K for y
          using that s xK f_le p'(3) by fastforce
      qed
    qed
    moreover have 0 ≤ i − integral (cbox a b) (f r) i − integral (cbox a b) (f
r) < e/4
      using r by auto
    ultimately show |(∑(x,K)∈𝒟. integral K (f (m x))) − i| < e/4
```

```
        using comb i'[of s] by auto
      qed
    qed
  qed
  with i integral_unique show ?thesis
    by blast
qed

lemma monotone_convergence_increasing:
  fixes f :: nat ⇒ 'n::euclidean_space ⇒ real
  assumes int_f: ⋀k. (f k) integrable_on S
    and ⋀k x. x ∈ S ⟹ (f k x) ≤ (f (Suc k) x)
    and fg: ⋀x. x ∈ S ⟹ ((λk. f k x) ⟶ g x) sequentially
    and bou: bounded (range (λk. integral S (f k)))
  shows g integrable_on S ∧ ((λk. integral S (f k)) ⟶ integral S g) sequentially
proof −
  have lem: g integrable_on S ∧ ((λk. integral S (f k)) ⟶ integral S g) sequentially
    if f0: ⋀k x. x ∈ S ⟹ 0 ≤ f k x
    and int_f: ⋀k. (f k) integrable_on S
    and le: ⋀k x. x ∈ S ⟹ f k x ≤ f (Suc k) x
    and lim: ⋀x. x ∈ S ⟹ ((λk. f k x) ⟶ g x) sequentially
    and bou: bounded (range(λk. integral S (f k)))
    for f :: nat ⇒ 'n::euclidean_space ⇒ real and g S
  proof −
    have fg: (f k x) ≤ (g x) if x ∈ S for x k
    proof −
      have ⋀xa. k ≤ xa ⟹ f k x ≤ f xa x
        using le  by (force intro: transitive_stepwise_le that)
      then show ?thesis
        using tendsto_lowerbound [OF lim [OF that]] eventually_sequentiallyI by
force
    qed
    obtain i where i: (λk. integral S (f k)) ⟶ i
      using bounded_increasing_convergent [OF bou] le int_f integral_le by blast
    have i': (integral S (f k)) ≤ i for k
    proof −
      have ⋀k. ⋀x. x ∈ S ⟹ ∀ n≥k. f k x ≤ f n x
        using le  by (force intro: transitive_stepwise_le)
      then show ?thesis
      using tendsto_lowerbound [OF i eventually_sequentiallyI trivial_limit_sequentially]
        by (meson int_f integral_le)
    qed
    let ?f = (λk x. if x ∈ S then f k x else 0)
    let ?g = (λx. if x ∈ S then g x else 0)
    have int: ?f k integrable_on cbox a b for a b k
      by (simp add: int_f integrable_altD(1))
    have int': ⋀k a b. f k integrable_on cbox a b ∩ S
      using int by (simp add: Int_commute integrable_restrict_Int)
```

```
  have g: ?g integrable_on cbox a b ∧
        (λk. integral (cbox a b) (?f k)) ⟶ integral (cbox a b) ?g for a b
  proof (rule monotone_convergence_interval)
    have norm (integral (cbox a b) (?f k)) ≤ norm (integral S (f k)) for k
    proof −
      have 0 ≤ integral (cbox a b) (?f k)
        by (metis (no_types) integral_nonneg Int_iff f0 inf_commute integral_restrict_Int int')
      moreover have 0 ≤ integral S (f k)
        by (simp add: integral_nonneg f0 int_f)
      moreover have integral (S ∩ cbox a b) (f k) ≤ integral S (f k)
        by (metis f0 inf_commute int' int_f integral_subset_le le_inf_iff order_refl)
      ultimately show ?thesis
        by (simp add: integral_restrict_Int)
    qed
    moreover obtain B where ⋀x. x ∈ range (λk. integral S (f k)) ⟹ norm x ≤ B
      using bou unfolding bounded_iff by blast
    ultimately show bounded (range (λk. integral (cbox a b) (?f k)))
      unfolding bounded_iff by (blast intro: order_trans)
  qed (use int le lim in auto)
  moreover have ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶ norm (integral (cbox a b) ?g − i) < e
    if 0 < e for e
  proof −
    have e/4>0
      using that by auto
    with LIMSEQ_D [OF i] obtain N where N: ⋀n. n ≥ N ⟹ norm (integral S (f n) − i) < e/4
      by metis
    with int_f[of N, unfolded has_integral_integral has_integral_alt'[of f N]]
    obtain B where 0 < B and B:
      ⋀a b. ball 0 B ⊆ cbox a b ⟹ norm (integral (cbox a b) (?f N) − integral S (f N)) < e/4
      by (meson ‹0 < e/4›)
    have norm (integral (cbox a b) ?g − i) < e if ab: ball 0 B ⊆ cbox a b for a b
    proof −
      obtain M where M: ⋀n. n ≥ M ⟹ abs (integral (cbox a b) (?f n) − integral (cbox a b) ?g) < e/2
        using ‹e > 0› g by (fastforce simp add: dest!: LIMSEQ_D [where r = e/2])
      have ∗: ⋀α β g. ⟦|α − i| < e/2; |β − g| < e/2; α ≤ β; β ≤ i⟧ ⟹ |g − i| < e
        unfolding real_inner_1_right by arith
      show norm (integral (cbox a b) ?g − i) < e
        unfolding real_norm_def
      proof (rule ∗)
        show |integral (cbox a b) (?f N) − i| < e/2
```

```
        proof (rule abs_triangle_half_l)
          show |integral (cbox a b) (?f N) − integral S (f N)| < e/2/2
            using B[OF ab] by simp
          show abs (i − integral S (f N)) < e/2/2
            using N by (simp add: abs_minus_commute)
        qed
        show |integral (cbox a b) (?f (M + N)) − integral (cbox a b) ?g| < e/2
          by (metis le_add1 M[of M + N])
        show integral (cbox a b) (?f N) ≤ integral (cbox a b) (?f (M + N))
        proof (intro ballI integral_le[OF int int])
          fix x assume x ∈ cbox a b
          have (f m x) ≤ (f n x) if x ∈ S n ≥ m for m n
          proof (rule transitive_stepwise_le [OF ‹n ≥ m› order_refl])
            show ⋀u y z. ⟦f u x ≤ f y x; f y x ≤ f z x⟧ ⟹ f u x ≤ f z x
              using dual_order.trans by blast
          qed (simp add: le ‹x ∈ S›)
          then show (?f N)x ≤ (?f (M+N))x
            by auto
        qed
        have integral (cbox a b ∩ S) (f (M + N)) ≤ integral S (f (M + N))
          by (metis Int_lower1 f0 inf_commute int' int_f integral_subset_le)
        then have integral (cbox a b) (?f (M + N)) ≤ integral S (f (M + N))
          by (metis (no_types) inf_commute integral_restrict_Int)
        also have ... ≤ i
          using i'[of M + N] by auto
        finally show integral (cbox a b) (?f (M + N)) ≤ i .
      qed
    qed
    then show ?thesis
      using ‹0 < B› by blast
  qed
  ultimately have (g has_integral i) S
    unfolding has_integral_alt' by auto
  then show ?thesis
    using has_integral_integrable_integral i integral_unique by metis
qed

have sub: ⋀k. integral S (λx. f k x − f 0 x) = integral S (f k) − integral S (f 0)
  by (simp add: integral_diff int_f)
have ∗: ⋀x m n. x ∈ S ⟹ n≥m ⟹ f m x ≤ f n x
  using assms(2) by (force intro: transitive_stepwise_le)
have gf: (λx. g x − f 0 x) integrable_on S ∧ ((λk. integral S (λx. f (Suc k) x −
f 0 x)) ⟶
  integral S (λx. g x − f 0 x)) sequentially
proof (rule lem)
  show ⋀k. (λx. f (Suc k) x − f 0 x) integrable_on S
    by (simp add: integrable_diff int_f)
  show (λk. f (Suc k) x − f 0 x) ⟶ g x − f 0 x if x ∈ S for x
  proof −
```

```
    have (λn. f (Suc n) x) ⟶ g x
      using LIMSEQ_ignore_initial_segment[OF fg[OF ‹x ∈ S›], of 1] by simp
    then show ?thesis
      by (simp add: tendsto_diff)
  qed
  show bounded (range (λk. integral S (λx. f (Suc k) x - f 0 x)))
  proof -
    obtain B where B: ⋀k. norm (integral S (f k)) ≤ B
      using  bou by (auto simp: bounded_iff)
    then have norm (integral S (λx. f (Suc k) x - f 0 x))
          ≤ B + norm (integral S (f 0)) for k
      unfolding sub by (meson add_le_cancel_right norm_triangle_le_diff)
    then show ?thesis
      unfolding bounded_iff by blast
  qed
 qed (use ∗ in auto)
 then have (λx. integral S (λxa. f (Suc x) xa - f 0 xa) + integral S (f 0))
          ⟶ integral S (λx. g x - f 0 x) + integral S (f 0)
   by (auto simp add: tendsto_add)
 moreover have (λx. g x - f 0 x + f 0 x) integrable_on S
   using gf integrable_add int_f [of 0] by metis
 ultimately show ?thesis
   by (simp add: integral_diff int_f LIMSEQ_imp_Suc sub)
qed

lemma has_integral_monotone_convergence_increasing:
 fixes f :: nat ⇒ 'a::euclidean_space ⇒ real
 assumes f: ⋀k. (f k has_integral x k) s
 assumes ⋀k x. x ∈ s ⟹ f k x ≤ f (Suc k) x
 assumes ⋀x. x ∈ s ⟹ (λk. f k x) ⟶ g x
 assumes x ⟶ x'
 shows (g has_integral x') s
proof -
 have x_eq: x = (λi. integral s (f i))
   by (simp add: integral_unique[OF f])
 then have x: range(λk. integral s (f k)) = range x
   by auto
 have ∗: g integrable_on s ∧ (λk. integral s (f k)) ⟶ integral s g
 proof (intro monotone_convergence_increasing allI ballI assms)
   show bounded (range(λk. integral s (f k)))
     using x convergent_imp_bounded assms by metis
 qed (use f in auto)
 then have integral s g = x'
   by (intro LIMSEQ_unique[OF _ ‹x ⟶ x'›]) (simp add: x_eq)
 with ∗ show ?thesis
   by (simp add: has_integral_integral)
qed

lemma monotone_convergence_decreasing:
```

fixes $f :: nat \Rightarrow 'n::euclidean\_space \Rightarrow real$
assumes $intf$: $\bigwedge k.\ (f\ k)\ integrable\_on\ S$
and le: $\bigwedge k\ x.\ x \in S \Longrightarrow f\ (Suc\ k)\ x \leq f\ k\ x$
and fg: $\bigwedge x.\ x \in S \Longrightarrow ((\lambda k.\ f\ k\ x) \longrightarrow g\ x)\ sequentially$
and bou: $bounded\ (range(\lambda k.\ integral\ S\ (f\ k)))$
shows $g\ integrable\_on\ S \wedge (\lambda k.\ integral\ S\ (f\ k)) \longrightarrow integral\ S\ g$
proof −
have $*$: $range(\lambda k.\ integral\ S\ (\lambda x.\ -\ f\ k\ x)) = (*_R)\ (-\ 1)\ `(range(\lambda k.\ integral\ S\ (f\ k)))$
by $force$
have $(\lambda x.\ -\ g\ x)\ integrable\_on\ S \wedge (\lambda k.\ integral\ S\ (\lambda x.\ -\ f\ k\ x)) \longrightarrow integral\ S\ (\lambda x.\ -\ g\ x)$
proof $(rule\ monotone\_convergence\_increasing)$
show $\bigwedge k.\ (\lambda x.\ -\ f\ k\ x)\ integrable\_on\ S$
by $(blast\ intro: integrable\_neg\ intf)$
show $\bigwedge k\ x.\ x \in S \Longrightarrow -\ f\ k\ x \leq -\ f\ (Suc\ k)\ x$
by $(simp\ add: le)$
show $\bigwedge x.\ x \in S \Longrightarrow (\lambda k.\ -\ f\ k\ x) \longrightarrow -\ g\ x$
by $(simp\ add: fg\ tendsto\_minus)$
show $bounded\ (range(\lambda k.\ integral\ S\ (\lambda x.\ -\ f\ k\ x)))$
using $*\ bou\ bounded\_scaling$ **by** $auto$
qed
then show $?thesis$
by $(force\ dest: integrable\_neg\ tendsto\_minus)$
qed

lemma $integral\_norm\_bound\_integral$:
fixes $f :: 'n::euclidean\_space \Rightarrow 'a::banach$
assumes int_f: $f\ integrable\_on\ S$
and int_g: $g\ integrable\_on\ S$
and le_g: $\bigwedge x.\ x \in S \Longrightarrow norm\ (f\ x) \leq g\ x$
shows $norm\ (integral\ S\ f) \leq integral\ S\ g$
proof −
have $norm$: $norm\ \eta \leq y + e$
if $norm\ \zeta \leq x$ **and** $|x - y| < e/2$ **and** $norm\ (\zeta - \eta) < e/2$
for $e\ x\ y$ **and** $\zeta\ \eta :: 'a$
proof −
have $norm\ (\eta - \zeta) < e/2$
by $(metis\ norm\_minus\_commute\ that(3))$
moreover have $x \leq y + e/2$
using $that(2)$ **by** $linarith$
ultimately show $?thesis$
using $that(1)\ le\_less\_trans[OF\ norm\_triangle\_sub[of\ \eta\ \zeta]]$ **by** $(auto\ simp: less\_imp\_le)$
qed
have lem: $norm\ (integral(cbox\ a\ b)\ f) \leq integral\ (cbox\ a\ b)\ g$
if f: $f\ integrable\_on\ cbox\ a\ b$
and g: $g\ integrable\_on\ cbox\ a\ b$
and nle: $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow norm\ (f\ x) \leq g\ x$

```
    for f :: 'n ⇒ 'a and g a b
  proof (rule field_le_epsilon)
    fix e :: real
    assume e > 0
    then have e: e/2 > 0
      by auto
    with integrable_integral[OF f,unfolded has_integral[of f]]
    obtain γ where γ: gauge γ
            ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ γ fine 𝒟
          ⟹ norm ((∑(x, k)∈𝒟. content k *R f x) − integral (cbox a b) f) < e/2
      by meson
    moreover
    from integrable_integral[OF g,unfolded has_integral[of g]] e
    obtain δ where δ: gauge δ
            ⋀𝒟. 𝒟 tagged_division_of cbox a b ∧ δ fine 𝒟
          ⟹ norm ((∑(x, k)∈𝒟. content k *R g x) − integral (cbox a b) g) < e/2
      by meson
    ultimately have gauge (λx. γ x ∩ δ x)
      using gauge_Int by blast
    with fine_division_exists obtain 𝒟
      where p: 𝒟 tagged_division_of cbox a b (λx. γ x ∩ δ x) fine 𝒟
      by metis
    have γ fine 𝒟 δ fine 𝒟
      using fine_Int p(2) by blast+
    show norm (integral (cbox a b) f) ≤ integral (cbox a b) g + e
    proof (rule norm)
      have norm (content K *R f x) ≤ content K *R g x if (x, K) ∈ 𝒟 for x K
      proof−
        have K: x ∈ K K ⊆ cbox a b
          using ‹(x, K) ∈ 𝒟› p(1) by blast+
        obtain u v where K = cbox u v
          using ‹(x, K) ∈ 𝒟› p(1) by blast
        moreover have content K * norm (f x) ≤ content K * g x
          by (meson K(1) K(2) content_pos_le mult_left_mono nle subsetD)
        then show ?thesis
          by simp
      qed
      then show norm (∑(x, k)∈𝒟. content k *R f x) ≤ (∑(x, k)∈𝒟. content k
*R g x)
        by (simp add: sum_norm_le split_def)
      show norm ((∑(x, k)∈𝒟. content k *R f x) − integral (cbox a b) f) < e/2
        using ‹γ fine 𝒟› γ p(1) by simp
      show |(∑(x, k)∈𝒟. content k *R g x) − integral (cbox a b) g| < e/2
        using ‹δ fine 𝒟› δ p(1) by simp
    qed
  qed
  show ?thesis
  proof (rule field_le_epsilon)
    fix e :: real
```

```
    assume e > 0
    then have e: e/2 > 0
      by auto
    let ?f = (λx. if x ∈ S then f x else 0)
    let ?g = (λx. if x ∈ S then g x else 0)
    have f: ?f integrable_on cbox a b and g: ?g integrable_on cbox a b for a b
      using int_f int_g integrable_altD by auto
    obtain Bf where 0 < Bf
      and Bf: ⋀a b. ball 0 Bf ⊆ cbox a b ⟹
        ∃z. (?f has_integral z) (cbox a b) ∧ norm (z − integral S f) < e/2
       using integrable_integral [OF int_f,unfolded has_integral'[of f]] e that by
blast
    obtain Bg where 0 < Bg
      and Bg: ⋀a b. ball 0 Bg ⊆ cbox a b ⟹
        ∃z. (?g has_integral z) (cbox a b) ∧ norm (z − integral S g) < e/2
       using integrable_integral [OF int_g,unfolded has_integral'[of g]] e that by
blast
    obtain a b::'n where ab: ball 0 Bf ∪ ball 0 Bg ⊆ cbox a b
     using ball_max_Un by (metis bounded_ball bounded_subset_cbox_symmetric)
    have ball 0 Bf ⊆ cbox a b
      using ab by auto
    with Bf obtain z where int_fz: (?f has_integral z) (cbox a b) and z: norm
(z − integral S f) < e/2
      by meson
    have ball 0 Bg ⊆ cbox a b
      using ab by auto
   with Bg obtain w where int_gw: (?g has_integral w) (cbox a b) and w: norm
(w − integral S g) < e/2
      by meson
    show norm (integral S f) ≤ integral S g + e
    proof (rule norm)
      show norm (integral (cbox a b) ?f) ≤ integral (cbox a b) ?g
        by (simp add: le_g lem[OF f g, of a b])
      show |integral (cbox a b) ?g − integral S g| < e/2
        using int_gw integral_unique w by auto
      show norm (integral (cbox a b) ?f − integral S f) < e/2
        using int_fz integral_unique z by blast
    qed
  qed
qed

lemma continuous_on_imp_absolutely_integrable_on:
  fixes f::real ⇒ 'a::banach
  shows continuous_on {a..b} f ⟹
    norm (integral {a..b} f) ≤ integral {a..b} (λx. norm (f x))
 by (intro integral_norm_bound_integral integrable_continuous_real continuous_on_norm)
auto

lemma integral_bound:
```

fixes f::*real* ⇒ $'a$::*banach*
assumes $a \le b$
assumes *continuous_on* $\{a\ ..\ b\}$ f
assumes $\bigwedge t.\ t \in \{a\ ..\ b\} \Longrightarrow$ *norm* $(f\ t) \le B$
shows *norm* (*integral* $\{a\ ..\ b\}$ f) $\le B * (b - a)$
proof −
note *continuous_on_imp_absolutely_integrable_on*[*OF assms*(*2*)]
also have *integral* $\{a..b\}$ $(\lambda x.\ norm\ (f\ x)) \le$ *integral* $\{a..b\}$ $(\lambda\_.\ B)$
by (*rule integral_le*)
(*auto intro*!: *integrable_continuous_real continuous_intros assms*)
also have $\ldots = B * (b - a)$ **using** *assms* **by** *simp*
finally show *?thesis* **.**
qed

lemma *integral_norm_bound_integral_component*:
fixes f :: $'n$::*euclidean_space* ⇒ $'a$::*banach*
fixes g :: $'n$ ⇒ $'b$::*euclidean_space*
assumes f: f *integrable_on* S **and** g: g *integrable_on* S
and fg: $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \le (g\ x) \cdot k$
shows *norm* (*integral* S f) $\le$ (*integral* S g)$\cdot k$
proof −
have *norm* (*integral* S f) $\le$ *integral* S $((\lambda x.\ x \cdot k) \circ g)$
using *integral_norm_bound_integral*[*OF f integrable_linear*[*OF g*]]
by (*simp add*: *bounded_linear_inner_left fg*)
then show *?thesis*
unfolding *o_def integral_component_eq*[*OF g*] **.**
qed

lemma *has_integral_norm_bound_integral_component*:
fixes f :: $'n$::*euclidean_space* ⇒ $'a$::*banach*
fixes g :: $'n$ ⇒ $'b$::*euclidean_space*
assumes (f *has_integral* i) S **and** (g *has_integral* j) S
and $\bigwedge x.\ x \in S \Longrightarrow$ *norm* $(f\ x) \le (g\ x) \cdot k$
shows *norm* $i \le j \cdot k$
by (*metis assms has_integral_integrable integral_norm_bound_integral_component integral_unique*)

lemma *uniformly_convergent_improper_integral*:
fixes f :: $'b$ ⇒ *real* ⇒ $'a$ :: {*banach*}
assumes *deriv*: $\bigwedge x.\ x \ge a \Longrightarrow$ (G *has_field_derivative* $g\ x$) (*at* x *within* $\{a..\}$)
assumes *integrable*: $\bigwedge a'\ b\ x.\ x \in A \Longrightarrow a' \ge a \Longrightarrow b \ge a' \Longrightarrow f\ x$ *integrable_on* $\{a'..b\}$
assumes G: *convergent* G
assumes *le*: $\bigwedge y\ x.\ y \in A \Longrightarrow x \ge a \Longrightarrow$ *norm* $(f\ y\ x) \le g\ x$
shows *uniformly_convergent_on* A $(\lambda b\ x.$ *integral* $\{a..b\}$ $(f\ x))$
proof (*intro Cauchy_uniformly_convergent uniformly_Cauchy_onI'*, *goal_cases*)
case (*1* ε)
from G **have** *Cauchy* G

```
    by (auto intro!: convergent_Cauchy)
  with 1 obtain M where M: dist (G (real m)) (G (real n)) < ε if m ≥ M n ≥
M for m n
    by (force simp: Cauchy_def)
  define M' where M' = max (nat ⌈a⌉) M

  show ?case
  proof (rule exI[of _ M'], safe, goal_cases)
    case (1 x m n)
    have M': M' ≥ a M' ≥ M unfolding M'_def by linarith+
    have int_g: (g has_integral (G (real n) − G (real m))) {real m..real n}
      using 1 M' by (intro fundamental_theorem_of_calculus)
                    (auto simp: has_real_derivative_iff_has_vector_derivative
[symmetric]
                    intro!: DERIV_subset[OF deriv])
    have int_f: f x integrable_on {a'..real n} if a' ≥ a for a'
      using that 1 by (cases a' ≤ real n) (auto intro: integrable)

    have dist (integral {a..real m} (f x)) (integral {a..real n} (f x)) =
          norm (integral {a..real n} (f x) − integral {a..real m} (f x))
      by (simp add: dist_norm norm_minus_commute)
    also have integral {a..real m} (f x) + integral {real m..real n} (f x) =
              integral {a..real n} (f x)
      using M' and 1 by (intro integral_combine int_f) auto
    hence integral {a..real n} (f x) − integral {a..real m} (f x) =
          integral {real m..real n} (f x)
      by (simp add: algebra_simps)
    also have norm ... ≤ integral {real m..real n} g
      using le 1 M' int_f int_g by (intro integral_norm_bound_integral) auto
    also from int_g have integral {real m..real n} g = G (real n) − G (real m)
      by (simp add: has_integral_iff)
    also have ... ≤ dist (G m) (G n)
      by (simp add: dist_norm)
    also from 1 and M' have ... < ε
      by (intro M) auto
    finally show ?case .
  qed
qed

lemma uniformly_convergent_improper_integral':
  fixes f :: 'b ⇒ real ⇒ 'a :: {banach, real_normed_algebra}
  assumes deriv: ⋀x. x ≥ a ⟹ (G has_field_derivative g x) (at x within {a..})
  assumes integrable: ⋀a' b x. x ∈ A ⟹ a' ≥ a ⟹ b ≥ a' ⟹ f x integrable_on
{a'..b}
  assumes G: convergent G
  assumes le: eventually (λx. ∀y∈A. norm (f y x) ≤ g x) at_top
  shows   uniformly_convergent_on A (λb x. integral {a..b} (f x))
proof −
  from le obtain a'' where le: ⋀y x. y ∈ A ⟹ x ≥ a'' ⟹ norm (f y x) ≤ g x
```

by (*auto simp*: *eventually_at_top_linorder*)
define a' **where** $a' = max\ a\ a''$

have *uniformly_convergent_on A* ($\lambda b\ x.$ *integral* $\{a'..real\ b\}$ $(f\ x)$)
proof (*rule uniformly_convergent_improper_integral*)
fix t **assume** t: $t \geq a'$
hence (G *has_field_derivative* $g\ t$) (*at* t *within* $\{a..\}$)
by (*intro deriv*) (*auto simp*: a'*_def*)
moreover have $\{a'..\} \subseteq \{a..\}$ **unfolding** a'*_def* **by** *auto*
ultimately show (G *has_field_derivative* $g\ t$) (*at* t *within* $\{a'..\}$)
by (*rule DERIV_subset*)
qed (*insert le*, *auto simp*: a'*_def intro*: *integrable* G)
hence *uniformly_convergent_on A* ($\lambda b\ x.$ *integral* $\{a..a'\}$ $(f\ x)$ + *integral* $\{a'..real\ b\}$ $(f\ x)$)
(**is** *?P*) **by** (*intro uniformly_convergent_add*) *auto*
also have *eventually* ($\lambda x.\ \forall y \in A.$ *integral* $\{a..a'\}$ $(f\ y)$ + *integral* $\{a'..x\}$ $(f\ y)$ = *integral* $\{a..x\}$ $(f\ y)$) *sequentially*
by (*intro eventually_mono* [*OF eventually_ge_at_top*[*of nat* $\lceil a' \rceil$]] *ballI integral_combine*)
(*auto simp*: a'*_def intro*: *integrable*)
hence *?P* $\longleftrightarrow$ *?thesis*
by (*intro uniformly_convergent_cong*) *simp_all*
finally show *?thesis* **.**
qed

7.14.42 differentiation under the integral sign

lemma *integral_continuous_on_param*:
fixes f::'a::*topological_space* $\Rightarrow$ 'b::*euclidean_space* $\Rightarrow$ 'c::*banach*
assumes *cont_fx*: *continuous_on* $(U \times cbox\ a\ b)$ $(\lambda(x,\ t).\ f\ x\ t)$
shows *continuous_on* U $(\lambda x.\ integral\ (cbox\ a\ b)\ (f\ x))$
proof *cases*
assume *content* $(cbox\ a\ b) \neq 0$
then have *ne*: $cbox\ a\ b \neq \{\}$ **by** *auto*

note [*continuous_intros*] =
continuous_on_compose2[*OF cont_fx*, **where** f=$\lambda y.\ Pair\ x\ y$ **for** x,
unfolded split_beta fst_conv snd_conv]

show *?thesis*
unfolding *continuous_on_def*
proof (*intro strip tendstoI*)
fix e'::*real* **and** x
assume $e' > 0$
define e **where** $e = e'\ /\ (content\ (cbox\ a\ b) + 1)$
have $e > 0$ **using** ‹$e' > 0$› **by** (*auto simp*: *e_def intro*!: *divide_pos_pos add_nonneg_pos*)
assume $x \in U$
from *continuous_on_prod_compactE*[*OF cont_fx compact_cbox* ‹$x \in U$› ‹*0*

```
< e›]
    obtain X0 where X0: x ∈ X0 open X0
      and fx_bound: ⋀y t. y ∈ X0 ∩ U ⟹ t ∈ cbox a b ⟹ norm (f y t − f x t)
≤ e
      unfolding split_beta fst_conv snd_conv dist_norm
      by metis
    have ∀F y in at x within U. y ∈ X0 ∩ U
      using X0(1) X0(2) eventually_at_topological by auto
   then show ∀F y in at x within U. dist (integral (cbox a b) (f y)) (integral (cbox
a b) (f x)) < e'
    proof eventually_elim
      case (elim y)
      have dist (integral (cbox a b) (f y)) (integral (cbox a b) (f x)) =
        norm (integral (cbox a b) (λt. f y t − f x t))
        using elim ‹x ∈ U›
        unfolding dist_norm
        by (subst integral_diff)
          (auto intro!: integrable_continuous continuous_intros)
      also have ... ≤ e * content (cbox a b)
        using elim ‹x ∈ U›
        by (intro integrable_bound)
          (auto intro!: fx_bound ‹x ∈ U › less_imp_le[OF ‹0 < e›]
            integrable_continuous continuous_intros)
      also have ... < e'
        using ‹0 < e'› ‹e > 0›
        by (auto simp: e_def field_split_simps)
      finally show dist (integral (cbox a b) (f y)) (integral (cbox a b) (f x)) < e' .
    qed
  qed
qed (auto intro!: continuous_on_const)

lemma leibniz_rule:
  fixes f::'a::banach ⇒ 'b::euclidean_space ⇒ 'c::banach
  assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹
    ((λx. f x t) has_derivative blinfun_apply (fx x t)) (at x within U)
  assumes integrable_f2: ⋀x. x ∈ U ⟹ f x integrable_on cbox a b
  assumes cont_fx: continuous_on (U × (cbox a b)) (λ(x, t). fx x t)
  assumes [intro]: x0 ∈ U
  assumes convex U
  shows
    ((λx. integral (cbox a b) (f x)) has_derivative integral (cbox a b) (fx x0)) (at x0
within U)
    (is (?F has_derivative ?dF) _)
proof cases
  assume content (cbox a b) ≠ 0
  then have ne: cbox a b ≠ {} by auto
  note [continuous_intros] =
    continuous_on_compose2[OF cont_fx, where f=λy. Pair x y for x,
      unfolded split_beta fst_conv snd_conv]
```

```
  show ?thesis
  proof (intro has_derivativeI bounded_linear_scaleR_left tendstoI, fold norm_conv_dist)
    have cont_f1: ⋀t. t ∈ cbox a b ⟹ continuous_on U (λx. f x t)
    by (auto simp: continuous_on_eq_continuous_within intro!: has_derivative_continuous
fx)
    note [continuous_intros] = continuous_on_compose2[OF cont_f1]
    fix e'::real
    assume e' > 0
    define e where e = e' / (content (cbox a b) + 1)
     have e > 0 using ‹e' > 0› by (auto simp: e_def intro!: divide_pos_pos
add_nonneg_pos)
    from continuous_on_prod_compactE[OF cont_fx compact_cbox ‹x0 ∈ U› ‹e
> 0›]
    obtain X0 where X0: x0 ∈ X0 open X0
      and fx_bound: ⋀x t. x ∈ X0 ∩ U ⟹ t ∈ cbox a b ⟹ norm (fx x t − fx x0
t) ≤ e
      unfolding split_beta fst_conv snd_conv
      by (metis dist_norm)

    note eventually_closed_segment[OF ‹open X0› ‹x0 ∈ X0›, of U]
    moreover
    have ∀F x in at x0 within U. x ∈ X0
      using ‹open X0› ‹x0 ∈ X0› eventually_at_topological by blast
    moreover have ∀F x in at x0 within U. x ≠ x0
      by (auto simp: eventually_at_filter)
    moreover have ∀F x in at x0 within U. x ∈ U
      by (auto simp: eventually_at_filter)
    ultimately
   show ∀F x in at x0 within U. norm ((?F x − ?F x0 − ?dF (x − x0)) /R norm
(x − x0)) < e'
    proof eventually_elim
      case (elim x)
      from elim have 0 < norm (x − x0) by simp
      have closed_segment x0 x ⊆ U
        by (simp add: assms closed_segment_subset elim(4))
      from elim have [intro]: x ∈ U by auto
      have ?F x − ?F x0 − ?dF (x − x0) =
        integral (cbox a b) (λy. f x y − f x0 y − fx x0 y (x − x0))
        (is _ = ?id)
        using ‹x ≠ x0›
        by (subst blinfun_apply_integral integral_diff,
            auto intro!: integrable_diff integrable_f2 continuous_intros
              intro: integrable_continuous)+
      also
      {
        fix t assume t: t ∈ (cbox a b)
        then have deriv:
          ((λx. f x t) has_derivative (fx y t)) (at y within X0 ∩ U)
          if y ∈ X0 ∩ U for y
```

```
      using fx has_derivative_subset that by fastforce
    have seg: ⋀t. t ∈ {0..1} ⟹ x0 + t *R (x − x0) ∈ X0 ∩ U
      using ‹closed_segment x0 x ⊆ U›
        ‹closed_segment x0 x ⊆ X0›
      by (force simp: closed_segment_def algebra_simps)
    have ⋀x. x ∈ X0 ∩ U ⟹ onorm (blinfun_apply (fx x t) − (fx x0 t)) ≤ e
      using fx_bound t
     by (auto simp add: norm_blinfun_def fun_diff_def blinfun.bilinear_simps[symmetric])
    from differentiable_bound_linearization[OF seg deriv this] X0
    have norm (f x t − f x0 t − fx x0 t (x − x0)) ≤ e * norm (x − x0)
      by (auto simp add: ac_simps)
   }
   then have norm ?id ≤ integral (cbox a b) (λ_. e * norm (x − x0))
    by (intro integral_norm_bound_integral)
      (auto intro!: continuous_intros integrable_diff integrable_f2
        intro: integrable_continuous)
   also have ... = content (cbox a b) * e * norm (x − x0)
    by simp
   also have ... < e' * norm (x − x0)
   proof (intro mult_strict_right_mono[OF _ ‹0 < norm (x − x0)›])
    show content (cbox a b) * e < e'
      using ‹e' > 0› by (simp add: divide_simps e_def not_less)
   qed
   finally have norm (?F x − ?F x0 − ?dF (x − x0)) < e' * norm (x − x0) .
   then show ?case
    by (auto simp: divide_simps)
  qed
 qed (rule blinfun.bounded_linear_right)
qed (auto intro!: derivative_eq_intros simp: blinfun.bilinear_simps)

lemma has_vector_derivative_eq_has_derivative_blinfun:
 (f has_vector_derivative f') (at x within U) ⟷
  (f has_derivative blinfun_scaleR_left f') (at x within U)
 by (simp add: has_vector_derivative_def)

lemma leibniz_rule_vector_derivative:
 fixes f::real ⇒ 'b::euclidean_space ⇒ 'c::banach
 assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹
     ((λx. f x t) has_vector_derivative (fx x t)) (at x within U)
 assumes integrable_f2: ⋀x. x ∈ U ⟹ (f x) integrable_on cbox a b
 assumes cont_fx: continuous_on (U × cbox a b) (λ(x, t). fx x t)
 assumes U: x0 ∈ U convex U
 shows ((λx. integral (cbox a b) (f x)) has_vector_derivative integral (cbox a b)
(fx x0))
     (at x0 within U)
proof −
 note [continuous_intros] =
   continuous_on_compose2[OF cont_fx, where f=λy. Pair x y for x,
     unfolded split_beta fst_conv snd_conv]
```

```
  show ?thesis
    unfolding has_vector_derivative_eq_has_derivative_blinfun
  proof (rule has_derivative_eq_rhs [OF leibniz_rule[OF _ integrable_f2 _ U]])
    show continuous_on (U × cbox a b) (λ(x, t). blinfun_scaleR_left (fx x t))
      using cont_fx by (auto simp: split_beta intro!: continuous_intros)
    show blinfun_apply (integral (cbox a b) (λt. blinfun_scaleR_left (fx x0 t))) =
          blinfun_apply (blinfun_scaleR_left (integral (cbox a b) (fx x0)))
    by (subst integral_linear[symmetric])
       (auto simp: has_vector_derivative_def o_def
         intro!: integrable_continuous U continuous_intros bounded_linear_intros)
    qed (use fx in ‹auto simp: has_vector_derivative_def›)
qed

lemma has_field_derivative_eq_has_derivative_blinfun:
  (f has_field_derivative f') (at x within U) ⟷ (f has_derivative blinfun_mult_right
f') (at x within U)
  by (simp add: has_field_derivative_def)

lemma leibniz_rule_field_derivative:
  fixes f::'a::{real_normed_field, banach} ⇒ 'b::euclidean_space ⇒ 'a
  assumes fx: ⋀x t. x ∈ U ⟹ t ∈ cbox a b ⟹ ((λx. f x t) has_field_derivative
fx x t) (at x within U)
  assumes integrable_f2: ⋀x. x ∈ U ⟹ (f x) integrable_on cbox a b
  assumes cont_fx: continuous_on (U × (cbox a b)) (λ(x, t). fx x t)
  assumes U: x0 ∈ U convex U
  shows ((λx. integral (cbox a b) (f x)) has_field_derivative integral (cbox a b) (fx
x0)) (at x0 within U)
proof −
  note [continuous_intros] =
    continuous_on_compose2[OF cont_fx, where f=λy. Pair x y for x,
      unfolded split_beta fst_conv snd_conv]
  have ∗: blinfun_mult_right (integral (cbox a b) (fx x0)) =
    integral (cbox a b) (λt. blinfun_mult_right (fx x0 t))
    by (subst integral_linear[symmetric])
      (auto simp: has_vector_derivative_def o_def
        intro!: integrable_continuous U continuous_intros bounded_linear_intros)
  show ?thesis
    unfolding has_field_derivative_eq_has_derivative_blinfun
  proof (rule has_derivative_eq_rhs [OF leibniz_rule[OF _ integrable_f2 _ U,
where fx=λx t. blinfun_mult_right (fx x t)]])
    show continuous_on (U × cbox a b) (λ(x, t). blinfun_mult_right (fx x t))
      using cont_fx by (auto simp: split_beta intro!: continuous_intros)
    show blinfun_apply (integral (cbox a b) (λt. blinfun_mult_right (fx x0 t))) =
          blinfun_apply (blinfun_mult_right (integral (cbox a b) (fx x0)))
      by (subst integral_linear[symmetric])
        (auto simp: has_vector_derivative_def o_def
          intro!: integrable_continuous U continuous_intros bounded_linear_intros)
  qed (use fx in ‹auto simp: has_field_derivative_def›)
qed
```

lemma *leibniz_rule_field_differentiable*:
fixes *f*::*'a*::{*real_normed_field*, *banach*} ⇒ *'b*::*euclidean_space* ⇒ *'a*
assumes ⋀*x t*. *x* ∈ *U* ⟹ *t* ∈ *cbox a b* ⟹ ((λ*x*. *f x t*) *has_field_derivative fx x t*) (*at x within U*)
assumes ⋀*x*. *x* ∈ *U* ⟹ (*f x*) *integrable_on cbox a b*
assumes *continuous_on* (*U* × (*cbox a b*)) (λ(*x*, *t*). *fx x t*)
assumes *x0* ∈ *U convex U*
shows (λ*x*. *integral* (*cbox a b*) (*f x*)) *field_differentiable at x0 within U*
using *leibniz_rule_field_derivative*[*OF assms*] **by** (*auto simp*: *field_differentiable_def*)

7.14.43 Exchange uniform limit and integral

lemma *uniform_limit_integral_cbox*:
fixes *f*::*'a* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*banach*
assumes *u*: *uniform_limit* (*cbox a b*) *f g F*
assumes *c*: ⋀*n*. *continuous_on* (*cbox a b*) (*f n*)
assumes [*simp*]: *F* ≠ *bot*
obtains *I J* **where**
⋀*n*. (*f n has_integral I n*) (*cbox a b*)
(*g has_integral J*) (*cbox a b*)
(*I* ⟶ *J*) *F*
proof −
have *fi*[*simp*]: *f n integrable_on* (*cbox a b*) **for** *n*
by (*auto intro*!: *integrable_continuous assms*)
then obtain *I* **where** *I*: ⋀*n*. (*f n has_integral I n*) (*cbox a b*)
unfolding *integrable_on_def* **by** *metis*

moreover
have *gi*[*simp*]: *g integrable_on* (*cbox a b*)
by (*auto intro*!: *integrable_continuous uniform_limit_theorem*[*OF _ u*] *eventuallyI c*)
then obtain *J* **where** *J*: (*g has_integral J*) (*cbox a b*)
by *blast*

moreover
have (*I* ⟶ *J*) *F*
proof *cases*
assume *content* (*cbox a b*) = *0*
hence *I* = (λ_. *0*) *J* = *0*
by (*auto intro*!: *has_integral_unique I J*)
thus *?thesis* **by** *simp*
next
assume *content_nonzero*: *content* (*cbox a b*) ≠ *0*
show *?thesis*
proof (*rule tendstoI*)
fix *e*::*real*
assume *e* > *0*
define *e'* **where** *e'* = *e*/*2*

```
      with ‹e > 0› have e' > 0 by simp
      then have ∀F n in F. ∀x∈cbox a b. norm (f n x − g x) < e' / content (cbox
a b)
          using u content_nonzero by (auto simp: uniform_limit_iff dist_norm
zero_less_measure_iff)
      then show ∀F n in F. dist (I n) J < e
      proof eventually_elim
        case (elim n)
        have I n = integral (cbox a b) (f n)
            J = integral (cbox a b) g
          using I[of n] J by (simp_all add: integral_unique)
        then have dist (I n) J = norm (integral (cbox a b) (λx. f n x − g x))
          by (simp add: integral_diff dist_norm)
        also have ... ≤ integral (cbox a b) (λx. (e' / content (cbox a b)))
          using elim
          by (intro integral_norm_bound_integral) (auto intro!: integrable_diff)
        also have ... < e
          using ‹0 < e› by (simp add: e'_def)
        finally show ?case .
      qed
    qed
  qed
  ultimately show ?thesis ..
qed

lemma uniform_limit_integral:
  fixes f::'a ⇒ 'b::ordered_euclidean_space ⇒ 'c::banach
  assumes u: uniform_limit {a..b} f g F
  assumes c: ⋀n. continuous_on {a..b} (f n)
  assumes [simp]: F ≠ bot
  obtains I J where
    ⋀n. (f n has_integral I n) {a..b}
    (g has_integral J) {a..b}
    (I ⟶ J) F
  by (metis interval_cbox assms uniform_limit_integral_cbox)
```

7.14.44 Integration by parts

```
lemma integration_by_parts_interior_strong:
  fixes prod :: _ ⇒ _ ⇒ 'b :: banach
  assumes bilinear: bounded_bilinear (prod)
  assumes s: finite s and le: a ≤ b
  assumes cont [continuous_intros]: continuous_on {a..b} f continuous_on {a..b}
g
  assumes deriv: ⋀x. x∈{a<..<b} − s ⟹ (f has_vector_derivative f' x) (at x)
                 ⋀x. x∈{a<..<b} − s ⟹ (g has_vector_derivative g' x) (at x)
  assumes int: ((λx. prod (f x) (g' x)) has_integral
                (prod (f b) (g b) − prod (f a) (g a) − y)) {a..b}
  shows   ((λx. prod (f' x) (g x)) has_integral y) {a..b}
```

proof −
interpret *bounded_bilinear prod* **by** *fact*
have ((λ*x*. *prod* (*f x*) (*g′ x*) + *prod* (*f′ x*) (*g x*)) *has_integral*
(*prod* (*f b*) (*g b*) − *prod* (*f a*) (*g a*))) {*a..b*}
using *deriv* **by** (*intro fundamental_theorem_of_calculus_interior_strong*[*OF s le*])
(*auto intro*!: *continuous_intros continuous_on has_vector_derivative*)
from *has_integral_diff*[*OF this int*] **show** *?thesis* **by** (*simp add*: *algebra_simps*)
qed

lemma *integration_by_parts_interior*:
fixes *prod* :: _ ⇒ _ ⇒ *′b* :: *banach*
assumes *bounded_bilinear* (*prod*) *a* ≤ *b*
continuous_on {*a..b*} *f continuous_on* {*a..b*} *g*
assumes ⋀*x*. *x*∈{*a*<..<*b*} ⟹ (*f has_vector_derivative f′ x*) (*at x*)
⋀*x*. *x*∈{*a*<..<*b*} ⟹ (*g has_vector_derivative g′ x*) (*at x*)
assumes ((λ*x*. *prod* (*f x*) (*g′ x*)) *has_integral* (*prod* (*f b*) (*g b*) − *prod* (*f a*) (*g a*) − *y*)) {*a..b*}
shows ((λ*x*. *prod* (*f′ x*) (*g x*)) *has_integral y*) {*a..b*}
by (*rule integration_by_parts_interior_strong*[*of* _ {} _ _ *f g f′ g′*]) (*use assms* **in** *simp_all*)

lemma *integration_by_parts*:
fixes *prod* :: _ ⇒ _ ⇒ *′b* :: *banach*
assumes *bounded_bilinear* (*prod*) *a* ≤ *b*
continuous_on {*a..b*} *f continuous_on* {*a..b*} *g*
assumes ⋀*x*. *x*∈{*a..b*} ⟹ (*f has_vector_derivative f′ x*) (*at x*)
⋀*x*. *x*∈{*a..b*} ⟹ (*g has_vector_derivative g′ x*) (*at x*)
assumes ((λ*x*. *prod* (*f x*) (*g′ x*)) *has_integral* (*prod* (*f b*) (*g b*) − *prod* (*f a*) (*g a*) − *y*)) {*a..b*}
shows ((λ*x*. *prod* (*f′ x*) (*g x*)) *has_integral y*) {*a..b*}
by (*rule integration_by_parts_interior*[*of* _ _ _ *f g f′ g′*]) (*use assms* **in** *simp_all*)

lemma *integrable_by_parts_interior_strong*:
fixes *prod* :: _ ⇒ _ ⇒ *′b* :: *banach*
assumes *bilinear*: *bounded_bilinear* (*prod*)
assumes *s*: *finite s* **and** *le*: *a* ≤ *b*
assumes *cont* [*continuous_intros*]: *continuous_on* {*a..b*} *f continuous_on* {*a..b*} *g*
assumes *deriv*: ⋀*x*. *x*∈{*a*<..<*b*} − *s* ⟹ (*f has_vector_derivative f′ x*) (*at x*)
⋀*x*. *x*∈{*a*<..<*b*} − *s* ⟹ (*g has_vector_derivative g′ x*) (*at x*)
assumes *int*: (λ*x*. *prod* (*f x*) (*g′ x*)) *integrable_on* {*a..b*}
shows (λ*x*. *prod* (*f′ x*) (*g x*)) *integrable_on* {*a..b*}
proof −
from *int* **obtain** *I* **where** ((λ*x*. *prod* (*f x*) (*g′ x*)) *has_integral I*) {*a..b*}
unfolding *integrable_on_def* **by** *blast*
hence ((λ*x*. *prod* (*f x*) (*g′ x*)) *has_integral* (*prod* (*f b*) (*g b*) − *prod* (*f a*) (*g a*) − (*prod* (*f b*) (*g b*) − *prod* (*f a*) (*g a*) − *I*))) {*a..b*} **by** *simp*

from *integration_by_parts_interior_strong*[*OF assms*(1−7) *this*]
show *?thesis* **unfolding** *integrable_on_def* **by** *blast*
qed

lemma *integrable_by_parts_interior*:
fixes *prod* :: _ ⇒ _ ⇒ 'b :: *banach*
assumes *bounded_bilinear* (*prod*) *a* ≤ *b*
continuous_on {*a*..*b*} *f continuous_on* {*a*..*b*} *g*
assumes ⋀*x*. *x*∈{*a*<..<*b*} ⟹ (*f has_vector_derivative f'* *x*) (*at x*)
⋀*x*. *x*∈{*a*<..<*b*} ⟹ (*g has_vector_derivative g'* *x*) (*at x*)
assumes (λ*x*. *prod* (*f x*) (*g' x*)) *integrable_on* {*a*..*b*}
shows (λ*x*. *prod* (*f' x*) (*g x*)) *integrable_on* {*a*..*b*}
by (*rule integrable_by_parts_interior_strong*[*of* _ {} _ _ *f g f' g'*]) (*use assms*
in *simp_all*)

lemma *integrable_by_parts*:
fixes *prod* :: _ ⇒ _ ⇒ 'b :: *banach*
assumes *bounded_bilinear* (*prod*) *a* ≤ *b*
continuous_on {*a*..*b*} *f continuous_on* {*a*..*b*} *g*
assumes ⋀*x*. *x*∈{*a*..*b*} ⟹ (*f has_vector_derivative f'* *x*) (*at x*)
⋀*x*. *x*∈{*a*..*b*} ⟹ (*g has_vector_derivative g'* *x*) (*at x*)
assumes (λ*x*. *prod* (*f x*) (*g' x*)) *integrable_on* {*a*..*b*}
shows (λ*x*. *prod* (*f' x*) (*g x*)) *integrable_on* {*a*..*b*}
by (*rule integrable_by_parts_interior_strong*[*of* _ {} _ _ *f g f' g'*]) (*use assms*
in *simp_all*)

7.14.45 Integration by substitution

lemma *has_integral_substitution_general*:
fixes *f* :: *real* ⇒ 'a::*euclidean_space* **and** *g* :: *real* ⇒ *real*
assumes *s*: *finite s* **and** *le*: *a* ≤ *b*
and *subset*: *g* ' {*a*..*b*} ⊆ {*c*..*d*}
and *f* [*continuous_intros*]: *continuous_on* {*c*..*d*} *f*
and *g* [*continuous_intros*]: *continuous_on* {*a*..*b*} *g*
and *deriv* [*derivative_intros*]:
⋀*x*. *x* ∈ {*a*..*b*} − *s* ⟹ (*g has_field_derivative g'* *x*) (*at x within* {*a*..*b*})
shows ((λ*x*. *g'* *x* ∗*R* *f* (*g x*)) *has_integral* (*integral* {*g a*..*g b*} *f* − *integral* {*g*
b..*g a*} *f*)) {*a*..*b*}
proof −
let *?F* = λ*x*. *integral* {*c*..*g x*} *f*
have *cont_int*: *continuous_on* {*a*..*b*} *?F*
by (*rule continuous_on_compose2*[*OF* _ *g subset*] *indefinite_integral_continuous_1*
f integrable_continuous_real)+
have *deriv*: (((λ*x*. *integral* {*c*..*x*} *f*) ∘ *g*) *has_vector_derivative g'* *x* ∗*R* *f* (*g x*))
(*at x within* {*a*..*b*}) **if** *x* ∈ {*a*..*b*} − *s* **for** *x*
proof (*rule has_vector_derivative_eq_rhs* [*OF vector_diff_chain_within refl*])
show (*g has_vector_derivative g'* *x*) (*at x within* {*a*..*b*})
using *deriv has_real_derivative_iff_has_vector_derivative that* **by** *blast*
show ((λ*x*. *integral* {*c*..*x*} *f*) *has_vector_derivative f* (*g x*))

```
        (at (g x) within g ' {a..b})
    using that le subset
   by (blast intro: has_vector_derivative_within_subset integral_has_vector_derivative
f)
  qed
  have deriv: (?F has_vector_derivative g' x *R f (g x))
                (at x) if x ∈ {a..b} - (s ∪ {a,b}) for x
    using deriv[of x] that by (simp add: at_within_Icc_at o_def)
  have ((λx. g' x *R f (g x)) has_integral (?F b - ?F a)) {a..b}
    using le cont_int s deriv cont_int
     by (intro fundamental_theorem_of_calculus_interior_strong[of s ∪ {a,b}])
simp_all
  also
  from subset have g x ∈ {c..d} if x ∈ {a..b} for x using that by blast
  from this[of a] this[of b] le have cd: c ≤ g a g b ≤ d c ≤ g b g a ≤ d by auto
  have integral {c..g b} f - integral {c..g a} f = integral {g a..g b} f - integral
{g b..g a} f
  proof cases
    assume g a ≤ g b
    note le = le this
    from cd have integral {c..g a} f + integral {g a..g b} f = integral {c..g b} f
    by (intro integral_combine integrable_continuous_real continuous_on_subset[OF
f] le) simp_all
    with le show ?thesis
      by (cases g a = g b) (simp_all add: algebra_simps)
  next
    assume less: ¬g a ≤ g b
    then have g a ≥ g b by simp
    note le = le this
    from cd have integral {c..g b} f + integral {g b..g a} f = integral {c..g a} f
    by (intro integral_combine integrable_continuous_real continuous_on_subset[OF
f] le) simp_all
    with less show ?thesis
      by (simp_all add: algebra_simps)
  qed
  finally show ?thesis .
qed

lemma has_integral_substitution_strong:
  fixes f :: real ⇒ 'a::euclidean_space and g :: real ⇒ real
  assumes s: finite s and le: a ≤ b g a ≤ g b
    and subset: g ' {a..b} ⊆ {c..d}
    and f [continuous_intros]: continuous_on {c..d} f
    and g [continuous_intros]: continuous_on {a..b} g
    and deriv [derivative_intros]:
    ⋀x. x ∈ {a..b} - s ⟹ (g has_field_derivative g' x) (at x within {a..b})
  shows ((λx. g' x *R f (g x)) has_integral (integral {g a..g b} f)) {a..b}
  using has_integral_substitution_general[OF s le(1) subset f g deriv] le(2)
  by (cases g a = g b) auto
```

lemma *has_integral_substitution*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$ **and** $g :: real \Rightarrow real$
assumes $a \leq b$ $g\ a \leq g\ b$ g ' $\{a..b\} \subseteq \{c..d\}$
and *continuous_on* $\{c..d\}$ f
and $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g$ *has_field_derivative* $g'\ x)$ *(at x within* $\{a..b\})$
shows $((\lambda x.\ g'\ x *_R f\ (g\ x))$ *has_integral* $(integral\ \{g\ a..g\ b\}\ f))\ \{a..b\}$
by (*intro has_integral_substitution_strong*[*of* $\{\}$ a b g c d] *assms*)
(*auto intro*: *DERIV_continuous_on assms*)

lemma *integral_shift*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
assumes *cont*: *continuous_on* $\{a + c..b + c\}$ f
shows $integral\ \{a..b\}\ (f \circ (\lambda x.\ x + c)) = integral\ \{a + c..b + c\}\ f$
proof (*cases* $a \leq b$)
case *True*
have $((\lambda x.\ 1 *_R f\ (x + c))$ *has_integral integral* $\{a+c..b+c\}\ f)\ \{a..b\}$
using *True cont*
by (*intro has_integral_substitution*[**where** $c = a + c$ **and** $d = b + c$])
(*auto intro*!: *derivative_eq_intros*)
thus *?thesis* **by** (*simp add*: *has_integral_iff o_def*)
qed *auto*

7.14.46 Compute a double integral using iterated integrals and switching the order of integration

lemma *continuous_on_imp_integrable_on_Pair1*:
fixes $f :: \_ \Rightarrow 'b::banach$
assumes *con*: *continuous_on* $(cbox\ (a,c)\ (b,d))\ f$ **and** x: $x \in cbox\ a\ b$
shows $(\lambda y.\ f\ (x,\ y))$ *integrable_on* $(cbox\ c\ d)$
proof −
have $f \circ (\lambda y.\ (x,\ y))$ *integrable_on* $(cbox\ c\ d)$
proof (*intro integrable_continuous continuous_on_compose* [*OF* _ *continuous_on_subset* [*OF con*]])
show *continuous_on* $(cbox\ c\ d)$ $(Pair\ x)$
by (*simp add*: *continuous_on_Pair*)
show $Pair\ x$ ' $cbox\ c\ d \subseteq cbox\ (a,c)\ (b,d)$
using x **by** *blast*
qed
then show *?thesis*
by (*simp add*: *o_def*)
qed

lemma *integral_integrable_2dim*:
fixes $f :: ('a::euclidean\_space * 'b::euclidean\_space) \Rightarrow 'c::banach$
assumes *continuous_on* $(cbox\ (a,c)\ (b,d))\ f$
shows $(\lambda x.\ integral\ (cbox\ c\ d)\ (\lambda y.\ f\ (x,y)))$ *integrable_on cbox a b*
proof (*cases content*$(cbox\ c\ d) = 0$)
case *True*

```
  then show ?thesis
    by (simp add: True integrable_const)
next
  case False
  have uc: uniformly_continuous_on (cbox (a,c) (b,d)) f
    by (simp add: assms compact_cbox compact_uniformly_continuous)
  { fix x::'a and e::real
    assume x: x ∈ cbox a b and e: 0 < e
    then have e2_gt: 0 < e/2 / content (cbox c d) and e2_less: e/2 / content
(cbox c d) * content (cbox c d) < e
      by (auto simp: False content_lt_nz e)
    then obtain dd
    where dd: ⋀x x'. ⟦x∈cbox (a, c) (b, d); x'∈cbox (a, c) (b, d); norm (x' − x)
< dd⟧
                    ⟹ norm (f x' − f x) ≤ e/(2 * content (cbox c d))  dd>0
      using uc [unfolded uniformly_continuous_on_def, THEN spec, of e/(2 *
content (cbox c d))]
      by (auto simp: dist_norm intro: less_imp_le)
   have ∃ delta>0. ∀ x'∈cbox a b. norm (x' − x) < delta ⟶ norm (integral (cbox
c d) (λu. f (x', u) − f (x, u))) < e
      using dd e2_gt assms x
      apply (rule_tac x=dd in exI)
      apply clarify
      apply (rule le_less_trans [OF integrable_bound e2_less])
     apply (auto intro: integrable_diff continuous_on_imp_integrable_on_Pair1)
      done
  } note * = this
  show ?thesis
  proof (rule integrable_continuous)
    show continuous_on (cbox a b) (λx. integral (cbox c d) (λy. f (x, y)))
      by (simp add: * continuous_on_iff dist_norm integral_diff [symmetric]
continuous_on_imp_integrable_on_Pair1 [OF assms])
  qed
qed

lemma integral_split:
  fixes f :: 'a::euclidean_space ⇒ 'b::{real_normed_vector,complete_space}
  assumes f: f integrable_on (cbox a b)
      and k: k ∈ Basis
  shows integral (cbox a b) f =
          integral (cbox a b ∩ {x. x•k ≤ c}) f +
          integral (cbox a b ∩ {x. x•k ≥ c}) f
  using k f
 by (auto simp: has_integral_integral intro: integral_unique [OF has_integral_split])

lemma integral_swap_operativeI:
  fixes f :: ('a::euclidean_space * 'b::euclidean_space) ⇒ 'c::banach
  assumes f: continuous_on s f and e: 0 < e
    shows operative conj True
```

```
        (λk. ∀ a b c d.
            cbox (a,c) (b,d) ⊆ k ∧ cbox (a,c) (b,d) ⊆ s
            ⟶ norm(integral (cbox (a,c) (b,d)) f −
                    integral (cbox a b) (λx. integral (cbox c d) (λy. f((x,y)))))
               ≤ e * content (cbox (a,c) (b,d)))
proof (standard, auto)
  fix a::'a and c::'b and b::'a and d::'b and u::'a and v::'a and w::'b and z::'b
  assume *: box (a, c) (b, d) = {}
    and cb1: cbox (u, w) (v, z) ⊆ cbox (a, c) (b, d)
    and cb2: cbox (u, w) (v, z) ⊆ s
  then have c0: content (cbox (a, c) (b, d)) = 0
    using * unfolding content_eq_0_interior by simp
  have c0': content (cbox (u, w) (v, z)) = 0
    by (fact content_0_subset [OF c0 cb1])
  show norm (integral (cbox (u,w) (v,z)) f − integral (cbox u v) (λx. integral (cbox
w z) (λy. f (x, y))))
         ≤ e * content (cbox (u,w) (v,z))
    using content_cbox_pair_eq0_D [OF c0']
    by (force simp add: c0')
next
  fix a::'a and c::'b and b::'a and d::'b
  and M::real and i::'a and j::'b
  and u::'a and v::'a and w::'b and z::'b
  assume ij: (i,j) ∈ Basis
    and n1: ∀ a' b' c' d'.
              cbox (a',c') (b',d') ⊆ cbox (a,c) (b,d) ∧
             cbox (a',c') (b',d') ⊆ {x. x • (i,j) ≤ M} ∧ cbox (a',c') (b',d') ⊆ s ⟶
               norm (integral (cbox (a',c') (b',d')) f − integral (cbox a' b') (λx.
integral (cbox c' d') (λy. f (x,y))))
              ≤ e * content (cbox (a',c') (b',d'))
    and n2: ∀ a' b' c' d'.
              cbox (a',c') (b',d') ⊆ cbox (a,c) (b,d) ∧
             cbox (a',c') (b',d') ⊆ {x. M ≤ x • (i,j)} ∧ cbox (a',c') (b',d') ⊆ s ⟶
               norm (integral (cbox (a',c') (b',d')) f − integral (cbox a' b') (λx.
integral (cbox c' d') (λy. f (x,y))))
              ≤ e * content (cbox (a',c') (b',d'))
    and subs: cbox (u,w) (v,z) ⊆ cbox (a,c) (b,d)  cbox (u,w) (v,z) ⊆ s
  have fcont: continuous_on (cbox (u, w) (v, z)) f
    using assms(1) continuous_on_subset subs(2) by blast
  then have fint: f integrable_on cbox (u, w) (v, z)
    by (metis integrable_continuous)
  consider i ∈ Basis j=0 | j ∈ Basis i=0 using ij
    by (auto simp: Euclidean_Space.Basis_prod_def)
  then show norm (integral (cbox (u,w) (v,z)) f − integral (cbox u v) (λx. integral
(cbox w z) (λy. f (x,y))))
           ≤ e * content (cbox (u,w) (v,z)) (is ?normle)
  proof cases
    case 1
    then have e: e * content (cbox (u, w) (v, z)) =
```

```
                e * (content (cbox u v ∩ {x. x • i ≤ M}) * content (cbox w z)) +
                e * (content (cbox u v ∩ {x. M ≤ x • i}) * content (cbox w z))
    by (simp add: content_split [where c=M] content_Pair algebra_simps)
  have *: integral (cbox u v) (λx. integral (cbox w z) (λy. f (x, y))) =
            integral (cbox u v ∩ {x. x • i ≤ M}) (λx. integral (cbox w z) (λy. f
(x, y))) +
            integral (cbox u v ∩ {x. M ≤ x • i}) (λx. integral (cbox w z) (λy. f
(x, y)))
    using 1 f subs integral_integrable_2dim continuous_on_subset
    by (blast intro: integral_split)
  show ?normle
    apply (rule norm_diff2 [OF integral_split [where c=M, OF fint ij] * e])
    using 1 subs
  apply (simp_all add: cbox_Pair_eq setcomp_dot1 [of λu. M≤u] setcomp_dot1
[of λu. u≤M] Sigma_Int_Paircomp1)
    apply (simp_all add: interval_split ij flip: cbox_Pair_eq content_Pair)
    apply (force simp flip: interval_split intro!: n1 [rule_format])
    apply (force simp flip: interval_split intro!: n2 [rule_format])
    done
next
  case 2
  then have e: e * content (cbox (u, w) (v, z)) =
            e * (content (cbox u v) * content (cbox w z ∩ {x. x • j ≤ M})) +
            e * (content (cbox u v) * content (cbox w z ∩ {x. M ≤ x • j}))
    by (simp add: content_split [where c=M] content_Pair algebra_simps)
  have (λx. integral (cbox w z ∩ {x. x • j ≤ M}) (λy. f (x, y))) integrable_on
cbox u v
      (λx. integral (cbox w z ∩ {x. M ≤ x • j}) (λy. f (x, y))) integrable_on cbox
u v
    using 2 subs
    apply (simp_all add: interval_split)
     apply (rule integral_integrable_2dim [OF continuous_on_subset [OF f]];
auto simp: cbox_Pair_eq interval_split [symmetric])+
    done
  with 2 have *: integral (cbox u v) (λx. integral (cbox w z) (λy. f (x, y))) =
            integral (cbox u v) (λx. integral (cbox w z ∩ {x. x • j ≤ M}) (λy. f
(x, y))) +
            integral (cbox u v) (λx. integral (cbox w z ∩ {x. M ≤ x • j}) (λy. f
(x, y)))
    by (simp add: integral_add [symmetric] integral_split [symmetric]
                  continuous_on_imp_integrable_on_Pair1 [OF fcont] cong:
integral_cong)
  show ?normle
    apply (rule norm_diff2 [OF integral_split [where c=M, OF fint ij] * e])
    using 2 subs
  apply (simp_all add: cbox_Pair_eq setcomp_dot2 [of λu. M≤u] setcomp_dot2
[of λu. u≤M] Sigma_Int_Paircomp2)
    apply (simp_all add: interval_split ij flip: cbox_Pair_eq content_Pair)
    apply (force simp flip: interval_split intro!: n1 [rule_format])
```

```
    apply (force simp flip: interval_split intro!: n2 [rule_format])
    done
 qed
qed

lemma integral_Pair_const:
   integral (cbox (a,c) (b,d)) (λx. k) =
    integral (cbox a b) (λx. integral (cbox c d) (λy. k))
 by (simp add: content_Pair)

lemma integral_prod_continuous:
 fixes f :: ('a::euclidean_space * 'b::euclidean_space) ⇒ 'c::banach
 assumes continuous_on (cbox (a, c) (b, d)) f (is continuous_on ?CBOX f)
   shows integral (cbox (a, c) (b, d)) f = integral (cbox a b) (λx. integral (cbox c
d) (λy. f (x, y)))
proof (cases content ?CBOX = 0)
 case True
 then show ?thesis
   by (auto simp: content_Pair)
next
 case False
 then have cbp: content ?CBOX > 0
   using content_lt_nz by blast
 have norm (integral ?CBOX f − integral (cbox a b) (λx. integral (cbox c d) (λy.
f (x,y)))) = 0
 proof (rule dense_eq0_I, simp)
   fix e :: real
   assume 0 < e
   with ‹content ?CBOX > 0› have 0 < e/content ?CBOX
     by simp
   have f_int_acbd: f integrable_on ?CBOX
     by (rule integrable_continuous [OF assms])
   { fix p
     assume p: p division_of ?CBOX
     then have finite p
       by blast
     define e' where e' = e/content ?CBOX
     with ‹0 < e› ‹0 < e/content ?CBOX›
     have 0 < e'
       by simp
     interpret operative conj True
         λk. ∀ a' b' c' d'.
             cbox (a', c') (b', d') ⊆ k ∧ cbox (a', c') (b', d') ⊆ ?CBOX
             ⟶ norm (integral (cbox (a', c') (b', d')) f −
                     integral (cbox a' b') (λx. integral (cbox c' d') (λy. f ((x, y)))))
                 ≤ e' * content (cbox (a', c') (b', d'))
       using assms ‹0 < e'› by (rule integral_swap_operativeI)
     have norm (integral ?CBOX f − integral (cbox a b) (λx. integral (cbox c d)
(λy. f (x, y))))
```

```
        ≤ e' * content ?CBOX
       if ⋀t u v w z. t ∈ p ⟹ cbox (u, w) (v, z) ⊆ t ⟹ cbox (u, w) (v, z) ⊆
?CBOX
        ⟹ norm (integral (cbox (u, w) (v, z)) f −
           integral (cbox u v) (λx. integral (cbox w z) (λy. f (x, y))))
           ≤ e' * content (cbox (u, w) (v, z))
        using that division [of p (a, c) (b, d)] p ‹finite p› by (auto simp add:
comm_monoid_set_F_and)
     with False have norm (integral ?CBOX f − integral (cbox a b) (λx. integral
(cbox c d) (λy. f (x, y))))
        ≤ e
       if ⋀t u v w z. t ∈ p ⟹ cbox (u, w) (v, z) ⊆ t ⟹ cbox (u, w) (v, z) ⊆
?CBOX
        ⟹ norm (integral (cbox (u, w) (v, z)) f −
           integral (cbox u v) (λx. integral (cbox w z) (λy. f (x, y))))
           ≤ e * content (cbox (u, w) (v, z)) / content ?CBOX
       using that by (simp add: e'_def)
   } note op_acbd = this
   { fix k::real and 𝒟 and u::'a and v w and z::'b and t1 t2 l
     assume k: 0 < k
       and nf: ⋀x y u v.
            ⟦x ∈ cbox a b; y ∈ cbox c d; u ∈ cbox a b; v∈cbox c d; norm (u−x,
v−y) < k⟧
            ⟹ norm (f(u,v) − f(x,y)) < e/(2 * (content ?CBOX))
       and p_acbd: 𝒟 tagged_division_of cbox (a,c) (b,d)
       and fine: (λx. ball x k) fine 𝒟  ((t1,t2), l) ∈ 𝒟
       and uwvz_sub: cbox (u,w) (v,z) ⊆ l cbox (u,w) (v,z) ⊆ cbox (a,c) (b,d)
     have t: t1 ∈ cbox a b t2 ∈ cbox c d
       by (meson fine p_acbd cbox_Pair_iff tag_in_interval)+
     have ls: l ⊆ ball (t1,t2) k
       using fine by (simp add: fine_def Ball_def)
     { fix x1 x2
       assume xuvwz: x1 ∈ cbox u v x2 ∈ cbox w z
       then have x: x1 ∈ cbox a b x2 ∈ cbox c d
         using uwvz_sub by auto
       have norm (x1 − t1, x2 − t2) = norm (t1 − x1, t2 − x2)
         by (simp add: norm_Pair norm_minus_commute)
       also have norm (t1 − x1, t2 − x2) < k
         using xuvwz ls uwvz_sub unfolding ball_def
         by (force simp add: cbox_Pair_eq dist_norm )
       finally have norm (f (x1,x2) − f (t1,t2)) ≤ e/content ?CBOX/2
         using nf [OF t x]  by simp
     } note nf' = this
     have f_int_uwvz: f integrable_on cbox (u,w) (v,z)
       using f_int_acbd uwvz_sub integrable_on_subcbox by blast
     have f_int_uv: ⋀x. x ∈ cbox u v ⟹ (λy. f (x,y)) integrable_on cbox w z
       using assms continuous_on_subset uwvz_sub
       by (blast intro: continuous_on_imp_integrable_on_Pair1)
    have 1: norm (integral (cbox (u,w) (v,z)) f − integral (cbox (u,w) (v,z)) (λx.
```

f (*t1*,*t2*)))
≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) / *content ?CBOX*/2
using *cbp* ‹0 < *e*/*content ?CBOX*› *nf'*
apply (*simp only*: *integral_diff* [*symmetric*] *f_int_uwvz integrable_const*)
apply (*auto simp*: *integrable_diff f_int_uwvz integrable_const intro*: *order_trans* [*OF integrable_bound* [*of e*/*content ?CBOX*/2]])
done
have *int_integrable*: (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*))) *integrable_on cbox u v*
using *assms integral_integrable_2dim continuous_on_subset uwvz_sub*(2) **by** *blast*
have *normint_wz*:
⋀*x*. *x* ∈ *cbox u v* ⟹
norm (*integral* (*cbox w z*) (λ*y*. *f* (*x*, *y*)) − *integral* (*cbox w z*) (λ*y*. *f* (*t1*, *t2*)))
≤ *e* ∗ *content* (*cbox w z*) / *content* (*cbox* (*a*, *c*) (*b*, *d*))/2
using *cbp* ‹0 < *e*/*content ?CBOX*› *nf'*
apply (*simp only*: *integral_diff* [*symmetric*] *f_int_uv integrable_const*)
apply (*auto simp*: *integrable_diff f_int_uv integrable_const intro*: *order_trans* [*OF integrable_bound* [*of e*/*content ?CBOX*/2]])
done
have *norm* (*integral* (*cbox u v*)
(λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*,*y*)) − *integral* (*cbox w z*) (λ*y*. *f* (*t1*,*t2*))))
≤ *e* ∗ *content* (*cbox w z*) / *content ?CBOX*/2 ∗ *content* (*cbox u v*)
using *cbp* ‹0 < *e*/*content ?CBOX*›
apply (*intro integrable_bound* [*OF _ _ normint_wz*])
apply (*auto simp*: *field_split_simps integrable_diff int_integrable integrable_const*)
done
also have ... ≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) / *content ?CBOX*/2
by (*simp add*: *content_Pair field_split_simps*)
finally have 2: *norm* (*integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*,*y*))) −
integral (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*t1*,*t2*))))
≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) / *content ?CBOX*/2
by (*simp only*: *integral_diff* [*symmetric*] *int_integrable integrable_const*)
have *norm_xx*: ⟦*x'* = *y'*; *norm*(*x* − *x'*) ≤ *e*/2; *norm*(*y* − *y'*) ≤ *e*/2⟧ ⟹ *norm*(*x* − *y*) ≤ *e* **for** *x*::'*c* **and** *y x' y' e*
using *norm_triangle_mono* [*of x*−*y' e*/2 *y'*−*y e*/2] *field_sum_of_halves*
by (*simp add*: *norm_minus_commute*)
have *norm* (*integral* (*cbox* (*u*,*w*) (*v*,*z*)) *f* − *integral* (*cbox u v*) (λ*x*. *integral* (*cbox w z*) (λ*y*. *f* (*x*,*y*))))
≤ *e* ∗ *content* (*cbox* (*u*,*w*) (*v*,*z*)) / *content ?CBOX*
by (*rule norm_xx* [*OF integral_Pair_const 1 2*])
} **note** ∗ = *this*
have *norm* (*integral ?CBOX f* − *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (λ*y*. *f* (*x*,*y*)))) ≤ *e*
if ∀ *x*∈*?CBOX*. ∀ *x'*∈*?CBOX*. *norm* (*x'* − *x*) < *k* ⟶ *norm* (*f x'* − *f x*) < *e*

```
/(2 * content (?CBOX)) 0 < k for k
    proof −
      obtain p where ptag: p tagged_division_of cbox (a, c) (b, d)
                and fine: (λx. ball x k) fine p
        using fine_division_exists ‹0 < k› by blast
      show ?thesis
        using that fine ptag ‹0 < k›
      by (auto simp: * intro: op_acbd [OF division_of_tagged_division [OF ptag]])
    qed
    then show norm (integral ?CBOX f − integral (cbox a b) (λx. integral (cbox
c d) (λy. f (x,y)))) ≤ e
      using compact_uniformly_continuous [OF assms compact_cbox]
      apply (simp add: uniformly_continuous_on_def dist_norm)
      apply (drule_tac x=e/2 / content?CBOX in spec)
      using cbp ‹0 < e› by (auto simp: zero_less_mult_iff)
  qed
  then show ?thesis
    by simp
qed

lemma integral_swap_2dim:
  fixes f :: ['a::euclidean_space, 'b::euclidean_space] ⇒ 'c::banach
  assumes continuous_on (cbox (a,c) (b,d)) (λ(x,y). f x y)
    shows integral (cbox (a, c) (b, d)) (λ(x, y). f x y) = integral (cbox (c, a) (d,
b)) (λ(x, y). f y x)
proof −
  have ((λ(x, y). f x y) has_integral integral (cbox (c, a) (d, b)) (λ(x, y). f y x))
(prod.swap ‘ (cbox (c, a) (d, b)))
  proof (rule has_integral_twiddle [of 1 prod.swap prod.swap λ(x,y). f y x integral
(cbox (c, a) (d, b)) (λ(x, y). f y x), simplified])
    show ⋀u v. content (prod.swap ‘ cbox u v) = content (cbox u v)
      by (metis content_Pair mult.commute old.prod.exhaust swap_cbox_Pair)
    show ((λ(x, y). f y x) has_integral integral (cbox (c, a) (d, b)) (λ(x, y). f y x))
(cbox (c, a) (d, b))
    by (simp add: assms integrable_continuous integrable_integral swap_continuous)
  qed (use isCont_swap in ‹fastforce+›)
 then show ?thesis
   by force
qed

theorem integral_swap_continuous:
  fixes f :: ['a::euclidean_space, 'b::euclidean_space] ⇒ 'c::banach
  assumes continuous_on (cbox (a,c) (b,d)) (λ(x,y). f x y)
    shows integral (cbox a b) (λx. integral (cbox c d) (f x)) =
          integral (cbox c d) (λy. integral (cbox a b) (λx. f x y))
proof −
  have integral (cbox a b) (λx. integral (cbox c d) (f x)) = integral (cbox (a, c) (b,
d)) (λ(x, y). f x y)
    using integral_prod_continuous [OF assms] by auto
```

```
  also have ... = integral (cbox (c, a) (d, b)) (λ(x, y). f y x)
    by (rule integral_swap_2dim [OF assms])
  also have ... = integral (cbox c d) (λy. integral (cbox a b) (λx. f x y))
    using integral_prod_continuous [OF swap_continuous] assms
    by auto
  finally show ?thesis .
qed
```

7.14.47 Definite integrals for exponential and power function

```
lemma has_integral_exp_minus_to_infinity:
  assumes a: a > 0
  shows   ((λx::real. exp (−a∗x)) has_integral exp (−a∗c)/a) {c..}
proof −
  define f where f = (λk x. if x ∈ {c..real k} then exp (−a∗x) else 0)
  {
    fix k :: nat assume k: of_nat k ≥ c
    from k a
    have ((λx. exp (−a∗x)) has_integral (−exp (−a∗real k)/a − (−exp (−a∗c)/a)))
{c..real k}
      by (intro fundamental_theorem_of_calculus)
         (auto intro!: derivative_eq_intros
               simp: has_real_derivative_iff_has_vector_derivative [symmetric])
    hence (f k has_integral (exp (−a∗c)/a − exp (−a∗real k)/a)) {c..} unfolding
f_def
      by (subst has_integral_restrict) simp_all
  } note has_integral_f = this

  have [simp]: f k = (λ_. 0) if of_nat k < c for k using that by (auto simp:
fun_eq_iff f_def)
  have integral_f: integral {c..} (f k) =
                     (if real k ≥ c then exp (−a∗c)/a − exp (−a∗real k)/a else 0)
    for k using integral_unique[OF has_integral_f[of k]] by simp

  have A: (λx. exp (−a∗x)) integrable_on {c..} ∧
           (λk. integral {c..} (f k)) ⟶ integral {c..} (λx. exp (−a∗x))
  proof (intro monotone_convergence_increasing allI ballI)
    fix k ::nat
    have (λx. exp (−a∗x)) integrable_on {c..of_real k}
    unfolding f_def by (auto intro!: continuous_intros integrable_continuous_real)
    hence  (f k) integrable_on {c..of_real k}
      by (rule integrable_eq) (simp add: f_def)
    then show f k integrable_on {c..}
      by (rule integrable_on_superset) (auto simp: f_def)
  next
    fix x assume x: x ∈ {c..}
   have sequentially ≤ principal {nat ⌈x⌉..} unfolding at_top_def by (simp add:
Inf_lower)
    also have {nat ⌈x⌉..} ⊆ {k. x ≤ real k} by auto
```

```
    also have inf (principal ...) (principal {k. ¬x ≤ real k}) =
                principal ({k. x ≤ real k} ∩ {k. ¬x ≤ real k})
      by simp
    also have {k. x ≤ real k} ∩ {k. ¬x ≤ real k} = {} by blast
    finally have inf sequentially (principal {k. ¬x ≤ real k}) = bot
      by (simp add: inf.coboundedI1 bot_unique)
    with x show (λk. f k x) ⟶ exp (−a∗x) unfolding f_def
      by (intro filterlim_If) auto
  next
    have |integral {c..} (f k)| ≤ exp (−a∗c)/a for k
    proof (cases c > of_nat k)
      case False
      hence abs (integral {c..} (f k)) = abs (exp (− (a ∗ c)) / a − exp (− (a ∗ real
k)) / a)
        by (simp add: integral_f)
      also have abs (exp (− (a ∗ c)) / a − exp (− (a ∗ real k)) / a) =
                  exp (− (a ∗ c)) / a − exp (− (a ∗ real k)) / a
        using False a by (intro abs_of_nonneg) (simp_all add: field_simps)
      also have ... ≤ exp (− a ∗ c) / a using a by simp
      finally show ?thesis .
    qed (insert a, simp_all add: integral_f)
    thus bounded (range(λk. integral {c..} (f k)))
      by (intro boundedI[of _ exp (−a∗c)/a]) auto
  qed (auto simp: f_def)
  have (λk. exp (−a∗c)/a − exp (−a ∗ of_nat k)/a) ⟶ exp (−a∗c)/a − 0/a
    by (intro tendsto_intros filterlim_compose[OF exp_at_bot]
      filterlim_tendsto_neg_mult_at_bot[OF tendsto_const] filterlim_real_sequentially)+
      (use a in simp_all)
  moreover
   from eventually_gt_at_top[of nat ⌈c⌉] have eventually (λk. of_nat k > c)
sequentially
    by eventually_elim linarith
  hence eventually (λk. exp (−a∗c)/a − exp (−a ∗ of_nat k)/a = integral {c..}
(f k)) sequentially
    by eventually_elim (simp add: integral_f)
  ultimately have (λk. integral {c..} (f k)) ⟶ exp (−a∗c)/a − 0/a
    by (rule Lim_transform_eventually)
  from LIMSEQ_unique[OF conjunct2[OF A] this]
  have integral {c..} (λx. exp (−a∗x)) = exp (−a∗c)/a by simp
  with conjunct1[OF A] show ?thesis
    by (simp add: has_integral_integral)
qed

lemma integrable_on_exp_minus_to_infinity: a > 0 ⟹ (λx. exp (−a∗x) :: real)
integrable_on {c..}
 using has_integral_exp_minus_to_infinity[of a c] unfolding integrable_on_def
by blast

lemma has_integral_powr_from_0:
```

```
  assumes a: a > (-1::real) and c: c ≥ 0
  shows   ((λx. x powr a) has_integral (c powr (a+1) / (a+1))) {0..c}
proof (cases c = 0)
  case False
  define f where f = (λk x. if x ∈ {inverse (of_nat (Suc k))..c} then x powr a
else 0)
  define F where F = (λk. if inverse (of_nat (Suc k)) ≤ c then
                            c powr (a+1)/(a+1) - inverse (real (Suc k)) powr
(a+1)/(a+1) else 0)
  have has_integral_f: (f k has_integral F k) {0..c} for k::nat
  proof (cases inverse (of_nat (Suc k)) ≤ c)
    case True
    have x: x > 0 if x ≥ inverse (1 + real k) for x
      by (smt (verit) that inverse_Suc of_nat_Suc)
    hence ((λx. x powr a) has_integral c powr (a + 1) / (a + 1) -
              inverse (real (Suc k)) powr (a + 1) / (a + 1)) {inverse (real (Suc
k))..c}
      using True a by (intro fundamental_theorem_of_calculus)
      (auto intro!: derivative_eq_intros continuous_on_powr' continuous_on_const
          simp: has_real_derivative_iff_has_vector_derivative [symmetric])
   with True show ?thesis unfolding f_def F_def by (subst has_integral_restrict)
simp_all
  next
    case False
    thus ?thesis unfolding f_def F_def
      by (subst has_integral_restrict) auto
  qed
  then have integral_f: integral {0..c} (f k) = F k for k
    by blast

  have A: (λx. x powr a) integrable_on {0..c} ∧
         (λk. integral {0..c} (f k)) ⟶ integral {0..c} (λx. x powr a)
  proof (intro monotone_convergence_increasing ballI allI)
    fix k from has_integral_f[of k] show f k integrable_on {0..c}
      by (auto simp: integrable_on_def)
  next
    fix k :: nat and x :: real
    {
      assume x: inverse (real (Suc k)) ≤ x
      then have inverse (real (Suc (Suc k))) ≤ x
        using dual_order.trans by fastforce
    }
    thus f k x ≤ f (Suc k) x by (auto simp: f_def simp del: of_nat_Suc)
  next
    fix x assume x: x ∈ {0..c}
    show (λk. f k x) ⟶ x powr a
    proof (cases x = 0)
      case False
      with x have x > 0 by simp
```

```
    from order_tendstoD(2)[OF LIMSEQ_inverse_real_of_nat this]
      have eventually (λk. x powr a = f k x) sequentially
      by eventually_elim (insert x, simp add: f_def)
    moreover have (λ_. x powr a) ⟶ x powr a by simp
    ultimately show ?thesis by (blast intro: Lim_transform_eventually)
  qed (simp_all add: f_def)
 next
  {
    fix k
    from a have F k ≤ c powr (a + 1) / (a + 1)
      by (auto simp add: F_def divide_simps)
    also from a have F k ≥ 0
    by (auto simp: F_def divide_simps simp del: of_nat_Suc intro!: powr_mono2)
    hence F k = abs (F k) by simp
    finally have abs (F k) ≤  c powr (a + 1) / (a + 1) .
  }
  thus bounded (range(λk. integral {0..c} (f k)))
    by (intro boundedI[of _ c powr (a+1) / (a+1)]) (auto simp: integral_f)
 qed

 from False c have c > 0 by simp
 from order_tendstoD(2)[OF LIMSEQ_inverse_real_of_nat this]
   have eventually (λk. c powr (a + 1) / (a + 1) − inverse (real (Suc k)) powr
(a+1) / (a+1) =
         integral {0..c} (f k)) sequentially
   by eventually_elim (simp add: integral_f F_def)
 moreover have (λk. c powr (a + 1) / (a + 1) − inverse (real (Suc k)) powr
(a + 1) / (a + 1))
               ⟶ c powr (a + 1) / (a + 1) − 0 powr (a + 1) / (a + 1)
   using a by (intro tendsto_intros LIMSEQ_inverse_real_of_nat) auto
 hence (λk. c powr (a + 1) / (a + 1) − inverse (real (Suc k)) powr (a + 1) /
(a + 1))
       ⟶ c powr (a + 1) / (a + 1) by simp
 ultimately have (λk. integral {0..c} (f k)) ⟶ c powr (a+1) / (a+1)
   by (blast intro: Lim_transform_eventually)
 with A have integral {0..c} (λx. x powr a) = c powr (a+1) / (a+1)
   by (blast intro: LIMSEQ_unique)
 with A show ?thesis by (simp add: has_integral_integral)
qed (simp_all add: has_integral_refl)

lemma integrable_on_powr_from_0:
 assumes a: a > (−1::real) and c: c ≥ 0
 shows   (λx. x powr a) integrable_on {0..c}
 using has_integral_powr_from_0[OF assms] unfolding integrable_on_def by
blast

lemma has_integral_powr_to_inf:
 fixes a e :: real
 assumes e < −1 a > 0
```

```
  shows   ((λx. x powr e) has_integral -(a powr (e + 1)) / (e + 1)) {a..}
proof -
  define f :: nat ⇒ real ⇒ real where f = (λn x. if x ∈ {a..n} then x powr e else
0)
  define F where F = (λx. x powr (e + 1) / (e + 1))

  have has_integral_f: (f n has_integral (F n - F a)) {a..}
    if n: n ≥ a for n :: nat
  proof -
    from n assms have ((λx. x powr e) has_integral (F n - F a)) {a..n}
    by (intro fundamental_theorem_of_calculus) (auto intro!: derivative_eq_intros
        simp: has_real_derivative_iff_has_vector_derivative [symmetric] F_def)
    hence (f n has_integral (F n - F a)) {a..n}
      by (rule has_integral_eq [rotated]) (simp add: f_def)
    thus (f n has_integral (F n - F a)) {a..}
      by (rule has_integral_on_superset) (auto simp: f_def)
  qed
  have integral_f: integral {a..} (f n) = (if n ≥ a then F n - F a else 0) for n ::
nat
  proof (cases n ≥ a)
    case True
    with has_integral_f[OF this] show ?thesis by (simp add: integral_unique)
  next
    case False
    have (f n has_integral 0) {a} by (rule has_integral_refl)
    hence (f n has_integral 0) {a..}
      using False f_def by force
    with False show ?thesis by (simp add: integral_unique)
  qed

  have ∗: (λx. x powr e) integrable_on {a..} ∧
          (λn. integral {a..} (f n)) ⟶ integral {a..} (λx. x powr e)
  proof (intro monotone_convergence_increasing allI ballI)
    fix n :: nat
    from assms have (λx. x powr e) integrable_on {a..n}
      by (auto intro!: integrable_continuous_real continuous_intros)
    hence f n integrable_on {a..n}
      by (rule integrable_eq) (auto simp: f_def)
    thus f n integrable_on {a..}
      by (rule integrable_on_superset) (auto simp: f_def)
  next
    fix n :: nat and x :: real
    show f n x ≤ f (Suc n) x by (auto simp: f_def)
  next
    fix x :: real assume x: x ∈ {a..}
    from filterlim_real_sequentially
      have eventually (λn. real n ≥ x) at_top
      by (simp add: filterlim_at_top)
    with x have eventually (λn. f n x = x powr e) at_top
```

```
    by (auto elim!: eventually_mono simp: f_def)
  thus (λn. f n x) ⟶ x powr e by (simp add: tendsto_eventually)
 next
  have norm (integral {a..} (f n)) ≤ −F a for n :: nat
  proof (cases n ≥ a)
   case True
   with assms have a powr (e + 1) ≥ n powr (e + 1)
    by (intro powr_mono2') simp_all
   with assms show ?thesis by (auto simp: divide_simps F_def integral_f)
  qed (use assms in ‹simp add: integral_f F_def field_split_simps›)
  thus bounded (range(λk. integral {a..} (f k)))
   unfolding bounded_iff by (intro exI[of _ −F a]) auto
 qed

 from filterlim_real_sequentially
  have eventually (λn. real n ≥ a) at_top
  by (simp add: filterlim_at_top)
 hence eventually (λn. F n − F a = integral {a..} (f n)) at_top
  by eventually_elim (simp add: integral_f)
 moreover have (λn. F n − F a) ⟶ 0 / (e + 1) − F a unfolding F_def
  by (insert assms, (rule tendsto_intros filterlim_compose[OF tendsto_neg_powr]
       filterlim_ident filterlim_real_sequentially | simp)+)
 hence (λn. F n − F a) ⟶ −F a by simp
 ultimately have (λn. integral {a..} (f n)) ⟶ −F a by (blast intro: Lim_transform_eventually)
 then have integral {a..} (λx. x powr e) = −F a
  using ∗ LIMSEQ_unique by blast
 with ∗ show ?thesis
  by (simp add: has_integral_integral F_def)
qed

lemma has_integral_inverse_power_to_inf:
 fixes a :: real and n :: nat
 assumes n > 1 a > 0
 shows  ((λx. 1 / x ^ n) has_integral 1 / (real (n − 1) ∗ a ^ (n − 1))) {a..}
proof −
 from assms have real_of_int (−int n) < −1 by simp
 note has_integral_powr_to_inf[OF this ‹a > 0›]
 also have − (a powr (real_of_int (− int n) + 1)) / (real_of_int (− int n) +
1) =
              1 / (real (n − 1) ∗ a powr (real (n − 1))) using assms
  by (simp add: field_split_simps powr_add [symmetric] of_nat_diff)
 also from assms have a powr (real (n − 1)) = a ^ (n − 1)
  by (intro powr_realpow)
 finally show ?thesis
  by (rule has_integral_eq [rotated])
    (insert assms, simp_all add: powr_minus powr_realpow field_split_simps)
qed
```

Adaption to ordered Euclidean spaces and the Cartesian Euclidean space

lemma *integral_component_eq_cart*[*simp*]:
fixes $f :: 'n::euclidean\_space \Rightarrow real^{\wedge}'m$
assumes *f integrable_on s*
shows *integral s* $(\lambda x.\ f\ x\ \$\ k) = integral\ s\ f\ \$\ k$
using *integral_linear*[*OF assms*(*1*) *bounded_linear_vec_nth*,*unfolded o_def*] .

lemma *content_closed_interval*:
fixes $a :: 'a::ordered\_euclidean\_space$
assumes $a \leq b$
shows $content\ \{a..b\} = (\prod i{\in}Basis.\ b{\cdot}i - a{\cdot}i)$
using *content_cbox*[*of a b*] *assms* **by** (*simp add*: *cbox_interval eucl_le*[**where** $'a='a$])

lemma *integrable_const_ivl*[*intro*]:
fixes $a::'a::ordered\_euclidean\_space$
shows $(\lambda x.\ c)$ *integrable_on* $\{a..b\}$
unfolding *cbox_interval*[*symmetric*] **by** (*rule integrable_const*)

lemma *integrable_on_subinterval*:
fixes $f :: 'n::ordered\_euclidean\_space \Rightarrow 'a::banach$
assumes *f integrable_on S* $\{a..b\} \subseteq S$
shows *f integrable_on* $\{a..b\}$
using *integrable_on_subcbox*[*of f S a b*] *assms* **by** (*simp add*: *cbox_interval*)

end

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Chapter 8

Kronecker's Theorem with Applications

theory *Kronecker_Approximation_Theorem*

imports *Complex_Transcendental Henstock_Kurzweil_Integration*
HOL−Real_Asymp.Real_Asymp

begin

8.1 Dirichlet's Approximation Theorem

Simultaneous version. From Hardy and Wright, An Introduction to the Theory of Numbers, page 170.

theorem *Dirichlet_approx_simult*:
fixes $\vartheta :: nat \Rightarrow real$ **and** $N\ n :: nat$
assumes $N > 0$
obtains $q\ p$ **where** $0<q\ q \leq int\ (N\hat{}n)$
and $\bigwedge i.\ i<n \Longrightarrow |of\_int\ q * \vartheta\ i - of\_int(p\ i)| < 1/N$
proof −
have *lessN*: $nat\ \lfloor x * real\ N \rfloor < N$ **if** $0 \leq x\ x < 1$ **for** x
proof −
have $\lfloor x * real\ N \rfloor < N$
using *that* **by** (*simp add*: *assms floor_less_iff*)
with *assms* **show** *?thesis* **by** *linarith*
qed
define *interv* **where** $interv \equiv \lambda k.\ \{real\ k/N..< Suc\ k/N\}$
define *fracs* **where** $fracs \equiv \lambda k.\ map\ (\lambda i.\ frac\ (real\ k * \vartheta\ i))\ [0..<n]$
define X **where** $X \equiv fracs\ `\ \{..N\hat{}n\}$
define Y **where** $Y \equiv set\ (List.n\_lists\ n\ (map\ interv\ [0..<N]))$
have *interv_iff*: $interv\ k = interv\ k' \longleftrightarrow k=k'$ **for** $k\ k'$
using *assms* **by** (*auto simp*: *interv_def Ico_eq_Ico divide_strict_right_mono*)
have *in_interv*: $x \in interv\ (nat\ \lfloor x * real\ N \rfloor)$ **if** $x \geq 0$ **for** x
using *that assms* **by** (*simp add*: *interv_def divide_simps*) *linarith*

```
  have False
    if non: ∀ a b. b ≤ N^n ⟶ a < b ⟶ ¬(∀ i<n. |frac (real b * ϑ i) − frac (real
a * ϑ i)| < 1/N)
  proof −
    have inj_on (λk. map (λi. frac (k * ϑ i)) [0..<n]) {..N^n}
      using that assms by (intro linorder_inj_onI) fastforce+
    then have caX: card X = Suc (N^n)
      by (simp add: X_def fracs_def card_image)
    have caY: card Y ≤ N^n
     by (metis Y_def card_length diff_zero length_map length_n_lists length_upt)
    define f where f ≡ λl. map (λx. interv (nat ⌊x * real N⌋)) l
    have f ‘ X ⊆ Y
       by (auto simp: f_def Y_def X_def fracs_def o_def set_n_lists frac_lt_1
lessN)
    then have ¬ inj_on f X
      using Y_def caX caY card_inj_on_le not_less_eq_eq by fastforce
    then obtain x x′ where x≠x′ x ∈ X x′ ∈ X and eq: f x = f x′
      by (auto simp: inj_on_def)
    then obtain c c′ where c: c ≠ c′ c ≤ N^n c′ ≤ N^n
          and xeq: x = fracs c and xeq′: x′ = fracs c′
      unfolding X_def by (metis atMost_iff image_iff)
    have [simp]: length x′ = n
      by (auto simp: xeq′ fracs_def)
    have ge0: x′!i ≥ 0 if i<n for i
      using that ‹x′ ∈ X› by (auto simp: X_def fracs_def)
    have f x ∈ Y
      using ‹f ‘ X ⊆ Y› ‹x ∈ X› by blast
    define k where k ≡ map (λr. nat ⌊r * real N⌋) x
    have |frac (real c * ϑ i) − frac (real c′ * ϑ i)| < 1/N if i < n for i
    proof −
      have k: x!i ∈ interv(k!i)
          using ‹i<n› assms by (auto simp: divide_simps k_def interv_def xeq
fracs_def) linarith
      then have k′: x′!i ∈ interv(k!i)
        using ‹i<n› eq assms ge0[OF ‹i<n›]
        by (auto simp: k_def f_def divide_simps map_equality_iff in_interv in-
terv_iff)
      with assms k show ?thesis
      using ‹i<n› by (auto simp add: xeq xeq′ fracs_def interv_def add_divide_distrib)
    qed
    then show False
      by (metis c non nat_neq_iff abs_minus_commute)
  qed
  then obtain a b where a<b b ≤ N^n and ∗: ⋀i. i<n ⟹ |frac (real b * ϑ i)
− frac (real a * ϑ i)| < 1/N
    by blast
  let ?k = b−a
  let ?h = λi. ⌊b * ϑ i⌋ − ⌊a * ϑ i⌋
  show ?thesis
```

```
  proof
    show int (b − a) ≤ int (N ^ n)
      using ‹b ≤ N ^ n› by auto
  next
    fix i
    assume i<n
    have frac (b ∗ ϑ i) − frac (a ∗ ϑ i) = ?k ∗ ϑ i − ?h i
      using ‹a < b› by (simp add: frac_def left_diff_distrib' of_nat_diff)
    then show |of_int ?k ∗ ϑ i − ?h i| < 1/N
      by (metis ∗ ‹i < n› of_int_of_nat_eq)
  qed (use ‹a < b› in auto)
qed
```

Theorem 7.1

```
corollary Dirichlet_approx:
  fixes ϑ:: real and N:: nat
  assumes N > 0
  obtains h k where 0 < k k ≤ int N |of_int k ∗ ϑ − of_int h| < 1/N
  by (rule Dirichlet_approx_simult [OF assms, where n=1 and ϑ=λ_. ϑ]) auto
```

Theorem 7.2

```
corollary Dirichlet_approx_coprime:
  fixes ϑ:: real and N:: nat
  assumes N > 0
  obtains h k where coprime h k 0 < k k ≤ int N |of_int k ∗ ϑ − of_int h| <
1/N
proof −
  obtain h' k' where k': 0 < k' k' ≤ int N and ∗: |of_int k' ∗ ϑ − of_int h'| <
1/N
    by (meson Dirichlet_approx assms)
  let ?d = gcd h' k'
  show thesis
  proof (cases ?d = 1)
    case True
    with k' ∗ that show ?thesis by auto
  next
    case False
    then have 1: ?d > 1
      by (smt (verit, ccfv_threshold) ‹k'>0› gcd_pos_int)
    let ?k = k' div ?d
    let ?h = h' div ?d
    show ?thesis
    proof
      show coprime (?h::int) ?k
        by (metis 1 div_gcd_coprime gcd_eq_0_iff not_one_less_zero)
      show k0: 0 < ?k
        using ‹k'>0› pos_imp_zdiv_pos_iff by force
      show ?k ≤ int N
        using k' 1 int_div_less_self by fastforce
```

```
    have |ϑ − of_int ?h / of_int ?k| = |ϑ − of_int h' / of_int k'|
      by (simp add: real_of_int_div)
    also have ... < 1 / of_int (N * k')
      using k' * by (simp add: field_simps)
    also have ... < 1 / of_int (N * ?k)
      using assms ‹k'>0› k0 1
      by (simp add: frac_less2 int_div_less_self)
    finally show |of_int ?k * ϑ − of_int ?h| < 1 / real N
      using k0 k' by (simp add: field_simps)
  qed
 qed
qed

definition approx_set :: real ⇒ (int × int) set
 where approx_set ϑ ≡
   {(h, k) | h k::int. k > 0 ∧ coprime h k ∧ |ϑ − of_int h / of_int k| < 1/k^2}
 for ϑ::real
```

Theorem 7.3

```
lemma approx_set_nonempty: approx_set ϑ ≠ {}
proof −
 have |ϑ − of_int ⌊ϑ⌋ / of_int 1| < 1 / (of_int 1)^2
   by simp linarith
 then have (⌊ϑ⌋, 1) ∈ approx_set ϑ
   by (auto simp: approx_set_def)
 then show ?thesis
   by blast
qed
```

Theorem 7.4(c)

```
theorem infinite_approx_set:
 assumes infinite (approx_set ϑ)
 shows ∃ h k. (h,k) ∈ approx_set ϑ ∧ k > K
proof (rule ccontr, simp add: not_less)
 assume Kb [rule_format]: ∀ h k. (h, k) ∈ approx_set ϑ ⟶ k ≤ K
 define H where H ≡ 1 + |K * ϑ|
 have k0: k > 0 if (h,k) ∈ approx_set ϑ for h k
   using approx_set_def that by blast
 have Hb: of_int |h| < H if (h,k) ∈ approx_set ϑ for h k
 proof −
   have ∗: |k * ϑ − h| < 1
   proof −
     have |k * ϑ − h| < 1 / k
       using that by (auto simp: field_simps approx_set_def eval_nat_numeral)
     also have ... ≤ 1
       using divide_le_eq_1 by fastforce
     finally show ?thesis .
   qed
   have k > 0
```

```
    using approx_set_def that by blast
  have of_int |h| ≤ |k * ϑ - h| + |k * ϑ|
    by force
  also have ... < 1 + |k * ϑ|
    using * that by simp
  also have ... ≤ H
   using Kb [OF that] ‹k>0› by (auto simp add: H_def abs_if mult_right_mono
mult_less_0_iff)
  finally show ?thesis .
 qed
 have approx_set ϑ ⊆ {-(ceiling H)..ceiling H} × {0..K}
  using Hb Kb k0
  apply (simp add: subset_iff)
  by (smt (verit, best) ceiling_add_of_int less_ceiling_iff)
 then have finite (approx_set ϑ)
  by (meson finite_SigmaI finite_atLeastAtMost_int finite_subset)
 then show False
  using assms by blast
qed
```

Theorem 7.4(b,d)

```
theorem rational_iff_finite_approx_set:
 shows ϑ ∈ ℚ ⟷ finite (approx_set ϑ)
proof
 assume ϑ ∈ ℚ
 then obtain a b where eq: ϑ = of_int a / of_int b and b>0 and coprime a b
  by (meson Rats_cases')
 then have ab: (a,b) ∈ approx_set ϑ
  by (auto simp: approx_set_def)
 show finite (approx_set ϑ)
 proof (rule ccontr)
  assume infinite (approx_set ϑ)
  then obtain h k where (h,k) ∈ approx_set ϑ k > b k>0
   using infinite_approx_set by (smt (verit, best))
  then have coprime h k and hk: |a/b - h/k| < 1 / (of_int k)^2
   by (auto simp: approx_set_def eq)
  have False if a * k = h * b
  proof -
   have b dvd k
    using that ‹coprime a b›
    by (metis coprime_commute coprime_dvd_mult_right_iff dvd_triv_right)
   then obtain d where k = b * d
    by (metis dvd_def)
   then have h = a * d
    using ‹0 < b› that by force
   then show False
    using ‹0 < b› ‹b < k› ‹coprime h k› ‹k = b * d› by auto
  qed
  then have 0: 0 < |a*k - b*h|
```

```
    by auto
  have |a*k - b*h| < b/k
    using ‹k>0› ‹b>0› hk by (simp add: field_simps eval_nat_numeral)
  also have ... < 1
    by (simp add: ‹0 < k› ‹b < k›)
  finally show False
    using 0 by linarith
 qed
next
 assume fin: finite (approx_set ϑ)
 show ϑ ∈ ℚ
 proof (rule ccontr)
  assume irr: ϑ ∉ ℚ
  define A where A ≡ ((λ(h,k). |ϑ - h/k|) ` approx_set ϑ)
  let ?α = Min A
  have 0 ∉ A
    using irr by (auto simp: A_def approx_set_def)
  moreover have ∀x∈A. x≥0 and A: finite A A ≠ {}
    using approx_set_nonempty by (auto simp: A_def fin)
  ultimately have α: ?α > 0
    using Min_in by force
  then obtain N where N: real N > 1 / ?α
    using reals_Archimedean2 by blast
  have 0 < 1 / ?α
    using α by auto
  also have ... < real N
    by (fact N)
  finally have N > 0
    by simp
  from N have 1/N < ?α
    by (simp add: α divide_less_eq mult.commute)
  then obtain h k where coprime h k 0 < k k ≤ int N |of_int k * ϑ - of_int
h| < 1/N
    by (metis Dirichlet_approx_coprime α N divide_less_0_1_iff less_le not_less_iff_gr_or_eq
of_nat_0_le_iff of_nat_le_iff of_nat_0)
  then have §: |ϑ - h/k| < 1 / (k*N)
    by (simp add: field_simps)
  also have ... ≤ 1/k^2
    using ‹k ≤ int N› by (simp add: eval_nat_numeral divide_simps)
  finally have hk_in: (h,k) ∈ approx_set ϑ
    using ‹0 < k› ‹coprime h k› by (auto simp: approx_set_def)
  then have |ϑ - h/k| ∈ A
    by (auto simp: A_def)
  moreover have 1 / real_of_int (k * int N) < ?α
  proof -
    have 1 / real_of_int (k * int N) = 1 / real N / of_int k
      by simp
    also have ... < ?α / of_int k
      using ‹k > 0› α ‹N > 0› N by (intro divide_strict_right_mono) (auto
```

```
simp: field_simps)
      also have ... ≤ ?α / 1
        using α ‹k > 0› by (intro divide_left_mono) auto
      finally show ?thesis
        by simp
    qed
    ultimately show False using A § by auto
  qed
qed
```

No formalisation of Liouville's Approximation Theorem because this is already in the AFP as Liouville_Numbers. Apostol's Theorem 7.5 should be exactly the theorem liouville_irrational_algebraic. There is a minor discrepancy in the definition of "Liouville number" between Apostol and Eberl: he requires the denominator to be positive, while Eberl require it to exceed 1.

8.2 Kronecker's Approximation Theorem: the One-dimensional Case

```
lemma frac_int_mult:
  assumes m > 0 and le: 1−frac r ≤ 1/m
  shows 1 − frac (of_int m ∗ r) = m ∗ (1 − frac r)
proof −
  have frac (of_int m ∗ r) = 1 − m ∗ (1 − frac r)
  proof (subst frac_unique_iff, intro conjI)
    show of_int m ∗ r − (1 − of_int m ∗ (1 − frac r)) ∈ ℤ
      by (simp add: algebra_simps frac_def)
  qed (use assms in ‹auto simp: divide_simps mult_ac frac_lt_1›)
  then show ?thesis
    by simp
qed
```

Concrete statement of Theorem 7.7, and the real proof

```
theorem Kronecker_approx_1_explicit:
  fixes ϑ :: real
  assumes ϑ ∉ ℚ and α: 0 ≤ α α ≤ 1 and ε > 0
  obtains k where k>0 |frac(real k ∗ ϑ) − α| < ε
proof −
  obtain N::nat where 1/N < ε N > 0
    by (metis ‹ε > 0› gr_zeroI inverse_eq_divide real_arch_inverse)
  then obtain h k where 0 < k and hk: |of_int k ∗ ϑ − of_int h| < ε
    using Dirichlet_approx that by (metis less_trans)
  have irrat: of_int n ∗ ϑ ∈ ℚ ⟹ n = 0 for n
    by (metis Rats_divide Rats_of_int assms(1) nonzero_mult_div_cancel_left
of_int_0_eq_iff)
  then consider of_int k ∗ ϑ < of_int h | of_int k ∗ ϑ > of_int h
```

```
    by (metis Rats_of_int ‹0 < k› less_irrefl linorder_neqE_linordered_idom)
  then show thesis
  proof cases
    case 1
    define ξ where ξ ≡ 1 − frac (of_int k ∗ ϑ)
    have pos: ξ > 0
      by (simp add: ξ_def frac_lt_1)
    define N where N ≡ ⌊1/ξ⌋
    have N > 0
      by (simp add: N_def ξ_def frac_lt_1)
    have False if 1/ξ ∈ ℤ
    proof −
      from that of_int_ceiling
      obtain r where r: of_int r = 1/ξ by blast
      then obtain s where s: of_int k ∗ ϑ = of_int s + 1 − 1/r
        by (simp add: ξ_def frac_def)
      from r pos s ‹k > 0› have ϑ = (of_int s + 1 − 1/r) / k
        by (auto simp: field_simps)
      with assms show False
        by simp
    qed
    then have N0: N < 1/ξ
      unfolding N_def by (metis Ints_of_int floor_correct less_le)
    then have N2: 1/(N+1) < ξ
    unfolding N_def by (smt (verit) divide_less_0_iff divide_less_eq floor_correct
mult.commute pos)
    have ξ ∗ (N+1) > 1
      by (smt (verit) N2 ‹0 < N› of_int_1_less_iff pos_divide_less_eq)
    then have ex: ∃m. int m ≤ N+1 ∧ 1−α < m ∗ ξ
    by (smt (verit, best) ‹0 < N› ‹0 ≤ α› floor_of_int floor_of_nat mult.commute
of_nat_nat)
    define m where m ≡ LEAST m. int m ≤ N+1 ∧ 1−α < m ∗ ξ
    have m: int m ≤ N+1 ∧ 1−α < m ∗ ξ
      using LeastI_ex [OF ex] unfolding m_def by blast
    have m > 0
      using m gr0I ‹α ≤ 1› by force
    have kϑ: ξ < ε
      using hk 1 by (smt (verit, best) floor_eq_iff frac_def ξ_def)
    show thesis
    proof (cases m=1)
      case True
      then have |frac (real (nat k) ∗ ϑ) − α| < ε
        using m ‹α ≤ 1› ‹0 < k› ξ_def kϑ by force
      with ‹0 < k› zero_less_nat_eq that show thesis by blast
    next
      case False
      with ‹0 < m› have m>1 by linarith
      have ξ < 1 / N
      by (metis N0 ‹0 < N› mult_of_int_commute of_int_pos pos pos_less_divide_eq)
```

```
    also have ... ≤ 1 / (real m − 1)
      using ‹m > 1› m by (simp add: divide_simps)
    finally have ξ < 1 / (real m − 1) .
    then have m1_eq: (int m − 1) * ξ = 1 − frac (of_int ((int m − 1) * k) *
ϑ)
      using frac_int_mult [of (int m − 1) k * ϑ] ‹1 < m›
      by (simp add: ξ_def mult.assoc)
    then
    have m1_eq': frac (of_int ((int m − 1) * k) * ϑ) = 1 − (int m − 1) * ξ
      by simp
    moreover have (m − Suc 0) * ξ ≤ 1−α
      using Least_le [where k=m−Suc 0] m
    by (smt (verit, best) Suc_n_not_le_n Suc_pred ‹0 < m› m_def of_nat_le_iff)
    ultimately have leα: α ≤ frac (of_int ((int m − 1) * k) * ϑ)
      by (simp add: Suc_leI ‹0 < m› of_nat_diff)
    moreover have m * ξ + frac (of_int ((int m − 1) * k) * ϑ) = ξ + 1
      by (subst m1_eq') (simp add: ξ_def algebra_simps)
    ultimately have |frac ((int (m − 1) * k) * ϑ) − α| < ε
      by (smt (verit, best) One_nat_def Suc_leI ‹0 < m› int_ops(2) kϑ m
of_nat_diff)
    with that show thesis
      by (metis ‹0 < k› ‹1 < m› mult_pos_pos of_int_of_nat_eq of_nat_mult
pos_int_cases zero_less_diff)
  qed
next
  case 2
  define ξ where ξ ≡ frac (of_int k * ϑ)
  have recip_frac: False if 1/ξ ∈ ℤ
  proof −
    have frac (of_int k * ϑ) ∈ ℚ
      using that unfolding ξ_def
        by (metis Ints_cases Rats_1 Rats_divide Rats_of_int div_by_1 di-
vide_divide_eq_right mult_cancel_right2)
    then show False
      using ‹0 < k› frac_in_Rats_iff irrat by blast
  qed
  have pos: ξ > 0
    by (metis ξ_def Ints_0 division_ring_divide_zero frac_unique_iff less_le
recip_frac)
  define N where N ≡ ⌊1 / ξ⌋
  have N > 0
    unfolding N_def
    by (smt (verit) ξ_def divide_less_eq_1_pos floor_less_one frac_lt_1 pos)
  have N0: N < 1 / ξ
    unfolding N_def by (metis Ints_of_int floor_eq_iff less_le recip_frac)
  then have N2: 1/(N+1) < ξ
    unfolding N_def
  by (smt (verit, best) divide_less_0_iff divide_less_eq floor_correct mult.commute
pos)
```

```
  have ξ * (N+1) > 1
    by (smt (verit) N2 ‹0 < N› of_int_1_less_iff pos_divide_less_eq)
  then have ex: ∃m. int m ≤ N+1 ∧ α < m * ξ
     by (smt (verit, best) mult.commute ‹α ≤ 1› ‹0 < N› of_int_of_nat_eq
pos_int_cases)
  define m where m ≡ LEAST m. int m ≤ N+1 ∧ α < m * ξ
  have m: int m ≤ N+1 ∧ α < m * ξ
    using LeastI_ex [OF ex] unfolding m_def by blast
  have m > 0
    using ‹0 ≤ α› m gr0I by force
  have kϑ: ξ < ε
    using hk 2 unfolding ξ_def by (smt (verit, best) floor_eq_iff frac_def)
  have mk_eq: frac (of_int (m*k) * ϑ) = m * frac (of_int k * ϑ)
    if m>0 frac (of_int k * ϑ) < 1/m for m k
  proof (subst frac_unique_iff , intro conjI)
    show real_of_int (m * k) * ϑ - real_of_int m * frac (real_of_int k * ϑ) ∈
ℤ
      by (simp add: algebra_simps frac_def)
  qed (use that in ‹auto simp: divide_simps mult_ac›)
  show thesis
  proof (cases m=1)
    case True
    then have |frac (real (nat k) * ϑ) - α| < ε
      using m α ‹0 < k› ξ_def kϑ by force
    with ‹0 < k› zero_less_nat_eq that show ?thesis by blast
  next
    case False
    with ‹0 < m› have m>1 by linarith
    with ‹0 < k› have mk_pos:(m - Suc 0) * nat k > 0 by force
    have real_of_int (int m - 1) < 1 / frac (real_of_int k * ϑ)
      using N0 ξ_def m by force
    then
    have m1_eq: (int m - 1) * ξ = frac (((int m - 1) * k) * ϑ)
      using m mk_eq [of int m-1 k] pos ‹m>1› by (simp add: divide_simps
mult_ac ξ_def)
    moreover have (m - Suc 0) * ξ ≤ α
      using Least_le [where k=m-Suc 0] m
    by (smt (verit, best) Suc_n_not_le_n Suc_pred ‹0 < m› m_def of_nat_le_iff)
    ultimately have leα: frac (of_int ((int m - 1) * k) * ϑ) ≤ α
      by (simp add: Suc_leI ‹0 < m› of_nat_diff)
    moreover have (m * ξ - frac (of_int ((int m - 1) * k) * ϑ)) < ε
     by (metis m1_eq add_diff_cancel_left' diff_add_cancel kϑ left_diff_distrib'

          mult_cancel_right2 of_int_1 of_int_diff of_int_of_nat_eq)
    ultimately have |frac (real( (m - 1) * nat k) * ϑ) - α| < ε
      using ‹0 < k› ‹0 < m› by simp (smt (verit, best) One_nat_def Suc_leI
m of_nat_1 of_nat_diff)
    with ‹m > 0› that show thesis
      using mk_pos One_nat_def by presburger
```

```
    qed
  qed
qed
```

Theorem 7.7 expressed more abstractly using *closure*

```
corollary Kronecker_approx_1:
  fixes ϑ :: real
  assumes ϑ ∉ ℚ
  shows closure (range (λn. frac (real n * ϑ))) = {0..1}  (is ?C = _)
proof −
  have ∃k>0. |frac(real k * ϑ) − α| < ε if 0 ≤ α α ≤ 1 ε > 0 for α ε
    by (meson Kronecker_approx_1_explicit assms that)
  then have x ∈ ?C if 0 ≤ x x ≤ 1 for x :: real
    using that by (auto simp add: closure_approachable dist_real_def)
  moreover
  have (range (λn. frac (real n * ϑ))) ⊆ {0..1}
    by (smt (verit) atLeastAtMost_iff frac_unique_iff image_subset_iff)
  then have ?C ⊆ {0..1}
    by (simp add: closure_minimal)
  ultimately show ?thesis by auto
qed
```

The next theorem removes the restriction $0 \leq \alpha \leq 1$.

Theorem 7.8

```
corollary sequence_of_fractional_parts_is_dense:
  fixes ϑ :: real
  assumes ϑ ∉ ℚ ε > 0
  obtains h k where k > 0 |of_int k * ϑ − of_int h − α| < ε
proof −
  obtain k where k>0 |frac(real k * ϑ) − frac α| < ε
   by (metis Kronecker_approx_1_explicit assms frac_ge_0 frac_lt_1 less_le_not_le)
  then have |real_of_int k * ϑ − real_of_int (⌊k * ϑ⌋ − ⌊α⌋) − α| < ε
    by (auto simp: frac_def)
  then show thesis
    by (meson ‹0 < k› of_nat_0_less_iff that)
qed
```

8.3 Extension of Kronecker's Theorem to Simultaneous Approximation

8.3.1 Towards Lemma 1

```
lemma integral_exp:
  assumes  T ≥ 0 a≠0
  shows integral {0..T} (λt. exp(a * complex_of_real t)) = (exp(a * of_real T)
− 1) / a
proof −
```

```
  have ⋀t. t ∈ {0..T} ⟹ ((λx. exp (a * x) / a) has_vector_derivative exp (a *
t)) (at t within {0..T})
    using assms
   by (intro derivative_eq_intros has_complex_derivative_imp_has_vector_derivative
[unfolded o_def] | simp)+
  then have ((λt. exp(a * of_real t)) has_integral exp(a * complex_of_real T)/a
- exp(a * of_real 0)/a)  {0..T}
    by (meson fundamental_theorem_of_calculus ‹T ≥ 0›)
  then show ?thesis
    by (simp add: diff_divide_distrib integral_unique)
qed

lemma Kronecker_simult_aux1:
  fixes η:: nat ⇒ real and c:: nat ⇒ complex and N::nat
  assumes inj: inj_on η {..N} and k ≤ N
  defines f ≡ λt. ∑r≤N. c r * exp(i * of_real t * η r)
  shows ((λT. (1/T) * integral {0..T} (λt. f t * exp(-i * of_real t * η k))) ⟶
c k) at_top
proof -
  define F where F ≡ λk t. f t * exp(-i * of_real t * η k)
  have f: F k = (λt. ∑r≤N. c r * exp(i * (η r - η k) * of_real t))
    by (simp add: F_def f_def sum_distrib_left field_simps exp_diff exp_minus)
  have *: integral {0..T} (F k)
      = c k * T + (∑r ∈ {..N}-{k}. c r * integral {0..T} (λt. exp(i * (η r - η
k) * of_real t)))
    if T > 0 for T
    using ‹k ≤ N› that
   by (simp add: f integral_sum integrable_continuous_interval continuous_intros
Groups_Big.sum_diff scaleR_conv_of_real)

  have **: (1/T) * integral {0..T} (F k)
      = c k + (∑r ∈ {..N}-{k}. c r * (exp(i * (η r - η k) * of_real T) - 1) /
(i * (η r - η k) * of_real T))
    if T > 0 for T
  proof -
   have I: integral {0..T} (λt. exp (i * (complex_of_real t * η r) - i * (complex_of_real
t * η k)))
          = (exp(i * (η r - η k) * T) - 1) / (i * (η r - η k))
      if r ≤ N r ≠ k for r
      using that ‹k ≤ N› inj ‹T > 0› integral_exp [of T i * (η r - η k)]
      by (simp add: inj_on_eq_iff algebra_simps)
    show ?thesis
      using that by (subst *) (auto simp add: algebra_simps sum_divide_distrib
I)
  qed
  have ((λT. c r * (exp(i * (η r - η k) * of_real T) - 1) / (i * (η r - η k) *
of_real T)) ⟶ 0) at_top
    for r
  proof -
```

```
    have ((λx. (cos ((η r − η k) ∗ x) − 1) / x) ⟶ 0) at_top
        ((λx. sin ((η r − η k) ∗ x) / x) ⟶ 0) at_top
      by real_asymp+
    hence ((λT. (exp (i ∗ (η r − η k) ∗ of_real T) − 1) / of_real T) ⟶ 0)
at_top
      by (simp add: tendsto_complex_iff Re_exp Im_exp)
    from tendsto_mult[OF this tendsto_const[of c r / (i ∗ (η r − η k))]] show
?thesis
      by (simp add: field_simps)
  qed
  then have ((λT. c k + (∑ r ∈ {..N}−{k}. c r ∗ (exp(i ∗ (η r − η k) ∗ of_real
T) − 1) /
                (i ∗ (η r − η k) ∗ of_real T))) ⟶ c k + 0) at_top
    by (intro tendsto_add tendsto_null_sum) auto
  also have ?this ⟷ ?thesis
  proof (rule filterlim_cong)
    show ∀ F x in at_top. c k + (∑ r∈{..N} − {k}. c r ∗ (exp (i ∗ of_real (η r −
η k) ∗ of_real x) − 1) /
           (i ∗ of_real (η r − η k) ∗ of_real x)) =
         1 / of_real x ∗ integral {0..x} (λt. f t ∗ exp (− i ∗ of_real t ∗ of_real (η
k)))
      using eventually_gt_at_top[of 0]
    proof eventually_elim
      case (elim T)
      show ?case
        using ∗∗[of T] elim by (simp add: F_def)
    qed
  qed auto
  finally show ?thesis .
qed
```

the version of Lemma 1 that we actually need

```
lemma Kronecker_simult_aux1_strict:
  fixes η:: nat ⇒ real and c:: nat ⇒ complex and N::nat
  assumes η: inj_on η {..<N} 0 ∉ η ‘ {..<N} and k < N
  defines f ≡ λt. 1 + (∑ r<N. c r ∗ exp(i ∗ of_real t ∗ η r))
  shows ((λT. (1/T) ∗ integral {0..T} (λt. f t ∗ exp(−i ∗ of_real t ∗ η k))) ⟶
c k) at_top
proof −
  define c′ where c′ ≡ case_nat 1 c
  define η′ where η′ ≡ case_nat 0 η
  define f′ where f′ ≡ λt. (∑ r≤N. c′ r ∗ exp(i ∗ of_real t ∗ η′ r))
  have inj_on η′ {..N}
    using η by (auto simp: η′_def inj_on_def split: nat.split_asm)
  moreover have Suc k ≤ N
    using ‹k < N› by auto
  ultimately have ((λT. 1 / T ∗ integral {0..T} (λt. (∑ r≤N. c′ r ∗ exp (i ∗
of_real t ∗ η′ r)) ∗
                  exp (− i ∗ t ∗ η′ (Suc k)))) ⟶ c′ (Suc k))
```

```
    at_top
  by (intro Kronecker_simult_aux1)
 moreover have (∑ r≤N. c' r * exp (i * of_real t * η' r)) = 1 + (∑ r<N. c r
* exp (i * of_real t * η r)) for t
  by (simp add: c'_def η'_def sum.atMost_shift)
 ultimately show ?thesis
  by (simp add: f_def c'_def η'_def)
qed
```

8.3.2 Towards Lemma 2

```
lemma cos_monotone_aux: ⟦|2 * pi * of_int i + x| ≤ y; y ≤ pi⟧ ⟹ cos y ≤
cos x
 by (metis add.commute abs_ge_zero cos_abs_real cos_monotone_0_pi_le sin_cos_eq_iff)

lemma Figure7_1:
 assumes 0 ≤ ε ε ≤ |x| |x| ≤ pi
 shows cmod (1 + exp (i * x)) ≤ cmod (1 + exp (i * ε))
proof −
 have norm_eq: sqrt (2 * (1 + cos t)) = cmod (1 + cis t) for t
  by (simp add: norm_complex_def power2_eq_square algebra_simps)
 have cos |x| ≤ cos ε
  by (rule cos_monotone_0_pi_le) (use assms in auto)
 hence sqrt (2 * (1 + cos |x|)) ≤ sqrt (2 * (1 + cos ε))
  by simp
 also have cos |x| = cos x
  by simp
 finally show ?thesis
  unfolding norm_eq by (simp add: cis_conv_exp)
qed

lemma Kronecker_simult_aux2:
 fixes α:: nat ⇒ real and ϑ:: nat ⇒ real and n::nat
 defines F ≡ λt. 1 + (∑ r<n. exp(i * of_real (2 * pi * (t * ϑ r − α r))))
 defines L ≡ Sup (range (norm ∘ F))
 shows (∀ε>0. ∃t h. ∀r<n. |t * ϑ r − α r − of_int (h r)| < ε) ⟷ L = Suc
n (is ?lhs = _)
proof
 assume ?lhs
 then have ⋀k. ∃t h. ∀r<n. |t * ϑ r − α r − of_int (h r)| < 1 / (2 * pi * Suc
k)
  by simp
 then obtain t h where th: ⋀k r. r<n ⟹ |t k * ϑ r − α r − of_int (h k r)|
< 1 / (2 * pi * Suc k)
  by metis
 have Fle: ⋀x. cmod (F x) ≤ real (Suc n)
  by (force simp: F_def intro: order_trans [OF norm_triangle_ineq] order_trans
[OF norm_sum])
 then have boundedF: bounded (range F)
```

```
    by (metis bounded_iff rangeE)
  have leL: 1 + n * cos(1 / Suc k) ≤ L for k
  proof −
    have ∗: cos (1 / Suc k) ≤ cos (2 * pi * (t k * ϑ r − α r)) if r<n for r
    proof (rule cos_monotone_aux)
      have |2 * pi * (− h k r) + 2 * pi * (t k * ϑ r − α r)| ≤ |t k * ϑ r − α r −
h k r| * 2 * pi
        by (simp add: algebra_simps abs_mult_pos abs_mult_pos')
      also have ... ≤ 1 / real (Suc k)
        using th [OF that, of k] by (simp add: divide_simps)
      finally show |2 * pi * real_of_int (− h k r) + 2 * pi * (t k * ϑ r − α r)|
≤ 1 / real (Suc k) .
      have 1 / real (Suc k) ≤ 1
        by auto
      then show 1 / real (Suc k) ≤ pi
        using pi_ge_two by linarith
    qed
    have 1 + n * cos(1 / Suc k) = 1 + (∑ r<n. cos(1 / Suc k))
      by simp
    also have ... ≤ 1 + (∑ r<n. cos (2 * pi * (t k * ϑ r − α r)))
      using ∗ by (intro add_mono sum_mono) auto
    also have ... ≤ Re (F(t k))
      by (simp add: F_def Re_exp)
    also have ... ≤ norm (F(t k))
      by (simp add: complex_Re_le_cmod)
    also have ... ≤ L
      by (simp add: L_def boundedF bounded_norm_le_SUP_norm)
    finally show ?thesis .
  qed
  show L = Suc n
  proof (rule antisym)
    show L ≤ Suc n
      using Fle by (auto simp add: L_def intro: cSup_least)
    have ((λk. 1 + real n * cos (1 / real (Suc k))) ⟶ 1 + real n) at_top
      by real_asymp
    with LIMSEQ_le_const2 leL show Suc n ≤ L
      by fastforce
  qed
next
  assume L: L = Suc n
  show ?lhs
  proof (rule ccontr)
    assume ¬ ?lhs
    then obtain e0 where e0>0 and e0: ⋀t h. ∃k<n. |t * ϑ k − α k − of_int
(h k)| ≥ e0
      by (force simp: not_less)
    define ε where ε ≡ min e0 (pi/4)
    have ε: ⋀t h. ∃k<n. |t * ϑ k − α k − of_int (h k)| ≥ ε
      unfolding ε_def using e0 min.coboundedI1 by blast
```

```
  have ε_bounds: ε > 0 ε < pi ε ≤ pi/4
    using ‹e0 > 0› by (auto simp: ε_def min_def)
  with sin_gt_zero have sin ε > 0
    by blast
  then have ¬ collinear{0, 1, exp (i * ε)}
   using collinear_iff_Reals cis.sel cis_conv_exp complex_is_Real_iff by force
  then have norm (1 + exp (i * ε)) < 2
    using norm_triangle_eq_imp_collinear
   by (smt (verit) linorder_not_le norm_exp_i_times norm_one norm_triangle_lt)
  then obtain δ where δ > 0 and δ: norm (1 + exp (i * ε)) = 2 - δ
    by (smt (verit, best))
  have norm (F t) ≤ n+1-δ for t
  proof -
    define h where h ≡ λr. round (t * ϑ r - α r)
    define X where X ≡ λr. t * ϑ r - α r - h r
    have exp (i * (of_int j * (of_real pi * 2))) = 1 for j
      by (metis add_0 exp_plus_2pin exp_zero)
    then have exp_X: exp (i * (2 * of_real pi * of_real (X r)))
                   = exp (i * of_real (2 * pi * (t * ϑ r - α r))) for r
      by (simp add: X_def right_diff_distrib exp_add exp_diff mult.commute)
    have norm_pr_12: X r ∈ {-1/2..<1/2} for r
      apply (simp add: X_def h_def)
   by (smt (verit, best) abs_of_nonneg half_bounded_equal of_int_round_abs_le
of_int_round_gt)
    obtain k where k<n and 1: |X k| ≥ ε
      using X_def ε [of t h] by auto
    have 2: 2*pi ≥ 1
      using pi_ge_two by auto
    have |2 * pi * X k| ≥ ε
      using mult_mono [OF 2 1] pi_ge_zero by linarith
    moreover
    have |2 * pi * X k| ≤ pi
      using norm_pr_12 [of k] apply (simp add: abs_if field_simps)
      by (smt (verit, best) divide_le_eq_1_pos divide_minus_left m2pi_less_pi
nonzero_mult_div_cancel_left)
    ultimately
    have *: norm (1 + exp (i * of_real (2 * pi * X k))) ≤ norm (1 + exp (i *
ε))
      using Figure7_1[of ε 2 * pi * X k] ε_bounds by linarith
    have norm (F t) = norm (1 + (∑ r<n. exp(i * of_real (2 * pi * (X r)))))
      by (auto simp: F_def exp_X)
      also have ... ≤ norm (1 + exp(i * of_real (2 * pi * X k)) + (∑ r ∈
{..<n}-{k}. exp(i * of_real (2 * pi * X r))))
      by (simp add: comm_monoid_add_class.sum.remove [where x=k] ‹k < n›
algebra_simps)
      also have ... ≤ norm (1 + exp(i * of_real (2 * pi * X k))) + (∑ r ∈
{..<n}-{k}. norm (exp(i * of_real (2 * pi * X r))))
      by (intro norm_triangle_mono sum_norm_le order_refl)
    also have ... ≤ (2 - δ) + (n - 1)
```

```
      using ‹k < n› δ
      by simp (metis * mult_2 of_real_add of_real_mult)
    also have ... = n + 1 - δ
      using ‹k < n› by simp
    finally show ?thesis .
  qed
  then have L ≤ 1 + real n - δ
    by (auto simp: L_def intro: cSup_least)
  with L ‹δ > 0› show False
    by linarith
 qed
qed
```

8.3.3 Towards lemma 3

The text doesn't say so, but the generated polynomial needs to be "clean": all coefficients nonzero, and with no exponent string mentioned more than once. And no constant terms (with all exponents zero).

Some tools for combining integer-indexed sequences or splitting them into parts

```
lemma less_sum_nat_iff:
  fixes b::nat and k::nat
  shows b < (∑ i<k. a i) ⟷ (∃ j<k. (∑ i<j. a i) ≤ b ∧ b < (∑ i<j. a i) + a
j)
proof (induction k arbitrary: b)
  case (Suc k)
  then show ?case
    by (simp add: less_Suc_eq) (metis not_less trans_less_add1)
qed auto

lemma less_sum_nat_iff_uniq:
  fixes b::nat and k::nat
  shows b < (∑ i<k. a i) ⟷ (∃!j. j<k ∧ (∑ i<j. a i) ≤ b ∧ b < (∑ i<j. a i)
+ a j)
  unfolding less_sum_nat_iff by (meson leD less_sum_nat_iff nat_neq_iff)

definition part :: (nat ⇒ nat) ⇒ nat ⇒ nat ⇒ nat
  where part a k b ≡ (THE j. j<k ∧ (∑ i<j. a i) ≤ b ∧ b < (∑ i<j. a i) + a j)

lemma part:
  b < (∑ i<k. a i) ⟷ (let j = part a k b in j < k ∧ (∑ i < j. a i) ≤ b ∧ b <
(∑ i < j. a i) + a j)
  (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
   unfolding less_sum_nat_iff_uniq part_def by (metis (no_types, lifting) theI')
qed (auto simp: less_sum_nat_iff Let_def)
```

lemma *part_Suc*: *part a (Suc k) b* = (*if sum a {..<k} ≤ b ∧ b < sum a {..<k} + a k then k else part a k b*)
using *leD*
by (*fastforce simp*: *part_def less_Suc_eq less_sum_nat_iff conj_disj_distribR cong*: *conj_cong*)

The polynomial expansions used in Lemma 3

definition *gpoly* :: [*nat*, *nat*⇒*complex*, *nat*, *nat*⇒*nat*, [*nat*,*nat*]⇒*nat*] ⇒ *complex*
where *gpoly n x N a r* ≡ (∑ *k*<*N*. *a k* ∗ (∏ *i*<*n*. *x i* ^ *r k i*))

lemma *gpoly_cong*:
assumes ⋀*k*. *k* < *N* ⟹ *a*′ *k* = *a k* ⋀*k i*. ⟦*k* < *N*; *i*<*n*⟧ ⟹ *r*′ *k i* = *r k i*
shows *gpoly n x N a r* = *gpoly n x N a*′ *r*′
using *assms* **by** (*auto simp*: *gpoly_def*)

lemma *gpoly_inc*: *gpoly n x N a r* = *gpoly* (*Suc n*) *x N a* (λ*k*. (*r k*)(*n*:=*0*))
by (*simp add*: *gpoly_def algebra_simps sum_distrib_left*)

lemma *monom_times_gpoly*: *gpoly n x N a r* ∗ *x n* ^ *q* = *gpoly* (*Suc n*) *x N a* (λ*k*. (*r k*)(*n*:=*q*))
by (*simp add*: *gpoly_def algebra_simps sum_distrib_left*)

lemma *const_times_gpoly*: *e* ∗ *gpoly n x N a r* = *gpoly n x N* ((∗)*e* ∘ *a*) *r*
by (*simp add*: *gpoly_def sum_distrib_left mult.assoc*)

lemma *one_plus_gpoly*: *1* + *gpoly n x N a r* = *gpoly n x* (*Suc N*) (*a*(*N*:=*1*)) (*r*(*N*:=(λ_. *0*)))
by (*simp add*: *gpoly_def*)

lemma *gpoly_add*:
gpoly n x N a r + *gpoly n x N*′ *a*′ *r*′
= *gpoly n x* (*N*+*N*′) (λ*k*. *if k*<*N then a k else a*′ (*k*−*N*)) (λ*k*. *if k*<*N then r k else r*′ (*k*−*N*))
proof −
have {..<*N*+*N*′} = {..<*N*} ∪ {*N*..<*N*+*N*′} {..<*N*} ∩ {*N*..<*N* + *N*′} = {}
by *auto*
then show *?thesis*
by (*simp add*: *gpoly_def sum.union_disjoint sum.atLeastLessThan_shift_0*[*of* _ *N*] *atLeast0LessThan*)
qed

lemma *gpoly_sum*:
fixes *n Nf af rf p*
defines *N* ≡ *sum Nf* {..<*p*}
defines *a* ≡ λ*k*. *let q* = (*part Nf p k*) *in af q* (*k* − *sum Nf* {..<*q*})
defines *r* ≡ λ*k*. *let q* = (*part Nf p k*) *in rf q* (*k* − *sum Nf* {..<*q*})
shows (∑ *q*<*p*. *gpoly n x* (*Nf q*) (*af q*) (*rf q*)) = *gpoly n x N a r*
unfolding *N_def a_def r_def*

proof (*induction p*)
case *0*
then show *?case*
by (*simp add*: *gpoly_def*)
next
case (*Suc p*)
then show *?case*
by (*auto simp*: *gpoly_add Let_def part_Suc intro*: *gpoly_cong*)
qed

For excluding constant terms: with all exponents zero

definition *nontriv_exponents* :: [*nat*, *nat*, [*nat*,*nat*]⇒*nat*] ⇒ *bool*
where *nontriv_exponents n N r* ≡ ∀ *k*<*N*. ∃ *i*<*n*. *r k i* ≠ *0*

lemma *nontriv_exponents_add*:
⟦*nontriv_exponents n M r*; *nontriv_exponents* (*Suc n*) *N r'*⟧ ⟹
nontriv_exponents (*Suc n*) (*M* + *N*) (λ*k*. *if k*<*M then r k else r'* (*k*−*M*))
by (*force simp add*: *nontriv_exponents_def less_Suc_eq*)

lemma *nontriv_exponents_sum*:
assumes ⋀*q*. *q* < *p* ⟹ *nontriv_exponents n* (*N q*) (*r q*)
shows *nontriv_exponents n* (*sum N* {..<*p*}) (λ*k*. *let q* = *part N p k in r q* (*k* − *sum N* {..<*q*}))
using *assms* **by** (*simp add*: *nontriv_exponents_def less_diff_conv2 part Let_def*)

definition *uniq_exponents* :: [*nat*, *nat*, [*nat*,*nat*]⇒*nat*] ⇒ *bool*
where *uniq_exponents n N r* ≡ ∀ *k*<*N*. ∀ *k'*<*k*. ∃ *i*<*n*. *r k i* ≠ *r k' i*

lemma *uniq_exponents_inj*: *uniq_exponents n N r* ⟹ *inj_on r* {..<*N*}
by (*smt* (*verit*, *best*) *inj_on_def lessThan_iff linorder_cases uniq_exponents_def*)

All coefficients must be nonzero

definition *nonzero_coeffs* :: [*nat*, *nat*⇒*nat*] ⇒ *bool*
where *nonzero_coeffs N a* ≡ ∀ *k*<*N*. *a k* ≠ *0*

Any polynomial expansion can be cleaned up, removing zero coeffs, etc.

lemma *gpoly_uniq_exponents*:
obtains *N' a' r'*
where ⋀*x*. *gpoly n x N a r* = *gpoly n x N' a' r'*
uniq_exponents n N' r' nonzero_coeffs N' a' N' ≤ *N*
nontriv_exponents n N r ⟹ *nontriv_exponents n N' r'*
proof (*cases* ∀ *k*<*N*. *a k* = *0*)
case *True*
show *thesis*
proof
show *gpoly n x N a r* = *gpoly n x 0 a r* **for** *x*
by (*auto simp*: *gpoly_def True*)
qed (*auto simp*: *uniq_exponents_def nonzero_coeffs_def nontriv_exponents_def*)
next

```
  case False
  define cut where cut f i ≡ if i<n then f i else (0::nat) for f i
  define R where R ≡ cut ‘ r ‘ ({..<N} ∩ {k. a k > 0})
  define N' where N' ≡ card R
  define r' where r' ≡ from_nat_into R
  define a' where a' ≡ λk'. ∑ k<N. if r' k' = cut (r k) then a k else 0
  have finite R
    using R_def by blast
  then have bij: bij_betw r' {..<N'} R
    using bij_betw_from_nat_into_finite N'_def r'_def by blast
  have 1: card {i. i < N' ∧ r' i = cut (r k)} = Suc 0
    if k < N a k > 0 for k
  proof −
    have card {i. i < N' ∧ r' i = cut (r k)} ≤ Suc 0
    using bij by (simp add: card_le_Suc0_iff_eq bij_betw_iff_bijections Ball_def)
metis
    moreover have card {i. i < N' ∧ r' i = cut (r k)} > 0
      using bij that by (auto simp: card_gt_0_iff bij_betw_iff_bijections R_def)
    ultimately show card {i. i < N' ∧ r' i = cut (r k)} = Suc 0
      using that by auto
  qed
  show thesis
  proof
    have ∃ i<n. r' k i ≠ r' k' i if k < N' and k' < k for k k'
    proof −
      have k' < N'
        using order.strict_trans that by blast
      then have r' k ≠ r' k'
        by (metis bij bij_betw_iff_bijections lessThan_iff nat_neq_iff that)
      moreover obtain sk sk' where r' k = cut sk r' k' = cut sk'
       by (metis R_def ‹k < N'› ‹k' < N'› bij bij_betwE image_iff lessThan_iff)
      ultimately show ?thesis
        using local.cut_def by force
    qed
    then show uniq_exponents n N' r'
      by (auto simp: uniq_exponents_def R_def)
    have R ⊆ (cut ∘ r) ‘ {..<N}
      by (auto simp: R_def)
    then show N' ≤ N
      unfolding N'_def by (metis card_lessThan finite_lessThan surj_card_le)
    show gpoly n x N a r = gpoly n x N' a' r' for x
    proof −
      have a k * (∏ i<n. x i ^ r k i)
        = (∑ i<N'. (if r' i = cut (r k) then of_nat (a k) else of_nat 0) * (∏ j<n.
x j ^ r' i j))
        if k<N for k
        using that
        by (casesa k = 0)
           (simp_all add: if_distrib [of λv. v * _] 1 cut_def flip: sum.inter_filter
```

```
cong: if_cong)
      then show ?thesis
           by (simp add: gpoly_def a'_def sum_distrib_right sum.swap [where
A={..<N'}] if_distrib [of of_nat])
    qed
    show nontriv_exponents n N' r' if ne: nontriv_exponents n N r
    proof −
      have ∃i<n. 0 < r' k' i if k' < N' for k'
      proof −
        have r' k' ∈ R
          using bij bij_betwE that by blast
        then obtain k where k<N and k: r' k' = cut (r k)
          using R_def by blast
        with ne obtain i where i<n r k i > 0
          by (auto simp: nontriv_exponents_def)
        then show ?thesis
          using k local.cut_def by auto
      qed
      then show ?thesis
        by (simp add: nontriv_exponents_def)
    qed
    have 0 < a' k' if k' < N' for k'
    proof −
      have r' k' ∈ R
        using bij bij_betwE that by blast
      then obtain k where k<N a k > 0 r' k' = cut (r k)
        using R_def by blast
      then have False if a' k' = 0
        using that by (force simp add: a'_def Ball_def )
      then show ?thesis
        by blast
    qed
    then show nonzero_coeffs N' a'
      by (auto simp: nonzero_coeffs_def)
  qed
qed


lemma Kronecker_simult_aux3:
  ∃N a r. (∀x. (1 + (∑i<n. x i))^p = 1 + gpoly n x N a r) ∧ Suc N ≤ (p+1)^n
        ∧ nontriv_exponents n N r
proof (induction n arbitrary: p)
  case 0
  then show ?case
    by (auto simp: gpoly_def nontriv_exponents_def nonzero_coeffs_def)
next
  case (Suc n)
  then obtain Nf af rf
    where feq: ⋀q x. (1 + (∑i<n. x i)) ^ q = 1 + gpoly n x (Nf q) (af q) (rf q)
```

```
    and Nle: ⋀q. Suc (Nf q) ≤ (q+1)^n
    and nontriv: ⋀q. nontriv_exponents n (Nf q) (rf q)
   by metis
  define N where N ≡ Nf p + (∑ q<p. Suc (Nf q))
  define a where a ≡ λk. if k < Nf p then af p k
                         else let q = part (λt. Suc (Nf t)) p (k − Nf p)
                              in (p choose q) *
                                 (if k − Nf p − (∑ t<q. Suc (Nf t)) = Nf q then Suc 0
                                  else af q (k − Nf p − (∑ t<q. Suc(Nf t))))
  define r where r ≡ λk. if k < Nf p then (rf p k)(n := 0)
                                  else let q = part (λt. Suc (Nf t)) p (k − Nf p)
                                       in (if k − Nf p − (∑ t<q. Suc (Nf t)) = Nf q
then λ_. 0
                                           else rf q (k − Nf p − (∑ t<q. Suc(Nf t))))
(n := p−q)
  have peq: {..p} = insert p {..<p}
   by auto
  have nontriv_exponents n (Nf p) (λi. (rf p i)(n := 0))
     ⋀q. q<p ⟹ nontriv_exponents (Suc n) (Suc (Nf q)) (λk. (if k = Nf q then
λ_. 0 else rf q k) (n := p − q))
    using nontriv by (fastforce simp: nontriv_exponents_def)+
  then have nontriv_exponents (Suc n) N r
   unfolding N_def r_def by (intro nontriv_exponents_add nontriv_exponents_sum)
  moreover
  { fix x :: nat ⇒ complex
    have (1 + (∑ i < Suc n. x i)) ^ p = (1 + (∑ i<n. x i) + x n) ^ p
     by (simp add: add_ac)
    also have ... = (∑ q≤p. (p choose q) * (1 + (∑ i<n. x i))^q * x n^(p−q))
     by (simp add: binomial_ring)
    also have ... = (∑ q≤p. (p choose q) * (1 + gpoly n x (Nf q) (af q) (rf q)) *
x n^(p−q))
     by (simp add: feq)
    also have ... = 1 + (gpoly n x (Nf p) (af p) (rf p) + (∑ q<p. (p choose q) *
(1 + gpoly n x (Nf q) (af q) (rf q)) * x n^(p−q)))
     by (simp add: algebra_simps sum.distrib peq)
    also have ... = 1 + gpoly (Suc n) x N a r
     apply (subst one_plus_gpoly)
     apply (simp add: const_times_gpoly monom_times_gpoly gpoly_sum)
     apply (simp add: gpoly_inc [where n=n] gpoly_add N_def a_def r_def)
     done
    finally have (1 + sum x {..<Suc n}) ^ p = 1 + gpoly (Suc n) x N a r .
  }
  moreover have Suc N ≤ (p + 1) ^ Suc n
  proof −
    have Suc N = (∑ q≤p. Suc (Nf q))
     by (simp add: N_def peq)
    also have ... ≤ (∑ q≤p. (q+1)^n)
     by (meson Nle sum_mono)
    also have ... ≤ (∑ q≤p. (p+1)^n)
```

```
    by (force intro!: sum_mono power_mono)
  also have ... ≤ (p + 1) ^ Suc n
    by simp
  finally show Suc N ≤ (p + 1) ^ Suc n .
 qed
 ultimately show ?case
  by blast
qed

lemma Kronecker_simult_aux3_uniq_exponents:
 obtains N a r where ⋀x. (1 + (∑ i<n. x i))^p = 1 + gpoly n x N a r Suc N
≤ (p+1)^n
                    nontriv_exponents n N r uniq_exponents n N r nonzero_coeffs
N a
proof −
 obtain N0 a0 r0 where ⋀x. (1 + (∑ r<n. x r)) ^ p = 1 + gpoly n x N0 a0 r0
  and Suc N0 ≤ (p+1)^n nontriv_exponents n N0 r0
  using Kronecker_simult_aux3 by blast
 with le_trans Suc_le_mono gpoly_uniq_exponents [of n N0 a0 r0] that show
thesis
  by (metis (no_types, lifting))
qed
```

8.3.4 And finally Kroncker's theorem itself

Statement of Theorem 7.9

```
theorem Kronecker_thm_1:
 fixes α ϑ:: nat ⇒ real and n:: nat
 assumes indp: module.independent (λr. (*) (real_of_int r)) (ϑ ' {..<n})
  and injϑ: inj_on ϑ {..<n} and ε > 0
 obtains t h where ⋀i. i < n ⟹ |t * ϑ i − of_int (h i) − α i| < ε
proof (cases n>0)
 case False then show ?thesis
  using that by blast
next
 case True
 interpret Modules.module (λr. (*) (real_of_int r))
  by (simp add: Modules.module.intro distrib_left mult.commute)
 define F where F ≡ λt. 1 + (∑ i<n. exp(ı * of_real (2 * pi * (t * ϑ i − α
i))))
 define L where L ≡ Sup (range (norm ∘ F))
 have [continuous_intros]: 0 < T ⟹ continuous_on {0..T} F for T
  unfolding F_def by (intro continuous_intros)
 have nft_Sucn: norm (F t) ≤ Suc n for t
   unfolding F_def by (rule norm_triangle_le order_trans [OF norm_sum] |
simp)+
 then have L_le: L ≤ Suc n
  by (simp add: L_def cSUP_least)
 have nft_L: norm (F t) ≤ L for t
```

```
    by (metis L_def UNIV_I bdd_aboveI2 cSUP_upper nft_Sucn o_apply)
  have L = Suc n
  proof −
    { fix p::nat
      assume p>0
      obtain N a r where 3: ⋀x. (1 + (∑ r<n. x r)) ^ p = 1 + gpoly n x N a r
            and SucN: Suc N ≤ (p+1)^n
            and nontriv: nontriv_exponents n N r and uniq: uniq_exponents n N r
            and apos: nonzero_coeffs N a
        using Kronecker_simult_aux3_uniq_exponents by blast
      have N ≠ 0
      proof
        assume N = 0
        have 2^p = (1::complex)
        using 3 [of (λ_. 0)(0:=1)] True ‹p>0› ‹N = 0› by (simp add: gpoly_def)
        then have 2 ^ p = Suc 0
       by (metis of_nat_1 One_nat_def of_nat_eq_iff of_nat_numeral of_nat_power)
        with ‹0 < p› show False by force
      qed
      define x where x ≡ λt r. exp(i * of_real (2 * pi * (t * ϑ r − α r)))
      define f where f ≡ λt. (F t) ^ p
      have feq: f t = 1 + gpoly n (x t) N a r for t
        unfolding f_def F_def x_def by (simp flip: 3)
      define c where c ≡ λk. a k / cis (∑ i<n. (pi * (2 * (α i * real (r k i)))))
      define η where η ≡ λk. 2 * pi * (∑ i<n. r k i * ϑ i)
     define INTT where INTT ≡ λk T. (1/T) * integral {0..T} (λt. f t * exp(−i
* of_real t * η k))
      have a k * (∏ i<n. x t i ^ r k i) = c k * exp (i * t * η k) if k<N for k t
          apply (simp add: x_def η_def sum_distrib_left flip: exp_of_nat_mult
exp_sum)
       apply (simp add: algebra_simps sum_subtractf exp_diff c_def sum_distrib_left
cis_conv_exp)
        done
      then have fdef: f t = 1 + (∑ k<N. c k * exp(i * of_real t * η k)) for t
        by (simp add: feq gpoly_def)
      have nzero: ϑ i ≠ 0 if i<n for i
        using indp that local.dependent_zero by force
      have ind_disj: ⋀u. (∀ x<n. u (ϑ x) = 0) ∨ (∑ v ∈ ϑ`{..<n}. of_int (u v) *
v) ≠ 0
        using indp by (auto simp: dependent_finite)
      have injη: inj_on η {..<N}
      proof
        fix k k'
        assume k: k ∈ {..<N} k' ∈ {..<N} and η k = η k'
        then have eq: (∑ i<n. real (r k i) * ϑ i) = (∑ i<n. real (r k' i) * ϑ i)
          by (auto simp: η_def)
        define f where f ≡ λz. let i = inv_into {..<n} ϑ z in int (r k i) − int (r
k' i)
        show k = k'
```

```
        using ind_disj [of f] injϑ uniq eq k
       apply (simp add: f_def Let_def sum.reindex sum_subtractf algebra_simps
uniq_exponents_def)
        by (metis not_less_iff_gr_or_eq)
    qed
    moreover have 0 ∉ η ` {..<N}
     unfolding η_def
    proof clarsimp
     fix k
     assume *: (∑ i<n. real (r k i) * ϑ i) = 0 k < N
     define f where f ≡ λz. let i = inv_into {..<n} ϑ z in int (r k i)
     obtain i where i<n and r k i > 0
     by (meson ‹k < N› nontriv nontriv_exponents_def zero_less_iff_neq_zero)
     with * nzero show False
       using ind_disj [of f] injϑ by (simp add: f_def sum.reindex)
    qed
    ultimately have 15: (INTT k ⟶ c k) at_top if k<N for k
      unfolding fdef INTT_def using Kronecker_simult_aux1_strict that by
presburger
    have norm_c: norm (c k) ≤ L^p if k<N for k
    proof (intro tendsto_le [of _ λ_. L^p])
     show ((norm ∘ INTT k) ⟶ cmod (c k)) at_top
       using that 15 by (simp add: o_def tendsto_norm)
     have norm (INTT k T) ≤ L^p if  T ≥ 0 for T::real
     proof −
       have 0 ≤ L ^ p
         by (meson nft_L norm_ge_zero order_trans zero_le_power)
       have norm (integral {0..T} (λt. f t * exp (− (i *  t * η k))))
            ≤ integral {0..T} (λt. L^p) (is ?L ≤ ?R)  if T>0
       proof −
         have ?L ≤ integral {0..T} (λt. norm (f t * exp (− (i *  t * η k))))
       unfolding f_def by (intro continuous_on_imp_absolutely_integrable_on
continuous_intros that)
         also have ... ≤ ?R
           unfolding f_def
          by (intro integral_le continuous_intros integrable_continuous_interval
that
              | simp add: nft_L norm_mult norm_power power_mono)+
         finally show ?thesis .
       qed
       with ‹T ≥ 0› have norm (INTT k T) ≤ (1/T) * integral {0..T} (λt. L
^ p)
           by (auto simp add: INTT_def norm_divide divide_simps simp del:
integral_const_real)
       also have ... ≤ L ^ p
         using ‹T ≥ 0› ‹0 ≤ L ^ p› by simp
       finally show ?thesis .
     qed
     then show ∀F x in at_top. (norm ∘ INTT k) x ≤ L ^ p
```

```
        using eventually_at_top_linorder that by fastforce
    qed auto
    then have (∑k<N. cmod (c k)) ≤ N * L^p
      by (metis sum_bounded_above card_lessThan lessThan_iff)
    moreover
    have Re((1 + (∑r<n. 1)) ^ p) = Re (1 + gpoly n (λ_. 1) N a r)
      using 3 [of λ_. 1] by metis
    then have 14: 1 + (∑k<N. norm (c k)) = (1 + real n) ^ p
      by (simp add: c_def norm_divide gpoly_def)
    moreover
    have L^p ≥ 1
      using norm_c [of 0] ‹N ≠ 0› apos
      by (force simp add: c_def norm_divide nonzero_coeffs_def)
    ultimately have *: (1 + real n) ^ p ≤ Suc N * L^p
      by (simp add: algebra_simps)
    have [simp]: L>0
       using ‹1 ≤ L ^ p› ‹0 < p› by (smt (verit, best) nft_L norm_ge_zero
power_eq_0_iff)
    have Suc n ^ p ≤ (p+1)^n * L^p
      by (smt (verit, best) * mult.commute ‹1 ≤ L ^ p› SucN mult_left_mono
of_nat_1 of_nat_add of_nat_power_eq_of_nat_cancel_iff of_nat_power_le_of_nat_cancel_iff
plus_1_eq_Suc)
    then have (Suc n ^ p) powr (1/p) ≤ ((p+1)^n * L^p) powr (1/p)
      by (simp add: powr_mono2)
    then have (Suc n) ≤ ((p+1)^n) powr (1/p) * L
        using ‹p > 0› ‹0 < L› by (simp add: powr_powr powr_mult flip:
powr_realpow)
    also have ... = ((p+1) powr n) powr (1/p) * L
      by (simp add: powr_realpow)
    also have ... = (p+1) powr (n/p) * L
      by (simp add: powr_powr)
    finally have (n + 1) / L ≤ (p+1) powr (n/p)
      by (simp add: divide_simps)
    then have ln ((n + 1) / L) ≤ ln (real (p + 1) powr (real n / real p))
      by (simp add: flip: ln_powr)
    also have ... ≤ (n/p) * ln(p+1)
      by (simp add: powr_def)
    finally have ln ((n + 1) / L) ≤ (n/p) * ln(p+1) ∧ L > 0
      by fastforce
  } note * = this
  then have [simp]: L > 0
    by blast
  have 0: (λp. (n/p) * ln(p+1)) ⟶ 0
    by real_asymp
  have ln (real (n + 1) / L) ≤ 0
   using * eventually_at_top_dense by (intro tendsto_lowerbound [OF 0]) auto
  then have n+1 ≤ L
    using ‹0 < L› by (simp add: ln_div)
  then show ?thesis
```

```
    using L_le by linarith
  qed
  with Kronecker_simult_aux2 [of n ϑ α] ‹ε > 0› that show thesis
    by (auto simp: F_def L_def add.commute diff_diff_eq)
qed
```

Theorem 7.10

```
corollary Kronecker_thm_2:
  fixes α ϑ :: nat ⇒ real and n :: nat
  assumes indp: module.independent (λr x. of_int r * x) (ϑ ` {..n})
    and injϑ: inj_on ϑ {..n} and [simp]: ϑ n = 1 and ε > 0
  obtains k m where ⋀i. i < n ⟹ |of_int k * ϑ i − of_int (m i) − α i| < ε
proof −
  interpret Modules.module (λr. (*) (real_of_int r))
    by (simp add: Modules.module.intro distrib_left mult.commute)
  have one_in_ϑ: 1 ∈ ϑ ` {..n}
    unfolding ‹ϑ n = 1›[symmetric] by blast

  have not_in_Ints: ϑ i ∉ ℤ if i: i < n for i
  proof
    assume ϑ i ∈ ℤ
    then obtain m where m: ϑ i = of_int m
      by (auto elim!: Ints_cases)
    have not_one: ϑ i ≠ 1
      using inj_onD[OF injϑ, of i n] i by auto
    define u :: real ⇒ int where u = (λ_. 0)(ϑ i := 1, 1 := −m)
    show False
     using independentD[OF indp, of ϑ ` {i, n} u ϑ i] ‹i < n› not_one one_in_ϑ
      by (auto simp: u_def simp flip: m)
  qed

  have injϑ': inj_on (frac ∘ ϑ) {..n}
  proof (rule linorder_inj_onI')
    fix i j assume ij: i ∈ {..n} j ∈ {..n} i < j
    show (frac ∘ ϑ) i ≠ (frac ∘ ϑ) j
    proof (cases j = n)
      case True
      thus ?thesis
        using not_in_Ints[of i] ij by auto
    next
      case False
      hence j < n
        using ij by auto
      have inj_on ϑ (set [i, j, n])
        using injϑ by (rule inj_on_subset) (use ij in auto)
      moreover have distinct [i, j, n]
        using ‹j < n› ij by auto
      ultimately have distinct (map ϑ [i, j, n])
        unfolding distinct_map by blast
```

```
      hence distinct: distinct [ϑ i, ϑ j, 1]
        by simp

      show (frac ∘ ϑ) i ≠ (frac ∘ ϑ) j
      proof
        assume eq: (frac ∘ ϑ) i = (frac ∘ ϑ) j
        define u :: real ⇒ int where u = (λ_. 0)(ϑ i := 1, ϑ j := −1, 1 := ⌊ϑ j⌋
− ⌊ϑ i⌋)
        have (∑ v∈{ϑ i, ϑ j, 1}. real_of_int (u v) ∗ v) = frac (ϑ i) − frac (ϑ j)
          using distinct by (simp add: u_def frac_def)
        also have ... = 0
          using eq by simp
        finally have eq0: (∑ v∈{ϑ i, ϑ j, 1}. real_of_int (u v) ∗ v) = 0 .
        show False
          using independentD[OF indp _ _ eq0, of ϑ i] one_in_ϑ ij distinct
          by (auto simp: u_def)
      qed
    qed
  qed

  have frac (ϑ n) = 0
    by auto
  then have ϑno_int: of_int r ∉ ϑ ‘ {..<n} for r
    using injϑ′ frac_of_int
    apply (simp add: inj_on_def Ball_def)
   by (metis ‹frac (ϑ n) = 0› frac_of_int imageE le_less lessThan_iff less_irrefl)
  define ϑ′ where ϑ′ ≡ (frac ∘ ϑ)(n:=1)
  have [simp]: {..<Suc n} ∩ {x. x ≠ n} = {..<n}
    by auto
  have ϑimage[simp]: ϑ ‘ {..n} = insert 1 (ϑ ‘ {..<n})
    using lessThan_Suc lessThan_Suc_atMost by force
  have module.independent (λr. (∗) (of_int r)) (ϑ′ ‘ {..<Suc n})
    unfolding dependent_explicit ϑ′_def
  proof clarsimp
    fix T u v
    assume T: T ⊆ insert 1 ((λi. frac (ϑ i)) ‘ {..<n})
      and finite T
      and uv_eq0: (∑ v∈T. of_int (u v) ∗ v) = 0
      and v ∈ T
    define invf where invf ≡ inv_into {..<n} (frac ∘ ϑ)
    have inj_on (λx. frac (ϑ x)) {..<n}
      using injϑ′ by (auto simp: inj_on_def)
    then have invf [simp]: invf (frac (ϑ i)) = i if i<n for i
      using frac_lt_1 [of ϑ i] that by (auto simp: invf_def o_def inv_into_f_eq
[where x=i])
    define N where N ≡ invf ‘ (T − {1})
    have Nsub: N ⊆ {..n} and finite N
      using T ‹finite T› by (auto simp: N_def subset_iff)
    have inj_invf: inj_on invf (T − {1})
```

using *ϑno_int* [*of 1*] ‹*ϑ n = 1*› *inv_into_injective T*
by (*fastforce simp*: *inj_on_def invf_def*)
have *invf_iff*: *invf t = i* ⟷ (*i*<*n* ∧ *t = frac* (*ϑ i*)) **if** *t* ∈ *T*−{*1*} **for** *i t*
using *that T* **by** *auto*
have *sumN*: (∑ *i*∈*N*. *f i*) = (∑ *x*∈*T* − {*1*}. *f* (*invf x*)) **for** *f* :: *nat* ⇒ *int*
using *inj_invf T* **by** (*simp add*: *N_def sum.reindex*)
define *T′* **where** *T′* ≡ *insert 1* (*ϑ* ‘ *N*)
have [*simp*]: *finite T′ 1* ∈ *T′*
using *T′_def N_def* ‹*finite T*› **by** *auto*
have *T′sub*: *T′* ⊆ *ϑ* ‘ {..*n*}
using *Nsub T′_def ϑimage* **by** *blast*
have *in_N_not1*: *x* ∈ *N* ⟹ *ϑ x* ≠ *1* **for** *x*
using *ϑno_int* [*of 1*] **by** (*metis N_def image_iff invf_iff lessThan_iff of_int_1*)
define *u′* **where** *u′* = (*u* ∘ *frac*)(*1*:=(*if 1*∈*T then u 1 else 0*) + (∑ *i*∈*N*. − ⌊*ϑ i*⌋ ∗ *u* (*frac* (*ϑ i*))))
have (∑ *v*∈*T′*. *real_of_int* (*u′ v*) ∗ *v*) = *u′ 1* + (∑ *v* ∈ *ϑ* ‘ *N*. *real_of_int* (*u′ v*) ∗ *v*)
using ‹*finite N*› **by** (*simp add*: *T′_def image_iff in_N_not1*)
also have ... = *u′ 1* + *sum* ((λ*v*. *real_of_int* (*u′ v*) ∗ *v*) ∘ *ϑ*) *N*
by (*smt* (*verit*) *N_def* ‹*finite N*› *image_iff invf_iff sum.reindex_nontrivial*)
also have ... = *u′ 1* + (∑ *i*∈*N*. *of_int* ((*u* ∘ *frac*) (*ϑ i*)) ∗ *ϑ i*)
by (*auto simp add*: *u′_def in_N_not1*)
also have ... = *u′ 1* + (∑ *i*∈*N*. *of_int* ((*u* ∘ *frac*) (*ϑ i*)) ∗ (*floor* (*ϑ i*) + *frac*(*ϑ i*)))
by (*simp add*: *frac_def cong*: *if_cong*)
also have ... = (∑ *v*∈*T*. *of_int* (*u v*) ∗ *v*)
proof (*cases 1* ∈ *T*)
case *True*
then have *T1*: (∑ *v*∈*T*. *real_of_int* (*u v*) ∗ *v*) = *u 1* + (∑ *v*∈*T*−{*1*}. *real_of_int* (*u v*) ∗ *v*)
by (*simp add*: ‹*finite T*› *sum.remove*)
show *?thesis*
using *inj_invf True T* **unfolding** *N_def u′_def*
by (*auto simp*: *add.assoc distrib_left sum.reindex T1 simp flip*: *sum.distrib intro*!: *sum.cong*)
next
case *False*
then show *?thesis*
using *inj_invf T* **unfolding** *N_def u′_def*
by (*auto simp*: *algebra_simps sum.reindex simp flip*: *sum.distrib intro*!: *sum.cong*)
qed
also have ... = *0*
using *uv_eq0* **by** *blast*
finally have *0*: (∑ *v*∈*T′*. *real_of_int* (*u′ v*) ∗ *v*) = *0* **.**
have *u v* = *0* **if** *T′0*: ⋀*v*. *v*∈*T′* ⟹ *u′ v* = *0*
proof −
have [*simp*]: *u t* = *0* **if** *t* ∈ *T* − {*1*} **for** *t*

```
      proof −
        have ϑ (invf t) ∈ T'
          using N_def T'_def that by blast
        then show ?thesis
          using that T'0 [of ϑ (invf t)]
           by (smt (verit, best) in_N_not1 N_def fun_upd_other imageI invf_iff
o_apply u'_def)
      qed
      show ?thesis
      proof (cases v = 1)
        case True
        then have 1 ∈ T
          using ‹v ∈ T› by blast
       have (∑ v∈T. real_of_int (u v) * v) = u 1 + (∑ v∈T − {1}. real_of_int
(u v) * v)
          using True ‹finite T› ‹v ∈ T› mk_disjoint_insert by fastforce
        then have 0 = u 1
          using uv_eq0 by auto
        with True show ?thesis by presburger
      next
        case False
        then have ϑ (invf v) ∈ ϑ ‘ N
          using N_def ‹v ∈ T› by blast
        then show ?thesis
            using that [of ϑ (invf v)] False ‹v ∈ T› T by (force simp: T'_def
in_N_not1 u'_def)
      qed
    qed
    with indp T'sub ‹finite T'› 0 show u v = 0
      unfolding dependent_explicit by blast
  qed
  moreover have inj_on ϑ' {..<Suc n}
    using injϑ'
    unfolding ϑ'_def inj_on_def
   by (metis comp_def frac_lt_1 fun_upd_other fun_upd_same lessThan_Suc_atMost
less_irrefl)
  ultimately obtain t h where th: ⋀i. i < Suc n ⟹ |t * ϑ' i − of_int (h i) −
(α(n:=0)) i| < ε/2
    using Kronecker_thm_1 [of ϑ' Suc n ε/2] lessThan_Suc_atMost assms using
half_gt_zero by blast
  define k where k = h n
  define m where m ≡ λi. h i + k * ⌊ϑ i⌋
  show thesis
  proof
    fix i assume i < n
    have |k * frac (ϑ i) − h i − α i| < ε
    proof −
      have |k * frac (ϑ i) − h i − α i| = |t * frac (ϑ i) − h i − α i + (k−t) *
frac (ϑ i)|
```

```
      by (simp add: algebra_simps)
    also have ... ≤ |t * frac (ϑ i) - h i - α i| + |(k-t) * frac (ϑ i)|
      by linarith
    also have ... ≤ |t * frac (ϑ i) - h i - α i| + |k-t|
      using frac_lt_1 [of ϑ i] le_less by (fastforce simp add: abs_mult)
    also have ... < ε
      using th[of i] th[of n] ‹i<n›
      by (simp add: k_def ϑ'_def) (smt (verit, best))
    finally show ?thesis .
  qed
  then show |k * ϑ i - m i - α i| < ε
    by (simp add: algebra_simps frac_def m_def)
 qed
qed



end
```

8.4 Bernstein-Weierstrass and Stone-Weierstrass

By L C Paulson (2015)

```
theory Weierstrass_Theorems
imports Uniform_Limit Path_Connected Derivative
begin
```

8.4.1 Bernstein polynomials

```
definition Bernstein :: [nat,nat,real] ⇒ real where
  Bernstein n k x ≡ of_nat (n choose k) * x^k * (1 - x)^(n - k)

lemma Bernstein_nonneg: ⟦0 ≤ x; x ≤ 1⟧ ⟹ 0 ≤ Bernstein n k x
  by (simp add: Bernstein_def)

lemma Bernstein_pos: ⟦0 < x; x < 1; k ≤ n⟧ ⟹ 0 < Bernstein n k x
  by (simp add: Bernstein_def)

lemma sum_Bernstein [simp]: (∑k≤n. Bernstein n k x) = 1
  using binomial_ring [of x 1-x n]
  by (simp add: Bernstein_def)

lemma binomial_deriv1:
   (∑k≤n. (of_nat k * of_nat (n choose k)) * a^(k-1) * b^(n-k)) = real_of_nat
n * (a+b)^(n-1)
  apply (rule DERIV_unique [where f = λa. (a+b)^n and x=a])
  apply (subst binomial_ring)
  apply (rule derivative_eq_intros sum.cong | simp add: atMost_atLeast0)+
  done
```

lemma *binomial_deriv2*:
(∑k≤n. (*of_nat* k * *of_nat* (k−1) * *of_nat* (n *choose* k)) * a^(k−2) * b^(n−k)) =
of_nat n * *of_nat* (n−1) * (a+b::*real*)^(n−2)
apply (*rule DERIV_unique* [**where** f = λa. *of_nat* n * (a+b::*real*)^(n−1) **and** x=a])
apply (*subst binomial_deriv1* [*symmetric*])
apply (*rule derivative_eq_intros sum.cong* | *simp add*: *Num.numeral_2_eq_2*)+
done

lemma *sum_k_Bernstein* [*simp*]: (∑k≤n. *real* k * *Bernstein* n k x) = *of_nat* n * x
apply (*subst binomial_deriv1* [*of* n x 1−x, *simplified*, *symmetric*])
apply (*simp add*: *sum_distrib_right*)
apply (*auto simp*: *Bernstein_def algebra_simps power_eq_if intro*!: *sum.cong*)
done

lemma *sum_kk_Bernstein* [*simp*]: (∑k≤n. *real* k * (*real* k − 1) * *Bernstein* n k x) = *real* n * (*real* n − 1) * x^2
proof −
have (∑k≤n. *real* k * (*real* k − 1) * *Bernstein* n k x) =
(∑k≤n. *real* k * *real* (k − *Suc* 0) * *real* (n *choose* k) * x^(k − 2) * (1 − x)^(n − k) * x^2)
proof (*rule sum.cong* [*OF refl*], *simp*)
fix k
assume k ≤ n
then consider k = 0 | k = 1 | k′ **where** k = *Suc* (*Suc* k′)
by (*metis One_nat_def not0_implies_Suc*)
then show k = 0 ∨
(*real* k − 1) * *Bernstein* n k x =
real (k − *Suc* 0) *
(*real* (n *choose* k) * (x^(k − 2) * ((1 − x)^(n − k) * x^2)))
by *cases* (*auto simp add*: *Bernstein_def power2_eq_square algebra_simps*)
qed
also have ... = *real_of_nat* n * *real_of_nat* (n − *Suc* 0) * x^2
by (*subst binomial_deriv2* [*of* n x 1−x, *simplified*, *symmetric*]) (*simp add*: *sum_distrib_right*)
also have ... = n * (n − 1) * x^2
by *auto*
finally show *?thesis*
by *auto*
qed

8.4.2 Explicit Bernstein version of the 1D Weierstrass approximation theorem

theorem *Bernstein_Weierstrass*:
fixes f :: *real* ⇒ *real*

```
  assumes contf: continuous_on {0..1} f and e: 0 < e
    shows ∃N. ∀n x. N ≤ n ∧ x ∈ {0..1}
                  ⟶ |f x − (∑k≤n. f(k/n) * Bernstein n k x)| < e
proof −
  have bounded (f ‘ {0..1})
    using compact_continuous_image compact_imp_bounded contf by blast
  then obtain M where M: ⋀x. 0 ≤ x ⟹ x ≤ 1 ⟹ |f x| ≤ M
    by (force simp add: bounded_iff)
  then have 0 ≤ M by force
  have ucontf: uniformly_continuous_on {0..1} f
    using compact_uniformly_continuous contf by blast
  then obtain d where d: d>0 ⋀x x'. ⟦ x ∈ {0..1}; x' ∈ {0..1}; |x' − x| < d⟧
⟹ |f x' − f x| < e/2
     apply (rule uniformly_continuous_onE [where e = e/2])
     using e by (auto simp: dist_norm)
  { fix n::nat and x::real
    assume n: Suc (nat⌈4*M/(e*d^2)⌉) ≤ n and x: 0 ≤ x x ≤ 1
    have 0 < n using n by simp
    have ed0: − (e * d^2) < 0
      using e ‹0<d› by simp
    also have ... ≤ M * 4
      using ‹0≤M› by simp
    finally have [simp]: real_of_int (nat ⌈4 * M / (e * d^2)⌉) = real_of_int ⌈4 *
M / (e * d^2)⌉
      using ‹0≤M› e ‹0<d›
      by (simp add: field_simps)
    have 4*M/(e*d^2) + 1 ≤ real (Suc (nat⌈4*M/(e*d^2)⌉))
      by (simp add: real_nat_ceiling_ge)
    also have ... ≤ real n
      using n by (simp add: field_simps)
    finally have nbig: 4*M/(e*d^2) + 1 ≤ real n .
    have sum_bern: (∑k≤n. (x − k/n)^2 * Bernstein n k x) = x * (1 − x) / n
    proof −
      have *: ⋀a b x::real. (a − b)^2 * x = a * (a − 1) * x + (1 − 2 * b) * a * x
+ b * b * x
        by (simp add: algebra_simps power2_eq_square)
      have (∑k≤n. (k − n * x)^2 * Bernstein n k x) = n * x * (1 − x)
        apply (simp add: * sum.distrib)
        apply (simp flip: sum_distrib_left add: mult.assoc)
        apply (simp add: algebra_simps power2_eq_square)
        done
      then have (∑k≤n. (k − n * x)^2 * Bernstein n k x)/n^2 = x * (1 − x) / n
        by (simp add: power2_eq_square)
      then show ?thesis
      using n by (simp add: sum_divide_distrib field_split_simps power2_commute)
    qed
    { fix k
      assume k: k ≤ n
      then have kn: 0 ≤ k / n k / n ≤ 1
```

```
      by (auto simp: field_split_simps)
    consider (lessd) |x − k / n| < d | (ged) d ≤ |x − k / n|
      by linarith
    then have |(f x − f (k/n))| ≤ e/2 + 2 * M / d^2 * (x − k/n)^2
    proof cases
      case lessd
      then have |(f x − f (k/n))| < e/2
        using d x kn by (simp add: abs_minus_commute)
      also have ... ≤ (e/2 + 2 * M / d^2 * (x − k/n)^2)
        using ‹M≥0› d by simp
      finally show ?thesis by simp
    next
      case ged
      then have dle: d^2 ≤ (x − k/n)^2
        by (metis d(1) less_eq_real_def power2_abs power_mono)
      have §: 1 ≤ (x − real k / real n)^2 / d^2
        using dle ‹d>0› by auto
      have |(f x − f (k/n))| ≤ |f x| + |f (k/n)|
        by (rule abs_triangle_ineq4)
      also have ... ≤ M+M
        by (meson M add_mono_thms_linordered_semiring(1) kn x)
      also have ... ≤ 2 * M * ((x − k/n)^2 / d^2)
        using § ‹M≥0› mult_left_mono by fastforce
      also have ... ≤ e/2 + 2 * M / d^2 * (x − k/n)^2
        using e  by simp
      finally show ?thesis .
      qed
  } note * = this
  have |f x − (∑ k≤n. f(k / n) * Bernstein n k x)| ≤ |∑ k≤n. (f x − f(k / n))
* Bernstein n k x|
    by (simp add: sum_subtractf sum_distrib_left [symmetric] algebra_simps)
  also have ... ≤ (∑ k≤n. |(f x − f(k / n)) * Bernstein n k x|)
    by (rule sum_abs)
  also have ... ≤ (∑ k≤n. (e/2 + (2 * M / d^2) * (x − k / n)^2) * Bernstein n
k x)
    using *
     by (force simp add: abs_mult Bernstein_nonneg x mult_right_mono intro:
sum_mono)
  also have ... ≤ e/2 + (2 * M) / (d^2 * n)
     unfolding sum.distrib Rings.semiring_class.distrib_right sum_distrib_left
[symmetric] mult.assoc sum_bern
   using ‹d>0› x by (simp add: divide_simps ‹M≥0› mult_le_one mult_left_le)
  also have ... < e
    using ‹d>0› nbig e ‹n>0›
    apply (simp add: field_split_simps)
    using ed0 by linarith
  finally have |f x − (∑ k≤n. f (real k / real n) * Bernstein n k x)| < e .
 }
 then show ?thesis
```

by *auto*
qed

8.4.3 General Stone-Weierstrass theorem

Source: Bruno Brosowski and Frank Deutsch. An Elementary Proof of the Stone-Weierstrass Theorem. Proceedings of the American Mathematical Society Volume 81, Number 1, January 1981. DOI: 10.2307/2043993 https://www.jstor.org/stable/2043993

locale *function_ring_on* =
fixes R :: ('a::*t2_space* ⇒ *real*) *set* **and** S :: 'a *set*
assumes *compact*: *compact* S
assumes *continuous*: $f \in R \Longrightarrow$ *continuous_on* S f
assumes *add*: $f \in R \Longrightarrow g \in R \Longrightarrow (\lambda x.\ f\ x + g\ x) \in R$
assumes *mult*: $f \in R \Longrightarrow g \in R \Longrightarrow (\lambda x.\ f\ x * g\ x) \in R$
assumes *const*: $(\lambda\_.\ c) \in R$
assumes *separable*: $x \in S \Longrightarrow y \in S \Longrightarrow x \neq y \Longrightarrow \exists f{\in}R.\ f\ x \neq f\ y$

begin
lemma *minus*: $f \in R \Longrightarrow (\lambda x.\ -f\ x) \in R$
by (*frule mult* [*OF const* [*of* −*1*]]) *simp*

lemma *diff*: $f \in R \Longrightarrow g \in R \Longrightarrow (\lambda x.\ f\ x - g\ x) \in R$
unfolding *diff_conv_add_uminus* **by** (*metis add minus*)

lemma *power*: $f \in R \Longrightarrow (\lambda x.\ f\ x\hat{}n) \in R$
by (*induct n*) (*auto simp*: *const mult*)

lemma *sum*: $[\![$*finite* I; $\bigwedge i.\ i \in I \Longrightarrow f\ i \in R]\!] \Longrightarrow (\lambda x.\ \sum i \in I.\ f\ i\ x) \in R$
by (*induct I rule*: *finite_induct*; *simp add*: *const add*)

lemma *prod*: $[\![$*finite* I; $\bigwedge i.\ i \in I \Longrightarrow f\ i \in R]\!] \Longrightarrow (\lambda x.\ \prod i \in I.\ f\ i\ x) \in R$
by (*induct I rule*: *finite_induct*; *simp add*: *const mult*)

definition *normf* :: ('a::*t2_space* ⇒ *real*) ⇒ *real*
where *normf* $f \equiv$ *SUP* $x{\in}S.\ |f\ x|$

lemma *normf_upper*:
assumes *continuous_on* S f $x \in S$ **shows** $|f\ x| \leq$ *normf* f
proof −
have *bdd_above* $((\lambda x.\ |f\ x|)\ `\ S)$
by (*simp add*: *assms*(*1*) *bounded_imp_bdd_above compact compact_continuous_image compact_imp_bounded continuous_on_rabs*)
then show *?thesis*
using *assms cSUP_upper normf_def* **by** *fastforce*
qed

lemma *normf_least*: $S \neq \{\} \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow |f\ x| \leq M) \Longrightarrow$ *normf* $f \leq M$

by (*simp add*: *normf_def cSUP_least*)

end

lemma (**in** *function_ring_on*) *one*:
assumes *U*: *open U* **and** *t0*: *t0* ∈ *S t0* ∈ *U* **and** *t1*: *t1* ∈ *S*−*U*
shows ∃ *V*. *open V* ∧ *t0* ∈ *V* ∧ *S* ∩ *V* ⊆ *U* ∧
(∀ *e*>*0*. ∃ *f* ∈ *R*. *f* ‘ *S* ⊆ {*0*..*1*} ∧ (∀ *t* ∈ *S* ∩ *V*. *f t* < *e*) ∧ (∀ *t* ∈ *S* − *U*. *f t* > *1* − *e*))
proof −
have ∃ *pt* ∈ *R*. *pt t0* = *0* ∧ *pt t* > *0* ∧ *pt* ‘ *S* ⊆ {*0*..*1*} **if** *t*: *t* ∈ *S* − *U* **for** *t*
proof −
have *t* ≠ *t0* **using** *t t0* **by** *auto*
then obtain *g* **where** *g*: *g* ∈ *R g t* ≠ *g t0*
using *separable t0* **by** (*metis Diff_subset subset_eq t*)
define *h* **where** [*abs_def*]: *h x* = *g x* − *g t0* **for** *x*
have *h* ∈ *R*
unfolding *h_def* **by** (*fast intro*: *g const diff*)
then have *hsq*: (λ*w*. (*h w*)^2) ∈ *R*
by (*simp add*: *power2_eq_square mult*)
have *h t* ≠ *h t0*
by (*simp add*: *h_def g*)
then have *h t* ≠ *0*
by (*simp add*: *h_def*)
then have *ht2*: *0* < (*h t*)^2
by *simp*
also have ... ≤ *normf* (λ*w*. (*h w*)^2)
using *t normf_upper* [**where** *x*=*t*] *continuous* [*OF hsq*] **by** *force*
finally have *nfp*: *0* < *normf* (λ*w*. (*h w*)^2) **.**
define *p* **where** [*abs_def*]: *p x* = (*1* / *normf* (λ*w*. (*h w*)^2)) ∗ (*h x*)^2 **for** *x*
have *p* ∈ *R*
unfolding *p_def* **by** (*fast intro*: *hsq const mult*)
moreover have *p t0* = *0*
by (*simp add*: *p_def h_def*)
moreover have *p t* > *0*
using *nfp ht2* **by** (*simp add*: *p_def*)
moreover have ⋀*x*. *x* ∈ *S* ⟹ *p x* ∈ {*0*..*1*}
using *nfp normf_upper* [*OF continuous* [*OF hsq*]] **by** (*auto simp*: *p_def*)
ultimately show ∃ *pt* ∈ *R*. *pt t0* = *0* ∧ *pt t* > *0* ∧ *pt* ‘ *S* ⊆ {*0*..*1*}
by *auto*
qed
then obtain *pf* **where** *pf*: ⋀*t*. *t* ∈ *S*−*U* ⟹ *pf t* ∈ *R* ∧ *pf t t0* = *0* ∧ *pf t t* > *0*
and *pf01*: ⋀*t*. *t* ∈ *S*−*U* ⟹ *pf t* ‘ *S* ⊆ {*0*..*1*}
by *metis*
have *com_sU*: *compact* (*S*−*U*)
using *compact closed_Int_compact U* **by** (*simp add*: *Diff_eq compact_Int_closed open_closed*)
have ⋀*t*. *t* ∈ *S*−*U* ⟹ ∃ *A*. *open A* ∧ *A* ∩ *S* = {*x*∈*S*. *0* < *pf t x*}

```
    apply (rule open_Collect_positive)
    by (metis pf continuous)
  then obtain Uf where Uf: ⋀t. t ∈ S−U ⟹ open (Uf t) ∧ (Uf t) ∩ S = {x∈S.
0 < pf t x}
    by metis
  then have open_Uf: ⋀t. t ∈ S−U ⟹ open (Uf t)
    by blast
  have tUft: ⋀t. t ∈ S−U ⟹ t ∈ Uf t
    using pf Uf by blast
  then have ∗: S−U ⊆ (⋃x ∈ S−U. Uf x)
    by blast
  obtain subU where subU: subU ⊆ S − U finite subU S − U ⊆ (⋃x ∈ subU.
Uf x)
    by (blast intro: that compactE_image [OF com_sU open_Uf ∗])
  then have [simp]: subU ≠ {}
    using t1 by auto
  then have cardp: card subU > 0 using subU
    by (simp add: card_gt_0_iff)
  define p where [abs_def]: p x = (1 / card subU) ∗ (∑ t ∈ subU. pf t x) for x
  have pR: p ∈ R
    unfolding p_def using subU pf by (fast intro: pf const mult sum)
  have pt0 [simp]: p t0 = 0
    using subU pf by (auto simp: p_def intro: sum.neutral)
  have pt_pos: p t > 0 if t: t ∈ S−U for t
  proof −
    obtain i where i: i ∈ subU t ∈ Uf i using subU t by blast
    show ?thesis
      using subU i t
      apply (clarsimp simp: p_def field_split_simps)
      apply (rule sum_pos2 [OF ‹finite subU›])
      using Uf t pf01 apply auto
      apply (force elim!: subsetCE)
      done
  qed
  have p01: p x ∈ {0..1} if t: x ∈ S for x
  proof −
    have 0 ≤ p x
      using subU cardp t pf01
      by (fastforce simp add: p_def field_split_simps intro: sum_nonneg)
    moreover have p x ≤ 1
      using subU cardp t
      apply (simp add: p_def field_split_simps)
      apply (rule sum_bounded_above [where 'a=real and K=1, simplified])
      using pf01 by force
    ultimately show ?thesis
      by auto
  qed
  have compact (p ‘ (S−U))
    by (meson Diff_subset com_sU compact_continuous_image continuous con-
```

```
tinuous_on_subset pR)
  then have open (− (p ‘ (S−U)))
    by (simp add: compact_imp_closed open_Compl)
  moreover have 0 ∈ − (p ‘ (S−U))
    by (metis (no_types) ComplI image_iff not_less_iff_gr_or_eq pt_pos)
  ultimately obtain delta0 where delta0: delta0 > 0 ball 0 delta0 ⊆ − (p ‘
(S−U))
    by (auto simp: elim!: openE)
  then have pt_delta: ⋀x. x ∈ S−U ⟹ p x ≥ delta0
    by (force simp: ball_def dist_norm dest: p01)
  define δ where δ = delta0/2
  have delta0 ≤ 1 using delta0 p01 [of t1] t1
     by (force simp: ball_def dist_norm dest: p01)
  with delta0 have δ01: 0 < δ δ < 1
    by (auto simp: δ_def)
  have pt_δ: ⋀x. x ∈ S−U ⟹ p x ≥ δ
    using pt_delta delta0 by (force simp: δ_def)
  have ∃ A. open A ∧ A ∩ S = {x∈S. p x < δ/2}
   by (rule open_Collect_less_Int [OF continuous [OF pR] continuous_on_const])
  then obtain V where V: open V V ∩ S = {x∈S. p x < δ/2}
    by blast
  define k where k = nat⌊1/δ⌋ + 1
  have k>0  by (simp add: k_def)
  have k−1 ≤ 1/δ
    using δ01 by (simp add: k_def)
  with δ01 have k ≤ (1+δ)/δ
    by (auto simp: algebra_simps add_divide_distrib)
  also have ... < 2/δ
    using δ01 by (auto simp: field_split_simps)
  finally have k2δ: k < 2/δ .
  have 1/δ < k
    using δ01 unfolding k_def by linarith
  with δ01 k2δ have kδ: 1 < k∗δ k∗δ < 2
    by (auto simp: field_split_simps)
  define q where [abs_def]: q n t = (1 − p t^n)^(k^n) for n t
  have qR: q n ∈ R for n
    by (simp add: q_def const diff power pR)
  have q01: ⋀n t. t ∈ S ⟹ q n t ∈ {0..1}
    using p01 by (simp add: q_def power_le_one algebra_simps)
  have qt0 [simp]: ⋀n. n>0 ⟹ q n t0 = 1
    using t0 pf by (simp add: q_def power_0_left)
  { fix t and n::nat
    assume t: t ∈ S ∩ V
    with ‹k>0› V have k ∗ p t < k ∗ δ / 2
      by force
    then have 1 − (k ∗ δ / 2)^n ≤ 1 − (k ∗ p t)^n
      using ‹k>0› p01 t by (simp add: power_mono)
    also have ... ≤ q n t
      using Bernoulli_inequality [of − ((p t)^n) k^n]
```

```
    apply (simp add: q_def)
    by (metis IntE atLeastAtMost_iff p01 power_le_one power_mult_distrib t)
  finally have 1 - (k * δ / 2)^n ≤ q n t .
 } note limitV = this
 { fix t and n::nat
  assume t: t ∈ S - U
  with ‹k>0› U have k * δ ≤ k * p t
    by (simp add: pt_δ)
  with kδ have kpt: 1 < k * p t
    by (blast intro: less_le_trans)
  have ptn_pos: 0 < p t^n
    using pt_pos [OF t] by simp
  have ptn_le: p t^n ≤ 1
    by (meson DiffE atLeastAtMost_iff p01 power_le_one t)
  have q n t = (1/(k^n * (p t)^n)) * (1 - p t^n)^(k^n) * k^n * (p t)^n
    using pt_pos [OF t] ‹k>0› by (simp add: q_def)
  also have ... ≤ (1/(k * (p t))^n) * (1 - p t^n)^(k^n) * (1 + k^n * (p t)^n)
    using pt_pos [OF t] ‹k>0›
    by (simp add: divide_simps mult_left_mono ptn_le)
  also have ... ≤ (1/(k * (p t))^n) * (1 - p t^n)^(k^n) * (1 + (p t)^n)^(k^n)
  proof (rule mult_left_mono [OF Bernoulli_inequality])
    show 0 ≤ 1 / (real k * p t)^n * (1 - p t^n)^k^n
      using ptn_pos ptn_le by (auto simp: power_mult_distrib)
  qed (use ptn_pos in auto)
  also have ... = (1/(k * (p t))^n) * (1 - p t^(2*n))^(k^n)
    using pt_pos [OF t] ‹k>0›
   by (simp add: algebra_simps power_mult power2_eq_square flip: power_mult_distrib)
  also have ... ≤ (1/(k * (p t))^n) * 1
    using pt_pos ‹k>0› p01 power_le_one t
    by (intro mult_left_mono [OF power_le_one]) auto
  also have ... ≤ (1 / (k*δ))^n
    using ‹k>0› δ01  power_mono pt_δ t
    by (fastforce simp: field_simps)
  finally have q n t ≤ (1 / (real k * δ))^n  .
 } note limitNonU = this
 define NN
   where NN e = 1 + nat ⌈max (ln e / ln (real k * δ / 2)) (- ln e / ln (real k
* δ))⌉ for e
 have NN: of_nat (NN e) > ln e / ln (real k * δ / 2)  of_nat (NN e) > - ln e
/ ln (real k * δ)
           if 0<e for e
    unfolding NN_def  by linarith+
  have NN1: (k * δ / 2)^NN e < e if e>0 for e
  proof -
    have ln ((real k * δ / 2)^NN e) = real (NN e) * ln (real k * δ / 2)
      by (simp add: ‹δ>0› ‹0 < k› ln_realpow)
    also have ... < ln e
      using NN kδ that by (force simp add: field_simps)
    finally show ?thesis
```

```
      by (simp add: ‹δ>0› ‹0 < k› that)
    qed
  have NN0: (1/(k*δ))^(NN e) < e if e>0 for e
  proof −
    have 0 < ln (real k) + ln δ
      using δ01(1) ‹0 < k› kδ(1) ln_gt_zero ln_mult by fastforce
    then have real (NN e) * ln (1 / (real k * δ)) < ln e
       using kδ(1) NN(2) [of e] ‹0 < δ› ‹0 < k› that by (simp add: ln_div
divide_simps)
    then have exp (real (NN e) * ln (1 / (real k * δ))) < e
      by (metis exp_less_mono exp_ln that)
    then show ?thesis
      by (simp add: δ01(1) ‹0 < k› exp_of_nat_mult)
  qed
  { fix t and e::real
    assume e>0
    have t ∈ S ∩ V ⟹ 1 − q (NN e) t < e t ∈ S − U ⟹ q (NN e) t < e
    proof −
      assume t: t ∈ S ∩ V
      show 1 − q (NN e) t < e
        by (metis add.commute diff_le_eq not_le limitV [OF t] less_le_trans [OF
NN1 [OF ‹e>0›]])
    next
      assume t: t ∈ S − U
      show q (NN e) t < e
       using  limitNonU [OF t] less_le_trans [OF NN0 [OF ‹e>0›]] not_le by
blast
    qed
  } then have ⋀e. e > 0 ⟹ ∃f∈R. f ‘ S ⊆ {0..1} ∧ (∀t ∈ S ∩ V. f t < e) ∧
(∀t ∈ S − U. 1 − e < f t)
    using q01
    by (rule_tac x=λx. 1 − q (NN e) x in bexI) (auto simp: algebra_simps intro:
diff const qR)
  moreover have t0V: t0 ∈ V  S ∩ V ⊆ U
    using pt_δ t0 U V δ01  by fastforce+
  ultimately show ?thesis using V t0V
    by blast
qed
```

Non-trivial case, with A and B both non-empty

```
lemma (in function_ring_on) two_special:
  assumes A: closed A A ⊆ S a ∈ A
      and B: closed B B ⊆ S b ∈ B
      and disj: A ∩ B = {}
      and e: 0 < e e < 1
    shows ∃f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀x ∈ A. f x < e) ∧ (∀x ∈ B. f x > 1 − e)
proof −
  { fix w
    assume w ∈ A
```

```
    then have open ( − B) b ∈ S w ∉ B w ∈ S
      using assms by auto
    then have ∃ V. open V ∧ w ∈ V ∧ S ∩ V ⊆ −B ∧
              (∀ e>0. ∃ f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀ x ∈ S ∩ V. f x < e) ∧ (∀ x ∈ S
∩ B. f x > 1 − e))
      using one [of −B w b] assms ‹w ∈ A› by simp
  }
  then obtain Vf where Vf:
        ⋀w. w ∈ A ⟹ open (Vf w) ∧ w ∈ Vf w ∧ S ∩ Vf w ⊆ −B ∧
                    (∀ e>0. ∃ f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀ x ∈ S ∩ Vf w. f x < e) ∧
(∀ x ∈ S ∩ B. f x > 1 − e))
    by metis
  then have open_Vf: ⋀w. w ∈ A ⟹ open (Vf w)
    by blast
  have tVft: ⋀w. w ∈ A ⟹ w ∈ Vf w
    using Vf by blast
  then have sum_max_0: A ⊆ (⋃ x ∈ A. Vf x)
    by blast
  have com_A: compact A using A
    by (metis compact compact_Int_closed inf.absorb_iff2)
  obtain subA where subA: subA ⊆ A finite subA A ⊆ (⋃ x ∈ subA. Vf x)
    by (blast intro: that compactE_image [OF com_A open_Vf sum_max_0])
  then have [simp]: subA ≠ {}
    using ‹a ∈ A› by auto
  then have cardp: card subA > 0 using subA
    by (simp add: card_gt_0_iff)
  have ⋀w. w ∈ A ⟹ ∃ f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀ x ∈ S ∩ Vf w. f x < e / card
subA) ∧ (∀ x ∈ S ∩ B. f x > 1 − e / card subA)
    using Vf e cardp by simp
  then obtain ff where ff:
        ⋀w. w ∈ A ⟹ ff w ∈ R ∧ ff w ‘ S ⊆ {0..1} ∧
                    (∀ x ∈ S ∩ Vf w. ff w x < e / card subA) ∧ (∀ x ∈ S ∩ B. ff w
x > 1 − e / card subA)
    by metis
  define pff where [abs_def]: pff x = (∏ w ∈ subA. ff w x) for x
  have pffR: pff ∈ R
    unfolding pff_def using subA ff by (auto simp: intro: prod)
  moreover
  have pff01: pff x ∈ {0..1} if t: x ∈ S for x
  proof −
    have 0 ≤ pff x
      using subA cardp t ff
     by (fastforce simp: pff_def field_split_simps sum_nonneg intro: prod_nonneg)
    moreover have pff x ≤ 1
      using subA cardp t ff
    by (fastforce simp add: pff_def field_split_simps sum_nonneg intro: prod_mono
[where g = λx. 1, simplified])
    ultimately show ?thesis
      by auto
```

```
  qed
  moreover
  { fix v x
    assume v: v ∈ subA and x: x ∈ Vf v x ∈ S
    from subA v have pff x = ff v x * (∏ w ∈ subA − {v}. ff w x)
      unfolding pff_def  by (metis prod.remove)
    also have ... ≤ ff v x * 1
    proof −
      have ⋀i. i ∈ subA − {v} ⟹ 0 ≤ ff i x ∧ ff i x ≤ 1
      by (metis Diff_subset atLeastAtMost_iff ff image_subset_iff subA(1) subsetD
x(2))
      moreover have 0 ≤ ff v x
        using ff subA(1) v x(2) by fastforce
      ultimately show ?thesis
        by (metis mult_left_mono prod_mono [where g = λx. 1, simplified])
    qed
    also have ... < e / card subA
      using ff subA(1) v x by auto
    also have ... ≤ e
      using cardp e by (simp add: field_split_simps)
    finally have pff x < e .
  }
  then have ⋀x. x ∈ A ⟹ pff x < e
    using A Vf subA by (metis UN_E contra_subsetD)
  moreover
  { fix x
    assume x: x ∈ B
    then have x ∈ S
      using B by auto
    have 1 − e ≤ (1 − e / card subA) ^ card subA
      using Bernoulli_inequality [of −e / card subA card subA] e cardp
      by (auto simp: field_simps)
    also have ... = (∏ w ∈ subA. 1 − e / card subA)
      by (simp add: subA(2))
    also have ... < pff x
    proof −
      have ⋀i. i ∈ subA ⟹ e / real (card subA) ≤ 1 ∧ 1 − e / real (card subA)
< ff i x
        using e ‹B ⊆ S› ff subA(1) x by (force simp: field_split_simps)
      then show ?thesis
        using prod_mono_strict[of _ subA λx. 1 − e / card subA ] subA
        unfolding pff_def by (smt (verit, best) UN_E assms(3) subsetD)
    qed
    finally have 1 − e < pff x .
  }
  ultimately show ?thesis by blast
qed

lemma (in function_ring_on) two:
```

```
  assumes A: closed A A ⊆ S
      and B: closed B B ⊆ S
      and disj: A ∩ B = {}
      and e: 0 < e e < 1
    shows ∃f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀x ∈ A. f x < e) ∧ (∀x ∈ B. f x > 1 − e)
proof (cases A ≠ {} ∧ B ≠ {})
  case True then show ?thesis
    using assms
    by (force simp flip: ex_in_conv intro!: two_special)
next
  case False
  then consider A={} | B={} by force
  then show ?thesis
  proof cases
    case 1
    with e show ?thesis
      by (rule_tac x=λx. 1 in bexI) (auto simp: const)
  next
    case 2
    with e show ?thesis
      by (rule_tac x=λx. 0 in bexI) (auto simp: const)
  qed
qed
```

The special case where f is non-negative and $e < 1 \ / \ (3::'a)$

```
lemma (in function_ring_on) Stone_Weierstrass_special:
  assumes f: continuous_on S f and fpos: ⋀x. x ∈ S ⟹ f x ≥ 0
      and e: 0 < e e < 1/3
  shows ∃g ∈ R. ∀x∈S. |f x − g x| < 2∗e
proof −
  define n where n = 1 + nat ⌈normf f / e⌉
  define A where A j = {x ∈ S. f x ≤ (j − 1/3)∗e} for j :: nat
  define B where B j = {x ∈ S. f x ≥ (j + 1/3)∗e} for j :: nat
  have ngt: (n−1) ∗ e ≥ normf f
    using e pos_divide_le_eq real_nat_ceiling_ge[of normf f / e]
    by (fastforce simp add: divide_simps n_def)
  moreover have n≥1
    by (simp_all add: n_def)
  ultimately have ge_fx: (n−1) ∗ e ≥ f x if x ∈ S for x
    using f normf_upper that by fastforce
  have closed S
    by (simp add: compact compact_imp_closed)
  { fix j
    have closed (A j) A j ⊆ S
    using ‹closed S› continuous_on_closed_Collect_le [OF f continuous_on_const]
      by (simp_all add: A_def Collect_restrict)
    moreover have closed (B j) B j ⊆ S
     using ‹closed S› continuous_on_closed_Collect_le [OF continuous_on_const
f]
```

```
    by (simp_all add: B_def Collect_restrict)
   moreover have (A j) ∩ (B j) = {}
    using e by (auto simp: A_def B_def field_simps)
   ultimately have ∃f ∈ R. f ‘ S ⊆ {0..1} ∧ (∀x ∈ A j. f x < e/n) ∧ (∀x ∈ B
j. f x > 1 − e/n)
    using e ‹1 ≤ n› by (auto intro: two)
  }
  then obtain xf where xfR: ⋀j. xf j ∈ R and xf01: ⋀j. xf j ‘ S ⊆ {0..1}
                  and xfA: ⋀x j. x ∈ A j ⟹ xf j x < e/n
                  and xfB: ⋀x j. x ∈ B j ⟹ xf j x > 1 − e/n
   by metis
  define g where [abs_def]: g x = e ∗ (∑ i≤n. xf i x) for x
  have gR: g ∈ R
   unfolding g_def by (fast intro: mult const sum xfR)
  have gge0: ⋀x. x ∈ S ⟹ g x ≥ 0
   using e xf01 by (simp add: g_def zero_le_mult_iff image_subset_iff sum_nonneg)
  have A0: A 0 = {}
   using fpos e by (fastforce simp: A_def)
  have An: A n = S
    using e ngt ‹n≥1› f normf_upper by (fastforce simp: A_def field_simps
of_nat_diff)
  have Asub: A j ⊆ A i if i≥j for i j
   using e that by (force simp: A_def intro: order_trans)
  { fix t
   assume t: t ∈ S
   define j where j = (LEAST j. t ∈ A j)
   have jn: j ≤ n
    using t An by (simp add: Least_le j_def)
   have Aj: t ∈ A j
    using t An by (fastforce simp add: j_def intro: LeastI)
   then have Ai: t ∈ A i if i≥j for i
    using Asub [OF that] by blast
   then have fj1: f t ≤ (j − 1/3)∗e
    by (simp add: A_def)
   then have Anj: t ∉ A i if i<j for i
    using  Aj ‹i<j› not_less_Least by (fastforce simp add: j_def)
   have j1: 1 ≤ j
    using A0 Aj j_def not_less_eq_eq by (fastforce simp add: j_def)
   then have Anj: t ∉ A (j−1)
    using Least_le by (fastforce simp add: j_def)
   then have fj2: (j − 4/3)∗e < f t
    using j1 t  by (simp add: A_def of_nat_diff)
   have xf_le1: ⋀i. xf i t ≤ 1
    using xf01 t by force
   have g t = e ∗ (∑ i≤n. xf i t)
    by (simp add: g_def flip: distrib_left)
   also have ... = e ∗ (∑ i ∈ {..<j} ∪ {j..n}. xf i t)
    by (simp add: ivl_disj_un_one(4) jn)
   also have ... = e ∗ (∑ i<j. xf i t) + e ∗ (∑ i=j..n. xf i t)
```

```
    by (simp add: distrib_left ivl_disj_int sum.union_disjoint)
  also have ... ≤ e*j + e * ((Suc n − j)*e/n)
  proof (intro add_mono mult_left_mono)
    show (∑ i<j. xf i t) ≤ j
    by (rule sum_bounded_above [OF xf_le1, where A = lessThan j, simplified])
    have xf i t ≤ e/n if i≥j for i
      using xfA [OF Ai] that by (simp add: less_eq_real_def)
    then show (∑ i = j..n. xf i t) ≤ real (Suc n − j) * e / real n
      using sum_bounded_above [of {j..n} λi. xf i t]
      by fastforce
  qed (use e in auto)
  also have ... ≤ j*e + e*(n − j + 1)*e/n
    using ‹1 ≤ n› e  by (simp add: field_simps del: of_nat_Suc)
  also have ... ≤ j*e + e*e
    using ‹1 ≤ n› e j1 by (simp add: field_simps del: of_nat_Suc)
  also have ... < (j + 1/3)*e
    using e by (auto simp: field_simps)
  finally have gj1: g t < (j + 1 / 3) * e .
  have gj2: (j − 4/3)*e < g t
  proof (cases 2 ≤ j)
    case False
    then have j=1 using j1 by simp
    with t gge0 e show ?thesis by force
  next
    case True
    then have (j − 4/3)*e < (j−1)*e − e^2
      using e by (auto simp: of_nat_diff algebra_simps power2_eq_square)
    also have ... < (j−1)*e − ((j − 1)/n) * e^2
    proof −
      have (j − 1) / n < 1
        using j1 jn by fastforce
      with ‹e>0› show ?thesis
        by (smt (verit, best) mult_less_cancel_right2 zero_less_power)
    qed
    also have ... = e * (j−1) * (1 − e/n)
      by (simp add: power2_eq_square field_simps)
    also have ... ≤ e * (∑ i≤j−2. xf i t)
    proof −
      { fix i
        assume i+2 ≤ j
        then obtain d where i+2+d = j
          using le_Suc_ex that by blast
        then have t ∈ B i
          using Anj e ge_fx [OF t] ‹1 ≤ n› fpos [OF t] t
          unfolding A_def B_def
           by (auto simp add: field_simps of_nat_diff not_le intro: order_trans
[of _ e * 2 + e * d * 3 + e * i * 3])
        then have xf i t > 1 − e/n
          by (rule xfB)
```

```
      }
      moreover have real (j − Suc 0) ∗ (1 − e / real n) ≤ real (card {..j − 2})
∗ (1 − e / real n)
        using Suc_diff_le True by fastforce
      ultimately show ?thesis
       using e True by (auto intro: order_trans [OF _ sum_bounded_below [OF
less_imp_le]])
    qed
    also have ... ≤ g t
      using jn e xf01 t
     by (auto intro!: Groups_Big.sum_mono2 simp add: g_def zero_le_mult_iff
image_subset_iff sum_nonneg)
    finally show ?thesis .
   qed
   have |f t − g t| < 2 ∗ e
    using fj1 fj2 gj1 gj2 by (simp add: abs_less_iff field_simps)
 }
 then show ?thesis
   by (rule_tac x=g in bexI) (auto intro: gR)
qed
```

The "unpretentious" formulation

```
proposition (in function_ring_on) Stone_Weierstrass_basic:
 assumes f: continuous_on S f and e: e > 0
 shows ∃ g ∈ R. ∀ x∈S. |f x − g x| < e
proof −
 have ∃ g ∈ R. ∀ x∈S. |(f x + normf f) − g x| < 2 ∗ min (e/2) (1/4)
 proof (rule Stone_Weierstrass_special)
   show continuous_on S (λx. f x + normf f)
    by (force intro: Limits.continuous_on_add [OF f Topological_Spaces.continuous_on_const])
   show ⋀x. x ∈ S ⟹ 0 ≤ f x + normf f
    using normf_upper [OF f] by force
 qed (use e in auto)
 then obtain g where g ∈ R ∀ x∈S. |g x − (f x + normf f)| < e
   by force
 then show ?thesis
   by (rule_tac x=λx. g x − normf f in bexI) (auto simp: algebra_simps intro:
diff const)
qed


theorem (in function_ring_on) Stone_Weierstrass:
 assumes f: continuous_on S f
 shows ∃ F∈UNIV → R. LIM n sequentially. F n :> uniformly_on S f
proof −
 define h where h ≡ λn::nat. SOME g. g ∈ R ∧ (∀ x∈S. |f x − g x| < 1 / (1 +
n))
 show ?thesis
 proof
```

```
    { fix e::real
      assume e: 0 < e
      then obtain N::nat where N: 0 < N 0 < inverse N inverse N < e
        by (auto simp: real_arch_inverse [of e])
      { fix n :: nat and x :: 'a and g :: 'a ⇒ real
        assume n: N ≤ n  ∀x∈S. |f x − g x| < 1 / (1 + real n)
        assume x: x ∈ S
        have ¬ real (Suc n) < inverse e
          using ‹N ≤ n› N using less_imp_inverse_less by force
        then have 1 / (1 + real n) ≤ e
          using e by (simp add: field_simps)
        then have |f x − g x| < e
          using n(2) x by auto
      }
      then have ∀F n in sequentially. ∀x∈S. |f x − h n x| < e
        unfolding h_def
        by (force intro: someI2_bex [OF Stone_Weierstrass_basic [OF f]] eventu-
ally_sequentiallyI [of N])
    }
    then show uniform_limit S h f sequentially
    unfolding uniform_limit_iff by (auto simp: dist_norm abs_minus_commute)
    show h ∈ UNIV → R
      unfolding h_def by (force intro: someI2_bex [OF Stone_Weierstrass_basic
[OF f]])
  qed
qed
```

A HOL Light formulation

```
corollary Stone_Weierstrass_HOL:
  fixes R :: ('a::t2_space ⇒ real) set and S :: 'a set
  assumes compact S  ⋀c. P(λx. c::real)
          ⋀f. P f ⟹ continuous_on S f
          ⋀f g. P(f) ∧ P(g) ⟹ P(λx. f x + g x)  ⋀f g. P(f) ∧ P(g) ⟹ P(λx. f
x * g x)
          ⋀x y. x ∈ S ∧ y ∈ S ∧ x ≠ y ⟹ ∃f. P(f) ∧ f x ≠ f y
          continuous_on S f
        0 < e
    shows ∃g. P(g) ∧ (∀x ∈ S. |f x − g x| < e)
proof −
  interpret PR: function_ring_on Collect P
    by unfold_locales (use assms in auto)
  show ?thesis
    using PR.Stone_Weierstrass_basic [OF ‹continuous_on S f› ‹0 < e›]
    by blast
qed
```

8.4.4 Polynomial functions

inductive *real_polynomial_function* :: (*'a*::*real_normed_vector* ⇒ *real*) ⇒ *bool*
where
  linear: *bounded_linear f* ⟹ *real_polynomial_function f*
 | *const*: *real_polynomial_function* (λ*x*. *c*)
 | *add*: ⟦*real_polynomial_function f*; *real_polynomial_function g*⟧ ⟹ *real_polynomial_function* (λ*x*. *f x* + *g x*)
 | *mult*: ⟦*real_polynomial_function f*; *real_polynomial_function g*⟧ ⟹ *real_polynomial_function* (λ*x*. *f x* ∗ *g x*)

declare *real_polynomial_function.intros* [*intro*]

definition *polynomial_function* :: (*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*) ⇒ *bool*
 where
  polynomial_function p ≡ (∀*f*. *bounded_linear f* ⟶ *real_polynomial_function* (*f* ∘ *p*))

lemma *real_polynomial_function_eq*: *real_polynomial_function p* = *polynomial_function p*
unfolding *polynomial_function_def*
proof
 assume *real_polynomial_function p*
 then show ∀*f*. *bounded_linear f* ⟶ *real_polynomial_function* (*f* ∘ *p*)
 proof (*induction p rule*: *real_polynomial_function.induct*)
  case (*linear h*) **then show** *?case*
   by (*auto simp*: *bounded_linear_compose real_polynomial_function.linear*)
 next
  case (*const h*) **then show** *?case*
   by (*simp add*: *real_polynomial_function.const*)
 next
  case (*add h*) **then show** *?case*
  by (*force simp add*: *bounded_linear_def linear_add real_polynomial_function.add*)
 next
  case (*mult h*) **then show** *?case*
  by (*force simp add*: *real_bounded_linear const real_polynomial_function.mult*)
 qed
next
  assume [*rule_format*, *OF bounded_linear_ident*]: ∀*f*. *bounded_linear f* ⟶ *real_polynomial_function* (*f* ∘ *p*)
 then show *real_polynomial_function p*
  by (*simp add*: *o_def*)
qed

lemma *polynomial_function_const* [*iff*]: *polynomial_function* (λ*x*. *c*)
 by (*simp add*: *polynomial_function_def o_def const*)

lemma *polynomial_function_bounded_linear*:
 bounded_linear f ⟹ *polynomial_function f*

by (*simp add*: *polynomial_function_def o_def bounded_linear_compose real_polynomial_function.linear*)

lemma *polynomial_function_id* [*iff*]: *polynomial_function*(λ*x*. *x*)
by (*simp add*: *polynomial_function_bounded_linear*)

lemma *polynomial_function_add* [*intro*]:
⟦*polynomial_function f*; *polynomial_function g*⟧ ⟹ *polynomial_function* (λ*x*. *f x* + *g x*)
by (*auto simp*: *polynomial_function_def bounded_linear_def linear_add real_polynomial_function.add o_def*)

lemma *polynomial_function_mult* [*intro*]:
assumes *f*: *polynomial_function f* **and** *g*: *polynomial_function g*
shows *polynomial_function* (λ*x*. *f x* ∗$_R$ *g x*)
proof −
have *real_polynomial_function* (λ*x*. *h* (*g x*)) **if** *bounded_linear h* **for** *h*
using *g that* **unfolding** *polynomial_function_def o_def bounded_linear_def*
by (*auto simp*: *real_polynomial_function_eq*)
moreover have *real_polynomial_function f*
by (*simp add*: *f real_polynomial_function_eq*)
ultimately show *?thesis*
unfolding *polynomial_function_def bounded_linear_def o_def*
by (*auto simp*: *linear.scaleR*)
qed

lemma *polynomial_function_cmul* [*intro*]:
assumes *f*: *polynomial_function f*
shows *polynomial_function* (λ*x*. *c* ∗$_R$ *f x*)
by (*rule polynomial_function_mult* [*OF polynomial_function_const f*])

lemma *polynomial_function_minus* [*intro*]:
assumes *f*: *polynomial_function f*
shows *polynomial_function* (λ*x*. − *f x*)
using *polynomial_function_cmul* [*OF f*, *of* −*1*] **by** *simp*

lemma *polynomial_function_diff* [*intro*]:
⟦*polynomial_function f*; *polynomial_function g*⟧ ⟹ *polynomial_function* (λ*x*. *f x* − *g x*)
unfolding *add_uminus_conv_diff* [*symmetric*]
by (*metis polynomial_function_add polynomial_function_minus*)

lemma *polynomial_function_sum* [*intro*]:
⟦*finite I*; ⋀*i*. *i* ∈ *I* ⟹ *polynomial_function* (λ*x*. *f x i*)⟧ ⟹ *polynomial_function* (λ*x*. *sum* (*f x*) *I*)
by (*induct I rule*: *finite_induct*) *auto*

lemma *real_polynomial_function_minus* [*intro*]:
real_polynomial_function f ⟹ *real_polynomial_function* (λ*x*. − *f x*)
using *polynomial_function_minus* [*of f*]

by (*simp add*: *real_polynomial_function_eq*)

lemma *real_polynomial_function_diff* [*intro*]:
⟦*real_polynomial_function f*; *real_polynomial_function g*⟧ ⟹ *real_polynomial_function* (λ*x*. *f x* − *g x*)
using *polynomial_function_diff* [*of f*]
by (*simp add*: *real_polynomial_function_eq*)

lemma *real_polynomial_function_divide* [*intro*]:
assumes *real_polynomial_function p* **shows** *real_polynomial_function* (λ*x*. *p x* / *c*)
proof −
have *real_polynomial_function* (λ*x*. *p x* ∗ *Fields.inverse c*)
using *assms* **by** *auto*
then show *?thesis*
by (*simp add*: *divide_inverse*)
qed

lemma *real_polynomial_function_sum* [*intro*]:
⟦*finite I*; ⋀*i*. *i* ∈ *I* ⟹ *real_polynomial_function* (λ*x*. *f x i*)⟧ ⟹ *real_polynomial_function* (λ*x*. *sum* (*f x*) *I*)
using *polynomial_function_sum* [*of I f*]
by (*simp add*: *real_polynomial_function_eq*)

lemma *real_polynomial_function_prod* [*intro*]:
⟦*finite I*; ⋀*i*. *i* ∈ *I* ⟹ *real_polynomial_function* (λ*x*. *f x i*)⟧ ⟹ *real_polynomial_function* (λ*x*. *prod* (*f x*) *I*)
by (*induct I rule*: *finite_induct*) *auto*

lemma *real_polynomial_function_gchoose*:
obtains *p* **where** *real_polynomial_function p* ⋀*x*. *x gchoose r* = *p x*
proof
show *real_polynomial_function* (λ*x*. (∏ *i* = *0*..<*r*. *x* − *real i*) / *fact r*)
by *force*
qed (*simp add*: *gbinomial_prod_rev*)

lemma *real_polynomial_function_power* [*intro*]:
real_polynomial_function f ⟹ *real_polynomial_function* (λ*x*. *f x*^*n*)
by (*induct n*) (*simp_all add*: *const mult*)

lemma *real_polynomial_function_compose* [*intro*]:
assumes *f*: *polynomial_function f* **and** *g*: *real_polynomial_function g*
shows *real_polynomial_function* (*g o f*)
using *g*
proof (*induction g rule*: *real_polynomial_function.induct*)
case (*linear f*)
then show *?case*
using *f polynomial_function_def* **by** *blast*
next

```
  case (add f g)
  then show ?case
    using f add by (auto simp: polynomial_function_def)
next
  case (mult f g)
  then show ?case
  using f mult by (auto simp: polynomial_function_def)
qed auto

lemma polynomial_function_compose [intro]:
  assumes f: polynomial_function f and g: polynomial_function g
    shows polynomial_function (g o f)
  using g real_polynomial_function_compose [OF f]
  by (auto simp: polynomial_function_def o_def)

lemma sum_max_0:
  fixes x::real
  shows (∑ i≤max m n. x^i * (if i ≤ m then a i else 0)) = (∑ i≤m. x^i * a i)
proof −
  have (∑ i≤max m n. x^i * (if i ≤ m then a i else 0)) = (∑ i≤max m n. (if i ≤
m then x^i * a i else 0))
    by (auto simp: algebra_simps intro: sum.cong)
  also have ... = (∑ i≤m. (if i ≤ m then x^i * a i else 0))
    by (rule sum.mono_neutral_right) auto
  also have ... = (∑ i≤m. x^i * a i)
    by (auto simp: algebra_simps intro: sum.cong)
  finally show ?thesis .
qed

lemma real_polynomial_function_imp_sum:
  assumes real_polynomial_function f
    shows ∃ a n::nat. f = (λx. ∑ i≤n. a i * x^i)
using assms
proof (induct f)
  case (linear f)
  then obtain c where f: f = (λx. x * c)
    by (auto simp add: real_bounded_linear)
  have x * c = (∑ i≤1. (if i = 0 then 0 else c) * x^i) for x
    by (simp add: mult_ac)
  with f show ?case
    by fastforce
next
  case (const c)
  have c = (∑ i≤0. c * x^i) for x
    by auto
  then show ?case
    by fastforce
  case (add f1 f2)
  then obtain a1 n1 a2 n2 where
```

```
    f1 = (λx. ∑ i≤n1. a1 i * x^i) f2 = (λx. ∑ i≤n2. a2 i * x^i)
    by auto
  then have f1 x + f2 x = (∑ i≤max n1 n2. ((if i ≤ n1 then a1 i else 0) + (if i
≤ n2 then a2 i else 0)) * x^i)
      for x
     using sum_max_0 [where m=n1 and n=n2] sum_max_0 [where m=n2
and n=n1]
    by (simp add: sum.distrib algebra_simps max.commute)
  then show ?case
    by force
  case (mult f1 f2)
  then obtain a1 n1 a2 n2 where
    f1 = (λx. ∑ i≤n1. a1 i * x^i) f2 = (λx. ∑ i≤n2. a2 i * x^i)
    by auto
  then obtain b1 b2 where
    f1 = (λx. ∑ i≤n1. b1 i * x^i) f2 = (λx. ∑ i≤n2. b2 i * x^i)
    b1 = (λi. if i≤n1 then a1 i else 0) b2 = (λi. if i≤n2 then a2 i else 0)
    by auto
  then have f1 x * f2 x = (∑ i≤n1 + n2. (∑ k≤i. b1 k * b2 (i − k)) * x ^ i)
for x
   using polynomial_product [of n1 b1 n2 b2] by (simp add: Set_Interval.atLeast0AtMost)
  then show ?case
    by force
qed

lemma real_polynomial_function_iff_sum:
     real_polynomial_function f ⟷ (∃ a n. f = (λx. ∑ i≤n. a i * x^i))  (is ?lhs
= ?rhs)
proof
  assume ?lhs then show ?rhs
    by (metis real_polynomial_function_imp_sum)
next
  assume ?rhs then show ?lhs
   by (auto simp: linear mult const real_polynomial_function_power real_polynomial_function_sum)
qed

lemma polynomial_function_iff_Basis_inner:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
   shows polynomial_function f ⟷ (∀ b∈Basis. real_polynomial_function (λx.
inner (f x) b))
        (is ?lhs = ?rhs)
unfolding polynomial_function_def
proof (intro iffI allI impI)
  assume ∀ h. bounded_linear h ⟶ real_polynomial_function (h ∘ f)
  then show ?rhs
    by (force simp add: bounded_linear_inner_left o_def)
next
  fix h :: 'b ⇒ real
 assume rp: ∀ b∈Basis. real_polynomial_function (λx. f x • b) and h: bounded_linear
```

```
h
  have real_polynomial_function (h ∘ (λx. ∑ b∈Basis. (f x • b) *R b))
    using rp
    by (force simp: real_polynomial_function_eq polynomial_function_mult
          intro!: real_polynomial_function_compose [OF _ linear [OF h]])
  then show real_polynomial_function (h ∘ f)
    by (simp add: euclidean_representation_sum_fun)
qed
```

8.4.5 Stone-Weierstrass theorem for polynomial functions

First, we need to show that they are continuous, differentiable and separable.

```
lemma continuous_real_polymonial_function:
  assumes real_polynomial_function f
    shows continuous (at x) f
using assms
by (induct f) (auto simp: linear_continuous_at)

lemma continuous_polymonial_function:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
  assumes polynomial_function f
    shows continuous (at x) f
proof (rule euclidean_isCont)
  show ⋀b. b ∈ Basis ⟹ isCont (λx. (f x • b) *R b) x
  using assms continuous_real_polymonial_function
  by (force simp: polynomial_function_iff_Basis_inner intro: isCont_scaleR)
qed

lemma continuous_on_polymonial_function:
  fixes f :: 'a::real_normed_vector ⇒ 'b::euclidean_space
  assumes polynomial_function f
    shows continuous_on S f
 using continuous_polymonial_function [OF assms] continuous_at_imp_continuous_on
  by blast

lemma has_real_derivative_polynomial_function:
  assumes real_polynomial_function p
    shows ∃ p'. real_polynomial_function p' ∧
                (∀ x. (p has_real_derivative (p' x)) (at x))
using assms
proof (induct p)
  case (linear p)
  then show ?case
    by (force simp: real_bounded_linear const intro!: derivative_eq_intros)
next
  case (const c)
  show ?case
    by (rule_tac x=λx. 0 in exI) auto
  case (add f1 f2)
```

```
  then obtain p1 p2 where
    real_polynomial_function p1 ⋀x. (f1 has_real_derivative p1 x) (at x)
    real_polynomial_function p2 ⋀x. (f2 has_real_derivative p2 x) (at x)
    by auto
  then show ?case
    by (rule_tac x=λx. p1 x + p2 x in exI) (auto intro!: derivative_eq_intros)
  case (mult f1 f2)
  then obtain p1 p2 where
    real_polynomial_function p1 ⋀x. (f1 has_real_derivative p1 x) (at x)
    real_polynomial_function p2 ⋀x. (f2 has_real_derivative p2 x) (at x)
    by auto
  then show ?case
    using mult
     by (rule_tac x=λx. f1 x * p2 x + f2 x * p1 x in exI) (auto intro!: deriva-
tive_eq_intros)
qed

lemma has_vector_derivative_polynomial_function:
  fixes p :: real ⇒ 'a::euclidean_space
  assumes polynomial_function p
  obtains p' where polynomial_function p' ⋀x. (p has_vector_derivative (p' x))
(at x)
proof −
  { fix b :: 'a
    assume b ∈ Basis
    then
    obtain p' where p': real_polynomial_function p' and pd: ⋀x. ((λx. p x • b)
has_real_derivative p' x) (at x)
    using assms [unfolded polynomial_function_iff_Basis_inner] has_real_derivative_polynomial_func
      by blast
    have polynomial_function (λx. p' x *R b)
      using ‹b ∈ Basis› p' const [where 'a=real and c=0]
      by (simp add: polynomial_function_iff_Basis_inner inner_Basis)
   then have ∃q. polynomial_function q ∧ (∀x. ((λu. (p u • b) *R b) has_vector_derivative
q x) (at x))
      by (fastforce intro: derivative_eq_intros pd)
  }
  then obtain qf where qf:
      ⋀b. b ∈ Basis ⟹ polynomial_function (qf b)
     ⋀b x. b ∈ Basis ⟹ ((λu. (p u • b) *R b) has_vector_derivative qf b x) (at x)
    by metis
  show ?thesis
  proof
    show ⋀x. (p has_vector_derivative (∑b∈Basis. qf b x)) (at x)
      apply (subst euclidean_representation_sum_fun [of p, symmetric])
      by (auto intro: has_vector_derivative_sum qf)
  qed (force intro: qf)
qed
```

lemma *real_polynomial_function_separable*:
fixes x :: *'a::euclidean_space*
assumes $x \neq y$ **shows** $\exists f.$ *real_polynomial_function* $f \wedge f\ x \neq f\ y$
proof −
have *real_polynomial_function* $(\lambda u.\ \sum b{\in}Basis.\ (inner\ (x{-}u)\ b)\hat{}2)$
proof (*rule real_polynomial_function_sum*)
show $\bigwedge i.\ i \in Basis \Longrightarrow$ *real_polynomial_function* $(\lambda u.\ ((x - u) \cdot i)^2)$
by (*auto simp*: *algebra_simps real_polynomial_function_diff const linear bounded_linear_inner_left*)
qed *auto*
moreover have $(\sum b{\in}Basis.\ ((x - y) \cdot b)^2) \neq 0$
using *assms* **by** (*force simp add*: *euclidean_eq_iff* [*of x y*] *sum_nonneg_eq_0_iff algebra_simps*)
ultimately show *?thesis*
by *auto*
qed

lemma *Stone_Weierstrass_real_polynomial_function*:
fixes f :: *'a::euclidean_space* ⇒ *real*
assumes *compact S continuous_on S f* $0 < e$
obtains g **where** *real_polynomial_function* g $\bigwedge x.\ x \in S \Longrightarrow |f\ x - g\ x| < e$
proof −
interpret *PR*: *function_ring_on Collect real_polynomial_function*
proof *unfold_locales*
qed (*use assms continuous_on_polymonial_function real_polynomial_function_eq*

in ‹*auto intro*: *real_polynomial_function_separable*›)
show *?thesis*
using *PR.Stone_Weierstrass_basic* [*OF* ‹*continuous_on S f*› ‹$0 < e$›] *that* **by** *blast*
qed

theorem *Stone_Weierstrass_polynomial_function*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *S*: *compact S*
and *f*: *continuous_on S f*
and *e*: $0 < e$
shows $\exists g.$ *polynomial_function* $g \wedge (\forall x \in S.\ norm(f\ x - g\ x) < e)$
proof −
{ fix b :: *'b*
assume $b \in Basis$
have $\exists p.$ *real_polynomial_function* $p \wedge (\forall x \in S.\ |f\ x \cdot b - p\ x| < e\ /\ DIM('b))$
proof (*rule Stone_Weierstrass_real_polynomial_function* [*OF S _*, *of* $\lambda x.\ f\ x \cdot b\ e\ /$ *card Basis*])
show *continuous_on S* $(\lambda x.\ f\ x \cdot b)$
using *f* **by** (*auto intro*: *continuous_intros*)
qed (*use e* **in** *auto*)
}
then obtain *pf* **where** *pf*:

```
    ⋀b. b ∈ Basis ⟹ real_polynomial_function (pf b) ∧ (∀x ∈ S. |f x ∙ b − pf
b x| < e / DIM('b))
   by metis
  let ?g = λx. ∑ b∈Basis. pf b x *R b
  { fix x
   assume x ∈ S
   have norm (∑ b∈Basis. (f x ∙ b) *R b − pf b x *R b) ≤ (∑ b∈Basis. norm ((f
x ∙ b) *R b − pf b x *R b))
     by (rule norm_sum)
   also have ... < of_nat DIM('b) * (e / DIM('b))
   proof (rule sum_bounded_above_strict)
    show ⋀i. i ∈ Basis ⟹ norm ((f x ∙ i) *R i − pf i x *R i) < e / real DIM('b)
     by (simp add: Real_Vector_Spaces.scaleR_diff_left [symmetric] pf ‹x ∈ S›)
   qed (rule DIM_positive)
   also have ... = e
    by (simp add: field_simps)
   finally have norm (∑ b∈Basis. (f x ∙ b) *R b − pf b x *R b) < e .
  }
  then have ∀x∈S. norm ((∑ b∈Basis. (f x ∙ b) *R b) − ?g x) < e
   by (auto simp flip: sum_subtractf)
  moreover
  have polynomial_function ?g
   using pf by (simp add: polynomial_function_sum polynomial_function_mult
real_polynomial_function_eq)
  ultimately show ?thesis
   using euclidean_representation_sum_fun [of f] by (metis (no_types, lifting))
qed

proposition Stone_Weierstrass_uniform_limit:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes S: compact S
   and f: continuous_on S f
  obtains g where uniform_limit S g f sequentially ⋀n. polynomial_function (g
n)
proof −
  have pos: inverse (Suc n) > 0 for n by auto
  obtain g where g: ⋀n. polynomial_function (g n) ⋀x n. x ∈ S ⟹ norm(f x
− g n x) < inverse (Suc n)
   using Stone_Weierstrass_polynomial_function[OF S f pos]
   by metis
  have uniform_limit S g f sequentially
  proof (rule uniform_limitI)
   fix e::real assume 0 < e
   with LIMSEQ_inverse_real_of_nat have ∀F n in sequentially. inverse (Suc
n) < e
    by (rule order_tendstoD)
   moreover have ∀F n in sequentially. ∀x∈S. dist (g n x) (f x) < inverse (Suc
n)
    using g by (simp add: dist_norm norm_minus_commute)
```

```
    ultimately show ∀F n in sequentially. ∀x∈S. dist (g n x) (f x) < e
      by (eventually_elim) auto
  qed
  then show ?thesis using g(1) ..
qed
```

8.4.6 Polynomial functions as paths

One application is to pick a smooth approximation to a path, or just pick a smooth path anyway in an open connected set

```
lemma path_polynomial_function:
    fixes g :: real ⇒ 'b::euclidean_space
    shows polynomial_function g ⟹ path g
  by (simp add: path_def continuous_on_polymonial_function)

lemma path_approx_polynomial_function:
    fixes g :: real ⇒ 'b::euclidean_space
    assumes path g 0 < e
    obtains p where polynomial_function p pathstart p = pathstart g pathfinish p
= pathfinish g
                    ⋀t. t ∈ {0..1} ⟹ norm(p t − g t) < e
proof −
  obtain q where poq: polynomial_function q and noq: ⋀x. x ∈ {0..1} ⟹ norm
(g x − q x) < e/4
    using Stone_Weierstrass_polynomial_function [of {0..1} g e/4] assms
    by (auto simp: path_def)
  define pf where pf ≡ λt. q t + (g 0 − q 0) + t *R (g 1 − q 1 − (g 0 − q 0))
  show thesis
  proof
    show polynomial_function pf
      by (force simp add: poq pf_def)
    show norm (pf t − g t) < e
      if t ∈ {0..1} for t
    proof −
      have *: norm (((q t − g t) + (g 0 − q 0)) + (t *R (g 1 − q 1) + t *R (q 0
− g 0))) < (e/4 + e/4) + (e/4+e/4)
      proof (intro Real_Vector_Spaces.norm_add_less)
        show norm (q t − g t) < e / 4
          by (metis noq norm_minus_commute that)
        show norm (t *R (g 1 − q 1)) < e / 4
          using noq that le_less_trans [OF mult_left_le_one_le noq]
          by auto
        show norm (t *R (q 0 − g 0)) < e / 4
          using noq that le_less_trans [OF mult_left_le_one_le noq]
          by simp (metis norm_minus_commute order_refl zero_le_one)
      qed (use noq norm_minus_commute that in auto)
      then show ?thesis
        by (auto simp add: algebra_simps pf_def)
    qed
```

qed (*auto simp add*: *path_defs pf_def*)
qed

proposition *connected_open_polynomial_connected*:
fixes S :: *'a*::*euclidean_space set*
assumes S: *open S connected S*
and $x \in S$ $y \in S$
shows $\exists g.$ *polynomial_function g* $\wedge$ *path_image g* $\subseteq S$ $\wedge$ *pathstart g* $= x$ $\wedge$ *pathfinish g* $= y$
proof −
have *path_connected S* **using** *assms*
by (*simp add*: *connected_open_path_connected*)
with ‹$x \in S$› ‹$y \in S$› **obtain** p **where** p: *path p path_image p* $\subseteq S$ *pathstart p* $= x$ *pathfinish p* $= y$
by (*force simp*: *path_connected_def*)
have $\exists e.$ $0 < e \wedge (\forall x \in$ *path_image p*. *ball x e* $\subseteq S)$
proof (*cases S* $=$ *UNIV*)
case *True* **then show** *?thesis*
by (*simp add*: *gt_ex*)
next
case *False*
show *?thesis*
proof (*intro exI conjI ballI*)
show $\bigwedge x.$ $x \in$ *path_image p* $\Longrightarrow$ *ball x* (*setdist* (*path_image p*) $(-S)) \subseteq S$
using *setdist_le_dist* [*of* _ *path_image p* _ $-S$] **by** *fastforce*
show $0 <$ *setdist* (*path_image p*) $(-S)$
using *S p False*
by (*fastforce simp add*: *setdist_gt_0_compact_closed compact_path_image open_closed*)
qed
qed
then obtain e **where** $0 < e$ **and** *eb*: $\bigwedge x.$ $x \in$ *path_image p* $\Longrightarrow$ *ball x e* $\subseteq S$
by *auto*
obtain *pf* **where** *polynomial_function pf* **and** *pf*: *pathstart pf* $=$ *pathstart p* *pathfinish pf* $=$ *pathfinish p*
and *pf_e*: $\bigwedge t.$ $t \in \{0..1\} \Longrightarrow$ *norm*(*pf t* $-$ *p t*) $< e$
using *path_approx_polynomial_function* [*OF* ‹*path p*› ‹$0 < e$›] **by** *blast*
show *?thesis*
proof (*intro exI conjI*)
show *polynomial_function pf*
by *fact*
show *pathstart pf* $= x$ *pathfinish pf* $= y$
by (*simp_all add*: *p pf*)
show *path_image pf* $\subseteq S$
unfolding *path_image_def*
proof *clarsimp*
fix *x'*::*real*
assume $0 \le x'$ $x' \le 1$
then have *dist* (*p x'*) (*pf x'*) $< e$

```
      by (metis atLeastAtMost_iff dist_commute dist_norm pf_e)
    then show pf x' ∈ S
      by (metis ‹0 ≤ x'› ‹x' ≤ 1› atLeastAtMost_iff eb imageI mem_ball
path_image_def subset_iff)
  qed
 qed
qed

lemma differentiable_componentwise_within:
  f differentiable (at a within S) ⟷
   (∀i ∈ Basis. (λx. f x • i) differentiable at a within S)
proof −
 { assume ∀i∈Basis. ∃D. ((λx. f x • i) has_derivative D) (at a within S)
  then obtain f' where f':
      ⋀i. i ∈ Basis ⟹ ((λx. f x • i) has_derivative f' i) (at a within S)
    by metis
  have eq: (λx. (∑j∈Basis. f' j x *R j) • i) = f' i if i ∈ Basis for i
    using that by (simp add: inner_add_left inner_add_right)
  have ∃D. ∀i∈Basis. ((λx. f x • i) has_derivative (λx. D x • i)) (at a within S)
    apply (rule_tac x=λx::'a. (∑j∈Basis. f' j x *R j) :: 'b in exI)
    apply (simp add: eq f')
    done
 }
 then show ?thesis
  apply (simp add: differentiable_def)
  using has_derivative_componentwise_within
  by blast
qed

lemma polynomial_function_inner [intro]:
 fixes i :: 'a::euclidean_space
 shows polynomial_function g ⟹ polynomial_function (λx. g x • i)
 apply (subst euclidean_representation [where x=i, symmetric])
 apply (force simp: inner_sum_right polynomial_function_iff_Basis_inner poly-
nomial_function_sum)
 done
```

Differentiability of real and vector polynomial functions.

```
lemma differentiable_at_real_polynomial_function:
  real_polynomial_function f ⟹ f differentiable (at a within S)
 by (induction f rule: real_polynomial_function.induct)
    (simp_all add: bounded_linear_imp_differentiable)

lemma differentiable_on_real_polynomial_function:
  real_polynomial_function p ⟹ p differentiable_on S
by (simp add: differentiable_at_imp_differentiable_on differentiable_at_real_polynomial_function)

lemma differentiable_at_polynomial_function:
 fixes f :: _ ⇒ 'a::euclidean_space
```

shows *polynomial_function f* $\Longrightarrow$ *f differentiable* (*at a within S*)
by (*metis differentiable_at_real_polynomial_function polynomial_function_iff_Basis_inner differentiable_componentwise_within*)

lemma *differentiable_on_polynomial_function*:
fixes *f* :: _ $\Rightarrow$ 'a::*euclidean_space*
shows *polynomial_function f* $\Longrightarrow$ *f differentiable_on S*
by (*simp add*: *differentiable_at_polynomial_function differentiable_on_def*)

lemma *vector_eq_dot_span*:
assumes $x \in$ *span B* $y \in$ *span B* **and** *i*: $\bigwedge i.\ i \in B \Longrightarrow i \cdot x = i \cdot y$
shows $x = y$
proof −
have $\bigwedge i.\ i \in B \Longrightarrow$ *orthogonal* $(x - y)$ *i*
by (*simp add*: *i inner_commute inner_diff_right orthogonal_def*)
moreover have $x - y \in$ *span B*
by (*simp add*: *assms span_diff*)
ultimately have $x - y = 0$
using *orthogonal_to_span orthogonal_self* **by** *blast*
then show *?thesis* **by** *simp*
qed

lemma *orthonormal_basis_expand*:
assumes *B*: *pairwise orthogonal B*
and *1*: $\bigwedge i.\ i \in B \Longrightarrow$ *norm i* = *1*
and $x \in$ *span B*
and *finite B*
shows $(\sum i{\in}B.\ (x \cdot i) *_R i) = x$
proof (*rule vector_eq_dot_span* [*OF* _ ‹$x \in$ *span B*›])
show $(\sum i{\in}B.\ (x \cdot i) *_R i) \in$ *span B*
by (*simp add*: *span_clauses span_sum*)
show $i \cdot (\sum i{\in}B.\ (x \cdot i) *_R i) = i \cdot x$ **if** $i \in B$ **for** *i*
proof −
have [*simp*]: $i \cdot j =$ (*if j* = *i then 1 else 0*) **if** $j \in B$ **for** *j*
using *B 1 that* ‹$i \in B$›
by (*force simp*: *norm_eq_1 orthogonal_def pairwise_def*)
have $i \cdot (\sum i{\in}B.\ (x \cdot i) *_R i) = (\sum j{\in}B.\ x \cdot j * (i \cdot j))$
by (*simp add*: *inner_sum_right*)
also have ... $= (\sum j{\in}B.$ *if j* = *i then* $x \cdot i$ *else 0*)
by (*rule sum.cong*; *simp*)
also have ... $= i \cdot x$
by (*simp add*: ‹*finite B*› *that inner_commute*)
finally show *?thesis* .
qed
qed

theorem *Stone_Weierstrass_polynomial_function_subspace*:
fixes *f* :: 'a::*euclidean_space* $\Rightarrow$ 'b::*euclidean_space*

```
  assumes compact S
      and contf: continuous_on S f
      and 0 < e
      and subspace T f ‘ S ⊆ T
    obtains g where polynomial_function g g ‘ S ⊆ T
                    ⋀x. x ∈ S ⟹ norm(f x − g x) < e
proof −
  obtain B where B ⊆ T and orthB: pairwise orthogonal B
            and B1: ⋀x. x ∈ B ⟹ norm x = 1
            and independent B and cardB: card B = dim T
            and spanB: span B = T
    using orthonormal_basis_subspace ‹subspace T› by metis
  then have finite B
    by (simp add: independent_imp_finite)
  then obtain n::nat and b where B = b ‘ {i. i < n} inj_on b {i. i < n}
    using finite_imp_nat_seg_image_inj_on by metis
  with cardB have n = card B dim T = n
    by (auto simp: card_image)
  have fx: (∑ i∈B. (f x • i) *R i) = f x if x ∈ S for x
    by (metis (no_types, lifting) B1 ‹finite B› assms(5) image_subset_iff orthB
orthonormal_basis_expand spanB sum.cong that)
  have cont: continuous_on S (λx. ∑ i∈B. (f x • i) *R i)
    by (intro continuous_intros contf)
  obtain g where polynomial_function g
          and g: ⋀x. x ∈ S ⟹ norm ((∑ i∈B. (f x • i) *R i) − g x) < e / (n+2)
    using Stone_Weierstrass_polynomial_function [OF ‹compact S› cont, of e /
real (n + 2)] ‹0 < e›
    by auto
  with fx have g: ⋀x. x ∈ S ⟹ norm (f x − g x) < e / (n+2)
    by auto
  show ?thesis
  proof
    show polynomial_function (λx. ∑ i∈B. (g x • i) *R i)
      using ‹polynomial_function g› by (force intro: ‹finite B›)
    show (λx. ∑ i∈B. (g x • i) *R i) ‘ S ⊆ T
      using ‹B ⊆ T›
      by (blast intro: subspace_sum subspace_mul ‹subspace T›)
    show norm (f x − (∑ i∈B. (g x • i) *R i)) < e if x ∈ S for x
    proof −
      have orth': pairwise (λi j. orthogonal ((f x • i) *R i − (g x • i) *R i)
                                             ((f x • j) *R j − (g x • j) *R j)) B
        by (auto simp: orthogonal_def inner_diff_right inner_diff_left intro: pair-
wise_mono [OF orthB])
      then have (norm (∑ i∈B. (f x • i) *R i − (g x • i) *R i))^2 =
                (∑ i∈B. (norm ((f x • i) *R i − (g x • i) *R i))^2)
        by (simp add: norm_sum_Pythagorean [OF ‹finite B› orth'])
      also have ... = (∑ i∈B. (norm (((f x − g x) • i) *R i))^2)
        by (simp add: algebra_simps)
      also have ... ≤ (∑ i∈B. (norm (f x − g x))^2)
```

```
    proof −
      have ⋀i. i ∈ B ⟹ ((f x − g x) · i)² ≤ (norm (f x − g x))²
        by (metis B1 Cauchy_Schwarz_ineq inner_commute mult.left_neutral
norm_eq_1 power2_norm_eq_inner)
      then show ?thesis
        by (intro sum_mono) (simp add: sum_mono B1)
    qed
    also have ... = n * norm (f x − g x)^2
      by (simp add: ‹n = card B›)
    also have ... ≤ n * (e / (n+2))^2
    proof (rule mult_left_mono)
      show (norm (f x − g x))² ≤ (e / real (n + 2))²
      by (meson dual_order.order_iff_strict g norm_ge_zero power_mono that)
    qed auto
    also have ... ≤ e^2 / (n+2)
      using ‹0 < e› by (simp add: divide_simps power2_eq_square)
    also have ... < e^2
      using ‹0 < e› by (simp add: divide_simps)
    finally have (norm (∑i∈B. (f x · i) *R i − (g x · i) *R i))² < e^2 .
    then have (norm (∑i∈B. (f x · i) *R i − (g x · i) *R i)) < e
      by (simp add: ‹0 < e› norm_lt_square power2_norm_eq_inner)
    then show ?thesis
      using fx that by (simp add: sum_subtractf)
  qed
 qed
qed


hide_fact linear add mult const

end
```

8.5 Radon-Nikodým Derivative

```
theory Radon_Nikodym
imports Bochner_Integration
begin

definition diff_measure :: 'a measure ⇒ 'a measure ⇒ 'a measure
where
  diff_measure M N = measure_of (space M) (sets M) (λA. emeasure M A −
emeasure N A)

lemma
  shows space_diff_measure[simp]: space (diff_measure M N) = space M
    and sets_diff_measure[simp]: sets (diff_measure M N) = sets M
  by (auto simp: diff_measure_def)

lemma emeasure_diff_measure:
```

assumes *fin*: *finite_measure M finite_measure N* **and** *sets_eq*: *sets M* = *sets N*
assumes *pos*: ⋀*A*. *A* ∈ *sets M* ⟹ *emeasure N A* ≤ *emeasure M A* **and** *A*: *A* ∈ *sets M*
shows *emeasure* (*diff_measure M N*) *A* = *emeasure M A* − *emeasure N A* (**is** _ = *?μ A*)
unfolding *diff_measure_def*
proof (*rule emeasure_measure_of_sigma*)
show *sigma_algebra* (*space M*) (*sets M*) **..**
show *positive* (*sets M*) *?μ*
using *pos* **by** (*simp add*: *positive_def*)
show *countably_additive* (*sets M*) *?μ*
proof (*rule countably_additiveI*)
fix *A* :: *nat* ⇒ _ **assume** *A*: *range A* ⊆ *sets M* **and** *disjoint_family A*
then have *suminf*:
(∑ *i*. *emeasure M* (*A i*)) = *emeasure M* (⋃ *i*. *A i*)
(∑ *i*. *emeasure N* (*A i*)) = *emeasure N* (⋃ *i*. *A i*)
by (*simp_all add*: *suminf_emeasure sets_eq*)
with *A* **have** (∑ *i*. *emeasure M* (*A i*) − *emeasure N* (*A i*)) =
(∑ *i*. *emeasure M* (*A i*)) − (∑ *i*. *emeasure N* (*A i*))
using *fin pos*[*of A* _]
by (*intro ennreal_suminf_minus*)
(*auto simp*: *sets_eq finite_measure.emeasure_eq_measure suminf_emeasure*)
then show (∑ *i*. *emeasure M* (*A i*) − *emeasure N* (*A i*)) =
emeasure M (⋃ *i*. *A i*) − *emeasure N* (⋃ *i*. *A i*)
by (*simp add*: *suminf*)
qed
qed *fact*

An equivalent characterization of sigma-finite spaces is the existence of integrable positive functions (or, still equivalently, the existence of a probability measure which is in the same measure class as the original measure).

proposition (**in** *sigma_finite_measure*) *obtain_positive_integrable_function*:
obtains *f*::*'a* ⇒ *real* **where**
f ∈ *borel_measurable M*
⋀*x*. *f x* > *0*
⋀*x*. *f x* ≤ *1*
integrable M f
proof −
obtain *A* :: *nat* ⇒ *'a set* **where** *range A* ⊆ *sets M* (⋃ *i*. *A i*) = *space M* ⋀*i*. *emeasure M* (*A i*) ≠ ∞
using *sigma_finite* **by** *auto*
then have [*measurable*]:*A n* ∈ *sets M* **for** *n* **by** *auto*
define *g* **where** *g* = (λ*x*. (∑ *n*. (*1*/*2*)^(*Suc n*) ∗ *indicator* (*A n*) *x* / (*1*+ *measure M* (*A n*))))
have [*measurable*]: *g* ∈ *borel_measurable M* **unfolding** *g_def* **by** *auto*
have ∗: *summable* (λ*n*. (*1*/*2*)^(*Suc n*) ∗ *indicator* (*A n*) *x* / (*1*+ *measure M* (*A n*))) **for** *x*
apply (*rule summable_comparison_test'*[*of* λ*n*. (*1*/*2*)^(*Suc n*) *0*])
using *power_half_series summable_def* **by** (*auto simp add*: *indicator_def*

```
divide_simps)
  have g x ≤ (∑ n. (1/2)^(Suc n)) for x unfolding g_def
    apply (rule suminf_le) using * power_half_series summable_def by (auto
simp add: indicator_def divide_simps)
  then have g_le_1: g x ≤ 1 for x using power_half_series sums_unique by
fastforce

  have g_pos: g x > 0 if x ∈ space M for x
  unfolding g_def proof (subst suminf_pos_iff[OF *[of x]], auto)
   obtain i where x ∈ A i using ‹(⋃ i. A i) = space M› ‹x ∈ space M› by auto
  then have 0 < (1 / 2) ^ Suc i * indicator (A i) x / (1 + Sigma_Algebra.measure
M (A i))
     unfolding indicator_def apply (auto simp add: divide_simps) using mea-
sure_nonneg[of M A i]
    by (auto, meson add_nonneg_nonneg linorder_not_le mult_nonneg_nonneg
zero_le_numeral zero_le_one zero_le_power)
  then show ∃ i. 0 < (1 / 2) ^ i * indicator (A i) x / (2 + 2 * Sigma_Algebra.measure
M (A i))
    by auto
  qed

  have integrable M g
  unfolding g_def proof (rule integrable_suminf)
   fix n
  show integrable M (λx. (1 / 2) ^ Suc n * indicator (A n) x / (1 + Sigma_Algebra.measure
M (A n)))
    using ‹emeasure M (A n) ≠ ∞›
   by (auto intro!: integrable_mult_right integrable_divide_zero integrable_real_indicator
simp add: top.not_eq_extremum)
  next
   show AE x in M. summable (λn. norm ((1 / 2) ^ Suc n * indicator (A n) x
/ (1 + Sigma_Algebra.measure M (A n))))
    using * by auto
   show summable (λn. (∫ x. norm ((1 / 2) ^ Suc n * indicator (A n) x / (1 +
Sigma_Algebra.measure M (A n))) ∂M))
    apply (rule summable_comparison_test'[of λn. (1/2)^(Suc n) 0], auto)
    using power_half_series summable_def apply auto[1]
     apply (auto simp add: field_split_simps) using measure_nonneg[of M]
not_less by fastforce
  qed

  define f where f = (λx. if x ∈ space M then g x else 1)
  have f x > 0 for x unfolding f_def using g_pos by auto
  moreover have f x ≤ 1 for x unfolding f_def using g_le_1 by auto
 moreover have [measurable]: f ∈ borel_measurable M unfolding f_def by auto
  moreover have integrable M f
   apply (subst integrable_cong[of _ _ _ g]) unfolding f_def using ‹integrable
M g› by auto
  ultimately show (⋀f. f ∈ borel_measurable M ⟹ (⋀x. 0 < f x) ⟹ (⋀x. f
```

```
x ≤ 1) ⟹ integrable M f ⟹ thesis) ⟹ thesis
    by (meson that)
qed

lemma (in sigma_finite_measure) Ex_finite_integrable_function:
  ∃h∈borel_measurable M. integral^N M h ≠ ∞ ∧ (∀x∈space M. 0 < h x ∧ h x <
∞)
proof −
  obtain A :: nat ⇒ 'a set where
    range[measurable]: range A ⊆ sets M and
    space: (⋃i. A i) = space M and
    measure: ⋀i. emeasure M (A i) ≠ ∞ and
    disjoint: disjoint_family A
    using sigma_finite_disjoint by blast
  let ?B = λi. 2^Suc i ∗ emeasure M (A i)
  have [measurable]: ⋀i. A i ∈ sets M
    using range by fastforce+
  have ∀i. ∃x. 0 < x ∧ x < inverse (?B i)
  proof
    fix i show ∃x. 0 < x ∧ x < inverse (?B i)
      using measure[of i]
    by (auto intro!: dense simp: ennreal_inverse_positive ennreal_mult_eq_top_iff
power_eq_top_ennreal)
  qed
  from choice[OF this] obtain n where n: ⋀i. 0 < n i
    ⋀i. n i < inverse (2^Suc i ∗ emeasure M (A i)) by auto
  { fix i have 0 ≤ n i using n(1)[of i] by auto } note pos = this
  let ?h = λx. ∑i. n i ∗ indicator (A i) x
  show ?thesis
  proof (safe intro!: bexI[of _ ?h] del: notI)
    have integral^N M ?h = (∑i. n i ∗ emeasure M (A i)) using pos
      by (simp add: nn_integral_suminf nn_integral_cmult_indicator)
    also have ... ≤ (∑i. ennreal ((1/2)^Suc i))
    proof (intro suminf_le allI)
      fix N
      have n N ∗ emeasure M (A N) ≤ inverse (2^Suc N ∗ emeasure M (A N))
∗ emeasure M (A N)
        using n[of N] by (intro mult_right_mono) auto
      also have ... = (1/2)^Suc N ∗ (inverse (emeasure M (A N)) ∗ emeasure
M (A N))
        using measure[of N]
      by (simp add: ennreal_inverse_power divide_ennreal_def ennreal_inverse_mult
                    power_eq_top_ennreal less_top[symmetric] mult_ac
                del: power_Suc)
      also have ... ≤ inverse (ennreal 2) ^ Suc N
        using measure[of N]
        by (cases emeasure M (A N); cases emeasure M (A N) = 0)
          (auto simp: inverse_ennreal ennreal_mult[symmetric] divide_ennreal_def
simp del: power_Suc)
```

```
    also have ... = ennreal (inverse 2 ^ Suc N)
      by (subst ennreal_power[symmetric], simp) (simp add: inverse_ennreal)
    finally show n N * emeasure M (A N) ≤ ennreal ((1/2)^Suc N)
      by simp
  qed auto
  also have ... < top
    unfolding less_top[symmetric]
    by (rule ennreal_suminf_neq_top)
       (auto simp: summable_geometric summable_Suc_iff simp del: power_Suc)
  finally show integral^N M ?h ≠ ∞
    by (auto simp: top_unique)
next
  { fix x assume x ∈ space M
    then obtain i where x ∈ A i using space[symmetric] by auto
    with disjoint n have ?h x = n i
      by (auto intro!: suminf_cmult_indicator intro: less_imp_le)
  then show 0 < ?h x and ?h x < ∞ using n[of i] by (auto simp: less_top[symmetric])
}
  note pos = this
qed measurable
qed
```

8.5.1 Absolutely continuous

```
definition absolutely_continuous :: 'a measure ⇒ 'a measure ⇒ bool where
  absolutely_continuous M N ⟷ null_sets M ⊆ null_sets N

lemma absolutely_continuousI_count_space: absolutely_continuous (count_space
A) M
  unfolding absolutely_continuous_def by (auto simp: null_sets_count_space)

lemma absolutely_continuousI_density:
  f ∈ borel_measurable M ⟹ absolutely_continuous M (density M f)
 by (force simp add: absolutely_continuous_def null_sets_density_iff dest: AE_not_in)

lemma absolutely_continuousI_point_measure_finite:
  (⋀x. ⟦ x ∈ A ; f x ≤ 0 ⟧ ⟹ g x ≤ 0) ⟹ absolutely_continuous (point_measure
A f) (point_measure A g)
 unfolding absolutely_continuous_def by (force simp: null_sets_point_measure_iff)

lemma absolutely_continuousD:
  absolutely_continuous M N ⟹ A ∈ sets M ⟹ emeasure M A = 0 ⟹ emeasure
N A = 0
  by (auto simp: absolutely_continuous_def null_sets_def)

lemma absolutely_continuous_AE:
  assumes sets_eq: sets M' = sets M
    and absolutely_continuous M M' AE x in M. P x
   shows AE x in M'. P x
```

proof −
from ‹*AE x in M. P x*› **obtain** *N* **where** *N*: *N* ∈ *null_sets M* {*x*∈*space M*. ¬ *P x*} ⊆ *N*
unfolding *eventually_ae_filter* **by** *auto*
show *AE x in M′. P x*
proof (*rule AE_I′*)
show {*x*∈*space M′*. ¬ *P x*} ⊆ *N* **using** *sets_eq_imp_space_eq*[*OF sets_eq*] *N*(*2*) **by** *simp*
from ‹*absolutely_continuous M M′*› **show** *N* ∈ *null_sets M′*
using *N* **unfolding** *absolutely_continuous_def sets_eq null_sets_def* **by** *auto*
qed
qed

8.5.2 Existence of the Radon-Nikodym derivative

proposition
(**in** *finite_measure*) *Radon_Nikodym_finite_measure*:
assumes *finite_measure N* **and** *sets_eq*[*simp*]: *sets N* = *sets M*
assumes *absolutely_continuous M N*
shows ∃*f* ∈ *borel_measurable M*. *density M f* = *N*
proof −
interpret *N*: *finite_measure N* **by** *fact*
define *G* **where** *G* = {*g* ∈ *borel_measurable M*. ∀ *A*∈*sets M*. (∫⁺*x*. *g x* ∗ *indicator A x* ∂*M*) ≤ *N A*}
have [*measurable_dest*]: *f* ∈ *G* ⟹ *f* ∈ *borel_measurable M*
and *G_D*: ⋀*A*. *f* ∈ *G* ⟹ *A* ∈ *sets M* ⟹ (∫⁺*x*. *f x* ∗ *indicator A x* ∂*M*) ≤ *N A* **for** *f*
by (*auto simp*: *G_def*)
note *this*[*measurable_dest*]
have (λ*x*. *0*) ∈ *G* **unfolding** *G_def* **by** *auto*
hence *G* ≠ {} **by** *auto*
{ fix *f g* **assume** *f*[*measurable*]: *f* ∈ *G* **and** *g*[*measurable*]: *g* ∈ *G*
have (λ*x*. *max* (*g x*) (*f x*)) ∈ *G* (**is** *?max* ∈ *G*) **unfolding** *G_def*
proof *safe*
let *?A* = {*x* ∈ *space M*. *f x* ≤ *g x*}
have *?A* ∈ *sets M* **using** *f g* **unfolding** *G_def* **by** *auto*
fix *A* **assume** [*measurable*]: *A* ∈ *sets M*
have *union*: ((*?A* ∩ *A*) ∪ ((*space M* − *?A*) ∩ *A*)) = *A*
using *sets.sets_into_space*[*OF* ‹*A* ∈ *sets M*›] **by** *auto*
have ⋀*x*. *x* ∈ *space M* ⟹ *max* (*g x*) (*f x*) ∗ *indicator A x* =
g x ∗ *indicator* (*?A* ∩ *A*) *x* + *f x* ∗ *indicator* ((*space M* − *?A*) ∩ *A*) *x*
by (*auto simp*: *indicator_def max_def*)
hence (∫⁺*x*. *max* (*g x*) (*f x*) ∗ *indicator A x* ∂*M*) =
(∫⁺*x*. *g x* ∗ *indicator* (*?A* ∩ *A*) *x* ∂*M*) +
(∫⁺*x*. *f x* ∗ *indicator* ((*space M* − *?A*) ∩ *A*) *x* ∂*M*)
by (*auto cong*: *nn_integral_cong intro*!: *nn_integral_add*)
also have ... ≤ *N* (*?A* ∩ *A*) + *N* ((*space M* − *?A*) ∩ *A*)
using *f g* **unfolding** *G_def* **by** (*auto intro*!: *add_mono*)

```
    also have ... = N A
      using union by (subst plus_emeasure) auto
    finally show (∫⁺x. max (g x) (f x) * indicator A x ∂M) ≤ N A .
  qed auto }
 note max_in_G = this
 { fix f assume  incseq f and f: ⋀i. f i ∈ G
  then have [measurable]: ⋀i. f i ∈ borel_measurable M by (auto simp: G_def)
  have (λx. SUP i. f i x) ∈ G unfolding G_def
  proof safe
    show (λx. SUP i. f i x) ∈ borel_measurable M by measurable
  next
    fix A assume A ∈ sets M
    have (∫⁺x. (SUP i. f i x) * indicator A x ∂M) =
      (∫⁺x. (SUP i. f i x * indicator A x) ∂M)
      by (intro nn_integral_cong) (simp split: split_indicator)
    also have ... = (SUP i. (∫⁺x. f i x * indicator A x ∂M))
      using ‹incseq f› f ‹A ∈ sets M›
      by (intro nn_integral_monotone_convergence_SUP)
         (auto simp: G_def incseq_Suc_iff le_fun_def split: split_indicator)
    finally show (∫⁺x. (SUP i. f i x) * indicator A x ∂M) ≤ N A
      using f ‹A ∈ sets M› by (auto intro!: SUP_least simp: G_D)
  qed }
 note SUP_in_G = this
 let ?y = SUP g ∈ G. integral^N M g
 have y_le: ?y ≤ N (space M) unfolding G_def
 proof (safe intro!: SUP_least)
  fix g assume ∀A∈sets M. (∫⁺x. g x * indicator A x ∂M) ≤ N A
  from this[THEN bspec, OF sets.top] show integral^N M g ≤ N (space M)
    by (simp cong: nn_integral_cong)
 qed
 from ennreal_SUP_countable_SUP [OF ‹G ≠ {}›, of integral^N M]
 obtain ys :: nat ⇒ ennreal
   where ys: range ys ⊆ integral^N M ‘ G ∧ Sup (integral^N M ‘ G) = Sup (range
ys)
   by auto
 then have ∀n. ∃g. g∈G ∧ integral^N M g = ys n
 proof safe
   fix n assume range ys ⊆ integral^N M ‘ G
   hence ys n ∈ integral^N M ‘ G by auto
   thus ∃g. g∈G ∧ integral^N M g = ys n by auto
 qed
 from choice[OF this] obtain gs where ⋀i. gs i ∈ G ⋀n. integral^N M (gs n) =
ys n by auto
 hence y_eq: ?y = (SUP i. integral^N M (gs i)) using ys by auto
 let ?g = λi x. Max ((λn. gs n x) ‘ {..i})
 define f where [abs_def]: f x = (SUP i. ?g i x) for x
 let ?F = λA x. f x * indicator A x
 have gs_not_empty: ⋀i x. (λn. gs n x) ‘ {..i} ≠ {} by auto
 { fix i have ?g i ∈ G
```

```
    proof (induct i)
      case 0 thus ?case by simp fact
    next
      case (Suc i)
      with Suc gs_not_empty ‹gs (Suc i) ∈ G› show ?case
        by (auto simp add: atMost_Suc intro!: max_in_G)
    qed }
  note g_in_G = this
  have incseq ?g using gs_not_empty
    by (auto intro!: incseq_SucI le_funI simp add: atMost_Suc)

  from SUP_in_G[OF this g_in_G] have [measurable]: f ∈ G unfolding f_def
.
  then have [measurable]: f ∈ borel_measurable M unfolding G_def by auto

  have integral^N M f = (SUP i. integral^N M (?g i)) unfolding f_def
   using g_in_G ‹incseq ?g› by (auto intro!: nn_integral_monotone_convergence_SUP
simp: G_def)
  also have ... = ?y
  proof (rule antisym)
    show (SUP i. integral^N M (?g i)) ≤ ?y
      using g_in_G by (auto intro: SUP_mono)
    show ?y ≤ (SUP i. integral^N M (?g i)) unfolding y_eq
      by (auto intro!: SUP_mono nn_integral_mono Max_ge)
  qed
  finally have int_f_eq_y: integral^N M f = ?y .

  have upper_bound: ∀ A∈sets M. N A ≤ density M f A
  proof (rule ccontr)
    assume ¬ ?thesis
    then obtain A where A[measurable]: A ∈ sets M and f_less_N: density M f
A < N A
      by (auto simp: not_le)
    then have pos_A: 0 < M A
      using ‹absolutely_continuous M N›[THEN absolutely_continuousD, OF A]
      by (auto simp: zero_less_iff_neq_zero)

    define b where b = (N A − density M f A) / M A / 2
    with f_less_N pos_A have 0 < b b ≠ top
    by (auto intro!: diff_gr0_ennreal simp: zero_less_iff_neq_zero diff_eq_0_iff_ennreal
ennreal_divide_eq_top_iff)

    let ?f = λx. f x + b
    have nn_integral M f ≠ top
      using ‹f ∈ G›[THEN G_D, of space M] by (auto simp: top_unique cong:
nn_integral_cong)
    with ‹b ≠ top› interpret Mf: finite_measure density M ?f
      by (intro finite_measureI)
         (auto simp: field_simps mult_indicator_subset ennreal_mult_eq_top_iff
```

```
            emeasure_density nn_integral_cmult_indicator nn_integral_add
          cong: nn_integral_cong)

  from unsigned_Hahn_decomposition[of density M ?f N A]
  obtain Y where [measurable]: Y ∈ sets M and [simp]: Y ⊆ A
    and Y1: ⋀C. C ∈ sets M ⟹ C ⊆ Y ⟹ density M ?f C ≤ N C
    and Y2: ⋀C. C ∈ sets M ⟹ C ⊆ A ⟹ C ∩ Y = {} ⟹ N C ≤ density
M ?f C
    by auto

  let ?f' = λx. f x + b * indicator Y x
  have M Y ≠ 0
  proof
    assume M Y = 0
    then have N Y = 0
      using ‹absolutely_continuous M N›[THEN absolutely_continuousD, of Y]
by auto
    then have N A = N (A - Y)
      by (subst emeasure_Diff) auto
    also have ... ≤ density M ?f (A - Y)
      by (rule Y2) auto
    also have ... ≤ density M ?f A - density M ?f Y
      by (subst emeasure_Diff) auto
    also have ... ≤ density M ?f A - 0
      by (intro ennreal_minus_mono) auto
    also have density M ?f A = b * M A + density M f A
    by (simp add: emeasure_density field_simps mult_indicator_subset nn_integral_add
nn_integral_cmult_indicator)
    also have ... < N A
      using f_less_N pos_A
      by (cases density M f A; cases M A; cases N A)
          (auto simp: b_def ennreal_less_iff ennreal_minus divide_ennreal en-
nreal_numeral[symmetric]
                ennreal_plus[symmetric] ennreal_mult[symmetric] field_simps
              simp del: ennreal_numeral ennreal_plus)
    finally show False
      by simp
  qed
  then have nn_integral M f < nn_integral M ?f'
    using ‹0 < b› ‹nn_integral M f ≠ top›
  by (simp add: nn_integral_add nn_integral_cmult_indicator ennreal_zero_less_mult_iff
zero_less_iff_neq_zero)
  moreover
  have ?f' ∈ G
    unfolding G_def
  proof safe
    fix X assume [measurable]: X ∈ sets M
    have (∫⁺ x. ?f' x * indicator X x ∂M) = density M f (X - Y) + density
M ?f (X ∩ Y)
```

by (*auto simp add*: *emeasure_density nn_integral_add*[*symmetric*] *split*: *split_indicator intro*!: *nn_integral_cong*)
also have ... $\leq N\ (X - Y) + N\ (X \cap Y)$
using *G_D*[*OF* ‹*f* ∈ *G*›] **by** (*intro add_mono Y1*) (*auto simp*: *emeasure_density*)
also have ... = *N X*
by (*subst plus_emeasure*) (*auto intro*!: *arg_cong2*[**where** *f*=*emeasure*])
finally show $(\int^+ x.\ ?f'\ x * \mathit{indicator}\ X\ x\ \partial M) \leq N\ X$ **.**
qed *simp*
then have *nn_integral M ?f'* ≤ *?y*
by (*rule SUP_upper*)
ultimately show *False*
by (*simp add*: *int_f_eq_y*)
qed
show *?thesis*
proof (*intro bexI*[*of* _ *f*] *measure_eqI conjI antisym*)
fix *A* **assume** *A* ∈ *sets* (*density M f*) **then show** *emeasure* (*density M f*) *A* ≤ *emeasure N A*
by (*auto simp*: *emeasure_density intro*!: *G_D*[*OF* ‹*f* ∈ *G*›])
next
fix *A* **assume** *A*: *A* ∈ *sets* (*density M f*) **then show** *emeasure N A* ≤ *emeasure* (*density M f*) *A*
using *upper_bound* **by** *auto*
qed *auto*
qed

lemma (**in** *finite_measure*) *split_space_into_finite_sets_and_rest*:
assumes *ac*: *absolutely_continuous M N* **and** *sets_eq*[*simp*]: *sets N* = *sets M*
shows ∃ *B*::*nat*⇒'*a set*. *disjoint_family B* ∧ *range B* ⊆ *sets M* ∧ (∀ *i*. *N* (*B i*) ≠ ∞) ∧
(∀ *A*∈*sets M*. *A* ∩ (⋃ *i*. *B i*) = {} ⟶ (*emeasure M A* = *0* ∧ *N A* = *0*) ∨ (*emeasure M A* > *0* ∧ *N A* = ∞))
proof −
let *?Q* = {*Q*∈*sets M*. *N Q* ≠ ∞}
let *?a* = *SUP Q*∈*?Q*. *emeasure M Q*
have {} ∈ *?Q* **by** *auto*
then have *Q_not_empty*: *?Q* ≠ {} **by** *blast*
have *?a* ≤ *emeasure M* (*space M*) **using** *sets.sets_into_space*
by (*auto intro*!: *SUP_least emeasure_mono*)
then have *?a* ≠ ∞
using *finite_emeasure_space*
by (*auto simp*: *less_top*[*symmetric*] *top_unique simp del*: *SUP_eq_top_iff Sup_eq_top_iff*)
from *ennreal_SUP_countable_SUP* [*OF Q_not_empty*, *of emeasure M*]
obtain *Q''* **where** *range Q''* ⊆ *emeasure M* ‘ *?Q* **and** *a*: *?a* = (*SUP i*::*nat*. *Q'' i*)
by *auto*
then have ∀ *i*. ∃ *Q'*. *Q'' i* = *emeasure M Q'* ∧ *Q'* ∈ *?Q* **by** *auto*
from *choice*[*OF this*] **obtain** *Q'* **where** *Q'*: ⋀*i*. *Q'' i* = *emeasure M* (*Q' i*) ⋀*i*.

```
Q' i ∈ ?Q
    by auto
  then have a_Lim: ?a = (SUP i. emeasure M (Q' i)) using a by simp
  let ?O = λn. ⋃i≤n. Q' i
  have Union: (SUP i. emeasure M (?O i)) = emeasure M (⋃i. ?O i)
  proof (rule SUP_emeasure_incseq[of ?O])
    show range ?O ⊆ sets M using Q' by auto
    show incseq ?O by (fastforce intro!: incseq_SucI)
  qed
  have Q'_sets[measurable]: ⋀i. Q' i ∈ sets M using Q' by auto
  have O_sets: ⋀i. ?O i ∈ sets M using Q' by auto
  then have O_in_G: ⋀i. ?O i ∈ ?Q
  proof (safe del: notI)
    fix i have Q' ‘ {..i} ⊆ sets M using Q' by auto
    then have N (?O i) ≤ (∑ i≤i. N (Q' i))
      by (simp add: emeasure_subadditive_finite)
    also have ... < ∞ using Q' by (simp add: less_top)
    finally show N (?O i) ≠ ∞ by simp
  qed auto
  have O_mono: ⋀n. ?O n ⊆ ?O (Suc n) by fastforce
  have a_eq: ?a = emeasure M (⋃i. ?O i) unfolding Union[symmetric]
  proof (rule antisym)
    show ?a ≤ (SUP i. emeasure M (?O i)) unfolding a_Lim
      using Q' by (auto intro!: SUP_mono emeasure_mono)
    show (SUP i. emeasure M (?O i)) ≤ ?a
    proof (safe intro!: Sup_mono, unfold bex_simps)
      fix i
      have ∗: (⋃(Q' ‘ {..i})) = ?O i by auto
      then show ∃x. (x ∈ sets M ∧ N x ≠ ∞) ∧
        emeasure M (⋃(Q' ‘ {..i})) ≤ emeasure M x
        using O_in_G[of i] by (auto intro!: exI[of _ ?O i])
    qed
  qed
  let ?O_0 = (⋃i. ?O i)
  have ?O_0 ∈ sets M using Q' by auto
  have disjointed Q' i ∈ sets M for i
    using sets.range_disjointed_sets[of Q' M] using Q'_sets by (auto simp: subset_eq)
  note Q_sets = this
  show ?thesis
  proof (intro bexI exI conjI ballI impI allI)
    show disjoint_family (disjointed Q')
      by (rule disjoint_family_disjointed)
    show range (disjointed Q') ⊆ sets M
      using Q'_sets by (intro sets.range_disjointed_sets) auto
    { fix A assume A: A ∈ sets M A ∩ (⋃i. disjointed Q' i) = {}
      then have A1: A ∩ (⋃i. Q' i) = {}
        unfolding UN_disjointed_eq by auto
      show emeasure M A = 0 ∧ N A = 0 ∨ 0 < emeasure M A ∧ N A = ∞
```

```
    proof (rule disjCI, simp)
      assume *: emeasure M A = 0 ∨ N A ≠ top
      show emeasure M A = 0 ∧ N A = 0
      proof (cases emeasure M A = 0)
        case True
        with ac A have N A = 0
          unfolding absolutely_continuous_def by auto
        with True show ?thesis by simp
      next
        case False
        with * have N A ≠ ∞ by auto
        with A have emeasure M ?O_0 + emeasure M A = emeasure M (?O_0
∪ A)
          using Q' A1 by (auto intro!: plus_emeasure simp: set_eq_iff)
        also have ... = (SUP i. emeasure M (?O i ∪ A))
      proof (rule SUP_emeasure_incseq[of λi. ?O i ∪ A, symmetric, simplified])
          show range (λi. ?O i ∪ A) ⊆ sets M
            using ‹N A ≠ ∞› O_sets A by auto
        qed (fastforce intro!: incseq_SucI)
        also have ... ≤ ?a
        proof (safe intro!: SUP_least)
          fix i have ?O i ∪ A ∈ ?Q
          proof (safe del: notI)
            show ?O i ∪ A ∈ sets M using O_sets A by auto
            from O_in_G[of i] have N (?O i ∪ A) ≤ N (?O i) + N A
              using emeasure_subadditive[of ?O i N A] A O_sets by auto
            with O_in_G[of i] show N (?O i ∪ A) ≠ ∞
              using ‹N A ≠ ∞› by (auto simp: top_unique)
          qed
          then show emeasure M (?O i ∪ A) ≤ ?a by (rule SUP_upper)
        qed
        finally have emeasure M A = 0
          unfolding a_eq using measure_nonneg[of M A] by (simp add: emea-
sure_eq_measure)
        with ‹emeasure M A ≠ 0› show ?thesis by auto
      qed
    qed }
  { fix i
    have N (disjointed Q' i) ≤ N (Q' i)
      by (auto intro!: emeasure_mono simp: disjointed_def)
    then show N (disjointed Q' i) ≠ ∞
      using Q'(2)[of i] by (auto simp: top_unique) }
 qed
qed

proposition (in finite_measure) Radon_Nikodym_finite_measure_infinite:
 assumes absolutely_continuous M N and sets_eq: sets N = sets M
 shows ∃f∈borel_measurable M. density M f = N
proof −
```

from *split_space_into_finite_sets_and_rest*[*OF assms*]
obtain $Q :: nat \Rightarrow 'a\ set$
where *Q*: *disjoint_family Q range Q* $\subseteq$ *sets M*
and *in_Q0*: $\bigwedge A.\ A \in sets\ M \Longrightarrow A \cap (\bigcup i.\ Q\ i) = \{\} \Longrightarrow emeasure\ M\ A = 0 \wedge N\ A = 0 \vee 0 < emeasure\ M\ A \wedge N\ A = \infty$
and *Q_fin*: $\bigwedge i.\ N\ (Q\ i) \neq \infty$ **by** *force*
from *Q* **have** *Q_sets*: $\bigwedge i.\ Q\ i \in sets\ M$ **by** *auto*
let *?N* = $\lambda i.\ density\ N\ (indicator\ (Q\ i))$ **and** *?M* = $\lambda i.\ density\ M\ (indicator\ (Q\ i))$
have $\forall\, i.\ \exists f{\in}borel\_measurable\ (?M\ i).\ density\ (?M\ i)\ f = ?N\ i$
proof (*intro allI finite_measure.Radon_Nikodym_finite_measure*)
fix *i*
from *Q* **show** *finite_measure* (*?M i*)
by (*auto intro*!: *finite_measureI cong*: *nn_integral_cong*
simp add: *emeasure_density subset_eq sets_eq*)
from *Q* **have** *emeasure* (*?N i*) (*space N*) = *emeasure N* (*Q i*)
by (*simp add*: *sets_eq*[*symmetric*] *emeasure_density subset_eq cong*: *nn_integral_cong*)
with *Q_fin* **show** *finite_measure* (*?N i*)
by (*auto intro*!: *finite_measureI*)
show *sets* (*?N i*) = *sets* (*?M i*) **by** (*simp add*: *sets_eq*)
have [*measurable*]: $\bigwedge A.\ A \in sets\ M \Longrightarrow A \in sets\ N$ **by** (*simp add*: *sets_eq*)
show *absolutely_continuous* (*?M i*) (*?N i*)
using ‹*absolutely_continuous M N*› ‹*Q i* $\in$ *sets M*›
by (*auto simp*: *absolutely_continuous_def null_sets_density_iff sets_eq*
intro!: *absolutely_continuous_AE*[*OF sets_eq*])
qed
from *choice*[*OF this*[*unfolded Bex_def*]]
obtain *f* **where** *borel*: $\bigwedge i.\ f\ i \in borel\_measurable\ M\ \bigwedge i\ x.\ 0 \leq f\ i\ x$
and *f_density*: $\bigwedge i.\ density\ (?M\ i)\ (f\ i) = ?N\ i$
by *force*
{ fix *A i* **assume** *A*: *A* $\in$ *sets M*
with *Q borel* **have** $(\int^+ x.\ f\ i\ x * indicator\ (Q\ i \cap A)\ x\ \partial M) = emeasure\ (density\ (?M\ i)\ (f\ i))\ A$
by (*auto simp add*: *emeasure_density nn_integral_density subset_eq*
intro!: *nn_integral_cong split*: *split_indicator*)
also have ... = *emeasure N* (*Q i* $\cap$ *A*)
using *A Q* **by** (*simp add*: *f_density emeasure_restricted subset_eq sets_eq*)
finally have *emeasure N* (*Q i* $\cap$ *A*) = $(\int^+ x.\ f\ i\ x * indicator\ (Q\ i \cap A)\ x\ \partial M)$ **.. }**
note *integral_eq* = *this*
let *?f* = $\lambda x.\ (\sum i.\ f\ i\ x * indicator\ (Q\ i)\ x) + \infty * indicator\ (space\ M - (\bigcup i.\ Q\ i))\ x$
show *?thesis*
proof (*safe intro*!: *bexI*[*of _ ?f*])
show *?f* $\in$ *borel_measurable M* **using** *borel Q_sets*
by (*auto intro*!: *measurable_If*)
show *density M ?f* = *N*
proof (*rule measure_eqI*)
fix *A* **assume** *A* $\in$ *sets* (*density M ?f*)

then have $A \in$ *sets* M **by** *simp*
have *Qi*: $\bigwedge i.\ Q\ i \in$ *sets* M **using** Q **by** *auto*
have [*intro*,*simp*]: $\bigwedge i.\ (\lambda x.\ f\ i\ x * \mathit{indicator}\ (Q\ i \cap A)\ x) \in \mathit{borel\_measurable}\ M$
$\bigwedge i.\ AE\ x\ \mathit{in}\ M.\ 0 \le f\ i\ x * \mathit{indicator}\ (Q\ i \cap A)\ x$
using *borel Qi* ‹$A \in$ *sets* M› **by** *auto*
have $(\int^+ x.\ ?f\ x * \mathit{indicator}\ A\ x\ \partial M) = (\int^+ x.\ (\sum i.\ f\ i\ x * \mathit{indicator}\ (Q\ i \cap A)\ x) + \infty * \mathit{indicator}\ ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A)\ x\ \partial M)$
using *borel* **by** (*intro nn_integral_cong*) (*auto simp*: *indicator_def*)
also have $\ldots = (\int^+ x.\ (\sum i.\ f\ i\ x * \mathit{indicator}\ (Q\ i \cap A)\ x)\ \partial M) + \infty * \mathit{emeasure}\ M\ ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A)$
using *borel Qi* ‹$A \in$ *sets* M›
by (*subst nn_integral_add*)
(*auto simp add*: *nn_integral_cmult_indicator sets.Int intro*!: *suminf_0_le*)
also have $\ldots = (\sum i.\ N\ (Q\ i \cap A)) + \infty * \mathit{emeasure}\ M\ ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A)$
by (*subst integral_eq*[*OF* ‹$A \in$ *sets* M›], *subst nn_integral_suminf*) *auto*
finally have $(\int^+ x.\ ?f\ x * \mathit{indicator}\ A\ x\ \partial M) = (\sum i.\ N\ (Q\ i \cap A)) + \infty * \mathit{emeasure}\ M\ ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A)$ **.**
moreover have $(\sum i.\ N\ (Q\ i \cap A)) = N\ ((\bigcup i.\ Q\ i) \cap A)$
using Q *Q_sets* ‹$A \in$ *sets* M›
by (*subst suminf_emeasure*) (*auto simp*: *disjoint_family_on_def sets_eq*)
moreover
have $(\mathit{space}\ M - (\bigcup x.\ Q\ x)) \cap A \cap (\bigcup x.\ Q\ x) = \{\}$
by *auto*
then have $\infty * \mathit{emeasure}\ M\ ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A) = N\ ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A)$
using *in_Q0*[*of* $(\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A$] ‹$A \in$ *sets* M› Q **by** (*auto simp*: *ennreal_top_mult*)
moreover have $(\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A \in \mathit{sets}\ M\ ((\bigcup i.\ Q\ i) \cap A) \in \mathit{sets}\ M$
using *Q_sets* ‹$A \in$ *sets* M› **by** *auto*
moreover have $((\bigcup i.\ Q\ i) \cap A) \cup ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A) = A\ ((\bigcup i.\ Q\ i) \cap A) \cap ((\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A) = \{\}$
using ‹$A \in$ *sets* M› *sets.sets_into_space* **by** *auto*
ultimately have $N\ A = (\int^+ x.\ ?f\ x * \mathit{indicator}\ A\ x\ \partial M)$
using *plus_emeasure*[*of* $(\bigcup i.\ Q\ i) \cap A\ N\ (\mathit{space}\ M - (\bigcup i.\ Q\ i)) \cap A$] **by** (*simp add*: *sets_eq*)
with ‹$A \in$ *sets* M› *borel* Q **show** *emeasure* (*density* M *?f*) $A = N\ A$
by (*auto simp*: *subset_eq emeasure_density*)
qed (*simp add*: *sets_eq*)
qed
qed

theorem (**in** *sigma_finite_measure*) *Radon_Nikodym*:
assumes *ac*: *absolutely_continuous* $M\ N$ **assumes** *sets_eq*: *sets* $N =$ *sets* M
shows $\exists f \in \mathit{borel\_measurable}\ M.\ \mathit{density}\ M\ f = N$
proof −
from *Ex_finite_integrable_function*

obtain h **where** *finite*: $integral^N\ M\ h \neq \infty$ **and**
borel: $h \in borel\_measurable\ M$ **and**
nn: $\bigwedge x.\ 0 \leq h\ x$ **and**
pos: $\bigwedge x.\ x \in space\ M \Longrightarrow 0 < h\ x$ **and**
$\bigwedge x.\ x \in space\ M \Longrightarrow h\ x < \infty$ **by** *auto*
let $?T = \lambda A.\ (\int^+ x.\ h\ x * indicator\ A\ x\ \partial M)$
let $?MT = density\ M\ h$
from *borel finite nn* **interpret** T: *finite_measure* $?MT$
by (*auto intro*!: *finite_measureI cong*: *nn_integral_cong simp*: *emeasure_density*)
have *absolutely_continuous* $?MT\ N$ *sets* $N = sets\ ?MT$
proof (*unfold absolutely_continuous_def*, *safe*)
fix A **assume** $A \in null\_sets\ ?MT$
with *borel* **have** $A \in sets\ M\ AE\ x\ in\ M.\ x \in A \longrightarrow h\ x \leq 0$
by (*auto simp add*: *null_sets_density_iff*)
with *pos sets.sets_into_space* **have** $AE\ x\ in\ M.\ x \notin A$
by (*elim eventually_mono*) (*auto simp*: *not_le*[*symmetric*])
then have $A \in null\_sets\ M$
using ‹$A \in sets\ M$› **by** (*simp add*: *AE_iff_null_sets*)
with *ac* **show** $A \in null\_sets\ N$
by (*auto simp*: *absolutely_continuous_def*)
qed (*auto simp add*: *sets_eq*)
from *T.Radon_Nikodym_finite_measure_infinite*[*OF this*]
obtain f **where** *f_borel*: $f \in borel\_measurable\ M\ density\ ?MT\ f = N$ **by** *auto*
with *nn borel* **show** *?thesis*
by (*auto intro*!: *bexI*[*of* _ $\lambda x.\ h\ x * f\ x$] *simp*: *density_density_eq*)
qed

8.5.3 Uniqueness of densities

lemma *finite_density_unique*:
assumes *borel*: $f \in borel\_measurable\ M\ g \in borel\_measurable\ M$
assumes *pos*: $AE\ x\ in\ M.\ 0 \leq f\ x\ AE\ x\ in\ M.\ 0 \leq g\ x$
and *fin*: $integral^N\ M\ f \neq \infty$
shows $density\ M\ f = density\ M\ g \longleftrightarrow (AE\ x\ in\ M.\ f\ x = g\ x)$
proof (*intro iffI ballI*)
fix A **assume** *eq*: $AE\ x\ in\ M.\ f\ x = g\ x$
with *borel* **show** $density\ M\ f = density\ M\ g$
by (*auto intro*: *density_cong*)
next
let $?P = \lambda f\ A.\ \int^+ x.\ f\ x * indicator\ A\ x\ \partial M$
assume $density\ M\ f = density\ M\ g$
with *borel* **have** *eq*: $\forall A \in sets\ M.\ ?P\ f\ A = ?P\ g\ A$
by (*simp add*: *emeasure_density*[*symmetric*])
from *this*[*THEN bspec*, *OF sets.top*] *fin*
have *g_fin*: $integral^N\ M\ g \neq \infty$ **by** (*simp cong*: *nn_integral_cong*)
{ fix $f\ g$ **assume** *borel*: $f \in borel\_measurable\ M\ g \in borel\_measurable\ M$
and *pos*: $AE\ x\ in\ M.\ 0 \leq f\ x\ AE\ x\ in\ M.\ 0 \leq g\ x$
and *g_fin*: $integral^N\ M\ g \neq \infty$ **and** *eq*: $\forall A \in sets\ M.\ ?P\ f\ A = ?P\ g\ A$
let $?N = \{x \in space\ M.\ g\ x < f\ x\}$

```
    have N: ?N ∈ sets M using borel by simp
    have ?P g ?N ≤ integral^N M g using pos
      by (intro nn_integral_mono_AE) (auto split: split_indicator)
    then have Pg_fin: ?P g ?N ≠ ∞ using g_fin by (auto simp: top_unique)
    have ?P (λx. (f x - g x)) ?N = (∫⁺x. f x * indicator ?N x - g x * indicator
?N x ∂M)
      by (auto intro!: nn_integral_cong simp: indicator_def)
    also have ... = ?P f ?N - ?P g ?N
    proof (rule nn_integral_diff)
      show (λx. f x * indicator ?N x) ∈ borel_measurable M (λx. g x * indicator
?N x) ∈ borel_measurable M
        using borel N by auto
      show AE x in M. g x * indicator ?N x ≤ f x * indicator ?N x
        using pos by (auto split: split_indicator)
    qed fact
    also have ... = 0
      unfolding eq[THEN bspec, OF N] using Pg_fin by auto
    finally have AE x in M. f x ≤ g x
      using pos borel nn_integral_PInf_AE[OF borel(2) g_fin]
      by (subst (asm) nn_integral_0_iff_AE)
         (auto split: split_indicator simp: not_less ennreal_minus_eq_0) }
  from this[OF borel pos g_fin eq] this[OF borel(2,1) pos(2,1) fin] eq
  show AE x in M. f x = g x by auto
qed

lemma (in finite_measure) density_unique_finite_measure:
  assumes borel: f ∈ borel_measurable M f' ∈ borel_measurable M
  assumes pos: AE x in M. 0 ≤ f x AE x in M. 0 ≤ f' x
  assumes f: ⋀A. A ∈ sets M ⟹ (∫⁺x. f x * indicator A x ∂M) = (∫⁺x. f' x
* indicator A x ∂M)
    (is ⋀A. A ∈ sets M ⟹ ?P f A = ?P f' A)
  shows AE x in M. f x = f' x
proof -
  let ?D = λf. density M f
  let ?N = λA. ?P f A and ?N' = λA. ?P f' A
  let ?f = λA x. f x * indicator A x and ?f' = λA x. f' x * indicator A x

  have ac: absolutely_continuous M (density M f) sets (density M f) = sets M
    using borel by (auto intro!: absolutely_continuousI_density)
  from split_space_into_finite_sets_and_rest[OF this]
  obtain Q :: nat ⇒ 'a set
    where Q: disjoint_family Q range Q ⊆ sets M
    and in_Q0: ⋀A. A ∈ sets M ⟹ A ∩ (⋃i. Q i) = {} ⟹ emeasure M A =
0 ∧ ?D f A = 0 ∨ 0 < emeasure M A ∧ ?D f A = ∞
    and Q_fin: ⋀i. ?D f (Q i) ≠ ∞ by force
  with borel pos have in_Q0: ⋀A. A ∈ sets M ⟹ A ∩ (⋃i. Q i) = {} ⟹
emeasure M A = 0 ∧ ?N A = 0 ∨ 0 < emeasure M A ∧ ?N A = ∞
    and Q_fin: ⋀i. ?N (Q i) ≠ ∞ by (auto simp: emeasure_density subset_eq)
```

```
  from Q have Q_sets[measurable]: ⋀i. Q i ∈ sets M by auto
  let ?D = {x∈space M. f x ≠ f' x}
  have ?D ∈ sets M using borel by auto
 have ∗: ⋀i x A. ⋀y::ennreal. y ∗ indicator (Q i) x ∗ indicator A x = y ∗ indicator
(Q i ∩ A) x
    unfolding indicator_def by auto
  have ∀ i. AE x in M. ?f (Q i) x = ?f' (Q i) x using borel Q_fin Q pos
    by (intro finite_density_unique[THEN iffD1] allI)
       (auto intro!: f measure_eqI simp: emeasure_density ∗ subset_eq)
  moreover have AE x in M. ?f (space M − (⋃i. Q i)) x = ?f' (space M −
(⋃i. Q i)) x
  proof (rule AE_I')
    { fix f :: 'a ⇒ ennreal assume borel: f ∈ borel_measurable M
        and eq: ⋀A. A ∈ sets M ⟹ ?N A = (∫+x. f x ∗ indicator A x ∂M)
      let ?A = λi. (space M − (⋃i. Q i)) ∩ {x ∈ space M. f x < (i::nat)}
      have (⋃i. ?A i) ∈ null_sets M
      proof (rule null_sets_UN)
        fix i ::nat have ?A i ∈ sets M
          using borel by auto
        have ?N (?A i) ≤ (∫+x. (i::ennreal) ∗ indicator (?A i) x ∂M)
          unfolding eq[OF ‹?A i ∈ sets M›]
          by (auto intro!: nn_integral_mono simp: indicator_def)
        also have ... = i ∗ emeasure M (?A i)
          using ‹?A i ∈ sets M› by (auto intro!: nn_integral_cmult_indicator)
         also have ... < ∞ using emeasure_real[of ?A i] by (auto simp: en-
nreal_mult_less_top of_nat_less_top)
        finally have ?N (?A i) ≠ ∞ by simp
        then show ?A i ∈ null_sets M using in_Q0[OF ‹?A i ∈ sets M›] ‹?A i
∈ sets M› by auto
      qed
      also have (⋃i. ?A i) = (space M − (⋃i. Q i)) ∩ {x∈space M. f x ≠ ∞}
        by (auto simp: ennreal_Ex_less_of_nat less_top[symmetric])
      finally have (space M − (⋃i. Q i)) ∩ {x∈space M. f x ≠ ∞} ∈ null_sets
M by simp }
    from this[OF borel(1) refl] this[OF borel(2) f]
    have (space M − (⋃i. Q i)) ∩ {x∈space M. f x ≠ ∞} ∈ null_sets M (space
M − (⋃i. Q i)) ∩ {x∈space M. f' x ≠ ∞} ∈ null_sets M by simp_all
   then show ((space M − (⋃i. Q i)) ∩ {x∈space M. f x ≠ ∞}) ∪ ((space M −
(⋃i. Q i)) ∩ {x∈space M. f' x ≠ ∞}) ∈ null_sets M by (rule null_sets.Un)
    show {x ∈ space M. ?f (space M − (⋃i. Q i)) x ≠ ?f' (space M − (⋃i. Q
i)) x} ⊆
      ((space M − (⋃i. Q i)) ∩ {x∈space M. f x ≠ ∞}) ∪ ((space M − (⋃i. Q
i)) ∩ {x∈space M. f' x ≠ ∞}) by (auto simp: indicator_def)
  qed
  moreover have AE x in M. (?f (space M − (⋃i. Q i)) x = ?f' (space M −
(⋃i. Q i)) x) ⟶ (∀ i. ?f (Q i) x = ?f' (Q i) x) ⟶
    ?f (space M) x = ?f' (space M) x
    by (auto simp: indicator_def)
  ultimately have AE x in M. ?f (space M) x = ?f' (space M) x
```

```
    unfolding AE_all_countable[symmetric]
    by eventually_elim (auto split: if_split_asm simp: indicator_def)
  then show AE x in M. f x = f' x by auto
qed

proposition (in sigma_finite_measure) density_unique:
  assumes f: f ∈ borel_measurable M
  assumes f': f' ∈ borel_measurable M
  assumes density_eq: density M f = density M f'
  shows AE x in M. f x = f' x
proof −
  obtain h where h_borel: h ∈ borel_measurable M
    and fin: integral^N M h ≠ ∞ and pos: ⋀x. x ∈ space M ⟹ 0 < h x ∧ h x <
∞ ⋀x. 0 ≤ h x
    using Ex_finite_integrable_function by auto
  then have h_nn: AE x in M. 0 ≤ h x by auto
  let ?H = density M h
  interpret h: finite_measure ?H
    using fin h_borel pos
     by (intro finite_measureI) (simp cong: nn_integral_cong emeasure_density
add: fin)
  let ?fM = density M f
  let ?f'M = density M f'
  { fix A assume A ∈ sets M
    then have {x ∈ space M. h x * indicator A x ≠ 0} = A
      using pos(1) sets.sets_into_space by (force simp: indicator_def)
    then have (∫+x. h x * indicator A x ∂M) = 0 ⟷ A ∈ null_sets M
      using h_borel ‹A ∈ sets M› h_nn by (subst nn_integral_0_iff) auto }
  note h_null_sets = this
  { fix A assume A ∈ sets M
     have (∫+x. f x * (h x * indicator A x) ∂M) = (∫+x. h x * indicator A x
∂?fM)
      using ‹A ∈ sets M› h_borel h_nn f f'
      by (intro nn_integral_density[symmetric]) auto
    also have ... = (∫+x. h x * indicator A x ∂?f'M)
      by (simp_all add: density_eq)
    also have ... = (∫+x. f' x * (h x * indicator A x) ∂M)
      using ‹A ∈ sets M› h_borel h_nn f f'
      by (intro nn_integral_density) auto
    finally have (∫+x. h x * (f x * indicator A x) ∂M) = (∫+x. h x * (f' x *
indicator A x) ∂M)
      by (simp add: ac_simps)
    then have (∫+x. (f x * indicator A x) ∂?H) = (∫+x. (f' x * indicator A x)
∂?H)
      using ‹A ∈ sets M› h_borel h_nn f f'
      by (subst (asm) (1 2) nn_integral_density[symmetric]) auto }
  then have AE x in ?H. f x = f' x using h_borel h_nn f f'
    by (intro h.density_unique_finite_measure absolutely_continuous_AE[of M])
auto
```

```
  with AE_space[of M] pos show AE x in M. f x = f' x
    unfolding AE_density[OF h_borel] by auto
qed

lemma (in sigma_finite_measure) density_unique_iff:
  assumes f: f ∈ borel_measurable M and f': f' ∈ borel_measurable M
  shows density M f = density M f' ⟷ (AE x in M. f x = f' x)
  using density_unique[OF assms] density_cong[OF f f'] by auto

lemma sigma_finite_density_unique:
  assumes borel: f ∈ borel_measurable M g ∈ borel_measurable M
  and fin: sigma_finite_measure (density M f)
  shows density M f = density M g ⟷ (AE x in M. f x = g x)
proof
  assume AE x in M. f x = g x with borel show density M f = density M g
    by (auto intro: density_cong)
next
  assume eq: density M f = density M g
  interpret f: sigma_finite_measure density M f by fact
  from f.sigma_finite_incseq obtain A where cover: range A ⊆ sets (density M
f)
    ⋃ (range A) = space (density M f)
    ⋀i. emeasure (density M f) (A i) ≠ ∞
    incseq A
    by auto
  have AE x in M. ∀ i. x ∈ A i ⟶ f x = g x
    unfolding AE_all_countable
  proof
    fix i
    have density (density M f) (indicator (A i)) = density (density M g) (indicator
(A i))
      unfolding eq ..
    moreover have (∫⁺x. f x * indicator (A i) x ∂M) ≠ ∞
      using cover(1) cover(3)[of i] borel by (auto simp: emeasure_density sub-
set_eq)
    ultimately have AE x in M. f x * indicator (A i) x = g x * indicator (A i) x
      using borel cover(1)
     by (intro finite_density_unique[THEN iffD1]) (auto simp: density_density_eq
subset_eq)
    then show AE x in M. x ∈ A i ⟶ f x = g x
      by auto
  qed
  with AE_space show AE x in M. f x = g x
    apply eventually_elim
    using cover(2)[symmetric]
    apply auto
    done
qed
```

lemma (**in** *sigma_finite_measure*) *sigma_finite_iff_density_finite'*:
assumes *f*: $f \in$ *borel_measurable M*
shows *sigma_finite_measure* (*density M f*) $\longleftrightarrow$ (*AE x in M*. $f\ x \neq \infty$)
(**is** *sigma_finite_measure ?N* $\longleftrightarrow$ _)
proof
assume *sigma_finite_measure ?N*
then interpret *N*: *sigma_finite_measure ?N* .
from *N.Ex_finite_integrable_function* **obtain** *h* **where**
h: $h \in$ *borel_measurable M* $\wedge$ *integral*^{N} *?N h* $\neq \infty$ **and**
fin: $\forall x \in$ *space M*. $0 < h\ x \wedge h\ x < \infty$
by *auto*
have *AE x in M*. $f\ x * h\ x \neq \infty$
proof (*rule AE_I'*)
have *integral*^{N} *?N h* = $(\int^+ x.\ f\ x * h\ x\ \partial M)$
using *f h* **by** (*auto intro*!: *nn_integral_density*)
then have $(\int^+ x.\ f\ x * h\ x\ \partial M) \neq \infty$
using *h*(*2*) **by** *simp*
then show $(\lambda x.\ f\ x * h\ x) -` \{\infty\} \cap$ *space M* $\in$ *null_sets M*
using *f h*(*1*) **by** (*auto intro*!: *nn_integral_PInf*[*unfolded infinity_ennreal_def*] *borel_measurable_vimage*)
qed *auto*
then show *AE x in M*. $f\ x \neq \infty$
using *fin* **by** (*auto elim*!: *AE_Ball_mp simp*: *less_top ennreal_mult_less_top*)
next
assume *AE*: *AE x in M*. $f\ x \neq \infty$
from *sigma_finite* **obtain** *Q* :: *nat* $\Rightarrow$ *'a set*
where *Q*: *range Q* $\subseteq$ *sets M* $\bigcup$ (*range Q*) = *space M* $\bigwedge i$. *emeasure M* (*Q i*) $\neq \infty$
by *auto*
define *A* **where** *A i* =
$f -` (case\ i\ of\ 0 \Rightarrow \{\infty\} \mid Suc\ n \Rightarrow \{..\ ennreal(of\_nat\ (Suc\ n))\}) \cap$ *space M*
for *i*
{ fix *i j* **have** *A i* $\cap$ *Q j* $\in$ *sets M*
unfolding *A_def* **using** *f Q*
apply (*rule_tac sets.Int*)
by (*cases i*) (*auto intro*: *measurable_sets*[*OF f*(*1*)]) **}**
note *A_in_sets* = *this*

show *sigma_finite_measure ?N*
proof (*standard*, *intro exI conjI ballI*)
show *countable* (*range* (λ(*i*, *j*). *A i* $\cap$ *Q j*))
by *auto*
show *range* (λ(*i*, *j*). *A i* $\cap$ *Q j*) $\subseteq$ *sets* (*density M f*)
using *A_in_sets* **by** *auto*
next
have $\bigcup$(*range* (λ(*i*, *j*). *A i* $\cap$ *Q j*)) = $(\bigcup i\ j.\ A\ i \cap Q\ j)$
by *auto*
also have ... = $(\bigcup i.\ A\ i) \cap$ *space M* **using** *Q* **by** *auto*
also have $(\bigcup i.\ A\ i)$ = *space M*

```
    proof safe
      fix x assume x: x ∈ space M
      show x ∈ (⋃ i. A i)
      proof (cases f x rule: ennreal_cases)
        case top with x show ?thesis unfolding A_def by (auto intro: exI[of _
0])
      next
        case (real r)
        with ennreal_Ex_less_of_nat[of f x] obtain n :: nat where f x < n
          by auto
        also have n < (Suc n :: ennreal)
          by simp
        finally show ?thesis
         using x real by (auto simp: A_def ennreal_of_nat_eq_real_of_nat intro!:
exI[of _ Suc n])
      qed
    qed (auto simp: A_def)
    finally show ⋃(range (λ(i, j). A i ∩ Q j)) = space ?N by simp
  next
    fix X assume X ∈ range (λ(i, j). A i ∩ Q j)
    then obtain i j where [simp]:X = A i ∩ Q j by auto
    have (∫⁺x. f x * indicator (A i ∩ Q j) x ∂M) ≠ ∞
    proof (cases i)
      case 0
      have AE x in M. f x * indicator (A i ∩ Q j) x = 0
        using AE by (auto simp: A_def ‹i = 0›)
      from nn_integral_cong_AE[OF this] show ?thesis by simp
    next
      case (Suc n)
      then have (∫⁺x. f x * indicator (A i ∩ Q j) x ∂M) ≤
        (∫⁺x. (Suc n :: ennreal) * indicator (Q j) x ∂M)
      by (auto intro!: nn_integral_mono simp: indicator_def A_def ennreal_of_nat_eq_real_of_nat)
      also have ... = Suc n * emeasure M (Q j)
        using Q by (auto intro!: nn_integral_cmult_indicator)
      also have ... < ∞
        using Q by (auto simp: ennreal_mult_less_top less_top of_nat_less_top)
      finally show ?thesis by simp
    qed
    then show emeasure ?N X ≠ ∞
      using A_in_sets Q f by (auto simp: emeasure_density)
  qed
qed

lemma (in sigma_finite_measure) sigma_finite_iff_density_finite:
  f ∈ borel_measurable M ⟹ sigma_finite_measure (density M f) ⟷ (AE x in
M. f x ≠ ∞)
  by (subst sigma_finite_iff_density_finite')
     (auto simp: max_def intro!: measurable_If)
```

8.5.4 Radon-Nikodym derivative

definition *RN_deriv* :: *'a measure* ⇒ *'a measure* ⇒ *'a* ⇒ *ennreal* **where**
RN_deriv M N =
(*if* ∃*f*. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*
then SOME f. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*
else (λ_. *0*))

lemma *RN_derivI*:
assumes *f* ∈ *borel_measurable M density M f* = *N*
shows *density M* (*RN_deriv M N*) = *N*
proof −
have ∗: ∃*f*. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*
using *assms* **by** *auto*
then have *density M* (*SOME f*. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*)
= *N*
by (*rule someI2_ex*) *auto*
with ∗ **show** *?thesis*
by (*auto simp*: *RN_deriv_def*)
qed

lemma *borel_measurable_RN_deriv*[*measurable*]: *RN_deriv M N* ∈ *borel_measurable*
M
proof −
{ assume *ex*: ∃*f*. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*
have *1*: (*SOME f*. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*) ∈ *borel_measurable*
M
using *ex* **by** (*rule someI2_ex*) *auto* **}**
from *this* **show** *?thesis*
by (*auto simp*: *RN_deriv_def*)
qed

lemma *density_RN_deriv_density*:
assumes *f*: *f* ∈ *borel_measurable M*
shows *density M* (*RN_deriv M* (*density M f*)) = *density M f*
by (*rule RN_derivI*[*OF f*]) *simp*

lemma (**in** *sigma_finite_measure*) *density_RN_deriv*:
absolutely_continuous M N ⟹ *sets N* = *sets M* ⟹ *density M* (*RN_deriv M*
N) = *N*
by (*metis RN_derivI Radon_Nikodym*)

lemma (**in** *sigma_finite_measure*) *RN_deriv_nn_integral*:
assumes *N*: *absolutely_continuous M N sets N* = *sets M*
and *f*: *f* ∈ *borel_measurable M*
shows *integral*^{N} *N f* = (∫$^+$*x*. *RN_deriv M N x* ∗ *f x* ∂*M*)
proof −
have *integral*^{N} *N f* = *integral*^{N} (*density M* (*RN_deriv M N*)) *f*
using *N* **by** (*simp add*: *density_RN_deriv*)
also have ... = (∫$^+$*x*. *RN_deriv M N x* ∗ *f x* ∂*M*)

using *f* **by** (*simp add*: *nn_integral_density*)
finally show *?thesis* **by** *simp*
qed

lemma (**in** *sigma_finite_measure*) *RN_deriv_unique*:
assumes *f*: $f \in$ *borel_measurable M*
and *eq*: *density M f* $= N$
shows *AE x in M*. $f\ x =$ *RN_deriv M N x*
unfolding *eq*[*symmetric*]
by (*intro density_unique_iff*[*THEN iffD1*] *f borel_measurable_RN_deriv density_RN_deriv_density*[*symmetric*])

lemma *RN_deriv_unique_sigma_finite*:
assumes *f*: $f \in$ *borel_measurable M*
and *eq*: *density M f* $= N$ **and** *fin*: *sigma_finite_measure N*
shows *AE x in M*. $f\ x =$ *RN_deriv M N x*
using *fin* **unfolding** *eq*[*symmetric*]
by (*intro sigma_finite_density_unique*[*THEN iffD1*] *f borel_measurable_RN_deriv density_RN_deriv_density*[*symmetric*])

lemma (**in** *sigma_finite_measure*) *RN_deriv_distr*:
fixes $T :: 'a \Rightarrow 'b$
assumes *T*: $T \in$ *measurable M M'* **and** *T'*: $T' \in$ *measurable M' M*
and *inv*: $\forall x \in$*space M*. $T'\ (T\ x) = x$
and *ac*[*simp*]: *absolutely_continuous* (*distr M M' T*) (*distr N M' T*)
and *N*: *sets N* = *sets M*
shows *AE x in M*. *RN_deriv* (*distr M M' T*) (*distr N M' T*) $(T\ x) =$ *RN_deriv M N x*
proof (*rule RN_deriv_unique*)
have [*simp*]: *sets N* = *sets M* **by** *fact*
note *sets_eq_imp_space_eq*[*OF N*, *simp*]
have *measurable_N*[*simp*]: $\bigwedge M'$. *measurable N M'* = *measurable M M'* **by** (*auto simp*: *measurable_def*)
{ fix *A* **assume** $A \in$ *sets M*
with *inv T T' sets.sets_into_space*[*OF this*]
have $T -` T' -` A \cap T -` space\ M' \cap space\ M = A$
by (*auto simp*: *measurable_def*) **}**
note *eq* = *this*[*simp*]
{ fix *A* **assume** $A \in$ *sets M*
with *inv T T' sets.sets_into_space*[*OF this*]
have $(T' \circ T) -` A \cap space\ M = A$
by (*auto simp*: *measurable_def*) **}**
note *eq2* = *this*[*simp*]
let *?M'* = *distr M M' T* **and** *?N'* = *distr N M' T*
interpret *M'*: *sigma_finite_measure ?M'*
proof
from *sigma_finite_countable* **obtain** *F*
where *F*: *countable F* $\wedge$ $F \subseteq$ *sets M* $\wedge$ $\bigcup F =$ *space M* $\wedge$ $(\forall a \in F.$ *emeasure M a* $\neq \infty)$ **..**

show $\exists A.$ *countable* $A \wedge A \subseteq$ *sets* (*distr M M′ T*) $\wedge \bigcup A =$ *space* (*distr M M′ T*) $\wedge$ ($\forall a{\in}A.$ *emeasure* (*distr M M′ T*) $a \neq \infty$)
proof (*intro exI conjI ballI*)
show $*$: ($\lambda A.$ *T′* $-`$ *A* $\cap$ *space ?M′*) ` *F* $\subseteq$ *sets ?M′*
using *F T′* **by** (*auto simp*: *measurable_def*)
show $\bigcup$((($\lambda A.$ *T′* $-`$ *A* $\cap$ *space ?M′*)`*F*) = *space ?M′*
using *F T′*[*THEN measurable_space*] **by** (*auto simp*: *set_eq_iff*)
next
fix *X* **assume** *X* $\in$ (($\lambda A.$ *T′* $-`$ *A* $\cap$ *space ?M′*)`*F*
then obtain *A* **where** [*simp*]: *X* = *T′* $-`$ *A* $\cap$ *space ?M′* **and** *A* $\in$ *F* **by** *auto*
have *X* $\in$ *sets M′* **using** *F T′* ‹*A*∈*F*› **by** *auto*
moreover
have *Fi*: *A* $\in$ *sets M* **using** *F* ‹*A*∈*F*› **by** *auto*
ultimately show *emeasure ?M′ X* $\neq \infty$
using *F T T′* ‹*A*∈*F*› **by** (*simp add*: *emeasure_distr*)
qed (*use F* **in** *auto*)
qed
have (*RN_deriv ?M′ ?N′*) $\circ$ *T* $\in$ *borel_measurable M*
using *T ac* **by** *measurable*
then show ($\lambda x.$ *RN_deriv ?M′ ?N′* (*T x*)) $\in$ *borel_measurable M*
by (*simp add*: *comp_def*)

have *N* = *distr N M* (*T′* $\circ$ *T*)
by (*subst measure_of_of_measure*[*of N*, *symmetric*])
(*auto simp add*: *distr_def sets.sigma_sets_eq intro*!: *measure_of_eq sets.space_closed*)
also have ... = *distr* (*distr N M′ T*) *M T′*
using *T T′* **by** (*simp add*: *distr_distr*)
also have ... = *distr* (*density* (*distr M M′ T*) (*RN_deriv* (*distr M M′ T*) (*distr N M′ T*))) *M T′*
using *ac* **by** (*simp add*: *M′.density_RN_deriv*)
also have ... = *density M* (*RN_deriv* (*distr M M′ T*) (*distr N M′ T*) $\circ$ *T*)
by (*simp add*: *distr_density_distr*[*OF T T′*, *OF inv*])
finally show *density M* ($\lambda x.$ *RN_deriv* (*distr M M′ T*) (*distr N M′ T*) (*T x*)) = *N*
by (*simp add*: *comp_def*)
qed

lemma (**in** *sigma_finite_measure*) *RN_deriv_finite*:
assumes *N*: *sigma_finite_measure N* **and** *ac*: *absolutely_continuous M N sets N* = *sets M*
shows *AE x in M*. *RN_deriv M N x* $\neq \infty$
proof −
interpret *N*: *sigma_finite_measure N* **by** *fact*
from *N* **show** *?thesis*
using *sigma_finite_iff_density_finite*[*OF borel_measurable_RN_deriv*, *of N*] *density_RN_deriv*[*OF ac*]
by *simp*
qed

lemma (**in** *sigma_finite_measure*)
  assumes *N*: *sigma_finite_measure N* **and** *ac*: *absolutely_continuous M N sets N = sets M*
    and *f*: *f ∈ borel_measurable M*
  shows *RN_deriv_integrable*: *integrable N f ⟷*
      integrable M (*λx. ennreal* (*RN_deriv M N x*) ∗ *f x*) (**is** *?integrable*)
    and *RN_deriv_integral*: *integral*^{L} *N f* = (*∫ x. ennreal* (*RN_deriv M N x*) ∗ *f x ∂M*) (**is** *?integral*)
proof −
  note *ac*(*2*)[*simp*] **and** *sets_eq_imp_space_eq*[*OF ac*(*2*), *simp*]
  interpret *N*: *sigma_finite_measure N* **by** *fact*

  have *eq*: *density M* (*RN_deriv M N*) = *density M* (*λx. ennreal* (*RN_deriv M N x*))
  proof (*rule density_cong*)
    from *RN_deriv_finite*[*OF assms*(*1*,*2*,*3*)]
    show *AE x in M. RN_deriv M N x = ennreal* (*ennreal* (*RN_deriv M N x*))
      by *eventually_elim* (*auto simp*: *less_top*)
  qed (*insert ac*, *auto*)

  show *?integrable*
    apply (*subst density_RN_deriv*[*OF ac*, *symmetric*])
    unfolding *eq*
    apply (*intro integrable_real_density f AE_I2 ennreal_nonneg*)
    apply (*insert ac*, *auto*)
    done

  show *?integral*
    apply (*subst density_RN_deriv*[*OF ac*, *symmetric*])
    unfolding *eq*
    apply (*intro integral_real_density f AE_I2 ennreal_nonneg*)
    apply (*insert ac*, *auto*)
    done
qed

proposition (**in** *sigma_finite_measure*) *real_RN_deriv*:
  assumes *finite_measure N*
  assumes *ac*: *absolutely_continuous M N sets N = sets M*
  obtains *D* **where** *D ∈ borel_measurable M*
    and *AE x in M. RN_deriv M N x = ennreal* (*D x*)
    and *AE x in N. 0 < D x*
    and *⋀x. 0 ≤ D x*
proof
  interpret *N*: *finite_measure N* **by** *fact*

  note *RN* = *borel_measurable_RN_deriv density_RN_deriv*[*OF ac*]

  let *?RN* = *λt.* {*x ∈ space M. RN_deriv M N x = t*}

show (λx. *enn2real* (*RN_deriv M N x*)) ∈ *borel_measurable M*
using *RN* **by** *auto*

have *N* (*?RN* ∞) = (∫⁺ *x*. *RN_deriv M N x* ∗ *indicator* (*?RN* ∞) *x* ∂*M*)
using *RN*(*1*) **by** (*subst RN*(*2*)[*symmetric*]) (*auto simp*: *emeasure_density*)
also have ... = (∫⁺ *x*. ∞ ∗ *indicator* (*?RN* ∞) *x* ∂*M*)
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def*)
also have ... = ∞ ∗ *emeasure M* (*?RN* ∞)
using *RN* **by** (*intro nn_integral_cmult_indicator*) *auto*
finally have *eq*: *N* (*?RN* ∞) = ∞ ∗ *emeasure M* (*?RN* ∞) **.**
moreover
have *emeasure M* (*?RN* ∞) = *0*
proof (*rule ccontr*)
assume *emeasure M* {*x* ∈ *space M*. *RN_deriv M N x* = ∞} ≠ *0*
then have *0* < *emeasure M* {*x* ∈ *space M*. *RN_deriv M N x* = ∞}
by (*auto simp*: *zero_less_iff_neq_zero*)
with *eq* **have** *N* (*?RN* ∞) = ∞ **by** (*simp add*: *ennreal_mult_eq_top_iff*)
with *N.emeasure_finite*[*of ?RN* ∞] *RN* **show** *False* **by** *auto*
qed
ultimately have *AE x in M*. *RN_deriv M N x* < ∞
using *RN* **by** (*intro AE_iff_measurable*[*THEN iffD2*]) (*auto simp*: *less_top*[*symmetric*])
then show *AE x in M*. *RN_deriv M N x* = *ennreal* (*enn2real* (*RN_deriv M N x*))
by *auto*
then have *eq*: *AE x in N*. *RN_deriv M N x* = *ennreal* (*enn2real* (*RN_deriv M N x*))
using *ac absolutely_continuous_AE* **by** *auto*

have *N* (*?RN 0*) = (∫⁺ *x*. *RN_deriv M N x* ∗ *indicator* (*?RN 0*) *x* ∂*M*)
by (*subst RN*(*2*)[*symmetric*]) (*auto simp*: *emeasure_density*)
also have ... = (∫⁺ *x*. *0* ∂*M*)
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def*)
finally have *AE x in N*. *RN_deriv M N x* ≠ *0*
using *RN* **by** (*subst AE_iff_measurable*[*OF _ refl*]) (*auto simp*: *ac cong*: *sets_eq_imp_space_eq*)
with *eq* **show** *AE x in N*. *0* < *enn2real* (*RN_deriv M N x*)
by (*auto simp*: *enn2real_positive_iff less_top*[*symmetric*] *zero_less_iff_neq_zero*)
qed (*rule enn2real_nonneg*)

lemma (**in** *sigma_finite_measure*) *RN_deriv_singleton*:
assumes *ac*: *absolutely_continuous M N sets N* = *sets M*
and *x*: {*x*} ∈ *sets M*
shows *N* {*x*} = *RN_deriv M N x* ∗ *emeasure M* {*x*}
proof −
from ‹{*x*} ∈ *sets M*›
have *density M* (*RN_deriv M N*) {*x*} = (∫⁺*w*. *RN_deriv M N x* ∗ *indicator* {*x*} *w* ∂*M*)

```
    by (auto simp: indicator_def emeasure_density intro!: nn_integral_cong)
  with x density_RN_deriv[OF ac] show ?thesis
    by (auto simp: max_def)
qed

end
```

Chapter 9

Integrals over a Set

theory *Set_Integral*
imports *Radon_Nikodym*
begin

9.1 Notation

definition *set_borel_measurable M A f* ≡ (λ*x*. *indicator A x* ∗$_R$ *f x*) ∈ *borel_measurable M*

definition *set_integrable M A f* ≡ *integrable M* (λ*x*. *indicator A x* ∗$_R$ *f x*)

definition *set_lebesgue_integral M A f* ≡ *lebesgue_integral M* (λ*x*. *indicator A x* ∗$_R$ *f x*)

syntax
_ascii_set_lebesgue_integral :: *pttrn* ⇒ *'a set* ⇒ *'a measure* ⇒ *real* ⇒ *real*
(‹(‹*indent=4 notation=‹binder LINT››LINT* (_):(_)/|(_)./ _)› [*0,0,0,10*] *10*)
syntax_consts
_ascii_set_lebesgue_integral == *set_lebesgue_integral*
translations
LINT x:*A*|*M*. *f* == *CONST set_lebesgue_integral M A* (λ*x*. *f*)

9.2 Basic properties

lemma *set_integrable_cong*:
assumes *M* = *M'* *A* = *A'* ⋀*x*. *x* ∈ *A* ⟹ *f x* = *f' x*
shows *set_integrable M A f* = *set_integrable M' A' f'*
proof −
have (λ*x*. *indicator A x* ∗$_R$ *f x*) = (λ*x*. *indicator A' x* ∗$_R$ *f' x*)
using *assms* **by** (*auto simp*: *indicator_def of_bool_def*)
thus *?thesis* **by** (*simp add*: *set_integrable_def assms*)
qed

lemma *set_borel_measurable_sets*:
fixes *f* :: _ ⇒ _::*real_normed_vector*
assumes *set_borel_measurable M X f B* ∈ *sets borel X* ∈ *sets M*
shows *f* −‘ *B* ∩ *X* ∈ *sets M*
proof −
have *f* ∈ *borel_measurable* (*restrict_space M X*)
using *assms* **unfolding** *set_borel_measurable_def* **by** (*subst borel_measurable_restrict_space_iff*) *auto*
then have *f* −‘ *B* ∩ *space* (*restrict_space M X*) ∈ *sets* (*restrict_space M X*)
by (*rule measurable_sets*) *fact*
with ‹*X* ∈ *sets M*› **show** *?thesis*
by (*subst* (*asm*) *sets_restrict_space_iff*) (*auto simp*: *space_restrict_space*)
qed

lemma *set_integrable_bound*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
and *g* :: *'a* ⇒ *'c*::{*banach*, *second_countable_topology*}
assumes *set_integrable M A f set_borel_measurable M A g*
assumes *AE x in M*. *x* ∈ *A* ⟶ *norm* (*g x*) ≤ *norm* (*f x*)
shows *set_integrable M A g*
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound*)
from *assms*(*1*) **show** *integrable M* (λ*x*. *indicator A x* ∗*R f x*)
by (*simp add*: *set_integrable_def*)
from *assms*(*2*) **show** (λ*x*. *indicat_real A x* ∗*R g x*) ∈ *borel_measurable M*
by (*simp add*: *set_borel_measurable_def*)
from *assms*(*3*) **show** *AE x in M*. *norm* (*indicat_real A x* ∗*R g x*) ≤ *norm* (*indicat_real A x* ∗*R f x*)
by *eventually_elim* (*simp add*: *indicator_def*)
qed

lemma *set_lebesgue_integral_zero* [*simp*]: *set_lebesgue_integral M A* (λ*x*. *0*) = *0*
by (*auto simp*: *set_lebesgue_integral_def*)

lemma *set_lebesgue_integral_cong*:
assumes *A* ∈ *sets M* **and** ∀ *x*. *x* ∈ *A* ⟶ *f x* = *g x*
shows (*LINT x*:*A*|*M*. *f x*) = (*LINT x*:*A*|*M*. *g x*)
unfolding *set_lebesgue_integral_def*
using *assms*
by (*metis indicator_simps*(*2*) *real_vector.scale_zero_left*)

lemma *set_lebesgue_integral_cong_AE*:
assumes [*measurable*]: *A* ∈ *sets M f* ∈ *borel_measurable M g* ∈ *borel_measurable M*
assumes *AE x* ∈ *A in M*. *f x* = *g x*
shows (*LINT x*:*A*|*M*. *f x*) = (*LINT x*:*A*|*M*. *g x*)
proof−
have *AE x in M*. *indicator A x* ∗*R f x* = *indicator A x* ∗*R g x*

using *assms* **by** *auto*
thus *?thesis*
unfolding *set_lebesgue_integral_def* **by** (*intro integral_cong_AE*) *auto*
qed

lemma *set_integrable_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹
AE x ∈ *A in M*. *f x* = *g x* ⟹ *A* ∈ *sets M* ⟹
set_integrable M A f = *set_integrable M A g*
unfolding *set_integrable_def*
by (*rule integrable_cong_AE*) *auto*

lemma *set_integrable_subset*:
fixes *M A B* **and** *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes *set_integrable M A f B* ∈ *sets M B* ⊆ *A*
shows *set_integrable M B f*
proof −
have *set_integrable M B* (λ*x*. *indicator A x* ∗$_R$ *f x*)
using *assms integrable_mult_indicator set_integrable_def* **by** *blast*
with ‹*B* ⊆ *A*› **show** *?thesis*
unfolding *set_integrable_def*
by (*simp add*: *indicator_inter_arith*[*symmetric*] *Int_absorb2*)
qed

lemma *set_integrable_restrict_space*:
fixes *f* :: ′*a* ⇒ ′*b*::{*banach*, *second_countable_topology*}
assumes *f*: *set_integrable M S f* **and** *T*: *T* ∈ *sets* (*restrict_space M S*)
shows *set_integrable M T f*
proof −
obtain *T*′ **where** *T_eq*: *T* = *S* ∩ *T*′ **and** *T*′ ∈ *sets M*
using *T* **by** (*auto simp*: *sets_restrict_space*)
have ‹*integrable M* (λ*x*. *indicator T*′ *x* ∗$_R$ (*indicator S x* ∗$_R$ *f x*))›
using ‹*T*′ ∈ *sets M*› *f integrable_mult_indicator set_integrable_def* **by** *blast*
then show *?thesis*
unfolding *set_integrable_def*
unfolding *T_eq indicator_inter_arith* **by** (*simp add*: *ac_simps*)
qed

lemma *set_integral_scaleR_left*:
assumes *A* ∈ *sets M c* ≠ *0* ⟹ *integrable M f*
shows (*LINT t*:*A*|*M*. *f t* ∗$_R$ *c*) = (*LINT t*:*A*|*M*. *f t*) ∗$_R$ *c*
unfolding *set_lebesgue_integral_def*
using *integrable_mult_indicator*[*OF assms*]
by (*subst integral_scaleR_left*[*symmetric*], *auto*)

lemma *set_integral_scaleR_right* [*simp*]: (*LINT t*:*A*|*M*. *a* ∗$_R$ *f t*) = *a* ∗$_R$ (*LINT*

t:$A|M.\ f\ t$)
unfolding *set_lebesgue_integral_def*
by (*subst integral_scaleR_right*[*symmetric*]) (*auto intro*!: *Bochner_Integration.integral_cong*)

lemma *set_integral_mult_right* [*simp*]:
fixes $a :: 'a$::{*real_normed_field*, *second_countable_topology*}
shows $(LINT\ t{:}A|M.\ a * f\ t) = a * (LINT\ t{:}A|M.\ f\ t)$
unfolding *set_lebesgue_integral_def*
by (*subst integral_mult_right_zero*[*symmetric*]) *auto*

lemma *set_integral_mult_left* [*simp*]:
fixes $a :: 'a$::{*real_normed_field*, *second_countable_topology*}
shows $(LINT\ t{:}A|M.\ f\ t * a) = (LINT\ t{:}A|M.\ f\ t) * a$
unfolding *set_lebesgue_integral_def*
by (*subst integral_mult_left_zero*[*symmetric*]) *auto*

lemma *set_integral_divide_zero* [*simp*]:
fixes $a :: 'a$::{*real_normed_field*, *field*, *second_countable_topology*}
shows $(LINT\ t{:}A|M.\ f\ t\ /\ a) = (LINT\ t{:}A|M.\ f\ t)\ /\ a$
unfolding *set_lebesgue_integral_def*
by (*subst integral_divide_zero*[*symmetric*], *intro Bochner_Integration.integral_cong*)
(*auto split*: *split_indicator*)

lemma *set_integrable_scaleR_right* [*simp*, *intro*]:
shows $(a \neq 0 \Longrightarrow set\_integrable\ M\ A\ f) \Longrightarrow set\_integrable\ M\ A\ (\lambda t.\ a *_R f\ t)$
unfolding *set_integrable_def*
unfolding *scaleR_left_commute* **by** (*rule integrable_scaleR_right*)

lemma *set_integrable_scaleR_left* [*simp*, *intro*]:
fixes $a :: \_ :: \{banach, second\_countable\_topology\}$
shows $(a \neq 0 \Longrightarrow set\_integrable\ M\ A\ f) \Longrightarrow set\_integrable\ M\ A\ (\lambda t.\ f\ t *_R a)$
unfolding *set_integrable_def*
using *integrable_scaleR_left*[*of a M* $\lambda x.$ *indicator A x* $*_R$ *f x*] **by** *simp*

lemma *set_integrable_mult_right* [*simp*, *intro*]:
fixes $a :: 'a$::{*real_normed_field*, *second_countable_topology*}
shows $(a \neq 0 \Longrightarrow set\_integrable\ M\ A\ f) \Longrightarrow set\_integrable\ M\ A\ (\lambda t.\ a * f\ t)$
unfolding *set_integrable_def*
using *integrable_mult_right*[*of a M* $\lambda x.$ *indicator A x* $*_R$ *f x*] **by** *simp*

lemma *set_integrable_mult_right_iff* [*simp*]:
fixes $a :: 'a$::{*real_normed_field*, *second_countable_topology*}
assumes $a \neq 0$
shows $set\_integrable\ M\ A\ (\lambda t.\ a * f\ t) \longleftrightarrow set\_integrable\ M\ A\ f$
proof
assume $set\_integrable\ M\ A\ (\lambda t.\ a * f\ t)$
then have $set\_integrable\ M\ A\ (\lambda t.\ 1/a * (a * f\ t))$
using *set_integrable_mult_right* **by** *blast*
then show $set\_integrable\ M\ A\ f$

using *assms* **by** *auto*
qed *auto*

lemma *set_integrable_mult_left* [*simp*, *intro*]:
fixes *a* :: *'a*::{*real_normed_field*, *second_countable_topology*}
shows (*a* ≠ *0* ⟹ *set_integrable M A f*) ⟹ *set_integrable M A* (λ*t*. *f t* ∗ *a*)
unfolding *set_integrable_def*
using *integrable_mult_left*[*of a M* λ*x*. *indicator A x* ∗*R f x*] **by** *simp*

lemma *set_integrable_mult_left_iff* [*simp*]:
fixes *a* :: *'a*::{*real_normed_field*, *second_countable_topology*}
assumes *a* ≠ *0*
shows *set_integrable M A* (λ*t*. *f t* ∗ *a*) ⟷ *set_integrable M A f*
using *assms* **by** (*subst set_integrable_mult_right_iff* [*symmetric*]) (*auto simp*: *mult.commute*)

lemma *set_integrable_divide* [*simp*, *intro*]:
fixes *a* :: *'a*::{*real_normed_field*, *field*, *second_countable_topology*}
assumes *a* ≠ *0* ⟹ *set_integrable M A f*
shows *set_integrable M A* (λ*t*. *f t* / *a*)
proof −
have *integrable M* (λ*x*. *indicator A x* ∗*R f x* / *a*)
using *assms* **unfolding** *set_integrable_def* **by** (*rule integrable_divide_zero*)
also have (λ*x*. *indicator A x* ∗*R f x* / *a*) = (λ*x*. *indicator A x* ∗*R* (*f x* / *a*))
by (*auto split*: *split_indicator*)
finally show *?thesis*
unfolding *set_integrable_def* **.**
qed

lemma *set_integrable_mult_divide_iff* [*simp*]:
fixes *a* :: *'a*::{*real_normed_field*, *second_countable_topology*}
assumes *a* ≠ *0*
shows *set_integrable M A* (λ*t*. *f t* / *a*) ⟷ *set_integrable M A f*
by (*simp add*: *divide_inverse assms*)

lemma *set_integral_add* [*simp*, *intro*]:
fixes *f g* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes *set_integrable M A f set_integrable M A g*
shows *set_integrable M A* (λ*x*. *f x* + *g x*)
and (*LINT x*:*A*|*M*. *f x* + *g x*) = (*LINT x*:*A*|*M*. *f x*) + (*LINT x*:*A*|*M*. *g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def* **by** (*simp_all add*: *scaleR_add_right*)

lemma *set_integral_diff* [*simp*, *intro*]:
assumes *set_integrable M A f set_integrable M A g*
shows *set_integrable M A* (λ*x*. *f x* − *g x*) **and** (*LINT x*:*A*|*M*. *f x* − *g x*) = (*LINT x*:*A*|*M*. *f x*) − (*LINT x*:*A*|*M*. *g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def* **by** (*simp_all add*: *scaleR_diff_right*)

lemma *set_integral_uminus*: *set_integrable M A f* $\Longrightarrow$ (*LINT x*:*A*|*M*. − *f x*) = − (*LINT x*:*A*|*M*. *f x*)
unfolding *set_integrable_def set_lebesgue_integral_def*
by (*subst integral_minus*[*symmetric*]) *simp_all*

lemma *set_integral_complex_of_real*:
(*LINT x*:*A*|*M*. *complex_of_real* (*f x*)) = *of_real* (*LINT x*:*A*|*M*. *f x*)
unfolding *set_lebesgue_integral_def*
by (*subst integral_complex_of_real*[*symmetric*])
(*auto intro*!: *Bochner_Integration.integral_cong split*: *split_indicator*)

lemma *set_integral_mono*:
fixes *f g* :: _ ⇒ *real*
assumes *set_integrable M A f set_integrable M A g*
$\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$
shows (*LINT x*:*A*|*M*. *f x*) ≤ (*LINT x*:*A*|*M*. *g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def*
by (*auto intro*: *integral_mono split*: *split_indicator*)

lemma *set_integral_mono_AE*:
fixes *f g* :: _ ⇒ *real*
assumes *set_integrable M A f set_integrable M A g*
AE x ∈ *A in M*. *f x* ≤ *g x*
shows (*LINT x*:*A*|*M*. *f x*) ≤ (*LINT x*:*A*|*M*. *g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def*
by (*auto intro*: *integral_mono_AE split*: *split_indicator*)

lemma *set_integrable_abs*: *set_integrable M A f* $\Longrightarrow$ *set_integrable M A* ($\lambda x.$ |*f x*| :: *real*)
using *integrable_abs*[*of M* $\lambda x.$ *f x* ∗ *indicator A x*]**unfolding** *set_integrable_def*
by (*simp add*: *abs_mult ac_simps*)

lemma *set_integrable_abs_iff*:
fixes *f* :: _ ⇒ *real*
shows *set_borel_measurable M A f* $\Longrightarrow$ *set_integrable M A* ($\lambda x.$ |*f x*|) = *set_integrable M A f*
unfolding *set_integrable_def set_borel_measurable_def*
by (*subst* (*2*) *integrable_abs_iff*[*symmetric*]) (*simp_all add*: *abs_mult ac_simps*)

lemma *set_integrable_abs_iff'*:
fixes *f* :: _ ⇒ *real*
shows *f* ∈ *borel_measurable M* $\Longrightarrow$ *A* ∈ *sets M* $\Longrightarrow$
set_integrable M A ($\lambda x.$ |*f x*|) = *set_integrable M A f*
by (*simp add*: *set_borel_measurable_def set_integrable_abs_iff*)

lemma *set_integrable_discrete_difference*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *countable X*

assumes *diff*: $(A - B) \cup (B - A) \subseteq X$
assumes $\bigwedge x.\ x \in X \Longrightarrow$ *emeasure M* $\{x\} = 0$ $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in$ *sets M*
shows *set_integrable M A f* $\longleftrightarrow$ *set_integrable M B f*
unfolding *set_integrable_def*
proof (*rule integrable_discrete_difference*[**where** X=X])
show $\bigwedge x.\ x \in$ *space M* $\Longrightarrow x \notin X \Longrightarrow$ *indicator A x* $*_R$ *f x* = *indicator B x* $*_R$ *f x*
using *diff* **by** (*auto split*: *split_indicator*)
qed *fact*+

lemma *set_integral_discrete_difference*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*banach*, *second_countable_topology*}
assumes *countable X*
assumes *diff*: $(A - B) \cup (B - A) \subseteq X$
assumes $\bigwedge x.\ x \in X \Longrightarrow$ *emeasure M* $\{x\} = 0$ $\bigwedge x.\ x \in X \Longrightarrow \{x\} \in$ *sets M*
shows *set_lebesgue_integral M A f* = *set_lebesgue_integral M B f*
unfolding *set_lebesgue_integral_def*
proof (*rule integral_discrete_difference*[**where** X=X])
show $\bigwedge x.\ x \in$ *space M* $\Longrightarrow x \notin X \Longrightarrow$ *indicator A x* $*_R$ *f x* = *indicator B x* $*_R$ *f x*
using *diff* **by** (*auto split*: *split_indicator*)
qed *fact*+

lemma *set_integrable_Un*:
fixes *f g* :: _ $\Rightarrow$ _ :: {*banach*, *second_countable_topology*}
assumes *f_A*: *set_integrable M A f* **and** *f_B*: *set_integrable M B f*
and [*measurable*]: *A* $\in$ *sets M B* $\in$ *sets M*
shows *set_integrable M* $(A \cup B)$ *f*
proof −
have *set_integrable M* $(A - B)$ *f*
using *f_A* **by** (*rule set_integrable_subset*) *auto*
with *f_B* **have** *integrable M* $(\lambda x.$ *indicator* $(A - B)$ *x* $*_R$ *f x* + *indicator B x* $*_R$ *f x*)
unfolding *set_integrable_def* **using** *integrable_add* **by** *blast*
then show *?thesis*
unfolding *set_integrable_def*
by (*rule integrable_cong*[*THEN iffD1*, *rotated 2*]) (*auto split*: *split_indicator*)
qed

lemma *set_integrable_empty* [*simp*]: *set_integrable M* {} *f*
by (*auto simp*: *set_integrable_def*)

lemma *set_integrable_UN*:
fixes *f* :: _ $\Rightarrow$ _ :: {*banach*, *second_countable_topology*}
assumes *finite I* $\bigwedge i.\ i \in I \Longrightarrow$ *set_integrable M* $(A\ i)$ *f*
$\bigwedge i.\ i \in I \Longrightarrow A\ i \in$ *sets M*
shows *set_integrable M* $(\bigcup i \in I.\ A\ i)$ *f*
using *assms*
by (*induct I*) (*auto simp*: *set_integrable_Un sets.finite_UN*)

lemma *set_integral_Un*:
fixes $f :: \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
assumes $A \cap B = \{\}$
and *set_integrable M A f*
and *set_integrable M B f*
shows $(LINT\ x{:}A \cup B|M.\ f\ x) = (LINT\ x{:}A|M.\ f\ x) + (LINT\ x{:}B|M.\ f\ x)$
using *assms*
unfolding *set_integrable_def set_lebesgue_integral_def*
by (*auto simp add*: *indicator_union_arith indicator_inter_arith*[*symmetric*] *scaleR_add_left*)

lemma *set_integral_cong_set*:
fixes $f :: \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
assumes *set_borel_measurable M A f set_borel_measurable M B f*
and *ae*: *AE x in M*. $x \in A \longleftrightarrow x \in B$
shows $(LINT\ x{:}B|M.\ f\ x) = (LINT\ x{:}A|M.\ f\ x)$
unfolding *set_lebesgue_integral_def*
proof (*rule integral_cong_AE*)
show *AE x in M*. *indicator B x* $*_R$ *f x = indicator A x* $*_R$ *f x*
using *ae* **by** (*auto simp*: *subset_eq split*: *split_indicator*)
qed (*use assms* **in** ‹*auto simp*: *set_borel_measurable_def*›)

proposition *set_borel_measurable_subset*:
fixes $f :: \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
assumes [*measurable*]: *set_borel_measurable M A f B* $\in$ *sets M* **and** $B \subseteq A$
shows *set_borel_measurable M B f*
proof−
have *set_borel_measurable M B* (λx. *indicator A x* $*_R$ *f x*)
using *assms* **unfolding** *set_borel_measurable_def*
using *borel_measurable_indicator borel_measurable_scaleR* **by** *blast*
moreover have (λx. *indicator B x* $*_R$ *indicator A x* $*_R$ *f x*) = (λx. *indicator B x* $*_R$ *f x*)
using ‹$B \subseteq A$› **by** (*auto simp*: *fun_eq_iff split*: *split_indicator*)
ultimately show *?thesis*
unfolding *set_borel_measurable_def* **by** *simp*
qed

lemma *set_integral_Un_AE*:
fixes $f :: \_ \Rightarrow \_ :: \{banach, second\_countable\_topology\}$
assumes *ae*: *AE x in M*. $\neg\ (x \in A \wedge x \in B)$ **and** [*measurable*]: $A \in$ *sets M B* $\in$ *sets M*
and *set_integrable M A f*
and *set_integrable M B f*
shows $(LINT\ x{:}A \cup B|M.\ f\ x) = (LINT\ x{:}A|M.\ f\ x) + (LINT\ x{:}B|M.\ f\ x)$
proof −
have *f*: *set_integrable M* $(A \cup B)$ *f*
by (*intro set_integrable_Un assms*)
then have *f'*: *set_borel_measurable M* $(A \cup B)$ *f*
using *integrable_iff_bounded set_borel_measurable_def set_integrable_def* **by**

```
  blast
  have (LINT x:A∪B|M. f x) = (LINT x:(A − A ∩ B) ∪ (B − A ∩ B)|M. f x)
  proof (rule set_integral_cong_set)
    show AE x in M. (x ∈ A − A ∩ B ∪ (B − A ∩ B)) = (x ∈ A ∪ B)
      using ae by auto
    show set_borel_measurable M (A − A ∩ B ∪ (B − A ∩ B)) f
      using f' by (rule set_borel_measurable_subset) auto
  qed fact
  also have ... = (LINT x:(A − A ∩ B)|M. f x) + (LINT x:(B − A ∩ B)|M. f
x)
    by (auto intro!: set_integral_Un set_integrable_subset[OF f])
  also have ... = (LINT x:A|M. f x) + (LINT x:B|M. f x)
    using ae
    by (intro arg_cong2[where f=(+)] set_integral_cong_set)
       (auto intro!: set_borel_measurable_subset[OF f'])
  finally show ?thesis .
qed

lemma set_integral_finite_Union:
  fixes f :: _ ⇒ _ :: {banach, second_countable_topology}
  assumes finite I disjoint_family_on A I
    and ⋀i. i ∈ I ⟹ set_integrable M (A i) f ⋀i. i ∈ I ⟹ A i ∈ sets M
  shows (LINT x:(⋃i∈I. A i)|M. f x) = (∑ i∈I. LINT x:A i|M. f x)
  using assms
proof induction
  case (insert x F)
  then have A x ∩ ⋃(A ` F) = {}
    by (meson disjoint_family_on_insert)
  with insert show ?case
     by (simp add: set_integral_Un set_integrable_Un set_integrable_UN dis-
joint_family_on_insert)
qed (simp add: set_lebesgue_integral_def)


lemma pos_integrable_to_top:
  fixes l::real
  assumes ⋀i. A i ∈ sets M mono A
  assumes nneg: ⋀x i. x ∈ A i ⟹ 0 ≤ f x
  and intgbl: ⋀i::nat. set_integrable M (A i) f
  and lim: (λi::nat. LINT x:A i|M. f x) ⟶ l
shows set_integrable M (⋃ i. A i) f
  unfolding set_integrable_def
  apply (rule integrable_monotone_convergence[where f = λi::nat. λx. indicator
(A i) x *R f x and x = l])
     apply (rule intgbl [unfolded set_integrable_def])
    prefer 3 apply (rule lim [unfolded set_lebesgue_integral_def])
   apply (rule AE_I2)
  using ‹mono A› apply (auto simp: mono_def nneg split: split_indicator) []
proof (rule AE_I2)
```

```
  { fix x assume x ∈ space M
    show (λi. indicator (A i) x *R f x) ⟶ indicator (⋃i. A i) x *R f x
    proof cases
      assume ∃i. x ∈ A i
      then obtain i where x ∈ A i ..
      then have ∀F i in sequentially. x ∈ A i
       using ‹x ∈ A i› ‹mono A› by (auto simp: eventually_sequentially mono_def)
       with eventually_mono have ∀F i in sequentially. indicat_real (A i) x *R f
x = indicat_real (⋃ (range A)) x *R f x
        by fastforce
      then show ?thesis
        by (intro tendsto_eventually)
    qed auto }
  then show (λx. indicator (⋃i. A i) x *R f x) ∈ borel_measurable M
    apply (rule borel_measurable_LIMSEQ_real)
     apply assumption
    using intgbl set_integrable_def by blast
qed
```

Proof from Royden, *Real Analysis*, p. 91.

```
lemma lebesgue_integral_countable_add:
  fixes f :: _ ⇒ 'a :: {banach, second_countable_topology}
  assumes meas[intro]: ⋀i::nat. A i ∈ sets M
    and disj: ⋀i j. i ≠ j ⟹ A i ∩ A j = {}
    and intgbl: set_integrable M (⋃i. A i) f
  shows (LINT x:(⋃i. A i)|M. f x) = (∑ i. (LINT x:(A i)|M. f x))
    unfolding set_lebesgue_integral_def
proof (subst integral_suminf[symmetric])
  show int_A: integrable M (λx. indicat_real (A i) x *R f x) for i
    using intgbl unfolding set_integrable_def [symmetric]
    by (rule set_integrable_subset) auto
  { fix x assume x ∈ space M
    have (λi. indicator (A i) x *R f x) sums (indicator (⋃i. A i) x *R f x)
      by (intro sums_scaleR_left indicator_sums) fact }
  note sums = this

  have norm_f: ⋀i. set_integrable M (A i) (λx. norm (f x))
    using int_A[THEN integrable_norm] unfolding set_integrable_def by auto

  show AE x in M. summable (λi. norm (indicator (A i) x *R f x))
   using disj by (intro AE_I2) (auto intro!: summable_mult2 sums_summable[OF
indicator_sums])

  show summable (λi. LINT x|M. norm (indicator (A i) x *R f x))
  proof (rule summableI_nonneg_bounded)
    fix n
    show 0 ≤ LINT x|M. norm (indicator (A n) x *R f x)
      using norm_f by (auto intro!: integral_nonneg_AE)
```

have $(\sum i{<}n.\ LINT\ x|M.\ norm\ (indicator\ (A\ i)\ x *_R f\ x)) = (\sum i{<}n.\ LINT\ x{:}A\ i|M.\ norm\ (f\ x))$
by (*simp add*: *abs_mult set_lebesgue_integral_def*)
also have $\ldots = set\_lebesgue\_integral\ M\ (\bigcup i{<}n.\ A\ i)\ (\lambda x.\ norm\ (f\ x))$
using *norm_f*
by (*subst set_integral_finite_Union*) (*auto simp*: *disjoint_family_on_def disj*)
also have $\ldots \le set\_lebesgue\_integral\ M\ (\bigcup i.\ A\ i)\ (\lambda x.\ norm\ (f\ x))$
using *intgbl*[*unfolded set_integrable_def*, *THEN integrable_norm*] *norm_f*
unfolding *set_lebesgue_integral_def set_integrable_def*
apply (*intro integral_mono set_integrable_UN*[*of* {..<*n*}, *unfolded set_integrable_def*])
apply (*auto split*: *split_indicator*)
done
finally show $(\sum i{<}n.\ LINT\ x|M.\ norm\ (indicator\ (A\ i)\ x *_R f\ x)) \le set\_lebesgue\_integral\ M\ (\bigcup i.\ A\ i)\ (\lambda x.\ norm\ (f\ x))$
by *simp*
qed
show $LINT\ x|M.\ indicator\ (\bigcup (A\ `\ UNIV))\ x *_R f\ x = LINT\ x|M.\ (\sum i.\ indicator\ (A\ i)\ x *_R f\ x)$
by (*metis* (*no_types*, *lifting*) *integral_cong sums sums_unique*)
qed

lemma *set_integral_cont_up*:
fixes $f :: \_ \Rightarrow 'a :: \{banach, second\_countable\_topology\}$
assumes [*measurable*]: $\bigwedge i.\ A\ i \in sets\ M$ **and** *A*: *incseq A*
and *intgbl*: *set_integrable M* $(\bigcup i.\ A\ i)\ f$
shows $(\lambda i.\ LINT\ x{:}(A\ i)|M.\ f\ x) \longrightarrow (LINT\ x{:}(\bigcup i.\ A\ i)|M.\ f\ x)$
unfolding *set_lebesgue_integral_def*
proof (*intro integral_dominated_convergence*[**where** $w{=}\lambda x.\ indicator\ (\bigcup i.\ A\ i)\ x *_R norm\ (f\ x)$])
have *int_A*: $\bigwedge i.\ set\_integrable\ M\ (A\ i)\ f$
using *intgbl* **by** (*rule set_integrable_subset*) *auto*
show $\bigwedge i.\ (\lambda x.\ indicator\ (A\ i)\ x *_R f\ x) \in borel\_measurable\ M$
using *int_A integrable_iff_bounded set_integrable_def* **by** *blast*
show $(\lambda x.\ indicator\ (\bigcup (A\ `\ UNIV))\ x *_R f\ x) \in borel\_measurable\ M$
using *integrable_iff_bounded intgbl set_integrable_def* **by** *blast*
show *integrable M* $(\lambda x.\ indicator\ (\bigcup i.\ A\ i)\ x *_R norm\ (f\ x))$
using *int_A intgbl integrable_norm* **unfolding** *set_integrable_def*
by *fastforce*
{ fix *x i* **assume** $x \in A\ i$
with *A* **have** $(\lambda xa.\ indicator\ (A\ xa)\ x{::}real) \longrightarrow 1 \longleftrightarrow (\lambda xa.\ 1{::}real) \longrightarrow 1$
by (*intro filterlim_cong refl*)
(*fastforce simp*: *eventually_sequentially incseq_def subset_eq intro*!: *exI*[*of* _ *i*]) **}**
then show $AE\ x\ in\ M.\ (\lambda i.\ indicator\ (A\ i)\ x *_R f\ x) \longrightarrow indicator\ (\bigcup i.\ A\ i)\ x *_R f\ x$
by (*intro AE_I2 tendsto_intros*) (*auto split*: *split_indicator*)
qed (*auto split*: *split_indicator*)

lemma *set_integral_cont_down*:
fixes $f :: \_ \Rightarrow 'a :: \{banach, second\_countable\_topology\}$
assumes [*measurable*]: $\bigwedge i.\ A\ i \in sets\ M$ **and** A: *decseq A*
and *int0*: *set_integrable M* $(A\ 0)\ f$
shows $(\lambda i{::}nat.\ LINT\ x{:}(A\ i)|M.\ f\ x) \longrightarrow (LINT\ x{:}(\bigcap i.\ A\ i)|M.\ f\ x)$
unfolding *set_lebesgue_integral_def*
proof (*rule integral_dominated_convergence*)
have *int_A*: $\bigwedge i.$ *set_integrable M* $(A\ i)\ f$
using *int0* **by** (*rule set_integrable_subset*) (*insert A*, *auto simp*: *decseq_def*)
have *integrable M* $(\lambda c.\ norm\ (indicat\_real\ (A\ 0)\ c *_R f\ c))$
by (*metis* (*no_types*) *int0 integrable_norm set_integrable_def*)
then show *integrable M* $(\lambda x.\ indicator\ (A\ 0)\ x *_R norm\ (f\ x))$
by *force*
have *set_integrable M* $(\bigcap i.\ A\ i)\ f$
using *int0* **by** (*rule set_integrable_subset*) (*insert A*, *auto simp*: *decseq_def*)
with *int_A* **show** $(\lambda x.\ indicat\_real\ (\bigcap (A\ `\ UNIV))\ x *_R f\ x) \in borel\_measurable\ M$
$\bigwedge i.\ (\lambda x.\ indicat\_real\ (A\ i)\ x *_R f\ x) \in borel\_measurable\ M$
by (*auto simp*: *set_integrable_def*)
show $\bigwedge i.$ *AE x in M*. $norm\ (indicator\ (A\ i)\ x *_R f\ x) \leq indicator\ (A\ 0)\ x *_R norm\ (f\ x)$
using *A* **by** (*auto split*: *split_indicator simp*: *decseq_def*)
{ fix *x i* **assume** $x \in space\ M\ x \notin A\ i$
with *A* **have** $(\lambda i.\ indicator\ (A\ i)\ x{::}real) \longrightarrow 0 \longleftrightarrow (\lambda i.\ 0{::}real) \longrightarrow 0$
by (*intro filterlim_cong refl*)
(*auto split*: *split_indicator simp*: *eventually_sequentially decseq_def intro*!: *exI*[*of* _ *i*]) **}**
then show *AE x in M*. $(\lambda i.\ indicator\ (A\ i)\ x *_R f\ x) \longrightarrow indicator\ (\bigcap i.\ A\ i)\ x *_R f\ x$
by (*intro AE_I2 tendsto_intros*) (*auto split*: *split_indicator*)
qed

lemma *set_integral_at_point*:
fixes $a :: real$
assumes *set_integrable M* $\{a\}\ f$
and [*simp*]: $\{a\} \in sets\ M$ **and** $(emeasure\ M)\ \{a\} \neq \infty$
shows $(LINT\ x{:}\{a\} \mid M.\ f\ x) = f\ a * measure\ M\ \{a\}$
proof−
have *set_lebesgue_integral M* $\{a\}\ f =$ *set_lebesgue_integral M* $\{a\}\ (\%x.\ f\ a)$
by (*intro set_lebesgue_integral_cong*) *simp_all*
then show *?thesis* **using** *assms*
unfolding *set_lebesgue_integral_def* **by** *simp*
qed

9.3 Complex integrals

abbreviation *complex_integrable* :: $'a\ measure \Rightarrow ('a \Rightarrow complex) \Rightarrow bool$ **where**

complex_integrable M f ≡ integrable M f

abbreviation *complex_lebesgue_integral* :: *'a measure ⇒ ('a ⇒ complex) ⇒ complex* (‹*integral*^{C}›) **where**
integral^C *M f* == *integral*^{L} *M f*

syntax
_complex_lebesgue_integral :: *pttrn ⇒ complex ⇒ 'a measure ⇒ complex*
(‹(‹*open_block notation=‹binder integral››* $\int^C$ *_. _ ∂_*)› [*0,0*] *110*)
syntax_consts
_complex_lebesgue_integral == *complex_lebesgue_integral*
translations
$\int^C x.\ f\ \partial M$ == *CONST complex_lebesgue_integral M* (*λx. f*)

syntax
_ascii_complex_lebesgue_integral :: *pttrn ⇒ 'a measure ⇒ real ⇒ real*
(‹(‹*indent=3 notation=‹binder CLINT››CLINT _|_. _*)› [*0,0,10*] *10*)
syntax_consts
_ascii_complex_lebesgue_integral == *complex_lebesgue_integral*
translations
CLINT x|M. f == *CONST complex_lebesgue_integral M* (*λx. f*)

lemma *complex_integrable_cnj* [*simp*]:
complex_integrable M (*λx. cnj* (*f x*)) ⟷ *complex_integrable M f*
proof
assume *complex_integrable M* (*λx. cnj* (*f x*))
then have *complex_integrable M* (*λx. cnj* (*cnj* (*f x*)))
by (*rule integrable_cnj*)
then show *complex_integrable M f*
by *simp*
qed *simp*

lemma *complex_of_real_integrable_eq*:
complex_integrable M (*λx. complex_of_real* (*f x*)) ⟷ *integrable M f*
proof
assume *complex_integrable M* (*λx. complex_of_real* (*f x*))
then have *integrable M* (*λx. Re* (*complex_of_real* (*f x*)))
by (*rule integrable_Re*)
then show *integrable M f*
by *simp*
qed *simp*

abbreviation *complex_set_integrable* :: *'a measure ⇒ 'a set ⇒ ('a ⇒ complex) ⇒ bool* **where**
complex_set_integrable M A f ≡ *set_integrable M A f*

abbreviation *complex_set_lebesgue_integral* :: *'a measure ⇒ 'a set ⇒ ('a ⇒ complex) ⇒ complex* **where**

complex_set_lebesgue_integral M A f ≡ set_lebesgue_integral M A f

syntax
_ascii_complex_set_lebesgue_integral :: pttrn ⇒ 'a set ⇒ 'a measure ⇒ real ⇒ real
(‹(‹*indent=4 notation=‹binder CLINT››CLINT _:_|_. _*)› [*0,0,0,10*] *10*)
syntax_consts
_ascii_complex_set_lebesgue_integral == complex_set_lebesgue_integral
translations
CLINT x:A|M. f == CONST complex_set_lebesgue_integral M A (λx. f)

lemma *set_measurable_continuous_on_ivl*:
assumes *continuous_on {a..b} (f :: real ⇒ real)*
shows *set_borel_measurable borel {a..b} f*
unfolding *set_borel_measurable_def*
by (*rule borel_measurable_continuous_on_indicator*[*OF _ assms*]) *simp*

9.4 NN Set Integrals

This notation is from Sébastien Gouëzel: His use is not directly in line with the notations in this file, they are more in line with sum, and more readable he thinks.

abbreviation *set_nn_integral M A f ≡ nn_integral M (λx. f x * indicator A x)*

syntax
_set_nn_integral :: pttrn => 'a set => 'a measure => ereal => ereal
(‹(‹*notation=‹binder integral›*›∫⁺((_)∈(_)./ _)/∂_)› [*0,0,0,110*] *10*)
_set_lebesgue_integral :: pttrn => 'a set => 'a measure => ereal => ereal
(‹(‹*notation=‹binder integral›*›∫((_)∈(_)./ _)/∂_)› [*0,0,0,110*] *10*)
syntax_consts
_set_nn_integral == set_nn_integral **and**
_set_lebesgue_integral == set_lebesgue_integral
translations
∫⁺x ∈ A. f ∂M == CONST set_nn_integral M A (λx. f)
∫ x ∈ A. f ∂M == CONST set_lebesgue_integral M A (λx. f)

lemma *set_nn_integral_cong*:
assumes *M = M' A = B ⋀x. x ∈ space M ∩ A ⟹ f x = g x*
shows *set_nn_integral M A f = set_nn_integral M' B g*
by (*metis* (*mono_tags*, *lifting*) *IntI assms indicator_simps*(*2*) *mult_eq_0_iff nn_integral_cong*)

lemma *nn_integral_disjoint_pair*:
assumes [*measurable*]: *f ∈ borel_measurable M*
B ∈ sets M C ∈ sets M
B ∩ C = {}
shows *(∫⁺x ∈ B ∪ C. f x ∂M) = (∫⁺x ∈ B. f x ∂M) + (∫⁺x ∈ C. f x ∂M)*
proof −

have *mes*: $\bigwedge D.\ D \in sets\ M \Longrightarrow (\lambda x.\ f\ x * indicator\ D\ x) \in borel\_measurable\ M$
by *simp*
have *pos*: $\bigwedge D.\ AE\ x\ in\ M.\ f\ x * indicator\ D\ x \geq 0$ **using** *assms*(2) **by** *auto*
have $\bigwedge x.\ f\ x * indicator\ (B \cup C)\ x = f\ x * indicator\ B\ x + f\ x * indicator\ C\ x$
using *assms*(4)
by (*auto split*: *split_indicator*)
then have $(\int^+ x.\ f\ x * indicator\ (B \cup C)\ x\ \partial M) = (\int^+ x.\ f\ x * indicator\ B\ x + f\ x * indicator\ C\ x\ \partial M)$
by *simp*
also have $\ldots = (\int^+ x.\ f\ x * indicator\ B\ x\ \partial M) + (\int^+ x.\ f\ x * indicator\ C\ x\ \partial M)$
by (*rule nn_integral_add*) (*auto simp add*: *assms mes pos*)
finally show *?thesis* **by** *simp*
qed

lemma *nn_integral_disjoint_pair_countspace*:
assumes $B \cap C = \{\}$
shows $(\int^+ x \in B \cup C.\ f\ x\ \partial count\_space\ UNIV) = (\int^+ x \in B.\ f\ x\ \partial count\_space\ UNIV) + (\int^+ x \in C.\ f\ x\ \partial count\_space\ UNIV)$
by (*rule nn_integral_disjoint_pair*) (*simp_all add*: *assms*)

lemma *nn_integral_null_delta*:
assumes $A \in sets\ M\ B \in sets\ M$
$(A - B) \cup (B - A) \in null\_sets\ M$
shows $(\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in B.\ f\ x\ \partial M)$
proof (*rule nn_integral_cong_AE*)
have $*$: $AE\ a\ in\ M.\ a \notin (A - B) \cup (B - A)$
using *assms*(3) *AE_not_in* **by** *blast*
then show ‹$AE\ x\ in\ M.\ f\ x * indicator\ A\ x = f\ x * indicator\ B\ x$›
by *auto*
qed

proposition *nn_integral_disjoint_family*:
assumes [*measurable*]: $f \in borel\_measurable\ M\ \bigwedge(n::nat).\ B\ n \in sets\ M$
and *disjoint_family B*
shows $(\int^+ x \in (\bigcup n.\ B\ n).\ f\ x\ \partial M) = (\sum n.\ (\int^+ x \in B\ n.\ f\ x\ \partial M))$
proof −
have $(\int^+ x.\ (\sum n.\ f\ x * indicator\ (B\ n)\ x)\ \partial M) = (\sum n.\ (\int^+ x.\ f\ x * indicator\ (B\ n)\ x\ \partial M))$
by (*rule nn_integral_suminf*) *simp*
moreover have $(\sum n.\ f\ x * indicator\ (B\ n)\ x) = f\ x * indicator\ (\bigcup n.\ B\ n)\ x$
for x
proof (*cases*)
assume $x \in (\bigcup n.\ B\ n)$
then obtain n **where** $x \in B\ n$ **by** *blast*
have *a*: *finite* $\{n\}$ **by** *simp*
have $\bigwedge i.\ i \neq n \Longrightarrow x \notin B\ i$ **using** ‹$x \in B\ n$› *assms*(3) *disjoint_family_on_def*
by (*metis IntI UNIV_I empty_iff*)
then have $\bigwedge i.\ i \notin \{n\} \Longrightarrow indicator\ (B\ i)\ x = (0::ennreal)$ **using** *indicator_def*

by *simp*
  then have *b*: $\bigwedge i.\ i \notin \{n\} \Longrightarrow f\ x * \mathit{indicator}\ (B\ i)\ x = (0::\mathit{ennreal})$ **by** *simp*

  define *h* **where** $h = (\lambda i.\ f\ x * \mathit{indicator}\ (B\ i)\ x)$
  then have $\bigwedge i.\ i \notin \{n\} \Longrightarrow h\ i = 0$ **using** *b* **by** *simp*
  then have $(\sum i.\ h\ i) = (\sum i{\in}\{n\}.\ h\ i)$
    by (*metis sums_unique*[*OF sums_finite*[*OF a*]])
  then have $(\sum i.\ h\ i) = h\ n$ **by** *simp*
  then have $(\sum n.\ f\ x * \mathit{indicator}\ (B\ n)\ x) = f\ x * \mathit{indicator}\ (B\ n)\ x$ **using** *h_def* **by** *simp*
  then have $(\sum n.\ f\ x * \mathit{indicator}\ (B\ n)\ x) = f\ x$ **using** ‹$x \in B\ n$› *indicator_def* **by** *simp*
  then show *?thesis* **using** ‹$x \in (\bigcup n.\ B\ n)$› **by** *auto*
 next
  assume $x \notin (\bigcup n.\ B\ n)$
  then have $\bigwedge n.\ f\ x * \mathit{indicator}\ (B\ n)\ x = 0$ **by** *simp*
  have $(\sum n.\ f\ x * \mathit{indicator}\ (B\ n)\ x) = 0$
    by (*simp add*: ‹$\bigwedge n.\ f\ x * \mathit{indicator}\ (B\ n)\ x = 0$›)
  then show *?thesis* **using** ‹$x \notin (\bigcup n.\ B\ n)$› **by** *auto*
 qed
 ultimately show *?thesis* **by** *simp*
qed

lemma *nn_set_integral_add*:
 assumes [*measurable*]: $f \in \mathit{borel\_measurable}\ M\ g \in \mathit{borel\_measurable}\ M$
      $A \in \mathit{sets}\ M$
 shows $(\int^+ x \in A.\ (f\ x + g\ x)\ \partial M) = (\int^+ x \in A.\ f\ x\ \partial M) + (\int^+ x \in A.\ g\ x\ \partial M)$
proof −
 have $(\int^+ x \in A.\ (f\ x + g\ x)\ \partial M) = (\int^+ x.\ (f\ x * \mathit{indicator}\ A\ x + g\ x * \mathit{indicator}\ A\ x)\ \partial M)$
   by (*auto simp add*: *indicator_def intro*!: *nn_integral_cong*)
 also have $\ldots = (\int^+ x.\ f\ x * \mathit{indicator}\ A\ x\ \partial M) + (\int^+ x.\ g\ x * \mathit{indicator}\ A\ x\ \partial M)$
   apply (*rule nn_integral_add*) **using** *assms*(*1*) *assms*(*2*) **by** *auto*
 finally show *?thesis* **by** *simp*
qed

lemma *nn_set_integral_cong*:
 assumes *AE x in M*. $f\ x = g\ x$
 shows $(\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in A.\ g\ x\ \partial M)$
apply (*rule nn_integral_cong_AE*) **using** *assms*(*1*) **by** *auto*

lemma *nn_set_integral_set_mono*:
 $A \subseteq B \Longrightarrow (\int^+ x \in A.\ f\ x\ \partial M) \leq (\int^+ x \in B.\ f\ x\ \partial M)$
by (*auto intro*!: *nn_integral_mono split*: *split_indicator*)

lemma *nn_set_integral_mono*:
 assumes [*measurable*]: $f \in \mathit{borel\_measurable}\ M\ g \in \mathit{borel\_measurable}\ M$
      $A \in \mathit{sets}\ M$

and *AE x∈A in M. f x ≤ g x*
shows $(\int^+ x \in A.\ f\ x\ \partial M) \leq (\int^+ x \in A.\ g\ x\ \partial M)$
by (*auto intro*!: *nn_integral_mono_AE split*: *split_indicator simp*: *assms*)

lemma *nn_set_integral_space* [*simp*]:
shows $(\int^+ x \in \mathit{space}\ M.\ f\ x\ \partial M) = (\int^+ x.\ f\ x\ \partial M)$
by (*metis* (*mono_tags*, *lifting*) *indicator_simps*(*1*) *mult.right_neutral nn_integral_cong*)

lemma *nn_integral_count_compose_inj*:
assumes *inj_on g A*
shows $(\int^+ x \in A.\ f\ (g\ x)\ \partial \mathit{count\_space}\ \mathit{UNIV}) = (\int^+ y \in g`A.\ f\ y\ \partial \mathit{count\_space}\ \mathit{UNIV})$
proof −
have $(\int^+ x \in A.\ f\ (g\ x)\ \partial \mathit{count\_space}\ \mathit{UNIV}) = (\int^+ x.\ f\ (g\ x)\ \partial \mathit{count\_space}\ A)$
by (*auto simp add*: *nn_integral_count_space_indicator*[*symmetric*])
also have $\ldots = (\int^+ y.\ f\ y\ \partial \mathit{count\_space}\ (g`A))$
by (*simp add*: *assms nn_integral_bij_count_space inj_on_imp_bij_betw*)
also have $\ldots = (\int^+ y \in g`A.\ f\ y\ \partial \mathit{count\_space}\ \mathit{UNIV})$
by (*auto simp add*: *nn_integral_count_space_indicator*[*symmetric*])
finally show *?thesis* **by** *simp*
qed

lemma *nn_integral_count_compose_bij*:
assumes *bij_betw g A B*
shows $(\int^+ x \in A.\ f\ (g\ x)\ \partial \mathit{count\_space}\ \mathit{UNIV}) = (\int^+ y \in B.\ f\ y\ \partial \mathit{count\_space}\ \mathit{UNIV})$
proof −
have *inj_on g A* **using** *assms bij_betw_def* **by** *auto*
then have $(\int^+ x \in A.\ f\ (g\ x)\ \partial \mathit{count\_space}\ \mathit{UNIV}) = (\int^+ y \in g`A.\ f\ y\ \partial \mathit{count\_space}\ \mathit{UNIV})$
by (*rule nn_integral_count_compose_inj*)
then show *?thesis* **using** *assms* **by** (*simp add*: *bij_betw_def*)
qed

lemma *set_integral_null_delta*:
fixes *f*::_ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes [*measurable*]: *integrable M f A ∈ sets M B ∈ sets M*
and *null*: $(A - B) \cup (B - A) \in \mathit{null\_sets}\ M$
shows $(\int x \in A.\ f\ x\ \partial M) = (\int x \in B.\ f\ x\ \partial M)$
proof (*rule set_integral_cong_set*)
have ∗: *AE a in M. a ∉ (A − B) ∪ (B − A)*
using *null AE_not_in* **by** *blast*
then show *AE x in M.* $(x \in B) = (x \in A)$
by *auto*
qed (*simp_all add*: *set_borel_measurable_def*)

lemma *set_integral_space*:
assumes *integrable M f*
shows $(\int x \in \mathit{space}\ M.\ f\ x\ \partial M) = (\int x.\ f\ x\ \partial M)$

by (*metis* (*no_types*, *lifting*) *indicator_simps*(*1*) *integral_cong scaleR_one set_lebesgue_integral_def*)

lemma *null_if_pos_func_has_zero_nn_int*:
fixes $f::'a \Rightarrow ennreal$
assumes [*measurable*]: $f \in borel\_measurable\ M\ A \in sets\ M$
and $AE\ x{\in}A\ in\ M.\ f\ x > 0\ (\int^+ x{\in}A.\ f\ x\ \partial M) = 0$
shows $A \in null\_sets\ M$
proof −
have $AE\ x\ in\ M.\ f\ x * indicator\ A\ x = 0$
by (*subst nn_integral_0_iff_AE*[*symmetric*], *auto simp add*: *assms*(*4*))
then have $AE\ x{\in}A\ in\ M.\ False$ **using** *assms*(*3*) **by** *auto*
then show $A \in null\_sets\ M$ **using** *assms*(*2*) **by** (*simp add*: *AE_iff_null_sets*)
qed

lemma *null_if_pos_func_has_zero_int*:
assumes [*measurable*]: $integrable\ M\ f\ A \in sets\ M$
and $AE\ x{\in}A\ in\ M.\ f\ x > 0\ (\int x{\in}A.\ f\ x\ \partial M) = (0::real)$
shows $A \in null\_sets\ M$
proof −
have $AE\ x\ in\ M.\ indicator\ A\ x * f\ x = 0$
apply (*subst integral_nonneg_eq_0_iff_AE*[*symmetric*])
using *assms integrable_mult_indicator*[*OF* ‹$A \in sets\ M$› *assms*(*1*)]
by (*auto simp*: *set_lebesgue_integral_def*)
then have $AE\ x{\in}A\ in\ M.\ f\ x = 0$ **by** *auto*
then have $AE\ x{\in}A\ in\ M.\ False$ **using** *assms*(*3*) **by** *auto*
then show $A \in null\_sets\ M$ **using** *assms*(*2*) **by** (*simp add*: *AE_iff_null_sets*)
qed

The next lemma is a variant of *density_unique*. Note that it uses the notation for nonnegative set integrals introduced earlier.

lemma (**in** *sigma_finite_measure*) *density_unique2*:
assumes [*measurable*]: $f \in borel\_measurable\ M\ f' \in borel\_measurable\ M$
assumes *density_eq*: $\bigwedge A.\ A \in sets\ M \Longrightarrow (\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in A.\ f'\ x\ \partial M)$
shows $AE\ x\ in\ M.\ f\ x = f'\ x$
proof (*rule density_unique*)
show $density\ M\ f = density\ M\ f'$
by (*intro measure_eqI*) (*auto simp*: *emeasure_density intro*!: *density_eq*)
qed (*auto simp add*: *assms*)

The next lemma implies the same statement for Banach-space valued functions using Hahn-Banach theorem and linear forms. Since they are not yet easily available, I only formulate it for real-valued functions.

lemma *density_unique_real*:
fixes $f\ f'::\_ \Rightarrow real$
assumes M[*measurable*]: $integrable\ M\ f\ integrable\ M\ f'$
assumes *density_eq*: $\bigwedge A.\ A \in sets\ M \Longrightarrow (\int x \in A.\ f\ x\ \partial M) = (\int x \in A.\ f'\ x\ \partial M)$
shows $AE\ x\ in\ M.\ f\ x = f'\ x$

proof −
define A **where** $A = \{x \in \mathit{space}\ M.\ f\ x < f'\ x\}$
then have [*measurable*]: $A \in \mathit{sets}\ M$ **by** *simp*
have $(\int x \in A.\ (f'\ x - f\ x)\ \partial M) = (\int x \in A.\ f'\ x\ \partial M) - (\int x \in A.\ f\ x\ \partial M)$
using ‹$A \in \mathit{sets}\ M$› M *integrable_mult_indicator set_integrable_def* **by** *blast*
then have $(\int x \in A.\ (f'\ x - f\ x)\ \partial M) = 0$ **using** *assms*(*3*) **by** *simp*
then have $A \in \mathit{null\_sets}\ M$
using *A_def null_if_pos_func_has_zero_int*[**where** *?f* = $\lambda x.\ f'\ x - f\ x$ **and** *?A* = A] *assms* **by** *auto*
then have *AE x in M*. $x \notin A$ **by** (*simp add*: *AE_not_in*)
then have $*$: *AE x in M*. $f'\ x \leq f\ x$ **unfolding** *A_def* **by** *auto*

define B **where** $B = \{x \in \mathit{space}\ M.\ f'\ x < f\ x\}$
then have [*measurable*]: $B \in \mathit{sets}\ M$ **by** *simp*
have $(\int x \in B.\ (f\ x - f'\ x)\ \partial M) = (\int x \in B.\ f\ x\ \partial M) - (\int x \in B.\ f'\ x\ \partial M)$
using ‹$B \in \mathit{sets}\ M$› M *integrable_mult_indicator set_integrable_def* **by** *blast*
then have $(\int x \in B.\ (f\ x - f'\ x)\ \partial M) = 0$ **using** *assms*(*3*) **by** *simp*
then have $B \in \mathit{null\_sets}\ M$
using *B_def null_if_pos_func_has_zero_int*[**where** *?f* = $\lambda x.\ f\ x - f'\ x$ **and** *?A* = B] *assms* **by** *auto*
then have *AE x in M*. $x \notin B$ **by** (*simp add*: *AE_not_in*)
then have *AE x in M*. $f'\ x \geq f\ x$ **unfolding** *B_def* **by** *auto*
then show *?thesis* **using** $*$ **by** *auto*
qed

9.5 Scheffé's lemma

The next lemma shows that L^1 convergence of a sequence of functions follows from almost everywhere convergence and the weaker condition of the convergence of the integrated norms (or even just the nontrivial inequality about them). Useful in a lot of contexts! This statement (or its variations) are known as Scheffe lemma.

The formalization is more painful as one should jump back and forth between reals and ereals and justify all the time positivity or integrability (thankfully, measurability is handled more or less automatically).

proposition *Scheffe_lemma1*:
assumes $\bigwedge n.$ *integrable M* (*F n*) *integrable M f*
AE x in M. $(\lambda n.\ F\ n\ x) \longrightarrow f\ x$
limsup $(\lambda n.\ \int^+ x.\ \mathit{norm}(F\ n\ x)\ \partial M) \leq (\int^+ x.\ \mathit{norm}(f\ x)\ \partial M)$
shows $(\lambda n.\ \int^+ x.\ \mathit{norm}(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$
proof −
have [*measurable*]: $\bigwedge n.\ F\ n \in \mathit{borel\_measurable}\ M\ f \in \mathit{borel\_measurable}\ M$
using *assms*(*1*) *assms*(*2*) **by** *simp_all*
define G **where** $G = (\lambda n\ x.\ \mathit{norm}(f\ x) + \mathit{norm}(F\ n\ x) - \mathit{norm}(F\ n\ x - f\ x))$
have [*measurable*]: $\bigwedge n.\ G\ n \in \mathit{borel\_measurable}\ M$ **unfolding** *G_def* **by** *simp*
have *G_pos*[*simp*]: $\bigwedge n\ x.\ G\ n\ x \geq 0$
unfolding *G_def* **by** (*metis ge_iff_diff_ge_0 norm_minus_commute norm_triangle_ineq4*)

have *finint*: $(\int^+ x.\ norm(f\ x)\ \partial M) \neq \infty$
using *has_bochner_integral_implies_finite_norm*[*OF has_bochner_integral_integrable*[*OF* ‹*integrable M f*›]]
by *simp*
then have *fin2*: $2 * (\int^+ x.\ norm(f\ x)\ \partial M) \neq \infty$
by (*auto simp*: *ennreal_mult_eq_top_iff*)

{
fix x **assume** $*$: $(\lambda n.\ F\ n\ x) \longrightarrow f\ x$
then have $(\lambda n.\ norm(F\ n\ x)) \longrightarrow norm(f\ x)$ **using** *tendsto_norm* **by** *blast*
moreover have $(\lambda n.\ norm(F\ n\ x - f\ x)) \longrightarrow 0$ **using** $*$ *Lim_null tendsto_norm_zero_iff* **by** *fastforce*
ultimately have *a*: $(\lambda n.\ norm(F\ n\ x) - norm(F\ n\ x - f\ x)) \longrightarrow norm(f\ x)$ **using** *tendsto_diff* **by** *fastforce*
have $(\lambda n.\ norm(f\ x) + (norm(F\ n\ x) - norm(F\ n\ x - f\ x))) \longrightarrow norm(f\ x) + norm(f\ x)$
by (*rule tendsto_add*) (*auto simp add*: *a*)
moreover have $\bigwedge n.\ G\ n\ x = norm(f\ x) + (norm(F\ n\ x) - norm(F\ n\ x - f\ x))$ **unfolding** *G_def* **by** *simp*
ultimately have $(\lambda n.\ G\ n\ x) \longrightarrow 2 * norm(f\ x)$ **by** *simp*
then have $(\lambda n.\ ennreal(G\ n\ x)) \longrightarrow ennreal(2 * norm(f\ x))$ **by** *simp*
then have *liminf* $(\lambda n.\ ennreal(G\ n\ x)) = ennreal(2 * norm(f\ x))$
using *sequentially_bot tendsto_iff_Liminf_eq_Limsup* **by** *blast*
}
then have *AE x in M*. *liminf* $(\lambda n.\ ennreal(G\ n\ x)) = ennreal(2 * norm(f\ x))$ **using** *assms*(*3*) **by** *auto*
then have $(\int^+ x.\ liminf\ (\lambda n.\ ennreal\ (G\ n\ x))\ \partial M) = (\int^+ x.\ 2 * ennreal(norm(f\ x))\ \partial M)$
by (*simp add*: *nn_integral_cong_AE ennreal_mult*)
also have $\ldots = 2 * (\int^+ x.\ norm(f\ x)\ \partial M)$ **by** (*rule nn_integral_cmult*) *auto*
finally have *int_liminf*: $(\int^+ x.\ liminf\ (\lambda n.\ ennreal\ (G\ n\ x))\ \partial M) = 2 * (\int^+ x.\ norm(f\ x)\ \partial M)$
by *simp*

have $(\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M) = (\int^+ x.\ norm(f\ x)\ \partial M) + (\int^+ x.\ norm(F\ n\ x)\ \partial M)$ **for** *n*
by (*rule nn_integral_add*) (*auto simp add*: *assms*)
then have *limsup* $(\lambda n.\ (\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M)) =$
limsup $(\lambda n.\ (\int^+ x.\ norm(f\ x)\ \partial M) + (\int^+ x.\ norm(F\ n\ x)\ \partial M))$
by *simp*
also have $\ldots = (\int^+ x.\ norm(f\ x)\ \partial M) + limsup\ (\lambda n.\ (\int^+ x.\ norm(F\ n\ x)\ \partial M))$
by (*rule Limsup_const_add*, *auto simp add*: *finint*)
also have $\ldots \leq (\int^+ x.\ norm(f\ x)\ \partial M) + (\int^+ x.\ norm(f\ x)\ \partial M)$
using *assms*(*4*) **by** (*simp add*: *add_left_mono*)
also have $\ldots = 2 * (\int^+ x.\ norm(f\ x)\ \partial M)$
unfolding *one_add_one*[*symmetric*] *distrib_right* **by** *simp*
ultimately have *a*: *limsup* $(\lambda n.\ (\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M)) \leq$

$2 * (\int^+ x.\ norm(f\ x)\ \partial M)$ **by** *simp*

have *le*: *ennreal* $(norm\ (F\ n\ x - f\ x)) \leq ennreal\ (norm\ (f\ x)) + ennreal\ (norm\ (F\ n\ x))$ **for** $n\ x$
by (*simp add*: *norm_minus_commute norm_triangle_ineq4 ennreal_minus flip*: *ennreal_plus*)
then have *le2*: $(\int^+ x.\ ennreal\ (norm\ (F\ n\ x - f\ x))\ \partial M) \leq (\int^+ x.\ ennreal\ (norm\ (f\ x)) + ennreal\ (norm\ (F\ n\ x))\ \partial M)$ **for** n
by (*rule nn_integral_mono*)

have $2 * (\int^+ x.\ norm(f\ x)\ \partial M) = (\int^+ x.\ liminf\ (\lambda n.\ ennreal\ (G\ n\ x))\ \partial M)$
by (*simp add*: *int_liminf*)
also have $\ldots \leq liminf\ (\lambda n.\ (\int^+ x.\ G\ n\ x\ \partial M))$
by (*rule nn_integral_liminf*) *auto*
also have $liminf\ (\lambda n.\ (\int^+ x.\ G\ n\ x\ \partial M)) =$
$liminf\ (\lambda n.\ (\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M) - (\int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M))$
proof (*intro arg_cong*[**where** f=*liminf*] *ext*)
fix n
have $\bigwedge x.\ ennreal(G\ n\ x) = ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x)) - ennreal(norm(F\ n\ x - f\ x))$
unfolding *G_def* **by** (*simp add*: *ennreal_minus flip*: *ennreal_plus*)
moreover have $(\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x)) - ennreal(norm(F\ n\ x - f\ x))\ \partial M)$
$= (\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M) - (\int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M)$
proof (*rule nn_integral_diff*)
from *le* **show** *AE x in M*. *ennreal* $(norm\ (F\ n\ x - f\ x)) \leq ennreal\ (norm\ (f\ x)) + ennreal\ (norm\ (F\ n\ x))$
by *simp*
from *le2* **have** $(\int^+ x.\ ennreal\ (norm\ (F\ n\ x - f\ x))\ \partial M) < \infty$ **using** *assms*(*1*) *assms*(*2*)
by (*metis has_bochner_integral_implies_finite_norm integrable.simps Bochner_Integration.integrable_diff*)
then show $(\int^+ x.\ ennreal\ (norm\ (F\ n\ x - f\ x))\ \partial M) \neq \infty$ **by** *simp*
qed (*auto simp add*: *assms*)
ultimately show $(\int^+ x.\ G\ n\ x\ \partial M) = (\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M) - (\int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M)$
by *simp*
qed
finally have $2 * (\int^+ x.\ norm(f\ x)\ \partial M) + limsup\ (\lambda n.\ (\int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M)) \leq$
$liminf\ (\lambda n.\ (\int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M) - (\int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M)) +$
$limsup\ (\lambda n.\ (\int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M))$
by (*intro add_mono*) *auto*
also have $\ldots \leq (limsup\ (\lambda n.\ \int^+ x.\ ennreal(norm(f\ x)) + ennreal(norm(F\ n\ x))\ \partial M) - limsup\ (\lambda n.\ \int^+ x.\ norm\ (F\ n\ x - f\ x)\ \partial M)) +$
$limsup\ (\lambda n.\ (\int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M))$

by (*intro add_mono liminf_minus_ennreal le2*) *auto*
also have ... = *limsup* (λ*n*. (∫⁺*x*. *ennreal*(*norm*(*f x*)) + *ennreal*(*norm*(*F n x*)) ∂*M*))
by (*intro diff_add_cancel_ennreal Limsup_mono always_eventually allI le2*)
also have ... ≤ *2* ∗ (∫⁺*x*. *norm*(*f x*) ∂*M*)
by *fact*
finally have *limsup* (λ*n*. (∫⁺*x*. *norm*(*F n x* − *f x*) ∂*M*)) = *0*
using *fin2* **by** *simp*
then show *?thesis*
by (*rule tendsto_0_if_Limsup_eq_0_ennreal*)
qed

proposition *Scheffe_lemma2*:
fixes *F*::*nat* ⇒ 'a ⇒ 'b::{*banach*, *second_countable_topology*}
assumes ⋀ *n*::*nat*. *F n* ∈ *borel_measurable M integrable M f*
AE x in M. (λ*n*. *F n x*) ⟶ *f x*
⋀*n*. (∫⁺ *x*. *norm*(*F n x*) ∂*M*) ≤ (∫⁺ *x*. *norm*(*f x*) ∂*M*)
shows (λ*n*. ∫⁺ *x*. *norm*(*F n x* − *f x*) ∂*M*) ⟶ *0*
proof (*rule Scheffe_lemma1*)
fix *n*::*nat*
have (∫⁺ *x*. *norm*(*f x*) ∂*M*) < ∞ **using** *assms*(*2*) **by** (*metis has_bochner_integral_implies_finite_norm integrable.cases*)
then have (∫⁺ *x*. *norm*(*F n x*) ∂*M*) < ∞ **using** *assms*(*4*)[*of n*] **by** *auto*
then show *integrable M* (*F n*) **by** (*subst integrable_iff_bounded*, *simp add*: *assms*(*1*)[*of n*])
qed (*auto simp add*: *assms Limsup_bounded*)

lemma *tendsto_set_lebesgue_integral_at_right*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ 'a :: {*banach*,*second_countable_topology*}
assumes *a* < *b* **and** *sets*: ⋀*a'*. *a'* ∈ {*a*<..*b*} ⟹ {*a'*..*b*} ∈ *sets M*
and *set_integrable M* {*a*<..*b*} *f*
shows ((λ*a'*. *set_lebesgue_integral M* {*a'*..*b*} *f*) ⟶
set_lebesgue_integral M {*a*<..*b*} *f*) (*at_right a*)
proof (*rule tendsto_at_right_sequentially*[*OF assms*(*1*)], *goal_cases*)
case (*1 S*)
have *eq*: (⋃ *n*. {*S n*..*b*}) = {*a*<..*b*}
proof *safe*
fix *x n* **assume** *x* ∈ {*S n*..*b*}
with *1*(*1*,*2*)[*of n*] **show** *x* ∈ {*a*<..*b*} **by** *auto*
next
fix *x* **assume** *x* ∈ {*a*<..*b*}
with *order_tendstoD*[*OF* ‹*S* ⟶ *a*›, *of x*] **show** *x* ∈ (⋃ *n*. {*S n*..*b*})
by (*force simp*: *eventually_at_top_linorder dest*: *less_imp_le*)
qed
have (λ*n*. *set_lebesgue_integral M* {*S n*..*b*} *f*) ⟶ *set_lebesgue_integral M* (⋃ *n*. {*S n*..*b*}) *f*
by (*rule set_integral_cont_up*) (*insert assms 1*, *auto simp*: *eq incseq_def decseq_def less_imp_le*)
with *eq* **show** *?case* **by** *simp*

qed

9.6 Convergence of integrals over an interval

The next lemmas relate convergence of integrals over an interval to improper integrals.

lemma *tendsto_set_lebesgue_integral_at_left*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ *'a* :: {*banach*,*second_countable_topology*}
assumes *a* < *b* **and** *sets*: ⋀*b'*. *b'* ∈ {*a*..<*b*} ⟹ {*a*..*b'*} ∈ *sets M*
and *set_integrable M* {*a*..<*b*} *f*
shows ((λ*b'*. *set_lebesgue_integral M* {*a*..*b'*} *f*) ⟶
set_lebesgue_integral M {*a*..<*b*} *f*) (*at_left b*)
proof (*rule tendsto_at_left_sequentially*[*OF assms*(*1*)], *goal_cases*)
case (*1 S*)
have *eq*: (⋃ *n*. {*a*..*S n*}) = {*a*..<*b*}
proof *safe*
fix *x n* **assume** *x* ∈ {*a*..*S n*}
with *1*(*1*,*2*)[*of n*] **show** *x* ∈ {*a*..<*b*} **by** *auto*
next
fix *x* **assume** *x* ∈ {*a*..<*b*}
with *order_tendstoD*[*OF* ‹*S* ⟶ *b*›, *of x*] **show** *x* ∈ (⋃ *n*. {*a*..*S n*})
by (*force simp*: *eventually_at_top_linorder dest*: *less_imp_le*)
qed
have (λ*n*. *set_lebesgue_integral M* {*a*..*S n*} *f*) ⟶ *set_lebesgue_integral M* (⋃ *n*. {*a*..*S n*}) *f*
by (*rule set_integral_cont_up*) (*insert assms 1*, *auto simp*: *eq incseq_def decseq_def less_imp_le*)
with *eq* **show** *?case* **by** *simp*
qed

proposition *tendsto_set_lebesgue_integral_at_top*:
fixes *f* :: *real* ⇒ *'a*::{*banach*, *second_countable_topology*}
assumes *sets*: ⋀*b*. *b* ≥ *a* ⟹ {*a*..*b*} ∈ *sets M*
and *int*: *set_integrable M* {*a*..} *f*
shows ((λ*b*. *set_lebesgue_integral M* {*a*..*b*} *f*) ⟶ *set_lebesgue_integral M* {*a*..} *f*) *at_top*
proof (*rule tendsto_at_topI_sequentially*)
fix *X* :: *nat* ⇒ *real* **assume** *filterlim X at_top sequentially*
show (λ*n*. *set_lebesgue_integral M* {*a*..*X n*} *f*) ⟶ *set_lebesgue_integral M* {*a*..} *f*
unfolding *set_lebesgue_integral_def*
proof (*rule integral_dominated_convergence*)
show *integrable M* (λ*x*. *indicat_real* {*a*..} *x* ∗*R* *norm* (*f x*))
using *integrable_norm*[*OF int*[*unfolded set_integrable_def*]] **by** *simp*
show *AE x in M*. (λ*n*. *indicator* {*a*..*X n*} *x* ∗*R* *f x*) ⟶ *indicat_real* {*a*..} *x* ∗*R* *f x*
proof
fix *x*

```
    from ‹filterlim X at_top sequentially›
    have eventually (λn. x ≤ X n) sequentially
      unfolding filterlim_at_top_ge[where c=x] by auto
    then show (λn. indicator {a..X n} x *R f x) ⟶ indicat_real {a..} x *R
f x
        by (intro tendsto_eventually) (auto split: split_indicator elim!: eventu-
ally_mono)
  qed
  fix n show AE x in M. norm (indicator {a..X n} x *R f x) ≤
                        indicator {a..} x *R norm (f x)
    by (auto split: split_indicator)
 next
  from int show (λx. indicat_real {a..} x *R f x) ∈ borel_measurable M
    by (simp add: set_integrable_def)
 next
  fix n :: nat
  from sets have {a..X n} ∈ sets M by (cases X n ≥ a) auto
  with int have set_integrable M {a..X n} f
    by (rule set_integrable_subset) auto
  thus (λx. indicat_real {a..X n} x *R f x) ∈ borel_measurable M
    by (simp add: set_integrable_def)
 qed
qed

proposition tendsto_set_lebesgue_integral_at_bot:
 fixes f :: real ⇒ 'a::{banach, second_countable_topology}
 assumes sets: ⋀a. a ≤ b ⟹ {a..b} ∈ sets M
    and int: set_integrable M {..b} f
  shows ((λa. set_lebesgue_integral M {a..b} f) ⟶ set_lebesgue_integral M
{..b} f) at_bot
proof (rule tendsto_at_botI_sequentially)
 fix X :: nat ⇒ real assume filterlim X at_bot sequentially
 show (λn. set_lebesgue_integral M {X n..b} f) ⟶ set_lebesgue_integral M
{..b} f
  unfolding set_lebesgue_integral_def
 proof (rule integral_dominated_convergence)
  show integrable M (λx. indicat_real {..b} x *R norm (f x))
    using integrable_norm[OF int[unfolded set_integrable_def]] by simp
  show AE x in M. (λn. indicator {X n..b} x *R f x) ⟶ indicat_real {..b}
x *R f x
  proof
    fix x
    from ‹filterlim X at_bot sequentially›
    have eventually (λn. x ≥ X n) sequentially
      unfolding filterlim_at_bot_le[where c=x] by auto
    then show (λn. indicator {X n..b} x *R f x) ⟶ indicat_real {..b} x *R
f x
        by (intro tendsto_eventually) (auto split: split_indicator elim!: eventu-
ally_mono)
```

```
    qed
    fix n show AE x in M. norm (indicator {X n..b} x *R f x) ≤
                              indicator {..b} x *R norm (f x)
      by (auto split: split_indicator)
  next
    from int show (λx. indicat_real {..b} x *R f x) ∈ borel_measurable M
      by (simp add: set_integrable_def)
  next
    fix n :: nat
    from sets have {X n..b} ∈ sets M by (cases X n ≤ b) auto
    with int have set_integrable M {X n..b} f
      by (rule set_integrable_subset) auto
    thus (λx. indicat_real {X n..b} x *R f x) ∈ borel_measurable M
      by (simp add: set_integrable_def)
  qed
qed



theorem integral_Markov_inequality':
  fixes u :: 'a ⇒ real
  assumes [measurable]: set_integrable M A u and A ∈ sets M
  assumes AE x in M. x ∈ A ⟶ u x ≥ 0 and 0 < (c::real)
  shows emeasure M {x∈A. u x ≥ c} ≤ (1/c::real) * (∫ x∈A. u x ∂M)
proof −
  have (λx. u x * indicator A x) ∈ borel_measurable M
    using assms by (auto simp: set_integrable_def mult_ac)
  hence (λx. ennreal (u x * indicator A x)) ∈ borel_measurable M
    by measurable
  also have (λx. ennreal (u x * indicator A x)) = (λx. ennreal (u x) * indicator
A x)
    by (intro ext) (auto simp: indicator_def)
  finally have meas: ... ∈ borel_measurable M .
  from assms(3) have AE: AE x in M. 0 ≤ u x * indicator A x
    by eventually_elim (auto simp: indicator_def)
  have nonneg: set_lebesgue_integral M A u ≥ 0
    unfolding set_lebesgue_integral_def
   by (intro Bochner_Integration.integral_nonneg_AE eventually_mono[OF AE])
(auto simp: mult_ac)

  have A: A ⊆ space M
    using ‹A ∈ sets M› by (simp add: sets.sets_into_space)
  have {x ∈ A. u x ≥ c} = {x ∈ A. ennreal(1/c) * u x ≥ 1}
    using ‹c>0› A by (auto simp: ennreal_mult'[symmetric])
  then have emeasure M {x ∈ A. u x ≥ c} = emeasure M ({x ∈ A. ennreal(1/c)
* u x ≥ 1})
    by simp
  also have ... ≤ ennreal(1/c) * (∫⁺ x. ennreal(u x) * indicator A x ∂M)
    by (intro nn_integral_Markov_inequality meas assms)
```

```
  also have (∫⁺ x. ennreal(u x) * indicator A x ∂M) = ennreal (set_lebesgue_integral
M A u)
     unfolding set_lebesgue_integral_def nn_integral_set_ennreal using assms
AE
   by (subst nn_integral_eq_integral) (simp_all add: mult_ac set_integrable_def)
  finally show ?thesis
    using ‹c > 0› nonneg by (subst ennreal_mult) auto
qed

theorem integral_Markov_inequality'_measure:
  assumes [measurable]: set_integrable M A u and A ∈ sets M
     and AE x in M. x ∈ A ⟶ 0 ≤ u x 0 < (c::real)
  shows measure M {x∈A. u x ≥ c} ≤ (∫ x∈A. u x ∂M) / c
proof −
  have nonneg: set_lebesgue_integral M A u ≥ 0
    unfolding set_lebesgue_integral_def
   by (intro Bochner_Integration.integral_nonneg_AE eventually_mono[OF assms(3)])
      (auto simp: mult_ac)
  have le: emeasure M {x∈A. u x ≥ c} ≤ ennreal ((1/c) * (∫ x∈A. u x ∂M))
    by (rule integral_Markov_inequality') (use assms in auto)
  also have ... < top
    by auto
  finally have ennreal (measure M {x∈A. u x ≥ c}) = emeasure M {x∈A. u x ≥
c}
    by (intro emeasure_eq_ennreal_measure [symmetric]) auto
  also note le
  finally show ?thesis using nonneg
    by (subst (asm) ennreal_le_iff)
       (auto intro!: divide_nonneg_pos Bochner_Integration.integral_nonneg_AE
assms)
qed

theorem (in finite_measure) Chernoff_ineq_ge:
  assumes s: s > 0
  assumes integrable: set_integrable M A (λx. exp (s * f x)) and A ∈ sets M
  shows   measure M {x∈A. f x ≥ a} ≤ exp (−s * a) * (∫ x∈A. exp (s * f x) ∂M)
proof −
  have {x∈A. f x ≥ a} = {x∈A. exp (s * f x) ≥ exp (s * a)}
    using s by auto
  also have measure M ... ≤ set_lebesgue_integral M A (λx. exp (s * f x)) / exp
(s * a)
    by (intro integral_Markov_inequality'_measure assms) auto
  finally show ?thesis
    by (simp add: exp_minus field_simps)
qed

theorem (in finite_measure) Chernoff_ineq_le:
  assumes s: s > 0
  assumes integrable: set_integrable M A (λx. exp (−s * f x)) and A ∈ sets M
```

shows *measure M* $\{x{\in}A.\ f\,x \le a\} \le exp\ (s * a) * (\int x{\in}A.\ exp\ (-s * f\,x)\ \partial M)$
proof −
have $\{x{\in}A.\ f\,x \le a\} = \{x{\in}A.\ exp\ (-s * f\,x) \ge exp\ (-s * a)\}$
using *s* **by** *auto*
also have *measure M* ... $\le$ *set_lebesgue_integral M A* $(\lambda x.\ exp\ (-s * f\,x))$ / *exp* $(-s * a)$
by (*intro integral_Markov_inequality'_measure assms*) *auto*
finally show *?thesis*
by (*simp add*: *exp_minus field_simps*)
qed

9.7 Integrable Simple Functions

This section is from the Martingales AFP entry, by Ata Keskin, TU München

We restate some basic results concerning Bochner-integrable functions.

lemma *integrable_implies_simple_function_sequence*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes *integrable M f*
obtains *s* **where** $\bigwedge i.$ *simple_function M* (*s i*)
and $\bigwedge i.$ *emeasure M* $\{y \in$ *space M*. *s i y* $\neq 0\} \neq \infty$
and $\bigwedge x.\ x \in$ *space M* $\Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\,x$
and $\bigwedge x\ i.\ x \in$ *space M* $\Longrightarrow$ *norm* $(s\ i\ x) \le 2 *$ *norm* $(f\,x)$
proof−
have *f*: *f* ∈ *borel_measurable M* $(\int^+ x.$ *norm* $(f\ x)\ \partial M) < \infty$ **using** *assms* **unfolding** *integrable_iff_bounded* **by** *auto*
obtain *s* **where** *s*: $\bigwedge i.$ *simple_function M* (*s i*) $\bigwedge x.\ x \in$ *space M* $\Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x\ \bigwedge i\ x.\ x \in$ *space M* $\Longrightarrow$ *norm* $(s\ i\ x) \le 2 *$ *norm* $(f\ x)$ **using** *borel_measurable_implies_sequence_metric*[*OF f*(*1*)] **unfolding** *norm_conv_dist* **by** *metis*
{
fix *i*
have $(\int^+ x.$ *norm* $(s\ i\ x)\ \partial M) \le (\int^+ x.$ *ennreal* $(2 *$ *norm* $(f\ x))\ \partial M)$ **using** *s* **by** (*intro nn_integral_mono*, *auto*)
also have ... $< \infty$ **using** *f* **by** (*simp add*: *nn_integral_cmult ennreal_mult_less_top ennreal_mult*)
finally have *sbi*: *Bochner_Integration.simple_bochner_integrable M* (*s i*) **using** *s* **by** (*intro simple_bochner_integrableI_bounded*) *auto*
hence *emeasure M* $\{y \in$ *space M*. *s i y* $\neq 0\} \neq \infty$ **by** (*auto intro*: *integrableI_simple_bochner_integrable simple_bochner_integrable.cases*)
}
thus *?thesis* **using** *that s* **by** *blast*
qed

Simple functions can be represented by sums of indicator functions.

lemma *simple_function_indicator_representation_banach*:
fixes *f* ::*'a* ⇒ *'b* :: {*second_countable_topology*, *banach*}
assumes *simple_function M f x* ∈ *space M*

shows $f\ x = (\sum y \in f\ `\ space\ M.\ indicator\ (f\ -`\ \{y\} \cap space\ M)\ x *_R y)$
(**is** $?l = ?r$)
proof −
have $?r = (\sum y \in f\ `\ space\ M.\ (if\ y = f\ x\ then\ indicator\ (f\ -`\ \{y\} \cap space\ M)\ x *_R y\ else\ 0))$
by (*auto intro*!: *sum.cong*)
also have $\ldots = indicator\ (f\ -`\ \{f\ x\} \cap space\ M)\ x *_R f\ x$ **using** *assms* **by** (*auto dest*: *simple_functionD*)
also have $\ldots = f\ x$ **using** *assms* **by** (*auto simp*: *indicator_def*)
finally show *?thesis* **by** *auto*
qed

lemma *simple_function_indicator_representation_AE*:
fixes $f :: 'a \Rightarrow 'b :: \{second\_countable\_topology, banach\}$
assumes *simple_function M f*
shows *AE x in M*. $f\ x = (\sum y \in f\ `\ space\ M.\ indicator\ (f\ -`\ \{y\} \cap space\ M)\ x *_R y)$
by (*metis* (*mono_tags*, *lifting*) *AE_I2 simple_function_indicator_representation_banach assms*)

lemmas *simple_function_scaleR*[*intro*] = *simple_function_compose2*[**where** $h=(*_R)$]
lemmas *integrable_simple_function* = *simple_bochner_integrable.intros*[*THEN has_bochner_integral_*
THEN integrable.intros]

Induction rule for simple integrable functions.

lemma *integrable_simple_function_induct*[*consumes 2*, *case_names cong indicator add*, *induct set*: *simple_function*]:
fixes $f :: 'a \Rightarrow 'b :: \{second\_countable\_topology, banach\}$
assumes *f*: *simple_function M f emeasure M* $\{y \in space\ M.\ f\ y \neq 0\} \neq \infty$
assumes *cong*: $\bigwedge f\ g.\ simple\_function\ M\ f \Longrightarrow emeasure\ M\ \{y \in space\ M.\ f\ y \neq 0\} \neq \infty \Longrightarrow simple\_function\ M\ g \Longrightarrow emeasure\ M\ \{y \in space\ M.\ g\ y \neq 0\} \neq \infty \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *indicator*: $\bigwedge A\ y.\ A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow P\ (\lambda x.\ indicator\ A\ x *_R y)$
assumes *add*: $\bigwedge f\ g.\ simple\_function\ M\ f \Longrightarrow emeasure\ M\ \{y \in space\ M.\ f\ y \neq 0\} \neq \infty \Longrightarrow simple\_function\ M\ g \Longrightarrow emeasure\ M\ \{y \in space\ M.\ g\ y \neq 0\} \neq \infty \Longrightarrow (\bigwedge z.\ z \in space\ M \Longrightarrow norm\ (f\ z + g\ z) = norm\ (f\ z) + norm\ (g\ z)) \Longrightarrow P\ f \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
shows $P\ f$
proof−
let $?f = \lambda x.\ (\sum y \in f\ `\ space\ M.\ indicat\_real\ (f\ -`\ \{y\} \cap space\ M)\ x *_R y)$
have *f_ae_eq*: $f\ x = ?f\ x$ **if** $x \in space\ M$ **for** x **using** *simple_function_indicator_representation_banach f*(*1*) *that*] **.**
moreover have *emeasure M* $\{y \in space\ M.\ ?f\ y \neq 0\} \neq \infty$ **by** (*metis* (*no_types*,

lifting) *Collect_cong calculation f*(*2*))
moreover have *P* (λ*x*. ∑ *y*∈*S*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *x* ∗*R y*)
simple_function M (λ*x*. ∑ *y*∈*S*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *x* ∗*R y*)
emeasure M {*y* ∈ *space M*. (∑ *x*∈*S*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x*) ≠ *0*} ≠ ∞
if *S* ⊆ *f* ‘ *space M* **for** *S* **using** *simple_functionD*(*1*)[*OF assms*(*1*), *THEN rev_finite_subset*, *OF that*] *that*
proof (*induction rule*: *finite_induct*)
case *empty*
{
case *1*
then show *?case* **using** *indicator*[*of* {}] **by** *force*
next
case *2*
then show *?case* **by** *force*
next
case *3*
then show *?case* **by** *force*
}
next
case (*insert x F*)
have (*f* −‘ {*x*} ∩ *space M*) ⊆ {*y* ∈ *space M*. *f y* ≠ *0*} **if** *x* ≠ *0* **using** *that* **by** *blast*
moreover have {*y* ∈ *space M*. *f y* ≠ *0*} = *space M* − (*f* −‘ {*0*} ∩ *space M*) **by** *blast*
moreover have *space M* − (*f* −‘ {*0*} ∩ *space M*) ∈ *sets M* **using** *simple_functionD*(*2*)[*OF f*(*1*)] **by** *blast*
ultimately have *fin_0*: *emeasure M* (*f* −‘ {*x*} ∩ *space M*) < ∞ **if** *x* ≠ *0* **using** *that* **by** (*metis emeasure_mono f*(*2*) *infinity_ennreal_def top.not_eq_extremum top_unique*)
hence *fin_1*: *emeasure M* {*y* ∈ *space M*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x* ≠ *0*} ≠ ∞ **if** *x* ≠ *0* **by** (*metis* (*mono_tags*, *lifting*) *emeasure_mono f*(*1*) *indicator_simps*(*2*) *linorder_not_less mem_Collect_eq scaleR_eq_0_iff simple_functionD*(*2*) *subsetI that*)

have ∗: (∑ *y*∈*insert x F*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *xa* ∗*R y*) = (∑ *y*∈*F*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *xa* ∗*R y*) + *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *xa* ∗*R x* **for** *xa* **by** (*metis* (*no_types*, *lifting*) *Diff_empty Diff_insert0 add.commute insert.hyps*(*1*) *insert.hyps*(*2*) *sum.insert_remove*)
have ∗∗: {*y* ∈ *space M*. (∑ *x*∈*insert x F*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x*) ≠ *0*} ⊆ {*y* ∈ *space M*. (∑ *x*∈*F*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x*) ≠ *0*} ∪ {*y* ∈ *space M*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x* ≠ *0*} **unfolding** ∗ **by** *fastforce*
{
case *1*
hence *x*: *x* ∈ *f* ‘ *space M* **and** *F*: *F* ⊆ *f* ‘ *space M* **by** *auto*
show *?case*
proof (*cases x* = *0*)

case *True*
then show *?thesis* **unfolding** $*$ **using** *insert(3)*[*OF F*] **by** *simp*
next
case *False*
have *norm_argument*: *norm* ((∑ *y*∈*F*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *z* $*_R$ *y*) + *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *z* $*_R$ *x*) = *norm* (∑ *y*∈*F*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *z* $*_R$ *y*) + *norm* (*indicat_real* (*f* −‘ {*x*} ∩ *space M*) *z* $*_R$ *x*) **if** *z*: *z* ∈ *space M* **for** *z*
proof (*cases f z* = *x*)
case *True*
have *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *z* $*_R$ *y* = *0* **if** *y* ∈ *F* **for** *y*
using *True insert(2) z that 1* **unfolding** *indicator_def* **by** *force*
hence (∑ *y*∈*F*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *z* $*_R$ *y*) = *0* **by** (*meson sum.neutral*)
then show *?thesis* **by** *force*
next
case *False*
then show *?thesis* **by** *force*
qed
show *?thesis*
using *False simple_functionD(2)*[*OF f(1)*] *insert(3,5)*[*OF F*] *simple_function_scaleR fin_0 fin_1*
by (*subst* $*$, *subst add*, *subst simple_function_sum*) (*blast intro*: *norm_argument indicator*)+
qed
next
case *2*
hence *x*: *x* ∈ *f* ‘ *space M* **and** *F*: *F* ⊆ *f* ‘ *space M* **by** *auto*
show *?case*
proof (*cases x* = *0*)
case *True*
then show *?thesis* **unfolding** $*$ **using** *insert(4)*[*OF F*] **by** *simp*
next
case *False*
then show *?thesis* **unfolding** $*$ **using** *insert(4)*[*OF F*] *simple_functionD(2)*[*OF f(1)*] **by** *fast*
qed
next
case *3*
hence *x*: *x* ∈ *f* ‘ *space M* **and** *F*: *F* ⊆ *f* ‘ *space M* **by** *auto*
show *?case*
proof (*cases x* = *0*)
case *True*
then show *?thesis* **unfolding** $*$ **using** *insert(5)*[*OF F*] **by** *simp*
next
case *False*
have *emeasure M* {*y* ∈ *space M*. (∑ *x*∈*insert x F*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* $*_R$ *x*) ≠ *0*} ≤ *emeasure M* ({*y* ∈ *space M*. (∑ *x*∈*F*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* $*_R$ *x*) ≠ *0*} ∪ {*y* ∈ *space M*. *indicat_real* (*f* −‘ {*x*} ∩ *space*

M) *y* $*_R$ *x* $\neq$ *0*})
using ** *simple_functionD*(*2*)[*OF insert*(*4*)[*OF F*]] *simple_functionD*(*2*)[*OF f*(*1*)] **by** (*intro emeasure_mono*, *force*+)
also have ... $\leq$ *emeasure M* {*y* $\in$ *space M*. ($\sum$ *x*$\in$*F*. *indicat_real* (*f* $-$‘ {*x*} $\cap$ *space M*) *y* $*_R$ *x*) $\neq$ *0*} + *emeasure M* {*y* $\in$ *space M*. *indicat_real* (*f* $-$‘ {*x*} $\cap$ *space M*) *y* $*_R$ *x* $\neq$ *0*}
using *simple_functionD*(*2*)[*OF insert*(*4*)[*OF F*]] *simple_functionD*(*2*)[*OF f*(*1*)] **by** (*intro emeasure_subadditive*, *force*+)
also have ... $<$ ∞ **using** *insert*(*5*)[*OF F*] *fin_1*[*OF False*] **by** (*simp add*: *less_top*)
finally show *?thesis* **by** *simp*
qed
}
qed
moreover have *simple_function M* (λ*x*. $\sum$ *y*$\in$*f* ‘ *space M*. *indicat_real* (*f* $-$‘ {*y*} $\cap$ *space M*) *x* $*_R$ *y*) **using** *calculation* **by** *blast*
moreover have *P* (λ*x*. $\sum$ *y*$\in$*f* ‘ *space M*. *indicat_real* (*f* $-$‘ {*y*} $\cap$ *space M*) *x* $*_R$ *y*) **using** *calculation* **by** *blast*
ultimately show *?thesis* **by** (*intro cong*[*OF* _ _ *f*(*1*,*2*)], *blast*, *presburger*+)
qed

Induction rule for non-negative simple integrable functions

lemma *integrable_simple_function_induct_nn*[*consumes 3*, *case_names cong indicator add*, *induct set*: *simple_function*]:
fixes *f* :: *'a* $\Rightarrow$ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *f*: *simple_function M f emeasure M* {*y* $\in$ *space M*. *f y* $\neq$ *0*} $\neq$ ∞ $\bigwedge$*x*. *x* $\in$ *space M* $\longrightarrow$ *f x* $\geq$ *0*
assumes *cong*: $\bigwedge$*f g*. *simple_function M f* $\Longrightarrow$ *emeasure M* {*y* $\in$ *space M*. *f y* $\neq$ *0*} $\neq$ ∞ $\Longrightarrow$ ($\bigwedge$*x*. *x* $\in$ *space M* $\Longrightarrow$ *f x* $\geq$ *0*) $\Longrightarrow$ *simple_function M g* $\Longrightarrow$ *emeasure M* {*y* $\in$ *space M*. *g y* $\neq$ *0*} $\neq$ ∞ $\Longrightarrow$ ($\bigwedge$*x*. *x* $\in$ *space M* $\Longrightarrow$ *g x* $\geq$ *0*) $\Longrightarrow$ ($\bigwedge$*x*. *x* $\in$ *space M* $\Longrightarrow$ *f x* = *g x*) $\Longrightarrow$ *P f* $\Longrightarrow$ *P g*
assumes *indicator*: $\bigwedge$*A y*. *y* $\geq$ *0* $\Longrightarrow$ *A* $\in$ *sets M* $\Longrightarrow$ *emeasure M A* $<$ ∞ $\Longrightarrow$ *P* (λ*x*. *indicator A x* $*_R$ *y*)
assumes *add*: $\bigwedge$*f g*. ($\bigwedge$*x*. *x* $\in$ *space M* $\Longrightarrow$ *f x* $\geq$ *0*) $\Longrightarrow$ *simple_function M f* $\Longrightarrow$ *emeasure M* {*y* $\in$ *space M*. *f y* $\neq$ *0*} $\neq$ ∞ $\Longrightarrow$
($\bigwedge$*x*. *x* $\in$ *space M* $\Longrightarrow$ *g x* $\geq$ *0*) $\Longrightarrow$ *simple_function M g* $\Longrightarrow$ *emeasure M* {*y* $\in$ *space M*. *g y* $\neq$ *0*} $\neq$ ∞ $\Longrightarrow$
($\bigwedge$*z*. *z* $\in$ *space M* $\Longrightarrow$ *norm* (*f z* + *g z*) = *norm* (*f z*) + *norm* (*g z*)) $\Longrightarrow$
P f $\Longrightarrow$ *P g* $\Longrightarrow$ *P* (λ*x*. *f x* + *g x*)
shows *P f*
proof$-$
let *?f* = λ*x*. ($\sum$ *y*$\in$*f* ‘ *space M*. *indicat_real* (*f* $-$‘ {*y*} $\cap$ *space M*) *x* $*_R$ *y*)
have *f_ae_eq*: *f x* = *?f x* **if** *x* $\in$ *space M* **for** *x* **using** *simple_function_indicator_representation_banach*[*OF f*(*1*) *that*] **.**
moreover have *emeasure M* {*y* $\in$ *space M*. *?f y* $\neq$ *0*} $\neq$ ∞ **by** (*metis* (*no_types*, *lifting*) *Collect_cong calculation f*(*2*))
moreover have *P* (λ*x*. $\sum$ *y*$\in$*S*. *indicat_real* (*f* $-$‘ {*y*} $\cap$ *space M*) *x* $*_R$ *y*)

simple_function M (λ*x*. ∑ *y*∈*S*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *x* ∗*R y*)
emeasure M {*y* ∈ *space M*. (∑ *x*∈*S*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x*) ≠ *0*} ≠ ∞
⋀*x*. *x* ∈ *space M* ⟹ *0* ≤ (∑ *y*∈*S*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *x* ∗*R y*)
if *S* ⊆ *f* ‘ *space M* **for** *S* **using** *simple_functionD*(*1*)[*OF assms*(*1*), *THEN rev_finite_subset*, *OF that*] *that*
proof (*induction rule*: *finite_subset_induct*′)
case *empty*
{
case *1*
then show *?case* **using** *indicator*[*of 0* {}] **by** *force*
next
case *2*
then show *?case* **by** *force*
next
case *3*
then show *?case* **by** *force*
next
case *4*
then show *?case* **by** *force*
}
next
case (*insert x F*)
have (*f* −‘ {*x*} ∩ *space M*) ⊆ {*y* ∈ *space M*. *f y* ≠ *0*} **if** *x* ≠ *0*
using *that* **by** *blast*
moreover have {*y* ∈ *space M*. *f y* ≠ *0*} = *space M* − (*f* −‘ {*0*} ∩ *space M*)
by *blast*
moreover have *space M* − (*f* −‘ {*0*} ∩ *space M*) ∈ *sets M*
using *simple_functionD*(*2*)[*OF f*(*1*)] **by** *blast*
ultimately have *fin_0*: *emeasure M* (*f* −‘ {*x*} ∩ *space M*) < ∞ **if** *x* ≠ *0*
using *that* **by** (*metis emeasure_mono f*(*2*) *infinity_ennreal_def top.not_eq_extremum top_unique*)
hence *fin_1*: *emeasure M* {*y* ∈ *space M*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x* ≠ *0*} ≠ ∞ **if** *x* ≠ *0*
by (*metis* (*mono_tags*, *lifting*) *emeasure_mono f*(*1*) *indicator_simps*(*2*) *linorder_not_less mem_Collect_eq scaleR_eq_0_iff simple_functionD*(*2*) *subsetI that*)

have *nonneg_x*: *x* ≥ *0* **using** *insert f* **by** *blast*
have ∗: (∑ *y*∈*insert x F*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *xa* ∗*R y*) = (∑ *y*∈*F*. *indicat_real* (*f* −‘ {*y*} ∩ *space M*) *xa* ∗*R y*) + *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *xa* ∗*R x* **for** *xa* **by** (*metis* (*no_types*, *lifting*) *add.commute insert.hyps*(*1*) *insert.hyps*(*4*) *sum.insert*)
have ∗∗: {*y* ∈ *space M*. (∑ *x*∈*insert x F*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x*) ≠ *0*} ⊆ {*y* ∈ *space M*. (∑ *x*∈*F*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x*) ≠ *0*} ∪ {*y* ∈ *space M*. *indicat_real* (*f* −‘ {*x*} ∩ *space M*) *y* ∗*R x* ≠ *0*} **unfolding** ∗ **by** *fastforce*

```
  {
    case 1
    show ?case
    proof (cases x = 0)
      case True
      then show ?thesis unfolding * using insert by simp
    next
      case False
      have norm_argument: norm ((∑y∈F. indicat_real (f -` {y} ∩ space M)
z *R y) + indicat_real (f -` {x} ∩ space M) z *R x)
          = norm (∑y∈F. indicat_real (f -` {y} ∩ space M) z *R y) + norm
(indicat_real (f -` {x} ∩ space M) z *R x)
        if z: z ∈ space M for z
      proof (cases f z = x)
        case True
        have indicat_real (f -` {y} ∩ space M) z *R y = 0 if y ∈ F for y
          using True insert z that 1 unfolding indicator_def by force
        hence (∑y∈F. indicat_real (f -` {y} ∩ space M) z *R y) = 0
          by (meson sum.neutral)
        thus ?thesis by force
      qed (force)
      show ?thesis using False fin_0 fin_1 f norm_argument
        by (subst *, subst add, presburger add: insert, intro insert, intro insert,
fastforce simp add: indicator_def intro!: insert(2) f(3), auto intro!: indicator insert
nonneg_x)
    qed
  next
    case 2
    show ?case
    proof (cases x = 0)
      case True
      then show ?thesis unfolding * using insert by simp
    next
      case False
     then show ?thesis unfolding * using insert simple_functionD(2)[OF f(1)]
by fast
    qed
  next
    case 3
    show ?case
    proof (cases x = 0)
      case True
      then show ?thesis unfolding * using insert by simp
    next
      case False
      have emeasure M {y ∈ space M. (∑x∈insert x F. indicat_real (f -` {x}
∩ space M) y *R x) ≠ 0}
        ≤ emeasure M ({y ∈ space M. (∑x∈F. indicat_real (f -` {x} ∩ space
M) y *R x) ≠ 0} ∪ {y ∈ space M. indicat_real (f -` {x} ∩ space M) y *R x ≠
```

```
0})
          using ∗∗ simple_functionD(2)[OF insert(6)] simple_functionD(2)[OF
f(1)] insert.IH(2) by (intro emeasure_mono, blast, simp)
        also have ... ≤ emeasure M {y ∈ space M. (∑x∈F. indicat_real (f −‘
{x} ∩ space M) y ∗R x) ≠ 0} + emeasure M {y ∈ space M. indicat_real (f −‘
{x} ∩ space M) y ∗R x ≠ 0}
          using simple_functionD(2)[OF insert(6)] simple_functionD(2)[OF f(1)]
by (intro emeasure_subadditive, force+)
       also have ... < ∞ using insert(7) fin_1[OF False] by (simp add: less_top)
        finally show ?thesis by simp
      qed
    next
      case (4 ξ)
    thus ?case using insert nonneg_x f(3) by (auto simp add: scaleR_nonneg_nonneg
intro: sum_nonneg)
    }
  qed
  moreover have simple_function M (λx. ∑y∈f ‘ space M. indicat_real (f −‘
{y} ∩ space M) x ∗R y)
    using calculation by blast
  moreover have P (λx. ∑y∈f ‘ space M. indicat_real (f −‘ {y} ∩ space M) x
∗R y)
    using calculation by blast
  moreover have ⋀x. x ∈ space M ⟹ 0 ≤ f x using f(3) by simp
  ultimately show ?thesis
    by (smt (verit) Collect_cong f(1) local.cong)
qed

lemma finite_nn_integral_imp_ae_finite:
  fixes f :: 'a ⇒ ennreal
  assumes f ∈ borel_measurable M (∫+x. f x ∂M) < ∞
  shows AE x in M. f x < ∞
proof (rule ccontr, goal_cases)
  case 1
  let ?A = space M ∩ {x. f x = ∞}
  have ∗: emeasure M ?A > 0 using 1 assms(1)
   by (metis (mono_tags, lifting) assms(2) eventually_mono infinity_ennreal_def
nn_integral_noteq_infinite top.not_eq_extremum)
  have (∫+x. f x ∗ indicator ?A x ∂M) = (∫+x. ∞ ∗ indicator ?A x ∂M)
   by (metis (mono_tags, lifting) indicator_inter_arith indicator_simps(2) mem_Collect_eq
mult_eq_0_iff)
  also have ... = ∞ ∗ emeasure M ?A
    using assms(1) by (intro nn_integral_cmult_indicator, simp)
  also have ... = ∞
    using ∗ by fastforce
  finally have (∫+x. f x ∗ indicator ?A x ∂M) = ∞ .
  moreover have (∫+x. f x ∗ indicator ?A x ∂M) ≤ (∫+x. f x ∂M)
    by (intro nn_integral_mono, simp add: indicator_def)
  ultimately show ?case using assms(2) by simp
```

qed

Convergence in L1-Norm implies existence of a subsequence which convergences almost everywhere. This lemma is easier to use than the existing one in *HOL−Analysis.Bochner_Integration*

lemma *cauchy_L1_AE_cauchy_subseq*:
fixes $s :: nat \Rightarrow 'a \Rightarrow 'b::\{banach, second\_countable\_topology\}$
assumes [*measurable*]: $\bigwedge n.\ integrable\ M\ (s\ n)$
and $\bigwedge e.\ e > 0 \Longrightarrow \exists N.\ \forall i{\geq}N.\ \forall j{\geq}N.\ LINT\ x|M.\ norm\ (s\ i\ x - s\ j\ x) < e$
obtains r **where** *strict_mono r AE x in M. Cauchy* $(\lambda i.\ s\ (r\ i)\ x)$
proof−
have $\exists r.\ \forall n.\ (\forall i{\geq}r\ n.\ \forall j{\geq}\ r\ n.\ LINT\ x|M.\ norm\ (s\ i\ x - s\ j\ x) < (1\ /\ 2)\ \hat{}\ n) \wedge (r\ (Suc\ n) > r\ n)$
proof (*intro dependent_nat_choice*, *goal_cases*)
case *1*
then show *?case* **using** *assms*(*2*) **by** *presburger*
next
case (*2 x n*)
obtain N **where** $*$: $LINT\ x|M.\ norm\ (s\ i\ x - s\ j\ x) < (1\ /\ 2)\ \hat{}\ Suc\ n$ **if** $i \geq N\ j \geq N$ **for** $i\ j$
using *assms*(*2*)[*of* $(1\ /\ 2)\ \hat{}\ Suc\ n$] **by** *auto*
{
fix $i\ j$ **assume** $i \geq max\ N\ (Suc\ x)\ j \geq max\ N\ (Suc\ x)$
hence $integral^L\ M\ (\lambda x.\ norm\ (s\ i\ x - s\ j\ x)) < (1\ /\ 2)\ \hat{}\ Suc\ n$ **using** $*$ **by** *fastforce*
}
then show *?case* **by** *fastforce*
qed
then obtain r **where** *strict_mono*: *strict_mono r* **and** $\forall i{\geq}r\ n.\ \forall j{\geq}\ r\ n.\ LINT\ x|M.\ norm\ (s\ i\ x - s\ j\ x) < (1\ /\ 2)\ \hat{}\ n$ **for** n
using *strict_mono_Suc_iff* **by** *blast*
hence *r_is*: $LINT\ x|M.\ norm\ (s\ (r\ (Suc\ n))\ x - s\ (r\ n)\ x) < (1\ /\ 2)\ \hat{}\ n$ **for** n **by** (*simp add*: *strict_mono_leD*)

define g **where** $g = (\lambda n\ x.\ (\sum i{\leq}n.\ ennreal\ (norm\ (s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x))))$
define g' **where** $g' = (\lambda x.\ \sum i.\ ennreal\ (norm\ (s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x)))$

have *integrable_g*: $(\int^+ x.\ g\ n\ x\ \partial M) < 2$ **for** n
proof −
have $(\int^+ x.\ g\ n\ x\ \partial M) = (\int^+ x.\ (\sum i{\leq}n.\ ennreal\ (norm\ (s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x)))\ \partial M)$
using *g_def* **by** *simp*
also have $\ldots = (\sum i{\leq}n.\ (\int^+ x.\ ennreal\ (norm\ (s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x))\ \partial M))$
by (*intro nn_integral_sum*, *simp*)
also have $\ldots = (\sum i{\leq}n.\ LINT\ x|M.\ norm\ (s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x))$
unfolding *dist_norm* **using** *assms*(*1*) **by** (*subst nn_integral_eq_integral*[*OF integrable_norm*], *auto*)

also have ... < *ennreal* ($\sum i{\leq}n.$ (*1 / 2*) ^ *i*)
by (*intro ennreal_lessI*[*OF sum_pos sum_strict_mono*[*OF finite_atMost _ r_is*]], *auto*)
also have ... ≤ *ennreal 2*
unfolding *sum_gp0*[*of 1 / 2 n*] **by** (*intro ennreal_leI*, *auto*)
finally show ($\int^+ x.\ g\ n\ x\ \partial M$) < *2* **by** *simp*
qed

have *integrable_g'*: ($\int^+ x.\ g'\ x\ \partial M$) ≤ *2*
proof −
have *incseq* (λ*n. g n x*) **for** *x* **by** (*intro incseq_SucI*, *auto simp add*: *g_def ennreal_leI*)
hence *convergent* (λ*n. g n x*) **for** *x*
unfolding *convergent_def* **using** *LIMSEQ_incseq_SUP* **by** *fast*
hence (λ*n. g n x*) ⟶ *g' x* **for** *x*
unfolding *g_def g'_def* **by** (*intro summable_iff_convergent'*[*THEN iffD2*, *THEN summable_LIMSEQ'*], *blast*)
hence ($\int^+ x.\ g'\ x\ \partial M$) = ($\int^+ x.$ *liminf* (λ*n. g n x*) ∂M) **by** (*metis lim_imp_Liminf trivial_limit_sequentially*)
also have ... ≤ *liminf* (λ*n.* $\int^+ x.\ g\ n\ x\ \partial M$) **by** (*intro nn_integral_liminf*, *simp add*: *g_def*)
also have ... ≤ *liminf* (λ*n. 2*) **using** *integrable_g* **by** (*intro Liminf_mono*) (*simp add*: *order_le_less*)
also have ... = *2*
using *sequentially_bot tendsto_iff_Liminf_eq_Limsup* **by** *blast*
finally show *?thesis* **.**
qed
hence *AE x in M. g' x* < ∞
by (*intro finite_nn_integral_imp_ae_finite*) (*auto simp add*: *order_le_less_trans g'_def*)
moreover have *summable* (λ*i. norm* (*s* (*r* (*Suc i*)) *x* − *s* (*r i*) *x*)) **if** *g' x* ≠ ∞ **for** *x*
using *that* **unfolding** *g'_def* **by** (*intro summable_suminf_not_top*) *fastforce*+

ultimately have *ae_summable*: *AE x in M. summable* (λ*i. s* (*r* (*Suc i*)) *x* − *s* (*r i*) *x*)
using *summable_norm_cancel* **unfolding** *dist_norm* **by** *force*

{
fix *x* **assume** *summable* (λ*i. s* (*r* (*Suc i*)) *x* − *s* (*r i*) *x*)
hence (λ*n.* $\sum i{<}n.$ *s* (*r* (*Suc i*)) *x* − *s* (*r i*) *x*) ⟶ ($\sum i.$ *s* (*r* (*Suc i*)) *x* − *s* (*r i*) *x*)
using *summable_LIMSEQ* **by** *blast*
moreover have (λ*n.* ($\sum i{<}n.$ *s* (*r* (*Suc i*)) *x* − *s* (*r i*) *x*)) = (λ*n. s* (*r n*) *x* − *s* (*r 0*) *x*)
using *sum_lessThan_telescope* **by** *fastforce*
ultimately have (λ*n. s* (*r n*) *x* − *s* (*r 0*) *x*) ⟶ ($\sum i.$ *s* (*r* (*Suc i*)) *x* − *s* (*r i*) *x*) **by** *argo*
hence (λ*n. s* (*r n*) *x* − *s* (*r 0*) *x* + *s* (*r 0*) *x*) ⟶ ($\sum i.$ *s* (*r* (*Suc i*)) *x* −

s (r i) x) + s (r 0) x
by (*intro isCont_tendsto_compose*[*of _ λz. z + s (r 0) x*], *auto*)
hence *Cauchy* (*λn. s (r n) x*) **by** (*simp add*: *LIMSEQ_imp_Cauchy*)
}
hence *AE x in M. Cauchy* (*λi. s (r i) x*) **using** *ae_summable* **by** *fast*
thus *?thesis* **by** (*rule that*[*OF strict_mono*(*1*)])
qed

9.7.1 Totally Ordered Banach Spaces

lemma *integrable_max*[*simp*, *intro*]:
fixes *f* :: *'a ⇒ 'b* :: {*second_countable_topology*, *banach*, *linorder_topology*}
assumes *fg*[*measurable*]: *integrable M f integrable M g*
shows *integrable M* (*λx. max* (*f x*) (*g x*))
proof (*rule Bochner_Integration.integrable_bound*)
{
fix *x y* :: *'b*
have *norm* (*max x y*) ≤ *max* (*norm x*) (*norm y*) **by** *linarith*
also have *. . .* ≤ *norm x + norm y* **by** *simp*
finally have *norm* (*max x y*) ≤ *norm* (*norm x + norm y*) **by** *simp*
}
thus *AE x in M. norm* (*max* (*f x*) (*g x*)) ≤ *norm* (*norm* (*f x*) + *norm* (*g x*)) **by** *simp*
qed (*auto intro*: *Bochner_Integration.integrable_add*[*OF integrable_norm*[*OF fg*(*1*)] *integrable_norm*[*OF fg*(*2*)]])

lemma *integrable_min*[*simp*, *intro*]:
fixes *f* :: *'a ⇒ 'b* :: {*second_countable_topology*, *banach*, *linorder_topology*}
assumes [*measurable*]: *integrable M f integrable M g*
shows *integrable M* (*λx. min* (*f x*) (*g x*))
proof −
have *norm* (*min* (*f x*) (*g x*)) ≤ *norm* (*f x*) ∨ *norm* (*min* (*f x*) (*g x*)) ≤ *norm* (*g x*) **for** *x* **by** *linarith*
thus *?thesis*
by (*intro integrable_bound*[*OF integrable_max*[*OF integrable_norm*(*1*,*1*), *OF assms*]], *auto*)
qed

Restatement of *integral_nonneg_AE* for functions taking values in a Banach space.

lemma *integral_nonneg_AE_banach*:
fixes *f* :: *'a ⇒ 'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes [*measurable*]: *f* ∈ *borel_measurable M* **and** *nonneg*: *AE x in M. 0* ≤ *f x*
shows *0* ≤ *integral*^{L} *M f*
proof *cases*
assume *integrable*: *integrable M f*
hence *max*: (*λx. max 0* (*f x*)) ∈ *borel_measurable M* ⋀*x. 0* ≤ *max 0* (*f x*)

integrable M ($\lambda x.$ *max 0* (*f x*)) **by** *auto*
hence *0* ≤ *integral*^{L} *M* ($\lambda x.$ *max 0* (*f x*))
proof −
obtain *s* **where** ∗: $\bigwedge i.$ *simple_function M* (*s i*)
$\bigwedge i.$ *emeasure M* {*y* ∈ *space M*. *s i y* ≠ *0*} ≠ ∞
$\bigwedge x.$ *x* ∈ *space M* ⟹ ($\lambda i.$ *s i x*) ⟶ *f x*
$\bigwedge x\ i.$ *x* ∈ *space M* ⟹ *norm* (*s i x*) ≤ *2* ∗ *norm* (*f x*)
using *integrable_implies_simple_function_sequence*[*OF integrable*] **by** *blast*
have *simple*: $\bigwedge i.$ *simple_function M* ($\lambda x.$ *max 0* (*s i x*))
using ∗ **by** *fast*
have $\bigwedge i.$ {*y* ∈ *space M*. *max 0* (*s i y*) ≠ *0*} ⊆ {*y* ∈ *space M*. *s i y* ≠ *0*}
unfolding *max_def* **by** *force*
moreover have $\bigwedge i.$ {*y* ∈ *space M*. *s i y* ≠ *0*} ∈ *sets M*
using ∗ **by** *measurable*
ultimately have $\bigwedge i.$ *emeasure M* {*y* ∈ *space M*. *max 0* (*s i y*) ≠ *0*} ≤ *emeasure M* {*y* ∈ *space M*. *s i y* ≠ *0*}
by (*rule emeasure_mono*)
hence ∗∗:$\bigwedge i.$ *emeasure M* {*y* ∈ *space M*. *max 0* (*s i y*) ≠ *0*} ≠ ∞
using ∗(*2*) **by** (*auto intro*: *order.strict_trans1 simp add*: *top.not_eq_extremum*)
have $\bigwedge x.$ *x* ∈ *space M* ⟹ ($\lambda i.$ *max 0* (*s i x*)) ⟶ *max 0* (*f x*)
using ∗(*3*) *tendsto_max* **by** *blast*
moreover have $\bigwedge x\ i.$ *x* ∈ *space M* ⟹ *norm* (*max 0* (*s i x*)) ≤ *norm* (*2* ∗$_R$ *f x*)
using ∗(*4*) **unfolding** *max_def* **by** *auto*
ultimately have *tendsto*: ($\lambda i.$ *integral*^{L} *M* ($\lambda x.$ *max 0* (*s i x*))) ⟶ *integral*^{L} *M* ($\lambda x.$ *max 0* (*f x*))
using *borel_measurable_simple_function simple integrable* **by** (*intro integral_dominated_convergence*[*OF max*(*1*) *_ integrable_norm*[*OF integrable_scaleR_right*], *of _ 2 f*], *blast*+)
{
fix *h* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assume *simple_function M h emeasure M* {*y* ∈ *space M*. *h y* ≠ *0*} ≠ ∞ $\bigwedge x.$ *x* ∈ *space M* ⟶ *h x* ≥ *0*
hence ∗: *integral*^{L} *M h* ≥ *0*
proof (*induct rule*: *integrable_simple_function_induct_nn*)
case (*cong f g*)
then show *?case* **using** *Bochner_Integration.integral_cong* **by** *force*
next
case (*indicator A y*)
hence *A* ≠ {} ⟹ *y* ≥ *0* **using** *sets.sets_into_space* **by** *fastforce*
then show *?case* **using** *indicator* **by** (*cases A* = {}, *auto simp add*: *scaleR_nonneg_nonneg*)
next
case (*add f g*)
then show *?case* **by** (*simp add*: *integrable_simple_function*)
qed
}
thus *?thesis* **using** *LIMSEQ_le_const*[*OF tendsto*, *of 0*] ∗∗ *simple* **by** *fastforce*

qed
also have ... = $integral^L\ M\ f$ **using** *nonneg* **by** (*auto intro*: *integral_cong_AE*)
finally show *?thesis* **.**
qed (*simp add*: *not_integrable_integral_eq*)

lemma *integral_mono_AE_banach*:
fixes *f g* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *integrable M f integrable M g AE x in M*. *f x* ≤ *g x*
shows $integral^L\ M\ f \leq integral^L\ M\ g$
using *integral_nonneg_AE_banach*[*of* λ*x*. *g x* − *f x*] *assms Bochner_Integration.integral_diff*[*OF assms*(*1*,*2*)] **by** *force*

lemma *integral_mono_banach*:
fixes *f g* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *integrable M f integrable M g* ⋀*x*. *x* ∈ *space M* ⟹ *f x* ≤ *g x*
shows $integral^L\ M\ f \leq integral^L\ M\ g$
using *integral_mono_AE_banach assms* **by** *blast*

9.7.2 Auxiliary Lemmas for Set Integrals

lemma *nn_set_integral_eq_set_integral*:
assumes [*measurable*]:*integrable M f*
and *AE x* ∈ *A in M*. *0* ≤ *f x A* ∈ *sets M*
shows (∫ $^+$*x*∈*A*. *f x* ∂*M*) = (∫ *x* ∈ *A*. *f x* ∂*M*)
proof−
have (∫ $^+$*x*. *indicator A x* ∗$_R$ *f x* ∂*M*) = (∫ *x* ∈ *A*. *f x* ∂*M*)
unfolding *set_lebesgue_integral_def*
using *assms*(*2*) **by** (*intro nn_integral_eq_integral*[*of* _ λ*x*. *indicat_real A x* ∗$_R$ *f x*], *blast intro*: *assms integrable_mult_indicator*, *fastforce*)
moreover have (∫ $^+$*x*. *indicator A x* ∗$_R$ *f x* ∂*M*) = (∫ $^+$*x*∈*A*. *f x* ∂*M*)
by (*metis ennreal_0 indicator_simps*(*1*) *indicator_simps*(*2*) *mult.commute mult_1 mult_zero_left real_scaleR_def*)
ultimately show *?thesis* **by** *argo*
qed

lemma *set_integral_restrict_space*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes Ω ∩ *space M* ∈ *sets M*
shows *set_lebesgue_integral* (*restrict_space M* Ω) *A f* = *set_lebesgue_integral M A* (λ*x*. *indicator* Ω *x* ∗$_R$ *f x*)
unfolding *set_lebesgue_integral_def*
by (*subst integral_restrict_space*, *auto intro*!: *integrable_mult_indicator assms simp*: *mult.commute*)

lemma *set_integral_const*:
fixes *c* :: *'b*::{*banach*, *second_countable_topology*}
assumes *A* ∈ *sets M emeasure M A* ≠ ∞

shows *set_lebesgue_integral M A* (λ_. *c*) = *measure M A* ∗$_R$ *c*
unfolding *set_lebesgue_integral_def*
using *assms* **by** (*metis has_bochner_integral_indicator has_bochner_integral_integral_eq infinity_ennreal_def less_top*)

lemma *set_integral_mono_banach*:
fixes *f g* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *set_integrable M A f set_integrable M A g*
⋀*x*. *x* ∈ *A* ⟹ *f x* ≤ *g x*
shows (*LINT x*:*A*|*M*. *f x*) ≤ (*LINT x*:*A*|*M*. *g x*)
using *assms* **unfolding** *set_integrable_def set_lebesgue_integral_def*
by (*auto intro*: *integral_mono_banach split*: *split_indicator*)

lemma *set_integral_mono_AE_banach*:
fixes *f g* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *set_integrable M A f set_integrable M A g AE x*∈*A in M*. *f x* ≤ *g x*
shows *set_lebesgue_integral M A f* ≤ *set_lebesgue_integral M A g* **using** *assms* **unfolding** *set_lebesgue_integral_def* **by** (*auto simp add*: *set_integrable_def intro*!: *integral_mono_AE_banach*[*of M* λ*x*. *indicator A x* ∗$_R$ *f x* λ*x*. *indicator A x* ∗$_R$ *g x*], *simp add*: *indicator_def*)

9.7.3 Integrability and Measurability of the Diameter

context
fixes *s* :: *nat* ⇒ *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*} **and** *M*
assumes *bounded*: ⋀*x*. *x* ∈ *space M* ⟹ *bounded* (*range* (λ*i*. *s i x*))
begin

lemma *borel_measurable_diameter*:
assumes [*measurable*]: ⋀*i*. (*s i*) ∈ *borel_measurable M*
shows (λ*x*. *diameter* {*s i x* |*i*. *n* ≤ *i*}) ∈ *borel_measurable M*
proof −
have {*s i x* |*i*. *N* ≤ *i*} ≠ {} **for** *x N* **by** *blast*
hence *diameter_SUP*: *diameter* {*s i x* |*i*. *N* ≤ *i*} = (*SUP* (*i*, *j*) ∈ {*N*..} × {*N*..}. *dist* (*s i x*) (*s j x*)) **for** *x N* **unfolding** *diameter_def* **by** (*auto intro*!: *arg_cong*[*of* _ _ *Sup*])

have *case_prod dist* ‘ ({*s i x* |*i*. *n* ≤ *i*} × {*s i x* |*i*. *n* ≤ *i*}) = ((λ(*i*, *j*). *dist* (*s i x*) (*s j x*)) ‘ ({*n*..} × {*n*..})) **for** *x* **by** *fast*
hence ∗: (λ*x*. *diameter* {*s i x* |*i*. *n* ≤ *i*}) = (λ*x*. *Sup* ((λ(*i*, *j*). *dist* (*s i x*) (*s j x*)) ‘ ({*n*..} × {*n*..}))) **using** *diameter_SUP* **by** (*simp add*: *case_prod_beta′*)

have *bounded* ((λ(*i*, *j*). *dist* (*s i x*) (*s j x*)) ‘ ({*n*..} × {*n*..})) **if** *x* ∈ *space M* **for** *x* **by** (*rule bounded_imp_dist_bounded*[*OF bounded*, *OF that*])
hence *bdd*: *bdd_above* ((λ(*i*, *j*). *dist* (*s i x*) (*s j x*)) ‘ ({*n*..} × {*n*..})) **if** *x* ∈ *space M* **for** *x* **using** *that bounded_imp_bdd_above* **by** *presburger*
have *fst p* ∈ *borel_measurable M snd p* ∈ *borel_measurable M* **if** *p* ∈ *s* ‘ {*n*..} ×

s ‘ $\{n..\}$ **for** p **using** *that* **by** *fastforce+*
hence $(\lambda x.\ fst\ p\ x - snd\ p\ x) \in borel\_measurable\ M$ **if** $p \in s$ ‘ $\{n..\} \times s$ ‘ $\{n..\}$ **for** p **using** *that borel_measurable_diff* **by** *simp*
hence $(\lambda x.\ case\ p\ of\ (f, g) \Rightarrow dist\ (f\ x)\ (g\ x)) \in borel\_measurable\ M$ **if** $p \in s$ ‘ $\{n..\} \times s$ ‘ $\{n..\}$ **for** p **unfolding** *dist_norm* **using** *that* **by** *measurable*
moreover have *countable* $(s$ ‘ $\{n..\} \times s$ ‘ $\{n..\})$ **by** (*intro countable_SIGMA countable_image, auto*)
ultimately show *?thesis* **unfolding** $*$ **by** (*auto intro*!: *borel_measurable_cSUP bdd*)
qed

lemma *integrable_bound_diameter*:
fixes $f :: 'a \Rightarrow real$
assumes *integrable M f*
and [*measurable*]: $\bigwedge i.\ (s\ i) \in borel\_measurable\ M$
and $\bigwedge x\ i.\ x \in space\ M \Longrightarrow norm\ (s\ i\ x) \leq f\ x$
shows *integrable M* $(\lambda x.\ diameter\ \{s\ i\ x\ |i.\ n \leq i\})$
proof −
have $\{s\ i\ x\ |i.\ N \leq i\} \neq \{\}$ **for** $x\ N$ **by** *blast*
hence *diameter_SUP*: $diameter\ \{s\ i\ x\ |i.\ N \leq i\} = (SUP\ (i, j) \in \{N..\} \times \{N..\}.\ dist\ (s\ i\ x)\ (s\ j\ x))$ **for** $x\ N$ **unfolding** *diameter_def* **by** (*auto intro*!: *arg_cong*[*of* _ _ *Sup*])
{
fix x **assume** x: $x \in space\ M$
let $?S = (\lambda(i, j).\ dist\ (s\ i\ x)\ (s\ j\ x))$ ‘ $(\{n..\} \times \{n..\})$
have *case_prod dist* ‘ $(\{s\ i\ x\ |i.\ n \leq i\} \times \{s\ i\ x\ |i.\ n \leq i\}) = (\lambda(i, j).\ dist\ (s\ i\ x)\ (s\ j\ x))$ ‘ $(\{n..\} \times \{n..\})$ **by** *fast*
hence $*$: $diameter\ \{s\ i\ x\ |i.\ n \leq i\} = Sup\ ?S$ **using** *diameter_SUP* **by** (*simp add*: *case_prod_beta'*)

have *bounded ?S* **by** (*rule bounded_imp_dist_bounded*[*OF bounded*[*OF x*]])
hence *Sup_S_nonneg*:$0 \leq Sup\ ?S$ **by** (*auto intro*!: *cSup_upper2 x bounded_imp_bdd_above*)

have $dist\ (s\ i\ x)\ (s\ j\ x) \leq 2 * f\ x$ **for** $i\ j$ **by** (*intro dist_triangle2*[*THEN order_trans, of _ 0*]) (*metis norm_conv_dist assms*(*3*) *x add_mono mult_2*)
hence $\forall c \in ?S.\ c \leq 2 * f\ x$ **by** *force*
hence $Sup\ ?S \leq 2 * f\ x$ **by** (*intro cSup_least, auto*)
hence $norm\ (Sup\ ?S) \leq 2 * norm\ (f\ x)$ **using** *Sup_S_nonneg* **by** *auto*
also have $\ldots = norm\ (2 *_R f\ x)$ **by** *simp*
finally have $norm\ (diameter\ \{s\ i\ x\ |i.\ n \leq i\}) \leq norm\ (2 *_R f\ x)$ **unfolding** $*$.
}
hence *AE x in M*. $norm\ (diameter\ \{s\ i\ x\ |i.\ n \leq i\}) \leq norm\ (2 *_R f\ x)$ **by** *blast*
thus *integrable M* $(\lambda x.\ diameter\ \{s\ i\ x\ |i.\ n \leq i\})$ **using** *borel_measurable_diameter* **by** (*intro Bochner_Integration.integrable_bound*[*OF assms*(*1*)[*THEN integrable_scaleR_right*[*of 2*]]], *measurable*)
qed
end

9.7.4 Averaging Theorem

We aim to lift results from the real case to arbitrary Banach spaces. Our fundamental tool in this regard will be the averaging theorem. The proof of this theorem is due to Serge Lang (Real and Functional Analysis) [5]. The theorem allows us to make statements about a function's value almost everywhere, depending on the value its integral takes on various sets of the measure space.

Before we introduce and prove the averaging theorem, we will first show the following lemma which is crucial for our proof. While not stated exactly in this manner, our proof makes use of the characterization of second-countable topological spaces given in the book General Topology by Ryszard Engelking (Theorem 4.1.15) [4]. (Engelking's book *General Topology*)

lemma *balls_countable_basis*:
obtains $D :: 'a :: \{metric\_space, second\_countable\_topology\}$ *set*
where *topological_basis* (*case_prod ball* ‘ $(D \times (\mathbb{Q} \cap \{0<..\}))$)
and *countable D*
and $D \neq \{\}$
proof −
obtain $D :: 'a$ *set* **where** *dense_subset*: *countable D* $D \neq \{\}$ $[\![open\ U;\ U \neq \{\}]\!] \Longrightarrow \exists y \in D.\ y \in U$ **for** *U* **using** *countable_dense_exists* **by** *blast*
have *topological_basis* (*case_prod ball* ‘ $(D \times (\mathbb{Q} \cap \{0<..\}))$)
proof (*intro topological_basis_iff*[*THEN iffD2*], *fast*, *clarify*)
fix *U* **and** $x :: 'a$ **assume** *asm*: *open U x* ∈ *U*
obtain *e* **where** *e*: *e* > *0 ball x e* ⊆ *U* **using** *asm openE* **by** *blast*
obtain *y* **where** *y*: *y* ∈ *D y* ∈ *ball x* (*e* / *3*) **using** *dense_subset*(*3*)[*OF open_ball*, *of x e* / *3*] *centre_in_ball*[*THEN iffD2*, *OF divide_pos_pos*[*OF e*(*1*), *of 3*]] **by** *force*
obtain *r* **where** *r*: *r* ∈ ℚ ∩ {*e*/*3*<..<*e*/*2*} **unfolding** *Rats_def* **using** *of_rat_dense*[*OF divide_strict_left_mono*[*OF _ e*(*1*)], *of 2 3*] **by** *auto*

have ∗: *x* ∈ *ball y r* **using** *r y* **by** (*simp add*: *dist_commute*)
hence *ball y r* ⊆ *U* **using** *r* **by** (*intro order_trans*[*OF _ e*(*2*)], *simp*, *metric*)
moreover have *ball y r* ∈ (*case_prod ball* ‘ $(D \times (\mathbb{Q} \cap \{0<..\}))$) **using** *y*(*1*) *r* **by** *force*
ultimately show $\exists B' \in$(*case_prod ball* ‘ $(D \times (\mathbb{Q} \cap \{0<..\}))$). $x \in B' \wedge B' \subseteq U$ **using** ∗ **by** *meson*
qed
thus *?thesis* **using** *that dense_subset* **by** *blast*
qed

context *sigma_finite_measure*
begin

To show statements concerning σ-finite measure spaces, one usually shows the statement for finite measure spaces and uses a limiting argument to show it for the σ-finite case. The following induction scheme formalizes this.

lemma *sigma_finite_measure_induct*[*case_names finite_measure*, *consumes 0*]:
assumes $\bigwedge(N :: 'a\ measure)\ \Omega.\ finite\_measure\ N \Longrightarrow N = restrict\_space\ M\ \Omega \Longrightarrow \Omega \in sets\ M \Longrightarrow emeasure\ N\ \Omega \neq \infty \Longrightarrow emeasure\ N\ \Omega \neq 0 \Longrightarrow almost\_everywhere\ N\ Q$
and [*measurable*]: *Measurable.pred M Q*
shows *almost_everywhere M Q*
proof −
have ∗: *almost_everywhere N Q* **if** *finite_measure N N = restrict_space M Ω Ω ∈ sets M emeasure N Ω ≠ ∞* **for** *N Ω* **using** *that* **by** (*cases emeasure N Ω = 0*, *auto intro*: *emeasure_0_AE assms*(*1*))

obtain *A* :: *nat* ⇒ *'a set* **where** *A*: *range A ⊆ sets M* $(\bigcup i.\ A\ i) = space\ M$ **and** *emeasure_finite*: *emeasure M* (*A i*) ≠ ∞ **for** *i* **using** *sigma_finite* **by** *metis*
note *A*(*1*)[*measurable*]
have *space_restr*: *space* (*restrict_space M* (*A i*)) = *A i* **for** *i* **unfolding** *space_restrict_space* **by** *simp*
{
fix *i*
have ∗: $\{x \in A\ i \cap space\ M.\ Q\ x\} = \{x \in space\ M.\ Q\ x\} \cap (A\ i)$ **by** *fast*
have *Measurable.pred* (*restrict_space M* (*A i*)) *Q* **using** *A* **by** (*intro measurableI*, *auto simp add*: *space_restr intro*!: *sets_restrict_space_iff*[*THEN iffD2*], *measurable*, *auto*)
}
note *this*[*measurable*]
{
fix *i*
have *finite_measure* (*restrict_space M* (*A i*)) **using** *emeasure_finite* **by** (*intro finite_measureI*, *subst space_restr*, *subst emeasure_restrict_space*, *auto*)
hence *emeasure* (*restrict_space M* (*A i*)) $\{x \in A\ i.\ \neg Q\ x\} = 0$ **using** *emeasure_finite* **by** (*intro AE_iff_measurable*[*THEN iffD1*, *OF _ _ ∗*], *measurable*, *subst space_restr*[*symmetric*], *intro sets.top*, *auto simp add*: *emeasure_restrict_space*)
hence *emeasure M* $\{x \in A\ i.\ \neg\ Q\ x\} = 0$ **by** (*subst emeasure_restrict_space*[*symmetric*], *auto*)
}
hence *emeasure M* $(\bigcup i.\ \{x \in A\ i.\ \neg\ Q\ x\}) = 0$ **by** (*intro emeasure_UN_eq_0*, *auto*)
moreover have $(\bigcup i.\ \{x \in A\ i.\ \neg\ Q\ x\}) = \{x \in space\ M.\ \neg\ Q\ x\}$ **using** *A* **by** *auto*
ultimately show *?thesis* **by** (*intro AE_iff_measurable*[*THEN iffD2*], *auto*)
qed

Real Functional Analysis - Lang

The Averaging Theorem allows us to make statements concerning how a function behaves almost everywhere, depending on its behaviour on average.

lemma *averaging_theorem*:

fixes f::_ $\Rightarrow$ 'b::{*second_countable_topology*, *banach*}
assumes [*measurable*]: *integrable M f*
and *closed*: *closed S*
and $\bigwedge A$. *A* $\in$ *sets M* $\Longrightarrow$ *measure M A* > *0* $\Longrightarrow$ (*1* / *measure M A*) $*_R$ *set_lebesgue_integral M A f* $\in$ *S*
shows *AE x in M*. *f x* $\in$ *S*
proof (*induct rule*: *sigma_finite_measure_induct*)
case (*finite_measure N* Ω)

interpret *finite_measure N* **by** (*rule finite_measure*)

have *integrable*[*measurable*]: *integrable N f* **using** *assms finite_measure* **by** (*auto simp*: *integrable_restrict_space integrable_mult_indicator*)
have *average*: (*1* / *Sigma_Algebra.measure N A*) $*_R$ *set_lebesgue_integral N A f* $\in$ *S* **if** *A* $\in$ *sets N measure N A* > *0* **for** *A*
proof −
have $*$: *A* $\in$ *sets M* **using** *that* **by** (*simp add*: *sets_restrict_space_iff finite_measure*)
have *A* = *A* $\cap$ Ω **by** (*metis finite_measure*(*2*,*3*) *inf.orderE sets.sets_into_space space_restrict_space that*(*1*))
hence *set_lebesgue_integral N A f* = *set_lebesgue_integral M A f* **unfolding** *finite_measure* **by** (*subst set_integral_restrict_space*, *auto simp add*: *finite_measure set_lebesgue_integral_def indicator_inter_arith*[*symmetric*])
moreover have *measure N A* = *measure M A* **using** *that* **by** (*auto intro*!: *measure_restrict_space simp add*: *finite_measure sets_restrict_space_iff*)
ultimately show *?thesis* **using** *that* $*$ *assms*(*3*) **by** *presburger*
qed

obtain *D* :: 'b *set* **where** *balls_basis*: *topological_basis* (*case_prod ball* ‘ (*D* × (ℚ ∩ {*0*<..}))) **and** *countable_D*: *countable D* **using** *balls_countable_basis* **by** *blast*
have *countable_balls*: *countable* (*case_prod ball* ‘ (*D* × (ℚ ∩ {*0*<..}))) **using** *countable_rat countable_D* **by** *blast*

obtain *B* **where** *B_balls*: *B* $\subseteq$ *case_prod ball* ‘ (*D* × (ℚ ∩ {*0*<..})) $\bigcup$ *B* = −*S* **using** *topological_basis*[*THEN iffD1*, *OF balls_basis*] *open_Compl*[*OF assms*(*2*)] **by** *meson*
hence *countable_B*: *countable B* **using** *countable_balls countable_subset* **by** *fast*

define *b* **where** *b* = *from_nat_into* (*B* ∪ {{}})
have *B* ∪ {{}} ≠ {} **by** *simp*
have *range_b*: *range b* = *B* ∪ {{}} **using** *countable_B* **by** (*auto simp add*: *b_def intro*!: *range_from_nat_into*)
have *open_b*: *open* (*b i*) **for** *i* **unfolding** *b_def* **using** *B_balls open_ball from_nat_into*[*of B* ∪ {{}} *i*] **by** *force*
have *Union_range_b*: $\bigcup$(*range b*) = −*S* **using** *B_balls range_b* **by** *simp*

{
fix *v r* **assume** *ball_in_Compl*: *ball v r* $\subseteq$ −*S*
define *A* **where** *A* = *f* −‘ *ball v r* ∩ *space N*

have *dist_less*: *dist* (*f x*) *v* < *r* **if** *x* ∈ *A* **for** *x* **using** *that* **unfolding** *A_def vimage_def* **by** (*simp add*: *dist_commute*)
hence *AE_less*: *AE x* ∈ *A in N*. *norm* (*f x* − *v*) < *r* **by** (*auto simp add*: *dist_norm*)
have ∗: *A* ∈ *sets N* **unfolding** *A_def* **by** *simp*
have *emeasure N A* = *0*
proof −
{
assume *asm*: *emeasure N A* > *0*
hence *measure_pos*: *measure N A* > *0* **unfolding** *emeasure_eq_measure* **by** *simp*
hence (*1* / *measure N A*) ∗$_R$ *set_lebesgue_integral N A f* − *v*
= (*1* / *measure N A*) ∗$_R$ *set_lebesgue_integral N A* (λ*x*. *f x* − *v*)
using *integrable integrable_const* ∗
by (*subst set_integral_diff*(*2*), *auto simp add*: *set_integrable_def set_integral_const*[*OF* ∗] *algebra_simps intro*!: *integrable_mult_indicator*)
moreover have *norm* (∫ *x*∈*A*. (*f x* − *v*)∂*N*) ≤ (∫ *x*∈*A*. *norm* (*f x* − *v*)∂*N*)
using ∗ **by** (*auto intro*!: *integral_norm_bound*[*of N* λ*x*. *indicator A x* ∗$_R$ (*f x* − *v*), *THEN order_trans*] *integrable_mult_indicator integrable simp add*: *set_lebesgue_integral_def*)
ultimately have *norm* ((*1* / *measure N A*) ∗$_R$ *set_lebesgue_integral N A f* − *v*)
≤ *set_lebesgue_integral N A* (λ*x*. *norm* (*f x* − *v*)) / *measure N A* **using** *asm* **by** (*auto intro*: *divide_right_mono*)
also have ... < *set_lebesgue_integral N A* (λ*x*. *r*) / *measure N A*
unfolding *set_lebesgue_integral_def*
using *asm* ∗ *integrable integrable_const AE_less measure_pos*
by (*intro divide_strict_right_mono integral_less_AE*[*of* _ _ *A*] *integrable_mult_indicator*)
(*fastforce simp add*: *dist_less dist_norm indicator_def*)+
also have ... = *r* **using** ∗ *measure_pos* **by** (*simp add*: *set_integral_const*)
finally have *dist* ((*1* / *measure N A*) ∗$_R$ *set_lebesgue_integral N A f*) *v* < *r* **by** (*subst dist_norm*)
hence *False* **using** *average*[*OF* ∗ *measure_pos*] **by** (*metis ComplD dist_commute in_mono mem_ball ball_in_Compl*)
}
thus *?thesis* **by** *fastforce*
qed
}
note ∗ = *this*
{
fix *b*′ **assume** *b*′ ∈ *B*
hence *ball_subset_Compl*: *b*′ ⊆ −*S* **and** *ball_radius_pos*: ∃ *v* ∈ *D*. ∃ *r*>*0*. *b*′ = *ball v r* **using** *B_balls* **by** (*blast*, *fast*)
}
note ∗∗ = *this*
hence *emeasure N* (*f* −‘ *b i* ∩ *space N*) = *0* **for** *i* **by** (*cases b i* = {}, *simp*) (*metis UnE singletonD* ∗ *range_b*[*THEN eq_refl*, *THEN range_subsetD*])

hence *emeasure N* ($\bigcup i.\ f\ -`\ b\ i \cap$ *space N*) = *0* **using** *open_b* **by** (*intro emeasure_UN_eq_0*) *fastforce+*
moreover have ($\bigcup i.\ f\ -`\ b\ i \cap$ *space N*) = *f* $-`$ ($\bigcup$(*range b*)) $\cap$ *space N* **by** *blast*
ultimately have *emeasure N* (*f* $-`$ (−*S*) $\cap$ *space N*) = *0* **using** *Union_range_b* **by** *argo*
hence *AE x in N. f x* $\notin$ −*S* **using** *open_Compl*[*OF assms*(*2*)] **by** (*intro AE_iff_measurable*[*THEN iffD2*], *auto*)
thus *?case* **by** *force*
qed (*simp add*: *pred_sets2*[*OF borel_closed*] *assms*(*2*))

lemma *density_zero*:
fixes *f*::*'a* ⇒ *'b*::{*second_countable_topology*, *banach*}
assumes *integrable M f*
and *density_0*: $\bigwedge$*A*. *A* ∈ *sets M* ⟹ *set_lebesgue_integral M A f* = *0*
shows *AE x in M. f x* = *0*
using *averaging_theorem*[*OF assms*(*1*), *of* {*0*}] *assms*(*2*)
by (*simp add*: *scaleR_nonneg_nonneg*)

The following lemma shows that densities are unique in Banach spaces.

lemma *density_unique_banach*:
fixes *f f'*::*'a* ⇒ *'b*::{*second_countable_topology*, *banach*}
assumes *integrable M f integrable M f'*
and *density_eq*: $\bigwedge$*A*. *A* ∈ *sets M* ⟹ *set_lebesgue_integral M A f* = *set_lebesgue_integral M A f'*
shows *AE x in M. f x* = *f' x*
proof−
{
fix *A* **assume** *asm*: *A* ∈ *sets M*
hence *LINT x*|*M. indicat_real A x* ∗$_R$ (*f x* − *f' x*) = *0* **using** *density_eq assms*(*1*,*2*) **by** (*simp add*: *set_lebesgue_integral_def algebra_simps Bochner_Integration.integral_diff*[*O integrable_mult_indicator*(*1*,*1*)])
}
thus *?thesis* **using** *density_zero*[*OF Bochner_Integration.integrable_diff*[*OF assms*(*1*,*2*)]] **by** (*simp add*: *set_lebesgue_integral_def*)
qed

lemma *density_nonneg*:
fixes *f*::_ ⇒ *'b*::{*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *integrable M f*
and $\bigwedge$*A*. *A* ∈ *sets M* ⟹ *set_lebesgue_integral M A f* ≥ *0*
shows *AE x in M. f x* ≥ *0*
using *averaging_theorem*[*OF assms*(*1*), *of* {*0*..}, *OF closed_atLeast*] *assms*(*2*)
by (*simp add*: *scaleR_nonneg_nonneg*)

corollary *integral_nonneg_eq_0_iff_AE_banach*:
fixes *f* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *f*[*measurable*]: *integrable M f* **and** *nonneg*: *AE x in M. 0* ≤ *f x*

shows $integral^L\ M\ f = 0 \longleftrightarrow (AE\ x\ in\ M.\ f\ x = 0)$
proof
assume $*$: $integral^L\ M\ f = 0$
{
fix A **assume** *asm*: $A \in sets\ M$
have $0 \leq integral^L\ M\ (\lambda x.\ indicator\ A\ x *_R f\ x)$ **using** *nonneg* **by** (*subst integral_zero*[*of M*, *symmetric*], *intro integral_mono_AE_banach integrable_mult_indicator asm f integrable_zero*, *auto simp add*: *indicator_def*)
moreover have $\ldots \leq integral^L\ M\ f$ **using** *nonneg* **by** (*intro integral_mono_AE_banach integrable_mult_indicator asm f*, *auto simp add*: *indicator_def*)
ultimately have *set_lebesgue_integral M A f* $= 0$ **unfolding** *set_lebesgue_integral_def* **using** $*$ **by** *force*
}
thus $AE\ x\ in\ M.\ f\ x = 0$ **by** (*intro density_zero f*, *blast*)
qed (*auto simp add*: *integral_eq_zero_AE*)

corollary *integral_eq_mono_AE_eq_AE*:
fixes $f\ g :: 'a \Rightarrow 'b :: \{$*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*$\}$
assumes *integrable M f integrable M g* $integral^L\ M\ f = integral^L\ M\ g$ $AE\ x\ in\ M.\ f\ x \leq g\ x$
shows $AE\ x\ in\ M.\ f\ x = g\ x$
proof −
define h **where** $h = (\lambda x.\ g\ x - f\ x)$
have $AE\ x\ in\ M.\ h\ x = 0$ **unfolding** *h_def* **using** *assms* **by** (*subst integral_nonneg_eq_0_iff_AE_banach*[*symmetric*]) *auto*
then show *?thesis* **unfolding** *h_def* **by** *auto*
qed

end

end

9.8 Homeomorphism Theorems

theory *Homeomorphism*
imports *Homotopy*
begin

lemma *homeomorphic_spheres'*:
fixes $a :: 'a :: euclidean\_space$ **and** $b :: 'b :: euclidean\_space$
assumes $0 < \delta$ **and** *dimeq*: $DIM('a) = DIM('b)$
shows (*sphere a* δ) *homeomorphic* (*sphere b* δ)
proof −
obtain $f :: 'a \Rightarrow 'b$ **and** g **where** *linear f linear g*
and *fg*: $\bigwedge x.\ norm(f\ x) = norm\ x\ \bigwedge y.\ norm(g\ y) = norm\ y\ \bigwedge x.\ g(f\ x) = x\ \bigwedge y.\ f(g\ y) = y$
by (*blast intro*: *isomorphisms_UNIV_UNIV* [*OF dimeq*])
then have *continuous_on UNIV f continuous_on UNIV g*

using *linear_continuous_on linear_linear* **by** *blast+*
then show *?thesis*
unfolding *homeomorphic_minimal*
apply(*rule_tac x*=$\lambda x.\ b + f(x - a)$ **in** *exI*)
apply(*rule_tac x*=$\lambda x.\ a + g(x - b)$ **in** *exI*)
using *assms*
apply (*force intro*: *continuous_intros*
continuous_on_compose2 [*of* _ *f*] *continuous_on_compose2* [*of* _ *g*] *simp*: *dist_commute dist_norm fg*)
done
qed

lemma *homeomorphic_spheres_gen*:
fixes *a* :: *'a*::*euclidean_space* **and** *b* :: *'b*::*euclidean_space*
assumes $0 < r$ $0 < s$ *DIM*(*'a*::*euclidean_space*) = *DIM*(*'b*::*euclidean_space*)
shows (*sphere a r homeomorphic sphere b s*)
using *assms homeomorphic_trans* [*OF homeomorphic_spheres homeomorphic_spheres'*]
by *auto*

9.8.1 Homeomorphism of all convex compact sets with nonempty interior

proposition
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S* **and** *0*: $0 \in$ *rel_interior S*
and *star*: $\bigwedge x.\ x \in S \Longrightarrow$ *open_segment 0 x* $\subseteq$ *rel_interior S*
shows *starlike_compact_projective1_0*:
S − *rel_interior S homeomorphic sphere 0 1* ∩ *affine hull S*
(**is** *?SMINUS homeomorphic ?SPHER*)
and *starlike_compact_projective2_0*:
S homeomorphic cball 0 1 ∩ *affine hull S*
(**is** *S homeomorphic ?CBALL*)
proof −
have *starI*: $(u *_R x) \in$ *rel_interior S* **if** $x \in S$ $0 \le u$ $u < 1$ **for** *x u*
proof (*cases x=0* ∨ *u=0*)
case *True* **with** *0* **show** *?thesis* **by** *force*
next
case *False* **with** *that* **show** *?thesis*
by (*auto simp*: *in_segment intro*: *star* [*THEN subsetD*])
qed
have $0 \in S$ **using** *assms rel_interior_subset* **by** *auto*
define *proj* **where** *proj* ≡ λx::*'a*. $x /_R$ *norm x*
have *eqI*: $x = y$ **if** *proj x* = *proj y norm x* = *norm y* **for** *x y*
using *that* **by** (*force simp*: *proj_def*)
then have *iff_eq*: $\bigwedge x\ y.\ (proj\ x = proj\ y \wedge norm\ x = norm\ y) \longleftrightarrow x = y$
by *blast*
have *projI*: $x \in$ *affine hull S* $\Longrightarrow$ *proj x* $\in$ *affine hull S* **for** *x*
by (*metis* ‹$0 \in S$› *affine_hull_span_0 hull_inc span_mul proj_def*)
have *nproj1* [*simp*]: $x \neq 0 \Longrightarrow norm(proj\ x) = 1$ **for** *x*

```
    by (simp add: proj_def)
  have proj0_iff [simp]: proj x = 0 ⟷ x = 0 for x
    by (simp add: proj_def)
  have cont_proj: continuous_on (UNIV - {0}) proj
    unfolding proj_def by (rule continuous_intros | force)+
  have proj_spherI: ⋀x. ⟦x ∈ affine hull S; x ≠ 0⟧ ⟹ proj x ∈ ?SPHER
    by (simp add: projI)
  have bounded S closed S
    using ‹compact S› compact_eq_bounded_closed by blast+
  have inj_on_proj: inj_on proj (S - rel_interior S)
  proof
    fix x y
    assume x: x ∈ S - rel_interior S
       and y: y ∈ S - rel_interior S and eq: proj x = proj y
    then have xynot: x ≠ 0 y ≠ 0 x ∈ S y ∈ S x ∉ rel_interior S y ∉ rel_interior
S
      using 0 by auto
    consider norm x = norm y | norm x < norm y | norm x > norm y by linarith
    then show x = y
    proof cases
      assume norm x = norm y
        with iff_eq eq show x = y by blast
    next
      assume *: norm x < norm y
      have x /R norm x = norm x *R (x /R norm x) /R norm (norm x *R (x /R
norm x))
        by force
      then have proj ((norm x / norm y) *R y) = proj x
        by (metis (no_types) divide_inverse local.proj_def eq scaleR_scaleR)
      then have [simp]: (norm x / norm y) *R y = x
        by (rule eqI) (simp add: ‹y ≠ 0›)
      have no: 0 ≤ norm x / norm y norm x / norm y < 1
        using * by (auto simp: field_split_simps)
      then show x = y
        using starI [OF ‹y ∈ S› no] xynot by auto
    next
      assume *: norm x > norm y
      have y /R norm y = norm y *R (y /R norm y) /R norm (norm y *R (y /R
norm y))
        by force
      then have proj ((norm y / norm x) *R x) = proj y
        by (metis (no_types) divide_inverse local.proj_def eq scaleR_scaleR)
      then have [simp]: (norm y / norm x) *R x = y
        by (rule eqI) (simp add: ‹x ≠ 0›)
      have no: 0 ≤ norm y / norm x norm y / norm x < 1
        using * by (auto simp: field_split_simps)
      then show x = y
        using starI [OF ‹x ∈ S› no] xynot by auto
    qed
```

qed
have $\exists$ *surf*. *homeomorphism* (*S* − *rel_interior S*) *?SPHER proj surf*
proof (*rule homeomorphism_compact*)
show *compact* (*S* − *rel_interior S*)
using ‹*compact S*› *compact_rel_boundary* **by** *blast*
show *continuous_on* (*S* − *rel_interior S*) *proj*
using *0* **by** (*blast intro*: *continuous_on_subset* [*OF cont_proj*])
show *proj* ‘ (*S* − *rel_interior S*) = *?SPHER*
proof
show *proj* ‘ (*S* − *rel_interior S*) ⊆ *?SPHER*
using *0* **by** (*force simp*: *hull_inc projI intro*: *nproj1*)
show *?SPHER* ⊆ *proj* ‘ (*S* − *rel_interior S*)
proof (*clarsimp simp*: *proj_def*)
fix *x*
assume *x* ∈ *affine hull S* **and** *nox*: *norm x* = *1*
then have *x* ≠ *0* **by** *auto*
obtain *d* **where** *0* < *d* **and** *dx*: (*d* ∗*R x*) ∈ *rel_frontier S*
and *ri*: ⋀*e*. ⟦*0* ≤ *e*; *e* < *d*⟧ ⟹ (*e* ∗*R x*) ∈ *rel_interior S*
using *ray_to_rel_frontier* [*OF* ‹*bounded S*› *0*] ‹*x* ∈ *affine hull S*› ‹*x* ≠ *0*› **by** *auto*
show *x* ∈ (λ*x*. *x* /*R norm x*) ‘ (*S* − *rel_interior S*)
proof
show *x* = *d* ∗*R x* /*R norm* (*d* ∗*R x*)
using ‹*0* < *d*› **by** (*auto simp*: *nox*)
show *d* ∗*R x* ∈ *S* − *rel_interior S*
using *dx* ‹*closed S*› **by** (*auto simp*: *rel_frontier_def*)
qed
qed
qed
qed (*rule inj_on_proj*)
then obtain *surf* **where** *surf*: *homeomorphism* (*S* − *rel_interior S*) *?SPHER proj surf*
by *blast*
then have *cont_surf*: *continuous_on* (*proj* ‘ (*S* − *rel_interior S*)) *surf*
by (*auto simp*: *homeomorphism_def*)
have *surf_nz*: ⋀*x*. *x* ∈ *?SPHER* ⟹ *surf x* ≠ *0*
by (*metis 0 DiffE homeomorphism_def imageI surf*)
have *cont_nosp*: *continuous_on* (*?SPHER*) (λ*x*. *norm x* ∗*R* ((*surf o proj*) *x*))
proof (*intro continuous_intros*)
show *continuous_on* (*sphere 0 1* ∩ *affine hull S*) *proj*
by (*rule continuous_on_subset* [*OF cont_proj*], *force*)
show *continuous_on* (*proj* ‘ (*sphere 0 1* ∩ *affine hull S*)) *surf*
by (*intro continuous_on_subset* [*OF cont_surf*]) (*force simp*: *homeomorphism_image1* [*OF surf*] *dest*: *proj_spherI*)
qed
have *surfpS*: ⋀*x*. ⟦*norm x* = *1*; *x* ∈ *affine hull S*⟧ ⟹ *surf* (*proj x*) ∈ *S*
by (*metis* (*full_types*) *DiffE* ‹*0* ∈ *S*› *homeomorphism_def image_eqI norm_zero proj_spherI real_vector.scale_zero_left scaleR_one surf*)
have ∗: ∃ *y*. *norm y* = *1* ∧ *y* ∈ *affine hull S* ∧ *x* = *surf* (*proj y*)

```
        if x ∈ S x ∉ rel_interior S for x
  proof −
    have proj x ∈ ?SPHER
      by (metis (full_types) 0 hull_inc proj_spherI that)
    moreover have surf (proj x) = x
      by (metis Diff_iff homeomorphism_def surf that)
    ultimately show ?thesis
     by (metis ‹⋀x. x ∈ ?SPHER ⟹ surf x ≠ 0› hull_inc inverse_1 local.proj_def
norm_sgn projI scaleR_one sgn_div_norm that(1))
  qed
  have surfp_notin: ⋀x. ⟦norm x = 1; x ∈ affine hull S⟧ ⟹ surf (proj x) ∉
rel_interior S
     by (metis (full_types) DiffE one_neq_zero homeomorphism_def image_eqI
norm_zero proj_spherI surf)
  have no_sp_im: (λx. norm x *R surf (proj x)) ‘ (?SPHER) = S − rel_interior
S
    by (auto simp: surfpS image_def Bex_def surfp_notin *)
  have inj_spher: inj_on (λx. norm x *R surf (proj x)) ?SPHER
  proof
    fix x y
    assume xy: x ∈ ?SPHER y ∈ ?SPHER
       and eq:  norm x *R surf (proj x) = norm y *R surf (proj y)
    then have norm x = 1 norm y = 1 x ∈ affine hull S y ∈ affine hull S
      using 0 by auto
    with eq show x = y
      by (simp add: proj_def) (metis surf xy homeomorphism_def)
  qed
  have co01: compact ?SPHER
    by (simp add: compact_Int_closed)
  show ?SMINUS homeomorphic ?SPHER
    using homeomorphic_def surf by blast
  have proj_scaleR: ⋀a x. 0 < a ⟹ proj (a *R x) = proj x
    by (simp add: proj_def)
  have cont_sp0: continuous_on (affine hull S − {0}) (surf o proj)
  proof (rule continuous_on_compose [OF continuous_on_subset [OF cont_proj]])
    show continuous_on (proj ‘ (affine hull S − {0})) surf
    using homeomorphism_image1 proj_spherI surf by (intro continuous_on_subset
[OF cont_surf]) fastforce
  qed auto
  obtain B where B>0 and B: ⋀x. x ∈ S ⟹ norm x ≤ B
   by (metis compact_imp_bounded ‹compact S› bounded_pos_less less_eq_real_def)
  have cont_nosp: continuous (at x within ?CBALL) (λx. norm x *R surf (proj
x))
        if norm x ≤ 1 x ∈ affine hull S for x
  proof (cases x=0)
    case True
    have (norm ⟶ 0) (at 0 within cball 0 1 ∩ affine hull S)
      by (simp add: tendsto_norm_zero eventually_at)
    with True show ?thesis
```

```
    apply (simp add: continuous_within)
    apply (rule lim_null_scaleR_bounded [where B=B])
    using B ‹0 < B› local.proj_def projI surfpS by (auto simp: eventually_at)
  next
    case False
    then have ∀⇩F x in at x. (x ∈ affine hull S - {0}) = (x ∈ affine hull S)
      by (force simp: False eventually_at)
    moreover
    have continuous (at x within affine hull S - {0}) (λx. surf (proj x))
    using cont_sp0 False that by (auto simp add: continuous_on_eq_continuous_within)
    ultimately have ∗: continuous (at x within affine hull S) (λx. surf (proj x))
    by (simp add: continuous_within Lim_transform_within_set continuous_on_eq_continuous_within
    show ?thesis
    by (intro continuous_within_subset [where S = affine hull S, OF _ Int_lower2]
continuous_intros ∗)
  qed
  have cont_nosp2: continuous_on ?CBALL (λx. norm x ∗⇩R ((surf o proj) x))
    by (simp add: continuous_on_eq_continuous_within cont_nosp)
  have norm y ∗⇩R surf (proj y) ∈ S  if y ∈ cball 0 1 and yaff: y ∈ affine hull S
for y
  proof (cases y=0)
    case True then show ?thesis
      by (simp add: ‹0 ∈ S›)
  next
    case False
    then have norm y ∗⇩R surf (proj y) = norm y ∗⇩R surf (proj (y /⇩R norm y))
      by (simp add: proj_def)
    have norm y ≤ 1 using that by simp
    have surf (proj (y /⇩R norm y)) ∈ S
      using False local.proj_def nproj1 projI surfpS yaff by blast
    then have surf (proj y) ∈ S
      by (simp add: False proj_def)
    then show norm y ∗⇩R surf (proj y) ∈ S
    by (metis dual_order.antisym le_less_linear norm_ge_zero rel_interior_subset
scaleR_one
              starI subset_eq ‹norm y ≤ 1›)
  qed
  moreover have x ∈ (λx. norm x ∗⇩R surf (proj x)) ‘ (?CBALL) if x ∈ S for x
  proof (cases x=0)
    case True with that hull_inc  show ?thesis by fastforce
  next
    case False
    then have psp: proj (surf (proj x)) = proj x
      by (metis homeomorphism_def hull_inc proj_spherI surf that)
    have nxx: norm x ∗⇩R proj x = x
      by (simp add: False local.proj_def)
    have affineI: (1 / norm (surf (proj x))) ∗⇩R x ∈ affine hull S
      by (metis ‹0 ∈ S› affine_hull_span_0 hull_inc span_clauses(4) that)
    have sproj_nz: surf (proj x) ≠ 0
```

```
      by (metis False proj0_iff psp)
    then have proj x = proj (proj x)
      by (metis False nxx proj_scaleR zero_less_norm_iff)
    moreover have scaleproj: ⋀a r. r *R proj a = (r / norm a) *R a
      by (simp add: divide_inverse local.proj_def)
    ultimately have (norm (surf (proj x)) / norm x) *R x ∉ rel_interior S
      by (metis (no_types) sproj_nz divide_self_if hull_inc norm_eq_zero nproj1
projI psp scaleR_one surfp_notin that)
    then have (norm (surf (proj x)) / norm x) ≥ 1
     using starI [OF that] by (meson starI [OF that] le_less_linear norm_ge_zero
zero_le_divide_iff)
    then have nole: norm x ≤ norm (surf (proj x))
      by (simp add: le_divide_eq_1)
    let ?inx = x /R norm (surf (proj x))
    show ?thesis
    proof
      show x = norm ?inx *R surf (proj ?inx)
      by (simp add: field_simps) (metis inverse_eq_divide nxx positive_imp_inverse_positive
proj_scaleR psp scaleproj sproj_nz zero_less_norm_iff)
      qed (auto simp: field_split_simps nole affineI)
  qed
  ultimately have im_cball: (λx. norm x *R surf (proj x)) ‘ ?CBALL = S
    by blast
  have inj_cball: inj_on (λx. norm x *R surf (proj x)) ?CBALL
  proof
    fix x y
    assume x ∈ ?CBALL y ∈ ?CBALL
      and eq: norm x *R surf (proj x) = norm y *R surf (proj y)
    then have x: x ∈ affine hull S and y: y ∈ affine hull S
      using 0 by auto
    show x = y
    proof (cases x=0 ∨ y=0)
      case True then show x = y using eq proj_spherI surf_nz x y by force
    next
      case False
      with x y have speq: surf (proj x) = surf (proj y)
      by (metis eq homeomorphism_apply2 proj_scaleR proj_spherI surf zero_less_norm_iff)
      then have norm x = norm y
      by (metis ‹x ∈ affine hull S› ‹y ∈ affine hull S› eq proj_spherI real_vector.scale_cancel_right
surf_nz)
      moreover have proj x = proj y
        by (metis (no_types) False speq homeomorphism_apply2 proj_spherI surf x
y)
      ultimately show x = y
        using eq eqI by blast
    qed
  qed
  have co01: compact ?CBALL
    by (simp add: compact_Int_closed)
```

show *S homeomorphic ?CBALL*
using *homeomorphic_compact* [*OF co01 cont_nosp2* [*unfolded o_def*] *im_cball inj_cball*] *homeomorphic_sym* **by** *blast*
qed

corollary
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S* **and** *a*: *a* ∈ *rel_interior S*
and *star*: ⋀*x*. *x* ∈ *S* ⟹ *open_segment a x* ⊆ *rel_interior S*
shows *starlike_compact_projective1*:
S − *rel_interior S homeomorphic sphere a 1* ∩ *affine hull S*
and *starlike_compact_projective2*:
S homeomorphic cball a 1 ∩ *affine hull S*
proof −
have *1*: *compact* ((+) (−*a*) ‘ *S*) **by** (*meson assms compact_translation*)
have *2*: *0* ∈ *rel_interior* ((+) (−*a*) ‘ *S*)
using *a rel_interior_translation* [*of* − *a S*] **by** (*simp cong*: *image_cong_simp*)
have *3*: *open_segment 0 x* ⊆ *rel_interior* ((+) (−*a*) ‘ *S*) **if** *x* ∈ ((+) (−*a*) ‘ *S*) **for** *x*
proof −
have *x*+*a* ∈ *S* **using** *that* **by** *auto*
then have *open_segment a* (*x*+*a*) ⊆ *rel_interior S* **by** (*metis star*)
then show *?thesis* **using** *open_segment_translation* [*of a 0 x*]
using *rel_interior_translation* [*of* − *a S*] **by** (*fastforce simp add*: *ac_simps image_iff cong*: *image_cong_simp*)
qed
have *S* − *rel_interior S homeomorphic* ((+) (−*a*) ‘ *S*) − *rel_interior* ((+) (−*a*) ‘ *S*)
by (*metis rel_interior_translation translation_diff homeomorphic_translation*)
also have ... *homeomorphic sphere 0 1* ∩ *affine hull* ((+) (−*a*) ‘ *S*)
by (*rule starlike_compact_projective1_0* [*OF 1 2 3*])
also have ... = (+) (−*a*) ‘ (*sphere a 1* ∩ *affine hull S*)
by (*metis affine_hull_translation left_minus sphere_translation translation_Int*)
also have ... *homeomorphic sphere a 1* ∩ *affine hull S*
using *homeomorphic_translation homeomorphic_sym* **by** *blast*
finally show *S* − *rel_interior S homeomorphic sphere a 1* ∩ *affine hull S* **.**

have *S homeomorphic* ((+) (−*a*) ‘ *S*)
by (*metis homeomorphic_translation*)
also have ... *homeomorphic cball 0 1* ∩ *affine hull* ((+) (−*a*) ‘ *S*)
by (*rule starlike_compact_projective2_0* [*OF 1 2 3*])
also have ... = (+) (−*a*) ‘ (*cball a 1* ∩ *affine hull S*)
by (*metis affine_hull_translation left_minus cball_translation translation_Int*)
also have ... *homeomorphic cball a 1* ∩ *affine hull S*
using *homeomorphic_translation homeomorphic_sym* **by** *blast*
finally show *S homeomorphic cball a 1* ∩ *affine hull S* **.**
qed

corollary *starlike_compact_projective_special*:

```
  assumes compact S
    and cb01: cball (0::'a::euclidean_space) 1 ⊆ S
    and scale: ⋀x u. ⟦x ∈ S; 0 ≤ u; u < 1⟧ ⟹ u *R x ∈ S − frontier S
  shows S homeomorphic (cball (0::'a::euclidean_space) 1)
proof −
  have ball 0 1 ⊆ interior S
    using cb01 interior_cball interior_mono by blast
  then have 0: 0 ∈ rel_interior S
   by (meson centre_in_ball subsetD interior_subset_rel_interior le_numeral_extra(2)
not_le)
  have [simp]: affine hull S = UNIV
    using ‹ball 0 1 ⊆ interior S› by (auto intro!: affine_hull_nonempty_interior)
  have star: open_segment 0 x ⊆ rel_interior S if x ∈ S for x
  proof
    fix p assume p ∈ open_segment 0 x
    then obtain u where x ≠ 0 and u: 0 ≤ u u < 1 and p: u *R x = p
      by (auto simp: in_segment)
    then show p ∈ rel_interior S
    using scale [OF that u] closure_subset frontier_def interior_subset_rel_interior
by fastforce
  qed
  show ?thesis
    using starlike_compact_projective2_0 [OF ‹compact S› 0 star] by simp
qed

lemma homeomorphic_convex_lemma:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes convex S compact S convex T compact T
      and affeq: aff_dim S = aff_dim T
    shows (S − rel_interior S) homeomorphic (T − rel_interior T) ∧
          S homeomorphic T
proof (cases rel_interior S = {} ∨ rel_interior T = {})
  case True
    then show ?thesis
      by (metis Diff_empty affeq ‹convex S› ‹convex T› aff_dim_empty homeo-
morphic_empty rel_interior_eq_empty aff_dim_empty)
next
  case False
  then obtain a b where a: a ∈ rel_interior S and b: b ∈ rel_interior T by auto
  have starS: ⋀x. x ∈ S ⟹ open_segment a x ⊆ rel_interior S
    using rel_interior_closure_convex_segment
          a ‹convex S› closure_subset subsetCE by blast
  have starT: ⋀x. x ∈ T ⟹ open_segment b x ⊆ rel_interior T
    using rel_interior_closure_convex_segment
          b ‹convex T› closure_subset subsetCE by blast
  let ?aS = (+) (−a) ‘ S and ?bT = (+) (−b) ‘ T
  have 0: 0 ∈ affine hull ?aS 0 ∈ affine hull ?bT
    by (metis a b subsetD hull_inc image_eqI left_minus rel_interior_subset)+
  have subs: subspace (span ?aS) subspace (span ?bT)
```

by (*rule subspace_span*)+
moreover
have *dim* (*span* ((+) (− *a*) ‘ *S*)) = *dim* (*span* ((+) (− *b*) ‘ *T*))
by (*metis 0 aff_dim_translation_eq aff_dim_zero affeq dim_span nat_int*)
ultimately obtain *f g* **where** *linear f linear g*
and *fim*: *f* ‘ *span ?aS* = *span ?bT*
and *gim*: *g* ‘ *span ?bT* = *span ?aS*
and *fno*: ⋀*x*. *x* ∈ *span ?aS* ⟹ *norm*(*f x*) = *norm x*
and *gno*: ⋀*x*. *x* ∈ *span ?bT* ⟹ *norm*(*g x*) = *norm x*
and *gf*: ⋀*x*. *x* ∈ *span ?aS* ⟹ *g*(*f x*) = *x*
and *fg*: ⋀*x*. *x* ∈ *span ?bT* ⟹ *f*(*g x*) = *x*
by (*rule isometries_subspaces*) *blast*
have [*simp*]: *continuous_on A f* **for** *A*
using ‹*linear f*› *linear_conv_bounded_linear linear_continuous_on* **by** *blast*
have [*simp*]: *continuous_on B g* **for** *B*
using ‹*linear g*› *linear_conv_bounded_linear linear_continuous_on* **by** *blast*
have *eqspanS*: *affine hull ?aS* = *span ?aS*
by (*metis a affine_hull_span_0 subsetD hull_inc image_eqI left_minus rel_interior_subset*)
have *eqspanT*: *affine hull ?bT* = *span ?bT*
by (*metis b affine_hull_span_0 subsetD hull_inc image_eqI left_minus rel_interior_subset*)
have *S homeomorphic cball a 1* ∩ *affine hull S*
by (*rule starlike_compact_projective2* [*OF* ‹*compact S*› *a starS*])
also have ... *homeomorphic* (+) (−*a*) ‘ (*cball a 1* ∩ *affine hull S*)
by (*metis homeomorphic_translation*)
also have ... = *cball 0 1* ∩ (+) (−*a*) ‘ (*affine hull S*)
by (*auto simp*: *dist_norm*)
also have ... = *cball 0 1* ∩ *span ?aS*
using *eqspanS affine_hull_translation* **by** *blast*
also have ... *homeomorphic cball 0 1* ∩ *span ?bT*
proof (*rule homeomorphicI*)
show *fim1*: *f* ‘ (*cball 0 1* ∩ *span ?aS*) = *cball 0 1* ∩ *span ?bT*
proof
show *f* ‘ (*cball 0 1* ∩ *span ?aS*) ⊆ *cball 0 1* ∩ *span ?bT*
using *fim fno* **by** *auto*
show *cball 0 1* ∩ *span ?bT* ⊆ *f* ‘ (*cball 0 1* ∩ *span ?aS*)
by *clarify* (*metis IntI fg gim gno image_eqI mem_cball_0*)
qed
show *g* ‘ (*cball 0 1* ∩ *span ?bT*) = *cball 0 1* ∩ *span ?aS*
proof
show *g* ‘ (*cball 0 1* ∩ *span ?bT*) ⊆ *cball 0 1* ∩ *span ?aS*
using *gim gno* **by** *auto*
show *cball 0 1* ∩ *span ?aS* ⊆ *g* ‘ (*cball 0 1* ∩ *span ?bT*)
by *clarify* (*metis IntI fim1 gf image_eqI*)
qed
qed (*auto simp*: *fg gf*)
also have ... = *cball 0 1* ∩ (+) (−*b*) ‘ (*affine hull T*)
using *eqspanT affine_hull_translation* **by** *blast*
also have ... = (+) (−*b*) ‘ (*cball b 1* ∩ *affine hull T*)
by (*auto simp*: *dist_norm*)

also have ... *homeomorphic* (*cball b 1* ∩ *affine hull T*)
by (*metis homeomorphic_translation homeomorphic_sym*)
also have ... *homeomorphic T*
by (*metis starlike_compact_projective2* [*OF* ‹*compact T*› *b starT*] *homeomorphic_sym*)
finally have *1*: *S homeomorphic T* **.**

have *S* − *rel_interior S homeomorphic sphere a 1* ∩ *affine hull S*
by (*rule starlike_compact_projective1* [*OF* ‹*compact S*› *a starS*])
also have ... *homeomorphic* (+) (−*a*) ‘ (*sphere a 1* ∩ *affine hull S*)
by (*metis homeomorphic_translation*)
also have ... = *sphere 0 1* ∩ (+) (−*a*) ‘ (*affine hull S*)
by (*auto simp*: *dist_norm*)
also have ... = *sphere 0 1* ∩ *span ?aS*
using *eqspanS affine_hull_translation* **by** *blast*
also have ... *homeomorphic sphere 0 1* ∩ *span ?bT*
proof (*rule homeomorphicI*)
show *fim1*: *f* ‘ (*sphere 0 1* ∩ *span ?aS*) = *sphere 0 1* ∩ *span ?bT*
proof
show *f* ‘ (*sphere 0 1* ∩ *span ?aS*) ⊆ *sphere 0 1* ∩ *span ?bT*
using *fim fno* **by** *auto*
show *sphere 0 1* ∩ *span ?bT* ⊆ *f* ‘ (*sphere 0 1* ∩ *span ?aS*)
by *clarify* (*metis IntI fg gim gno image_eqI mem_sphere_0*)
qed
show *g* ‘ (*sphere 0 1* ∩ *span ?bT*) = *sphere 0 1* ∩ *span ?aS*
proof
show *g* ‘ (*sphere 0 1* ∩ *span ?bT*) ⊆ *sphere 0 1* ∩ *span ?aS*
using *gim gno* **by** *auto*
show *sphere 0 1* ∩ *span ?aS* ⊆ *g* ‘ (*sphere 0 1* ∩ *span ?bT*)
by *clarify* (*metis IntI fim1 gf image_eqI*)
qed
qed (*auto simp*: *fg gf*)
also have ... = *sphere 0 1* ∩ (+) (−*b*) ‘ (*affine hull T*)
using *eqspanT affine_hull_translation* **by** *blast*
also have ... = (+) (−*b*) ‘ (*sphere b 1* ∩ *affine hull T*)
by (*auto simp*: *dist_norm*)
also have ... *homeomorphic* (*sphere b 1* ∩ *affine hull T*)
by (*metis homeomorphic_translation homeomorphic_sym*)
also have ... *homeomorphic T* − *rel_interior T*
by (*metis starlike_compact_projective1* [*OF* ‹*compact T*› *b starT*] *homeomorphic_sym*)
finally have *2*: *S* − *rel_interior S homeomorphic T* − *rel_interior T* **.**
show *?thesis*
using *1 2* **by** *blast*
qed

lemma *homeomorphic_convex_compact_sets*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *convex S compact S convex T compact T*

and *affeq*: *aff_dim S = aff_dim T*
shows *S homeomorphic T*
using *homeomorphic_convex_lemma* [*OF assms*] *assms*
by (*auto simp*: *rel_frontier_def*)

lemma *homeomorphic_rel_frontiers_convex_bounded_sets*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *convex S bounded S convex T bounded T*
and *affeq*: *aff_dim S = aff_dim T*
shows *rel_frontier S homeomorphic rel_frontier T*
using *assms homeomorphic_convex_lemma* [*of closure S closure T*]
by (*simp add*: *rel_frontier_def convex_rel_interior_closure*)

9.8.2 Homeomorphisms between punctured spheres and affine sets

Including the famous stereoscopic projection of the 3-D sphere to the complex plane

The special case with centre 0 and radius 1

lemma *homeomorphic_punctured_affine_sphere_affine_01*:
assumes *b ∈ sphere 0 1 affine T 0 ∈ T b ∈ T affine p*
and *affT*: *aff_dim T = aff_dim p + 1*
shows (*sphere 0 1 ∩ T*) *− {b} homeomorphic p*
proof −
have [*simp*]: *norm b = 1 b•b = 1*
using *assms* **by** (*auto simp*: *norm_eq_1*)
have [*simp*]: *T ∩ {v. b•v = 0} ≠ {}*
using ‹*0 ∈ T*› **by** *auto*
have [*simp*]: *¬ T ⊆ {v. b•v = 0}*
using ‹*norm b = 1*› ‹*b ∈ T*› **by** *auto*
define *f* **where** *f ≡ λx. 2 *R b + (2 / (1 − b•x)) *R (x − b)*
define *g* **where** *g ≡ λy. b + (4 / (norm y ^ 2 + 4)) *R (y − 2 *R b)*
have *fg*[*simp*]: *⋀x. ⟦x ∈ T; b•x = 0⟧ ⟹ f (g x) = x*
unfolding *f_def g_def* **by** (*simp add*: *algebra_simps field_split_simps add_nonneg_eq_0_iff*)
have *no*: *(norm (f x))² = 4 * (1 + b • x) / (1 − b • x)*
if *norm x = 1* **and** *b • x ≠ 1* **for** *x*
using *that sum_sqs_eq* [*of 1 b • x*]
apply (*simp flip*: *dot_square_norm add*: *norm_eq_1 nonzero_eq_divide_eq*)
apply (*simp add*: *f_def vector_add_divide_simps inner_simps*)
apply (*auto simp add*: *field_split_simps inner_commute*)
done
have [*simp*]: *⋀u*::*real. 8 + u * (u * 8) = u * 16 ⟷ u=1*
by *algebra*
have *gf*[*simp*]: *⋀x. ⟦norm x = 1; b • x ≠ 1⟧ ⟹ g (f x) = x*
unfolding *g_def no* **by** (*auto simp*: *f_def field_split_simps*)
have *g1*: *norm (g x) = 1* **if** *x ∈ T* **and** *b • x = 0* **for** *x*
using *that*
apply (*simp only*: *g_def*)

apply (*rule power2_eq_imp_eq*)
apply (*simp_all add*: *dot_square_norm* [*symmetric*] *divide_simps vector_add_divide_simps*)
apply (*simp add*: *algebra_simps inner_commute*)
done
have *ne1*: $b \cdot g\ x \neq 1$ **if** $x \in T$ **and** $b \cdot x = 0$ **for** x
using *that* **unfolding** *g_def*
apply (*simp_all add*: *dot_square_norm* [*symmetric*] *divide_simps vector_add_divide_simps add_nonneg_eq_0_iff*)
apply (*auto simp*: *algebra_simps*)
done
have *subspace T*
by (*simp add*: *assms subspace_affine*)
have *gT*: $\bigwedge x.\ [\![x \in T;\ b \cdot x = 0]\!] \Longrightarrow g\ x \in T$
unfolding *g_def*
by (*blast intro*: ‹*subspace T*› ‹$b \in T$› *subspace_add subspace_mul subspace_diff*)
have $f\ `\ \{x.\ norm\ x = 1 \wedge b \cdot x \neq 1\} \subseteq \{x.\ b \cdot x = 0\}$
unfolding *f_def* **using** ‹*norm b* = *1*› *norm_eq_1*
by (*force simp*: *field_simps inner_add_right inner_diff_right*)
moreover have $f\ `\ T \subseteq T$
unfolding *f_def* **using** *assms* ‹*subspace T*›
by (*auto simp add*: *inner_add_right inner_diff_right mem_affine_3_minus subspace_mul*)
moreover have $\{x.\ b \cdot x = 0\} \cap T \subseteq f\ `\ (\{x.\ norm\ x = 1 \wedge b \cdot x \neq 1\} \cap T)$
by *clarify* (*metis* (*mono_tags*) *IntI ne1 fg gT g1 imageI mem_Collect_eq*)
ultimately have *imf*: $f\ `\ (\{x.\ norm\ x = 1 \wedge b \cdot x \neq 1\} \cap T) = \{x.\ b \cdot x = 0\} \cap T$
by *blast*
have *no4*: $\bigwedge y.\ b \cdot y = 0 \Longrightarrow norm\ ((y \cdot y + 4) *_R\ b + 4 *_R\ (y - 2 *_R\ b)) = y \cdot y + 4$
apply (*rule power2_eq_imp_eq*)
apply (*simp_all flip*: *dot_square_norm*)
apply (*auto simp*: *power2_eq_square algebra_simps inner_commute*)
done
have [*simp*]: $\bigwedge x.\ [\![norm\ x = 1;\ b \cdot x \neq 1]\!] \Longrightarrow b \cdot f\ x = 0$
by (*simp add*: *f_def algebra_simps field_split_simps*)
have [*simp*]: $\bigwedge x.\ [\![x \in T;\ norm\ x = 1;\ b \cdot x \neq 1]\!] \Longrightarrow f\ x \in T$
unfolding *f_def*
by (*blast intro*: ‹*subspace T*› ‹$b \in T$› *subspace_add subspace_mul subspace_diff*)
have $g\ `\ \{x.\ b \cdot x = 0\} \subseteq \{x.\ norm\ x = 1 \wedge b \cdot x \neq 1\}$
unfolding *g_def*
apply (*clarsimp simp*: *no4 vector_add_divide_simps divide_simps add_nonneg_eq_0_iff dot_square_norm* [*symmetric*])
apply (*auto simp*: *algebra_simps*)
done
moreover have $g\ `\ T \subseteq T$
unfolding *g_def*
by (*blast intro*: ‹*subspace T*› ‹$b \in T$› *subspace_add subspace_mul subspace_diff*)
moreover have $\{x.\ norm\ x = 1 \wedge b \cdot x \neq 1\} \cap T \subseteq g\ `\ (\{x.\ b \cdot x = 0\} \cap T)$
by *clarify* (*metis* (*mono_tags*, *lifting*) *IntI gf image_iff imf mem_Collect_eq*)

```
  ultimately have img: g ‘ ({x. b•x = 0} ∩ T) = {x. norm x = 1 ∧ b•x ≠ 1} ∩
T
    by blast
  have aff: affine ({x. b•x = 0} ∩ T)
    by (blast intro: affine_hyperplane assms)
  have contf: continuous_on ({x. norm x = 1 ∧ b•x ≠ 1} ∩ T) f
    unfolding f_def by (rule continuous_intros | force)+
  have contg: continuous_on ({x. b•x = 0} ∩ T) g
   unfolding g_def by (rule continuous_intros | force simp: add_nonneg_eq_0_iff)+
  have (sphere 0 1 ∩ T) − {b} = {x. norm x = 1 ∧ (b•x ≠ 1)} ∩ T
    using ‹norm b = 1› by (auto simp: norm_eq_1) (metis vector_eq ‹b•b = 1›)
  also have ... homeomorphic {x. b•x = 0} ∩ T
    by (rule homeomorphicI [OF imf img contf contg]) auto
  also have ... homeomorphic p
  proof (rule homeomorphic_affine_sets [OF aff ‹affine p›])
    show aff_dim ({x. b • x = 0} ∩ T) = aff_dim p
      by (simp add: Int_commute aff_dim_affine_Int_hyperplane [OF ‹affine T›]
affT)
  qed
  finally show ?thesis .
qed

theorem homeomorphic_punctured_affine_sphere_affine:
  fixes a :: 'a :: euclidean_space
  assumes 0 < r b ∈ sphere a r affine T a ∈ T b ∈ T affine p
      and aff: aff_dim T = aff_dim p + 1
    shows (sphere a r ∩ T) − {b} homeomorphic p
proof −
  have a ≠ b using assms by auto
  then have inj: inj (λx::'a. x /R norm (a − b))
    by (simp add: inj_on_def)
  have ((sphere a r ∩ T) − {b}) homeomorphic
        (+) (−a) ‘ ((sphere a r ∩ T) − {b})
    by (rule homeomorphic_translation)
  also have ... homeomorphic (∗R) (inverse r) ‘ (+) (− a) ‘ (sphere a r ∩ T −
{b})
     by (metis ‹0 < r› homeomorphic_scaling inverse_inverse_eq inverse_zero
less_irrefl)
  also have ... = sphere 0 1 ∩ ((∗R) (inverse r) ‘ (+) (− a) ‘ T) − {(b − a) /R
r}
    using assms by (auto simp: dist_norm norm_minus_commute divide_simps)
  also have ... homeomorphic p
     using assms affine_translation [symmetric, of − a] aff_dim_translation_eq
[of − a]
      by (intro homeomorphic_punctured_affine_sphere_affine_01) (auto simp:
dist_norm norm_minus_commute affine_scaling inj)
  finally show ?thesis .
qed
```

corollary *homeomorphic_punctured_sphere_affine*:
fixes $a :: 'a :: euclidean\_space$
assumes $0 < r$ **and** b: $b \in sphere\ a\ r$
and *affine* T **and** *affS*: $aff\_dim\ T + 1 = DIM('a)$
shows $(sphere\ a\ r - \{b\})\ homeomorphic\ T$
using *homeomorphic_punctured_affine_sphere_affine* [*of r b a UNIV T*] *assms*
by *auto*

corollary *homeomorphic_punctured_sphere_hyperplane*:
fixes $a :: 'a :: euclidean\_space$
assumes $0 < r$ **and** b: $b \in sphere\ a\ r$
and $c \neq 0$
shows $(sphere\ a\ r - \{b\})\ homeomorphic\ \{x::'a.\ c \cdot x = d\}$
using *assms*
by (*intro homeomorphic_punctured_sphere_affine*) (*auto simp*: *affine_hyperplane of_nat_diff*)

proposition *homeomorphic_punctured_sphere_affine_gen*:
fixes $a :: 'a :: euclidean\_space$
assumes *convex S bounded S* **and** a: $a \in rel\_frontier\ S$
and *affine T* **and** *affS*: $aff\_dim\ S = aff\_dim\ T + 1$
shows $rel\_frontier\ S - \{a\}\ homeomorphic\ T$
proof −
obtain $U :: 'a\ set$ **where** *affine U convex U* **and** *affdS*: $aff\_dim\ U = aff\_dim\ S$
using *choose_affine_subset* [*OF affine_UNIV aff_dim_geq*]
by (*meson aff_dim_affine_hull affine_affine_hull affine_imp_convex*)
have $S \neq \{\}$ **using** *assms* **by** *auto*
then obtain z **where** $z \in U$
by (*metis aff_dim_negative_iff equals0I affdS*)
then have *bne*: $ball\ z\ 1 \cap U \neq \{\}$ **by** *force*
then have [*simp*]: $aff\_dim(ball\ z\ 1 \cap U) = aff\_dim\ U$
using *aff_dim_convex_Int_open* [*OF ‹convex U› open_ball*]
by (*fastforce simp add*: *Int_commute*)
have *rel_frontier S homeomorphic rel_frontier* (*ball z 1* ∩ *U*)
by (*rule homeomorphic_rel_frontiers_convex_bounded_sets*)
(*auto simp*: *‹affine U› affine_imp_convex convex_Int affdS assms*)
also have ... = *sphere z 1* ∩ *U*
using *convex_affine_rel_frontier_Int* [*of ball z 1 U*]
by (*simp add*: *‹affine U› bne*)
finally have *rel_frontier S homeomorphic sphere z 1* ∩ *U* .
then obtain *h k* **where** *him*: $h\ `\ rel\_frontier\ S = sphere\ z\ 1 \cap U$
and *kim*: $k\ `\ (sphere\ z\ 1 \cap U) = rel\_frontier\ S$
and *hcon*: *continuous_on* (*rel_frontier S*) *h*
and *kcon*: *continuous_on* (*sphere z 1* ∩ *U*) *k*
and *kh*: $\bigwedge x.\ x \in rel\_frontier\ S \Longrightarrow k(h(x)) = x$
and *hk*: $\bigwedge y.\ y \in sphere\ z\ 1 \cap U \Longrightarrow h(k(y)) = y$
unfolding *homeomorphic_def homeomorphism_def* **by** *auto*
have $rel\_frontier\ S - \{a\}\ homeomorphic\ (sphere\ z\ 1 \cap U) - \{h\ a\}$
proof (*rule homeomorphicI*)

```
    show h: h ‘ (rel_frontier S − {a}) = sphere z 1 ∩ U − {h a}
      using him a kh by auto metis
    show k ‘ (sphere z 1 ∩ U − {h a}) = rel_frontier S − {a}
      by (force simp: h [symmetric] image_comp o_def kh)
  qed (auto intro: continuous_on_subset hcon kcon simp: kh hk)
  also have ... homeomorphic T
    by (rule homeomorphic_punctured_affine_sphere_affine)
       (use a him in ‹auto simp: affS affdS ‹affine T› ‹affine U› ‹z ∈ U››)
  finally show ?thesis .
qed
```

When dealing with AR, ANR and ANR later, it's useful to know that every set is homeomorphic to a closed subset of a convex set, and if the set is locally compact we can take the convex set to be the universe.

```
proposition homeomorphic_closedin_convex:
  fixes S :: 'm::euclidean_space set
  assumes aff_dim S < DIM('n)
  obtains U and T :: 'n::euclidean_space set
    where convex U U ≠ {} closedin (top_of_set U) T
          S homeomorphic T
proof (cases S = {})
  case True then show ?thesis
    by (rule_tac U=UNIV and T={} in that) auto
next
  case False
  then obtain a where a ∈ S by auto
  obtain i::'n where i: i ∈ Basis i ≠ 0
    using SOME_Basis Basis_zero by force
  have 0 ∈ affine hull ((+) (− a) ‘ S)
    by (simp add: ‹a ∈ S› hull_inc)
  then have dim ((+) (− a) ‘ S) = aff_dim ((+) (− a) ‘ S)
    by (simp add: aff_dim_zero)
  also have ... < DIM('n)
   by (simp add: aff_dim_translation_eq_subtract assms cong: image_cong_simp)
  finally have dd: dim ((+) (− a) ‘ S) < DIM('n)
    by linarith
  have span: span {x. i • x = 0} = {x. i • x = 0}
    using span_eq_iff [symmetric, of {x. i • x = 0}] subspace_hyperplane [of i]
by simp
  have dim ((+) (− a) ‘ S) ≤ dim {x. i • x = 0}
    using dd by (simp add: dim_hyperplane [OF ‹i ≠ 0›])
  then obtain T where subspace T and Tsub: T ⊆ {x. i • x = 0}
    and dimT: dim T = dim ((+) (− a) ‘ S)
    by (rule choose_subspace_of_subspace) (simp add: span)
  have subspace (span ((+) (− a) ‘ S))
    using subspace_span by blast
  then obtain h k where linear h linear k
              and heq: h ‘ span ((+) (− a) ‘ S) = T
              and keq:k ‘ T = span ((+) (− a) ‘ S)
```

```
            and hinv [simp]: ⋀x. x ∈ span ((+) (− a) ‘ S) ⟹ k(h x) = x
            and kinv [simp]: ⋀x. x ∈ T ⟹ h(k x) = x
    by (auto simp: dimT intro: isometries_subspaces [OF _ ‹subspace T›] dimT)
  have hcont: continuous_on A h and kcont: continuous_on B k for A B
    using ‹linear h› ‹linear k› linear_continuous_on linear_conv_bounded_linear
by blast+
  have ihhhh[simp]: ⋀x. x ∈ S ⟹ i • h (x − a) = 0
    using Tsub [THEN subsetD] heq span_superset by fastforce
  have sphere 0 1 − {i} homeomorphic {x. i • x = 0}
  proof (rule homeomorphic_punctured_sphere_affine)
    show affine {x. i • x = 0}
      by (auto simp: affine_hyperplane)
    show aff_dim {x. i • x = 0} + 1 = int DIM('n)
      using i by force
  qed (use i in auto)
  then obtain f g where fg: homeomorphism (sphere 0 1 − {i}) {x. i • x = 0} f
g
    by (force simp: homeomorphic_def)
  show ?thesis
  proof
    have h ‘ (+) (− a) ‘ S ⊆ T
      using heq span_superset span_linear_image by blast
    then have g ‘ h ‘ (+) (− a) ‘ S ⊆ g ‘ {x. i • x = 0}
      using Tsub by (simp add: image_mono)
    also have ... ⊆ sphere 0 1 − {i}
      by (simp add: fg [unfolded homeomorphism_def])
    finally have gh_sub_sph: (g ∘ h) ‘ (+) (− a) ‘ S ⊆ sphere 0 1 − {i}
      by (metis image_comp)
    then have gh_sub_cb: (g ∘ h) ‘ (+) (− a) ‘ S ⊆ cball 0 1
      by (metis Diff_subset order_trans sphere_cball)
    have [simp]: ⋀u. u ∈ S ⟹ norm (g (h (u − a))) = 1
      using gh_sub_sph [THEN subsetD] by (auto simp: o_def)
    show convex (ball 0 1 ∪ (g ∘ h) ‘ (+) (− a) ‘ S)
    by (meson ball_subset_cball convex_intermediate_ball gh_sub_cb sup.bounded_iff
sup.cobounded1)
   show closedin (top_of_set (ball 0 1 ∪ (g ∘ h) ‘ (+) (− a) ‘ S)) ((g ∘ h) ‘ (+)
(− a) ‘ S)
      unfolding closedin_closed
      by (rule_tac x=sphere 0 1 in exI) auto
    have ghcont: continuous_on ((λx. x − a) ‘ S) (λx. g (h x))
     by (rule continuous_on_compose2 [OF homeomorphism_cont2 [OF fg] hcont],
force)
    have kfcont: continuous_on ((λx. g (h (x − a))) ‘ S) (λx. k (f x))
    proof (rule continuous_on_compose2 [OF kcont])
      show continuous_on ((λx. g (h (x − a))) ‘ S) f
          using homeomorphism_cont1 [OF fg] gh_sub_sph by (fastforce intro:
continuous_on_subset)
    qed auto
    have S homeomorphic (+) (− a) ‘ S
```

by (*fact homeomorphic_translation*)
also have ... *homeomorphic* (*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S*
apply (*simp add*: *homeomorphic_def homeomorphism_def cong*: *image_cong_simp*)
apply (*rule_tac x*=*g* ∘ *h* **in** *exI*)
apply (*rule_tac x*=*k* ∘ *f* **in** *exI*)
apply (*auto simp*: *ghcont kfcont span_base homeomorphism_apply2* [*OF fg*] *image_comp cong*: *image_cong_simp*)
done
finally show *S homeomorphic* (*g* ∘ *h*) ‘ (+) (− *a*) ‘ *S* **.**
qed *auto*
qed

9.8.3 Locally compact sets in an open set

Locally compact sets are closed in an open set and are homeomorphic to an absolutely closed set if we have one more dimension to play with.

lemma *locally_compact_open_Int_closure*:
fixes *S* :: *'a* :: *metric_space set*
assumes *locally compact S*
obtains *T* **where** *open T S* = *T* ∩ *closure S*
proof −
have ∀ *x*∈*S*. ∃ *T v u*. *u* = *S* ∩ *T* ∧ *x* ∈ *u* ∧ *u* ⊆ *v* ∧ *v* ⊆ *S* ∧ *open T* ∧ *compact v*
by (*metis assms locally_compact openin_open*)
then obtain *t v* **where**
tv: ⋀*x*. *x* ∈ *S*
⟹ *v x* ⊆ *S* ∧ *open* (*t x*) ∧ *compact* (*v x*) ∧ (∃ *u*. *x* ∈ *u* ∧ *u* ⊆ *v x* ∧ *u* = *S* ∩ *t x*)
by *metis*
then have *o*: *open* (⋃ (*t* ‘ *S*))
by *blast*
have *S* = ⋃ (*v* ‘ *S*)
using *tv* **by** *blast*
also have ... = ⋃ (*t* ‘ *S*) ∩ *closure S*
proof
show ⋃ (*v* ‘ *S*) ⊆ ⋃ (*t* ‘ *S*) ∩ *closure S*
by *clarify* (*meson IntD2 IntI UN_I closure_subset subsetD tv*)
have *t x* ∩ *closure S* ⊆ *v x* **if** *x* ∈ *S* **for** *x*
proof −
have *t x* ∩ *closure S* ⊆ *closure* (*t x* ∩ *S*)
by (*simp add*: *open_Int_closure_subset that tv*)
also have ... ⊆ *v x*
by (*metis Int_commute closure_minimal compact_imp_closed that tv*)
finally show *?thesis* **.**
qed
then show ⋃ (*t* ‘ *S*) ∩ *closure S* ⊆ ⋃ (*v* ‘ *S*)
by *blast*
qed
finally have *e*: *S* = ⋃ (*t* ‘ *S*) ∩ *closure S* **.**

show *?thesis*
by (*rule that* [*OF o e*])
qed

lemma *locally_compact_closedin_open*:
fixes $S :: 'a :: metric\_space\ set$
assumes *locally compact S*
obtains *T* **where** *open T closedin* (*top_of_set T*) *S*
by (*metis locally_compact_open_Int_closure* [*OF assms*] *closed_closure closedin_closed_Int*)

lemma *locally_compact_homeomorphism_projection_closed*:
assumes *locally compact S*
obtains *T* **and** $f :: 'a \Rightarrow 'a :: euclidean\_space \times 'b :: euclidean\_space$
where *closed T homeomorphism S T f fst*
proof (*cases closed S*)
case *True*
show *?thesis*
proof
show *homeomorphism S* ($S \times \{0\}$) ($\lambda x.\ (x,\ 0)$) *fst*
by (*auto simp*: *homeomorphism_def continuous_intros*)
qed (*use True closed_Times* **in** *auto*)
next
case *False*
obtain *U* **where** *open U* **and** *US*: $U \cap closure\ S = S$
by (*metis locally_compact_open_Int_closure* [*OF assms*])
with *False* **have** *Ucomp*: $-U \neq \{\}$
using *closure_eq* **by** *auto*
have [*simp*]: *closure* $(-\ U) = -U$
by (*simp add*: ‹*open U*› *closed_Compl*)
define $f :: 'a \Rightarrow 'a \times 'b$ **where** $f \equiv \lambda x.\ (x,\ One\ /_R\ setdist\ \{x\}\ (-\ U))$
have *continuous_on U* ($\lambda x.\ (x,\ One\ /_R\ setdist\ \{x\}\ (-\ U))$)
proof (*intro continuous_intros continuous_on_setdist*)
show $\forall x \in U.\ setdist\ \{x\}\ (-\ U) \neq 0$
by (*simp add*: *Ucomp setdist_eq_0_sing_1*)
qed
then have *homU*: *homeomorphism U* (*f*‘*U*) *f fst*
by (*auto simp*: *f_def homeomorphism_def image_iff continuous_intros*)
have *cloS*: *closedin* (*top_of_set U*) *S*
by (*metis US closed_closure closedin_closed_Int*)
have *cont*: *isCont* (($\lambda x.\ setdist\ \{x\}\ (-\ U)$) *o fst*) *z* **for** $z :: 'a \times 'b$
by (*rule continuous_at_compose continuous_intros continuous_at_setdist*)+
have *setdist1D*: $setdist\ \{a\}\ (-\ U) *_R b = One \Longrightarrow setdist\ \{a\}\ (-\ U) \neq 0$ **for** *a*::*'a* **and** *b*::*'b*
by *force*
have $*$: $r *_R b = One \Longrightarrow b = (1\ /\ r) *_R One$ **for** *r* **and** *b*::*'b*
by (*metis One_non_0 nonzero_divide_eq_eq real_vector.scale_eq_0_iff real_vector.scale_scale scaleR_one*)

```
    have ⋀a b::'b. setdist {a} (- U) *R b = One ⟹ (a,b) ∈ (λx. (x, (1 / setdist
{x} (- U)) *R One)) ` U
     by (metis (mono_tags, lifting) * ComplI image_eqI setdist1D setdist_sing_in_set)
    then have f ` U = (λz. (setdist {fst z} (- U) *R snd z)) -` {One}
      by (auto simp: f_def setdist_eq_0_sing_1 field_simps Ucomp)
    then have clfU: closed (f ` U)
     by (force intro: continuous_intros cont [unfolded o_def] continuous_closed_vimage)
    have closed (f ` S)
     by (metis closedin_closed_trans [OF _ clfU] homeomorphism_imp_closed_map
[OF homU cloS])
    then show ?thesis
      by (metis US homU homeomorphism_of_subsets inf_sup_ord(1) that)
qed

lemma locally_compact_closed_Int_open:
  fixes S :: 'a :: euclidean_space set
  shows locally compact S ⟷ (∃ U V. closed U ∧ open V ∧ S = U ∩ V) (is
?lhs = ?rhs)
proof
  show ?lhs ⟹ ?rhs
    by (metis closed_closure inf_commute locally_compact_open_Int_closure)
  show ?rhs ⟹ ?lhs
   by (meson closed_imp_locally_compact locally_compact_Int open_imp_locally_compact)
qed

lemma lowerdim_embeddings:
  assumes  DIM('a) < DIM('b)
  obtains f :: 'a::euclidean_space*real ⇒ 'b::euclidean_space
      and g :: 'b ⇒ 'a*real
      and j :: 'b
  where linear f linear g ⋀z. g (f z) = z j ∈ Basis ⋀x. f(x,0) · j = 0
proof -
  let ?B = Basis :: ('a*real) set
  have b01: (0,1) ∈ ?B
    by (simp add: Basis_prod_def)
  have DIM('a * real) ≤ DIM('b)
    by (simp add: Suc_leI assms)
 then obtain basf :: 'a*real ⇒ 'b where sbf: basf ` ?B ⊆ Basis and injbf: inj_on
basf Basis
    by (metis finite_Basis card_le_inj)
  define basg:: 'b ⇒ 'a * real where
    basg ≡ λi. if i ∈ basf ` Basis then inv_into Basis basf i else (0,1)
  have bgf[simp]: basg (basf i) = i if i ∈ Basis for i
    using inv_into_f_f injbf that by (force simp: basg_def)
  have sbg: basg ` Basis ⊆ ?B
    by (force simp: basg_def injbf b01)
  define f :: 'a*real ⇒ 'b where f ≡ λu. ∑j∈Basis. (u · basg j) *R j
  define g :: 'b ⇒ 'a*real where g ≡ λz. (∑ i∈Basis. (z · basf i) *R i)
  show ?thesis
```

```
  proof
    show linear f
      unfolding f_def
      by (intro linear_compose_sum linearI ballI) (auto simp: algebra_simps)
    show linear g
      unfolding g_def
      by (intro linear_compose_sum linearI ballI) (auto simp: algebra_simps)
    have *: (∑ a ∈ Basis. a · basf b * (x · basg a)) = x · b if b ∈ Basis for x b
      using sbf that by auto
    show gf: g (f x) = x for x
    proof (rule euclidean_eqI)
      show ⋀b. b ∈ Basis ⟹ g (f x) · b = x · b
        using f_def g_def sbf by auto
    qed
    show basf(0,1) ∈ Basis
      using b01 sbf by auto
    then show f(x,0) · basf(0,1) = 0 for x
      unfolding f_def inner_sum_left
      using b01 inner_not_same_Basis
      by (fastforce intro: comm_monoid_add_class.sum.neutral)
  qed
qed

proposition locally_compact_homeomorphic_closed:
  fixes S :: 'a::euclidean_space set
  assumes locally compact S and dimlt: DIM('a) < DIM('b)
  obtains T :: 'b::euclidean_space set where closed T S homeomorphic T
proof −
  obtain U:: ('a*real)set and h
    where closed U and homU: homeomorphism S U h fst
    using locally_compact_homeomorphism_projection_closed assms by metis
  obtain f :: 'a*real ⇒ 'b and g :: 'b ⇒ 'a*real
    where linear f linear g and gf [simp]: ⋀z. g (f z) = z
    using lowerdim_embeddings [OF dimlt] by metis
  then have inj f
    by (metis injI)
  have gfU: g ‘ f ‘ U = U
    by (simp add: image_comp o_def)
  have S homeomorphic U
    using homU homeomorphic_def by blast
  also have ... homeomorphic f ‘ U
  proof (rule homeomorphicI [OF refl gfU])
    show continuous_on U f
    by (meson ‹inj f› ‹linear f› homeomorphism_cont2 linear_homeomorphism_image)
    show continuous_on (f ‘ U) g
      using ‹linear g› linear_continuous_on linear_conv_bounded_linear by blast
  qed (auto simp: o_def)
  finally show ?thesis
    using ‹closed U› ‹inj f› ‹linear f› closed_injective_linear_image that by blast
```

qed

lemma *homeomorphic_convex_compact_lemma*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *convex S*
and *compact S*
and *cball 0 1* $\subseteq$ *S*
shows *S homeomorphic* (*cball* (*0*::*'a*) *1*)
proof (*rule starlike_compact_projective_special*[*OF assms*(*2*−*3*)])
fix *x u*
assume $x \in S$ **and** $0 \leq u$ **and** $u < (1::real)$
have *open* (*ball* ($u *_R x$) (*1* − *u*))
by (*rule open_ball*)
moreover have $u *_R x \in ball\ (u *_R x)\ (1 - u)$
unfolding *centre_in_ball* **using** ‹*u* < *1*› **by** *simp*
moreover have *ball* ($u *_R x$) (*1* − *u*) $\subseteq$ *S*
proof
fix *y*
assume $y \in ball\ (u *_R x)\ (1 - u)$
then have *dist* ($u *_R x$) *y* < *1* − *u*
unfolding *mem_ball* **.**
with ‹*u* < *1*› **have** *inverse* (*1* − *u*) $*_R$ ($y - u *_R x$) $\in$ *cball 0 1*
by (*simp add*: *dist_norm inverse_eq_divide norm_minus_commute*)
with *assms*(*3*) **have** *inverse* (*1* − *u*) $*_R$ ($y - u *_R x$) $\in$ *S* **..**
with *assms*(*1*) **have** (*1* − *u*) $*_R$ (($y - u *_R x$) $/_R$ (*1* − *u*)) + $u *_R x \in S$
using ‹*x* $\in$ *S*› ‹*0* $\leq$ *u*› ‹*u* < *1*› [*THEN less_imp_le*] **by** (*rule convexD_alt*)
then show *y* $\in$ *S* **using** ‹*u* < *1*›
by *simp*
qed
ultimately have $u *_R x \in interior\ S$ **..**
then show $u *_R x \in S - frontier\ S$
using *frontier_def* **and** *interior_subset* **by** *auto*
qed

proposition *homeomorphic_convex_compact_cball*:
fixes *e* :: *real*
and $S :: 'a::euclidean\_space\ set$
assumes *S*: *convex S compact S interior S* $\neq$ {} **and** *e* > *0*
shows *S homeomorphic* (*cball* (*b*::*'a*) *e*)
proof (*rule homeomorphic_trans*[*OF _ homeomorphic_balls*(*2*)])
obtain *a* **where** *a* $\in$ *interior S*
using *assms* **by** *auto*
then show *S homeomorphic cball* (*0*::*'a*) *1*
by (*metis* (*no_types*) *aff_dim_cball S compact_cball convex_cball*
homeomorphic_convex_lemma interior_rel_interior_gen zero_less_one)
qed (*use* ‹*e*>*0*› **in** *auto*)

corollary *homeomorphic_convex_compact*:

fixes $S :: 'a::euclidean\_space\ set$
and $T :: 'a\ set$
assumes $convex\ S\ compact\ S\ interior\ S \neq \{\}$
and $convex\ T\ compact\ T\ interior\ T \neq \{\}$
shows $S\ homeomorphic\ T$
using *assms*
by (*meson zero_less_one homeomorphic_trans homeomorphic_convex_compact_cball homeomorphic_sym*)

lemma *homeomorphic_closed_intervals*:
fixes $a :: 'a::euclidean\_space$ **and** b **and** $c :: 'a::euclidean\_space$ **and** d
assumes $box\ a\ b \neq \{\}$ **and** $box\ c\ d \neq \{\}$
shows $(cbox\ a\ b)\ homeomorphic\ (cbox\ c\ d)$
by (*simp add*: *assms homeomorphic_convex_compact*)

lemma *homeomorphic_closed_intervals_real*:
fixes $a::real$ **and** b **and** $c::real$ **and** d
assumes $a<b$ **and** $c<d$
shows $\{a..b\}\ homeomorphic\ \{c..d\}$
using *assms* **by** (*auto intro*: *homeomorphic_convex_compact*)

9.8.4 Covering spaces and lifting results for them

definition *covering_space*
$:: 'a::topological\_space\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b::topological\_space\ set \Rightarrow bool$
where
$covering\_space\ c\ p\ S \equiv$
$continuous\_on\ c\ p \wedge p\ `\ c = S \wedge$
$(\forall x \in S.\ \exists T.\ x \in T \wedge openin\ (top\_of\_set\ S)\ T \wedge$
$(\exists v.\ \bigcup v = c \cap p -` T \wedge$
$(\forall u \in v.\ openin\ (top\_of\_set\ c)\ u) \wedge$
$pairwise\ disjnt\ v \wedge$
$(\forall u \in v.\ \exists q.\ homeomorphism\ u\ T\ p\ q)))$

lemma *covering_spaceI* [*intro?*]:
assumes $continuous\_on\ c\ p\ p\ `\ c = S$
$\bigwedge x.\ x \in S \Longrightarrow \exists T.\ x \in T \wedge openin\ (top\_of\_set\ S)\ T \wedge$
$(\exists v.\ \bigcup v = c \cap p -` T \wedge (\forall u \in v.\ openin\ (top\_of\_set\ c)\ u) \wedge$
$pairwise\ disjnt\ v \wedge (\forall u \in v.\ \exists q.\ homeomorphism\ u\ T\ p\ q))$
shows $covering\_space\ c\ p\ S$
using *assms* **unfolding** *covering_space_def* **by** *auto*

lemma *covering_space_imp_continuous*: $covering\_space\ c\ p\ S \Longrightarrow continuous\_on\ c\ p$
by (*simp add*: *covering_space_def*)

lemma *covering_space_imp_surjective*: $covering\_space\ c\ p\ S \Longrightarrow p\ `\ c = S$
by (*simp add*: *covering_space_def*)

lemma *homeomorphism_imp_covering_space*: *homeomorphism S T f g* $\Longrightarrow$ *covering_space S f T*
apply (*clarsimp simp add*: *homeomorphism_def covering_space_def*)
apply (*rule_tac x*=*T* **in** *exI*, *simp*)
apply (*rule_tac x*=$\{S\}$ **in** *exI*, *auto*)
done

lemma *covering_space_local_homeomorphism*:
assumes *covering_space c p S x* $\in$ *c*
obtains *T u q* **where** *x* $\in$ *T openin* (*top_of_set c*) *T*
p x $\in$ *u openin* (*top_of_set S*) *u*
homeomorphism T u p q
using *assms*
by (*clarsimp simp add*: *covering_space_def*) (*metis IntI UnionE vimage_eq*)

lemma *covering_space_local_homeomorphism_alt*:
assumes *p*: *covering_space c p S* **and** *y* $\in$ *S*
obtains *x T U q* **where** *p x* = *y*
x $\in$ *T openin* (*top_of_set c*) *T*
y $\in$ *U openin* (*top_of_set S*) *U*
homeomorphism T U p q
proof −
obtain *x* **where** *p x* = *y x* $\in$ *c*
using *assms covering_space_imp_surjective* **by** *blast*
show *?thesis*
using *that* ‹*p x* = *y*› **by** (*auto intro*: *covering_space_local_homeomorphism* [*OF p* ‹*x* $\in$ *c*›])
qed

proposition *covering_space_open_map*:
fixes *S* :: *'a* :: *metric_space set* **and** *T* :: *'b* :: *metric_space set*
assumes *p*: *covering_space c p S* **and** *T*: *openin* (*top_of_set c*) *T*
shows *openin* (*top_of_set S*) (*p* ‘ *T*)
proof −
have *pce*: *p* ‘ *c* = *S*
and *covs*:
$\bigwedge x.\ x \in S \Longrightarrow$
$\exists X\ VS.\ x \in X \wedge$ *openin* (*top_of_set S*) *X* $\wedge$
$\bigcup VS = c \cap p -$‘ *X* $\wedge$
$(\forall u \in VS.$ *openin* (*top_of_set c*) *u*) $\wedge$
pairwise disjnt VS $\wedge$
$(\forall u \in VS.\ \exists q.$ *homeomorphism u X p q*)
using *p* **by** (*auto simp*: *covering_space_def*)
have *T* $\subseteq$ *c* **by** (*metis openin_euclidean_subtopology_iff T*)
have $\exists X.$ *openin* (*top_of_set S*) *X* $\wedge$ *y* $\in$ *X* $\wedge$ *X* $\subseteq$ *p* ‘ *T*
if *y* $\in$ *p* ‘ *T* **for** *y*
proof −

```
    have y ∈ S using ‹T ⊆ c› pce that by blast
    obtain U VS where y ∈ U and U: openin (top_of_set S) U
                  and VS: ⋃ VS = c ∩ p -‘ U
                  and openVS: ∀ V ∈ VS. openin (top_of_set c) V
                  and homVS: ⋀V. V ∈ VS ⟹ ∃ q. homeomorphism V U p q
      using covs [OF ‹y ∈ S›] by auto
    obtain x where x ∈ c p x ∈ U x ∈ T p x = y
      using T [unfolded openin_euclidean_subtopology_iff] ‹y ∈ U› ‹y ∈ p ‘ T›
by blast
    with VS obtain V where x ∈ V V ∈ VS by auto
    then obtain q where q: homeomorphism V U p q using homVS by blast
    then have ptV: p ‘ (T ∩ V) = U ∩ q -‘ (T ∩ V)
      using VS ‹V ∈ VS› by (auto simp: homeomorphism_def)
    have ocv: openin (top_of_set c) V
      by (simp add: ‹V ∈ VS› openVS)
    have openin (top_of_set (q ‘ U)) (T ∩ V)
      using q unfolding homeomorphism_def
      by (metis T inf.absorb_iff2 ocv openin_imp_subset openin_subtopology_Int
subtopology_subtopology)
    then have openin (top_of_set U) (U ∩ q -‘ (T ∩ V))
      using continuous_on_open homeomorphism_def q by blast
    then have os: openin (top_of_set S) (U ∩ q -‘ (T ∩ V))
      using openin_trans [of U] by (simp add: Collect_conj_eq U)
    show ?thesis
    proof (intro exI conjI)
      show openin (top_of_set S) (p ‘ (T ∩ V))
        by (simp only: ptV os)
    qed (use ‹p x = y› ‹x ∈ V› ‹x ∈ T› in auto)
  qed
  with openin_subopen show ?thesis by blast
qed

lemma covering_space_lift_unique_gen:
  fixes f :: 'a::topological_space ⇒ 'b::topological_space
  fixes g1 :: 'a ⇒ 'c::real_normed_vector
  assumes cov: covering_space c p S
      and eq: g1 a = g2 a
      and f: continuous_on T f  f ∈ T → S
      and g1: continuous_on T g1  g1 ∈ T → c
      and fg1: ⋀x. x ∈ T ⟹ f x = p(g1 x)
      and g2: continuous_on T g2  g2 ∈ T → c
      and fg2: ⋀x. x ∈ T ⟹ f x = p(g2 x)
      and u_compt: U ∈ components T and a ∈ U x ∈ U
    shows g1 x = g2 x
proof -
  have U ⊆ T by (rule in_components_subset [OF u_compt])
  define G12 where G12 ≡ {x ∈ U. g1 x - g2 x = 0}
  have connected U by (rule in_components_connected [OF u_compt])
  have contu: continuous_on U g1 continuous_on U g2
```

```
      using ‹U ⊆ T› continuous_on_subset g1 g2 by blast+
  have o12: openin (top_of_set U) G12
  unfolding G12_def
  proof (subst openin_subopen, clarify)
    fix z
    assume z: z ∈ U g1 z − g2 z = 0
    obtain v w q where g1 z ∈ v and ocv: openin (top_of_set c) v
      and p (g1 z) ∈ w and osw: openin (top_of_set S) w
      and hom: homeomorphism v w p q
    proof (rule covering_space_local_homeomorphism [OF cov])
      show g1 z ∈ c
        using ‹U ⊆ T› ‹z ∈ U› g1(2) by blast
    qed auto
    have g2 z ∈ v using ‹g1 z ∈ v› z by auto
    have gg: U ∩ g −‘ v = U ∩ g −‘ (v ∩ g ‘ U) for g
      by auto
    have openin (top_of_set (g1 ‘ U)) (v ∩ g1 ‘ U)
      using ocv ‹U ⊆ T› g1 by (fastforce simp add: openin_open)
    then have 1: openin (top_of_set U) (U ∩ g1 −‘ v)
       unfolding gg by (blast intro: contu continuous_on_open [THEN iffD1,
rule_format])
    have openin (top_of_set (g2 ‘ U)) (v ∩ g2 ‘ U)
      using ocv ‹U ⊆ T› g2 by (fastforce simp add: openin_open)
    then have 2: openin (top_of_set U) (U ∩ g2 −‘ v)
       unfolding gg by (blast intro: contu continuous_on_open [THEN iffD1,
rule_format])
    let ?T = (U ∩ g1 −‘ v) ∩ (U ∩ g2 −‘ v)
    show ∃ T. openin (top_of_set U) T ∧ z ∈ T ∧ T ⊆ {z ∈ U. g1 z − g2 z =
0}
    proof (intro exI conjI)
      show openin (top_of_set U) ?T
        using 1 2 by blast
      show z ∈ ?T
        using z by (simp add: ‹g1 z ∈ v› ‹g2 z ∈ v›)
      show ?T ⊆ {z ∈ U. g1 z − g2 z = 0}
        using hom
        by (clarsimp simp: homeomorphism_def) (metis ‹U ⊆ T› fg1 fg2 subsetD)
    qed
  qed
  have c12: closedin (top_of_set U) G12
    unfolding G12_def
    by (intro continuous_intros continuous_closedin_preimage_constant contu)
  have G12 = {} ∨ G12 = U
     by (intro connected_clopen [THEN iffD1, rule_format] ‹connected U› conjI
o12 c12)
  with eq ‹a ∈ U› have ⋀x. x ∈ U ⟹ g1 x − g2 x = 0 by (auto simp: G12_def)
  then show ?thesis
    using ‹x ∈ U› by force
qed
```

proposition *covering_space_lift_unique*:
fixes $f :: 'a::topological\_space \Rightarrow 'b::topological\_space$
fixes $g1 :: 'a \Rightarrow 'c::real\_normed\_vector$
assumes *covering_space c p S*
$g1\ a = g2\ a$
continuous_on T f $f \in T \to S$
continuous_on T g1 $g1 \in T \to c$ $\bigwedge x.\ x \in T \Longrightarrow f\ x = p(g1\ x)$
continuous_on T g2 $g2 \in T \to c$ $\bigwedge x.\ x \in T \Longrightarrow f\ x = p(g2\ x)$
connected T $a \in T$ $x \in T$
shows $g1\ x = g2\ x$
using *covering_space_lift_unique_gen* [*of c p S*] *in_components_self assms ex_in_conv*
by *blast*

lemma *covering_space_locally*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *loc*: *locally* φ *C* **and** *cov*: *covering_space C p S*
and *pim*: $\bigwedge T.\ [\![T \subseteq C; \varphi\ T]\!] \Longrightarrow \psi(p\ `\ T)$
shows *locally* ψ *S*
proof −
have *locally* ψ $(p\ `\ C)$
proof (*rule locally_open_map_image* [*OF loc*])
show *continuous_on C p*
using *cov covering_space_imp_continuous* **by** *blast*
show $\bigwedge T.$ *openin* (*top_of_set C*) *T* $\Longrightarrow$ *openin* (*top_of_set* $(p\ `\ C)$) $(p\ `\ T)$
using *cov covering_space_imp_surjective covering_space_open_map* **by** *blast*
qed (*simp add*: *pim*)
then show *?thesis*
using *covering_space_imp_surjective* [*OF cov*] **by** *metis*
qed

proposition *covering_space_locally_eq*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *cov*: *covering_space C p S*
and *pim*: $\bigwedge T.\ [\![T \subseteq C; \varphi\ T]\!] \Longrightarrow \psi(p\ `\ T)$
and *qim*: $\bigwedge q\ U.\ [\![U \subseteq S;$ *continuous_on U q*; $\psi\ U]\!] \Longrightarrow \varphi(q\ `\ U)$
shows *locally* ψ *S* $\longleftrightarrow$ *locally* φ *C*
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
show *?rhs*
proof (*rule locallyI*)
fix *V x*
assume *V*: *openin* (*top_of_set C*) *V* **and** $x \in V$
have $p\ x \in p\ `\ C$
by (*metis IntE V* ‹$x \in V$› *imageI openin_open*)
then obtain *T* $\mathcal{V}$ **where** $p\ x \in T$

```
                    and opeT: openin (top_of_set S) T
                    and veq: ⋃𝒱 = C ∩ p -‘ T
                    and ope: ∀ U∈𝒱. openin (top_of_set C) U
                    and hom: ∀ U∈𝒱. ∃ q. homeomorphism U T p q
    using cov unfolding covering_space_def by (blast intro: that)
  have x ∈ ⋃𝒱
    using V veq ‹p x ∈ T› ‹x ∈ V› openin_imp_subset by fastforce
  then obtain U where x ∈ U U ∈ 𝒱
    by blast
 then obtain q where opeU: openin (top_of_set C) U and q: homeomorphism
U T p q
    using ope hom by blast
  with V have openin (top_of_set C) (U ∩ V)
    by blast
  then have UV: openin (top_of_set S) (p ‘ (U ∩ V))
    using cov covering_space_open_map by blast
  obtain W W' where opeW: openin (top_of_set S) W and ψ W' p x ∈ W
W ⊆ W' and W'sub: W' ⊆ p ‘ (U ∩ V)
    using locallyE [OF L UV] ‹x ∈ U› ‹x ∈ V› by blast
  then have W ⊆ T
  by (metis Int_lower1 q homeomorphism_image1 image_Int_subset order_trans)
  show ∃ U Z. openin (top_of_set C) U ∧
            φ Z ∧ x ∈ U ∧ U ⊆ Z ∧ Z ⊆ V
  proof (intro exI conjI)
    have openin (top_of_set T) W
     by (meson opeW opeT openin_imp_subset openin_subset_trans ‹W ⊆ T›)
    then have openin (top_of_set U) (q ‘ W)
     by (meson homeomorphism_imp_open_map homeomorphism_symD q)
    then show openin (top_of_set C) (q ‘ W)
     using opeU openin_trans by blast
    show φ (q ‘ W')
      by (metis (mono_tags, lifting) Int_subset_iff UV W'sub ‹ψ W'› continu-
ous_on_subset dual_order.trans homeomorphism_def image_Int_subset openin_imp_subset
q qim)
    show x ∈ q ‘ W
     by (metis ‹p x ∈ W› ‹x ∈ U› homeomorphism_def imageI q)
    show q ‘ W ⊆ q ‘ W'
     using ‹W ⊆ W'› by blast
    have W' ⊆ p ‘ V
     using W'sub by blast
    then show q ‘ W' ⊆ V
     using W'sub homeomorphism_apply1 [OF q] by auto
    qed
 qed
next
 assume ?rhs
 then show ?lhs
   using cov covering_space_locally pim by blast
qed
```

lemma *covering_space_locally_compact_eq*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *covering_space C p S*
shows *locally compact S* $\longleftrightarrow$ *locally compact C*
proof (*rule covering_space_locally_eq* [*OF assms*])
show $\bigwedge T.\ [\![T \subseteq C;\ compact\ T]\!] \Longrightarrow compact\ (p\ `\ T)$
by (*meson assms compact_continuous_image continuous_on_subset covering_space_imp_continuous*)
qed (*use compact_continuous_image* **in** *blast*)

lemma *covering_space_locally_connected_eq*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *covering_space C p S*
shows *locally connected S* $\longleftrightarrow$ *locally connected C*
proof (*rule covering_space_locally_eq* [*OF assms*])
show $\bigwedge T.\ [\![T \subseteq C;\ connected\ T]\!] \Longrightarrow connected\ (p\ `\ T)$
by (*meson connected_continuous_image assms continuous_on_subset covering_space_imp_continuous*)
qed (*use connected_continuous_image* **in** *blast*)

lemma *covering_space_locally_path_connected_eq*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *covering_space C p S*
shows *locally path_connected S* $\longleftrightarrow$ *locally path_connected C*
proof (*rule covering_space_locally_eq* [*OF assms*])
show $\bigwedge T.\ [\![T \subseteq C;\ path\_connected\ T]\!] \Longrightarrow path\_connected\ (p\ `\ T)$
by (*meson path_connected_continuous_image assms continuous_on_subset covering_space_imp_continuous*)
qed (*use path_connected_continuous_image* **in** *blast*)

lemma *covering_space_locally_compact*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *locally compact C covering_space C p S*
shows *locally compact S*
using *assms covering_space_locally_compact_eq* **by** *blast*

lemma *covering_space_locally_connected*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *locally connected C covering_space C p S*
shows *locally connected S*
using *assms covering_space_locally_connected_eq* **by** *blast*

lemma *covering_space_locally_path_connected*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes *locally path_connected C covering_space C p S*
shows *locally path_connected S*

using *assms covering_space_locally_path_connected_eq* **by** *blast*

proposition *covering_space_lift_homotopy*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
and $h :: real \times 'c::real\_normed\_vector \Rightarrow 'b$
assumes *cov*: *covering_space* $C\ p\ S$
and *conth*: *continuous_on* $(\{0..1\} \times U)\ h$
and *him*: $h \in (\{0..1\} \times U) \to S$
and *heq*: $\bigwedge y.\ y \in U \Longrightarrow h\ (0,y) = p(f\ y)$
and *contf*: *continuous_on* $U\ f$ **and** *fim*: $f \in U \to C$
obtains k **where** *continuous_on* $(\{0..1\} \times U)\ k$
$k \in (\{0..1\} \times U) \to C$
$\bigwedge y.\ y \in U \Longrightarrow k(0,\ y) = f\ y$
$\bigwedge z.\ z \in \{0..1\} \times U \Longrightarrow h\ z = p(k\ z)$
proof −
have $\exists V\ k.\ openin\ (top\_of\_set\ U)\ V \wedge y \in V \wedge$
$continuous\_on\ (\{0..1\} \times V)\ k \wedge k\ `\ (\{0..1\} \times V) \subseteq C \wedge$
$(\forall z \in V.\ k(0,\ z) = f\ z) \wedge (\forall z \in \{0..1\} \times V.\ h\ z = p(k\ z))$
if $y \in U$ **for** y
proof −
obtain *UU* **where** *UU*: $\bigwedge s.\ s \in S \Longrightarrow s \in (UU\ s) \wedge openin\ (top\_of\_set\ S)\ (UU\ s) \wedge$
$(\exists \mathcal{V}.\ \bigcup \mathcal{V} = C \cap p\ -`\ UU\ s \wedge$
$(\forall U \in \mathcal{V}.\ openin\ (top\_of\_set\ C)\ U) \wedge$
$pairwise\ disjnt\ \mathcal{V} \wedge$
$(\forall U \in \mathcal{V}.\ \exists q.\ homeomorphism\ U\ (UU\ s)\ p\ q))$
using *cov* **unfolding** *covering_space_def* **by** (*metis* (*mono_tags*))
then have *ope*: $\bigwedge s.\ s \in S \Longrightarrow s \in (UU\ s) \wedge openin\ (top\_of\_set\ S)\ (UU\ s)$
by *blast*
have $\exists k\ n\ i.\ open\ k \wedge open\ n \wedge$
$t \in k \wedge y \in n \wedge i \in S \wedge h\ `\ ((\{0..1\} \cap k) \times (U \cap n)) \subseteq UU\ i$ **if** $t \in \{0..1\}$ **for** t
proof −
have *hinS*: $h\ (t,\ y) \in S$
using ‹$y \in U$› *him that* **by** *blast*
then have $(t,y) \in (\{0..1\} \times U) \cap h\ -`\ UU(h(t,\ y))$
using ‹$y \in U$› ‹$t \in \{0..1\}$› **by** (*auto simp*: *ope*)
moreover have *ope_01U*: *openin* (*top_of_set* $(\{0..1\} \times U)$) $((\{0..1\} \times U) \cap h\ -`\ UU(h(t,\ y)))$
using *hinS ope continuous_on_open_gen* [*OF him*] *conth* **by** *blast*
ultimately obtain $V\ W$ **where** *openV*: *open* V **and** $t \in \{0..1\} \cap V\ t \in \{0..1\} \cap V$
and *openW*: *open* W **and** $y \in U\ y \in W$
and *VW*: $(\{0..1\} \cap V) \times (U \cap W) \subseteq ((\{0..1\} \times U) \cap h\ -`\ UU(h(t,\ y)))$
by (*rule Times_in_interior_subtopology*) (*auto simp*: *openin_open*)
then show *?thesis*
using *hinS* **by** *blast*
qed

```
  then obtain K NN X where
        K: ⋀t. t ∈ {0..1} ⟹ open (K t)
      and NN: ⋀t. t ∈ {0..1} ⟹ open (NN t)
      and inUS: ⋀t. t ∈ {0..1} ⟹ t ∈ K t ∧ y ∈ NN t ∧ X t ∈ S
      and him: ⋀t. t ∈ {0..1} ⟹ h ` (({0..1} ∩ K t) × (U ∩ NN t)) ⊆ UU
(X t)
  by (metis (mono_tags))
  obtain 𝒯 where 𝒯 ⊆ ((λi. K i × NN i)) ` {0..1} finite 𝒯 {0::real..1} × {y}
⊆ ⋃𝒯
  proof (rule compactE)
    show compact ({0::real..1} × {y})
      by (simp add: compact_Times)
    show {0..1} × {y} ⊆ (⋃i∈{0..1}. K i × NN i)
      using K inUS by auto
    show ⋀B. B ∈ (λi. K i × NN i) ` {0..1} ⟹ open B
      using K NN by (auto simp: open_Times)
  qed blast
  then obtain tk where tk ⊆ {0..1} finite tk
                   and tk: {0::real..1} × {y} ⊆ (⋃i ∈ tk. K i × NN i)
    by (metis (no_types, lifting) finite_subset_image)
  then have tk ≠ {}
    by auto
  define n where n = ⋂(NN ` tk)
  have y ∈ n open n
    using inUS NN ‹tk ⊆ {0..1}› ‹finite tk›
    by (auto simp: n_def open_INT subset_iff)
  obtain δ where 0 < δ and δ: ⋀T. ⟦T ⊆ {0..1}; diameter T < δ⟧ ⟹ ∃B∈K
` tk. T ⊆ B
  proof (rule Lebesgue_number_lemma [of {0..1} K ` tk])
    show K ` tk ≠ {}
      using ‹tk ≠ {}› by auto
    show {0..1} ⊆ ⋃(K ` tk)
      using tk by auto
    show ⋀B. B ∈ K ` tk ⟹ open B
      using ‹tk ⊆ {0..1}› K by auto
  qed auto
  obtain N::nat where N: N > 1 / δ
    using reals_Archimedean2 by blast
  then have N > 0
    using ‹0 < δ› order.asym by force
  have ∗: ∃ V k. openin (top_of_set U) V ∧ y ∈ V ∧
                 continuous_on ({0..of_nat n / N} × V) k ∧
                 k ` ({0..of_nat n / N} × V) ⊆ C ∧
                 (∀z∈V. k (0, z) = f z) ∧
                 (∀z∈{0..of_nat n / N} × V. h z = p (k z)) if n ≤ N for n
    using that
  proof (induction n)
    case 0
    show ?case
```

```
      apply (rule_tac x=U in exI)
      apply (rule_tac x=f ∘ snd in exI)
      apply (intro conjI ‹y ∈ U› continuous_intros continuous_on_subset [OF
contf])
      using fim  apply (auto simp: heq)
      done
    next
     case (Suc n)
     then obtain V k where opeUV: openin (top_of_set U) V
                       and y ∈ V
                       and contk: continuous_on ({0..n/N} × V) k
                       and kim: k ‘ ({0..n/N} × V) ⊆ C
                       and keq: ⋀z. z ∈ V ⟹ k (0, z) = f z
                       and heq: ⋀z. z ∈ {0..n/N} × V ⟹ h z = p (k z)
      using Suc_leD by auto
     have n ≤ N
      using Suc.prems by auto
     obtain t where t ∈ tk and t: {n/N .. (1 + real n) / N} ⊆ K t
     proof (rule bexE [OF δ])
      show {n/N .. (1 + real n) / N} ⊆ {0..1}
       using Suc.prems by (auto simp: field_split_simps)
      show diameter_less: diameter {n/N .. (1 + real n) / N} < δ
       using ‹0 < δ› N by (auto simp: field_split_simps)
     qed blast
     have t01: t ∈ {0..1}
      using ‹t ∈ tk› ‹tk ⊆ {0..1}› by blast
     obtain 𝒱 where 𝒱: ⋃𝒱 = C ∩ p −‘ UU (X t)
               and opeC: ⋀U. U ∈ 𝒱 ⟹ openin (top_of_set C) U
               and pairwise disjnt 𝒱
              and homuu: ⋀U. U ∈ 𝒱 ⟹ ∃q. homeomorphism U (UU (X t)) p q
      using inUS [OF t01] UU by meson
     have n_div_N_in: n/N ∈ {n/N .. (1 + real n) / N}
      using N by (auto simp: field_split_simps)
     with t have nN_in_kkt: n/N ∈ K t
      by blast
     have k (n/N, y) ∈ C ∩ p −‘ UU (X t)
     proof (simp, rule conjI)
      show k (n/N, y) ∈ C
       using ‹y ∈ V› kim keq by force
      have p (k (n/N, y)) = h (n/N, y)
       by (simp add: ‹y ∈ V› heq)
      also have ... ∈ h ‘ (({0..1} ∩ K t) × (U ∩ NN t))
       using ‹y ∈ V› t01 ‹n ≤ N›
       by (simp add: nN_in_kkt ‹y ∈ U› inUS field_split_simps)
      also have ... ⊆ UU (X t)
       using him t01 by blast
      finally show p (k (n/N, y)) ∈ UU (X t) .
     qed
     with 𝒱 have k (n/N, y) ∈ ⋃𝒱
```

```
      by blast
    then obtain W where W: k (n/N, y) ∈ W and W ∈ 𝒱
      by blast
    then obtain p' where opeC': openin (top_of_set C) W
                  and hom': homeomorphism W (UU (X t)) p p'
      using homuu opeC by blast
    then have W ⊆ C
      using openin_imp_subset by blast
    define W' where W' = UU(X t)
    have opeVW: openin (top_of_set V) (V ∩ (k ∘ Pair (n / N)) -` W)
    proof (rule continuous_openin_preimage [OF _ _ opeC'])
      show continuous_on V (k ∘ Pair (n/N))
        by (intro continuous_intros continuous_on_subset [OF contk], auto)
      show (k ∘ Pair (n/N)) ∈ V → C
        using kim by (auto simp: ‹y ∈ V› W)
    qed
    obtain N' where opeUN': openin (top_of_set U) N'
          and y ∈ N' and kimw: k ` ({(n/N)} × N') ⊆ W
    proof
      show openin (top_of_set U) (V ∩ (k ∘ Pair (n/N)) -` W)
        using opeUV opeVW openin_trans by blast
    qed (use ‹y ∈ V› W in ‹force+›)
    obtain Q Q' where opeUQ: openin (top_of_set U) Q
                and cloUQ': closedin (top_of_set U) Q'
                and y ∈ Q Q ⊆ Q'
                and Q': Q' ⊆ (U ∩ NN(t)) ∩ N' ∩ V
    proof -
      obtain VO VX where open VO open VX and VO: V = U ∩ VO and
VX: N' = U ∩ VX
        using opeUV opeUN' by (auto simp: openin_open)
      then have open (NN(t) ∩ VO ∩ VX)
        using NN t01 by blast
      then obtain e where e > 0 and e: cball y e ⊆ NN(t) ∩ VO ∩ VX
       by (metis Int_iff ‹N' = U ∩ VX› ‹V = U ∩ VO› ‹y ∈ N'› ‹y ∈ V› inUS
open_contains_cball t01)
      show ?thesis
      proof
        show openin (top_of_set U) (U ∩ ball y e)
          by blast
        show closedin (top_of_set U) (U ∩ cball y e)
          using e by (auto simp: closedin_closed)
      qed (use ‹y ∈ U› ‹e > 0› VO VX e in auto)
    qed
    then have y ∈ Q' Q ⊆ (U ∩ NN(t)) ∩ N' ∩ V
      by blast+
    have neq: {0..n/N} ∪ {n/N..(1 + real n) / N} = {0..(1 + real n) / N}
      apply (auto simp: field_split_simps)
    by (metis not_less of_nat_0_le_iff of_nat_0_less_iff order_trans zero_le_mult_iff)
    then have neqQ': {0..n/N} × Q' ∪ {n/N..(1 + real n) / N} × Q' = {0..(1
```

```
+ real n) / N} × Q'
      by blast
    have cont: continuous_on ({0..(1 + real n) / N} × Q') (λx. if x ∈ {0..n/N}
× Q' then k x else (p' ∘ h) x)
      unfolding neqQ' [symmetric]
   proof (rule continuous_on_cases_local, simp_all add: neqQ' del: comp_apply)
      have ∃ T. closed T ∧ {0..n/N} × Q' = {0..(1+n)/N} × Q' ∩ T
        using n_div_N_in
        by (rule_tac x={0 .. n/N} × UNIV in exI) (auto simp: closed_Times)
      then show closedin (top_of_set ({0..(1 + real n) / N} × Q')) ({0..n/N}
× Q')
        by (simp add: closedin_closed)
      have ∃ T. closed T ∧ {n/N..(1+n)/N} × Q' = {0..(1+n)/N} × Q' ∩ T
      by (rule_tac x={n/N..(1+n)/N} × UNIV in exI) (auto simp: closed_Times
order_trans [rotated])
      then show closedin (top_of_set ({0..(1 + real n) / N} × Q')) ({n/N..(1
+ real n) / N} × Q')
        by (simp add: closedin_closed)
      show continuous_on ({0..n/N} × Q') k
        using Q' by (auto intro: continuous_on_subset [OF contk])
      have continuous_on ({n/N..(1 + real n) / N} × Q') h
      proof (rule continuous_on_subset [OF conth])
        show {n/N..(1 + real n) / N} × Q' ⊆ {0..1} × U
        proof (clarsimp, intro conjI)
          fix a b
          assume b ∈ Q' and a: n/N ≤ a a ≤ (1 + real n) / N
          have 0 ≤ n/N (1 + real n) / N ≤ 1
            using a Suc.prems by (auto simp: divide_simps)
          with a show 0 ≤ a  a ≤ 1
            by linarith+
          show b ∈ U
            using ‹b ∈ Q'› cloUQ' closedin_imp_subset by blast
        qed
      qed
      moreover have continuous_on (h ' ({n/N..(1 + real n) / N} × Q')) p'
     proof (rule continuous_on_subset [OF homeomorphism_cont2 [OF hom']])
        have h ' ({n/N..(1 + real n) / N} × Q') ⊆ h ' (({0..1} ∩ K t) × (U ∩
NN t))
        proof (rule image_mono)
          show {n/N..(1 + real n) / N} × Q' ⊆ ({0..1} ∩ K t) × (U ∩ NN t)
          proof (clarsimp, intro conjI)
            fix a::real and b
            assume b ∈ Q' n/N ≤ a a ≤ (1 + real n) / N
            show 0 ≤ a
                 by (meson ‹n/N ≤ a› divide_nonneg_nonneg of_nat_0_le_iff
order_trans)
            show a ≤ 1
              using Suc.prems ‹a ≤ (1 + real n) / N› order_trans by force
            show a ∈ K t
```

```
            using ‹a ≤ (1 + real n) / N› ‹n/N ≤ a› t by auto
          show b ∈ U
            using ‹b ∈ Q'› cloUQ' closedin_imp_subset by blast
          show b ∈ NN t
            using Q' ‹b ∈ Q'› by auto
        qed
      qed
      with him show h ` ({n/N..(1 + real n) / N} × Q') ⊆ UU (X t)
        using t01 by blast
    qed
    ultimately show continuous_on ({n/N..(1 + real n) / N} × Q') (p' ∘ h)
      by (rule continuous_on_compose)
    have k (n/N, b) = p' (h (n/N, b)) if b ∈ Q' for b
    proof −
      have k (n/N, b) ∈ W
        using that Q' kimw  by force
      then have k (n/N, b) = p' (p (k (n/N, b)))
        by (simp add:  homeomorphism_apply1 [OF hom'])
      then show ?thesis
        using Q' that by (force simp: heq)
    qed
    then show ⋀x. x ∈ {n/N..(1 + real n) / N} × Q' ∧
          x ∈ {0..n/N} × Q' ⟹ k x = (p' ∘ h) x
      by auto
  qed
  have h_in_UU: h (x, y) ∈ UU (X t) if y ∈ Q ¬ x ≤ n/N 0 ≤ x x ≤ (1 +
real n) / N for x y
  proof −
    have x ≤ 1
      using Suc.prems that order_trans by force
    moreover have x ∈ K t
      by (meson atLeastAtMost_iff le_less not_le subset_eq t that)
    moreover have y ∈ U
      using ‹y ∈ Q› opeUQ openin_imp_subset by blast
    moreover have y ∈ NN t
      using Q' ‹Q ⊆ Q'› ‹y ∈ Q› by auto
    ultimately have (x, y) ∈ (({0..1} ∩ K t) × (U ∩ NN t))
      using that by auto
    then have h (x, y) ∈ h ` (({0..1} ∩ K t) × (U ∩ NN t))
      by blast
    also have ... ⊆ UU (X t)
      by (metis him t01)
    finally show ?thesis .
  qed
  let ?k = (λx. if x ∈ {0..n/N} × Q' then k x else (p' ∘ h) x)
  show ?case
  proof (intro exI conjI)
    show continuous_on ({0..real (Suc n) / N} × Q) ?k
      using ‹Q ⊆ Q'› by (auto intro: continuous_on_subset [OF cont])
```

```
      have ⋀x y. ⟦x ≤ n/N; y ∈ Q'; 0 ≤ x⟧ ⟹ k (x, y) ∈ C
        using kim Q' by force
       moreover have p' (h (x, y)) ∈ C if y ∈ Q ¬ x ≤ n/N 0 ≤ x x ≤ (1 +
real n) / N for x y
      proof (rule ‹W ⊆ C› [THEN subsetD])
        show p' (h (x, y)) ∈ W
          using homeomorphism_image2 [OF hom', symmetric]  h_in_UU  Q'
‹Q ⊆ Q'› ‹W ⊆ C› that by auto
      qed
      ultimately show ?k ` ({0..real (Suc n) / N} × Q) ⊆ C
        using Q' ‹Q ⊆ Q'› by force
      show ∀z∈Q. ?k (0, z) = f z
        using Q' keq  ‹Q ⊆ Q'› by auto
      show ∀z ∈ {0..real (Suc n) / N} × Q. h z = p(?k z)
        using ‹Q ⊆ U ∩ NN t ∩ N' ∩ V› heq Q' ‹Q ⊆ Q'›
          by (auto simp: homeomorphism_apply2 [OF hom'] dest: h_in_UU)
      qed (auto simp: ‹y ∈ Q› opeUQ)
    qed
    show ?thesis
      using ∗[OF order_refl] N ‹0 < δ› by (simp add: split: if_split_asm)
  qed
  then obtain V fs where opeV: ⋀y. y ∈ U ⟹ openin (top_of_set U) (V y)
        and V: ⋀y. y ∈ U ⟹ y ∈ V y
        and contfs: ⋀y. y ∈ U ⟹ continuous_on ({0..1} × V y) (fs y)
        and ∗: ⋀y. y ∈ U ⟹ (fs y) ` ({0..1} × V y) ⊆ C ∧
                            (∀z ∈ V y. fs y (0, z) = f z) ∧
                            (∀z ∈ {0..1} × V y. h z = p(fs y z))
    by (metis (mono_tags))
  then have VU: ⋀y. y ∈ U ⟹ V y ⊆ U
    by (meson openin_imp_subset)
  obtain k where contk: continuous_on ({0..1} × U) k
          and k: ⋀x i. ⟦i ∈ U; x ∈ {0..1} × U ∩ {0..1} × V i⟧ ⟹ k x = fs i x
  proof (rule pasting_lemma_exists)
    let ?X = top_of_set ({0..1::real} × U)
    show topspace ?X ⊆ (⋃i∈U. {0..1} × V i)
      using V by force
    show ⋀i. i ∈ U ⟹ openin (top_of_set ({0..1} × U)) ({0..1} × V i)
      by (simp add: Abstract_Topology.openin_Times opeV)
    show ⋀i. i ∈ U ⟹ continuous_map
           (subtopology (top_of_set ({0..1} × U)) ({0..1} × V i)) euclidean (fs
i)
      by (metis contfs subtopology_subtopology continuous_map_iff_continuous
Times_Int_Times VU inf.absorb_iff2 inf.idem)
    show fs i x = fs j x  if i ∈ U j ∈ U and x: x ∈ topspace ?X ∩ {0..1} × V i
∩ {0..1} × V j
      for i j x
    proof −
      obtain u y where x = (u, y) y ∈ V i y ∈ V j 0 ≤ u u ≤ 1
        using x by auto
```

```
      show ?thesis
      proof (rule covering_space_lift_unique [OF cov, of _ (0,y) _ {0..1} × {y}
h])
        show fs i (0, y) = fs j (0, y)
          using∗V by (simp add: ‹y ∈ V i› ‹y ∈ V j› that)
        show conth_y: continuous_on ({0..1} × {y}) h
            using VU ‹y ∈ V j› that by (auto intro: continuous_on_subset [OF
conth])
        show h ∈ ({0..1} × {y}) → S
          using ‹y ∈ V i› assms(3) VU that by fastforce
        show continuous_on ({0..1} × {y}) (fs i)
          using continuous_on_subset [OF contfs] ‹i ∈ U›
          by (simp add: ‹y ∈ V i› subset_iff)
        show fs i ∈ ({0..1} × {y}) → C
          using ∗ ‹y ∈ V i› ‹i ∈ U› by fastforce
        show ⋀x. x ∈ {0..1} × {y} ⟹ h x = p (fs i x)
          using ∗ ‹y ∈ V i› ‹i ∈ U› by blast
        show continuous_on ({0..1} × {y}) (fs j)
          using continuous_on_subset [OF contfs] ‹j ∈ U›
          by (simp add: ‹y ∈ V j› subset_iff)
        show fs j ∈ ({0..1} × {y}) → C
          using ∗ ‹y ∈ V j› ‹j ∈ U› by fastforce
        show ⋀x. x ∈ {0..1} × {y} ⟹ h x = p (fs j x)
          using ∗ ‹y ∈ V j› ‹j ∈ U› by blast
        show connected ({0..1::real} × {y})
          using connected_Icc connected_Times connected_sing by blast
        show (0, y) ∈ {0..1::real} × {y}
          by force
        show x ∈ {0..1} × {y}
          using ‹x = (u, y)› x by blast
      qed
    qed
  qed force
  show ?thesis
  proof
    show k ∈ ({0..1} × U) → C
      using V∗k VU by fastforce
    show ⋀y. y ∈ U ⟹ k (0, y) = f y
      by (simp add: V∗k)
    show ⋀z. z ∈ {0..1} × U ⟹ h z = p (k z)
      using V∗k by auto
  qed (auto simp: contk)
qed

corollary covering_space_lift_homotopy_alt:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
    and h :: 'c::real_normed_vector × real ⇒ 'b
  assumes cov: covering_space C p S
      and conth: continuous_on (U × {0..1}) h
```

```
    and him: h ∈ (U × {0..1}) → S
    and heq: ⋀y. y ∈ U ⟹ h (y,0) = p(f y)
    and contf: continuous_on U f and fim: f ∈ U → C
  obtains k where continuous_on (U × {0..1}) k
               k ∈ (U × {0..1}) → C
               ⋀y. y ∈ U ⟹ k(y, 0) = f y
               ⋀z. z ∈ U × {0..1} ⟹ h z = p(k z)
proof −
  have continuous_on ({0..1} × U) (h ∘ (λz. (snd z, fst z)))
    by (intro continuous_intros continuous_on_subset [OF conth]) auto
  then obtain k where contk: continuous_on ({0..1} × U) k
               and kim:  k ‘ ({0..1} × U) ⊆ C
               and k0: ⋀y. y ∈ U ⟹ k(0, y) = f y
               and heqp: ⋀z. z ∈ {0..1} × U ⟹ (h ∘ (λz. Pair (snd z) (fst z)))
z = p(k z)
    apply (rule covering_space_lift_homotopy [OF cov _ _ _ contf fim])
    using him  by (auto simp: contf heq)
  show ?thesis
  proof
    show continuous_on (U × {0..1}) (k ∘ (λz. (snd z, fst z)))
      by (intro continuous_intros continuous_on_subset [OF contk]) auto
  qed (use kim heqp in ‹auto simp: k0›)
qed

corollary covering_space_lift_homotopic_function:
 fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector and g:: 'c::real_normed_vector
⇒ 'a
  assumes cov: covering_space C p S
    and contg: continuous_on U g
    and gim: g ∈ U → C
    and pgeq: ⋀y. y ∈ U ⟹ p(g y) = f y
    and hom: homotopic_with_canon (λx. True) U S f f'
   obtains g' where continuous_on U g' image g' U ⊆ C ⋀y. y ∈ U ⟹ p(g'
y) = f' y
proof −
  obtain h where conth: continuous_on ({0..1::real} × U) h
          and him: h ∈ ({0..1} × U) → S
          and h0:  ⋀x. h(0, x) = f x
          and h1: ⋀x. h(1, x) = f' x
    using hom by (auto simp: homotopic_with_def)
  have ⋀y. y ∈ U ⟹ h (0, y) = p (g y)
    by (simp add: h0 pgeq)
  then obtain k where contk: continuous_on ({0..1} × U) k
               and kim: k ‘ ({0..1} × U) ⊆ C
               and k0: ⋀y. y ∈ U ⟹ k(0, y) = g y
               and heq: ⋀z. z ∈ {0..1} × U ⟹ h z = p(k z)
   using covering_space_lift_homotopy [OF cov conth him _ contg gim] by (metis
image_subset_iff_funcset)
  show ?thesis
```

```
  proof
    show continuous_on U (k ∘ Pair 1)
     by (meson contk atLeastAtMost_iff continuous_on_o_Pair order_refl zero_le_one)
    show (k ∘ Pair 1) ‘ U ⊆ C
      using kim by auto
    show ⋀y. y ∈ U ⟹ p ((k ∘ Pair 1) y) = f' y
      by (auto simp: h1 heq [symmetric])
  qed
qed

corollary covering_space_lift_inessential_function:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector and U :: 'c::real_normed_vector
set
  assumes cov: covering_space C p S
      and hom: homotopic_with_canon (λx. True) U S f (λx. a)
  obtains g where continuous_on U g g ‘ U ⊆ C ⋀y. y ∈ U ⟹ p(g y) = f y
proof (cases U = {})
  case True
  then show ?thesis
    using that continuous_on_empty by blast
next
  case False
  then obtain b where b: b ∈ C p b = a
    using covering_space_imp_surjective [OF cov] homotopic_with_imp_subset2
[OF hom]
    by auto
  then have gim: (λy. b) ∈ U → C
    by blast
  show ?thesis
  proof (rule covering_space_lift_homotopic_function [OF cov continuous_on_const
gim])
    show ⋀y. y ∈ U ⟹ p b = a
      using b by auto
  qed (use that homotopic_with_symD [OF hom] in auto)
qed
```

9.8.5 Lifting of general functions to covering space

```
proposition covering_space_lift_path_strong:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
    and f :: 'c::real_normed_vector ⇒ 'b
  assumes cov: covering_space C p S and a ∈ C
      and path g and pag: path_image g ⊆ S and pas: pathstart g = p a
    obtains h where path h path_image h ⊆ C pathstart h = a
              and ⋀t. t ∈ {0..1} ⟹ p(h t) = g t
proof −
  obtain k:: real × 'c ⇒ 'a
    where contk: continuous_on ({0..1} × {undefined}) k
      and kim: k ‘ ({0..1} × {undefined}) ⊆ C
```

```
    and k0:  k (0, undefined) = a
    and pk: ⋀z. z ∈ {0..1} × {undefined} ⟹ p(k z) = (g ∘ fst) z
  proof (rule covering_space_lift_homotopy [OF cov, of {undefined} g ∘ fst])
   show continuous_on ({0..1::real} × {undefined::'c}) (g ∘ fst)
    using ‹path g› by (intro continuous_intros) (simp add: path_def)
   show (g ∘ fst) ∈ ({0..1} × {undefined}) → S
    using pag by (auto simp: path_image_def)
   show (g ∘ fst) (0, y) = p a if y ∈ {undefined} for y::'c
    by (metis comp_def fst_conv pas pathstart_def)
  qed (use assms in auto)
  show ?thesis
  proof
   show path (k ∘ (λt. Pair t undefined))
    unfolding path_def
    by (intro continuous_on_compose continuous_intros continuous_on_subset
[OF contk]) auto
   show path_image (k ∘ (λt. (t, undefined))) ⊆ C
    using kim by (auto simp: path_image_def)
   show pathstart (k ∘ (λt. (t, undefined))) = a
    by (auto simp: pathstart_def k0)
   show ⋀t. t ∈ {0..1} ⟹ p ((k ∘ (λt. (t, undefined))) t) = g t
    by (auto simp: pk)
  qed
qed

corollary covering_space_lift_path:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes cov: covering_space C p S and path g and pig: path_image g ⊆ S
  obtains h where path h path_image h ⊆ C ⋀t. t ∈ {0..1} ⟹ p(h t) = g t
proof −
  obtain a where a ∈ C pathstart g = p a
   by (metis pig cov covering_space_imp_surjective imageE pathstart_in_path_image
subsetCE)
  show ?thesis
   using covering_space_lift_path_strong [OF cov ‹a ∈ C› ‹path g› pig]
   by (metis ‹pathstart g = p a› that)
qed


proposition covering_space_lift_homotopic_paths:
  fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
  assumes cov: covering_space C p S
      and path g1 and pig1: path_image g1 ⊆ S
      and path g2 and pig2: path_image g2 ⊆ S
      and hom: homotopic_paths S g1 g2
      and path h1 and pih1: path_image h1 ⊆ C and ph1: ⋀t. t ∈ {0..1} ⟹
p(h1 t) = g1 t
      and path h2 and pih2: path_image h2 ⊆ C and ph2: ⋀t. t ∈ {0..1} ⟹
p(h2 t) = g2 t
```

```
    and h1h2: pathstart h1 = pathstart h2
  shows homotopic_paths C h1 h2
proof −
  obtain h :: real × real ⇒ 'b
    where conth: continuous_on ({0..1} × {0..1}) h
      and him: h ∈ ({0..1} × {0..1}) → S
      and h0: ⋀x. h (0, x) = g1 x and h1: ⋀x. h (1, x) = g2 x
      and heq0: ⋀t. t ∈ {0..1} ⟹ h (t, 0) = g1 0
      and heq1: ⋀t. t ∈ {0..1} ⟹ h (t, 1) = g1 1
    using hom by (auto simp: homotopic_paths_def homotopic_with_def path-
start_def pathfinish_def image_subset_iff_funcset)
  obtain k where contk: continuous_on ({0..1} × {0..1}) k
           and kim: k ∈ ({0..1} × {0..1}) → C
           and kh2: ⋀y. y ∈ {0..1} ⟹ k (y, 0) = h2 0
           and hpk: ⋀z. z ∈ {0..1} × {0..1} ⟹ h z = p (k z)
  proof (rule covering_space_lift_homotopy_alt [OF cov conth him])
    show ⋀y. y ∈ {0..1} ⟹ h (y, 0) = p (h2 0)
    by (metis atLeastAtMost_iff h1h2 heq0 order_refl pathstart_def ph1 zero_le_one)
  qed (use path_image_def pih2 in ‹fastforce+›)
  have contg1: continuous_on {0..1} g1 and contg2: continuous_on {0..1} g2
    using ‹path g1› ‹path g2› path_def by blast+
  have g1im: g1 ∈ {0..1} → S and g2im: g2 ∈ {0..1} → S
    using path_image_def pig1 pig2 by auto
  have conth1: continuous_on {0..1} h1 and conth2: continuous_on {0..1} h2
    using ‹path h1› ‹path h2› path_def by blast+
  have h1im: h1 ∈ {0..1} → C and h2im: h2 ∈ {0..1} → C
    using path_image_def pih1 pih2 by auto
  show ?thesis
    unfolding homotopic_paths pathstart_def pathfinish_def
  proof (intro exI conjI ballI)
    show keqh1: k(0, x) = h1 x if x ∈ {0..1} for x
    proof (rule covering_space_lift_unique [OF cov _ contg1 g1im])
      show k (0,0) = h1 0
       by (metis atLeastAtMost_iff h1h2 kh2 order_refl pathstart_def zero_le_one)
      show continuous_on {0..1} (λa. k (0, a))
        by (intro continuous_intros continuous_on_compose2 [OF contk]) auto
      show ⋀x. x ∈ {0..1} ⟹ g1 x = p (k (0, x))
      by (metis atLeastAtMost_iff h0 hpk zero_le_one mem_Sigma_iff order_refl)
    qed (use conth1 h1im kim that in ‹auto simp: ph1›)
    show k(1, x) = h2 x if x ∈ {0..1} for x
    proof (rule covering_space_lift_unique [OF cov _ contg2 g2im])
      show k (1,0) = h2 0
        by (metis atLeastAtMost_iff kh2 order_refl zero_le_one)
      show continuous_on {0..1} (λa. k (1, a))
        by (intro continuous_intros continuous_on_compose2 [OF contk]) auto
      show ⋀x. x ∈ {0..1} ⟹ g2 x = p (k (1, x))
      by (metis atLeastAtMost_iff h1 hpk mem_Sigma_iff order_refl zero_le_one)
    qed (use conth2 h2im kim that in ‹auto simp: ph2›)
    show ⋀t. t ∈ {0..1} ⟹ (k ∘ Pair t) 0 = h1 0
```

```
    by (metis comp_apply h1h2 kh2 pathstart_def)
  show (k ∘ Pair t) 1 = h1 1 if t ∈ {0..1} for t
  proof (rule covering_space_lift_unique
        [OF cov, of λa. (k ∘ Pair a) 1 0 λa. h1 1 {0..1}  λx. g1 1])
    show (k ∘ Pair 0) 1 = h1 1
      using keqh1 by auto
    show continuous_on {0..1} (λa. (k ∘ Pair a) 1)
      by (auto intro!: continuous_intros continuous_on_compose2 [OF contk])
    show ⋀x. x ∈ {0..1} ⟹ g1 1 = p ((k ∘ Pair x) 1)
      using heq1 hpk by auto
  qed (use contk kim g1im h1im that in ‹auto simp: ph1›)
 qed (use contk kim in auto)
qed


corollary covering_space_monodromy:
 fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
 assumes cov: covering_space C p S
    and path g1 and pig1: path_image g1 ⊆ S
    and path g2 and pig2: path_image g2 ⊆ S
    and hom: homotopic_paths S g1 g2
     and path h1 and pih1: path_image h1 ⊆ C and ph1: ⋀t. t ∈ {0..1} ⟹
p(h1 t) = g1 t
     and path h2 and pih2: path_image h2 ⊆ C and ph2: ⋀t. t ∈ {0..1} ⟹
p(h2 t) = g2 t
    and h1h2: pathstart h1 = pathstart h2
  shows pathfinish h1 = pathfinish h2
 using covering_space_lift_homotopic_paths [OF assms] homotopic_paths_imp_pathfinish
 by blast


corollary covering_space_lift_homotopic_path:
 fixes p :: 'a::real_normed_vector ⇒ 'b::real_normed_vector
 assumes cov: covering_space C p S
    and hom: homotopic_paths S f f'
    and path g and pig: path_image g ⊆ C
    and a: pathstart g = a and b: pathfinish g = b
    and pgeq: ⋀t. t ∈ {0..1} ⟹ p(g t) = f t
 obtains g' where path g' path_image g' ⊆ C
                  pathstart g' = a pathfinish g' = b ⋀t. t ∈ {0..1} ⟹ p(g' t) = f' t
proof (rule covering_space_lift_path_strong [OF cov, of a f'])
 show a ∈ C
   using a pig by auto
 show path f' path_image f' ⊆ S
  using hom homotopic_paths_imp_path homotopic_paths_imp_subset by blast+
 show pathstart f' = p a
  by (metis a atLeastAtMost_iff hom homotopic_paths_imp_pathstart order_refl
pathstart_def pgeq zero_le_one)
qed (metis (mono_tags, lifting) assms cov covering_space_monodromy hom ho-
```

motopic_paths_imp_path homotopic_paths_imp_subset pgeq pig)

proposition *covering_space_lift_general*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
and $f :: 'c::real\_normed\_vector \Rightarrow 'b$
assumes *cov*: *covering_space C p S* **and** $a \in C$ $z \in U$
and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
and *feq*: $f\ z = p\ a$
and *hom*: $\bigwedge r.\ [\![path\ r;\ path\_image\ r \subseteq U;\ pathstart\ r = z;\ pathfinish\ r = z]\!]$
$\Longrightarrow \exists q.\ path\ q \wedge path\_image\ q \subseteq C \wedge$
$pathstart\ q = a \wedge pathfinish\ q = a \wedge$
$homotopic\_paths\ S\ (f \circ r)\ (p \circ q)$
obtains *g* **where** *continuous_on U g* $g \in U \to C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$
proof −
have $*$: $\exists g\ h.\ path\ g \wedge path\_image\ g \subseteq U \wedge$
$pathstart\ g = z \wedge pathfinish\ g = y \wedge$
$path\ h \wedge path\_image\ h \subseteq C \wedge pathstart\ h = a \wedge$
$(\forall t \in \{0..1\}.\ p(h\ t) = f(g\ t))$
if $y \in U$ **for** *y*
proof −
obtain *g* **where** *path g path_image g* $\subseteq U$ **and** *pastg*: *pathstart g* $= z$
and *pafig*: *pathfinish g* $= y$
using *U* ‹$z \in U$› ‹$y \in U$› **by** (*force simp*: *path_connected_def*)
obtain *h* **where** *path h path_image h* $\subseteq C$ *pathstart h* $= a$
and $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h\ t) = (f \circ g)\ t$
proof (*rule covering_space_lift_path_strong* [*OF cov* ‹$a \in C$›])
show *path* $(f \circ g)$
using ‹*path g*› ‹*path_image g* $\subseteq U$› *contf continuous_on_subset path_continuous_image* **by** *blast*
show *path_image* $(f \circ g) \subseteq S$
by (*metis* ‹*path_image g* $\subseteq U$› *fim image_mono path_image_compose subset_trans image_subset_iff_funcset*)
show *pathstart* $(f \circ g) = p\ a$
by (*simp add*: *feq pastg pathstart_compose*)
qed *auto*
then show *?thesis*
by (*metis* ‹*path g*› ‹*path_image g* $\subseteq U$› *comp_apply pafig pastg*)
qed
have $\exists l.\ \forall g\ h.\ path\ g \wedge path\_image\ g \subseteq U \wedge pathstart\ g = z \wedge pathfinish\ g = y \wedge$
$path\ h \wedge path\_image\ h \subseteq C \wedge pathstart\ h = a \wedge$
$(\forall t \in \{0..1\}.\ p(h\ t) = f(g\ t)) \longrightarrow pathfinish\ h = l$ **for** *y*
proof −
have *pathfinish h* $=$ *pathfinish h'*
if *g*: *path g path_image g* $\subseteq U$ *pathstart g* $= z$ *pathfinish g* $= y$
and *h*: *path h path_image h* $\subseteq C$ *pathstart h* $= a$

```
          and phg: ⋀t. t ∈ {0..1} ⟹ p(h t) = f(g t)
          and g': path g' path_image g' ⊆ U pathstart g' = z pathfinish g' = y
          and h': path h' path_image h' ⊆ C pathstart h' = a
          and phg': ⋀t. t ∈ {0..1} ⟹ p(h' t) = f(g' t)
        for g h g' h'
    proof −
      obtain q where path q and piq: path_image q ⊆ C and pastq: pathstart q
= a and pafiq: pathfinish q = a
              and homS: homotopic_paths S (f ∘ g +++ reversepath g') (p ∘ q)
      using g g' hom [of g +++ reversepath g'] by (auto simp: subset_path_image_join)
            have papq: path (p ∘ q)
              using homS homotopic_paths_imp_path by blast
            have pipq: path_image (p ∘ q) ⊆ S
              using homS homotopic_paths_imp_subset by blast
      obtain q' where path q' path_image q' ⊆ C
              and pathstart q' = pathstart q pathfinish q' = pathfinish q
              and pq'_eq: ⋀t. t ∈ {0..1} ⟹ p (q' t) = (f ∘ g +++ reversepath
g') t
        using covering_space_lift_homotopic_path [OF cov homotopic_paths_sym
[OF homS] ‹path q› piq refl refl]
        by auto
      have q' t = (h ∘ (∗R) 2) t if 0 ≤ t t ≤ 1/2 for t
      proof (rule covering_space_lift_unique [OF cov, of q' 0 h ∘ (∗R) 2 {0..1/2}
f ∘ g ∘ (∗R) 2 t])
        show q' 0 = (h ∘ (∗R) 2) 0
          by (metis ‹pathstart q' = pathstart q› comp_def h(3) pastq pathstart_def
pth_4(2))
        show continuous_on {0..1/2} (f ∘ g ∘ (∗R) 2)
        proof (intro continuous_intros continuous_on_path [OF ‹path g›] contin-
uous_on_subset [OF contf])
          show g ‘ (∗R) 2 ‘ {0..1/2} ⊆ U
            using g path_image_def by fastforce
        qed auto
        show (f ∘ g ∘ (∗R) 2) ∈ {0..1/2} → S
        using g(2) fim by (fastforce simp: path_image_def image_subset_iff_funcset)
        show (h ∘ (∗R) 2) ∈ {0..1/2} → C
          using h path_image_def by fastforce
        show q' ∈ {0..1/2} → C
          using ‹path_image q' ⊆ C› path_image_def by fastforce
        show ⋀x. x ∈ {0..1/2} ⟹ (f ∘ g ∘ (∗R) 2) x = p (q' x)
          by (auto simp: joinpaths_def pq'_eq)
        show ⋀x. x ∈ {0..1/2} ⟹ (f ∘ g ∘ (∗R) 2) x = p ((h ∘ (∗R) 2) x)
          by (simp add: phg)
        show continuous_on {0..1/2} q'
          by (simp add: continuous_on_path ‹path q'›)
        show continuous_on {0..1/2} (h ∘ (∗R) 2)
          by (intro continuous_intros continuous_on_path [OF ‹path h›]) auto
      qed (use that in auto)
      moreover have q' t = (reversepath h' ∘ (λt. 2 ∗R t − 1)) t if 1/2 < t t ≤
```

```
1 for t
    proof (rule covering_space_lift_unique [OF cov, of q' 1 reversepath h' ∘ (λt. 2 *R t − 1) {1/2<..1} f ∘ reversepath g' ∘ (λt. 2 *R t − 1) t])
      show q' 1 = (reversepath h' ∘ (λt. 2 *R t − 1)) 1
        using h' ‹pathfinish q' = pathfinish q› pafiq
        by (simp add: pathstart_def pathfinish_def reversepath_def)
      show continuous_on {1/2<..1} (f ∘ reversepath g' ∘ (λt. 2 *R t − 1))
         proof (intro continuous_intros continuous_on_path ‹path g'› continuous_on_subset [OF contf])
        show reversepath g' ` (λt. 2 *R t − 1) ` {1/2<..1} ⊆ U
          using g' by (auto simp: path_image_def reversepath_def)
      qed (use g' in auto)
      show (f ∘ reversepath g' ∘ (λt. 2 *R t − 1)) ∈ {1/2<..1} → S
           using g'(2) path_image_def fim by (auto simp: image_subset_iff path_image_def reversepath_def)
      show q' ∈ {1/2<..1} → C
        using ‹path_image q' ⊆ C› path_image_def by fastforce
      show (reversepath h' ∘ (λt. 2 *R t − 1)) ∈ {1/2<..1} → C
        using h' by (simp add: path_image_def reversepath_def subset_eq)
      show ⋀x. x ∈ {1/2<..1} ⟹ (f ∘ reversepath g' ∘ (λt. 2 *R t − 1)) x = p (q' x)
        by (auto simp: joinpaths_def pq'_eq)
      show ⋀x. x ∈ {1/2<..1} ⟹
            (f ∘ reversepath g' ∘ (λt. 2 *R t − 1)) x = p ((reversepath h' ∘ (λt. 2 *R t − 1)) x)
        by (simp add: phg' reversepath_def)
      show continuous_on {1/2<..1} q'
        by (auto intro: continuous_on_path [OF ‹path q'›])
      show continuous_on {1/2<..1} (reversepath h' ∘ (λt. 2 *R t − 1))
         by (intro continuous_intros continuous_on_path ‹path h'›) (use h' in auto)
    qed (use that in auto)
    ultimately have q' t = (h +++ reversepath h') t if 0 ≤ t t ≤ 1 for t
      using that by (simp add: joinpaths_def)
    then have path(h +++ reversepath h')
      by (auto intro: path_eq [OF ‹path q'›])
    then show ?thesis
      by (auto simp: ‹path h› ‹path h'›)
  qed
  then show ?thesis by metis
 qed
 then obtain l :: 'c ⇒ 'a
       where l: ⋀y g h. ⟦path g; path_image g ⊆ U; pathstart g = z; pathfinish g = y;
                         path h; path_image h ⊆ C; pathstart h = a;
                         ⋀t. t ∈ {0..1} ⟹ p(h t) = f(g t)⟧ ⟹ pathfinish h = l y
  by metis
 show ?thesis
 proof
```

```
    show pleq: p (l y) = f y if y ∈ U for y
     using*[OF ‹y ∈ U›] by (metis l atLeastAtMost_iff order_refl pathfinish_def
zero_le_one)
    show l z = a
     using l [of linepath z z z linepath a a] by (auto simp: assms)
    show LC: l ∈ U → C
     by (clarify dest!: *) (metis (full_types) l pathfinish_in_path_image subsetCE)
    have ∃ T. openin (top_of_set U) T ∧ y ∈ T ∧ T ⊆ U ∩ l -‘ X
        if X: openin (top_of_set C) X and y ∈ U l y ∈ X for X y
    proof -
     have X ⊆ C
      using X openin_euclidean_subtopology_iff by blast
     have f y ∈ S
      using fim ‹y ∈ U› by blast
     then obtain W V
           where WV: f y ∈ W ∧ openin (top_of_set S) W ∧
                     (⋃V = C ∩ p -‘ W ∧
                      (∀ U ∈ V. openin (top_of_set C) U) ∧
                      pairwise disjnt V ∧
                      (∀ U ∈ V. ∃ q. homeomorphism U W p q))
      using cov by (force simp: covering_space_def)
     then have l y ∈ ⋃V
      using ‹X ⊆ C› pleq that by auto
     then obtain W' where l y ∈ W' and W' ∈ V
      by blast
     with WV obtain p' where opeCW': openin (top_of_set C) W'
                         and homUW': homeomorphism W' W p p'
      by blast
     then have contp': continuous_on W p' and p'im: p' ‘ W ⊆ W'
      using homUW' homeomorphism_image2 homeomorphism_cont2 by fast-
force+
     obtain V where y ∈ V y ∈ U and fimW: f ‘ V ⊆ W V ⊆ U
               and path_connected V and opeUV: openin (top_of_set U) V
     proof -
      have openin (top_of_set U) (U ∩ f -‘ W)
       using WV contf continuous_on_open_gen fim by auto
      then obtain UO where openin (top_of_set U) UO ∧ path_connected UO
∧ y ∈ UO ∧ UO ⊆ U ∩ f -‘ W
       using U WV ‹y ∈ U› unfolding locally_path_connected by (meson IntI
vimage_eq)
      then show ?thesis
       by (meson ‹y ∈ U› image_subset_iff_subset_vimage le_inf_iff that)
     qed
     have W' ⊆ C W ⊆ S
      using opeCW' WV openin_imp_subset by auto
     have p'im: p' ‘ W ⊆ W'
      using homUW' homeomorphism_image2 by fastforce
     show ?thesis
     proof (intro exI conjI)
```

have *openin* (*top_of_set S*) (*W* ∩ *p′* −‘ (*W′* ∩ *X*))
proof (*rule openin_trans*)
show *openin* (*top_of_set W*) (*W* ∩ *p′* −‘ (*W′* ∩ *X*))
using *X* ‹*W′* ⊆ *C*›
by (*metis continuous_on_open contp′ homUW′ homeomorphism_image2 inf.assoc inf.orderE openin_open*)
show *openin* (*top_of_set S*) *W*
using *WV* **by** *blast*
qed
then show *openin* (*top_of_set U*) (*V* ∩ (*U* ∩ (*f* −‘ (*W* ∩ (*p′* −‘ (*W′* ∩ *X*))))))
by (*blast intro*: *opeUV openin_subtopology_self continuous_openin_preimage* [*OF contf fim*])
have *p′* (*f y*) ∈ *X*
using ‹*l y* ∈ *W′*› *homeomorphism_apply1* [*OF homUW′*] *pleq* ‹*y* ∈ *U*› ‹*l y* ∈ *X*› **by** *fastforce*
then show *y* ∈ *V* ∩ (*U* ∩ *f* −‘ (*W* ∩ *p′* −‘ (*W′* ∩ *X*)))
using ‹*y* ∈ *U*› ‹*y* ∈ *V*› *WV p′im* **by** *auto*
show *V* ∩ (*U* ∩ *f* −‘ (*W* ∩ *p′* −‘ (*W′* ∩ *X*))) ⊆ *U* ∩ *l* −‘ *X*
proof (*intro subsetI IntI*; *clarify*)
fix *y′*
assume *y′*: *y′* ∈ *V y′* ∈ *U f y′* ∈ *W p′* (*f y′*) ∈ *W′ p′* (*f y′*) ∈ *X*
then obtain *γ* **where** *path γ path_image γ* ⊆ *V pathstart γ* = *y pathfinish γ* = *y′*
by (*meson* ‹*path_connected V*› ‹*y* ∈ *V*› *path_connected_def*)
obtain *pp qq* **where** *pp*: *path pp path_image pp* ⊆ *U pathstart pp* = *z pathfinish pp* = *y*
and *qq*: *path qq path_image qq* ⊆ *C pathstart qq* = *a*
and *pqqeq*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*qq t*) = *f*(*pp t*)
using∗[*OF* ‹*y* ∈ *U*›] **by** *blast*
have *finW*: ⋀*x*. ⟦*0* ≤ *x*; *x* ≤ *1*⟧ ⟹ *f* (*γ x*) ∈ *W*
using ‹*path_image γ* ⊆ *V*› **by** (*auto simp*: *image_subset_iff path_image_def fimW* [*THEN subsetD*])
have *pathfinish* (*qq* +++ (*p′* ∘ *f* ∘ *γ*)) = *l y′*
proof (*rule l* [*of pp* +++ *γ y′ qq* +++ (*p′* ∘ *f* ∘ *γ*)])
show *path* (*pp* +++ *γ*)
by (*simp add*: ‹*path γ*› ‹*path pp*› ‹*pathfinish pp* = *y*› ‹*pathstart γ* = *y*›)
show *path_image* (*pp* +++ *γ*) ⊆ *U*
using ‹*V* ⊆ *U*› ‹*path_image γ* ⊆ *V*› ‹*path_image pp* ⊆ *U*› *not_in_path_image_join* **by** *blast*
show *pathstart* (*pp* +++ *γ*) = *z*
by (*simp add*: ‹*pathstart pp* = *z*›)
show *pathfinish* (*pp* +++ *γ*) = *y′*
by (*simp add*: ‹*pathfinish γ* = *y′*›)
have *pathfinish qq* = *l y*
using ‹*path pp*› ‹*path qq*› ‹*path_image pp* ⊆ *U*› ‹*path_image qq* ⊆ *C*› ‹*pathfinish pp* = *y*› ‹*pathstart pp* = *z*› ‹*pathstart qq* = *a*› *l pqqeq* **by** *blast*
also have ... = *p′* (*f y*)
using ‹*l y* ∈ *W′*› *homUW′ homeomorphism_apply1 pleq that*(*2*) **by**

```
fastforce
          finally have pathfinish qq = p' (f y) .
          then have paqq: pathfinish qq = pathstart (p' ∘ f ∘ γ)
            by (simp add: ‹pathstart γ = y› pathstart_compose)
          have continuous_on (path_image γ) (p' ∘ f)
          proof (rule continuous_on_compose)
            show continuous_on (path_image γ) f
               using ‹path_image γ ⊆ V› ‹V ⊆ U› contf continuous_on_subset
by blast
            show continuous_on (f ‘ path_image γ) p'
            proof (rule continuous_on_subset [OF contp'])
              show f ‘ path_image γ ⊆ W
                by (auto simp: path_image_def pathfinish_def pathstart_def finW)
            qed
          qed
          then show path (qq +++ (p' ∘ f ∘ γ))
            using ‹path γ› ‹path qq› paqq path_continuous_image path_join_imp
by blast
          show path_image (qq +++ (p' ∘ f ∘ γ)) ⊆ C
          proof (rule subset_path_image_join)
            show path_image qq ⊆ C
              by (simp add: ‹path_image qq ⊆ C›)
            show path_image (p' ∘ f ∘ γ) ⊆ C
             by (metis ‹W' ⊆ C› ‹path_image γ ⊆ V› dual_order.trans fimW(1)
image_comp image_mono p'im path_image_compose)
          qed
          show pathstart (qq +++ (p' ∘ f ∘ γ)) = a
            by (simp add: ‹pathstart qq = a›)
          show p ((qq +++ (p' ∘ f ∘ γ)) ξ) = f ((pp +++ γ) ξ) if ξ: ξ ∈ {0..1}
for ξ
          proof (simp add: joinpaths_def, safe)
            show p (qq (2*ξ)) = f (pp (2*ξ)) if ξ*2 ≤ 1
              using ‹ξ ∈ {0..1}› pqqeq that by auto
            show p (p' (f (γ (2*ξ − 1)))) = f (γ (2*ξ − 1)) if ¬ ξ*2 ≤ 1
              using that ξ by (auto intro: homeomorphism_apply2 [OF homUW'
finW])
          qed
        qed
        with ‹pathfinish γ = y'›  ‹p' (f y') ∈ X› show y' ∈ l −‘ X
          unfolding pathfinish_join by (simp add: pathfinish_def)
      qed
    qed
  qed
  then show continuous_on U l
    using vimage_eq openin_subopen continuous_on_open_gen [OF LC]
   by (metis IntD1 IntD2 vimage_eq openin_subopen continuous_on_open_gen
[OF LC])
 qed
qed
```

corollary *covering_space_lift_stronger*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
and $f :: 'c::real\_normed\_vector \Rightarrow 'b$
assumes *cov*: *covering_space C p S* $a \in C$ $z \in U$
and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
and *feq*: $f\ z = p\ a$
and *hom*: $\bigwedge r.\ [\![path\ r; path\_image\ r \subseteq U; pathstart\ r = z; pathfinish\ r = z]\!] \Longrightarrow \exists b.\ homotopic\_paths\ S\ (f \circ r)\ (linepath\ b\ b)$
obtains *g* **where** *continuous_on U g* $g \in U \to C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$
proof (*rule covering_space_lift_general* [*OF cov U contf fim feq*])
fix *r*
assume *path r path_image r* $\subseteq U$ *pathstart r* $= z$ *pathfinish r* $= z$
then obtain *b* **where** *b*: *homotopic_paths S* $(f \circ r)$ (*linepath b b*)
using *hom* **by** *blast*
then have f (*pathstart r*) $= b$
by (*metis homotopic_paths_imp_pathstart pathstart_compose pathstart_linepath*)
then have *homotopic_paths S* $(f \circ r)$ (*linepath* $(f\ z)$ $(f\ z)$)
by (*simp add*: *b* ‹*pathstart r* $= z$›)
then have *homotopic_paths S* $(f \circ r)$ $(p \circ linepath\ a\ a)$
by (*simp add*: *o_def feq linepath_def*)
then show $\exists q.$ *path q* $\wedge$
path_image q $\subseteq C$ $\wedge$
pathstart q $= a$ $\wedge$ *pathfinish q* $= a$ $\wedge$ *homotopic_paths S* $(f \circ r)$ $(p \circ q)$
by (*force simp*: ‹$a \in C$›)
qed *auto*

corollary *covering_space_lift_strong*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
and $f :: 'c::real\_normed\_vector \Rightarrow 'b$
assumes *cov*: *covering_space C p S* $a \in C$ $z \in U$
and *scU*: *simply_connected U* **and** *lpcU*: *locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
and *feq*: $f\ z = p\ a$
obtains *g* **where** *continuous_on U g* $g \in U \to C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$
proof (*rule covering_space_lift_stronger* [*OF cov _ lpcU contf fim feq*])
show *path_connected U*
using *scU simply_connected_eq_contractible_loop_some* **by** *blast*
fix *r*
assume *r*: *path r path_image r* $\subseteq U$ *pathstart r* $= z$ *pathfinish r* $= z$
have *linepath* $(f\ z)$ $(f\ z) = f \circ linepath\ z\ z$
by (*simp add*: *o_def linepath_def*)
then have *homotopic_paths S* $(f \circ r)$ (*linepath* $(f\ z)$ $(f\ z)$)
by (*metis r contf fim homotopic_paths_continuous_image scU simply_connected_eq_contractible_path*)
then show $\exists b.$ *homotopic_paths S* $(f \circ r)$ (*linepath b b*)

by *blast*
qed *blast*

corollary *covering_space_lift*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
and $f :: 'c::real\_normed\_vector \Rightarrow 'b$
assumes *cov*: *covering_space C p S*
and *U*: *simply_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
obtains *g* **where** *continuous_on U g g* $\in U \to C$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$
proof (*cases* $U = \{\}$)
case *True*
with *that* **show** *?thesis* **by** *auto*
next
case *False*
then obtain *z* **where** $z \in U$ **by** *blast*
then obtain *a* **where** $a \in C\ f\ z = p\ a$
by (*metis cov covering_space_imp_surjective fim image_iff Pi_iff*)
then show *?thesis*
by (*metis that covering_space_lift_strong* [*OF cov _* ‹$z \in U$› *U contf fim*])
qed

9.8.6 Homeomorphisms of arc images

lemma *homeomorphism_arc*:
fixes *g* :: *real* $\Rightarrow$ $'a::t2\_space$
assumes *arc g*
obtains *h* **where** *homeomorphism* $\{0..1\}$ (*path_image g*) *g h*
using *assms* **by** (*force simp*: *arc_def homeomorphism_compact path_def path_image_def*)

lemma *homeomorphic_arc_image_interval*:
fixes *g* :: *real* $\Rightarrow$ $'a::t2\_space$ **and** $a::real$
assumes *arc g* $a < b$
shows (*path_image g*) *homeomorphic* $\{a..b\}$
proof −
have (*path_image g*) *homeomorphic* $\{0..1::real\}$
by (*meson assms*(*1*) *homeomorphic_def homeomorphic_sym homeomorphism_arc*)
also have ... *homeomorphic* $\{a..b\}$
using *assms* **by** (*force intro*: *homeomorphic_closed_intervals_real*)
finally show *?thesis* **.**
qed

lemma *homeomorphic_arc_images*:
fixes *g* :: *real* $\Rightarrow$ $'a::t2\_space$ **and** *h* :: *real* $\Rightarrow$ $'b::t2\_space$
assumes *arc g arc h*
shows (*path_image g*) *homeomorphic* (*path_image h*)
proof −
have (*path_image g*) *homeomorphic* $\{0..1::real\}$
by (*meson assms homeomorphic_def homeomorphic_sym homeomorphism_arc*)

also have ... *homeomorphic* (*path_image h*)
by (*meson assms homeomorphic_def homeomorphism_arc*)
finally show *?thesis* **.**
qed

end

theory *Equivalence_Lebesgue_Henstock_Integration*
imports
Lebesgue_Measure
Henstock_Kurzweil_Integration
Complete_Measure
Set_Integral
Homeomorphism
Cartesian_Euclidean_Space
begin

lemma *LIMSEQ_if_less*: $(\lambda k.\ \textit{if } i < k \textit{ then } a \textit{ else } b) \longrightarrow a$
by (*rule_tac k=Suc i* **in** *LIMSEQ_offset*) *auto*

Note that the rhs is an implication. This lemma plays a specific role in one proof.

lemma *le_left_mono*: $x \leq y \Longrightarrow y \leq a \longrightarrow x \leq (a::'a::preorder)$
by (*auto intro*: *order_trans*)

lemma *ball_trans*:
assumes $y \in \textit{ball } z\ q\ r + q \leq s$ **shows** $\textit{ball } y\ r \subseteq \textit{ball } z\ s$
using *assms* **by** *metric*

lemma *has_integral_implies_lebesgue_measurable_cbox*:
fixes $f :: 'a :: \textit{euclidean\_space} \Rightarrow \textit{real}$
assumes *f*: $(f \textit{ has\_integral } I)\ (\textit{cbox } x\ y)$
shows $f \in \textit{lebesgue\_on } (\textit{cbox } x\ y) \rightarrow_M \textit{borel}$
proof (*rule cld_measure.borel_measurable_cld*)
let $?L = \textit{lebesgue\_on } (\textit{cbox } x\ y)$
let $?\mu = \textit{emeasure } ?L$
let $?\mu' = \textit{outer\_measure\_of } ?L$
interpret *L*: *finite_measure ?L*
proof
show $?\mu\ (\textit{space } ?L) \neq \infty$
by (*simp add*: *emeasure_restrict_space space_restrict_space emeasure_lborel_cbox_eq*)
qed

show *cld_measure ?L*
proof
fix *B A* **assume** $B \subseteq A\ A \in \textit{null\_sets } ?L$
then show $B \in \textit{sets } ?L$
using *null_sets_completion_subset*[*OF* ‹$B \subseteq A$›, *of lborel*]

```
    by (auto simp add: null_sets_restrict_space sets_restrict_space_iff intro: )
  next
    fix A assume A ⊆ space ?L ⋀B. B ∈ sets ?L ⟹ ?μ B < ∞ ⟹ A ∩ B ∈ sets ?L
    from this(1) this(2)[of space ?L] show A ∈ sets ?L
      by (auto simp: Int_absorb2 less_top[symmetric])
  qed auto
  then interpret cld_measure ?L
    .

  have content_eq_L: A ∈ sets borel ⟹ A ⊆ cbox x y ⟹ content A = measure ?L A for A
    by (subst measure_restrict_space) (auto simp: measure_def)

  fix E and a b :: real assume E ∈ sets ?L a < b 0 < ?μ E ?μ E < ∞
  then obtain M :: real where ?μ E = M 0 < M
    by (cases ?μ E) auto
  define e where e = M / (4 + 2 / (b - a))
  from ‹a < b› ‹0<M› have 0 < e
    by (auto intro!: divide_pos_pos simp: field_simps e_def)

  have e < M / (3 + 2 / (b - a))
    using ‹a < b› ‹0 < M›
    unfolding e_def by (intro divide_strict_left_mono add_strict_right_mono mult_pos_pos) (auto simp: field_simps)
  then have 2 * e < (b - a) * (M - e * 3)
    using ‹0<M› ‹0 < e› ‹a < b› by (simp add: field_simps)

  have e_less_M: e < M / 1
    unfolding e_def using ‹a < b› ‹0<M› by (intro divide_strict_left_mono) (auto simp: field_simps)

  obtain d
    where gauge d
      and integral_f: ∀ p. p tagged_division_of cbox x y ∧ d fine p ⟶
        norm ((∑ (x,k) ∈ p. content k *R f x) - I) < e
    using ‹0<e› f unfolding has_integral by auto

  define C where C X m = X ∩ {x. ball x (1/Suc m) ⊆ d x} for X m
  have incseq (C X) for X
    unfolding C_def [abs_def]
    by (intro monoI Collect_mono conj_mono imp_refl le_left_mono subset_ball divide_left_mono Int_mono) auto

  { fix X assume X ⊆ space ?L and eq: ?μ' X = ?μ E
    have (SUP m. outer_measure_of ?L (C X m)) = outer_measure_of ?L (⋃ m. C X m)
      using ‹X ⊆ space ?L› by (intro SUP_outer_measure_of_incseq ‹incseq (C X)›) (auto simp: C_def)
```

```
    also have (⋃ m. C X m) = X
    proof −
      { fix x
        obtain e where 0 < e ball x e ⊆ d x
         using gaugeD[OF ‹gauge d›, of x] unfolding open_contains_ball by auto
        moreover
        obtain n where 1 / (1 + real n) < e
          using reals_Archimedean[OF ‹0<e›] by (auto simp: inverse_eq_divide)
        then have ball x (1 / (1 + real n)) ⊆ ball x e
          by (intro subset_ball) auto
        ultimately have ∃ n. ball x (1 / (1 + real n)) ⊆ d x
          by blast }
      then show ?thesis
        by (auto simp: C_def)
    qed
    finally have (SUP m. outer_measure_of ?L (C X m)) = ?μ E
      using eq by auto
    also have ... > M − e
      using ‹0 < M› ‹?μ E = M› ‹0<e› by (auto intro!: ennreal_lessI)
    finally have ∃ m. M − e < outer_measure_of ?L (C X m)
      unfolding less_SUP_iff by auto }
  note C = this

  let ?E = {x∈E. f x ≤ a} and ?F = {x∈E. b ≤ f x}

  have ¬ (?μ' ?E = ?μ E ∧ ?μ' ?F = ?μ E)
  proof
    assume eq: ?μ' ?E = ?μ E ∧ ?μ' ?F = ?μ E
    with C[of ?E] C[of ?F] ‹E ∈ sets ?L›[THEN sets.sets_into_space] obtain ma
mb
      where M − e < outer_measure_of ?L (C ?E ma) M − e < outer_measure_of
?L (C ?F mb)
      by auto
    moreover define m where m = max ma mb
    ultimately have M_minus_e: M − e < outer_measure_of ?L (C ?E m) M
− e < outer_measure_of ?L (C ?F m)
      using
        incseqD[OF ‹incseq (C ?E)›, of ma m, THEN outer_measure_of_mono]
        incseqD[OF ‹incseq (C ?F)›, of mb m, THEN outer_measure_of_mono]
      by (auto intro: less_le_trans)
    define d' where d' x = d x ∩ ball x (1 / (3 * Suc m)) for x
    have gauge d'
      unfolding d'_def by (intro gauge_Int ‹gauge d› gauge_ball) auto
    then obtain p where p: p tagged_division_of cbox x y d' fine p
      by (rule fine_division_exists)
    then have d fine p
      unfolding d'_def[abs_def] fine_def by auto

    define s where s = {(x::'a, k). k ∩ (C ?E m) ≠ {} ∧ k ∩ (C ?F m) ≠ {}}
```

```
    define T where T E k = (SOME x. x ∈ k ∩ C E m) for E k
    let ?A = (λ(x, k). (T ?E k, k)) ‘ (p ∩ s) ∪ (p − s)
    let ?B = (λ(x, k). (T ?F k, k)) ‘ (p ∩ s) ∪ (p − s)

    { fix X assume X_eq: X = ?E ∨ X = ?F
      let ?T = (λ(x, k). (T X k, k))
      let ?p = ?T ‘ (p ∩ s) ∪ (p − s)

      have in_s: (x, k) ∈ s ⟹ T X k ∈ k ∩ C X m for x k
        using someI_ex[of λx. x ∈ k ∩ C X m] X_eq unfolding ex_in_conv by
(auto simp: T_def s_def)

      { fix x k assume (x, k) ∈ p (x, k) ∈ s
        have k: k ⊆ ball x (1 / (3 * Suc m))
          using ‹d' fine p›[THEN fineD, OF ‹(x, k) ∈ p›] by (auto simp: d'_def)
        then have x ∈ ball (T X k) (1 / (3 * Suc m))
        using in_s[OF ‹(x, k) ∈ s›] by (auto simp: C_def subset_eq dist_commute)
        then have ball x (1 / (3 * Suc m)) ⊆ ball (T X k) (1 / Suc m)
          by (rule ball_trans) (auto simp: field_split_simps)
        with k in_s[OF ‹(x, k) ∈ s›] have k ⊆ d (T X k)
          by (auto simp: C_def) }
      then have d fine ?p
        using ‹d fine p› by (auto intro!: fineI)
      moreover
      have ?p tagged_division_of cbox x y
      proof (rule tagged_division_ofI)
        show finite ?p
          using p(1) by auto
      next
        fix z k assume *: (z, k) ∈ ?p
        then consider (z, k) ∈ p (z, k) ∉ s
          | x' where (x', k) ∈ p (x', k) ∈ s z = T X k
          by (auto simp: T_def)
        then have z ∈ k ∧ k ⊆ cbox x y ∧ (∃ a b. k = cbox a b)
          using p(1) by cases (auto dest: in_s)
        then show z ∈ k k ⊆ cbox x y ∃ a b. k = cbox a b
          by auto
      next
        fix z k z' k' assume (z, k) ∈ ?p (z', k') ∈ ?p (z, k) ≠ (z', k')
        with tagged_division_ofD(5)[OF p(1), of _ k _ k']
        show interior k ∩ interior k' = {}
          by (auto simp: T_def dest: in_s)
      next
        have {k. ∃ x. (x, k) ∈ ?p} = {k. ∃ x. (x, k) ∈ p}
          by (auto simp: T_def image_iff Bex_def)
        then show ⋃ {k. ∃ x. (x, k) ∈ ?p} = cbox x y
          using p(1) by auto
      qed
      ultimately have I: norm ((∑(x,k) ∈ ?p. content k *R f x) − I) < e
```

using *integral_f* **by** *auto*

have $(\sum (x,k) \in ?p.\ content\ k *_R f\ x) =$
$(\sum (x,k) \in ?T\ `\ (p \cap s).\ content\ k *_R f\ x) + (\sum (x,k) \in p - s.\ content\ k *_R f\ x)$
using *p(1)*[*THEN tagged_division_ofD(1)*]
by (*safe intro*!: *sum.union_inter_neutral*) (*auto simp*: *s_def T_def*)
also have $(\sum (x,k) \in ?T\ `\ (p \cap s).\ content\ k *_R f\ x) = (\sum (x,k) \in p \cap s.\ content\ k *_R f\ (T\ X\ k))$
proof (*subst sum.reindex_nontrivial*, *safe*)
fix *x1 x2 k* **assume** *1*: $(x1,\ k) \in p\ (x1,\ k) \in s$ **and** *2*: $(x2,\ k) \in p\ (x2,\ k) \in s$
and *eq*: $content\ k *_R f\ (T\ X\ k) \neq 0$
with *tagged_division_ofD(5)*[*OF p(1)*, *of x1 k x2 k*] *tagged_division_ofD(4)*[*OF p(1)*, *of x1 k*]
show *x1* = *x2*
by (*auto simp*: *content_eq_0_interior*)
qed (*use p* **in** ‹*auto intro*!: *sum.cong*›)
finally have *eq*: $(\sum (x,k) \in ?p.\ content\ k *_R f\ x) =$
$(\sum (x,k) \in p \cap s.\ content\ k *_R f\ (T\ X\ k)) + (\sum (x,k) \in p - s.\ content\ k *_R f\ x)$ **.**

have *in_T*: $(x,\ k) \in s \Longrightarrow T\ X\ k \in X$ **for** *x k*
using *in_s*[*of x k*] **by** (*auto simp*: *C_def*)

note *I eq in_T* **}**
note *parts* = *this*

have *p_in_L*: $(x,\ k) \in p \Longrightarrow k \in sets\ ?L$ **for** *x k*
using *tagged_division_ofD(3, 4)*[*OF p(1)*, *of x k*] **by** (*auto simp*: *sets_restrict_space*)

have [*simp*]: *finite p*
using *tagged_division_ofD(1)*[*OF p(1)*] **.**

have $(M - 3*e) * (b - a) \leq (\sum (x,k) \in p \cap s.\ content\ k) * (b - a)$
proof (*intro mult_right_mono*)
have *fin*: $?\mu\ (E \cap \bigcup \{k \in snd`p.\ k \cap C\ X\ m = \{\}\}) < \infty$ **for** *X*
using ‹$?\mu\ E < \infty$› **by** (*rule le_less_trans*[*rotated*]) (*auto intro*!: *emeasure_mono* ‹$E \in sets\ ?L$›)
have *sets*: $(E \cap \bigcup \{k \in snd`p.\ k \cap C\ X\ m = \{\}\}) \in sets\ ?L$ **for** *X*
using *tagged_division_ofD(1)*[*OF p(1)*] **by** (*intro sets.Diff* ‹$E \in sets\ ?L$› *sets.finite_Union sets.Int*) (*auto intro*: *p_in_L*)
{ fix *X* **assume** $X \subseteq E$ $M - e < ?\mu'\ (C\ X\ m)$
have $M - e \leq ?\mu'\ (C\ X\ m)$
by (*rule less_imp_le*) *fact*
also have $\ldots \leq ?\mu'\ (E - (E \cap \bigcup \{k \in snd`p.\ k \cap C\ X\ m = \{\}\}))$
proof (*intro outer_measure_of_mono subsetI*)
fix *v* **assume** $v \in C\ X\ m$
then have $v \in cbox\ x\ y$ $v \in E$

```
          using ‹E ⊆ space ?L› ‹X ⊆ E› by (auto simp: space_restrict_space
C_def)
        then obtain z k where (z, k) ∈ p v ∈ k
          using tagged_division_ofD(6)[OF p(1), symmetric] by auto
        then show v ∈ E − E ∩ (⋃{k∈snd‘p. k ∩ C X m = {}})
          using ‹v ∈ C X m› ‹v ∈ E› by auto
      qed
      also have ... = ?μ E − ?μ (E ∩ ⋃{k∈snd‘p. k ∩ C X m = {}})
        using ‹E ∈ sets ?L› fin[of X] sets[of X] by (auto intro!: emeasure_Diff)
      finally have ?μ (E ∩ ⋃{k∈snd‘p. k ∩ C X m = {}}) ≤ e
        using ‹0 < e› e_less_M
        by (cases ?μ (E ∩ ⋃{k∈snd‘p. k ∩ C X m = {}})) (auto simp add: ‹?μ
E = M› ennreal_minus ennreal_le_iff2)
      note this }
    note upper_bound = this

    have ?μ (E ∩ ⋃(snd‘(p − s))) =
      ?μ ((E ∩ ⋃{k∈snd‘p. k ∩ C ?E m = {}}) ∪ (E ∩ ⋃{k∈snd‘p. k ∩ C ?F
m = {}}))
      by (intro arg_cong[where f=?μ]) (auto simp: s_def image_def Bex_def)
     also have ... ≤ ?μ (E ∩ ⋃{k∈snd‘p. k ∩ C ?E m = {}}) + ?μ (E ∩
⋃{k∈snd‘p. k ∩ C ?F m = {}})
      using sets[of ?E] sets[of ?F] M_minus_e by (intro emeasure_subadditive)
auto
    also have ... ≤ e + ennreal e
       using upper_bound[of ?E] upper_bound[of ?F] M_minus_e by (intro
add_mono) auto
    finally have ?μ E − 2∗e ≤ ?μ (E − (E ∩ ⋃(snd‘(p − s))))
      using ‹0 < e› ‹E ∈ sets ?L› tagged_division_ofD(1)[OF p(1)]
      by (subst emeasure_Diff)
         (auto simp: top_unique simp flip: ennreal_plus
               intro!: sets.Int sets.finite_UN ennreal_mono_minus intro: p_in_L)
    also have ... ≤ ?μ (⋃x∈p ∩ s. snd x)
    proof (safe intro!: emeasure_mono subsetI)
      fix v assume v ∈ E and not: v ∉ (⋃x∈p ∩ s. snd x)
      then have v ∈ cbox x y
        using ‹E ⊆ space ?L› by (auto simp: space_restrict_space)
      then obtain z k where (z, k) ∈ p v ∈ k
        using tagged_division_ofD(6)[OF p(1), symmetric] by auto
      with not show v ∈ ⋃(snd ‘ (p − s))
        by (auto intro!: bexI[of _ (z, k)] elim: ballE[of _ _ (z, k)])
    qed (auto intro!: sets.Int sets.finite_UN ennreal_mono_minus intro: p_in_L)
    also have ... = measure ?L (⋃x∈p ∩ s. snd x)
      by (auto intro!: emeasure_eq_ennreal_measure)
    finally have M − 2 ∗ e ≤ measure ?L (⋃x∈p ∩ s. snd x)
      unfolding ‹?μ E = M› using ‹0 < e› by (simp add: ennreal_minus)
    also have measure ?L (⋃x∈p ∩ s. snd x) = content (⋃x∈p ∩ s. snd x)
      using tagged_division_ofD(1,3,4) [OF p(1)]
      by (intro content_eq_L[symmetric])
```

```
        (fastforce intro!: sets.finite_UN UN_least del: subsetI)+
    also have content (⋃x∈p ∩ s. snd x) ≤ (∑k∈p ∩ s. content (snd k))
    using p(1) by (auto simp: emeasure_lborel_cbox_eq intro!: measure_subadditive_finite
                    dest!: p(1)[THEN tagged_division_ofD(4)])
    finally show M − 3 * e ≤ (∑(x, y)∈p ∩ s. content y)
      using ‹0 < e› by (simp add: split_beta)
  qed (use ‹a < b› in auto)
  also have ... = (∑(x,k) ∈ p ∩ s. content k * (b − a))
    by (simp add: sum_distrib_right split_beta')
  also have ... ≤ (∑(x,k) ∈ p ∩ s. content k * (f (T ?F k) − f (T ?E k)))
    using parts(3) by (auto intro!: sum_mono mult_left_mono diff_mono)
  also have ... = (∑(x,k) ∈ p ∩ s. content k * f (T ?F k)) − (∑(x,k) ∈ p ∩
s. content k * f (T ?E k))
    by (auto intro!: sum.cong simp: field_simps sum_subtractf[symmetric])
  also have ... = (∑(x,k) ∈ ?B. content k *R f x) − (∑(x,k) ∈ ?A. content k
*R f x)
    by (subst (1 2) parts) auto
   also have ... ≤ norm ((∑(x,k) ∈ ?B. content k *R f x) − (∑(x,k) ∈ ?A.
content k *R f x))
    by auto
  also have ... ≤ e + e
    using parts(1)[of ?E] parts(1)[of ?F] by (intro norm_diff_triangle_le[of _
I]) auto
  finally show False
    using ‹2 * e < (b − a) * (M − e * 3)› by (auto simp: field_simps)
 qed
 moreover have ?μ' ?E ≤ ?μ E ?μ' ?F ≤ ?μ E
  unfolding outer_measure_of_eq[OF ‹E ∈ sets ?L›, symmetric] by (auto intro!:
outer_measure_of_mono)
 ultimately show min (?μ' ?E) (?μ' ?F) < ?μ E
  unfolding min_less_iff_disj by (auto simp: less_le)
qed

lemma has_integral_implies_lebesgue_measurable_real:
 fixes f :: 'a :: euclidean_space ⇒ real
 assumes f: (f has_integral I) Ω
 shows (λx. f x * indicator Ω x) ∈ lebesgue →M borel
proof −
 define B :: nat ⇒ 'a set where B n = cbox (− real n *R One) (real n *R One)
for n
 show (λx. f x * indicator Ω x) ∈ lebesgue →M borel
 proof (rule measurable_piecewise_restrict)
  have (⋃n. box (− real n *R One) (real n *R One)) ⊆ ⋃(B ‘ UNIV)
   unfolding B_def by (intro UN_mono box_subset_cbox order_refl)
  then show countable (range B) space lebesgue ⊆ ⋃(B ‘ UNIV)
   by (auto simp: B_def UN_box_eq_UNIV)
 next
  fix Ω' assume Ω' ∈ range B
  then obtain n where Ω': Ω' = B n by auto
```

```
    then show Ω' ∩ space lebesgue ∈ sets lebesgue
      by (auto simp: B_def)

    have f integrable_on Ω
      using f by auto
    then have (λx. f x * indicator Ω x) integrable_on Ω
      by (auto simp: integrable_on_def cong: has_integral_cong)
    then have (λx. f x * indicator Ω x) integrable_on (Ω ∪ B n)
      by (rule integrable_on_superset) auto
    then have (λx. f x * indicator Ω x) integrable_on B n
      unfolding B_def by (rule integrable_on_subcbox) auto
    then show (λx. f x * indicator Ω x) ∈ lebesgue_on Ω' →M borel
    unfolding B_def Ω' by (auto intro: has_integral_implies_lebesgue_measurable_cbox
simp: integrable_on_def)
  qed
qed

lemma has_integral_implies_lebesgue_measurable:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f: (f has_integral I) Ω
  shows (λx. indicator Ω x *R f x) ∈ lebesgue →M borel
proof (intro borel_measurable_euclidean_space[where 'c='b, THEN iffD2] ballI)
  fix i :: 'b assume i ∈ Basis
  have (λx. (f x • i) * indicator Ω x) ∈ borel_measurable (completion lborel)
    using has_integral_linear[OF f bounded_linear_inner_left, of i]
   by (intro has_integral_implies_lebesgue_measurable_real) (auto simp: comp_def)
  then show (λx. indicator Ω x *R f x • i) ∈ borel_measurable (completion lborel)
    by (simp add: ac_simps)
qed
```

9.8.7 Equivalence Lebesgue integral on *lborel* and HK-integral

```
lemma has_integral_measure_lborel:
  fixes A :: 'a::euclidean_space set
  assumes A[measurable]: A ∈ sets borel and finite: emeasure lborel A < ∞
  shows ((λx. 1) has_integral measure lborel A) A
proof −
  { fix l u :: 'a
    have ((λx. 1) has_integral measure lborel (box l u)) (box l u)
    proof cases
      assume ∀ b∈Basis. l • b ≤ u • b
      then show ?thesis
        using has_integral_const[of 1::real l u]
      by (simp flip: has_integral_restrict[OF box_subset_cbox] add: has_integral_spike_interior)
    next
      assume ¬ (∀ b∈Basis. l • b ≤ u • b)
      then have box l u = {}
        unfolding box_eq_empty by (auto simp: not_le intro: less_imp_le)
      then show ?thesis
```

```
      by simp
  qed }
 note has_integral_box = this

 { fix a b :: 'a let ?M = λA. measure lborel (A ∩ box a b)
  have Int_stable (range (λ(a, b). box a b))
   by (auto simp: Int_stable_def box_Int_box)
  moreover have (range (λ(a, b). box a b)) ⊆ Pow UNIV
   by auto
  moreover have A ∈ sigma_sets UNIV (range (λ(a, b). box a b))
    using A unfolding borel_eq_box by simp
  ultimately have ((λx. 1) has_integral ?M A) (A ∩ box a b)
  proof (induction rule: sigma_sets_induct_disjoint)
   case (basic A) then show ?case
    by (auto simp: box_Int_box has_integral_box)
  next
   case empty then show ?case
    by simp
  next
   case (compl A)
   then have [measurable]: A ∈ sets borel
    by (simp add: borel_eq_box)

   have ((λx. 1) has_integral ?M (box a b)) (box a b)
    by (simp add: has_integral_box)
   moreover have ((λx. if x ∈ A ∩ box a b then 1 else 0) has_integral ?M A)
(box a b)
    by (subst has_integral_restrict) (auto intro: compl)
   ultimately have ((λx. 1 − (if x ∈ A ∩ box a b then 1 else 0)) has_integral
?M (box a b) − ?M A) (box a b)
    by (rule has_integral_diff)
   then have ((λx. (if x ∈ (UNIV − A) ∩ box a b then 1 else 0)) has_integral
?M (box a b) − ?M A) (box a b)
    by (rule has_integral_cong[THEN iffD1, rotated 1]) auto
  then have ((λx. 1) has_integral ?M (box a b) − ?M A) ((UNIV − A) ∩ box
a b)
    by (subst (asm) has_integral_restrict) auto
   also have ?M (box a b) − ?M A = ?M (UNIV − A)
     by (subst measure_Diff[symmetric]) (auto simp: emeasure_lborel_box_eq
Diff_Int_distrib2)
   finally show ?case .
  next
   case (union F)
   then have [measurable]: ⋀i. F i ∈ sets borel
    by (simp add: borel_eq_box subset_eq)
   have ((λx. if x ∈ ⋃(F ‘ UNIV) ∩ box a b then 1 else 0) has_integral ?M
(⋃ i. F i)) (box a b)
   proof (rule has_integral_monotone_convergence_increasing)
    let ?f = λk x. ∑ i<k. if x ∈ F i ∩ box a b then 1 else 0 :: real
```

```
      show ⋀k. (?f k has_integral (∑ i<k. ?M (F i))) (box a b)
        using union.IH by (auto intro!: has_integral_sum simp del: Int_iff)
      show ⋀k x. ?f k x ≤ ?f (Suc k) x
        by (intro sum_mono2) auto
      from union(1) have ∗: ⋀x i j. x ∈ F i ⟹ x ∈ F j ⟷ j = i
        by (auto simp add: disjoint_family_on_def)
      show (λk. ?f k x) ⟶ (if x ∈ ⋃(F ` UNIV) ∩ box a b then 1 else 0) for
x
      by (auto simp: ∗ sum.If_cases Iio_Int_singleton if_distrib LIMSEQ_if_less
cong: if_cong)
      have ∗: emeasure lborel ((⋃x. F x) ∩ box a b) ≤ emeasure lborel (box a b)
        by (intro emeasure_mono) auto

      with union(1) show (λk. ∑ i<k. ?M (F i)) ⟶ ?M (⋃ i. F i)
        unfolding sums_def[symmetric] UN_extend_simps
        by (intro measure_UNION) (auto simp: disjoint_family_on_def emea-
sure_lborel_box_eq top_unique)
    qed
    then show ?case
      by (subst (asm) has_integral_restrict) auto
  qed }
 note ∗ = this

 show ?thesis
 proof (rule has_integral_monotone_convergence_increasing)
  let ?B = λn::nat. box (− real n ∗R One) (real n ∗R One) :: 'a set
  let ?f = λn::nat. λx. if x ∈ A ∩ ?B n then 1 else 0 :: real
  let ?M = λn. measure lborel (A ∩ ?B n)

  show ⋀n::nat. (?f n has_integral ?M n) A
    using ∗ by (subst has_integral_restrict) simp_all
  show ⋀k x. ?f k x ≤ ?f (Suc k) x
    by (auto simp: box_def)
  { fix x assume x ∈ A
   moreover have (λk. indicator (A ∩ ?B k) x :: real) ⟶ indicator (⋃k::nat.
A ∩ ?B k) x
      by (intro LIMSEQ_indicator_incseq) (auto simp: incseq_def box_def)
    ultimately show (λk. if x ∈ A ∩ ?B k then 1 else 0::real) ⟶ 1
      by (simp add: indicator_def of_bool_def UN_box_eq_UNIV) }

  have (λn. emeasure lborel (A ∩ ?B n)) ⟶ emeasure lborel (⋃n::nat. A ∩
?B n)
    by (intro Lim_emeasure_incseq) (auto simp: incseq_def box_def)
  also have (λn. emeasure lborel (A ∩ ?B n)) = (λn. measure lborel (A ∩ ?B n))
  proof (intro ext emeasure_eq_ennreal_measure)
    fix n have emeasure lborel (A ∩ ?B n) ≤ emeasure lborel (?B n)
      by (intro emeasure_mono) auto
    then show emeasure lborel (A ∩ ?B n) ≠ top
      by (auto simp: top_unique)
```

```
    qed
    finally show (λn. measure lborel (A ∩ ?B n)) ⟶ measure lborel A
      using emeasure_eq_ennreal_measure[of lborel A] finite
      by (simp add: UN_box_eq_UNIV less_top)
  qed
qed

lemma nn_integral_has_integral:
  fixes f::'a::euclidean_space ⇒ real
  assumes f: f ∈ borel_measurable borel ⋀x. 0 ≤ f x (∫⁺x. f x ∂lborel) = ennreal
r 0 ≤ r
  shows (f has_integral r) UNIV
using f proof (induct f arbitrary: r rule: borel_measurable_induct_real)
  case (set A)
  then have ((λx. 1) has_integral measure lborel A) A
    by (intro has_integral_measure_lborel) (auto simp: ennreal_indicator)
  with set show ?case
   by (simp add: ennreal_indicator measure_def) (simp add: indicator_def of_bool_def)
next
  case (mult g c)
  then have ennreal c * (∫⁺ x. g x ∂lborel) = ennreal r
    by (subst nn_integral_cmult[symmetric]) (auto simp: ennreal_mult)
  with ‹0 ≤ r› ‹0 ≤ c›
  obtain r' where (c = 0 ∧ r = 0) ∨ (0 ≤ r' ∧ (∫⁺ x. ennreal (g x) ∂lborel) =
ennreal r' ∧ r = c * r')
    by (cases ∫⁺ x. ennreal (g x) ∂lborel rule: ennreal_cases)
       (auto split: if_split_asm simp: ennreal_mult_top ennreal_mult[symmetric])
  with mult show ?case
    by (auto intro!: has_integral_cmult_real)
next
  case (add g h)
  then have (∫⁺ x. h x + g x ∂lborel) = (∫⁺ x. h x ∂lborel) + (∫⁺ x. g x ∂lborel)
    by (simp add: nn_integral_add)
  with add obtain a b where 0 ≤ a 0 ≤ b (∫⁺ x. h x ∂lborel) = ennreal a (∫⁺
x. g x ∂lborel) = ennreal b r = a + b
    by (cases ∫⁺ x. h x ∂lborel ∫⁺ x. g x ∂lborel rule: ennreal2_cases)
       (auto simp: add_top nn_integral_add top_add simp flip: ennreal_plus)
  with add show ?case
    by (auto intro!: has_integral_add)
next
  case (seq U)
  note seq(1)[measurable] and f[measurable]

  have U_le_f: U i x ≤ f x for i x
   by (metis (no_types) LIMSEQ_le_const UNIV_I incseq_def le_fun_def seq.hyps(4)
seq.hyps(5) space_borel)

  { fix i
    have (∫⁺x. U i x ∂lborel) ≤ (∫⁺x. f x ∂lborel)
```

```
    using seq(2) f(2) U_le_f by (intro nn_integral_mono) simp
  then obtain p where (∫⁺x. U i x ∂lborel) = ennreal p p ≤ r 0 ≤ p
    using seq(6) ‹0≤r› by (cases ∫⁺x. U i x ∂lborel rule: ennreal_cases) (auto simp: top_unique)
  moreover note seq
  ultimately have ∃p. (∫⁺x. U i x ∂lborel) = ennreal p ∧ 0 ≤ p ∧ p ≤ r ∧ (U i has_integral p) UNIV
    by auto }
 then obtain p where p: ⋀i. (∫⁺x. ennreal (U i x) ∂lborel) = ennreal (p i)
  and bnd: ⋀i. p i ≤ r ⋀i. 0 ≤ p i
  and U_int: ⋀i.(U i has_integral (p i)) UNIV by metis

 have int_eq: ⋀i. integral UNIV (U i) = p i using U_int by (rule integral_unique)

 have *: f integrable_on UNIV ∧ (λk. integral UNIV (U k)) ⟶ integral UNIV f
 proof (rule monotone_convergence_increasing)
  show ⋀k. U k integrable_on UNIV using U_int by auto
  show ⋀k x. x∈UNIV ⟹ U k x ≤ U (Suc k) x using ‹incseq U› by (auto simp: incseq_def le_fun_def)
  then show bounded (range (λk. integral UNIV (U k)))
    using bnd int_eq by (auto simp: bounded_real intro!: exI[of _ r])
  show ⋀x. x∈UNIV ⟹ (λk. U k x) ⟶ f x
    using seq by auto
 qed
 moreover have (λi. (∫⁺x. U i x ∂lborel)) ⟶ (∫⁺x. f x ∂lborel)
  using seq f(2) U_le_f by (intro nn_integral_dominated_convergence[where w=f]) auto
 ultimately have integral UNIV f = r
  by (auto simp add: bnd int_eq p seq intro: LIMSEQ_unique)
 with * show ?case
  by (simp add: has_integral_integral)
qed

lemma nn_integral_lborel_eq_integral:
 fixes f::'a::euclidean_space ⇒ real
 assumes f: f ∈ borel_measurable borel ⋀x. 0 ≤ f x (∫⁺x. f x ∂lborel) < ∞
 shows (∫⁺x. f x ∂lborel) = integral UNIV f
proof −
 from f(3) obtain r where r: (∫⁺x. f x ∂lborel) = ennreal r 0 ≤ r
  by (cases ∫⁺x. f x ∂lborel rule: ennreal_cases) auto
 then show ?thesis
  using nn_integral_has_integral[OF f(1,2) r] by (simp add: integral_unique)
qed

lemma nn_integral_integrable_on:
 fixes f::'a::euclidean_space ⇒ real
 assumes f: f ∈ borel_measurable borel ⋀x. 0 ≤ f x (∫⁺x. f x ∂lborel) < ∞
 shows f integrable_on UNIV
```

```
proof −
  from f(3) obtain r where r: (∫⁺x. f x ∂lborel) = ennreal r 0 ≤ r
    by (cases ∫⁺x. f x ∂lborel rule: ennreal_cases) auto
  then show ?thesis
   by (intro has_integral_integrable[where i=r] nn_integral_has_integral[where
r=r] f)
qed

lemma nn_integral_has_integral_lborel:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes f_borel: f ∈ borel_measurable borel and nonneg: ⋀x. 0 ≤ f x
  assumes I: (f has_integral I) UNIV
  shows integral^N lborel f = I
proof −
  from f_borel have (λx. ennreal (f x)) ∈ borel_measurable lborel by auto
  from borel_measurable_implies_simple_function_sequence'[OF this]
  obtain F where F: ⋀i. simple_function lborel (F i) incseq F
                    ⋀i x. F i x < top ⋀x. (SUP i. F i x) = ennreal (f x)
    by blast
  then have [measurable]: ⋀i. F i ∈ borel_measurable lborel
    by (metis borel_measurable_simple_function)
  let ?B = λi::nat. box (− (real i *_R One)) (real i *_R One) :: 'a set

  have 0 ≤ I
    using I by (rule has_integral_nonneg) (simp add: nonneg)

  have F_le_f: enn2real (F i x) ≤ f x for i x
    using F(3,4)[where x=x] nonneg SUP_upper[of i UNIV λi. F i x]
    by (cases F i x rule: ennreal_cases) auto
  let ?F = λi x. F i x * indicator (?B i) x
  have (∫⁺ x. ennreal (f x) ∂lborel) = (SUP i. integral^N lborel (λx. ?F i x))
  proof (subst nn_integral_monotone_convergence_SUP[symmetric])
    { fix x
      obtain j where j: x ∈ ?B j
        using UN_box_eq_UNIV by auto

      have ennreal (f x) = (SUP i. F i x)
        using F(4)[of x] nonneg[of x] by (simp add: max_def)
      also have ... = (SUP i. ?F i x)
      proof (rule SUP_eq)
        fix i show ∃j∈UNIV. F i x ≤ ?F j x
          using j F(2)
          by (intro bexI[of _ max i j])
             (auto split: split_max split_indicator simp: incseq_def le_fun_def
box_def)
      qed (auto intro!: F split: split_indicator)
      finally have ennreal (f x) = (SUP i. ?F i x) . }
    then show (∫⁺ x. ennreal (f x) ∂lborel) = (∫⁺ x. (SUP i. ?F i x) ∂lborel)
      by simp
```

```
  qed (insert F, auto simp: incseq_def le_fun_def box_def split: split_indicator)
  also have ... ≤ ennreal I
  proof (rule SUP_least)
    fix i :: nat
    have finite_F: (∫⁺ x. ennreal (enn2real (F i x) * indicator (?B i) x) ∂lborel)
< ∞
    proof (rule nn_integral_bound_simple_function)
      have emeasure lborel {x ∈ space lborel. ennreal (enn2real (F i x) * indicator
(?B i) x) ≠ 0} ≤
        emeasure lborel (?B i)
        by (intro emeasure_mono) (auto split: split_indicator)
       then show emeasure lborel {x ∈ space lborel. ennreal (enn2real (F i x) *
indicator (?B i) x) ≠ 0} < ∞
        by (auto simp: less_top[symmetric] top_unique)
    qed (auto split: split_indicator
         intro!: F simple_function_compose1[where g=enn2real] simple_function_ennreal)

    have int_F: (λx. enn2real (F i x) * indicator (?B i) x) integrable_on UNIV
      using F(4) finite_F
    by (intro nn_integral_integrable_on) (auto split: split_indicator simp: enn2real_nonneg)

    have (∫⁺ x. F i x * indicator (?B i) x ∂lborel) =
      (∫⁺ x. ennreal (enn2real (F i x) * indicator (?B i) x) ∂lborel)
      using F(3,4)
    by (intro nn_integral_cong) (auto simp: image_iff eq_commute split: split_indicator)
    also have ... = ennreal (integral UNIV (λx. enn2real (F i x) * indicator (?B
i) x))
      using F
      by (intro nn_integral_lborel_eq_integral[OF _ _ finite_F])
         (auto split: split_indicator intro: enn2real_nonneg)
    also have ... ≤ ennreal I
      by (auto intro!: has_integral_le[OF integrable_integral[OF int_F] I] nonneg
F_le_f
              simp: ‹0 ≤ I› split: split_indicator )
    finally show (∫⁺ x. F i x * indicator (?B i) x ∂lborel) ≤ ennreal I .
  qed
  finally have (∫⁺ x. ennreal (f x) ∂lborel) < ∞
    by (auto simp: less_top[symmetric] top_unique)
  from nn_integral_lborel_eq_integral[OF assms(1,2) this] I show ?thesis
    by (simp add: integral_unique)
qed

lemma has_integral_iff_emeasure_lborel:
  fixes A :: 'a::euclidean_space set
  assumes A[measurable]: A ∈ sets borel and [simp]: 0 ≤ r
  shows ((λx. 1) has_integral r) A ⟷ emeasure lborel A = ennreal r
proof (cases emeasure lborel A = ∞)
  case emeasure_A: True
  have ¬ (λx. 1::real) integrable_on A
```

```
  proof
    assume int: (λx. 1::real) integrable_on A
    then have (indicator A::'a ⇒ real) integrable_on UNIV
      unfolding indicator_def of_bool_def integrable_restrict_UNIV .
    then obtain r where ((indicator A::'a⇒real) has_integral r) UNIV
      by auto
    from nn_integral_has_integral_lborel[OF _ _ this] emeasure_A show False
      by (simp add: ennreal_indicator)
  qed
  with emeasure_A show ?thesis
    by auto
next
  case False
  then have ((λx. 1) has_integral measure lborel A) A
    by (simp add: has_integral_measure_lborel less_top)
  with False show ?thesis
    by (auto simp: emeasure_eq_ennreal_measure has_integral_unique)
qed

lemma ennreal_max_0: ennreal (max 0 x) = ennreal x
  by (auto simp: max_def ennreal_neg)

lemma has_integral_integral_real:
  fixes f::'a::euclidean_space ⇒ real
  assumes f: integrable lborel f
  shows (f has_integral (integral^L lborel f)) UNIV
proof −
  from integrableE[OF f] obtain r q
    where 0 ≤ r 0 ≤ q
      and r: (∫⁺ x. ennreal (max 0 (f x)) ∂lborel) = ennreal r
      and q: (∫⁺ x. ennreal (max 0 (− f x)) ∂lborel) = ennreal q
      and f: f ∈ borel_measurable lborel and eq: integral^L lborel f = r − q
    unfolding ennreal_max_0 by auto
   then have ((λx. max 0 (f x)) has_integral r) UNIV ((λx. max 0 (− f x))
has_integral q) UNIV
    using nn_integral_has_integral[OF _ _ r] nn_integral_has_integral[OF _ _
q] by auto
  note has_integral_diff[OF this]
  moreover have (λx. max 0 (f x) − max 0 (− f x)) = f
    by auto
  ultimately show ?thesis
    by (simp add: eq)
qed

lemma has_integral_AE:
  assumes ae: AE x in lborel. x ∈ Ω ⟶ f x = g x
  shows (f has_integral x) Ω = (g has_integral x) Ω
proof −
  from ae obtain N
```

where N: $N \in$ *sets borel emeasure lborel* $N = 0$ $\{x.\ \neg\ (x \in \Omega \longrightarrow f\ x = g\ x)\} \subseteq N$
by (*auto elim*!: *AE_E*)
then have *not_N*: *AE x in lborel*. $x \notin N$
by (*simp add*: *AE_iff_measurable*)
show *?thesis*
proof (*rule has_integral_spike_eq*[*symmetric*])
show $\bigwedge x.\ x \in \Omega - N \Longrightarrow f\ x = g\ x$ **using** *N*(*3*) **by** *auto*
show *negligible N*
unfolding *negligible_def*
proof (*intro allI*)
fix *a b* :: *'a*
let *?F* = λx::*'a*. *if x* $\in$ *cbox a b then indicator N x else 0* :: *real*
have *integrable lborel ?F* = *integrable lborel* (λx::*'a*. *0*::*real*)
using *not_N N*(*1*) **by** (*intro integrable_cong_AE*) *auto*
moreover have (*LINT x*|*lborel*. *?F x*) = (*LINT x*::*'a*|*lborel*. *0*::*real*)
using *not_N N*(*1*) **by** (*intro integral_cong_AE*) *auto*
ultimately have (*?F has_integral 0*) *UNIV*
using *has_integral_integral_real*[*of ?F*] **by** *simp*
then show (*indicator N has_integral* (*0*::*real*)) (*cbox a b*)
unfolding *has_integral_restrict_UNIV* **.**
qed
qed
qed

lemma *nn_integral_has_integral_lebesgue*:
fixes *f* :: *'a*::*euclidean_space* $\Rightarrow$ *real*
assumes *nonneg*: $\bigwedge x.\ x \in \Omega \Longrightarrow 0 \le f\ x$ **and** *I*: (*f has_integral I*) Ω
shows $integral^N$ *lborel* ($\lambda x.$ *indicator* Ω *x* $*$ *f x*) = *I*
proof −
from *I* **have** ($\lambda x.$ *indicator* Ω *x* $*_R$ *f x*) $\in$ *lebesgue* $\rightarrow_M$ *borel*
by (*rule has_integral_implies_lebesgue_measurable*)
then obtain *f'* :: *'a* $\Rightarrow$ *real*
where [*measurable*]: *f'* $\in$ *borel* $\rightarrow_M$ *borel* **and** *eq*: *AE x in lborel*. *indicator* Ω *x* $*$ *f x* = *f' x*
by (*auto dest*: *completion_ex_borel_measurable_real*)

from *I* **have** (($\lambda x.$ *abs* (*indicator* Ω *x* $*$ *f x*)) *has_integral I*) *UNIV*
using *nonneg* **by** (*simp add*: *indicator_def of_bool_def if_distrib*[*of* $\lambda x.$ *x* $*$ *f y* **for** *y*] *cong*: *if_cong*)
also have (($\lambda x.$ *abs* (*indicator* Ω *x* $*$ *f x*)) *has_integral I*) *UNIV* $\longleftrightarrow$ (($\lambda x.$ *abs* (*f' x*)) *has_integral I*) *UNIV*
using *eq* **by** (*intro has_integral_AE*) *auto*
finally have $integral^N$ *lborel* ($\lambda x.$ *abs* (*f' x*)) = *I*
by (*rule nn_integral_has_integral_lborel*[*rotated 2*]) *auto*
also have $integral^N$ *lborel* ($\lambda x.$ *abs* (*f' x*)) = $integral^N$ *lborel* ($\lambda x.$ *abs* (*indicator* Ω *x* $*$ *f x*))
using *eq* **by** (*intro nn_integral_cong_AE*) *auto*
also have ($\lambda x.$ *abs* (*indicator* Ω *x* $*$ *f x*)) = ($\lambda x.$ *indicator* Ω *x* $*$ *f x*)

using *nonneg* **by** (*auto simp*: *indicator_def fun_eq_iff*)
finally show *?thesis* **.**
qed

lemma *has_integral_iff_nn_integral_lebesgue*:
assumes *f*: $\bigwedge x.\ 0 \le f\ x$
shows (*f has_integral r*) *UNIV* $\longleftrightarrow$ ($f \in$ *lebesgue* $\to_M$ *borel* $\wedge$ *integral*^{N} *lebesgue* $f = r \wedge 0 \le r$) (**is** *?I* = *?N*)
proof
assume *?I*
have $0 \le r$
using *has_integral_nonneg*[*OF* ‹*?I*›] *f* **by** *auto*
then show *?N*
using *nn_integral_has_integral_lebesgue*[*OF f* ‹*?I*›]
has_integral_implies_lebesgue_measurable[*OF* ‹*?I*›]
by (*auto simp*: *nn_integral_completion*)
next
assume *?N*
then obtain f' **where** f': $f' \in$ *borel* $\to_M$ *borel AE x in lborel*. $f\ x = f'\ x$
by (*auto dest*: *completion_ex_borel_measurable_real*)
moreover have $(\int^+ x.$ *ennreal* $|f'\ x|\ \partial$*lborel*$) = (\int^+ x.$ *ennreal* $|f\ x|\ \partial$*lborel*$)$
using f' **by** (*intro nn_integral_cong_AE*) *auto*
moreover have $((\lambda x.\ |f'\ x|)$ *has_integral r*$)$ *UNIV* $\longleftrightarrow$ $((\lambda x.\ |f\ x|)$ *has_integral r*$)$ *UNIV*
using f' **by** (*intro has_integral_AE*) *auto*
moreover note *nn_integral_has_integral*[*of* $\lambda x.\ |f'\ x|$ *r*] ‹*?N*›
ultimately show *?I*
using *f* **by** (*auto simp*: *nn_integral_completion*)
qed

lemma *set_nn_integral_lborel_eq_integral*:
fixes *f*::*'a*::*euclidean_space* $\Rightarrow$ *real*
assumes *set_borel_measurable borel A f*
assumes $\bigwedge x.\ x \in A \Longrightarrow 0 \le f\ x\ (\int^+ x{\in}A.\ f\ x\ \partial$*lborel*$) < \infty$
shows $(\int^+ x{\in}A.\ f\ x\ \partial$*lborel*$)$ = *integral A f*
proof −
have *eq*: $(\int^+ x{\in}A.\ f\ x\ \partial$*lborel*$) = (\int^+ x.$ *ennreal* (*indicator A x* $*$ *f x*) ∂*lborel*$)$
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def*)
also have ... = *integral UNIV* ($\lambda x.$ *indicator A x* $*$ *f x*)
using *assms eq* **by** (*intro nn_integral_lborel_eq_integral*)
(*auto simp*: *indicator_def set_borel_measurable_def*)
also have *integral UNIV* ($\lambda x.$ *indicator A x* $*$ *f x*) = *integral A* ($\lambda x.$ *indicator A x* $*$ *f x*)
by (*rule integral_spike_set*) (*auto intro*: *empty_imp_negligible*)

also have ... = *integral A f*
by (*rule integral_cong*) (*auto simp*: *indicator_def*)
finally show *?thesis* **.**
qed

lemma *nn_integral_has_integral_lebesgue'*:
fixes $f :: 'a::euclidean\_space \Rightarrow real$
assumes *nonneg*: $\bigwedge x.\ x \in \Omega \Longrightarrow 0 \le f\ x$ **and** I: $(f\ has\_integral\ I)\ \Omega$
shows $integral^N\ lborel\ (\lambda x.\ ennreal\ (f\ x) * indicator\ \Omega\ x) = ennreal\ I$
proof −
have $integral^N\ lborel\ (\lambda x.\ ennreal\ (f\ x) * indicator\ \Omega\ x) =$
$integral^N\ lborel\ (\lambda x.\ ennreal\ (indicator\ \Omega\ x * f\ x))$
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def*)
also have $\ldots = ennreal\ I$
using *assms* **by** (*intro nn_integral_has_integral_lebesgue*)
finally show *?thesis* **.**
qed

context
fixes $f::'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
begin

lemma *has_integral_integral_lborel*:
assumes f: *integrable lborel f*
shows $(f\ has\_integral\ (integral^L\ lborel\ f))\ UNIV$
proof −
have $((\lambda x.\ \sum b{\in}Basis.\ (f\ x \cdot b) *_R b)\ has\_integral\ (\sum b{\in}Basis.\ integral^L\ lborel\ (\lambda x.\ f\ x \cdot b) *_R b))\ UNIV$
using *f* **by** (*intro has_integral_sum finite_Basis ballI has_integral_scaleR_left has_integral_integral_real*) *auto*
also have *eq_f*: $(\lambda x.\ \sum b{\in}Basis.\ (f\ x \cdot b) *_R b) = f$
by (*simp add*: *fun_eq_iff euclidean_representation*)
also have $(\sum b{\in}Basis.\ integral^L\ lborel\ (\lambda x.\ f\ x \cdot b) *_R b) = integral^L\ lborel\ f$
using *f* **by** (*subst* (*2*) *eq_f*[*symmetric*]) *simp*
finally show *?thesis* **.**
qed

lemma *integrable_on_lborel*: *integrable lborel f* $\Longrightarrow$ *f integrable_on UNIV*
using *has_integral_integral_lborel* **by** *auto*

lemma *integral_lborel*: *integrable lborel f* $\Longrightarrow$ *integral UNIV f* $= (\int x.\ f\ x\ \partial lborel)$
using *has_integral_integral_lborel* **by** *auto*

end

context
begin

private lemma *has_integral_integral_lebesgue_real*:
fixes $f :: 'a::euclidean\_space \Rightarrow real$
assumes f: *integrable lebesgue f*
shows $(f\ has\_integral\ (integral^L\ lebesgue\ f))\ UNIV$
proof −

obtain f' **where** f': $f' \in borel \rightarrow_M borel$ *AE* x *in lborel*. $f\,x = f'\,x$
using *completion_ex_borel_measurable_real*[*OF borel_measurable_integrable*[*OF* f]] **by** *auto*
moreover have $(\int^+ x.\ ennreal\ (norm\ (f\,x))\ \partial lborel) = (\int^+ x.\ ennreal\ (norm\ (f'\,x))\ \partial lborel)$
using f' **by** (*intro nn_integral_cong_AE*) *auto*
ultimately have *integrable lborel* f'
using f **by** (*auto simp*: *integrable_iff_bounded nn_integral_completion cong*: *nn_integral_cong_AE*)
note *has_integral_integral_real*[*OF this*]
moreover have $integral^L\ lebesgue\ f = integral^L\ lebesgue\ f'$
using f' f **by** (*intro integral_cong_AE*) (*auto intro*: *AE_completion measurable_completion*)
moreover have $integral^L\ lebesgue\ f' = integral^L\ lborel\ f'$
using f' **by** (*simp add*: *integral_completion*)
moreover have $(f'\ has\_integral\ integral^L\ lborel\ f')\ UNIV \longleftrightarrow (f\ has\_integral\ integral^L\ lborel\ f')\ UNIV$
using f' **by** (*intro has_integral_AE*) *auto*
ultimately show *?thesis*
by *auto*
qed

lemma *has_integral_integral_lebesgue*:
fixes $f :: 'a{::}euclidean\_space \Rightarrow 'b{::}euclidean\_space$
assumes f: *integrable lebesgue f*
shows $(f\ has\_integral\ (integral^L\ lebesgue\ f))\ UNIV$
proof −
have $((\lambda x.\ \sum b{\in}Basis.\ (f\,x \cdot b) *_R b)\ has\_integral\ (\sum b{\in}Basis.\ integral^L\ lebesgue\ (\lambda x.\ f\,x \cdot b) *_R b))\ UNIV$
using f **by** (*intro has_integral_sum finite_Basis ballI has_integral_scaleR_left has_integral_integral_lebesgue_real*) *auto*
also have *eq_f*: $(\lambda x.\ \sum b{\in}Basis.\ (f\,x \cdot b) *_R b) = f$
by (*simp add*: *fun_eq_iff euclidean_representation*)
also have $(\sum b{\in}Basis.\ integral^L\ lebesgue\ (\lambda x.\ f\,x \cdot b) *_R b) = integral^L\ lebesgue\ f$
using f **by** (*subst* (*2*) *eq_f*[*symmetric*]) *simp*
finally show *?thesis* **.**
qed

lemma *has_integral_integral_lebesgue_on*:
fixes $f :: 'a{::}euclidean\_space \Rightarrow 'b{::}euclidean\_space$
assumes *integrable* (*lebesgue_on S*) f $S \in sets\ lebesgue$
shows $(f\ has\_integral\ (integral^L\ (lebesgue\_on\ S)\ f))\ S$
proof −
let *?f* $= \lambda x.$ *if* $x \in S$ *then* $f\,x$ *else* 0
have *integrable lebesgue* $(\lambda x.\ indicat\_real\ S\ x *_R f\,x)$
using *indicator_scaleR_eq_if* [*of S _ f*] *assms*
by (*metis* (*full_types*) *integrable_restrict_space sets.Int_space_eq2*)
then have *integrable lebesgue ?f*

using *indicator_scaleR_eq_if* [*of S _ f*] *assms* **by** *auto*
then have (*?f has_integral* (*integral*^{L} *lebesgue ?f*)) *UNIV*
by (*rule has_integral_integral_lebesgue*)
then have (*f has_integral* (*integral*^{L} *lebesgue ?f*)) *S*
using *has_integral_restrict_UNIV* **by** *blast*
moreover
have *S* ∩ *space lebesgue* ∈ *sets lebesgue*
by (*simp add*: *assms*)
then have (*integral*^{L} *lebesgue ?f*) = (*integral*^{L} (*lebesgue_on S*) *f*)
by (*simp add*: *integral_restrict_space indicator_scaleR_eq_if*)
ultimately show *?thesis*
by *auto*
qed

lemma *lebesgue_integral_eq_integral*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *integrable* (*lebesgue_on S*) *f S* ∈ *sets lebesgue*
shows *integral*^{L} (*lebesgue_on S*) *f* = *integral S f*
by (*metis has_integral_integral_lebesgue_on assms integral_unique*)

lemma *integrable_on_lebesgue*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows *integrable lebesgue f* ⟹ *f integrable_on UNIV*
using *has_integral_integral_lebesgue* **by** *auto*

lemma *integral_lebesgue*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows *integrable lebesgue f* ⟹ *integral UNIV f* = (∫ *x*. *f x* ∂*lebesgue*)
using *has_integral_integral_lebesgue* **by** *auto*

end

9.8.8 Absolute integrability (this is the same as Lebesgue integrability)

syntax
__lebesgue_borel_integral :: *pttrn* ⇒ *real* ⇒ *real*
(‹(‹*indent=2 notation=‹binder LBINT›‹LBINT _./ _*)› [*0,10*] *10*)
__set_lebesgue_borel_integral :: *pttrn* ⇒ *real set* ⇒ *real* ⇒ *real*
(‹(‹*indent=3 notation=‹binder LBINT›‹LBINT _:_./ _*)› [*0,0,10*] *10*)
syntax_consts
__lebesgue_borel_integral ⇌ *lebesgue_integral* **and**
__set_lebesgue_borel_integral ⇌ *set_lebesgue_integral*
translations
LBINT x. f == *CONST lebesgue_integral CONST lborel* (λ*x*. *f*)
LBINT x:*A. f* == *CONST set_lebesgue_integral CONST lborel A* (λ*x*. *f*)

lemma *set_integral_reflect*:
fixes *S* **and** *f* :: *real* ⇒ *'a* :: {*banach*, *second_countable_topology*}

shows $(LBINT\ x : S.\ f\ x) = (LBINT\ x : \{x.\ -x \in S\}.\ f\ (-x))$
unfolding *set_lebesgue_integral_def*
by (*subst lborel_integral_real_affine*[**where** *c*=−*1* **and** *t*=*0*])
(*auto intro*!: *Bochner_Integration.integral_cong split*: *split_indicator*)

lemma *borel_integrable_atLeastAtMost'*:
fixes $f :: real \Rightarrow 'a::\{banach, second\_countable\_topology\}$
assumes *f*: *continuous_on* $\{a..b\}$ *f*
shows *set_integrable lborel* $\{a..b\}$ *f*
unfolding *set_integrable_def*
by (*intro borel_integrable_compact compact_Icc f*)

lemma *integral_FTC_atLeastAtMost*:
fixes $f :: real \Rightarrow 'a :: euclidean\_space$
assumes $a \le b$
and *F*: $\bigwedge x.\ a \le x \Longrightarrow x \le b \Longrightarrow (F\ has\_vector\_derivative\ f\ x)\ (at\ x\ within\ \{a\ ..\ b\})$
and *f*: *continuous_on* $\{a\ ..\ b\}$ *f*
shows $integral^L\ lborel\ (\lambda x.\ indicator\ \{a\ ..\ b\}\ x *_R f\ x) = F\ b - F\ a$
proof −
let *?f* = $\lambda x.\ indicator\ \{a\ ..\ b\}\ x *_R f\ x$
have $(?f\ has\_integral\ (\int x.\ ?f\ x\ \partial lborel))\ UNIV$
using *borel_integrable_atLeastAtMost'*[*OF f*]
unfolding *set_integrable_def* **by** (*rule has_integral_integral_lborel*)
moreover
have $(f\ has\_integral\ F\ b - F\ a)\ \{a\ ..\ b\}$
by (*intro fundamental_theorem_of_calculus ballI assms*) *auto*
then have $(?f\ has\_integral\ F\ b - F\ a)\ \{a\ ..\ b\}$
by (*subst has_integral_cong*[**where** *g*=*f*]) *auto*
then have $(?f\ has\_integral\ F\ b - F\ a)\ UNIV$
by (*intro has_integral_on_superset*[**where** *T*=*UNIV* **and** *S*=$\{a..b\}$]) *auto*
ultimately show $integral^L\ lborel\ ?f = F\ b - F\ a$
by (*rule has_integral_unique*)
qed

lemma *set_borel_integral_eq_integral*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *set_integrable lborel S f*
shows *f integrable_on S* $(LINT\ x : S \mid lborel.\ f\ x) = integral\ S\ f$
proof −
let *?f* = $\lambda x.\ indicator\ S\ x *_R f\ x$
have $(?f\ has\_integral\ (LINT\ x : S \mid lborel.\ f\ x))\ UNIV$
using *assms has_integral_integral_lborel*
unfolding *set_integrable_def set_lebesgue_integral_def* **by** *blast*
hence *1*: $(f\ has\_integral\ (set\_lebesgue\_integral\ lborel\ S\ f))\ S$
by (*simp add*: *indicator_scaleR_eq_if*)
thus *f integrable_on S*
by (*auto simp add*: *integrable_on_def*)
with *1* **have** $(f\ has\_integral\ (integral\ S\ f))\ S$

by (*intro integrable_integral*, *auto simp add*: *integrable_on_def*)
thus (*LINT x* : *S* | *lborel*. *f x*) = *integral S f*
by (*intro has_integral_unique* [*OF 1*])
qed

lemma *has_integral_set_lebesgue*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *set_integrable lebesgue S f*
shows (*f has_integral* (*LINT x*:*S*|*lebesgue*. *f x*)) *S*
using *has_integral_integral_lebesgue f*
by (*fastforce simp add*: *set_integrable_def set_lebesgue_integral_def indicator_def of_bool_def if_distrib*[*of* λ*x*. *x* ∗*R f* _] *cong*: *if_cong*)

lemma *set_lebesgue_integral_eq_integral*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *set_integrable lebesgue S f*
shows *f integrable_on S* (*LINT x*:*S* | *lebesgue*. *f x*) = *integral S f*
using *has_integral_set_lebesgue*[*OF f*] **by** (*auto simp*: *integral_unique integrable_on_def*)

lemma *lmeasurable_iff_has_integral*:
S ∈ *lmeasurable* ⟷ ((*indicator S*) *has_integral measure lebesgue S*) *UNIV*
by (*subst has_integral_iff_nn_integral_lebesgue*)
(*auto simp*: *ennreal_indicator emeasure_eq_measure2 borel_measurable_indicator_iff intro*!: *fmeasurableI*)

abbreviation
absolutely_integrable_on :: (*'a*::*euclidean_space* ⇒ *'b*::{*banach*, *second_countable_topology*}) ⇒ *'a set* ⇒ *bool*
(**infixr** ‹*absolutely'_integrable'_on*› *46*)
where *f absolutely_integrable_on s* ≡ *set_integrable lebesgue s f*

lemma *absolutely_integrable_zero* [*simp*]: (λ*x*. *0*) *absolutely_integrable_on S*
by (*simp add*: *set_integrable_def*)

lemma *absolutely_integrable_on_def*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows *f absolutely_integrable_on S* ⟷ *f integrable_on S* ∧ (λ*x*. *norm* (*f x*)) *integrable_on S*
proof *safe*
assume *f*: *f absolutely_integrable_on S*
then have *nf*: *integrable lebesgue* (λ*x*. *norm* (*indicator S x* ∗*R f x*))
using *integrable_norm set_integrable_def* **by** *blast*
show *f integrable_on S*
by (*rule set_lebesgue_integral_eq_integral*[*OF f*])
have (λ*x*. *norm* (*indicator S x* ∗*R f x*)) = (λ*x*. *if x* ∈ *S then norm* (*f x*) *else 0*)
by *auto*
with *integrable_on_lebesgue*[*OF nf*] **show** (λ*x*. *norm* (*f x*)) *integrable_on S*

by (*simp add*: *integrable_restrict_UNIV*)
next
assume *f*: *f integrable_on S* **and** *nf*: (λ*x*. *norm* (*f x*)) *integrable_on S*
show *f absolutely_integrable_on S*
unfolding *set_integrable_def*
proof (*rule integrableI_bounded*)
show (λ*x*. *indicator S x* ∗*R f x*) ∈ *borel_measurable lebesgue*
using *f has_integral_implies_lebesgue_measurable*[*of f _ S*] **by** (*auto simp*: *integrable_on_def*)
show (∫+ *x*. *ennreal* (*norm* (*indicator S x* ∗*R f x*)) ∂*lebesgue*) < ∞
using *nf nn_integral_has_integral_lebesgue*[*of _* λ*x*. *norm* (*f x*)]
by (*auto simp*: *integrable_on_def nn_integral_completion*)
qed
qed

lemma *integrable_on_lebesgue_on*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *integrable* (*lebesgue_on S*) *f* **and** *S*: *S* ∈ *sets lebesgue*
shows *f integrable_on S*
proof −
have *integrable lebesgue* (λ*x*. *indicator S x* ∗*R f x*)
using *S f inf_top.comm_neutral integrable_restrict_space* **by** *blast*
then show *?thesis*
using *absolutely_integrable_on_def set_integrable_def* **by** *blast*
qed

lemma *absolutely_integrable_imp_integrable*:
assumes *f absolutely_integrable_on S S* ∈ *sets lebesgue*
shows *integrable* (*lebesgue_on S*) *f*
by (*meson assms integrable_restrict_space set_integrable_def sets.Int sets.top*)

lemma *absolutely_integrable_on_null* [*intro*]:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows *content* (*cbox a b*) = *0* ⟹ *f absolutely_integrable_on* (*cbox a b*)
by (*auto simp*: *absolutely_integrable_on_def*)

lemma *absolutely_integrable_on_open_interval*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
shows *f absolutely_integrable_on box a b* ⟷
f absolutely_integrable_on cbox a b
by (*auto simp*: *integrable_on_open_interval absolutely_integrable_on_def*)

lemma *absolutely_integrable_restrict_UNIV*:
(λ*x*. *if x* ∈ *S then f x else 0*) *absolutely_integrable_on UNIV* ⟷ *f absolutely_integrable_on S*
unfolding *set_integrable_def*
by (*intro arg_cong2*[**where** *f*=*integrable*]) *auto*

lemma *absolutely_integrable_onI*:

fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
shows f *integrable_on* $S \Longrightarrow (\lambda x.\ norm\ (f\ x))$ *integrable_on* $S \Longrightarrow f$ *absolutely_integrable_on* S
unfolding *absolutely_integrable_on_def* **by** *auto*

lemma *nonnegative_absolutely_integrable_1*:
fixes $f :: 'a :: euclidean\_space \Rightarrow real$
assumes f: f *integrable_on* A **and** $\bigwedge x.\ x \in A \Longrightarrow 0 \le f\ x$
shows f *absolutely_integrable_on* A
by (*rule absolutely_integrable_onI* [*OF f*]) (*use assms* **in** ‹*simp add*: *integrable_eq*›)

lemma *absolutely_integrable_on_iff_nonneg*:
fixes $f :: 'a :: euclidean\_space \Rightarrow real$
assumes $\bigwedge x.\ x \in S \Longrightarrow 0 \le f\ x$ **shows** f *absolutely_integrable_on* $S \longleftrightarrow f$ *integrable_on* S
proof −
{ assume f *integrable_on* S
then have $(\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)$ *integrable_on UNIV*
by (*simp add*: *integrable_restrict_UNIV*)
then have $(\lambda x.\ if\ x \in S\ then\ f\ x\ else\ 0)$ *absolutely_integrable_on UNIV*
using ‹f *integrable_on* S› *absolutely_integrable_restrict_UNIV assms nonnegative_absolutely_integrable_1* **by** *blast*
then have f *absolutely_integrable_on* S
using *absolutely_integrable_restrict_UNIV* **by** *blast*
}
then show *?thesis*
unfolding *absolutely_integrable_on_def* **by** *auto*
qed

lemma *absolutely_integrable_on_scaleR_iff*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
shows
$(\lambda x.\ c *_R f\ x)$ *absolutely_integrable_on* $S \longleftrightarrow$
$c = 0 \vee f$ *absolutely_integrable_on* S
proof (*cases c=0*)
case *False*
then show *?thesis*
unfolding *absolutely_integrable_on_def*
by (*simp add*: *norm_mult*)
qed *auto*

lemma *absolutely_integrable_spike*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes f *absolutely_integrable_on* T **and** S: *negligible* S $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
shows g *absolutely_integrable_on* T
using *assms integrable_spike*
unfolding *absolutely_integrable_on_def* **by** *metis*

lemma *absolutely_integrable_negligible*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *negligible S*
shows *f absolutely_integrable_on S*
using *assms* **by** (*simp add*: *absolutely_integrable_on_def integrable_negligible*)

lemma *absolutely_integrable_spike_eq*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *negligible S* $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
shows (*f absolutely_integrable_on T* $\longleftrightarrow$ *g absolutely_integrable_on T*)
using *assms* **by** (*blast intro*: *absolutely_integrable_spike sym*)

lemma *absolutely_integrable_spike_set_eq*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *negligible* $\{x \in S - T.\ f\ x \neq 0\}$ *negligible* $\{x \in T - S.\ f\ x \neq 0\}$
shows (*f absolutely_integrable_on S* $\longleftrightarrow$ *f absolutely_integrable_on T*)
proof −
have ($\lambda x.$ *if* $x \in S$ *then f x else 0*) *absolutely_integrable_on UNIV* $\longleftrightarrow$
($\lambda x.$ *if* $x \in T$ *then f x else 0*) *absolutely_integrable_on UNIV*
proof (*rule absolutely_integrable_spike_eq*)
show *negligible* ($\{x \in S - T.\ f\ x \neq 0\} \cup \{x \in T - S.\ f\ x \neq 0\}$)
by (*rule negligible_Un* [*OF assms*])
qed *auto*
with *absolutely_integrable_restrict_UNIV* **show** *?thesis*
by *blast*
qed

lemma *absolutely_integrable_spike_set*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *f*: *f absolutely_integrable_on S* **and** *neg*: *negligible* $\{x \in S - T.\ f\ x \neq 0\}$ *negligible* $\{x \in T - S.\ f\ x \neq 0\}$
shows *f absolutely_integrable_on T*
using *absolutely_integrable_spike_set_eq f neg* **by** *blast*

lemma *absolutely_integrable_reflect*[*simp*]:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
shows ($\lambda x.\ f(-x)$) *absolutely_integrable_on cbox* $(-b)$ $(-a)$ $\longleftrightarrow$ *f absolutely_integrable_on cbox a b*
unfolding *absolutely_integrable_on_def*
by (*metis* (*mono_tags*, *lifting*) *integrable_eq integrable_reflect*)

lemma *absolutely_integrable_reflect_real*[*simp*]:
fixes $f :: real \Rightarrow 'b::euclidean\_space$
shows ($\lambda x.\ f(-x)$) *absolutely_integrable_on* $\{-b\ ..\ -a\}$ $\longleftrightarrow$ *f absolutely_integrable_on* $\{a..b::real\}$
unfolding *box_real*[*symmetric*] **by** (*rule absolutely_integrable_reflect*)

lemma *absolutely_integrable_on_subcbox*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$

shows $[\![f$ *absolutely_integrable_on* S; *cbox* a $b \subseteq S]\!] \Longrightarrow f$ *absolutely_integrable_on cbox* a b
by (*meson absolutely_integrable_on_def integrable_on_subcbox*)

lemma *absolutely_integrable_on_subinterval*:
fixes f :: *real* $\Rightarrow$ $'b$::*euclidean_space*
shows $[\![f$ *absolutely_integrable_on* S; $\{a..b\} \subseteq S]\!] \Longrightarrow f$ *absolutely_integrable_on* $\{a..b\}$
using *absolutely_integrable_on_subcbox* **by** *fastforce*

lemma *integrable_subinterval*:
fixes f :: *real* $\Rightarrow$ $'a$::*euclidean_space*
assumes *integrable* (*lebesgue_on* $\{a..b\}$) f
and $\{c..d\} \subseteq \{a..b\}$
shows *integrable* (*lebesgue_on* $\{c..d\}$) f
proof (*rule absolutely_integrable_imp_integrable*)
show f *absolutely_integrable_on* $\{c..d\}$
proof −
have f *integrable_on* $\{c..d\}$
using *assms integrable_on_lebesgue_on integrable_on_subinterval* **by** *fastforce*
moreover have ($\lambda x.$ *norm* (f x)) *integrable_on* $\{c..d\}$
proof (*rule integrable_on_subinterval*)
show ($\lambda x.$ *norm* (f x)) *integrable_on* $\{a..b\}$
by (*simp add*: *assms integrable_on_lebesgue_on*)
qed (*use assms* **in** *auto*)
ultimately show *?thesis*
by (*auto simp*: *absolutely_integrable_on_def*)
qed
qed *auto*

lemma *indefinite_integral_continuous_real*:
fixes f :: *real* $\Rightarrow$ $'a$::*euclidean_space*
assumes *integrable* (*lebesgue_on* $\{a..b\}$) f
shows *continuous_on* $\{a..b\}$ ($\lambda x.$ *integral*^{L} (*lebesgue_on* $\{a..x\}$) f)
proof −
have f *integrable_on* $\{a..b\}$
by (*simp add*: *assms integrable_on_lebesgue_on*)
then have *continuous_on* $\{a..b\}$ ($\lambda x.$ *integral* $\{a..x\}$ f)
using *indefinite_integral_continuous_1* **by** *blast*
moreover have *integral*^{L} (*lebesgue_on* $\{a..x\}$) f = *integral* $\{a..x\}$ f **if** $a \leq x$ $x \leq b$ **for** x
proof −
have $\{a..x\} \subseteq \{a..b\}$
using *that* **by** *auto*
then have *integrable* (*lebesgue_on* $\{a..x\}$) f
using *integrable_subinterval assms* **by** *blast*
then show *integral*^{L} (*lebesgue_on* $\{a..x\}$) f = *integral* $\{a..x\}$ f
by (*simp add*: *lebesgue_integral_eq_integral*)

qed
ultimately show *?thesis*
by (*metis* (*no_types*, *lifting*) *atLeastAtMost_iff continuous_on_cong*)
qed

lemma *lmeasurable_iff_integrable_on*: $S \in lmeasurable \longleftrightarrow (\lambda x.\ 1::real)$ *integrable_on S*
by (*subst absolutely_integrable_on_iff_nonneg*[*symmetric*])
(*simp_all add*: *lmeasurable_iff_integrable set_integrable_def*)

lemma *lmeasure_integral_UNIV*: $S \in lmeasurable \Longrightarrow$ *measure lebesgue S* = *integral UNIV* (*indicator S*)
by (*simp add*: *lmeasurable_iff_has_integral integral_unique*)

lemma *lmeasure_integral*: $S \in lmeasurable \Longrightarrow$ *measure lebesgue S* = *integral S* $(\lambda x.\ 1::real)$
by (*fastforce simp add*: *lmeasure_integral_UNIV indicator_def* [*abs_def*] *of_bool_def lmeasurable_iff_integrable_on*)

lemma *integrable_on_const*: $S \in lmeasurable \Longrightarrow (\lambda x.\ c)$ *integrable_on S*
unfolding *lmeasurable_iff_integrable*
by (*metis* (*mono_tags*, *lifting*) *integrable_eq integrable_on_scaleR_left lmeasurable_iff_integrable lmeasurable_iff_integrable_on scaleR_one*)

lemma *integral_indicator*:
assumes $(S \cap T) \in lmeasurable$
shows *integral T* (*indicator S*) = *measure lebesgue* $(S \cap T)$
proof −
have *integral UNIV* (*indicator* $(S \cap T)$) = *integral UNIV* $(\lambda a.$ *if* $a \in S \cap T$ *then* $1::real$ *else* $0)$
by (*simp add*: *indicator_def* [*abs_def*] *of_bool_def*)
moreover have (*indicator* $(S \cap T)$ *has_integral measure lebesgue* $(S \cap T)$) *UNIV*
using *assms* **by** (*simp add*: *lmeasurable_iff_has_integral*)
ultimately have *integral UNIV* $(\lambda x.$ *if* $x \in S \cap T$ *then 1 else 0*) = *measure lebesgue* $(S \cap T)$
by (*metis* (*no_types*) *integral_unique*)
moreover have *integral T* $(\lambda a.$ *if* $a \in S$ *then* $1::real$ *else 0*) = *integral* $(S \cap T \cap UNIV)$ $(\lambda a.\ 1)$
by (*simp add*: *Henstock_Kurzweil_Integration.integral_restrict_Int*)
moreover have *integral T* (*indicat_real S*) = *integral T* $(\lambda a.$ *if* $a \in S$ *then 1 else 0*)
by (*simp add*: *indicator_def* [*abs_def*] *of_bool_def*)
ultimately show *?thesis*
by (*simp add*: *assms lmeasure_integral*)
qed

lemma *measurable_integrable*:
fixes *S* :: *'a*::*euclidean_space set*

shows $S \in lmeasurable \longleftrightarrow (indicat\_real\ S)\ integrable\_on\ UNIV$
by (*auto simp*: *lmeasurable_iff_integrable absolutely_integrable_on_iff_nonneg* [*symmetric*] *set_integrable_def*)

lemma *integrable_on_indicator*:
fixes $S :: 'a::euclidean\_space\ set$
shows *indicat_real S integrable_on T* $\longleftrightarrow$ $(S \cap T) \in lmeasurable$
unfolding *measurable_integrable*
unfolding *integrable_restrict_UNIV* [*of T*, *symmetric*]
by (*fastforce simp add*: *indicator_def elim*: *integrable_eq*)

lemma
assumes $\mathcal{D}$: $\mathcal{D}$ *division_of S*
shows *lmeasurable_division*: $S \in lmeasurable$ (**is** *?l*)
and *content_division*: $(\sum k{\in}\mathcal{D}.\ measure\ lebesgue\ k) = measure\ lebesgue\ S$ (**is** *?m*)
proof −
{ fix *d1 d2* **assume** $*$: $d1 \in \mathcal{D}\ d2 \in \mathcal{D}\ d1 \neq d2$
then obtain *a b c d* **where** *d1 = cbox a b d2 = cbox c d*
using *division_ofD*(*4*)[*OF* $\mathcal{D}$] **by** *blast*
with *division_ofD*(*5*)[*OF* $\mathcal{D}$ $*$]
have *d1* $\in$ *sets lborel d2* $\in$ *sets lborel d1* $\cap$ *d2* $\subseteq$ (*cbox a b* − *box a b*) $\cup$ (*cbox c d* − *box c d*)
by *auto*
moreover have (*cbox a b* − *box a b*) $\cup$ (*cbox c d* − *box c d*) $\in$ *null_sets lborel*
by (*intro null_sets.Un null_sets_cbox_Diff_box*)
ultimately have *d1* $\cap$ *d2* $\in$ *null_sets lborel*
by (*blast intro*: *null_sets_subset*) **}**
then show *?l ?m*
unfolding *division_ofD*(*6*)[*OF* $\mathcal{D}$, *symmetric*]
using *division_ofD*(*1*,*4*)[*OF* $\mathcal{D}$]
by (*auto intro*!: *measure_Union_AE*[*symmetric*] *simp*: *completion.AE_iff_null_sets Int_def*[*symmetric*] *pairwise_def null_sets_def*)
qed

lemma *has_measure_limit*:
assumes $S \in lmeasurable\ e > 0$
obtains B **where** $B > 0$
$\bigwedge a\ b.\ ball\ 0\ B \subseteq cbox\ a\ b \Longrightarrow |measure\ lebesgue\ (S \cap cbox\ a\ b) - measure\ lebesgue\ S| < e$
using *assms* **unfolding** *lmeasurable_iff_has_integral has_integral_alt'*
by (*force simp*: *integral_indicator integrable_on_indicator*)

lemma *lmeasurable_iff_indicator_has_integral*:
fixes $S :: 'a::euclidean\_space\ set$
shows $S \in lmeasurable \wedge m = measure\ lebesgue\ S \longleftrightarrow (indicat\_real\ S\ has\_integral\ m)\ UNIV$
by (*metis has_integral_iff lmeasurable_iff_has_integral measurable_integrable*)

lemma *has_measure_limit_iff*:
fixes *f* :: *'n::euclidean_space* ⇒ *'a::banach*
shows *S* ∈ *lmeasurable* ∧ *m* = *measure lebesgue S* ⟷
(∀ *e*>*0*. ∃ *B*>*0*. ∀ *a b*. *ball 0 B* ⊆ *cbox a b* ⟶
(*S* ∩ *cbox a b*) ∈ *lmeasurable* ∧ |*measure lebesgue* (*S* ∩ *cbox a b*) − *m*|
< *e*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
by (*meson has_measure_limit fmeasurable.Int lmeasurable_cbox*)
next
assume *RHS* [*rule_format*]: *?rhs*
then show *?lhs*
apply (*simp add*: *lmeasurable_iff_indicator_has_integral has_integral'* [**where**
i=*m*])
by (*metis* (*full_types*) *integral_indicator integrable_integral integrable_on_indicator*)
qed

9.8.9 Applications to Negligibility

lemma *negligible_iff_null_sets*: *negligible S* ⟷ *S* ∈ *null_sets lebesgue*
proof
assume *negligible S*
then have (*indicator S has_integral* (*0*::*real*)) *UNIV*
by (*auto simp*: *negligible*)
then show *S* ∈ *null_sets lebesgue*
by (*subst* (*asm*) *has_integral_iff_nn_integral_lebesgue*)
(*auto simp*: *borel_measurable_indicator_iff nn_integral_0_iff_AE AE_iff_null_sets*
indicator_eq_0_iff)
next
assume *S*: *S* ∈ *null_sets lebesgue*
show *negligible S*
unfolding *negligible_def*
proof (*safe intro*!: *has_integral_iff_nn_integral_lebesgue*[*THEN iffD2*]
has_integral_restrict_UNIV[**where** *s*=*cbox _ _*, *THEN iffD1*])
fix *a b*
show (λ*x*. *if x* ∈ *cbox a b then indicator S x else 0*) ∈ *lebesgue* →$_M$ *borel*
using *S* **by** (*auto intro*!: *measurable_If*)
then show (∫$^+$ *x*. *ennreal* (*if x* ∈ *cbox a b then indicator S x else 0*) ∂*lebesgue*)
= *ennreal 0*
using *S*[*THEN AE_not_in*] **by** (*auto intro*!: *nn_integral_0_iff_AE*[*THEN*
iffD2])
qed *auto*
qed

corollary *eventually_ae_filter_negligible*:
eventually P (*ae_filter lebesgue*) ⟷ (∃ *N*. *negligible N* ∧ {*x*. ¬ *P x*} ⊆ *N*)
by (*auto simp*: *completion.AE_iff_null_sets negligible_iff_null_sets null_sets_completion_subset*)

lemma *starlike_negligible*:

```
  assumes closed S
      and eq1: ⋀c x. (a + c *R x) ∈ S ⟹ 0 ≤ c ⟹ a + x ∈ S ⟹ c = 1
    shows negligible S
proof −
  have negligible ((+) (−a) ‘ S)
  proof (subst negligible_on_intervals, intro allI)
    fix u v
    show negligible ((+) (− a) ‘ S ∩ cbox u v)
        using ‹closed S› eq1 by (auto simp add: negligible_iff_null_sets alge-
bra_simps
          intro!: closed_translation_subtract starlike_negligible_compact cong: im-
age_cong_simp)
       (metis add_diff_eq diff_add_cancel scale_right_diff_distrib)
  qed
  then show ?thesis
    by (rule negligible_translation_rev)
qed

lemma starlike_negligible_strong:
  assumes closed S
      and star: ⋀c x. ⟦0 ≤ c; c < 1; a+x ∈ S⟧ ⟹ a + c *R x ∉ S
    shows negligible S
proof −
  show ?thesis
  proof (rule starlike_negligible [OF ‹closed S›, of a])
    fix c x
    assume cx: a + c *R x ∈ S 0 ≤ c a + x ∈ S
    with star have ¬ (c < 1) by auto
    moreover have ¬ (c > 1)
      using star [of 1/c c *R x] cx by force
    ultimately show c = 1 by arith
  qed
qed

lemma negligible_hyperplane:
  assumes a ≠ 0 ∨ b ≠ 0 shows negligible {x. a · x = b}
proof −
  obtain x where x: a · x ≠ b
    using assms by (metis euclidean_all_zero_iff inner_zero_right)
  moreover have c = 1 if a · (x + c *R w) = b a · (x + w) = b for c w
    using that
    by (metis (no_types, opaque_lifting) add_diff_eq diff_0 diff_minus_eq_add
inner_diff_right inner_scaleR_right mult_cancel_right2 right_minus_eq x)
  ultimately
  show ?thesis
    using starlike_negligible [OF closed_hyperplane, of x] by simp
qed

lemma negligible_lowdim:
```

```
  fixes S :: 'N :: euclidean_space set
  assumes dim S < DIM('N)
    shows negligible S
proof −
  obtain a where a ≠ 0 and a: span S ⊆ {x. a • x = 0}
    using lowdim_subset_hyperplane [OF assms] by blast
  have negligible (span S)
    using ‹a ≠ 0› a negligible_hyperplane by (blast intro: negligible_subset)
  then show ?thesis
    using span_base by (blast intro: negligible_subset)
qed

proposition negligible_convex_frontier:
  fixes S :: 'N :: euclidean_space set
  assumes convex S
    shows negligible(frontier S)
proof −
  have nf: negligible(frontier S) if convex S 0 ∈ S for S :: 'N set
  proof −
    obtain B where B ⊆ S and indB: independent B
              and spanB: S ⊆ span B and cardB: card B = dim S
      by (metis basis_exists)
    consider dim S < DIM('N) | dim S = DIM('N)
      using dim_subset_UNIV le_eq_less_or_eq by auto
    then show ?thesis
    proof cases
      case 1
      show ?thesis
        by (rule negligible_subset [of closure S])
           (simp_all add: frontier_def negligible_lowdim 1)
    next
      case 2
      obtain a where a ∈ interior (convex hull insert 0 B)
      proof (rule interior_simplex_nonempty [OF indB])
        show finite B
          by (simp add: indB independent_imp_finite)
        show card B = DIM('N)
          by (simp add: cardB 2)
      qed
      then have a: a ∈ interior S
        by (metis ‹B ⊆ S› ‹0 ∈ S› ‹convex S› insert_absorb insert_subset inte-
rior_mono subset_hull)
      show ?thesis
      proof (rule starlike_negligible_strong [where a=a])
        fix c::real and x
        have eq: a + c *R x = (a + x) − (1 − c) *R ((a + x) − a)
          by (simp add: algebra_simps)
        assume 0 ≤ c c < 1 a + x ∈ frontier S
        then show a + c *R x ∉ frontier S
```

```
        using eq mem_interior_closure_convex_shrink [OF ‹convex S› a, of _
1−c]
       unfolding frontier_def
        by (metis Diff_iff add_diff_cancel_left' add_diff_eq diff_gt_0_iff_gt
group_cancel.rule0 not_le)
     qed auto
   qed
  qed
  show ?thesis
  proof (cases S = {})
   case True then show ?thesis by auto
  next
   case False
   then obtain a where a ∈ S by auto
   show ?thesis
    using nf [of (λx. −a + x) ‘ S]
   by (metis ‹a ∈ S› add.left_inverse assms convex_translation_eq frontier_translation
            image_eqI negligible_translation_rev)
  qed
qed

corollary negligible_sphere: negligible (sphere a e)
  using frontier_cball negligible_convex_frontier convex_cball
  by (blast intro: negligible_subset)

lemma non_negligible_UNIV [simp]: ¬ negligible UNIV
  unfolding negligible_iff_null_sets by (auto simp: null_sets_def)

lemma negligible_interval:
  negligible (cbox a b) ⟷ box a b = {} negligible (box a b) ⟷ box a b = {}
  by (auto simp: negligible_iff_null_sets null_sets_def prod_nonneg inner_diff_left
box_eq_empty
              not_le emeasure_lborel_cbox_eq emeasure_lborel_box_eq
          intro: eq_refl antisym less_imp_le)

proposition open_not_negligible:
  assumes open S S ≠ {}
  shows ¬ negligible S
proof
  assume neg: negligible S
  obtain a where a ∈ S
   using ‹S ≠ {}› by blast
  then obtain e where e > 0 cball a e ⊆ S
   using ‹open S› open_contains_cball_eq by blast
  let ?p = a − (e / DIM('a)) *R One let ?q = a + (e / DIM('a)) *R One
  have cbox ?p ?q ⊆ cball a e
  proof (clarsimp simp: mem_box dist_norm)
   fix x
   assume ∀ i∈Basis. ?p · i ≤ x · i ∧ x · i ≤ ?q · i
```

then have *ax*: $|(a - x) \cdot i| \leq e \,/\, real\ DIM('a)$ **if** $i \in Basis$ **for** i
using *that* **by** (*auto simp*: *algebra_simps*)
have *norm* $(a - x) \leq (\sum i{\in}Basis.\ |(a - x) \cdot i|)$
by (*rule norm_le_l1*)
also have $\ldots \leq DIM('a) * (e \,/\, real\ DIM('a))$
by (*intro sum_bounded_above ax*)
also have $\ldots = e$
by *simp*
finally show *norm* $(a - x) \leq e$ **.**
qed
then have *negligible* (*cbox ?p ?q*)
by (*meson* ‹*cball a e* ⊆ *S*› *neg negligible_subset*)
with ‹$e > 0$› **show** *False*
by (*simp add*: *negligible_interval box_eq_empty algebra_simps field_split_simps mult_le_0_iff*)
qed

lemma *negligible_convex_interior*:
convex S ⟹ (*negligible S* ⟷ *interior S* = {})
by (*metis Diff_empty closure_subset frontier_def interior_subset negligible_convex_frontier negligible_subset open_interior open_not_negligible*)

lemma *measure_eq_0_null_sets*: $S \in null\_sets\ M \Longrightarrow measure\ M\ S = 0$
by (*auto simp*: *measure_def null_sets_def*)

lemma *negligible_imp_measure0*: *negligible S* ⟹ *measure lebesgue S* = *0*
by (*simp add*: *measure_eq_0_null_sets negligible_iff_null_sets*)

lemma *negligible_iff_emeasure0*: $S \in sets\ lebesgue \Longrightarrow$ (*negligible S* ⟷ *emeasure lebesgue S* = *0*)
by (*auto simp*: *measure_eq_0_null_sets negligible_iff_null_sets*)

lemma *negligible_iff_measure0*: $S \in lmeasurable \Longrightarrow$ (*negligible S* ⟷ *measure lebesgue S* = *0*)
by (*metis* (*full_types*) *completion.null_sets_outer negligible_iff_null_sets negligible_imp_measure0 order_refl*)

lemma *negligible_imp_sets*: *negligible S* ⟹ $S \in sets\ lebesgue$
by (*simp add*: *negligible_iff_null_sets null_setsD2*)

lemma *negligible_imp_measurable*: *negligible S* ⟹ $S \in lmeasurable$
by (*simp add*: *fmeasurableI_null_sets negligible_iff_null_sets*)

lemma *negligible_iff_measure*: *negligible S* ⟷ $S \in lmeasurable \wedge$ *measure lebesgue S* = *0*
by (*fastforce simp*: *negligible_iff_measure0 negligible_imp_measurable dest*: *negligible_imp_measure0*)

lemma *negligible_outer*:

negligible S $\longleftrightarrow$ ($\forall$ *e>0*. $\exists$ *T*. *S* $\subseteq$ *T* $\wedge$ *T* $\in$ *lmeasurable* $\wedge$ *measure lebesgue T* $<$ *e*) (**is** _ = *?rhs*)
proof
assume *negligible S* **then show** *?rhs*
by (*metis negligible_iff_measure order_refl*)
next
assume *?rhs* **then show** *negligible S*
by (*meson completion.null_sets_outer negligible_iff_null_sets*)
qed

lemma *negligible_outer_le*:
negligible S $\longleftrightarrow$ ($\forall$ *e>0*. $\exists$ *T*. *S* $\subseteq$ *T* $\wedge$ *T* $\in$ *lmeasurable* $\wedge$ *measure lebesgue T* $\leq$ *e*) (**is** _ = *?rhs*)
proof
assume *negligible S* **then show** *?rhs*
by (*metis dual_order.strict_implies_order negligible_imp_measurable negligible_imp_measure0 order_refl*)
next
assume *?rhs* **then show** *negligible S*
by (*metis le_less_trans negligible_outer field_lbound_gt_zero*)
qed

lemma *negligible_UNIV*: *negligible S* $\longleftrightarrow$ (*indicat_real S has_integral 0*) *UNIV* (**is** _=*?rhs*)
by (*metis lmeasurable_iff_indicator_has_integral negligible_iff_measure*)

lemma *sets_negligible_symdiff*:
$[\![$*S* $\in$ *sets lebesgue*; *negligible*((*S* $-$ *T*) $\cup$ (*T* $-$ *S*))$]\!]$ $\Longrightarrow$ *T* $\in$ *sets lebesgue*
by (*metis Diff_Diff_Int Int_Diff_Un inf_commute negligible_Un_eq negligible_imp_sets sets.Diff sets.Un*)

lemma *lmeasurable_negligible_symdiff*:
$[\![$*S* $\in$ *lmeasurable*; *negligible*((*S* $-$ *T*) $\cup$ (*T* $-$ *S*))$]\!]$ $\Longrightarrow$ *T* $\in$ *lmeasurable*
using *integrable_spike_set_eq lmeasurable_iff_integrable_on* **by** *blast*

lemma *measure_Un3_negligible*:
assumes *meas*: *S* $\in$ *lmeasurable T* $\in$ *lmeasurable U* $\in$ *lmeasurable*
and *neg*: *negligible*(*S* $\cap$ *T*) *negligible*(*S* $\cap$ *U*) *negligible*(*T* $\cap$ *U*) **and** *V*: *S* $\cup$ *T* $\cup$ *U* = *V*
shows *measure lebesgue V* = *measure lebesgue S* + *measure lebesgue T* + *measure lebesgue U*
proof $-$
have [*simp*]: *measure lebesgue* (*S* $\cap$ *T*) = *0*
using *neg*(*1*) *negligible_imp_measure0* **by** *blast*
have [*simp*]: *measure lebesgue* (*S* $\cap$ *U* $\cup$ *T* $\cap$ *U*) = *0*
using *neg*(*2*) *neg*(*3*) *negligible_Un negligible_imp_measure0* **by** *blast*
have *measure lebesgue V* = *measure lebesgue* (*S* $\cup$ *T* $\cup$ *U*)
using *V* **by** *simp*

also have ... = *measure lebesgue S + measure lebesgue T + measure lebesgue U*
by (*simp add*: *measure_Un3 meas fmeasurable.Un Int_Un_distrib2*)
finally show *?thesis* **.**
qed

lemma *measure_translate_add*:
assumes *meas*: $S \in$ *lmeasurable* $T \in$ *lmeasurable*
and *U*: $S \cup ((+)a \text{ ` } T) = U$ **and** *neg*: *negligible*$(S \cap ((+)a \text{ ` } T))$
shows *measure lebesgue S + measure lebesgue T = measure lebesgue U*
proof −
have [*simp*]: *measure lebesgue* $(S \cap (+)\ a \text{ ` } T) = 0$
using *neg negligible_imp_measure0* **by** *blast*
have *measure lebesgue* $(S \cup ((+)a \text{ ` } T))$ = *measure lebesgue S + measure lebesgue T*
by (*simp add*: *measure_Un3 meas measurable_translation measure_translation fmeasurable.Un*)
then show *?thesis*
using *U* **by** *auto*
qed

lemma *measure_negligible_symdiff*:
assumes *S*: $S \in$ *lmeasurable*
and *neg*: *negligible* $(S - T \cup (T - S))$
shows *measure lebesgue T = measure lebesgue S*
proof −
have *measure lebesgue* $(S - T) = 0$
using *neg negligible_Un_eq negligible_imp_measure0* **by** *blast*
then show *?thesis*
by (*metis S Un_commute add.right_neutral lmeasurable_negligible_symdiff measure_Un2 neg negligible_Un_eq negligible_imp_measure0*)
qed

lemma *measure_closure*:
assumes *bounded S* **and** *neg*: *negligible* (*frontier S*)
shows *measure lebesgue* (*closure S*) = *measure lebesgue S*
proof −
have *measure lebesgue* (*frontier S*) = *0*
by (*metis neg negligible_imp_measure0*)
then show *?thesis*
by (*metis assms lmeasurable_iff_integrable_on eq_iff_diff_eq_0 has_integral_interior integrable_on_def integral_unique lmeasurable_interior lmeasure_integral measure_frontier*)
qed

lemma *measure_interior*:
⟦*bounded S*; *negligible*(*frontier S*)⟧ ⟹ *measure lebesgue* (*interior S*) = *measure lebesgue S*
using *measure_closure measure_frontier negligible_imp_measure0* **by** *fastforce*

lemma *measurable_Jordan*:
assumes *bounded S* **and** *neg*: *negligible* (*frontier S*)
shows *S* ∈ *lmeasurable*
proof −
have *closure S* ∈ *lmeasurable*
by (*metis lmeasurable_closure* ‹*bounded S*›)
moreover have *interior S* ∈ *lmeasurable*
by (*simp add*: *lmeasurable_interior* ‹*bounded S*›)
moreover have *interior S* ⊆ *S*
by (*simp add*: *interior_subset*)
ultimately show *?thesis*
using *assms* **by** (*metis* (*full_types*) *closure_subset completion.complete_sets_sandwich_fmeasurable measure_closure measure_interior*)
qed

lemma *measurable_convex*: ⟦*convex S*; *bounded S*⟧ ⟹ *S* ∈ *lmeasurable*
by (*simp add*: *measurable_Jordan negligible_convex_frontier*)

lemma *content_cball_conv_ball*: *content* (*cball c r*) = *content* (*ball c r*)
proof −
have *ball c r* − *cball c r* ∪ (*cball c r* − *ball c r*) = *sphere c r*
by *auto*
hence *measure lebesgue* (*cball c r*) = *measure lebesgue* (*ball c r*)
using *negligible_sphere*[*of c r*] **by** (*intro measure_negligible_symdiff*) *simp_all*
thus *?thesis* **by** *simp*
qed

9.8.10 Negligibility of image under non-injective linear map

lemma *negligible_Union_nat*:
assumes ⋀*n*::*nat*. *negligible*(*S n*)
shows *negligible*(⋃ *n*. *S n*)
proof −
have *negligible* (⋃ *m*≤*k*. *S m*) **for** *k*
using *assms* **by** *blast*
then have *0*: *integral UNIV* (*indicat_real* (⋃ *m*≤*k*. *S m*)) = *0*
and *1*: (*indicat_real* (⋃ *m*≤*k*. *S m*)) *integrable_on UNIV* **for** *k*
by (*auto simp*: *negligible has_integral_iff*)
have *2*: ⋀*k x*. *indicat_real* (⋃ *m*≤*k*. *S m*) *x* ≤ (*indicat_real* (⋃ *m*≤*Suc k*. *S m*) *x*)
by (*auto simp add*: *indicator_def*)
have *3*: ⋀*x*. (λ*k*. *indicat_real* (⋃ *m*≤*k*. *S m*) *x*) ⟶ (*indicat_real* (⋃ *n*. *S n*) *x*)
by (*force simp*: *indicator_def eventually_sequentially intro*: *tendsto_eventually*)
have *4*: *bounded* (*range* (λ*k*. *integral UNIV* (*indicat_real* (⋃ *m*≤*k*. *S m*))))
by (*simp add*: *0*)
have ∗: *indicat_real* (⋃ *n*. *S n*) *integrable_on UNIV* ∧
(λ*k*. *integral UNIV* (*indicat_real* (⋃ *m*≤*k*. *S m*))) ⟶ (*integral UNIV* (*indicat_real* (⋃ *n*. *S n*)))

```
    by (intro monotone_convergence_increasing 1 2 3 4)
  then have integral UNIV (indicat_real (⋃n. S n)) = (0::real)
    using LIMSEQ_unique by (auto simp: 0)
  then show ?thesis
    using ∗ by (simp add: negligible_UNIV has_integral_iff)
qed


lemma negligible_linear_singular_image:
  fixes f :: 'n::euclidean_space ⇒ 'n
  assumes linear f ¬ inj f
  shows negligible (f ‘ S)
proof −
  obtain a where a ≠ 0 ⋀S. f ‘ S ⊆ {x. a · x = 0}
    using assms linear_singular_image_hyperplane by blast
  then show negligible (f ‘ S)
    by (metis negligible_hyperplane negligible_subset)
qed

lemma measure_negligible_finite_Union:
  assumes finite ℱ
    and meas: ⋀S. S ∈ ℱ ⟹ S ∈ lmeasurable
    and djointish: pairwise (λS T. negligible (S ∩ T)) ℱ
  shows measure lebesgue (⋃ℱ) = (∑S∈ℱ. measure lebesgue S)
  using assms
proof (induction)
  case empty
  then show ?case
    by auto
next
  case (insert S ℱ)
  then have S ∈ lmeasurable ⋃ℱ ∈ lmeasurable pairwise (λS T. negligible (S ∩
T)) ℱ
     by (simp_all add: fmeasurable.finite_Union insert.hyps(1) insert.prems(1)
pairwise_insert subsetI)
  then show ?case
  proof (simp add: measure_Un3 insert)
    have ∗: ⋀T. T ∈ (∩) S ‘ ℱ ⟹ negligible T
      using insert by (force simp: pairwise_def)
    have negligible(S ∩ ⋃ℱ)
      unfolding Int_Union
      by (rule negligible_Union) (simp_all add: ∗ insert.hyps(1))
    then show measure lebesgue (S ∩ ⋃ℱ) = 0
      using negligible_imp_measure0 by blast
  qed
qed

lemma measure_negligible_finite_Union_image:
  assumes finite S
```

and *meas*: $\bigwedge x.\ x \in S \Longrightarrow f\ x \in lmeasurable$
and *djointish*: *pairwise* $(\lambda x\ y.\ negligible\ (f\ x \cap f\ y))\ S$
shows *measure lebesgue* $(\bigcup (f\ `\ S)) = (\sum x{\in}S.\ measure\ lebesgue\ (f\ x))$
proof −
have *measure lebesgue* $(\bigcup (f\ `\ S)) = sum\ (measure\ lebesgue)\ (f\ `\ S)$
using *assms* **by** (*auto simp*: *pairwise_mono pairwise_image intro*: *measure_negligible_finite_Union*)
also have $\ldots = sum\ (measure\ lebesgue \circ f)\ S$
using *djointish* [*unfolded pairwise_def*] **by** (*metis inf.idem negligible_imp_measure0 sum.reindex_nontrivial* [*OF* ‹*finite S*›])
also have $\ldots = (\sum x{\in}S.\ measure\ lebesgue\ (f\ x))$
by *simp*
finally show *?thesis* **.**
qed

9.8.11 Negligibility of a Lipschitz image of a negligible set

The bound will be eliminated by a sort of onion argument

lemma *locally_Lipschitz_negl_bounded*:
fixes $f :: 'M{::}euclidean\_space \Rightarrow 'N{::}euclidean\_space$
assumes *MleN*: $DIM('M) \le DIM('N)\ 0 < B$ *bounded S negligible S*
and *lips*: $\bigwedge x.\ x \in S$
$\Longrightarrow \exists T.\ open\ T \wedge x \in T \wedge$
$(\forall y \in S \cap T.\ norm(f\ y - f\ x) \le B * norm(y - x))$
shows *negligible* $(f\ `\ S)$
unfolding *negligible_iff_null_sets*
proof (*clarsimp simp*: *completion.null_sets_outer*)
fix *e*::*real*
assume $0 < e$
have $S \in lmeasurable$
using ‹*negligible S*› **by** (*simp add*: *negligible_iff_null_sets fmeasurableI_null_sets*)
then have $S \in sets\ lebesgue$
by *blast*
have *e22*: $0 < e/2\ /\ (2 * B * real\ DIM('M))\ \hat{}\ DIM('N)$
using ‹$0 < e$› ‹$0 < B$› **by** (*simp add*: *field_split_simps*)
obtain T **where** *open* $T\ S \subseteq T\ (T - S) \in lmeasurable$
$measure\ lebesgue\ (T - S) < e/2\ /\ (2 * B * DIM('M))\ \hat{}\ DIM('N)$
using *sets_lebesgue_outer_open* [*OF* ‹$S \in sets\ lebesgue$› *e22*]
by (*metis emeasure_eq_measure2 ennreal_leI linorder_not_le*)
then have T: *measure lebesgue* $T \le e/2\ /\ (2 * B * DIM('M))\ \hat{}\ DIM('N)$
using ‹*negligible S*› **by** (*simp add*: *measure_Diff_null_set negligible_iff_null_sets*)
have $\exists r.\ 0 < r \wedge r \le 1/2 \wedge$
$(x \in S \longrightarrow (\forall y.\ norm(y - x) < r$
$\longrightarrow y \in T \wedge (y \in S \longrightarrow norm(f\ y - f\ x) \le B * norm(y - x))))$
for x
proof (*cases* $x \in S$)
case *True*
obtain U **where** *open* $U\ x \in U$ **and** U: $\bigwedge y.\ y \in S \cap U \Longrightarrow norm(f\ y - f\ x) \le B * norm(y - x)$
using *lips* [*OF* ‹$x \in S$›] **by** *auto*

```
    have x ∈ T ∩ U
      using ‹S ⊆ T› ‹x ∈ U› ‹x ∈ S› by auto
    then obtain ε where 0 < ε ball x ε ⊆ T ∩ U
      by (metis ‹open T› ‹open U› openE open_Int)
    then show ?thesis
    by (rule_tac x=min (1/2) ε in exI) (simp add: U dist_norm norm_minus_commute
subset_iff)
  next
    case False
    then show ?thesis
      by (rule_tac x=1/4 in exI) auto
  qed
  then obtain R where R12: ⋀x. 0 < R x ∧ R x ≤ 1/2
              and RT: ⋀x y. ⟦x ∈ S; norm(y − x) < R x⟧ ⟹ y ∈ T
              and RB: ⋀x y. ⟦x ∈ S; y ∈ S; norm(y − x) < R x⟧ ⟹ norm(f y −
f x) ≤ B ∗ norm(y − x)
    by metis+
  then have gaugeR: gauge (λx. ball x (R x))
    by (simp add: gauge_def)
  obtain c where c: S ⊆ cbox (−c ∗R One) (c ∗R One) box (−c ∗R One:: 'M) (c
∗R One) ≠ {}
  proof −
    obtain B where B: ⋀x. x ∈ S ⟹ norm x ≤ B
      using ‹bounded S› bounded_iff by blast
    show ?thesis
    proof (rule_tac c = abs B + 1 in that)
      show S ⊆ cbox (− (|B| + 1) ∗R One) ((|B| + 1) ∗R One)
        using norm_bound_Basis_le Basis_le_norm
        by (fastforce simp: mem_box dest!: B intro: order_trans)
      show box (− (|B| + 1) ∗R One) ((|B| + 1) ∗R One) ≠ {}
        by (simp add: box_eq_empty(1))
    qed
  qed
  obtain 𝒟 where countable 𝒟
     and Dsub: ⋃𝒟 ⊆ cbox (−c ∗R One) (c ∗R One)
     and cbox: ⋀K. K ∈ 𝒟 ⟹ interior K ≠ {} ∧ (∃ c d. K = cbox c d)
     and pw:   pairwise (λA B. interior A ∩ interior B = {}) 𝒟
     and Ksub: ⋀K. K ∈ 𝒟 ⟹ ∃ x ∈ S ∩ K. K ⊆ (λx. ball x (R x)) x
     and exN:  ⋀u v. cbox u v ∈ 𝒟 ⟹ ∃ n. ∀ i ∈ Basis. v • i − u • i = (2∗c) /
2^n
     and S ⊆ ⋃𝒟
    using covering_lemma [OF c gaugeR] by force
  have ∃ u v z. K = cbox u v ∧ box u v ≠ {} ∧ z ∈ S ∧ z ∈ cbox u v ∧
              cbox u v ⊆ ball z (R z) if K ∈ 𝒟 for K
  proof −
    obtain u v where K = cbox u v
      using ‹K ∈ 𝒟› cbox by blast
    with that show ?thesis
      by (metis Int_iff interior_cbox cbox Ksub)
```

```
  qed
  then obtain uf vf zf
    where uvz: ⋀K. K ∈ 𝒟 ⟹
              K = cbox (uf K) (vf K) ∧ box (uf K) (vf K) ≠ {} ∧ zf K ∈ S ∧
              zf K ∈ cbox (uf K) (vf K) ∧ cbox (uf K) (vf K) ⊆ ball (zf K) (R (zf
K))
     by metis
  define prj1 where prj1 ≡ λx::'M. x • (SOME i. i ∈ Basis)
  define fbx where fbx ≡ λD. cbox (f(zf D) − (B * DIM('M) * (prj1(vf D − uf
D))) *R One::'N)
                                  (f(zf D) + (B * DIM('M) * prj1(vf D − uf D)) *R
One)
  have vu_pos: 0 < prj1 (vf X − uf X) if X ∈ 𝒟 for X
     using uvz [OF that] by (simp add: prj1_def box_ne_empty SOME_Basis
inner_diff_left)
  have prj1_idem: prj1 (vf X − uf X) = (vf X − uf X) • i if  X ∈ 𝒟 i ∈ Basis
for X i
  proof −
    have cbox (uf X) (vf X) ∈ 𝒟
      using uvz ‹X ∈ 𝒟› by auto
    with exN obtain n where ⋀i. i ∈ Basis ⟹ vf X • i − uf X • i = (2*c) /
2^n
      by blast
    then show ?thesis
      by (simp add: ‹i ∈ Basis› SOME_Basis inner_diff prj1_def)
  qed
  have countbl: countable (fbx ` 𝒟)
    using ‹countable 𝒟› by blast
  have (∑ k∈fbx`𝒟'. measure lebesgue k) ≤ e/2 if 𝒟' ⊆ 𝒟 finite 𝒟' for 𝒟'
  proof −
    have BM_ge0: 0 ≤ B * (DIM('M) * prj1 (vf X − uf X)) if X ∈ 𝒟' for X
      using ‹0 < B› ‹𝒟' ⊆ 𝒟› that vu_pos by fastforce
    have {} ∉ 𝒟'
      using cbox ‹𝒟' ⊆ 𝒟› interior_empty by blast
    have (∑ k∈fbx`𝒟'. measure lebesgue k) ≤ sum (measure lebesgue o fbx) 𝒟'
      by (rule sum_image_le [OF ‹finite 𝒟'›]) (force simp: fbx_def)
   also have ... ≤ (∑ X∈𝒟'. (2 * B * DIM('M)) ^ DIM('N) * measure lebesgue
X)
    proof (rule sum_mono)
      fix X assume X ∈ 𝒟'
      then have X ∈ 𝒟 using ‹𝒟' ⊆ 𝒟› by blast
      then have ufvf: cbox (uf X) (vf X) = X
        using uvz by blast
      have prj1 (vf X − uf X) ^ DIM('M) = (∏ i::'M ∈ Basis. prj1 (vf X − uf
X))
        by (rule prod_constant [symmetric])
      also have ... = (∏ i∈Basis. vf X • i − uf X • i)
        by (simp add: ‹X ∈ 𝒟› inner_diff_left prj1_idem cong: prod.cong)
      finally have prj1_eq: prj1 (vf X − uf X) ^ DIM('M) = (∏ i∈Basis. vf X •
```

```
i − uf X · i) .
    have uf X ∈ cbox (uf X) (vf X) vf X ∈ cbox (uf X) (vf X)
      using uvz [OF ‹X ∈ 𝒟›] by (force simp: mem_box)+
    moreover have cbox (uf X) (vf X) ⊆ ball (zf X) (1/2)
      by (meson R12 order_trans subset_ball uvz [OF ‹X ∈ 𝒟›])
    ultimately have uf X ∈ ball (zf X) (1/2)  vf X ∈ ball (zf X) (1/2)
      by auto
    then have dist (vf X) (uf X) ≤ 1
      unfolding mem_ball
      by (metis dist_commute dist_triangle_half_l dual_order.order_iff_strict)
    then have 1: prj1 (vf X − uf X) ≤ 1
      unfolding prj1_def dist_norm using Basis_le_norm SOME_Basis or-
der_trans by fastforce
    have 0: 0 ≤ prj1 (vf X − uf X)
      using ‹X ∈ 𝒟› prj1_def vu_pos by fastforce
   have (measure lebesgue ∘ fbx) X ≤ (2 * B * DIM('M)) ^ DIM('N) * content
(cbox (uf X) (vf X))
     apply (simp add: fbx_def content_cbox_cases algebra_simps BM_ge0 ‹X
∈ 𝒟'› ‹0 < B› flip: prj1_eq)
      using MleN 0 1 uvz ‹X ∈ 𝒟›
      by (fastforce simp add: box_ne_empty power_decreasing)
    also have ... = (2 * B * DIM('M)) ^ DIM('N) * measure lebesgue X
      by (subst (3) ufvf[symmetric]) simp
    finally show (measure lebesgue ∘ fbx) X ≤ (2 * B * DIM('M)) ^ DIM('N)
* measure lebesgue X .
  qed
  also have ... = (2 * B * DIM('M)) ^ DIM('N) * sum (measure lebesgue) 𝒟'
    by (simp add: sum_distrib_left)
  also have ... ≤ e/2
  proof −
    have ⋀K. K ∈ 𝒟' ⟹ ∃a b. K = cbox a b
      using cbox that by blast
    then have div: 𝒟' division_of ⋃𝒟'
      using pairwise_subset [OF pw ‹𝒟' ⊆ 𝒟›] unfolding pairwise_def
      by (force simp: ‹finite 𝒟'› ‹{} ∉ 𝒟'› division_of_def)
    have le_meaT: measure lebesgue (⋃𝒟') ≤ measure lebesgue T
    proof (rule measure_mono_fmeasurable)
      show (⋃𝒟') ∈ sets lebesgue
        using div lmeasurable_division by auto
      have ⋃𝒟' ⊆ ⋃𝒟
        using ‹𝒟' ⊆ 𝒟› by blast
      also have ... ⊆ T
      proof (clarify)
        fix x D
        assume x ∈ D D ∈ 𝒟
        show x ∈ T
          using Ksub [OF ‹D ∈ 𝒟›]
          by (metis ‹x ∈ D› Int_iff dist_norm mem_ball norm_minus_commute
subsetD RT)
```

```
      qed
      finally show ⋃𝒟' ⊆ T .
      show T ∈ lmeasurable
          using ‹S ∈ lmeasurable› ‹S ⊆ T› ‹T − S ∈ lmeasurable› fmeasurable_Diff_D by blast
    qed
    have sum (measure lebesgue) 𝒟' = sum content 𝒟'
      using ‹𝒟' ⊆ 𝒟› cbox by (force intro: sum.cong)
    then have (2 * B * DIM('M)) ^ DIM('N) * sum (measure lebesgue) 𝒟' =
            (2 * B * real DIM('M)) ^ DIM('N) * measure lebesgue (⋃𝒟')
      using content_division [OF div] by auto
    also have ... ≤ (2 * B * real DIM('M)) ^ DIM('N) * measure lebesgue T
      using ‹0 < B›
      by (intro mult_left_mono [OF le_meaT]) (force simp add: algebra_simps)
    also have ... ≤ e/2
      using T ‹0 < B› by (simp add: field_simps)
    finally show ?thesis .
  qed
  finally show ?thesis .
 qed
 then have e2: sum (measure lebesgue) 𝒢 ≤ e/2 if 𝒢 ⊆ fbx ` 𝒟 finite 𝒢 for 𝒢
  by (metis finite_subset_image that)
 show ∃W∈lmeasurable. f ` S ⊆ W ∧ measure lebesgue W < e
 proof (intro bexI conjI)
  have ∃X∈𝒟. f y ∈ fbx X if y ∈ S for y
  proof −
   obtain X where y ∈ X X ∈ 𝒟
    using ‹S ⊆ ⋃𝒟› ‹y ∈ S› by auto
   then have y: y ∈ ball(zf X) (R(zf X))
    using uvz by fastforce
   have conj_le_eq: z − b ≤ y ∧ y ≤ z + b ⟷ abs(y − z) ≤ b for z y b::real
    by auto
   have yin: y ∈ cbox (uf X) (vf X) and zin: (zf X) ∈ cbox (uf X) (vf X)
    using uvz ‹X ∈ 𝒟› ‹y ∈ X› by auto
   have norm (y − zf X) ≤ (∑ i∈Basis. |(y − zf X) • i|)
    by (rule norm_le_l1)
   also have ... ≤ real DIM('M) * prj1 (vf X − uf X)
   proof (rule sum_bounded_above)
    fix j::'M assume j: j ∈ Basis
    show |(y − zf X) • j| ≤ prj1 (vf X − uf X)
     using yin zin j
     by (fastforce simp add: mem_box prj1_idem [OF ‹X ∈ 𝒟› j] inner_diff_left)
   qed
   finally have nole: norm (y − zf X) ≤ DIM('M) * prj1 (vf X − uf X)
    by simp
   have fle: |f y • i − f(zf X) • i| ≤ B * DIM('M) * prj1 (vf X − uf X) if i ∈ Basis for i
   proof −
    have |f y • i − f (zf X) • i| = |(f y − f (zf X)) • i|
```

```
        by (simp add: algebra_simps)
      also have ... ≤ norm (f y - f (zf X))
        by (simp add: Basis_le_norm that)
      also have ... ≤ B * norm(y - zf X)
        by (metis uvz RB ‹X ∈ 𝒟› dist_commute dist_norm mem_ball ‹y ∈ S›
y)
      also have ... ≤ B * real DIM('M) * prj1 (vf X - uf X)
        using ‹0 < B› by (simp add: nole)
      finally show ?thesis .
    qed
    show ?thesis
      by (rule_tac x=X in bexI)
       (auto simp: fbx_def prj1_idem mem_box conj_le_eq inner_add inner_diff
fle ‹X ∈ 𝒟›)
  qed
  then show f ‘ S ⊆ (⋃D∈𝒟. fbx D) by auto
 next
  have 1: ⋀D. D ∈ 𝒟 ⟹ fbx D ∈ lmeasurable
    by (auto simp: fbx_def)
  have 2: I' ⊆ 𝒟 ⟹ finite I' ⟹ measure lebesgue (⋃D∈I'. fbx D) ≤ e/2 for
I'
    by (rule order_trans[OF measure_Union_le e2]) (auto simp: fbx_def)
  show (⋃D∈𝒟. fbx D) ∈ lmeasurable
    by (intro fmeasurable_UN_bound[OF ‹countable 𝒟› 1 2])
  have measure lebesgue (⋃D∈𝒟. fbx D) ≤ e/2
    by (intro measure_UN_bound[OF ‹countable 𝒟› 1 2])
  then show measure lebesgue (⋃D∈𝒟. fbx D) < e
    using ‹0 < e› by linarith
 qed
qed

proposition negligible_locally_Lipschitz_image:
 fixes f :: 'M::euclidean_space ⇒ 'N::euclidean_space
 assumes MleN: DIM('M) ≤ DIM('N) negligible S
    and lips: ⋀x. x ∈ S
                ⟹ ∃ T B. open T ∧ x ∈ T ∧
                     (∀ y ∈ S ∩ T. norm(f y - f x) ≤ B * norm(y - x))
  shows negligible (f ‘ S)
proof -
 let ?S = λn. ({x ∈ S. norm x ≤ n ∧
                    (∃ T. open T ∧ x ∈ T ∧
                       (∀ y∈S ∩ T. norm (f y - f x) ≤ (real n + 1) * norm (y
- x)))})
 have negfn: f ‘ ?S n ∈ null_sets lebesgue for n::nat
   unfolding negligible_iff_null_sets[symmetric]
   apply (rule_tac B = real n + 1 in locally_Lipschitz_negl_bounded)
   by (auto simp: MleN bounded_iff intro: negligible_subset [OF ‹negligible S›])
 have S = (⋃n. ?S n)
 proof (intro set_eqI iffI)
```

```
    fix x assume x ∈ S
    with lips obtain T B where T: open T x ∈ T
                          and B: ⋀y. y ∈ S ∩ T ⟹ norm(f y − f x) ≤ B * norm(y
− x)
      by metis+
    have no: norm (f y − f x) ≤ (nat ⌈max B (norm x)⌉ + 1) * norm (y − x) if
y ∈ S ∩ T for y
    proof −
      have B * norm(y − x) ≤ (nat ⌈max B (norm x)⌉ + 1) * norm (y − x)
      by (meson max.cobounded1 mult_right_mono nat_ceiling_le_eq nat_le_iff_add
norm_ge_zero order_trans)
      then show ?thesis
        using B order_trans that by blast
    qed
    have norm x ≤ real (nat ⌈max B (norm x)⌉)
      by linarith
    then have x ∈ ?S (nat (ceiling (max B (norm x))))
      using T no by (force simp: ‹x ∈ S› algebra_simps)
    then show x ∈ (⋃ n. ?S n) by force
  qed auto
  then show ?thesis
   by (rule ssubst) (auto simp: image_Union negligible_iff_null_sets intro: negfn)
qed

corollary negligible_differentiable_image_negligible:
  fixes f :: 'M::euclidean_space ⇒ 'N::euclidean_space
  assumes MleN: DIM('M) ≤ DIM('N) negligible S
      and diff_f: f differentiable_on S
    shows negligible (f ‘ S)
proof −
  have ∃ T B. open T ∧ x ∈ T ∧ (∀ y ∈ S ∩ T. norm(f y − f x) ≤ B * norm(y
− x))
        if x ∈ S for x
  proof −
    obtain f' where linear f'
      and f': ⋀e. e>0 ⟹
                    ∃ d>0. ∀ y∈S. norm (y − x) < d ⟶
                           norm (f y − f x − f' (y − x)) ≤ e * norm (y − x)
      using diff_f ‹x ∈ S›
    by (auto simp: linear_linear differentiable_on_def differentiable_def has_derivative_within_alt)
    obtain B where B > 0 and B: ∀ x. norm (f' x) ≤ B * norm x
      using linear_bounded_pos ‹linear f'› by blast
    obtain d where d>0
           and d: ⋀y. ⟦y ∈ S; norm (y − x) < d⟧ ⟹
                      norm (f y − f x − f' (y − x)) ≤ norm (y − x)
      using f' [of 1] by (force simp:)
    show ?thesis
    proof (intro exI conjI ballI)
      show norm (f y − f x) ≤ (B + 1) * norm (y − x)
```

```
      if y ∈ S ∩ ball x d for y
    proof −
     have norm (f y − f x) − B ∗ norm (y − x) ≤ norm (f y − f x) − norm (f′
(y − x))
       by (simp add: B)
      also have ... ≤ norm (f y − f x − f′ (y − x))
       by (rule norm_triangle_ineq2)
      also have ... ≤ norm (y − x)
       by (metis IntE d dist_norm mem_ball norm_minus_commute that)
      finally show ?thesis
       by (simp add: algebra_simps)
    qed
   qed (use ‹d>0› in auto)
  qed
  with negligible_locally_Lipschitz_image assms show ?thesis by metis
qed

corollary negligible_differentiable_image_lowdim:
  fixes f :: 'M::euclidean_space ⇒ 'N::euclidean_space
  assumes MlessN: DIM('M) < DIM('N) and diff_f: f differentiable_on S
    shows negligible (f ‘ S)
proof −
  have x ≤ DIM('M) ⟹ x ≤ DIM('N) for x
    using MlessN by linarith
  obtain lift :: 'M ∗ real ⇒ 'N and drop :: 'N ⇒ 'M ∗ real and j :: 'N
    where linear lift linear drop and dropl [simp]: ⋀z. drop (lift z) = z
      and j ∈ Basis and j: ⋀x. lift(x,0) • j = 0
    using lowerdim_embeddings [OF MlessN] by metis
  have negligible ((λx. lift (x, 0)) ‘ S)
  proof −
    have negligible {x. x•j = 0}
      by (metis ‹j ∈ Basis› negligible_standard_hyperplane)
    moreover have (λm. lift (m, 0)) ‘ S ⊆ {n. n • j = 0}
      using j by force
    ultimately show ?thesis
      using negligible_subset by auto
  qed
  moreover
  have f ∘ fst ∘ drop differentiable_on (λx. lift (x, 0)) ‘ S
    using diff_f
    apply (clarsimp simp add: differentiable_on_def)
   apply (intro differentiable_chain_within linear_imp_differentiable [OF ‹linear
drop›]
            linear_imp_differentiable [OF linear_fst])
    apply (force simp: image_comp o_def)
    done
  moreover
  have f = f ∘ fst ∘ drop ∘ (λx. lift (x, 0))
    by (simp add: o_def)
```

ultimately show *?thesis*
by (*metis* (*no_types*) *image_comp negligible_differentiable_image_negligible order_refl*)
qed

9.8.12 Measurability of countable unions and intersections of various kinds.

lemma
assumes *S*: $\bigwedge n.\ S\ n \in lmeasurable$
and *leB*: $\bigwedge n.\ measure\ lebesgue\ (S\ n) \le B$
and *nest*: $\bigwedge n.\ S\ n \subseteq S(Suc\ n)$
shows *measurable_nested_Union*: $(\bigcup n.\ S\ n) \in lmeasurable$
and *measure_nested_Union*: $(\lambda n.\ measure\ lebesgue\ (S\ n)) \longrightarrow measure\ lebesgue\ (\bigcup n.\ S\ n)$ (**is** *?Lim*)
proof −
have *indicat_real* $(\bigcup\ (range\ S))$ *integrable_on UNIV* $\wedge$
$(\lambda n.\ integral\ UNIV\ (indicat\_real\ (S\ n)))$
$\longrightarrow$ *integral UNIV* (*indicat_real* $(\bigcup\ (range\ S))$)
proof (*rule monotone_convergence_increasing*)
show $\bigwedge n.\ (indicat\_real\ (S\ n))\ integrable\_on\ UNIV$
using *S measurable_integrable* **by** *blast*
show $\bigwedge n\ x{::}'a.\ indicat\_real\ (S\ n)\ x \le (indicat\_real\ (S\ (Suc\ n))\ x)$
by (*simp add*: *indicator_leI nest rev_subsetD*)
have $\bigwedge x.\ (\exists n.\ x \in S\ n) \longrightarrow (\forall_F\ n\ in\ sequentially.\ x \in S\ n)$
by (*metis eventually_sequentiallyI lift_Suc_mono_le nest subsetCE*)
then
show $\bigwedge x.\ (\lambda n.\ indicat\_real\ (S\ n)\ x) \longrightarrow (indicat\_real\ (\bigcup(S\ `\ UNIV))\ x)$
by (*simp add*: *indicator_def tendsto_eventually*)
show *bounded* (*range* $(\lambda n.\ integral\ UNIV\ (indicat\_real\ (S\ n))))$
using *leB* **by** (*auto simp*: *lmeasure_integral_UNIV* [*symmetric*] *S bounded_iff*)
qed
then have $(\bigcup n.\ S\ n) \in lmeasurable \wedge ?Lim$
by (*simp add*: *lmeasure_integral_UNIV S cong*: *conj_cong*) (*simp add*: *measurable_integrable*)
then show $(\bigcup n.\ S\ n) \in lmeasurable\ ?Lim$
by *auto*
qed

lemma
assumes *S*: $\bigwedge n.\ S\ n \in lmeasurable$
and *djointish*: *pairwise* $(\lambda m\ n.\ negligible\ (S\ m \cap S\ n))\ UNIV$
and *leB*: $\bigwedge n.\ (\sum k{\le}n.\ measure\ lebesgue\ (S\ k)) \le B$
shows *measurable_countable_negligible_Union*: $(\bigcup n.\ S\ n) \in lmeasurable$
and *measure_countable_negligible_Union*: $(\lambda n.\ (measure\ lebesgue\ (S\ n)))$ *sums measure lebesgue* $(\bigcup n.\ S\ n)$ (**is** *?Sums*)
proof −
have *1*: $\bigcup\ (S\ `\ \{..n\}) \in lmeasurable$ **for** *n*
using *S* **by** *blast*

```
  have 2: measure lebesgue (⋃ (S ` {..n})) ≤ B for n
  proof −
    have measure lebesgue (⋃ (S ` {..n})) ≤ (∑k≤n. measure lebesgue (S k))
      by (simp add: S fmeasurableD measure_UNION_le)
    with leB show ?thesis
      using order_trans by blast
  qed
  have 3: ⋀n. ⋃ (S ` {..n}) ⊆ ⋃ (S ` {..Suc n})
    by (simp add: SUP_subset_mono)
  have eqS: (⋃n. S n) = (⋃n. ⋃ (S ` {..n}))
    using atLeastAtMost_iff by blast
  also have (⋃n. ⋃ (S ` {..n})) ∈ lmeasurable
    by (intro measurable_nested_Union [OF 1 2] 3)
  finally show (⋃n. S n) ∈ lmeasurable .
  have eqm: (∑i≤n. measure lebesgue (S i)) = measure lebesgue (⋃ (S ` {..n}))
for n
    using assms by (simp add: measure_negligible_finite_Union_image pair-
wise_mono)
  have (λn. (measure lebesgue (S n))) sums measure lebesgue (⋃n. ⋃ (S ` {..n}))
    by (simp add: sums_def' eqm atLeast0AtMost) (intro measure_nested_Union
[OF 1 2] 3)
  then show ?Sums
    by (simp add: eqS)
qed

lemma negligible_countable_Union [intro]:
  assumes countable ℱ and meas: ⋀S. S ∈ ℱ ⟹ negligible S
  shows negligible (⋃ℱ)
proof (cases ℱ = {})
  case False
  then show ?thesis
    by (metis from_nat_into range_from_nat_into assms negligible_Union_nat)
qed simp

lemma
  assumes S: ⋀n. (S n) ∈ lmeasurable
    and djointish: pairwise (λm n. negligible (S m ∩ S n)) UNIV
    and bo: bounded (⋃n. S n)
  shows measurable_countable_negligible_Union_bounded: (⋃n. S n) ∈ lmeasur-
able
  and   measure_countable_negligible_Union_bounded:   (λn. (measure lebesgue
(S n))) sums measure lebesgue (⋃n. S n) (is ?Sums)
proof −
  obtain a b where ab: (⋃n. S n) ⊆ cbox a b
    using bo bounded_subset_cbox_symmetric by metis
  then have B: (∑k≤n. measure lebesgue (S k)) ≤ measure lebesgue (cbox a b)
for n
  proof −
    have (∑k≤n. measure lebesgue (S k)) = measure lebesgue (⋃ (S ` {..n}))
```

using *measure_negligible_finite_Union_image* [*OF _ _ pairwise_subset*] *djointish*
by (*metis S finite_atMost subset_UNIV*)
also have ... ≤ *measure lebesgue* (*cbox a b*)
proof (*rule measure_mono_fmeasurable*)
show ⋃ (*S* ‘ {..*n*}) ∈ *sets lebesgue* **using** *S* **by** *blast*
qed (*use ab* **in** *auto*)
finally show *?thesis* **.**
qed
show (⋃ *n*. *S n*) ∈ *lmeasurable*
by (*rule measurable_countable_negligible_Union* [*OF S djointish B*])
show *?Sums*
by (*rule measure_countable_negligible_Union* [*OF S djointish B*])
qed

lemma *measure_countable_Union_approachable*:
assumes *countable* 𝒟 *e* > *0* **and** *measD*: ⋀*d*. *d* ∈ 𝒟 ⟹ *d* ∈ *lmeasurable*
and *B*: ⋀*D'*. ⟦*D'* ⊆ 𝒟; *finite D'*⟧ ⟹ *measure lebesgue* (⋃ *D'*) ≤ *B*
obtains *D'* **where** *D'* ⊆ 𝒟 *finite D' measure lebesgue* (⋃ 𝒟) − *e* < *measure lebesgue* (⋃ *D'*)
proof (*cases* 𝒟 = {})
case *True*
then show *?thesis*
by (*simp add*: ‹*e* > *0*› *that*)
next
case *False*
let *?S* = λ*n*. ⋃ *k* ≤ *n*. *from_nat_into* 𝒟 *k*
have (λ*n*. *measure lebesgue* (*?S n*)) ⟶ *measure lebesgue* (⋃ *n*. *?S n*)
proof (*rule measure_nested_Union*)
show *?S n* ∈ *lmeasurable* **for** *n*
by (*simp add*: *False fmeasurable.finite_UN from_nat_into measD*)
show *measure lebesgue* (*?S n*) ≤ *B* **for** *n*
by (*metis* (*mono_tags*, *lifting*) *B False finite_atMost finite_imageI from_nat_into image_iff subsetI*)
show *?S n* ⊆ *?S* (*Suc n*) **for** *n*
by *force*
qed
then obtain *N* **where** *N*: ⋀*n*. *n* ≥ *N* ⟹ *dist* (*measure lebesgue* (*?S n*)) (*measure lebesgue* (⋃ *n*. *?S n*)) < *e*
using *metric_LIMSEQ_D* ‹*e* > *0*› **by** *blast*
show *?thesis*
proof
show *from_nat_into* 𝒟 ‘ {..*N*} ⊆ 𝒟
by (*auto simp*: *False from_nat_into*)
have *eq*: (⋃ *n*. ⋃ *k*≤*n*. *from_nat_into* 𝒟 *k*) = (⋃ 𝒟)
using ‹*countable* 𝒟› *False*
by (*auto intro*: *from_nat_into dest*: *from_nat_into_surj* [*OF* ‹*countable* 𝒟›])
show *measure lebesgue* (⋃ 𝒟) − *e* < *measure lebesgue* (⋃ (*from_nat_into* 𝒟 ‘ {..*N*}))

using *N* [*OF order_refl*]
by (*auto simp*: *eq algebra_simps dist_norm*)
qed *auto*
qed

9.8.13 Negligibility is a local property

lemma *locally_negligible_alt*:
negligible S ⟷ (∀ *x* ∈ *S*. ∃ *U*. *openin* (*top_of_set S*) *U* ∧ *x* ∈ *U* ∧ *negligible U*)
(**is** _ = *?rhs*)
proof
assume *negligible S*
then show *?rhs*
using *openin_subtopology_self* **by** *blast*
next
assume *?rhs*
then obtain *U* **where** *ope*: ⋀*x*. *x* ∈ *S* ⟹ *openin* (*top_of_set S*) (*U x*)
and *cov*: ⋀*x*. *x* ∈ *S* ⟹ *x* ∈ *U x*
and *neg*: ⋀*x*. *x* ∈ *S* ⟹ *negligible* (*U x*)
by *metis*
obtain *F* **where** *F* ⊆ *U* ‘ *S countable F* **and** *eq*: ⋃*F* = ⋃(*U* ‘ *S*)
using *ope* **by** (*force intro*: *Lindelof_openin* [*of U* ‘ *S S*])
then have *negligible* (⋃*F*)
by (*metis imageE neg negligible_countable_Union subset_eq*)
with *eq* **have** *negligible* (⋃(*U* ‘ *S*))
by *metis*
moreover have *S* ⊆ ⋃(*U* ‘ *S*)
using *cov* **by** *blast*
ultimately show *negligible S*
using *negligible_subset* **by** *blast*
qed

lemma *locally_negligible*: *locally negligible S* ⟷ *negligible S*
unfolding *locally_def*
by (*metis locally_negligible_alt negligible_subset openin_imp_subset openin_subtopology_self*)

9.8.14 Integral bounds

lemma *set_integral_norm_bound*:
fixes *f* :: _ ⇒ *'a* :: {*banach*, *second_countable_topology*}
shows *set_integrable M k f* ⟹ *norm* (*LINT x*:*k*|*M*. *f x*) ≤ (*LINT x*:*k*|*M*. *norm* (*f x*))
using *integral_norm_bound*[*of M* λ*x*. *indicator k x* ∗*R f x*] **by** (*simp add*: *set_lebesgue_integral_def*)

lemma *set_integral_finite_UN_AE*:
fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes *finite I*
and *ae*: ⋀*i j*. *i* ∈ *I* ⟹ *j* ∈ *I* ⟹ *AE x in M*. (*x* ∈ *A i* ∧ *x* ∈ *A j*) ⟶ *i* = *j*

```
    and [measurable]: ⋀i. i ∈ I ⟹ A i ∈ sets M
    and f: ⋀i. i ∈ I ⟹ set_integrable M (A i) f
  shows (LINT x:(⋃ i∈I. A i)|M. f x) = (∑ i∈I. LINT x:A i|M. f x)
  using ‹finite I› order_refl[of I]
proof (induction I rule: finite_subset_induct')
  case (insert i I')
  have AE x in M. (∀ j∈I'. x ∈ A i ⟶ x ∉ A j)
  proof (intro AE_ball_countable[THEN iffD2] ballI)
    fix j assume j ∈ I'
    with ‹I' ⊆ I› ‹i ∉ I'› have i ≠ j j ∈ I
      by auto
    then show AE x in M. x ∈ A i ⟶ x ∉ A j
      using ae[of i j] ‹i ∈ I› by auto
  qed (use ‹finite I'› in ‹rule countable_finite›)
  then have AE x∈A i in M. ∀ xa∈I'. x ∉ A xa
    by auto
  with insert.hyps insert.IH[symmetric]
  show ?case
    by (auto intro!: set_integral_Un_AE sets.finite_UN f set_integrable_UN)
qed (simp add: set_lebesgue_integral_def)

lemma set_integrable_norm:
  fixes f :: 'a ⇒ 'b::{banach, second_countable_topology}
  assumes f: set_integrable M k f shows set_integrable M k (λx. norm (f x))
  using integrable_norm f by (force simp add: set_integrable_def)

lemma absolutely_integrable_bounded_variation:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f: f absolutely_integrable_on UNIV
  obtains B where ∀ d. d division_of (⋃ d) ⟶ sum (λk. norm (integral k f)) d
≤ B
proof (rule that[of integral UNIV (λx. norm (f x))]; safe)
  fix d :: 'a set set assume d: d division_of ⋃ d
  have ∗: k ∈ d ⟹ f absolutely_integrable_on k for k
    using f[THEN set_integrable_subset, of k] division_ofD(2,4)[OF d, of k] by
auto
  note d' = division_ofD[OF d]
  have (∑ k∈d. norm (integral k f)) = (∑ k∈d. norm (LINT x:k|lebesgue. f x))
   by (intro sum.cong refl arg_cong[where f=norm] set_lebesgue_integral_eq_integral(2)[symmetric]
∗)
  also have ... ≤ (∑ k∈d. LINT x:k|lebesgue. norm (f x))
    by (intro sum_mono set_integral_norm_bound ∗)
  also have ... = (∑ k∈d. integral k (λx. norm (f x)))
   by (intro sum.cong refl set_lebesgue_integral_eq_integral(2) set_integrable_norm
∗)
  also have ... ≤ integral (⋃ d) (λx. norm (f x))
   using integrable_on_subdivision[OF d] assms f unfolding absolutely_integrable_on_def
    by (subst integral_combine_division_topdown[OF _ d]) auto
  also have ... ≤ integral UNIV (λx. norm (f x))
```

using *integrable_on_subdivision*[*OF d*] *assms* **unfolding** *absolutely_integrable_on_def*
by (*intro integral_subset_le*) *auto*
finally show $(\sum k{\in}d.\ norm\ (integral\ k\ f)) \le integral\ UNIV\ (\lambda x.\ norm\ (f\ x))$ **.**
qed

lemma *absdiff_norm_less*:
assumes $sum\ (\lambda x.\ norm\ (f\ x - g\ x))\ S < e$
shows $|sum\ (\lambda x.\ norm(f\ x))\ S - sum\ (\lambda x.\ norm(g\ x))\ S| < e$ (**is** *?lhs* $< e$)
proof −
have $?lhs \le (\sum i{\in}S.\ |norm\ (f\ i) - norm\ (g\ i)|)$
by (*metis* (*no_types*) *sum_abs sum_subtractf*)
also have ... $\le (\sum x{\in}S.\ norm\ (f\ x - g\ x))$
by (*simp add*: *norm_triangle_ineq3 sum_mono*)
also have ... $< e$
using *assms*(*1*) **by** *blast*
finally show *?thesis* **.**
qed

proposition *bounded_variation_absolutely_integrable_interval*:
fixes $f :: 'n{::}euclidean\_space \Rightarrow 'm{::}euclidean\_space$
assumes *f*: *f integrable_on cbox a b*
and ∗: $\bigwedge d.\ d\ division\_of\ (cbox\ a\ b) \Longrightarrow sum\ (\lambda K.\ norm(integral\ K\ f))\ d \le B$
shows *f absolutely_integrable_on cbox a b*
proof −
let *?f* $= \lambda d.\ \sum K{\in}d.\ norm\ (integral\ K\ f)$ **and** *?D* $= \{d.\ d\ division\_of\ (cbox\ a\ b)\}$
have *D_1*: *?D* $\neq \{\}$
by (*rule elementary_interval*[*of a b*]) *auto*
have *D_2*: *bdd_above* (*?f*‘*?D*)
by (*metis* ∗ *mem_Collect_eq bdd_aboveI2*)
note *D* = *D_1 D_2*
let *?S* = *SUP x*∈*?D*. *?f x*
have ∗: $\exists \gamma.\ gauge\ \gamma \wedge$
$(\forall p.\ p\ tagged\_division\_of\ cbox\ a\ b \wedge$
$\gamma\ fine\ p \longrightarrow$
$norm\ ((\sum (x,k) \in p.\ content\ k *_R norm\ (f\ x)) - ?S) < e)$
if *e*: $e > 0$ **for** *e*
proof −
have $?S - e/2 <$ *?S* **using** ‹$e > 0$› **by** *simp*
then obtain *d* **where** *d*: *d division_of* (*cbox a b*) $?S - e/2 < (\sum k{\in}d.\ norm\ (integral\ k\ f))$
unfolding *less_cSUP_iff*[*OF D*] **by** *auto*
note $d' = division\_ofD[OF\ this(1)]$

have $\exists e{>}0.\ \forall i{\in}d.\ x \notin i \longrightarrow ball\ x\ e \cap i = \{\}$ **for** *x*
proof −
have $\exists d'{>}0.\ \forall x'{\in}\bigcup\{i \in d.\ x \notin i\}.\ d' \le dist\ x\ x'$
proof (*rule separate_point_closed*)
show *closed* $(\bigcup\{i \in d.\ x \notin i\})$

```
        using d' by force
      show x ∉ ⋃{i ∈ d. x ∉ i}
        by auto
    qed
    then show ?thesis
      by force
  qed
  then obtain k where k: ⋀x. 0 < k x ⋀i x. ⟦i ∈ d; x ∉ i⟧ ⟹ ball x (k x) ∩
i = {}
    by metis
  have e/2 > 0
    using e by auto
  with Henstock_lemma[OF f]
  obtain γ where g: gauge γ
    ⋀p. ⟦p tagged_partial_division_of cbox a b; γ fine p⟧
          ⟹ (∑(x,k) ∈ p. norm (content k *R f x − integral k f)) < e/2
    by (metis (no_types, lifting))
  let ?g = λx. γ x ∩ ball x (k x)
  show ?thesis
  proof (intro exI conjI allI impI)
    show gauge ?g
      using g(1) k(1) by (auto simp: gauge_def)
  next
    fix p
    assume p tagged_division_of (cbox a b) ∧ ?g fine p
    then have p: p tagged_division_of cbox a b γ fine p (λx. ball x (k x)) fine p
      by (auto simp: fine_Int)
    note p' = tagged_division_ofD[OF p(1)]
    define p' where p' = {(x,k) | x k. ∃i l. x ∈ i ∧ i ∈ d ∧ (x,l) ∈ p ∧ k = i ∩
l}
    have gp': γ fine p'
      using p(2) by (auto simp: p'_def fine_def)
    have p'': p' tagged_division_of (cbox a b)
    proof (rule tagged_division_ofI)
      show finite p'
      proof (rule finite_subset)
        show p' ⊆ (λ(k, x, l). (x, k ∩ l)) ` (d × p)
          by (force simp: p'_def image_iff)
        show finite ((λ(k, x, l). (x, k ∩ l)) ` (d × p))
          by (simp add: d'(1) p'(1))
      qed
    next
      fix x K
      assume (x, K) ∈ p'
      then have ∃i l. x ∈ i ∧ i ∈ d ∧ (x, l) ∈ p ∧ K = i ∩ l
        unfolding p'_def by auto
      then obtain i l where il: x ∈ i i ∈ d (x, l) ∈ p K = i ∩ l by blast
      show x ∈ K and K ⊆ cbox a b
        using p'(2−3)[OF il(3)] il by auto
```

```
      show ∃ a b. K = cbox a b
      unfolding il using p'(4)[OF il(3)] d'(4)[OF il(2)] by (meson Int_interval)
    next
      fix x1 K1
      assume (x1, K1) ∈ p'
      then have ∃ i l. x1 ∈ i ∧ i ∈ d ∧ (x1, l) ∈ p ∧ K1 = i ∩ l
        unfolding p'_def by auto
      then obtain i1 l1 where il1: x1 ∈ i1 i1 ∈ d (x1, l1) ∈ p K1 = i1 ∩ l1
by blast
      fix x2 K2
      assume (x2,K2) ∈ p'
      then have ∃ i l. x2 ∈ i ∧ i ∈ d ∧ (x2, l) ∈ p ∧ K2 = i ∩ l
        unfolding p'_def by auto
      then obtain i2 l2 where il2: x2 ∈ i2 i2 ∈ d (x2, l2) ∈ p K2 = i2 ∩ l2
by blast
      assume (x1, K1) ≠ (x2, K2)
      then have interior i1 ∩ interior i2 = {} ∨ interior l1 ∩ interior l2 = {}
        using d'(5)[OF il1(2) il2(2)] p'(5)[OF il1(3) il2(3)]  by (auto simp: il1
il2)
      then show interior K1 ∩ interior K2 = {}
        unfolding il1 il2 by auto
    next
      have *: ∀ (x, X) ∈ p'. X ⊆ cbox a b
        unfolding p'_def using d' by blast
      show ⋃ {K. ∃ x. (x, K) ∈ p'} = cbox a b
      proof
        show ⋃ {k. ∃ x. (x, k) ∈ p'} ⊆ cbox a b
          using * by auto
      next
        show cbox a b ⊆ ⋃ {k. ∃ x. (x, k) ∈ p'}
        proof
          fix y
          assume y: y ∈ cbox a b
          obtain x L where xl: (x, L) ∈ p y ∈ L
            using y unfolding p'(6)[symmetric] by auto
          obtain I where i: I ∈ d y ∈ I
            using y unfolding d'(6)[symmetric] by auto
          have x ∈ I
            using fineD[OF p(3) xl(1)] using k(2) i xl by auto
          then show y ∈ ⋃ {K. ∃ x. (x, K) ∈ p'}
          proof −
            obtain x l where xl: (x, l) ∈ p y ∈ l
              using y unfolding p'(6)[symmetric] by auto
            obtain i where i: i ∈ d y ∈ i
              using y unfolding d'(6)[symmetric] by auto
            have x ∈ i
              using fineD[OF p(3) xl(1)] using k(2) i xl by auto
            then show ?thesis
            unfolding p'_def by (rule_tac X=i ∩ l in UnionI) (use i xl in auto)
```

```
        qed
      qed
    qed
  qed
  then have sum_less_e2: (∑(x,K) ∈ p'. norm (content K *R f x − integral K f)) < e/2
    using g(2) gp' tagged_division_of_def by blast

  have in_p': (x, I ∩ L) ∈ p' if x: (x, L) ∈ p I ∈ d and y: y ∈ I y ∈ L
    for x I L y
  proof −
    have x ∈ I
      using fineD[OF p(3) that(1)] k(2)[OF ‹I ∈ d›] y by auto
    with x have (∃ i l. x ∈ i ∧ i ∈ d ∧ (x, l) ∈ p ∧ I ∩ L = i ∩ l)
      by blast
    then have (x, I ∩ L) ∈ p'
      by (simp add: p'_def)
    with y show ?thesis by auto
  qed
  moreover
  have Ex_p_p': ∃ y i l. (x, K) = (y, i ∩ l) ∧ (y, l) ∈ p ∧ i ∈ d ∧ i ∩ l ≠ {}
    if xK: (x,K) ∈ p' for x K
  proof −
    obtain i l where il: x ∈ i i ∈ d (x, l) ∈ p K = i ∩ l
      using xK unfolding p'_def by auto
    then show ?thesis
      using p'(2) by fastforce
  qed
  ultimately have p'alt: p' = {(x, I ∩ L) | x I L. (x,L) ∈ p ∧ I ∈ d ∧ I ∩ L ≠ {}}
    by auto
  have sum_p': (∑(x,K) ∈ p'. norm (integral K f)) = (∑k∈snd ‘ p'. norm (integral k f))
  proof (rule sum.over_tagged_division_lemma[OF p'(1)])
    show ⋀u v. box u v = {} ⟹ norm (integral (cbox u v) f) = 0
      by (auto intro: integral_null simp: content_eq_0_interior)
  qed
  have snd_p_div: snd ‘ p division_of cbox a b
    by (rule division_of_tagged_division[OF p(1)])
  note snd_p = division_ofD[OF snd_p_div]
  have fin_d_sndp: finite (d × snd ‘ p)
    by (simp add: d'(1) snd_p(1))

  have *: ⋀sni sni' sf sf'. ⟦|sf' − sni'| < e/2; ?S − e/2 < sni; sni' ≤ ?S;
                sni ≤ sni'; sf' = sf⟧ ⟹ |sf − ?S| < e
    by arith
  show norm ((∑(x,k) ∈ p. content k *R norm (f x)) − ?S) < e
    unfolding real_norm_def
  proof (rule *)
```

show $|(\sum(x,K)\in p'.\ norm\ (content\ K *_R f\ x)) - (\sum(x,k)\in p'.\ norm\ (integral\ k\ f))| < e/2$
using p'' *sum_less_e2* **unfolding** *split_def* **by** (*force intro*!: *absdiff_norm_less*)
show $(\sum(x,k) \in p'.\ norm\ (integral\ k\ f)) \leq ?S$
by (*auto simp*: *sum_p' division_of_tagged_division*[*OF* p''] *D intro*!: *cSUP_upper*)
show $(\sum k\in d.\ norm\ (integral\ k\ f)) \leq (\sum(x,k) \in p'.\ norm\ (integral\ k\ f))$
proof −
have $*$: $\{k \cap l \mid k\ l.\ k \in d \wedge l \in snd\ `\ p\} = (\lambda(k,l).\ k \cap l)\ `\ (d \times snd\ `\ p)$
by *auto*
have $(\sum K\in d.\ norm\ (integral\ K\ f)) \leq (\sum i\in d.\ \sum l\in snd\ `\ p.\ norm\ (integral\ (i \cap l)\ f))$
proof (*rule sum_mono*)
fix K **assume** k: $K \in d$
from $d'(4)$[*OF this*] **obtain** $u\ v$ **where** uv: $K = cbox\ u\ v$ **by** *metis*
define d' **where** $d' = \{cbox\ u\ v \cap l \mid l.\ l \in snd\ `\ p \wedge\ cbox\ u\ v \cap l \neq \{\}\}$
have *uvab*: $cbox\ u\ v \subseteq cbox\ a\ b$
using $d(1)$ k uv **by** *blast*
have d'*_div*: d' *division_of cbox u v*
unfolding d'*_def* **by** (*rule division_inter_1* [*OF snd_p_div uvab*])
moreover have $norm\ (\sum i\in d'.\ integral\ i\ f) \leq (\sum k\in d'.\ norm\ (integral\ k\ f))$
by (*simp add*: *sum_norm_le*)
moreover have *f integrable_on K*
using *f integrable_on_subcbox uv uvab* **by** *blast*
moreover have d' *division_of K*
using d'*_div uv* **by** *blast*
ultimately have $norm\ (integral\ K\ f) \leq sum\ (\lambda k.\ norm\ (integral\ k\ f))\ d'$
by (*simp add*: *integral_combine_division_topdown*)
also have $\ldots = (\sum I\in\{K \cap L \mid L.\ L \in snd\ `\ p\}.\ norm\ (integral\ I\ f))$
proof (*rule sum.mono_neutral_left*)
show *finite* $\{K \cap L \mid L.\ L \in snd\ `\ p\}$
by (*simp add*: *snd_p*(1))
show $\forall i\in\{K \cap L \mid L.\ L \in snd\ `\ p\} - d'.\ norm\ (integral\ i\ f) = 0$
$d' \subseteq \{K \cap L \mid L.\ L \in snd\ `\ p\}$
using d'*_def image_eqI uv* **by** *auto*
qed
also have $\ldots = (\sum l\in snd\ `\ p.\ norm\ (integral\ (K \cap l)\ f))$
unfolding *Setcompr_eq_image*
proof (*rule sum.reindex_nontrivial* [*unfolded o_def*])
show *finite* $(snd\ `\ p)$
using *snd_p*(1) **by** *blast*
show $norm\ (integral\ (K \cap l)\ f) = 0$
if $l \in snd\ `\ p\ y \in snd\ `\ p\ l \neq y\ K \cap l = K \cap y$ **for** $l\ y$
proof −
have $interior\ (K \cap l) \subseteq interior\ (l \cap y)$
by (*metis Int_lower2 interior_mono le_inf_iff that*(4))
then have $interior\ (K \cap l) = \{\}$

```
            by (simp add: snd_p(5) that)
          moreover from d'(4)[OF k] snd_p(4)[OF that(1)]
          obtain u1 v1 u2 v2
            where uv: K = cbox u1 u2 l = cbox v1 v2 by metis
          ultimately show ?thesis
            using that integral_null
            unfolding uv Int_interval content_eq_0_interior
            by (metis (mono_tags, lifting) norm_eq_zero)
        qed
      qed
      finally show norm (integral K f) ≤ (∑ l∈snd ‘ p. norm (integral (K ∩
l) f)) .
    qed
    also have ... = (∑ (i,l) ∈ d × snd ‘ p. norm (integral (i∩l) f))
      by (simp add: sum.cartesian_product)
    also have ... = (∑ x ∈ d × snd ‘ p. norm (integral (case_prod (∩) x) f))
      by (force simp: split_def intro!: sum.cong)
    also have ... = (∑ k∈{i ∩ l |i l. i ∈ d ∧ l ∈ snd ‘ p}. norm (integral k
f))
    proof −
      have eq0: (integral (l1 ∩ k1) f) = 0
        if l1 ∩ k1 = l2 ∩ k2 (l1, k1) ≠ (l2, k2)
           l1 ∈ d (j1,k1) ∈ p l2 ∈ d (j2,k2) ∈ p
        for l1 l2 k1 k2 j1 j2
      proof −
        obtain u1 v1 u2 v2 where uv: l1 = cbox u1 u2 k1 = cbox v1 v2
          using ‹(j1, k1) ∈ p› ‹l1 ∈ d› d'(4) p'(4) by blast
        have l1 ≠ l2 ∨ k1 ≠ k2
          using that by auto
        then have interior k1 ∩ interior k2 = {} ∨ interior l1 ∩ interior l2
= {}
          by (meson d'(5) old.prod.inject p'(5) that(3) that(4) that(5) that(6))
        moreover have interior (l1 ∩ k1) = interior (l2 ∩ k2)
          by (simp add: that(1))
        ultimately have interior(l1 ∩ k1) = {}
          by auto
        then show ?thesis
        unfolding uv Int_interval content_eq_0_interior[symmetric] by auto
      qed
      show ?thesis
        unfolding ∗
          apply (rule sum.reindex_nontrivial [OF fin_d_sndp, symmetric,
unfolded o_def])
        apply clarsimp
        by (metis eq0 fst_conv snd_conv)
    qed
    also have ... = (∑ (x,k) ∈ p'. norm (integral k f))
      unfolding sum_p'
    proof (rule sum.mono_neutral_right)
```

```
          show finite {i ∩ l |i l. i ∈ d ∧ l ∈ snd ‘ p}
            by (metis ∗ finite_imageI[OF fin_d_sndp])
          show snd ‘ p′ ⊆ {i ∩ l |i l. i ∈ d ∧ l ∈ snd ‘ p}
            by (clarsimp simp: p′_def) (metis image_eqI snd_conv)
          show ∀ i∈{i ∩ l |i l. i ∈ d ∧ l ∈ snd ‘ p} − snd ‘ p′. norm (integral i f)
= 0
              by clarsimp (metis Henstock_Kurzweil_Integration.integral_empty
disjoint_iff image_eqI in_p′ snd_conv)
        qed
        finally show ?thesis .
      qed
      show (∑(x,k) ∈ p′. norm (content k ∗R f x)) = (∑(x,k) ∈ p. content k ∗R
norm (f x))
      proof −
        let ?S = {(x, i ∩ l) |x i l. (x, l) ∈ p ∧ i ∈ d}
        have ∗: ?S = (λ(xl,i). (fst xl, snd xl ∩ i)) ‘ (p × d)
          by force
        have fin_pd: finite (p × d)
          using finite_cartesian_product[OF p′(1) d′(1)] by metis
        have (∑(x,k) ∈ p′. norm (content k ∗R f x)) = (∑(x,k) ∈ ?S. |content
k| ∗ norm (f x))
          unfolding norm_scaleR
        proof (rule sum.mono_neutral_left)
          show finite {(x, i ∩ l) |x i l. (x, l) ∈ p ∧ i ∈ d}
            by (simp add: ∗ fin_pd)
        qed (use p′alt in ‹force+›)
        also have ... = (∑((x,l),i)∈p × d. |content (l ∩ i)| ∗ norm (f x))
        proof −
          have |content (l1 ∩ k1)| ∗ norm (f x1) = 0
            if (x1, l1) ∈ p (x2, l2) ∈ p k1 ∈ d k2 ∈ d
              x1 = x2 l1 ∩ k1 = l2 ∩ k2 x1 ≠ x2 ∨ l1 ≠ l2 ∨ k1 ≠ k2
            for x1 l1 k1 x2 l2 k2
          proof −
            obtain u1 v1 u2 v2 where uv: k1 = cbox u1 u2 l1 = cbox v1 v2
              by (meson ‹(x1, l1) ∈ p› ‹k1 ∈ d› d(1) division_ofD(4) p′(4))
            have l1 ≠ l2 ∨ k1 ≠ k2
              using that by auto
             then have interior k1 ∩ interior k2 = {} ∨ interior l1 ∩ interior l2
= {}
              using that p′(5) d′(5) by (metis snd_conv)
            moreover have interior (l1 ∩ k1) = interior (l2 ∩ k2)
              unfolding that ..
            ultimately have interior (l1 ∩ k1) = {}
              by auto
            then show |content (l1 ∩ k1)| ∗ norm (f x1) = 0
            unfolding uv Int_interval content_eq_0_interior[symmetric] by auto
          qed
          then show ?thesis
            unfolding ∗
```

```
      apply (subst sum.reindex_nontrivial [OF fin_pd])
      unfolding split_paired_all o_def split_def prod.inject
      by force+
    qed
    also have ... = (∑(x,k) ∈ p. content k *R norm (f x))
    proof −
      have sumeq: (∑i∈d. content (l ∩ i) * norm (f x)) = content l * norm
(f x)
        if (x, l) ∈ p for x l
      proof −
        note xl = p'(2−4)[OF that]
        then obtain u v where uv: l = cbox u v by blast
        have (∑i∈d. |content (l ∩ i)|) = (∑k∈d. content (k ∩ cbox u v))
          by (simp add: Int_commute uv)
        also have ... = sum content {k ∩ cbox u v| k. k ∈ d}
        proof −
          have eq0: content (k ∩ cbox u v) = 0
            if k ∈ d y ∈ d k ≠ y and eq: k ∩ cbox u v = y ∩ cbox u v for k y
          proof −
            from d'(4)[OF that(1)] d'(4)[OF that(2)]
            obtain α β where α: k ∩ cbox u v = cbox α β
              by (meson Int_interval)
            have {} = interior ((k ∩ y) ∩ cbox u v)
              by (simp add: d'(5) that)
            also have ... = interior (y ∩ (k ∩ cbox u v))
              by auto
            also have ... = interior (k ∩ cbox u v)
              unfolding eq by auto
            finally show ?thesis
              unfolding α content_eq_0_interior ..
          qed
          then show ?thesis
            unfolding Setcompr_eq_image
            by (fastforce intro: sum.reindex_nontrivial [OF ‹finite d›, unfolded
o_def, symmetric])
        qed
        also have ... = sum content {cbox u v ∩ k |k. k ∈ d ∧ cbox u v ∩ k
≠ {}}
        proof (rule sum.mono_neutral_right)
          show finite {k ∩ cbox u v |k. k ∈ d}
            by (simp add: d'(1))
        qed (fastforce simp: inf.commute)+
        finally have (∑i∈d. |content (l ∩ i)|) = content (cbox u v)
          using additive_content_division[OF division_inter_1[OF d(1)]] uv
xl(2) by auto
        then show (∑i∈d. content (l ∩ i) * norm (f x)) = content l * norm
(f x)
          unfolding sum_distrib_right[symmetric] using uv by auto
      qed
```

```
          show ?thesis
          by (subst sum_Sigma_product[symmetric]) (auto intro!: sumeq sum.cong
p' d')
        qed
        finally show ?thesis .
      qed
    qed (rule d)
  qed
 qed
 then show ?thesis
  using absolutely_integrable_onI [OF f has_integral_integrable] has_integral[of
_ ?S]
  by blast
qed


lemma bounded_variation_absolutely_integrable:
 fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
 assumes f integrable_on UNIV
  and ∀ d. d division_of (⋃ d) ⟶ sum (λk. norm (integral k f)) d ≤ B
 shows f absolutely_integrable_on UNIV
proof (rule absolutely_integrable_onI, fact)
 let ?f = λD. ∑ k∈D. norm (integral k f) and ?D = {d. d division_of (⋃ d)}
 define SDF where SDF ≡ SUP d∈?D. ?f d
 have D_1: ?D ≠ {}
  by (rule elementary_interval) auto
 have D_2: bdd_above (?f`?D)
  using assms(2) by auto
 have f_int: ⋀a b. f absolutely_integrable_on cbox a b
  using assms integrable_on_subcbox
  by (blast intro!: bounded_variation_absolutely_integrable_interval)
 have ∃ B>0. ∀ a b. ball 0 B ⊆ cbox a b ⟶
                |integral (cbox a b) (λx. norm (f x)) − SDF| < e
  if 0 < e for e
 proof −
  have ∃ y ∈ ?f ` ?D. ¬ y ≤ SDF − e
  proof (rule ccontr)
   assume ¬ ?thesis
   then have SDF ≤ SDF − e
    unfolding SDF_def
    by (metis (mono_tags) D_1 cSUP_least image_eqI)
   then show False
    using that by auto
  qed
  then obtain d K where ddiv: d division_of ⋃ d and K = ?f d SDF − e < K
   by (auto simp add: image_iff not_le)
  then have d: SDF − e < ?f d
   by auto
  note d'=division_ofD[OF ddiv]
```

```
    have bounded (⋃ d)
      using ddiv by blast
    then obtain K where K: 0 < K ∀ x∈⋃ d. norm x ≤ K
      using bounded_pos by blast
    show ?thesis
    proof (intro conjI impI allI exI)
      fix a b :: 'n
      assume ab: ball 0 (K + 1) ⊆ cbox a b
      have *: ⋀s s1. ⟦SDF − e < s1; s1 ≤ s; s < SDF + e⟧ ⟹ |s − SDF| < e
        by arith
      show |integral (cbox a b) (λx. norm (f x)) − SDF| < e
        unfolding real_norm_def
      proof (rule * [OF d])
        have ?f d ≤ sum (λk. integral k (λx. norm (f x))) d
        proof (intro sum_mono)
          fix k assume k ∈ d
          with d'(4) f_int show norm (integral k f) ≤ integral k (λx. norm (f x))
         by (force simp: absolutely_integrable_on_def integral_norm_bound_integral)
        qed
        also have ... = integral (⋃ d) (λx. norm (f x))
            by (metis (full_types) absolutely_integrable_on_def d'(4) ddiv f_int
integral_combine_division_bottomup)
        also have ... ≤ integral (cbox a b) (λx. norm (f x))
        proof −
          have ⋃ d ⊆ cbox a b
            using K(2) ab by fastforce
          then show ?thesis
              using integrable_on_subdivision[OF ddiv] f_int[of a b] unfolding
absolutely_integrable_on_def
            by (auto intro!: integral_subset_le)
        qed
        finally show ?f d ≤ integral (cbox a b) (λx. norm (f x)) .
      next
        have e/2>0
          using ‹e > 0› by auto
        moreover
        have f: f integrable_on cbox a b (λx. norm (f x)) integrable_on cbox a b
          using f_int by (auto simp: absolutely_integrable_on_def)
        ultimately obtain d1 where gauge d1
          and d1: ⋀p. ⟦p tagged_division_of (cbox a b); d1 fine p⟧ ⟹
           norm ((∑(x,k) ∈ p. content k *R norm (f x)) − integral (cbox a b) (λx.
norm (f x))) < e/2
          unfolding has_integral_integral has_integral by meson
        obtain d2 where gauge d2
          and d2: ⋀p. ⟦p tagged_partial_division_of (cbox a b); d2 fine p⟧ ⟹
            (∑(x,k) ∈ p. norm (content k *R f x − integral k f)) < e/2
          by (blast intro: Henstock_lemma [OF f(1) ‹e/2>0›])
        obtain p where
          p: p tagged_division_of (cbox a b) d1 fine p d2 fine p
```

```
        by (rule fine_division_exists [OF gauge_Int [OF ‹gauge d1› ‹gauge d2›],
of a b])
          (auto simp add: fine_Int)
       have *: ⋀sf sf' si di. ⟦sf' = sf; si ≤ SDF; |sf − si| < e/2;
                  |sf' − di| < e/2⟧ ⟹ di < SDF + e
        by arith
       have integral (cbox a b) (λx. norm (f x)) < SDF + e
       proof (rule *)
        show |(∑(x,k)∈p. norm (content k *R f x)) − (∑(x,k)∈p. norm (integral
k f))| < e/2
          unfolding split_def
         proof (rule absdiff_norm_less)
          show (∑p∈p. norm (content (snd p) *R f (fst p) − integral (snd p) f))
< e/2
           using d2[of p] p(1,3) by (auto simp: tagged_division_of_def split_def)
         qed
         show |(∑(x,k) ∈ p. content k *R norm (f x)) − integral (cbox a b) (λx.
norm(f x))| < e/2
           using d1[OF p(1,2)] by (simp only: real_norm_def)
           show (∑(x,k) ∈ p. content k *R norm (f x)) = (∑(x,k) ∈ p. norm
(content k *R f x))
           by (auto simp: split_paired_all sum.cong [OF refl])
         have (∑(x,k) ∈ p. norm (integral k f)) = (∑k∈snd ‘ p. norm (integral k
f))
           apply (rule sum.over_tagged_division_lemma[OF p(1)])
        by (metis Henstock_Kurzweil_Integration.integral_empty integral_open_interval
norm_zero)
         also have ... ≤ SDF
           using partial_division_of_tagged_division[of p cbox a b] p(1)
        by (auto simp: SDF_def tagged_partial_division_of_def intro!: cSUP_upper2
D_1 D_2)
         finally show (∑(x,k) ∈ p. norm (integral k f)) ≤ SDF .
       qed
       then show integral (cbox a b) (λx. norm (f x)) < SDF + e
        by simp
     qed
   qed (use K in auto)
  qed
  moreover have ⋀a b. (λx. norm (f x)) integrable_on cbox a b
   using absolutely_integrable_on_def f_int by auto
  ultimately
  have ((λx. norm (f x)) has_integral SDF) UNIV
   by (auto simp: has_integral_alt')
  then show (λx. norm (f x)) integrable_on UNIV
   by blast
qed
```

9.8.15 Outer and inner approximation of measurable sets by well-behaved sets.

proposition *measurable_outer_intervals_bounded*:
assumes $S \in$ *lmeasurable* $S \subseteq$ *cbox a b* $e > 0$
obtains $\mathcal{D}$
where *countable* $\mathcal{D}$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq$ *cbox a b* $\wedge\ K \neq \{\} \wedge (\exists\, c\ d.\ K =$ *cbox c d*$)$
pairwise $(\lambda A\ B.$ *interior* $A \cap$ *interior* $B = \{\})\ \mathcal{D}$
$\bigwedge u\ v.$ *cbox u v* $\in \mathcal{D} \Longrightarrow \exists\, n.\ \forall\, i \in$ *Basis*. $v \cdot i - u \cdot i = (b \cdot i - a \cdot i)/2\hat{}n$
$\bigwedge K.\ [\![K \in \mathcal{D};$ *box a b* $\neq \{\}]\!] \Longrightarrow$ *interior* $K \neq \{\}$
$S \subseteq \bigcup \mathcal{D}\ \bigcup \mathcal{D} \in$ *lmeasurable measure lebesgue* $(\bigcup \mathcal{D}) \leq$ *measure lebesgue* $S + e$
proof (*cases box a b* $= \{\}$)
case *True*
show *?thesis*
proof (*cases cbox a b* $= \{\}$)
case *True*
with *assms* **have** [*simp*]: $S = \{\}$
by *auto*
show *?thesis*
proof
show *countable* $\{\}$
by *simp*
qed (*use* ‹$e > 0$› **in** *auto*)
next
case *False*
show *?thesis*
proof
show *countable* $\{$*cbox a b*$\}$
by *simp*
show $\bigwedge u\ v.$ *cbox u v* $\in \{$*cbox a b*$\} \Longrightarrow \exists\, n.\ \forall\, i \in$ *Basis*. $v \cdot i - u \cdot i = (b \cdot i - a \cdot i)/2\ \hat{}\ n$
using *False* **by** (*force simp*: *eq_cbox intro*: *exI* [**where** *x=0*])
show *measure lebesgue* $(\bigcup \{$*cbox a b*$\}) \leq$ *measure lebesgue* $S + e$
using *assms* **by** (*simp add*: *sum_content.box_empty_imp* [*OF True*])
qed (*use assms* ‹*cbox a b* $\neq \{\}$› **in** *auto*)
qed
next
case *False*
let $?\mu =$ *measure lebesgue*
have $S \cap$ *cbox a b* $\in$ *lmeasurable*
using ‹$S \in$ *lmeasurable*› **by** *blast*
then have *indS_int*: (*indicator S has_integral* $(?\mu\ S))$ (*cbox a b*)
by (*metis integral_indicator* ‹$S \subseteq$ *cbox a b*› *has_integral_integrable_integral inf.orderE integrable_on_indicator*)
with ‹$e > 0$› **obtain** γ **where** *gauge* γ **and** γ:
$\bigwedge \mathcal{D}.\ [\![\mathcal{D}$ *tagged_division_of* (*cbox a b*)$;\ \gamma$ *fine* $\mathcal{D}]\!] \Longrightarrow$ *norm* $((\sum (x,K) \in \mathcal{D}.$ *content*$(K) *_R$ *indicator S x*$) - ?\mu\ S) < e$
by (*force simp*: *has_integral*)

have *inteq*: *integral* (*cbox a b*) (*indicat_real S*) = *integral UNIV* (*indicator S*)
using *assms* **by** (*metis has_integral_iff indS_int lmeasure_integral_UNIV*)
obtain $\mathcal{D}$ **where** $\mathcal{D}$: *countable* $\mathcal{D}$ $\bigcup\mathcal{D} \subseteq$ *cbox a b*
and *cbox*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow$ *interior K* $\neq \{\} \wedge (\exists\, c\ d.\ K =$ *cbox c d*)
and *djointish*: *pairwise* ($\lambda A\ B.$ *interior A* $\cap$ *interior B* $= \{\}$) $\mathcal{D}$
and *covered*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists\, x \in S \cap K.\ K \subseteq \gamma\ x$
and *close*: $\bigwedge u\ v.$ *cbox u v* $\in \mathcal{D} \Longrightarrow \exists\, n.\ \forall\, i \in$ *Basis*. $v{\cdot}i - u{\cdot}i = (b{\cdot}i - a{\cdot}i)/2\hat{}n$
and *covers*: $S \subseteq \bigcup\mathcal{D}$
using *covering_lemma* [*of S a b* γ] ‹*gauge* γ› ‹*box a b* $\neq \{\}$› *assms* **by** *force*
show *?thesis*
proof
show $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq$ *cbox a b* $\wedge\ K \neq \{\} \wedge (\exists\, c\ d.\ K =$ *cbox c d*)
by (*meson Sup_le_iff* $\mathcal{D}(2)$ *cbox interior_empty*)
have *negl_int*: *negligible*($K \cap L$) **if** $K \in \mathcal{D}$ $L \in \mathcal{D}$ $K \neq L$ **for** K L
proof −
have *interior K* $\cap$ *interior L* $= \{\}$
using *djointish pairwiseD that* **by** *fastforce*
moreover obtain *u v x y* **where** $K =$ *cbox u v* $L =$ *cbox x y*
using *cbox* ‹$K \in \mathcal{D}$› ‹$L \in \mathcal{D}$› **by** *blast*
ultimately show *?thesis*
by (*simp add*: *Int_interval box_Int_box negligible_interval*(*1*))
qed
have *fincase*: $\bigcup\mathcal{F} \in$ *lmeasurable* $\wedge$ *?μ* ($\bigcup\mathcal{F}$) $\leq$ *?μ S* + *e* **if** *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{D}$ **for** $\mathcal{F}$
proof −
obtain *t* **where** *t*: $\bigwedge K.\ K \in \mathcal{F} \Longrightarrow t\ K \in S \cap K \wedge K \subseteq \gamma(t\ K)$
using *covered* ‹$\mathcal{F} \subseteq \mathcal{D}$› *subsetD* **by** *metis*
have $\forall\, K \in \mathcal{F}.\ \forall\, L \in \mathcal{F}.\ K \neq L \longrightarrow$ *interior K* $\cap$ *interior L* $= \{\}$
using *that djointish* **by** (*simp add*: *pairwise_def*) (*metis subsetD*)
with *cbox that* $\mathcal{D}$ **have** $\mathcal{F}$*div*: $\mathcal{F}$ *division_of* ($\bigcup\mathcal{F}$)
by (*fastforce simp*: *division_of_def dest*: *cbox*)
then have *1*: $\bigcup\mathcal{F} \in$ *lmeasurable*
by *blast*
have *norme*: $\bigwedge p.$ ⟦*p tagged_division_of cbox a b*; γ *fine p*⟧
$\Longrightarrow$ *norm* (($\sum (x,K) \in p.$ *content K* $*$ *indicator S x*) − *integral* (*cbox a b*) (*indicator S*)) $<$ *e*
by (*auto simp*: *lmeasure_integral_UNIV assms inteq dest*: γ)
have $\forall\, x\ K\ y\ L.\ (x,K) \in (\lambda K.\ (t\ K,K))$ ‘ $\mathcal{F} \wedge (y,L) \in (\lambda K.\ (t\ K,K))$ ‘ $\mathcal{F} \wedge (x,K) \neq (y,L) \longrightarrow$ *interior K* $\cap$ *interior L* $= \{\}$
using *that djointish* **by** (*clarsimp simp*: *pairwise_def*) (*metis subsetD*)
with *that* $\mathcal{D}$ **have** *tagged*: $(\lambda K.\ (t\ K,\ K))$ ‘ $\mathcal{F}$ *tagged_partial_division_of cbox a b*
by (*auto simp*: *tagged_partial_division_of_def dest*: *t cbox*)
have *fine*: γ *fine* $(\lambda K.\ (t\ K,\ K))$ ‘ $\mathcal{F}$
using *t* **by** (*auto simp*: *fine_def*)
have $*$: $y \leq$ *?μ S* $\Longrightarrow |x - y| \leq e \Longrightarrow x \leq$ *?μ S* + *e* **for** *x y*
by *arith*
have *?μ* ($\bigcup\mathcal{F}$) $\leq$ *?μ S* + *e*

```
    proof (rule *)
      have (∑ K∈ℱ. ?μ (K ∩ S)) = ?μ (⋃ C∈ℱ. C ∩ S)
       proof (rule measure_negligible_finite_Union_image [OF ‹finite ℱ›, sym-
metric])
        show ⋀K. K ∈ ℱ ⟹ K ∩ S ∈ lmeasurable
          using ℱdiv ‹S ∈ lmeasurable› by blast
        show pairwise (λK y. negligible (K ∩ S ∩ (y ∩ S))) ℱ
          unfolding pairwise_def
         by (metis inf.commute inf_sup_aci(3) negligible_Int subsetCE negl_int
‹ℱ ⊆ 𝒟›)
       qed
       also have ... = ?μ (⋃ℱ ∩ S)
         by simp
       also have ... ≤ ?μ S
      by (simp add: 1 ‹S ∈ lmeasurable› fmeasurableD measure_mono_fmeasurable
sets.Int)
       finally show (∑ K∈ℱ. ?μ (K ∩ S)) ≤ ?μ S .
     next
       have ?μ (⋃ℱ) = sum ?μ ℱ
         by (metis ℱdiv content_division)
       also have ... = (∑ K∈ℱ. content K)
         using ℱdiv by (force intro: sum.cong)
       also have ... = (∑ x∈ℱ. content x * indicator S (t x))
         using t by auto
       finally have eq1: ?μ (⋃ℱ) = (∑ x∈ℱ. content x * indicator S (t x)) .
       have eq2: (∑ K∈ℱ. ?μ (K ∩ S)) = (∑ K∈ℱ. integral K (indicator S))
         apply (rule sum.cong [OF refl])
           by (metis integral_indicator ℱdiv ‹S ∈ lmeasurable› division_ofD(4)
fmeasurable.Int inf.commute lmeasurable_cbox)
      have |∑ (x,K)∈(λK. (t K, K)) ` ℱ. content K * indicator S x − integral K
(indicator S)| ≤ e
         using Henstock_lemma_part1 [of indicator S::'a⇒real, OF _ ‹e > 0›
‹gauge γ› _ tagged fine]
          indS_int norme by auto
       then show |?μ (⋃ℱ) − (∑ K∈ℱ. ?μ (K ∩ S))| ≤ e
        by (simp add: eq1 eq2 comm_monoid_add_class.sum.reindex inj_on_def
sum_subtractf)
     qed
     with 1 show ?thesis by blast
   qed
   have ⋃𝒟 ∈ lmeasurable ∧ ?μ (⋃𝒟) ≤ ?μ S + e
   proof (cases finite 𝒟)
     case True
     with fincase show ?thesis
       by blast
   next
     case False
     let ?T = from_nat_into 𝒟
     have T: bij_betw ?T UNIV 𝒟
```

by (*simp add*: *False* $\mathcal{D}(1)$ *bij_betw_from_nat_into*)
have *TM*: $\bigwedge n.$ *?T n* $\in$ *lmeasurable*
by (*metis False cbox finite.emptyI from_nat_into lmeasurable_cbox*)
have *TN*: $\bigwedge m\ n.\ m \neq n \Longrightarrow$ *negligible* (*?T m* $\cap$ *?T n*)
by (*simp add*: *False* $\mathcal{D}(1)$ *from_nat_into infinite_imp_nonempty negl_int*)
have *TB*: $(\sum k{\leq}n.$ *?μ* (*?T k*)) $\leq$ *?μ S* + *e* **for** *n*
proof −
have $(\sum k{\leq}n.$ *?μ* (*?T k*)) = *?μ* $(\bigcup$ (*?T* ‘ {..*n*}))
by (*simp add*: *pairwise_def TM TN measure_negligible_finite_Union_image*)
also have *?μ* $(\bigcup$ (*?T* ‘ {..*n*})) $\leq$ *?μ S* + *e*
using *fincase* [*of ?T* ‘ {..*n*}] *T* **by** (*auto simp*: *bij_betw_def*)
finally show *?thesis* **.**
qed
have $\bigcup\mathcal{D} \in$ *lmeasurable*
by (*metis lmeasurable_compact T* $\mathcal{D}(2)$ *bij_betw_def cbox compact_cbox countable_Un_Int*(1) *fmeasurableD fmeasurableI2 rangeI*)
moreover
have *?μ* $(\bigcup x.$ *from_nat_into* $\mathcal{D}$ *x*) $\leq$ *?μ S* + *e*
proof (*rule measure_countable_Union_le* [*OF TM*])
show *?μ* $(\bigcup x{\leq}n.$ *from_nat_into* $\mathcal{D}$ *x*) $\leq$ *?μ S* + *e* **for** *n*
by (*metis* (*mono_tags*, *lifting*) *False fincase finite.emptyI finite_atMost finite_imageI from_nat_into imageE subsetI*)
qed
ultimately show *?thesis* **by** (*metis T bij_betw_def*)
qed
then show $\bigcup\mathcal{D} \in$ *lmeasurable measure lebesgue* $(\bigcup\mathcal{D}) \leq$ *?μ S* + *e* **by** *blast*+
qed (*use* $\mathcal{D}$ *cbox djointish close covers* **in** *auto*)
qed

9.8.16 Transformation of measure by linear maps

lemma *emeasure_lebesgue_ball_conv_unit_ball*:
fixes *c* :: *'a* :: *euclidean_space*
assumes *r* $\geq$ *0*
shows *emeasure lebesgue* (*ball c r*) =
ennreal (*r* ^ *DIM*(*'a*)) $*$ *emeasure lebesgue* (*ball* (*0* :: *'a*) *1*)
proof (*cases r* = *0*)
case *False*
with *assms* **have** *r*: *r* > *0* **by** *auto*
have *emeasure lebesgue* ((λ*x*. *c* + *x*) ‘ (λ*x*. *r* $*_R$ *x*) ‘ *ball* (*0* :: *'a*) *1*) =
r ^ *DIM*(*'a*) $*$ *emeasure lebesgue* (*ball* (*0* :: *'a*) *1*)
unfolding *image_image* **using** *emeasure_lebesgue_affine*[*of r c ball 0 1*] *assms*
by (*simp add*: *add_ac*)
also have (λ*x*. *r* $*_R$ *x*) ‘ *ball 0 1* = *ball* (*0* :: *'a*) *r*
using *r* **by** (*subst ball_scale*) *auto*
also have (λ*x*. *c* + *x*) ‘ ... = *ball c r*
by (*subst image_add_ball*) (*simp_all add*: *algebra_simps*)
finally show *?thesis* **by** *simp*
qed *auto*

lemma *content_ball_conv_unit_ball*:
fixes $c :: 'a :: euclidean\_space$
assumes $r \geq 0$
shows $content\ (ball\ c\ r) = r\ \widehat{}\ DIM('a) * content\ (ball\ (0 :: 'a)\ 1)$
proof −
have $ennreal\ (content\ (ball\ c\ r)) = emeasure\ lebesgue\ (ball\ c\ r)$
using *emeasure_lborel_ball_finite*[*of c r*] **by** (*subst emeasure_eq_ennreal_measure*) *auto*
also have $\ldots = ennreal\ (r\ \widehat{}\ DIM('a)) * emeasure\ lebesgue\ (ball\ (0 :: 'a)\ 1)$
using *assms* **by** (*intro emeasure_lebesgue_ball_conv_unit_ball*) *auto*
also have $\ldots = ennreal\ (r\ \widehat{}\ DIM('a) * content\ (ball\ (0::'a)\ 1))$
using *emeasure_lborel_ball_finite*[*of 0::'a 1*] *assms*
by (*subst emeasure_eq_ennreal_measure*) (*auto simp*: *ennreal_mult'*)
finally show *?thesis*
using *assms* **by** (*subst* (*asm*) *ennreal_inj*) *auto*
qed

lemma *measurable_linear_image_interval*:
$linear\ f \Longrightarrow f\ `\ (cbox\ a\ b) \in lmeasurable$
by (*metis bounded_linear_image linear_linear bounded_cbox closure_bounded_linear_image closure_cbox compact_closure lmeasurable_compact*)

proposition *measure_linear_sufficient*:
fixes $f :: 'n::euclidean\_space \Rightarrow 'n$
assumes *linear f* **and** S: $S \in lmeasurable$
and *im*: $\bigwedge a\ b.\ measure\ lebesgue\ (f\ `\ (cbox\ a\ b)) = m * measure\ lebesgue\ (cbox\ a\ b)$
shows $f\ `\ S \in lmeasurable \wedge m * measure\ lebesgue\ S = measure\ lebesgue\ (f\ `\ S)$
using *le_less_linear* [*of 0 m*]
proof
assume $m < 0$
then show *?thesis*
using *im* [*of 0 One*] **by** *auto*
next
assume $m \geq 0$
let $?\mu = measure\ lebesgue$
show *?thesis*
proof (*cases inj f*)
case *False*
then have $?\mu\ (f\ `\ S) = 0$
using ‹*linear f*› *negligible_imp_measure0 negligible_linear_singular_image* **by** *blast*
then have $m * ?\mu\ (cbox\ 0\ (One)) = 0$
by (*metis False* ‹*linear f*› *cbox_borel content_unit im measure_completion negligible_imp_measure0 negligible_linear_singular_image sets_lborel*)
then show *?thesis*
using ‹*linear f*› *negligible_linear_singular_image negligible_imp_measure0 False*

```
    by (auto simp: lmeasurable_iff_has_integral negligible_UNIV)
  next
    case True
    then obtain h where linear h and hf: ⋀x. h (f x) = x and fh: ⋀x. f (h x)
= x
      using ‹linear f› linear_injective_isomorphism by blast
    have fBS: (f ‘ S) ∈ lmeasurable ∧ m * ?μ S = ?μ (f ‘ S)
      if bounded S S ∈ lmeasurable for S
    proof −
      obtain a b where S ⊆ cbox a b
        using ‹bounded S› bounded_subset_cbox_symmetric by metis
      have fUD: (f ‘ ⋃𝒟) ∈ lmeasurable ∧ ?μ (f ‘ ⋃𝒟) = (m * ?μ (⋃𝒟))
        if countable 𝒟
          and cbox: ⋀K. K ∈ 𝒟 ⟹ K ⊆ cbox a b ∧ K ≠ {} ∧ (∃ c d. K = cbox c
d)
          and intint: pairwise (λA B. interior A ∩ interior B = {}) 𝒟
        for 𝒟
      proof −
        have conv: ⋀K. K ∈ 𝒟 ⟹ convex K
          using cbox convex_box(1) by blast
        have neg: negligible (g ‘ K ∩ g ‘ L) if linear g K ∈ 𝒟 L ∈ 𝒟 K ≠ L
          for K L and g :: 'n⇒'n
        proof (cases inj g)
          case True
          have negligible (frontier(g ‘ K ∩ g ‘ L) ∪ interior(g ‘ K ∩ g ‘ L))
          proof (rule negligible_Un)
            show negligible (frontier (g ‘ K ∩ g ‘ L))
                  by (simp add: negligible_convex_frontier convex_Int conv con-
vex_linear_image that)
          next
            have ∀ p N. pairwise p N = (∀ Na. (Na::'n set) ∈ N ⟶ (∀ Nb. Nb ∈ N
∧ Na ≠ Nb ⟶ p Na Nb))
              by (metis pairwise_def)
            then have interior K ∩ interior L = {}
              using intint that(2) that(3) that(4) by presburger
            then show negligible (interior (g ‘ K ∩ g ‘ L))
              by (metis True empty_imp_negligible image_Int image_empty inte-
rior_Int interior_injective_linear_image that(1))
          qed
          moreover have g ‘ K ∩ g ‘ L ⊆ frontier (g ‘ K ∩ g ‘ L) ∪ interior (g ‘
K ∩ g ‘ L)
            by (metis Diff_partition Int_commute calculation closure_Un_frontier
frontier_def inf.absorb_iff2 inf_bot_right inf_sup_absorb negligible_Un_eq open_interior
open_not_negligible sup_commute)
          ultimately show ?thesis
            by (rule negligible_subset)
        next
          case False
          then show ?thesis
```

```
      by (simp add: negligible_Int negligible_linear_singular_image ‹linear g›)
    qed
    have negf: negligible ((f ‘ K) ∩ (f ‘ L))
    and negid: negligible (K ∩ L) if K ∈ 𝒟 L ∈ 𝒟 K ≠ L for K L
      using neg [OF ‹linear f›] neg [OF linear_id] that by auto
    show ?thesis
    proof (cases finite 𝒟)
      case True
      then have ?μ (⋃x∈𝒟. f ‘ x) = (∑x∈𝒟. ?μ (f ‘ x))
        using ‹linear f› cbox measurable_linear_image_interval negf
        by (blast intro: measure_negligible_finite_Union_image [unfolded pairwise_def])
      also have ... = (∑k∈𝒟. m * ?μ k)
        by (metis (no_types, lifting) cbox im sum.cong)
      also have ... = m * ?μ (⋃𝒟)
        unfolding sum_distrib_left [symmetric]
        by (metis True cbox lmeasurable_cbox measure_negligible_finite_Union [unfolded pairwise_def] negid)
      finally show ?thesis
          by (metis True ‹linear f› cbox image_Union fmeasurable.finite_UN measurable_linear_image_interval)
    next
      case False
     with ‹countable 𝒟› obtain X :: nat ⇒ 'n set where S: bij_betw X UNIV 𝒟
        using bij_betw_from_nat_into by blast
      then have eq: (⋃𝒟) = (⋃n. X n) (f ‘ ⋃𝒟) = (⋃n. f ‘ X n)
        by (auto simp: bij_betw_def)
      have meas: ⋀K. K ∈ 𝒟 ⟹ K ∈ lmeasurable
        using cbox by blast
      with S have 1: ⋀n. X n ∈ lmeasurable
        by (auto simp: bij_betw_def)
      have 2: pairwise (λm n. negligible (X m ∩ X n)) UNIV
          using S unfolding bij_betw_def pairwise_def by (metis injD negid range_eqI)
      have bounded (⋃𝒟)
        by (meson Sup_least bounded_cbox bounded_subset cbox)
      then have 3: bounded (⋃n. X n)
        using S unfolding bij_betw_def by blast
      have (⋃n. X n) ∈ lmeasurable
        by (rule measurable_countable_negligible_Union_bounded [OF 1 2 3])
      with S have f1: ⋀n. f ‘ (X n) ∈ lmeasurable
     unfolding bij_betw_def by (metis assms(1) cbox measurable_linear_image_interval rangeI)
      have f2: pairwise (λm n. negligible (f ‘ (X m) ∩ f ‘ (X n))) UNIV
          using S unfolding bij_betw_def pairwise_def by (metis injD negf rangeI)
      have bounded (⋃𝒟)
        by (meson Sup_least bounded_cbox bounded_subset cbox)
```

```
        then have f3: bounded (⋃n. f ‘ X n)
          using S unfolding bij_betw_def
          by (metis bounded_linear_image linear_linear assms(1) image_Union
range_composition)
        have (λn. ?μ (X n)) sums ?μ (⋃n. X n)
          by (rule measure_countable_negligible_Union_bounded [OF 1 2 3])
        have meq: ?μ (⋃n. f ‘ X n) = m * ?μ (⋃(X ‘ UNIV))
      proof (rule sums_unique2 [OF measure_countable_negligible_Union_bounded
[OF f1 f2 f3]])
          have m: ⋀n. ?μ (f ‘ X n) = (m * ?μ (X n))
            using S unfolding bij_betw_def by (metis cbox im rangeI)
          show (λn. ?μ (f ‘ X n)) sums (m * ?μ (⋃(X ‘ UNIV)))
            unfolding m
              using measure_countable_negligible_Union_bounded [OF 1 2 3]
sums_mult by blast
        qed
        show ?thesis
          using measurable_countable_negligible_Union_bounded [OF f1 f2 f3]
meq
          by (auto simp: eq [symmetric])
      qed
    qed
    show ?thesis
      unfolding completion.fmeasurable_measure_inner_outer_le
    proof (intro conjI allI impI)
      fix e :: real
      assume e > 0
      have 1: cbox a b − S ∈ lmeasurable
        by (simp add: fmeasurable.Diff that)
      have 2: 0 < e / (1 + |m|)
        using ‹e > 0› by (simp add: field_split_simps abs_add_one_gt_zero)
      obtain 𝒟
        where countable 𝒟
          and cbox: ⋀K. K ∈ 𝒟 ⟹ K ⊆ cbox a b ∧ K ≠ {} ∧ (∃c d. K = cbox
c d)
          and intdisj: pairwise (λA B. interior A ∩ interior B = {}) 𝒟
          and DD: cbox a b − S ⊆ ⋃𝒟 ⋃𝒟 ∈ lmeasurable
          and le: ?μ (⋃𝒟) ≤ ?μ (cbox a b − S) + e/(1 + |m|)
        by (rule measurable_outer_intervals_bounded [of cbox a b − S a b e/(1
+ |m|)]; use 1 2 pairwise_def in force)
      show ∃T ∈ lmeasurable. T ⊆ f ‘ S ∧ m * ?μ S − e ≤ ?μ T
      proof (intro bexI conjI)
        show f ‘ (cbox a b) − f ‘ (⋃𝒟) ⊆ f ‘ S
          using ‹cbox a b − S ⊆ ⋃𝒟› by force
        have m * ?μ S − e ≤ m * (?μ S − e / (1 + |m|))
          using ‹m ≥ 0› ‹e > 0› by (simp add: field_simps)
        also have ... ≤ ?μ (f ‘ cbox a b) − ?μ (f ‘ (⋃𝒟))
        proof −
          have ?μ (cbox a b − S) = ?μ (cbox a b) − ?μ S
```

```
          by (simp add: measurable_measure_Diff ‹S ⊆ cbox a b› fmeasurableD
that(2))
        then have (?μ S − e / (1 + m)) ≤ (content (cbox a b) − ?μ (⋃ 𝒟))
          using ‹m ≥ 0› le by auto
        then show ?thesis
          using ‹m ≥ 0› ‹e > 0›
         by (simp add: mult_left_mono im fUD [OF ‹countable 𝒟› cbox intdisj]
flip: right_diff_distrib)
      qed
      also have ... = ?μ (f ‘ cbox a b − f ‘ ⋃𝒟)
      proof (rule measurable_measure_Diff [symmetric])
        show f ‘ cbox a b ∈ lmeasurable
          by (simp add: assms(1) measurable_linear_image_interval)
        show f ‘ ⋃ 𝒟 ∈ sets lebesgue
          by (simp add: ‹countable 𝒟› cbox fUD fmeasurableD intdisj)
        show f ‘ ⋃ 𝒟 ⊆ f ‘ cbox a b
          by (simp add: Sup_le_iff cbox image_mono)
      qed
      finally show m * ?μ S − e ≤ ?μ (f ‘ cbox a b − f ‘ ⋃𝒟) .
      show f ‘ cbox a b − f ‘ ⋃𝒟 ∈ lmeasurable
       by (simp add: fUD ‹countable 𝒟› ‹linear f› cbox fmeasurable.Diff intdisj
measurable_linear_image_interval)
    qed
  next
    fix e :: real
    assume e > 0
    have em: 0 < e / (1 + |m|)
      using ‹e > 0› by (simp add: field_split_simps abs_add_one_gt_zero)
    obtain 𝒟
      where countable 𝒟
        and cbox: ⋀K. K ∈ 𝒟 ⟹ K ⊆ cbox a b ∧ K ≠ {} ∧ (∃ c d. K = cbox
c d)
        and intdisj: pairwise (λA B. interior A ∩ interior B = {}) 𝒟
        and DD: S ⊆ ⋃𝒟 ⋃𝒟 ∈ lmeasurable
        and le: ?μ (⋃𝒟) ≤ ?μ S + e/(1 + |m|)
      by (rule measurable_outer_intervals_bounded [of S a b e/(1 + |m|)]; use
‹S ∈ lmeasurable› ‹S ⊆ cbox a b› em in force)
    show ∃ U ∈ lmeasurable. f ‘ S ⊆ U ∧ ?μ U ≤ m * ?μ S + e
    proof (intro bexI conjI)
      show f ‘ S ⊆ f ‘ (⋃𝒟)
        by (simp add: DD(1) image_mono)
      have ?μ (f ‘ ⋃𝒟) ≤ m * (?μ S + e / (1 + |m|))
        using ‹m ≥ 0› le mult_left_mono
       by (auto simp: fUD ‹countable 𝒟› ‹linear f› cbox fmeasurable.Diff intdisj
measurable_linear_image_interval)
      also have ... ≤ m * ?μ S + e
         using ‹m ≥ 0› ‹e > 0› by (simp add: fUD [OF ‹countable 𝒟› cbox
intdisj] field_simps)
      finally show ?μ (f ‘ ⋃𝒟) ≤ m * ?μ S + e .
```

```
        show f ‘ ⋃ 𝒟 ∈ lmeasurable
          by (simp add: ‹countable 𝒟› cbox fUD intdisj)
      qed
    qed
  qed
  show ?thesis
    unfolding has_measure_limit_iff
  proof (intro allI impI)
    fix e :: real
    assume e > 0
    obtain B where B > 0 and B:
      ⋀a b. ball 0 B ⊆ cbox a b ⟹ |?μ (S ∩ cbox a b) − ?μ S| < e / (1 + |m|)
     using has_measure_limit [OF S] ‹e > 0› by (metis abs_add_one_gt_zero
zero_less_divide_iff)
    obtain c d::'n where cd: ball 0 B ⊆ cbox c d
      by (metis bounded_subset_cbox_symmetric bounded_ball)
    with B have less: |?μ (S ∩ cbox c d) − ?μ S| < e / (1 + |m|) .
    obtain D where D > 0 and D: cbox c d ⊆ ball 0 D
      by (metis bounded_cbox bounded_subset_ballD)
    obtain C where C > 0 and C: ⋀x. norm (f x) ≤ C * norm x
      using linear_bounded_pos ‹linear f› by blast
    have f ‘ S ∩ cbox a b ∈ lmeasurable ∧
          |?μ (f ‘ S ∩ cbox a b) − m * ?μ S| < e
      if ball 0 (D*C) ⊆ cbox a b for a b
    proof −
      have bounded (S ∩ h ‘ cbox a b)
      by (simp add: bounded_linear_image linear_linear ‹linear h› bounded_Int)
      moreover have Shab: S ∩ h ‘ cbox a b ∈ lmeasurable
     by (simp add: S ‹linear h› fmeasurable.Int measurable_linear_image_interval)
      moreover have fim: f ‘ (S ∩ h ‘ (cbox a b)) = (f ‘ S) ∩ cbox a b
        by (auto simp: hf rev_image_eqI fh)
      ultimately have 1: (f ‘ S) ∩ cbox a b ∈ lmeasurable
            and 2: ?μ ((f ‘ S) ∩ cbox a b) = m * ?μ (S ∩ h ‘ cbox a b)
        using fBS [of S ∩ (h ‘ (cbox a b))] by auto
      have ∗: ⟦|z − m| < e; z ≤ w; w ≤ m⟧ ⟹ |w − m| ≤ e
        for w z m and e::real by auto
      have meas_adiff: |?μ (S ∩ h ‘ cbox a b) − ?μ S| ≤ e / (1 + |m|)
      proof (rule ∗ [OF less])
        show ?μ (S ∩ cbox c d) ≤ ?μ (S ∩ h ‘ cbox a b)
        proof (rule measure_mono_fmeasurable [OF _ _ Shab])
          have f ‘ ball 0 D ⊆ ball 0 (C * D)
            using C ‹C > 0›
            apply (clarsimp simp: algebra_simps)
          by (meson le_less_trans linordered_comm_semiring_strict_class.comm_mult_strict_left_mono)
          then have f ‘ ball 0 D ⊆ cbox a b
            by (metis mult.commute order_trans that)
          have ball 0 D ⊆ h ‘ cbox a b
            by (metis ‹f ‘ ball 0 D ⊆ cbox a b› hf image_subset_iff subsetI)
          then show S ∩ cbox c d ⊆ S ∩ h ‘ cbox a b
```

```
          using D by blast
        next
          show S ∩ cbox c d ∈ sets lebesgue
            using S fmeasurable_cbox by blast
        qed
      next
        show ?μ (S ∩ h ‘ cbox a b) ≤ ?μ S
          by (simp add: S Shab fmeasurableD measure_mono_fmeasurable)
      qed
      have |?μ (f ‘ S ∩ cbox a b) − m ∗ ?μ S| ≤ |?μ S − ?μ (S ∩ h ‘ cbox a b)|
∗ m
        by (metis 2 ‹m ≥ 0› abs_minus_commute abs_mult_pos mult.commute
order_refl right_diff_distrib′)
      also have ... ≤ e / (1 + m) ∗ m
        by (metis ‹m ≥ 0› abs_minus_commute abs_of_nonneg meas_adiff
mult.commute mult_left_mono)
      also have ... < e
        using ‹e > 0› ‹m ≥ 0› by (simp add: field_simps)
      finally have |?μ (f ‘ S ∩ cbox a b) − m ∗ ?μ S| < e .
      with 1 show ?thesis by auto
    qed
    then show ∃B>0. ∀a b. ball 0 B ⊆ cbox a b ⟶
                          f ‘ S ∩ cbox a b ∈ lmeasurable ∧
                          |?μ (f ‘ S ∩ cbox a b) − m ∗ ?μ S| < e
    using ‹C>0› ‹D>0› by (metis mult_zero_left mult_less_cancel_right_pos)
  qed
 qed
qed
```

9.8.17 Lemmas about absolute integrability

```
lemma absolutely_integrable_linear:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
    and h :: 'n::euclidean_space ⇒ 'p::euclidean_space
   shows f absolutely_integrable_on s ⟹ bounded_linear h ⟹ (h ∘ f) abso-
lutely_integrable_on s
  using integrable_bounded_linear[of h lebesgue λx. indicator s x ∗R f x]
  by (simp add: linear_simps[of h] set_integrable_def)

lemma absolutely_integrable_sum:
  fixes f :: 'a ⇒ 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes finite T and ⋀a. a ∈ T ⟹ (f a) absolutely_integrable_on S
  shows (λx. sum (λa. f a x) T) absolutely_integrable_on S
  using assms by induction auto

lemma absolutely_integrable_integrable_bound:
  fixes f :: 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes le: ⋀x. x∈S ⟹ norm (f x) ≤ g x and f: f integrable_on S and g: g
integrable_on S
```

shows *f absolutely_integrable_on S*
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound*)
have *g absolutely_integrable_on S*
unfolding *absolutely_integrable_on_def*
proof
show (λ*x*. *norm* (*g x*)) *integrable_on S*
using *le norm_ge_zero*[*of f _*]
by (*intro integrable_spike_finite*[*OF _ _ g*, *of* {}])
(*auto intro*!: *abs_of_nonneg intro*: *order_trans simp del*: *norm_ge_zero*)
qed *fact*
then show *integrable lebesgue* (λ*x*. *indicat_real S x* $*_R$ *g x*)
by (*simp add*: *set_integrable_def*)
show (λ*x*. *indicat_real S x* $*_R$ *f x*) ∈ *borel_measurable lebesgue*
using *f* **by** (*auto intro*: *has_integral_implies_lebesgue_measurable simp*: *integrable_on_def*)
qed (*use le* **in** ‹*force intro*!: *always_eventually split*: *split_indicator*›)

corollary *absolutely_integrable_on_const* [*simp*]:
fixes *c* :: *'a*::*euclidean_space*
assumes *S* ∈ *lmeasurable*
shows (λ*x*. *c*) *absolutely_integrable_on S*
by (*metis* (*full_types*) *assms absolutely_integrable_integrable_bound integrable_on_const order_refl*)

lemma *absolutely_integrable_continuous*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
shows *continuous_on* (*cbox a b*) *f* ⟹ *f absolutely_integrable_on cbox a b*
using *absolutely_integrable_integrable_bound*
by (*simp add*: *absolutely_integrable_on_def continuous_on_norm integrable_continuous*)

lemma *absolutely_integrable_continuous_real*:
fixes *f* :: *real* ⇒ *'b*::*euclidean_space*
shows *continuous_on* {*a*..*b*} *f* ⟹ *f absolutely_integrable_on* {*a*..*b*}
by (*metis absolutely_integrable_continuous box_real*(*2*))

lemma *continuous_imp_integrable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *continuous_on* (*cbox a b*) *f*
shows *integrable* (*lebesgue_on* (*cbox a b*)) *f*
proof −
have *f absolutely_integrable_on cbox a b*
by (*simp add*: *absolutely_integrable_continuous assms*)
then show *?thesis*
by (*simp add*: *integrable_restrict_space set_integrable_def*)
qed

lemma *continuous_imp_integrable_real*:
fixes *f* :: *real* ⇒ *'b*::*euclidean_space*

assumes *continuous_on* $\{a..b\}$ f
shows *integrable* (*lebesgue_on* $\{a..b\}$) f
by (*metis assms continuous_imp_integrable interval_cbox*)

9.8.18 Componentwise

proposition *absolutely_integrable_componentwise_iff*:
shows f *absolutely_integrable_on* $A \longleftrightarrow (\forall\, b{\in}Basis.\ (\lambda x.\ f\, x \cdot b)$ *absolutely_integrable_on* $A)$
proof −
have $*$: $(\lambda x.\ norm\ (f\, x))$ *integrable_on* $A \longleftrightarrow (\forall\, b{\in}Basis.\ (\lambda x.\ norm\ (f\, x \cdot b))$ *integrable_on* $A)$ (**is** *?lhs* = *?rhs*)
if f *integrable_on* A
proof
assume *?lhs*
then show *?rhs*
by (*metis absolutely_integrable_on_def Topology_Euclidean_Space.norm_nth_le absolutely_integrable_integrable_bound integrable_component that*)
next
assume *R*: *?rhs*
have f *absolutely_integrable_on* A
proof (*rule absolutely_integrable_integrable_bound*)
show $(\lambda x.\ \sum i{\in}Basis.\ norm\ (f\, x \cdot i))$ *integrable_on* A
using *R* **by** (*force intro*: *integrable_sum*)
qed (*use that norm_le_l1* **in** *auto*)
then show *?lhs*
using *absolutely_integrable_on_def* **by** *auto*
qed
show *?thesis*
unfolding *absolutely_integrable_on_def*
by (*simp add*: *integrable_componentwise_iff* [*symmetric*] *ball_conj_distrib* $*$ *cong*: *conj_cong*)
qed

lemma *absolutely_integrable_componentwise*:
shows $(\bigwedge b.\ b \in Basis \Longrightarrow (\lambda x.\ f\, x \cdot b)$ *absolutely_integrable_on* $A) \Longrightarrow f$ *absolutely_integrable_on* A
using *absolutely_integrable_componentwise_iff* **by** *blast*

lemma *absolutely_integrable_component*:
f *absolutely_integrable_on* $A \Longrightarrow (\lambda x.\ f\, x \cdot (b :: 'b :: euclidean\_space))$ *absolutely_integrable_on* A
by (*drule absolutely_integrable_linear*[*OF* _ *bounded_linear_inner_left*[*of b*]]) (*simp add*: *o_def*)

lemma *absolutely_integrable_scaleR_left*:
fixes $f :: 'n{::}euclidean\_space \Rightarrow 'm{::}euclidean\_space$
assumes f *absolutely_integrable_on* S

```
  shows (λx. c *R f x) absolutely_integrable_on S
proof −
  have (λx. c *R x) o f absolutely_integrable_on S
   by (simp add: absolutely_integrable_linear assms bounded_linear_scaleR_right)
  then show ?thesis
    using assms by blast
qed

lemma absolutely_integrable_scaleR_right:
  assumes f absolutely_integrable_on S
  shows (λx. f x *R c) absolutely_integrable_on S
  using assms by blast

lemma absolutely_integrable_norm:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f absolutely_integrable_on S
  shows (norm o f) absolutely_integrable_on S
  using assms by (simp add: absolutely_integrable_on_def o_def)

lemma absolutely_integrable_abs:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f absolutely_integrable_on S
  shows (λx. ∑ i∈Basis. |f x • i| *R i) absolutely_integrable_on S
        (is ?g absolutely_integrable_on S)
proof −
  have *: (λy. ∑ j∈Basis. if j = i then y *R j else 0) ∘
          (λx. norm (∑ j∈Basis. if j = i then (x • i) *R j else 0)) ∘ f
          absolutely_integrable_on S
        if i ∈ Basis for i
  proof −
    have bounded_linear (λy. ∑ j∈Basis. if j = i then y *R j else 0)
      by (simp add: linear_linear algebra_simps linearI)
    moreover have (λx. norm (∑ j∈Basis. if j = i then (x • i) *R j else 0)) ∘ f
                  absolutely_integrable_on S
      using assms ‹i ∈ Basis›
      unfolding o_def
      by (intro absolutely_integrable_norm [unfolded o_def])
         (auto simp: algebra_simps dest: absolutely_integrable_component)
    ultimately show ?thesis
      by (subst comp_assoc) (blast intro: absolutely_integrable_linear)
  qed
  have eq: ?g =
        (λx. ∑ i∈Basis. ((λy. ∑ j∈Basis. if j = i then y *R j else 0) ∘
              (λx. norm(∑ j∈Basis. if j = i then (x • i) *R j else 0)) ∘ f) x)
    by (simp)
  show ?thesis
    unfolding eq
    by (rule absolutely_integrable_sum) (force simp: intro!: *)+
qed
```

lemma *abs_absolutely_integrableI_1*:
fixes $f :: 'a :: euclidean\_space \Rightarrow real$
assumes f: f *integrable_on* A **and** $(\lambda x.\ |f\ x|)$ *integrable_on* A
shows f *absolutely_integrable_on* A
by (*rule absolutely_integrable_integrable_bound* [*OF* _ *assms*]) *auto*

lemma *abs_absolutely_integrableI*:
assumes f: f *integrable_on* S **and** *fcomp*: $(\lambda x.\ \sum i{\in}Basis.\ |f\ x \cdot i| *_R i)$ *integrable_on* S
shows f *absolutely_integrable_on* S
proof −
have $(\lambda x.\ (f\ x \cdot i) *_R i)$ *absolutely_integrable_on* S **if** $i \in Basis$ **for** i
proof −
have $(\lambda x.\ |f\ x \cdot i|)$ *integrable_on* S
using *assms integrable_component* [*OF fcomp*, **where** $y{=}i$] *that* **by** *simp*
then have $(\lambda x.\ f\ x \cdot i)$ *absolutely_integrable_on* S
using *abs_absolutely_integrableI_1 f integrable_component* **by** *blast*
then show *?thesis*
by (*rule absolutely_integrable_scaleR_right*)
qed
then have $(\lambda x.\ \sum i{\in}Basis.\ (f\ x \cdot i) *_R i)$ *absolutely_integrable_on* S
by (*simp add*: *absolutely_integrable_sum*)
then show *?thesis*
by (*simp add*: *euclidean_representation*)
qed

lemma *absolutely_integrable_abs_iff*:
f *absolutely_integrable_on* $S \longleftrightarrow$
f *integrable_on* $S \wedge (\lambda x.\ \sum i{\in}Basis.\ |f\ x \cdot i| *_R i)$ *integrable_on* S
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **then show** *?rhs*
using *absolutely_integrable_abs absolutely_integrable_on_def* **by** *blast*
next
assume *?rhs*
moreover
have $(\lambda x.\ if\ x \in S\ then\ \sum i{\in}Basis.\ |f\ x \cdot i| *_R i\ else\ 0) = (\lambda x.\ \sum i{\in}Basis.\ |(if\ x \in S\ then\ f\ x\ else\ 0) \cdot i| *_R i)$
by *force*
ultimately show *?lhs*
by (*simp only*: *absolutely_integrable_restrict_UNIV* [*of* S, *symmetric*] *integrable_restrict_UNIV* [*of* S, *symmetric*] *abs_absolutely_integrableI*)
qed

lemma *absolutely_integrable_max*:
fixes $f :: 'n{::}euclidean\_space \Rightarrow 'm{::}euclidean\_space$

assumes *f absolutely_integrable_on S g absolutely_integrable_on S*
shows $(\lambda x.\ \sum i{\in}Basis.\ max\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i)$
absolutely_integrable_on S
proof −
have $(\lambda x.\ \sum i{\in}Basis.\ max\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i) =$
$(\lambda x.\ (1/2) *_R (f\ x + g\ x + (\sum i{\in}Basis.\ |f\ x \cdot i - g\ x \cdot i| *_R i)))$
proof (*rule ext*)
fix *x*
have $(\sum i{\in}Basis.\ max\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i) = (\sum i{\in}Basis.\ ((f\ x \cdot i + g\ x \cdot i + |f\ x \cdot i - g\ x \cdot i|)\ /\ 2) *_R i)$
by (*force intro*: *sum.cong*)
also have $... = (1\ /\ 2) *_R (\sum i{\in}Basis.\ (f\ x \cdot i + g\ x \cdot i + |f\ x \cdot i - g\ x \cdot i|) *_R i)$
by (*simp add*: *scaleR_right.sum*)
also have $... = (1\ /\ 2) *_R (f\ x + g\ x + (\sum i{\in}Basis.\ |f\ x \cdot i - g\ x \cdot i| *_R i))$
by (*simp add*: *sum.distrib algebra_simps euclidean_representation*)
finally
show $(\sum i{\in}Basis.\ max\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i) =$
$(1\ /\ 2) *_R (f\ x + g\ x + (\sum i{\in}Basis.\ |f\ x \cdot i - g\ x \cdot i| *_R i))$.
qed
moreover have $(\lambda x.\ (1\ /\ 2) *_R (f\ x + g\ x + (\sum i{\in}Basis.\ |f\ x \cdot i - g\ x \cdot i| *_R i)))$
absolutely_integrable_on S
using *absolutely_integrable_abs* [*OF set_integral_diff*(*1*) [*OF assms*]]
by (*intro set_integral_add absolutely_integrable_scaleR_left assms*) (*simp add*: *algebra_simps*)
ultimately show *?thesis* **by** *metis*
qed

corollary *absolutely_integrable_max_1*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *real*
assumes *f absolutely_integrable_on S g absolutely_integrable_on S*
shows $(\lambda x.\ max\ (f\ x)\ (g\ x))$ *absolutely_integrable_on S*
using *absolutely_integrable_max* [*OF assms*] **by** *simp*

lemma *absolutely_integrable_min*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'m*::*euclidean_space*
assumes *f absolutely_integrable_on S g absolutely_integrable_on S*
shows $(\lambda x.\ \sum i{\in}Basis.\ min\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i)$
absolutely_integrable_on S
proof −
have $(\lambda x.\ \sum i{\in}Basis.\ min\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i) =$
$(\lambda x.\ (1/2) *_R (f\ x + g\ x - (\sum i{\in}Basis.\ |f\ x \cdot i - g\ x \cdot i| *_R i)))$
proof (*rule ext*)
fix *x*
have $(\sum i{\in}Basis.\ min\ (f\ x \cdot i)\ (g\ x \cdot i) *_R i) = (\sum i{\in}Basis.\ ((f\ x \cdot i + g\ x \cdot i - |f\ x \cdot i - g\ x \cdot i|)\ /\ 2) *_R i)$
by (*force intro*: *sum.cong*)
also have $... = (1\ /\ 2) *_R (\sum i{\in}Basis.\ (f\ x \cdot i + g\ x \cdot i - |f\ x \cdot i - g\ x \cdot i|)$

```
*R i)
      by (simp add: scaleR_right.sum)
    also have ... = (1 / 2) *R (f x + g x - (∑ i∈Basis. |f x • i - g x • i| *R i))
    by (simp add: sum.distrib sum_subtractf algebra_simps euclidean_representation)
    finally
    show (∑ i∈Basis. min (f x • i) (g x • i) *R i) =
         (1 / 2) *R (f x + g x - (∑ i∈Basis. |f x • i - g x • i| *R i)) .
  qed
  moreover have (λx. (1 / 2) *R (f x + g x - (∑ i∈Basis. |f x • i - g x • i| *R
i)))
                absolutely_integrable_on S
    using absolutely_integrable_abs [OF set_integral_diff(1) [OF assms]]
    by (intro set_integral_add set_integral_diff absolutely_integrable_scaleR_left
assms)
       (simp add: algebra_simps)
  ultimately show ?thesis by metis
qed

corollary absolutely_integrable_min_1:
  fixes f :: 'n::euclidean_space ⇒ real
  assumes f absolutely_integrable_on S g absolutely_integrable_on S
   shows (λx. min (f x) (g x)) absolutely_integrable_on S
  using absolutely_integrable_min [OF assms] by simp

lemma nonnegative_absolutely_integrable:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f integrable_on A and comp: ⋀x b. ⟦x ∈ A; b ∈ Basis⟧ ⟹ 0 ≤ f x • b
  shows f absolutely_integrable_on A
proof -
  have (λx. (f x • i) *R i) absolutely_integrable_on A if i ∈ Basis for i
  proof -
    have (λx. f x • i) integrable_on A
      by (simp add: assms(1) integrable_component)
    then have (λx. f x • i) absolutely_integrable_on A
      by (metis that comp nonnegative_absolutely_integrable_1)
    then show ?thesis
      by (rule absolutely_integrable_scaleR_right)
  qed
  then have (λx. ∑ i∈Basis. (f x • i) *R i) absolutely_integrable_on A
    by (simp add: absolutely_integrable_sum)
  then show ?thesis
    by (simp add: euclidean_representation)
qed

lemma absolutely_integrable_component_ubound:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes f: f integrable_on A and g: g absolutely_integrable_on A
      and comp: ⋀x b. ⟦x ∈ A; b ∈ Basis⟧ ⟹ f x • b ≤ g x • b
  shows f absolutely_integrable_on A
```

proof −
have $(\lambda x.\ g\ x - (g\ x - f\ x))$ *absolutely_integrable_on A*
proof (*rule set_integral_diff* [*OF g nonnegative_absolutely_integrable*])
show $(\lambda x.\ g\ x - f\ x)$ *integrable_on A*
using *Henstock_Kurzweil_Integration.integrable_diff absolutely_integrable_on_def*
f g **by** *blast*
qed (*simp add*: *comp inner_diff_left*)
then show *?thesis*
by *simp*
qed

lemma *absolutely_integrable_component_lbound*:
fixes *f* :: *'a* :: *euclidean_space* ⇒ *'b* :: *euclidean_space*
assumes *f*: *f absolutely_integrable_on A* **and** *g*: *g integrable_on A*
and *comp*: $\bigwedge x\ b.\ [\![x \in A;\ b \in Basis]\!] \Longrightarrow f\ x \cdot b \leq g\ x \cdot b$
shows *g absolutely_integrable_on A*
proof −
have $(\lambda x.\ f\ x + (g\ x - f\ x))$ *absolutely_integrable_on A*
proof (*rule set_integral_add* [*OF f nonnegative_absolutely_integrable*])
show $(\lambda x.\ g\ x - f\ x)$ *integrable_on A*
using *Henstock_Kurzweil_Integration.integrable_diff absolutely_integrable_on_def*
f g **by** *blast*
qed (*simp add*: *comp inner_diff_left*)
then show *?thesis*
by *simp*
qed

lemma *integrable_on_1_iff*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real^1*
shows *f integrable_on S* ⟷ $(\lambda x.\ f\ x\ \$\ 1)$ *integrable_on S*
by (*auto simp*: *integrable_componentwise_iff* [*of f*] *Basis_vec_def cart_eq_inner_axis*)

lemma *integral_on_1_eq*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real^1*
shows *integral S f* = *vec* (*integral S* $(\lambda x.\ f\ x\ \$\ 1)$)
by (*cases f integrable_on S*) (*simp_all add*: *integrable_on_1_iff vec_eq_iff not_integrable_integral*)

lemma *absolutely_integrable_on_1_iff*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real^1*
shows *f absolutely_integrable_on S* ⟷ $(\lambda x.\ f\ x\ \$\ 1)$ *absolutely_integrable_on S*
unfolding *absolutely_integrable_on_def*
by (*auto simp*: *integrable_on_1_iff norm_real*)

lemma *absolutely_integrable_absolutely_integrable_lbound*:
fixes *f* :: *'m*::*euclidean_space* ⇒ *real*
assumes *f*: *f integrable_on S* **and** *g*: *g absolutely_integrable_on S*
and ∗: $\bigwedge x.\ x \in S \Longrightarrow g\ x \leq f\ x$
shows *f absolutely_integrable_on S*

```
  by (rule absolutely_integrable_component_lbound [OF g f]) (simp add: *)

lemma absolutely_integrable_absolutely_integrable_ubound:
  fixes f :: 'm::euclidean_space ⇒ real
  assumes fg: f integrable_on S g absolutely_integrable_on S
    and *: ⋀x. x ∈ S ⟹ f x ≤ g x
  shows f absolutely_integrable_on S
  by (rule absolutely_integrable_component_ubound [OF fg]) (simp add: *)

lemma has_integral_vec1_I_cbox:
  fixes f :: real^1 ⇒ 'a::real_normed_vector
  assumes (f has_integral y) (cbox a b)
  shows ((f ∘ vec) has_integral y) {a$1..b$1}
proof −
  have ((λx. f(vec x)) has_integral (1 / 1) *_R y) ((λx. x $ 1) ‘ cbox a b)
  proof (rule has_integral_twiddle)
    show ∃ w z::real^1. vec ‘ cbox u v = cbox w z
         content (vec ‘ cbox u v :: (real^1) set) = 1 * content (cbox u v) for u v
      unfolding vec_cbox_1_eq
      by (auto simp: content_cbox_if_cart interval_eq_empty_cart)
    show ∃ w z. (λx. x $ 1) ‘ cbox u v = cbox w z for u v :: real^1
      using vec_nth_cbox_1_eq by blast
  qed (auto simp: continuous_vec assms)
  then show ?thesis
    by (simp add: o_def)
qed

lemma has_integral_vec1_I:
  fixes f :: real^1 ⇒ 'a::real_normed_vector
  assumes (f has_integral y) S
  shows (f ∘ vec has_integral y) ((λx. x $ 1) ‘ S)
proof −
  have *: ∃ z. ((λx. if x ∈ (λx. x $ 1) ‘ S then (f ∘ vec) x else 0) has_integral z)
{a..b} ∧ norm (z − y) < e
    if int: ⋀a b. ball 0 B ⊆ cbox a b ⟹
                  (∃ z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b) ∧
norm (z − y) < e)
      and B: ball 0 B ⊆ {a..b} for e B a b
  proof −
    have [simp]: (∃ y∈S. x = y $ 1) ⟷ vec x ∈ S for x
      by force
    have B': ball (0::real^1) B ⊆ cbox (vec a) (vec b)
        using B by (simp add: Basis_vec_def cart_eq_inner_axis [symmetric]
mem_box norm_real subset_iff)
    show ?thesis
        using int [OF B'] by (auto simp: image_iff o_def cong: if_cong dest!:
has_integral_vec1_I_cbox)
  qed
  show ?thesis
```

```
    using assms
    apply (subst has_integral_alt)
    apply (subst (asm) has_integral_alt)
    apply (simp add: has_integral_vec1_I_cbox split: if_split_asm)
    subgoal by (metis vector_one_nth)
    subgoal
      apply (erule all_forward imp_forward ex_forward asm_rl)+
      by (blast intro!: *)+
    done
qed

lemma has_integral_vec1_nth_cbox:
  fixes f :: real ⇒ 'a::real_normed_vector
  assumes (f has_integral y) {a..b}
  shows ((λx::real^1. f(x$1)) has_integral y) (cbox (vec a) (vec b))
proof −
  have ((λx::real^1. f(x$1)) has_integral (1 / 1) *R y) (vec ‘ cbox a b)
  proof (rule has_integral_twiddle)
    show ∃w z::real. (λx. x $ 1) ‘ cbox u v = cbox w z
         content ((λx. x $ 1) ‘ cbox u v) = 1 * content (cbox u v) for u v::real^1
     unfolding vec_cbox_1_eq by (auto simp: content_cbox_if_cart interval_eq_empty_cart)
    show ∃w z::real^1. vec ‘ cbox u v = cbox w z for u v :: real
      using vec_cbox_1_eq by auto
  qed (auto simp: continuous_vec assms)
  then show ?thesis
    using vec_cbox_1_eq by auto
qed

lemma has_integral_vec1_D_cbox:
  fixes f :: real^1 ⇒ 'a::real_normed_vector
  assumes ((f ∘ vec) has_integral y) {a$1..b$1}
  shows (f has_integral y) (cbox a b)
 by (metis (mono_tags, lifting) assms comp_apply has_integral_eq has_integral_vec1_nth_cbox
vector_one_nth)

lemma has_integral_vec1_D:
  fixes f :: real^1 ⇒ 'a::real_normed_vector
  assumes ((f ∘ vec) has_integral y) ((λx. x $ 1) ‘ S)
  shows (f has_integral y) S
proof −
  have *: ∃z. ((λx. if x ∈ S then f x else 0) has_integral z) (cbox a b) ∧ norm (z
− y) < e
    if int: ⋀a b. ball 0 B ⊆ {a..b} ⟹
                          (∃z. ((λx. if x ∈ (λx. x $ 1) ‘ S then (f ∘ vec) x else 0)
has_integral z) {a..b} ∧ norm (z − y) < e)
      and B: ball 0 B ⊆ cbox a b for e B and a b::real^1
  proof −
    have B': ball 0 B ⊆ {a$1..b$1}
    proof (clarsimp)
```

```
    fix t
    assume |t| < B then show a $ 1 ≤ t ∧ t ≤ b $ 1
      using subsetD [OF B]
        by (metis (mono_tags, opaque_lifting) mem_ball_0 mem_box_cart(2)
norm_real vec_component)
  qed
  have eq: (λx. if vec x ∈ S then f (vec x) else 0) = (λx. if x ∈ S then f x else
0) ∘ vec
    by force
  have [simp]: (∃ y∈S. x = y $ 1) ⟷ vec x ∈ S for x
    by force
  show ?thesis
  using int [OF B'] by (auto simp: image_iff eq cong: if_cong dest!: has_integral_vec1_D_cbox)
qed
 show ?thesis
  using assms
  apply (subst has_integral_alt)
  apply (subst (asm) has_integral_alt)
 apply (simp add: has_integral_vec1_D_cbox eq_cbox split: if_split_asm, blast)
  apply (intro conjI impI)
  subgoal by (metis vector_one_nth)
  apply (erule thin_rl)
  apply (erule all_forward ex_forward conj_forward)+
    by (blast intro!: *)+
qed


lemma integral_vec1_eq:
 fixes f :: real^1 ⇒ 'a::real_normed_vector
 shows integral S f = integral ((λx. x $ 1) ' S) (f ∘ vec)
 using has_integral_vec1_I [of f] has_integral_vec1_D [of f]
 by (metis has_integral_iff not_integrable_integral)

lemma absolutely_integrable_drop:
 fixes f :: real^1 ⇒ 'b::euclidean_space
 shows f absolutely_integrable_on S ⟷ (f ∘ vec) absolutely_integrable_on (λx.
x $ 1) ' S
 unfolding absolutely_integrable_on_def integrable_on_def
proof safe
 fix y r
 assume (f has_integral y) S ((λx. norm (f x)) has_integral r) S
 then show ∃ y. (f ∘ vec has_integral y) ((λx. x $ 1) ' S)
         ∃ y. ((λx. norm ((f ∘ vec) x)) has_integral y) ((λx. x $ 1) ' S)
   by (force simp: o_def dest!: has_integral_vec1_I)+
next
 fix y :: 'b and r :: real
 assume (f ∘ vec has_integral y) ((λx. x $ 1) ' S)
       ((λx. norm ((f ∘ vec) x)) has_integral r) ((λx. x $ 1) ' S)
 then show ∃ y. (f has_integral y) S ∃ y. ((λx. norm (f x)) has_integral y) S
```

by (*force simp*: *o_def intro*: *has_integral_vec1_D*)+
qed

9.8.19 Dominated convergence

lemma *dominated_convergence*:
fixes $f :: nat \Rightarrow 'n::euclidean\_space \Rightarrow 'm::euclidean\_space$
assumes f: $\bigwedge k.\ (f\ k)$ *integrable_on* S **and** h: h *integrable_on* S
and *le*: $\bigwedge k\ x.\ x \in S \Longrightarrow norm\ (f\ k\ x) \leq h\ x$
and *conv*: $\bigwedge x.\ x \in S \Longrightarrow (\lambda k.\ f\ k\ x) \longrightarrow g\ x$
shows g *integrable_on* S $(\lambda k.\ integral\ S\ (f\ k)) \longrightarrow integral\ S\ g$
proof −
have *3*: h *absolutely_integrable_on* S
unfolding *absolutely_integrable_on_def*
proof
show $(\lambda x.\ norm\ (h\ x))$ *integrable_on* S
proof (*intro integrable_spike_finite*[*OF* _ _ h, *of* {}] *ballI*)
fix x **assume** $x \in S - \{\}$ **then show** $norm\ (h\ x) = h\ x$
by (*metis Diff_empty abs_of_nonneg bot_set_def le norm_ge_zero order_trans real_norm_def*)
qed *auto*
qed *fact*
have *2*: *set_borel_measurable lebesgue* S $(f\ k)$ **for** k
unfolding *set_borel_measurable_def*
using f **by** (*auto intro*: *has_integral_implies_lebesgue_measurable simp*: *integrable_on_def*)
then have *1*: *set_borel_measurable lebesgue* S g
unfolding *set_borel_measurable_def*
by (*rule borel_measurable_LIMSEQ_metric*) (*use conv* **in** ‹*auto split*: *split_indicator*›)
have *4*: *AE x in lebesgue*. $(\lambda i.\ indicator\ S\ x *_R f\ i\ x) \longrightarrow indicator\ S\ x *_R g\ x$
AE x in lebesgue. $norm\ (indicator\ S\ x *_R f\ k\ x) \leq indicator\ S\ x *_R h\ x$ **for** k
using *conv le* **by** (*auto intro*!: *always_eventually split*: *split_indicator*)
have g: g *absolutely_integrable_on* S
using *1 2 3 4* **unfolding** *set_borel_measurable_def set_integrable_def*
by (*rule integrable_dominated_convergence*)
then show g *integrable_on* S
by (*auto simp*: *absolutely_integrable_on_def*)
have $(\lambda k.\ (LINT\ x{:}S|lebesgue.\ f\ k\ x)) \longrightarrow (LINT\ x{:}S|lebesgue.\ g\ x)$
unfolding *set_borel_measurable_def set_lebesgue_integral_def*
using *1 2 3 4* **unfolding** *set_borel_measurable_def set_lebesgue_integral_def set_integrable_def*
by (*rule integral_dominated_convergence*)
then show $(\lambda k.\ integral\ S\ (f\ k)) \longrightarrow integral\ S\ g$
using g *absolutely_integrable_integrable_bound*[*OF le f h*]
by (*subst* (*asm*) (*1 2*) *set_lebesgue_integral_eq_integral*) *auto*
qed

lemma *has_integral_dominated_convergence*:

```
  fixes f :: nat ⇒ 'n::euclidean_space ⇒ 'm::euclidean_space
  assumes ⋀k. (f k has_integral y k) S h integrable_on S
    ⋀k. ∀x∈S. norm (f k x) ≤ h x ∀x∈S. (λk. f k x) ⟶ g x
    and x: y ⟶ x
  shows (g has_integral x) S
proof −
  have int_f: ⋀k. (f k) integrable_on S
    using assms by (auto simp: integrable_on_def)
  have (g has_integral (integral S g)) S
   by (metis assms(2−4) dominated_convergence(1) has_integral_integral int_f)
  moreover have integral S g = x
  proof (rule LIMSEQ_unique)
    show (λi. integral S (f i)) ⟶ x
      using integral_unique[OF assms(1)] x by simp
    show (λi. integral S (f i)) ⟶ integral S g
      by (metis assms(2) assms(3) assms(4) dominated_convergence(2) int_f)
  qed
  ultimately show ?thesis
    by simp
qed

lemma dominated_convergence_integrable_1:
  fixes f :: nat ⇒ 'n::euclidean_space ⇒ real
  assumes f: ⋀k. f k absolutely_integrable_on S
    and h: h integrable_on S
    and normg: ⋀x. x ∈ S ⟹ norm(g x) ≤ (h x)
    and lim: ⋀x. x ∈ S ⟹ (λk. f k x) ⟶ g x
 shows g integrable_on S
proof −
  have habs: h absolutely_integrable_on S
   using h normg nonnegative_absolutely_integrable_1 norm_ge_zero order_trans
by blast
  let ?f = λn x. (min (max (− h x) (f n x)) (h x))
  have h0: h x ≥ 0 if x ∈ S for x
    using normg that by force
  have leh: norm (?f k x) ≤ h x if x ∈ S for k x
    using h0 that by force
  have limf: (λk. ?f k x) ⟶ g x if x ∈ S for x
  proof −
    have ⋀e y. |f y x − g x| < e ⟹ |min (max (− h x) (f y x)) (h x) − g x| < e
      using h0 [OF that] normg [OF that] by simp
    then show ?thesis
      using lim [OF that] by (auto simp add: tendsto_iff dist_norm elim!: eventu-
ally_mono)
  qed
  show ?thesis
  proof (rule dominated_convergence [of ?f S h g])
    have (λx. − h x) absolutely_integrable_on S
      using habs unfolding set_integrable_def by auto
```

then show *?f k integrable_on S* **for** *k*
by (*intro set_lebesgue_integral_eq_integral absolutely_integrable_min_1 absolutely_integrable_max_1 f habs*)
qed (*use assms leh limf* **in** *auto*)
qed

lemma *dominated_convergence_integrable*:
fixes *f* :: *nat* ⇒ *'n::euclidean_space* ⇒ *'m::euclidean_space*
assumes *f*: ⋀*k*. *f k absolutely_integrable_on S*
and *h*: *h integrable_on S*
and *normg*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*g x*) ≤ (*h x*)
and *lim*: ⋀*x*. *x* ∈ *S* ⟹ (λ*k*. *f k x*) ⟶ *g x*
shows *g integrable_on S*
using *f*
unfolding *integrable_componentwise_iff* [*of g*] *absolutely_integrable_componentwise_iff* [**where** *f* = *f k* **for** *k*]
proof *clarify*
fix *b* :: *'m*
assume *fb* [*rule_format*]: ⋀*k*. ∀ *b*∈*Basis*. (λ*x*. *f k x* ∙ *b*) *absolutely_integrable_on S* **and** *b*: *b* ∈ *Basis*
show (λ*x*. *g x* ∙ *b*) *integrable_on S*
proof (*rule dominated_convergence_integrable_1* [*OF fb h*])
fix *x*
assume *x* ∈ *S*
show *norm* (*g x* ∙ *b*) ≤ *h x*
using *norm_nth_le* ‹*x* ∈ *S*› *b normg order.trans* **by** *blast*
show (λ*k*. *f k x* ∙ *b*) ⟶ *g x* ∙ *b*
using ‹*x* ∈ *S*› *b lim tendsto_componentwise_iff* **by** *fastforce*
qed (*use b* **in** *auto*)
qed

lemma *dominated_convergence_absolutely_integrable*:
fixes *f* :: *nat* ⇒ *'n::euclidean_space* ⇒ *'m::euclidean_space*
assumes *f*: ⋀*k*. *f k absolutely_integrable_on S*
and *h*: *h integrable_on S*
and *normg*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*g x*) ≤ (*h x*)
and *lim*: ⋀*x*. *x* ∈ *S* ⟹ (λ*k*. *f k x*) ⟶ *g x*
shows *g absolutely_integrable_on S*
proof −
have *g integrable_on S*
by (*rule dominated_convergence_integrable* [*OF assms*])
with *assms* **show** *?thesis*
by (*blast intro*: *absolutely_integrable_integrable_bound* [**where** *g*=*h*])
qed

proposition *integral_countable_UN*:
fixes *f* :: *real^'m* ⇒ *real^'n*
assumes *f*: *f absolutely_integrable_on* (⋃(*range s*))

and *s*: $\bigwedge m.\ s\ m \in sets\ lebesgue$
shows $\bigwedge n.\ f\ absolutely\_integrable\_on\ (\bigcup m{\leq}n.\ s\ m)$
and $(\lambda n.\ integral\ (\bigcup m{\leq}n.\ s\ m)\ f) \longrightarrow integral\ (\bigcup (s\ `\ UNIV))\ f$ (**is** *?F* $\longrightarrow$ *?I*)
proof −
show *fU*: *f absolutely_integrable_on* $(\bigcup m{\leq}n.\ s\ m)$ **for** *n*
using *assms* **by** (*blast intro*: *set_integrable_subset* [*OF f*])
have *fint*: *f integrable_on* $(\bigcup\ (range\ s))$
using *absolutely_integrable_on_def f* **by** *blast*
let *?h* = $\lambda x.$ *if* $x \in \bigcup (s\ `\ UNIV)$ *then norm*(*f x*) *else 0*
have $(\lambda n.\ integral\ UNIV\ (\lambda x.\ if\ x \in (\bigcup m{\leq}n.\ s\ m)\ then\ f\ x\ else\ 0))$
$\longrightarrow integral\ UNIV\ (\lambda x.\ if\ x \in \bigcup (s\ `\ UNIV)\ then\ f\ x\ else\ 0)$
proof (*rule dominated_convergence*)
show $(\lambda x.\ if\ x \in (\bigcup m{\leq}n.\ s\ m)\ then\ f\ x\ else\ 0)\ integrable\_on\ UNIV$ **for** *n*
unfolding *integrable_restrict_UNIV*
using *fU absolutely_integrable_on_def* **by** *blast*
show $(\lambda x.\ if\ x \in \bigcup (s\ `\ UNIV)\ then\ norm(f\ x)\ else\ 0)\ integrable\_on\ UNIV$
by (*metis* (*no_types*) *absolutely_integrable_on_def f integrable_restrict_UNIV*)
show $\bigwedge x.\ (\lambda n.\ if\ x \in (\bigcup m{\leq}n.\ s\ m)\ then\ f\ x\ else\ 0)$
$\longrightarrow (if\ x \in \bigcup (s\ `\ UNIV)\ then\ f\ x\ else\ 0)$
by (*force intro*: *tendsto_eventually eventually_sequentiallyI*)
qed *auto*
then show *?F* $\longrightarrow$ *?I*
by (*simp only*: *integral_restrict_UNIV*)
qed

9.8.20 Fundamental Theorem of Calculus for the Lebesgue integral

For the positive integral we replace continuity with Borel-measurability.

lemma
fixes *f* :: *real* ⇒ *real*
assumes [*measurable*]: *f* ∈ *borel_measurable borel*
assumes *f*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow DERIV\ F\ x :> f\ x$ $\bigwedge x.\ x \in \{a..b\} \Longrightarrow 0 \leq f\ x$ **and** $a \leq b$
shows *nn_integral_FTC_Icc*: $(\int^+ x.\ ennreal\ (f\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) = F\ b - F\ a$ (**is** *?nn*)
and *has_bochner_integral_FTC_Icc_nonneg*:
has_bochner_integral lborel $(\lambda x.\ f\ x * indicator\ \{a\ ..\ b\}\ x)\ (F\ b - F\ a)$ (**is** *?has*)
and *integral_FTC_Icc_nonneg*: $(\int x.\ f\ x * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) = F\ b - F\ a$ (**is** *?eq*)
and *integrable_FTC_Icc_nonneg*: *integrable lborel* $(\lambda x.\ f\ x * indicator\ \{a\ ..\ b\}\ x)$ (**is** *?int*)
proof −
have ∗: $(\lambda x.\ f\ x * indicator\ \{a..b\}\ x) \in borel\_measurable\ borel$ $\bigwedge x.\ 0 \leq f\ x * indicator\ \{a..b\}\ x$
using *f*(*2*) **by** (*auto split*: *split_indicator*)

have *F_mono*: $a \leq x \Longrightarrow x \leq y \Longrightarrow y \leq b \Longrightarrow F\ x \leq F\ y$ **for** *x y*
using *f* **by** (*intro DERIV_nonneg_imp_nondecreasing*[*of x y F*]) (*auto intro*: *order_trans*)

have (*f has_integral F b − F a*) {*a..b*}
by (*intro fundamental_theorem_of_calculus*)
(*auto simp*: *has_real_derivative_iff_has_vector_derivative*[*symmetric*]
intro: *has_field_derivative_subset*[*OF f*(*1*)] ‹*a* ≤ *b*›)
then have *i*: ((λ*x*. *f x* ∗ *indicator* {*a* .. *b*} *x*) *has_integral F b − F a*) *UNIV*
unfolding *indicator_def of_bool_def if_distrib*[**where** *f*=λ*x*. *a* ∗ *x* **for** *a*]
by (*simp cong del*: *if_weak_cong del*: *atLeastAtMost_iff*)
then have *nn*: (∫$^+$*x*. *f x* ∗ *indicator* {*a* .. *b*} *x* ∂*lborel*) = *F b − F a*
by (*rule nn_integral_has_integral_lborel*[*OF* ∗])
then show *?has*
by (*rule has_bochner_integral_nn_integral*[*rotated 3*]) (*simp_all add*: ∗ *F_mono* ‹*a* ≤ *b*›)
then show *?eq ?int*
unfolding *has_bochner_integral_iff* **by** *auto*
show *?nn*
by (*subst nn*[*symmetric*])
(*auto intro*!: *nn_integral_cong simp add*: *ennreal_mult f split*: *split_indicator*)
qed

lemma
fixes *f* :: *real* ⇒ *'a* :: *euclidean_space*
assumes *a* ≤ *b*
assumes ⋀*x*. *a* ≤ *x* ⟹ *x* ≤ *b* ⟹ (*F has_vector_derivative f x*) (*at x within* {*a* .. *b*})
assumes *cont*: *continuous_on* {*a* .. *b*} *f*
shows *has_bochner_integral_FTC_Icc*:
has_bochner_integral lborel (λ*x*. *indicator* {*a* .. *b*} *x* ∗$_R$ *f x*) (*F b − F a*) (**is** *?has*)
and *integral_FTC_Icc*: (∫ *x*. *indicator* {*a* .. *b*} *x* ∗$_R$ *f x* ∂*lborel*) = *F b − F a* (**is** *?eq*)
proof −
let *?f* = λ*x*. *indicator* {*a* .. *b*} *x* ∗$_R$ *f x*
have *int*: *integrable lborel ?f*
using *borel_integrable_compact*[*OF _ cont*] **by** *auto*
have (*f has_integral F b − F a*) {*a..b*}
using *assms*(*1*,*2*) **by** (*intro fundamental_theorem_of_calculus*) *auto*
moreover
have (*f has_integral integral*^{L} *lborel ?f*) {*a..b*}
using *has_integral_integral_lborel*[*OF int*]
unfolding *indicator_def of_bool_def if_distrib*[**where** *f*=λ*x*. *x* ∗$_R$ *a* **for** *a*]
by (*simp cong del*: *if_weak_cong del*: *atLeastAtMost_iff*)
ultimately show *?eq*
by (*auto dest*: *has_integral_unique*)
then show *?has*
using *int* **by** (*auto simp*: *has_bochner_integral_iff*)

qed

lemma
fixes $f :: real \Rightarrow real$
assumes $a \leq b$
assumes *deriv*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow DERIV\ F\ x :> f\ x$
assumes *cont*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow isCont\ f\ x$
shows *has_bochner_integral_FTC_Icc_real*:
has_bochner_integral lborel $(\lambda x.\ f\ x * indicator\ \{a\ ..\ b\}\ x)\ (F\ b - F\ a)$ (**is** *?has*)
and *integral_FTC_Icc_real*: $(\int x.\ f\ x * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) = F\ b - F\ a$ (**is** *?eq*)
proof −
have *1*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow (F\ has\_vector\_derivative\ f\ x)\ (at\ x\ within\ \{a\ ..\ b\})$
unfolding *has_real_derivative_iff_has_vector_derivative*[*symmetric*]
using *deriv* **by** (*auto intro*: *DERIV_subset*)
have *2*: *continuous_on* $\{a\ ..\ b\}\ f$
using *cont* **by** (*intro continuous_at_imp_continuous_on*) *auto*
show *?has ?eq*
using *has_bochner_integral_FTC_Icc*[*OF* ‹$a \leq b$› *1 2*] *integral_FTC_Icc*[*OF* ‹$a \leq b$› *1 2*]
by (*auto simp*: *mult.commute*)
qed

lemma *nn_integral_FTC_atLeast*:
fixes $f :: real \Rightarrow real$
assumes *f_borel*: $f \in borel\_measurable\ borel$
assumes *f*: $\bigwedge x.\ a \leq x \Longrightarrow DERIV\ F\ x :> f\ x$
assumes *nonneg*: $\bigwedge x.\ a \leq x \Longrightarrow 0 \leq f\ x$
assumes *lim*: $(F \longrightarrow T)\ at\_top$
shows $(\int^+ x.\ ennreal\ (f\ x) * indicator\ \{a\ ..\}\ x\ \partial lborel) = T - F\ a$
proof −
let *?f* $= \lambda(i::nat)\ (x::real).\ ennreal\ (f\ x) * indicator\ \{a..a + real\ i\}\ x$
let *?fR* $= \lambda x.\ ennreal\ (f\ x) * indicator\ \{a\ ..\}\ x$

have *F_mono*: $a \leq x \Longrightarrow x \leq y \Longrightarrow F\ x \leq F\ y$ **for** *x y*
using *f nonneg* **by** (*intro DERIV_nonneg_imp_nondecreasing*[*of x y F*]) (*auto intro*: *order_trans*)
then have *F_le_T*: $a \leq x \Longrightarrow F\ x \leq T$ **for** *x*
by (*intro tendsto_lowerbound*[*OF lim*])
(*auto simp*: *eventually_at_top_linorder*)

have $(SUP\ i.\ ?f\ i\ x) = ?fR\ x$ **for** *x*
proof (*rule LIMSEQ_unique*[*OF LIMSEQ_SUP*])
obtain *n* **where** $x - a < real\ n$
using *reals_Archimedean2*[*of x − a*] **..**
then have *eventually* $(\lambda n.\ ?f\ n\ x = ?fR\ x)\ sequentially$
by (*auto simp*: *frequently_def intro*!: *eventually_sequentiallyI*[**where** *c*=*n*]

split: *split_indicator*)
then show $(\lambda n.\ ?f\ n\ x) \longrightarrow ?fR\ x$
by (*rule tendsto_eventually*)
qed (*auto simp*: *nonneg incseq_def le_fun_def split*: *split_indicator*)
then have $integral^N\ lborel\ ?fR = (\int^+ x.\ (SUP\ i.\ ?f\ i\ x)\ \partial lborel)$
by *simp*
also have $\ldots = (SUP\ i.\ (\int^+ x.\ ?f\ i\ x\ \partial lborel))$
proof (*rule nn_integral_monotone_convergence_SUP*)
show *incseq ?f*
using *nonneg* **by** (*auto simp*: *incseq_def le_fun_def split*: *split_indicator*)
show $\bigwedge i.\ (?f\ i) \in borel\_measurable\ lborel$
using *f_borel* **by** *auto*
qed
also have $\ldots = (SUP\ i.\ ennreal\ (F\ (a + real\ i) - F\ a))$
by (*subst nn_integral_FTC_Icc*[*OF f_borel f nonneg*]) *auto*
also have $\ldots = T - F\ a$
proof (*rule LIMSEQ_unique*[*OF LIMSEQ_SUP*])
have $(\lambda x.\ F\ (a + real\ x)) \longrightarrow T$
by (*auto intro*: *filterlim_compose*[*OF lim filterlim_tendsto_add_at_top*] *filterlim_real_sequentially*)
then show $(\lambda n.\ ennreal\ (F\ (a + real\ n) - F\ a)) \longrightarrow ennreal\ (T - F\ a)$
by (*simp add*: *F_mono F_le_T tendsto_diff*)
qed (*auto simp*: *incseq_def intro*!: *ennreal_le_iff*[*THEN iffD2*] *F_mono*)
finally show *?thesis* **.**
qed

lemma *integral_power*:
$a \le b \Longrightarrow (\int x.\ x\hat{}k * indicator\ \{a..b\}\ x\ \partial lborel) = (b\hat{}Suc\ k - a\hat{}Suc\ k)\ /\ Suc\ k$
proof (*subst integral_FTC_Icc_real*)
fix *x* **show** $DERIV\ (\lambda x.\ x\hat{}Suc\ k\ /\ Suc\ k)\ x :> x\hat{}k$
by (*intro derivative_eq_intros*) *auto*
qed (*auto simp*: *field_simps simp del*: *of_nat_Suc*)

9.8.21 Integration by parts

lemma *integral_by_parts_integrable*:
fixes *f g F G*::*real* ⇒ *real*
assumes $a \le b$
assumes *cont_f*[*intro*]: $!!x.\ a \le x \Longrightarrow x \le b \Longrightarrow isCont\ f\ x$
assumes *cont_g*[*intro*]: $!!x.\ a \le x \Longrightarrow x \le b \Longrightarrow isCont\ g\ x$
assumes [*intro*]: $!!x.\ DERIV\ F\ x :> f\ x$
assumes [*intro*]: $!!x.\ DERIV\ G\ x :> g\ x$
shows $integrable\ lborel\ (\lambda x.((F\ x) * (g\ x) + (f\ x) * (G\ x)) * indicator\ \{a\ ..\ b\}\ x)$
by (*auto intro*!: *borel_integrable_atLeastAtMost continuous_intros*) (*auto intro*!: *DERIV_isCont*)

lemma *integral_by_parts*:
fixes *f g F G*::*real* ⇒ *real*

assumes [*arith*]: $a \leq b$
assumes *cont_f*[*intro*]: $!!x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow isCont\ f\ x$
assumes *cont_g*[*intro*]: $!!x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow isCont\ g\ x$
assumes [*intro*]: $!!x.\ DERIV\ F\ x :> f\ x$
assumes [*intro*]: $!!x.\ DERIV\ G\ x :> g\ x$
shows $(\int x.\ (F\ x * g\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel)$
$= F\ b * G\ b - F\ a * G\ a - \int x.\ (f\ x * G\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel$
proof−
have $(\int x.\ (F\ x * g\ x + f\ x * G\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel)$
$= (LBINT\ x.\ F\ x * g\ x * indicat\_real\ \{a..b\}\ x + f\ x * G\ x * indicat\_real\ \{a..b\}\ x)$
by (*meson vector_space_over_itself.scale_left_distrib*)
also have $... = (\int x.\ (F\ x * g\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) + \int x.\ (f\ x * G\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel$
proof (*intro Bochner_Integration.integral_add borel_integrable_atLeastAtMost cont_f cont_g continuous_intros*)
show $\bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow isCont\ F\ x\ \bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow isCont\ G\ x$
using *DERIV_isCont* **by** *blast*+
qed
finally have $(\int x.\ (F\ x * g\ x + f\ x * G\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) =$
$(\int x.\ (F\ x * g\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) + \int x.\ (f\ x * G\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel$ **.**
moreover have $(\int x.\ (F\ x * g\ x + f\ x * G\ x) * indicator\ \{a\ ..\ b\}\ x\ \partial lborel) = F\ b * G\ b - F\ a * G\ a$
proof (*intro integral_FTC_Icc_real derivative_eq_intros cont_f cont_g continuous_intros*)
show $\bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow isCont\ F\ x\ \bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow isCont\ G\ x$
using *DERIV_isCont* **by** *blast*+
qed *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *integral_by_parts'*:
fixes *f g F G*::*real* ⇒ *real*
assumes $a \leq b$
assumes $!!x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow isCont\ f\ x$
assumes $!!x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow isCont\ g\ x$
assumes $!!x.\ DERIV\ F\ x :> f\ x$
assumes $!!x.\ DERIV\ G\ x :> g\ x$
shows $(\int x.\ indicator\ \{a\ ..\ b\}\ x *_R (F\ x * g\ x)\ \partial lborel)$
$= F\ b * G\ b - F\ a * G\ a - \int x.\ indicator\ \{a\ ..\ b\}\ x *_R (f\ x * G\ x)\ \partial lborel$
using *integral_by_parts*[*OF assms*] **by** (*simp add*: *ac_simps*)

lemma *has_bochner_integral_even_function*:
fixes *f* :: *real* ⇒ *'a* :: {*banach*, *second_countable_topology*}
assumes *f*: *has_bochner_integral lborel* (λ*x*. *indicator* {*0*..} *x* *$_R$ *f x*) *x*
assumes *even*: $\bigwedge x.\ f\ (-x) = f\ x$
shows *has_bochner_integral lborel f* (*2* *$_R$ *x*)

proof −
have *indicator*: $\bigwedge x$::*real*. *indicator* {..*0*} (− *x*) = *indicator* {*0*..} *x*
by (*auto split*: *split_indicator*)
have *has_bochner_integral lborel* (λ*x*. *indicator* {.. *0*} *x* ∗$_R$ *f x*) *x*
by (*subst lborel_has_bochner_integral_real_affine_iff*[**where** *c*=−*1* **and** *t*=*0*])
(*auto simp*: *indicator even f*)
with *f* **have** *has_bochner_integral lborel* (λ*x*. *indicator* {*0*..} *x* ∗$_R$ *f x* + *indicator* {.. *0*} *x* ∗$_R$ *f x*) (*x* + *x*)
by (*rule has_bochner_integral_add*)
then have *has_bochner_integral lborel f* (*x* + *x*)
by (*rule has_bochner_integral_discrete_difference*[**where** *X*={*0*}, *THEN iffD1*, *rotated 4*])
(*auto split*: *split_indicator*)
then show *?thesis*
by (*simp add*: *scaleR_2*)
qed

lemma *has_bochner_integral_odd_function*:
fixes *f* :: *real* ⇒ *'a* :: {*banach*, *second_countable_topology*}
assumes *f*: *has_bochner_integral lborel* (λ*x*. *indicator* {*0*..} *x* ∗$_R$ *f x*) *x*
assumes *odd*: $\bigwedge x$. *f* (− *x*) = − *f x*
shows *has_bochner_integral lborel f 0*
proof −
have *indicator*: $\bigwedge x$::*real*. *indicator* {..*0*} (− *x*) = *indicator* {*0*..} *x*
by (*auto split*: *split_indicator*)
have *has_bochner_integral lborel* (λ*x*. − *indicator* {.. *0*} *x* ∗$_R$ *f x*) *x*
by (*subst lborel_has_bochner_integral_real_affine_iff*[**where** *c*=−*1* **and** *t*=*0*])
(*auto simp*: *indicator odd f*)
from *has_bochner_integral_minus*[*OF this*]
have *has_bochner_integral lborel* (λ*x*. *indicator* {.. *0*} *x* ∗$_R$ *f x*) (− *x*)
by *simp*
with *f* **have** *has_bochner_integral lborel* (λ*x*. *indicator* {*0*..} *x* ∗$_R$ *f x* + *indicator* {.. *0*} *x* ∗$_R$ *f x*) (*x* + − *x*)
by (*rule has_bochner_integral_add*)
then have *has_bochner_integral lborel f* (*x* + − *x*)
by (*rule has_bochner_integral_discrete_difference*[**where** *X*={*0*}, *THEN iffD1*, *rotated 4*])
(*auto split*: *split_indicator*)
then show *?thesis*
by *simp*
qed

9.8.22 A non-negative continuous function whose integral is zero must be zero

lemma *has_integral_0_closure_imp_0*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real*
assumes *f*: *continuous_on* (*closure S*) *f*
and *nonneg_interior*: $\bigwedge x$. *x* ∈ *S* ⟹ *0* ≤ *f x*

```
    and pos: 0 < emeasure lborel S
    and finite: emeasure lborel S < ∞
    and regular: emeasure lborel (closure S) = emeasure lborel S
    and opn: open S
  assumes int: (f has_integral 0) (closure S)
  assumes x: x ∈ closure S
  shows f x = 0
proof −
  have zero: emeasure lborel (frontier S) = 0
    using finite closure_subset regular
    unfolding frontier_def
    by (subst emeasure_Diff) (auto simp: frontier_def interior_open ‹open S› )
  have nonneg: 0 ≤ f x if x ∈ closure S for x
    using continuous_ge_on_closure[OF f that nonneg_interior] by simp
  have 0 = integral (closure S) f
    by (blast intro: int sym)
  also
  note intl = has_integral_integrable[OF int]
  have af: f absolutely_integrable_on (closure S)
    using nonneg
    by (intro absolutely_integrable_onI intl integrable_eq[OF intl]) simp
  then have integral (closure S) f = set_lebesgue_integral lebesgue (closure S) f
    by (intro set_lebesgue_integral_eq_integral(2)[symmetric])
  also have ... = 0 ⟷ (AE x in lebesgue. indicator (closure S) x *R f x = 0)
    unfolding set_lebesgue_integral_def
  proof (rule integral_nonneg_eq_0_iff_AE)
    show integrable lebesgue (λx. indicat_real (closure S) x *R f x)
      by (metis af set_integrable_def)
  qed (use nonneg in ‹auto simp: indicator_def›)
  also have ... ⟷ (AE x in lebesgue. x ∈ {x. x ∈ closure S ⟶ f x = 0})
    by (auto simp: indicator_def)
  finally have (AE x in lebesgue. x ∈ {x. x ∈ closure S ⟶ f x = 0}) by simp
  moreover have (AE x in lebesgue. x ∈ − frontier S)
    using zero
   by (auto simp: eventually_ae_filter null_sets_def intro!: exI[where x=frontier
S])
   ultimately have ae: AE x ∈ S in lebesgue. x ∈ {x ∈ closure S. f x = 0} (is
?th)
    by eventually_elim (use closure_subset in ‹auto simp: ›)
  have closed {0::real} by simp
  with continuous_on_closed_vimage[OF closed_closure, of S f] f
  have closed (f −‘ {0} ∩ closure S) by blast
  then have closed {x ∈ closure S. f x = 0} by (auto simp: vimage_def Int_def
conj_commute)
  with ‹open S› have x ∈ {x ∈ closure S. f x = 0} if x ∈ S for x using ae that
    by (rule mem_closed_if_AE_lebesgue_open)
  then have f x = 0 if x ∈ S for x using that by auto
  from continuous_constant_on_closure[OF f this ‹x ∈ closure S›]
  show f x = 0 .
```

qed

lemma *has_integral_0_cbox_imp_0*:
fixes $f :: 'a::euclidean\_space \Rightarrow real$
assumes *continuous_on* $(cbox\ a\ b)\ f$ **and** $\bigwedge x.\ x \in box\ a\ b \Longrightarrow 0 \le f\ x$
assumes $(f\ has\_integral\ 0)\ (cbox\ a\ b)$
assumes *ne*: $box\ a\ b \neq \{\}$
assumes *x*: $x \in cbox\ a\ b$
shows $f\ x = 0$
proof −
have $0 <$ *emeasure lborel* $(box\ a\ b)$
using *ne x* **unfolding** *emeasure_lborel_box_eq*
by (*force intro*!: *prod_pos simp*: *mem_box algebra_simps*)
then show *?thesis* **using** *assms*
by (*intro has_integral_0_closure_imp_0*[*of box a b f x*])
(*auto simp*: *emeasure_lborel_box_eq emeasure_lborel_cbox_eq algebra_simps mem_box*)
qed

corollary *integral_cbox_eq_0_iff*:
fixes $f :: 'a::euclidean\_space \Rightarrow real$
assumes *continuous_on* $(cbox\ a\ b)\ f$ **and** $box\ a\ b \neq \{\}$
and $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow f\ x \ge 0$
shows *integral* $(cbox\ a\ b)\ f = 0 \longleftrightarrow (\forall x \in cbox\ a\ b.\ f\ x = 0)$ (**is** *?lhs* = *?rhs*)
proof
assume *int0*: *?lhs*
show *?rhs*
using *has_integral_0_cbox_imp_0* [*of a b f*] *assms*
by (*metis box_subset_cbox eq_integralD int0 integrable_continuous subsetD*)
next
assume *?rhs* **then show** *?lhs*
by (*meson has_integral_is_0_cbox integral_unique*)
qed

lemma *integral_eq_0_iff*:
fixes $f :: real \Rightarrow real$
assumes *continuous_on* $\{a..b\}\ f$ **and** $a < b$
and $\bigwedge x.\ x \in \{a..b\} \Longrightarrow f\ x \ge 0$
shows *integral* $\{a..b\}\ f = 0 \longleftrightarrow (\forall x \in \{a..b\}.\ f\ x = 0)$
using *integral_cbox_eq_0_iff* [*of a b f*] *assms* **by** *simp*

lemma *integralL_eq_0_iff*:
fixes $f :: real \Rightarrow real$
assumes *contf*: *continuous_on* $\{a..b\}\ f$ **and** $a < b$
and $\bigwedge x.\ x \in \{a..b\} \Longrightarrow f\ x \ge 0$
shows $integral^L\ (lebesgue\_on\ \{a..b\})\ f = 0 \longleftrightarrow (\forall x \in \{a..b\}.\ f\ x = 0)$
using *integral_eq_0_iff* [*OF assms*]
by (*simp add*: *contf continuous_imp_integrable_real lebesgue_integral_eq_integral*)

In fact, strict inequality is required only at a single point within the box.

lemma *integral_less*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *real*
assumes *cont*: *continuous_on* (*cbox a b*) *f continuous_on* (*cbox a b*) *g* **and** *box a b* ≠ {}
and *fg*: ⋀*x*. *x* ∈ *box a b* ⟹ *f x* < *g x*
shows *integral* (*cbox a b*) *f* < *integral* (*cbox a b*) *g*
proof −
obtain *int*: *f integrable_on* (*cbox a b*) *g integrable_on* (*cbox a b*)
using *cont integrable_continuous* **by** *blast*
then have *integral* (*cbox a b*) *f* ≤ *integral* (*cbox a b*) *g*
by (*metis fg integrable_on_open_interval integral_le integral_open_interval less_eq_real_def*)
moreover have *integral* (*cbox a b*) *f* ≠ *integral* (*cbox a b*) *g*
proof (*rule ccontr*)
assume ¬ *integral* (*cbox a b*) *f* ≠ *integral* (*cbox a b*) *g*
then have *0*: ((λ*x*. *g x* − *f x*) *has_integral 0*) (*cbox a b*)
by (*metis* (*full_types*) *cancel_comm_monoid_add_class.diff_cancel has_integral_diff int integrable_integral*)
have *cgf*: *continuous_on* (*cbox a b*) (λ*x*. *g x* − *f x*)
using *cont continuous_on_diff* **by** *blast*
show *False*
using *has_integral_0_cbox_imp_0* [*OF cgf _ 0*] *assms*(*3*) *box_subset_cbox fg less_eq_real_def* **by** *fastforce*
qed
ultimately show *?thesis*
by *linarith*
qed

lemma *integral_less_real*:
fixes *f* :: *real* ⇒ *real*
assumes *continuous_on* {*a..b*} *f continuous_on* {*a..b*} *g* **and** {*a<..<b*} ≠ {}
and ⋀*x*. *x* ∈ {*a<..<b*} ⟹ *f x* < *g x*
shows *integral* {*a..b*} *f* < *integral* {*a..b*} *g*
by (*metis assms box_real integral_less*)

9.8.23 Various common equivalent forms of function measurability

lemma *indicator_sum_eq*:
fixes *m*::*real* **and** *f* :: *'a* ⇒ *real*
assumes |*m*| ≤ *2* ^ (*2*∗*n*) *m*/*2*^*n* ≤ *f x f x* < (*m*+*1*)/*2*^*n m* ∈ ℤ
shows (∑ *k*::*real* | *k* ∈ ℤ ∧ |*k*| ≤ *2* ^ (*2*∗*n*).
k/*2*^*n* ∗ *indicator* {*y*. *k*/*2*^*n* ≤ *f y* ∧ *f y* < (*k*+*1*)/*2*^*n*} *x*) = *m*/*2*^*n*
(**is** *sum ?f ?S* = _)
proof −
have *sum ?f ?S* = *sum* (λ*k*. *k*/*2*^*n* ∗ *indicator* {*y*. *k*/*2*^*n* ≤ *f y* ∧ *f y* < (*k*+*1*)/*2*^*n*} *x*) {*m*}
proof (*rule comm_monoid_add_class.sum.mono_neutral_right*)
show *finite ?S*

```
    by (rule finite_abs_int_segment)
  show {m} ⊆ {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}
    using assms by auto
  show ∀ i∈{k ∈ ℤ. |k| ≤ 2 ^ (2*n)} - {m}. ?f i = 0
   using assms by (auto simp: indicator_def Ints_def abs_le_iff field_split_simps)
 qed
 also have ... = m/2^n
  using assms by (auto simp: indicator_def not_less)
 finally show ?thesis .
qed

lemma measurable_on_sf_limit_lemma1:
 fixes f :: 'a::euclidean_space ⇒ real
 assumes ⋀a b. {x ∈ S. a ≤ f x ∧ f x < b} ∈ sets (lebesgue_on S)
 obtains g where ⋀n. g n ∈ borel_measurable (lebesgue_on S)
                 ⋀n. finite(range (g n))
                 ⋀x. (λn. g n x) ⟶ f x
proof
 show (λx. sum (λk::real. k/2^n * indicator {y. k/2^n ≤ f y ∧ f y < (k+1)/2^n}
x)
              {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}) ∈ borel_measurable (lebesgue_on S)
       (is ?g ∈ _)  for n
 proof −
  have ⋀k. ⟦k ∈ ℤ; |k| ≤ 2 ^ (2*n)⟧
       ⟹ Measurable.pred (lebesgue_on S) (λx. k / (2^n) ≤ f x ∧ f x < (k+1)
/ (2^n))
    using assms by (force simp: pred_def space_restrict_space)
  then show ?thesis
    by (simp add: field_class.field_divide_inverse)
 qed
 show finite (range (?g n)) for n
 proof −
  have range (?g n) ⊆ (λk. k/2^n) ` {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}
  proof clarify
   fix x
   show ?g n x  ∈ (λk. k/2^n) ` {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}
    proof (cases ∃ k::real. k ∈ ℤ ∧ |k| ≤ 2 ^ (2*n) ∧ k/2^n ≤ (f x) ∧ (f x) <
(k+1)/2^n)
     case True
     then show ?thesis
      apply clarify
      by (subst indicator_sum_eq) auto
    next
     case False
     then have ?g n x = 0 by auto
     then show ?thesis by force
    qed
  qed
  moreover have finite ((λk::real. (k/2^n)) ` {k ∈ ℤ. |k| ≤ 2 ^ (2*n)})
```

```
    by (simp add: finite_abs_int_segment)
   ultimately show ?thesis
    using finite_subset by blast
  qed
  show (λn. ?g n x) ⟶ f x for x
  proof (rule LIMSEQ_I)
   fix e::real
   assume e > 0
   obtain N1 where N1: |f x| < 2 ^ N1
    using real_arch_pow by fastforce
   obtain N2 where N2: (1/2) ^ N2 < e
    using real_arch_pow_inv ‹e > 0› by force
   have norm (?g n x − f x) < e if n: n ≥ max N1 N2 for n
   proof −
    define m where m ≡ floor(2^n * (f x))
    have 1 ≤ |2^n| * e
     using n N2 ‹e > 0› less_eq_real_def less_le_trans by (fastforce simp add:
field_split_simps)
    then have *: ⟦x ≤ y; y < x + 1⟧ ⟹ abs(x − y) < |2^n| * e for x y::real
     by linarith
    have |2^n| * |m/2^n − f x| = |2^n * (m/2^n − f x)|
     by (simp add: abs_mult)
    also have ... = |real_of_int ⌊2^n * f x⌋ − f x * 2^n|
     by (simp add: algebra_simps m_def)
    also have ... < |2^n| * e
     by (rule *; simp add: mult.commute)
    finally have |2^n| * |m/2^n − f x| < |2^n| * e .
    then have me: |m/2^n − f x| < e
     by simp
    have |real_of_int m| ≤ 2 ^ (2*n)
    proof (cases f x < 0)
     case True
     then have −⌊f x⌋ ≤ ⌊(2::real) ^ N1⌋
      using N1 le_floor_iff minus_le_iff by fastforce
     with n True have |real_of_int⌊f x⌋| ≤ 2 ^ N1
      by linarith
     also have ... ≤ 2^n
      using n by (simp add: m_def)
     finally have |real_of_int ⌊f x⌋| * 2^n ≤ 2^n * 2^n
      by simp
     moreover
     have |real_of_int ⌊2^n * f x⌋| ≤ |real_of_int ⌊f x⌋| * 2^n
     proof −
      have |real_of_int ⌊2^n * f x⌋| = − (real_of_int ⌊2^n * f x⌋)
       using True by (simp add: abs_if mult_less_0_iff)
      also have ... ≤ − (real_of_int (⌊(2::real) ^ n⌋ * ⌊f x⌋))
       using le_mult_floor_Ints [of (2::real)^n] by simp
      also have ... ≤ |real_of_int ⌊f x⌋| * 2^n
       using True
```

```
        by simp
      finally show ?thesis .
    qed
    ultimately show ?thesis
    by (metis (no_types, opaque_lifting) m_def order_trans power2_eq_square
power_even_eq)
  next
    case False
    with n N1 have f x ≤ 2^n
      by (simp add: not_less) (meson less_eq_real_def one_le_numeral or-
der_trans power_increasing)
    moreover have 0 ≤ m
      using False m_def by force
    ultimately show ?thesis
     by (metis abs_of_nonneg floor_mono le_floor_iff m_def of_int_0_le_iff
power2_eq_square power_mult mult_le_cancel_right_pos zero_less_numeral mult.commute
zero_less_power)
  qed
  then have ?g n x = m/2^n
    by (rule indicator_sum_eq) (auto simp add: m_def field_split_simps,
linarith)
  then have norm (?g n x − f x) = norm (m/2^n − f x)
    by simp
  also have ... < e
    by (simp add: me)
  finally show ?thesis .
 qed
 then show ∃no. ∀n≥no. norm (?g n x − f x) < e
  by blast
 qed
qed


lemma borel_measurable_simple_function_limit:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 shows f ∈ borel_measurable (lebesgue_on S) ⟷
      (∃g. (∀n. (g n) ∈ borel_measurable (lebesgue_on S)) ∧
          (∀n. finite (range (g n))) ∧ (∀x. (λn. g n x) ⟶ f x))
proof −
 have ∃g. (∀n. (g n) ∈ borel_measurable (lebesgue_on S)) ∧
         (∀n. finite (range (g n))) ∧ (∀x. (λn. g n x) ⟶ f x)
    if f: ⋀a i. i ∈ Basis ⟹ {x ∈ S. f x • i < a} ∈ sets (lebesgue_on S)
 proof −
  have ∃g. (∀n. (g n) ∈ borel_measurable (lebesgue_on S)) ∧
              (∀n. finite(image (g n) UNIV)) ∧
              (∀x. ((λn. g n x) ⟶ f x • i)) if i ∈ Basis for i
  proof (rule measurable_on_sf_limit_lemma1 [of S λx. f x • i])
   show {x ∈ S. a ≤ f x • i ∧ f x • i < b} ∈ sets (lebesgue_on S) for a b
   proof −
```

```
      have {x ∈ S. a ≤ f x • i ∧ f x • i < b} = {x ∈ S. f x • i < b} - {x ∈ S. a
> f x • i}
        by auto
      also have ... ∈ sets (lebesgue_on S)
        using f that by blast
      finally show ?thesis .
    qed
  qed blast
  then obtain g where g:
      ⋀i n. i ∈ Basis ⟹ g i n ∈ borel_measurable (lebesgue_on S)
      ⋀i n. i ∈ Basis ⟹ finite(range (g i n))
      ⋀i x. i ∈ Basis ⟹ ((λn. g i n x) ⟶ f x • i)
    by metis
  show ?thesis
  proof (intro conjI allI exI)
    show (λx. ∑i∈Basis. g i n x *R i) ∈ borel_measurable (lebesgue_on S) for
n
      by (intro borel_measurable_sum borel_measurable_scaleR) (auto intro: g)
    show finite (range (λx. ∑i∈Basis. g i n x *R i)) for n
    proof -
     have range (λx. ∑i∈Basis. g i n x *R i) ⊆ (λh. ∑i∈Basis. h i *R i) ` PiE
Basis (λi. range (g i n))
      proof clarify
        fix x
       show (∑i∈Basis. g i n x *R i) ∈ (λh. ∑i∈Basis. h i *R i) ` (ΠE i∈Basis.
range (g i n))
          by (rule_tac x=λi∈Basis. g i n x in image_eqI) auto
      qed
      moreover have finite(PiE Basis (λi. range (g i n)))
        by (simp add: g finite_PiE)
      ultimately show ?thesis
        by (metis (mono_tags, lifting) finite_surj)
    qed
    show (λn. ∑i∈Basis. g i n x *R i) ⟶ f x for x
    proof -
      have (λn. ∑i∈Basis. g i n x *R i) ⟶ (∑i∈Basis. (f x • i) *R i)
        by (auto intro!: tendsto_sum tendsto_scaleR g)
      moreover have (∑i∈Basis. (f x • i) *R i) = f x
        using euclidean_representation by blast
      ultimately show ?thesis
        by metis
    qed
  qed
qed
moreover have f ∈ borel_measurable (lebesgue_on S)
          if meas_g: ⋀n. g n ∈ borel_measurable (lebesgue_on S)
            and fin: ⋀n. finite (range (g n))
            and to_f: ⋀x. (λn. g n x) ⟶ f x for  g
  by (rule borel_measurable_LIMSEQ_metric [OF meas_g to_f])
```

ultimately show *?thesis*
using *borel_measurable_vimage_halfspace_component_lt* **by** *blast*
qed

9.8.24 Lebesgue sets and continuous images

proposition *lebesgue_regular_inner*:
assumes $S \in$ *sets lebesgue*
obtains $K\ C$ **where** *negligible* K $\bigwedge n$::*nat*. *compact*$(C\ n)$ $S = (\bigcup n.\ C\ n) \cup K$
proof −
have $\exists T.$ *closed* $T \wedge T \subseteq S \wedge (S - T) \in$ *lmeasurable* $\wedge$ *emeasure lebesgue* $(S - T) <$ *ennreal* $((1/2)\hat{}n)$ **for** n
using *sets_lebesgue_inner_closed assms*
by (*metis sets_lebesgue_inner_closed zero_less_divide_1_iff zero_less_numeral zero_less_power*)
then obtain C **where** *clo*: $\bigwedge n.$ *closed* $(C\ n)$ **and** *subS*: $\bigwedge n.\ C\ n \subseteq S$
and *mea*: $\bigwedge n.\ (S - C\ n) \in$ *lmeasurable*
and *less*: $\bigwedge n.$ *emeasure lebesgue* $(S - C\ n) <$ *ennreal* $((1/2)\hat{}n)$
by *metis*
have $\exists F.\ (\forall n$::*nat*. *compact*$(F\ n)) \wedge (\bigcup n.\ F\ n) = C\ m$ **for** m::*nat*
by (*metis clo closed_Union_compact_subsets*)
then obtain $D :: [nat,nat] \Rightarrow$ *'a set* **where** D: $\bigwedge m\ n.$ *compact*$(D\ m\ n)$ $\bigwedge m.$ $(\bigcup n.\ D\ m\ n) = C\ m$
by *metis*
let *?C* = *from_nat_into* $(\bigcup m.$ *range* $(D\ m))$
have *countable* $(\bigcup m.$ *range* $(D\ m))$
by *blast*
have *range* (*from_nat_into* $(\bigcup m.$ *range* $(D\ m))) = (\bigcup m.$ *range* $(D\ m))$
using *range_from_nat_into* **by** *simp*
then have *CD*: $\exists m\ n.$ *?C* $k = D\ m\ n$ **for** k
by (*metis* (*mono_tags*, *lifting*) *UN_iff rangeE range_eqI*)
show *thesis*
proof
show *negligible* $(S - (\bigcup n.\ C\ n))$
proof (*clarsimp simp*: *negligible_outer_le*)
fix e :: *real*
assume $e > 0$
then obtain n **where** n: $(1/2)\hat{}n < e$
using *real_arch_pow_inv* [*of e 1/2*] **by** *auto*
show $\exists T.\ S - (\bigcup n.\ C\ n) \subseteq T \wedge T \in$ *lmeasurable* $\wedge$ *measure lebesgue* $T \leq e$
proof (*intro exI conjI*)
show $S - (\bigcup n.\ C\ n) \subseteq S - C\ n$
by *blast*
show $S - C\ n \in$ *lmeasurable*
by (*simp add*: *mea*)
show *measure lebesgue* $(S - C\ n) \leq e$
using *less* [*of n*] n
by (*simp add*: *emeasure_eq_measure2 less_le mea*)
qed

```
    qed
    show compact (?C n) for n
      using CD D by metis
    show S = (⋃ n. ?C n) ∪ (S − (⋃ n. C n)) (is _ = ?rhs)
    proof
      show S ⊆ ?rhs
        using D by fastforce
      show ?rhs ⊆ S
        using subS D CD by auto (metis Sup_upper range_eqI subsetCE)
    qed
  qed
qed

lemma sets_lebesgue_continuous_image:
  assumes T: T ∈ sets lebesgue and contf: continuous_on S f
    and negim: ⋀T. ⟦negligible T; T ⊆ S⟧ ⟹ negligible(f ‘ T) and T ⊆ S
 shows f ‘ T ∈ sets lebesgue
proof −
  obtain K C where negligible K and com: ⋀n::nat. compact(C n) and Teq: T
= (⋃ n. C n) ∪ K
    using lebesgue_regular_inner [OF T] by metis
  then have comf: ⋀n::nat. compact(f ‘ C n)
    by (metis Un_subset_iff Union_upper ‹T ⊆ S› compact_continuous_image
contf continuous_on_subset rangeI)
  have ((⋃ n. f ‘ C n) ∪ f ‘ K) ∈ sets lebesgue
  proof (rule sets.Un)
    have K ⊆ S
      using Teq ‹T ⊆ S› by blast
    show (⋃ n. f ‘ C n) ∈ sets lebesgue
    proof (rule sets.countable_Union)
      show range (λn. f ‘ C n) ⊆ sets lebesgue
        using borel_compact comf by (auto simp: borel_compact)
    qed auto
    show f ‘ K ∈ sets lebesgue
      by (simp add: ‹K ⊆ S› ‹negligible K› negim negligible_imp_sets)
  qed
  then show ?thesis
    by (simp add: Teq image_Un image_Union)
qed

lemma differentiable_image_in_sets_lebesgue:
  fixes f :: 'm::euclidean_space ⇒ 'n::euclidean_space
  assumes S: S ∈ sets lebesgue and dim: DIM('m) ≤ DIM('n) and f: f differen-
tiable_on S
  shows f‘S ∈ sets lebesgue
proof (rule sets_lebesgue_continuous_image [OF S])
  show continuous_on S f
    by (meson differentiable_imp_continuous_on f)
  show ⋀T. ⟦negligible T; T ⊆ S⟧ ⟹ negligible (f ‘ T)
```

using *differentiable_on_subset f*
by (*auto simp*: *intro*!: *negligible_differentiable_image_negligible* [*OF dim*])
qed *auto*

lemma *sets_lebesgue_on_continuous_image*:
assumes *S*: $S \in$ *sets lebesgue* **and** *X*: $X \in$ *sets* (*lebesgue_on S*) **and** *contf*: *continuous_on S f*
and *negim*: $\bigwedge T.\ [\![\mathit{negligible}\ T;\ T \subseteq S]\!] \Longrightarrow \mathit{negligible}(f\ `\ T)$
shows $f\ `\ X \in$ *sets* (*lebesgue_on* ($f\ `\ S$))
proof −
have $X \subseteq S$
by (*metis S X sets.Int_space_eq2 sets_restrict_space_iff*)
moreover have $f\ `\ S \in$ *sets lebesgue*
using *S contf negim sets_lebesgue_continuous_image* **by** *blast*
moreover have $f\ `\ X \in$ *sets lebesgue*
by (*metis S X contf negim sets_lebesgue_continuous_image sets_restrict_space_iff space_restrict_space space_restrict_space2*)
ultimately show *?thesis*
by (*auto simp*: *sets_restrict_space_iff*)
qed

lemma *differentiable_image_in_sets_lebesgue_on*:
fixes $f :: 'm{::}\mathit{euclidean\_space} \Rightarrow 'n{::}\mathit{euclidean\_space}$
assumes *S*: $S \in$ *sets lebesgue* **and** *X*: $X \in$ *sets* (*lebesgue_on S*) **and** *dim*: $\mathit{DIM}('m) \leq \mathit{DIM}('n)$
and *f*: *f differentiable_on S*
shows $f\ `\ X \in$ *sets* (*lebesgue_on* ($f$`$S$))
proof (*rule sets_lebesgue_on_continuous_image* [*OF S X*])
show *continuous_on S f*
by (*meson differentiable_imp_continuous_on f*)
show $\bigwedge T.\ [\![\mathit{negligible}\ T;\ T \subseteq S]\!] \Longrightarrow \mathit{negligible}\ (f\ `\ T)$
using *differentiable_on_subset f*
by (*auto simp*: *intro*!: *negligible_differentiable_image_negligible* [*OF dim*])
qed

9.8.25 Affine lemmas

lemma *borel_measurable_affine*:
fixes $f :: 'a :: \mathit{euclidean\_space} \Rightarrow 'b :: \mathit{euclidean\_space}$
assumes *f*: $f \in$ *borel_measurable lebesgue* **and** $c \neq 0$
shows $(\lambda x.\ f(t + c *_R x)) \in$ *borel_measurable lebesgue*
proof −
{ fix *a b*
have $\{x.\ f\ x \in \mathit{cbox}\ a\ b\} \in$ *sets lebesgue*
using *f cbox_borel lebesgue_measurable_vimage_borel* **by** *blast*
then have $(\lambda x.\ (x - t)\ /_R\ c)$ `$\{x.\ f\ x \in \mathit{cbox}\ a\ b\} \in$ *sets lebesgue*
proof (*rule differentiable_image_in_sets_lebesgue*)
show $(\lambda x.\ (x - t)\ /_R\ c)$ *differentiable_on* $\{x.\ f\ x \in \mathit{cbox}\ a\ b\}$
unfolding *differentiable_on_def differentiable_def*

```
      by (rule ‹c ≠ 0› derivative_eq_intros strip exI | simp)+
    qed auto
    moreover
    have {x. f(t + c *R x) ∈ cbox a b} = (λx. (x−t) /R c) ‘ {x. f x ∈ cbox a b}
      using ‹c ≠ 0› by (auto simp: image_def)
    ultimately have {x. f(t + c *R x) ∈ cbox a b} ∈ sets lebesgue
      by (auto simp: borel_measurable_vimage_closed_interval) }
  then show ?thesis
   by (subst lebesgue_on_UNIV_eq [symmetric]; auto simp: borel_measurable_vimage_closed_interval)
qed

lemma lebesgue_integrable_real_affine:
  fixes f :: real ⇒ 'a :: euclidean_space
  assumes f: integrable lebesgue f and c ≠ 0
  shows integrable lebesgue (λx. f(t + c ∗ x))
proof −
  have (λx. norm (f x)) ∈ borel_measurable lebesgue
    by (simp add: borel_measurable_integrable f)
  then show ?thesis
    using assms borel_measurable_affine [of f c]
    unfolding integrable_iff_bounded
   by (subst (asm) nn_integral_real_affine_lebesgue[where c=c and t=t]) (auto
simp: ennreal_mult_less_top)
qed

lemma lebesgue_integrable_real_affine_iff:
  fixes f :: real ⇒ 'a :: euclidean_space
  shows c ≠ 0 ⟹ integrable lebesgue (λx. f(t + c ∗ x)) ⟷ integrable lebesgue f
  using lebesgue_integrable_real_affine[of f c t]
        lebesgue_integrable_real_affine[of λx. f(t + c ∗ x) 1/c −t/c]
  by (auto simp: field_simps)

lemma lebesgue_integral_real_affine:
  fixes f :: real ⇒ 'a :: euclidean_space and c :: real
   assumes c: c ≠ 0 shows (∫ x. f x ∂ lebesgue) = |c| *R (∫ x. f(t + c ∗ x)
∂lebesgue)
proof cases
  have (λx. t + c ∗ x) ∈ lebesgue →M lebesgue
    using lebesgue_affine_measurable[where c= λx::real. c] ‹c ≠ 0› by simp
  moreover
  assume integrable lebesgue f
  ultimately show ?thesis
    by (subst lebesgue_real_affine[OF c, of t]) (auto simp: integral_density inte-
gral_distr)
next
  assume ¬ integrable lebesgue f with c show ?thesis
    by (simp add: lebesgue_integrable_real_affine_iff not_integrable_integral_eq)
qed
```

lemma *has_bochner_integral_lebesgue_real_affine_iff*:
fixes $i :: 'a :: euclidean\_space$
shows $c \neq 0 \Longrightarrow$
$has\_bochner\_integral\ lebesgue\ f\ i \longleftrightarrow$
$has\_bochner\_integral\ lebesgue\ (\lambda x.\ f(t + c * x))\ (i\ /_R\ |c|)$
unfolding *has_bochner_integral_iff lebesgue_integrable_real_affine_iff*
by (*simp_all add*: *lebesgue_integral_real_affine*[*symmetric*] *divideR_right cong*: *conj_cong*)

lemma *has_bochner_integral_reflect_real_lemma*[*intro*]:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
assumes *has_bochner_integral* (*lebesgue_on* $\{a..b\}$) $f\ i$
shows *has_bochner_integral* (*lebesgue_on* $\{-b..-a\}$) $(\lambda x.\ f(-x))\ i$
proof −
have *eq*: *indicat_real* $\{a..b\}\ (-\ x) *_R f(-\ x) =$ *indicat_real* $\{-\ b..-\ a\}\ x *_R$
$f(-\ x)$ **for** x
by (*auto simp*: *indicator_def*)
have *i*: *has_bochner_integral lebesgue* $(\lambda x.$ *indicator* $\{a..b\}\ x *_R f\ x)\ i$
using *assms* **by** (*auto simp*: *has_bochner_integral_restrict_space*)
then have *has_bochner_integral lebesgue* $(\lambda x.$ *indicator* $\{-b..-a\}\ x *_R f(-x))$
i
using *has_bochner_integral_lebesgue_real_affine_iff* [*of* -1 $(\lambda x.$ *indicator*
$\{a..b\}\ x *_R f\ x)\ i\ 0$]
by (*auto simp*: *eq*)
then show *?thesis*
by (*auto simp*: *has_bochner_integral_restrict_space*)
qed

lemma *has_bochner_integral_reflect_real*[*simp*]:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
shows *has_bochner_integral* (*lebesgue_on* $\{-b..-a\}$) $(\lambda x.\ f(-x))\ i \longleftrightarrow$ *has_bochner_integral*
(*lebesgue_on* $\{a..b\}$) $f\ i$
by (*auto simp*: *dest*: *has_bochner_integral_reflect_real_lemma*)

lemma *integrable_reflect_real*[*simp*]:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
shows *integrable* (*lebesgue_on* $\{-b..-a\}$) $(\lambda x.\ f(-x)) \longleftrightarrow$ *integrable* (*lebesgue_on*
$\{a..b\}$) f
by (*metis has_bochner_integral_iff has_bochner_integral_reflect_real*)

lemma *integral_reflect_real*[*simp*]:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
shows $integral^L$ (*lebesgue_on* $\{-b\ ..\ -a\}$) $(\lambda x.\ f(-x)) = integral^L$ (*lebesgue_on*
$\{a..b::real\}$) f
using *has_bochner_integral_reflect_real* [*of b a f*]
by (*metis has_bochner_integral_iff not_integrable_integral_eq*)

9.8.26 More results on integrability

lemma *integrable_on_all_intervals_UNIV*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::banach$
assumes *intf*: $\bigwedge a\ b.\ f$ *integrable_on cbox a b*
and *normf*: $\bigwedge x.\ norm(f\ x) \leq g\ x$ **and** *g*: *g integrable_on UNIV*
shows *f integrable_on UNIV*
proof −
have *intg*: $(\forall\ a\ b.\ g$ *integrable_on cbox a b*$)$
and *gle_e*: $\forall\ e{>}0.\ \exists\ B{>}0.\ \forall\ a\ b\ c\ d.$
ball 0 B $\subseteq$ *cbox a b* $\wedge$ *cbox a b* $\subseteq$ *cbox c d* $\longrightarrow$
$|$*integral* (*cbox a b*) *g* − *integral* (*cbox c d*) *g*$|$
$<$ *e*
using *g*
by (*auto simp*: *integrable_alt_subset* [*of* _ *UNIV*] *intf*)
have *le*: *norm* (*integral* (*cbox a b*) *f* − *integral* (*cbox c d*) *f*) ≤ $|$*integral* (*cbox a b*) *g* − *integral* (*cbox c d*) *g*$|$
if *cbox a b* ⊆ *cbox c d* **for** *a b c d*
proof −
have *norm* (*integral* (*cbox a b*) *f* − *integral* (*cbox c d*) *f*) = *norm* (*integral* (*cbox c d* − *cbox a b*) *f*)
using *intf that* **by** (*simp add*: *norm_minus_commute integral_setdiff*)
also have ... ≤ *integral* (*cbox c d* − *cbox a b*) *g*
proof (*rule integral_norm_bound_integral* [*OF* _ _ *normf*])
show *f integrable_on cbox c d* − *cbox a b g integrable_on cbox c d* − *cbox a b*
by (*meson integrable_integral integrable_setdiff intf intg negligible_setdiff that*)+
qed
also have ... = *integral* (*cbox c d*) *g* − *integral* (*cbox a b*) *g*
using *intg that* **by** (*simp add*: *integral_setdiff*)
also have ... ≤ $|$*integral* (*cbox a b*) *g* − *integral* (*cbox c d*) *g*$|$
by *simp*
finally show *?thesis* **.**
qed
show *?thesis*
using *gle_e*
apply (*simp add*: *integrable_alt_subset* [*of* _ *UNIV*] *intf*)
apply (*erule imp_forward all_forward ex_forward asm_rl*)+
by (*meson not_less order_trans le*)
qed

lemma *integrable_on_all_intervals_integrable_bound*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::banach$
assumes *intf*: $\bigwedge a\ b.\ (\lambda x.$ *if* $x \in S$ *then f x else 0*$)$ *integrable_on cbox a b*
and *normf*: $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq g\ x$ **and** *g*: *g integrable_on S*
shows *f integrable_on S*
using *integrable_on_all_intervals_UNIV* [*OF intf*, *of* $(\lambda x.$ *if* $x \in S$ *then g x else 0*$)$]
by (*simp add*: *g integrable_restrict_UNIV normf*)

lemma *measurable_bounded_lemma*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f* ∈ *borel_measurable lebesgue* **and** *g*: *g integrable_on cbox a b*
and *normf*: ⋀*x*. *x* ∈ *cbox a b* ⟹ *norm*(*f x*) ≤ *g x*
shows *f integrable_on cbox a b*
proof −
have *g absolutely_integrable_on cbox a b*
by (*metis* (*full_types*) *add_increasing g le_add_same_cancel1 nonnegative_absolutely_integrable_1 norm_ge_zero normf*)
then have *integrable* (*lebesgue_on* (*cbox a b*)) *g*
by (*simp add*: *integrable_restrict_space set_integrable_def*)
then have *integrable* (*lebesgue_on* (*cbox a b*)) *f*
proof (*rule Bochner_Integration.integrable_bound*)
show *AE x in lebesgue_on* (*cbox a b*). *norm* (*f x*) ≤ *norm* (*g x*)
by (*rule AE_I2*) (*auto intro*: *normf order_trans*)
qed (*simp add*: *f measurable_restrict_space1*)
then show *?thesis*
by (*simp add*: *integrable_on_lebesgue_on*)
qed

proposition *measurable_bounded_by_integrable_imp_integrable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f* ∈ *borel_measurable* (*lebesgue_on S*) **and** *g*: *g integrable_on S*
and *normf*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) ≤ *g x* **and** *S*: *S* ∈ *sets lebesgue*
shows *f integrable_on S*
proof (*rule integrable_on_all_intervals_integrable_bound* [*OF* _ *normf g*])
show (λ*x*. *if x* ∈ *S then f x else 0*) *integrable_on cbox a b* **for** *a b*
proof (*rule measurable_bounded_lemma*)
show (λ*x*. *if x* ∈ *S then f x else 0*) ∈ *borel_measurable lebesgue*
by (*simp add*: *S borel_measurable_if f*)
show (λ*x*. *if x* ∈ *S then g x else 0*) *integrable_on cbox a b*
by (*simp add*: *g integrable_altD*(*1*))
show *norm* (*if x* ∈ *S then f x else 0*) ≤ (*if x* ∈ *S then g x else 0*) **for** *x*
using *normf* **by** *simp*
qed
qed

lemma *measurable_bounded_by_integrable_imp_lebesgue_integrable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f* ∈ *borel_measurable* (*lebesgue_on S*) **and** *g*: *integrable* (*lebesgue_on S*) *g*
and *normf*: ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) ≤ *g x* **and** *S*: *S* ∈ *sets lebesgue*
shows *integrable* (*lebesgue_on S*) *f*
proof −
have *f absolutely_integrable_on S*
by (*metis* (*no_types*) *S absolutely_integrable_integrable_bound f g integrable_on_lebesgue_on measurable_bounded_by_integrable_imp_integrable normf*)
then show *?thesis*
by (*simp add*: *S integrable_restrict_space set_integrable_def*)

qed

lemma *measurable_bounded_by_integrable_imp_integrable_real*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real*
assumes *f* ∈ *borel_measurable* (*lebesgue_on S*) *g integrable_on S* ⋀*x*. *x* ∈ *S* ⟹ *abs*(*f x*) ≤ *g x S* ∈ *sets lebesgue*
shows *f integrable_on S*
using *measurable_bounded_by_integrable_imp_integrable* [*of f S g*] *assms* **by** *simp*

9.8.27 Relation between Borel measurability and integrability.

lemma *integrable_imp_measurable_weak*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S* ∈ *sets lebesgue f integrable_on S*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*)
by (*metis* (*mono_tags*, *lifting*) *assms has_integral_implies_lebesgue_measurable borel_measurable_restrict_space_iff integrable_on_def sets.Int_space_eq2*)

lemma *integrable_imp_measurable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f integrable_on S*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*)
proof −
have (*UNIV*::*'a set*) ∈ *sets lborel*
by *simp*
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *assms borel_measurable_if_D integrable_imp_measurable_weak integrable_restrict_UNIV lebesgue_on_UNIV_eq sets_lebesgue_on_refl*)
qed

lemma *integrable_iff_integrable_on*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S* ∈ *sets lebesgue* (∫$^+$ *x*. *ennreal* (*norm* (*f x*)) ∂*lebesgue_on S*) < ∞
shows *integrable* (*lebesgue_on S*) *f* ⟷ *f integrable_on S*
using *assms integrable_iff_bounded integrable_imp_measurable integrable_on_lebesgue_on* **by** *blast*

lemma *absolutely_integrable_measurable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S* ∈ *sets lebesgue*
shows *f absolutely_integrable_on S* ⟷ *f* ∈ *borel_measurable* (*lebesgue_on S*) ∧ *integrable* (*lebesgue_on S*) (*norm* ∘ *f*)
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then have *f* ∈ *borel_measurable* (*lebesgue_on S*)
by (*simp add*: *absolutely_integrable_on_def integrable_imp_measurable*)

then show *?rhs*
using *assms set_integrable_norm* [*of lebesgue S f*] *L*
by (*simp add*: *integrable_restrict_space set_integrable_def*)
next
assume *?rhs* **then show** *?lhs*
using *assms integrable_on_lebesgue_on*
by (*metis absolutely_integrable_integrable_bound comp_def eq_iff measurable_bounded_by_integrable_imp_integrable*)
qed

lemma *absolutely_integrable_measurable_real*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real*
assumes *S* ∈ *sets lebesgue*
shows *f absolutely_integrable_on S* ⟷
f ∈ *borel_measurable* (*lebesgue_on S*) ∧ *integrable* (*lebesgue_on S*) (λ*x*. |*f x*|)
by (*simp add*: *absolutely_integrable_measurable assms o_def*)

lemma *absolutely_integrable_measurable_real'*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real*
assumes *S* ∈ *sets lebesgue*
shows *f absolutely_integrable_on S* ⟷ *f* ∈ *borel_measurable* (*lebesgue_on S*) ∧ (λ*x*. |*f x*|) *integrable_on S*
by (*metis abs_absolutely_integrableI_1 absolutely_integrable_measurable_real assms*
measurable_bounded_by_integrable_imp_integrable order_refl real_norm_def set_integrable_abs set_lebesgue_integral_eq_integral(*1*))

lemma *absolutely_integrable_imp_borel_measurable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f absolutely_integrable_on S S* ∈ *sets lebesgue*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*)
using *absolutely_integrable_measurable assms* **by** *blast*

lemma *measurable_bounded_by_integrable_imp_absolutely_integrable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f* ∈ *borel_measurable* (*lebesgue_on S*) *S* ∈ *sets lebesgue*
and *g integrable_on S* **and** ⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x*) ≤ (*g x*)
shows *f absolutely_integrable_on S*
using *assms absolutely_integrable_integrable_bound measurable_bounded_by_integrable_imp_integrable*
by *blast*

proposition *negligible_differentiable_vimage*:
fixes *f* :: *'a* ⇒ *'a*::*euclidean_space*
assumes *negligible T*
and *f'*: ⋀*x*. *x* ∈ *S* ⟹ *inj*(*f' x*)
and *derf*: ⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f' x*) (*at x within S*)
shows *negligible* {*x* ∈ *S*. *f x* ∈ *T*}
proof −

```
  define U where
    U ≡ λn::nat. {x ∈ S. ∀ y. y ∈ S ∧ norm(y − x) < 1/n
                  ⟶ norm(y − x) ≤ n ∗ norm(f y − f x)}
  have negligible {x ∈ U n. f x ∈ T} if n > 0 for n
  proof (subst locally_negligible_alt, clarify)
    fix a
    assume a: a ∈ U n and fa: f a ∈ T
    define V where V ≡ {x. x ∈ U n ∧ f x ∈ T} ∩ ball a (1 / n / 2)
    show ∃ V. openin (top_of_set {x ∈ U n. f x ∈ T}) V ∧ a ∈ V ∧ negligible V
    proof (intro exI conjI)
      have noxy: norm(x − y) ≤ n ∗ norm(f x − f y) if x ∈ V y ∈ V for x y
        using that unfolding U_def V_def mem_Collect_eq Int_iff mem_ball
dist_norm
        by (meson norm_triangle_half_r)
      then have inj_on f V
        by (force simp: inj_on_def)
      then obtain g where g: ⋀x. x ∈ V ⟹ g(f x) = x
        by (metis inv_into_f_f)
      have ∃ T' B. open T' ∧ f x ∈ T' ∧
                (∀ y∈f ‘ V ∩ T ∩ T'. norm (g y − g (f x)) ≤ B ∗ norm (y − f x))
        if f x ∈ T x ∈ V for x
        using that noxy
        by (rule_tac x = ball (f x) 1 in exI) (force simp: g)
      then have negligible (g ‘ (f ‘ V ∩ T))
      by (force simp: ‹negligible T› negligible_Int intro!: negligible_locally_Lipschitz_image)
      moreover have V ⊆ g ‘ (f ‘ V ∩ T)
        by (force simp: g image_iff V_def)
      ultimately show negligible V
        by (rule negligible_subset)
    qed (use a fa V_def that in auto)
  qed
  with negligible_countable_Union have negligible (⋃ n ∈ {0<..}. {x. x ∈ U n ∧
f x ∈ T})
    by auto
  moreover have {x ∈ S. f x ∈ T} ⊆ (⋃ n ∈ {0<..}. {x. x ∈ U n ∧ f x ∈ T})
  proof clarsimp
    fix x
    assume x ∈ S and f x ∈ T
    then obtain inj: inj(f' x) and der: (f has_derivative f' x) (at x within S)
      using assms by metis
    moreover have linear(f' x)
      and eps: ⋀ε. ε > 0 ⟹ ∃ δ>0. ∀ y∈S. norm (y − x) < δ ⟶
                    norm (f y − f x − f' x (y − x)) ≤ ε ∗ norm (y − x)
      using der by (auto simp: has_derivative_within_alt linear_linear)
    ultimately obtain g where linear g and g: g ∘ f' x = id
      using linear_injective_left_inverse by metis
    then obtain B where B > 0 and B: ⋀z. B ∗ norm z ≤ norm(f' x z)
      using linear_invertible_bounded_below_pos ‹linear (f' x)› by blast
    then obtain i where i ≠ 0 and i: 1 / real i < B
```

```
    by (metis (full_types) inverse_eq_divide real_arch_invD)
  then obtain δ where δ > 0
      and δ: ⋀y. ⟦y∈S; norm (y − x) < δ⟧ ⟹
            norm (f y − f x − f' x (y − x)) ≤ (B − 1 / real i) * norm (y − x)
    using eps [of B − 1/i] by auto
  then obtain j where j ≠ 0 and j: inverse (real j) < δ
    using real_arch_inverse by blast
  have norm (y − x)/(max i j) ≤ norm (f y − f x)
    if y ∈ S and less: norm (y − x) < 1 / (max i j) for y
  proof −
    have 1 / real (max i j) < δ
      using j ‹j ≠ 0› ‹0 < δ›
      by (auto simp: field_split_simps max_mult_distrib_left of_nat_max)
  then have norm (y − x) < δ
    using less by linarith
  with δ ‹y ∈ S› have le: norm (f y − f x − f' x (y − x)) ≤ B * norm (y − x)
− norm (y − x)/i
    by (auto simp: algebra_simps)
  have norm (y − x) / real (max i j) ≤ norm (y − x) / real i
   using ‹i ≠ 0› ‹j ≠ 0› by (simp add: field_split_simps max_mult_distrib_left
of_nat_max less_max_iff_disj)
  also have ... ≤ norm (f y − f x)
    using B [of y−x] le norm_triangle_ineq3 [of f y − f x f' x (y − x)]
    by linarith
  finally show ?thesis .
 qed
 with ‹x ∈ S› ‹i ≠ 0› ‹j ≠ 0› show ∃ n∈{0<..}. x ∈ U n
  by (rule_tac x=max i j in bexI) (auto simp: field_simps U_def less_max_iff_disj)
qed
 ultimately show ?thesis
   by (rule negligible_subset)
qed

lemma absolutely_integrable_Un:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes S: f absolutely_integrable_on S and T: f absolutely_integrable_on T
 shows f absolutely_integrable_on (S ∪ T)
proof −
 have [simp]: {x. (if x ∈ A then f x else 0) ≠ 0} = {x ∈ A. f x ≠ 0} for A
   by auto
 let ?ST = {x ∈ S. f x ≠ 0} ∩ {x ∈ T. f x ≠ 0}
 have ?ST ∈ sets lebesgue
 proof (rule Sigma_Algebra.sets.Int)
   have f integrable_on S
     using S absolutely_integrable_on_def by blast
   then have (λx. if x ∈ S then f x else 0) integrable_on UNIV
     by (simp add: integrable_restrict_UNIV)
  then have borel: (λx. if x ∈ S then f x else 0) ∈ borel_measurable (lebesgue_on
UNIV)
```

```
    using integrable_imp_measurable lebesgue_on_UNIV_eq by blast
  then show {x ∈ S. f x ≠ 0} ∈ sets lebesgue
    unfolding borel_measurable_vimage_open
    by (rule allE [where x = -{0}]) auto
next
  have f integrable_on T
    using T absolutely_integrable_on_def by blast
  then have (λx. if x ∈ T then f x else 0) integrable_on UNIV
    by (simp add: integrable_restrict_UNIV)
 then have borel: (λx. if x ∈ T then f x else 0) ∈ borel_measurable (lebesgue_on
UNIV)
    using integrable_imp_measurable lebesgue_on_UNIV_eq by blast
  then show {x ∈ T. f x ≠ 0} ∈ sets lebesgue
    unfolding borel_measurable_vimage_open
    by (rule allE [where x = -{0}]) auto
qed
then have f absolutely_integrable_on ?ST
  by (rule set_integrable_subset [OF S]) auto
then have Int: (λx. if x ∈ ?ST then f x else 0) absolutely_integrable_on UNIV
  using absolutely_integrable_restrict_UNIV by blast
have (λx. if x ∈ S then f x else 0) absolutely_integrable_on UNIV
     (λx. if x ∈ T then f x else 0) absolutely_integrable_on UNIV
  using S T absolutely_integrable_restrict_UNIV by blast+
then have (λx. (if x ∈ S then f x else 0) + (if x ∈ T then f x else 0)) abso-
lutely_integrable_on UNIV
  by (rule set_integral_add)
then have (λx. ((if x ∈ S then f x else 0) + (if x ∈ T then f x else 0)) - (if x
∈ ?ST then f x else 0)) absolutely_integrable_on UNIV
  using Int by (rule set_integral_diff)
then have (λx. if x ∈ S ∪ T then f x else 0) absolutely_integrable_on UNIV
  by (rule absolutely_integrable_spike) (auto intro: empty_imp_negligible)
then show ?thesis
  unfolding absolutely_integrable_restrict_UNIV .
qed

lemma absolutely_integrable_on_combine:
fixes f :: real ⇒ 'a::euclidean_space
assumes f absolutely_integrable_on {a..c}
  and f absolutely_integrable_on {c..b}
  and a ≤ c
  and c ≤ b
shows f absolutely_integrable_on {a..b}
by (metis absolutely_integrable_Un assms ivl_disj_un_two_touch(4))

lemma uniform_limit_set_lebesgue_integral_at_top:
fixes f :: 'a ⇒ real ⇒ 'b::{banach, second_countable_topology}
  and g :: real ⇒ real
assumes bound: ⋀x y. x ∈ A ⟹ y ≥ a ⟹ norm (f x y) ≤ g y
assumes integrable: set_integrable M {a..} g
```

assumes *measurable*: $\bigwedge x.\ x \in A \Longrightarrow$ *set_borel_measurable M* $\{a..\}$ $(f\ x)$
assumes *sets borel* $\subseteq$ *sets M*
shows *uniform_limit A* $(\lambda b\ x.$ *LINT* $y{:}\{a..b\}|M.\ f\ x\ y)$ $(\lambda x.$ *LINT* $y{:}\{a..\}|M.$ $f\ x\ y)$ *at_top*
proof (*cases* $A = \{\}$)
case *False*
then obtain x **where** x: $x \in A$ **by** *auto*
have *g_nonneg*: $g\ y \geq 0$ **if** $y \geq a$ **for** y
proof −
have $0 \leq$ *norm* $(f\ x\ y)$ **by** *simp*
also have $\ldots \leq g\ y$ **using** *bound*[*OF x that*] **by** *simp*
finally show *?thesis* **.**
qed

have *integrable'*: *set_integrable M* $\{a..\}$ $(\lambda y.\ f\ x\ y)$ **if** $x \in A$ **for** x
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound*)
show *integrable M* $(\lambda x.$ *indicator* $\{a..\}\ x * g\ x)$
using *integrable* **by** (*simp add*: *set_integrable_def*)
show $(\lambda y.$ *indicat_real* $\{a..\}\ y *_R f\ x\ y) \in$ *borel_measurable M* **using** *measurable*[*OF that*]
by (*simp add*: *set_borel_measurable_def*)
show *AE y in M. norm* (*indicat_real* $\{a..\}\ y *_R f\ x\ y) \leq$ *norm* (*indicat_real* $\{a..\}\ y * g\ y$)
using *bound*[*OF that*] **by** (*intro AE_I2*) (*auto simp*: *indicator_def g_nonneg*)
qed

show *?thesis*
proof (*rule uniform_limitI*)
fix e :: *real* **assume** e: $e > 0$
have *sets* [*intro*]: $A \in$ *sets M* **if** $A \in$ *sets borel* **for** A
using *that assms* **by** *blast*

have $((\lambda b.$ *LINT* $y{:}\{a..b\}|M.\ g\ y) \longrightarrow$ (*LINT* $y{:}\{a..\}|M.\ g\ y))$ *at_top*
by (*intro tendsto_set_lebesgue_integral_at_top assms sets*) *auto*
with e **obtain** *b0* :: *real* **where** *b0*: $\forall\, b{\geq}b0.$ $|($*LINT* $y{:}\{a..\}|M.\ g\ y) - ($*LINT* $y{:}\{a..b\}|M.\ g\ y)| < e$
by (*auto simp*: *tendsto_iff eventually_at_top_linorder dist_real_def abs_minus_commute*)
define b **where** $b =$ *max a b0*
have $a \leq b$ **by** (*simp add*: *b_def*)
from *b0* **have** $|($*LINT* $y{:}\{a..\}|M.\ g\ y) - ($*LINT* $y{:}\{a..b\}|M.\ g\ y)| < e$
by (*auto simp*: *b_def*)
also have $\{a..\} = \{a..b\} \cup \{b{<}..\}$ **by** (*auto simp*: *b_def*)
also have $|($*LINT* $y{:}\ldots|M.\ g\ y) - ($*LINT* $y{:}\{a..b\}|M.\ g\ y)| = |($*LINT* $y{:}\{b{<}..\}|M.\ g\ y)|$
using $\langle a \leq b \rangle$ **by** (*subst set_integral_Un*) (*auto intro*!: *set_integrable_subset*[*OF integrable*])
also have (*LINT* $y{:}\{b{<}..\}|M.\ g\ y) \geq 0$
using *g_nonneg* $\langle a \leq b \rangle$ **unfolding** *set_lebesgue_integral_def*

```
    by (intro Bochner_Integration.integral_nonneg) (auto simp: indicator_def)
  hence |(LINT y:{b<..}|M. g y)| = (LINT y:{b<..}|M. g y) by simp
  finally have less: (LINT y:{b<..}|M. g y) < e .

  have eventually (λb. b ≥ b0) at_top by (rule eventually_ge_at_top)
 moreover have eventually (λb. b ≥ a) at_top by (rule eventually_ge_at_top)
  ultimately show eventually (λb. ∀x∈A.
                    dist (LINT y:{a..b}|M. f x y) (LINT y:{a..}|M. f x y) < e)
at_top
  proof eventually_elim
    case (elim b)
    show ?case
    proof
      fix x assume x: x ∈ A
      have dist (LINT y:{a..b}|M. f x y) (LINT y:{a..}|M. f x y) =
            norm ((LINT y:{a..}|M. f x y) − (LINT y:{a..b}|M. f x y))
        by (simp add: dist_norm norm_minus_commute)
      also have {a..} = {a..b} ∪ {b<..} using elim by auto
       also have (LINT y:...|M. f x y) − (LINT y:{a..b}|M. f x y) = (LINT
y:{b<..}|M. f x y)
        using elim x
         by (subst set_integral_Un) (auto intro!: set_integrable_subset[OF inte-
grable'])
      also have norm ... ≤ (LINT y:{b<..}|M. norm (f x y)) using elim x
       by (intro set_integral_norm_bound set_integrable_subset[OF integrable'])
auto
      also have ... ≤ (LINT y:{b<..}|M. g y) using elim x bound g_nonneg
      by (intro set_integral_mono set_integrable_norm set_integrable_subset[OF
integrable']
                set_integrable_subset[OF integrable]) auto
      also have (LINT y:{b<..}|M. g y) ≥ 0
        using g_nonneg ‹a ≤ b› unfolding set_lebesgue_integral_def
       by (intro Bochner_Integration.integral_nonneg) (auto simp: indicator_def)
      hence (LINT y:{b<..}|M. g y) = |(LINT y:{b<..}|M. g y)| by simp
      also have ... = |(LINT y:{a..b} ∪ {b<..}|M. g y) − (LINT y:{a..b}|M. g
y)|
      using elim by (subst set_integral_Un) (auto intro!: set_integrable_subset[OF
integrable])
      also have {a..b} ∪ {b<..} = {a..} using elim by auto
      also have |(LINT y:{a..}|M. g y) − (LINT y:{a..b}|M. g y)| < e
        using b0 elim by blast
      finally show dist (LINT y:{a..b}|M. f x y) (LINT y:{a..}|M. f x y) < e .
    qed
   qed
  qed
qed auto
```

Differentiability of inverse function (most basic form)

proposition *has_derivative_inverse_within*:
fixes f :: *'a::real_normed_vector* ⇒ *'b::euclidean_space*
assumes *der_f*: $(f$ *has_derivative* $f')$ $(at\ a\ within\ S)$
and *cont_g*: *continuous* $(at\ (f\ a)\ within\ f\ `\ S)\ g$
and $a \in S$ *linear* g' **and** *id*: $g' \circ f' = id$
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$
shows $(g$ *has_derivative* $g')$ $(at\ (f\ a)\ within\ f\ `\ S)$
proof −
have [*simp*]: $g'\ (f'\ x) = x$ **for** x
by (*simp add*: *local.id pointfree_idE*)
have *bounded_linear* f'
and f': $\bigwedge e.\ e{>}0 \Longrightarrow \exists\, d{>}0.\ \forall\, y{\in}S.\ norm\ (y - a) < d \longrightarrow$
$norm\ (f\ y - f\ a - f'\ (y - a)) \leq e * norm\ (y - a)$
using *der_f* **by** (*auto simp*: *has_derivative_within_alt*)
obtain C **where** $C > 0$ **and** C: $\bigwedge x.\ norm\ (g'\ x) \leq C * norm\ x$
using *linear_bounded_pos* [*OF* ‹*linear* g'›] **by** *metis*
obtain $B\ k$ **where** $B > 0\ k > 0$
and *Bk*: $\bigwedge x.\ [\![x \in S;\ norm(f\ x - f\ a) < k]\!] \Longrightarrow norm(x - a) \leq B * norm(f\ x$
$- f\ a)$
proof −
obtain B **where** $B > 0$ **and** B: $\bigwedge x.\ B * norm\ x \leq norm\ (f'\ x)$
using *linear_inj_bounded_below_pos* [*of* f'] ‹*linear* g'› *id der_f has_derivative_linear*
linear_invertible_bounded_below_pos **by** *blast*
then obtain d **where** $d{>}0$
and d: $\bigwedge y.\ [\![y \in S;\ norm\ (y - a) < d]\!] \Longrightarrow$
$norm\ (f\ y - f\ a - f'\ (y - a)) \leq B\ /\ 2 * norm\ (y - a)$
using f' [*of* $B/2$] **by** *auto*
then obtain e **where** $e > 0$
and e: $\bigwedge x.\ [\![x \in S;\ norm\ (f\ x - f\ a) < e]\!] \Longrightarrow norm\ (g\ (f\ x) - g\ (f\ a)) < d$
using *cont_g* **by** (*auto simp*: *continuous_within_eps_delta dist_norm*)
show *thesis*
proof
show $2/B > 0$
using ‹$B > 0$› **by** *simp*
show $norm\ (x - a) \leq 2\ /\ B * norm\ (f\ x - f\ a)$
if $x \in S\ norm\ (f\ x - f\ a) < e$ **for** x
proof −
have *xa*: $norm\ (x - a) < d$
using e [*OF that*] *gf* **by** (*simp add*: ‹$a \in S$› *that*)
have ∗: $[\![norm(y - f') \leq B\ /\ 2 * norm\ x;\ B * norm\ x \leq norm\ f']\!]$
$\Longrightarrow norm\ y \geq B\ /\ 2 * norm\ x$ **for** $y\ f'$::*'b* **and** x::*'a*
using *norm_triangle_ineq3* [*of* $y\ f'$] **by** *linarith*
show *?thesis*
using ∗ [*OF* d [*OF* ‹$x \in S$› *xa*] B] ‹$B > 0$› **by** (*simp add*: *field_simps*)
qed
qed (*use* ‹$e > 0$› **in** *auto*)
qed
show *?thesis*

```
    unfolding has_derivative_within_alt
  proof (intro conjI impI allI)
    show bounded_linear g'
      using ‹linear g'› by (simp add: linear_linear)
  next
    fix e :: real
    assume e > 0
    then obtain d where d>0
      and d: ⋀y. ⟦y ∈ S; norm (y - a) < d⟧ ⟹
                 norm (f y - f a - f' (y - a)) ≤ e / (B * C) * norm (y - a)
      using f' [of e / (B * C)] ‹B > 0› ‹C > 0› by auto
    have norm (x - a - g' (f x - f a)) ≤ e * norm (f x - f a)
     if x ∈ S and lt_k: norm (f x - f a) < k and lt_dB: norm (f x - f a) < d/B
for x
    proof -
      have norm (x - a) ≤ B * norm(f x - f a)
        using Bk lt_k ‹x ∈ S› by blast
      also have ... < d
        by (metis ‹0 < B› lt_dB mult.commute pos_less_divide_eq)
      finally have lt_d: norm (x - a) < d .
      have norm (x - a - g' (f x - f a)) ≤ norm(g'(f x - f a - (f' (x - a))))
        by (simp add: linear_diff [OF ‹linear g'›] norm_minus_commute)
      also have ... ≤ C * norm (f x - f a - f' (x - a))
        using C by blast
      also have ... ≤ e * norm (f x - f a)
      proof -
        have norm (f x - f a - f' (x - a)) ≤ e / (B * C) * norm (x - a)
          using d [OF ‹x ∈ S› lt_d] .
        also have ... ≤ (norm (f x - f a) * e) / C
          using ‹B > 0› ‹C > 0› ‹e > 0› by (simp add: field_simps Bk lt_k ‹x ∈
S›)
        finally show ?thesis
          using ‹C > 0› by (simp add: field_simps)
      qed
    finally show ?thesis .
    qed
    with ‹k > 0› ‹B > 0› ‹d > 0› ‹a ∈ S›
    show ∃ d>0. ∀ y∈f ' S.
              norm (y - f a) < d ⟶
              norm (g y - g (f a) - g' (y - f a)) ≤ e * norm (y - f a)
      by (rule_tac x=min k (d / B) in exI) (auto simp: gf)
  qed
qed

end
```

9.9 Harmonic Numbers

theory *Harmonic_Numbers*

imports
Complex_Transcendental
Summation_Tests
begin

The definition of the Harmonic Numbers and the Euler-Mascheroni constant. Also provides a reasonably accurate approximation of *ln 2* and the Euler-Mascheroni constant.

9.9.1 The Harmonic numbers

definition *harm* :: *nat* ⇒ *'a* :: *real_normed_field* **where**
harm n = (∑ *k*=*1*..*n*. *inverse* (*of_nat k*))

lemma *harm_altdef*: *harm n* = (∑ *k*<*n*. *inverse* (*of_nat* (*Suc k*)))
unfolding *harm_def* **by** (*induction n*) *simp_all*

lemma *harm_Suc*: *harm* (*Suc n*) = *harm n* + *inverse* (*of_nat* (*Suc n*))
by (*simp add*: *harm_def*)

lemma *harm_nonneg*: *harm n* ≥ (*0* :: *'a* :: {*real_normed_field*,*linordered_field*})
unfolding *harm_def* **by** (*intro sum_nonneg*) *simp_all*

lemma *harm_pos*: *n* > *0* ⟹ *harm n* > (*0* :: *'a* :: {*real_normed_field*,*linordered_field*})
unfolding *harm_def* **by** (*intro sum_pos*) *simp_all*

lemma *harm_mono*: *m* ≤ *n* ⟹ *harm m* ≤ (*harm n* :: *'a* :: {*real_normed_field*,*linordered_field*})
by(*simp add*: *harm_def sum_mono2*)

lemma *of_real_harm*: *of_real* (*harm n*) = *harm n*
unfolding *harm_def* **by** *simp*

lemma *abs_harm* [*simp*]: (*abs* (*harm n*) :: *real*) = *harm n*
using *harm_nonneg*[*of n*] **by** (*rule abs_of_nonneg*)

lemma *norm_harm*: *norm* (*harm n*) = *harm n*
by (*subst of_real_harm* [*symmetric*]) (*simp add*: *harm_nonneg*)

lemma *harm_expand*:
harm 0 = *0*
harm (*Suc 0*) = *1*
harm (*numeral n*) = *harm* (*pred_numeral n*) + *inverse* (*numeral n*)
proof −
have *numeral n* = *Suc* (*pred_numeral n*) **by** *simp*
also have *harm* ... = *harm* (*pred_numeral n*) + *inverse* (*numeral n*)
by (*subst harm_Suc*, *subst numeral_eq_Suc*[*symmetric*]) *simp*
finally show *harm* (*numeral n*) = *harm* (*pred_numeral n*) + *inverse* (*numeral n*) .
qed (*simp_all add*: *harm_def*)

theorem *not_convergent_harm*: ¬*convergent* (*harm* :: *nat* ⇒ *'a* :: *real_normed_field*)
proof −
have *convergent* (λ*n*. *norm* (*harm n* :: *'a*)) ⟷
convergent (*harm* :: *nat* ⇒ *real*) **by** (*simp add*: *norm_harm*)
also have ... ⟷ *convergent* (λ*n*. ∑ *k*=*Suc 0*..*Suc n*. *inverse* (*of_nat k*) :: *real*)
unfolding *harm_def*[*abs_def*] **by** (*subst convergent_Suc_iff*) *simp_all*
also have ... ⟷ *convergent* (λ*n*. ∑ *k*≤*n*. *inverse* (*of_nat* (*Suc k*)) :: *real*)
by (*subst sum.shift_bounds_cl_Suc_ivl*) (*simp add*: *atLeast0AtMost*)
also have ... ⟷ *summable* (λ*n*. *inverse* (*of_nat n*) :: *real*)
by (*subst summable_Suc_iff* [*symmetric*]) (*simp add*: *summable_iff_convergent'*)
also have ¬... **by** (*rule not_summable_harmonic*)
finally show *?thesis* **by** (*blast dest*: *convergent_norm*)
qed

lemma *harm_pos_iff* [*simp*]: *harm n* > (*0* :: *'a* :: {*real_normed_field*,*linordered_field*})
⟷ *n* > *0*
by (*rule iffI*, *cases n*, *simp add*: *harm_expand*, *simp*, *rule harm_pos*)

lemma *ln_diff_le_inverse*:
assumes *x* ≥ (*1*::*real*)
shows *ln* (*x* + *1*) − *ln x* < *1* / *x*
proof −
from *assms* **have** ∃ *z*>*x*. *z* < *x* + *1* ∧ *ln* (*x* + *1*) − *ln x* = (*x* + *1* − *x*) ∗
inverse z
by (*intro MVT2*) (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
then obtain *z* **where** *z*: *z* > *x z* < *x* + *1 ln* (*x* + *1*) − *ln x* = *inverse z* **by** *auto*
have *ln* (*x* + *1*) − *ln x* = *inverse z* **by** *fact*
also from *z*(*1*,*2*) *assms* **have** ... < *1* / *x* **by** (*simp add*: *field_simps*)
finally show *?thesis* **.**
qed

lemma *ln_le_harm*: *ln* (*real n* + *1*) ≤ (*harm n* :: *real*)
proof (*induction n*)
fix *n* **assume** *IH*: *ln* (*real n* + *1*) ≤ *harm n*
have *ln* (*real* (*Suc n*) + *1*) = *ln* (*real n* + *1*) + (*ln* (*real n* + *2*) − *ln* (*real n* +
1)) **by** *simp*
also have (*ln* (*real n* + *2*) − *ln* (*real n* + *1*)) ≤ *1* / *real* (*Suc n*)
using *ln_diff_le_inverse*[*of real n* + *1*] **by** (*simp add*: *add_ac*)
also note *IH*
also have *harm n* + *1* / *real* (*Suc n*) = *harm* (*Suc n*) **by** (*simp add*: *harm_Suc*
field_simps)
finally show *ln* (*real* (*Suc n*) + *1*) ≤ *harm* (*Suc n*) **by** − *simp*
qed (*simp_all add*: *harm_def*)

lemma *harm_at_top*: *filterlim* (*harm* :: *nat* ⇒ *real*) *at_top sequentially*
proof (*rule filterlim_at_top_mono*)
show *eventually* (λ*n*. *harm n* ≥ *ln* (*real* (*Suc n*))) *at_top*
using *ln_le_harm* **by** (*intro always_eventually allI*) (*simp_all add*: *add_ac*)

show *filterlim* (λ*n*. *ln* (*real* (*Suc n*))) *at_top sequentially*
by (*intro filterlim_compose*[*OF ln_at_top*] *filterlim_compose*[*OF filterlim_real_sequentially*]
filterlim_Suc)
qed

9.9.2 The Euler-Mascheroni constant

The limit of the difference between the partial harmonic sum and the natural logarithm (approximately 0.577216). This value occurs e.g. in the definition of the Gamma function.

definition *euler_mascheroni* :: *'a* :: *real_normed_algebra_1* **where**
euler_mascheroni = *of_real* (*lim* (λ*n*. *harm n* − *ln* (*of_nat n*)))

lemma *of_real_euler_mascheroni* [*simp*]: *of_real euler_mascheroni* = *euler_mascheroni*
by (*simp add*: *euler_mascheroni_def*)

lemma *harm_ge_ln*: *harm n* ≥ *ln* (*real n* + *1*)
proof −
have *ln* (*n* + *1*) = (∑*j*<*n*. *ln* (*real* (*Suc j* + *1*)) − *ln* (*real* (*j* + *1*)))
by (*subst sum_lessThan_telescope*) *auto*
also have ... ≤ (∑*j*<*n*. *1* / (*Suc j*))
proof (*intro sum_mono*, *clarify*)
fix *j* **assume** *j*: *j* < *n*
have ∃ξ. ξ > *real j* + *1* ∧ ξ < *real j* + *2* ∧
ln (*real j* + *2*) − *ln* (*real j* + *1*) = (*real j* + *2* − (*real j* + *1*)) ∗ (*1* / ξ)
by (*intro MVT2*) (*auto intro*!: *derivative_eq_intros*)
then obtain ξ :: *real*
where ξ: ξ ∈ {*real j* + *1*..*real j* + *2*} *ln* (*real j* + *2*) − *ln* (*real j* + *1*) = *1*
/ ξ
by *auto*
note ξ(*2*)
also have *1* / ξ ≤ *1* / (*Suc j*)
using ξ(*1*) **by** (*auto simp*: *field_simps*)
finally show *ln* (*real* (*Suc j* + *1*)) − *ln* (*real* (*j* + *1*)) ≤ *1* / (*Suc j*)
by (*simp add*: *add_ac*)
qed
also have ... = *harm n*
by (*simp add*: *harm_altdef field_simps*)
finally show *?thesis* **by** (*simp add*: *add_ac*)
qed

lemma *decseq_harm_diff_ln*: *decseq* (λ*n*. *harm* (*Suc n*) − *ln* (*Suc n*))
proof (*rule decseq_SucI*)
fix *m* :: *nat*
define *n* **where** *n* = *Suc m*
have *n* > *0* **by** (*simp add*: *n_def*)
have *convex_on* {*0*<..} (λ*x* :: *real*. −*ln x*)
by (*rule convex_on_realI*[**where** *f'* = λ*x*. −*1*/*x*])
(*auto intro*!: *derivative_eq_intros simp*: *field_simps*)

hence $(-1 \,/\, (n + 1)) * (real\ n - real\ (n + 1)) \leq (-\ ln\ (real\ n)) - (-ln\ (real\ (n + 1)))$
using ‹$n > 0$› **by** (*intro convex_on_imp_above_tangent*[**where** $A = \{0<..\}$])
(*auto intro*!: *derivative_eq_intros simp*: *interior_open*)
thus $harm\ (Suc\ n) - ln\ (Suc\ n) \leq harm\ n - ln\ n$
by (*auto simp*: *harm_Suc field_simps*)
qed

lemma *euler_mascheroni_sequence_nonneg*:
assumes $n > 0$
shows $harm\ n - ln\ (real\ n) \geq (0 :: real)$
proof −
have $ln\ (real\ n) \leq ln\ (real\ n + 1)$
using *assms* **by** *simp*
also have $\ldots \leq harm\ n$
by (*rule harm_ge_ln*)
finally show *?thesis* **by** *simp*
qed

lemma *euler_mascheroni_convergent*: $convergent\ (\lambda n.\ harm\ n - ln\ n)$
proof −
have $harm\ (Suc\ n) - ln\ (real\ (Suc\ n)) \geq 0$ **for** $n :: nat$
using *euler_mascheroni_sequence_nonneg*[*of Suc n*] **by** *simp*
hence $convergent\ (\lambda n.\ harm\ (Suc\ n) - ln\ (Suc\ n))$
by (*intro Bseq_monoseq_convergent decseq_bounded*[*of _ 0*] *decseq_harm_diff_ln decseq_imp_monoseq*)
auto
thus *?thesis*
by (*subst* (*asm*) *convergent_Suc_iff*)
qed

lemma *euler_mascheroni_sequence_decreasing*:
$m > 0 \Longrightarrow m \leq n \Longrightarrow harm\ n - ln\ (of\_nat\ n) \leq harm\ m - ln\ (of\_nat\ m :: real)$
using *decseqD*[*OF decseq_harm_diff_ln*, *of m − 1 n − 1*] **by** *simp*

lemma *euler_mascheroni_LIMSEQ*:
$(\lambda n.\ harm\ n - ln\ (of\_nat\ n) :: real) \longrightarrow euler\_mascheroni$
unfolding *euler_mascheroni_def*
by (*simp add*: *convergent_LIMSEQ_iff* [*symmetric*] *euler_mascheroni_convergent*)

lemma *euler_mascheroni_LIMSEQ_of_real*:
$(\lambda n.\ of\_real\ (harm\ n - ln\ (of\_nat\ n))) \longrightarrow$
$(euler\_mascheroni :: 'a :: \{real\_normed\_algebra\_1, topological\_space\})$
proof −
have $(\lambda n.\ of\_real\ (harm\ n - ln\ (of\_nat\ n))) \longrightarrow (of\_real\ (euler\_mascheroni) :: 'a)$
by (*intro tendsto_of_real euler_mascheroni_LIMSEQ*)
thus *?thesis* **by** *simp*

qed

lemma *euler_mascheroni_sum_real*:
(λ*n*. *inverse* (*of_nat* (*n*+1)) + *ln* (*of_nat* (*n*+1)) − *ln* (*of_nat* (*n*+2)) :: *real*)
sums euler_mascheroni
using *sums_add*[*OF telescope_sums*[*OF LIMSEQ_Suc*[*OF euler_mascheroni_LIMSEQ*]]
telescope_sums'[*OF LIMSEQ_inverse_real_of_nat*]]
by (*simp_all add*: *harm_def algebra_simps*)

lemma *euler_mascheroni_sum*:
(λ*n*. *inverse* (*of_nat* (*n*+1)) + *of_real* (*ln* (*of_nat* (*n*+1))) − *of_real* (*ln* (*of_nat* (*n*+2))))
sums (*euler_mascheroni* :: *'a* :: {*banach*, *real_normed_field*})
proof −
have (λ*n*. *of_real* (*inverse* (*of_nat* (*n*+1)) + *ln* (*of_nat* (*n*+1)) − *ln* (*of_nat* (*n*+2))))
sums (*of_real euler_mascheroni* :: *'a* :: {*banach*, *real_normed_field*})
by (*subst sums_of_real_iff*) (*rule euler_mascheroni_sum_real*)
thus *?thesis* **by** *simp*
qed

theorem *alternating_harmonic_series_sums*: (λ*k*. (−1)^*k* / *real_of_nat* (*Suc k*)) *sums ln 2*
proof −
let *?f* = λ*n*. *harm n* − *ln* (*real_of_nat n*)
let *?g* = λ*n*. *if even n then 0 else* (2::*real*)
let *?em* = λ*n*. *harm n* − *ln* (*real_of_nat n*)
have *eventually* (λ*n*. *?em* (2∗*n*) − *?em n* + *ln 2* = (∑*k*<2∗*n*. (−1)^*k* / *real_of_nat* (*Suc k*))) *at_top*
using *eventually_gt_at_top*[*of 0*::*nat*]
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
have (∑*k*<2∗*n*. (−1)^*k* / *real_of_nat* (*Suc k*)) =
(∑*k*<2∗*n*. ((−1)^*k* + *?g k*) / *of_nat* (*Suc k*)) − (∑*k*<2∗*n*. *?g k* / *of_nat* (*Suc k*))
by (*simp add*: *sum.distrib algebra_simps divide_inverse*)
also have (∑*k*<2∗*n*. ((−1)^*k* + *?g k*) / *real_of_nat* (*Suc k*)) = *harm* (2∗*n*)
unfolding *harm_altdef* **by** (*intro sum.cong*) (*auto simp*: *field_simps*)
also have (∑*k*<2∗*n*. *?g k* / *real_of_nat* (*Suc k*)) = (∑*k*|*k*<2∗*n* ∧ *odd k*. *?g k* / *of_nat* (*Suc k*))
by (*intro sum.mono_neutral_right*) *auto*
also have ... = (∑*k*|*k*<2∗*n* ∧ *odd k*. *2* / (*real_of_nat* (*Suc k*)))
by (*intro sum.cong*) *auto*
also have (∑*k*|*k*<2∗*n* ∧ *odd k*. *2* / (*real_of_nat* (*Suc k*))) = *harm n*
unfolding *harm_altdef*
by (*intro sum.reindex_cong*[*of* λ*n*. *2*∗*n*+*1*]) (*auto simp*: *inj_on_def field_simps elim*!: *oddE*)
also have *harm* (2∗*n*) − *harm n* = *?em* (2∗*n*) − *?em n* + *ln 2* **using** *n*
by (*simp_all add*: *algebra_simps ln_mult*)

finally show *?em (2∗n) − ?em n + ln 2 = (∑k<2∗n. (−1)^k / real_of_nat (Suc k))* **..**
qed
moreover have *(λn. ?em (2∗n) − ?em n + ln (2::real)) ⟶ euler_mascheroni − euler_mascheroni + ln 2*
by *(intro tendsto_intros euler_mascheroni_LIMSEQ filterlim_compose[OF euler_mascheroni_LIMSEQ] filterlim_subseq) (auto simp: strict_mono_def)*
hence *(λn. ?em (2∗n) − ?em n + ln (2::real)) ⟶ ln 2* **by** *simp*
ultimately have *(λn. (∑k<2∗n. (−1)^k / real_of_nat (Suc k))) ⟶ ln 2*
by *(blast intro: Lim_transform_eventually)*

moreover have *summable (λk. (−1)^k ∗ inverse (real_of_nat (Suc k)))*
using *LIMSEQ_inverse_real_of_nat*
by *(intro summable_Leibniz(1) decseq_imp_monoseq decseq_SucI) simp_all*
hence *A: (λn. ∑k<n. (−1)^k / real_of_nat (Suc k)) ⟶ (∑k. (−1)^k / real_of_nat (Suc k))*
by *(simp add: summable_sums_iff divide_inverse sums_def)*
from *filterlim_compose[OF this filterlim_subseq[of (∗) (2::nat)]]*
have *(λn. ∑k<2∗n. (−1)^k / real_of_nat (Suc k)) ⟶ (∑k. (−1)^k / real_of_nat (Suc k))*
by *(simp add: strict_mono_def)*
ultimately have *(∑k. (− 1) ^ k / real_of_nat (Suc k)) = ln 2* **by** *(intro LIMSEQ_unique)*
with *A* **show** *?thesis* **by** *(simp add: sums_def)*
qed

lemma *alternating_harmonic_series_sums'*:
(λk. inverse (real_of_nat (2∗k+1)) − inverse (real_of_nat (2∗k+2))) sums ln 2
unfolding *sums_def*
proof *(rule Lim_transform_eventually)*
show *(λn. ∑k<2∗n. (−1)^k / (real_of_nat (Suc k))) ⟶ ln 2*
using *alternating_harmonic_series_sums* **unfolding** *sums_def*
by *(rule filterlim_compose) (rule mult_nat_left_at_top, simp)*
show *eventually (λn. (∑k<2∗n. (−1)^k / (real_of_nat (Suc k))) = (∑k<n. inverse (real_of_nat (2∗k+1)) − inverse (real_of_nat (2∗k+2)))) sequentially*
proof *(intro always_eventually allI)*
fix *n :: nat*
show *(∑k<2∗n. (−1)^k / (real_of_nat (Suc k))) = (∑k<n. inverse (real_of_nat (2∗k+1)) − inverse (real_of_nat (2∗k+2)))*
by *(induction n) (simp_all add: inverse_eq_divide)*
qed
qed

9.9.3 Bounds on the Euler-Mascheroni constant

lemma *ln_inverse_approx_le*:
assumes (*x*::*real*) > *0* *a* > *0*
shows *ln* (*x* + *a*) − *ln x* ≤ *a* ∗ (*inverse x* + *inverse* (*x* + *a*))/*2* (**is** _ ≤ *?A*)
proof −
define *f′* **where** *f′* = (*inverse* (*x* + *a*) − *inverse x*)/*a*
let *?f* = λ*t*. (*t* − *x*)^2 ∗ *f′* / *2* + *t* ∗ *inverse x*
let *?F* = λ*t*. (*t* − *x*)^2 ∗ *f′* / *2* + *t* ∗ *inverse x*

have *deriv*: ∃ *D*. ((λ*x*. *?F x* − *ln x*) *has_field_derivative D*) (*at* ξ) ∧ *D* ≥ *0*
if ξ ≥ *x* ξ ≤ *x* + *a* **for** ξ
proof −
from *that assms* **have** *t*: *0* ≤ (ξ − *x*) / *a* (ξ − *x*) / *a* ≤ *1* **by** *simp_all*
have *inverse* ξ = *inverse* ((*1* − (ξ − *x*) / *a*) ∗*R* *x* + ((ξ − *x*) / *a*) ∗*R* (*x* + *a*)) (**is** _ = *?A*)
using *assms* **by** (*simp add*: *field_simps*)
also from *assms* **have** *convex_on* {*x*..*x*+*a*} *inverse* **by** (*intro convex_on_inverse*) *auto*
from *convex_onD_Icc*[*OF this* _ *t*] *assms*
have *?A* ≤ (*1* − (ξ − *x*) / *a*) ∗ *inverse x* + (ξ − *x*) / *a* ∗ *inverse* (*x* + *a*) **by** *simp*
also have ... = (ξ − *x*) ∗ *f′* + *inverse x* **using** *assms*
by (*simp add*: *f′_def divide_simps*) (*simp add*: *field_simps*)
finally have *?f* ξ − *1* / ξ ≥ *0* **by** (*simp add*: *field_simps*)
moreover have ((λ*x*. *?F x* − *ln x*) *has_field_derivative ?f* ξ − *1* / ξ) (*at* ξ)
using *that assms* **by** (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
ultimately show *?thesis* **by** *blast*
qed
have *?F x* − *ln x* ≤ *?F* (*x* + *a*) − *ln* (*x* + *a*)
by (*rule DERIV_nonneg_imp_nondecreasing*[*of x x* + *a*, *OF* _ *deriv*]) (*use assms* **in** *auto*)
thus *?thesis*
using *assms* **by** (*simp add*: *f′_def divide_simps*) (*simp add*: *algebra_simps power2_eq_square*)*?*
qed

lemma *ln_inverse_approx_ge*:
assumes (*x*::*real*) > *0* *x* < *y*
shows *ln y* − *ln x* ≥ *2* ∗ (*y* − *x*) / (*x* + *y*) (**is** _ ≥ *?A*)
proof −
define *m* **where** *m* = (*x*+*y*)/*2*
define *f′* **where** *f′* = −*inverse* (*m*^2)
from *assms* **have** *m*: *m* > *0* **by** (*simp add*: *m_def*)
let *?F* = λ*t*. (*t* − *m*)^2 ∗ *f′* / *2* + *t* / *m*
let *?f* = λ*t*. (*t* − *m*) ∗ *f′* + *inverse m*

have *deriv*: ∃ *D*. ((λ*x*. *ln x* − *?F x*) *has_field_derivative D*) (*at* ξ) ∧ *D* ≥ *0*
if ξ ≥ *x* ξ ≤ *y* **for** ξ
proof −

```
    from that assms have inverse ξ − inverse m ≥ f′ ∗ (ξ − m)
      by (intro convex_on_imp_above_tangent[of {0<..}] convex_on_inverse)
         (auto simp: m_def interior_open f′_def power2_eq_square intro!: derivative_eq_intros)
    hence 1 / ξ − ?f ξ ≥ 0 by (simp add: field_simps f′_def)
    moreover have ((λx. ln x − ?F x) has_field_derivative 1 / ξ − ?f ξ) (at ξ)
      using that assms m by (auto intro!: derivative_eq_intros simp: field_simps)
    ultimately show ?thesis by blast
  qed
  have ln x − ?F x ≤ ln y − ?F y
    by (rule DERIV_nonneg_imp_nondecreasing[of x y, OF _ deriv]) (use assms in auto)
  hence ln y − ln x ≥ ?F y − ?F x
    by (simp add: algebra_simps)
  also have ?F y − ?F x = ?A
     using assms by (simp add: f′_def m_def divide_simps) (simp add: algebra_simps power2_eq_square)
  finally show ?thesis .
qed

lemma euler_mascheroni_lower:
            euler_mascheroni ≥ harm (Suc n) − ln (real_of_nat (n + 2)) + 1/real_of_nat (2 ∗ (n + 2))
    and euler_mascheroni_upper:
            euler_mascheroni ≤ harm (Suc n) − ln (real_of_nat (n + 2)) + 1/real_of_nat (2 ∗ (n + 1))
proof −
  define D :: _ ⇒ real
     where D n = inverse (of_nat (n+1)) + ln (of_nat (n+1)) − ln (of_nat (n+2)) for n
  let ?g = λn. ln (of_nat (n+2)) − ln (of_nat (n+1)) − inverse (of_nat (n+1)) :: real
  define inv where [abs_def]: inv n = inverse (real_of_nat n) for n
  fix n :: nat
 note summable = sums_summable[OF euler_mascheroni_sum_real, folded D_def]
  have sums: (λk. (inv (Suc (k + (n+1))) − inv (Suc (Suc k + (n+1))))/2) sums ((inv (Suc (0 + (n+1))) − 0)/2)
    unfolding inv_def
   by (intro sums_divide telescope_sums′ LIMSEQ_ignore_initial_segment LIMSEQ_inverse_real_of_nat)
  have sums′: (λk. (inv (Suc (k + n)) − inv (Suc (Suc k + n)))/2) sums ((inv (Suc (0 + n)) − 0)/2)
    unfolding inv_def
   by (intro sums_divide telescope_sums′ LIMSEQ_ignore_initial_segment LIMSEQ_inverse_real_of_nat)
  from euler_mascheroni_sum_real have euler_mascheroni = (∑ k. D k)
    by (simp add: sums_iff D_def)
  also have ... = (∑ k. D (k + Suc n)) + (∑ k≤n. D k)
    by (subst suminf_split_initial_segment[OF summable, of Suc n],
```

subst lessThan_Suc_atMost) *simp*
finally have *sum*: $(\sum k{\le}n.\ D\ k) - \mathit{euler\_mascheroni} = -(\sum k.\ D\ (k + \mathit{Suc}\ n))$
by *simp*

note *sum*
also have $\ldots \le -(\sum k.\ (\mathit{inv}\ (k + \mathit{Suc}\ n + 1) - \mathit{inv}\ (k + \mathit{Suc}\ n + 2))\ /\ 2)$
proof (*intro le_imp_neg_le suminf_le allI summable_ignore_initial_segment*[*OF summable*])
fix k' :: *nat*
define k **where** $k = k' + \mathit{Suc}\ n$
hence *k*: $k > 0$ **by** (*simp add*: *k_def*)
have *real_of_nat* $(k{+}1) > 0$ **by** (*simp add*: *k_def*)
with *ln_inverse_approx_le*[*OF this zero_less_one*]
have *ln* (*of_nat k* + 2) − *ln* (*of_nat k* + 1) ≤ (*inv* (*k*+1) + *inv* (*k*+2))/2
by (*simp add*: *inv_def add_ac*)
hence (*inv* (*k*+1) − *inv* (*k*+2))/2 ≤ *inv* (*k*+1) + *ln* (*of_nat* (*k*+1)) − *ln* (*of_nat* (*k*+2))
by (*simp add*: *field_simps*)
also have $\ldots = D\ k$ **unfolding** *D_def inv_def* **..**
finally show $D\ (k' + \mathit{Suc}\ n) \ge (\mathit{inv}\ (k' + \mathit{Suc}\ n + 1) - \mathit{inv}\ (k' + \mathit{Suc}\ n + 2))\ /\ 2$
by (*simp add*: *k_def*)
from *sums_summable*[*OF sums*]
show *summable* (λ*k*. (*inv* (*k* + *Suc n* + 1) − *inv* (*k* + *Suc n* + 2))/2) **by** *simp*
qed
also from *sums* **have** $\ldots = -\mathit{inv}\ (n{+}2)\ /\ 2$ **by** (*simp add*: *sums_iff*)
finally have *euler_mascheroni* $\ge (\sum k{\le}n.\ D\ k) + 1\ /\ (\mathit{of\_nat}\ (2 * (n{+}2)))$
by (*simp add*: *inv_def field_simps*)
also have $(\sum k{\le}n.\ D\ k) = \mathit{harm}\ (\mathit{Suc}\ n) - (\sum k{\le}n.\ \mathit{ln}\ (\mathit{real\_of\_nat}\ (\mathit{Suc}\ k{+}1)) - \mathit{ln}\ (\mathit{of\_nat}\ (k{+}1)))$
unfolding *harm_altdef D_def* **by** (*subst lessThan_Suc_atMost*) (*simp add*: *sum.distrib sum_subtractf*)
also have $(\sum k{\le}n.\ \mathit{ln}\ (\mathit{real\_of\_nat}\ (\mathit{Suc}\ k{+}1)) - \mathit{ln}\ (\mathit{of\_nat}\ (k{+}1))) = \mathit{ln}\ (\mathit{of\_nat}\ (n{+}2))$
by (*subst atLeast0AtMost* [*symmetric*], *subst sum_Suc_diff*) *simp_all*
finally show *euler_mascheroni* ≥ *harm* (*Suc n*) − *ln* (*real_of_nat* (*n* + 2)) + 1/*real_of_nat* (2 ∗ (*n* + 2))
by *simp*

note *sum*
also have $-(\sum k.\ D\ (k + \mathit{Suc}\ n)) \ge -(\sum k.\ (\mathit{inv}\ (\mathit{Suc}\ (k + n)) - \mathit{inv}\ (\mathit{Suc}\ (\mathit{Suc}\ k + n)))/2)$
proof (*intro le_imp_neg_le suminf_le allI summable_ignore_initial_segment*[*OF summable*])
fix k' :: *nat*
define k **where** $k = k' + \mathit{Suc}\ n$
hence *k*: $k > 0$ **by** (*simp add*: *k_def*)
have *real_of_nat* $(k{+}1) > 0$ **by** (*simp add*: *k_def*)

```
    from ln_inverse_approx_ge[of of_nat k + 1 of_nat k + 2]
      have 2 / (2 * real_of_nat k + 3) ≤ ln (of_nat (k+2)) − ln (real_of_nat
(k+1))
      by (simp add: add_ac)
    hence D k ≤ 1 / real_of_nat (k+1) − 2 / (2 * real_of_nat k + 3)
      by (simp add: D_def inverse_eq_divide inv_def)
     also have ... = inv ((k+1)*(2*k+3)) unfolding inv_def by (simp add:
field_simps)
    also have ... ≤ inv (2*k*(k+1)) unfolding inv_def using k
      by (intro le_imp_inverse_le)
         (simp add: algebra_simps, simp del: of_nat_add)
    also have ... = (inv k − inv (k+1))/2 unfolding inv_def using k
      by (simp add: divide_simps del: of_nat_mult) (simp add: algebra_simps)
   finally show D k ≤ (inv (Suc (k' + n)) − inv (Suc (Suc k' + n)))/2 unfolding
k_def by simp
  next
    from sums_summable[OF sums']
      show summable (λk. (inv (Suc (k + n)) − inv (Suc (Suc k + n)))/2) by
simp
  qed
  also from sums' have (∑ k. (inv (Suc (k + n)) − inv (Suc (Suc k + n)))/2)
= inv (n+1)/2
    by (simp add: sums_iff)
  finally have euler_mascheroni ≤ (∑ k≤n. D k) + 1 / of_nat (2 * (n+1))
    by (simp add: inv_def field_simps)
 also have (∑ k≤n. D k) = harm (Suc n) − (∑ k≤n. ln (real_of_nat (Suc k+1))
− ln (of_nat (k+1)))
     unfolding harm_altdef D_def by (subst lessThan_Suc_atMost) (simp add:
sum.distrib sum_subtractf)
   also have (∑ k≤n. ln (real_of_nat (Suc k+1)) − ln (of_nat (k+1))) = ln
(of_nat (n+2))
    by (subst atLeast0AtMost [symmetric], subst sum_Suc_diff) simp_all
  finally show euler_mascheroni ≤ harm (Suc n) − ln (real_of_nat (n + 2)) +
1/real_of_nat (2 * (n + 1))
    by simp
qed

lemma euler_mascheroni_pos: euler_mascheroni > (0::real)
  using euler_mascheroni_lower[of 0] ln_2_less_1 by (simp add: harm_def)

context
begin

private lemma ln_approx_aux:
  fixes n :: nat and x :: real
  defines y ≡ (x−1)/(x+1)
  assumes x: x > 0 x ≠ 1
 shows inverse (2*y^(2*n+1)) * (ln x − (∑ k<n. 2*y^(2*k+1) / of_nat (2*k+1)))
∈
```

```
        {0..(1 / (1 - y^2) / of_nat (2*n+1))}
proof -
  from x have norm_y: norm y < 1 unfolding y_def by simp
  from power_strict_mono[OF this, of 2] have norm_y': norm y^2 < 1 by simp

  let ?f = λk. 2 * y ^ (2*k+1) / of_nat (2*k+1)
  note sums = ln_series_quadratic[OF x(1)]
  define c where c = inverse (2*y^(2*n+1))
  let ?d = c * (ln x - (∑k<n. ?f k))
  have ⋀k. y^2^k / of_nat (2*(k+n)+1) ≤ y^2 ^ k / of_nat (2*n+1)
    by (intro divide_left_mono mult_right_mono mult_pos_pos zero_le_power[of
y^2]) simp_all
  moreover {
    have (λk. ?f (k + n)) sums (ln x - (∑k<n. ?f k))
      using sums_split_initial_segment[OF sums] by (simp add: y_def)
    hence (λk. c * ?f (k + n)) sums ?d by (rule sums_mult)
    also have (λk. c * (2*y^(2*(k+n)+1) / of_nat (2*(k+n)+1))) =
                (λk. (c * (2*y^(2*n+1))) * ((y^2)^k / of_nat (2*(k+n)+1)))
      by (simp only: ring_distribs power_add power_mult) (simp add: mult_ac)
    also from x have c * (2*y^(2*n+1)) = 1 by (simp add: c_def y_def)
    finally have (λk. (y^2)^k / of_nat (2*(k+n)+1)) sums ?d by simp
  } note sums' = this
  moreover from norm_y' have (λk. (y^2)^k / of_nat (2*n+1)) sums (1 / (1
- y^2) / of_nat (2*n+1))
    by (intro sums_divide geometric_sums) (simp_all add: norm_power)
  ultimately have ?d ≤ (1 / (1 - y^2) / of_nat (2*n+1)) by (rule sums_le)
  moreover have c * (ln x - (∑k<n. 2 * y ^ (2 * k + 1) / real_of_nat (2 * k
+ 1))) ≥ 0
    by (intro sums_le[OF _ sums_zero sums']) simp_all
  ultimately show ?thesis unfolding c_def by simp
qed

lemma
  fixes n :: nat and x :: real
  defines y ≡ (x-1)/(x+1)
  defines approx ≡ (∑k<n. 2*y^(2*k+1) / of_nat (2*k+1))
  defines d ≡ y^(2*n+1) / (1 - y^2) / of_nat (2*n+1)
  assumes x: x > 1
  shows   ln_approx_bounds: ln x ∈ {approx..approx + 2*d}
  and     ln_approx_abs:    abs (ln x - (approx + d)) ≤ d
proof -
  define c where c = 2*y^(2*n+1)
  from x have c_pos: c > 0 unfolding c_def y_def
    by (intro mult_pos_pos zero_less_power) simp_all
  have A: inverse c * (ln x - (∑k<n. 2*y^(2*k+1) / of_nat (2*k+1))) ∈
              {0.. (1 / (1 - y^2) / of_nat (2*n+1))} using assms unfolding
y_def c_def
    by (intro ln_approx_aux) simp_all
  hence inverse c * (ln x - (∑k<n. 2*y^(2*k+1)/of_nat (2*k+1))) ≤ (1 /
```

```
(1−y^2) / of_nat (2∗n+1))
    by simp
  hence (ln x − (∑k<n. 2∗y^(2∗k+1) / of_nat (2∗k+1))) / c ≤ (1 / (1 −
y^2) / of_nat (2∗n+1))
    by (auto simp add: field_split_simps)
  with c_pos have ln x ≤ c / (1 − y^2) / of_nat (2∗n+1) + approx
    by (subst (asm) pos_divide_le_eq) (simp_all add: mult_ac approx_def)
  moreover {
    from A c_pos have 0 ≤ c ∗ (inverse c ∗ (ln x − (∑k<n. 2∗y^(2∗k+1) /
of_nat (2∗k+1))))
      by (intro mult_nonneg_nonneg[of c]) simp_all
    also have ... = (c ∗ inverse c) ∗ (ln x − (∑k<n. 2∗y^(2∗k+1) / of_nat
(2∗k+1)))
      by (simp add: mult_ac)
    also from c_pos have c ∗ inverse c = 1 by simp
    finally have ln x ≥ approx by (simp add: approx_def)
  }
  ultimately show ln x ∈ {approx..approx + 2∗d} by (simp add: c_def d_def)
  thus abs (ln x − (approx + d)) ≤ d by auto
qed

end

lemma euler_mascheroni_bounds:
  fixes n :: nat assumes n ≥ 1 defines t ≡ harm n − ln (of_nat (Suc n)) :: real
  shows euler_mascheroni ∈ {t + inverse (of_nat (2∗(n+1)))..t + inverse (of_nat
(2∗n))}
  using assms euler_mascheroni_upper[of n−1] euler_mascheroni_lower[of n−1]
  unfolding t_def by (cases n) (simp_all add: harm_Suc t_def inverse_eq_divide)

lemma euler_mascheroni_bounds′:
  fixes n :: nat assumes n ≥ 1 ln (real_of_nat (Suc n)) ∈ {l<..<u}
  shows euler_mascheroni ∈
          {harm n − u + inverse (of_nat (2∗(n+1)))<..<harm n − l + inverse
(of_nat (2∗n))}
  using euler_mascheroni_bounds[OF assms(1)] assms(2) by auto
```

Approximation of *ln* (*2*::′*a*). The lower bound is accurate to about 0.03; the upper bound is accurate to about 0.0015.

```
lemma ln2_ge_two_thirds: 2/3 ≤ ln (2::real)
  and ln2_le_25_over_36: ln (2::real) ≤ 25/36
  using ln_approx_bounds[of 2 1, simplified, simplified eval_nat_numeral, sim-
plified] by simp_all
```

Approximation of the Euler-Mascheroni constant. The lower bound is accurate to about 0.0015; the upper bound is accurate to about 0.015.

```
lemma euler_mascheroni_gt_19_over_33: (euler_mascheroni :: real) > 19/33
(is ?th1)
```

and *euler_mascheroni_less_13_over_22*: (*euler_mascheroni* :: *real*) < *13/22* (**is** *?th2*)
proof −
have *ln* (*real* (*Suc 7*)) = *3* ∗ *ln 2* **by** (*simp add*: *ln_powr* [*symmetric*])
also from *ln_approx_bounds*[*of 2 3*] **have** ... ∈ {*3∗307/443*<..<*3∗4615/6658*}
by (*simp add*: *eval_nat_numeral*)
finally have *ln* (*real* (*Suc 7*)) ∈ ... .
from *euler_mascheroni_bounds'*[*OF* _ *this*] **have** *?th1* ∧ *?th2* **by** (*simp_all add*: *harm_expand*)
thus *?th1 ?th2* **by** *blast*+
qed

end

9.10 The Gamma Function

theory *Gamma_Function*
imports
Equivalence_Lebesgue_Henstock_Integration
Summation_Tests
Harmonic_Numbers
HOL−Library.Nonpos_Ints
HOL−Library.Periodic_Fun
begin

Several equivalent definitions of the Gamma function and its most important properties. Also contains the definition and some properties of the log-Gamma function and the Digamma function and the other Polygamma functions.

Based on the Gamma function, we also prove the Weierstraß product form of the sin function and, based on this, the solution of the Basel problem (the sum over all *1 / real* (n^2).

lemma *pochhammer_eq_0_imp_nonpos_Int*:
pochhammer (*x*::'*a*::*field_char_0*) *n* = *0* ⟹ *x* ∈ $\mathbb{Z}_{\leq 0}$
by (*auto simp*: *pochhammer_eq_0_iff*)

lemma *closed_nonpos_Ints* [*simp*]: *closed* ($\mathbb{Z}_{\leq 0}$ :: '*a* :: *real_normed_algebra_1 set*)
proof −
have $\mathbb{Z}_{\leq 0}$ = (*of_int* ' {*n*. *n* ≤ *0*} :: '*a set*)
by (*auto elim*!: *nonpos_Ints_cases intro*!: *nonpos_Ints_of_int*)
also have *closed* ... **by** (*rule closed_of_int_image*)
finally show *?thesis* .
qed

lemma *plus_one_in_nonpos_Ints_imp*: *z* + *1* ∈ $\mathbb{Z}_{\leq 0}$ ⟹ *z* ∈ $\mathbb{Z}_{\leq 0}$
using *nonpos_Ints_diff_Nats*[*of z+1 1*] **by** *simp_all*

lemma *of_int_in_nonpos_Ints_iff*:
(*of_int n* :: *'a* :: *ring_char_0*) ∈ $\mathbb{Z}_{\leq 0}$ ⟷ *n* ≤ *0*
by (*auto simp*: *nonpos_Ints_def*)

lemma *one_plus_of_int_in_nonpos_Ints_iff*:
(*1* + *of_int n* :: *'a* :: *ring_char_0*) ∈ $\mathbb{Z}_{\leq 0}$ ⟷ *n* ≤ *−1*
proof −
have *1* + *of_int n* = (*of_int* (*n* + *1*) :: *'a*) **by** *simp*
also have ... ∈ $\mathbb{Z}_{\leq 0}$ ⟷ *n* + *1* ≤ *0* **by** (*subst of_int_in_nonpos_Ints_iff*) *simp_all*
also have ... ⟷ *n* ≤ *−1* **by** *presburger*
finally show *?thesis* **.**
qed

lemma *one_minus_of_nat_in_nonpos_Ints_iff*:
(*1* − *of_nat n* :: *'a* :: *ring_char_0*) ∈ $\mathbb{Z}_{\leq 0}$ ⟷ *n* > *0*
proof −
have (*1* − *of_nat n* :: *'a*) = *of_int* (*1* − *int n*) **by** *simp*
also have ... ∈ $\mathbb{Z}_{\leq 0}$ ⟷ *n* > *0* **by** (*subst of_int_in_nonpos_Ints_iff*) *presburger*
finally show *?thesis* **.**
qed

lemma *fraction_not_in_ints*:
assumes ¬(*n dvd m*) *n* ≠ *0*
shows *of_int m* / *of_int n* ∉ ($\mathbb{Z}$ :: *'a* :: {*division_ring*,*ring_char_0*} *set*)
proof
assume *of_int m* / (*of_int n* :: *'a*) ∈ $\mathbb{Z}$
then obtain *k* **where** *of_int m* / *of_int n* = (*of_int k* :: *'a*) **by** (*elim Ints_cases*)
with *assms* **have** *of_int m* = (*of_int* (*k* ∗ *n*) :: *'a*) **by** (*auto simp add*: *field_split_simps*)
hence *m* = *k* ∗ *n* **by** (*subst* (*asm*) *of_int_eq_iff*)
hence *n dvd m* **by** *simp*
with *assms*(*1*) **show** *False* **by** *contradiction*
qed

lemma *fraction_not_in_nats*:
assumes ¬*n dvd m n* ≠ *0*
shows *of_int m* / *of_int n* ∉ ($\mathbb{N}$ :: *'a* :: {*division_ring*,*ring_char_0*} *set*)
proof
assume *of_int m* / *of_int n* ∈ ($\mathbb{N}$ :: *'a set*)
also note *Nats_subset_Ints*
finally have *of_int m* / *of_int n* ∈ ($\mathbb{Z}$ :: *'a set*) **.**
moreover have *of_int m* / *of_int n* ∉ ($\mathbb{Z}$ :: *'a set*)
using *assms* **by** (*intro fraction_not_in_ints*)
ultimately show *False* **by** *contradiction*
qed

lemma *not_in_Ints_imp_not_in_nonpos_Ints*: *z* ∉ $\mathbb{Z}$ ⟹ *z* ∉ $\mathbb{Z}_{\leq 0}$
by (*auto simp*: *Ints_def nonpos_Ints_def*)

lemma *double_in_nonpos_Ints_imp*:
assumes $2 * (z :: 'a :: field\_char\_0) \in \mathbb{Z}_{\leq 0}$
shows $z \in \mathbb{Z}_{\leq 0} \vee z + 1/2 \in \mathbb{Z}_{\leq 0}$
proof−
from *assms* **obtain** *k* **where** *k*: $2 * z = -$ *of_nat k* **by** (*elim nonpos_Ints_cases'*)
thus *?thesis* **by** (*cases even k*) (*auto elim*!: *evenE oddE simp*: *field_simps*)
qed

lemma *sin_series*: $(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n+1))\ *_R\ z\hat{}(2*n+1))$ *sums sin z*
proof −
from *sin_converges*[*of z*] **have** $(\lambda n.\ sin\_coeff\ n\ *_R\ z\hat{}n)$ *sums sin z* **.**
also have $(\lambda n.\ sin\_coeff\ n\ *_R\ z\hat{}n)$ *sums sin z* $\longleftrightarrow$
$(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n+1))\ *_R\ z\hat{}(2*n+1))$ *sums sin z*
by (*subst sums_mono_reindex*[*of* $\lambda n.\ 2*n+1$, *symmetric*])
(*auto simp*: *sin_coeff_def strict_mono_def ac_simps elim*!: *oddE*)
finally show *?thesis* **.**
qed

lemma *cos_series*: $(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n))\ *_R\ z\hat{}(2*n))$ *sums cos z*
proof −
from *cos_converges*[*of z*] **have** $(\lambda n.\ cos\_coeff\ n\ *_R\ z\hat{}n)$ *sums cos z* **.**
also have $(\lambda n.\ cos\_coeff\ n\ *_R\ z\hat{}n)$ *sums cos z* $\longleftrightarrow$
$(\lambda n.\ ((-1)\hat{}n\ /\ fact\ (2*n))\ *_R\ z\hat{}(2*n))$ *sums cos z*
by (*subst sums_mono_reindex*[*of* $\lambda n.\ 2*n$, *symmetric*])
(*auto simp*: *cos_coeff_def strict_mono_def ac_simps elim*!: *evenE*)
finally show *?thesis* **.**
qed

lemma *sin_z_over_z_series*:
fixes $z :: 'a :: \{real\_normed\_field, banach\}$
assumes $z \neq 0$
shows $(\lambda n.\ (-1)\hat{}n\ /\ fact\ (2*n+1) * z\hat{}(2*n))$ *sums* $(sin\ z\ /\ z)$
proof −
from *sin_series*[*of z*] **have** $(\lambda n.\ z * ((-1)\hat{}n\ /\ fact\ (2*n+1)) * z\hat{}(2*n))$ *sums sin z*
by (*simp add*: *field_simps scaleR_conv_of_real*)
from *sums_mult*[*OF this*, *of inverse z*] **and** *assms* **show** *?thesis*
by (*simp add*: *field_simps*)
qed

lemma *sin_z_over_z_series'*:
fixes $z :: 'a :: \{real\_normed\_field, banach\}$
assumes $z \neq 0$
shows $(\lambda n.\ sin\_coeff\ (n+1)\ *_R\ z\hat{}n)$ *sums* $(sin\ z\ /\ z)$
proof −
from *sums_split_initial_segment*[*OF sin_converges*[*of z*], *of 1*]
have $(\lambda n.\ z * (sin\_coeff\ (n+1)\ *_R\ z\ \hat{}\ n))$ *sums sin z* **by** *simp*

from *sums_mult*[*OF this*, *of inverse z*] *assms* **show** *?thesis* **by** (*simp add*: *field_simps*)
qed

lemma *has_field_derivative_sin_z_over_z*:
fixes *A* :: *'a* :: {*real_normed_field*,*banach*} *set*
shows ((λ*z*. *if z* = *0 then 1 else sin z* / *z*) *has_field_derivative 0*) (*at 0 within A*)
(**is** (*?f has_field_derivative ?f'*) _)
proof (*rule has_field_derivative_at_within*)
have ((λ*z*::*'a*. ∑ *n*. *of_real* (*sin_coeff* (*n*+*1*)) ∗ *z*^*n*)
has_field_derivative (∑ *n*. *diffs* (λ*n*. *of_real* (*sin_coeff* (*n*+*1*))) *n* ∗ *0*^*n*)) (*at 0*)
proof (*rule termdiffs_strong*)
from *summable_ignore_initial_segment*[*OF sums_summable*[*OF sin_converges*[*of 1*::*'a*]], *of 1*]
show *summable* (λ*n*. *of_real* (*sin_coeff* (*n*+*1*)) ∗ (*1*::*'a*)^*n*) **by** (*simp add*: *of_real_def*)
qed *simp*
also have (λ*z*::*'a*. ∑ *n*. *of_real* (*sin_coeff* (*n*+*1*)) ∗ *z*^*n*) = *?f*
proof
fix *z*
show (∑ *n*. *of_real* (*sin_coeff* (*n*+*1*)) ∗ *z*^*n*) = *?f z*
by (*cases z* = *0*) (*insert sin_z_over_z_series'*[*of z*],
simp_all add: *scaleR_conv_of_real sums_iff sin_coeff_def*)
qed
also have (∑ *n*. *diffs* (λ*n*. *of_real* (*sin_coeff* (*n* + *1*))) *n* ∗ (*0*::*'a*) ^ *n*) =
diffs (λ*n*. *of_real* (*sin_coeff* (*Suc n*))) *0* **by** *simp*
also have ... = *0* **by** (*simp add*: *sin_coeff_def diffs_def*)
finally show ((λ*z*::*'a*. *if z* = *0 then 1 else sin z* / *z*) *has_field_derivative 0*) (*at 0*) .
qed

lemma *round_Re_minimises_norm*:
norm ((*z*::*complex*) − *of_int m*) ≥ *norm* (*z* − *of_int* (*round* (*Re z*)))
proof −
let *?n* = *round* (*Re z*)
have *norm* (*z* − *of_int ?n*) = *sqrt* ((*Re z* − *of_int ?n*)^2 + (*Im z*)^2)
by (*simp add*: *cmod_def*)
also have |*Re z* − *of_int ?n*| ≤ |*Re z* − *of_int m*| **by** (*rule round_diff_minimal*)
hence *sqrt* ((*Re z* − *of_int ?n*)^2 + (*Im z*)^2) ≤ *sqrt* ((*Re z* − *of_int m*)^2 + (*Im z*)^2)
by (*intro real_sqrt_le_mono add_mono*) (*simp_all add*: *abs_le_square_iff*)
also have ... = *norm* (*z* − *of_int m*) **by** (*simp add*: *cmod_def*)
finally show *?thesis* .
qed

lemma *Re_pos_in_ball*:
assumes *Re z* > *0 t* ∈ *ball z* (*Re z*/*2*)

shows *Re t > 0*
proof −
have *Re (z − t) ≤ norm (z − t)* **by** (*rule complex_Re_le_cmod*)
also from *assms* **have** *... < Re z / 2* **by** (*simp add*: *dist_complex_def*)
finally show *Re t > 0* **using** *assms* **by** *simp*
qed

lemma *no_nonpos_Int_in_ball_complex*:
assumes *Re z > 0 t ∈ ball z (Re z/2)*
shows *t ∉ ℤ≤0*
using *Re_pos_in_ball*[*OF assms*] **by** (*force elim*!: *nonpos_Ints_cases*)

lemma *no_nonpos_Int_in_ball*:
assumes *t ∈ ball z (dist z (round (Re z)))*
shows *t ∉ ℤ≤0*
proof
assume *t ∈ ℤ≤0*
then obtain *n* **where** *t = of_int n* **by** (*auto elim*!: *nonpos_Ints_cases*)
have *dist z (of_int n) ≤ dist z t + dist t (of_int n)* **by** (*rule dist_triangle*)
also from *assms* **have** *dist z t < dist z (round (Re z))* **by** *simp*
also have *... ≤ dist z (of_int n)*
using *round_Re_minimises_norm*[*of z*] **by** (*simp add*: *dist_complex_def*)
finally have *dist t (of_int n) > 0* **by** *simp*
with ‹*t = of_int n*› **show** *False* **by** *simp*
qed

lemma *no_nonpos_Int_in_ball'*:
assumes (*z :: 'a :: {euclidean_space,real_normed_algebra_1}*) *∉ ℤ≤0*
obtains *d* **where** *d > 0* ⋀*t*. *t ∈ ball z d ⟹ t ∉ ℤ≤0*
proof (*rule that*)
from *assms* **show** *setdist {z} ℤ≤0 > 0* **by** (*subst setdist_gt_0_compact_closed*)
auto
next
fix *t* **assume** *t ∈ ball z (setdist {z} ℤ≤0)*
thus *t ∉ ℤ≤0* **using** *setdist_le_dist*[*of z {z} t ℤ≤0*] **by** *force*
qed

lemma *no_nonpos_Real_in_ball*:
assumes *z*: *z ∉ ℝ≤0* **and** *t*: *t ∈ ball z (if Im z = 0 then Re z / 2 else abs (Im z) / 2)*
shows *t ∉ ℝ≤0*
using *z*
proof (*cases Im z = 0*)
assume *A*: *Im z = 0*
with *z* **have** *Re z > 0* **by** (*force simp add*: *complex_nonpos_Reals_iff*)
with *t A Re_pos_in_ball*[*of z t*] **show** *?thesis* **by** (*force simp add*: *complex_nonpos_Reals_iff*)
next
assume *A*: *Im z ≠ 0*
have *abs (Im z) − abs (Im t) ≤ abs (Im z − Im t)* **by** *linarith*

also have ... = *abs* (*Im* (*z* − *t*)) **by** *simp*
also have ... ≤ *norm* (*z* − *t*) **by** (*rule abs_Im_le_cmod*)
also from *A t* **have** ... ≤ *abs* (*Im z*) / *2* **by** (*simp add*: *dist_complex_def*)
finally have *abs* (*Im t*) > *0* **using** *A* **by** *simp*
thus *?thesis* **by** (*force simp add*: *complex_nonpos_Reals_iff*)
qed

9.10.1 The Euler form and the logarithmic Gamma function

We define the Gamma function by first defining its multiplicative inverse *rGamma*. This is more convenient because *rGamma* is entire, which makes proofs of its properties more convenient because one does not have to watch out for discontinuities. (e.g. *rGamma* fulfils *rGamma z* = *z* ∗ *rGamma* (*z* + *1*) everywhere, whereas the Γ function does not fulfil the analogous equation on the non-positive integers)

We define the Γ function (resp. its reciprocale) in the Euler form. This form has the advantage that it is a relatively simple limit that converges everywhere. The limit at the poles is 0 (due to division by 0). The functional equation *Gamma* (*z* + *1*) = *z* ∗ *Gamma z* follows immediately from the definition.

definition *Gamma_series* :: ('*a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ '*a* **where**
Gamma_series z n = *fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))) / *pochhammer z* (*n*+*1*)

definition *Gamma_series'* :: ('*a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ '*a* **where**
Gamma_series' z n = *fact* (*n* − *1*) ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))) / *pochhammer z n*

definition *rGamma_series* :: ('*a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ '*a* **where**
rGamma_series z n = *pochhammer z* (*n*+*1*) / (*fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))))

lemma *Gamma_series_altdef*: *Gamma_series z n* = *inverse* (*rGamma_series z n*)
and *rGamma_series_altdef*: *rGamma_series z n* = *inverse* (*Gamma_series z n*)
unfolding *Gamma_series_def rGamma_series_def* **by** *simp_all*

lemma *rGamma_series_minus_of_nat*:
eventually (λ*n*. *rGamma_series* (− *of_nat k*) *n* = *0*) *sequentially*
using *eventually_ge_at_top*[*of k*]
by *eventually_elim* (*auto simp*: *rGamma_series_def pochhammer_of_nat_eq_0_iff*)

lemma *Gamma_series_minus_of_nat*:
eventually (λ*n*. *Gamma_series* (− *of_nat k*) *n* = *0*) *sequentially*

using *eventually_ge_at_top*[*of k*]
by *eventually_elim* (*auto simp*: *Gamma_series_def pochhammer_of_nat_eq_0_iff*)

lemma *Gamma_series'_minus_of_nat*:
eventually (λ*n*. *Gamma_series'* (− *of_nat k*) *n* = *0*) *sequentially*
using *eventually_gt_at_top*[*of k*]
by *eventually_elim* (*auto simp*: *Gamma_series'_def pochhammer_of_nat_eq_0_iff*)

lemma *rGamma_series_nonpos_Ints_LIMSEQ*: *z* ∈ ℤ≤0 ⟹ *rGamma_series z* ⟶ *0*
by (*elim nonpos_Ints_cases'*, *hypsubst*, *subst tendsto_cong*, *rule rGamma_series_minus_of_nat*, *simp*)

lemma *Gamma_series_nonpos_Ints_LIMSEQ*: *z* ∈ ℤ≤0 ⟹ *Gamma_series z* ⟶ *0*
by (*elim nonpos_Ints_cases'*, *hypsubst*, *subst tendsto_cong*, *rule Gamma_series_minus_of_nat*, *simp*)

lemma *Gamma_series'_nonpos_Ints_LIMSEQ*: *z* ∈ ℤ≤0 ⟹ *Gamma_series' z* ⟶ *0*
by (*elim nonpos_Ints_cases'*, *hypsubst*, *subst tendsto_cong*, *rule Gamma_series'_minus_of_nat*, *simp*)

lemma *Gamma_series_Gamma_series'*:
assumes *z*: *z* ∉ ℤ≤0
shows (λ*n*. *Gamma_series' z n* / *Gamma_series z n*) ⟶ *1*
proof (*rule Lim_transform_eventually*)
from *eventually_gt_at_top*[*of 0*::*nat*]
show *eventually* (λ*n*. *z* / *of_nat n* + *1* = *Gamma_series' z n* / *Gamma_series z n*) *sequentially*
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
from *n z* **have** *Gamma_series' z n* / *Gamma_series z n* = (*z* + *of_nat n*) / *of_nat n*
by (*cases n*, *simp*)
(*auto simp add*: *Gamma_series_def Gamma_series'_def pochhammer_rec'*
dest: *pochhammer_eq_0_imp_nonpos_Int plus_of_nat_eq_0_imp*)
also from *n* **have** ... = *z* / *of_nat n* + *1* **by** (*simp add*: *field_split_simps*)
finally show *z* / *of_nat n* + *1* = *Gamma_series' z n* / *Gamma_series z n* ..
qed
have (λ*x*. *z* / *of_nat x*) ⟶ *0*
by (*rule tendsto_norm_zero_cancel*)
(*insert tendsto_mult*[*OF tendsto_const*[*of norm z*] *lim_inverse_n*],
simp add: *norm_divide inverse_eq_divide*)
from *tendsto_add*[*OF this tendsto_const*[*of 1*]] **show** (λ*n*. *z* / *of_nat n* + *1*) ⟶ *1* **by** *simp*
qed

We now show that the series that defines the Γ function in the Euler form

converges and that the function defined by it is continuous on the complex halfspace with positive real part.

We do this by showing that the logarithm of the Euler series is continuous and converges locally uniformly, which means that the log-Gamma function defined by its limit is also continuous.

This will later allow us to lift holomorphicity and continuity from the log-Gamma function to the inverse of the Gamma function, and from that to the Gamma function itself.

definition *ln_Gamma_series* :: ('*a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ *nat* ⇒ '*a* **where**
ln_Gamma_series z n = *z* ∗ *ln* (*of_nat n*) − *ln z* − (∑ *k*=*1*..*n*. *ln* (*z* / *of_nat k* + *1*))

definition *ln_Gamma_series'* :: ('*a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ *nat* ⇒ '*a* **where**
ln_Gamma_series' z n =
− *euler_mascheroni*∗*z* − *ln z* + (∑ *k*=*1*..*n*. *z* / *of_nat n* − *ln* (*z* / *of_nat k* + *1*))

definition *ln_Gamma* :: ('*a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ '*a* **where**
ln_Gamma z = *lim* (*ln_Gamma_series z*)

We now show that the log-Gamma series converges locally uniformly for all complex numbers except the non-positive integers. We do this by proving that the series is locally Cauchy.

context
begin

private lemma *ln_Gamma_series_complex_converges_aux*:
fixes *z* :: *complex* **and** *k* :: *nat*
assumes *z*: *z* ≠ *0* **and** *k*: *of_nat k* ≥ *2*∗*norm z k* ≥ *2*
shows *norm* (*z* ∗ *ln* (*1* − *1*/*of_nat k*) + *ln* (*z*/*of_nat k* + *1*)) ≤ *2*∗(*norm z* + *norm z*^*2*) / *of_nat k*^*2*
proof −
let *?k* = *of_nat k* :: *complex* **and** *?z* = *norm z*
have *z* ∗*ln* (*1* − *1*/*?k*) + *ln* (*z*/*?k*+*1*) = *z*∗(*ln* (*1* − *1*/*?k* :: *complex*) + *1*/*?k*) + (*ln* (*1*+*z*/*?k*) − *z*/*?k*)
by (*simp add*: *algebra_simps*)
also have *norm* ... ≤ *?z* ∗ *norm* (*ln* (*1*−*1*/*?k*) + *1*/*?k* :: *complex*) + *norm* (*ln* (*1*+*z*/*?k*) − *z*/*?k*)
by (*subst norm_mult* [*symmetric*], *rule norm_triangle_ineq*)
also have *norm* (*Ln* (*1* + −*1*/*?k*) − (−*1*/*?k*)) ≤ (*norm* (−*1*/*?k*))^*2* / (*1* − *norm*(−*1*/*?k*))
using *k* **by** (*intro Ln_approx_linear*) (*simp add*: *norm_divide*)
hence *?z* ∗ *norm* (*ln* (*1*−*1*/*?k*) + *1*/*?k*) ≤ *?z* ∗ ((*norm* (*1*/*?k*))^*2* / (*1* − *norm* (*1*/*?k*)))
by (*intro mult_left_mono*) *simp_all*

```
  also have ... ≤ (?z * (of_nat k / (of_nat k - 1))) / of_nat k^2 using k
    by (simp add: field_simps power2_eq_square norm_divide)
  also have ... ≤ (?z * 2) / of_nat k^2 using k
    by (intro divide_right_mono mult_left_mono) (simp_all add: field_simps)
  also have norm (ln (1+z/?k) - z/?k) ≤ norm (z/?k)^2 / (1 - norm (z/?k))
using k
    by (intro Ln_approx_linear) (simp add: norm_divide)
  hence norm (ln (1+z/?k) - z/?k) ≤ ?z^2 / of_nat k^2 / (1 - ?z / of_nat k)
    by (simp add: field_simps norm_divide)
  also have ... ≤ (?z^2 * (of_nat k / (of_nat k - ?z))) / of_nat k^2 using k
    by (simp add: field_simps power2_eq_square)
  also have ... ≤ (?z^2 * 2) / of_nat k^2 using k
    by (intro divide_right_mono mult_left_mono) (simp_all add: field_simps)
  also note add_divide_distrib [symmetric]
  finally show ?thesis by (simp only: distrib_left mult.commute)
qed

lemma ln_Gamma_series_complex_converges:
  assumes z: z ∉ ℤ≤0
  assumes d: d > 0 ⋀n. n ∈ ℤ≤0 ⟹ norm (z - of_int n) > d
  shows uniformly_convergent_on (ball z d) (λn z. ln_Gamma_series z n :: com-
plex)
proof (intro Cauchy_uniformly_convergent uniformly_Cauchy_onI')
  fix e :: real assume e: e > 0
  define e'' where e'' = (SUP t∈ball z d. norm t + norm t^2)
  define e' where e' = e / (2*e'')
  have bounded ((λt. norm t + norm t^2) ` cball z d)
   by (intro compact_imp_bounded compact_continuous_image) (auto intro!: con-
tinuous_intros)
  hence bounded ((λt. norm t + norm t^2) ` ball z d) by (rule bounded_subset)
auto
 hence bdd: bdd_above ((λt. norm t + norm t^2) ` ball z d) by (rule bounded_imp_bdd_above)

  with z d(1) d(2)[of -1] have e''_pos: e'' > 0 unfolding e''_def
    by (subst less_cSUP_iff) (auto intro!: add_pos_nonneg bexI[of _ z])
  have e'': norm t + norm t^2 ≤ e'' if t ∈ ball z d for t unfolding e''_def using
that
    by (rule cSUP_upper[OF _ bdd])
  from e z e''_pos have e': e' > 0 unfolding e'_def
   by (intro divide_pos_pos mult_pos_pos add_pos_pos) (simp_all add: field_simps)

  have summable (λk. inverse ((real_of_nat k)^2))
    by (rule inverse_power_summable) simp
  from summable_partial_sum_bound[OF this e']
  obtain M where M: ⋀m n. M ≤ m ⟹ norm (∑ k = m..n. inverse ((real k)^2))
< e'
    by auto

  define N where N = max 2 (max (nat ⌈2 * (norm z + d)⌉) M)
```

```
  {
    from d have ⌈2 * (cmod z + d)⌉ ≥ ⌈0::real⌉
     by (intro ceiling_mono mult_nonneg_nonneg add_nonneg_nonneg) simp_all
    hence 2 * (norm z + d) ≤ of_nat (nat ⌈2 * (norm z + d)⌉) unfolding N_def
      by (simp_all)
    also have ... ≤ of_nat N unfolding N_def
      by (subst of_nat_le_iff) (rule max.coboundedI2, rule max.cobounded1)
    finally have of_nat N ≥ 2 * (norm z + d) .
    moreover have N ≥ 2 N ≥ M unfolding N_def by simp_all
    moreover have (∑k=m..n. 1/(of_nat k)^2) < e' if m ≥ N for m n
      using M[OF order.trans[OF ‹N ≥ M› that]] unfolding real_norm_def
     by (subst (asm) abs_of_nonneg) (auto intro: sum_nonneg simp: field_split_simps)
    moreover note calculation
  } note N = this

  show ∃M. ∀t∈ball z d. ∀m≥M. ∀n>m. dist (ln_Gamma_series t m) (ln_Gamma_series
t n) < e
    unfolding dist_complex_def
  proof (intro exI[of _ N] ballI allI impI)
    fix t m n assume t: t ∈ ball z d and mn: m ≥ N n > m
     from d(2)[of 0] t have 0 < dist z 0 − dist z t by (simp add: field_simps
dist_complex_def)
    also have dist z 0 − dist z t ≤ dist 0 t using dist_triangle[of 0 z t]
      by (simp add: dist_commute)
    finally have t_nz: t ≠ 0 by auto

    have norm t ≤ norm z + norm (t − z) by (rule norm_triangle_sub)
   also from t have norm (t − z) < d by (simp add: dist_complex_def norm_minus_commute)
    also have 2 * (norm z + d) ≤ of_nat N by (rule N)
    also have N ≤ m by (rule mn)
    finally have norm_t: 2 * norm t < of_nat m by simp

    have ln_Gamma_series t m − ln_Gamma_series t n =
            (−(t * Ln (of_nat n)) − (−(t * Ln (of_nat m)))) +
            ((∑k=1..n. Ln (t / of_nat k + 1)) − (∑k=1..m. Ln (t / of_nat k
+ 1)))
      by (simp add: ln_Gamma_series_def algebra_simps)
    also have (∑k=1..n. Ln (t / of_nat k + 1)) − (∑k=1..m. Ln (t / of_nat
k + 1)) =
                (∑k∈{1..n}−{1..m}. Ln (t / of_nat k + 1)) using mn
      by (simp add: sum_diff)
    also from mn have {1..n}−{1..m} = {Suc m..n} by fastforce
    also have −(t * Ln (of_nat n)) − (−(t * Ln (of_nat m))) =
                  (∑k = Suc m..n. t * Ln (of_nat (k − 1)) − t * Ln (of_nat k))
using mn
      by (subst sum_telescope'' [symmetric]) simp_all
    also have ... = (∑k = Suc m..n. t * Ln (of_nat (k − 1) / of_nat k)) using
mn N
      by (intro sum_cong_Suc)
```

```
    (simp_all del: of_nat_Suc add: field_simps Ln_of_nat Ln_of_nat_over_of_nat)
  also have of_nat (k - 1) / of_nat k = 1 - 1 / (of_nat k :: complex) if k ∈
{Suc m..n} for k
    using that of_nat_eq_0_iff[of Suc i for i] by (cases k) (simp_all add:
field_split_simps)
  hence (∑k = Suc m..n. t * Ln (of_nat (k - 1) / of_nat k)) =
        (∑k = Suc m..n. t * Ln (1 - 1 / of_nat k)) using mn N
    by (intro sum.cong) simp_all
  also note sum.distrib [symmetric]
  also have norm (∑k=Suc m..n. t * Ln (1 - 1/of_nat k) + Ln (t/of_nat k
+ 1)) ≤
    (∑k=Suc m..n. 2 * (norm t + (norm t)^2) / (real_of_nat k)^2) using t_nz
N(2) mn norm_t
  by (intro order.trans[OF norm_sum sum_mono[OF ln_Gamma_series_complex_converges_aux]])
simp_all
  also have ... ≤ 2 * (norm t + norm t^2) * (∑k=Suc m..n. 1 / (of_nat k)^2)
    by (simp add: sum_distrib_left)
  also have ... < 2 * (norm t + norm t^2) * e' using mn z t_nz
    by (intro mult_strict_left_mono N mult_pos_pos add_pos_pos) simp_all
  also from e''_pos have ... = e * ((cmod t + (cmod t)^2) / e'')
    by (simp add: e'_def field_simps power2_eq_square)
  also from e''[OF t] e''_pos e
    have ... ≤ e * 1 by (intro mult_left_mono) (simp_all add: field_simps)
  finally show norm (ln_Gamma_series t m - ln_Gamma_series t n) < e by
simp
 qed
qed

end

lemma ln_Gamma_series_complex_converges':
 assumes z: (z :: complex) ∉ ℤ≤0
 shows ∃d>0. uniformly_convergent_on (ball z d) (λn z. ln_Gamma_series z
n)
proof -
 define d' where d' = Re z
 define d where d = (if d' > 0 then d' / 2 else norm (z - of_int (round d')) /
2)
 have of_int (round d') ∈ ℤ≤0 if d' ≤ 0 using that
   by (intro nonpos_Ints_of_int) (simp_all add: round_def)
 with assms have d_pos: d > 0 unfolding d_def by (force simp: not_less)

 have d < cmod (z - of_int n) if n ∈ ℤ≤0 for n
 proof (cases Re z > 0)
   case True
   from nonpos_Ints_nonpos[OF that] have n: n ≤ 0 by simp
   from True have d = Re z/2 by (simp add: d_def d'_def)
   also from n True have ... < Re (z - of_int n) by simp
   also have ... ≤ norm (z - of_int n) by (rule complex_Re_le_cmod)
```

```
    finally show ?thesis .
  next
    case False
    with assms nonpos_Ints_of_int[of round (Re z)]
      have z ≠ of_int (round d') by (auto simp: not_less)
     with False have d < norm (z − of_int (round d')) by (simp add: d_def
d'_def)
   also have ... ≤ norm (z − of_int n) unfolding d'_def by (rule round_Re_minimises_norm)
    finally show ?thesis .
  qed
  hence conv: uniformly_convergent_on (ball z d) (λn z. ln_Gamma_series z n)
    by (intro ln_Gamma_series_complex_converges d_pos z) simp_all
  from d_pos conv show ?thesis by blast
qed

lemma ln_Gamma_series_complex_converges'': (z :: complex) ∉ ℤ≤0 ⟹ con-
vergent (ln_Gamma_series z)
 by (drule ln_Gamma_series_complex_converges') (auto intro: uniformly_convergent_imp_convergent

theorem ln_Gamma_complex_LIMSEQ: (z :: complex) ∉ ℤ≤0 ⟹ ln_Gamma_series
z ⟶ ln_Gamma z
 using ln_Gamma_series_complex_converges'' by (simp add: convergent_LIMSEQ_iff
ln_Gamma_def)

lemma exp_ln_Gamma_series_complex:
  assumes n > 0 z ∉ ℤ≤0
  shows   exp (ln_Gamma_series z n :: complex) = Gamma_series z n
proof −
  from assms obtain m where m: n = Suc m by (cases n) blast
  from assms have z ≠ 0 by (intro notI) auto
  with assms have exp (ln_Gamma_series z n) =
        (of_nat n) powr z / (z * (∏k=1..n. exp (Ln (z / of_nat k + 1))))
    unfolding ln_Gamma_series_def powr_def by (simp add: exp_diff exp_sum)
  also from assms have (∏k=1..n. exp (Ln (z / of_nat k + 1))) = (∏k=1..n.
z / of_nat k + 1)
   by (intro prod.cong[OF refl], subst exp_Ln) (auto simp: field_simps plus_of_nat_eq_0_imp)
  also have ... = (∏k=1..n. z + k) / fact n
    by (simp add: fact_prod)
    (subst prod_dividef [symmetric], simp_all add: field_simps)
  also from m have z * ... = (∏k=0..n. z + k) / fact n
   by (simp add: prod.atLeast0_atMost_Suc_shift prod.atLeast_Suc_atMost_Suc_shift
del: prod.cl_ivl_Suc)
  also have (∏k=0..n. z + k) = pochhammer z (Suc n)
    unfolding pochhammer_prod
   by (simp add: prod.atLeast0_atMost_Suc atLeastLessThanSuc_atLeastAtMost)
  also have of_nat n powr z / (pochhammer z (Suc n) / fact n) = Gamma_series
z n
    unfolding Gamma_series_def using assms by (simp add: field_split_simps
powr_def)
```

```
  finally show ?thesis .
qed


lemma ln_Gamma_series'_aux:
  assumes (z::complex) ∉ ℤ≤0
  shows   (λk. z / of_nat (Suc k) − ln (1 + z / of_nat (Suc k))) sums
            (ln_Gamma z + euler_mascheroni ∗ z + ln z) (is ?f sums ?s)
unfolding sums_def
proof (rule Lim_transform)
  show (λn. ln_Gamma_series z n + of_real (harm n − ln (of_nat n)) ∗ z + ln
z) ⟶ ?s
    (is ?g ⟶ _)
   by (intro tendsto_intros ln_Gamma_complex_LIMSEQ euler_mascheroni_LIMSEQ_of_real
assms)

  have A: eventually (λn. (∑ k<n. ?f k) − ?g n = 0) sequentially
    using eventually_gt_at_top[of 0::nat]
  proof eventually_elim
    fix n :: nat assume n: n > 0
    have (∑ k<n. ?f k) = (∑ k=1..n. z / of_nat k − ln (1 + z / of_nat k))
       by (subst atLeast0LessThan [symmetric], subst sum.shift_bounds_Suc_ivl
[symmetric],
         subst atLeastLessThanSuc_atLeastAtMost) simp_all
    also have ... = z ∗ of_real (harm n) − (∑ k=1..n. ln (1 + z / of_nat k))
      by (simp add: harm_def sum_subtractf sum_distrib_left divide_inverse)
    also from n have ... − ?g n = 0
      by (simp add: ln_Gamma_series_def sum_subtractf algebra_simps)
    finally show (∑ k<n. ?f k) − ?g n = 0 .
  qed
   show (λn. (∑ k<n. ?f k) − ?g n) ⟶ 0 by (subst tendsto_cong[OF A])
simp_all
qed


lemma uniformly_summable_deriv_ln_Gamma:
  assumes z: (z :: 'a :: {real_normed_field,banach}) ≠ 0 and d: d > 0 d ≤ norm
z/2
  shows uniformly_convergent_on (ball z d)
          (λk z. ∑ i<k. inverse (of_nat (Suc i)) − inverse (z + of_nat (Suc i)))
         (is uniformly_convergent_on _ (λk z. ∑ i<k. ?f i z))
proof (rule Weierstrass_m_test'_ev)
  {
    fix t assume t: t ∈ ball z d
    have norm z = norm (t + (z − t)) by simp
   have norm (t + (z − t)) ≤ norm t + norm (z − t) by (rule norm_triangle_ineq)
    also from t d have norm (z − t) < norm z / 2 by (simp add: dist_norm)
    finally have A: norm t > norm z / 2 using z by (simp add: field_simps)
```

```
    have norm t = norm (z + (t - z)) by simp
    also have ... ≤ norm z + norm (t - z) by (rule norm_triangle_ineq)
    also from t d have norm (t - z) ≤ norm z / 2 by (simp add: dist_norm
norm_minus_commute)
    also from z have ... < norm z by simp
    finally have B: norm t < 2 * norm z by simp
    note A B
  } note ball = this

  show eventually (λn. ∀t∈ball z d. norm (?f n t) ≤ 4 * norm z * inverse (of_nat
(Suc n)^2)) sequentially
    using eventually_gt_at_top apply eventually_elim
  proof safe
    fix t :: 'a assume t: t ∈ ball z d
    from z ball[OF t] have t_nz: t ≠ 0 by auto
    fix n :: nat assume n: n > nat ⌈4 * norm z⌉
    from ball[OF t] t_nz have 4 * norm z > 2 * norm t by simp
    also from n have ...  < of_nat n by linarith
    finally have n: of_nat n > 2 * norm t .
    hence of_nat n > norm t by simp
    hence t': t ≠ -of_nat (Suc n) by (intro notI) (simp del: of_nat_Suc)

    with t_nz have ?f n t = 1 / (of_nat (Suc n) * (1 + of_nat (Suc n)/t))
      by (simp add: field_split_simps eq_neg_iff_add_eq_0 del: of_nat_Suc)
    also have norm ... = inverse (of_nat (Suc n)) * inverse (norm (of_nat (Suc
n)/t + 1))
      by (simp add: norm_divide norm_mult field_split_simps del: of_nat_Suc)
    also {
      from z t_nz ball[OF t] have of_nat (Suc n) / (4 * norm z) ≤ of_nat (Suc
n) / (2 * norm t)
        by (intro divide_left_mono mult_pos_pos) simp_all
      also have ... < norm (of_nat (Suc n) / t) - norm (1 :: 'a)
        using t_nz n by (simp add: field_simps norm_divide del: of_nat_Suc)
      also have ... ≤ norm (of_nat (Suc n)/t + 1) by (rule norm_diff_ineq)
      finally have inverse (norm (of_nat (Suc n)/t + 1)) ≤ 4 * norm z / of_nat
(Suc n)
      using z by (simp add: field_split_simps norm_divide mult_ac del: of_nat_Suc)
    }
    also have inverse (real_of_nat (Suc n)) * (4 * norm z / real_of_nat (Suc
n)) =
                4 * norm z * inverse (of_nat (Suc n)^2)
              by (simp add: field_split_simps power2_eq_square del: of_nat_Suc)
    finally show norm (?f n t) ≤ 4 * norm z * inverse (of_nat (Suc n)^2)
      by (simp del: of_nat_Suc)
  qed
next
  show summable (λn. 4 * norm z * inverse ((of_nat (Suc n))^2))
   by (subst summable_Suc_iff) (simp add: summable_mult inverse_power_summable)
qed
```

9.10.2 The Polygamma functions

lemma *summable_deriv_ln_Gamma*:
  $z \neq (0 :: 'a :: \{real\_normed\_field, banach\}) \Longrightarrow$
    summable $(\lambda n.\ inverse\ (of\_nat\ (Suc\ n)) - inverse\ (z + of\_nat\ (Suc\ n)))$
  unfolding *summable_iff_convergent*
  by (*rule uniformly_convergent_imp_convergent*,
    rule uniformly_summable_deriv_ln_Gamma[*of z norm z/2*]) *simp_all*

definition *Polygamma* :: *nat* ⇒ ('*a* :: {*real_normed_field*,*banach*}) ⇒ '*a* **where**
  Polygamma n z = (*if n* = *0 then*
    $(\sum k.\ inverse\ (of\_nat\ (Suc\ k)) - inverse\ (z + of\_nat\ k)) - euler\_mascheroni$
else
    $(-1)\hat{}Suc\ n * fact\ n * (\sum k.\ inverse\ ((z + of\_nat\ k)\hat{}Suc\ n))$)

abbreviation *Digamma* :: ('*a* :: {*real_normed_field*,*banach*}) ⇒ '*a* **where**
  Digamma ≡ *Polygamma 0*

lemma *Digamma_def*:
  $Digamma\ z = (\sum k.\ inverse\ (of\_nat\ (Suc\ k)) - inverse\ (z + of\_nat\ k)) - euler\_mascheroni$
  by (*simp add*: *Polygamma_def*)

lemma *summable_Digamma*:
  assumes $(z :: 'a :: \{real\_normed\_field, banach\}) \neq 0$
  shows *summable* $(\lambda n.\ inverse\ (of\_nat\ (Suc\ n)) - inverse\ (z + of\_nat\ n))$
proof −
  have *sums*: $(\lambda n.\ inverse\ (z + of\_nat\ (Suc\ n)) - inverse\ (z + of\_nat\ n))\ sums$
                $(0 - inverse\ (z + of\_nat\ 0))$
    by (*intro telescope_sums filterlim_compose*[*OF tendsto_inverse_0*]
          tendsto_add_filterlim_at_infinity[*OF tendsto_const*] *tendsto_of_nat*)
  from *summable_add*[*OF summable_deriv_ln_Gamma*[*OF assms*] *sums_summable*[*OF sums*]]
    show *summable* $(\lambda n.\ inverse\ (of\_nat\ (Suc\ n)) - inverse\ (z + of\_nat\ n))$ **by** *simp*
qed

lemma *summable_offset*:
  assumes *summable* $(\lambda n.\ f\ (n + k) :: 'a :: real\_normed\_vector)$
  shows *summable f*
proof −
  from *assms* **have** *convergent* $(\lambda m.\ \sum n{<}m.\ f\ (n + k))$
    using *summable_iff_convergent* **by** *blast*
  hence *convergent* $(\lambda m.\ (\sum n{<}k.\ f\ n) + (\sum n{<}m.\ f\ (n + k)))$
    by (*intro convergent_add convergent_const*)
  also have $(\lambda m.\ (\sum n{<}k.\ f\ n) + (\sum n{<}m.\ f\ (n + k))) = (\lambda m.\ \sum n{<}m{+}k.\ f\ n)$
  proof
    fix *m* :: *nat*
    have $\{..<m+k\} = \{..<k\} \cup \{k..<m+k\}$ **by** *auto*

```
    also have (∑ n∈.... f n) = (∑ n<k. f n) + (∑ n=k..<m+k. f n)
      by (rule sum.union_disjoint) auto
    also have (∑ n=k..<m+k. f n) = (∑ n=0..<m+k−k. f (n + k))
      using sum.shift_bounds_nat_ivl [of f 0 k m] by simp
    finally show (∑ n<k. f n) + (∑ n<m. f (n + k)) = (∑ n<m+k. f n) by
(simp add: atLeast0LessThan)
  qed
  finally have (λa. sum f {..<a}) ⟶ lim (λm. sum f {..<m + k})
    by (auto simp: convergent_LIMSEQ_iff dest: LIMSEQ_offset)
  thus ?thesis by (auto simp: summable_iff_convergent convergent_def)
qed

lemma Polygamma_converges:
  fixes z :: 'a :: {real_normed_field,banach}
  assumes z: z ≠ 0 and n: n ≥ 2
  shows uniformly_convergent_on (ball z d) (λk z. ∑ i<k. inverse ((z + of_nat
i) ^ n))
proof (rule Weierstrass_m_test'_ev)
  define e where e = (1 + d / norm z)
  define m where m = nat ⌈norm z * e⌉
  {
    fix t assume t: t ∈ ball z d
    have norm t = norm (z + (t − z)) by simp
    also have ... ≤ norm z + norm (t − z) by (rule norm_triangle_ineq)
   also from t have norm (t − z) < d by (simp add: dist_norm norm_minus_commute)
    finally have norm t < norm z * e using z by (simp add: divide_simps e_def)
  } note ball = this

  show eventually (λk. ∀ t∈ball z d. norm (inverse ((t + of_nat k) ^ n)) ≤
          inverse (of_nat (k − m) ^ n)) sequentially
    using eventually_gt_at_top[of m] apply eventually_elim
  proof (intro ballI)
    fix k :: nat and t :: 'a assume k: k > m and t: t ∈ ball z d
    from k have real_of_nat (k − m) = of_nat k − of_nat m by (simp add:
of_nat_diff)
    also have ... ≤ norm (of_nat k :: 'a) − norm z * e
      unfolding m_def by (subst norm_of_nat) linarith
    also from ball[OF t] have ... ≤ norm (of_nat k :: 'a) − norm t by simp
    also have ... ≤ norm (of_nat k + t) by (rule norm_diff_ineq)
    finally have inverse ((norm (t + of_nat k)) ^ n) ≤ inverse (real_of_nat (k −
m) ^ n) using k n
        by (intro le_imp_inverse_le power_mono) (simp_all add: add_ac del:
of_nat_Suc)
    thus norm (inverse ((t + of_nat k) ^ n)) ≤ inverse (of_nat (k − m) ^ n)
      by (simp add: norm_inverse norm_power power_inverse)
  qed

  have summable (λk. inverse ((real_of_nat k) ^ n))
    using inverse_power_summable[of n] n by simp
```

hence *summable* (λ*k*. *inverse* ((*real_of_nat* (*k* + *m* − *m*))^*n*)) **by** *simp*
thus *summable* (λ*k*. *inverse* ((*real_of_nat* (*k* − *m*))^*n*)) **by** (*rule summable_offset*)
qed

lemma *Polygamma_converges'*:
fixes *z* :: '*a* :: {*real_normed_field*,*banach*}
assumes *z*: *z* ≠ *0* **and** *n*: *n* ≥ *2*
shows *summable* (λ*k*. *inverse* ((*z* + *of_nat k*)^*n*))
using *uniformly_convergent_imp_convergent*[*OF Polygamma_converges*[*OF assms*, *of 1*], *of z*]
by (*simp add*: *summable_iff_convergent*)

theorem *Digamma_LIMSEQ*:
fixes *z* :: '*a* :: {*banach*,*real_normed_field*}
assumes *z*: *z* ≠ *0*
shows (λ*m*. *of_real* (*ln* (*real m*)) − (∑ *n*<*m*. *inverse* (*z* + *of_nat n*))) ⟶ *Digamma z*
proof −
have (λ*n*. *of_real* (*ln* (*real n* / (*real* (*Suc n*))))) ⟶ (*of_real* (*ln 1*) :: '*a*)
by (*intro tendsto_intros LIMSEQ_n_over_Suc_n*) *simp_all*
hence (λ*n*. *of_real* (*ln* (*real n* / (*real n* + *1*)))) ⟶ (*0* :: '*a*) **by** (*simp add*: *add_ac*)
hence *lim*: (λ*n*. *of_real* (*ln* (*real n*)) − *of_real* (*ln* (*real n* + *1*))) ⟶ (*0*::'*a*)
proof (*rule Lim_transform_eventually*)
show *eventually* (λ*n*. *of_real* (*ln* (*real n* / (*real n* + *1*))) =
of_real (*ln* (*real n*)) − (*of_real* (*ln* (*real n* + *1*)) :: '*a*)) *at_top*
using *eventually_gt_at_top*[*of 0*::*nat*] **by** *eventually_elim* (*simp add*: *ln_div*)
qed

from *summable_Digamma*[*OF z*]
have (λ*n*. *inverse* (*of_nat* (*n*+*1*)) − *inverse* (*z* + *of_nat n*))
sums (*Digamma z* + *euler_mascheroni*)
by (*simp add*: *Digamma_def summable_sums*)
from *sums_diff*[*OF this euler_mascheroni_sum*]
have (λ*n*. *of_real* (*ln* (*real* (*Suc n*) + *1*)) − *of_real* (*ln* (*real n* + *1*)) − *inverse* (*z* + *of_nat n*))
sums Digamma z **by** (*simp add*: *add_ac*)
hence (λ*m*. (∑ *n*<*m*. *of_real* (*ln* (*real* (*Suc n*) + *1*)) − *of_real* (*ln* (*real n* + *1*))) −
(∑ *n*<*m*. *inverse* (*z* + *of_nat n*))) ⟶ *Digamma z*
by (*simp add*: *sums_def sum_subtractf*)
also have (λ*m*. (∑ *n*<*m*. *of_real* (*ln* (*real* (*Suc n*) + *1*)) − *of_real* (*ln* (*real n* + *1*)))) =
(λ*m*. *of_real* (*ln* (*m* + *1*)) :: '*a*)
by (*subst sum_lessThan_telescope*) *simp_all*
finally show *?thesis* **by** (*rule Lim_transform*) (*insert lim*, *simp*)
qed

theorem *Polygamma_LIMSEQ*:

```
  fixes z :: 'a :: {banach,real_normed_field}
  assumes z ≠ 0 and n > 0
  shows   (λk. inverse ((z + of_nat k)^Suc n)) sums ((−1) ^ Suc n * Polygamma
n z / fact n)
  using Polygamma_converges'[OF assms(1), of Suc n] assms(2)
  by (simp add: sums_iff Polygamma_def)

theorem has_field_derivative_ln_Gamma_complex [derivative_intros]:
  fixes z :: complex
  assumes z: z ∉ ℝ_≤0
  shows   (ln_Gamma has_field_derivative Digamma z) (at z)
proof −
  have not_nonpos_Int [simp]: t ∉ ℤ_≤0 if Re t > 0 for t
    using that by (auto elim!: nonpos_Ints_cases')
  from z have z': z ∉ ℤ_≤0 and z'': z ≠ 0 using nonpos_Ints_subset_nonpos_Reals
nonpos_Reals_zero_I
    by blast+
  let ?f' = λz k. inverse (of_nat (Suc k)) − inverse (z + of_nat (Suc k))
  let ?f = λz k. z / of_nat (Suc k) − ln (1 + z / of_nat (Suc k)) and ?F' = λz.
∑n. ?f' z n
  define d where d = min (norm z/2) (if Im z = 0 then Re z / 2 else abs (Im z)
/ 2)
  from z have d: d > 0 norm z/2 ≥ d by (auto simp add: complex_nonpos_Reals_iff
d_def)
  have ball: Im t = 0 ⟶ Re t > 0 if dist z t < d for t
    using no_nonpos_Real_in_ball[OF z, of t] that unfolding d_def by (force
simp add: complex_nonpos_Reals_iff)
  have sums: (λn. inverse (z + of_nat (Suc n)) − inverse (z + of_nat n)) sums
                  (0 − inverse (z + of_nat 0))
    by (intro telescope_sums filterlim_compose[OF tendsto_inverse_0]
          tendsto_add_filterlim_at_infinity[OF tendsto_const] tendsto_of_nat)

  have ((λz. ∑n. ?f z n) has_field_derivative ?F' z) (at z)
    using d z ln_Gamma_series'_aux[OF z']
   apply (intro has_field_derivative_series'(2)[of ball z d _ _ z] uniformly_summable_deriv_ln_Gamm
   apply (auto intro!: derivative_eq_intros add_pos_pos mult_pos_pos dest!: ball
          simp: field_simps sums_iff nonpos_Reals_divide_of_nat_iff
          simp del: of_nat_Suc)
    apply (auto simp add: complex_nonpos_Reals_iff)
    done
  with z have ((λz. (∑k. ?f z k) − euler_mascheroni * z − Ln z) has_field_derivative
                ?F' z − euler_mascheroni − inverse z) (at z)
    by (force intro!: derivative_eq_intros simp: Digamma_def)
  also have ?F' z − euler_mascheroni − inverse z = (?F' z + −inverse z) −
euler_mascheroni by simp
  also from sums have −inverse z = (∑n. inverse (z + of_nat (Suc n)) −
inverse (z + of_nat n))
    by (simp add: sums_iff)
  also from sums summable_deriv_ln_Gamma[OF z'']
```

have *?F' z + ... = (∑n. inverse (of_nat (Suc n)) − inverse (z + of_nat n))*
by (*subst suminf_add*) (*simp_all add*: *add_ac sums_iff*)
also have *... − euler_mascheroni = Digamma z* **by** (*simp add*: *Digamma_def*)
finally have ((*λz*. (*∑k*. *?f z k*) − *euler_mascheroni* ∗ *z* − *Ln z*)
has_field_derivative Digamma z) (*at z*) **.**
moreover from *eventually_nhds_ball*[*OF d*(*1*), *of z*]
have *eventually* (*λz*. *ln_Gamma z* = (*∑k*. *?f z k*) − *euler_mascheroni* ∗ *z* − *Ln z*) (*nhds z*)
proof *eventually_elim*
fix *t* **assume** *t* ∈ *ball z d*
hence *t* ∉ $\mathbb{Z}_{\leq 0}$ **by** (*auto dest*!: *ball elim*!: *nonpos_Ints_cases*)
from *ln_Gamma_series'_aux*[*OF this*]
show *ln_Gamma t* = (*∑k*. *?f t k*) − *euler_mascheroni* ∗ *t* − *Ln t* **by** (*simp add*: *sums_iff*)
qed
ultimately show *?thesis* **by** (*subst DERIV_cong_ev*[*OF refl _ refl*])
qed

declare *has_field_derivative_ln_Gamma_complex*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *Digamma_1* [*simp*]: *Digamma* (*1* :: *'a* :: {*real_normed_field*,*banach*}) = − *euler_mascheroni*
by (*simp add*: *Digamma_def*)

lemma *Digamma_plus1*:
assumes *z* ≠ *0*
shows *Digamma* (*z*+*1*) = *Digamma z* + *1*/*z*
proof −
have *sums*: (*λk*. *inverse* (*z* + *of_nat k*) − *inverse* (*z* + *of_nat* (*Suc k*)))
sums (*inverse* (*z* + *of_nat* *0*) − *0*)
by (*intro telescope_sums'*[*OF filterlim_compose*[*OF tendsto_inverse_0*]]
tendsto_add_filterlim_at_infinity[*OF tendsto_const*] *tendsto_of_nat*)
have *Digamma* (*z*+*1*) = (*∑k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat* (*Suc k*))) −
euler_mascheroni (**is** *_* = *suminf ?f* − *_*) **by** (*simp add*: *Digamma_def add_ac*)
also have *suminf ?f* = (*∑k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) +
(*∑k*. *inverse* (*z* + *of_nat k*) − *inverse* (*z* + *of_nat* (*Suc k*)))
using *summable_Digamma*[*OF assms*] *sums* **by** (*subst suminf_add*) (*simp_all add*: *add_ac sums_iff*)
also have (*∑k*. *inverse* (*z* + *of_nat k*) − *inverse* (*z* + *of_nat* (*Suc k*))) = *1*/*z*
using *sums* **by** (*simp add*: *sums_iff inverse_eq_divide*)
finally show *?thesis* **by** (*simp add*: *Digamma_def*[*of z*])
qed

theorem *Polygamma_plus1*:
assumes $z \neq 0$
shows *Polygamma n (z + 1) = Polygamma n z + (−1)^n * fact n / (z ^ Suc n)*
proof (*cases n = 0*)
assume *n*: $n \neq 0$
let *?f* = λk. *inverse ((z + of_nat k) ^ Suc n)*
have *Polygamma n (z + 1) = (−1) ^ Suc n * fact n * (∑ k. ?f (k+1))*
using *n* **by** (*simp add*: *Polygamma_def add_ac*)
also have *(∑ k. ?f (k+1)) + (∑ k<1. ?f k) = (∑ k. ?f k)*
using *Polygamma_converges'*[*OF assms*, *of Suc n*] *n*
by (*subst suminf_split_initial_segment* [*symmetric*]) *simp_all*
hence *(∑ k. ?f (k+1)) = (∑ k. ?f k) − inverse (z ^ Suc n)* **by** (*simp add*: *algebra_simps*)
also have *(−1) ^ Suc n * fact n * ((∑ k. ?f k) − inverse (z ^ Suc n)) =*
*Polygamma n z + (−1)^n * fact n / (z ^ Suc n)* **using** *n*
by (*simp add*: *inverse_eq_divide algebra_simps Polygamma_def*)
finally show *?thesis* **.**
qed (*insert assms*, *simp add*: *Digamma_plus1 inverse_eq_divide*)

theorem *Digamma_of_nat*:
Digamma (of_nat (Suc n) :: 'a :: {real_normed_field,banach}) = harm n − euler_mascheroni
proof (*induction n*)
case (*Suc n*)
have *Digamma (of_nat (Suc (Suc n)) :: 'a) = Digamma (of_nat (Suc n) + 1)* **by** *simp*
also have *... = Digamma (of_nat (Suc n)) + inverse (of_nat (Suc n))*
by (*subst Digamma_plus1*) (*simp_all add*: *inverse_eq_divide del*: *of_nat_Suc*)
also have *Digamma (of_nat (Suc n) :: 'a) = harm n − euler_mascheroni* **by** (*rule Suc*)
also have *... + inverse (of_nat (Suc n)) = harm (Suc n) − euler_mascheroni*
by (*simp add*: *harm_Suc*)
finally show *?case* **.**
qed (*simp add*: *harm_def*)

lemma *Digamma_numeral*: *Digamma (numeral n) = harm (pred_numeral n) − euler_mascheroni*
by (*subst of_nat_numeral*[*symmetric*], *subst numeral_eq_Suc*, *subst Digamma_of_nat*) (*rule refl*)

lemma *Polygamma_of_real*: $x \neq 0 \Longrightarrow$ *Polygamma n (of_real x) = of_real (Polygamma n x)*
unfolding *Polygamma_def* **using** *summable_Digamma*[*of x*] *Polygamma_converges'*[*of x Suc n*]
by (*simp_all add*: *suminf_of_real*)

lemma *Polygamma_Real*: $z \in \mathbb{R} \Longrightarrow z \neq 0 \Longrightarrow$ *Polygamma n z* $\in \mathbb{R}$
by (*elim Reals_cases*, *hypsubst*, *subst Polygamma_of_real*) *simp_all*

lemma *Digamma_half_integer*:
Digamma (*of_nat n* + *1/2* :: *'a* :: {*real_normed_field*,*banach*}) =
(∑ *k*<*n*. *2* / (*of_nat* (*2*∗*k*+*1*))) − *euler_mascheroni* − *of_real* (*2* ∗ *ln 2*)
proof (*induction n*)
case *0*
have *Digamma* (*1/2* :: *'a*) = *of_real* (*Digamma* (*1/2*)) **by** (*simp add*: *Polygamma_of_real* [*symmetric*])
also have *Digamma* (*1/2*::*real*) =
(∑ *k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*of_nat k* + *1/2*)) − *euler_mascheroni*
by (*simp add*: *Digamma_def add_ac*)
also have (∑ *k*. *inverse* (*of_nat* (*Suc k*) :: *real*) − *inverse* (*of_nat k* + *1/2*)) =
(∑ *k*. *inverse* (*1/2*) ∗ (*inverse* (*2* ∗ *of_nat* (*Suc k*)) − *inverse* (*2* ∗ *of_nat k* + *1*)))
by (*simp_all add*: *add_ac inverse_mult_distrib*[*symmetric*] *ring_distribs del*: *inverse_divide*)
also have ... = − *2* ∗ *ln 2* **using** *sums_minus*[*OF alternating_harmonic_series_sums'*]
by (*subst suminf_mult*) (*simp_all add*: *algebra_simps sums_iff*)
finally show *?case* **by** *simp*
next
case (*Suc n*)
have *nz*: *2* ∗ *of_nat n* + (*1*:: *'a*) ≠ *0*
using *of_nat_neq_0*[*of 2*∗*n*] **by** (*simp only*: *of_nat_Suc*) (*simp add*: *add_ac*)
hence *nz'*: *of_nat n* + (*1/2*::*'a*) ≠ *0* **by** (*simp add*: *field_simps*)
have *Digamma* (*of_nat* (*Suc n*) + *1/2* :: *'a*) = *Digamma* (*of_nat n* + *1/2* + *1*) **by** *simp*
also from *nz'* **have** ... = *Digamma* (*of_nat n* + *1/2*) + *1* / (*of_nat n* + *1/2*)
by (*rule Digamma_plus1*)
also from *nz nz'* **have** *1* / (*of_nat n* + *1/2* :: *'a*) = *2* / (*2* ∗ *of_nat n* + *1*)
by (*subst divide_eq_eq*) *simp_all*
also note *Suc*
finally show *?case* **by** (*simp add*: *add_ac*)
qed

lemma *Digamma_one_half*: *Digamma* (*1/2*) = − *euler_mascheroni* − *of_real* (*2* ∗ *ln 2*)
using *Digamma_half_integer*[*of 0*] **by** *simp*

lemma *Digamma_real_three_halves_pos*: *Digamma* (*3/2* :: *real*) > *0*
proof −
have −*Digamma* (*3/2* :: *real*) = −*Digamma* (*of_nat 1* + *1/2*) **by** *simp*
also have ... = *2* ∗ *ln 2* + *euler_mascheroni* − *2* **by** (*subst Digamma_half_integer*) *simp*
also note *euler_mascheroni_less_13_over_22*
also note *ln2_le_25_over_36*
finally show *?thesis* **by** *simp*
qed

theorem *has_field_derivative_Polygamma* [*derivative_intros*]:
fixes $z :: 'a :: \{real\_normed\_field, euclidean\_space\}$
assumes z: $z \notin \mathbb{Z}_{\leq 0}$
shows (*Polygamma n has_field_derivative Polygamma* (*Suc n*) *z*) (*at z within A*)
proof (*rule has_field_derivative_at_within*, *cases n = 0*)
assume n: $n = 0$
let *?f* = $\lambda k\ z.$ *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)
let *?F* = $\lambda z. \sum k.$ *?f k z* **and** *?f'* = $\lambda k\ z.$ *inverse* ((z + *of_nat k*)^2)
from *no_nonpos_Int_in_ball'*[*OF z*] **obtain** d **where** d: $0 < d$ $\bigwedge t.$ $t \in$ *ball z d* $\Longrightarrow$ $t \notin \mathbb{Z}_{\leq 0}$
by *auto*
from *z* **have** *summable*: *summable* ($\lambda k.$ *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*))
by (*intro summable_Digamma*) *force*
from *z* **have** *conv*: *uniformly_convergent_on* (*ball z d*) ($\lambda k\ z. \sum i<k.$ *inverse* ((z + *of_nat i*)^2))
by (*intro Polygamma_converges*) *auto*
with d **have** *summable* ($\lambda k.$ *inverse* ((z + *of_nat k*)^2)) **unfolding** *summable_iff_convergent*
by (*auto dest*!: *uniformly_convergent_imp_convergent simp*: *summable_iff_convergent*)

have (*?F has_field_derivative* ($\sum k.$ *?f' k z*)) (*at z*)
proof (*rule has_field_derivative_series'*[*of ball z d _ _ z*])
fix k :: *nat* **and** t :: $'a$ **assume** t: $t \in$ *ball z d*
from t *d*(*2*)[*of t*] **show** (($\lambda z.$ *?f k z*) *has_field_derivative ?f' k t*) (*at t within ball z d*)
by (*auto intro*!: *derivative_eq_intros simp*: *power2_eq_square simp del*: *of_nat_Suc*
dest!: *plus_of_nat_eq_0_imp elim*!: *nonpos_Ints_cases*)
qed (*insert d*(*1*) *summable conv*, (*assumption*|*simp*)+)
with *z* **show** (*Polygamma n has_field_derivative Polygamma* (*Suc n*) *z*) (*at z*)
unfolding *Digamma_def* [*abs_def*] *Polygamma_def* [*abs_def*] **using** n
by (*force simp*: *power2_eq_square intro*!: *derivative_eq_intros*)
next
assume n: $n \neq 0$
from *z* **have** z': $z \neq 0$ **by** *auto*
from *no_nonpos_Int_in_ball'*[*OF z*] **obtain** d **where** d: $0 < d$ $\bigwedge t.$ $t \in$ *ball z d* $\Longrightarrow$ $t \notin \mathbb{Z}_{\leq 0}$
by *auto*
define n' **where** $n' =$ *Suc n*
from n **have** n': $n' \geq 2$ **by** (*simp add*: n'*_def*)
have (($\lambda z. \sum k.$ *inverse* ((z + *of_nat k*) ^ n')) *has_field_derivative*
($\sum k.$ − *of_nat n'* ∗ *inverse* ((z + *of_nat k*) ^ (n'+1)))) (*at z*)
proof (*rule has_field_derivative_series'*[*of ball z d _ _ z*])
fix k :: *nat* **and** t :: $'a$ **assume** t: $t \in$ *ball z d*
with d **have** t': $t \notin \mathbb{Z}_{\leq 0}$ $t \neq 0$ **by** *auto*
show (($\lambda a.$ *inverse* ((a + *of_nat k*) ^ n')) *has_field_derivative*

```
            − of_nat n′ * inverse ((t + of_nat k) ^(n′+1))) (at t within ball z
d) using t′
    by (fastforce intro!: derivative_eq_intros simp: divide_simps power_diff dest:
plus_of_nat_eq_0_imp)
  next
    have uniformly_convergent_on (ball z d)
          (λk z. (− of_nat n′ :: 'a) * (∑ i<k. inverse ((z + of_nat i) ^(n′+1))))
      using z′ n by (intro uniformly_convergent_mult Polygamma_converges)
(simp_all add: n′_def)
    thus uniformly_convergent_on (ball z d)
          (λk z. ∑ i<k. − of_nat n′ * inverse ((z + of_nat i :: 'a) ^(n′+1)))
      by (subst (asm) sum_distrib_left) simp
  qed (insert Polygamma_converges′[OF z′ n′] d, simp_all)
  also have (∑ k. − of_nat n′ * inverse ((z + of_nat k) ^(n′ + 1))) =
            (− of_nat n′) * (∑ k. inverse ((z + of_nat k) ^(n′ + 1)))
    using Polygamma_converges′[OF z′, of n′+1] n′ by (subst suminf_mult)
simp_all
  finally have ((λz. ∑ k. inverse ((z + of_nat k) ^n′)) has_field_derivative
                − of_nat n′ * (∑ k. inverse ((z + of_nat k) ^(n′ + 1)))) (at z) .
  from DERIV_cmult[OF this, of (−1)^Suc n * fact n :: 'a]
    show (Polygamma n has_field_derivative Polygamma (Suc n) z) (at z)
      unfolding n′_def Polygamma_def[abs_def] using n by (simp add: alge-
bra_simps)
qed

declare has_field_derivative_Polygamma[THEN DERIV_chain2, derivative_intros]

lemma isCont_Polygamma [continuous_intros]:
  fixes f :: _ ⇒ 'a :: {real_normed_field,euclidean_space}
  shows isCont f z ⟹ f z ∉ ℤ≤0 ⟹ isCont (λx. Polygamma n (f x)) z
  by (rule isCont_o2[OF _ DERIV_isCont[OF has_field_derivative_Polygamma]])

lemma continuous_on_Polygamma:
  A ∩ ℤ≤0 = {} ⟹ continuous_on A (Polygamma n :: _ ⇒ 'a :: {real_normed_field,euclidean_space})
  by (intro continuous_at_imp_continuous_on isCont_Polygamma[OF continu-
ous_ident] ballI) blast

lemma isCont_ln_Gamma_complex [continuous_intros]:
  fixes f :: 'a::t2_space ⇒ complex
  shows isCont f z ⟹ f z ∉ ℝ≤0 ⟹ isCont (λz. ln_Gamma (f z)) z
  by (rule isCont_o2[OF _ DERIV_isCont[OF has_field_derivative_ln_Gamma_complex]])

lemma continuous_on_ln_Gamma_complex [continuous_intros]:
  fixes A :: complex set
  shows A ∩ ℝ≤0 = {} ⟹ continuous_on A ln_Gamma
  by (intro continuous_at_imp_continuous_on ballI isCont_ln_Gamma_complex[OF
continuous_ident])
     fastforce
```

lemma *deriv_Polygamma*:
assumes $z \notin \mathbb{Z}_{\leq 0}$
shows *deriv* (*Polygamma m*) *z* =
Polygamma (*Suc m*) (*z* :: *'a* :: {*real_normed_field*,*euclidean_space*})
by (*intro DERIV_imp_deriv has_field_derivative_Polygamma assms*)
thm *has_field_derivative_Polygamma*

lemma *higher_deriv_Polygamma*:
assumes $z \notin \mathbb{Z}_{\leq 0}$
shows (*deriv* ^^ *n*) (*Polygamma m*) *z* =
Polygamma (*m* + *n*) (*z* :: *'a* :: {*real_normed_field*,*euclidean_space*})
proof −
have *eventually* (λ*u*. (*deriv* ^^ *n*) (*Polygamma m*) *u* = *Polygamma* (*m* + *n*) *u*) (*nhds z*)
proof (*induction n*)
case (*Suc n*)
from *Suc.IH* **have** *eventually* (λ*z*. *eventually* (λ*u*. (*deriv* ^^ *n*) (*Polygamma m*) *u* = *Polygamma* (*m* + *n*) *u*) (*nhds z*)) (*nhds z*)
by (*simp add*: *eventually_eventually*)
hence *eventually* (λ*z*. *deriv* ((*deriv* ^^ *n*) (*Polygamma m*)) *z* =
deriv (*Polygamma* (*m* + *n*)) *z*) (*nhds z*)
by *eventually_elim* (*intro deriv_cong_ev refl*)
moreover have *eventually* (λ*z*. *z* ∈ *UNIV* − $\mathbb{Z}_{\leq 0}$) (*nhds z*) **using** *assms*
by (*intro eventually_nhds_in_open open_Diff open_UNIV*) *auto*
ultimately show *?case* **by** *eventually_elim* (*simp_all add*: *deriv_Polygamma*)
qed *simp_all*
thus *?thesis* **by** (*rule eventually_nhds_x_imp_x*)
qed

lemma *deriv_ln_Gamma_complex*:
assumes $z \notin \mathbb{R}_{\leq 0}$
shows *deriv ln_Gamma z* = *Digamma* (*z* :: *complex*)
by (*intro DERIV_imp_deriv has_field_derivative_ln_Gamma_complex assms*)

lemma *higher_deriv_ln_Gamma_complex*:
assumes (*x*::*complex*) $\notin \mathbb{R}_{\leq 0}$
shows (*deriv* ^^ *j*) *ln_Gamma x* = (*if j* = *0 then ln_Gamma x else Polygamma* (*j* − *1*) *x*)
proof (*cases j*)
case (*Suc j'*)
have (*deriv* ^^ *j'*) (*deriv ln_Gamma*) *x* = (*deriv* ^^ *j'*) *Digamma x*
using *eventually_nhds_in_open*[*of UNIV* − $\mathbb{R}_{\leq 0}$ *x*] *assms*
by (*intro higher_deriv_cong_ev refl*)
(*auto elim*!: *eventually_mono simp*: *open_Diff deriv_ln_Gamma_complex*)
also have ... = *Polygamma j' x* **using** *assms*
by (*subst higher_deriv_Polygamma*)
(*auto elim*!: *nonpos_Ints_cases simp*: *complex_nonpos_Reals_iff*)
finally show *?thesis* **using** *Suc* **by** (*simp del*: *funpow.simps add*: *funpow_Suc_right*)

qed *simp_all*

We define a type class that captures all the fundamental properties of the inverse of the Gamma function and defines the Gamma function upon that. This allows us to instantiate the type class both for the reals and for the complex numbers with a minimal amount of proof duplication.

class *Gamma* = *real_normed_field* + *complete_space* +
fixes *rGamma* :: *'a* ⇒ *'a*
assumes *rGamma_eq_zero_iff_aux*: *rGamma z* = *0* ⟷ (∃ *n*. *z* = − *of_nat n*)
assumes *differentiable_rGamma_aux1*:
(⋀*n*. *z* ≠ − *of_nat n*) ⟹
let d = (*THE d*. (λ*n*. ∑*k*<*n*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) ⟶ *d*) − *scaleR euler_mascheroni 1*
in filterlim (λ*y*. (*rGamma y* − *rGamma z* + *rGamma z* ∗ *d* ∗ (*y* − *z*)) /$_R$ *norm* (*y* − *z*)) (*nhds 0*) (*at z*)
assumes *differentiable_rGamma_aux2*:
let z = − *of_nat n*
in filterlim (λ*y*. (*rGamma y* − *rGamma z* − (−*1*)^*n* ∗ (*prod of_nat* {*1*..*n*}) ∗ (*y* − *z*)) /$_R$ *norm* (*y* − *z*)) (*nhds 0*) (*at z*)
assumes *rGamma_series_aux*: (⋀*n*. *z* ≠ − *of_nat n*) ⟹
let fact' = (λ*n*. *prod of_nat* {*1*..*n*});
exp = (λ*x*. *THE e*. (λ*n*. ∑*k*<*n*. *x*^*k* /$_R$ *fact k*) ⟶ *e*);
pochhammer' = (λ*a n*. (∏*n* = *0*..*n*. *a* + *of_nat n*))
in filterlim (λ*n*. *pochhammer' z n* / (*fact' n* ∗ *exp* (*z* ∗ (*ln* (*of_nat n*) ∗$_R$ *1*)))) (*nhds* (*rGamma z*)) *sequentially*
begin
subclass *banach* **..**
end

definition *Gamma z* = *inverse* (*rGamma z*)

9.10.3 Basic properties

lemma *Gamma_nonpos_Int*: *z* ∈ ℤ$_{≤0}$ ⟹ *Gamma z* = *0*
and *rGamma_nonpos_Int*: *z* ∈ ℤ$_{≤0}$ ⟹ *rGamma z* = *0*
using *rGamma_eq_zero_iff_aux*[*of z*] **unfolding** *Gamma_def* **by** (*auto elim*!: *nonpos_Ints_cases'*)

lemma *Gamma_nonzero*: *z* ∉ ℤ$_{≤0}$ ⟹ *Gamma z* ≠ *0*
and *rGamma_nonzero*: *z* ∉ ℤ$_{≤0}$ ⟹ *rGamma z* ≠ *0*
using *rGamma_eq_zero_iff_aux*[*of z*] **unfolding** *Gamma_def* **by** (*auto elim*!: *nonpos_Ints_cases'*)

lemma *Gamma_eq_zero_iff*: *Gamma z* = *0* ⟷ *z* ∈ ℤ$_{≤0}$
and *rGamma_eq_zero_iff*: *rGamma z* = *0* ⟷ *z* ∈ ℤ$_{≤0}$

using *rGamma_eq_zero_iff_aux*[*of z*] **unfolding** *Gamma_def* **by** (*auto elim*!: *nonpos_Ints_cases'*)

lemma *rGamma_inverse_Gamma*: *rGamma z* = *inverse* (*Gamma z*)
unfolding *Gamma_def* **by** *simp*

lemma *rGamma_series_LIMSEQ* [*tendsto_intros*]:
rGamma_series z ⟶ *rGamma z*
proof (*cases z* ∈ $\mathbb{Z}_{\leq 0}$)
case *False*
hence *z* ≠ − *of_nat n* **for** *n* **by** *auto*
from *rGamma_series_aux*[*OF this*] **show** *?thesis*
by (*simp add*: *rGamma_series_def*[*abs_def*] *fact_prod pochhammer_Suc_prod exp_def of_real_def*[*symmetric*] *suminf_def sums_def*[*abs_def*] *atLeast0AtMost*)
qed (*insert rGamma_eq_zero_iff*[*of z*], *simp_all add*: *rGamma_series_nonpos_Ints_LIMSEQ*)

theorem *Gamma_series_LIMSEQ* [*tendsto_intros*]:
Gamma_series z ⟶ *Gamma z*
proof (*cases z* ∈ $\mathbb{Z}_{\leq 0}$)
case *False*
hence (λ*n*. *inverse* (*rGamma_series z n*)) ⟶ *inverse* (*rGamma z*)
by (*intro tendsto_intros*) (*simp_all add*: *rGamma_eq_zero_iff*)
also have (λ*n*. *inverse* (*rGamma_series z n*)) = *Gamma_series z*
by (*simp add*: *rGamma_series_def Gamma_series_def*[*abs_def*])
finally show *?thesis* **by** (*simp add*: *Gamma_def*)
qed (*insert Gamma_eq_zero_iff*[*of z*], *simp_all add*: *Gamma_series_nonpos_Ints_LIMSEQ*)

lemma *Gamma_altdef*: *Gamma z* = *lim* (*Gamma_series z*)
using *Gamma_series_LIMSEQ*[*of z*] **by** (*simp add*: *limI*)

lemma *rGamma_1* [*simp*]: *rGamma 1* = *1*
proof −
have *A*: *eventually* (λ*n*. *rGamma_series 1 n* = *of_nat* (*Suc n*) / *of_nat n*) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
by (*force elim*!: *eventually_mono simp*: *rGamma_series_def exp_of_real pochhammer_fact field_split_simps pochhammer_rec' dest*!: *pochhammer_eq_0_imp_nonpos_Int*)
have *rGamma_series 1* ⟶ *1* **by** (*subst tendsto_cong*[*OF A*]) (*rule LIMSEQ_Suc_n_over_n*)
moreover have *rGamma_series 1* ⟶ *rGamma 1* **by** (*rule tendsto_intros*)
ultimately show *?thesis* **by** (*intro LIMSEQ_unique*)
qed

lemma *rGamma_plus1*: *z* ∗ *rGamma* (*z* + *1*) = *rGamma z*
proof −
let *?f* = λ*n*. (*z* + *1*) ∗ *inverse* (*of_nat n*) + *1*
have *eventually* (λ*n*. *?f n* ∗ *rGamma_series z n* = *z* ∗ *rGamma_series* (*z* + *1*)

```
n) sequentially
    using eventually_gt_at_top[of 0::nat]
  proof eventually_elim
    fix n :: nat assume n: n > 0
    hence z * rGamma_series (z + 1) n = inverse (of_nat n) *
            pochhammer z (Suc (Suc n)) / (fact n * exp (z * of_real (ln (of_nat
n))))
        by (subst pochhammer_rec) (simp add: rGamma_series_def field_simps
exp_add exp_of_real)
    also from n have ... = ?f n * rGamma_series z n
    by (subst pochhammer_rec') (simp_all add: field_split_simps rGamma_series_def)
    finally show ?f n * rGamma_series z n = z * rGamma_series (z + 1) n ..
  qed
  moreover have (λn. ?f n * rGamma_series z n) ⟶ ((z+1) * 0 + 1) *
rGamma z
    by (intro tendsto_intros lim_inverse_n)
  hence (λn. ?f n * rGamma_series z n) ⟶ rGamma z by simp
  ultimately have (λn. z * rGamma_series (z + 1) n) ⟶ rGamma z
    by (blast intro: Lim_transform_eventually)
  moreover have (λn. z * rGamma_series (z + 1) n) ⟶ z * rGamma (z +
1)
    by (intro tendsto_intros)
  ultimately show z * rGamma (z + 1) = rGamma z using LIMSEQ_unique
by blast
qed


lemma pochhammer_rGamma: rGamma z = pochhammer z n * rGamma (z +
of_nat n)
proof (induction n arbitrary: z)
  case (Suc n z)
  have rGamma z = pochhammer z n * rGamma (z + of_nat n) by (rule Suc.IH)
  also note rGamma_plus1 [symmetric]
  finally show ?case by (simp add: add_ac pochhammer_rec')
qed simp_all

theorem Gamma_plus1: z ∉ ℤ≤0 ⟹ Gamma (z + 1) = z * Gamma z
  using rGamma_plus1[of z] by (simp add: rGamma_inverse_Gamma field_simps
Gamma_eq_zero_iff)

theorem pochhammer_Gamma: z ∉ ℤ≤0 ⟹ pochhammer z n = Gamma (z +
of_nat n) / Gamma z
  using pochhammer_rGamma[of z]
  by (simp add: rGamma_inverse_Gamma Gamma_eq_zero_iff field_simps)

lemma Gamma_0 [simp]: Gamma 0 = 0
  and rGamma_0 [simp]: rGamma 0 = 0
  and Gamma_neg_1 [simp]: Gamma (− 1) = 0
  and rGamma_neg_1 [simp]: rGamma (− 1) = 0
```

and *Gamma_neg_numeral* [*simp*]: *Gamma* (− *numeral n*) = *0*
and *rGamma_neg_numeral* [*simp*]: *rGamma* (− *numeral n*) = *0*
and *Gamma_neg_of_nat* [*simp*]: *Gamma* (− *of_nat m*) = *0*
and *rGamma_neg_of_nat* [*simp*]: *rGamma* (− *of_nat m*) = *0*
by (*simp_all add*: *rGamma_eq_zero_iff Gamma_eq_zero_iff*)

lemma *Gamma_1* [*simp*]: *Gamma 1* = *1* **unfolding** *Gamma_def* **by** *simp*

theorem *Gamma_fact*: *Gamma* (*1* + *of_nat n*) = *fact n*
by (*simp add*: *pochhammer_fact pochhammer_Gamma of_nat_in_nonpos_Ints_iff flip*: *of_nat_Suc*)

lemma *Gamma_numeral*: *Gamma* (*numeral n*) = *fact* (*pred_numeral n*)
by (*subst of_nat_numeral*[*symmetric*], *subst numeral_eq_Suc*,
subst of_nat_Suc, *subst Gamma_fact*) (*rule refl*)

lemma *Gamma_of_int*: *Gamma* (*of_int n*) = (*if n* > *0 then fact* (*nat* (*n* − *1*)) *else 0*)
proof (*cases n* > *0*)
case *True*
hence *Gamma* (*of_int n*) = *Gamma* (*of_nat* (*Suc* (*nat* (*n* − *1*)))) **by** (*subst of_nat_Suc*) *simp_all*
with *True* **show** *?thesis* **by** (*subst* (*asm*) *of_nat_Suc*, *subst* (*asm*) *Gamma_fact*) *simp*
qed (*simp_all add*: *Gamma_eq_zero_iff nonpos_Ints_of_int*)

lemma *rGamma_of_int*: *rGamma* (*of_int n*) = (*if n* > *0 then inverse* (*fact* (*nat* (*n* − *1*))) *else 0*)
by (*simp add*: *Gamma_of_int rGamma_inverse_Gamma*)

lemma *Gamma_seriesI*:
assumes (λ*n*. *g n* / *Gamma_series z n*) ⟶ *1*
shows *g* ⟶ *Gamma z*
proof (*rule Lim_transform_eventually*)
have *1*/*2* > (*0*::*real*) **by** *simp*
from *tendstoD*[*OF assms*, *OF this*]
show *eventually* (λ*n*. *g n* / *Gamma_series z n* ∗ *Gamma_series z n* = *g n*) *sequentially*
by (*force elim*!: *eventually_mono simp*: *dist_real_def*)
from *assms* **have** (λ*n*. *g n* / *Gamma_series z n* ∗ *Gamma_series z n*) ⟶ *1* ∗ *Gamma z*
by (*intro tendsto_intros*)
thus (λ*n*. *g n* / *Gamma_series z n* ∗ *Gamma_series z n*) ⟶ *Gamma z* **by** *simp*
qed

lemma *Gamma_seriesI'*:
assumes *f* ⟶ *rGamma z*
assumes (λ*n*. *g n* ∗ *f n*) ⟶ *1*

assumes $z \notin \mathbb{Z}_{\leq 0}$
shows $g \longrightarrow Gamma\ z$
proof (*rule Lim_transform_eventually*)
have $1/2 > (0::real)$ **by** *simp*
from *tendstoD*[*OF assms*(2), *OF this*] **show** *eventually* ($\lambda n.\ g\ n * f\ n\ /\ f\ n = g\ n$) *sequentially*
by (*force elim*!: *eventually_mono simp*: *dist_real_def*)
from *assms* **have** ($\lambda n.\ g\ n * f\ n\ /\ f\ n$) $\longrightarrow$ *1 / rGamma z*
by (*intro tendsto_divide assms*) (*simp_all add*: *rGamma_eq_zero_iff*)
thus ($\lambda n.\ g\ n * f\ n\ /\ f\ n$) $\longrightarrow$ *Gamma z* **by** (*simp add*: *Gamma_def divide_inverse*)
qed

lemma *Gamma_series'_LIMSEQ*: *Gamma_series' z* $\longrightarrow$ *Gamma z*
by (*cases* $z \in \mathbb{Z}_{\leq 0}$) (*simp_all add*: *Gamma_nonpos_Int Gamma_seriesI*[*OF Gamma_series_Gamma_series'*]
Gamma_series'_nonpos_Ints_LIMSEQ[*of z*])

9.10.4 Differentiability

lemma *has_field_derivative_rGamma_no_nonpos_int*:
assumes $z \notin \mathbb{Z}_{\leq 0}$
shows (*rGamma has_field_derivative −rGamma z * Digamma z*) (*at z within A*)
proof (*rule has_field_derivative_at_within*)
from *assms* **have** $z \neq -\ of\_nat\ n$ **for** *n* **by** *auto*
from *differentiable_rGamma_aux1*[*OF this*]
show (*rGamma has_field_derivative −rGamma z * Digamma z*) (*at z*)
unfolding *Digamma_def suminf_def sums_def*[*abs_def*]
has_field_derivative_def has_derivative_def netlimit_at
by (*simp add*: *Let_def bounded_linear_mult_right mult_ac of_real_def* [*symmetric*])
qed

lemma *has_field_derivative_rGamma_nonpos_int*:
(*rGamma has_field_derivative* $(-1)\hat{}n * fact\ n$) (*at* $(-\ of\_nat\ n)$ *within A*)
apply (*rule has_field_derivative_at_within*)
using *differentiable_rGamma_aux2*[*of n*]
unfolding *Let_def has_field_derivative_def has_derivative_def netlimit_at*
by (*simp only*: *bounded_linear_mult_right mult_ac of_real_def* [*symmetric*] *fact_prod*) *simp*

lemma *has_field_derivative_rGamma* [*derivative_intros*]:
(*rGamma has_field_derivative* (*if* $z \in \mathbb{Z}_{\leq 0}$ *then* $(-1)\hat{}(nat\ \lfloor norm\ z \rfloor) * fact\ (nat\ \lfloor norm\ z \rfloor)$
else $-rGamma\ z * Digamma\ z$)) (*at z within A*)
using *has_field_derivative_rGamma_no_nonpos_int*[*of z A*]
has_field_derivative_rGamma_nonpos_int[*of nat* $\lfloor norm\ z \rfloor$ *A*]
by (*auto elim*!: *nonpos_Ints_cases'*)

declare *has_field_derivative_rGamma_no_nonpos_int* [*THEN DERIV_chain2*, *derivative_intros*]
declare *has_field_derivative_rGamma* [*THEN DERIV_chain2*, *derivative_intros*]
declare *has_field_derivative_rGamma_nonpos_int* [*derivative_intros*]
declare *has_field_derivative_rGamma_no_nonpos_int* [*derivative_intros*]
declare *has_field_derivative_rGamma* [*derivative_intros*]

theorem *has_field_derivative_Gamma* [*derivative_intros*]:
  $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ (*Gamma has_field_derivative Gamma z* $*$ *Digamma z*) (*at z within A*)
  unfolding *Gamma_def* [*abs_def*]
  by (*fastforce intro*!: *derivative_eq_intros simp*: *rGamma_eq_zero_iff*)

declare *has_field_derivative_Gamma*[*THEN DERIV_chain2*, *derivative_intros*]

hide_fact *rGamma_eq_zero_iff_aux differentiable_rGamma_aux1 differentiable_rGamma_aux2 differentiable_rGamma_aux2 rGamma_series_aux Gamma_class.rGamma_eq_zero_iff_aux*

lemma *continuous_on_rGamma* [*continuous_intros*]: *continuous_on A rGamma*
  by (*rule DERIV_continuous_on has_field_derivative_rGamma*)+

lemma *continuous_on_Gamma* [*continuous_intros*]: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *continuous_on A Gamma*
  by (*rule DERIV_continuous_on has_field_derivative_Gamma*)+ *blast*

lemma *isCont_rGamma* [*continuous_intros*]:
  isCont f z $\Longrightarrow$ *isCont* (λx. *rGamma* (*f x*)) *z*
  by (*rule isCont_o2*[*OF _ DERIV_isCont*[*OF has_field_derivative_rGamma*]])

lemma *isCont_Gamma* [*continuous_intros*]:
  isCont f z $\Longrightarrow$ *f z* $\notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *isCont* (λx. *Gamma* (*f x*)) *z*
  by (*rule isCont_o2*[*OF _ DERIV_isCont*[*OF has_field_derivative_Gamma*]])

9.10.5 The complex Gamma function

instantiation *complex* :: *Gamma*
begin

definition *rGamma_complex* :: *complex* $\Rightarrow$ *complex* **where**
  rGamma_complex z = *lim* (*rGamma_series z*)

lemma *rGamma_series_complex_converges*:
    convergent (*rGamma_series* (*z* :: *complex*)) (**is** *?thesis1*)
  and *rGamma_complex_altdef*:
    rGamma z = (*if z* $\in \mathbb{Z}_{\leq 0}$ *then 0 else exp* ($-$*ln_Gamma z*)) (**is** *?thesis2*)
proof $-$
  have *?thesis1* $\wedge$ *?thesis2*
  proof (*cases z* $\in \mathbb{Z}_{\leq 0}$)

```
    case False
    have rGamma_series z ⟶ exp (− ln_Gamma z)
    proof (rule Lim_transform_eventually)
      from ln_Gamma_series_complex_converges'[OF False]
    obtain d where 0 < d uniformly_convergent_on (ball z d) (λn z. ln_Gamma_series
z n)
        by auto
      from this(1) uniformly_convergent_imp_convergent[OF this(2), of z]
       have ln_Gamma_series z ⟶ lim (ln_Gamma_series z) by (simp add:
convergent_LIMSEQ_iff)
      thus (λn. exp (−ln_Gamma_series z n)) ⟶ exp (− ln_Gamma z)
         unfolding convergent_def ln_Gamma_def by (intro tendsto_exp tend-
sto_minus)
     from eventually_gt_at_top[of 0::nat] exp_ln_Gamma_series_complex False
        show eventually (λn. exp (−ln_Gamma_series z n) = rGamma_series z
n) sequentially
          by (force elim!: eventually_mono simp: exp_minus Gamma_series_def
rGamma_series_def)
    qed
    with False show ?thesis
      by (auto simp: convergent_def rGamma_complex_def intro!: limI)
  next
    case True
    then obtain k where z = − of_nat k by (erule nonpos_Ints_cases')
    also have rGamma_series ... ⟶ 0
     by (subst tendsto_cong[OF rGamma_series_minus_of_nat]) (simp_all add:
convergent_const)
    finally show ?thesis using True
      by (auto simp: rGamma_complex_def convergent_def intro!: limI)
  qed
  thus ?thesis1 ?thesis2 by blast+
qed

context
begin


private lemma rGamma_complex_plus1: z * rGamma (z + 1) = rGamma (z ::
complex)
proof −
  let ?f = λn. (z + 1) * inverse (of_nat n) + 1
  have eventually (λn. ?f n * rGamma_series z n = z * rGamma_series (z + 1)
n) sequentially
    using eventually_gt_at_top[of 0::nat]
  proof eventually_elim
    fix n :: nat assume n: n > 0
    hence z * rGamma_series (z + 1) n = inverse (of_nat n) *
            pochhammer z (Suc (Suc n)) / (fact n * exp (z * of_real (ln (of_nat
n))))
```

```
      by (subst pochhammer_rec) (simp add: rGamma_series_def field_simps
exp_add exp_of_real)
    also from n have ... = ?f n * rGamma_series z n
    by (subst pochhammer_rec') (simp_all add: field_split_simps rGamma_series_def
add_ac)
    finally show ?f n * rGamma_series z n = z * rGamma_series (z + 1) n ..
  qed
  moreover have (λn. ?f n * rGamma_series z n) ⟶ ((z+1) * 0 + 1) *
rGamma z
    using rGamma_series_complex_converges
    by (intro tendsto_intros lim_inverse_n)
       (simp_all add: convergent_LIMSEQ_iff rGamma_complex_def)
  hence (λn. ?f n * rGamma_series z n) ⟶ rGamma z by simp
  ultimately have (λn. z * rGamma_series (z + 1) n) ⟶ rGamma z
    by (blast intro: Lim_transform_eventually)
  moreover have (λn. z * rGamma_series (z + 1) n) ⟶ z * rGamma (z +
1)
    using rGamma_series_complex_converges
   by (auto intro!: tendsto_mult simp: rGamma_complex_def convergent_LIMSEQ_iff)
  ultimately show z * rGamma (z + 1) = rGamma z using LIMSEQ_unique
by blast
qed

private lemma has_field_derivative_rGamma_complex_no_nonpos_Int:
  assumes (z :: complex) ∉ ℤ≤0
  shows   (rGamma has_field_derivative - rGamma z * Digamma z) (at z)
proof -
  have diff: (rGamma has_field_derivative - rGamma z * Digamma z) (at z) if
Re z > 0 for z
  proof (subst DERIV_cong_ev[OF refl _ refl])
    from that have eventually (λt. t ∈ ball z (Re z/2)) (nhds z)
      by (intro eventually_nhds_in_nhd) simp_all
    thus eventually (λt. rGamma t = exp (- ln_Gamma t)) (nhds z)
      using no_nonpos_Int_in_ball_complex[OF that]
      by (auto elim!: eventually_mono simp: rGamma_complex_altdef)
  next
    have z ∉ ℝ≤0 using that by (simp add: complex_nonpos_Reals_iff)
    with that show ((λt. exp (- ln_Gamma t)) has_field_derivative (-rGamma
z * Digamma z)) (at z)
   by (force elim!: nonpos_Ints_cases intro!: derivative_eq_intros simp: rGamma_complex_altdef)
  qed

  from assms show (rGamma has_field_derivative - rGamma z * Digamma z)
(at z)
  proof (induction nat ⌊1 - Re z⌋ arbitrary: z)
    case (Suc n z)
    from Suc.prems have z: z ≠ 0 by auto
    from Suc.hyps have n = nat ⌊- Re z⌋ by linarith
    hence A: n = nat ⌊1 - Re (z + 1)⌋ by simp
```

```
  from Suc.prems have B: z + 1 ∉ ℤ≤0 by (force dest: plus_one_in_nonpos_Ints_imp)

  have ((λz. z * (rGamma ∘ (λz. z + 1)) z) has_field_derivative
    −rGamma (z + 1) * (Digamma (z + 1) * z − 1)) (at z)
      by (rule derivative_eq_intros DERIV_chain Suc refl A B)+ (simp add:
algebra_simps)
  also have (λz. z * (rGamma ∘ (λz. z + 1 :: complex)) z) = rGamma
    by (simp add: rGamma_complex_plus1)
  also from z have Digamma (z + 1) * z − 1 = z * Digamma z
    by (subst Digamma_plus1) (simp_all add: field_simps)
  also have −rGamma (z + 1) * (z * Digamma z) = −rGamma z * Digamma z
    by (simp add: rGamma_complex_plus1[of z, symmetric])
  finally show ?case .
 qed (intro diff, simp)
qed

private lemma rGamma_complex_1: rGamma (1 :: complex) = 1
proof −
  have A: eventually (λn. rGamma_series 1 n = of_nat (Suc n) / of_nat n)
sequentially
    using eventually_gt_at_top[of 0::nat]
  by (force elim!: eventually_mono simp: rGamma_series_def exp_of_real pochham-
mer_fact
          field_split_simps pochhammer_rec' dest!: pochhammer_eq_0_imp_nonpos_Int)
  have rGamma_series 1 ⟶ 1 by (subst tendsto_cong[OF A]) (rule LIM-
SEQ_Suc_n_over_n)
  thus rGamma 1 = (1 :: complex) unfolding rGamma_complex_def by (rule
limI)
qed

private lemma has_field_derivative_rGamma_complex_nonpos_Int:
  (rGamma has_field_derivative (−1)^n * fact n) (at (− of_nat n :: complex))
proof (induction n)
  case 0
  have A: (0::complex) + 1 ∉ ℤ≤0 by simp
  have ((λz. z * (rGamma ∘ (λz. z + 1 :: complex)) z) has_field_derivative 1)
(at 0)
    by (rule derivative_eq_intros DERIV_chain refl
          has_field_derivative_rGamma_complex_no_nonpos_Int A)+ (simp
add: rGamma_complex_1)
    thus ?case by (simp add: rGamma_complex_plus1)
next
  case (Suc n)
  hence A: (rGamma has_field_derivative (−1)^n * fact n)
          (at (− of_nat (Suc n) + 1 :: complex)) by simp
   have ((λz. z * (rGamma ∘ (λz. z + 1 :: complex)) z) has_field_derivative
         (− 1) ^ Suc n * fact (Suc n)) (at (− of_nat (Suc n)))
     by (rule derivative_eq_intros refl A DERIV_chain)+
       (simp add: algebra_simps rGamma_complex_altdef)
```

thus *?case* **by** (*simp add*: *rGamma_complex_plus1*)
qed

instance proof
fix *z* :: *complex* **show** (*rGamma z = 0*) ⟷ (∃ *n*. *z* = − *of_nat n*)
by (*auto simp*: *rGamma_complex_altdef elim*!: *nonpos_Ints_cases′*)
next
fix *z* :: *complex* **assume** ⋀*n*. *z* ≠ − *of_nat n*
hence *z* ∉ ℤ≤0 **by** (*auto elim*!: *nonpos_Ints_cases′*)
from *has_field_derivative_rGamma_complex_no_nonpos_Int*[*OF this*]
show *let d* = (*THE d*. (λ*n*. ∑*k*<*n*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*))
⟶ *d*) − *euler_mascheroni* ∗*R* *1 in* (λ*y*. (*rGamma y* − *rGamma z* +
rGamma z ∗ *d* ∗ (*y* − *z*)) /*R* *cmod* (*y* − *z*)) −*z*→ *0*
by (*simp add*: *has_field_derivative_def has_derivative_def Digamma_def sums_def* [*abs_def*]
of_real_def[*symmetric*] *suminf_def*)
next
fix *n* :: *nat*
from *has_field_derivative_rGamma_complex_nonpos_Int*[*of n*]
show *let z* = − *of_nat n in* (λ*y*. (*rGamma y* − *rGamma z* − (− *1*) ^ *n* ∗ *prod of_nat* {*1*..*n*} ∗
(*y* − *z*)) /*R* *cmod* (*y* − *z*)) −*z*→ *0*
by (*simp add*: *has_field_derivative_def has_derivative_def fact_prod Let_def*)
next
fix *z* :: *complex*
from *rGamma_series_complex_converges*[*of z*] **have** *rGamma_series z* ⟶ *rGamma z*
by (*simp add*: *convergent_LIMSEQ_iff rGamma_complex_def*)
thus *let fact′* = λ*n*. *prod of_nat* {*1*..*n*};
exp = λ*x*. *THE e*. (λ*n*. ∑*k*<*n*. *x* ^ *k* /*R* *fact k*) ⟶ *e*;
pochhammer′ = λ*a n*. ∏*n* = *0*..*n*. *a* + *of_nat n*
in (λ*n*. *pochhammer′ z n* / (*fact′ n* ∗ *exp* (*z* ∗ *ln* (*real_of_nat n*) ∗*R* *1*)))
⟶ *rGamma z*
by (*simp add*: *fact_prod pochhammer_Suc_prod rGamma_series_def* [*abs_def*] *exp_def*
of_real_def [*symmetric*] *suminf_def sums_def* [*abs_def*] *atLeast0AtMost*)
qed

end
end

lemma *Gamma_complex_altdef*:
Gamma z = (*if z* ∈ ℤ≤0 *then 0 else exp* (*ln_Gamma* (*z* :: *complex*)))
unfolding *Gamma_def rGamma_complex_altdef* **by** (*simp add*: *exp_minus*)

lemma *cnj_rGamma*: *cnj* (*rGamma z*) = *rGamma* (*cnj z*)
proof −
have *rGamma_series* (*cnj z*) = (λ*n*. *cnj* (*rGamma_series z n*))
by (*intro ext*) (*simp_all add*: *rGamma_series_def exp_cnj*)
also have ... ⟶ *cnj* (*rGamma z*) **by** (*intro tendsto_cnj tendsto_intros*)
finally show *?thesis* **unfolding** *rGamma_complex_def* **by** (*intro sym*[*OF limI*])
qed

lemma *cnj_Gamma*: *cnj* (*Gamma z*) = *Gamma* (*cnj z*)
unfolding *Gamma_def* **by** (*simp add*: *cnj_rGamma*)

lemma *Gamma_complex_real*:
z ∈ ℝ ⟹ *Gamma z* ∈ (ℝ :: *complex set*) **and** *rGamma_complex_real*: *z* ∈ ℝ
⟹ *rGamma z* ∈ ℝ
by (*simp_all add*: *Reals_cnj_iff cnj_Gamma cnj_rGamma*)

lemma *field_differentiable_rGamma*: *rGamma field_differentiable* (*at z within A*)
using *has_field_derivative_rGamma*[*of z*] **unfolding** *field_differentiable_def*
by *blast*

lemma *holomorphic_rGamma* [*holomorphic_intros*]: *rGamma holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_rGamma*)

lemma *holomorphic_rGamma′* [*holomorphic_intros*]:
assumes *f holomorphic_on A*
shows (λ*x*. *rGamma* (*f x*)) *holomorphic_on A*
proof −
have *rGamma* ∘ *f holomorphic_on A* **using** *assms*
by (*intro holomorphic_on_compose assms holomorphic_rGamma*)
thus *?thesis* **by** (*simp only*: *o_def*)
qed

lemma *analytic_rGamma*: *rGamma analytic_on A*
unfolding *analytic_on_def* **by** (*auto intro*!: *exI*[*of _ 1*] *holomorphic_rGamma*)

lemma *field_differentiable_Gamma*: *z* ∉ ℤ≤0 ⟹ *Gamma field_differentiable* (*at z within A*)
using *has_field_derivative_Gamma*[*of z*] **unfolding** *field_differentiable_def* **by** *auto*

lemma *holomorphic_Gamma* [*holomorphic_intros*]: *A* ∩ ℤ≤0 = {} ⟹ *Gamma holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_Gamma*)

lemma *holomorphic_Gamma′* [*holomorphic_intros*]:
assumes *f holomorphic_on A* **and** ⋀*x*. *x* ∈ *A* ⟹ *f x* ∉ ℤ≤0
shows (λ*x*. *Gamma* (*f x*)) *holomorphic_on A*
proof −

have *Gamma* ∘ *f holomorphic_on A* **using** *assms*
by (*intro holomorphic_on_compose assms holomorphic_Gamma*) *auto*
thus *?thesis* **by** (*simp only*: *o_def*)
qed

lemma *analytic_Gamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Gamma analytic_on A*
by (*rule analytic_on_subset*[*of* _ *UNIV* − $\mathbb{Z}_{\leq 0}$], *subst analytic_on_open*)
(*auto intro*!: *holomorphic_Gamma*)

lemma *field_differentiable_ln_Gamma_complex*:
$z \notin \mathbb{R}_{\leq 0} \Longrightarrow$ *ln_Gamma field_differentiable* (*at* (*z*::*complex*) *within A*)
by (*rule field_differentiable_within_subset*[*of* _ _ *UNIV*])
(*force simp*: *field_differentiable_def intro*!: *derivative_intros*)+

lemma *holomorphic_ln_Gamma* [*holomorphic_intros*]: $A \cap \mathbb{R}_{\leq 0} = \{\} \Longrightarrow$ *ln_Gamma holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_ln_Gamma_complex*)

lemma *analytic_ln_Gamma*: $A \cap \mathbb{R}_{\leq 0} = \{\} \Longrightarrow$ *ln_Gamma analytic_on A*
by (*rule analytic_on_subset*[*of* _ *UNIV* − $\mathbb{R}_{\leq 0}$], *subst analytic_on_open*)
(*auto intro*!: *holomorphic_ln_Gamma*)

lemma *has_field_derivative_rGamma_complex'* [*derivative_intros*]:
(*rGamma has_field_derivative* (*if* $z \in \mathbb{Z}_{\leq 0}$ *then* $(-1)\hat{}(nat\ \lfloor -Re\ z \rfloor) * fact\ (nat\ \lfloor -Re\ z \rfloor)$ *else*
$-rGamma\ z * Digamma\ z$)) (*at z within A*)
using *has_field_derivative_rGamma*[*of z*] **by** (*auto elim*!: *nonpos_Ints_cases'*)

declare *has_field_derivative_rGamma_complex'*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *field_differentiable_Polygamma*:
fixes *z* :: *complex*
shows
$z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ *Polygamma n field_differentiable* (*at z within A*)
using *has_field_derivative_Polygamma*[*of z n*] **unfolding** *field_differentiable_def*
by *auto*

lemma *holomorphic_on_Polygamma* [*holomorphic_intros*]: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Polygamma n holomorphic_on A*
unfolding *holomorphic_on_def* **by** (*auto intro*!: *field_differentiable_Polygamma*)

lemma *analytic_on_Polygamma*: $A \cap \mathbb{Z}_{\leq 0} = \{\} \Longrightarrow$ *Polygamma n analytic_on A*
by (*rule analytic_on_subset*[*of* _ *UNIV* − $\mathbb{Z}_{\leq 0}$], *subst analytic_on_open*)
(*auto intro*!: *holomorphic_on_Polygamma*)

lemma *analytic_on_rGamma* [*analytic_intros*]: *f analytic_on A* $\Longrightarrow$ (λw. *rGamma* (*f w*)) *analytic_on A*
using *analytic_on_compose*[*OF _ analytic_rGamma, of f A*] **by** (*simp add*: *o_def*)

lemma *analytic_on_ln_Gamma* [*analytic_intros*]:
f analytic_on A $\Longrightarrow$ ($\bigwedge z.\ z \in A \Longrightarrow f\ z \notin \mathbb{R}_{\leq 0}$) $\Longrightarrow$ (λw. *ln_Gamma* (*f w*)) *analytic_on A*
by (*rule analytic_on_compose*[*OF _ analytic_ln_Gamma, unfolded o_def*]) (*auto simp*: *o_def*)

lemma *Polygamma_plus_of_nat*:
assumes $\forall k{<}m.\ z \neq -of\_nat\ k$
shows *Polygamma n* (*z* + *of_nat m*) =
Polygamma n z + (−1) ^ *n* * *fact n* * ($\sum k{<}m.$ *1* / (*z* + *of_nat k*) ^ *Suc n*)
using *assms*
proof (*induction m*)
case (*Suc m*)
have *Polygamma n* (*z* + *of_nat* (*Suc m*)) = *Polygamma n* (*z* + *of_nat m* + *1*)
by (*simp add*: *add_ac*)
also have ... = *Polygamma n* (*z* + *of_nat m*) + (−1) ^ *n* * *fact n* * (*1* / ((*z* + *of_nat m*) ^ *Suc n*))
using *Suc.prems* **by** (*subst Polygamma_plus1*) (*auto simp*: *add_eq_0_iff2*)
also have *Polygamma n* (*z* + *of_nat m*) =
Polygamma n z + (−1) ^ *n* * ($\sum k{<}m.$ *1* / (*z* + *of_nat k*) ^ *Suc n*) * *fact n*
using *Suc.prems* **by** (*subst Suc.IH*) *auto*
finally show *?case*
by (*simp add*: *algebra_simps*)
qed *auto*

lemma *tendsto_Gamma* [*tendsto_intros*]:
assumes (*f* ⟶ *c*) *F c* $\notin \mathbb{Z}_{\leq 0}$
shows ((λz. *Gamma* (*f z*)) ⟶ *Gamma c*) *F*
by (*intro isCont_tendsto_compose*[*OF _ assms*(*1*)] *continuous_intros assms*)

lemma *tendsto_Polygamma* [*tendsto_intros*]:
fixes *f* :: _ ⇒ *'a* :: {*real_normed_field*,*euclidean_space*}
assumes (*f* ⟶ *c*) *F c* $\notin \mathbb{Z}_{\leq 0}$
shows ((λz. *Polygamma n* (*f z*)) ⟶ *Polygamma n c*) *F*
by (*intro isCont_tendsto_compose*[*OF _ assms*(*1*)] *continuous_intros assms*)

lemma *analytic_on_Gamma'* [*analytic_intros*]:
assumes *f analytic_on A* $\forall x{\in}A.\ f\ x \notin \mathbb{Z}_{\leq 0}$
shows (λz. *Gamma* (*f z*)) *analytic_on A*
using *analytic_on_compose_gen*[*OF assms*(*1*) *analytic_Gamma*[*of f ' A*]] *assms*(*2*)

by (*auto simp*: *o_def*)

lemma *analytic_on_Polygamma'* [*analytic_intros*]:
assumes *f analytic_on A* $\forall x{\in}A.\ f\ x \notin \mathbb{Z}_{\leq 0}$
shows $(\lambda z.\ Polygamma\ n\ (f\ z))$ *analytic_on A*
using *analytic_on_compose_gen*[*OF assms*(*1*) *analytic_on_Polygamma*[*of f ' A n*]] *assms*(*2*)
by (*auto simp*: *o_def*)

9.10.6 The real Gamma function

lemma *rGamma_series_real*:
eventually $(\lambda n.\ rGamma\_series\ x\ n = Re\ (rGamma\_series\ (of\_real\ x)\ n))$ *sequentially*
using *eventually_gt_at_top*[*of 0* :: *nat*]
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: $n > 0$
have $Re\ (rGamma\_series\ (of\_real\ x)\ n) =$
$Re\ (of\_real\ (pochhammer\ x\ (Suc\ n))\ /\ (fact\ n * exp\ (of\_real\ (x * ln\ (real\_of\_nat\ n)))))$
using *n* **by** (*simp add*: *rGamma_series_def powr_def pochhammer_of_real*)
also from *n* **have** $\ldots = Re\ (of\_real\ ((pochhammer\ x\ (Suc\ n))\ /$
$(fact\ n * (exp\ (x * ln\ (real\_of\_nat\ n))))))$
by (*subst exp_of_real*) *simp*
also from *n* **have** $\ldots = rGamma\_series\ x\ n$
by (*subst Re_complex_of_real*) (*simp add*: *rGamma_series_def powr_def*)
finally show $rGamma\_series\ x\ n = Re\ (rGamma\_series\ (of\_real\ x)\ n)$ **..**
qed

instantiation *real* :: *Gamma*
begin

definition $rGamma\_real\ x = Re\ (rGamma\ (of\_real\ x :: complex))$

instance proof
fix *x* :: *real*
have $rGamma\ x = Re\ (rGamma\ (of\_real\ x))$ **by** (*simp add*: *rGamma_real_def*)
also have $of\_real\ \ldots = rGamma\ (of\_real\ x :: complex)$
by (*intro of_real_Re rGamma_complex_real*) *simp_all*
also have $\ldots = 0 \longleftrightarrow x \in \mathbb{Z}_{\leq 0}$ **by** (*simp add*: *rGamma_eq_zero_iff of_real_in_nonpos_Ints_iff*)
also have $\ldots \longleftrightarrow (\exists n.\ x = -\ of\_nat\ n)$ **by** (*auto elim*!: *nonpos_Ints_cases'*)
finally show $(rGamma\ x) = 0 \longleftrightarrow (\exists n.\ x = -\ real\_of\_nat\ n)$ **by** *simp*
next
fix *x* :: *real* **assume** $\bigwedge n.\ x \neq -\ of\_nat\ n$
hence *x*: $complex\_of\_real\ x \notin \mathbb{Z}_{\leq 0}$
by (*subst of_real_in_nonpos_Ints_iff*) (*auto elim*!: *nonpos_Ints_cases'*)
then have $x \neq 0$ **by** *auto*
with *x* **have** (*rGamma has_field_derivative* $-\ rGamma\ x * Digamma\ x$) (*at x*)
by (*fastforce intro*!: *derivative_eq_intros has_vector_derivative_real_field*

simp: *Polygamma_of_real rGamma_real_def* [*abs_def*])
thus *let d* = (*THE d*. (λ*n*. ∑*k*<*n*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*x* + *of_nat k*)) ⟶ *d*) − *euler_mascheroni* ∗$_R$ *1 in* (λ*y*. (*rGamma y* − *rGamma x* + *rGamma x* ∗ *d* ∗ (*y* − *x*)) /$_R$ *norm* (*y* − *x*)) −*x*→ *0*
by (*simp add*: *has_field_derivative_def has_derivative_def Digamma_def sums_def* [*abs_def*] *of_real_def*[*symmetric*] *suminf_def*)
next
fix *n* :: *nat*
have (*rGamma has_field_derivative* (−*1*)^*n* ∗ *fact n*) (*at* (− *of_nat n* :: *real*))
by (*fastforce intro*!: *derivative_eq_intros has_vector_derivative_real_field simp*: *Polygamma_of_real rGamma_real_def* [*abs_def*])
thus *let x* = − *of_nat n in* (λ*y*. (*rGamma y* − *rGamma x* − (− *1*) ^ *n* ∗ *prod of_nat* {*1*..*n*} ∗ (*y* − *x*)) /$_R$ *norm* (*y* − *x*)) −*x*::*real*→ *0*
by (*simp add*: *has_field_derivative_def has_derivative_def fact_prod Let_def*)
next
fix *x* :: *real*
have *rGamma_series x* ⟶ *rGamma x*
proof (*rule Lim_transform_eventually*)
show (λ*n*. *Re* (*rGamma_series* (*of_real x*) *n*)) ⟶ *rGamma x* **unfolding** *rGamma_real_def*
by (*intro tendsto_intros*)
qed (*insert rGamma_series_real*, *simp add*: *eq_commute*)
thus *let fact*′ = λ*n*. *prod of_nat* {*1*..*n*}; *exp* = λ*x*. *THE e*. (λ*n*. ∑*k*<*n*. *x* ^ *k* /$_R$ *fact k*) ⟶ *e*; *pochhammer*′ = λ*a n*. ∏*n* = *0*..*n*. *a* + *of_nat n* *in* (λ*n*. *pochhammer*′ *x n* / (*fact*′ *n* ∗ *exp* (*x* ∗ *ln* (*real_of_nat n*) ∗$_R$ *1*))) ⟶ *rGamma x*
by (*simp add*: *fact_prod pochhammer_Suc_prod rGamma_series_def* [*abs_def*] *exp_def of_real_def* [*symmetric*] *suminf_def sums_def* [*abs_def*] *atLeast0AtMost*)
qed

end

lemma *rGamma_complex_of_real*: *rGamma* (*complex_of_real x*) = *complex_of_real* (*rGamma x*)
unfolding *rGamma_real_def* **using** *rGamma_complex_real* **by** *simp*

lemma *Gamma_complex_of_real*: *Gamma* (*complex_of_real x*) = *complex_of_real* (*Gamma x*)
unfolding *Gamma_def* **by** (*simp add*: *rGamma_complex_of_real*)

lemma *rGamma_real_altdef*: *rGamma x* = *lim* (*rGamma_series* (*x* :: *real*))

by (*rule sym*, *rule limI*, *rule tendsto_intros*)

lemma *Gamma_real_altdef1*: *Gamma x* = *lim* (*Gamma_series* (*x* :: *real*))
by (*rule sym*, *rule limI*, *rule tendsto_intros*)

lemma *Gamma_real_altdef2*: *Gamma x* = *Re* (*Gamma* (*of_real x*))
using *rGamma_complex_real*[*OF Reals_of_real*[*of x*]]
by (*elim Reals_cases*)
(*simp only*: *Gamma_def rGamma_real_def of_real_inverse*[*symmetric*] *Re_complex_of_real*)

lemma *ln_Gamma_series_complex_of_real*:
x > *0* ⟹ *n* > *0* ⟹ *ln_Gamma_series* (*complex_of_real x*) *n* = *of_real* (*ln_Gamma_series x n*)
proof −
assume *xn*: *x* > *0 n* > *0*
have *Ln* (*complex_of_real x* / *of_nat k* + *1*) = *of_real* (*ln* (*x* / *of_nat k* + *1*)) **if** *k* ≥ *1* **for** *k*
using *that xn* **by** (*subst Ln_of_real* [*symmetric*]) (*auto intro*!: *add_nonneg_pos simp*: *field_simps*)
with *xn* **show** *?thesis* **by** (*simp add*: *ln_Gamma_series_def Ln_of_real*)
qed

lemma *ln_Gamma_real_converges*:
assumes (*x*::*real*) > *0*
shows *convergent* (*ln_Gamma_series x*)
proof −
have (λ*n*. *ln_Gamma_series* (*complex_of_real x*) *n*) ⟶ *ln_Gamma* (*of_real x*) **using** *assms*
by (*intro ln_Gamma_complex_LIMSEQ*) (*auto simp*: *of_real_in_nonpos_Ints_iff*)
moreover from *eventually_gt_at_top*[*of 0*::*nat*]
have *eventually* (λ*n*. *complex_of_real* (*ln_Gamma_series x n*) =
ln_Gamma_series (*complex_of_real x*) *n*) *sequentially*
by *eventually_elim* (*simp add*: *ln_Gamma_series_complex_of_real assms*)
ultimately have (λ*n*. *complex_of_real* (*ln_Gamma_series x n*)) ⟶ *ln_Gamma* (*of_real x*)
by (*subst tendsto_cong*) *assumption*+
from *tendsto_Re*[*OF this*] **show** *?thesis* **by** (*auto simp*: *convergent_def*)
qed

lemma *ln_Gamma_real_LIMSEQ*: (*x*::*real*) > *0* ⟹ *ln_Gamma_series x* ⟶ *ln_Gamma x*
using *ln_Gamma_real_converges*[*of x*] **unfolding** *ln_Gamma_def* **by** (*simp add*: *convergent_LIMSEQ_iff*)

lemma *ln_Gamma_complex_of_real*: *x* > *0* ⟹ *ln_Gamma* (*complex_of_real x*) = *of_real* (*ln_Gamma x*)
proof (*unfold ln_Gamma_def*, *rule limI*, *rule Lim_transform_eventually*)
assume *x*: *x* > *0*
show *eventually* (λ*n*. *of_real* (*ln_Gamma_series x n*) =

ln_Gamma_series (complex_of_real x) n) sequentially
using *eventually_gt_at_top[of 0::nat]*
by *eventually_elim (simp add: ln_Gamma_series_complex_of_real x)*
qed *(intro tendsto_of_real, insert ln_Gamma_real_LIMSEQ[of x], simp add: ln_Gamma_def)*

lemma *Gamma_real_pos_exp: x > (0 :: real) ⟹ Gamma x = exp (ln_Gamma x)*
by *(auto simp: Gamma_real_altdef2 Gamma_complex_altdef of_real_in_nonpos_Ints_iff ln_Gamma_complex_of_real exp_of_real)*

lemma *ln_Gamma_real_pos: x > 0 ⟹ ln_Gamma x = ln (Gamma x :: real)*
unfolding *Gamma_real_pos_exp* **by** *simp*

lemma *ln_Gamma_complex_conv_fact: n > 0 ⟹ ln_Gamma (of_nat n :: complex) = ln (fact (n − 1))*
using *ln_Gamma_complex_of_real[of real n] Gamma_fact[of n − 1, where 'a = real]*
by *(simp add: ln_Gamma_real_pos of_nat_diff Ln_of_real [symmetric])*

lemma *ln_Gamma_real_conv_fact: n > 0 ⟹ ln_Gamma (real n) = ln (fact (n − 1))*
using *Gamma_fact[of n − 1, where 'a = real]*
by *(simp add: ln_Gamma_real_pos of_nat_diff Ln_of_real [symmetric])*

lemma *Gamma_real_pos [simp, intro]: x > (0::real) ⟹ Gamma x > 0*
by *(simp add: Gamma_real_pos_exp)*

lemma *Gamma_real_nonneg [simp, intro]: x > (0::real) ⟹ Gamma x ≥ 0*
by *(simp add: Gamma_real_pos_exp)*

lemma *has_field_derivative_ln_Gamma_real [derivative_intros]:*
assumes *x: x > (0::real)*
shows *(ln_Gamma has_field_derivative Digamma x) (at x)*
proof *(subst DERIV_cong_ev[OF refl _ refl])*
from *assms* **show** *((Re ∘ ln_Gamma ∘ complex_of_real) has_field_derivative Digamma x) (at x)*
by *(auto intro!: derivative_eq_intros has_vector_derivative_real_field simp: Polygamma_of_real o_def)*
from *eventually_nhds_in_nhd[of x {0<..}] assms*
show *eventually (λy. ln_Gamma y = (Re ∘ ln_Gamma ∘ of_real) y) (nhds x)*
by *(auto elim!: eventually_mono simp: ln_Gamma_complex_of_real interior_open)*
qed

lemma *field_differentiable_ln_Gamma_real:*
x > 0 ⟹ ln_Gamma field_differentiable (at (x::real) within A)
by *(rule field_differentiable_within_subset[of _ _ UNIV])*
(auto simp: field_differentiable_def intro!: derivative_intros)+

declare *has_field_derivative_ln_Gamma_real*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *deriv_ln_Gamma_real*:
assumes $z > 0$
shows *deriv ln_Gamma z* = *Digamma* (*z* :: *real*)
by (*intro DERIV_imp_deriv has_field_derivative_ln_Gamma_real assms*)

lemma *higher_deriv_ln_Gamma_real*:
assumes (*x*::*real*) > *0*
shows (*deriv* ^^ *j*) *ln_Gamma x* = (*if j* = *0 then ln_Gamma x else Polygamma* (*j* − *1*) *x*)
proof (*cases j*)
case (*Suc j'*)
have (*deriv* ^^ *j'*) (*deriv ln_Gamma*) *x* = (*deriv* ^^ *j'*) *Digamma x*
using *eventually_nhds_in_open*[*of* {*0*<..} *x*] *assms*
by (*intro higher_deriv_cong_ev refl*)
(*auto elim*!: *eventually_mono simp*: *open_Diff deriv_ln_Gamma_real*)
also have ... = *Polygamma j' x* **using** *assms*
by (*subst higher_deriv_Polygamma*)
(*auto elim*!: *nonpos_Ints_cases simp*: *complex_nonpos_Reals_iff*)
finally show *?thesis* **using** *Suc* **by** (*simp del*: *funpow.simps add*: *funpow_Suc_right*)
qed *simp_all*

lemma *higher_deriv_ln_Gamma_complex_of_real*:
assumes (*x* :: *real*) > *0*
shows (*deriv* ^^ *j*) *ln_Gamma* (*complex_of_real x*) = *of_real* ((*deriv* ^^ *j*) *ln_Gamma x*)
using *assms*
by (*auto simp*: *higher_deriv_ln_Gamma_real higher_deriv_ln_Gamma_complex*
ln_Gamma_complex_of_real Polygamma_of_real)

lemma *has_field_derivative_rGamma_real'* [*derivative_intros*]:
(*rGamma has_field_derivative* (*if x* ∈ $\mathbb{Z}_{\leq 0}$ *then* (−*1*)^(*nat* ⌊−*x*⌋) ∗ *fact* (*nat* ⌊−*x*⌋) *else*
−*rGamma x* ∗ *Digamma x*)) (*at x within A*)
using *has_field_derivative_rGamma*[*of x*] **by** (*force elim*!: *nonpos_Ints_cases'*)

declare *has_field_derivative_rGamma_real'*[*THEN DERIV_chain2*, *derivative_intros*]

lemma *Polygamma_real_odd_pos*:
assumes (*x*::*real*) ∉ $\mathbb{Z}_{\leq 0}$ *odd n*
shows *Polygamma n x* > *0*
proof −
from *assms* **have** *x* ≠ *0* **by** *auto*
with *assms* **show** *?thesis*
unfolding *Polygamma_def* **using** *Polygamma_converges'*[*of x Suc n*]
by (*auto simp*: *zero_less_power_eq simp del*: *power_Suc*
dest: *plus_of_nat_eq_0_imp intro*!: *mult_pos_pos suminf_pos*)

qed

lemma *Polygamma_real_even_neg*:
assumes (*x*::*real*) > *0* *n* > *0* *even n*
shows *Polygamma n x* < *0*
using *assms* **unfolding** *Polygamma_def* **using** *Polygamma_converges'*[*of x Suc n*]
by (*auto intro*!: *mult_pos_pos suminf_pos*)

lemma *Polygamma_real_strict_mono*:
assumes *x* > *0* *x* < (*y*::*real*) *even n*
shows *Polygamma n x* < *Polygamma n y*
proof −
have ∃ξ. *x* < ξ ∧ ξ < *y* ∧ *Polygamma n y* − *Polygamma n x* = (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ
using *assms* **by** (*intro MVT2 derivative_intros impI allI*) (*auto elim*!: *nonpos_Ints_cases*)
then obtain ξ
where ξ: *x* < ξ ξ < *y*
and *Polygamma*: *Polygamma n y* − *Polygamma n x* = (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ
by *auto*
note *Polygamma*
also from ξ *assms* **have** (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ > *0*
by (*intro mult_pos_pos Polygamma_real_odd_pos*) (*auto elim*!: *nonpos_Ints_cases*)
finally show *?thesis* **by** *simp*
qed

lemma *Polygamma_real_strict_antimono*:
assumes *x* > *0* *x* < (*y*::*real*) *odd n*
shows *Polygamma n x* > *Polygamma n y*
proof −
have ∃ξ. *x* < ξ ∧ ξ < *y* ∧ *Polygamma n y* − *Polygamma n x* = (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ
using *assms* **by** (*intro MVT2 derivative_intros impI allI*) (*auto elim*!: *nonpos_Ints_cases*)
then obtain ξ
where ξ: *x* < ξ ξ < *y*
and *Polygamma*: *Polygamma n y* − *Polygamma n x* = (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ
by *auto*
note *Polygamma*
also from ξ *assms* **have** (*y* − *x*) ∗ *Polygamma* (*Suc n*) ξ < *0*
by (*intro mult_pos_neg Polygamma_real_even_neg*) *simp_all*
finally show *?thesis* **by** *simp*
qed

lemma *Polygamma_real_mono*:
assumes *x* > *0* *x* ≤ (*y*::*real*) *even n*

shows *Polygamma n x ≤ Polygamma n y*
using *Polygamma_real_strict_mono*[*OF assms*(*1*) _ *assms*(*3*), *of y*] *assms*(*2*)
by (*cases x = y*) *simp_all*

lemma *Digamma_real_strict_mono*: (*0*::*real*) < *x* ⟹ *x* < *y* ⟹ *Digamma x* < *Digamma y*
by (*rule Polygamma_real_strict_mono*) *simp_all*

lemma *Digamma_real_mono*: (*0*::*real*) < *x* ⟹ *x* ≤ *y* ⟹ *Digamma x* ≤ *Digamma y*
by (*rule Polygamma_real_mono*) *simp_all*

lemma *Digamma_real_ge_three_halves_pos*:
assumes *x* ≥ *3/2*
shows *Digamma* (*x* :: *real*) > *0*
proof −
have *0* < *Digamma* (*3/2* :: *real*) **by** (*fact Digamma_real_three_halves_pos*)
also from *assms* **have** ... ≤ *Digamma x* **by** (*intro Polygamma_real_mono*) *simp_all*
finally show *?thesis* **.**
qed

lemma *ln_Gamma_real_strict_mono*:
assumes *x* ≥ *3/2 x* < *y*
shows *ln_Gamma* (*x* :: *real*) < *ln_Gamma y*
proof −
have ∃ξ. *x* < ξ ∧ ξ < *y* ∧ *ln_Gamma y* − *ln_Gamma x* = (*y* − *x*) ∗ *Digamma* ξ
using *assms* **by** (*intro MVT2 derivative_intros impI allI*) (*auto elim*!: *non-pos_Ints_cases*)
then obtain ξ **where** ξ: *x* < ξ ξ < *y*
and *ln_Gamma*: *ln_Gamma y* − *ln_Gamma x* = (*y* − *x*) ∗ *Digamma* ξ
by *auto*
note *ln_Gamma*
also from ξ *assms* **have** (*y* − *x*) ∗ *Digamma* ξ > *0*
by (*intro mult_pos_pos Digamma_real_ge_three_halves_pos*) *simp_all*
finally show *?thesis* **by** *simp*
qed

lemma *Gamma_real_strict_mono*:
assumes *x* ≥ *3/2 x* < *y*
shows *Gamma* (*x* :: *real*) < *Gamma y*
proof −
from *Gamma_real_pos_exp*[*of x*] *assms* **have** *Gamma x* = *exp* (*ln_Gamma x*)
by *simp*
also have ... < *exp* (*ln_Gamma y*) **by** (*intro exp_less_mono ln_Gamma_real_strict_mono assms*)
also from *Gamma_real_pos_exp*[*of y*] *assms* **have** ... = *Gamma y* **by** *simp*
finally show *?thesis* **.**

qed

theorem *log_convex_Gamma_real*: *convex_on* {*0*<..} (*ln* ∘ *Gamma* :: *real* ⇒ *real*)
by (*rule convex_on_realI*[*of* _ _ *Digamma*])
(*auto intro*!: *derivative_eq_intros Polygamma_real_mono Gamma_real_pos*
simp: *o_def Gamma_eq_zero_iff elim*!: *nonpos_Ints_cases*′)

9.10.7 The uniqueness of the real Gamma function

The following is a proof of the Bohr–Mollerup theorem, which states that any log-convex function G on the positive reals that fulfils $G(1) = 1$ and satisfies the functional equation $G(x + 1) = x\ G(x)$ must be equal to the Gamma function. In principle, if G is a holomorphic complex function, one could then extend this from the positive reals to the entire complex plane (minus the non-positive integers, where the Gamma function is not defined).

context
fixes *G* :: *real* ⇒ *real*
assumes *G_1*: *G 1* = *1*
assumes *G_plus1*: *x* > *0* ⟹ *G* (*x* + *1*) = *x* ∗ *G x*
assumes *G_pos*: *x* > *0* ⟹ *G x* > *0*
assumes *log_convex_G*: *convex_on* {*0*<..} (*ln* ∘ *G*)
begin

private lemma *G_fact*: *G* (*of_nat n* + *1*) = *fact n*
using *G_plus1*[*of real n* + *1* **for** *n*]
by (*induction n*) (*simp_all add*: *G_1 G_plus1*)

private definition *S* :: *real* ⇒ *real* ⇒ *real* **where**
S x y = (*ln* (*G y*) − *ln* (*G x*)) / (*y* − *x*)

private lemma *S_eq*:
n ≥ *2* ⟹ *S* (*of_nat n*) (*of_nat n* + *x*) = (*ln* (*G* (*real n* + *x*)) − *ln* (*fact* (*n* − *1*))) / *x*
by (*subst G_fact* [*symmetric*]) (*simp add*: *S_def add_ac of_nat_diff*)

private lemma *G_lower*:
assumes *x*: *x* > *0* **and** *n*: *n* ≥ *1*
shows *Gamma_series x n* ≤ *G x*
proof −
have (*ln* ∘ *G*) (*real* (*Suc n*)) ≤ ((*ln* ∘ *G*) (*real* (*Suc n*) + *x*) −
(*ln* ∘ *G*) (*real* (*Suc n*) − *1*)) / (*real* (*Suc n*) + *x* − (*real* (*Suc n*) − *1*)) ∗
(*real* (*Suc n*) − (*real* (*Suc n*) − *1*)) + (*ln* ∘ *G*) (*real* (*Suc n*) − *1*)
using *x n* **by** (*intro convex_onD_Icc*′ *convex_on_subset*[*OF log_convex_G*])
auto
hence *S* (*of_nat n*) (*of_nat* (*Suc n*)) ≤ *S* (*of_nat* (*Suc n*)) (*of_nat* (*Suc n*) + *x*)
unfolding *S_def* **using** *x* **by** (*simp add*: *field_simps*)

```
  also have S (of_nat n) (of_nat (Suc n)) = ln (fact n) - ln (fact (n-1))
    unfolding S_def using n
    by (subst (1 2) G_fact [symmetric]) (simp_all add: add_ac of_nat_diff)
  also have ... = ln (fact n / fact (n-1)) by (subst ln_div) simp_all
  also from n have fact n / fact (n - 1) = n by (cases n) simp_all
  finally have x * ln (real n) + ln (fact n) ≤ ln (G (real (Suc n) + x))
    using x n by (subst (asm) S_eq) (simp_all add: field_simps)
  also have x * ln (real n) + ln (fact n) = ln (exp (x * ln (real n)) * fact n)
    using x by (simp add: ln_mult)
  finally have exp (x * ln (real n)) * fact n ≤ G (real (Suc n) + x) using x
    by (subst (asm) ln_le_cancel_iff) (simp_all add: G_pos)
  also have G (real (Suc n) + x) = pochhammer x (Suc n) * G x
    using G_plus1[of real (Suc n) + x for n] G_plus1[of x] x
    by (induction n) (simp_all add: pochhammer_Suc add_ac)
  finally show Gamma_series x n ≤ G x
    using x by (simp add: field_simps pochhammer_pos Gamma_series_def)
qed

private lemma G_upper:
  assumes x: x > 0 x ≤ 1 and n: n ≥ 2
  shows  G x ≤ Gamma_series x n * (1 + x / real n)
proof -
  have (ln ∘ G) (real n + x) ≤ ((ln ∘ G) (real n + 1) -
        (ln ∘ G) (real n)) / (real n + 1 - (real n)) *
         ((real n + x) - real n) + (ln ∘ G) (real n)
    using x n by (intro convex_onD_Icc' convex_on_subset[OF log_convex_G])
auto
  hence S (of_nat n) (of_nat n + x) ≤ S (of_nat n) (of_nat n + 1)
    unfolding S_def using x by (simp add: field_simps)
  also from n have S (of_nat n) (of_nat n + 1) = ln (fact n) - ln (fact (n-1))
    by (subst (1 2) G_fact [symmetric]) (simp add: S_def add_ac of_nat_diff)
  also have ... = ln (fact n / (fact (n-1))) using n by (subst ln_div) simp_all
  also from n have fact n / fact (n - 1) = n by (cases n) simp_all
  finally have ln (G (real n + x)) ≤ x * ln (real n) + ln (fact (n - 1))
    using x n by (subst (asm) S_eq) (simp_all add: field_simps)
  also have ... = ln (exp (x * ln (real n)) * fact (n - 1)) using x
    by (simp add: ln_mult)
  finally have G (real n + x) ≤ exp (x * ln (real n)) * fact (n - 1) using x
    by (subst (asm) ln_le_cancel_iff) (simp_all add: G_pos)
  also have G (real n + x) = pochhammer x n * G x
    using G_plus1[of real n + x for n] x
    by (induction n) (simp_all add: pochhammer_Suc add_ac)
  finally have G x ≤ exp (x * ln (real n)) * fact (n- 1) / pochhammer x n
    using x by (simp add: field_simps pochhammer_pos)
  also from n have fact (n - 1) = fact n / n by (cases n) simp_all
  also have exp (x * ln (real n)) * ... / pochhammer x n =
              Gamma_series x n * (1 + x / real n) using n x
    by (simp add: Gamma_series_def divide_simps pochhammer_Suc)
  finally show ?thesis .
```

qed

private lemma *G_eq_Gamma_aux*:
assumes *x*: $x > 0$ $x \leq 1$
shows $G\ x = \Gamma\ x$
proof (*rule antisym*)
show $G\ x \geq \Gamma\ x$
proof (*rule tendsto_upperbound*)
from *G_lower*[*of x*] **show** *eventually* (λn. *Gamma_series x n* $\leq G\ x$) *sequentially*
using *x* **by** (*auto intro*: *eventually_mono*[*OF eventually_ge_at_top*[*of* 1::*nat*]])
qed (*simp_all add*: *Gamma_series_LIMSEQ*)
next
show $G\ x \leq \Gamma\ x$
proof (*rule tendsto_lowerbound*)
have (λn. *Gamma_series x n* $*$ (1 + *x* / *real n*)) $\longrightarrow$ $\Gamma\ x * (1 + 0)$
by (*rule tendsto_intros real_tendsto_divide_at_top Gamma_series_LIMSEQ filterlim_real_sequentially*)+
thus (λn. *Gamma_series x n* $*$ (1 + *x* / *real n*)) $\longrightarrow$ $\Gamma\ x$ **by** *simp*
next
from *G_upper*[*of x*] **show** *eventually* (λn. *Gamma_series x n* $*$ (1 + *x* / *real n*) $\geq G\ x$) *sequentially*
using *x* **by** (*auto intro*: *eventually_mono*[*OF eventually_ge_at_top*[*of* 2::*nat*]])
qed *simp_all*
qed

theorem *Gamma_pos_real_unique*:
assumes *x*: $x > 0$
shows $G\ x = \Gamma\ x$
proof −
have *G_eq*: $G\ (real\ n + x) = \Gamma\ (real\ n + x)$ **if** $x \in \{0<..1\}$ **for** *n x* **using** *that*
proof (*induction n*)
case (*Suc n*)
from *Suc* **have** $x + real\ n > 0$ **by** *simp*
hence $x + real\ n \notin \mathbb{Z}_{\leq 0}$ **by** *auto*
with *Suc* **show** *?case* **using** *G_plus1*[*of real n* + *x*] *Gamma_plus1*[*of real n* + *x*]
by (*auto simp*: *add_ac*)
qed (*simp_all add*: *G_eq_Gamma_aux*)

show *?thesis*
proof (*cases frac x* = 0)
case *True*
hence *x* = *of_int* (*floor x*) **by** (*simp add*: *frac_def*)
with *x* **have** *x_eq*: *x* = *of_nat* (*nat* (*floor x*) − 1) + 1 **by** *simp*
show *?thesis* **by** (*subst* (1 2) *x_eq*, *rule G_eq*) *simp_all*
next

case *False*
from *assms* **have** *x_eq*: *x = of_nat (nat (floor x)) + frac x*
by (*simp add*: *frac_def*)
have *frac_le_1*: *frac x ≤ 1* **unfolding** *frac_def* **by** *linarith*
show *?thesis*
by (*subst (1 2) x_eq*, *rule G_eq*, *insert False frac_le_1*) *simp_all*
qed
qed

end

9.10.8 The Beta function

definition *Beta* **where** *Beta a b = Gamma a * Gamma b / Gamma (a + b)*

lemma *Beta_altdef*: *Beta a b = Gamma a * Gamma b * rGamma (a + b)*
by (*simp add*: *inverse_eq_divide Beta_def Gamma_def*)

lemma *Beta_commute*: *Beta a b = Beta b a*
unfolding *Beta_def* **by** (*simp add*: *ac_simps*)

lemma *has_field_derivative_Beta1* [*derivative_intros*]:
assumes *x ∉ ℤ≤0 x + y ∉ ℤ≤0*
shows *((λx. Beta x y) has_field_derivative (Beta x y * (Digamma x − Digamma (x + y))))*
(at x within A) **unfolding** *Beta_altdef*
by (*rule DERIV_cong*, (*rule derivative_intros assms*)+) (*simp add*: *algebra_simps*)

lemma *Beta_pole1*: *x ∈ ℤ≤0 ⟹ Beta x y = 0*
by (*auto simp add*: *Beta_def elim*!: *nonpos_Ints_cases′*)

lemma *Beta_pole2*: *y ∈ ℤ≤0 ⟹ Beta x y = 0*
by (*auto simp add*: *Beta_def elim*!: *nonpos_Ints_cases′*)

lemma *Beta_zero*: *x + y ∈ ℤ≤0 ⟹ Beta x y = 0*
by (*auto simp add*: *Beta_def elim*!: *nonpos_Ints_cases′*)

lemma *has_field_derivative_Beta2* [*derivative_intros*]:
assumes *y ∉ ℤ≤0 x + y ∉ ℤ≤0*
shows *((λy. Beta x y) has_field_derivative (Beta x y * (Digamma y − Digamma (x + y))))*
(at y within A)
using *has_field_derivative_Beta1*[*of y x A*] *assms* **by** (*simp add*: *Beta_commute add_ac*)

theorem *Beta_plus1_plus1*:
assumes *x ∉ ℤ≤0 y ∉ ℤ≤0*
shows *Beta (x + 1) y + Beta x (y + 1) = Beta x y*
proof −

have *Beta (x + 1) y + Beta x (y + 1) =*
*(Gamma (x + 1) * Gamma y + Gamma x * Gamma (y + 1)) * rGamma ((x + y) + 1)*
by (*simp add*: *Beta_altdef add_divide_distrib algebra_simps*)
also have *... = (Gamma x * Gamma y) * ((x + y) * rGamma ((x + y) + 1))*
by (*subst assms*[*THEN Gamma_plus1*])+ (*simp add*: *algebra_simps*)
also from *assms* **have** *... = Beta x y* **unfolding** *Beta_altdef* **by** (*subst rGamma_plus1*) *simp*
finally show *?thesis* **.**
qed

theorem *Beta_plus1_left*:
assumes $x \notin \mathbb{Z}_{\leq 0}$
shows $(x + y) * Beta\ (x + 1)\ y = x * Beta\ x\ y$
proof −
have $(x + y) * Beta\ (x + 1)\ y = Gamma\ (x + 1) * Gamma\ y * ((x + y) * rGamma\ ((x + y) + 1))$
unfolding *Beta_altdef* **by** (*simp only*: *ac_simps*)
also have *... = x * Beta x y* **unfolding** *Beta_altdef*
by (*subst assms*[*THEN Gamma_plus1*] *rGamma_plus1*)+ (*simp only*: *ac_simps*)
finally show *?thesis* **.**
qed

theorem *Beta_plus1_right*:
assumes $y \notin \mathbb{Z}_{\leq 0}$
shows $(x + y) * Beta\ x\ (y + 1) = y * Beta\ x\ y$
using *Beta_plus1_left*[*of y x*] *assms* **by** (*simp_all add*: *Beta_commute add.commute*)

lemma *Gamma_Gamma_Beta*:
assumes $x + y \notin \mathbb{Z}_{\leq 0}$
shows $Gamma\ x * Gamma\ y = Beta\ x\ y * Gamma\ (x + y)$
unfolding *Beta_altdef* **using** *assms Gamma_eq_zero_iff*[*of x+y*]
by (*simp add*: *rGamma_inverse_Gamma*)

9.10.9 Legendre duplication theorem

context
begin

private lemma *Gamma_legendre_duplication_aux*:
fixes $z :: 'a :: Gamma$
assumes $z \notin \mathbb{Z}_{\leq 0}$ $z + 1/2 \notin \mathbb{Z}_{\leq 0}$
shows $Gamma\ z * Gamma\ (z + 1/2) = exp\ ((1 - 2*z) * of\_real\ (ln\ 2)) * Gamma\ (1/2) * Gamma\ (2*z)$
proof −
let *?powr* $= \lambda b\ a.\ exp\ (a * of\_real\ (ln\ (of\_nat\ b)))$
let *?h* $= \lambda n.\ (fact\ (n-1))^2\ /\ fact\ (2*n-1) * of\_nat\ (2\hat{}(2*n)) *$
$exp\ (1/2 * of\_real\ (ln\ (real\_of\_nat\ n)))$
{

fix $z :: 'a$ **assume** z: $z \notin \mathbb{Z}_{\leq 0}$ $z + 1/2 \notin \mathbb{Z}_{\leq 0}$
let $?g = \lambda n.$ $?powr\ 2\ (2{*}z) * Gamma\_series'\ z\ n * Gamma\_series'\ (z + 1/2)\ n\ /$
$Gamma\_series'\ (2{*}z)\ (2{*}n)$
have *eventually* $(\lambda n.\ ?g\ n = ?h\ n)$ *sequentially* **using** *eventually_gt_at_top*
proof *eventually_elim*
fix $n :: nat$ **assume** n: $n > 0$
let $?f = fact\ (n - 1) :: 'a$ **and** $?f' = fact\ (2{*}n - 1) :: 'a$
have A: $exp\ t * exp\ t = exp\ (2{*}t :: 'a)$ **for** t **by** (*subst exp_add* [*symmetric*]) *simp*
have A: $Gamma\_series'\ z\ n * Gamma\_series'\ (z + 1/2)\ n = ?f\hat{}2 * ?powr\ n\ (2{*}z + 1/2)\ /$
$(pochhammer\ z\ n * pochhammer\ (z + 1/2)\ n)$
by (*simp add*: *Gamma_series'_def exp_add ring_distribs power2_eq_square A mult_ac*)
have B: $Gamma\_series'\ (2{*}z)\ (2{*}n) =$
$?f' * ?powr\ 2\ (2{*}z) * ?powr\ n\ (2{*}z)\ /$
$(of\_nat\ (2\hat{}(2{*}n)) * pochhammer\ z\ n * pochhammer\ (z{+}1/2)\ n)$ **using** n
by (*simp add*: *Gamma_series'_def ln_mult exp_add ring_distribs pochhammer_double*)
from z **have** $pochhammer\ z\ n \neq 0$ **by** (*auto dest*: *pochhammer_eq_0_imp_nonpos_Int*)
moreover from z **have** $pochhammer\ (z + 1/2)\ n \neq 0$ **by** (*auto dest*: *pochhammer_eq_0_imp_nonpos_Int*)
ultimately have $?powr\ 2\ (2{*}z) * (Gamma\_series'\ z\ n * Gamma\_series'\ (z + 1/2)\ n)\ /\ Gamma\_series'\ (2{*}z)\ (2{*}n) =$
$?f\hat{}2\ /\ ?f' * of\_nat\ (2\hat{}(2{*}n)) * (?powr\ n\ ((4{*}z + 1)/2) * ?powr\ n\ (-2{*}z))$
using n **unfolding** A B **by** (*simp add*: *field_split_simps exp_minus*)
also have $?powr\ n\ ((4{*}z + 1)/2) * ?powr\ n\ (-2{*}z) = ?powr\ n\ (1/2)$
by (*simp add*: *algebra_simps exp_add*[*symmetric*] *add_divide_distrib*)
finally show $?g\ n = ?h\ n$ **by** (*simp only*: *mult_ac*)
qed

moreover from z *double_in_nonpos_Ints_imp*[*of z*] **have** $2 * z \notin \mathbb{Z}_{\leq 0}$ **by** *auto*
hence $?g \longrightarrow ?powr\ 2\ (2{*}z) * Gamma\ z * Gamma\ (z{+}1/2)\ /\ Gamma\ (2{*}z)$
using *LIMSEQ_subseq_LIMSEQ*[*OF Gamma_series'_LIMSEQ*, *of* $(*)2\ 2{*}z$]
by (*intro tendsto_intros Gamma_series'_LIMSEQ*)
(*simp_all add*: *o_def strict_mono_def Gamma_eq_zero_iff*)
ultimately have $?h \longrightarrow ?powr\ 2\ (2{*}z) * Gamma\ z * Gamma\ (z{+}1/2)\ /\ Gamma\ (2{*}z)$
by (*blast intro*: *Lim_transform_eventually*)
} note *lim* = *this*

from *assms double_in_nonpos_Ints_imp*[*of z*] **have** z': $2 * z \notin \mathbb{Z}_{\leq 0}$ **by** *auto*
from *fraction_not_in_ints*[*of 2 1*] **have** $(1/2 :: 'a) \notin \mathbb{Z}_{\leq 0}$
by (*intro not_in_Ints_imp_not_in_nonpos_Ints*) *simp_all*
with *lim*[*of* $1/2 :: 'a$] **have** $?h \longrightarrow 2 * Gamma\ (1/2 :: 'a)$ **by** (*simp add*:

exp_of_real)
from *LIMSEQ_unique*[*OF this lim*[*OF assms*]] *z'* **show** *?thesis*
by (*simp add*: *field_split_simps Gamma_eq_zero_iff ring_distribs exp_diff exp_of_real*)
qed

The following lemma is somewhat annoying. With a little bit of complex analysis (Cauchy's integral theorem, to be exact), this would be completely trivial. However, we want to avoid depending on the complex analysis session at this point, so we prove it the hard way.

private lemma *Gamma_reflection_aux*:
defines $h \equiv \lambda z{::}complex.\ if\ z \in \mathbb{Z}\ then\ 0\ else$
$(of\_real\ pi * cot\ (of\_real\ pi{*}z) + Digamma\ z - Digamma\ (1 - z))$
defines $a \equiv complex\_of\_real\ pi$
obtains h' **where** *continuous_on UNIV h'* $\bigwedge z.\ (h\ has\_field\_derivative\ (h'\ z))\ (at\ z)$
proof −
define *f* **where** $f\ n = a * of\_real\ (cos\_coeff\ (n{+}1) - sin\_coeff\ (n{+}2))$ **for** *n*
define *F* **where** $F\ z = (if\ z = 0\ then\ 0\ else\ (cos\ (a{*}z) - sin\ (a{*}z)/(a{*}z))\ /\ z)$ **for** *z*
define *g* **where** $g\ n = complex\_of\_real\ (sin\_coeff\ (n{+}1))$ **for** *n*
define *G* **where** $G\ z = (if\ z = 0\ then\ 1\ else\ sin\ (a{*}z)/(a{*}z))$ **for** *z*
have *a_nz*: $a \neq 0$ **unfolding** *a_def* **by** *simp*

have $(\lambda n.\ f\ n * (a{*}z)\hat{}n)\ sums\ (F\ z) \wedge (\lambda n.\ g\ n * (a{*}z)\hat{}n)\ sums\ (G\ z)$
if $abs\ (Re\ z) < 1$ **for** *z*
proof (*cases* $z = 0$; *rule conjI*)
assume $z \neq 0$
note $z = this\ that$

from *z* **have** *sin_nz*: $sin\ (a{*}z) \neq 0$ **unfolding** *a_def* **by** (*auto simp*: *sin_eq_0*)
have $(\lambda n.\ of\_real\ (sin\_coeff\ n) * (a{*}z)\hat{}n)\ sums\ (sin\ (a{*}z))$ **using** *sin_converges*[*of* $a{*}z$]
by (*simp add*: *scaleR_conv_of_real*)
from *sums_split_initial_segment*[*OF this*, *of 1*]
have $(\lambda n.\ (a{*}z) * of\_real\ (sin\_coeff\ (n{+}1)) * (a{*}z)\hat{}n)\ sums\ (sin\ (a{*}z))$ **by** (*simp add*: *mult_ac*)
from *sums_mult*[*OF this*, *of inverse* $(a{*}z)$] *z a_nz*
have *A*: $(\lambda n.\ g\ n * (a{*}z)\hat{}n)\ sums\ (sin\ (a{*}z)/(a{*}z))$
by (*simp add*: *field_simps g_def*)
with *z* **show** $(\lambda n.\ g\ n * (a{*}z)\hat{}n)\ sums\ (G\ z)$ **by** (*simp add*: *G_def*)
from *A z a_nz sin_nz* **have** *g_nz*: $(\sum n.\ g\ n * (a{*}z)\hat{}n) \neq 0$ **by** (*simp add*: *sums_iff g_def*)

have [*simp*]: $sin\_coeff\ (Suc\ 0) = 1$ **by** (*simp add*: *sin_coeff_def*)
from *sums_split_initial_segment*[*OF sums_diff*[*OF cos_converges*[*of* $a{*}z$] *A*], *of 1*]
have $(\lambda n.\ z * f\ n * (a{*}z)\hat{}n)\ sums\ (cos\ (a{*}z) - sin\ (a{*}z)\ /\ (a{*}z))$
by (*simp add*: *mult_ac scaleR_conv_of_real ring_distribs f_def g_def*)

```
    from sums_mult[OF this, of inverse z] z assms
      show (λn. f n * (a*z)^n) sums (F z) by (simp add: divide_simps mult_ac
f_def F_def)
  next
    assume z: z = 0
    have (λn. f n * (a * z) ^ n) sums f 0 using powser_sums_zero[of f] z by simp
    with z show (λn. f n * (a * z) ^ n) sums (F z)
      by (simp add: f_def F_def sin_coeff_def cos_coeff_def)
    have (λn. g n * (a * z) ^ n) sums g 0 using powser_sums_zero[of g] z by
simp
    with z show (λn. g n * (a * z) ^ n) sums (G z)
      by (simp add: g_def G_def sin_coeff_def cos_coeff_def)
  qed
  note sums = conjunct1[OF this] conjunct2[OF this]

  define h2 where [abs_def]:
    h2 z = (∑ n. f n * (a*z)^n) / (∑ n. g n * (a*z)^n) + Digamma (1 + z) -
Digamma (1 - z) for z
  define POWSER where [abs_def]: POWSER f z = (∑ n. f n * (z^n :: complex))
for f z
  define POWSER' where [abs_def]: POWSER' f z = (∑ n. diffs f n * (z^n))
for f and z :: complex
  define h2' where [abs_def]:
    h2' z = a * (POWSER g (a*z) * POWSER' f (a*z) - POWSER f (a*z) *
POWSER' g (a*z)) /
      (POWSER g (a*z))^2 + Polygamma 1 (1 + z) + Polygamma 1 (1 - z) for
z

  have h_eq: h t = h2 t if abs (Re t) < 1 for t
  proof -
    from that have t: t ∈ ℤ ⟷ t = 0 by (auto elim!: Ints_cases)
    hence h t = a*cot (a*t) - 1/t + Digamma (1 + t) - Digamma (1 - t)
      unfolding h_def using Digamma_plus1[of t] by (force simp: field_simps
a_def)
    also have a*cot (a*t) - 1/t = (F t) / (G t)
      using t by (auto simp add: divide_simps sin_eq_0 cot_def a_def F_def
G_def)
    also have ... = (∑ n. f n * (a*t)^n) / (∑ n. g n * (a*t)^n)
      using sums[of t] that by (simp add: sums_iff)
    finally show h t = h2 t by (simp only: h2_def)
  qed

  let ?A = {z. abs (Re z) < 1}
  have open ({z. Re z < 1} ∩ {z. Re z > -1})
    using open_halfspace_Re_gt open_halfspace_Re_lt by auto
  also have ({z. Re z < 1} ∩ {z. Re z > -1}) = {z. abs (Re z) < 1} by auto
  finally have open_A: open ?A .
  hence [simp]: interior ?A = ?A by (simp add: interior_open)
```

have *summable_f*: *summable* (λ*n*. *f n* ∗ *z*^*n*) **for** *z*
by (*rule powser_inside*, *rule sums_summable*, *rule sums*[*of* i ∗ *of_real* (*norm z* + *1*) / *a*])
(*simp_all add*: *norm_mult a_def del*: *of_real_add*)
have *summable_g*: *summable* (λ*n*. *g n* ∗ *z*^*n*) **for** *z*
by (*rule powser_inside*, *rule sums_summable*, *rule sums*[*of* i ∗ *of_real* (*norm z* + *1*) / *a*])
(*simp_all add*: *norm_mult a_def del*: *of_real_add*)
have *summable_fg'*: *summable* (λ*n*. *diffs f n* ∗ *z*^*n*) *summable* (λ*n*. *diffs g n* ∗ *z*^*n*) **for** *z*
by (*intro termdiff_converges_all summable_f summable_g*)+
have (*POWSER f has_field_derivative* (*POWSER' f z*)) (*at z*)
(*POWSER g has_field_derivative* (*POWSER' g z*)) (*at z*) **for** *z*
unfolding *POWSER_def POWSER'_def*
by (*intro termdiffs_strong_converges_everywhere summable_f summable_g*)+
note *derivs* = *this*[*THEN DERIV_chain2*[*OF _ DERIV_cmult*[*OF DERIV_ident*]], *unfolded POWSER_def*]
have *isCont* (*POWSER f*) *z isCont* (*POWSER g*) *z isCont* (*POWSER' f*) *z isCont* (*POWSER' g*) *z*
for *z* **unfolding** *POWSER_def POWSER'_def*
by (*intro isCont_powser_converges_everywhere summable_f summable_g summable_fg'*)+
note *cont* = *this*[*THEN isCont_o2*[*rotated*], *unfolded POWSER_def POWSER'_def*]

{
fix *z* :: *complex* **assume** *z*: *abs* (*Re z*) < *1*
define *d* **where** *d* = i ∗ *of_real* (*norm z* + *1*)
have *d*: *abs* (*Re d*) < *1 norm z* < *norm d* **by** (*simp_all add*: *d_def norm_mult del*: *of_real_add*)
have *eventually* (λ*z*. *h z* = *h2 z*) (*nhds z*)
using *eventually_nhds_in_nhd*[*of z ?A*] **using** *h_eq z*
by (*auto elim*!: *eventually_mono*)

moreover from *sums*(*2*)[*OF z*] *z* **have** *nz*: (∑ *n*. *g n* ∗ (*a* ∗ *z*) ^ *n*) ≠ *0*
unfolding *G_def* **by** (*auto simp*: *sums_iff sin_eq_0 a_def*)
have *A*: *z* ∈ ℤ ⟷ *z* = *0* **using** *z* **by** (*auto elim*!: *Ints_cases*)
have *no_int*: *1* + *z* ∈ ℤ ⟷ *z* = *0* **using** *z Ints_diff*[*of 1*+*z 1*] *A*
by (*auto elim*!: *nonpos_Ints_cases*)
have *no_int'*: *1* − *z* ∈ ℤ ⟷ *z* = *0* **using** *z Ints_diff*[*of 1 1*−*z*] *A*
by (*auto elim*!: *nonpos_Ints_cases*)
from *no_int no_int'* **have** *no_int*: *1* − *z* ∉ ℤ≤0 *1* + *z* ∉ ℤ≤0 **by** *auto*
have (*h2 has_field_derivative h2' z*) (*at z*) **unfolding** *h2_def*
by (*rule DERIV_cong*, (*rule derivative_intros refl derivs*[*unfolded POWSER_def*] *nz no_int*)+)
(*auto simp*: *h2'_def POWSER_def field_simps power2_eq_square*)
ultimately have *deriv*: (*h has_field_derivative h2' z*) (*at z*)
by (*subst DERIV_cong_ev*[*OF refl _ refl*])

from *sums*(*2*)[*OF z*] *z* **have** (∑ *n*. *g n* ∗ (*a* ∗ *z*) ^ *n*) ≠ *0*
unfolding *G_def* **by** (*auto simp*: *sums_iff a_def sin_eq_0*)

```
  hence isCont h2' z using no_int unfolding h2'_def[abs_def] POWSER_def
POWSER'_def
    by (intro continuous_intros cont
          continuous_on_compose2[OF _ continuous_on_Polygamma[of {z. Re
z > 0}]]) auto
  note deriv and this
 } note A = this

 interpret h: periodic_fun_simple' h
 proof
  fix z :: complex
  show h (z + 1) = h z
  proof (cases z ∈ ℤ)
   assume z: z ∉ ℤ
   hence A: z + 1 ∉ ℤ z ≠ 0 using Ints_diff[of z+1 1] by auto
   hence Digamma (z + 1) − Digamma (−z) = Digamma z − Digamma (−z
+ 1)
     by (subst (1 2) Digamma_plus1) simp_all
   with A z show h (z + 1) = h z
    by (simp add: h_def sin_plus_pi cos_plus_pi ring_distribs cot_def)
  qed (simp add: h_def)
 qed

 have h2'_eq: h2' (z − 1) = h2' z if z: Re z > 0 Re z < 1 for z
 proof −
  have ((λz. h (z − 1)) has_field_derivative h2' (z − 1)) (at z)
   by (rule DERIV_cong, rule DERIV_chain'[OF _ A(1)])
     (insert z, auto intro!: derivative_eq_intros)
  hence (h has_field_derivative h2' (z − 1)) (at z) by (subst (asm) h.minus_1)
   moreover from z have (h has_field_derivative h2' z) (at z) by (intro A)
simp_all
  ultimately show h2' (z − 1) = h2' z by (rule DERIV_unique)
 qed

 define h2'' where h2'' z = h2' (z − of_int ⌊Re z⌋) for z
 have deriv: (h has_field_derivative h2'' z) (at z) for z
 proof −
  fix z :: complex
  have B: |Re z − real_of_int ⌊Re z⌋| < 1 by linarith
  have ((λt. h (t − of_int ⌊Re z⌋)) has_field_derivative h2'' z) (at z)
   unfolding h2''_def by (rule DERIV_cong, rule DERIV_chain'[OF _ A(1)])
                         (insert B, auto intro!: derivative_intros)
  thus (h has_field_derivative h2'' z) (at z) by (simp add: h.minus_of_int)
 qed

 have cont: continuous_on UNIV h2''
 proof (intro continuous_at_imp_continuous_on ballI)
  fix z :: complex
  define r where r = ⌊Re z⌋
```

```
    define A where A = {t. of_int r − 1 < Re t ∧ Re t < of_int r + 1}
    have continuous_on A (λt. h2' (t − of_int r)) unfolding A_def
      by (intro continuous_at_imp_continuous_on isCont_o2[OF _ A(2)] ballI
continuous_intros)
         (simp_all add: abs_real_def)
    moreover have h2'' t = h2' (t − of_int r) if t: t ∈ A for t
    proof (cases Re t ≥ of_int r)
      case True
      from t have of_int r − 1 < Re t Re t < of_int r + 1 by (simp_all add:
A_def)
      with True have ⌊Re t⌋ = ⌊Re z⌋ unfolding r_def by linarith
      thus ?thesis by (auto simp: r_def h2''_def)
    next
      case False
      from t have t: of_int r − 1 < Re t Re t < of_int r + 1 by (simp_all add:
A_def)
      with False have t': ⌊Re t⌋ = ⌊Re z⌋ − 1 unfolding r_def by linarith
      moreover from t False have h2' (t − of_int r + 1 − 1) = h2' (t − of_int
r + 1)
        by (intro h2'_eq) simp_all
      ultimately show ?thesis by (auto simp: r_def h2''_def algebra_simps t')
    qed
    ultimately have continuous_on A h2'' by (subst continuous_on_cong[OF
refl])
    moreover {
      have open ({t. of_int r − 1 < Re t} ∩ {t. of_int r + 1 > Re t})
        by (intro open_Int open_halfspace_Re_gt open_halfspace_Re_lt)
      also have {t. of_int r − 1 < Re t} ∩ {t. of_int r + 1 > Re t} = A
        unfolding A_def by blast
      finally have open A .
    }
    ultimately have C: isCont h2'' t if t ∈ A for t using that
      by (subst (asm) continuous_on_eq_continuous_at) auto
    have of_int r − 1 < Re z Re z < of_int r + 1 unfolding r_def by linarith+
    thus isCont h2'' z by (intro C) (simp_all add: A_def)
  qed

  from that[OF cont deriv] show ?thesis .
qed

lemma Gamma_reflection_complex:
  fixes z :: complex
  shows Gamma z * Gamma (1 − z) = of_real pi / sin (of_real pi * z)
proof −
  let ?g = λz::complex. Gamma z * Gamma (1 − z) * sin (of_real pi * z)
  define g where [abs_def]: g z = (if z ∈ ℤ then of_real pi else ?g z) for z ::
complex
  let ?h = λz::complex. (of_real pi * cot (of_real pi*z) + Digamma z − Digamma
(1 − z))
```

define h **where** [*abs_def*]: $h\ z = (\mathit{if}\ z \in \mathbb{Z}\ \mathit{then}\ 0\ \mathit{else}\ ?h\ z)$ **for** $z :: \mathit{complex}$

— g is periodic with period 1.
interpret g: *periodic_fun_simple'* g
proof
fix $z :: \mathit{complex}$
show $g\ (z + 1) = g\ z$
proof (*cases* $z \in \mathbb{Z}$)
case *False*
hence $z * g\ z = z * \mathit{Beta}\ z\ (-\ z + 1) * \mathit{sin}\ (\mathit{of\_real}\ \mathit{pi} * z)$ **by** (*simp add*: *g_def Beta_def*)
also have $z * \mathit{Beta}\ z\ (-\ z + 1) = (z + 1 + -z) * \mathit{Beta}\ (z + 1)\ (-\ z + 1)$
using *False Ints_diff*[*of 1 1* $-\ z$] *nonpos_Ints_subset_Ints*
by (*subst Beta_plus1_left* [*symmetric*]) *auto*
also have $\ldots * \mathit{sin}\ (\mathit{of\_real}\ \mathit{pi} * z) = z * (\mathit{Beta}\ (z + 1)\ (-z) * \mathit{sin}\ (\mathit{of\_real}\ \mathit{pi} * (z + 1)))$
using *False Ints_diff*[*of z+1 1*] *Ints_minus*[*of* $-z$] *nonpos_Ints_subset_Ints*
by (*subst Beta_plus1_right*) (*auto simp*: *ring_distribs sin_plus_pi*)
also from *False* **have** $\mathit{Beta}\ (z + 1)\ (-z) * \mathit{sin}\ (\mathit{of\_real}\ \mathit{pi} * (z + 1)) = g\ (z + 1)$
using *Ints_diff*[*of z+1 1*] **by** (*auto simp*: *g_def Beta_def*)
finally show $g\ (z + 1) = g\ z$ **using** *False* **by** (*subst* (*asm*) *mult_left_cancel*) *auto*
qed (*simp add*: *g_def*)
qed

— g is entire.
have *g_g'*: (*g has_field_derivative* $(h\ z * g\ z))$ (*at* z) **for** $z :: \mathit{complex}$
proof (*cases* $z \in \mathbb{Z}$)
let $?h' = \lambda z.\ \mathit{Beta}\ z\ (1 - z) * ((\mathit{Digamma}\ z - \mathit{Digamma}\ (1 - z)) * \mathit{sin}\ (z * \mathit{of\_real}\ \mathit{pi}) +$
$\mathit{of\_real}\ \mathit{pi} * \mathit{cos}\ (z * \mathit{of\_real}\ \mathit{pi}))$
case *False*
from *False* **have** *eventually* $(\lambda t.\ t \in \mathit{UNIV} - \mathbb{Z})$ (*nhds* z)
by (*intro eventually_nhds_in_open*) (*auto simp*: *open_Diff*)
hence *eventually* $(\lambda t.\ g\ t = ?g\ t)$ (*nhds* z) **by** *eventually_elim* (*simp add*: *g_def*)
moreover {
from *False Ints_diff*[*of 1 1*$-z$] **have** $1 - z \notin \mathbb{Z}$ **by** *auto*
hence (*?g has_field_derivative* $?h'\ z$) (*at* z) **using** *nonpos_Ints_subset_Ints*
by (*auto intro*!: *derivative_eq_intros simp*: *algebra_simps Beta_def*)
also from *False* **have** $\mathit{sin}\ (\mathit{of\_real}\ \mathit{pi} * z) \neq 0$ **by** (*subst sin_eq_0*) *auto*
hence $?h'\ z = h\ z * g\ z$
using *False* **unfolding** *g_def h_def cot_def* **by** (*simp add*: *field_simps Beta_def*)
finally have (*?g has_field_derivative* $(h\ z * g\ z)$) (*at* z) **.**
}
ultimately show *?thesis* **by** (*subst DERIV_cong_ev*[*OF refl _ refl*])
next

```
    case True
    then obtain n where z: z = of_int n by (auto elim!: Ints_cases)
    let ?t = (λz::complex. if z = 0 then 1 else sin z / z) ∘ (λz. of_real pi * z)
    have deriv_0: (g has_field_derivative 0) (at 0)
    proof (subst DERIV_cong_ev[OF refl _ refl])
      show eventually (λz. g z = of_real pi * Gamma (1 + z) * Gamma (1 - z)
* ?t z) (nhds 0)
        using eventually_nhds_ball[OF zero_less_one, of 0::complex]
      proof eventually_elim
        fix z :: complex assume z: z ∈ ball 0 1
        show g z = of_real pi * Gamma (1 + z) * Gamma (1 - z) * ?t z
        proof (cases z = 0)
          assume z': z ≠ 0
          with z have z'': z ∉ ℤ≤0 z ∉ ℤ by (auto elim!: Ints_cases)
          from Gamma_plus1[OF this(1)] have Gamma z = Gamma (z + 1) / z
by simp
          with z'' z' show ?thesis by (simp add: g_def ac_simps)
        qed (simp add: g_def)
      qed
      have (?t has_field_derivative (0 * of_real pi)) (at 0)
        using has_field_derivative_sin_z_over_z[of UNIV :: complex set]
        by (intro DERIV_chain) simp_all
    thus ((λz. of_real pi * Gamma (1 + z) * Gamma (1 - z) * ?t z) has_field_derivative
0) (at 0)
        by (auto intro!: derivative_eq_intros simp: o_def)
    qed

    have ((g ∘ (λx. x - of_int n)) has_field_derivative 0 * 1) (at (of_int n))
      using deriv_0 by (intro DERIV_chain) (auto intro!: derivative_eq_intros)
    also have g ∘ (λx. x - of_int n) = g by (intro ext) (simp add: g.minus_of_int)
    finally show (g has_field_derivative (h z * g z)) (at z) by (simp add: z h_def)
  qed

  have g_eq: g (z/2) * g ((z+1)/2) = Gamma (1/2)^2 * g z if Re z > -1 Re z
< 2 for z
  proof (cases z ∈ ℤ)
    case True
    with that have z = 0 ∨ z = 1 by (force elim!: Ints_cases)
    moreover have g 0 * g (1/2) = Gamma (1/2)^2 * g 0
     using fraction_not_in_ints[where 'a = complex, of 2 1] by (simp add: g_def
power2_eq_square)
    moreover have g (1/2) * g 1 = Gamma (1/2)^2 * g 1
      using fraction_not_in_ints[where 'a = complex, of 2 1]
      by (simp add: g_def power2_eq_square Beta_def algebra_simps)
    ultimately show ?thesis by force
  next
    case False
     hence z: z/2 ∉ ℤ (z+1)/2 ∉ ℤ using Ints_diff[of z+1 1] by (auto elim!:
Ints_cases)
```

```
    hence z': z/2 ∉ ℤ≤0 (z+1)/2 ∉ ℤ≤0 by (auto elim!: nonpos_Ints_cases)
    from z have 1−z/2 ∉ ℤ 1−((z+1)/2) ∉ ℤ
      using Ints_diff[of 1 1−z/2] Ints_diff[of 1 1−((z+1)/2)] by auto
   hence z'': 1−z/2 ∉ ℤ≤0 1−((z+1)/2) ∉ ℤ≤0 by (auto elim!: nonpos_Ints_cases)
    from z have g (z/2) * g ((z+1)/2) =
    (Gamma (z/2) * Gamma ((z+1)/2)) * (Gamma (1−z/2) * Gamma (1−((z+1)/2)))
*
      (sin (of_real pi * z/2) * sin (of_real pi * (z+1)/2))
      by (simp add: g_def)
    also from z' Gamma_legendre_duplication_aux[of z/2]
      have Gamma (z/2) * Gamma ((z+1)/2) = exp ((1−z) * of_real (ln 2)) *
Gamma (1/2) * Gamma z
      by (simp add: add_divide_distrib)
    also from z'' Gamma_legendre_duplication_aux[of 1−(z+1)/2]
      have Gamma (1−z/2) * Gamma (1−(z+1)/2) =
            Gamma (1−z) * Gamma (1/2) * exp (z * of_real (ln 2))
      by (simp add: add_divide_distrib ac_simps)
   finally have g (z/2) * g ((z+1)/2) = Gamma (1/2)^2 * (Gamma z * Gamma
(1−z) *
                  (2 * (sin (of_real pi*z/2) * sin (of_real pi*(z+1)/2))))
       by (simp add: add_ac power2_eq_square exp_add ring_distribs exp_diff
exp_of_real)
    also have sin (of_real pi*(z+1)/2) = cos (of_real pi*z/2)
      using cos_sin_eq[of − of_real pi * z/2, symmetric]
      by (simp add: ring_distribs add_divide_distrib ac_simps)
    also have 2 * (sin (of_real pi*z/2) * cos (of_real pi*z/2)) = sin (of_real pi
* z)
      by (subst sin_times_cos) (simp add: field_simps)
    also have Gamma z * Gamma (1 − z) * sin (complex_of_real pi * z) = g z
      using ‹z ∉ ℤ› by (simp add: g_def)
    finally show ?thesis .
  qed
  have g_eq: g (z/2) * g ((z+1)/2) = Gamma (1/2)^2 * g z for z
  proof −
    define r where r = ⌊Re z / 2⌋
    have Gamma (1/2)^2 * g z = Gamma (1/2)^2 * g (z − of_int (2*r)) by
(simp only: g.minus_of_int)
    also have of_int (2*r) = 2 * of_int r by simp
    also have Re z − 2 * of_int r > −1 Re z − 2 * of_int r < 2 unfolding
r_def by linarith+
    hence Gamma (1/2)^2 * g (z − 2 * of_int r) =
                g ((z − 2 * of_int r)/2) * g ((z − 2 * of_int r + 1)/2)
      unfolding r_def by (intro g_eq[symmetric]) simp_all
    also have (z − 2 * of_int r) / 2 = z/2 − of_int r by simp
    also have g ... = g (z/2) by (rule g.minus_of_int)
    also have (z − 2 * of_int r + 1) / 2 = (z + 1)/2 − of_int r by simp
    also have g ... = g ((z+1)/2) by (rule g.minus_of_int)
    finally show ?thesis ..
  qed
```

```
  have g_nz [simp]: g z ≠ 0 for z :: complex
  unfolding g_def using Ints_diff[of 1 1 − z]
    by (auto simp: Gamma_eq_zero_iff sin_eq_0 dest!: nonpos_Ints_Int)

  have h_eq: h z = (h (z/2) + h ((z+1)/2)) / 2 for z
  proof −
    have ((λt. g (t/2) * g ((t+1)/2)) has_field_derivative
                  (g (z/2) * g ((z+1)/2)) * ((h (z/2) + h ((z+1)/2)) / 2)) (at z)
      by (auto intro!: derivative_eq_intros g_g'[THEN DERIV_chain2] simp:
field_simps)
    hence ((λt. Gamma (1/2)^2 * g t) has_field_derivative
           Gamma (1/2)^2 * g z * ((h (z/2) + h ((z+1)/2)) / 2)) (at z)
      by (subst (1 2) g_eq[symmetric]) simp
    from DERIV_cmult[OF this, of inverse ((Gamma (1/2))^2)]
      have (g has_field_derivative (g z * ((h (z/2) + h ((z+1)/2))/2))) (at z)
      using fraction_not_in_ints[where 'a = complex, of 2 1]
    by (simp add: divide_simps Gamma_eq_zero_iff not_in_Ints_imp_not_in_nonpos_Ints)
    moreover have (g has_field_derivative (g z * h z)) (at z)
      using g_g'[of z] by (simp add: ac_simps)
    ultimately have g z * h z = g z * ((h (z/2) + h ((z+1)/2))/2)
      by (intro DERIV_unique)
    thus h z = (h (z/2) + h ((z+1)/2)) / 2 by simp
  qed

  obtain h' where h'_cont: continuous_on UNIV h' and
                  h_h': ⋀z. (h has_field_derivative h' z) (at z)
     unfolding h_def by (erule Gamma_reflection_aux)

  have h'_eq: h' z = (h' (z/2) + h' ((z+1)/2)) / 4 for z
  proof −
    have ((λt. (h (t/2) + h ((t+1)/2)) / 2) has_field_derivative
                    ((h' (z/2) + h' ((z+1)/2)) / 4)) (at z)
      by (fastforce intro!: derivative_eq_intros h_h'[THEN DERIV_chain2])
    hence (h has_field_derivative ((h' (z/2) + h' ((z+1)/2))/4)) (at z)
      by (subst (asm) h_eq[symmetric])
     from h_h' and this show h' z = (h' (z/2) + h' ((z+1)/2)) / 4 by (rule
DERIV_unique)
  qed

  have h'_zero: h' z = 0 for z
  proof −
    define m where m = max 1 |Re z|
    define B where B = {t. abs (Re t) ≤ m ∧ abs (Im t) ≤ abs (Im z)}
    have closed ({t. Re t ≥ −m} ∩ {t. Re t ≤ m} ∩
                 {t. Im t ≥ −|Im z|} ∩ {t. Im t ≤ |Im z|})
    (is closed ?B) by (intro closed_Int closed_halfspace_Re_ge closed_halfspace_Re_le
                            closed_halfspace_Im_ge closed_halfspace_Im_le)
    also have ?B = B unfolding B_def by fastforce
```

```
    finally have closed B .
    moreover have bounded B unfolding bounded_iff
    proof (intro ballI exI)
      fix t assume t: t ∈ B
      have norm t ≤ |Re t| + |Im t| by (rule cmod_le)
      also from t have |Re t| ≤ m unfolding B_def by blast
      also from t have |Im t| ≤ |Im z| unfolding B_def by blast
      finally show norm t ≤ m + |Im z| by − simp
    qed
    ultimately have compact: compact B by (subst compact_eq_bounded_closed)
blast

    define M where M = (SUP z∈B. norm (h' z))
    have compact (h' ‘ B)
       by (intro compact_continuous_image continuous_on_subset[OF h'_cont]
compact) blast+
    hence bdd: bdd_above ((λz. norm (h' z)) ‘ B)
      using bdd_above_norm[of h' ‘ B] by (simp add: image_comp o_def com-
pact_imp_bounded)
   have norm (h' z) ≤ M unfolding M_def by (intro cSUP_upper bdd) (simp_all
add: B_def m_def)
    also have M ≤ M/2
    proof (subst M_def, subst cSUP_le_iff)
      have z ∈ B unfolding B_def m_def by simp
      thus B ≠ {} by auto
    next
      show ∀ z∈B. norm (h' z) ≤ M/2
      proof
        fix t :: complex assume t: t ∈ B
        from h'_eq[of t] t have h' t = (h' (t/2) + h' ((t+1)/2)) / 4 by (simp)
        also have norm ... = norm (h' (t/2) + h' ((t+1)/2)) / 4 by simp
        also have norm (h' (t/2) + h' ((t+1)/2)) ≤ norm (h' (t/2)) + norm (h'
((t+1)/2))
          by (rule norm_triangle_ineq)
        also from t have abs (Re ((t + 1)/2)) ≤ m unfolding m_def B_def by
auto
        with t have t/2 ∈ B (t+1)/2 ∈ B unfolding B_def by auto
       hence norm (h' (t/2)) + norm (h' ((t+1)/2)) ≤ M + M unfolding M_def
          by (intro add_mono cSUP_upper bdd) (auto simp: B_def)
        also have (M + M) / 4 = M / 2 by simp
        finally show norm (h' t) ≤ M/2 by − simp_all
      qed
    qed (insert bdd, auto)
    hence M ≤ 0 by simp
    finally show h' z = 0 by simp
  qed
  have h_h'_2: (h has_field_derivative 0) (at z) for z
    using h_h'[of z] h'_zero[of z] by simp
```

```
  have g_real: g z ∈ ℝ if z ∈ ℝ for z
   unfolding g_def using that by (auto intro!: Reals_mult Gamma_complex_real)
  have h_real: h z ∈ ℝ if z ∈ ℝ for z
    unfolding h_def using that by (auto intro!: Reals_mult Reals_add Reals_diff
Polygamma_Real)
  have g_nz: g z ≠ 0 for z unfolding g_def using Ints_diff[of 1 1−z]
    by (auto simp: Gamma_eq_zero_iff sin_eq_0)

  from h'_zero h_h'_2 have ∃ c. ∀ z∈UNIV. h z = c
    by (intro has_field_derivative_zero_constant) (simp_all add: dist_0_norm)
  then obtain c where c: ⋀z. h z = c by auto
  have ∃ u. u ∈ closed_segment 0 1 ∧ Re (g 1) − Re (g 0) = Re (h u ∗ g u ∗ (1
− 0))
    by (intro complex_mvt_line g_g')
  then obtain u where u: u ∈ closed_segment 0 1 Re (g 1) − Re (g 0) = Re (h
u ∗ g u)
    by auto
  from u(1) have u': u ∈ ℝ unfolding closed_segment_def
    by (auto simp: scaleR_conv_of_real)
  from u' g_real[of u] g_nz[of u] have Re (g u) ≠ 0 by (auto elim!: Reals_cases)
  with u(2) c[of u] g_real[of u] g_nz[of u] u'
    have Re c = 0 by (simp add: complex_is_Real_iff g.of_1)
  with h_real[of 0] c[of 0] have c = 0 by (auto elim!: Reals_cases)
  with c have A: h z ∗ g z = 0 for z by simp
  hence (g has_field_derivative 0) (at z) for z using g_g'[of z] by simp
  hence ∃ c'. ∀ z∈UNIV. g z = c' by (intro has_field_derivative_zero_constant)
simp_all
  then obtain c' where c: ⋀z. g z = c' by (force)
  from this[of 0] have c' = pi unfolding g_def by simp
  with c have g z = pi by simp

  show ?thesis
  proof (cases z ∈ ℤ)
    case False
    with ‹g z = pi› show ?thesis by (auto simp: g_def divide_simps)
  next
    case True
    then obtain n where n: z = of_int n by (elim Ints_cases)
    with sin_eq_0[of of_real pi ∗ z] have sin (of_real pi ∗ z) = 0 by force
     moreover have of_int (1 − n) ∈ ℤ≤0 if n > 0 using that by (intro non-
pos_Ints_of_int) simp
    ultimately show ?thesis using n
      by (cases n ≤ 0) (auto simp: Gamma_eq_zero_iff nonpos_Ints_of_int)
  qed
qed

lemma rGamma_reflection_complex:
  rGamma z ∗ rGamma (1 − z :: complex) = sin (of_real pi ∗ z) / of_real pi
  using Gamma_reflection_complex[of z]
```

```
  by (simp add: Gamma_def field_split_simps split: if_split_asm)

lemma rGamma_reflection_complex':
  rGamma z * rGamma (- z :: complex) = -z * sin (of_real pi * z) / of_real pi
proof -
  have rGamma z * rGamma (-z) = -z * (rGamma z * rGamma (1 - z))
    using rGamma_plus1[of -z, symmetric] by simp
  also have rGamma z * rGamma (1 - z) = sin (of_real pi * z) / of_real pi
    by (rule rGamma_reflection_complex)
  finally show ?thesis by simp
qed

lemma Gamma_reflection_complex':
  Gamma z * Gamma (- z :: complex) = - of_real pi / (z * sin (of_real pi * z))
 using rGamma_reflection_complex'[of z] by (force simp add: Gamma_def field_split_simps)



lemma Gamma_one_half_real: Gamma (1/2 :: real) = sqrt pi
proof -
   from Gamma_reflection_complex[of 1/2] fraction_not_in_ints[where 'a =
complex, of 2 1]
   have Gamma (1/2 :: complex)^2 = of_real pi by (simp add: power2_eq_square)
  hence of_real pi = Gamma (complex_of_real (1/2))^2 by simp
 also have ... = of_real ((Gamma (1/2))^2) by (subst Gamma_complex_of_real)
simp_all
  finally have Gamma (1/2)^2 = pi by (subst (asm) of_real_eq_iff) simp_all
  moreover have Gamma (1/2 :: real) ≥ 0 using Gamma_real_pos[of 1/2] by
simp
  ultimately show ?thesis by (rule real_sqrt_unique [symmetric])
qed

lemma Gamma_one_half_complex: Gamma (1/2 :: complex) = of_real (sqrt pi)
proof -
  have Gamma (1/2 :: complex) = Gamma (of_real (1/2)) by simp
   also have ... = of_real (sqrt pi) by (simp only: Gamma_complex_of_real
Gamma_one_half_real)
  finally show ?thesis .
qed

theorem Gamma_legendre_duplication:
  fixes z :: complex
  assumes z ∉ ℤ≤0 z + 1/2 ∉ ℤ≤0
  shows Gamma z * Gamma (z + 1/2) =
            exp ((1 - 2*z) * of_real (ln 2)) * of_real (sqrt pi) * Gamma (2*z)
 using Gamma_legendre_duplication_aux[OF assms] by (simp add: Gamma_one_half_complex)

end
```

9.10.10 Limits and residues

The inverse of the Gamma function has simple zeros:

lemma *rGamma_zeros*:
(λz. *rGamma z* / (*z* + *of_nat n*)) − (− *of_nat n*) → ((−1)^*n* ∗ *fact n* :: 'a :: *Gamma*)
proof (*subst tendsto_cong*)
let *?f* = λz. *pochhammer z n* ∗ *rGamma* (*z* + *of_nat* (*Suc n*)) :: 'a
from *eventually_at_ball'*[*OF zero_less_one*, *of* − *of_nat n* :: 'a *UNIV*]
show *eventually* (λz. *rGamma z* / (*z* + *of_nat n*) = *?f z*) (*at* (− *of_nat n*))
by (*subst pochhammer_rGamma*[*of* _ *Suc n*])
(*auto elim*!: *eventually_mono simp*: *field_split_simps pochhammer_rec'* *eq_neg_iff_add_eq_0*)
have *isCont ?f* (− *of_nat n*) **by** (*intro continuous_intros*)
thus *?f* − (− *of_nat n*) → (− *1*) ^ *n* ∗ *fact n* **unfolding** *isCont_def*
by (*simp add*: *pochhammer_same*)
qed

The simple zeros of the inverse of the Gamma function correspond to simple poles of the Gamma function, and their residues can easily be computed from the limit we have just proven:

lemma *Gamma_poles*: *filterlim Gamma at_infinity* (*at* (− *of_nat n* :: 'a :: *Gamma*))
proof −
from *eventually_at_ball'*[*OF zero_less_one*, *of* − *of_nat n* :: 'a *UNIV*]
have *eventually* (λz. *rGamma z* ≠ (*0* :: 'a)) (*at* (− *of_nat n*))
by (*auto elim*!: *eventually_mono nonpos_Ints_cases'*
simp: *rGamma_eq_zero_iff dist_of_nat dist_minus*)
with *isCont_rGamma*[*of* − *of_nat n* :: 'a, *OF continuous_ident*]
have *filterlim* (λz. *inverse* (*rGamma z*) :: 'a) *at_infinity* (*at* (− *of_nat n*))
unfolding *isCont_def* **by** (*intro filterlim_compose*[*OF filterlim_inverse_at_infinity*])
(*simp_all add*: *filterlim_at*)
moreover have (λz. *inverse* (*rGamma z*) :: 'a) = *Gamma*
by (*intro ext*) (*simp add*: *rGamma_inverse_Gamma*)
ultimately show *?thesis* **by** (*simp only*:)
qed

lemma *Gamma_residues*:
(λz. *Gamma z* ∗ (*z* + *of_nat n*)) − (− *of_nat n*) → ((−1)^*n* / *fact n* :: 'a :: *Gamma*)
proof (*subst tendsto_cong*)
let *?c* = (− *1*) ^ *n* / *fact n* :: 'a
from *eventually_at_ball'*[*OF zero_less_one*, *of* − *of_nat n* :: 'a *UNIV*]
show *eventually* (λz. *Gamma z* ∗ (*z* + *of_nat n*) = *inverse* (*rGamma z* / (*z* + *of_nat n*)))
(*at* (− *of_nat n*))
by (*auto elim*!: *eventually_mono simp*: *field_split_simps rGamma_inverse_Gamma*)
have (λz. *inverse* (*rGamma z* / (*z* + *of_nat n*))) − (− *of_nat n*) →
inverse ((− *1*) ^ *n* ∗ *fact n* :: 'a)
by (*intro tendsto_intros rGamma_zeros*) *simp_all*

also have *inverse* ((− *1*) ^ *n* ∗ *fact n*) = *?c*
by (*simp_all add*: *field_simps flip*: *power_mult_distrib*)
finally show (λ*z*. *inverse* (*rGamma z* / (*z* + *of_nat n*))) − (− *of_nat n*) → *?c*
.
qed

9.10.11 Alternative definitions

Variant of the Euler form

definition *Gamma_series_euler'* **where**
Gamma_series_euler' z n =
inverse z ∗ (∏ *k*=*1*..*n*. *exp* (*z* ∗ *of_real* (*ln* (*1* + *inverse* (*of_nat k*)))) / (*1* + *z* / *of_nat k*))

context
begin
private lemma *Gamma_euler'_aux1*:
fixes *z* :: *'a* :: {*real_normed_field*,*banach*}
assumes *n*: *n* > *0*
shows *exp* (*z* ∗ *of_real* (*ln* (*of_nat n* + *1*))) = (∏ *k*=*1*..*n*. *exp* (*z* ∗ *of_real* (*ln* (*1* + *1* / *of_nat k*))))
proof −
have (∏ *k*=*1*..*n*. *exp* (*z* ∗ *of_real* (*ln* (*1* + *1* / *of_nat k*)))) =
exp (*z* ∗ *of_real* (∑ *k* = *1*..*n*. *ln* (*1* + *1* / *real_of_nat k*)))
by (*subst exp_sum* [*symmetric*]) (*simp_all add*: *sum_distrib_left*)
also have (∑ *k*=*1*..*n*. *ln* (*1* + *1* / *of_nat k*) :: *real*) = *ln* (∏ *k*=*1*..*n*. *1* + *1* / *real_of_nat k*)
by (*subst ln_prod* [*symmetric*]) (*auto simp*: *divide_simps*)
also have (∏ *k*=*1*..*n*. *1* + *1* / *of_nat k* :: *real*) = (∏ *k*=*1*..*n*. (*of_nat k* + *1*) / *of_nat k*)
by (*intro prod.cong*) (*simp_all add*: *field_split_simps*)
also have (∏ *k*=*1*..*n*. (*of_nat k* + *1*) / *of_nat k* :: *real*) = *of_nat n* + *1*
by (*induction n*) (*simp_all add*: *prod.nat_ivl_Suc' field_split_simps*)
finally show *?thesis* **..**
qed

theorem *Gamma_series_euler'*:
assumes *z*: (*z* :: *'a* :: *Gamma*) ∉ ℤ≤0
shows (λ*n*. *Gamma_series_euler' z n*) ⟶ *Gamma z*
proof (*rule Gamma_seriesI*, *rule Lim_transform_eventually*)
let *?f* = λ*n*. *fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n* + *1*))) / *pochhammer z* (*n* + *1*)
let *?r* = λ*n*. *?f n* / *Gamma_series z n*
let *?r'* = λ*n*. *exp* (*z* ∗ *of_real* (*ln* (*of_nat* (*Suc n*) / *of_nat n*)))
from *z* **have** *z'*: *z* ≠ *0* **by** *auto*

have *eventually* (λ*n*. *?r' n* = *?r n*) *sequentially*
using *z* **by** (*auto simp*: *field_split_simps Gamma_series_def ring_distribs exp_diff ln_div*

```
                intro: eventually_mono eventually_gt_at_top[of 0::nat] dest:
pochhammer_eq_0_imp_nonpos_Int)
  moreover have ?r' ⟶ exp (z * of_real (ln 1))
    by (intro tendsto_intros LIMSEQ_Suc_n_over_n) simp_all
  ultimately show ?r ⟶ 1 by (force intro: Lim_transform_eventually)

  from eventually_gt_at_top[of 0::nat]
    show eventually (λn. ?r n = Gamma_series_euler' z n / Gamma_series z n)
sequentially
  proof eventually_elim
    fix n :: nat assume n: n > 0
    from n z' have Gamma_series_euler' z n =
      exp (z * of_real (ln (of_nat n + 1))) / (z * (∏k=1..n. (1 + z / of_nat k)))
      by (subst Gamma_euler'_aux1)
         (simp_all add: Gamma_series_euler'_def prod.distrib
                        prod_inversef[symmetric] divide_inverse)
    also have (∏k=1..n. (1 + z / of_nat k)) = pochhammer (z + 1) n / fact n
    proof (cases n)
      case (Suc n')
      then show ?thesis
        unfolding pochhammer_prod fact_prod
       by (simp add:  atLeastLessThanSuc_atLeastAtMost field_simps prod_dividef

                  prod.atLeast_Suc_atMost_Suc_shift del: prod.cl_ivl_Suc)
    qed auto
    also have z * ... = pochhammer z (Suc n) / fact n by (simp add: pochham-
mer_rec)
    finally show ?r n = Gamma_series_euler' z n / Gamma_series z n by simp
  qed
qed

end
```

Weierstrass form

```
definition Gamma_series_Weierstrass :: 'a :: {banach,real_normed_field} ⇒ nat
⇒ 'a where
  Gamma_series_Weierstrass z n =
     exp (−euler_mascheroni * z) / z * (∏k=1..n. exp (z / of_nat k) / (1 + z /
of_nat k))

definition
  rGamma_series_Weierstrass :: 'a :: {banach,real_normed_field} ⇒ nat ⇒ 'a
where
  rGamma_series_Weierstrass z n =
     exp (euler_mascheroni * z) * z * (∏k=1..n. (1 + z / of_nat k) * exp (−z /
of_nat k))

lemma Gamma_series_Weierstrass_nonpos_Ints:
```

eventually (λ*k*. *Gamma_series_Weierstrass* (− *of_nat n*) *k* = *0*) *sequentially*
using *eventually_ge_at_top*[*of n*] **by** *eventually_elim* (*auto simp*: *Gamma_series_Weierstrass_def*)

lemma *rGamma_series_Weierstrass_nonpos_Ints*:
eventually (λ*k*. *rGamma_series_Weierstrass* (− *of_nat n*) *k* = *0*) *sequentially*
using *eventually_ge_at_top*[*of n*] **by** *eventually_elim* (*auto simp*: *rGamma_series_Weierstrass_def*)

theorem *Gamma_Weierstrass_complex*: *Gamma_series_Weierstrass z* ⟶
Gamma (*z* :: *complex*)
proof (*cases z* ∈ ℤ≤0)
case *True*
then obtain *n* **where** *z* = − *of_nat n* **by** (*elim nonpos_Ints_cases'*)
also from *True* **have** *Gamma_series_Weierstrass* ... ⟶ *Gamma z*
by (*simp add*: *tendsto_cong*[*OF Gamma_series_Weierstrass_nonpos_Ints*]
Gamma_nonpos_Int)
finally show *?thesis* **.**
next
case *False*
hence *z*: *z* ≠ *0* **by** *auto*
let *?f* = (λ*x*. ∏ *x* = *Suc 0..x*. *exp* (*z* / *of_nat x*) / (*1* + *z* / *of_nat x*))
have *A*: *exp* (*ln* (*1* + *z* / *of_nat n*)) = (*1* + *z* / *of_nat n*) **if** *n* ≥ *1* **for** *n* :: *nat*
using *False that* **by** (*subst exp_Ln*) (*auto simp*: *field_simps dest*!: *plus_of_nat_eq_0_imp*)
have (λ*n*. ∑ *k*=*1..n*. *z* / *of_nat k* − *ln* (*1* + *z* / *of_nat k*)) ⟶ *ln_Gamma*
z + *euler_mascheroni* ∗ *z* + *ln z*
using *ln_Gamma_series'_aux*[*OF False*]
by (*simp only*: *atLeastLessThanSuc_atLeastAtMost* [*symmetric*] *One_nat_def*
sum.shift_bounds_Suc_ivl sums_def atLeast0LessThan)
from *tendsto_exp*[*OF this*] *False z* **have** *?f* ⟶ *z* ∗ *exp* (*euler_mascheroni*
∗ *z*) ∗ *Gamma z*
by (*simp add*: *exp_add exp_sum exp_diff mult_ac Gamma_complex_altdef A*)
from *tendsto_mult*[*OF tendsto_const*[*of exp* (−*euler_mascheroni* ∗ *z*) / *z*] *this*]
z
show *Gamma_series_Weierstrass z* ⟶ *Gamma z*
by (*simp add*: *exp_minus field_split_simps Gamma_series_Weierstrass_def*
[*abs_def*])
qed

lemma *tendsto_complex_of_real_iff*: ((λ*x*. *complex_of_real* (*f x*)) ⟶ *of_real*
c) *F* = (*f* ⟶ *c*) *F*
by (*rule tendsto_of_real_iff*)

lemma *Gamma_Weierstrass_real*: *Gamma_series_Weierstrass x* ⟶ *Gamma*
(*x* :: *real*)
using *Gamma_Weierstrass_complex*[*of of_real x*] **unfolding** *Gamma_series_Weierstrass_def*[*abs_de*
by (*subst tendsto_complex_of_real_iff* [*symmetric*])
(*simp_all add*: *exp_of_real*[*symmetric*] *Gamma_complex_of_real*)

lemma *rGamma_Weierstrass_complex*: *rGamma_series_Weierstrass z* ⟶
rGamma (*z* :: *complex*)

proof (*cases* $z \in \mathbb{Z}_{\leq 0}$)
case *True*
then obtain n **where** $z = -$ *of_nat* n **by** (*elim nonpos_Ints_cases'*)
also from *True* **have** *rGamma_series_Weierstrass* ... $\longrightarrow$ *rGamma* z
by (*simp add*: *tendsto_cong*[*OF rGamma_series_Weierstrass_nonpos_Ints*] *rGamma_nonpos_Int*)
finally show *?thesis* **.**
next
case *False*
have *rGamma_series_Weierstrass* z = (λn. *inverse* (*Gamma_series_Weierstrass* z n))
by (*simp add*: *rGamma_series_Weierstrass_def*[*abs_def*] *Gamma_series_Weierstrass_def* *exp_minus divide_inverse prod_inversef*[*symmetric*] *mult_ac*)
also from *False* **have** ... $\longrightarrow$ *inverse* (*Gamma* z)
by (*intro tendsto_intros Gamma_Weierstrass_complex*) (*simp add*: *Gamma_eq_zero_iff*)
finally show *?thesis* **by** (*simp add*: *Gamma_def*)
qed

Binomial coefficient form

lemma *Gamma_gbinomial*:
(λn. ((z + *of_nat* n) *gchoose* n) * *exp* ($-z$ * *of_real* (*ln* (*of_nat* n)))) $\longrightarrow$ *rGamma* (z+1)
proof (*cases* z = 0)
case *False*
show *?thesis*
proof (*rule Lim_transform_eventually*)
let *?powr* = λa b. *exp* (b * *of_real* (*ln* (*of_nat* a)))
show *eventually* (λn. *rGamma_series* z n / z = ((z + *of_nat* n) *gchoose* n) * *?powr* n ($-z$)) *sequentially*
proof (*intro always_eventually allI*)
fix n :: *nat*
from *False* **have** ((z + *of_nat* n) *gchoose* n) = *pochhammer* z (*Suc* n) / z / *fact* n
by (*simp add*: *gbinomial_pochhammer'* *pochhammer_rec*)
also have *pochhammer* z (*Suc* n) / z / *fact* n * *?powr* n ($-z$) = *rGamma_series* z n / z
by (*simp add*: *rGamma_series_def field_split_simps exp_minus*)
finally show *rGamma_series* z n / z = ((z + *of_nat* n) *gchoose* n) * *?powr* n ($-z$) **..**
qed

from *False* **have** (λn. *rGamma_series* z n / z) $\longrightarrow$ *rGamma* z / z **by** (*intro tendsto_intros*)
also from *False* **have** *rGamma* z / z = *rGamma* (z + 1) **using** *rGamma_plus1*[*of* z]
by (*simp add*: *field_simps*)
finally show (λn. *rGamma_series* z n / z) $\longrightarrow$ *rGamma* (z+1) **.**
qed

qed (*simp_all add*: *binomial_gbinomial* [*symmetric*])

lemma *gbinomial_minus'*: (*a* + *of_nat b*) *gchoose b* = (− *1*) ^ *b* ∗ (− (*a* + *1*) *gchoose b*)
by (*subst gbinomial_minus*) (*simp add*: *power_mult_distrib* [*symmetric*])

lemma *gbinomial_asymptotic*:
fixes *z* :: *'a* :: *Gamma*
shows (λ*n*. (*z gchoose n*) / ((−*1*)^*n* / *exp* ((*z*+*1*) ∗ *of_real* (*ln* (*real n*))))) ⟶
inverse (*Gamma* (− *z*))
unfolding *rGamma_inverse_Gamma* [*symmetric*] **using** *Gamma_gbinomial*[*of* −*z*−*1*]
by (*subst* (*asm*) *gbinomial_minus'*)
(*simp add*: *add_ac mult_ac divide_inverse power_inverse* [*symmetric*])

lemma *fact_binomial_limit*:
(λ*n*. *of_nat* ((*k* + *n*) *choose n*) / *of_nat* (*n* ^ *k*) :: *'a* :: *Gamma*) ⟶ *1* / *fact k*
proof (*rule Lim_transform_eventually*)
have (λ*n*. *of_nat* ((*k* + *n*) *choose n*) / *of_real* (*exp* (*of_nat k* ∗ *ln* (*real_of_nat n*))))
⟶ *1* / *Gamma* (*of_nat* (*Suc k*) :: *'a*) (**is** *?f* ⟶ _)
using *Gamma_gbinomial*[*of of_nat k* :: *'a*]
by (*simp add*: *binomial_gbinomial Gamma_def field_split_simps exp_of_real* [*symmetric*] *exp_minus*)
also have *Gamma* (*of_nat* (*Suc k*)) = *fact k* **by** (*simp add*: *Gamma_fact*)
finally show *?f* ⟶ *1* / *fact k* **.**

show *eventually* (λ*n*. *?f n* = *of_nat* ((*k* + *n*) *choose n*) / *of_nat* (*n* ^ *k*)) *sequentially*
using *eventually_gt_at_top*[*of 0*::*nat*]
proof *eventually_elim*
fix *n* :: *nat* **assume** *n*: *n* > *0*
from *n* **have** *exp* (*real_of_nat k* ∗ *ln* (*real_of_nat n*)) = *real_of_nat* (*n*^*k*)
by (*simp add*: *exp_of_nat_mult*)
thus *?f n* = *of_nat* ((*k* + *n*) *choose n*) / *of_nat* (*n* ^ *k*) **by** *simp*
qed
qed

lemma *binomial_asymptotic'*:
(λ*n*. *of_nat* ((*k* + *n*) *choose n*) / (*of_nat* (*n* ^ *k*) / *fact k*) :: *'a* :: *Gamma*) ⟶ *1*
using *tendsto_mult*[*OF fact_binomial_limit*[*of k*] *tendsto_const*[*of fact k* :: *'a*]]
by *simp*

lemma *gbinomial_Beta*:
assumes *z* + *1* ∉ $\mathbb{Z}_{\leq 0}$
shows ((*z*::*'a*::*Gamma*) *gchoose n*) = *inverse* ((*z* + *1*) ∗ *Beta* (*z* − *of_nat n*

```
+ 1) (of_nat n + 1))
using assms
proof (induction n arbitrary: z)
  case 0
  hence z + 2 ∉ ℤ≤0
    using plus_one_in_nonpos_Ints_imp[of z+1] by (auto simp: add.commute)
  with 0 show ?case
     by (auto simp: Beta_def Gamma_eq_zero_iff Gamma_plus1 [symmetric]
add.commute)
next
  case (Suc n z)
  show ?case
  proof (cases z ∈ ℤ≤0)
    case True
    with Suc.prems have z = 0
    by (auto elim!: nonpos_Ints_cases simp: algebra_simps one_plus_of_int_in_nonpos_Ints_iff)
    show ?thesis
    proof (cases n = 0)
      case True
      with ‹z = 0› show ?thesis
        by (simp add: Beta_def Gamma_eq_zero_iff Gamma_plus1 [symmetric])
    next
      case False
      with ‹z = 0› show ?thesis
        by (simp_all add: Beta_pole1 one_minus_of_nat_in_nonpos_Ints_iff)
    qed
  next
    case False
    have (z gchoose (Suc n)) = ((z - 1 + 1) gchoose (Suc n)) by simp
    also have ... = (z - 1 gchoose n) * ((z - 1) + 1) / of_nat (Suc n)
      by (subst gbinomial_factors) (simp add: field_simps)
    also from False have ... = inverse (of_nat (Suc n) * Beta (z - of_nat n)
(of_nat (Suc n)))
      (is _ = inverse ?x) by (subst Suc.IH) (simp_all add: field_simps Beta_pole1)
   also have of_nat (Suc n) ∉ (ℤ≤0 :: 'a set) by (subst of_nat_in_nonpos_Ints_iff)
simp_all
    hence ?x = (z + 1) * Beta (z - of_nat (Suc n) + 1) (of_nat (Suc n) + 1)
      by (subst Beta_plus1_right [symmetric]) simp_all
    finally show ?thesis .
  qed
qed

theorem gbinomial_Gamma:
  assumes z + 1 ∉ ℤ≤0
  shows   (z gchoose n) = Gamma (z + 1) / (fact n * Gamma (z - of_nat n +
1))
proof -
  have (z gchoose n) = Gamma (z + 2) / (z + 1) / (fact n * Gamma (z - of_nat
n + 1))
```

by (*subst gbinomial_Beta*[*OF assms*]) (*simp_all add*: *Beta_def Gamma_fact* [*symmetric*] *add_ac*)
also from *assms* **have** *Gamma* (*z* + *2*) / (*z* + *1*) = *Gamma* (*z* + *1*)
using *Gamma_plus1*[*of z+1*] **by** (*auto simp add*: *field_split_simps*)
finally show *?thesis* **.**
qed

Integral form

lemma *integrable_on_powr_from_0'*:
assumes *a*: *a* > (−*1*::*real*) **and** *c*: *c* ≥ *0*
shows (λ*x*. *x powr a*) *integrable_on* {*0*<..*c*}
proof −
from *c* **have** ∗: {*0*<..*c*} − {*0*..*c*} = {} {*0*..*c*} − {*0*<..*c*} = {*0*} **by** *auto*
show *?thesis*
by (*rule integrable_spike_set* [*OF integrable_on_powr_from_0*[*OF a c*]]) (*simp_all add*: ∗)
qed

lemma *absolutely_integrable_Gamma_integral*:
assumes *Re z* > *0 a* > *0*
shows (λ*t*. *complex_of_real t powr* (*z* − *1*) / *of_real* (*exp* (*a* ∗ *t*)))
absolutely_integrable_on {*0*<..} (**is** *?f absolutely_integrable_on* _)
proof −
have ((λ*x*. (*Re z* − *1*) ∗ (*ln x* / *x*)) ⟶ (*Re z* − *1*) ∗ *0*) *at_top*
by (*intro tendsto_intros ln_x_over_x_tendsto_0*)
hence ((λ*x*. ((*Re z* − *1*) ∗ *ln x*) / *x*) ⟶ *0*) *at_top* **by** *simp*
from *order_tendstoD*(*2*)[*OF this*, *of a*/*2*] **and** ‹*a* > *0*›
have *eventually* (λ*x*. (*Re z* − *1*) ∗ *ln x* / *x* < *a*/*2*) *at_top* **by** *simp*
from *eventually_conj*[*OF this eventually_gt_at_top*[*of 0*]]
obtain *x0* **where** ∀ *x*≥*x0*. (*Re z* − *1*) ∗ *ln x* / *x* < *a*/*2* ∧ *x* > *0*
by (*auto simp*: *eventually_at_top_linorder*)
hence *x0* > *0* **by** *simp*
have *x powr* (*Re z* − *1*) / *exp* (*a* ∗ *x*) < *exp* (−(*a*/*2*) ∗ *x*) **if** *x* ≥ *x0* **for** *x*
proof −
from *that* **and** ‹∀ *x*≥*x0*. _› **have** *x*: (*Re z* − *1*) ∗ *ln x* / *x* < *a* / *2 x* > *0* **by** *auto*
have *x powr* (*Re z* − *1*) = *exp* ((*Re z* − *1*) ∗ *ln x*)
using ‹*x* > *0*› **by** (*simp add*: *powr_def*)
also from *x* **have** (*Re z* − *1*) ∗ *ln x* < (*a* ∗ *x*) / *2* **by** (*simp add*: *field_simps*)
finally show *?thesis* **by** (*simp add*: *field_simps exp_add* [*symmetric*])
qed
note *x0* = ‹*x0* > *0*› *this*

have *?f absolutely_integrable_on* ({*0*<..*x0*} ∪ {*x0*..})
proof (*rule set_integrable_Un*)
show *?f absolutely_integrable_on* {*0*<..*x0*}
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound* [*OF _ _ AE_I2*])

show *integrable lebesgue* (λ*x*. *indicat_real* {*0*<..*x0*} *x* *$_R$ *x powr* (*Re z* − *1*))

using *x0*(*1*) *assms*
by (*intro nonnegative_absolutely_integrable_1* [*unfolded set_integrable_def*] *integrable_on_powr_from_0'*) *auto*
show (λ*x*. *indicat_real* {*0*<..*x0*} *x* *$_R$ (*x powr* (*z* − *1*) / *exp* (*a* ∗ *x*))) ∈ *borel_measurable lebesgue*
by (*intro measurable_completion*)
(*auto intro*!: *borel_measurable_continuous_on_indicator continuous_intros*)
fix *x* :: *real*
have *x powr* (*Re z* − *1*) / *exp* (*a* ∗ *x*) ≤ *x powr* (*Re z* − *1*) / *1* **if** *x* ≥ *0*
using *that assms* **by** (*intro divide_left_mono*) *auto*
thus *norm* (*indicator* {*0*<..*x0*} *x* *$_R$ *?f x*) ≤
norm (*indicator* {*0*<..*x0*} *x* *$_R$ *x powr* (*Re z* − *1*))
by (*simp_all add*: *norm_divide norm_powr_real_powr indicator_def*)
qed
next
show *?f absolutely_integrable_on* {*x0*..}
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound* [*OF _ _ AE_I2*])
show *integrable lebesgue* (λ*x*. *indicat_real* {*x0*..} *x* *$_R$ *exp* (− (*a* / *2*) ∗ *x*))
using *assms*
by (*intro nonnegative_absolutely_integrable_1* [*unfolded set_integrable_def*] *integrable_on_exp_minus_to_infinity*) *auto*
show (λ*x*. *indicat_real* {*x0*..} *x* *$_R$ (*x powr* (*z* − *1*) / *exp* (*a* ∗ *x*))) ∈ *borel_measurable lebesgue* **using** *x0*(*1*)
by (*intro measurable_completion*)
(*auto intro*!: *borel_measurable_continuous_on_indicator continuous_intros*)
fix *x* :: *real*
show *norm* (*indicator* {*x0*..} *x* *$_R$ *?f x*) ≤
norm (*indicator* {*x0*..} *x* *$_R$ *exp* (−(*a*/*2*) ∗ *x*)) **using** *x0*
by (*auto simp*: *norm_divide norm_powr_real_powr indicator_def less_imp_le*)
qed
qed *auto*
also have {*0*<..*x0*} ∪ {*x0*..} = {*0*<..} **using** *x0*(*1*) **by** *auto*
finally show *?thesis* **.**
qed

lemma *integrable_Gamma_integral_bound*:
fixes *a c* :: *real*
assumes *a*: *a* > −*1* **and** *c*: *c* ≥ *0*
defines *f* ≡ λ*x*. *if x* ∈ {*0*..*c*} *then x powr a else exp* (−*x*/*2*)
shows *f integrable_on* {*0*..}
proof −
have *f integrable_on* {*0*..*c*}
by (*rule integrable_spike_finite*[*of* {}, *OF _ _ integrable_on_powr_from_0*[*of a c*]])
(*insert a c*, *simp_all add*: *f_def*)
moreover have *A*: (λ*x*. *exp* (−*x*/*2*)) *integrable_on* {*c*..}

```
    using integrable_on_exp_minus_to_infinity[of 1/2] by simp
  have f integrable_on {c..}
    by (rule integrable_spike_finite[of {c}, OF _ _ A]) (simp_all add: f_def)
  ultimately show f integrable_on {0..}
    by (rule integrable_Un') (insert c, auto simp: max_def)
qed

theorem Gamma_integral_complex:
  assumes z: Re z > 0
  shows   ((λt. of_real t powr (z − 1) / of_real (exp t)) has_integral Gamma z)
{0..}
proof −
  have A: ((λt. (of_real t) powr (z − 1) ∗ of_real ((1 − t) ^ n))
        has_integral (fact n / pochhammer z (n+1))) {0..1}
    if Re z > 0 for n z using that
  proof (induction n arbitrary: z)
    case 0
    have ((λt. complex_of_real t powr (z − 1)) has_integral
          (of_real 1 powr z / z − of_real 0 powr z / z)) {0..1} using 0
      by (intro fundamental_theorem_of_calculus_interior)
       (auto intro!: continuous_intros derivative_eq_intros has_vector_derivative_real_field)
    thus ?case by simp
  next
    case (Suc n)
    let ?f = λt. complex_of_real t powr z / z
    let ?f' = λt. complex_of_real t powr (z − 1)
    let ?g = λt. (1 − complex_of_real t) ^ Suc n
    let ?g' = λt. − ((1 − complex_of_real t) ^ n) ∗ of_nat (Suc n)
    have ((λt. ?f' t ∗ ?g t) has_integral
          (of_nat (Suc n)) ∗ fact n / pochhammer z (n+2)) {0..1}
      (is (_ has_integral ?I) _)
    proof (rule integration_by_parts_interior[where f' = ?f' and g = ?g])
      from Suc.prems show continuous_on {0..1} ?f continuous_on {0..1} ?g
        by (auto intro!: continuous_intros)
    next
      fix t :: real assume t: t ∈ {0<..<1}
      show (?f has_vector_derivative ?f' t) (at t) using t Suc.prems
        by (auto intro!: derivative_eq_intros has_vector_derivative_real_field)
      show (?g has_vector_derivative ?g' t) (at t)
          by (rule has_vector_derivative_real_field derivative_eq_intros refl)+
simp_all
    next
      from Suc.prems have [simp]: z ≠ 0 by auto
      from Suc.prems have A: Re (z + of_nat n) > 0 for n by simp
      have [simp]: z + of_nat n ≠ 0 z + 1 + of_nat n ≠ 0 for n
      using A[of n] A[of Suc n] by (auto simp add: add.assoc simp del: plus_complex.sel)
      have ((λx. of_real x powr z ∗ of_real ((1 − x) ^ n) ∗ (− of_nat (Suc n) /
z)) has_integral
            fact n / pochhammer (z+1) (n+1) ∗ (− of_nat (Suc n) / z)) {0..1}
```

```
      (is (?A has_integral ?B) _)
    using Suc.IH[of z+1] Suc.prems by (intro has_integral_mult_left) (simp_all
add: add_ac pochhammer_rec)
    also have ?A = (λt. ?f t * ?g' t) by (intro ext) (simp_all add: field_simps)
    also have ?B = − (of_nat (Suc n) * fact n / pochhammer z (n+2))
      by (simp add: field_split_simps pochhammer_rec
          prod.shift_bounds_cl_Suc_ivl del: of_nat_Suc)
    finally show ((λt. ?f t * ?g' t) has_integral (?f 1 * ?g 1 − ?f 0 * ?g 0 −
?I)) {0..1}
      by simp
  qed (simp_all add: bounded_bilinear_mult)
  thus ?case by simp
 qed

 have B: ((λt. if t ∈ {0..of_nat n} then
          of_real t powr (z − 1) * (1 − of_real t / of_nat n) ^ n else 0)
         has_integral (of_nat n powr z * fact n / pochhammer z (n+1))) {0..}
for n
 proof (cases n > 0)
  case [simp]: True
  hence [simp]: n ≠ 0 by auto
  with has_integral_affinity01[OF A[OF z, of n], of inverse (of_nat n) 0]
    have ((λx. (of_nat n − of_real x) ^ n * (of_real x / of_nat n) powr (z −
1) / of_nat n ^ n)
          has_integral fact n * of_nat n / pochhammer z (n+1)) ((λx. real n *
x) ‘ {0..1})
    (is (?f has_integral ?I) ?ivl) by (simp add: field_simps scaleR_conv_of_real)
  also from True have ((λx. real n*x) ‘ {0..1}) = {0..real n}
    by (subst image_mult_atLeastAtMost) simp_all
  also have ?f = (λx. (of_real x / of_nat n) powr (z − 1) * (1 − of_real x /
of_nat n) ^ n)
    using True by (intro ext) (simp add: field_simps)
  finally have ((λx. (of_real x / of_nat n) powr (z − 1) * (1 − of_real x /
of_nat n) ^ n)
                has_integral ?I) {0..real n} (is ?P) .
  also have ?P ⟷ ((λx. exp ((z − 1) * of_real (ln (x / of_nat n))) * (1 −
of_real x / of_nat n) ^ n)
                  has_integral ?I) {0..real n}
  by (intro has_integral_spike_finite_eq[of {0}]) (auto simp: powr_def Ln_of_real
[symmetric])
  also have ... ⟷ ((λx. exp ((z − 1) * of_real (ln x − ln (of_nat n))) * (1
− of_real x / of_nat n) ^ n)
                  has_integral ?I) {0..real n}
    by (intro has_integral_spike_finite_eq[of {0}]) (simp_all add: ln_div)
  finally have ... .
  note B = has_integral_mult_right[OF this, of exp ((z − 1) * ln (of_nat n))]
  have ((λx. exp ((z − 1) * of_real (ln x)) * (1 − of_real x / of_nat n) ^ n)
        has_integral (?I * exp ((z − 1) * ln (of_nat n)))) {0..real n} (is ?P)
    by (insert B, subst (asm) mult.assoc [symmetric], subst (asm) exp_add
```

```
[symmetric])
        (simp add: algebra_simps)
    also have ?P ⟷ ((λx. of_real x powr (z − 1) ∗ (1 − of_real x / of_nat n)
^ n)
           has_integral (?I ∗ exp ((z − 1) ∗ ln (of_nat n)))) {0..real n}
       by (intro has_integral_spike_finite_eq[of {0}]) (simp_all add: powr_def
Ln_of_real)
     also have fact n ∗ of_nat n / pochhammer z (n+1) ∗ exp ((z − 1) ∗ Ln
(of_nat n)) =
                (of_nat n powr z ∗ fact n / pochhammer z (n+1))
      by (auto simp add: powr_def algebra_simps exp_diff exp_of_real)
    finally show ?thesis by (subst has_integral_restrict) simp_all
  next
    case False
   thus ?thesis by (subst has_integral_restrict) (simp_all add: has_integral_refl)
  qed

  have eventually (λn. Gamma_series z n =
         of_nat n powr z ∗ fact n / pochhammer z (n+1)) sequentially
    using eventually_gt_at_top[of 0::nat]
    by eventually_elim (simp add: powr_def algebra_simps Gamma_series_def)
  from this and Gamma_series_LIMSEQ[of z]
   have C: (λk. of_nat k powr z ∗ fact k / pochhammer z (k+1)) ⟶ Gamma
z
    by (blast intro: Lim_transform_eventually)
  {
    fix x :: real assume x: x ≥ 0
    have lim_exp: (λk. (1 − x / real k) ^ k) ⟶ exp (−x)
      using tendsto_exp_limit_sequentially[of −x] by simp
    have (λk. of_real x powr (z − 1) ∗ of_real ((1 − x / of_nat k) ^ k))
          ⟶ of_real x powr (z − 1) ∗ of_real (exp (−x)) (is ?P)
      by (intro tendsto_intros lim_exp)
    also from eventually_gt_at_top[of nat ⌈x⌉]
      have eventually (λk. of_nat k > x) sequentially by eventually_elim linarith
    hence ?P ⟷ (λk. if x ≤ of_nat k then
                of_real x powr (z − 1) ∗ of_real ((1 − x / of_nat k) ^ k) else 0)
                  ⟶ of_real x powr (z − 1) ∗ of_real (exp (−x))
      by (intro tendsto_cong) (auto elim!: eventually_mono)
    finally have ... .
  }
  hence D: ∀x∈{0..}. (λk. if x ∈ {0..real k} then
             of_real x powr (z − 1) ∗ (1 − of_real x / of_nat k) ^ k else 0)
            ⟶ of_real x powr (z − 1) / of_real (exp x)
    by (simp add: exp_minus field_simps cong: if_cong)

  have ((λx. (Re z − 1) ∗ (ln x / x)) ⟶ (Re z − 1) ∗ 0) at_top
    by (intro tendsto_intros ln_x_over_x_tendsto_0)
  hence ((λx. ((Re z − 1) ∗ ln x) / x) ⟶ 0) at_top by simp
  from order_tendstoD(2)[OF this, of 1/2]
```

```
    have eventually (λx. (Re z − 1) * ln x / x < 1/2) at_top by simp
  from eventually_conj[OF this eventually_gt_at_top[of 0]]
    obtain x0 where ∀x≥x0. (Re z − 1) * ln x / x < 1/2 ∧ x > 0
    by (auto simp: eventually_at_top_linorder)
  hence x0: x0 > 0 ⋀x. x ≥ x0 ⟹ (Re z − 1) * ln x < x / 2 by auto

  define h where h = (λx. if x ∈ {0..x0} then x powr (Re z − 1) else exp (−x/2))
  have le_h: x powr (Re z − 1) * exp (−x) ≤ h x if x: x ≥ 0 for x
  proof (cases x > x0)
    case True
    from True x0(1) have x powr (Re z − 1) * exp (−x) = exp ((Re z − 1) * ln
x − x)
      by (simp add: powr_def exp_diff exp_minus field_simps exp_add)
    also from x0(2)[of x] True have ... < exp (−x/2)
      by (simp add: field_simps)
    finally show ?thesis using True by (auto simp add: h_def)
  next
    case False
    from x have x powr (Re z − 1) * exp (− x) ≤ x powr (Re z − 1) * 1
      by (intro mult_left_mono) simp_all
    with False show ?thesis by (auto simp add: h_def)
  qed

  have E: ∀x∈{0..}. cmod (if x ∈ {0..real k} then of_real x powr (z − 1) *
                (1 − complex_of_real x / of_nat k) ^ k else 0) ≤ h x
    (is ∀x∈_. ?f x ≤ _) for k
  proof safe
    fix x :: real assume x: x ≥ 0
    {
      fix x :: real and n :: nat assume x: x ≤ of_nat n
       have (1 − complex_of_real x / of_nat n) = complex_of_real ((1 − x /
of_nat n)) by simp
      also have norm ... = |(1 − x / real n)| by (subst norm_of_real) (rule refl)
     also from x have ... = (1 − x / real n) by (intro abs_of_nonneg) (simp_all
add: field_split_simps)
      finally have cmod (1 − complex_of_real x / of_nat n) = 1 − x / real n .
    } note D = this
    from D[of x k] x
      have ?f x ≤ (if of_nat k ≥ x ∧ k > 0 then x powr (Re z − 1) * (1 − x /
real k) ^ k else 0)
      by (auto simp: norm_mult norm_powr_real_powr norm_power intro!: mult_nonneg_nonneg)
    also have ... ≤ x powr (Re z − 1) * exp (−x)
      by (auto intro!: mult_left_mono exp_ge_one_minus_x_over_n_power_n)
    also from x have ... ≤ h x by (rule le_h)
    finally show ?f x ≤ h x .
  qed

  have F: h integrable_on {0..} unfolding h_def
    by (rule integrable_Gamma_integral_bound) (insert assms x0(1), simp_all)
```

```
  show ?thesis
    by (rule has_integral_dominated_convergence[OF B F E D C])
qed

lemma Gamma_integral_real:
  assumes x: x > (0 :: real)
  shows   ((λt. t powr (x - 1) / exp t) has_integral Gamma x) {0..}
proof -
  have A: ((λt. complex_of_real t powr (complex_of_real x - 1) /
        complex_of_real (exp t)) has_integral complex_of_real (Gamma x)) {0..}
    using Gamma_integral_complex[of x] assms by (simp_all add: Gamma_complex_of_real
powr_of_real)
    have ((λt. complex_of_real (t powr (x - 1) / exp t)) has_integral of_real
(Gamma x)) {0..}
    by (rule has_integral_eq[OF _ A]) (simp_all add: powr_of_real [symmetric])
  from has_integral_linear[OF this bounded_linear_Re] show ?thesis by (simp
add: o_def)
qed

lemma absolutely_integrable_Gamma_integral':
  assumes Re z > 0
 shows  (λt. complex_of_real t powr (z - 1) / of_real (exp t)) absolutely_integrable_on
{0<..}
 using absolutely_integrable_Gamma_integral [OF assms zero_less_one] by simp

lemma Gamma_integral_complex':
  assumes z: Re z > 0
  shows   ((λt. of_real t powr (z - 1) / of_real (exp t)) has_integral Gamma z)
{0<..}
proof -
  have ((λt. of_real t powr (z - 1) / of_real (exp t)) has_integral Gamma z)
{0..}
    by (rule Gamma_integral_complex) fact+
  hence ((λt. if t ∈ {0<..} then of_real t powr (z - 1) / of_real (exp t) else 0)
          has_integral Gamma z) {0..}
    by (rule has_integral_spike [of {0}, rotated 2]) auto
  also have ?this = ?thesis
    by (subst has_integral_restrict) auto
  finally show ?thesis .
qed

lemma Gamma_conv_nn_integral_real:
  assumes s > (0::real)
  shows   Gamma s = nn_integral lborel (λt. ennreal (indicator {0..} t * t powr
(s - 1) / exp t))
 using nn_integral_has_integral_lebesgue[OF _ Gamma_integral_real[OF assms]]
by simp

lemma integrable_Beta:
```

assumes $a > 0$ $b > (0::real)$
shows *set_integrable lborel* $\{0..1\}$ $(\lambda t.\ t\ powr\ (a - 1) * (1 - t)\ powr\ (b - 1))$
proof −
define C **where** $C = max\ 1\ ((1/2)\ powr\ (b - 1))$
define D **where** $D = max\ 1\ ((1/2)\ powr\ (a - 1))$
have C: $(1 - x)\ powr\ (b - 1) \leq C$ **if** $x \in \{0..1/2\}$ **for** x
proof (*cases* $b < 1$)
case *False*
with *that* **have** $(1 - x)\ powr\ (b - 1) \leq (1\ powr\ (b - 1))$ **by** (*intro powr_mono2*) *auto*
thus *?thesis* **by** (*auto simp*: *C_def*)
qed (*insert that*, *auto simp*: *max.coboundedI1 max.coboundedI2 powr_mono2' powr_mono2 C_def*)
have D: $x\ powr\ (a - 1) \leq D$ **if** $x \in \{1/2..1\}$ **for** x
proof (*cases* $a < 1$)
case *False*
with *that* **have** $x\ powr\ (a - 1) \leq (1\ powr\ (a - 1))$ **by** (*intro powr_mono2*) *auto*
thus *?thesis* **by** (*auto simp*: *D_def*)
next
case *True*
qed (*insert that*, *auto simp*: *max.coboundedI1 max.coboundedI2 powr_mono2' powr_mono2 D_def*)
have [*simp*]: $C \geq 0$ $D \geq 0$ **by** (*simp_all add*: *C_def D_def*)

have *I1*: *set_integrable lborel* $\{0..1/2\}$ $(\lambda t.\ t\ powr\ (a - 1) * (1 - t)\ powr\ (b - 1))$
unfolding *set_integrable_def*
proof (*rule Bochner_Integration.integrable_bound*[*OF _ _ AE_I2*])
have $(\lambda t.\ t\ powr\ (a - 1))$ *integrable_on* $\{0..1/2\}$
by (*rule integrable_on_powr_from_0*) (*use assms* **in** *auto*)
hence $(\lambda t.\ t\ powr\ (a - 1))$ *absolutely_integrable_on* $\{0..1/2\}$
by (*subst absolutely_integrable_on_iff_nonneg*) *auto*
from *integrable_mult_right*[*OF this* [*unfolded set_integrable_def*], *of C*]
show *integrable lborel* $(\lambda x.\ indicat\_real\ \{0..1/2\}\ x *_R (C * x\ powr\ (a - 1)))$
by (*subst* (*asm*) *integrable_completion*) (*auto simp*: *mult_ac*)
next
fix x :: *real*
have $x\ powr\ (a - 1) * (1 - x)\ powr\ (b - 1) \leq x\ powr\ (a - 1) * C$ **if** $x \in \{0..1/2\}$
using *that* **by** (*intro mult_left_mono powr_mono2 C*) *auto*
thus *norm* (*indicator* $\{0..1/2\}$ $x *_R (x\ powr\ (a - 1) * (1 - x)\ powr\ (b - 1))) \leq$
norm (*indicator* $\{0..1/2\}$ $x *_R (C * x\ powr\ (a - 1)))$
by (*auto simp*: *indicator_def abs_mult mult_ac*)
qed (*auto intro*!: *AE_I2 simp*: *indicator_def*)

have *I2*: *set_integrable lborel* $\{1/2..1\}$ $(\lambda t.\ t\ powr\ (a - 1) * (1 - t)\ powr\ (b$

```
− 1))
    unfolding set_integrable_def
  proof (rule Bochner_Integration.integrable_bound[OF _ _ AE_I2])
    have (λt. t powr (b − 1)) integrable_on {0..1/2}
      by (rule integrable_on_powr_from_0) (use assms in auto)
    hence (λt. t powr (b − 1)) integrable_on (cbox 0 (1/2)) by simp
    from integrable_affinity[OF this, of −1 1]
      have (λt. (1 − t) powr (b − 1)) integrable_on {1/2..1} by simp
    hence (λt. (1 − t) powr (b − 1)) absolutely_integrable_on {1/2..1}
      by (subst absolutely_integrable_on_iff_nonneg) auto
    from integrable_mult_right[OF this [unfolded set_integrable_def], of D]
    show integrable lborel (λx. indicat_real {1/2..1} x *R (D * (1 − x) powr (b
− 1)))
      by (subst (asm) integrable_completion) (auto simp: mult_ac)
  next
    fix x :: real
    have x powr (a − 1) * (1 − x) powr (b − 1) ≤ D * (1 − x) powr (b − 1) if
x ∈ {1/2..1}
      using that by (intro mult_right_mono powr_mono2 D) auto
    thus norm (indicator {1/2..1} x *R (x powr (a − 1) * (1 − x) powr (b −
1))) ≤
           norm (indicator {1/2..1} x *R (D * (1 − x) powr (b − 1)))
      by (auto simp: indicator_def abs_mult mult_ac)
  qed (auto intro!: AE_I2 simp: indicator_def)

  have set_integrable lborel ({0..1/2} ∪ {1/2..1}) (λt. t powr (a − 1) * (1 − t)
powr (b − 1))
    by (intro set_integrable_Un I1 I2) auto
  also have {0..1/2} ∪ {1/2..1} = {0..(1::real)} by auto
  finally show ?thesis .
qed

lemma integrable_Beta′:
  assumes a > 0 b > (0::real)
  shows   (λt. t powr (a − 1) * (1 − t) powr (b − 1)) integrable_on {0..1}
  using integrable_Beta[OF assms] by (rule set_borel_integral_eq_integral)

theorem has_integral_Beta_real:
  assumes a: a > 0 and b: b > (0 :: real)
  shows ((λt. t powr (a − 1) * (1 − t) powr (b − 1)) has_integral Beta a b)
{0..1}
proof −
  define B where B = integral {0..1} (λx. x powr (a − 1) * (1 − x) powr (b −
1))
  have [simp]: B ≥ 0 unfolding B_def using a b
    by (intro integral_nonneg integrable_Beta′) auto
  from a b have ennreal (Gamma a * Gamma b) =
    (∫+ t. ennreal (indicator {0..} t * t powr (a − 1) / exp t) ∂lborel) *
    (∫+ t. ennreal (indicator {0..} t * t powr (b − 1) / exp t) ∂lborel)
```

by (*subst ennreal_mult'*) (*simp_all add*: *Gamma_conv_nn_integral_real*)
also have ... = ($\int^+ t.$ $\int^+ u.$ *ennreal* (*indicator* {*0*..} *t* * *t powr* (*a* − *1*) / *exp t*) *
ennreal (*indicator* {*0*..} *u* * *u powr* (*b* − *1*) / *exp u*) ∂*lborel* ∂*lborel*)
by (*simp add*: *nn_integral_cmult nn_integral_multc*)
also have ... = ($\int^+ t.$ $\int^+ u.$ *ennreal* (*indicator* ({*0*..}×{*0*..}) (*t*,*u*) * *t powr* (*a* − *1*) * *u powr* (*b* − *1*)
/ *exp* (*t* + *u*)) ∂*lborel* ∂*lborel*)
by (*intro nn_integral_cong*)
(*auto simp*: *indicator_def divide_ennreal ennreal_mult'* [*symmetric*] *exp_add*)
also have ... = ($\int^+ t.$ $\int^+ u.$ *ennreal* (*indicator* ({*0*..}×{*t*..}) (*t*,*u*) * *t powr* (*a* − *1*) *
(*u* − *t*) *powr* (*b* − *1*) / *exp u*) ∂*lborel* ∂*lborel*)
proof (*rule nn_integral_cong*, *goal_cases*)
case (*1 t*)
have ($\int^+ u.$ *ennreal* (*indicator* ({*0*..}×{*0*..}) (*t*,*u*) * *t powr* (*a* − *1*) *
u powr (*b* − *1*) / *exp* (*t* + *u*)) ∂*distr lborel borel* ((+) (−*t*))) =
($\int^+ u.$ *ennreal* (*indicator* ({*0*..}×{*t*..}) (*t*,*u*) * *t powr* (*a* − *1*) *
(*u* − *t*) *powr* (*b* − *1*) / *exp u*) ∂*lborel*)
by (*subst nn_integral_distr*) (*auto intro*!: *nn_integral_cong simp*: *indicator_def*)
thus *?case* **by** (*subst* (*asm*) *lborel_distr_plus*)
qed
also have ... = ($\int^+ u.$ $\int^+ t.$ *ennreal* (*indicator* ({*0*..}×{*t*..}) (*t*,*u*) * *t powr* (*a* − *1*) *
(*u* − *t*) *powr* (*b* − *1*) / *exp u*) ∂*lborel* ∂*lborel*)
by (*subst lborel_pair.Fubini'*)
(*auto simp*: *case_prod_unfold indicator_def cong*: *measurable_cong_sets*)
also have ... = ($\int^+ u.$ $\int^+ t.$ *ennreal* (*indicator* {*0*..*u*} *t* * *t powr* (*a* − *1*) * (*u* − *t*) *powr* (*b* − *1*)) *
ennreal (*indicator* {*0*..} *u* / *exp u*) ∂*lborel* ∂*lborel*)
by (*intro nn_integral_cong*) (*auto simp*: *indicator_def ennreal_mult'* [*symmetric*])
also have ... = ($\int^+ u.$ ($\int^+ t.$ *ennreal* (*indicator* {*0*..*u*} *t* * *t powr* (*a* − *1*) * (*u* − *t*) *powr* (*b* − *1*))
∂*lborel*) * *ennreal* (*indicator* {*0*..} *u* / *exp u*) ∂*lborel*)
by (*subst nn_integral_multc* [*symmetric*]) *auto*
also have ... = ($\int^+ u.$ ($\int^+ t.$ *ennreal* (*indicator* {*0*..*u*} *t* * *t powr* (*a* − *1*) * (*u* − *t*) *powr* (*b* − *1*))
∂*lborel*) * *ennreal* (*indicator* {*0*<..} *u* / *exp u*) ∂*lborel*)
by (*intro nn_integral_cong_AE eventually_mono*[*OF AE_lborel_singleton*[*of 0*]])
(*auto simp*: *indicator_def*)
also have ... = ($\int^+ u.$ *ennreal B* * *ennreal* (*indicator* {*0*..} *u* / *exp u* * *u powr* (*a* + *b* − *1*)) ∂*lborel*)
proof (*intro nn_integral_cong*, *goal_cases*)
case (*1 u*)
show *?case*

```
    proof (cases u > 0)
      case True
      have (∫⁺t. ennreal (indicator {0..u} t * t powr (a − 1) * (u − t) powr (b
− 1)) ∂lborel) =
                (∫⁺t. ennreal (indicator {0..1} t * (u * t) powr (a − 1) * (u − u *
t) powr (b − 1))
                  ∂distr lborel borel ((*) (1 / u))) (is _ = nn_integral _ ?f)
        using True
        by (subst nn_integral_distr) (auto simp: indicator_def field_simps intro!:
nn_integral_cong)
      also have distr lborel borel ((*) (1 / u)) = density lborel (λ_. u)
        using ‹u > 0› by (subst lborel_distr_mult) auto
      also have nn_integral ... ?f = (∫⁺x. ennreal (indicator {0..1} x * (u * (u
* x) powr (a − 1) *
                                                  (u * (1 − x)) powr (b − 1))) ∂lborel) using
‹u > 0›
          by (subst nn_integral_density) (auto simp: ennreal_mult' [symmetric]
algebra_simps)
      also have ... = (∫⁺x. ennreal (u powr (a + b − 1)) *
                          ennreal (indicator {0..1} x * x powr (a − 1) *
                                   (1 − x) powr (b − 1)) ∂lborel) using ‹u > 0› a b
        by (intro nn_integral_cong)
           (auto simp: indicator_def powr_mult powr_add powr_diff mult_ac
ennreal_mult' [symmetric])
      also have ... = ennreal (u powr (a + b − 1)) *
                      (∫⁺x. ennreal (indicator {0..1} x * x powr (a − 1) *
                                     (1 − x) powr (b − 1)) ∂lborel)
        by (subst nn_integral_cmult) auto
      also have ((λx. x powr (a − 1) * (1 − x) powr (b − 1)) has_integral
                  integral {0..1} (λx. x powr (a − 1) * (1 − x) powr (b − 1)))
{0..1}
        using a b by (intro integrable_integral integrable_Beta')
      from nn_integral_has_integral_lebesgue[OF _ this] a b
        have (∫⁺x. ennreal (indicator {0..1} x * x powr (a − 1) *
                            (1 − x) powr (b − 1)) ∂lborel) = B by (simp add: mult_ac
B_def)
      finally show ?thesis using ‹u > 0› by (simp add: ennreal_mult' [symmetric]
mult_ac)
    qed auto
  qed
  also have ... = ennreal B * ennreal (Gamma (a + b))
    using a b by (subst nn_integral_cmult) (auto simp: Gamma_conv_nn_integral_real)
  also have ... = ennreal (B * Gamma (a + b))
    by (subst (1 2) mult.commute, intro ennreal_mult' [symmetric]) (use a b in
auto)
  finally have B = Beta a b using a b Gamma_real_pos[of a + b]
    by (subst (asm) ennreal_inj) (auto simp: field_simps Beta_def Gamma_eq_zero_iff)
  moreover have (λt. t powr (a − 1) * (1 − t) powr (b − 1)) integrable_on
{0..1}
```

by (*intro integrable_Beta′ a b*)
ultimately show *?thesis* **by** (*simp add*: *has_integral_iff B_def*)
qed

9.10.12 The Weierstraß product formula for the sine

theorem *sin_product_formula_complex*:
fixes *z* :: *complex*
shows (λ*n*. *of_real pi* ∗ *z* ∗ (∏ *k*=*1*..*n*. *1* − *z*^*2* / *of_nat k*^*2*)) ⟶ *sin* (*of_real pi* ∗ *z*)
proof −
let *?f* = *rGamma_series_Weierstrass*
have (λ*n*. (− *of_real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (− *z*) *n*))
⟶ (− *of_real pi* ∗ *inverse z*) ∗ (*rGamma z* ∗ *rGamma* (− *z*))
by (*intro tendsto_intros rGamma_Weierstrass_complex*)
also have (λ*n*. (− *of_real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (−*z*) *n*)) =
(λ*n*. *of_real pi* ∗ *z* ∗ (∏ *k*=*1*..*n*. *1* − *z*^*2* / *of_nat k* ^ *2*))
proof
fix *n* :: *nat*
have (− *of_real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (−*z*) *n*) =
of_real pi ∗ *z* ∗ (∏ *k*=*1*..*n*. (*of_nat k* − *z*) ∗ (*of_nat k* + *z*) / *of_nat k* ^ *2*)
by (*simp add*: *rGamma_series_Weierstrass_def mult_ac exp_minus*
divide_simps prod.distrib[*symmetric*] *power2_eq_square*)
also have (∏ *k*=*1*..*n*. (*of_nat k* − *z*) ∗ (*of_nat k* + *z*) / *of_nat k* ^ *2*) =
(∏ *k*=*1*..*n*. *1* − *z*^*2* / *of_nat k* ^ *2*)
by (*intro prod.cong*) (*simp_all add*: *power2_eq_square field_simps*)
finally show (− *of_real pi* ∗ *inverse z*) ∗ (*?f z n* ∗ *?f* (−*z*) *n*) = *of_real pi* ∗ *z* ∗ ...
by (*simp add*: *field_split_simps*)
qed
also have (− *of_real pi* ∗ *inverse z*) ∗ (*rGamma z* ∗ *rGamma* (− *z*)) = *sin* (*of_real pi* ∗ *z*)
by (*subst rGamma_reflection_complex′*) (*simp add*: *field_split_simps*)
finally show *?thesis* **.**
qed

lemma *sin_product_formula_real*:
(λ*n*. *pi* ∗ (*x*::*real*) ∗ (∏ *k*=*1*..*n*. *1* − *x*^*2* / *of_nat k*^*2*)) ⟶ *sin* (*pi* ∗ *x*)
proof −
from *sin_product_formula_complex*[*of of_real x*]
have (λ*n*. *of_real pi* ∗ *of_real x* ∗ (∏ *k*=*1*..*n*. *1* − (*of_real x*)^*2* / (*of_nat k*)^*2*))
⟶ *sin* (*of_real pi* ∗ *of_real x* :: *complex*) (**is** *?f* ⟶ *?y*) **.**
also have *?f* = (λ*n*. *of_real* (*pi* ∗ *x* ∗ (∏ *k*=*1*..*n*. *1* − *x*^*2* / (*of_nat k*^*2*))))
by *simp*
also have *?y* = *of_real* (*sin* (*pi* ∗ *x*)) **by** (*simp only*: *sin_of_real* [*symmetric*] *of_real_mult*)
finally show *?thesis* **by** (*subst* (*asm*) *tendsto_of_real_iff*)

qed

lemma *sin_product_formula_real'*:
assumes $x \neq (0::real)$
shows $(\lambda n.\ (\prod k{=}1..n.\ 1 - x\hat{}2\ /\ of\_nat\ k\hat{}2)) \longrightarrow sin\ (pi * x)\ /\ (pi * x)$
using *tendsto_divide*[*OF sin_product_formula_real*[*of x*] *tendsto_const*[*of pi * x*]] *assms*
by *simp*

theorem *wallis*: $(\lambda n.\ \prod k{=}1..n.\ (4{*}real\ k\hat{}2)\ /\ (4{*}real\ k\hat{}2 - 1)) \longrightarrow pi\ /\ 2$
proof −
from *tendsto_inverse*[*OF tendsto_mult*[*OF*
sin_product_formula_real[*of 1/2*] *tendsto_const*[*of 2/pi*]]]
have $(\lambda n.\ (\prod k{=}1..n.\ inverse\ (1 - (1/2)^2\ /\ (real\ k)^2))) \longrightarrow pi/2$
by (*simp add*: *prod_inversef* [*symmetric*])
also have $(\lambda n.\ (\prod k{=}1..n.\ inverse\ (1 - (1/2)^2\ /\ (real\ k)^2))) =$
$(\lambda n.\ (\prod k{=}1..n.\ (4{*}real\ k\hat{}2)/(4{*}real\ k\hat{}2 - 1)))$
by (*intro ext prod.cong refl*) (*simp add*: *field_split_simps*)
finally show *?thesis* **.**
qed

9.10.13 The Solution to the Basel problem

theorem *inverse_squares_sums*: $(\lambda n.\ 1\ /\ (n + 1)^2)\ sums\ (pi^2\ /\ 6)$
proof −
define *P* **where** $P\ x\ n = (\prod k{=}1..n.\ 1 - x\hat{}2\ /\ of\_nat\ k\hat{}2)$ **for** $x :: real$ **and** *n*
define *K* **where** $K = (\sum n.\ inverse\ (real\_of\_nat\ (Suc\ n))\hat{}2)$
define *f* **where** [*abs_def*]: $f\ x = (\sum n.\ P\ x\ n\ /\ of\_nat\ (Suc\ n)\hat{}2)$ **for** *x*
define *g* **where** [*abs_def*]: $g\ x = (1 - sin\ (pi * x)\ /\ (pi * x))$ **for** *x*

have *sums*: $(\lambda n.\ P\ x\ n\ /\ of\_nat\ (Suc\ n)\hat{}2)\ sums\ (if\ x = 0\ then\ K\ else\ g\ x\ /\ x\hat{}2)$ **for** *x*
proof (*cases* $x = 0$)
assume *x*: $x = 0$
have *summable* $(\lambda n.\ inverse\ ((real\_of\_nat\ (Suc\ n))^2))$
using *inverse_power_summable*[*of 2*] **by** (*subst summable_Suc_iff*) *simp*
thus *?thesis* **by** (*simp add*: *x g_def P_def K_def inverse_eq_divide power_divide summable_sums*)
next
assume *x*: $x \neq 0$
have $(\lambda n.\ P\ x\ n - P\ x\ (Suc\ n))\ sums\ (P\ x\ 0 - sin\ (pi * x)\ /\ (pi * x))$
unfolding *P_def* **using** *x* **by** (*intro telescope_sums' sin_product_formula_real'*)
also have $(\lambda n.\ P\ x\ n - P\ x\ (Suc\ n)) = (\lambda n.\ (x\hat{}2\ /\ of\_nat\ (Suc\ n)\hat{}2) * P\ x\ n)$
unfolding *P_def* **by** (*simp add*: *prod.nat_ivl_Suc' algebra_simps*)
also have $P\ x\ 0 = 1$ **by** (*simp add*: *P_def*)
finally have $(\lambda n.\ x^2\ /\ (of\_nat\ (Suc\ n))^2 * P\ x\ n)\ sums\ (1 - sin\ (pi * x)\ /\ (pi * x))$ **.**

```
  from sums_divide[OF this, of x^2] x show ?thesis unfolding g_def by simp
 qed

 have continuous_on (ball 0 1) f
 proof (rule uniform_limit_theorem; (intro always_eventually allI)?)
   show uniform_limit (ball 0 1) (λn x. ∑k<n. P x k / of_nat (Suc k)^2) f
sequentially
   proof (unfold f_def, rule Weierstrass_m_test)
     fix n :: nat and x :: real assume x: x ∈ ball 0 1
     {
       fix k :: nat assume k: k ≥ 1
       from x have x^2 < 1 by (auto simp: abs_square_less_1)
       also from k have ... ≤ of_nat k^2 by simp
       finally have (1 - x^2 / of_nat k^2) ∈ {0..1} using k
         by (simp_all add: field_simps del: of_nat_Suc)
     }
     hence (∏k=1..n. abs (1 - x^2 / of_nat k^2)) ≤ (∏k=1..n. 1) by (intro
prod_mono) simp
     thus norm (P x n / (of_nat (Suc n)^2)) ≤ 1 / of_nat (Suc n)^2
       unfolding P_def by (simp add: field_simps abs_prod del: of_nat_Suc)
    qed (subst summable_Suc_iff, insert inverse_power_summable[of 2], simp
add: inverse_eq_divide)
 qed (auto simp: P_def intro!: continuous_intros)
 hence isCont f 0 by (subst (asm) continuous_on_eq_continuous_at) simp_all
 hence (f - 0 → f 0) by (simp add: isCont_def)
 also have f 0 = K unfolding f_def P_def K_def by (simp add: inverse_eq_divide
power_divide)
 finally have f - 0 → K .

 moreover have f - 0 → pi^2 / 6
 proof (rule Lim_transform_eventually)
   define f' where [abs_def]: f' x = (∑n. - sin_coeff (n+3) * pi ^ (n+2) *
x^n) for x
   have eventually (λx. x ≠ (0::real)) (at 0)
     by (auto simp add: eventually_at intro!: exI[of _ 1])
   thus eventually (λx. f' x = f x) (at 0)
   proof eventually_elim
     fix x :: real assume x: x ≠ 0
      have sin_coeff 1 = (1 :: real) sin_coeff 2 = (0::real) by (simp_all add:
sin_coeff_def)
      with sums_split_initial_segment[OF sums_minus[OF sin_converges], of 3
pi*x]
     have (λn. - (sin_coeff (n+3) * (pi*x)^(n+3))) sums (pi * x - sin (pi*x))
       by (simp add: eval_nat_numeral)
     from sums_divide[OF this, of x^3 * pi] x
       have (λn. - (sin_coeff (n+3) * pi^(n+2) * x^n)) sums ((1 - sin (pi*x)
/ (pi*x)) / x^2)
       by (simp add: field_split_simps eval_nat_numeral)
     with x have (λn. - (sin_coeff (n+3) * pi^(n+2) * x^n)) sums (g x / x^2)
```

```
      by (simp add: g_def)
    hence f' x = g x / x^2 by (simp add: sums_iff f'_def)
   also have ... = f x using sums[of x] x by (simp add: sums_iff g_def f_def)
    finally show f' x = f x .
  qed

  have isCont f' 0 unfolding f'_def
  proof (intro isCont_powser_converges_everywhere)
    fix x :: real show summable (λn. -sin_coeff (n+3) * pi^(n+2) * x^n)
    proof (cases x = 0)
      assume x: x ≠ 0
    from summable_divide[OF sums_summable[OF sums_split_initial_segment[OF
          sin_converges[of pi*x]], of 3], of -pi*x^3] x
      show ?thesis by (simp add: field_split_simps eval_nat_numeral)
    qed (simp only: summable_0_powser)
  qed
  hence f' - 0 → f' 0 by (simp add: isCont_def)
  also have f' 0 = pi * pi / fact 3 unfolding f'_def
    by (subst powser_zero) (simp add: sin_coeff_def)
  finally show f' - 0 → pi^2 / 6 by (simp add: eval_nat_numeral)
 qed

 ultimately have K = pi^2 / 6 by (rule LIM_unique)
 moreover from inverse_power_summable[of 2]
  have summable (λn. (inverse (real_of_nat (Suc n)))^2)
  by (subst summable_Suc_iff) (simp add: power_inverse)
 ultimately show ?thesis unfolding K_def
  by (auto simp add: sums_iff power_divide inverse_eq_divide)
qed

end


theory Interval_Integral
 imports Equivalence_Lebesgue_Henstock_Integration
begin

definition einterval a b = {x. a < ereal x ∧ ereal x < b}

lemma einterval_eq[simp]:
 shows einterval_eq_Icc: einterval (ereal a) (ereal b) = {a <..< b}
   and einterval_eq_Ici: einterval (ereal a) ∞ = {a <..}
   and einterval_eq_Iic: einterval (- ∞) (ereal b) = {..< b}
   and einterval_eq_UNIV: einterval (- ∞) ∞ = UNIV
 by (auto simp: einterval_def)

lemma einterval_same: einterval a a = {}
 by (auto simp: einterval_def)
```

lemma *einterval_iff*: $x \in \mathit{einterval}\ a\ b \longleftrightarrow a < \mathit{ereal}\ x \wedge \mathit{ereal}\ x < b$
by (*simp add*: *einterval_def*)

lemma *einterval_nonempty*: $a < b \Longrightarrow \exists\, c.\ c \in \mathit{einterval}\ a\ b$
by (*cases a b rule*: *ereal2_cases*, *auto simp*: *einterval_def intro*!: *dense gt_ex lt_ex*)

lemma *open_einterval*[*simp*]: *open* (*einterval a b*)
by (*cases a b rule*: *ereal2_cases*)
(*auto simp*: *einterval_def intro*!: *open_Collect_conj open_Collect_less continuous_intros*)

lemma *borel_einterval*[*measurable*]: *einterval a b* $\in$ *sets borel*
unfolding *einterval_def* **by** *measurable*

9.10.14 Approximating a (possibly infinite) interval

lemma *filterlim_sup1*: (*LIM x F. f x :> G1*) $\Longrightarrow$ (*LIM x F. f x :> (sup G1 G2)*)
unfolding *filterlim_def* **by** (*auto intro*: *le_supI1*)

lemma *ereal_incseq_approx*:
fixes *a b* :: *ereal*
assumes $a < b$
obtains *X* :: *nat* $\Rightarrow$ *real* **where** *incseq X* $\bigwedge i.\ a < X\ i\ \bigwedge i.\ X\ i < b\ X \longrightarrow b$
proof (*cases b*)
case *PInf*
with ‹$a < b$› **have** $a = -\infty \vee (\exists\, r.\ a = \mathit{ereal}\ r)$
by (*cases a*) *auto*
moreover have $(\lambda x.\ \mathit{ereal}\ (\mathit{real}\ (\mathit{Suc}\ x))) \longrightarrow \infty$
by (*simp add*: *Lim_PInfty filterlim_sequentially_Suc*) (*metis le_SucI of_nat_Suc of_nat_mono order_trans real_arch_simple*)
moreover have $\bigwedge r.\ (\lambda x.\ \mathit{ereal}\ (r + \mathit{real}\ (\mathit{Suc}\ x))) \longrightarrow \infty$
by (*simp add*: *filterlim_sequentially_Suc Lim_PInfty*) (*metis add.commute diff_le_eq nat_ceiling_le_eq*)
ultimately show *thesis*
by (*intro that*[*of* $\lambda i.$ *real_of_ereal a* + *Suc i*])
(*auto simp*: *incseq_def PInf*)
next
case (*real b'*)
define *d* **where** $d = b' - (\mathit{if}\ a = -\infty\ \mathit{then}\ b' - 1\ \mathit{else}\ \mathit{real\_of\_ereal}\ a)$
with ‹$a < b$› **have** a': $0 < d$
by (*cases a*) (*auto simp*: *real*)
moreover
have $\bigwedge i\ r.\ r < b' \Longrightarrow (b' - r) * 1 < (b' - r) * \mathit{real}\ (\mathit{Suc}\ (\mathit{Suc}\ i))$
by (*intro mult_strict_left_mono*) *auto*
with ‹$a < b$› a' **have** $\bigwedge i.\ a < \mathit{ereal}\ (b' - d\ /\ \mathit{real}\ (\mathit{Suc}\ (\mathit{Suc}\ i)))$
by (*cases a*) (*auto simp*: *real d_def field_simps*)
moreover
have $(\lambda i.\ b' - d\ /\ \mathit{real}\ i) \longrightarrow b'$

```
    by (force intro: tendsto_eq_intros tendsto_divide_0[OF tendsto_const] filterlim_sup1
            simp: at_infinity_eq_at_top_bot filterlim_real_sequentially)
  then have (λi. b' − d / Suc (Suc i)) ⟶ b'
    by (blast intro: dest: filterlim_sequentially_Suc [THEN iffD2])
  ultimately show thesis
    by (intro that[of λi. b' − d / Suc (Suc i)])
       (auto simp: real incseq_def intro!: divide_left_mono)
qed (use ‹a < b› in auto)

lemma ereal_decseq_approx:
  fixes a b :: ereal
  assumes a < b
  obtains X :: nat ⇒ real where
    decseq X ⋀i. a < X i ⋀i. X i < b X ⟶ a
proof −
  have −b < −a using ‹a < b› by simp
  from ereal_incseq_approx[OF this] obtain X where
    incseq X
    ⋀i. − b < ereal (X i)
    ⋀i. ereal (X i) < − a
    (λx. ereal (X x)) ⟶ − a
    by auto
  then show thesis
    apply (intro that[of λi. − X i])
    apply (auto simp: decseq_def incseq_def simp flip: uminus_ereal.simps)
   apply (metis ereal_minus_less_minus ereal_uminus_uminus ereal_Lim_uminus)+
    done
qed

proposition einterval_Icc_approximation:
  fixes a b :: ereal
  assumes a < b
  obtains u l :: nat ⇒ real where
    einterval a b = (⋃ i. {l i .. u i})
    incseq u decseq l ⋀i. l i < u i ⋀i. a < l i ⋀i. u i < b
    l ⟶ a u ⟶ b
proof −
  from dense[OF ‹a < b›] obtain c where a < c c < b by safe
  from ereal_incseq_approx[OF ‹c < b›] obtain u where u:
    incseq u
    ⋀i. c < ereal (u i)
    ⋀i. ereal (u i) < b
    (λx. ereal (u x)) ⟶ b
    by auto
  from ereal_decseq_approx[OF ‹a < c›] obtain l where l:
    decseq l
    ⋀i. a < ereal (l i)
    ⋀i. ereal (l i) < c
```

```
    (λx. ereal (l x)) ⟶ a
    by auto
  have einterval a b = (⋃i. {l i .. u i})
  proof (auto simp: einterval_iff)
    fix x assume a < ereal x ereal x < b
    have eventually (λi. ereal (l i) < ereal x) sequentially
      using l(4) ‹a < ereal x› by (rule order_tendstoD)
    moreover
    have eventually (λi. ereal x < ereal (u i)) sequentially
      using u(4) ‹ereal x< b› by (rule order_tendstoD)
    ultimately have eventually (λi. l i < x ∧ x < u i) sequentially
      by eventually_elim auto
    then show ∃i. l i ≤ x ∧ x ≤ u i
      by (auto intro: less_imp_le simp: eventually_sequentially)
  next
    fix x i assume l i ≤ x x ≤ u i
    with ‹a < ereal (l i)› ‹ereal (u i) < b›
    show a < ereal x ereal x < b
      by (auto simp flip: ereal_less_eq(3))
  qed
  moreover { fix i from less_trans[OF ‹l i < c› ‹c < u i›] have l i < u i by
simp }
  ultimately show thesis
    by (simp add: l that u)
qed


definition interval_lebesgue_integral :: real measure ⇒ ereal ⇒ ereal ⇒ (real ⇒
'a) ⇒ 'a::{banach, second_countable_topology} where
  interval_lebesgue_integral M a b f =
    (if a ≤ b then (LINT x:einterval a b|M. f x) else − (LINT x:einterval b a|M. f
x))

syntax
  _ascii_interval_lebesgue_integral :: pttrn ⇒ real ⇒ real ⇒ real measure ⇒ real
⇒ real
  (‹(‹indent=5 notation=‹binder LINT››LINT _=_..__|_. _)› [0,60,60,61,100]
60)
syntax_consts
  _ascii_interval_lebesgue_integral == interval_lebesgue_integral
translations
  LINT x=a..b|M. f == CONST interval_lebesgue_integral M a b (λx. f)

definition interval_lebesgue_integrable :: real measure ⇒ ereal ⇒ ereal ⇒ (real
⇒ 'a::{banach, second_countable_topology}) ⇒ bool where
  interval_lebesgue_integrable M a b f =
    (if a ≤ b then set_integrable M (einterval a b) f else set_integrable M (einterval
b a) f)
```

syntax
_ascii_interval_lebesgue_borel_integral :: *pttrn* ⇒ *real* ⇒ *real* ⇒ *real* ⇒ *real*
(‹(‹*indent=4 notation=‹binder LBINT›*›*LBINT _=_.._. _*)› [*0,60,60,61*] *60*)
syntax_consts
_ascii_interval_lebesgue_borel_integral == *interval_lebesgue_integral*
translations
LBINT x=a..b. f == *CONST interval_lebesgue_integral CONST lborel a b* (λ*x*. *f*)

9.10.15 Basic properties of integration over an interval

lemma *interval_lebesgue_integral_cong*:
a ≤ *b* ⟹ (⋀*x*. *x* ∈ *einterval a b* ⟹ *f x* = *g x*) ⟹ *einterval a b* ∈ *sets M* ⟹
interval_lebesgue_integral M a b f = *interval_lebesgue_integral M a b g*
by (*auto intro*: *set_lebesgue_integral_cong simp*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_cong_AE*:
f ∈ *borel_measurable M* ⟹ *g* ∈ *borel_measurable M* ⟹
a ≤ *b* ⟹ *AE x* ∈ *einterval a b in M*. *f x* = *g x* ⟹ *einterval a b* ∈ *sets M* ⟹
interval_lebesgue_integral M a b f = *interval_lebesgue_integral M a b g*
by (*auto intro*: *set_lebesgue_integral_cong_AE simp*: *interval_lebesgue_integral_def*)

lemma *interval_integrable_mirror*:
shows *interval_lebesgue_integrable lborel a b* (λ*x*. *f* (−*x*)) ⟷
interval_lebesgue_integrable lborel (−*b*) (−*a*) *f*
proof −
have ∗: *indicator* (*einterval a b*) (− *x*) = (*indicator* (*einterval* (−*b*) (−*a*)) *x* :: *real*)
for *a b* :: *ereal* **and** *x* :: *real*
by (*cases a b rule*: *ereal2_cases*) (*auto simp*: *einterval_def split*: *split_indicator*)
show *?thesis*
unfolding *interval_lebesgue_integrable_def*
using *lborel_integrable_real_affine_iff*[*symmetric*, *of* −*1* λ*x*. *indicator* (*einterval* _ _) *x* ∗$_R$ *f x 0*]
by (*simp add*: ∗ *set_integrable_def*)
qed

lemma *interval_lebesgue_integral_add* [*intro*, *simp*]:
fixes *M a b f*
assumes *interval_lebesgue_integrable M a b f interval_lebesgue_integrable M a b g*
shows *interval_lebesgue_integrable M a b* (λ*x*. *f x* + *g x*)
and *interval_lebesgue_integral M a b* (λ*x*. *f x* + *g x*) =
interval_lebesgue_integral M a b f + *interval_lebesgue_integral M a b g*
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def field_simps*)

lemma *interval_lebesgue_integral_diff* [*intro*, *simp*]:
fixes *M a b f*

assumes *interval_lebesgue_integrable M a b f*
interval_lebesgue_integrable M a b g
shows *interval_lebesgue_integrable M a b* (λ*x*. *f x* − *g x*) **and**
interval_lebesgue_integral M a b (λ*x*. *f x* − *g x*) =
interval_lebesgue_integral M a b f − *interval_lebesgue_integral M a b g*
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def field_simps*)

lemma *interval_lebesgue_integrable_mult_right* [*intro*, *simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows (*c* ≠ *0* ⟹ *interval_lebesgue_integrable M a b f*) ⟹
interval_lebesgue_integrable M a b (λ*x*. *c* ∗ *f x*)
by (*simp add*: *interval_lebesgue_integrable_def*)

lemma *interval_lebesgue_integrable_mult_left* [*intro*, *simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows (*c* ≠ *0* ⟹ *interval_lebesgue_integrable M a b f*) ⟹
interval_lebesgue_integrable M a b (λ*x*. *f x* ∗ *c*)
by (*simp add*: *interval_lebesgue_integrable_def*)

lemma *interval_lebesgue_integrable_divide* [*intro*, *simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *field*, *second_countable_topology*}
shows (*c* ≠ *0* ⟹ *interval_lebesgue_integrable M a b f*) ⟹
interval_lebesgue_integrable M a b (λ*x*. *f x* / *c*)
by (*simp add*: *interval_lebesgue_integrable_def*)

lemma *interval_lebesgue_integral_mult_right* [*simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows *interval_lebesgue_integral M a b* (λ*x*. *c* ∗ *f x*) =
c ∗ *interval_lebesgue_integral M a b f*
by (*simp add*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_mult_left* [*simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *second_countable_topology*}
shows *interval_lebesgue_integral M a b* (λ*x*. *f x* ∗ *c*) =
interval_lebesgue_integral M a b f ∗ *c*
by (*simp add*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_divide* [*simp*]:
fixes *M a b c* **and** *f* :: *real* ⇒ *'a*::{*banach*, *real_normed_field*, *field*, *second_countable_topology*}
shows *interval_lebesgue_integral M a b* (λ*x*. *f x* / *c*) =
interval_lebesgue_integral M a b f / *c*
by (*simp add*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_uminus*:
interval_lebesgue_integral M a b (λ*x*. − *f x*) = − *interval_lebesgue_integral M a b f*
by (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def set_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_of_real*:
  interval_lebesgue_integral M a b (λx. *complex_of_real* (*f x*)) =
    of_real (*interval_lebesgue_integral M a b f*)
  unfolding *interval_lebesgue_integral_def*
  by (*auto simp*: *interval_lebesgue_integral_def set_integral_complex_of_real*)

lemma *interval_lebesgue_integral_le_eq*:
  fixes *a b f*
  assumes $a \leq b$
  shows *interval_lebesgue_integral M a b f* = (*LINT x* : *einterval a b* | *M*. *f x*)
  using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def*)

lemma *interval_lebesgue_integral_gt_eq*:
  fixes *a b f*
  assumes $a > b$
  shows *interval_lebesgue_integral M a b f* = $-$(*LINT x* : *einterval b a* | *M*. *f x*)
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def less_imp_le einterval_def*)

lemma *interval_lebesgue_integral_gt_eq'*:
  fixes *a b f*
  assumes $a > b$
  shows *interval_lebesgue_integral M a b f* = $-$ *interval_lebesgue_integral M b a f*
using *assms* **by** (*auto simp*: *interval_lebesgue_integral_def less_imp_le einterval_def*)

lemma *interval_integral_endpoints_same* [*simp*]: (*LBINT x*=*a*..*a*. *f x*) = *0*
  by (*simp add*: *interval_lebesgue_integral_def set_lebesgue_integral_def einterval_same*)

lemma *interval_integral_endpoints_reverse*: (*LBINT x*=*a*..*b*. *f x*) = $-$(*LBINT x*=*b*..*a*. *f x*)
  by (*cases a b rule*: *linorder_cases*) (*auto simp*: *interval_lebesgue_integral_def set_lebesgue_integral_def einterval_same*)

lemma *interval_integrable_endpoints_reverse*:
  interval_lebesgue_integrable lborel a b f $\longleftrightarrow$
    interval_lebesgue_integrable lborel b a f
  by (*cases a b rule*: *linorder_cases*) (*auto simp*: *interval_lebesgue_integrable_def einterval_same*)

lemma *interval_integral_reflect*:
  (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x*=$-$*b*..$-$*a*. *f* ($-x$))
proof (*induct a b rule*: *linorder_wlog*)
  case (*sym a b*) **then show** *?case*
   by (*auto simp*: *interval_lebesgue_integral_def interval_integrable_endpoints_reverse*
        split: *if_split_asm*)

next
case (*le a b*)
have (*LBINT x*:{*x*. − *x* ∈ *einterval a b*}. *f* (− *x*)) = (*LBINT x*:*einterval* (− *b*) (− *a*). *f* (− *x*))
unfolding *interval_lebesgue_integrable_def set_lebesgue_integral_def einterval_def*
by (*metis* (*lifting*) *ereal_less_uminus_reorder ereal_uminus_less_reorder indicator_simps mem_Collect_eq uminus_ereal.simps*(*1*))
then show *?case*
unfolding *interval_lebesgue_integral_def*
by (*subst set_integral_reflect*) (*simp add*: *le*)
qed

lemma *interval_lebesgue_integral_0_infty*:
interval_lebesgue_integrable M 0 ∞ *f* ⟷ *set_integrable M* {*0*<..} *f*
interval_lebesgue_integral M 0 ∞ *f* = (*LINT x*:{*0*<..}|*M*. *f x*)
unfolding *zero_ereal_def*
by (*auto simp*: *interval_lebesgue_integral_le_eq interval_lebesgue_integrable_def*)

lemma *interval_integral_to_infinity_eq*: (*LINT x*=*ereal a*..∞ | *M*. *f x*) = (*LINT x* : {*a*<..} | *M*. *f x*)
unfolding *interval_lebesgue_integral_def* **by** *auto*

proposition *interval_integrable_to_infinity_eq*: (*interval_lebesgue_integrable M a* ∞ *f*) =
(*set_integrable M* {*a*<..} *f*)
unfolding *interval_lebesgue_integrable_def* **by** *auto*

9.10.16 Basic properties of integration over an interval wrt lebesgue measure

lemma *interval_integral_zero* [*simp*]:
fixes *a b* :: *ereal*
shows (*LBINT x*=*a*..*b*. *0*) = *0*
unfolding *interval_lebesgue_integral_def set_lebesgue_integral_def einterval_eq*
by *simp*

lemma *interval_integral_const* [*intro*, *simp*]:
fixes *a b c* :: *real*
shows *interval_lebesgue_integrable lborel a b* (λ*x*. *c*) **and** (*LBINT x*=*a*..*b*. *c*) = *c* ∗ (*b* − *a*)
unfolding *interval_lebesgue_integral_def interval_lebesgue_integrable_def einterval_eq*
by (*auto simp*: *less_imp_le field_simps measure_def set_integrable_def set_lebesgue_integral_def*)

lemma *interval_integral_cong_AE*:
assumes [*measurable*]: *f* ∈ *borel_measurable borel g* ∈ *borel_measurable borel*
assumes *AE x* ∈ *einterval* (*min a b*) (*max a b*) *in lborel*. *f x* = *g x*
shows *interval_lebesgue_integral lborel a b f* = *interval_lebesgue_integral lborel*

a b g
using *assms*
by (*auto simp*: *interval_lebesgue_integral_def max_def min_def intro*!: *set_lebesgue_integral_cong_A*

lemma *interval_integral_cong*:
assumes $\bigwedge x.\ x \in einterval\ (min\ a\ b)\ (max\ a\ b) \Longrightarrow f\ x = g\ x$
shows *interval_lebesgue_integral lborel a b f = interval_lebesgue_integral lborel a b g*
using *assms* **by** (*simp add*: *interval_lebesgue_integral_def set_lebesgue_integral_cong*)

lemma *interval_lebesgue_integrable_cong_AE*:
$f \in borel\_measurable\ lborel \Longrightarrow g \in borel\_measurable\ lborel \Longrightarrow$
AE x ∈ einterval (min a b) (max a b) in lborel. f x = g x ⟹
interval_lebesgue_integrable lborel a b f = interval_lebesgue_integrable lborel a b g
apply (*simp add*: *interval_lebesgue_integrable_def*)
apply (*intro conjI impI set_integrable_cong_AE*)
apply (*auto simp*: *min_def max_def*)
done

lemma *interval_integrable_abs_iff*:
fixes *f* :: *real* ⇒ *real*
shows $f \in borel\_measurable\ lborel \Longrightarrow$
interval_lebesgue_integrable lborel a b (λx. |f x|) = interval_lebesgue_integrable lborel a b f
unfolding *interval_lebesgue_integrable_def*
by (*simp add*: *set_integrable_abs_iff'*)

lemma *interval_integral_Icc*:
fixes *a b* :: *real*
shows $a \le b \Longrightarrow (LBINT\ x{=}a..b.\ f\ x) = (LBINT\ x : \{a..b\}.\ f\ x)$
by (*auto intro*!: *set_integral_discrete_difference*[**where** *X*={*a*, *b*}]
simp add: *interval_lebesgue_integral_def*)

lemma *interval_integral_Icc'*:
$a \le b \Longrightarrow (LBINT\ x{=}a..b.\ f\ x) = (LBINT\ x : \{x.\ a \le ereal\ x \wedge ereal\ x \le b\}.\ f\ x)$
by (*auto intro*!: *set_integral_discrete_difference*[**where** *X*={*real_of_ereal a*, *real_of_ereal b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioc*:
$a \le b \Longrightarrow (LBINT\ x{=}a..b.\ f\ x) = (LBINT\ x : \{a<..b\}.\ f\ x)$
by (*auto intro*!: *set_integral_discrete_difference*[**where** *X*={*a*, *b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioc'*:
$a \le b \Longrightarrow (LBINT\ x{=}a..b.\ f\ x) = (LBINT\ x : \{x.\ a < ereal\ x \wedge ereal\ x \le b\}.\ f$

x)
by (*auto intro*!: *set_integral_discrete_difference*[**where** *X*={*real_of_ereal a*, *real_of_ereal b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ico*:
a ≤ *b* ⟹ (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x* : {*a*..<*b*}. *f x*)
by (*auto intro*!: *set_integral_discrete_difference*[**where** *X*={*a*, *b*}]
simp add: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioi*:
|*a*| < ∞ ⟹ (*LBINT x*=*a*..∞. *f x*) = (*LBINT x* : {*real_of_ereal a* <..}. *f x*)
by (*auto simp*: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_Ioo*:
a ≤ *b* ⟹ |*a*| < ∞ ==> |*b*| < ∞ ⟹ (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x* : {*real_of_ereal a* <..< *real_of_ereal b*}. *f x*)
by (*auto simp*: *interval_lebesgue_integral_def einterval_iff*)

lemma *interval_integral_discrete_difference*:
fixes *f* :: *real* ⇒ *'b*::{*banach*, *second_countable_topology*} **and** *a b* :: *ereal*
assumes *countable X*
and *eq*: ⋀*x*. *a* ≤ *b* ⟹ *a* < *x* ⟹ *x* < *b* ⟹ *x* ∉ *X* ⟹ *f x* = *g x*
and *anti_eq*: ⋀*x*. *b* ≤ *a* ⟹ *b* < *x* ⟹ *x* < *a* ⟹ *x* ∉ *X* ⟹ *f x* = *g x*
assumes ⋀*x*. *x* ∈ *X* ⟹ *emeasure M* {*x*} = *0* ⋀*x*. *x* ∈ *X* ⟹ {*x*} ∈ *sets M*
shows *interval_lebesgue_integral M a b f* = *interval_lebesgue_integral M a b g*
unfolding *interval_lebesgue_integral_def set_lebesgue_integral_def*
apply (*intro if_cong refl arg_cong*[**where** *f*=λ*x*. − *x*] *integral_discrete_difference*[*of X*] *assms*)
apply (*auto simp*: *eq anti_eq einterval_iff split*: *split_indicator*)
done

lemma *interval_integral_sum*:
fixes *a b c* :: *ereal*
assumes *integrable*: *interval_lebesgue_integrable lborel* (*min a* (*min b c*)) (*max a* (*max b c*)) *f*
shows (*LBINT x*=*a*..*b*. *f x*) + (*LBINT x*=*b*..*c*. *f x*) = (*LBINT x*=*a*..*c*. *f x*)
proof −
let *?I* = λ*a b*. *LBINT x*=*a*..*b*. *f x*
{ fix *a b c* :: *ereal* **assume** *interval_lebesgue_integrable lborel a c f a* ≤ *b b* ≤ *c*
then have *ord*: *a* ≤ *b b* ≤ *c a* ≤ *c* **and** *f'*: *set_integrable lborel* (*einterval a c*) *f*
by (*auto simp*: *interval_lebesgue_integrable_def*)
then have *f*: *set_borel_measurable borel* (*einterval a c*) *f*
unfolding *set_integrable_def set_borel_measurable_def*
by (*drule_tac borel_measurable_integrable*) *simp*
have (*LBINT x*:*einterval a c*. *f x*) = (*LBINT x*:*einterval a b* ∪ *einterval b c*. *f x*)
proof (*rule set_integral_cong_set*)

```
    show AE x in lborel. (x ∈ einterval a b ∪ einterval b c) = (x ∈ einterval a c)
      using AE_lborel_singleton[of real_of_ereal b] ord
      by (cases a b c rule: ereal3_cases) (auto simp: einterval_iff)
   show set_borel_measurable lborel (einterval a c) f set_borel_measurable lborel
(einterval a b ∪ einterval b c) f
      unfolding set_borel_measurable_def
    using ord by (auto simp: einterval_iff intro!: set_borel_measurable_subset[OF
f, unfolded set_borel_measurable_def])
  qed
  also have ... = (LBINT x:einterval a b. f x) + (LBINT x:einterval b c. f x)
    using ord
     by (intro set_integral_Un_AE) (auto intro!: set_integrable_subset[OF f'']
simp: einterval_iff not_less)
  finally have ?I a b + ?I b c = ?I a c
    using ord by (simp add: interval_lebesgue_integral_def)
 } note 1 = this
 { fix a b c :: ereal assume interval_lebesgue_integrable lborel a c f a ≤ b b ≤ c
  from 1[OF this] have ?I b c + ?I a b = ?I a c
    by (metis add.commute)
 } note 2 = this
 have 3: ⋀a b. b ≤ a ⟹ (LBINT x=a..b. f x) = - (LBINT x=b..a. f x)
  by (rule interval_integral_endpoints_reverse)
 show ?thesis
  using integrable
  apply (cases a b b c a c rule: linorder_le_cases[case_product linorder_le_cases
linorder_cases])
  apply simp_all
 by (simp_all add: min_absorb1 min_absorb2 max_absorb1 max_absorb2 field_simps
1 2 3)
qed

lemma interval_integrable_isCont:
 fixes a b and f :: real ⇒ 'a::{banach, second_countable_topology}
 shows (⋀x. min a b ≤ x ⟹ x ≤ max a b ⟹ isCont f x) ⟹
   interval_lebesgue_integrable lborel a b f
proof (induct a b rule: linorder_wlog)
 case (le a b) then show ?case
  unfolding interval_lebesgue_integrable_def set_integrable_def
  by (auto simp: interval_lebesgue_integrable_def
    intro!: set_integrable_subset[unfolded set_integrable_def, OF borel_integrable_compact[of
{a .. b}]]
      continuous_at_imp_continuous_on)
qed (auto intro: interval_integrable_endpoints_reverse[THEN iffD1])

lemma interval_integrable_continuous_on:
 fixes a b :: real and f
 assumes a ≤ b and continuous_on {a..b} f
 shows interval_lebesgue_integrable lborel a b f
using assms unfolding interval_lebesgue_integrable_def apply simp
```

by (*rule set_integrable_subset*, *rule borel_integrable_atLeastAtMost'* [*of a b*], *auto*)

lemma *interval_integral_eq_integral*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
shows $a \leq b \Longrightarrow set\_integrable\ lborel\ \{a..b\}\ f \Longrightarrow LBINT\ x{=}a..b.\ f\ x = integral\ \{a..b\}\ f$
by (*subst interval_integral_Icc*, *simp*) (*rule set_borel_integral_eq_integral*)

lemma *interval_integral_eq_integral'*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
shows $a \leq b \Longrightarrow set\_integrable\ lborel\ (einterval\ a\ b)\ f \Longrightarrow LBINT\ x{=}a..b.\ f\ x = integral\ (einterval\ a\ b)\ f$
by (*subst interval_lebesgue_integral_le_eq*, *simp*) (*rule set_borel_integral_eq_integral*)

9.10.17 General limit approximation arguments

proposition *interval_integral_Icc_approx_nonneg*:
fixes $a\ b :: ereal$
assumes $a < b$
fixes $u\ l :: nat \Rightarrow real$
assumes *approx*: $einterval\ a\ b = (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
$incseq\ u\ decseq\ l\ \bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$
fixes $f :: real \Rightarrow real$
assumes *f_integrable*: $\bigwedge i.\ set\_integrable\ lborel\ \{l\ i..u\ i\}\ f$
assumes *f_nonneg*: $AE\ x\ in\ lborel.\ a < ereal\ x \longrightarrow ereal\ x < b \longrightarrow 0 \leq f\ x$
assumes *f_measurable*: $set\_borel\_measurable\ lborel\ (einterval\ a\ b)\ f$
assumes *lbint_lim*: $(\lambda i.\ LBINT\ x{=}l\ i..\ u\ i.\ f\ x) \longrightarrow C$
shows
$set\_integrable\ lborel\ (einterval\ a\ b)\ f$
$(LBINT\ x{=}a..b.\ f\ x) = C$
proof −
have *1* [*unfolded set_integrable_def*]: $\bigwedge i.\ set\_integrable\ lborel\ \{l\ i..u\ i\}\ f$ **by** (*rule f_integrable*)
have *2*: $AE\ x\ in\ lborel.\ mono\ (\lambda n.\ indicator\ \{l\ n..u\ n\}\ x *_R f\ x)$
proof −
from *f_nonneg* **have** $AE\ x\ in\ lborel.\ \forall i.\ l\ i \leq x \longrightarrow x \leq u\ i \longrightarrow 0 \leq f\ x$
by *eventually_elim*
(*metis approx*(*5*) *approx*(*6*) *dual_order.strict_trans1 ereal_less_eq*(*3*) *le_less_trans*)
then show *?thesis*
apply *eventually_elim*
apply (*auto simp*: *mono_def split*: *split_indicator*)
apply (*metis approx*(*3*) *decseqD order_trans*)
apply (*metis approx*(*2*) *incseqD order_trans*)
done
qed
have *3*: $AE\ x\ in\ lborel.\ (\lambda i.\ indicator\ \{l\ i..u\ i\}\ x *_R f\ x) \longrightarrow indicator$

```
(einterval a b) x *R f x
  proof −
    { fix x i assume l i ≤ x x ≤ u i
      then have eventually (λi. l i ≤ x ∧ x ≤ u i) sequentially
        apply (auto simp: eventually_sequentially intro!: exI[of _ i])
        apply (metis approx(3) decseqD order_trans)
        apply (metis approx(2) incseqD order_trans)
        done
      then have eventually (λi. f x * indicator {l i..u i} x = f x) sequentially
        by eventually_elim auto }
    then show ?thesis
    unfolding approx(1) by (auto intro!: AE_I2 tendsto_eventually split: split_indicator)
  qed
  have 4: (λi. ∫ x. indicator {l i..u i} x *R f x ∂lborel) ⟶ C
   using lbint_lim by (simp add: interval_integral_Icc [unfolded set_lebesgue_integral_def]
approx less_imp_le)
  have 5: (λx. indicat_real (einterval a b) x *R f x) ∈ borel_measurable lborel
    using f_measurable set_borel_measurable_def by blast
  have (LBINT x=a..b. f x) = lebesgue_integral lborel (λx. indicator (einterval a
b) x *R f x)
   using assms by (simp add: interval_lebesgue_integral_def set_lebesgue_integral_def
less_imp_le)
  also have ... = C
    by (rule integral_monotone_convergence [OF 1 2 3 4 5])
  finally show (LBINT x=a..b. f x) = C .
  show set_integrable lborel (einterval a b) f
    unfolding set_integrable_def
    by (rule integrable_monotone_convergence[OF 1 2 3 4 5])
qed

proposition interval_integral_Icc_approx_integrable:
  fixes u l :: nat ⇒ real and a b :: ereal
  fixes f :: real ⇒ 'a::{banach, second_countable_topology}
  assumes a < b
  assumes  approx: einterval a b = (⋃i. {l i .. u i})
    incseq u decseq l ⋀i. l i < u i ⋀i. a < l i ⋀i. u i < b
    l ⟶ a u ⟶ b
  assumes f_integrable: set_integrable lborel (einterval a b) f
  shows (λi. LBINT x=l i.. u i. f x) ⟶ (LBINT x=a..b. f x)
proof −
  have (λi. LBINT x:{l i.. u i}. f x) ⟶ (LBINT x:einterval a b. f x)
    unfolding set_lebesgue_integral_def
  proof (rule integral_dominated_convergence)
    show integrable lborel (λx. norm (indicator (einterval a b) x *R f x))
      using f_integrable integrable_norm set_integrable_def by blast
    show (λx. indicat_real (einterval a b) x *R f x) ∈ borel_measurable lborel
      using f_integrable by (simp add: set_integrable_def)
    then show ⋀i. (λx. indicat_real {l i..u i} x *R f x) ∈ borel_measurable lborel
      by (rule set_borel_measurable_subset [unfolded set_borel_measurable_def])
```

(*auto simp*: *approx*)
show $\bigwedge i$. *AE x in lborel. norm* (*indicator* $\{l\ i..u\ i\}\ x *_R f\ x) \leq$ *norm* (*indicator* (*einterval a b*) $x *_R f\ x$)
by (*intro AE_I2*) (*auto simp*: *approx split*: *split_indicator*)
show *AE x in lborel.* (λi. *indicator* $\{l\ i..u\ i\}\ x *_R f\ x) \longrightarrow$ *indicator* (*einterval a b*) $x *_R f\ x$
proof (*intro AE_I2 tendsto_intros tendsto_eventually*)
fix *x*
{ fix *i* **assume** $l\ i \leq x\ x \leq u\ i$
with ‹*incseq u*›[*THEN incseqD*, *of i*] ‹*decseq l*›[*THEN decseqD*, *of i*]
have *eventually* (λi. $l\ i \leq x \wedge x \leq u\ i$) *sequentially*
by (*auto simp*: *eventually_sequentially decseq_def incseq_def intro*: *order_trans*) **}**
then show *eventually* (λxa. *indicator* $\{l\ xa..u\ xa\}\ x =$ (*indicator* (*einterval a b*) *x*::*real*)) *sequentially*
using *approx order_tendstoD*(*2*)[*OF* ‹$l \longrightarrow a$›, *of x*] *order_tendstoD*(*1*)[*OF* ‹$u \longrightarrow b$›, *of x*]
by (*auto split*: *split_indicator*)
qed
qed
with ‹$a < b$› ‹$\bigwedge i.\ l\ i < u\ i$› **show** *?thesis*
by (*simp add*: *interval_lebesgue_integral_le_eq*[*symmetric*] *interval_integral_Icc less_imp_le*)
qed

9.10.18 A slightly stronger Fundamental Theorem of Calculus

Three versions: first, for finite intervals, and then two versions for arbitrary intervals.

lemma *interval_integral_FTC_finite*:
fixes *f F* :: *real* ⇒ *'a*::*euclidean_space* **and** *a b* :: *real*
assumes *f*: *continuous_on* $\{min\ a\ b..max\ a\ b\}\ f$
assumes *F*: $\bigwedge x.\ min\ a\ b \leq x \Longrightarrow x \leq max\ a\ b \Longrightarrow$ (*F has_vector_derivative* (*f x*)) (*at x within* $\{min\ a\ b..max\ a\ b\}$)
shows (*LBINT x*=*a*..*b*. *f x*) = *F b* − *F a*
proof (*cases* $a \leq b$)
case *True*
have (*LBINT x*=*a*..*b*. *f x*) = (*LBINT x*. *indicat_real* $\{a..b\}\ x *_R f\ x$)
by (*simp add*: *True interval_integral_Icc set_lebesgue_integral_def*)
also have ... = *F b* − *F a*
proof (*rule integral_FTC_atLeastAtMost* [*OF True*])
show *continuous_on* $\{a..b\}\ f$
using *True f* **by** *linarith*
show $\bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow$ (*F has_vector_derivative f x*) (*at x within* $\{a..b\}$)
by (*metis F True max.commute max_absorb1 min_def*)
qed

```
  finally show ?thesis .
next
  case False
  then have b ≤ a
    by simp
  have − interval_lebesgue_integral lborel (ereal b) (ereal a) f = − (LBINT x.
indicat_real {b..a} x *R f x)
    by (simp add: ‹b ≤ a› interval_integral_Icc set_lebesgue_integral_def)
  also have ... = F b − F a
  proof (subst integral_FTC_atLeastAtMost [OF ‹b ≤ a›])
    show continuous_on {b..a} f
      using False f by linarith
    show ⋀x. ⟦b ≤ x; x ≤ a⟧
        ⟹ (F has_vector_derivative f x) (at x within {b..a})
      by (metis F False max_def min_def)
  qed auto
  finally show ?thesis
    by (metis interval_integral_endpoints_reverse)
qed


lemma interval_integral_FTC_nonneg:
  fixes f F :: real ⇒ real and a b :: ereal
  assumes a < b
  assumes F: ⋀x. a < ereal x ⟹ ereal x < b ⟹ DERIV F x :> f x
  assumes f: ⋀x. a < ereal x ⟹ ereal x < b ⟹ isCont f x
  assumes f_nonneg: AE x in lborel. a < ereal x ⟶ ereal x < b ⟶ 0 ≤ f x
  assumes A: ((F ∘ real_of_ereal) ⟶⟶ A) (at_right a)
  assumes B: ((F ∘ real_of_ereal) ⟶⟶ B) (at_left b)
  shows
    set_integrable lborel (einterval a b) f
    (LBINT x=a..b. f x) = B − A
proof −
  obtain u l where approx:
    einterval a b = (⋃ i. {l i .. u i})
    incseq u decseq l ⋀i. l i < u i ⋀i. a < l i ⋀i. u i < b
    l ⟶⟶ a u ⟶⟶ b
    by (blast intro: einterval_Icc_approximation[OF ‹a < b›])
  have aless[simp]: ⋀x i. l i ≤ x ⟹ a < ereal x
    by (rule order_less_le_trans, rule approx, force)
  have lessb[simp]: ⋀x i. x ≤ u i ⟹ ereal x < b
   by (rule order_le_less_trans, subst ereal_less_eq(3), assumption, rule approx)
  have cf: ⋀i. continuous_on {min (l i) (u i)..max (l i) (u i)} f
    using approx f by (intro continuous_at_imp_continuous_on strip) auto
  have FTCi: ⋀i. (LBINT x=l i..u i. f x) = F (u i) − F (l i)
    apply (intro interval_integral_FTC_finite cf DERIV_subset [OF F])
   by (smt (verit) F aless approx(4) has_real_derivative_iff_has_vector_derivative
has_vector_derivative_at_within lessb)
  have 1: ⋀i. set_integrable lborel {l i..u i} f
```

by (*meson aless lessb assms*(*3*) *atLeastAtMost_iff borel_integrable_atLeastAtMost′ continuous_at_imp_continuous_on*)
have *2*: *set_borel_measurable lborel* (*einterval a b*) *f*
unfolding *set_borel_measurable_def*
by (*auto simp del*: *real_scaleR_def intro*!: *borel_measurable_continuous_on_indicator simp*: *continuous_on_eq_continuous_at einterval_iff f*)
have (λ*x*. *F* (*l x*)) ⟶ *A*
using *A approx* **unfolding** *tendsto_at_iff_sequentially comp_def*
by (*force elim*!: *allE*[*of* _ λ*i*. *ereal* (*l i*)])
moreover have (λ*x*. *F* (*u x*)) ⟶ *B*
using *B approx* **unfolding** *tendsto_at_iff_sequentially comp_def*
by (*force elim*!: *allE*[*of* _ λ*i*. *ereal* (*u i*)])
ultimately have *3*: (λ*i*. *LBINT x*=*l i*..*u i*. *f x*) ⟶ *B* − *A*
by (*simp add*: *FTCi tendsto_diff*)
show (*LBINT x*=*a*..*b*. *f x*) = *B* − *A*
by (*rule interval_integral_Icc_approx_nonneg* [*OF* ‹*a* < *b*› *approx 1 f_nonneg 2 3*])
show *set_integrable lborel* (*einterval a b*) *f*
by (*rule interval_integral_Icc_approx_nonneg* [*OF* ‹*a* < *b*› *approx 1 f_nonneg 2 3*])
qed

theorem *interval_integral_FTC_integrable*:
fixes *f F* :: *real* ⇒ *'a*::*euclidean_space* **and** *a b* :: *ereal*
assumes *a* < *b*
assumes *F*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ (*F has_vector_derivative f x*) (*at x*)
assumes *f*: ⋀*x*. *a* < *ereal x* ⟹ *ereal x* < *b* ⟹ *isCont f x*
assumes *f_integrable*: *set_integrable lborel* (*einterval a b*) *f*
assumes *A*: ((*F* ∘ *real_of_ereal*) ⟶ *A*) (*at_right a*)
assumes *B*: ((*F* ∘ *real_of_ereal*) ⟶ *B*) (*at_left b*)
shows (*LBINT x*=*a*..*b*. *f x*) = *B* − *A*
proof −
obtain *u l* **where** *approx*:
einterval a b = (⋃*i*. {*l i* .. *u i*})
incseq u decseq l ⋀*i*. *l i* < *u i* ⋀*i*. *a* < *l i* ⋀*i*. *u i* < *b*
l ⟶ *a u* ⟶ *b*
by (*blast intro*: *einterval_Icc_approximation*[*OF* ‹*a* < *b*›])
have [*simp*]: ⋀*x i*. *l i* ≤ *x* ⟹ *a* < *ereal x*
by (*rule order_less_le_trans*, *rule approx*, *force*)
have [*simp*]: ⋀*x i*. *x* ≤ *u i* ⟹ *ereal x* < *b*
by (*rule order_le_less_trans*, *subst ereal_less_eq*(*3*), *assumption*, *rule approx*)
have *FTCi*: ⋀*i*. (*LBINT x*=*l i*..*u i*. *f x*) = *F* (*u i*) − *F* (*l i*)
using *assms approx*
by (*auto simp*: *less_imp_le min_def max_def*
intro!: *f continuous_at_imp_continuous_on interval_integral_FTC_finite*
intro: *has_vector_derivative_at_within*)
have (λ*i*. *LBINT x*=*l i*..*u i*. *f x*) ⟶ *B* − *A*
unfolding *FTCi*

```
  proof (intro tendsto_intros)
    show (λx. F (l x)) ⟶ A
      using A approx unfolding tendsto_at_iff_sequentially comp_def
      by (elim allE[of _ λi. ereal (l i)], auto)
    show (λx. F (u x)) ⟶ B
      using B approx unfolding tendsto_at_iff_sequentially comp_def
      by (elim allE[of _ λi. ereal (u i)], auto)
  qed
  moreover have (λi. LBINT x=l i..u i. f x) ⟶ (LBINT x=a..b. f x)
   by (rule interval_integral_Icc_approx_integrable [OF ‹a < b› approx f_integrable])
  ultimately show ?thesis
    by (elim LIMSEQ_unique)
qed



theorem interval_integral_FTC2:
  fixes a b c :: real and f :: real ⇒ 'a::euclidean_space
  assumes a ≤ c c ≤ b
  and contf: continuous_on {a..b} f
  fixes x :: real
  assumes a ≤ x and x ≤ b
  shows ((λu. LBINT y=c..u. f y) has_vector_derivative (f x)) (at x within {a..b})
proof −
  let ?F = (λu. LBINT y=a..u. f y)
  have intf: set_integrable lborel {a..b} f
    by (rule borel_integrable_atLeastAtMost', rule contf)
  have ((λu. integral {a..u} f) has_vector_derivative f x) (at x within {a..b})
    using ‹a ≤ x› ‹x ≤ b›
      by (auto intro: integral_has_vector_derivative continuous_on_subset [OF
contf])
   then have ((λu. integral {a..u} f) has_vector_derivative (f x)) (at x within
{a..b})
    by simp
  then have (?F has_vector_derivative (f x)) (at x within {a..b})
    by (rule has_vector_derivative_weaken)
     (auto intro!: assms interval_integral_eq_integral[symmetric] set_integrable_subset
[OF intf])
  then have ((λx. (LBINT y=c..a. f y) + ?F x) has_vector_derivative (f x)) (at
x within {a..b})
    by (auto intro!: derivative_eq_intros)
  then show ?thesis
  proof (rule has_vector_derivative_weaken)
    fix u assume u ∈ {a .. b}
    then show (LBINT y=c..a. f y) + (LBINT y=a..u. f y) = (LBINT y=c..u. f
y)
      using assms
      apply (intro interval_integral_sum)
    apply (auto simp: interval_lebesgue_integrable_def simp del: real_scaleR_def)
```

```
    by (rule set_integrable_subset [OF intf], auto simp: min_def max_def)
  qed (insert assms, auto)
qed

proposition einterval_antiderivative:
  fixes a b :: ereal and f :: real ⇒ 'a::euclidean_space
  assumes a < b and contf: ⋀x :: real. a < x ⟹ x < b ⟹ isCont f x
  shows ∃ F. ∀ x :: real. a < x ⟶ x < b ⟶ (F has_vector_derivative f x) (at x)
proof −
  from einterval_nonempty [OF ‹a < b›] obtain c :: real where [simp]: a < c c
< b
    by (auto simp: einterval_def)
  let ?F = (λu. LBINT y=c..u. f y)
  show ?thesis
  proof (rule exI, clarsimp)
    fix x :: real
    assume [simp]: a < x x < b
    have 1: a < min c x by simp
    from einterval_nonempty [OF 1] obtain d :: real where [simp]: a < d d < c
d < x
      by (auto simp: einterval_def)
    have 2: max c x < b by simp
    from einterval_nonempty [OF 2] obtain e :: real where [simp]: c < e x < e
e < b
      by (auto simp: einterval_def)
    have (?F has_vector_derivative f x) (at x within {d<..<e})
    proof (rule has_vector_derivative_within_subset [of _ _ _ {d..e}])
      have continuous_on {d..e} f
      proof (intro continuous_at_imp_continuous_on ballI contf; clarsimp)
        show ⋀x. ⟦d ≤ x; x ≤ e⟧ ⟹ a < ereal x
          using ‹a < ereal d› ereal_less_ereal_Ex by auto
        show ⋀x. ⟦d ≤ x; x ≤ e⟧ ⟹ ereal x < b
          using ‹ereal e < b› ereal_less_eq(3) le_less_trans by blast
      qed
      then show (?F has_vector_derivative f x) (at x within {d..e})
        by (intro interval_integral_FTC2) (use ‹d < c› ‹c < e› ‹d < x› ‹x < e›
in ‹linarith+›)
    qed auto
    then show (?F has_vector_derivative f x) (at x)
      by (force simp: has_vector_derivative_within_open [of _ {d<..<e}])
  qed
qed
```

9.10.19 The substitution theorem

Once again, three versions: first, for finite intervals, and then two versions for arbitrary intervals.

```
theorem interval_integral_substitution_finite:
  fixes a b :: real and f :: real ⇒ 'a::euclidean_space
```

assumes $a \leq b$
and *derivg*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow (g\ has\_real\_derivative\ (g'\ x))\ (at\ x\ within\ \{a..b\})$
and *contf* : $continuous\_on\ (g\ `\ \{a..b\})\ f$
and *contg'*: $continuous\_on\ \{a..b\}\ g'$
shows $(LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x)) = (LBINT\ y{=}g\ a..g\ b.\ f\ y)$
proof−
have *v_derivg*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow (g\ has\_vector\_derivative\ (g'\ x))\ (at\ x\ within\ \{a..b\})$
using *derivg* **unfolding** *has_real_derivative_iff_has_vector_derivative* **.**
then have *contg* [*simp*]: $continuous\_on\ \{a..b\}\ g$
by (*rule continuous_on_vector_derivative*) *auto*
have *1*: $\exists x{\in}\{a..b\}.\ u = g\ x$ **if** $min\ (g\ a)\ (g\ b) \leq u\ u \leq max\ (g\ a)\ (g\ b)$ **for** u
by (*cases* $g\ a \leq g\ b$) (*use that assms IVT' [of g a u b] IVT2' [of g b u a]* **in** ‹*auto simp*: *min_def max_def*›)
obtain $c\ d$ **where** *g_im*: $g\ `\ \{a..b\} = \{c..d\}$ **and** $c \leq d$
by (*metis continuous_image_closed_interval contg* ‹$a \leq b$›)
obtain F **where** *derivF*:
$\bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow (F\ has\_vector\_derivative\ (f\ (g\ x)))\ (at\ (g\ x)\ within\ (g\ `\ \{a..b\}))$
using *continuous_on_subset* [*OF contf*] *g_im*
by (*metis antiderivative_continuous atLeastAtMost_iff image_subset_iff set_eq_subset*)
have *contfg*: $continuous\_on\ \{a..b\}\ (\lambda x.\ f\ (g\ x))$
by (*blast intro*: *continuous_on_compose2 contf contg*)
have $continuous\_on\ \{a..b\}\ (\lambda x.\ g'\ x *_R f\ (g\ x))$
by (*auto intro*!: *continuous_on_scaleR contg' contfg*)
then have $(LBINT\ x.\ indicat\_real\ \{a..b\}\ x *_R g'\ x *_R f\ (g\ x)) = F\ (g\ b) - F\ (g\ a)$
using *integral_FTC_atLeastAtMost* [*OF* ‹$a \leq b$› *vector_diff_chain_within*[*OF v_derivg derivF*]]
by *force*
then have $LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x) = F\ (g\ b) - F\ (g\ a)$
by (*simp add*: *assms interval_integral_Icc set_lebesgue_integral_def*)
moreover have $LBINT\ y{=}(g\ a)..(g\ b).\ f\ y = F\ (g\ b) - F\ (g\ a)$
proof (*rule interval_integral_FTC_finite*)
show $continuous\_on\ \{min\ (g\ a)\ (g\ b)..max\ (g\ a)\ (g\ b)\}\ f$
by (*rule continuous_on_subset* [*OF contf*]) (*auto simp*: *image_def 1*)
show $(F\ has\_vector\_derivative\ f\ y)\ (at\ y\ within\ \{min\ (g\ a)\ (g\ b)..max\ (g\ a)\ (g\ b)\})$
if *y*: $min\ (g\ a)\ (g\ b) \leq y\ y \leq max\ (g\ a)\ (g\ b)$ **for** y
proof −
obtain x **where** $a \leq x\ x \leq b\ y = g\ x$
using *1 y* **by** *force*
then show *?thesis*
by (*auto simp*: *image_def intro*!: *1 has_vector_derivative_within_subset* [*OF derivF*])
qed
qed

ultimately show *?thesis* **by** *simp*
qed

theorem *interval_integral_substitution_integrable*:
fixes f :: *real* ⇒ *'a*::*euclidean_space* **and** a b u v :: *ereal*
assumes $a < b$
and *deriv_g*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow DERIV\ g\ x :> g'\ x$
and *contf*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ f\ (g\ x)$
and *contg'*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ g'\ x$
and *g'_nonneg*: $\bigwedge x.\ a \le ereal\ x \Longrightarrow ereal\ x \le b \Longrightarrow 0 \le g'\ x$
and *A*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow A)\ (at\_right\ a)$
and *B*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow B)\ (at\_left\ b)$
and *integrable*: *set_integrable lborel* $(einterval\ a\ b)\ (\lambda x.\ g'\ x *_R f\ (g\ x))$
and *integrable2*: *set_integrable lborel* $(einterval\ A\ B)\ (\lambda x.\ f\ x)$
shows $(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x))$
proof −
obtain u l **where** *approx* [*simp*]:
$einterval\ a\ b = (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
$incseq\ u\ decseq\ l\ \bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$
by (*blast intro*: *einterval_Icc_approximation*[*OF* ‹$a < b$›])
note *less_imp_le* [*simp*]
have [*simp*]: $\bigwedge x\ i.\ l\ i \le x \Longrightarrow a < ereal\ x$
by (*rule order_less_le_trans*, *rule approx*, *force*)
have [*simp*]: $\bigwedge x\ i.\ x \le u\ i \Longrightarrow ereal\ x < b$
by (*rule order_le_less_trans*, *subst ereal_less_eq*(*3*), *assumption*, *rule approx*)
then have *lessb*[*simp*]: $\bigwedge i.\ l\ i < b$
using *approx*(*4*) *less_eq_real_def* **by** *blast*
have [*simp*]: $\bigwedge i.\ a < u\ i$
by (*rule order_less_trans*, *rule approx*, *auto*, *rule approx*)
have *lle*[*simp*]: $\bigwedge i\ j.\ i \le j \Longrightarrow l\ j \le l\ i$ **by** (*rule decseqD*, *rule approx*)
have [*simp*]: $\bigwedge i\ j.\ i \le j \Longrightarrow u\ i \le u\ j$ **by** (*rule incseqD*, *rule approx*)
have *g_nondec* [*simp*]: $g\ x \le g\ y$ **if** $a < x\ x \le y\ y < b$ **for** $x\ y$
proof (*rule DERIV_nonneg_imp_nondecreasing* [*OF* ‹$x \le y$›], *intro exI conjI*)
show $\bigwedge u.\ x \le u \Longrightarrow u \le y \Longrightarrow (g\ has\_real\_derivative\ g'\ u)\ (at\ u)$
by (*meson deriv_g ereal_less_eq*(*3*) *le_less_trans less_le_trans that*)
show $\bigwedge u.\ x \le u \Longrightarrow u \le y \Longrightarrow 0 \le g'\ u$
by (*meson assms*(*5*) *dual_order.trans le_ereal_le less_imp_le order_refl that*)
qed
have $A \le B$ **and** *un*: $einterval\ A\ B = (\bigcup i.\ \{g(l\ i)<..<g(u\ i)\})$
proof −
have *A2*: $(\lambda i.\ g\ (l\ i)) \longrightarrow A$
using *A* **apply** (*auto simp*: *einterval_def tendsto_at_iff_sequentially comp_def*)
by (*drule_tac* $x = \lambda i.\ ereal\ (l\ i)$ **in** *spec*, *auto*)
hence *A3*: $\bigwedge i.\ g\ (l\ i) \ge A$
by (*intro decseq_ge*, *auto simp*: *decseq_def*)

```
  have B2: (λi. g (u i)) ⟶ B
  using B apply (auto simp: einterval_def tendsto_at_iff_sequentially comp_def)
    by (drule_tac x = λi. ereal (u i) in spec, auto)
  hence B3: ⋀i. g (u i) ≤ B
    by (intro incseq_le, auto simp: incseq_def)
  have ereal (g (l 0)) ≤ ereal (g (u 0))
    by auto
  then show A ≤ B
    by (meson A3 B3 order.trans)
  { fix x :: real
    assume A < x and x < B
   then have eventually (λi. ereal (g (l i)) < x ∧ x < ereal (g (u i))) sequentially
      by (fast intro: eventually_conj order_tendstoD A2 B2)
    hence ∃ i. g (l i) < x ∧ x < g (u i)
      by (simp add: eventually_sequentially, auto)
  } note AB = this
  show einterval A B = (⋃ i. {g(l i)<..<g(u i)})
  proof
    show einterval A B ⊆ (⋃ i. {g(l i)<..<g(u i)})
      by (auto simp: einterval_def AB)
    show (⋃ i. {g(l i)<..<g(u i)}) ⊆ einterval A B
    proof (clarsimp simp add: einterval_def, intro conjI)
      show ⋀x i. ⟦g (l i) < x; x < g (u i)⟧ ⟹ A < ereal x
        using A3 le_ereal_less by blast
      show ⋀x i. ⟦g (l i) < x; x < g (u i)⟧ ⟹ ereal x < B
        using B3 ereal_le_less by blast
    qed
  qed
qed

  have eq1: (LBINT x=l i.. u i. g' x *R f (g x)) = (LBINT y=g (l i)..g (u i). f
y) for i
    apply (rule interval_integral_substitution_finite [OF _ DERIV_subset [OF
deriv_g]])
    unfolding has_real_derivative_iff_has_vector_derivative[symmetric]
        apply (auto intro!: continuous_at_imp_continuous_on contf contg')
    done
  have (λi. LBINT x=l i..u i. g' x *R f (g x)) ⟶ (LBINT x=a..b. g' x *R f
(g x))
   using approx(4) ‹a < b› integrable interval_integral_Icc_approx_integrable by
fastforce
  hence 2: (λi. (LBINT y=g (l i)..g (u i). f y)) ⟶ (LBINT x=a..b. g' x *R f
(g x))
    by (simp add: eq1)
  have incseq: incseq (λi. {g (l i)<..<g (u i)})
    apply (clarsimp simp: incseq_def, intro conjI)
    using lessb lle approx(5) g_nondec le_less_trans apply blast
    by (force intro: less_le_trans)
  have (λi. set_lebesgue_integral lborel {g (l i)<..<g (u i)} f)
```

```
      ⟶ set_lebesgue_integral lborel (einterval A B) f
    unfolding un  by (rule set_integral_cont_up) (use incseq  integrable2 un in
auto)
  then have (λi. (LBINT y=g (l i)..g (u i). f y)) ⟶ (LBINT x = A..B. f x)
    by (simp add: interval_lebesgue_integral_le_eq ‹A ≤ B›)
  thus ?thesis by (intro LIMSEQ_unique [OF _ 2])
qed



theorem interval_integral_substitution_nonneg:
  fixes f g g'::: real ⇒ real and a b u v :: ereal
  assumes a < b
  and deriv_g: ⋀x. a < ereal x ⟹ ereal x < b ⟹ DERIV g x :> g' x
  and contf: ⋀x. a < ereal x ⟹ ereal x < b ⟹ isCont f (g x)
  and contg': ⋀x. a < ereal x ⟹ ereal x < b ⟹ isCont g' x
  and f_nonneg: ⋀x. a < ereal x ⟹ ereal x < b ⟹ 0 ≤ f (g x)
  and g'_nonneg: ⋀x. a ≤ ereal x ⟹ ereal x ≤ b ⟹ 0 ≤ g' x
  and A: ((ereal ∘ g ∘ real_of_ereal) ⟶ A) (at_right a)
  and B: ((ereal ∘ g ∘ real_of_ereal) ⟶ B) (at_left b)
  and integrable_fg: set_integrable lborel (einterval a b) (λx. f (g x) * g' x)
  shows
    set_integrable lborel (einterval A B) f
    (LBINT x=A..B. f x) = (LBINT x=a..b. (f (g x) * g' x))
proof −
  from einterval_Icc_approximation[OF ‹a < b›] obtain u l where approx [simp]:
    einterval a b = (⋃i. {l i..u i})
    incseq u
    decseq l
    ⋀i. l i < u i
    ⋀i. a < ereal (l i)
    ⋀i. ereal (u i) < b
    (λx. ereal (l x)) ⟶ a
    (λx. ereal (u x)) ⟶ b by this auto
  have aless[simp]: ⋀x i. l i ≤ x ⟹ a < ereal x
    by (rule order_less_le_trans, rule approx, force)
  have lessb[simp]: ⋀x i. x ≤ u i ⟹ ereal x < b
   by (rule order_le_less_trans, subst ereal_less_eq(3), assumption, rule approx)
  have llb[simp]: ⋀i. l i < b
    using lessb approx(4) less_eq_real_def by blast
  have alu[simp]: ⋀i. a < u i
    by (rule order_less_trans, rule approx, auto, rule approx)
  have [simp]: ⋀i j. i ≤ j ⟹ l j ≤ l i by (rule decseqD, rule approx)
  have uleu[simp]: ⋀i j. i ≤ j ⟹ u i ≤ u j by (rule incseqD, rule approx)
  have g_nondec [simp]: g x ≤ g y if a < x x ≤ y y < b for x y
  proof (rule DERIV_nonneg_imp_nondecreasing [OF ‹x ≤ y›], intro exI conjI)
    show ⋀u. x ≤ u ⟹ u ≤ y ⟹ (g has_real_derivative g' u) (at u)
      by (meson deriv_g ereal_less_eq(3) le_less_trans less_le_trans that)
    show ⋀u. x ≤ u ⟹ u ≤ y ⟹ 0 ≤ g' u
```

```
    by (meson g'_nonneg less_ereal.simps(1) less_trans not_less that)
  qed
  have A ≤ B and un: einterval A B = (⋃i. {g(l i)<..<g(u i)})
  proof −
    have A2: (λi. g (l i)) ⟶ A
      using A by (force simp: einterval_def tendsto_at_iff_sequentially comp_def
elim!: allE[where x = λi. ereal (l i)])
    hence A3: ⋀i. g (l i) ≥ A
      by (intro decseq_ge, auto simp: decseq_def)
    have B2: (λi. g (u i)) ⟶ B
      using B by (force simp: einterval_def tendsto_at_iff_sequentially comp_def
elim!: allE[where x = λi. ereal (u i)])
    hence B3: ⋀i. g (u i) ≤ B
      by (intro incseq_le, auto simp: incseq_def)
    have ereal (g (l 0)) ≤ ereal (g (u 0))
      by (auto simp: less_imp_le)
    then show A ≤ B
      by (meson A3 B3 order.trans)
    { fix x :: real
      assume A < x and x < B
     then have eventually (λi. ereal (g (l i)) < x ∧ x < ereal (g (u i))) sequentially
        by (fast intro: eventually_conj order_tendstoD A2 B2)
      hence ∃i. g (l i) < x ∧ x < g (u i)
        by (simp add: eventually_sequentially, auto)
    } note AB = this
    show einterval A B = (⋃i. {g(l i)<..<g(u i)})
    proof
      show einterval A B ⊆ (⋃i. {g (l i)<..<g (u i)})
        by (auto simp: einterval_def AB)
      show (⋃i. {g (l i)<..<g (u i)}) ⊆ einterval A B
       using A3 B3 by (force simp: einterval_def intro: le_ereal_less ereal_le_less)
    qed
  qed

  have eq1: (LBINT x=l i.. u i. (f (g x) * g' x)) = (LBINT y=g (l i)..g (u i). f
y) for i
  proof −
    have (LBINT x=l i.. u i. g' x *R f (g x)) = (LBINT y=g (l i)..g (u i). f y)
      apply (rule interval_integral_substitution_finite [OF _ DERIV_subset [OF
deriv_g]])
      unfolding has_real_derivative_iff_has_vector_derivative[symmetric]
        apply (auto simp: less_imp_le intro!: continuous_at_imp_continuous_on
contf contg')
      done
    then show ?thesis
      by (simp add: ac_simps)
  qed
  have incseq: incseq (λi. {g (l i)<..<g (u i)})
    apply (clarsimp simp: incseq_def, intro conjI)
```

apply (*meson llb antimono_def approx(3) approx(5) g_nondec le_less_trans*)
using *alu uleu approx(6) g_nondec less_le_trans* **by** *blast*
have *img*: $\exists c \geq l\ i.\ c \leq u\ i \wedge x = g\ c$ **if** $g\ (l\ i) \leq x\ x \leq g\ (u\ i)$ **for** $x\ i$
proof −
have *continuous_on* $\{l\ i..u\ i\}\ g$
by (*force intro*!: *DERIV_isCont deriv_g continuous_at_imp_continuous_on*)
with *that* **show** *?thesis*
using *IVT'* [*of g*] *approx(4) dual_order.strict_implies_order* **by** *blast*
qed
have *continuous_on* $\{g\ (l\ i)..g\ (u\ i)\}\ f$ **for** i
using *contf img* **by** (*force simp add*: *intro*!: *continuous_at_imp_continuous_on*)
then have *int_f*: $\bigwedge i.$ *set_integrable lborel* $\{g\ (l\ i)<..<g\ (u\ i)\}\ f$
by (*rule set_integrable_subset* [*OF borel_integrable_atLeastAtMost'*]) (*auto intro*: *less_imp_le*)
have *integrable*: *set_integrable lborel* $(\bigcup i.\ \{g\ (l\ i)<..<g\ (u\ i)\})\ f$
proof (*intro pos_integrable_to_top incseq int_f*)
let *?l* = $(LBINT\ x{=}a..b.\ f\ (g\ x) * g'\ x)$
have $(\lambda i.\ LBINT\ x{=}l\ i..u\ i.\ f\ (g\ x) * g'\ x) \longrightarrow$ *?l*
by (*intro assms interval_integral_Icc_approx_integrable* [*OF* ‹$a < b$› *approx*])
hence $(\lambda i.\ (LBINT\ y{=}g\ (l\ i)..g\ (u\ i).\ f\ y)) \longrightarrow$ *?l*
by (*simp add*: *eq1*)
then show $(\lambda i.$ *set_lebesgue_integral lborel* $\{g\ (l\ i)<..<g\ (u\ i)\}\ f) \longrightarrow$ *?l*
unfolding *interval_lebesgue_integral_def* **by** (*auto simp*: *less_imp_le*)
have $\bigwedge x\ i.\ g\ (l\ i) \leq x \Longrightarrow x \leq g\ (u\ i) \Longrightarrow 0 \leq f\ x$
using *aless f_nonneg img lessb* **by** *blast*
then show $\bigwedge x\ i.\ x \in \{g\ (l\ i)<..<g\ (u\ i)\} \Longrightarrow 0 \leq f\ x$
using *less_eq_real_def* **by** *auto*
qed (*auto simp*: *greaterThanLessThan_borel*)
thus *set_integrable lborel* (*einterval A B*) *f*
by (*simp add*: *un*)

have $(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x))$
proof (*rule interval_integral_substitution_integrable*)
show *set_integrable lborel* (*einterval a b*) $(\lambda x.\ g'\ x *_R f\ (g\ x))$
using *integrable_fg* **by** (*simp add*: *ac_simps*)
qed *fact+*
then show $(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ (f\ (g\ x) * g'\ x))$
by (*simp add*: *ac_simps*)
qed

syntax *_complex_lebesgue_borel_integral* :: *pttrn* ⇒ *real* ⇒ *complex*
(‹(‹*indent=2 notation=‹binder CLBINT››CLBINT* _. _)› [*0,60*] *60*)
syntax_consts
_complex_lebesgue_borel_integral == *complex_lebesgue_integral*
translations *CLBINT x. f* == *CONST complex_lebesgue_integral CONST lborel* $(\lambda x.\ f)$

syntax *_complex_set_lebesgue_borel_integral* :: *pttrn* ⇒ *real set* ⇒ *real* ⇒ *com-*

plex
(‹(‹*indent=3 notation=‹binder CLBINT››CLBINT _:_. _*)› [*0,60,61*] *60*)
syntax_consts
_complex_set_lebesgue_borel_integral == complex_set_lebesgue_integral
translations
CLBINT x:A. f == CONST complex_set_lebesgue_integral CONST lborel A (λx. f)

abbreviation *complex_interval_lebesgue_integral* ::
real measure ⇒ ereal ⇒ ereal ⇒ (real ⇒ complex) ⇒ complex **where**
complex_interval_lebesgue_integral M a b f ≡ interval_lebesgue_integral M a b f

abbreviation *complex_interval_lebesgue_integrable* ::
real measure ⇒ ereal ⇒ ereal ⇒ (real ⇒ complex) ⇒ bool **where**
complex_interval_lebesgue_integrable M a b f ≡ interval_lebesgue_integrable M a b f

syntax
_ascii_complex_interval_lebesgue_borel_integral :: pttrn ⇒ ereal ⇒ ereal ⇒ real ⇒ complex
(‹(‹*indent=4 notation=‹binder CLBINT››CLBINT _=_..._. _*)› [*0,60,60,61*] *60*)
syntax_consts
_ascii_complex_interval_lebesgue_borel_integral == complex_interval_lebesgue_integral
translations
CLBINT x=a..b. f == CONST complex_interval_lebesgue_integral CONST lborel a b (λx. f)

proposition *interval_integral_norm*:
fixes *f :: real ⇒ 'a :: {banach, second_countable_topology}*
shows *interval_lebesgue_integrable lborel a b f ⟹ a ≤ b ⟹*
norm (LBINT t=a..b. f t) ≤ LBINT t=a..b. norm (f t)
using *integral_norm_bound[of lborel λx. indicator (einterval a b) x *R f x]*
by (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def set_lebesgue_integral_def*)

proposition *interval_integral_norm2*:
interval_lebesgue_integrable lborel a b f ⟹
norm (LBINT t=a..b. f t) ≤ |LBINT t=a..b. norm (f t)|
proof (*induct a b rule*: *linorder_wlog*)
case (*sym a b*) **then show** *?case*
by (*simp add*: *interval_integral_endpoints_reverse[of a b] interval_integrable_endpoints_reverse[of a b]*)
next
case (*le a b*)
then have *|LBINT t=a..b. norm (f t)| = LBINT t=a..b. norm (f t)*
using *integrable_norm[of lborel λx. indicator (einterval a b) x *R f x]*
by (*auto simp*: *interval_lebesgue_integral_def interval_lebesgue_integrable_def*

set_lebesgue_integral_def
intro!: *integral_nonneg_AE abs_of_nonneg*)
then show *?case*
using *le* **by** (*simp add*: *interval_integral_norm*)
qed

lemma *integral_cos*: $t \neq 0 \Longrightarrow LBINT\ x{=}a..b.\ cos\ (t * x) = sin\ (t * b) / t - sin\ (t * a) / t$
apply (*intro interval_integral_FTC_finite continuous_intros*)
by (*auto intro*!: *derivative_eq_intros simp*: *has_real_derivative_iff_has_vector_derivative*[*symmetric*])

end

9.11 Integration by Substition for the Lebesgue Integral

theory *Lebesgue_Integral_Substitution*
imports *Interval_Integral*
begin

lemma *nn_integral_substitution_aux*:
fixes $f :: real \Rightarrow ennreal$
assumes *Mf*: $f \in borel\_measurable\ borel$
assumes *nonnegf*: $\bigwedge x.\ f\ x \geq 0$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_real\_derivative\ g'\ x)\ (at\ x)$
assumes *contg'*: $continuous\_on\ \{a..b\}\ g'$
assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes $a < b$
shows $(\int^+ x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x\ \partial lborel) =$
$(\int^+ x.\ f\ (g\ x) * g'\ x * indicator\ \{a..b\}\ x\ \partial lborel)$
proof−
from ‹$a < b$› **have** [*simp*]: $a \leq b$ **by** *simp*
from *derivg* **have** *contg*: $continuous\_on\ \{a..b\}\ g$ **by** (*rule has_real_derivative_imp_continuous_on*)
from *this* **and** *contg'* **have** *Mg*: $set\_borel\_measurable\ borel\ \{a..b\}\ g$ **and**
Mg': $set\_borel\_measurable\ borel\ \{a..b\}\ g'$
by (*simp_all only*: *set_measurable_continuous_on_ivl*)
from *derivg* **have** *derivg'*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
by (*simp only*: *has_real_derivative_iff_has_vector_derivative*)

have *real_ind*[*simp*]: $\bigwedge A\ x.\ enn2real\ (indicator\ A\ x) = indicator\ A\ x$
by (*auto split*: *split_indicator*)
have *ennreal_ind*[*simp*]: $\bigwedge A\ x.\ ennreal\ (indicator\ A\ x) = indicator\ A\ x$
by (*auto split*: *split_indicator*)
have [*simp*]: $\bigwedge x\ A.\ indicator\ A\ (g\ x) = indicator\ (g -` A)\ x$
by (*auto split*: *split_indicator*)

```
  from derivg derivg_nonneg have monog: ⋀x y. a ≤ x ⟹ x ≤ y ⟹ y ≤ b ⟹
g x ≤ g y
    by (rule deriv_nonneg_imp_mono) simp_all
  with monog have [simp]: g a ≤ g b by (auto intro: mono_onD)

  show ?thesis
  proof (induction rule: borel_measurable_induct[OF Mf, case_names cong set
mult add sup])
    case (cong f1 f2)
    from cong.hyps(3) have f1 = f2 by auto
    with cong show ?case by simp
  next
    case (set A)
    from set.hyps show ?case
    proof (induction rule: borel_set_induct)
      case empty
      thus ?case by simp
    next
      case (interval c d)
      {
        fix u v :: real assume asm: {u..v} ⊆ {g a..g b} u ≤ v

        obtain u' v' where u'v': {a..b} ∩ g-`{u..v} = {u'..v'} u' ≤ v' g u' = u g
v' = v
            using asm by (rule_tac continuous_interval_vimage_Int[OF contg
monog, of u v]) simp_all
        hence {u'..v'} ⊆ {a..b} {u'..v'} ⊆ g -` {u..v} by blast+
        with u'v'(2) have u' ∈ g -` {u..v} v' ∈ g -` {u..v} by auto
        from u'v'(1) have [simp]: {a..b} ∩ g -` {u..v} ∈ sets borel by simp

        have A: continuous_on {min u' v'..max u' v'} g'
          by (simp only: u'v' max_absorb2 min_absorb1)
            (intro continuous_on_subset[OF contg'], insert u'v', auto)
        have ⋀x. x ∈ {u'..v'} ⟹ (g has_real_derivative g' x) (at x within {u'..v'})
          using asm by (intro has_field_derivative_subset[OF derivg] subsetD[OF
‹{u'..v'} ⊆ {a..b}›]) auto
        hence B: ⋀x. min u' v' ≤ x ⟹ x ≤ max u' v' ⟹
                  (g has_vector_derivative g' x) (at x within {min u' v'..max u'
v'})
          by (simp only: u'v' max_absorb2 min_absorb1)
            (auto simp: has_real_derivative_iff_has_vector_derivative)
        have integrable lborel (λx. indicator ({a..b} ∩ g -` {u..v}) x *R g' x)
          using set_integrable_subset borel_integrable_atLeastAtMost'[OF contg']
          by (metis ‹{u'..v'} ⊆ {a..b}› eucl_ivals(5) set_integrable_def sets_lborel
u'v'(1))
        hence (∫+x. ennreal (g' x) * indicator ({a..b} ∩ g-`{u..v}) x ∂lborel) =
                (LBINT x:{a..b} ∩ g-`{u..v}. g' x)
          unfolding set_lebesgue_integral_def
```

```
        by (subst nn_integral_eq_integral[symmetric])
           (auto intro!: derivg_nonneg nn_integral_cong split: split_indicator)
      also from interval_integral_FTC_finite[OF A B]
        have (LBINT x:{a..b} ∩ g-`{u..v}. g' x) = v - u
          by (simp add: u'v' interval_integral_Icc ‹u ≤ v›)
       finally have (∫⁺ x. ennreal (g' x) * indicator ({a..b} ∩ g -` {u..v}) x ∂lborel) =
                    ennreal (v - u) .
    } note A = this

    have (∫⁺x. indicator {c..d} (g x) * ennreal (g' x) * indicator {a..b} x ∂lborel) =
          (∫⁺ x. ennreal (g' x) * indicator ({a..b} ∩ g -` {c..d}) x ∂lborel)
      by (intro nn_integral_cong) (simp split: split_indicator)
    also have {a..b} ∩ g-`{c..d} = {a..b} ∩ g-`{max (g a) c..min (g b) d}
      using ‹a ≤ b› ‹c ≤ d›
      by (auto intro!: monog intro: order.trans)
    also have (∫⁺ x. ennreal (g' x) * indicator ... x ∂lborel) =
      (if max (g a) c ≤ min (g b) d then min (g b) d - max (g a) c else 0)
       using ‹c ≤ d› by (simp add: A)
    also have ... = (∫⁺ x. indicator ({g a..g b} ∩ {c..d}) x ∂lborel)
      by (subst nn_integral_indicator) (auto intro!: measurable_sets Mg simp:)
    also have ... = (∫⁺ x. indicator {c..d} x * indicator {g a..g b} x ∂lborel)
      by (intro nn_integral_cong) (auto split: split_indicator)
    finally show ?case ..

    next

    case (compl A)
    note ‹A ∈ sets borel›[measurable]
    from emeasure_mono[of A ∩ {g a..g b} {g a..g b} lborel]
   have [simp]: emeasure lborel (A ∩ {g a..g b}) ≠ top by (auto simp: top_unique)
    have [simp]: g -` A ∩ {a..b} ∈ sets borel
      by (rule set_borel_measurable_sets[OF Mg]) auto
    have [simp]: g -` (-A) ∩ {a..b} ∈ sets borel
      by (rule set_borel_measurable_sets[OF Mg]) auto

    have (∫⁺x. indicator (-A) x * indicator {g a..g b} x ∂lborel) =
           (∫⁺x. indicator (-A ∩ {g a..g b}) x ∂lborel)
      by (rule nn_integral_cong) (simp split: split_indicator)
     also from compl have ... = emeasure lborel ({g a..g b} - A) using derivg_nonneg
      by (simp add: vimage_Compl diff_eq Int_commute[of -A])
    also have {g a..g b} - A = {g a..g b} - A ∩ {g a..g b} by blast
    also have emeasure lborel ... = g b - g a - emeasure lborel (A ∩ {g a..g b})
         using ‹A ∈ sets borel› by (subst emeasure_Diff) auto
   also have emeasure lborel (A ∩ {g a..g b}) =
                ∫⁺x. indicator A x * indicator {g a..g b} x ∂lborel
     using ‹A ∈ sets borel›
```

by (*subst nn_integral_indicator*[*symmetric*], *simp*, *intro nn_integral_cong*)
(*simp split*: *split_indicator*)
also have ... = $\int^+ x.$ *indicator* ($g - `A \cap \{a..b\}$) $x *$ *ennreal* ($g' x *$ *indicator* $\{a..b\}$ x) ∂*lborel* (**is** _ = *?I*)
by (*subst compl.IH*, *intro nn_integral_cong*) (*simp split*: *split_indicator*)
also have $g\ b - g\ a =$ (*LBINT* x:$\{a..b\}$. $g'\ x$) **using** *derivg'*
unfolding *set_lebesgue_integral_def*
by (*intro integral_FTC_atLeastAtMost*[*symmetric*])
(*auto intro*: *continuous_on_subset*[*OF contg'*] *has_field_derivative_subset*[*OF derivg*]
has_vector_derivative_at_within)
also have *ennreal* ... = ($\int^+ x.$ $g' x *$ *indicator* $\{a..b\}$ x ∂*lborel*)
using *borel_integrable_atLeastAtMost'*[*OF contg'*] **unfolding** *set_lebesgue_integral_def*
by (*subst nn_integral_eq_integral*)
(*simp_all add*: *mult.commute derivg_nonneg set_integrable_def split*: *split_indicator*)
also have *Mg''*: ($\lambda x.$ *indicator* ($g - `A \cap \{a..b\}$) $x *$ *ennreal* ($g' x *$ *indicator* $\{a..b\}$ x))
$\in$ *borel_measurable borel* **using** *Mg'*
by (*intro borel_measurable_times_ennreal borel_measurable_indicator*)
(*simp_all add*: *mult.commute set_borel_measurable_def*)
have *le*: ($\int^+ x.$ *indicator* ($g - `A \cap \{a..b\}$) $x *$ *ennreal* ($g' x *$ *indicator* $\{a..b\}$ x) ∂*lborel*) $\leq$
($\int^+ x.$ *ennreal* ($g' x$) $*$ *indicator* $\{a..b\}$ x ∂*lborel*)
by (*intro nn_integral_mono*) (*simp split*: *split_indicator add*: *derivg_nonneg*)
note *integrable* = *borel_integrable_atLeastAtMost'*[*OF contg'*]
with *le* **have** *notinf*: ($\int^+ x.$ *indicator* ($g - `A \cap \{a..b\}$) $x *$ *ennreal* ($g' x *$ *indicator* $\{a..b\}$ x) ∂*lborel*) $\neq$ *top*
by (*auto simp*: *real_integrable_def nn_integral_set_ennreal mult.commute top_unique set_integrable_def*)
have ($\int^+ x.$ $g' x *$ *indicator* $\{a..b\}$ x ∂*lborel*) $-$ *?I* =
$\int^+ x.$ *ennreal* ($g' x *$ *indicator* $\{a..b\}$ x) $-$
indicator ($g - `A \cap \{a..b\}$) $x *$ *ennreal* ($g' x *$ *indicator* $\{a..b\}$ x) ∂*lborel*
apply (*intro nn_integral_diff*[*symmetric*])
apply (*insert Mg'*, *simp add*: *mult.commute set_borel_measurable_def*) []
apply (*insert Mg''*, *simp*) []
apply (*simp split*: *split_indicator add*: *derivg_nonneg*)
apply (*rule notinf*)
apply (*simp split*: *split_indicator add*: *derivg_nonneg*)
done
also have ... = $\int^+ x.$ *indicator* ($-A$) ($g\ x$) $*$ *ennreal* ($g' x$) $*$ *indicator* $\{a..b\}$ x ∂*lborel*
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
finally show *?case* **.**

next
case (*union f*)
then have [*simp*]: $\bigwedge i.$ $\{a..b\} \cap g - `f\ i \in$ *sets borel*

by (*subst Int_commute*, *intro set_borel_measurable_sets*[*OF Mg*]) *auto*
have $g -` (\bigcup i.\ f\ i) \cap \{a..b\} = (\bigcup i.\ \{a..b\} \cap g -` f\ i)$ **by** *auto*
hence $g -` (\bigcup i.\ f\ i) \cap \{a..b\} \in$ *sets borel* **by** (*auto simp del*: *UN_simps*)

have $(\int^+ x.$ *indicator* $(\bigcup i.\ f\ i)\ x * $ *indicator* $\{g\ a..g\ b\}\ x\ \partial$*lborel*$) =$
$\int^+ x.$ *indicator* $(\bigcup i.\ \{g\ a..g\ b\} \cap f\ i)\ x\ \partial$*lborel*
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also from *union* **have** $\ldots =$ *emeasure lborel* $(\bigcup i.\ \{g\ a..g\ b\} \cap f\ i)$ **by** *simp*
also from *union* **have** $\ldots = (\sum i.$ *emeasure lborel* $(\{g\ a..g\ b\} \cap f\ i))$
by (*intro suminf_emeasure*[*symmetric*]) (*auto simp*: *disjoint_family_on_def*)
also from *union* **have** $\ldots = (\sum i.\ \int^+ x.$ *indicator* $(\{g\ a..g\ b\} \cap f\ i)\ x\ \partial$*lborel*$)$
by *simp*
also have $(\lambda i.\ \int^+ x.$ *indicator* $(\{g\ a..g\ b\} \cap f\ i)\ x\ \partial$*lborel*$) =$
$(\lambda i.\ \int^+ x.$ *indicator* $(f\ i)\ x * $ *indicator* $\{g\ a..g\ b\}\ x\ \partial$*lborel*$)$
by (*intro ext nn_integral_cong*) (*simp split*: *split_indicator*)
also from *union.IH* **have** $(\sum i.\ \int^+ x.$ *indicator* $(f\ i)\ x * $ *indicator* $\{g\ a..g\ b\}\ x\ \partial$*lborel*$) =$
$(\sum i.\ \int^+ x.$ *indicator* $(f\ i)\ (g\ x) *$ *ennreal* $(g'\ x) *$ *indicator* $\{a..b\}\ x\ \partial$*lborel*$)$ **by** *simp*
also have $(\lambda i.\ \int^+ x.$ *indicator* $(f\ i)\ (g\ x) *$ *ennreal* $(g'\ x) *$ *indicator* $\{a..b\}\ x\ \partial$*lborel*$) =$
$(\lambda i.\ \int^+ x.$ *ennreal* $(g'\ x *$ *indicator* $\{a..b\}\ x) *$ *indicator* $(\{a..b\} \cap g -` f\ i)\ x\ \partial$*lborel*$)$
by (*intro ext nn_integral_cong*) (*simp split*: *split_indicator*)
also have $(\sum i.\ \ldots\ i) = \int^+ x.\ (\sum i.$ *ennreal* $(g'\ x *$ *indicator* $\{a..b\}\ x) *$ *indicator* $(\{a..b\} \cap g -` f\ i)\ x)\ \partial$*lborel*
using *Mg'*
apply (*intro nn_integral_suminf*[*symmetric*])
apply (*rule borel_measurable_times_ennreal*, *simp add*: *mult.commute set_borel_measurable_def*)
apply (*rule borel_measurable_indicator*, *subst sets_lborel*)
apply (*simp_all split*: *split_indicator add*: *derivg_nonneg*)
done
also have $(\lambda x\ i.$ *ennreal* $(g'\ x *$ *indicator* $\{a..b\}\ x) *$ *indicator* $(\{a..b\} \cap g -` f\ i)\ x) =$
$(\lambda x\ i.$ *ennreal* $(g'\ x *$ *indicator* $\{a..b\}\ x) *$ *indicator* $(g -` f\ i)\ x)$
by (*intro ext*) (*simp split*: *split_indicator*)
also have $(\int^+ x.\ (\sum i.$ *ennreal* $(g'\ x *$ *indicator* $\{a..b\}\ x) *$ *indicator* $(g -` f\ i)\ x)\ \partial$*lborel*$) =$
$\int^+ x.$ *ennreal* $(g'\ x *$ *indicator* $\{a..b\}\ x) * (\sum i.$ *indicator* $(g -` f\ i)\ x)\ \partial$*lborel*
by (*intro nn_integral_cong*) (*auto split*: *split_indicator simp*: *derivg_nonneg*)
also from *union* **have** $(\lambda x.\ \sum i.$ *indicator* $(g -` f\ i)\ x ::$ *ennreal*$) = (\lambda x.$ *indicator* $(\bigcup i.\ g -` f\ i)\ x)$
by (*intro ext suminf_indicator*) (*auto simp*: *disjoint_family_on_def*)
also have $(\int^+ x.$ *ennreal* $(g'\ x *$ *indicator* $\{a..b\}\ x) * \ldots\ x\ \partial$*lborel*$) =$
$(\int^+ x.$ *indicator* $(\bigcup i.\ f\ i)\ (g\ x) *$ *ennreal* $(g'\ x) *$ *indicator* $\{a..b\}\ x\ \partial$*lborel*$)$
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)

```
    finally show ?case .
  qed

next
  case (mult f c)
    note Mf[measurable] = ‹f ∈ borel_measurable borel›
    let ?I = indicator {a..b}
   have (λx. f (g x * ?I x) * ennreal (g' x * ?I x)) ∈ borel_measurable borel using
Mg Mg'
      by (intro borel_measurable_times_ennreal measurable_compose[OF _ Mf])
         (simp_all add: mult.commute set_borel_measurable_def)
    also have (λx. f (g x * ?I x) * ennreal (g' x * ?I x)) = (λx. f (g x) * ennreal
(g' x) * ?I x)
      by (intro ext) (simp split: split_indicator)
   finally have Mf': (λx. f (g x) * ennreal (g' x) * ?I x) ∈ borel_measurable borel
.
    with mult show ?case
      by (subst (1 2 3) mult_ac, subst (1 2) nn_integral_cmult) (simp_all add:
mult_ac)

next
  case (add f2 f1)
    let ?I = indicator {a..b}
    {
      fix f :: real ⇒ ennreal assume Mf: f ∈ borel_measurable borel
      have (λx. f (g x * ?I x) * ennreal (g' x * ?I x)) ∈ borel_measurable borel
using Mg Mg'
        by (intro borel_measurable_times_ennreal measurable_compose[OF _ Mf])
           (simp_all add:  mult.commute set_borel_measurable_def)
     also have (λx. f (g x * ?I x) * ennreal (g' x * ?I x)) = (λx. f (g x) * ennreal
(g' x) * ?I x)
        by (intro ext) (simp split: split_indicator)
      finally have (λx. f (g x) * ennreal (g' x) * ?I x) ∈ borel_measurable borel .
    } note Mf' = this[OF ‹f1 ∈ borel_measurable borel›] this[OF ‹f2 ∈ borel_measurable
borel›]

    have (∫⁺ x. (f1 x + f2 x) * indicator {g a..g b} x ∂lborel) =
              (∫⁺ x. f1 x * indicator {g a..g b} x + f2 x * indicator {g a..g b} x
∂lborel)
      by (intro nn_integral_cong) (simp split: split_indicator)
    also from add have ... = (∫⁺ x. f1 (g x) * ennreal (g' x) * indicator {a..b}
x ∂lborel) +
                                 (∫⁺ x. f2 (g x) * ennreal (g' x) * indicator {a..b} x
∂lborel)
      by (simp_all add: nn_integral_add)
    also from add have ... = (∫⁺ x. f1 (g x) * ennreal (g' x) * indicator {a..b}
x +
                                  f2 (g x) * ennreal (g' x) * indicator {a..b} x ∂lborel)
      by (intro nn_integral_add[symmetric])
```

```
      (auto simp add: Mf' derivg_nonneg split: split_indicator)
  also have ... = ∫⁺ x. (f1 (g x) + f2 (g x)) * ennreal (g' x) * indicator {a..b}
x ∂lborel
    by (intro nn_integral_cong) (simp split: split_indicator add: distrib_right)
  finally show ?case .

next
 case (sup F)
 {
  fix i
  let ?I = indicator {a..b}
  have (λx. F i (g x * ?I x) * ennreal (g' x * ?I x)) ∈ borel_measurable borel
using Mg Mg'
    by (rule_tac borel_measurable_times_ennreal, rule_tac measurable_compose[OF
_ sup.hyps(1)])
       (simp_all add: mult.commute set_borel_measurable_def)
   also have (λx. F i (g x * ?I x) * ennreal (g' x * ?I x)) = (λx. F i (g x) *
ennreal (g' x) * ?I x)
    by (intro ext) (simp split: split_indicator)
   finally have ... ∈ borel_measurable borel .
 } note Mf' = this

  have (∫⁺x. (SUP i. F i x) * indicator {g a..g b} x ∂lborel) =
            ∫⁺x. (SUP i. F i x* indicator {g a..g b} x) ∂lborel
    by (intro nn_integral_cong) (simp split: split_indicator)
  also from sup have ... = (SUP i. ∫⁺x. F i x* indicator {g a..g b} x ∂lborel)
    by (intro nn_integral_monotone_convergence_SUP)
      (auto simp: incseq_def le_fun_def split: split_indicator)
  also from sup have ... = (SUP i. ∫⁺x. F i (g x) * ennreal (g' x) * indicator
{a..b} x ∂lborel)
    by simp
  also from sup have ... =  ∫⁺x. (SUP i. F i (g x) * ennreal (g' x) * indicator
{a..b} x) ∂lborel
    by (intro nn_integral_monotone_convergence_SUP[symmetric])
      (auto simp: incseq_def le_fun_def derivg_nonneg Mf' split: split_indicator
          intro!: mult_right_mono)
  also from sup have ... = ∫⁺x. (SUP i. F i (g x)) * ennreal (g' x) * indicator
{a..b} x ∂lborel
    by (subst mult.assoc, subst mult.commute, subst SUP_mult_left_ennreal)
      (auto split: split_indicator simp: derivg_nonneg mult_ac)
  finally show ?case by (simp add: image_comp)
 qed
qed

theorem nn_integral_substitution:
 fixes f :: real ⇒ real
 assumes Mf[measurable]: set_borel_measurable borel {g a..g b} f
 assumes derivg: ⋀x. x ∈ {a..b} ⟹ (g has_real_derivative g' x) (at x)
 assumes contg': continuous_on {a..b} g'
```

assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes $a \leq b$
shows $(\int^+ x.\ f\ x * \mathit{indicator}\ \{g\ a..g\ b\}\ x\ \partial \mathit{lborel}) =$
$(\int^+ x.\ f\ (g\ x) * g'\ x * \mathit{indicator}\ \{a..b\}\ x\ \partial \mathit{lborel})$
proof (*cases* $a = b$)
assume $a \neq b$
with ‹$a \leq b$› **have** $a < b$ **by** *auto*
let *?f'* $= \lambda x.\ f\ x * \mathit{indicator}\ \{g\ a..g\ b\}\ x$

from *derivg derivg_nonneg* **have** *monog*: $\bigwedge x\ y.\ a \leq x \Longrightarrow x \leq y \Longrightarrow y \leq b \Longrightarrow g\ x \leq g\ y$
by (*rule deriv_nonneg_imp_mono*) *simp_all*
have *bounds*: $\bigwedge x.\ x \geq a \Longrightarrow x \leq b \Longrightarrow g\ x \geq g\ a\ \bigwedge x.\ x \geq a \Longrightarrow x \leq b \Longrightarrow g\ x \leq g\ b$
by (*auto intro*: *monog*)

from *derivg_nonneg* **have** *nonneg*:
$\bigwedge f\ x.\ x \geq a \Longrightarrow x \leq b \Longrightarrow g'\ x \neq 0 \Longrightarrow f\ x * \mathit{ennreal}\ (g'\ x) \geq 0 \Longrightarrow f\ x \geq 0$
by (*force simp*: *field_simps*)
have *nonneg'*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow \neg\ 0 \leq f\ (g\ x) \Longrightarrow 0 \leq f\ (g\ x) * g'\ x \Longrightarrow g'\ x = 0$
by (*metis atLeastAtMost_iff derivg_nonneg eq_iff mult_eq_0_iff mult_le_0_iff*)

have $(\int^+ x.\ f\ x * \mathit{indicator}\ \{g\ a..g\ b\}\ x\ \partial \mathit{lborel}) =$
$(\int^+ x.\ \mathit{ennreal}\ (\mathit{?f'}\ x) * \mathit{indicator}\ \{g\ a..g\ b\}\ x\ \partial \mathit{lborel})$
by (*intro nn_integral_cong*)
(*auto split*: *split_indicator split_max simp*: *zero_ennreal.rep_eq ennreal_neg*)
also have ... $= \int^+ x.\ \mathit{?f'}\ (g\ x) * \mathit{ennreal}\ (g'\ x) * \mathit{indicator}\ \{a..b\}\ x\ \partial \mathit{lborel}$ **using** *Mf*
by (*subst nn_integral_substitution_aux*[*OF* _ _ *derivg contg' derivg_nonneg* ‹$a < b$›])
(*auto simp add*: *mult.commute set_borel_measurable_def*)
also have ... $= \int^+ x.\ f\ (g\ x) * \mathit{ennreal}\ (g'\ x) * \mathit{indicator}\ \{a..b\}\ x\ \partial \mathit{lborel}$
by (*intro nn_integral_cong*) (*auto split*: *split_indicator simp*: *max_def dest*: *bounds*)
also have ... $= \int^+ x.\ \mathit{ennreal}\ (f\ (g\ x) * g'\ x * \mathit{indicator}\ \{a..b\}\ x)\ \partial \mathit{lborel}$
by (*intro nn_integral_cong*) (*auto simp*: *mult.commute derivg_nonneg ennreal_mult' split*: *split_indicator*)
finally show *?thesis* **.**
qed *auto*

theorem *integral_substitution*:
assumes *integrable*: *set_integrable lborel* $\{g\ a..g\ b\}\ f$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ \mathit{has\_real\_derivative}\ g'\ x)\ (\mathit{at}\ x)$
assumes *contg'*: *continuous_on* $\{a..b\}\ g'$
assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes $a \leq b$
shows *set_integrable lborel* $\{a..b\}\ (\lambda x.\ f\ (g\ x) * g'\ x)$
and $(\mathit{LBINT}\ x.\ f\ x * \mathit{indicator}\ \{g\ a..g\ b\}\ x) = (\mathit{LBINT}\ x.\ f\ (g\ x) * g'\ x *$

indicator {*a*..*b*} *x*)
proof−
from *derivg* **have** *contg*: *continuous_on* {*a*..*b*} *g* **by** (*rule has_real_derivative_imp_continuous_on*)
with *contg*′ **have** *Mg*: *set_borel_measurable borel* {*a*..*b*} *g*
and *Mg*′: *set_borel_measurable borel* {*a*..*b*} *g*′
by (*simp_all only*: *set_measurable_continuous_on_ivl*)
from *derivg derivg_nonneg* **have** *monog*: ⋀*x y*. *a* ≤ *x* ⟹ *x* ≤ *y* ⟹ *y* ≤ *b* ⟹
g x ≤ *g y*
by (*rule deriv_nonneg_imp_mono*) *simp_all*

have (λ*x*. *ennreal* (*f x*) ∗ *indicator* {*g a*..*g b*} *x*) =
(λ*x*. *ennreal* (*f x* ∗ *indicator* {*g a*..*g b*} *x*))
by (*intro ext*) (*simp split*: *split_indicator*)
with *integrable* **have** *M1*: (λ*x*. *f x* ∗ *indicator* {*g a*..*g b*} *x*) ∈ *borel_measurable*
borel
by (*force simp*: *mult.commute set_integrable_def*)
from *integrable* **have** *M2*: (λ*x*. −*f x* ∗ *indicator* {*g a*..*g b*} *x*) ∈ *borel_measurable*
borel
by (*force simp*: *mult.commute set_integrable_def*)

have (*LBINT x*. (*f x* :: *real*) ∗ *indicator* {*g a*..*g b*} *x*) =
enn2real (∫⁺ *x*. *ennreal* (*f x*) ∗ *indicator* {*g a*..*g b*} *x* ∂*lborel*) −
enn2real (∫⁺ *x*. *ennreal* (− (*f x*)) ∗ *indicator* {*g a*..*g b*} *x* ∂*lborel*) **using**
integrable
unfolding *set_integrable_def*
by (*subst real_lebesgue_integral_def*) (*simp_all add*: *nn_integral_set_ennreal*
mult.commute)
also have ∗: (∫⁺*x*. *ennreal* (*f x*) ∗ *indicator* {*g a*..*g b*} *x* ∂*lborel*) =
(∫⁺*x*. *ennreal* (*f x* ∗ *indicator* {*g a*..*g b*} *x*) ∂*lborel*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also from *M1* ∗ **have** *A*: (∫⁺ *x*. *ennreal* (*f x* ∗ *indicator* {*g a*..*g b*} *x*) ∂*lborel*)
=
(∫⁺ *x*. *ennreal* (*f* (*g x*) ∗ *g*′ *x* ∗ *indicator* {*a*..*b*} *x*) ∂*lborel*)
by (*subst nn_integral_substitution*[*OF* _ *derivg contg*′ *derivg_nonneg* ‹*a* ≤ *b*›])
(*auto simp*: *nn_integral_set_ennreal mult.commute set_borel_measurable_def*)
also have ∗∗: (∫⁺ *x*. *ennreal* (− (*f x*)) ∗ *indicator* {*g a*..*g b*} *x* ∂*lborel*) =
(∫⁺ *x*. *ennreal* (− (*f x*) ∗ *indicator* {*g a*..*g b*} *x*) ∂*lborel*)
by (*intro nn_integral_cong*) (*simp split*: *split_indicator*)
also from *M2* ∗∗ **have** *B*: (∫⁺ *x*. *ennreal* (− (*f x*) ∗ *indicator* {*g a*..*g b*} *x*)
∂*lborel*) =
(∫⁺ *x*. *ennreal* (− (*f* (*g x*)) ∗ *g*′ *x* ∗ *indicator* {*a*..*b*} *x*) ∂*lborel*)
by (*subst nn_integral_substitution*[*OF* _ *derivg contg*′ *derivg_nonneg* ‹*a* ≤ *b*›])
(*auto simp*: *nn_integral_set_ennreal mult.commute set_borel_measurable_def*)

also {
from *integrable* **have** *Mf*: *set_borel_measurable borel* {*g a*..*g b*} *f*
unfolding *set_borel_measurable_def set_integrable_def* **by** *simp*
from *measurable_compose Mg Mf Mg*′ *borel_measurable_times*
have (λ*x*. *f* (*g x* ∗ *indicator* {*a*..*b*} *x*) ∗ *indicator* {*g a*..*g b*} (*g x* ∗ *indicator*

```
{a..b} x) *
                (g' x * indicator {a..b} x)) ∈ borel_measurable borel  (is ?f ∈ _)
    by (simp add: mult.commute set_borel_measurable_def)
  also have ?f = (λx. f (g x) * g' x * indicator {a..b} x)
    using monog by (intro ext) (auto split: split_indicator)
  finally show set_integrable lborel {a..b} (λx. f (g x) * g' x)
    using A B integrable unfolding real_integrable_def set_integrable_def
    by (simp_all add: nn_integral_set_ennreal mult.commute)
 } note integrable' = this

 have enn2real (∫+ x. ennreal (f (g x) * g' x * indicator {a..b} x) ∂lborel) -
              enn2real (∫+ x. ennreal (-f (g x) * g' x * indicator {a..b} x)
∂lborel) =
            (LBINT x. f (g x) * g' x * indicator {a..b} x)
   using integrable' unfolding set_integrable_def
   by (subst real_lebesgue_integral_def) (simp_all add: field_simps)
 finally show (LBINT x. f x * indicator {g a..g b} x) =
              (LBINT x. f (g x) * g' x * indicator {a..b} x) .
qed

theorem interval_integral_substitution:
 assumes integrable: set_integrable lborel {g a..g b} f
 assumes derivg: ⋀x. x ∈ {a..b} ⟹ (g has_real_derivative g' x) (at x)
 assumes contg': continuous_on {a..b} g'
 assumes derivg_nonneg: ⋀x. x ∈ {a..b} ⟹ g' x ≥ 0
 assumes a ≤ b
 shows set_integrable lborel {a..b} (λx. f (g x) * g' x)
   and (LBINT x=g a..g b. f x) = (LBINT x=a..b. f (g x) * g' x)
 apply (rule integral_substitution[OF assms], simp, simp)
 apply (subst (1 2) interval_integral_Icc, fact)
 apply (rule deriv_nonneg_imp_mono[OF derivg derivg_nonneg], simp, simp,
fact)
 using integral_substitution(2)[OF assms]
 apply (simp add: mult.commute set_lebesgue_integral_def)
 done

lemma set_borel_integrable_singleton[simp]: set_integrable lborel {x} (f :: real ⇒
real)
 unfolding set_integrable_def
 by (subst integrable_discrete_difference[where X={x} and g=λ_. 0]) auto

end
```

9.12 The Volume of an n-Dimensional Ball

```
theory Ball_Volume
 imports Gamma_Function Lebesgue_Integral_Substitution
begin
```

We define the volume of the unit ball in terms of the Gamma function. Note that the dimension need not be an integer; we also allow fractional dimensions, although we do not use this case or prove anything about it for now.

definition *unit_ball_vol* :: *real* ⇒ *real* **where**
unit_ball_vol n = *pi powr* (*n* / *2*) / *Gamma* (*n* / *2* + *1*)

lemma *unit_ball_vol_pos* [*simp*]: *n* ≥ *0* ⟹ *unit_ball_vol n* > *0*
by (*force simp*: *unit_ball_vol_def intro*: *divide_nonneg_pos*)

lemma *unit_ball_vol_nonneg* [*simp*]: *n* ≥ *0* ⟹ *unit_ball_vol n* ≥ *0*
by (*simp add*: *dual_order.strict_implies_order*)

We first need the value of the following integral, which is at the core of computing the measure of an *n* + *1*-dimensional ball in terms of the measure of an *n*-dimensional one.

lemma *emeasure_cball_aux_integral*:
(∫⁺*x*. *indicator* {−*1*..*1*} *x* ∗ *sqrt* (*1* − *x*²) ^ *n* ∂*lborel*) =
ennreal (*Beta* (*1* / *2*) (*real n* / *2* + *1*))
proof −
have ((λ*t*. *t powr* (−*1* / *2*) ∗ (*1* − *t*) *powr* (*real n* / *2*)) *has_integral*
Beta (*1* / *2*) (*real n* / *2* + *1*)) {*0*..*1*}
using *has_integral_Beta_real*[*of 1/2 n* / *2* + *1*] **by** *simp*
from *nn_integral_has_integral_lebesgue*[*OF* _ *this*] **have**
ennreal (*Beta* (*1* / *2*) (*real n* / *2* + *1*)) =
nn_integral lborel (λ*t*. *ennreal* (*t powr* (−*1* / *2*) ∗ (*1* − *t*) *powr* (*real n* / *2*) ∗
indicator {*0*^*2*..*1*^*2*} *t*))
by (*simp add*: *mult_ac ennreal_mult*′ *ennreal_indicator*)
also have ... = (∫⁺ *x*. *ennreal* (*x*² *powr* − (*1* / *2*) ∗ (*1* − *x*²) *powr* (*real n* / *2*) ∗ (*2* ∗ *x*) ∗
indicator {*0*..*1*} *x*) ∂*lborel*)
by (*subst nn_integral_substitution*[**where** *g* = λ*x*. *x* ^ *2* **and** *g*′ = λ*x*. *2* ∗ *x*])
(*auto intro*!: *derivative_eq_intros continuous_intros simp*: *set_borel_measurable_def*)
also have ... = (∫⁺ *x*. *2* ∗ *ennreal* ((*1* − *x*²) *powr* (*real n* / *2*) ∗ *indicator* {*0*..*1*} *x*) ∂*lborel*)
by (*intro nn_integral_cong_AE AE_I*[*of* _ _ {*0*}])
(*auto simp*: *indicator_def powr_minus powr_half_sqrt field_split_simps ennreal_mult*′)
also have ... = (∫⁺ *x*. *ennreal* ((*1* − *x*²) *powr* (*real n* / *2*) ∗ *indicator* {*0*..*1*} *x*) ∂*lborel*) +
(∫⁺ *x*. *ennreal* ((*1* − *x*²) *powr* (*real n* / *2*) ∗ *indicator* {*0*..*1*} *x*) ∂*lborel*)
(**is** _ = *?I* + _) **by** (*simp add*: *mult_2 nn_integral_add*)
also have *?I* = (∫⁺ *x*. *ennreal* ((*1* − *x*²) *powr* (*real n* / *2*) ∗ *indicator* {−*1*..*0*} *x*) ∂*lborel*)
by (*subst nn_integral_real_affine*[*of* _ −*1 0*])
(*auto simp*: *indicator_def intro*!: *nn_integral_cong*)

```
  hence ?I + ?I = ... + ?I by simp
  also have ... = (∫⁺ x. ennreal ((1 - x²) powr (real n / 2) *
                  (indicator {-1..0} x + indicator{0..1} x)) ∂lborel)
    by (subst nn_integral_add [symmetric]) (auto simp: algebra_simps)
  also have ... = (∫⁺ x. ennreal ((1 - x²) powr (real n / 2) * indicator {-1..1}
x) ∂lborel)
    by (intro nn_integral_cong_AE AE_I[of _ _ {0}]) (auto simp: indicator_def)
  also have ... = (∫⁺ x. ennreal (indicator {-1..1} x * sqrt (1 - x²) ^ n)
∂lborel)
    by (intro nn_integral_cong_AE AE_I[of _ _ {1, -1}])
       (auto simp: powr_half_sqrt [symmetric] indicator_def abs_square_le_1
       abs_square_eq_1 powr_def exp_of_nat_mult [symmetric] emeasure_lborel_countable)
  finally show ?thesis ..
qed

lemma real_sqrt_le_iff': x ≥ 0 ⟹ y ≥ 0 ⟹ sqrt x ≤ y ⟷ x ≤ y ^ 2
  using real_le_lsqrt sqrt_le_D by blast
```

Isabelle's type system makes it very difficult to do an induction over the dimension of a Euclidean space type, because the type would change in the inductive step. To avoid this problem, we instead formulate the problem in a more concrete way by unfolding the definition of the Euclidean norm.

```
lemma emeasure_cball_aux:
  assumes finite A r > 0
  shows   emeasure (Pi_M A (λ_. lborel))
          ({f. sqrt (∑ i∈A. (f i)²) ≤ r} ∩ space (Pi_M A (λ_. lborel))) =
          ennreal (unit_ball_vol (real (card A)) * r ^ card A)
  using assms
proof (induction arbitrary: r)
  case (empty r)
  thus ?case
    by (simp add: unit_ball_vol_def space_PiM)
next
  case (insert i A r)
  interpret product_sigma_finite λ_. lborel
    by standard
  have emeasure (Pi_M (insert i A) (λ_. lborel))
           ({f. sqrt (∑ i∈insert i A. (f i)²) ≤ r} ∩ space (Pi_M (insert i A) (λ_.
lborel))) =
       nn_integral (Pi_M (insert i A) (λ_. lborel))
         (indicator ({f. sqrt (∑ i∈insert i A. (f i)²) ≤ r} ∩
         space (Pi_M (insert i A) (λ_. lborel))))
    by (subst nn_integral_indicator) auto
  also have ... = (∫⁺ y. ∫⁺ x. indicator ({f. sqrt ((f i)² + (∑ i∈A. (f i)²)) ≤
r} ∩
                                  space (Pi_M (insert i A) (λ_. lborel))) (x(i := y))
                ∂Pi_M A (λ_. lborel) ∂lborel)
    using insert.prems insert.hyps by (subst product_nn_integral_insert_rev) auto
  also have ... = (∫⁺ (y::real). ∫⁺ x. indicator {-r..r} y * indicator ({f. sqrt
```

```
((∑ i∈A. (f i)²)) ≤
            sqrt (r ^ 2 − y ^ 2)} ∩ space (Pi_M A (λ_. lborel))) x ∂Pi_M A (λ_.
lborel) ∂lborel)
  proof (intro nn_integral_cong, goal_cases)
    case (1 y f)
    have ∗: y ∈ {−r..r} if y ^ 2 + c ≤ r ^ 2 c ≥ 0 for c
    proof −
      have y ^ 2 ≤ y ^ 2 + c using that by simp
      also have ... ≤ r ^ 2 by fact
      finally show ?thesis
        using ‹r > 0› by (simp add: power2_le_iff_abs_le abs_if split: if_splits)
    qed
    have (∑ x∈A. (if x = i then y else f x)²) = (∑ x∈A. (f x)²)
      using insert.hyps by (intro sum.cong) auto
    thus ?case using 1 ‹r > 0›
    by (auto simp: sum_nonneg real_sqrt_le_iff' indicator_def PiE_def space_PiM
dest!: ∗)
  qed
  also have ... = (∫⁺ (y::real). indicator {−r..r} y ∗ (∫⁺ x. indicator ({f. sqrt
((∑ i∈A. (f i)²))
                                ≤ sqrt (r ^ 2 − y ^ 2)} ∩ space (Pi_M A (λ_. lborel))) x
                  ∂Pi_M A (λ_. lborel)) ∂lborel) by (subst nn_integral_cmult) auto
  also have ... = (∫⁺ (y::real). indicator {−r..r} y ∗ emeasure (PiM A (λ_.
lborel))
       ({f. sqrt ((∑ i∈A. (f i)²)) ≤ sqrt (r ^ 2 − y ^ 2)} ∩ space (Pi_M A (λ_.
lborel))) ∂lborel)
    using ‹finite A› by (intro nn_integral_cong, subst nn_integral_indicator) auto
  also have ... = (∫⁺ (y::real). indicator {−r..r} y ∗ ennreal (unit_ball_vol (real
(card A)) ∗
                                    (sqrt (r ^ 2 − y ^ 2)) ^ card A) ∂lborel)
  proof (intro nn_integral_cong_AE, goal_cases)
    case 1
    have AE y in lborel. y ∉ {−r,r}
      by (intro AE_not_in countable_imp_null_set_lborel) auto
    thus ?case
    proof eventually_elim
      case (elim y)
      show ?case
      proof (cases y ∈ {−r<..<r})
        case True
        hence y² < r² by (subst real_sqrt_less_iff [symmetric]) auto
        thus ?thesis by (subst insert.IH) (auto)
      qed (insert elim, auto)
    qed
  qed
  also have ... = ennreal (unit_ball_vol (real (card A))) ∗
                  (∫⁺ (y::real). indicator {−r..r} y ∗ (sqrt (r ^ 2 − y ^ 2)) ^ card
A ∂lborel)
    by (subst nn_integral_cmult [symmetric])
```

```
    (auto simp: mult_ac ennreal_mult' [symmetric] indicator_def intro!: nn_integral_cong)
  also have (∫⁺ (y::real). indicator {-r..r} y * (sqrt (r ^ 2 - y ^ 2)) ^ card A
∂lborel) =
              (∫⁺ (y::real). r ^ card A * indicator {-1..1} y * (sqrt (1 - y ^ 2))
^ card A
              ∂(distr lborel borel ((*) (1/r)))) using ‹r > 0›
    by (subst nn_integral_distr)
    (auto simp: indicator_def field_simps real_sqrt_divide intro!: nn_integral_cong)
  also have ... = (∫⁺ x. ennreal (r ^ Suc (card A)) *
              (indicator {- 1..1} x * sqrt (1 - x²) ^ card A) ∂lborel) using ‹r >
0›
    by (subst lborel_distr_mult) (auto simp: nn_integral_density ennreal_mult'
[symmetric] mult_ac)
  also have ... = ennreal (r ^ Suc (card A)) * (∫⁺ x. indicator {- 1..1} x *
                  sqrt (1 - x²) ^ card A ∂lborel)
    by (subst nn_integral_cmult) auto
  also note emeasure_cball_aux_integral
  also have ennreal (unit_ball_vol (real (card A))) * (ennreal (r ^ Suc (card A))
*
                ennreal (Beta (1/2) (card A / 2 + 1))) =
              ennreal (unit_ball_vol (card A) * Beta (1/2) (card A / 2 + 1) * r ^
Suc (card A))
    using ‹r > 0› by (simp add: ennreal_mult' [symmetric] mult_ac)
  also have unit_ball_vol (card A) * Beta (1/2) (card A / 2 + 1) = unit_ball_vol
(Suc (card A))
    by (auto simp: unit_ball_vol_def Beta_def Gamma_eq_zero_iff field_simps
        Gamma_one_half_real powr_half_sqrt [symmetric] powr_add [symmetric])
  also have Suc (card A) = card (insert i A) using insert.hyps by simp
  finally show ?case .
qed
```

We now get the main theorem very easily by just applying the above lemma.

```
context
  fixes c :: 'a :: euclidean_space and r :: real
  assumes r: r ≥ 0
begin

theorem emeasure_cball:
  emeasure lborel (cball c r) = ennreal (unit_ball_vol (DIM('a)) * r ^ DIM('a))
proof (cases r = 0)
  case False
  with r have r: r > 0 by simp
  have (lborel :: 'a measure) =
        distr (Pi_M Basis (λ_. lborel)) borel (λf. ∑ b∈Basis. f b *_R b)
    by (rule lborel_eq)
  also have emeasure ... (cball 0 r) =
              emeasure (Pi_M Basis (λ_. lborel))
              ({y. dist 0 (∑ b∈Basis. y b *_R b :: 'a) ≤ r} ∩ space (Pi_M Basis (λ_.
lborel)))
```

```
    by (subst emeasure_distr) (auto simp: cball_def)
  also have {f. dist 0 (∑ b∈Basis. f b *R b :: 'a) ≤ r} = {f. sqrt (∑ i∈Basis. (f
i)^2) ≤ r}
    by (subst euclidean_dist_l2) (auto simp: L2_set_def)
  also have emeasure (Pi_M Basis (λ_. lborel)) (... ∩ space (Pi_M Basis (λ_.
lborel))) =
              ennreal (unit_ball_vol (real DIM('a)) * r ^ DIM('a))
    using r by (subst emeasure_cball_aux) simp_all
  also have emeasure lborel (cball 0 r :: 'a set) =
              emeasure (distr lborel borel (λx. c + x)) (cball c r)
   by (subst emeasure_distr) (auto simp: cball_def dist_norm norm_minus_commute)
  also have distr lborel borel (λx. c + x) = lborel
    using lborel_affine[of 1 c] by (simp add: density_1)
  finally show ?thesis .
qed auto

corollary content_cball:
  content (cball c r) = unit_ball_vol (DIM('a)) * r ^ DIM('a)
  by (simp add: measure_def emeasure_cball r)

corollary emeasure_ball:
  emeasure lborel (ball c r) = ennreal (unit_ball_vol (DIM('a)) * r ^ DIM('a))
proof −
  from negligible_sphere[of c r] have sphere c r ∈ null_sets lborel
   by (auto simp: null_sets_completion_iff negligible_iff_null_sets negligible_convex_frontier)
  hence emeasure lborel (ball c r ∪ sphere c r :: 'a set) = emeasure lborel (ball c
r :: 'a set)
    by (intro emeasure_Un_null_set) auto
  also have ball c r ∪ sphere c r = (cball c r :: 'a set) by auto
  also have emeasure lborel ... = ennreal (unit_ball_vol (real DIM('a)) * r ^
DIM('a))
    by (rule emeasure_cball)
  finally show ?thesis ..
qed

corollary content_ball:
  content (ball c r) = unit_ball_vol (DIM('a)) * r ^ DIM('a)
  by (simp add: measure_def r emeasure_ball)

end
```

Lastly, we now prove some nicer explicit formulas for the volume of the unit balls in the cases of even and odd integer dimensions.

```
lemma unit_ball_vol_even:
  unit_ball_vol (real (2 * n)) = pi ^ n / fact n
  by (simp add: unit_ball_vol_def add_ac powr_realpow Gamma_fact)

lemma unit_ball_vol_odd':
      unit_ball_vol (real (2 * n + 1)) = pi ^ n / pochhammer (1 / 2) (Suc n)
```

```
  and unit_ball_vol_odd:
        unit_ball_vol (real (2 * n + 1)) =
          (2 ^ (2 * Suc n) * fact (Suc n)) / fact (2 * Suc n) * pi ^ n
proof −
  have unit_ball_vol (real (2 * n + 1)) =
         pi powr (real n + 1 / 2) / Gamma (1 / 2 + real (Suc n))
    by (simp add: unit_ball_vol_def field_simps)
  also have pochhammer (1 / 2) (Suc n) = Gamma (1 / 2 + real (Suc n)) /
Gamma (1 / 2)
    by (intro pochhammer_Gamma) auto
  hence Gamma (1 / 2 + real (Suc n)) = sqrt pi * pochhammer (1 / 2) (Suc n)
    by (simp add: Gamma_one_half_real)
  also have pi powr (real n + 1 / 2) / ... = pi ^ n / pochhammer (1 / 2) (Suc
n)
    by (simp add: powr_add powr_half_sqrt powr_realpow)
  finally show unit_ball_vol (real (2 * n + 1)) = ... .
  also have pochhammer (1 / 2 :: real) (Suc n) =
              fact (2 * Suc n) / (2 ^ (2 * Suc n) * fact (Suc n))
    using fact_double[of Suc n, where ?'a = real] by (simp add: divide_simps
mult_ac)
  also have pi ^ n / ... = (2 ^ (2 * Suc n) * fact (Suc n)) / fact (2 * Suc n) *
pi ^ n
    by simp
  finally show unit_ball_vol (real (2 * n + 1)) = ... .
qed

lemma unit_ball_vol_numeral:
  unit_ball_vol (numeral (Num.Bit0 n)) = pi ^ numeral n / fact (numeral n) (is
?th1)
  unit_ball_vol (numeral (Num.Bit1 n)) = 2 ^ (2 * Suc (numeral n)) * fact (Suc
(numeral n)) /
    fact (2 * Suc (numeral n)) * pi ^ numeral n (is ?th2)
proof −
  have numeral (Num.Bit0 n) = (2 * numeral n :: nat)
    by (simp only: numeral_Bit0 mult_2 ring_distribs)
  also have unit_ball_vol ... = pi ^ numeral n / fact (numeral n)
    by (rule unit_ball_vol_even)
  finally show ?th1 by simp
next
  have numeral (Num.Bit1 n) = (2 * numeral n + 1 :: nat)
    by (simp only: numeral_Bit1 mult_2)
  also have unit_ball_vol ... = 2 ^ (2 * Suc (numeral n)) * fact (Suc (numeral
n)) /
                                fact (2 * Suc (numeral n)) * pi ^ numeral n
    by (rule unit_ball_vol_odd)
  finally show ?th2 by simp
qed

lemmas eval_unit_ball_vol = unit_ball_vol_numeral fact_numeral
```

Just for fun, we compute the volume of unit balls for a few dimensions.

lemma *unit_ball_vol_0* [*simp*]: *unit_ball_vol 0 = 1*
using *unit_ball_vol_even*[*of 0*] **by** *simp*

lemma *unit_ball_vol_1* [*simp*]: *unit_ball_vol 1 = 2*
using *unit_ball_vol_odd*[*of 0*] **by** *simp*

corollary
unit_ball_vol_2: *unit_ball_vol 2 = pi*
and *unit_ball_vol_3*: *unit_ball_vol 3 = 4 / 3 * pi*
and *unit_ball_vol_4*: *unit_ball_vol 4 = pi^2 / 2*
and *unit_ball_vol_5*: *unit_ball_vol 5 = 8 / 15 * pi^2*
by (*simp_all add*: *eval_unit_ball_vol*)

corollary *circle_area*:
*r ≥ 0 ⟹ content (ball c r :: (real ^ 2) set) = r ^ 2 * pi*
by (*simp add*: *content_ball unit_ball_vol_2*)

corollary *sphere_volume*:
*r ≥ 0 ⟹ content (ball c r :: (real ^ 3) set) = 4 / 3 * r ^ 3 * pi*
by (*simp add*: *content_ball unit_ball_vol_3*)

Useful equivalent forms

corollary *content_ball_eq_0_iff* [*simp*]: *content (ball c r) = 0 ⟷ r ≤ 0*
proof −
have *r > 0 ⟹ content (ball c r) > 0*
by (*simp add*: *content_ball unit_ball_vol_def*)
then show *?thesis*
by (*fastforce simp*: *ball_empty*)
qed

corollary *content_ball_gt_0_iff* [*simp*]: *0 < content (ball z r) ⟷ 0 < r*
by (*auto simp*: *zero_less_measure_iff*)

corollary *content_cball_eq_0_iff* [*simp*]: *content (cball c r) = 0 ⟷ r ≤ 0*
proof (*cases r = 0*)
case *False*
moreover have *r > 0 ⟹ content (cball c r) > 0*
by (*simp add*: *content_cball unit_ball_vol_def*)
ultimately show *?thesis*
by *fastforce*
qed *auto*

corollary *content_cball_gt_0_iff* [*simp*]: *0 < content (cball z r) ⟷ 0 < r*
by (*auto simp*: *zero_less_measure_iff*)

end

9.13 Integral Test for Summability

theory *Integral_Test*
imports *Henstock_Kurzweil_Integration*
begin

The integral test for summability. We show here that for a decreasing non-negative function, the infinite sum over that function evaluated at the natural numbers converges iff the corresponding integral converges.

As a useful side result, we also provide some results on the difference between the integral and the partial sum. (This is useful e.g. for the definition of the Euler-Mascheroni constant)

locale *antimono_fun_sum_integral_diff* =
  fixes f :: *real* ⇒ *real*
  assumes *dec*: $\bigwedge x\ y.\ x \geq 0 \Longrightarrow x \leq y \Longrightarrow f\ x \geq f\ y$
  assumes *nonneg*: $\bigwedge x.\ x \geq 0 \Longrightarrow f\ x \geq 0$
  assumes *cont*: *continuous_on* {0..} f
begin

definition *sum_integral_diff_series* $n = (\sum k{\leq}n.\ f\ (of\_nat\ k)) - (integral\ \{0..of\_nat\ n\}\ f)$

lemma *sum_integral_diff_series_nonneg*:
  sum_integral_diff_series $n \geq 0$
proof −
  note *int* = *integrable_continuous_real*[*OF continuous_on_subset*[*OF cont*]]
  let *?int* = $\lambda a\ b.$ *integral* {*of_nat a..of_nat b*} f
  have $-$*sum_integral_diff_series* n = *?int* 0 $n - (\sum k{\leq}n.\ f\ (of\_nat\ k))$
    by (*simp add*: *sum_integral_diff_series_def*)
  also have *?int* 0 $n = (\sum k{<}n.$ *?int* k (*Suc* k))
  proof (*induction* n)
    case (*Suc* n)
    have *?int* 0 (*Suc* n) = *?int* 0 n + *?int* n (*Suc* n)
      by (*intro integral_combine*[*symmetric*] *int*) *simp_all*
    with *Suc* **show** *?case* **by** *simp*
  qed *simp_all*
  also have ... ≤ $(\sum k{<}n.$ *integral* {*of_nat k..of_nat* (*Suc* k)} (λ_::*real*. f (*of_nat* k)))
    by (*intro sum_mono integral_le int*) (*auto intro*: *dec*)
  also have ... = $(\sum k{<}n.\ f\ (of\_nat\ k))$ **by** *simp*
  also have ... − $(\sum k{\leq}n.\ f\ (of\_nat\ k)) = -(\sum k{\in}\{..n\} - \{..<n\}.\ f\ (of\_nat\ k))$
    by (*subst sum_diff*) *auto*
  also have ... ≤ 0 **by** (*auto intro*!: *sum_nonneg nonneg*)
  finally show *sum_integral_diff_series* $n \geq 0$ **by** *simp*
qed

lemma *sum_integral_diff_series_antimono*:
  assumes $m \leq n$
  shows *sum_integral_diff_series* $m \geq$ *sum_integral_diff_series* n

proof −
let *?int* = λ*a b*. *integral* {*of_nat a*..*of_nat b*} *f*
note *int* = *integrable_continuous_real*[*OF continuous_on_subset*[*OF cont*]]
have *d_mono*: *sum_integral_diff_series* (*Suc n*) ≤ *sum_integral_diff_series n*
for *n*
proof −
fix *n* :: *nat*
have *sum_integral_diff_series* (*Suc n*) − *sum_integral_diff_series n* =
f (*of_nat* (*Suc n*)) + (*?int 0 n* − *?int 0* (*Suc n*))
unfolding *sum_integral_diff_series_def* **by** (*simp add*: *algebra_simps*)
also have *?int 0 n* − *?int 0* (*Suc n*) = −*?int n* (*Suc n*)
by (*subst integral_combine* [*symmetric*, *of of_nat 0 of_nat n of_nat* (*Suc n*)])
(*auto intro*!: *int simp*: *algebra_simps*)
also have *?int n* (*Suc n*) ≥ *integral* {*of_nat n*..*of_nat* (*Suc n*)} (λ_::*real*. *f* (*of_nat* (*Suc n*)))
by (*intro integral_le int*) (*auto intro*: *dec*)
hence *f* (*of_nat* (*Suc n*)) + −*?int n* (*Suc n*) ≤ *0* **by** (*simp add*: *algebra_simps*)
finally show *sum_integral_diff_series* (*Suc n*) ≤ *sum_integral_diff_series n*
by *simp*
qed
with *assms* **show** *?thesis*
by (*induction rule*: *inc_induct*) (*auto intro*: *order.trans*[*OF _ d_mono*])
qed

lemma *sum_integral_diff_series_Bseq*: *Bseq sum_integral_diff_series*
proof −
from *sum_integral_diff_series_nonneg* **and** *sum_integral_diff_series_antimono*
have *norm* (*sum_integral_diff_series n*) ≤ *sum_integral_diff_series 0* **for** *n*
by *simp*
thus *Bseq sum_integral_diff_series* **by** (*rule BseqI'*)
qed

lemma *sum_integral_diff_series_monoseq*: *monoseq sum_integral_diff_series*
using *sum_integral_diff_series_antimono* **unfolding** *monoseq_def* **by** *blast*

lemma *sum_integral_diff_series_convergent*: *convergent sum_integral_diff_series*
using *sum_integral_diff_series_Bseq sum_integral_diff_series_monoseq*
by (*blast intro*!: *Bseq_monoseq_convergent*)

theorem *integral_test*:
summable (λ*n*. *f* (*of_nat n*)) ⟷ *convergent* (λ*n*. *integral* {*0*..*of_nat n*} *f*)
proof −
have *summable* (λ*n*. *f* (*of_nat n*)) ⟷ *convergent* (λ*n*. ∑ *k*≤*n*. *f* (*of_nat k*))
by (*simp add*: *summable_iff_convergent'*)
also have ... ⟷ *convergent* (λ*n*. *integral* {*0*..*of_nat n*} *f*)
proof
assume *convergent* (λ*n*. ∑ *k*≤*n*. *f* (*of_nat k*))
from *convergent_diff*[*OF this sum_integral_diff_series_convergent*]

```
    show convergent (λn. integral {0..of_nat n} f)
      unfolding sum_integral_diff_series_def by simp
  next
    assume convergent (λn. integral {0..of_nat n} f)
    from convergent_add[OF this sum_integral_diff_series_convergent]
    show convergent (λn. ∑k≤n. f (of_nat k)) unfolding sum_integral_diff_series_def
by simp
  qed
  finally show ?thesis by simp
qed

end

end
```

9.14 Continuity of the indefinite integral; improper integral theorem

```
theory Improper_Integral
  imports Equivalence_Lebesgue_Henstock_Integration
begin
```

9.14.1 Equiintegrability

The definition here only really makes sense for an elementary set. We just use compact intervals in applications below.

```
definition equiintegrable_on (infixr ‹equiintegrable'_on› 46)
  where F equiintegrable_on I ≡
        (∀f ∈ F. f integrable_on I) ∧
        (∀e > 0. ∃γ. gauge γ ∧
                  (∀f 𝒟. f ∈ F ∧ 𝒟 tagged_division_of I ∧ γ fine 𝒟
                        ⟶ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f)
< e))

lemma equiintegrable_on_integrable:
    ⟦F equiintegrable_on I; f ∈ F⟧ ⟹ f integrable_on I
  using equiintegrable_on_def by metis

lemma equiintegrable_on_sing [simp]:
    {f} equiintegrable_on cbox a b ⟷ f integrable_on cbox a b
  by (simp add: equiintegrable_on_def has_integral_integral has_integral integrable_on_def)

lemma equiintegrable_on_subset: ⟦F equiintegrable_on I; G ⊆ F⟧ ⟹ G equiintegrable_on I
  unfolding equiintegrable_on_def Ball_def
  by (erule conj_forward imp_forward all_forward ex_forward | blast)+
```

lemma *equiintegrable_on_Un*:
assumes F *equiintegrable_on* I G *equiintegrable_on* I
shows $(F \cup G)$ *equiintegrable_on* I
unfolding *equiintegrable_on_def*
proof (*intro conjI impI allI*)
show $\forall f{\in}F \cup G.$ f *integrable_on* I
using *assms* **unfolding** *equiintegrable_on_def* **by** *blast*
show $\exists \gamma.$ *gauge* $\gamma \wedge$
$(\forall f\ \mathcal{D}.\ f \in F \cup G \wedge$
$\mathcal{D}$ *tagged_division_of* $I \wedge \gamma$ *fine* $\mathcal{D} \longrightarrow$
norm $((\sum (x,K) \in \mathcal{D}.$ *content* $K *_R f\ x) -$ *integral* $I\ f) < \varepsilon)$
if $\varepsilon > 0$ **for** ε
proof −
obtain $\gamma 1$ **where** *gauge* $\gamma 1$
and $\gamma 1$: $\bigwedge f\ \mathcal{D}.\ f \in F \wedge \mathcal{D}$ *tagged_division_of* $I \wedge \gamma 1$ *fine* $\mathcal{D}$
$\Longrightarrow$ *norm* $((\sum (x,K) \in \mathcal{D}.$ *content* $K *_R f\ x) -$ *integral* $I\ f) < \varepsilon$
using *assms* ‹$\varepsilon > 0$› **unfolding** *equiintegrable_on_def* **by** *auto*
obtain $\gamma 2$ **where** *gauge* $\gamma 2$
and $\gamma 2$: $\bigwedge f\ \mathcal{D}.\ f \in G \wedge \mathcal{D}$ *tagged_division_of* $I \wedge \gamma 2$ *fine* $\mathcal{D}$
$\Longrightarrow$ *norm* $((\sum (x,K) \in \mathcal{D}.$ *content* $K *_R f\ x) -$ *integral* $I\ f) < \varepsilon$
using *assms* ‹$\varepsilon > 0$› **unfolding** *equiintegrable_on_def* **by** *auto*
have *gauge* $(\lambda x.\ \gamma 1\ x \cap \gamma 2\ x)$
using ‹*gauge* $\gamma 1$› ‹*gauge* $\gamma 2$› **by** *blast*
moreover have $\forall f\ \mathcal{D}.\ f \in F \cup G \wedge \mathcal{D}$ *tagged_division_of* $I \wedge (\lambda x.\ \gamma 1\ x \cap \gamma 2\ x)$ *fine* $\mathcal{D} \longrightarrow$
norm $((\sum (x,K) \in \mathcal{D}.$ *content* $K *_R f\ x) -$ *integral* $I\ f) < \varepsilon$
using $\gamma 1$ $\gamma 2$ **by** (*auto simp*: *fine_Int*)
ultimately show *?thesis*
by (*intro exI conjI*) *assumption*+
qed
qed

lemma *equiintegrable_on_insert*:
assumes f *integrable_on cbox a b* F *equiintegrable_on cbox a b*
shows (*insert* f F) *equiintegrable_on cbox a b*
by (*metis assms equiintegrable_on_Un equiintegrable_on_sing insert_is_Un*)

lemma *equiintegrable_cmul*:
assumes F: F *equiintegrable_on* I
shows $(\bigcup c \in \{-k..k\}.\ \bigcup f \in F.\ \{(\lambda x.\ c *_R f\ x)\})$ *equiintegrable_on* I
unfolding *equiintegrable_on_def*
proof (*intro conjI impI allI ballI*)
show f *integrable_on* I
if $f \in (\bigcup c{\in}\{-\ k..k\}.\ \bigcup f{\in}F.\ \{\lambda x.\ c *_R f\ x\})$
for $f :: 'a \Rightarrow 'b$
using *that assms equiintegrable_on_integrable integrable_cmul* **by** *blast*
show $\exists \gamma.$ *gauge* $\gamma \wedge (\forall f\ \mathcal{D}.\ f \in (\bigcup c{\in}\{-\ k..k\}.\ \bigcup f{\in}F.\ \{\lambda x.\ c *_R f\ x\}) \wedge \mathcal{D}$

tagged_division_of I
        ∧ γ *fine* 𝒟 ⟶ *norm* ((∑ (*x*, *K*)∈𝒟. *content K* ∗*R f x*) − *integral I f*) < ε)
    if ε > *0* **for** ε
  proof −
    obtain γ **where** *gauge* γ
      and γ: ⋀*f* 𝒟. ⟦*f* ∈ *F*; 𝒟 *tagged_division_of I*; γ *fine* 𝒟⟧
                ⟹ *norm* ((∑ (*x*,*K*) ∈ 𝒟. *content K* ∗*R f x*) − *integral I f*) < ε / (|*k*| + *1*)
      using *assms* ‹ε > *0*› **unfolding** *equiintegrable_on_def*
      by (*metis add.commute add.right_neutral add_strict_mono divide_pos_pos norm_eq_zero real_norm_def zero_less_norm_iff zero_less_one*)
    moreover have *norm* ((∑ (*x*, *K*)∈𝒟. *content K* ∗*R c* ∗*R* (*f x*)) − *integral I* (λ*x*. *c* ∗*R f x*)) < ε
      if *c*: *c* ∈ {− *k*..*k*}
        and *f* ∈ *F* 𝒟 *tagged_division_of I* γ *fine* 𝒟
      for 𝒟 *c f*
    proof −
      have *norm* ((∑ *x*∈𝒟. *case x of* (*x*, *K*) ⇒ *content K* ∗*R c* ∗*R f x*) − *integral I* (λ*x*. *c* ∗*R f x*))
          = |*c*| ∗ *norm* ((∑ *x*∈𝒟. *case x of* (*x*, *K*) ⇒ *content K* ∗*R f x*) − *integral I f*)
            by (*simp add*: *algebra_simps scale_sum_right case_prod_unfold flip*: *norm_scaleR*)
      also have ... ≤ (|*k*| + *1*) ∗ *norm* ((∑ *x*∈𝒟. *case x of* (*x*, *K*) ⇒ *content K* ∗*R f x*) − *integral I f*)
        using *c* **by** (*auto simp*: *mult_right_mono*)
      also have ... < (|*k*| + *1*) ∗ (ε / (|*k*| + *1*))
        by (*rule mult_strict_left_mono*) (*use* γ *less_eq_real_def that* **in** *auto*)
      also have ... = ε
        by *auto*
      finally show *?thesis* **.**
    qed
    ultimately show *?thesis*
      by (*rule_tac x*=γ **in** *exI*) *auto*
  qed
qed

lemma *equiintegrable_add*:
  assumes *F*: *F equiintegrable_on I* **and** *G*: *G equiintegrable_on I*
  shows (⋃*f* ∈ *F*. ⋃*g* ∈ *G*. {(λ*x*. *f x* + *g x*)}) *equiintegrable_on I*
  unfolding *equiintegrable_on_def*
proof (*intro conjI impI allI ballI*)
  show *f integrable_on I*
    if *f* ∈ (⋃*f*∈*F*. ⋃*g*∈*G*. {λ*x*. *f x* + *g x*}) **for** *f*
    using *that equiintegrable_on_integrable assms* **by** (*auto intro*: *integrable_add*)
  show ∃γ. *gauge* γ ∧ (∀*f* 𝒟. *f* ∈ (⋃*f*∈*F*. ⋃*g*∈*G*. {λ*x*. *f x* + *g x*}) ∧ 𝒟 *tagged_division_of I*

```
        ∧ γ fine 𝒟 ⟶ norm ((∑(x, K)∈𝒟. content K *R f x) − integral I f) <
ε)
    if ε > 0 for ε
  proof −
    obtain γ1 where gauge γ1
      and γ1: ⋀f 𝒟. ⟦f ∈ F; 𝒟 tagged_division_of I; γ1 fine 𝒟⟧
                ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) < ε/2
    using assms ‹ε > 0› unfolding equiintegrable_on_def by (meson half_gt_zero_iff)
    obtain γ2 where  gauge γ2
      and γ2: ⋀g 𝒟. ⟦g ∈ G; 𝒟 tagged_division_of I; γ2 fine 𝒟⟧
                ⟹ norm ((∑(x,K) ∈ 𝒟. content K *R g x) − integral I g) < ε/2
    using assms ‹ε > 0› unfolding equiintegrable_on_def by (meson half_gt_zero_iff)
    have gauge (λx. γ1 x ∩ γ2 x)
      using ‹gauge γ1› ‹gauge γ2› by blast
    moreover have norm ((∑(x,K) ∈ 𝒟. content K *R h x) − integral I h) < ε
      if h: h ∈ (⋃f∈F. ⋃g∈G. {λx. f x + g x})
        and 𝒟: 𝒟 tagged_division_of I and fine: (λx. γ1 x ∩ γ2 x) fine 𝒟
      for h 𝒟
    proof −
      obtain f g where f ∈ F g ∈ G and heq: h = (λx. f x + g x)
        using h by blast
      then have int: f integrable_on I g integrable_on I
        using F G equiintegrable_on_def by blast+
      have norm ((∑(x,K) ∈ 𝒟. content K *R h x) − integral I h)
         = norm ((∑(x,K) ∈ 𝒟. content K *R f x + content K *R g x) − (integral
I f + integral I g))
        by (simp add: heq algebra_simps integral_add int)
      also have ... = norm (((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f +
(∑(x,K) ∈ 𝒟. content K *R g x) − integral I g))
        by (simp add: sum.distrib algebra_simps case_prod_unfold)
      also have ... ≤ norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral I f) +
norm ((∑(x,K) ∈ 𝒟. content K *R g x) − integral I g)
        by (metis (mono_tags) add_diff_eq norm_triangle_ineq)
      also have ... < ε/2 + ε/2
        using γ1 [OF ‹f ∈ F› 𝒟] γ2 [OF ‹g ∈ G› 𝒟] fine  by (simp add: fine_Int)
      finally show ?thesis by simp
    qed
    ultimately show ?thesis
      by meson
  qed
qed

lemma equiintegrable_minus:
  assumes F equiintegrable_on I
  shows (⋃f ∈ F. {(λx. − f x)}) equiintegrable_on I
  by (force intro: equiintegrable_on_subset [OF equiintegrable_cmul [OF assms,
of 1]])

lemma equiintegrable_diff:
```

assumes F: F *equiintegrable_on* I **and** G: G *equiintegrable_on* I
shows $(\bigcup f \in F.\ \bigcup g \in G.\ \{(\lambda x.\ f\ x - g\ x)\})$ *equiintegrable_on* I
by (*rule equiintegrable_on_subset* [*OF equiintegrable_add* [*OF F equiintegrable_minus* [*OF G*]]]) *auto*

lemma *equiintegrable_sum*:
fixes $F :: ('a::euclidean\_space \Rightarrow 'b::euclidean\_space)$ *set*
assumes F *equiintegrable_on cbox a b*
shows $(\bigcup I \in Collect\ finite.\ \bigcup c \in \{c.\ (\forall i \in I.\ c\ i \geq 0) \wedge sum\ c\ I = 1\}.$
$\bigcup f \in I \to F.\ \{(\lambda x.\ sum\ (\lambda i::'j.\ c\ i *_R f\ i\ x)\ I)\})$ *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on _*)
unfolding *equiintegrable_on_def*
proof (*intro conjI impI allI ballI*)
show f *integrable_on cbox a b* **if** $f \in ?G$ **for** f
using *that assms* **by** (*auto simp*: *equiintegrable_on_def intro*!: *integrable_sum integrable_cmul*)
show $\exists \gamma.\ gauge\ \gamma$
$\wedge\ (\forall g\ \mathcal{D}.\ g \in ?G \wedge \mathcal{D}\ tagged\_division\_of\ cbox\ a\ b \wedge \gamma\ fine\ \mathcal{D}$
$\longrightarrow norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ x) - integral\ (cbox\ a\ b)\ g) < \varepsilon)$
if $\varepsilon > 0$ **for** ε
proof −
obtain γ **where** *gauge* γ
and γ: $\bigwedge f\ \mathcal{D}.\ [\![f \in F; \mathcal{D}\ tagged\_division\_of\ cbox\ a\ b; \gamma\ fine\ \mathcal{D}]\!]$
$\Longrightarrow norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ x) - integral\ (cbox\ a\ b)\ f) < \varepsilon\ /\ 2$
using *assms* ‹$\varepsilon > 0$› **unfolding** *equiintegrable_on_def* **by** (*meson half_gt_zero_iff*)
moreover have $norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ x) - integral\ (cbox\ a\ b)\ g) < \varepsilon$
if g: $g \in ?G$
and $\mathcal{D}$: $\mathcal{D}\ tagged\_division\_of\ cbox\ a\ b$
and *fine*: $\gamma\ fine\ \mathcal{D}$
for $\mathcal{D}\ g$
proof −
obtain $I\ c\ f$ **where** *finite* I **and** *0*: $\bigwedge i::'j.\ i \in I \Longrightarrow 0 \leq c\ i$
and *1*: $sum\ c\ I = 1$ **and** f: $f \in I \to F$ **and** *geq*: $g = (\lambda x.\ \sum i \in I.\ c\ i *_R f\ i\ x)$
using g **by** *auto*
have *fi_int*: $f\ i$ *integrable_on cbox a b* **if** $i \in I$ **for** i
by (*metis Pi_iff assms equiintegrable_on_def f that*)
have $*$: $integral\ (cbox\ a\ b)\ (\lambda x.\ c\ i *_R f\ i\ x) = (\sum (x,\ K) \in \mathcal{D}.\ integral\ K\ (\lambda x.\ c\ i *_R f\ i\ x))$
if $i \in I$ **for** i
proof −
have $f\ i$ *integrable_on cbox a b*
by (*metis Pi_iff assms equiintegrable_on_def f that*)
then show *?thesis*

```
      by (intro 𝒟 integrable_cmul integral_combine_tagged_division_topdown)
    qed
    have finite 𝒟
      using 𝒟 by blast
    have swap: (∑(x,K)∈𝒟. content K *R (∑i∈I. c i *R f i x))
        = (∑i∈I. c i *R (∑(x,K)∈𝒟. content K *R f i x))
      by (simp add: scale_sum_right case_prod_unfold algebra_simps) (rule sum.swap)
    have norm ((∑(x, K)∈𝒟. content K *R g x) − integral (cbox a b) g)
        = norm ((∑i∈I. c i *R ((∑(x,K)∈𝒟. content K *R f i x) − integral (cbox a b) (f i))))
      unfolding geq swap
      by (simp add: scaleR_right.sum algebra_simps integral_sum fi_int integrable_cmul ‹finite I› sum_subtractf flip: sum_diff)
    also have ... ≤ (∑i∈I. c i * ε / 2)
    proof (rule sum_norm_le)
      show norm (c i *R ((∑(xa, K)∈𝒟. content K *R f i xa) − integral (cbox a b) (f i))) ≤ c i * ε / 2
        if i ∈ I for i
      proof −
        have norm ((∑(x, K)∈𝒟. content K *R f i x) − integral (cbox a b) (f i)) ≤ ε/2
          using γ [OF _ 𝒟 fine, of f i] funcset_mem [OF f] that by auto
        then show ?thesis
          using that by (auto simp: 0 mult.assoc intro: mult_left_mono)
      qed
    qed
    also have ... < ε
     using 1 ‹ε > 0› by (simp add: flip: sum_divide_distrib sum_distrib_right)
    finally show ?thesis .
  qed
  ultimately show ?thesis
    by (rule_tac x=γ in exI) auto
 qed
qed

corollary equiintegrable_sum_real:
 fixes F :: (real ⇒ 'b::euclidean_space) set
 assumes F equiintegrable_on {a..b}
 shows (⋃I ∈ Collect finite. ⋃c ∈ {c. (∀i ∈ I. c i ≥ 0) ∧ sum c I = 1}.
        ⋃f ∈ I → F. {(λx. sum (λi. c i *R f i x) I)})
       equiintegrable_on {a..b}
 using equiintegrable_sum [of F a b] assms by auto
```

Basic combining theorems for the interval of integration.

```
lemma equiintegrable_on_null [simp]:
  content(cbox a b) = 0 ⟹ F equiintegrable_on cbox a b
 unfolding equiintegrable_on_def
 by (metis diff_zero gauge_trivial integrable_on_null integral_null norm_zero
```

sum_content_null)

Main limit theorem for an equiintegrable sequence.

theorem *equiintegrable_limit*:
fixes $g :: 'a :: euclidean\_space \Rightarrow 'b :: banach$
assumes *feq*: *range f equiintegrable_on cbox a b*
and *to_g*: $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
shows *g integrable_on cbox a b* $\wedge$ $(\lambda n.\ integral\ (cbox\ a\ b)\ (f\ n)) \longrightarrow integral\ (cbox\ a\ b)\ g$
proof −
have *Cauchy* $(\lambda n.\ integral(cbox\ a\ b)\ (f\ n))$
proof (*clarsimp simp add*: *Cauchy_def*)
fix *e*::*real*
assume $0 < e$
then have *e3*: $0 < e/3$
by *simp*
then obtain γ **where** *gauge* γ
and γ: $\bigwedge n\ \mathcal{D}.\ [\![\mathcal{D}\ tagged\_division\_of\ cbox\ a\ b;\ \gamma\ fine\ \mathcal{D}]\!] \Longrightarrow norm((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ n\ x) - integral\ (cbox\ a\ b)\ (f\ n)) < e/3$
using *feq* **unfolding** *equiintegrable_on_def*
by (*meson image_eqI iso_tuple_UNIV_I*)
obtain $\mathcal{D}$ **where** $\mathcal{D}$: $\mathcal{D}$ *tagged_division_of* $(cbox\ a\ b)$ **and** γ *fine* $\mathcal{D}$ *finite* $\mathcal{D}$
by (*meson* ‹*gauge* γ› *fine_division_exists tagged_division_of_finite*)
with γ **have** δT: $\bigwedge n.\ dist\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ n\ x))\ (integral\ (cbox\ a\ b)\ (f\ n)) < e/3$
by (*force simp*: *dist_norm*)
have $(\lambda n.\ \sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ n\ x) \longrightarrow (\sum (x,K) \in \mathcal{D}.\ content\ K *_R g\ x)$
using $\mathcal{D}$ *to_g* **by** (*auto intro*!: *tendsto_sum tendsto_scaleR*)
then have *Cauchy* $(\lambda n.\ \sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ n\ x)$
by (*meson convergent_eq_Cauchy*)
with *e3* **obtain** *M* **where**
M: $\bigwedge m\ n.\ [\![m \geq M;\ n \geq M]\!] \Longrightarrow dist\ (\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ m\ x)\ (\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ n\ x) < e/3$
unfolding *Cauchy_def* **by** *blast*
have $\bigwedge m\ n.\ [\![m \geq M;\ n \geq M;\ dist\ (\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ m\ x)\ (\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ n\ x) < e/3]\!] \Longrightarrow dist\ (integral\ (cbox\ a\ b)\ (f\ m))\ (integral\ (cbox\ a\ b)\ (f\ n)) < e$
by (*metis* δT *dist_commute dist_triangle_third* [*OF* _ _ δT])
then show $\exists M.\ \forall m \geq M.\ \forall n \geq M.\ dist\ (integral\ (cbox\ a\ b)\ (f\ m))\ (integral\ (cbox\ a\ b)\ (f\ n)) < e$
using *M* **by** *auto*
qed
then obtain *L* **where** *L*: $(\lambda n.\ integral\ (cbox\ a\ b)\ (f\ n)) \longrightarrow L$
by (*meson convergent_eq_Cauchy*)
have $(g\ has\_integral\ L)\ (cbox\ a\ b)$

```
  proof (clarsimp simp: has_integral)
    fix e::real assume 0 < e
    then have e2: 0 < e/2
      by simp
    then obtain γ where gauge γ
      and γ: ⋀n 𝒟. ⟦𝒟 tagged_division_of cbox a b; γ fine 𝒟⟧
                ⟹ norm((∑(x,K)∈𝒟. content K *R f n x) − integral (cbox a b)
(f n)) < e/2
      using feq unfolding equiintegrable_on_def
      by (meson image_eqI iso_tuple_UNIV_I)
    moreover
    have norm ((∑(x,K)∈𝒟. content K *R g x) − L) < e
           if 𝒟 tagged_division_of cbox a b γ fine 𝒟 for 𝒟
    proof −
      have norm ((∑(x,K)∈𝒟. content K *R g x) − L) ≤ e/2
      proof (rule Lim_norm_ubound)
        show (λn. (∑(x,K)∈𝒟. content K *R f n x) − integral (cbox a b) (f n))
⟶ (∑(x,K)∈𝒟. content K *R g x) − L
          using to_g that L
          by (intro tendsto_diff tendsto_sum) (auto simp: tag_in_interval tend-
sto_scaleR)
        show ∀F n in sequentially.
               norm ((∑(x,K) ∈ 𝒟. content K *R f n x) − integral (cbox a b) (f
n)) ≤ e/2
          by (intro eventuallyI less_imp_le γ that)
      qed auto
      with ‹0 < e› show ?thesis
        by linarith
    qed
    ultimately
    show ∃γ. gauge γ ∧
           (∀𝒟. 𝒟 tagged_division_of cbox a b ∧ γ fine 𝒟 ⟶
                norm ((∑(x,K)∈𝒟. content K *R g x) − L) < e)
      by meson
  qed
  with L show ?thesis
   by (simp add: ‹(λn. integral (cbox a b) (f n)) ⟶ L› has_integral_integrable_integral)
qed


lemma equiintegrable_reflect:
  assumes F equiintegrable_on cbox a b
  shows (λf. f ∘ uminus) ` F equiintegrable_on cbox (−b) (−a)
proof −
  have §: ∃γ. gauge γ ∧
           (∀f 𝒟. f ∈ (λf. f ∘ uminus) ` F ∧ 𝒟 tagged_division_of cbox (− b) (−
a) ∧ γ fine 𝒟 ⟶
                norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral (cbox (− b) (−
a)) f) < e)
```

```
    if gauge γ and
      γ: ⋀f 𝒟. ⟦f ∈ F; 𝒟 tagged_division_of cbox a b; γ fine 𝒟⟧ ⟹
            norm ((∑(x,K) ∈ 𝒟. content K *R f x) − integral (cbox a b) f)
< e for e γ
  proof (intro exI, safe)
    show gauge (λx. uminus ‘ γ (−x))
      by (metis ‹gauge γ› gauge_reflect)
    show norm ((∑(x,K) ∈ 𝒟. content K *R (f ∘ uminus) x) − integral (cbox (−
b) (− a)) (f ∘ uminus)) < e
      if f ∈ F and tag: 𝒟 tagged_division_of cbox (− b) (− a)
        and fine: (λx. uminus ‘ γ (− x)) fine 𝒟 for f 𝒟
    proof −
      have 1: (λ(x,K). (− x, uminus ‘ K)) ‘ 𝒟 tagged_partial_division_of cbox a b
        if 𝒟 tagged_partial_division_of cbox (− b) (− a)
      proof −
        have − y ∈ cbox a b
          if ⋀x K. (x,K) ∈ 𝒟 ⟹ x ∈ K ∧ K ⊆ cbox (− b) (− a) ∧ (∃a b. K =
cbox a b)
             (x, Y) ∈ 𝒟 y ∈ Y for x Y y
        proof −
          have y ∈ uminus ‘ cbox a b
            using that by auto
          then show − y ∈ cbox a b
            by force
        qed
        with that show ?thesis
          by (fastforce simp: tagged_partial_division_of_def interior_negations
image_iff)
      qed
      have 2: ∃K. (∃x. (x,K) ∈ (λ(x,K). (− x, uminus ‘ K)) ‘ 𝒟) ∧ x ∈ K
          if ⋃{K. ∃x. (x,K) ∈ 𝒟} = cbox (− b) (− a) x ∈ cbox a b for x
      proof −
        have xm: x ∈ uminus ‘ ⋃{A. ∃a. (a, A) ∈ 𝒟}
          by (simp add: that)
        then obtain a X where −x ∈ X (a, X) ∈ 𝒟
          by auto
        then show ?thesis
          by (metis (no_types, lifting) add.inverse_inverse image_iff pair_imageI)
      qed
      have 3: ⋀x X y. ⟦𝒟 tagged_partial_division_of cbox (− b) (− a); (x, X) ∈
𝒟; y ∈ X⟧ ⟹ − y ∈ cbox a b
      by (metis (no_types, lifting) equation_minus_iff imageE subsetD tagged_partial_division_ofD(3)
uminus_interval_vector)
      have tag′: (λ(x,K). (− x, uminus ‘ K)) ‘ 𝒟 tagged_division_of cbox a b
        using tag  by (auto simp: tagged_division_of_def dest: 1 2 3)
      have fine′: γ fine (λ(x,K). (− x, uminus ‘ K)) ‘ 𝒟
        using fine by (fastforce simp: fine_def)
      have inj: inj_on (λ(x,K). (− x, uminus ‘ K)) 𝒟
        unfolding inj_on_def by force
```

```
    have eq: content (uminus ‘ I) = content I
          if I: (x, I) ∈ 𝒟 and fnz: f (− x) ≠ 0 for x I
    proof −
     obtain a b where I = cbox a b
     using tag I that by (force simp: tagged_division_of_def tagged_partial_division_of_def)
      then show ?thesis
       using content_image_affinity_cbox [of −1 0] by auto
    qed
    have (∑(x,K) ∈ (λ(x,K). (− x, uminus ‘ K)) ‘ 𝒟. content K *R f x) =
        (∑(x,K) ∈ 𝒟. content K *R f (− x))
     by (auto simp add: eq sum.reindex [OF inj] intro!: sum.cong)
    then show ?thesis
     using γ [OF ‹f ∈ F› tag′ fine′] integral_reflect
     by (metis (mono_tags, lifting) Henstock_Kurzweil_Integration.integral_cong
comp_apply split_def sum.cong)
   qed
  qed
   show ?thesis
    using assms
    apply (auto simp: equiintegrable_on_def)
    subgoal for f
     by (metis (mono_tags, lifting) comp_apply integrable_eq integrable_reflect)
    using § by fastforce
qed
```

9.14.2 Subinterval restrictions for equiintegrable families

First, some technical lemmas about minimizing a "flat" part of a sum over a division.

```
lemma lemma0:
  assumes i ∈ Basis
     shows content (cbox u v) / (interval_upperbound (cbox u v) • i − inter-
val_lowerbound (cbox u v) • i) =
          (if content (cbox u v) = 0 then 0
             else ∏j ∈ Basis − {i}. interval_upperbound (cbox u v) • j − inter-
val_lowerbound (cbox u v) • j)
proof (cases content (cbox u v) = 0)
  case True
  then show ?thesis by simp
next
  case False
  then show ?thesis
    using prod.subset_diff [of {i} Basis] assms
     by (force simp: content_cbox_if divide_simps split: if_split_asm)
qed


lemma content_division_lemma1:
  assumes div: 𝒟 division_of S and S: S ⊆ cbox a b and i: i ∈ Basis
```

```
    and mt: ⋀K. K ∈ 𝒟 ⟹ content K ≠ 0
    and disj: (∀K ∈ 𝒟. K ∩ {x. x · i = a · i} ≠ {}) ∨ (∀K ∈ 𝒟. K ∩ {x. x · i
= b · i} ≠ {})
  shows (b · i − a · i) ∗ (∑K∈𝒟. content K / (interval_upperbound K · i −
interval_lowerbound K · i))
        ≤ content(cbox a b)   (is ?lhs ≤ ?rhs)
proof −
 have finite 𝒟
  using div by blast
 define extend where
  extend ≡ λK. cbox (∑j ∈ Basis. if j = i then (a · i) ∗R i else (interval_lowerbound
K · j) ∗R j)
                    (∑j ∈ Basis. if j = i then (b · i) ∗R i else (interval_upperbound
K · j) ∗R j)
 have div_subset_cbox: ⋀K. K ∈ 𝒟 ⟹ K ⊆ cbox a b
  using S div by auto
 have ⋀K. K ∈ 𝒟 ⟹ K ≠ {}
  using div by blast
 have extend_cbox: ⋀K.  K ∈ 𝒟 ⟹ ∃a b. extend K = cbox a b
  using extend_def by blast
 have extend: extend K ≠ {} extend K ⊆ cbox a b if K: K ∈ 𝒟 for K
 proof −
  obtain u v where K: K = cbox u v K ≠ {} K ⊆ cbox a b
   using K cbox_division_memE [OF _ div] by (meson div_subset_cbox)
  with i show extend K ⊆ cbox a b
   by (auto simp: extend_def subset_box box_ne_empty)
  have a · i ≤ b · i
   using K by (metis bot.extremum_uniqueI box_ne_empty(1) i)
  with K show extend K ≠ {}
   by (simp add: extend_def i box_ne_empty)
 qed
 have int_extend_disjoint:
    interior(extend K1) ∩ interior(extend K2) = {} if K: K1 ∈ 𝒟 K2 ∈ 𝒟 K1
≠ K2 for K1 K2
 proof −
  obtain u v where K1: K1 = cbox u v K1 ≠ {} K1 ⊆ cbox a b
   using K cbox_division_memE [OF _ div] by (meson div_subset_cbox)
  obtain w z where K2: K2 = cbox w z K2 ≠ {} K2 ⊆ cbox a b
   using K cbox_division_memE [OF _ div] by (meson div_subset_cbox)
  have cboxes: cbox u v ∈ 𝒟 cbox w z ∈ 𝒟 cbox u v ≠ cbox w z
   using K1 K2 that by auto
  with div have interior (cbox u v) ∩ interior (cbox w z) = {}
   by blast
  moreover
  have ∃x. x ∈ box u v ∧ x ∈ box w z
     if x ∈ interior (extend K1) x ∈ interior (extend K2) for x
  proof −
   have a · i < x · i x · i < b · i
    and ux: ⋀k. k ∈ Basis − {i} ⟹ u · k < x · k
```

and *xv*: $\bigwedge k.\ k \in Basis - \{i\} \Longrightarrow x \cdot k < v \cdot k$
and *wx*: $\bigwedge k.\ k \in Basis - \{i\} \Longrightarrow w \cdot k < x \cdot k$
and *xz*: $\bigwedge k.\ k \in Basis - \{i\} \Longrightarrow x \cdot k < z \cdot k$
using *that K1 K2 i* **by** (*auto simp*: *extend_def box_ne_empty mem_box*)
have *box u v* $\neq \{\}$ *box w z* $\neq \{\}$
using *cboxes interior_cbox* **by** (*auto simp*: *content_eq_0_interior dest*: *mt*)
then obtain *q s*
where *q*: $\bigwedge k.\ k \in Basis \Longrightarrow w \cdot k < q \cdot k \wedge q \cdot k < z \cdot k$
and *s*: $\bigwedge k.\ k \in Basis \Longrightarrow u \cdot k < s \cdot k \wedge s \cdot k < v \cdot k$
by (*meson all_not_in_conv mem_box*(*1*))
show *?thesis* **using** *disj*
proof
assume $\forall K \in \mathcal{D}.\ K \cap \{x.\ x \cdot i = a \cdot i\} \neq \{\}$
then have *uva*: $(cbox\ u\ v) \cap \{x.\ x \cdot i = a \cdot i\} \neq \{\}$
and *wza*: $(cbox\ w\ z) \cap \{x.\ x \cdot i = a \cdot i\} \neq \{\}$
using *cboxes* **by** (*auto simp*: *content_eq_0_interior*)
then obtain *r t* **where** $r \cdot i = a \cdot i$ **and** *r*: $\bigwedge k.\ k \in Basis \Longrightarrow w \cdot k \leq r \cdot k \wedge r \cdot k \leq z \cdot k$
and $t \cdot i = a \cdot i$ **and** *t*: $\bigwedge k.\ k \in Basis \Longrightarrow u \cdot k \leq t \cdot k \wedge t \cdot k \leq v \cdot k$
by (*fastforce simp*: *mem_box*)
have *u*: $u \cdot i < q \cdot i$
using *i K2*(*1*) *K2*(*3*) ‹$t \cdot i = a \cdot i$› *q s t* [*OF i*] **by** (*force simp*: *subset_box*)
have *w*: $w \cdot i < s \cdot i$
using *i K1*(*1*) *K1*(*3*) ‹$r \cdot i = a \cdot i$› *s r* [*OF i*] **by** (*force simp*: *subset_box*)
define ξ **where** $\xi \equiv (\sum j \in Basis.\ if\ j = i\ then\ min\ (q \cdot i)\ (s \cdot i) *_R i\ else\ (x \cdot j) *_R j)$
have [*simp*]: $\xi \cdot j = (if\ j = i\ then\ min\ (q \cdot j)\ (s \cdot j)\ else\ x \cdot j)$ **if** $j \in Basis$ **for** *j*
unfolding *ξ_def*
by (*intro sum_if_inner that* ‹$i \in Basis$›)
show *?thesis*
proof (*intro exI conjI*)
have *min* $(q \cdot i)\ (s \cdot i) < v \cdot i$
using *i s* **by** *fastforce*
with ‹$i \in Basis$› *s u ux xv*
show $\xi \in box\ u\ v$
by (*force simp*: *mem_box*)
have *min* $(q \cdot i)\ (s \cdot i) < z \cdot i$
using *i q* **by** *force*
with ‹$i \in Basis$› *q w wx xz*
show $\xi \in box\ w\ z$
by (*force simp*: *mem_box*)
qed
next
assume $\forall K \in \mathcal{D}.\ K \cap \{x.\ x \cdot i = b \cdot i\} \neq \{\}$
then have *uva*: $(cbox\ u\ v) \cap \{x.\ x \cdot i = b \cdot i\} \neq \{\}$
and *wza*: $(cbox\ w\ z) \cap \{x.\ x \cdot i = b \cdot i\} \neq \{\}$
using *cboxes* **by** (*auto simp*: *content_eq_0_interior*)

```
      then obtain r t where r · i = b · i and r: ⋀k. k ∈ Basis ⟹ w · k ≤ r ·
k ∧ r · k ≤ z · k
                        and t · i = b · i and t: ⋀k. k ∈ Basis ⟹ u · k ≤ t · k ∧ t ·
k ≤ v · k
        by (fastforce simp: mem_box)
      have z: s · i < z · i
       using K1(1) K1(3) ‹r · i = b · i› r [OF i] i s  by (force simp: subset_box)
      have v: q · i < v · i
       using K2(1) K2(3) ‹t · i = b · i› t [OF i] i q  by (force simp: subset_box)
      define ξ where ξ ≡ (∑ j ∈ Basis. if j = i then max (q · i) (s · i) *R i else
(x · j) *R j)
      have [simp]: ξ · j = (if j = i then max (q · j) (s · j) else x · j) if j ∈ Basis
for j
        unfolding ξ_def
        by (intro sum_if_inner that ‹i ∈ Basis›)
      show ?thesis
      proof (intro exI conjI)
        show ξ ∈ box u v
          using ‹i ∈ Basis› s by (force simp: mem_box ux v xv)
        show ξ ∈ box w z
          using ‹i ∈ Basis› q by (force simp: mem_box wx xz z)
      qed
    qed
  qed
  ultimately show ?thesis by auto
 qed
 define interv_diff where interv_diff ≡ λK. λi::'a. interval_upperbound K · i
− interval_lowerbound K · i
 have ?lhs = (∑ K∈𝒟. (b · i − a · i) * content K / (interv_diff K i))
  by (simp add: sum_distrib_left interv_diff_def)
 also have ... = sum (content ∘ extend) 𝒟
 proof (rule sum.cong [OF refl])
  fix K assume K ∈ 𝒟
  then obtain u v where K: K = cbox u v cbox u v ≠ {} K ⊆ cbox a b
   using cbox_division_memE [OF _ div] div_subset_cbox by metis
  then have uv: u · i < v · i
   using mt [OF ‹K ∈ 𝒟›] ‹i ∈ Basis› content_eq_0 by fastforce
  have insert i (Basis ∩ −{i}) = Basis
   using ‹i ∈ Basis› by auto
  then have (b · i − a · i) * content K / (interv_diff K i)
          = (b · i − a · i) * (∏ i ∈ insert i (Basis ∩ −{i}). v · i − u · i) /
(interv_diff (cbox u v) i)
    using K box_ne_empty(1) content_cbox by fastforce
  also have ... = (∏ x∈Basis. if x = i then b · x − a · x
                else (interval_upperbound (cbox u v) − interval_lowerbound (cbox
u v)) · x)
    using ‹i ∈ Basis› K uv by (simp add: prod.If_cases interv_diff_def) (simp
add: algebra_simps)
  also have ... = (∏ k∈Basis.
```

$(\sum j{\in}Basis.\ if\ j = i\ then\ (b \cdot i - a \cdot i) *_R i$
$else\ ((interval\_upperbound\ (cbox\ u\ v) -$
$interval\_lowerbound\ (cbox\ u\ v)) \cdot j) *_R j) \cdot k)$
using ‹$i \in Basis$› **by** (*subst prod.cong* [*OF refl sum_if_inner*]; *simp*)
also have ... = $(\prod k{\in}Basis.$
$(\sum j{\in}Basis.\ if\ j = i\ then\ (b \cdot i) *_R i\ else\ (interval\_upperbound$
$(cbox\ u\ v) \cdot j) *_R j) \cdot k -$
$(\sum j{\in}Basis.\ if\ j = i\ then\ (a \cdot i) *_R i\ else\ (interval\_lowerbound$
$(cbox\ u\ v) \cdot j) *_R j) \cdot k)$
using ‹$i \in Basis$›
by (*intro prod.cong* [*OF refl*]) (*subst sum_if_inner*; *simp add*: *algebra_simps*)+
also have ... = $(content \circ extend)\ K$
using ‹$i \in Basis$› K *box_ne_empty* ‹$K \in \mathcal{D}$› *extend*(*1*)
by (*auto simp add*: *extend_def content_cbox_if*)
finally show $(b \cdot i - a \cdot i) * content\ K\ /\ (interv\_diff\ K\ i) = (content \circ$
$extend)\ K$ **.**
qed
also have ... = $sum\ content\ (extend\ `\ \mathcal{D})$
proof −
have $[\![K1 \in \mathcal{D};\ K2 \in \mathcal{D};\ K1 \neq K2;\ extend\ K1 = extend\ K2]\!] \Longrightarrow content$
$(extend\ K1) = 0$ **for** *K1 K2*
using *int_extend_disjoint* [*of K1 K2*] *extend_def* **by** (*simp add*: *con-*
tent_eq_0_interior)
then show *?thesis*
by (*simp add*: *comm_monoid_add_class.sum.reindex_nontrivial* [*OF* ‹*finite*
$\mathcal{D}$›])
qed
also have ... ≤ *?rhs*
proof (*rule subadditive_content_division*)
show $extend\ `\ \mathcal{D}\ division\_of \bigcup (extend\ `\ \mathcal{D})$
using *int_extend_disjoint* **by** (*auto simp*: *division_of_def* ‹*finite* $\mathcal{D}$› *extend*
extend_cbox)
show $\bigcup (extend\ `\ \mathcal{D}) \subseteq cbox\ a\ b$
using *extend* **by** *fastforce*
qed
finally show *?thesis* **.**
qed

proposition *sum_content_area_over_thin_division*:
assumes *div*: $\mathcal{D}\ division\_of\ S$ **and** S: $S \subseteq cbox\ a\ b$ **and** i: $i \in Basis$
and $a \cdot i \leq c\ c \leq b \cdot i$
and *nonmt*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \cap \{x.\ x \cdot i = c\} \neq \{\}$
shows $(b \cdot i - a \cdot i) * (\sum K{\in}\mathcal{D}.\ content\ K\ /\ (interval\_upperbound\ K \cdot i -$
$interval\_lowerbound\ K \cdot i))$
$\leq 2 * content(cbox\ a\ b)$
proof (*cases* $content(cbox\ a\ b) = 0$)
case *True*
have $(\sum K{\in}\mathcal{D}.\ content\ K\ /\ (interval\_upperbound\ K \cdot i - interval\_lowerbound$

$K \cdot i)) = 0$
using *S div* **by** (*force intro*!: *sum.neutral content_0_subset* [*OF True*])
then show *?thesis*
by (*auto simp*: *True*)
next
case *False*
then have *content*(*cbox a b*) > 0
using *zero_less_measure_iff* **by** *blast*
then have $a \cdot i < b \cdot i$ **if** $i \in$ *Basis* **for** i
using *content_pos_lt_eq that* **by** *blast*
have *finite* $\mathcal{D}$
using *div* **by** *blast*
define *Dlec* **where** *Dlec* $\equiv \{L \in (\lambda L.\ L \cap \{x.\ x \cdot i \le c\})\ `\ \mathcal{D}.\ content\ L \neq 0\}$
define *Dgec* **where** *Dgec* $\equiv \{L \in (\lambda L.\ L \cap \{x.\ x \cdot i \ge c\})\ `\ \mathcal{D}.\ content\ L \neq 0\}$
define a' **where** $a' \equiv (\sum j \in$ *Basis*. (*if* $j = i$ *then* c *else* $a \cdot j) *_R\ j)$
define b' **where** $b' \equiv (\sum j \in$ *Basis*. (*if* $j = i$ *then* c *else* $b \cdot j) *_R\ j)$
define *interv_diff* **where** *interv_diff* $\equiv \lambda K.\ \lambda i{::}'a.$ *interval_upperbound* $K \cdot i$ $-$ *interval_lowerbound* $K \cdot i$
have *Dlec_cbox*: $\bigwedge K.\ K \in$ *Dlec* $\Longrightarrow \exists a\ b.\ K =$ *cbox a b*
using *interval_split* [*OF i*] *div* **by** (*fastforce simp*: *Dlec_def division_of_def*)
then have *lec_is_cbox*: $[\![$*content* $(L \cap \{x.\ x \cdot i \le c\}) \neq 0;\ L \in \mathcal{D}]\!] \Longrightarrow \exists a\ b.$ $L \cap \{x.\ x \cdot i \le c\} =$ *cbox a b* **for** L
using *Dlec_def* **by** *blast*
have *Dgec_cbox*: $\bigwedge K.\ K \in$ *Dgec* $\Longrightarrow \exists a\ b.\ K =$ *cbox a b*
using *interval_split* [*OF i*] *div* **by** (*fastforce simp*: *Dgec_def division_of_def*)
then have *gec_is_cbox*: $[\![$*content* $(L \cap \{x.\ x \cdot i \ge c\}) \neq 0;\ L \in \mathcal{D}]\!] \Longrightarrow \exists a\ b.$ $L \cap \{x.\ x \cdot i \ge c\} =$ *cbox a b* **for** L
using *Dgec_def* **by** *blast*

have *zero_left*: $\bigwedge x\ y.\ [\![x \in \mathcal{D};\ y \in \mathcal{D};\ x \neq y;\ x \cap \{x.\ x \cdot i \le c\} = y \cap \{x.\ x \cdot i \le c\}]\!]$
$\Longrightarrow$ *content* $(y \cap \{x.\ x \cdot i \le c\}) = 0$
by (*metis division_split_left_inj* [*OF div*] *lec_is_cbox content_eq_0_interior*)
have *zero_right*: $\bigwedge x\ y.\ [\![x \in \mathcal{D};\ y \in \mathcal{D};\ x \neq y;\ x \cap \{x.\ c \le x \cdot i\} = y \cap \{x.\ c \le x \cdot i\}]\!]$
$\Longrightarrow$ *content* $(y \cap \{x.\ c \le x \cdot i\}) = 0$
by (*metis division_split_right_inj* [*OF div*] *gec_is_cbox content_eq_0_interior*)

have $(b' \cdot i - a \cdot i) * (\sum K \in$ *Dlec*. *content K* / *interv_diff K i*) $\le$ *content*(*cbox a b*$'$)
unfolding *interv_diff_def*
proof (*rule content_division_lemma1*)
show *Dlec division_of* $\bigcup$ *Dlec*
unfolding *division_of_def*
proof (*intro conjI ballI Dlec_cbox*)
show $\bigwedge K1\ K2.\ [\![K1 \in$ *Dlec*; *K2* $\in$ *Dlec*$]\!] \Longrightarrow K1 \neq K2 \longrightarrow$ *interior K1* $\cap$ *interior K2* $= \{\}$
by (*clarsimp simp*: *Dlec_def*) (*use div* **in** *auto*)
qed (*use* ‹*finite* $\mathcal{D}$› *Dlec_def* **in** *auto*)

show $\bigcup Dlec \subseteq cbox\ a\ b'$
using *Dlec_def div S* **by** (*auto simp*: *b'_def division_of_def mem_box*)
show $(\forall K{\in}Dlec.\ K \cap \{x.\ x \cdot i = a \cdot i\} \neq \{\}) \vee (\forall K{\in}Dlec.\ K \cap \{x.\ x \cdot i = b' \cdot i\} \neq \{\})$
using *nonmt* **by** (*fastforce simp*: *Dlec_def b'_def i*)
qed (*use i Dlec_def* **in** *auto*)
moreover
have $(\sum K{\in}Dlec.\ content\ K\ /\ (interv\_diff\ K\ i)) = (\sum K{\in}(\lambda K.\ K \cap \{x.\ x \cdot i \leq c\})\ `\ \mathcal{D}.\ content\ K\ /\ interv\_diff\ K\ i)$
unfolding *Dlec_def* **using** ‹*finite* $\mathcal{D}$› **by** (*auto simp*: *sum.mono_neutral_left*)
moreover have $\ldots =$
$(\sum K{\in}\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \leq c\}))\ K)$
by (*simp add*: *zero_left sum.reindex_nontrivial* [*OF* ‹*finite* $\mathcal{D}$›])
moreover have $(b' \cdot i - a \cdot i) = (c - a \cdot i)$
by (*simp add*: *b'_def i*)
ultimately
have *lec*: $(c - a \cdot i) * (\sum K{\in}\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \leq c\}))\ K)$
$\leq content(cbox\ a\ b')$
by *simp*

have $(b \cdot i - a' \cdot i) * (\sum K{\in}Dgec.\ content\ K\ /\ (interv\_diff\ K\ i)) \leq content(cbox\ a'\ b)$
unfolding *interv_diff_def*
proof (*rule content_division_lemma1*)
show *Dgec division_of* $\bigcup Dgec$
unfolding *division_of_def*
proof (*intro conjI ballI Dgec_cbox*)
show $\bigwedge K1\ K2.\ [\![K1 \in Dgec;\ K2 \in Dgec]\!] \Longrightarrow K1 \neq K2 \longrightarrow interior\ K1 \cap interior\ K2 = \{\}$
by (*clarsimp simp*: *Dgec_def*) (*use div* **in** *auto*)
qed (*use* ‹*finite* $\mathcal{D}$› *Dgec_def* **in** *auto*)
show $\bigcup Dgec \subseteq cbox\ a'\ b$
using *Dgec_def div S* **by** (*auto simp*: *a'_def division_of_def mem_box*)
show $(\forall K{\in}Dgec.\ K \cap \{x.\ x \cdot i = a' \cdot i\} \neq \{\}) \vee (\forall K{\in}Dgec.\ K \cap \{x.\ x \cdot i = b \cdot i\} \neq \{\})$
using *nonmt* **by** (*fastforce simp*: *Dgec_def a'_def i*)
qed (*use i Dgec_def* **in** *auto*)
moreover
have $(\sum K{\in}Dgec.\ content\ K\ /\ (interv\_diff\ K\ i)) = (\sum K{\in}(\lambda K.\ K \cap \{x.\ c \leq x \cdot i\})\ `\ \mathcal{D}.$
content K / interv_diff K i)
unfolding *Dgec_def* **using** ‹*finite* $\mathcal{D}$› **by** (*auto simp*: *sum.mono_neutral_left*)
moreover have $\ldots =$
$(\sum K{\in}\mathcal{D}.\ ((\lambda K.\ content\ K\ /\ (interv\_diff\ K\ i)) \circ ((\lambda K.\ K \cap \{x.\ x \cdot i \geq c\}))\ K)$
by (*simp add*: *zero_right sum.reindex_nontrivial* [*OF* ‹*finite* $\mathcal{D}$›])
moreover have $(b \cdot i - a' \cdot i) = (b \cdot i - c)$

```
    by (simp add: a'_def i)
  ultimately
  have gec: (b · i − c) ∗ (∑ K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK.
K ∩ {x. x · i ≥ c}))) K)
          ≤ content(cbox a' b)
    by simp

  show ?thesis
  proof (cases c = a · i ∨ c = b · i)
    case True
    then show ?thesis
    proof
      assume c: c = a · i
      moreover
      have (∑ j∈Basis. (if j = i then a · i else a · j) ∗R j) = a
        using euclidean_representation [of a] sum.cong [OF refl, of Basis λi. (a ·
i) ∗R i] by presburger
      ultimately have a' = a
        by (simp add: i a'_def cong: if_cong)
      then have content (cbox a' b) ≤ 2 ∗ content (cbox a b)  by simp
      moreover
      have eq: (∑ K∈𝒟. content (K ∩ {x. a · i ≤ x · i}) / interv_diff (K ∩ {x.
a · i ≤ x · i}) i)
            = (∑ K∈𝒟. content K / interv_diff K i)
        (is sum ?f _ = sum ?g _)
      proof (rule sum.cong [OF refl])
        fix K assume K ∈ 𝒟
        then have a · i ≤ x · i if x ∈ K for x
          by (metis S UnionI div division_ofD(6) i mem_box(2) subsetCE that)
        then have K ∩ {x. a · i ≤ x · i} = K
          by blast
        then show ?f K = ?g K
          by simp
      qed
      ultimately show ?thesis
        using gec c eq interv_diff_def by auto
    next
      assume c: c = b · i
      moreover have (∑ j∈Basis. (if j = i then b · i else b · j) ∗R j) = b
        using euclidean_representation [of b] sum.cong [OF refl, of Basis λi. (b ·
i) ∗R i] by presburger
      ultimately have b' = b
        by (simp add: i b'_def cong: if_cong)
      then have content (cbox a b') ≤ 2 ∗ content (cbox a b)  by simp
      moreover
      have eq: (∑ K∈𝒟. content (K ∩ {x. x · i ≤ b · i}) / interv_diff (K ∩ {x. x
· i ≤ b · i}) i)
            = (∑ K∈𝒟. content K / interv_diff K i)
             (is sum ?f _ = sum ?g _)
```

```
    proof (rule sum.cong [OF refl])
      fix K assume K ∈ 𝒟
      then have x · i ≤ b · i if x ∈ K for x
        by (metis S UnionI div division_ofD(6) i mem_box(2) subsetCE that)
      then have K ∩ {x. x · i ≤ b · i} = K
        by blast
      then show ?f K = ?g K
        by simp
    qed
    ultimately show ?thesis
      using lec c eq interv_diff_def by auto
  qed
 next
  case False
   have prod_if: (∏k∈Basis ∩ - {i}. f k) = (∏k∈Basis. f k) / f i if f i ≠
(0::real) for f
  proof -
    have f i * prod f (Basis ∩ - {i}) = prod f Basis
      using that mk_disjoint_insert [OF i]
     by (metis Int_insert_left_if0 finite_Basis finite_insert le_iff_inf order_refl
prod.insert subset_Compl_singleton)
    then show ?thesis
      by (metis nonzero_mult_div_cancel_left that)
  qed
  have abc: a · i < c c < b · i
    using False assms by auto
  then have (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK. K ∩ {x. x
· i ≤ c}))) K)
              ≤ content(cbox a b') / (c - a · i)
            (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK. K ∩ {x. x · i
≥ c}))) K)
              ≤ content(cbox a' b) / (b · i - c)
    using lec gec by (simp_all add: field_split_simps)
  moreover
  have (∑K∈𝒟. content K / (interv_diff K i))
       ≤ (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK. K ∩ {x. x · i
≤ c}))) K) +
         (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK. K ∩ {x. x · i ≥
c}))) K)
        (is ?lhs ≤ ?rhs)
  proof -
    have ?lhs ≤
         (∑K∈𝒟. ((λK. content K / (interv_diff K i)) ∘ ((λK. K ∩ {x. x · i ≤
c}))) K +
                  ((λK. content K / (interv_diff K i)) ∘ ((λK. K ∩ {x. x · i ≥
c}))) K)
        (is sum ?f _ ≤ sum ?g _)
    proof (rule sum_mono)
      fix K assume K ∈ 𝒟
```

then obtain $u\ v$ **where** uv: $K = cbox\ u\ v$
using div **by** $blast$
obtain $u'\ v'$ **where** uv': $cbox\ u\ v \cap \{x.\ x \cdot i \leq c\} = cbox\ u\ v'$
$cbox\ u\ v \cap \{x.\ c \leq x \cdot i\} = cbox\ u'\ v$
$\bigwedge k.\ k \in Basis \Longrightarrow u' \cdot k = (if\ k = i\ then\ max\ (u \cdot i)\ c\ else\ u \cdot k)$
$\bigwedge k.\ k \in Basis \Longrightarrow v' \cdot k = (if\ k = i\ then\ min\ (v \cdot i)\ c\ else\ v \cdot k)$
using i **by** ($auto\ simp$: $interval\_split$)
have $*$: $[\![content\ (cbox\ u\ v') = 0;\ content\ (cbox\ u'\ v) = 0]\!] \Longrightarrow content\ (cbox\ u\ v) = 0$
$content\ (cbox\ u'\ v) \neq 0 \Longrightarrow content\ (cbox\ u\ v) \neq 0$
$content\ (cbox\ u\ v') \neq 0 \Longrightarrow content\ (cbox\ u\ v) \neq 0$
using $i\ uv\ uv'$ **by** ($auto\ simp$: $content\_eq\_0\ le\_max\_iff\_disj\ min\_le\_iff\_disj$ $split$: $if\_split\_asm\ intro$: $order\_trans$)
have $uniq$: $\bigwedge j.\ [\![j \in Basis; \neg\ u \cdot j \leq v \cdot j]\!] \Longrightarrow j = i$
by ($metis\ \langle K \in \mathcal{D}\rangle\ box\_ne\_empty(1)\ div\ division\_of\_def\ uv$)
show $?f\ K \leq ?g\ K$
using $i\ uv\ uv'$ **by** ($auto\ simp\ add$: $interv\_diff\_def\ lemma0$ $dest$: $uniq\ *$ $intro!$: $prod\_nonneg$)
qed
also have $... = ?rhs$
by ($simp\ add$: $sum.distrib$)
finally show $?thesis$ **.**
qed
moreover have $content\ (cbox\ a\ b') / (c - a \cdot i) = content\ (cbox\ a\ b) / (b \cdot i - a \cdot i)$
using $i\ abc$
apply ($simp\ add$: $field\_simps\ a'\_def\ b'\_def\ measure\_lborel\_cbox\_eq\ inner\_diff$)
apply ($auto\ simp$: $if\_distrib\ if\_distrib$ [$of\ \lambda f.\ f\ x$ **for** x] $prod.If\_cases$ [$of\ Basis\ \lambda x.\ x = i$, $simplified$] $prod\_if\ field\_simps$)
done
moreover have $content\ (cbox\ a'\ b) / (b \cdot i - c) = content\ (cbox\ a\ b) / (b \cdot i - a \cdot i)$
using $i\ abc$
apply ($simp\ add$: $field\_simps\ a'\_def\ b'\_def\ measure\_lborel\_cbox\_eq\ inner\_diff$)
apply ($auto\ simp$: $if\_distrib\ prod.If\_cases$ [$of\ Basis\ \lambda x.\ x = i$, $simplified$] $prod\_if\ field\_simps$)
done
ultimately
have $(\sum K \in \mathcal{D}.\ content\ K\ /\ (interv\_diff\ K\ i)) \leq 2 * content\ (cbox\ a\ b) / (b \cdot i - a \cdot i)$
by $linarith$
then show $?thesis$
using $abc\ interv\_diff\_def$ **by** ($simp\ add$: $field\_split\_simps$)
qed
qed

proposition *bounded_equiintegral_over_thin_tagged_partial_division*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$ **and** $0 < \varepsilon$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \le norm(f\ x)$
obtains γ **where** *gauge* γ
$\bigwedge c\ i\ S\ h.\ [\![c \in cbox\ a\ b;\ i \in Basis;\ S\ tagged\_partial\_division\_of\ cbox\ a\ b;$
$\gamma\ fine\ S;\ h \in F;\ \bigwedge x\ K.\ (x,K) \in S \Longrightarrow (K \cap \{x.\ x \cdot i = c \cdot i\} \neq \{\})]\!]$
$\Longrightarrow (\sum (x,K) \in S.\ norm\ (integral\ K\ h)) < \varepsilon$
proof (*cases content*$(cbox\ a\ b) = 0$)
case *True*
show *?thesis*
proof
show *gauge* $(\lambda x.\ ball\ x\ 1)$
by (*simp add*: *gauge_trivial*)
show $(\sum (x,K) \in S.\ norm\ (integral\ K\ h)) < \varepsilon$
if S *tagged_partial_division_of cbox a b* $(\lambda x.\ ball\ x\ 1)$ *fine* S **for** S **and** $h :: 'a \Rightarrow 'b$
proof −
have $(\sum (x,K) \in S.\ norm\ (integral\ K\ h)) = 0$
using *that True content_0_subset*
by (*fastforce simp*: *tagged_partial_division_of_def intro*: *sum.neutral*)
with $\langle 0 < \varepsilon \rangle$ **show** *?thesis*
by *simp*
qed
qed
next
case *False*
then have *contab_gt0*: $content(cbox\ a\ b) > 0$
by (*simp add*: *zero_less_measure_iff*)
then have *a_less_b*: $\bigwedge i.\ i \in Basis \Longrightarrow a \cdot i < b \cdot i$
by (*auto simp*: *content_pos_lt_eq*)
obtain $\gamma 0$ **where** *gauge* $\gamma 0$
and $\gamma 0$: $\bigwedge S\ h.\ [\![S\ tagged\_partial\_division\_of\ cbox\ a\ b;\ \gamma 0\ fine\ S;\ h \in F]\!]$
$\Longrightarrow (\sum (x,K) \in S.\ norm\ (content\ K *_R h\ x - integral\ K\ h)) < \varepsilon/2$
proof −
obtain γ **where** *gauge* γ
and γ: $\bigwedge f\ \mathcal{D}.\ [\![f \in F;\ \mathcal{D}\ tagged\_division\_of\ cbox\ a\ b;\ \gamma\ fine\ \mathcal{D}]\!]$
$\Longrightarrow norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ x) - integral\ (cbox\ a\ b)\ f)$
$< \varepsilon/(5 * (Suc\ DIM('b)))$
proof −
have *e5*: $\varepsilon/(5 * (Suc\ DIM('b))) > 0$
using $\langle \varepsilon > 0 \rangle$ **by** *auto*
then show *?thesis*

```
      using F that by (auto simp: equiintegrable_on_def)
    qed
    show ?thesis
    proof
      show gauge γ
        by (rule ‹gauge γ›)
      show (∑(x,K) ∈ S. norm (content K *R h x − integral K h)) < ε/2
          if S tagged_partial_division_of cbox a b γ fine S h ∈ F for S h
      proof −
        have (∑(x,K) ∈ S. norm (content K *R h x − integral K h)) ≤ 2 * real
DIM('b) * (ε/(5 * Suc DIM('b)))
        proof (rule Henstock_lemma_part2 [of h a b])
          show h integrable_on cbox a b
            using that F equiintegrable_on_def by metis
          show gauge γ
            by (rule ‹gauge γ›)
        qed (use that ‹ε > 0› γ in auto)
        also have ... < ε/2
          using ‹ε > 0› by (simp add: divide_simps)
        finally show ?thesis .
      qed
    qed
  qed
  define γ where γ ≡ λx. γ0 x ∩
                       ball x ((ε/8 / (norm(f x) + 1)) * (INF m∈Basis. b · m − a
· m) / content(cbox a b))
  define interv_diff where interv_diff ≡ λK. λi::'a. interval_upperbound K · i
− interval_lowerbound K · i
  have 8 * content (cbox a b) + norm (f x) * (8 * content (cbox a b)) > 0 for x
    by (metis add.right_neutral add_pos_pos contab_gt0 mult_pos_pos mult_zero_left
norm_eq_zero zero_less_norm_iff zero_less_numeral)
  then have gauge (λx. ball x
                  (ε * (INF m∈Basis. b · m − a · m) / ((8 * norm (f x) + 8) *
content (cbox a b))))
    using ‹0 < content (cbox a b)› ‹0 < ε› a_less_b
    by (auto simp add: gauge_def field_split_simps add_nonneg_eq_0_iff fi-
nite_less_Inf_iff)
  then have gauge γ
    unfolding γ_def using ‹gauge γ0› gauge_Int by auto
  moreover
  have (∑(x,K) ∈ S. norm (integral K h)) < ε
      if c ∈ cbox a b i ∈ Basis and S: S tagged_partial_division_of cbox a b
        and γ fine S h ∈ F and ne: ⋀x K. (x,K) ∈ S ⟹ K ∩ {x. x · i = c · i}
≠ {} for c i S h
  proof −
    have cbox c b ⊆ cbox a b
      by (meson mem_box(2) order_refl subset_box(1) that(1))
    have finite S
      using S unfolding tagged_partial_division_of_def by blast
```

have *γ0 fine S* **and** *fineS*:
(λx. *ball x* (ε * (*INF m∈Basis*. *b* · *m* − *a* · *m*) / ((8 * *norm* (*f x*) + 8) * *content* (*cbox a b*)))) *fine S*
using ‹*γ fine S*› **by** (*auto simp*: *γ_def fine_Int*)
then have (∑(*x*,*K*) ∈ *S*. *norm* (*content K* *$_R$ *h x* − *integral K h*)) < *ε*/2
by (*intro γ0 that fineS*)
moreover have (∑(*x*,*K*) ∈ *S*. *norm* (*integral K h*) − *norm* (*content K* *$_R$ *h x* − *integral K h*)) ≤ *ε*/2
proof −
have (∑(*x*,*K*) ∈ *S*. *norm* (*integral K h*) − *norm* (*content K* *$_R$ *h x* − *integral K h*))
≤ (∑(*x*,*K*) ∈ *S*. *norm* (*content K* *$_R$ *h x*))
proof (*clarify intro*!: *sum_mono*)
fix *x K*
assume *xK*: (*x*,*K*) ∈ *S*
have *norm* (*integral K h*) − *norm* (*content K* *$_R$ *h x* − *integral K h*) ≤ *norm* (*integral K h* − (*integral K h* − *content K* *$_R$ *h x*))
by (*metis norm_minus_commute norm_triangle_ineq2*)
also have ... ≤ *norm* (*content K* *$_R$ *h x*)
by *simp*
finally show *norm* (*integral K h*) − *norm* (*content K* *$_R$ *h x* − *integral K h*) ≤ *norm* (*content K* *$_R$ *h x*) .
qed
also have ... ≤ (∑(*x*,*K*) ∈ *S*. *ε*/4 * (*b* · *i* − *a* · *i*) / *content* (*cbox a b*) * *content K* / *interv_diff K i*)
proof (*clarify intro*!: *sum_mono*)
fix *x K*
assume *xK*: (*x*,*K*) ∈ *S*
then have *x*: *x* ∈ *cbox a b*
using *S* **unfolding** *tagged_partial_division_of_def* **by** (*meson subset_iff*)
show *norm* (*content K* *$_R$ *h x*) ≤ *ε*/4 * (*b* · *i* − *a* · *i*) / *content* (*cbox a b*) * *content K* / *interv_diff K i*
proof (*cases content K* = 0)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then have *Kgt0*: *content K* > 0
using *zero_less_measure_iff* **by** *blast*
moreover
obtain *u v* **where** *uv*: *K* = *cbox u v*
using *S* ‹(*x*,*K*) ∈ *S*› **unfolding** *tagged_partial_division_of_def* **by** *blast*
then have *u_less_v*: ⋀*i*. *i* ∈ *Basis* ⟹ *u* · *i* < *v* · *i*
using *content_pos_lt_eq uv Kgt0* **by** *blast*
then have *dist_uv*: *dist u v* > 0
using *that* **by** *auto*
ultimately have *norm* (*h x*) ≤ (*ε* * (*b* · *i* − *a* · *i*)) / (4 * *content* (*cbox a b*) * *interv_diff K i*)

```
        proof −
          have dist x u < ε * (INF m∈Basis. b · m − a · m) / (4 * (norm (f x)
+ 1) * content (cbox a b)) / 2
               dist x v < ε * (INF m∈Basis. b · m − a · m) / (4 * (norm (f x) +
1) * content (cbox a b)) / 2
            using fineS u_less_v uv xK
            by (force simp: fine_def mem_box field_simps dest!: bspec)+
          moreover have ε * (INF m∈Basis. b · m − a · m) / (4 * (norm (f x)
+ 1) * content (cbox a b)) / 2
                  ≤ ε * (b · i − a · i) / (4 * (norm (f x) + 1) * content (cbox a b))
/ 2
          proof (intro mult_left_mono divide_right_mono)
            show (INF m∈Basis. b · m − a · m) ≤ b · i − a · i
              using ‹i ∈ Basis› by (auto intro!: cInf_le_finite)
          qed (use ‹0 < ε› in auto)
          ultimately
           have dist x u < ε * (b · i − a · i) / (4 * (norm (f x) + 1) * content
(cbox a b)) / 2
                dist x v < ε * (b · i − a · i) / (4 * (norm (f x) + 1) * content (cbox
a b)) / 2
            by linarith+
          then have duv: dist u v < ε * (b · i − a · i) / (4 * (norm (f x) + 1) *
content (cbox a b))
            using dist_triangle_half_r by blast
          have uvi: |v · i − u · i| ≤ norm (v − u)
          by (metis inner_commute inner_diff_right ‹i ∈ Basis› Basis_le_norm)
          have norm (h x) ≤ norm (f x)
            using x that by (auto simp: norm_f)
          also have ... < (norm (f x) + 1)
            by simp
          also have ... < ε * (b · i − a · i) / dist u v / (4 * content (cbox a b))
          proof −
            have 0 < norm (f x) + 1
              by (simp add: add.commute add_pos_nonneg)
            then show ?thesis
              using duv dist_uv contab_gt0
              by (simp only: mult_ac divide_simps) auto
          qed
          also have ... = ε * (b · i − a · i) / norm (v − u) / (4 * content (cbox
a b))
            by (simp add: dist_norm norm_minus_commute)
          also have ... ≤ ε * (b · i − a · i) / |v · i − u · i| / (4 * content (cbox
a b))
            proof (intro mult_right_mono divide_left_mono divide_right_mono
uvi)
            show norm (v − u) * |v · i − u · i| > 0
              using u_less_v [OF ‹i ∈ Basis›]
              by (auto simp: less_eq_real_def zero_less_mult_iff that)
            show ε * (b · i − a · i) ≥ 0
```

```
            using a_less_b ‹0 < ε› ‹i ∈ Basis› by force
          qed auto
          also have ... = ε * (b • i − a • i) / (4 * content (cbox a b) * interv_diff
K i)
            using uv False that(2) u_less_v interv_diff_def by fastforce
          finally show ?thesis by simp
        qed
        with Kgt0 have norm (content K *R h x) ≤ content K * ((ε/4 * (b • i
− a • i) / content (cbox a b)) / interv_diff K i)
          using mult_left_mono by fastforce
        also have ... = ε/4 * (b • i − a • i) / content (cbox a b) * content K /
interv_diff K i
          by (simp add: field_split_simps)
        finally show ?thesis .
      qed
    qed
    also have ... = (∑K∈snd ‘ S. ε/4 * (b • i − a • i) / content (cbox a b) *
content K / interv_diff K i)
      unfolding interv_diff_def
    apply (rule sum.over_tagged_division_lemma [OF tagged_partial_division_of_Union_self
[OF S]])
      apply (simp add: box_eq_empty(1) content_eq_0)
      done
    also have ... = ε/2 * ((b • i − a • i) / (2 * content (cbox a b)) * (∑K∈snd
‘ S. content K / interv_diff K i))
      by (simp add: interv_diff_def sum_distrib_left mult.assoc)
    also have ... ≤ (ε/2) * 1
    proof (rule mult_left_mono)
      have (b • i − a • i) * (∑K∈snd ‘ S. content K / interv_diff K i) ≤ 2 *
content (cbox a b)
        unfolding interv_diff_def
      proof (rule sum_content_area_over_thin_division)
        show snd ‘ S division_of ⋃(snd ‘ S)
      by (auto intro: S tagged_partial_division_of_Union_self division_of_tagged_division)
        show ⋃(snd ‘ S) ⊆ cbox a b
          using S unfolding tagged_partial_division_of_def by force
        show a • i ≤ c • i c • i ≤ b • i
          using mem_box(2) that by blast+
      qed (use that in auto)
      then show (b • i − a • i) / (2 * content (cbox a b)) * (∑K∈snd ‘ S.
content K / interv_diff K i) ≤ 1
        by (simp add: contab_gt0)
    qed (use ‹0 < ε› in auto)
    finally show ?thesis by simp
  qed
  then have (∑(x,K) ∈ S. norm (integral K h)) − (∑(x,K) ∈ S. norm (content
K *R h x − integral K h)) ≤ ε/2
    by (simp add: Groups_Big.sum_subtractf [symmetric])
  ultimately show (∑(x,K) ∈ S. norm (integral K h)) < ε
```

```
    by linarith
  qed
  ultimately show ?thesis using that by auto
qed



proposition equiintegrable_halfspace_restrictions_le:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes F: F equiintegrable_on cbox a b and f: f ∈ F
    and norm_f: ⋀h x. ⟦h ∈ F; x ∈ cbox a b⟧ ⟹ norm(h x) ≤ norm(f x)
  shows (⋃ i ∈ Basis. ⋃ c. ⋃ h ∈ F. {(λx. if x · i ≤ c then h x else 0)})
        equiintegrable_on cbox a b
proof (cases content(cbox a b) = 0)
  case True
  then show ?thesis by simp
next
  case False
  then have content(cbox a b) > 0
    using zero_less_measure_iff by blast
  then have a · i < b · i if i ∈ Basis for i
    using content_pos_lt_eq that by blast
  have int_F: f integrable_on cbox a b if f ∈ F for f
    using F that by (simp add: equiintegrable_on_def)
  let ?CI = λK h x. content K *R h x − integral K h
  show ?thesis
    unfolding equiintegrable_on_def
  proof (intro conjI; clarify)
    show int_lec: ⟦i ∈ Basis; h ∈ F⟧ ⟹ (λx. if x · i ≤ c then h x else 0)
integrable_on cbox a b for i c h
      using integrable_restrict_Int [of {x. x · i ≤ c} h]
      by (simp add: inf_commute int_F integrable_split(1))
    show ∃γ. gauge γ ∧
            (∀f T. f ∈ (⋃ i∈Basis. ⋃ c. ⋃h∈F. {λx. if x · i ≤ c then h x else 0})
∧
                  T tagged_division_of cbox a b ∧ γ fine T ⟶
                  norm ((∑ (x,K) ∈ T. content K *R f x) − integral (cbox a b) f)
< ε)
      if ε > 0 for ε
    proof −
      obtain γ0 where gauge γ0 and γ0:
        ⋀c i S h. ⟦c ∈ cbox a b; i ∈ Basis; S tagged_partial_division_of cbox a b;
                    γ0 fine S; h ∈ F; ⋀x K. (x,K) ∈ S ⟹ (K ∩ {x. x · i = c ·
i} ≠ {})⟧
                   ⟹ (∑ (x,K) ∈ S. norm (integral K h)) < ε/12
      proof (rule bounded_equiintegral_over_thin_tagged_partial_division [OF F
f, of ‹ε/12›])
        show ⋀h x. ⟦h ∈ F; x ∈ cbox a b⟧ ⟹ norm (h x) ≤ norm (f x)
          by (auto simp: norm_f)
```

qed (*use* ‹$\varepsilon > 0$› **in** *auto*)
obtain $\gamma 1$ **where** *gauge* $\gamma 1$
and $\gamma 1$: $\bigwedge h\ T.\ [\![h \in F;\ T$ *tagged_division_of cbox a b*; $\gamma 1$ *fine* $T]\!] \Longrightarrow$ *norm* $((\sum (x,K) \in T.$ *content* $K *_R h\ x) -$ *integral* (*cbox a b*) $h)$ $< \varepsilon/(7 * (Suc\ DIM('b)))$
proof −
have *e5*: $\varepsilon/(7 * (Suc\ DIM('b))) > 0$
using ‹$\varepsilon > 0$› **by** *auto*
then show *?thesis*
using *F that* **by** (*auto simp*: *equiintegrable_on_def*)
qed
have *h_less3*: $(\sum (x,K) \in T.$ *norm* (*?CI K h x*)) $< \varepsilon/3$
if T *tagged_partial_division_of cbox a b* $\gamma 1$ *fine* $T\ h \in F$ **for** $T\ h$
proof −
have $(\sum (x,K) \in T.$ *norm* (*?CI K h x*)) $\leq 2 *$ *real* $DIM('b) * (\varepsilon/(7 *$ *Suc* $DIM('b)))$
proof (*rule Henstock_lemma_part2* [*of h a b*])
show *h integrable_on cbox a b*
using *that F equiintegrable_on_def* **by** *metis*
qed (*use that* ‹$\varepsilon > 0$› ‹*gauge* $\gamma 1$› $\gamma 1$ **in** *auto*)
also have ... $< \varepsilon/3$
using ‹$\varepsilon > 0$› **by** (*simp add*: *divide_simps*)
finally show *?thesis* **.**
qed
have $*$: *norm* $((\sum (x,K) \in T.$ *content* $K *_R f\ x) -$ *integral* $(cbox\ a\ b)\ f) < \varepsilon$
if f: $f = (\lambda x.$ *if* $x \cdot i \leq c$ *then* $h\ x$ *else* $0)$
and T: T *tagged_division_of cbox a b*
and *fine*: $(\lambda x.\ \gamma 0\ x \cap \gamma 1\ x)$ *fine* T **and** $i \in Basis\ h \in F$ **for** $f\ T\ i\ c\ h$
proof (*cases* $a \cdot i \leq c \wedge c \leq b \cdot i$)
case *True*
have *finite* T
using T **by** *blast*
define T' **where** $T' \equiv \{(x,K) \in T.\ K \cap \{x.\ x \cdot i \leq c\} \neq \{\}\}$
then have $T' \subseteq T$
by *auto*
then have *finite* T'
using ‹*finite* T› *infinite_super* **by** *blast*
have T'*_tagged*: T' *tagged_partial_division_of cbox a b*
by (*meson* T ‹$T' \subseteq T$› *tagged_division_of_def tagged_partial_division_subset*)
have *fine'*: $\gamma 0$ *fine* T' $\gamma 1$ *fine* T'
using ‹$T' \subseteq T$› *fine_Int fine_subset fine* **by** *blast+*
have *int_KK'*: $(\sum (x,K) \in T.$ *integral* $K\ f) = (\sum (x,K) \in T'.$ *integral* $K\ f)$
proof (*rule sum.mono_neutral_right* [*OF* ‹*finite* T› ‹$T' \subseteq T$›])
show $\forall i \in T - T'.$ (*case* i *of* $(x,\ K) \Rightarrow$ *integral* $K\ f) = 0$
using f ‹*finite* T› ‹$T' \subseteq T$› *integral_restrict_Int* [*of* _ $\{x.\ x \cdot i \leq c\}\ h$]
by (*auto simp*: T'*_def Int_commute*)
qed
have $(\sum (x,K) \in T.$ *content* $K *_R f\ x) = (\sum (x,K) \in T'.$ *content* $K *_R f$

```
x)
      proof (rule sum.mono_neutral_right [OF ‹finite T› ‹T' ⊆ T›])
        show ∀ i ∈ T − T'. (case i of (x, K) ⇒ content K *R f x) = 0
          using T f ‹finite T› ‹T' ⊆ T› by (force simp: T'_def)
      qed
      moreover have norm ((∑(x,K) ∈ T'. content K *R f x) − integral (cbox
a b) f) < ε
      proof −
        have *: norm y < ε if norm x < ε/3 norm(x − y) ≤ 2 * ε/3 for x y::'b
        proof −
          have norm y ≤ norm x + norm(x − y)
            by (metis norm_minus_commute norm_triangle_sub)
          also have ... < ε/3 + 2*ε/3
            using that by linarith
          also have ... = ε
            by simp
          finally show ?thesis .
        qed
        have norm (∑(x,K) ∈ T'. ?CI K h x)
            ≤ (∑(x,K) ∈ T'. norm (?CI K h x))
          by (simp add: norm_sum split_def)
        also have ... < ε/3
          by (intro h_less3 T'_tagged fine' that)
        finally have norm (∑(x,K) ∈ T'. ?CI K h x) < ε/3 .
        moreover have integral (cbox a b) f = (∑(x,K) ∈ T. integral K f)
       using int_lec that by (auto simp: integral_combine_tagged_division_topdown)
        moreover have norm (∑(x,K) ∈ T'. ?CI K h x − ?CI K f x)
            ≤ 2*ε/3
        proof −
          define T'' where T'' ≡ {(x,K) ∈ T'. ¬ (K ⊆ {x. x • i ≤ c})}
          then have T'' ⊆ T'
            by auto
          then have finite T''
            using ‹finite T'› infinite_super by blast
          have T''_tagged: T'' tagged_partial_division_of cbox a b
            using T'_tagged ‹T'' ⊆ T'› tagged_partial_division_subset by blast
          have fine'': γ0 fine T'' γ1 fine T''
            using ‹T'' ⊆ T'› fine' by (blast intro: fine_subset)+
          have (∑(x,K) ∈ T'. ?CI K h x − ?CI K f x)
              = (∑(x,K) ∈ T''. ?CI K h x − ?CI K f x)
           proof (clarify intro!: sum.mono_neutral_right [OF ‹finite T'› ‹T'' ⊆
T'›])
            fix x K
            assume (x,K) ∈ T' (x,K) ∉ T''
            then have x ∈ K x • i ≤ c {x. x • i ≤ c} ∩ K = K
              using T''_def T'_tagged tagged_partial_division_of_def by blast+
            then show ?CI K h x − ?CI K f x = 0
              using integral_restrict_Int [of _ {x. x • i ≤ c} h] by (auto simp: f)
          qed
```

```
        moreover have norm (∑(x,K) ∈ T''. ?CI K h x − ?CI K f x) ≤ 2∗ε/3
         proof −
           define A where A ≡ {(x,K) ∈ T''. x • i ≤ c}
           define B where B ≡ {(x,K) ∈ T''. x • i > c}
           then have A ⊆ T'' B ⊆ T'' and disj: A ∩ B = {} and T''_eq: T''
= A ∪ B
             by (auto simp: A_def B_def)
           then have finite A finite B
             using ‹finite T''›  by (auto intro: finite_subset)
           have A_tagged: A tagged_partial_division_of cbox a b
            using T''_tagged ‹A ⊆ T''› tagged_partial_division_subset by blast
           have fineA: γ0 fine A γ1 fine A
             using ‹A ⊆ T''› fine'' by (blast intro: fine_subset)+
           have B_tagged: B tagged_partial_division_of cbox a b
            using T''_tagged ‹B ⊆ T''› tagged_partial_division_subset by blast
           have fineB: γ0 fine B γ1 fine B
             using ‹B ⊆ T''› fine'' by (blast intro: fine_subset)+
           have norm (∑(x,K) ∈ T''. ?CI K h x − ?CI K f x)
                   ≤ (∑(x,K) ∈ T''. norm (?CI K h x − ?CI K f x))
             by (simp add: norm_sum split_def)
           also have ... = (∑(x,K) ∈ A. norm (?CI K h x − ?CI K f x)) +
                           (∑(x,K) ∈ B. norm (?CI K h x − ?CI K f x))
             by (simp add: sum.union_disjoint T''_eq disj ‹finite A› ‹finite B›)
           also have ... = (∑(x,K) ∈ A. norm (integral K h − integral K f)) +
                           (∑(x,K) ∈ B. norm (?CI K h x + integral K f))
          by (auto simp: A_def B_def f norm_minus_commute intro!: sum.cong
arg_cong2 [where f= (+)])
           also have ... ≤ (∑(x,K)∈A. norm (integral K h)) +
                             (∑(x,K)∈(λ(x,K). (x,K ∩ {x. x • i ≤ c})) ‘ A. norm
(integral K h))
                         + ((∑(x,K)∈B. norm (?CI K h x)) +
                            (∑(x,K)∈B. norm (integral K h)) +
                             (∑(x,K)∈(λ(x,K). (x,K ∩ {x. c ≤ x • i})) ‘ B. norm
(integral K h)))
           proof (rule add_mono)
             show (∑(x,K)∈A. norm (integral K h − integral K f))
                   ≤ (∑(x,K)∈A. norm (integral K h)) +
                     (∑(x,K)∈(λ(x,K). (x,K ∩ {x. x • i ≤ c})) ‘ A.
                       norm (integral K h))
             proof (subst sum.reindex_nontrivial [OF ‹finite A›], clarsimp)
               fix x K L
               assume (x,K) ∈ A (x,L) ∈ A
                 and int_ne0: integral (L ∩ {x. x • i ≤ c}) h ≠ 0
                 and eq: K ∩ {x. x • i ≤ c} = L ∩ {x. x • i ≤ c}
               have False if K ≠ L
               proof −
                 obtain u v where uv: L = cbox u v
                   using T'_tagged ‹(x, L) ∈ A› ‹A ⊆ T''› ‹T'' ⊆ T'› by (blast
dest: tagged_partial_division_ofD)
```

```
                have interior (K ∩ {x. x • i ≤ c}) = {}
                  proof (rule tagged_division_split_left_inj [OF _ ‹(x,K) ∈ A›
‹(x,L) ∈ A›])
                  show A tagged_division_of ⋃(snd ‘ A)
                    using A_tagged tagged_partial_division_of_Union_self by
auto
                  show K ∩ {x. x • i ≤ c} = L ∩ {x. x • i ≤ c}
                    using eq ‹i ∈ Basis› by auto
                qed (use that in auto)
                then show False
              using interval_split [OF ‹i ∈ Basis›] int_ne0 content_eq_0_interior
eq uv by fastforce
              qed
              then show K = L by blast
            next
              show (∑(x,K) ∈ A. norm (integral K h − integral K f))
                    ≤ (∑(x,K) ∈ A. norm (integral K h)) +
                      sum ((λ(x,K). norm (integral K h)) ∘ (λ(x,K). (x,K ∩ {x.
x • i ≤ c}))) A
                using integral_restrict_Int [of _ {x. x • i ≤ c} h] f
                  by (auto simp: Int_commute A_def [symmetric] sum.distrib
[symmetric] intro!: sum_mono norm_triangle_ineq4)
              qed
            next
              show (∑(x,K)∈B. norm (?CI K h x + integral K f))
                ≤ (∑(x,K)∈B. norm (?CI K h x)) + (∑(x,K)∈B. norm (integral
K h)) +
                    (∑(x,K)∈(λ(x,K). (x,K ∩ {x. c ≤ x • i})) ‘ B. norm (integral
K h))
              proof (subst sum.reindex_nontrivial [OF ‹finite B›], clarsimp)
                fix x K L
                assume (x,K) ∈ B (x,L) ∈ B
                  and int_ne0: integral (L ∩ {x. c ≤ x • i}) h ≠ 0
                  and eq: K ∩ {x. c ≤ x • i} = L ∩ {x. c ≤ x • i}
                have False if K ≠ L
                proof −
                  obtain u v where uv: L = cbox u v
                    using T'_tagged ‹(x, L) ∈ B› ‹B ⊆ T''› ‹T'' ⊆ T'› by (blast
dest: tagged_partial_division_ofD)
                  have interior (K ∩ {x. c ≤ x • i}) = {}
                   proof (rule tagged_division_split_right_inj [OF _ ‹(x,K) ∈ B›
‹(x,L) ∈ B›])
                    show B tagged_division_of ⋃(snd ‘ B)
                      using B_tagged tagged_partial_division_of_Union_self by
auto
                    show K ∩ {x. c ≤ x • i} = L ∩ {x. c ≤ x • i}
                      using eq ‹i ∈ Basis› by auto
                  qed (use that in auto)
                  then show False
```

```
                using interval_split [OF ‹i ∈ Basis›] int_ne0
                  content_eq_0_interior eq uv by fastforce
              qed
              then show K = L by blast
            next
              show (∑(x,K) ∈ B. norm (?CI K h x + integral K f))
                  ≤ (∑(x,K) ∈ B. norm (?CI K h x)) +
                    (∑(x,K) ∈ B. norm (integral K h)) + sum ((λ(x,K). norm
(integral K h)) ∘ (λ(x,K). (x,K ∩ {x. c ≤ x • i}))) B
               proof (clarsimp simp: B_def [symmetric] sum.distrib [symmetric]
intro!: sum_mono)
                fix x K
                assume (x,K) ∈ B
                 have ∗: i = i1 + i2 ⟹ norm(c + i1) ≤ norm c + norm i +
norm(i2)
                  for i::'b and c i1 i2
                by (metis add.commute add.left_commute add_diff_cancel_right'
dual_order.refl norm_add_rule_thm norm_triangle_ineq4)
                obtain u v where uv: K = cbox u v
                  using T'_tagged ‹(x,K) ∈ B› ‹B ⊆ T''› ‹T'' ⊆ T'› by (blast
dest: tagged_partial_division_ofD)
                have huv: h integrable_on cbox u v
                proof (rule integrable_on_subcbox)
                  show cbox u v ⊆ cbox a b
               using B_tagged ‹(x,K) ∈ B› uv by (blast dest: tagged_partial_division_ofD)
                  show h integrable_on cbox a b
                    by (simp add: int_F ‹h ∈ F›)
                qed
               have integral K h = integral K f + integral (K ∩ {x. c ≤ x • i}) h
                    using integral_restrict_Int [of _ {x. x • i ≤ c} h] f uv ‹i ∈
Basis›
                 by (simp add: Int_commute integral_split [OF huv ‹i ∈ Basis›])
              then show norm (?CI K h x + integral K f)
                        ≤ norm (?CI K h x) + norm (integral K h) + norm (integral
(K ∩ {x. c ≤ x • i}) h)
                by (rule ∗)
            qed
          qed
        qed
        also have ... ≤ 2∗ε/3
        proof −
          have overlap: K ∩ {x. x • i = c} ≠ {} if (x,K) ∈ T'' for x K
          proof −
            obtain y y' where y: y' ∈ K c < y' • i y ∈ K y • i ≤ c
              using that  T''_def T'_def ‹(x,K) ∈ T''› by fastforce
            obtain u v where uv: K = cbox u v
           using T''_tagged ‹(x,K) ∈ T''› by (blast dest: tagged_partial_division_ofD)
            then have connected K
              by (simp add: is_interval_connected)
```

```
      then have (∃ z ∈ K. z · i = c)
        using y connected_ivt_component by fastforce
      then show ?thesis
        by fastforce
    qed
  have **: ⟦x < ε/12; y < ε/12; z ≤ ε/2⟧ ⟹ x + y + z ≤ 2 * ε/3 for
x y z
    by auto
  show ?thesis
  proof (rule **)
    have cb_ab: (∑ j ∈ Basis. if j = i then c *R i else (a · j) *R j) ∈
cbox a b
      using ‹i ∈ Basis› True ‹⋀i. i ∈ Basis ⟹ a · i < b · i›
      by (force simp add: mem_box sum_if_inner [where f = λj. c])
    show (∑ (x,K) ∈ A. norm (integral K h)) < ε/12
      using ‹i ∈ Basis› ‹A ⊆ T''› overlap
      by (force simp add: sum_if_inner [where f = λj. c]
          intro!: γ0 [OF cb_ab ‹i ∈ Basis› A_tagged fineA(1) ‹h ∈ F›])
    let ?F = λ(x,K). (x, K ∩ {x. x · i ≤ c})
    have 1: ?F ' A tagged_partial_division_of cbox a b
      unfolding tagged_partial_division_of_def
    proof (intro conjI strip)
      show ⋀x K. (x, K) ∈ ?F ' A ⟹ ∃ a b. K = cbox a b
        using A_tagged interval_split(1) [OF ‹i ∈ Basis›, of _ _ c]
        by (force dest: tagged_partial_division_ofD(4))
      show ⋀x K. (x, K) ∈ ?F ' A ⟹ x ∈ K
  using A_def A_tagged by (fastforce dest: tagged_partial_division_ofD)
  qed (use A_tagged in ‹fastforce dest: tagged_partial_division_ofD›)+
    have 2: γ0 fine (λ(x,K). (x,K ∩ {x. x · i ≤ c})) ' A
      using fineA(1) fine_def by fastforce
  show (∑ (x,K) ∈ (λ(x,K). (x,K ∩ {x. x · i ≤ c})) ' A. norm (integral
K h)) < ε/12
    using ‹i ∈ Basis› ‹A ⊆ T''› overlap
    by (force simp add: sum_if_inner [where f = λj. c]
        intro!: γ0 [OF cb_ab ‹i ∈ Basis› 1 2 ‹h ∈ F›])
  have *: ⟦x < ε/3; y < ε/12; z < ε/12⟧ ⟹ x + y + z ≤ ε/2 for x
y z
    by auto
  show (∑ (x,K) ∈ B. norm (?CI K h x)) +
       (∑ (x,K) ∈ B. norm (integral K h)) +
      (∑ (x,K) ∈ (λ(x,K). (x,K ∩ {x. c ≤ x · i})) ' B. norm (integral
K h))
       ≤ ε/2
  proof (rule *)
    show (∑ (x,K) ∈ B. norm (?CI K h x)) < ε/3
      by (intro h_less3 B_tagged fineB that)
    show (∑ (x,K) ∈ B. norm (integral K h)) < ε/12
    using ‹i ∈ Basis› ‹B ⊆ T''› overlap
    by (force simp add: sum_if_inner [where f = λj. c]
```

```
                intro!: γ0 [OF cb_ab ‹i ∈ Basis› B_tagged fineB(1) ‹h ∈ F›])
            let ?F = λ(x,K). (x, K ∩ {x. c ≤ x • i})
            have 1: ?F ‘ B tagged_partial_division_of cbox a b
              unfolding tagged_partial_division_of_def
            proof (intro conjI strip)
              show ⋀x K. (x, K) ∈ ?F ‘ B ⟹ ∃ a b. K = cbox a b
                using B_tagged interval_split(2) [OF ‹i ∈ Basis›, of _ _ c]
                by (force dest: tagged_partial_division_ofD(4))
              show ⋀x K. (x, K) ∈ ?F ‘ B ⟹ x ∈ K
          using B_def B_tagged by (fastforce dest: tagged_partial_division_ofD)
        qed (use B_tagged in ‹fastforce dest: tagged_partial_division_ofD›)+
            have 2: γ0 fine (λ(x,K). (x,K ∩ {x. c ≤ x • i})) ‘ B
              using fineB(1) fine_def by fastforce
               show (∑(x,K) ∈ (λ(x,K). (x,K ∩ {x. c ≤ x • i})) ‘ B. norm
(integral K h)) < ε/12
            using ‹i ∈ Basis› ‹A ⊆ T''› overlap
            by (force simp add: B_def sum_if_inner [where f = λj. c]
                intro!: γ0 [OF cb_ab ‹i ∈ Basis› 1 2 ‹h ∈ F›])
          qed
        qed
      qed
      finally show ?thesis .
    qed
    ultimately show ?thesis by metis
  qed
  ultimately show ?thesis
    by (simp add: sum_subtractf [symmetric] int_KK' *)
qed
  ultimately show ?thesis by metis
next
  case False
  then consider c < a • i | b • i < c
    by auto
  then show ?thesis
  proof cases
    case 1
    then have f0: f x = 0 if x ∈ cbox a b for x
      using that f ‹i ∈ Basis› mem_box(2) by force
    then have int_f0: integral (cbox a b) f = 0
      by (simp add: integral_cong)
    have f0_tag: f x = 0 if (x,K) ∈ T for x K
      using T f0 that by (meson tag_in_interval)
    then have (∑(x,K) ∈ T. content K *R f x) = 0
      by (metis (mono_tags, lifting) real_vector.scale_eq_0_iff split_conv
sum.neutral surj_pair)
    then show ?thesis
      using ‹0 < ε› by (simp add: int_f0)
next
    case 2
```

```
        then have fh: f x = h x if x ∈ cbox a b for x
          using that f ‹i ∈ Basis› mem_box(2) by force
        then have int_f: integral (cbox a b) f = integral (cbox a b) h
          using integral_cong by blast
        have fh_tag: f x = h x if (x,K) ∈ T for x K
          using T fh that by (meson tag_in_interval)
       then have fh: (∑(x,K) ∈ T. content K *R f x) = (∑(x,K) ∈ T. content
K *R h x)
          by (metis (mono_tags, lifting) split_cong sum.cong)
        show ?thesis
          unfolding fh int_f
        proof (rule less_trans [OF γ1])
          show γ1 fine T
            by (meson fine fine_Int)
          show ε / (7 * Suc DIM('b)) < ε
            using ‹0 < ε› by (force simp: divide_simps)+
        qed (use that in auto)
      qed
    qed
    have  gauge (λx. γ0 x ∩ γ1 x)
      by (simp add: ‹gauge γ0› ‹gauge γ1› gauge_Int)
    then show ?thesis
      by (auto intro: *)
  qed
 qed
qed


corollary equiintegrable_halfspace_restrictions_ge:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes F: F equiintegrable_on cbox a b and f: f ∈ F
   and norm_f: ⋀h x. ⟦h ∈ F; x ∈ cbox a b⟧ ⟹ norm(h x) ≤ norm(f x)
 shows (⋃i ∈ Basis. ⋃c. ⋃h ∈ F. {(λx. if x • i ≥ c then h x else 0)})
       equiintegrable_on cbox a b
proof −
 have *: (⋃i∈Basis. ⋃c. ⋃h∈(λf. f ∘ uminus) ‘ F. {λx. if x • i ≤ c then h x
else 0})
         equiintegrable_on  cbox (− b) (− a)
 proof (rule equiintegrable_halfspace_restrictions_le)
   show (λf. f ∘ uminus) ‘ F equiintegrable_on cbox (− b) (− a)
     using F equiintegrable_reflect by blast
   show f ∘ uminus ∈ (λf. f ∘ uminus) ‘ F
     using f by auto
   show ⋀h x. ⟦h ∈ (λf. f ∘ uminus) ‘ F; x ∈ cbox (− b) (− a)⟧ ⟹ norm (h x)
≤ norm ((f ∘ uminus) x)
     using f unfolding comp_def image_iff
       by (metis (no_types, lifting) equation_minus_iff imageE norm_f umi-
nus_interval_vector)
 qed
```

```
  have eq: (λf. f ∘ uminus) ‘
        (⋃i∈Basis. ⋃c. ⋃h∈F. {λx. if x • i ≤ c then (h ∘ uminus) x else 0}) =
         (⋃i∈Basis. ⋃c. ⋃h∈F. {λx. if c ≤ x • i then h x else 0})   (is ?lhs =
?rhs)
  proof
    show ?lhs ⊆ ?rhs
      using minus_le_iff by fastforce
    show ?rhs ⊆ ?lhs
    apply clarsimp
    apply (rule_tac x=λx. if c ≤ (−x) • i then h(−x) else 0 in image_eqI)
      using le_minus_iff by fastforce+
  qed
  show ?thesis
    using equiintegrable_reflect [OF ∗] by (auto simp: eq)
qed

corollary equiintegrable_halfspace_restrictions_lt:
  fixes f :: ′a::euclidean_space ⇒ ′b::euclidean_space
  assumes F: F equiintegrable_on cbox a b and f: f ∈ F
    and norm_f: ⋀h x. ⟦h ∈ F; x ∈ cbox a b⟧ ⟹ norm(h x) ≤ norm(f x)
  shows (⋃i ∈ Basis. ⋃c. ⋃h ∈ F. {(λx. if x • i < c then h x else 0)}) equiin-
tegrable_on cbox a b
        (is ?G equiintegrable_on cbox a b)
proof −
  have ∗: (⋃i∈Basis. ⋃c. ⋃h∈F. {λx. if c ≤ x • i then h x else 0}) equiinte-
grable_on cbox a b
    using equiintegrable_halfspace_restrictions_ge [OF F f] norm_f by auto
  have (λx. if x • i < c then h x else 0) = (λx. h x − (if c ≤ x • i then h x else 0))
    if i ∈ Basis h ∈ F for i c h
    using that by force
  then show ?thesis
    by (blast intro: equiintegrable_on_subset [OF equiintegrable_diff [OF F ∗]])
qed

corollary equiintegrable_halfspace_restrictions_gt:
  fixes f :: ′a::euclidean_space ⇒ ′b::euclidean_space
  assumes F: F equiintegrable_on cbox a b and f: f ∈ F
    and norm_f: ⋀h x. ⟦h ∈ F; x ∈ cbox a b⟧ ⟹ norm(h x) ≤ norm(f x)
  shows (⋃i ∈ Basis. ⋃c. ⋃h ∈ F. {(λx. if x • i > c then h x else 0)}) equiin-
tegrable_on cbox a b
        (is ?G equiintegrable_on cbox a b)
proof −
  have ∗: (⋃i∈Basis. ⋃c. ⋃h∈F. {λx. if c ≥ x • i then h x else 0}) equiinte-
grable_on cbox a b
    using equiintegrable_halfspace_restrictions_le [OF F f] norm_f by auto
  have (λx. if x • i > c then h x else 0) = (λx. h x − (if c ≥ x • i then h x else 0))
    if i ∈ Basis h ∈ F for i c h
    using that by force
  then show ?thesis
```

by (*blast intro*: *equiintegrable_on_subset* [*OF equiintegrable_diff* [*OF F* ∗]])
qed

proposition *equiintegrable_closed_interval_restrictions*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f integrable_on cbox a b*
shows (⋃ *c d*. {(λ*x*. *if x* ∈ *cbox c d then f x else 0*)}) *equiintegrable_on cbox a b*
proof −
let *?g* = λ*B c d x*. *if* ∀ *i*∈*B*. *c* · *i* ≤ *x* · *i* ∧ *x* · *i* ≤ *d* · *i then f x else 0*
have ∗: *insert f* (⋃ *c d*. {*?g B c d*}) *equiintegrable_on cbox a b* **if** *B* ⊆ *Basis* **for** *B*
proof −
have *finite B*
using *finite_Basis finite_subset* ‹*B* ⊆ *Basis*› **by** *blast*
then show *?thesis* **using** ‹*B* ⊆ *Basis*›
proof (*induction B*)
case *empty*
with *f* **show** *?case* **by** *auto*
next
case (*insert i B*)
then have *i* ∈ *Basis B* ⊆ *Basis*
by *auto*
have ∗: *norm* (*h x*) ≤ *norm* (*f x*)
if *h* ∈ *insert f* (⋃ *c d*. {*?g B c d*}) *x* ∈ *cbox a b* **for** *h x*
using *that* **by** *auto*
define *F* **where** *F* ≡ (⋃ *i*∈*Basis*.
⋃ ξ. ⋃ *h*∈*insert f* (⋃ *i*∈*Basis*. ⋃ ψ. ⋃ *h*∈*insert f* (⋃ *c d*. {*?g B c d*}).
{λ*x*. *if x* · *i* ≤ ψ *then h x else 0*}).
{λ*x*. *if* ξ ≤ *x* · *i then h x else 0*})
show *?case*
proof (*rule equiintegrable_on_subset*)
have *F equiintegrable_on cbox a b*
unfolding *F_def*
proof (*rule equiintegrable_halfspace_restrictions_ge*)
show *insert f* (⋃ *i*∈*Basis*. ⋃ ξ. ⋃ *h*∈*insert f* (⋃ *c d*. {*?g B c d*}).
{λ*x*. *if x* · *i* ≤ ξ *then h x else 0*}) *equiintegrable_on cbox a b*
by (*intro* ∗ *f equiintegrable_on_insert equiintegrable_halfspace_restrictions_le*
[*OF insert.IH insertI1*] ‹*B* ⊆ *Basis*›)
show *norm*(*h x*) ≤ *norm*(*f x*)
if *h* ∈ *insert f* (⋃ *i*∈*Basis*. ⋃ ξ. ⋃ *h*∈*insert f* (⋃ *c d*. {*?g B c d*}). {λ*x*.
if x · *i* ≤ ξ *then h x else 0*})
x ∈ *cbox a b* **for** *h x*
using *that* **by** *auto*
qed *auto*
then show *insert f F*
equiintegrable_on cbox a b
by (*blast intro*: *f equiintegrable_on_insert*)
show *insert f* (⋃ *c d*. {λ*x*. *if* ∀ *j*∈*insert i B*. *c* · *j* ≤ *x* · *j* ∧ *x* · *j* ≤ *d* · *j*
then f x else 0})

```
          ⊆ insert f F
        using ‹i ∈ Basis›
        apply clarify
        apply (simp add: F_def)
        apply (drule_tac x=i in bspec, assumption)
        apply (drule_tac x=c • i in spec, clarify)
        apply (drule_tac x=i in bspec, assumption)
        apply (drule_tac x=d • i in spec)
        apply (clarsimp simp: fun_eq_iff)
        apply (drule_tac x=c in spec)
        apply (drule_tac x=d in spec)
        apply (simp split: if_split_asm)
        done
    qed
  qed
 qed
 show ?thesis
  by (rule equiintegrable_on_subset [OF * [OF subset_refl]]) (auto simp: mem_box)
qed
```

9.14.3 Continuity of the indefinite integral

```
proposition indefinite_integral_continuous:
 fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
 assumes int_f: f integrable_on cbox a b
    and c: c ∈ cbox a b and d: d ∈ cbox a b 0 < ε
 obtains δ where 0 < δ
           ⋀c' d'. ⟦c' ∈ cbox a b; d' ∈ cbox a b; norm(c' − c) ≤ δ; norm(d' − d)
≤ δ⟧
                   ⟹ norm(integral(cbox c' d') f − integral(cbox c d) f) < ε
proof −
 { assume ∃ c' d'. c' ∈ cbox a b ∧ d' ∈ cbox a b ∧ norm(c' − c) ≤ δ ∧ norm(d'
− d) ≤ δ ∧
                  norm(integral(cbox c' d') f − integral(cbox c d) f) ≥ ε
                  (is ∃ c' d'. ?Φ c' d' δ) if 0 < δ for δ
   then have ∃ c' d'. ?Φ c' d' (1 / Suc n) for n
     by simp
   then obtain u v where ⋀n. ?Φ (u n) (v n) (1 / Suc n)
     by metis
   then have u: u n ∈ cbox a b and norm_u: norm(u n − c) ≤ 1 / Suc n
       and  v: v n ∈ cbox a b and norm_v: norm(v n − d) ≤ 1 / Suc n
       and ε: ε ≤ norm (integral (cbox (u n) (v n)) f − integral (cbox c d) f) for
n
     by blast+
   then have False
   proof −
     have uvn: cbox (u n) (v n) ⊆ cbox a b for n
       by (meson u v mem_box(2) subset_box(1))
     define S where S ≡ ⋃ i ∈ Basis. {x. x • i = c • i} ∪ {x. x • i = d • i}
```

```
    have negligible S
      unfolding S_def by force
    then have int_f': (λx. if x ∈ S then 0 else f x) integrable_on cbox a b
      by (force intro: integrable_spike assms)
    have get_n: ∃n. ∀m≥n. x ∈ cbox (u m) (v m) ⟷ x ∈ cbox c d if x: x ∉ S
for x
    proof −
      define ε where ε ≡ Min ((λi. min |x • i − c • i| |x • i − d • i|) ‘ Basis)
      have ε > 0
        using ‹x ∉ S› by (auto simp: S_def ε_def)
      then obtain n where n ≠ 0 and n: 1 / (real n) < ε
        by (metis inverse_eq_divide real_arch_inverse)
      have emin: ε ≤ min |x • i − c • i| |x • i − d • i| if i ∈ Basis for i
        unfolding ε_def
       by (meson Min.coboundedI euclidean_space_class.finite_Basis finite_imageI
image_iff that)
      have 1 / real (Suc n) < ε
        using n ‹n ≠ 0› ‹ε > 0› by (simp add: field_simps)
      have x ∈ cbox (u m) (v m) ⟷ x ∈ cbox c d if m ≥ n for m
      proof −
        have ∗: ⟦|u − c| ≤ n; |v − d| ≤ n; N < |x − c|; N < |x − d|; n ≤ N⟧
              ⟹ u ≤ x ∧ x ≤ v ⟷ c ≤ x ∧ x ≤ d for N n u v c d and x::real
          by linarith
        have (u m • i ≤ x • i ∧ x • i ≤ v m • i) = (c • i ≤ x • i ∧ x • i ≤ d • i)
          if i ∈ Basis for i
        proof (rule ∗)
          show |u m • i − c • i| ≤ 1 / Suc m
            using norm_u [of m]
              by (metis (full_types) order_trans Basis_le_norm inner_commute
inner_diff_right that)
          show |v m • i − d • i| ≤ 1 / real (Suc m)
            using norm_v [of m]
              by (metis (full_types) order_trans Basis_le_norm inner_commute
inner_diff_right that)
          show 1/n < |x • i − c • i| 1/n < |x • i − d • i|
            using n ‹n ≠ 0› emin [OF ‹i ∈ Basis›]
            by (simp_all add: inverse_eq_divide)
          show 1 / real (Suc m) ≤ 1 / real n
            using ‹n ≠ 0› ‹m ≥ n› by (simp add: field_split_simps)
        qed
        then show ?thesis by (simp add: mem_box)
      qed
      then show ?thesis by blast
    qed
    have 1: range (λn x. if x ∈ cbox (u n) (v n) then if x ∈ S then 0 else f x else
0) equiintegrable_on cbox a b
    by (blast intro: equiintegrable_on_subset [OF equiintegrable_closed_interval_restrictions
[OF int_f']])
    have 2: (λn. if x ∈ cbox (u n) (v n) then if x ∈ S then 0 else f x else 0)
```

```
        ⟶ (if x ∈ cbox c d then if x ∈ S then 0 else f x else 0) for x
    by (fastforce simp: dest: get_n intro: tendsto_eventually eventually_sequentiallyI)
    have [simp]: cbox c d ∩ cbox a b = cbox c d
      using c d by (force simp: mem_box)
    have [simp]: cbox (u n) (v n) ∩ cbox a b = cbox (u n) (v n) for n
      using u v by (fastforce simp: mem_box intro: order.trans)
    have ⋀y A. y ∈ A − S ⟹ f y = (λx. if x ∈ S then 0 else f x) y
      by simp
    then have ⋀A. integral A (λx. if x ∈ S then 0 else f (x)) = integral A (λx.
f (x))
      by (blast intro: integral_spike [OF ‹negligible S›])
    moreover
    obtain N where dist (integral (cbox (u N) (v N)) (λx. if x ∈ S then 0 else f
x))
                        (integral (cbox c d) (λx. if x ∈ S then 0 else f x)) < ε
    using equiintegrable_limit [OF 1 2] ‹0 < ε› by (force simp: integral_restrict_Int
lim_sequentially)
    ultimately have dist (integral (cbox (u N) (v N)) f) (integral (cbox c d) f)
< ε
      by simp
    then show False
      by (metis dist_norm not_le ε)
  qed
 }
 then show ?thesis
  by (meson not_le that)
qed

corollary indefinite_integral_uniformly_continuous:
 fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
 assumes f integrable_on cbox a b
 shows uniformly_continuous_on (cbox (Pair a a) (Pair b b)) (λy. integral (cbox
(fst y) (snd y)) f)
proof −
 show ?thesis
 proof (rule compact_uniformly_continuous, clarsimp simp add: continuous_on_iff)
   fix c d and ε::real
   assume c: c ∈ cbox a b and d: d ∈ cbox a b and 0 < ε
   obtain δ where 0 < δ and δ:
          ⋀c' d'. ⟦c' ∈ cbox a b; d' ∈ cbox a b; norm(c' − c) ≤ δ; norm(d' − d)
≤ δ⟧
                        ⟹ norm(integral(cbox c' d') f −
                                integral(cbox c d) f) < ε
    using indefinite_integral_continuous ‹0 < ε› assms c d by blast
   show ∃δ > 0. ∀x' ∈ cbox (a, a) (b, b).
            dist x' (c, d) < δ ⟶
            dist (integral (cbox (fst x') (snd x')) f)
                 (integral (cbox c d) f)
            < ε
```

```
    using ‹0 < δ›
   by (force simp: dist_norm intro: δ order_trans [OF norm_fst_le] order_trans
[OF norm_snd_le] less_imp_le)
 qed auto
qed


corollary bounded_integrals_over_subintervals:
 fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
 assumes f integrable_on cbox a b
 shows bounded {integral (cbox c d) f |c d. cbox c d ⊆ cbox a b}
proof −
 have bounded ((λy. integral (cbox (fst y) (snd y)) f) ‘ cbox (a, a) (b, b))
     (is bounded ?I)
   by (blast intro: bounded_cbox bounded_uniformly_continuous_image indefi-
nite_integral_uniformly_continuous [OF assms])
 then obtain B where B > 0 and B: ⋀x. x ∈ ?I ⟹ norm x ≤ B
   by (auto simp: bounded_pos)
 have norm x ≤ B if x = integral (cbox c d) f cbox c d ⊆ cbox a b for x c d
 proof (cases cbox c d = {})
   case True
   with ‹0 < B› that show ?thesis by auto
 next
   case False
   then have ∃ x ∈ cbox (a,a) (b,b). integral (cbox c d) f = integral (cbox (fst x)
(snd x)) f
   using that by (metis cbox_Pair_iff interval_subset_is_interval is_interval_cbox
prod.sel)
   then show ?thesis
     using B that(1) by blast
 qed
 then show ?thesis
   by (blast intro: boundedI)
qed
```

An existence theorem for "improper" integrals. Hake's theorem implies that if the integrals over subintervals have a limit, the integral exists. We only need to assume that the integrals are bounded, and we get absolute integrability, but we also need a (rather weak) bound assumption on the function.

```
theorem absolutely_integrable_improper:
 fixes f :: 'M::euclidean_space ⇒ 'N::euclidean_space
 assumes int_f: ⋀c d. cbox c d ⊆ box a b ⟹ f integrable_on cbox c d
     and bo: bounded {integral (cbox c d) f |c d. cbox c d ⊆ box a b}
     and absi: ⋀i. i ∈ Basis
         ⟹ ∃ g. g absolutely_integrable_on cbox a b ∧
               ((∀ x ∈ cbox a b. f x • i ≤ g x) ∨ (∀ x ∈ cbox a b. f x • i ≥ g x))
     shows f absolutely_integrable_on cbox a b
proof (cases content(cbox a b) = 0)
 case True
```

```
  then show ?thesis
    by auto
next
  case False
  then have pos: content(cbox a b) > 0
    using zero_less_measure_iff by blast
  show ?thesis
    unfolding absolutely_integrable_componentwise_iff [where f = f]
  proof
    fix j::'N
    assume j ∈ Basis
    then obtain g where absint_g: g absolutely_integrable_on cbox a b
                  and g: (∀ x ∈ cbox a b. f x • j ≤ g x) ∨ (∀ x ∈ cbox a b. f x • j ≥
g x)
      using absi by blast
    have int_gab: g integrable_on cbox a b
      using absint_g set_lebesgue_integral_eq_integral(1) by blast
    define α where α ≡ λk. a + (b − a) /R real k
    define β where β ≡ λk. b − (b − a) /R real k
    define I where I ≡ λk. cbox (α k) (β k)
    have ISuc_box: I (Suc n) ⊆ box a b for n
      using pos unfolding I_def
        by (intro subset_box_imp) (auto simp: α_def β_def content_pos_lt_eq
algebra_simps)
    have ISucSuc: I (Suc n) ⊆ I (Suc (Suc n)) for n
    proof −
      have ⋀i. i ∈ Basis
              ⟹ a • i / Suc n + b • i / (real n + 2)  ≤ b • i / Suc n + a • i /
(real n + 2)
        using pos
       by (simp add: content_pos_lt_eq divide_simps) (auto simp: algebra_simps)
      then show ?thesis
        unfolding I_def
         by (intro subset_box_imp) (auto simp: algebra_simps inverse_eq_divide
α_def β_def)
    qed
    have getN: ∃ N::nat. ∀ k. k ≥ N ⟶ x ∈ I k
      if x: x ∈ box a b for x
    proof −
      define Δ where Δ ≡ (⋃ i ∈ Basis. {((x − a) • i) / ((b − a) • i), (b − x) •
i / ((b − a) • i)})
      obtain N where N: real N > 1 / Inf Δ
        using reals_Archimedean2 by blast
      moreover have Δ: Inf Δ > 0
      using that by (auto simp: Δ_def finite_less_Inf_iff mem_box algebra_simps
divide_simps)
      ultimately have N > 0
        using of_nat_0_less_iff by fastforce
      show ?thesis
```

```
    proof (intro exI impI allI)
      fix k assume N ≤ k
      with ‹0 < N› have k > 0
        by linarith
      have xa_gt: (x − a) · i > ((b − a) · i) / (real k) if i ∈ Basis for i
      proof −
        have ∗: Inf Δ ≤ ((x − a) · i) / ((b − a) · i)
          unfolding Δ_def using that by (force intro: cInf_le_finite)
        have 1 / Inf Δ ≥ ((b − a) · i) / ((x − a) · i)
          using le_imp_inverse_le [OF ∗ Δ]
          by (simp add: field_simps)
        with N have k > ((b − a) · i) / ((x − a) · i)
          using ‹N ≤ k› by linarith
        with x that show ?thesis
          by (auto simp: mem_box algebra_simps field_split_simps)
      qed
      have bx_gt: (b − x) · i > ((b − a) · i) / k if i ∈ Basis for i
      proof −
        have ∗: Inf Δ ≤ ((b − x) · i) / ((b − a) · i)
          using that unfolding Δ_def by (force intro: cInf_le_finite)
        have 1 / Inf Δ ≥ ((b − a) · i) / ((b − x) · i)
          using le_imp_inverse_le [OF ∗ Δ]
          by (simp add: field_simps)
        with N have k > ((b − a) · i) / ((b − x) · i)
          using ‹N ≤ k› by linarith
        with x that show ?thesis
          by (auto simp: mem_box algebra_simps field_split_simps)
      qed
      show x ∈ I k
        using that Δ ‹k > 0› unfolding I_def
       by (auto simp: α_def β_def mem_box algebra_simps divide_inverse dest:
xa_gt bx_gt)
    qed
  qed
  obtain Bf where Bf: ⋀c d. cbox c d ⊆ box a b ⟹ norm (integral (cbox c d)
f) ≤ Bf
    using bo unfolding bounded_iff by blast
  obtain Bg where Bg:⋀c d. cbox c d ⊆ cbox a b ⟹ |integral (cbox c d) g| ≤
Bg
  using bounded_integrals_over_subintervals [OF int_gab] unfolding bounded_iff
real_norm_def by blast
  show (λx. f x · j) absolutely_integrable_on cbox a b
    using g
  proof     — A lot of duplication in the two proofs
    assume fg [rule_format]: ∀ x∈cbox a b. f x · j ≤ g x
    have (λx. (f x · j)) = (λx. g x − (g x − (f x · j)))
      by simp
    moreover have (λx. g x − (g x − (f x · j))) integrable_on cbox a b
    proof (rule Henstock_Kurzweil_Integration.integrable_diff [OF int_gab])
```

```
      define φ where φ ≡ λk x. if x ∈ I (Suc k) then g x − f x · j else 0
      have (λx. g x − f x · j) integrable_on box a b
      proof (rule monotone_convergence_increasing [of φ, THEN conjunct1])
        have ∗: I (Suc k) ∩ box a b = I (Suc k) for k
          using box_subset_cbox ISuc_box by fastforce
        show φ k integrable_on box a b for k
        proof −
          have I (Suc k) ⊆ cbox a b
            using ∗ box_subset_cbox by blast
          moreover have (λm. f m · j) integrable_on I (Suc k)
            by (metis ISuc_box I_def int_f integrable_component)
          ultimately have (λm. g m − f m · j) integrable_on I (Suc k)
           by (metis Henstock_Kurzweil_Integration.integrable_diff I_def int_gab
integrable_on_subcbox)
          then show ?thesis
            by (simp add: ∗ φ_def integrable_restrict_Int)
        qed
        show φ k x ≤ φ (Suc k) x if x ∈ box a b for k x
          using ISucSuc box_subset_cbox that by (force simp: φ_def intro!: fg)
        show (λk. φ k x) ⟶ g x − f x · j if x: x ∈ box a b for x
        proof (rule tendsto_eventually)
          obtain N::nat where N: ⋀k. k ≥ N ⟹ x ∈ I k
            using getN [OF x] by blast
          show ∀F k in sequentially. φ k x = g x − f x · j
          proof
            fix k::nat assume N ≤ k
            have x ∈ I (Suc k)
              by (metis ‹N ≤ k› le_Suc_eq N)
            then show φ k x = g x − f x · j
              by (simp add: φ_def)
          qed
        qed
        have |integral (box a b) (λx. if x ∈ I (Suc k) then g x − f x · j else 0)| ≤
Bg + Bf for k
        proof −
          have ABK_def [simp]: I (Suc k) ∩ box a b = I (Suc k)
            using ISuc_box by (simp add: Int_absorb2)
          have int_fI: f integrable_on I (Suc k)
            using ISuc_box I_def int_f by auto
          moreover
          have |integral (I (Suc k)) (λx. f x · j)| ≤ norm (integral (I (Suc k)) f)
            by (simp add: Basis_le_norm int_fI ‹j ∈ Basis›)
          with ISuc_box ABK_def have |integral (I (Suc k)) (λx. f x · j)| ≤ Bf
                by (metis Bf I_def ‹j ∈ Basis› int_fI integral_component_eq
norm_bound_Basis_le)
          ultimately
           have |integral (I (Suc k)) g − integral (I (Suc k)) (λx. f x · j)| ≤ Bg
+ Bf
            using ∗ box_subset_cbox unfolding I_def
```

```
          by (blast intro: Bg add_mono order_trans [OF abs_triangle_ineq4])
        moreover have g integrable_on I (Suc k)
              by (metis ISuc_box I_def int_gab integrable_on_open_interval
integrable_on_subcbox)
        moreover have (λx. f x · j) integrable_on I (Suc k)
          using int_fI by (simp add: integrable_component)
        ultimately show ?thesis
          by (simp add: integral_restrict_Int integral_diff)
      qed
      then show bounded (range (λk. integral (box a b) (φ k)))
        by (auto simp add: bounded_iff φ_def)
    qed
    then show (λx. g x − f x · j) integrable_on cbox a b
      by (simp add: integrable_on_open_interval)
  qed
  ultimately have (λx. f x · j) integrable_on cbox a b
    by auto
  then show ?thesis
   using absolutely_integrable_component_ubound [OF _ absint_g] fg by force
 next
  assume gf [rule_format]: ∀ x∈cbox a b. g x ≤ f x · j
  have (λx. (f x · j)) = (λx. ((f x · j) − g x) + g x)
    by simp
  moreover have (λx. (f x · j − g x) + g x) integrable_on cbox a b
  proof (rule Henstock_Kurzweil_Integration.integrable_add [OF _ int_gab])
    let ?φ = λk x. if x ∈ I(Suc k) then f x · j − g x else 0
    have (λx. f x · j − g x) integrable_on box a b
    proof (rule monotone_convergence_increasing [of ?φ, THEN conjunct1])
      have ∗: I (Suc k) ∩ box a b = I (Suc k) for k
        using box_subset_cbox ISuc_box by fastforce
      show ?φ k integrable_on box a b for k
      proof (simp add: integrable_restrict_Int integral_restrict_Int ∗)
        show (λx. f x · j − g x) integrable_on I (Suc k)
       by (metis ISuc_box Henstock_Kurzweil_Integration.integrable_diff I_def
int_f int_gab integrable_component integrable_on_open_interval integrable_on_subcbox)
      qed
      show ?φ k x ≤ ?φ (Suc k) x if x ∈ box a b for k x
        using ISucSuc box_subset_cbox that by (force simp: I_def intro!: gf)
      show (λk. ?φ k x) ⟶ f x · j − g x if x: x ∈ box a b for x
      proof (rule tendsto_eventually)
        obtain N::nat where N: ⋀k. k ≥ N ⟹ x ∈ I k
          using getN [OF x] by blast
        then show ∀F k in sequentially. ?φ k x = f x · j − g x
          by (metis (no_types, lifting) eventually_at_top_linorderI le_Suc_eq)
      qed
      have |integral (box a b)
                (λx. if x ∈ I (Suc k) then f x · j − g x else 0)| ≤ Bf + Bg for k
      proof −
        define ABK where ABK ≡ cbox (a + (b − a) /R (1 + real k)) (b −
```

```
(b - a) /_R (1 + real k))
          have ABK_eq [simp]: ABK ∩ box a b = ABK
            using * I_def α_def β_def ABK_def by auto
          have int_fI: f integrable_on ABK
            unfolding ABK_def
            using ISuc_box I_def α_def β_def int_f by force
          then have (λx. f x • j) integrable_on ABK
            by (simp add: integrable_component)
          moreover have g integrable_on ABK
              by (metis ABK_def ABK_eq IntE box_subset_cbox int_gab integrable_on_subcbox subset_eq)
          moreover
          have |integral ABK (λx. f x • j)| ≤ norm (integral ABK f)
            by (simp add: Basis_le_norm int_fI ‹j ∈ Basis›)
          then have |integral ABK (λx. f x • j)| ≤ Bf
            by (metis ABK_eq  ABK_def Bf IntE dual_order.trans subset_eq)
          ultimately show ?thesis
            using * box_subset_cbox
             apply (simp add: integral_restrict_Int integral_diff ABK_def I_def α_def β_def)
             by (blast intro: Bg add_mono order_trans [OF abs_triangle_ineq4])
        qed
        then show bounded (range (λk. integral (box a b) (?φ k)))
          by (auto simp add: bounded_iff)
      qed
      then show (λx. f x • j - g x) integrable_on cbox a b
        by (simp add: integrable_on_open_interval)
    qed
    ultimately have (λx. f x • j) integrable_on cbox a b
      by auto
    then show ?thesis
      using absint_g absolutely_integrable_absolutely_integrable_lbound gf by blast
  qed
 qed
qed
```

9.14.4 Second mean value theorem and corollaries

```
lemma level_approx:
 fixes f :: real ⇒ real and n::nat
 assumes f: ⋀x. x ∈ S ⟹ 0 ≤ f x ∧ f x ≤ 1 and x ∈ S n ≠ 0
 shows |f x - (∑ k = Suc 0..n. if k / n ≤ f x then inverse n else 0)| < inverse n
       (is ?lhs < _)
proof -
 have n * f x ≥ 0
   using assms by auto
 then obtain m::nat where m: floor(n * f x) = int m
   using nonneg_int_cases zero_le_floor by blast
```

```
  then have kn: real k / real n ≤ f x ⟷ k ≤ m for k
    using ‹n ≠ 0› by (simp add: field_split_simps) linarith
  then have Suc n / real n ≤ f x ⟷ Suc n ≤ m
    by blast
  have real n * f x ≤ real n
    by (simp add: ‹x ∈ S› f mult_left_le)
  then have m ≤ n
    using m by linarith
  have ?lhs = |f x − (∑ k ∈ {Suc 0..n} ∩ {..m}. inverse n)|
    by (subst sum.inter_restrict) (auto simp: kn)
  also have ... < inverse n
    using ‹m ≤ n› ‹n ≠ 0› m
    by (simp add: min_absorb2 field_split_simps) linarith
  finally show ?thesis .
qed


lemma SMVT_lemma2:
  fixes f :: real ⇒ real
  assumes f: f integrable_on {a..b}
    and g: ⋀x y. x ≤ y ⟹ g x ≤ g y
  shows (⋃ y::real. {λx. if g x ≥ y then f x else 0}) equiintegrable_on {a..b}
proof −
  have ffab: {f} equiintegrable_on {a..b}
    by (metis equiintegrable_on_sing f interval_cbox)
  then have ff: {f} equiintegrable_on (cbox a b)
    by simp
  have ge: (⋃ c. {λx. if x ≥ c then f x else 0}) equiintegrable_on {a..b}
    using equiintegrable_halfspace_restrictions_ge [OF ff] by auto
  have gt: (⋃ c. {λx. if x > c then f x else 0}) equiintegrable_on {a..b}
    using equiintegrable_halfspace_restrictions_gt [OF ff] by auto
  have 0: {(λx. 0)} equiintegrable_on {a..b}
    by (metis box_real(2) equiintegrable_on_sing integrable_0)
  have †: (λx. if g x ≥ y then f x else 0) ∈ {(λx. 0), f} ∪ (⋃ z. {λx. if z < x then
f x else 0}) ∪ (⋃ z. {λx. if z ≤ x then f x else 0})
    for y
  proof (cases (∀ x. g x ≥ y) ∨ (∀ x. ¬ (g x ≥ y)))
    let ?μ = Inf {x. g x ≥ y}
    case False
    have lower: ?μ ≤ x if g x ≥ y for x
    proof (rule cInf_lower)
      show x ∈ {x. y ≤ g x}
        using False by (auto simp: that)
      show bdd_below {x. y ≤ g x}
        by (metis False bdd_belowI dual_order.trans g linear mem_Collect_eq)
    qed
    have greatest: ?μ ≥ z if  (⋀x. g x ≥ y ⟹ z ≤ x) for z
      by (metis False cInf_greatest empty_iff mem_Collect_eq that)
    show ?thesis
```

```
    proof (cases g ?μ ≥ y)
      case True
      then obtain ζ where ζ: ⋀x. g x ≥ y ⟷ x ≥ ζ
        by (metis g lower order.trans)  — in fact y is Inf {x. y ≤ g x}
      then show ?thesis
        by (force simp: ζ)
    next
      case False
      have (y ≤ g x) ⟷ (?μ < x) for x
      proof
        show ?μ < x if y ≤ g x
          using that False less_eq_real_def lower by blast
        show y ≤ g x if ?μ < x
          by (metis g greatest le_less_trans that less_le_trans linear not_less)
      qed
      then obtain ζ where ζ: ⋀x. g x ≥ y ⟷ x > ζ ..
      then show ?thesis
        by (force simp: ζ)
    qed
  qed auto
  show ?thesis
     using † by (simp add: UN_subset_iff equiintegrable_on_subset [OF equiin-
tegrable_on_Un [OF gt equiintegrable_on_Un [OF ge equiintegrable_on_Un [OF
ffab 0]]]])
qed


lemma SMVT_lemma4:
  fixes f :: real ⇒ real
  assumes f: f integrable_on {a..b}
    and a ≤ b
    and g: ⋀x y. x ≤ y ⟹ g x ≤ g y
    and 01: ⋀x. ⟦a ≤ x; x ≤ b⟧ ⟹ 0 ≤ g x ∧ g x ≤ 1
  obtains c where a ≤ c c ≤ b ((λx. g x *R f x) has_integral integral {c..b} f)
{a..b}
proof −
  have connected ((λx. integral {x..b} f) ‘ {a..b})
   by (simp add: f indefinite_integral_continuous_1′ connected_continuous_image)
  moreover have compact ((λx. integral {x..b} f) ‘ {a..b})
   by (simp add: compact_continuous_image f indefinite_integral_continuous_1′)
   ultimately obtain m M where int_fab: (λx. integral {x..b} f) ‘ {a..b} =
{m..M}
    using connected_compact_interval_1 by meson
  have ∃c. c ∈ {a..b} ∧
             integral {c..b} f =
             integral {a..b} (λx. (∑k = 1..n. if g x ≥ real k / real n then inverse n
*R f x else 0)) for n
  proof (cases n=0)
    case True
```

```
    then show ?thesis
      using ‹a ≤ b› by auto
  next
    case False
    have (⋃c::real. {λx. if g x ≥ c then f x else 0}) equiintegrable_on {a..b}
      using SMVT_lemma2 [OF f g] .
    then have int: (λx. if g x ≥ c then f x else 0) integrable_on {a..b} for c
      by (simp add: equiintegrable_on_def)
    have int': (λx. if g x ≥ c then u * f x else 0) integrable_on {a..b} for c u
    proof −
      have (λx. if g x ≥ c then u * f x else 0) = (λx. u * (if g x ≥ c then f x else
0))
        by (force simp: if_distrib)
      then show ?thesis
        using integrable_on_cmult_left [OF int] by simp
    qed
   have ∃ d. d ∈ {a..b} ∧ integral {a..b} (λx. if g x ≥ y then f x else 0) = integral
{d..b} f for y
    proof −
      let ?X = {x. g x ≥ y}
      have ∗: ∃ a. ?X = {a..} ∨ ?X = {a<..}
        if 1: ?X ≠ {} and 2: ?X ≠ UNIV
      proof −
        let ?μ = Inf{x. g x ≥ y}
        have lower: ?μ ≤ x if g x ≥ y for x
        proof (rule cInf_lower)
          show x ∈ {x. y ≤ g x}
            using 1 2 by (auto simp: that)
          show bdd_below {x. y ≤ g x}
            unfolding bdd_below_def
         by (metis 2 UNIV_eq_I dual_order.trans g less_eq_real_def mem_Collect_eq
not_le)
        qed
        have greatest: ?μ ≥ z if ⋀x. g x ≥ y ⟹ z ≤ x for z
          by (metis cInf_greatest mem_Collect_eq that 1)
        show ?thesis
        proof (cases g ?μ ≥ y)
          case True
          then obtain ζ where ζ: ⋀x. g x ≥ y ⟷ x ≥ ζ
            by (metis g lower order.trans)  — in fact y is Inf {x. y ≤ g x}
          then show ?thesis
            by (force simp: ζ)
        next
          case False
          have (y ≤ g x) = (?μ < x) for x
          proof
            show ?μ < x if y ≤ g x
              using that False less_eq_real_def lower by blast
            show y ≤ g x if ?μ < x
```

```
          by (metis g greatest le_less_trans that less_le_trans linear not_less)
        qed
        then obtain ζ where ζ: ⋀x. g x ≥ y ⟷ x > ζ ..
        then show ?thesis
          by (force simp: ζ)
      qed
    qed
    then consider ?X = {} | ?X = UNIV | (intv) d where ?X = {d..} ∨ ?X
= {d<..}
      by metis
    then have ∃ d. d ∈ {a..b} ∧ integral {a..b} (λx. if x ∈ ?X then f x else 0) =
integral {d..b} f
    proof cases
      case (intv d)
      show ?thesis
      proof (cases d < a)
        case True
        with intv have integral {a..b} (λx. if y ≤ g x then f x else 0) = integral
{a..b} f
          by (intro Henstock_Kurzweil_Integration.integral_cong) force
        then show ?thesis
          by (rule_tac x=a in exI) (simp add: ‹a ≤ b›)
      next
        case False
        show ?thesis
        proof (cases b < d)
          case True
          have integral {a..b} (λx. if x ∈ {x. y ≤ g x} then f x else 0) = integral
{a..b} (λx. 0)
            by (rule Henstock_Kurzweil_Integration.integral_cong) (use intv True
in fastforce)
          then show ?thesis
            using ‹a ≤ b› by auto
        next
          case False
            with ‹¬ d < a› have eq: {d..} ∩ {a..b} = {d..b} {d<..} ∩ {a..b} =
{d<..b}
              by force+
          moreover have integral {d<..b} f = integral {d..b} f
          by (rule integral_spike_set [OF empty_imp_negligible negligible_subset
[OF negligible_sing [of d]]]) auto
          ultimately
          have integral {a..b} (λx. if x ∈ {x. y ≤ g x} then f x else 0) =  integral
{d..b} f
            unfolding integral_restrict_Int using intv by presburger
          moreover have d ∈ {a..b}
            using ‹¬ d < a› ‹a ≤ b› False by auto
          ultimately show ?thesis
            by auto
```

```
        qed
      qed
    qed (use ‹a ≤ b› in auto)
    then show ?thesis
      by auto
  qed
  then have ∀k. ∃d. d ∈ {a..b} ∧ integral {a..b} (λx. if real k / real n ≤ g x
then f x else 0) = integral {d..b} f
    by meson
  then obtain d where dab: ⋀k. d k ∈ {a..b}
    and deq: ⋀k::nat. integral {a..b} (λx. if k/n ≤ g x then f x else 0) = integral
{d k..b} f
    by metis
  have (∑k = 1..n. integral {a..b} (λx. if real k / real n ≤ g x then f x else 0))
/R n ∈ {m..M}
    unfolding scaleR_right.sum
  proof (intro conjI allI impI convex [THEN iffD1, rule_format])
    show integral {a..b} (λxa. if real k / real n ≤ g xa then f xa else 0) ∈ {m..M}
for k
      by (metis (no_types, lifting) deq image_eqI int_fab dab)
  qed (use ‹n ≠ 0› in auto)
  then have ∃c. c ∈ {a..b} ∧
          integral {c..b} f = inverse n *R (∑k = 1..n. integral {a..b} (λx. if g
x ≥ real k / real n then f x else 0))
    by (metis (no_types, lifting) int_fab imageE)
  then show ?thesis
  by (simp add: sum_distrib_left if_distrib integral_sum int' flip: integral_mult_right
cong: if_cong)
qed
then obtain c where cab: ⋀n. c n ∈ {a..b}
  and c: ⋀n. integral {c n..b} f = integral {a..b} (λx. (∑k = 1..n. if g x ≥ real
k / real n then f x /R n else 0))
  by metis
obtain d and σ :: nat⇒nat
  where d ∈ {a..b} and σ: strict_mono σ and d: (c ∘ σ) ⟶ d and non0:
⋀n. σ n ≥ Suc 0
proof −
  have compact{a..b}
    by auto
  with cab obtain d and s0
    where d ∈ {a..b} and s0: strict_mono s0 and tends: (c ∘ s0) ⟶ d
    unfolding compact_def
    using that by blast
  show thesis
  proof
    show d ∈ {a..b}
      by fact
    show strict_mono (s0 ∘ Suc)
      using s0 by (auto simp: strict_mono_def)
```

```
    show (c ∘ (s0 ∘ Suc)) ⟶ d
      by (metis tends LIMSEQ_subseq_LIMSEQ Suc_less_eq comp_assoc
strict_mono_def)
    show ⋀n. (s0 ∘ Suc) n ≥ Suc 0
     by (metis comp_apply le0 not_less_eq_eq old.nat.exhaust s0 seq_suble)
   qed
  qed
  define φ where φ ≡ λn x. ∑k = Suc 0..σ n. if k/(σ n) ≤ g x then f x /R (σ
n) else 0
  define ψ where ψ ≡ λn x. ∑k = Suc 0..σ n. if k/(σ n) ≤ g x then inverse (σ
n) else 0
  have **: (λx. g x *R f x) integrable_on cbox a b ∧
        (λn. integral (cbox a b) (φ n)) ⟶ integral (cbox a b) (λx. g x *R f x)
  proof (rule equiintegrable_limit)
   have †: ((λn. λx. (∑k = Suc 0..n. if k / n ≤ g x then inverse n *R f x else
0)) ‘ {Suc 0..}) equiintegrable_on {a..b}
   proof −
    have *: (⋃c::real. {λx. if g x ≥ c then f x else 0}) equiintegrable_on {a..b}
     using SMVT_lemma2 [OF f g] .
    show ?thesis
     apply (rule equiintegrable_on_subset [OF equiintegrable_sum_real [OF *]],
clarify)
      apply (rule_tac a={Suc 0..n} in UN_I, force)
      apply (rule_tac a=λk. inverse n in UN_I, auto)
      apply (rule_tac x=λk x. if real k / real n ≤ g x then f x else 0 in bexI)
       apply (force intro: sum.cong)+
      done
   qed
   show range φ equiintegrable_on cbox a b
    unfolding φ_def
    by (auto simp: non0 intro: equiintegrable_on_subset [OF †])
   show (λn. φ n x) ⟶ g x *R f x
    if x: x ∈ cbox a b for x
   proof −
    have eq: φ n x = ψ n x *R f x for n
     by (auto simp: φ_def ψ_def sum_distrib_right if_distrib intro: sum.cong)
    show ?thesis
     unfolding eq
    proof (rule tendsto_scaleR [OF _ tendsto_const])
     show (λn. ψ n x) ⟶ g x
      unfolding lim_sequentially dist_real_def
     proof (intro allI impI)
      fix e :: real
      assume e > 0
      then obtain N where N ≠ 0 0 < inverse (real N) and N: inverse (real
N) < e
       using real_arch_inverse by metis
      moreover have |ψ n x − g x| < inverse (real N) if n≥N for n
      proof −
```

```
          have |g x − ψ n x| < inverse (real (σ n))
            unfolding ψ_def
          proof (rule level_approx [of {a..b} g])
            show σ n ≠ 0
              by (metis Suc_n_not_le_n non0)
          qed (use x 01 non0 in auto)
          also have ... ≤ inverse N
           using seq_suble [OF σ] ‹N ≠ 0› non0 that by (auto intro: order_trans
simp: field_split_simps)
          finally show ?thesis
            by linarith
        qed
        ultimately show ∃ N. ∀ n≥N. |ψ n x − g x| < e
          using less_trans by blast
      qed
    qed
  qed
  show thesis
  proof
    show a ≤ d d ≤ b
      using ‹d ∈ {a..b}› atLeastAtMost_iff by blast+
    show ((λx. g x *R f x) has_integral integral {d..b} f) {a..b}
      unfolding has_integral_iff
    proof
      show (λx. g x *R f x) integrable_on {a..b}
        using ** by simp
      show integral {a..b} (λx. g x *R f x) = integral {d..b} f
      proof (rule tendsto_unique)
        show (λn. integral {c(σ n)..b} f) ⟶ integral {a..b} (λx. g x *R f x)
          using ** by (simp add: c φ_def)
        have continuous (at d within {a..b}) (λx. integral {x..b} f)
          using indefinite_integral_continuous_1' [OF f] ‹d ∈ {a..b}›
          by (simp add: continuous_on_eq_continuous_within)
        then show (λn. integral {c(σ n)..b} f) ⟶ integral {d..b} f
          using d cab unfolding o_def
          by (simp add: continuous_within_sequentially o_def)
      qed auto
    qed
  qed
qed


theorem second_mean_value_theorem_full:
  fixes f :: real ⇒ real
  assumes f: f integrable_on {a..b} and a ≤ b
    and g: ⋀x y. ⟦a ≤ x; x ≤ y; y ≤ b⟧ ⟹ g x ≤ g y
  obtains c where c ∈ {a..b}
    and ((λx. g x * f x) has_integral (g a * integral {a..c} f + g b * integral {c..b}
```

```
f)) {a..b}
proof −
  have gab: g a ≤ g b
    using ‹a ≤ b› g by blast
  then consider g a < g b | g a = g b
    by linarith
  then show thesis
  proof cases
    case 1
    define h where h ≡ λx. if x < a then 0 else if b < x then 1
                           else (g x − g a) / (g b − g a)
     obtain c where a ≤ c c ≤ b and c: ((λx. h x *R f x) has_integral integral
{c..b} f) {a..b}
    proof (rule SMVT_lemma4 [OF f ‹a ≤ b›, of h])
      show h x ≤ h y 0 ≤ h x ∧ h x ≤ 1  if x ≤ y for x y
        using that gab by (auto simp: divide_simps g h_def)
    qed
    show ?thesis
    proof
      show c ∈ {a..b}
        using ‹a ≤ c› ‹c ≤ b› by auto
      have I: ((λx. g x * f x − g a * f x) has_integral (g b − g a) * integral {c..b}
f) {a..b}
      proof (subst has_integral_cong)
        show g x * f x − g a * f x = (g b − g a) * h x *R f x
          if x ∈ {a..b} for x
          using 1 that by (simp add: h_def field_split_simps)
        show ((λx. (g b − g a) * h x *R f x) has_integral (g b − g a) * integral
{c..b} f) {a..b}
          using has_integral_mult_right [OF c, of g b − g a] .
      qed
      have II: ((λx. g a * f x) has_integral g a * integral {a..b} f) {a..b}
       using has_integral_mult_right [where c = g a, OF integrable_integral [OF
f]] .
       have ((λx. g x * f x) has_integral (g b − g a) * integral {c..b} f + g a *
integral {a..b} f) {a..b}
        using has_integral_add [OF I II] by simp
     then show ((λx. g x * f x) has_integral g a * integral {a..c} f + g b * integral
{c..b} f) {a..b}
       by (simp add: algebra_simps flip: integral_combine [OF ‹a ≤ c› ‹c ≤ b› f])
    qed
  next
    case 2
    show ?thesis
    proof
      show a ∈ {a..b}
        by (simp add: ‹a ≤ b›)
      have ((λx. g x * f x) has_integral g a * integral {a..b} f) {a..b}
      proof (rule has_integral_eq)
```

```
      show ((λx. g a * f x) has_integral g a * integral {a..b} f) {a..b}
        using f has_integral_mult_right by blast
      show g a * f x = g x * f x
        if x ∈ {a..b} for x
        by (metis atLeastAtMost_iff g less_eq_real_def not_le that 2)
    qed
   then show ((λx. g x * f x) has_integral g a * integral {a..a} f + g b * integral
{a..b} f) {a..b}
      by (simp add: 2)
  qed
 qed
qed


corollary second_mean_value_theorem:
  fixes f :: real ⇒ real
  assumes f: f integrable_on {a..b} and a ≤ b
   and g: ⋀x y. ⟦a ≤ x; x ≤ y; y ≤ b⟧ ⟹ g x ≤ g y
 obtains c where c ∈ {a..b}
            integral {a..b} (λx. g x * f x) = g a * integral {a..c} f + g b * integral
{c..b} f
    using second_mean_value_theorem_full [where g=g, OF assms]
    by (metis (full_types) integral_unique)

end
```

9.15 Continuous Extensions of Functions

```
theory Continuous_Extension
imports Starlike
begin
```

9.15.1 Partitions of unity subordinate to locally finite open coverings

A difference from HOL Light: all summations over infinite sets equal zero, so the "support" must be made explicit in the summation below!

```
proposition subordinate_partition_of_unity:
  fixes S :: 'a::metric_space set
  assumes S ⊆ ⋃C and opC: ⋀T. T ∈ C ⟹ open T
     and fin: ⋀x. x ∈ S ⟹ ∃ V. open V ∧ x ∈ V ∧ finite {U ∈ C. U ∩ V ≠ {}}
  obtains F :: ['a set, 'a] ⇒ real
    where ⋀U. U ∈ C ⟹ continuous_on S (F U) ∧ (∀x ∈ S. 0 ≤ F U x)
      and ⋀x U. ⟦U ∈ C; x ∈ S; x ∉ U⟧ ⟹ F U x = 0
      and ⋀x. x ∈ S ⟹ supp_sum (λW. F W x) C = 1
      and ⋀x. x ∈ S ⟹ ∃ V. open V ∧ x ∈ V ∧ finite {U ∈ C. ∃x∈V. F U x ≠
0}
proof (cases ∃ W. W ∈ C ∧ S ⊆ W)
```

```
  case True
    then obtain W where W ∈ 𝒞 S ⊆ W by metis
    then show ?thesis
      by (rule_tac F = λV x. if V = W then 1 else 0 in that) (auto simp:
supp_sum_def support_on_def)
next
  case False
    have nonneg: 0 ≤ supp_sum (λV. setdist {x} (S − V)) 𝒞 for x
      by (simp add: supp_sum_def sum_nonneg)
    have sd_pos: 0 < setdist {x} (S − V) if V ∈ 𝒞 x ∈ S x ∈ V for V x
    proof −
      have closedin (top_of_set S) (S − V)
        by (simp add: Diff_Diff_Int closedin_def opC openin_open_Int ‹V ∈ 𝒞›)
      with that False  setdist_pos_le [of {x} S − V]
      show ?thesis
        using setdist_gt_0_closedin by fastforce
    qed
    have ss_pos: 0 < supp_sum (λV. setdist {x} (S − V)) 𝒞 if x ∈ S for x
    proof −
      obtain U where U ∈ 𝒞 x ∈ U using ‹x ∈ S› ‹S ⊆ ⋃𝒞›
        by blast
      obtain V where open V x ∈ V finite {U ∈ 𝒞. U ∩ V ≠ {}}
        using ‹x ∈ S› fin by blast
      then have ∗: finite {A ∈ 𝒞. ¬ S ⊆ A ∧ x ∉ closure (S − A)}
        using closure_def that by (blast intro: rev_finite_subset)
      have x ∉ closure (S − U)
        using ‹U ∈ 𝒞› ‹x ∈ U› opC open_Int_closure_eq_empty by fastforce
      then show ?thesis
        apply (simp add: setdist_eq_0_sing_1 supp_sum_def support_on_def)
        apply (rule ordered_comm_monoid_add_class.sum_pos2 [OF ∗, of U])
        using ‹U ∈ 𝒞› ‹x ∈ U› False
        apply (auto simp: sd_pos that)
        done
    qed
    define F where
      F ≡ λW x. if x ∈ S then setdist {x} (S − W) / supp_sum (λV. setdist {x}
(S − V)) 𝒞 else 0
    show ?thesis
    proof (rule_tac F = F in that)
      have continuous_on S (F U) if U ∈ 𝒞 for U
      proof −
        have ∗: continuous_on S (λx. supp_sum (λV. setdist {x} (S − V)) 𝒞)
        proof (clarsimp simp add: continuous_on_eq_continuous_within)
          fix x assume x ∈ S
          then obtain X where open X and x: x ∈ S ∩ X and finX: finite {U ∈
𝒞. U ∩ X ≠ {}}
            using assms by blast
          then have OSX: openin (top_of_set S) (S ∩ X) by blast
          have sumeq: ⋀x. x ∈ S ∩ X ⟹
```

```
                (∑ V | V ∈ 𝒞 ∧ V ∩ X ≠ {}. setdist {x} (S − V))
                = supp_sum (λV. setdist {x} (S − V)) 𝒞
          apply (simp add: supp_sum_def)
          apply (rule sum.mono_neutral_right [OF finX])
          apply (auto simp: setdist_eq_0_sing_1 support_on_def subset_iff)
          apply (meson DiffI closure_subset disjoint_iff_not_equal subsetCE)
          done
        show continuous (at x within S) (λx. supp_sum (λV. setdist {x} (S −
V)) 𝒞)
          apply (rule continuous_transform_within_openin
              [where f = λx. (sum (λV. setdist {x} (S − V)) {V ∈ 𝒞. V ∩ X
≠ {}})
                  and S =S ∩ X])
       apply (rule continuous_intros continuous_at_setdist continuous_at_imp_continuous_at_within
OSX x)+
          apply (simp add: sumeq)
          done
       qed
       show ?thesis
         apply (simp add: F_def)
         apply (rule continuous_intros *)+
         using ss_pos apply force
         done
     qed
     moreover have ⟦U ∈ 𝒞; x ∈ S⟧ ⟹ 0 ≤ F U x for U x
       using nonneg [of x] by (simp add: F_def field_split_simps)
     ultimately show ⋀U. U ∈ 𝒞 ⟹ continuous_on S (F U) ∧ (∀x∈S. 0 ≤ F
U x)
       by metis
   next
     show ⋀x U. ⟦U ∈ 𝒞; x ∈ S; x ∉ U⟧ ⟹ F U x = 0
       by (simp add: setdist_eq_0_sing_1 closure_def F_def)
   next
     show supp_sum (λW. F W x) 𝒞 = 1 if x ∈ S for x
       using that ss_pos [OF that]
     by (simp add: F_def field_split_simps supp_sum_divide_distrib [symmetric])
   next
     show ∃V. open V ∧ x ∈ V ∧ finite {U ∈ 𝒞. ∃x∈V. F U x ≠ 0} if x ∈ S
for x
       using fin [OF that] that
     by (fastforce simp: setdist_eq_0_sing_1 closure_def F_def elim!: rev_finite_subset)
   qed
qed
```

9.15.2 Urysohn's Lemma for Euclidean Spaces

For Euclidean spaces the proof is easy using distances.

```
lemma Urysohn_both_ne:
  assumes US: closedin (top_of_set U) S
```

```
    and UT: closedin (top_of_set U) T
    and S ∩ T = {} S ≠ {} T ≠ {} a ≠ b
  obtains f :: 'a::euclidean_space ⇒ 'b::real_normed_vector
    where continuous_on U f
        ⋀x. x ∈ U ⟹ f x ∈ closed_segment a b
        ⋀x. x ∈ U ⟹ (f x = a ⟷ x ∈ S)
        ⋀x. x ∈ U ⟹ (f x = b ⟷ x ∈ T)
proof −
  have S0: ⋀x. x ∈ U ⟹ setdist {x} S = 0 ⟷ x ∈ S
    using ‹S ≠ {}› US setdist_eq_0_closedin by auto
  have T0: ⋀x. x ∈ U ⟹ setdist {x} T = 0 ⟷ x ∈ T
    using ‹T ≠ {}› UT setdist_eq_0_closedin by auto
  have sdpos: 0 < setdist {x} S + setdist {x} T if x ∈ U for x
  proof −
    have ¬ (setdist {x} S = 0 ∧ setdist {x} T = 0)
      using assms by (metis IntI empty_iff setdist_eq_0_closedin that)
    then show ?thesis
    by (metis add.left_neutral add.right_neutral add_pos_pos linorder_neqE_linordered_idom
not_le setdist_pos_le)
  qed
  define f where f ≡ λx. a + (setdist {x} S / (setdist {x} S + setdist {x} T))
*R (b − a)
  show ?thesis
  proof (rule_tac f = f in that)
    show continuous_on U f
      using sdpos unfolding f_def
      by (intro continuous_intros | force)+
    show f x ∈ closed_segment a b if x ∈ U for x
      unfolding f_def
      apply (simp add: closed_segment_def)
      apply (rule_tac x=(setdist {x} S / (setdist {x} S + setdist {x} T)) in exI)
      using sdpos that apply (simp add: algebra_simps)
      done
    show ⋀x. x ∈ U ⟹ (f x = a ⟷ x ∈ S)
      using S0 ‹a ≠ b› f_def sdpos by force
    show (f x = b ⟷ x ∈ T) if x ∈ U for x
    proof −
      have f x = b ⟷ (setdist {x} S / (setdist {x} S + setdist {x} T)) = 1
        unfolding f_def
        by (metis add_diff_cancel_left' ‹a ≠ b› diff_add_cancel eq_iff_diff_eq_0
scaleR_cancel_right scaleR_one)
      also have ... ⟷ setdist {x} T = 0 ∧ setdist {x} S ≠ 0
        using sdpos that
        by (simp add: field_split_simps) linarith
      also have ... ⟷ x ∈ T
        using ‹S ≠ {}› ‹T ≠ {}› ‹S ∩ T = {}› that
        by (force simp: S0 T0)
      finally show ?thesis .
    qed
```

```
  qed
qed

lemma Urysohn_local_strong_aux:
  assumes US: closedin (top_of_set U) S
      and UT: closedin (top_of_set U) T
      and S ∩ T = {} a ≠ b S ≠ {}
  obtains f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    where continuous_on U f
          ⋀x. x ∈ U ⟹ f x ∈ closed_segment a b
          ⋀x. x ∈ U ⟹ (f x = a ⟷ x ∈ S)
          ⋀x. x ∈ U ⟹ (f x = b ⟷ x ∈ T)
proof (cases T = {})
  case True show ?thesis
  proof (cases S = U)
    case True with ‹T = {}› ‹a ≠ b› show ?thesis
      by (rule_tac f = λx. a in that) (auto)
  next
    case False
    with US closedin_subset obtain c where c: c ∈ U c ∉ S
      by fastforce
    obtain f where f: continuous_on U f
      ⋀x. x ∈ U ⟹ f x ∈ closed_segment a (midpoint a b)
      ⋀x. x ∈ U ⟹ (f x = midpoint a b ⟷ x = c)
      ⋀x. x ∈ U ⟹ (f x = a ⟷ x ∈ S)
      apply (rule Urysohn_both_ne [of U S {c} a midpoint a b])
      using c ‹S ≠ {}› assms apply simp_all
      apply (metis midpoint_eq_endpoint)
      done
    show ?thesis
      apply (rule_tac f=f in that)
      using ‹S ≠ {}› ‹T = {}› f  ‹a ≠ b›
        apply simp_all
     apply (metis (no_types) closed_segment_commute csegment_midpoint_subset
midpoint_sym subset_iff)
    apply (metis closed_segment_commute midpoint_sym notin_segment_midpoint)
      done
  qed
next
  case False
  show ?thesis
    using Urysohn_both_ne [OF US UT ‹S ∩ T = {}› ‹S ≠ {}› ‹T ≠ {}› ‹a ≠
b›] that
    by blast
qed

proposition Urysohn_local_strong:
  assumes US: closedin (top_of_set U) S
      and UT: closedin (top_of_set U) T
```

```
    and S ∩ T = {} a ≠ b
  obtains f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    where continuous_on U f
          ⋀x. x ∈ U ⟹ f x ∈ closed_segment a b
          ⋀x. x ∈ U ⟹ (f x = a ⟷ x ∈ S)
          ⋀x. x ∈ U ⟹ (f x = b ⟷ x ∈ T)
proof (cases S = {})
  case True show ?thesis
  proof (cases T = {})
    case True show ?thesis
    proof (rule_tac f = λx. midpoint a b in that)
      show continuous_on U (λx. midpoint a b)
        by (intro continuous_intros)
      show midpoint a b ∈ closed_segment a b
        using csegment_midpoint_subset by blast
      show (midpoint a b = a) = (x ∈ S) for x
        using ‹S = {}› ‹a ≠ b› by simp
      show (midpoint a b = b) = (x ∈ T) for x
        using ‹T = {}› ‹a ≠ b› by simp
    qed
  next
    case False
    with Urysohn_local_strong_aux [OF UT US] assms show ?thesis
      by (smt (verit) True closed_segment_commute inf_bot_right that)
  qed
next
  case False
  with Urysohn_local_strong_aux [OF assms] show ?thesis
    using that by blast
qed

lemma Urysohn_local:
  assumes US: closedin (top_of_set U) S
    and UT: closedin (top_of_set U) T
    and S ∩ T = {}
  obtains f :: 'a::euclidean_space ⇒ 'b::euclidean_space
    where continuous_on U f
          ⋀x. x ∈ U ⟹ f x ∈ closed_segment a b
          ⋀x. x ∈ S ⟹ f x = a
          ⋀x. x ∈ T ⟹ f x = b
proof (cases a = b)
  case True then show ?thesis
    by (rule_tac f = λx. b in that) (auto)
next
  case False
  with Urysohn_local_strong [OF assms] show ?thesis
    by (smt (verit) US UT closedin_imp_subset subset_eq that)
qed
```

lemma *Urysohn_strong*:
assumes *US*: *closed S*
and *UT*: *closed T*
and $S \cap T = \{\}$ $a \neq b$
obtains *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
where *continuous_on UNIV f*
$\bigwedge x.\ f\ x \in$ *closed_segment a b*
$\bigwedge x.\ f\ x = a \longleftrightarrow x \in S$
$\bigwedge x.\ f\ x = b \longleftrightarrow x \in T$
using *assms* **by** (*auto intro*: *Urysohn_local_strong* [*of UNIV S T*])

proposition *Urysohn*:
assumes *US*: *closed S*
and *UT*: *closed T*
and $S \cap T = \{\}$
obtains *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
where *continuous_on UNIV f*
$\bigwedge x.\ f\ x \in$ *closed_segment a b*
$\bigwedge x.\ x \in S \Longrightarrow f\ x = a$
$\bigwedge x.\ x \in T \Longrightarrow f\ x = b$
using *assms* **by** (*auto intro*: *Urysohn_local* [*of UNIV S T a b*])

9.15.3 Dugundji's Extension Theorem and Tietze Variants

See [3].

theorem *Dugundji*:
fixes *f* :: *'a*::{*metric_space*,*second_countable_topology*} ⇒ *'b*::*real_inner*
assumes *convex C C* ≠ {}
and *cloin*: *closedin* (*top_of_set U*) *S*
and *contf*: *continuous_on S f* **and** *f* ‘ *S* ⊆ *C*
obtains *g* **where** *continuous_on U g g* ‘ *U* ⊆ *C*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof (*cases S* = {})
case *True* **show** *thesis*
proof
show *continuous_on U* (λ*x*. *SOME y*. *y* ∈ *C*)
by (*rule continuous_intros*)
show (λ*x*. *SOME y*. *y* ∈ *C*) ‘ *U* ⊆ *C*
by (*simp add*: ‹*C* ≠ {}› *image_subsetI some_in_eq*)
qed (*use True* **in** *auto*)
next
case *False*
then have *sd_pos*: $\bigwedge x.\ [\![x \in U;\ x \notin S]\!] \Longrightarrow 0 <$ *setdist* {*x*} *S*
using *setdist_eq_0_closedin* [*OF cloin*] *le_less setdist_pos_le* **by** *fastforce*
define $\mathcal{B}$ **where** $\mathcal{B} =$ {*ball x* (*setdist* {*x*} *S* / *2*) |*x*. *x* ∈ *U* − *S*}
have [*simp*]: $\bigwedge T.\ T \in \mathcal{B} \Longrightarrow$ *open T*
by (*auto simp*: $\mathcal{B}$*_def*)
have *USS*: $U - S \subseteq \bigcup \mathcal{B}$
by (*auto simp*: *sd_pos* $\mathcal{B}$*_def*)

obtain $\mathcal{C}$ **where** *USsub*: $U - S \subseteq \bigcup \mathcal{C}$
and *nbrhd*: $\bigwedge U.\ U \in \mathcal{C} \Longrightarrow$ *open* $U \wedge (\exists\, T.\ T \in \mathcal{B} \wedge U \subseteq T)$
and *fin*: $\bigwedge x.\ x \in U - S \Longrightarrow \exists\, V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U.\ U \in \mathcal{C} \wedge U \cap V \neq \{\}\}$
by (*rule paracompact* [*OF USS*]) *auto*
have $\exists\, v\ a.\ v \in U \wedge v \notin S \wedge a \in S \wedge$
$T \subseteq$ *ball* v (*setdist* $\{v\}$ S / 2) $\wedge$
dist v $a \leq 2 *$ *setdist* $\{v\}$ S **if** $T \in \mathcal{C}$ **for** T
proof −
obtain v **where** v: $T \subseteq$ *ball* v (*setdist* $\{v\}$ S / 2) $v \in U$ $v \notin S$
using ‹$T \in \mathcal{C}$› *nbrhd* **by** (*force simp*: $\mathcal{B}$_*def*)
then obtain a **where** $a \in S$ *dist* v $a <$ 2 $*$ *setdist* $\{v\}$ S
using *setdist_ltE* [*of* $\{v\}$ S 2 $*$ *setdist* $\{v\}$ S]
using *False sd_pos* **by** *force*
with v **show** *?thesis*
by *force*
qed
then obtain $\mathcal{V}$ $\mathcal{A}$ **where**
VA: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow \mathcal{V}\ T \in U \wedge \mathcal{V}\ T \notin S \wedge \mathcal{A}\ T \in S \wedge$
$T \subseteq$ *ball* ($\mathcal{V}$ T) (*setdist* $\{\mathcal{V}\ T\}$ S / 2) $\wedge$
dist ($\mathcal{V}$ T) ($\mathcal{A}$ T) $\leq 2 *$ *setdist* $\{\mathcal{V}\ T\}$ S
by *metis*
have *sdle*: *setdist* $\{\mathcal{V}\ T\}$ $S \leq 2 *$ *setdist* $\{v\}$ S **if** $T \in \mathcal{C}$ $v \in T$ **for** T v
using *setdist_Lipschitz* [*of* $\mathcal{V}$ T S v] *VA* [*OF* ‹$T \in \mathcal{C}$›] ‹$v \in T$› **by** *auto*
have *d6*: *dist* a ($\mathcal{A}$ T) $\leq 6 *$ *dist* a v **if** $T \in \mathcal{C}$ $v \in T$ $a \in S$ **for** T v a
proof −
have *dist* ($\mathcal{V}$ T) $v <$ *setdist* $\{\mathcal{V}\ T\}$ S / 2
using *that VA mem_ball* **by** *blast*
also have ... $\leq$ *setdist* $\{v\}$ S
using *sdle* [*OF* ‹$T \in \mathcal{C}$› ‹$v \in T$›] **by** *simp*
also have *vS*: *setdist* $\{v\}$ $S \leq$ *dist* a v
by (*simp add*: *setdist_le_dist setdist_sym* ‹$a \in S$›)
finally have *VTV*: *dist* ($\mathcal{V}$ T) $v <$ *dist* a v **.**
have *VTS*: *setdist* $\{\mathcal{V}\ T\}$ $S \leq 2 *$ *dist* a v
using *sdle that vS* **by** *force*
have *dist* a ($\mathcal{A}$ T) $\leq$ *dist* a v + *dist* v ($\mathcal{V}$ T) + *dist* ($\mathcal{V}$ T) ($\mathcal{A}$ T)
by (*metis add.commute add_le_cancel_left dist_commute dist_triangle2 dist_triangle_le*)
also have ... $\leq$ *dist* a v + *dist* a v + *dist* ($\mathcal{V}$ T) ($\mathcal{A}$ T)
using *VTV* **by** (*simp add*: *dist_commute*)
also have ... $\leq 2 *$ *dist* a v + 2 $*$ *setdist* $\{\mathcal{V}\ T\}$ S
using *VA* [*OF* ‹$T \in \mathcal{C}$›] **by** *auto*
finally show *?thesis*
using *VTS* **by** *linarith*
qed
obtain $H :: ['a\ set, 'a] \Rightarrow real$
where *Hcont*: $\bigwedge Z.\ Z \in \mathcal{C} \Longrightarrow$ *continuous_on* ($U-S$) (H Z)
and *Hge0*: $\bigwedge Z\ x.\ [\![Z \in \mathcal{C};\ x \in U-S]\!] \Longrightarrow 0 \leq H\ Z\ x$
and *Heq0*: $\bigwedge x\ Z.\ [\![Z \in \mathcal{C};\ x \in U-S;\ x \notin Z]\!] \Longrightarrow H\ Z\ x = 0$

```
    and H1: ⋀x. x ∈ U−S ⟹ supp_sum (λW. H W x) C = 1
    and Hfin: ⋀x. x ∈ U−S ⟹ ∃ V. open V ∧ x ∈ V ∧ finite {U ∈ C. ∃ x∈V.
H U x ≠ 0}
   apply (rule subordinate_partition_of_unity [OF USsub _ fin])
   using nbrhd by auto
 define g where g ≡ λx. if x ∈ S then f x else supp_sum (λT. H T x *R f(A
T)) C
 show ?thesis
 proof (rule that)
  show continuous_on U g
  proof (clarsimp simp: continuous_on_eq_continuous_within)
   fix a assume a ∈ U
   show continuous (at a within U) g
   proof (cases a ∈ S)
    case True show ?thesis
    proof (clarsimp simp add: continuous_within_topological)
     fix W
     assume open W g a ∈ W
     then obtain e where 0 < e and e: ball (f a) e ⊆ W
      using openE True g_def by auto
     have continuous (at a within S) f
      using True contf continuous_on_eq_continuous_within by blast
     then obtain d where 0 < d
                  and d: ⋀x. ⟦x ∈ S; dist x a < d⟧ ⟹ dist (f x) (f a) < e
      using continuous_within_eps_delta ‹0 < e› by force
     have g y ∈ ball (f a) e if y ∈ U and y: y ∈ ball a (d / 6) for y
     proof (cases y ∈ S)
      case True
      then have dist (f a) (f y) < e
     by (metis ball_divide_subset_numeral dist_commute in_mono mem_ball
y d)
      then show ?thesis
       by (simp add: True g_def)
     next
      case False
      have *: dist (f (A T)) (f a) < e if T ∈ C H T y ≠ 0 for T
      proof −
       have y ∈ T
        using Heq0 that False ‹y ∈ U› by blast
       have dist (A T) a < d
        using d6 [OF ‹T ∈ C› ‹y ∈ T› ‹a ∈ S›] y
        by (simp add: dist_commute mult.commute)
       then show ?thesis
        using VA [OF ‹T ∈ C›] by (auto simp: d)
      qed
      have supp_sum (λT. H T y *R f (A T)) C ∈ ball (f a) e
       apply (rule convex_supp_sum [OF convex_ball])
       apply (simp_all add: False H1 Hge0 ‹y ∈ U›)
       by (metis dist_commute *)
```

```
        then show ?thesis
          by (simp add: False g_def)
      qed
      then show ∃A. open A ∧ a ∈ A ∧ (∀y∈U. y ∈ A ⟶ g y ∈ W)
        apply (rule_tac x = ball a (d / 6) in exI)
        using e ‹0 < d› by fastforce
    qed
  next
    case False
    obtain N where N: open N a ∈ N
            and finN: finite {U ∈ 𝒞. ∃a∈N. H U a ≠ 0}
      using Hfin False ‹a ∈ U› by auto
    have oUS: openin (top_of_set U) (U - S)
      using cloin by (simp add: openin_diff)
    have HcontU: continuous (at a within U) (H T) if T ∈ 𝒞 for T
      using Hcont [OF ‹T ∈ 𝒞›] False ‹a ∈ U› ‹T ∈ 𝒞›
   apply (simp add: continuous_on_eq_continuous_within continuous_within)
      apply (rule Lim_transform_within_set)
      using oUS
        apply (force simp: eventually_at openin_contains_ball dist_commute
dest!: bspec)+
      done
    show ?thesis
    proof (rule continuous_transform_within_openin [OF _ oUS])
     show continuous (at a within U) (λx. supp_sum (λT. H T x *R f (𝒜 T))
𝒞)
      proof (rule continuous_transform_within_openin)
        show continuous (at a within U)
             (λx. ∑T∈{U ∈ 𝒞. ∃x∈N. H U x ≠ 0}. H T x *R f (𝒜 T))
          by (force intro: continuous_intros HcontU)+
      next
        show openin (top_of_set U) ((U - S) ∩ N)
          using N oUS openin_trans by blast
      next
        show a ∈ (U - S) ∩ N using False ‹a ∈ U› N by blast
      next
        show ⋀x. x ∈ (U - S) ∩ N ⟹
                  (∑ T ∈ {U ∈ 𝒞. ∃x∈N. H U x ≠ 0}. H T x *R f (𝒜 T))
                  = supp_sum (λT. H T x *R f (𝒜 T)) 𝒞
          by (auto simp: supp_sum_def support_on_def
                intro: sum.mono_neutral_right [OF finN])
      qed
    next
      show a ∈ U - S using False ‹a ∈ U› by blast
    next
      show ⋀x. x ∈ U - S ⟹ supp_sum (λT. H T x *R f (𝒜 T)) 𝒞 = g x
        by (simp add: g_def)
    qed
  qed
```

```
    qed
    show g ‘ U ⊆ C
      using ‹f ‘ S ⊆ C› VA
       by (fastforce simp: g_def Hge0 intro!: convex_supp_sum [OF ‹convex C›]
H1)
    show ⋀x. x ∈ S ⟹ g x = f x
      by (simp add: g_def)
  qed
qed


corollary Tietze:
  fixes f :: 'a::{metric_space,second_countable_topology} ⇒ 'b::real_inner
  assumes continuous_on S f
    and closedin (top_of_set U) S
    and 0 ≤ B
    and ⋀x. x ∈ S ⟹ norm(f x) ≤ B
  obtains g where continuous_on U g ⋀x. x ∈ S ⟹ g x = f x
    ⋀x. x ∈ U ⟹ norm(g x) ≤ B
  using assms by (auto simp: image_subset_iff intro: Dugundji [of cball 0 B U S
f])

corollary Tietze_closed_interval:
  fixes f :: 'a::{metric_space,second_countable_topology} ⇒ 'b::euclidean_space
  assumes continuous_on S f
    and closedin (top_of_set U) S
    and cbox a b ≠ {}
    and ⋀x. x ∈ S ⟹ f x ∈ cbox a b
  obtains g where continuous_on U g ⋀x. x ∈ S ⟹ g x = f x
    ⋀x. x ∈ U ⟹ g x ∈ cbox a b
  apply (rule Dugundji [of cbox a b U S f])
  using assms by auto

corollary Tietze_closed_interval_1:
  fixes f :: 'a::{metric_space,second_countable_topology} ⇒ real
  assumes continuous_on S f
    and closedin (top_of_set U) S
    and a ≤ b
    and ⋀x. x ∈ S ⟹ f x ∈ cbox a b
  obtains g where continuous_on U g ⋀x. x ∈ S ⟹ g x = f x
    ⋀x. x ∈ U ⟹ g x ∈ cbox a b
  apply (rule Dugundji [of cbox a b U S f])
  using assms by (auto simp: image_subset_iff)

corollary Tietze_open_interval:
  fixes f :: 'a::{metric_space,second_countable_topology} ⇒ 'b::euclidean_space
  assumes continuous_on S f
    and closedin (top_of_set U) S
    and box a b ≠ {}
```

and $\bigwedge x.\ x \in S \Longrightarrow f\ x \in box\ a\ b$
obtains g **where** *continuous_on* $U\ g$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ x \in U \Longrightarrow g\ x \in box\ a\ b$
apply (*rule Dugundji* [*of box a b U S f*])
using *assms* **by** *auto*

corollary *Tietze_open_interval_1*:
fixes f :: $'a$::{*metric_space*,*second_countable_topology*} ⇒ *real*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
and $a < b$
and *no*: $\bigwedge x.\ x \in S \Longrightarrow f\ x \in box\ a\ b$
obtains g **where** *continuous_on* $U\ g$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ x \in U \Longrightarrow g\ x \in box\ a\ b$
apply (*rule Dugundji* [*of box a b U S f*])
using *assms* **by** (*auto simp*: *image_subset_iff*)

corollary *Tietze_unbounded*:
fixes f :: $'a$::{*metric_space*,*second_countable_topology*} ⇒ $'b$::*real_inner*
assumes *continuous_on S f*
and *closedin* (*top_of_set U*) *S*
obtains g **where** *continuous_on* $U\ g$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
apply (*rule Dugundji* [*of UNIV U S f*])
using *assms* **by** *auto*

end

9.16 Equivalence Between Classical Borel Measurability and HOL Light's

theory *Equivalence_Measurable_On_Borel*
imports *Equivalence_Lebesgue_Henstock_Integration Improper_Integral Continuous_Extension*
begin

9.16.1 Austin's Lemma

lemma *Austin_Lemma*:
fixes $\mathcal{D}$ :: $'a$::*euclidean_space set set*
assumes *finite* $\mathcal{D}$ **and** $\mathcal{D}$: $\bigwedge D.\ D \in \mathcal{D} \Longrightarrow \exists k\ a\ b.\ D = cbox\ a\ b \wedge (\forall i \in Basis.\ b{\cdot}i - a{\cdot}i = k)$
obtains $\mathcal{C}$ **where** $\mathcal{C} \subseteq \mathcal{D}$ *pairwise disjnt* $\mathcal{C}$
measure lebesgue $(\bigcup \mathcal{C}) \geq$ *measure lebesgue* $(\bigcup \mathcal{D})$ / $3\ \widehat{}\ (DIM('a))$
using *assms*
proof (*induction card* $\mathcal{D}$ *arbitrary*: $\mathcal{D}$ *thesis rule*: *less_induct*)
case *less*
show *?case*
proof (*cases* $\mathcal{D} = \{\}$)

```
    case True
    then show thesis
      using less by auto
  next
    case False
   then have Max (Sigma_Algebra.measure lebesgue ` 𝒟) ∈ Sigma_Algebra.measure
lebesgue ` 𝒟
      using Max_in finite_imageI ‹finite 𝒟› by blast
     then obtain D where D ∈ 𝒟 and measure lebesgue D = Max (measure
lebesgue ` 𝒟)
      by auto
    then have D: ⋀C. C ∈ 𝒟 ⟹ measure lebesgue C ≤ measure lebesgue D
      by (simp add: ‹finite 𝒟›)
    let ?ℰ = {C. C ∈ 𝒟 − {D} ∧ disjnt C D}
    obtain 𝒟' where 𝒟'sub: 𝒟' ⊆ ?ℰ and 𝒟'dis: pairwise disjnt 𝒟'
     and 𝒟'm: measure lebesgue (⋃𝒟') ≥ measure lebesgue (⋃ ?ℰ) / 3 ^ (DIM('a))
    proof (rule less.hyps)
      have ∗: ?ℰ ⊂ 𝒟
        using ‹D ∈ 𝒟› by auto
      then show card ?ℰ < card 𝒟 finite ?ℰ
        by (auto simp: ‹finite 𝒟› psubset_card_mono)
      show ∃k a b. D = cbox a b ∧ (∀i∈Basis. b ∙ i − a ∙ i = k) if D ∈ ?ℰ for D
        using less.prems(3) that by auto
    qed
    then have [simp]: ⋃𝒟' − D = ⋃𝒟'
      by (auto simp: disjnt_iff)
    show ?thesis
    proof (rule less.prems)
      show insert D 𝒟' ⊆ 𝒟
        using 𝒟'sub ‹D ∈ 𝒟› by blast
      show disjoint (insert D 𝒟')
        using 𝒟'dis 𝒟'sub by (fastforce simp add: pairwise_def disjnt_sym)
       obtain a3 b3 where  m3: content (cbox a3 b3) = 3 ^ DIM('a) ∗ measure
lebesgue D
        and sub3: ⋀C. ⟦C ∈ 𝒟; ¬ disjnt C D⟧ ⟹ C ⊆ cbox a3 b3
      proof −
        obtain k a b where ab: D = cbox a b and k: ⋀i. i ∈ Basis ⟹ b∙i − a∙i
= k
          using less.prems ‹D ∈ 𝒟› by meson
        then have eqk: ⋀i. i ∈ Basis ⟹ a ∙ i ≤ b ∙ i ⟷ k ≥ 0
          by force
        show thesis
        proof
          let ?a = (a + b) /R 2 − (3/2) ∗R (b − a)
          let ?b = (a + b) /R 2 + (3/2) ∗R (b − a)
          have eq: (∏i∈Basis. b ∙ i ∗ 3 − a ∙ i ∗ 3) = (∏i∈Basis. b ∙ i − a ∙ i) ∗
3 ^ DIM('a)
         by (simp add: comm_monoid_mult_class.prod.distrib flip: left_diff_distrib
inner_diff_left)
```

show *content* (*cbox ?a ?b*) = *3* ^ *DIM*('*a*) * *measure lebesgue D*
by (*simp add*: *content_cbox_if box_eq_empty algebra_simps eq ab k*)
show *C* ⊆ *cbox ?a ?b* **if** *C* ∈ *D* **and** *CD*: ¬ *disjnt C D* **for** *C*
proof −
obtain *k*′ *a*′ *b*′ **where** *ab*′: *C* = *cbox a*′ *b*′ **and** *k*′: ⋀*i*. *i* ∈ *Basis* ⟹ *b*′•*i* − *a*′•*i* = *k*′
using *less.prems* ‹*C* ∈ *D*› **by** *meson*
then have *eqk*′: ⋀*i*. *i* ∈ *Basis* ⟹ *a*′ • *i* ≤ *b*′ • *i* ⟷ *k*′ ≥ *0*
by *force*
show *?thesis*
proof (*clarsimp simp add*: *disjoint_interval disjnt_def ab ab*′ *not_less subset_box algebra_simps*)
show *a* • *i* * *2* ≤ *a*′ • *i* + *b* • *i* ∧ *a* • *i* + *b*′ • *i* ≤ *b* • *i* * *2*
if * [*rule_format*]: ∀ *j*∈*Basis*. *a*′ • *j* ≤ *b*′ • *j* **and** *i* ∈ *Basis* **for** *i*
proof −
have *a*′ • *i* ≤ *b*′ • *i* ∧ *a* • *i* ≤ *b* • *i* ∧ *a* • *i* ≤ *b*′ • *i* ∧ *a*′ • *i* ≤ *b* • *i*
using ‹*i* ∈ *Basis*› *CD* **by** (*simp_all add*: *disjoint_interval disjnt_def ab ab*′ *not_less*)
then show *?thesis*
using *D* [*OF* ‹*C* ∈ *D*›] ‹*i* ∈ *Basis*›
apply (*simp add*: *ab ab*′ *k k*′ *eqk eqk*′ *content_cbox_cases*)
using *k k*′ **by** *fastforce*
qed
qed
qed
qed
qed
have *Dlm*: ⋀*D*. *D* ∈ *D* ⟹ *D* ∈ *lmeasurable*
using *less.prems*(*3*) **by** *blast*
have *measure lebesgue* (⋃*D*) ≤ *measure lebesgue* (*cbox a3 b3* ∪ (⋃*D* − *cbox a3 b3*))
proof (*rule measure_mono_fmeasurable*)
show ⋃*D* ∈ *sets lebesgue*
using *Dlm* ‹*finite D*› **by** *blast*
show *cbox a3 b3* ∪ (⋃*D* − *cbox a3 b3*) ∈ *lmeasurable*
by (*simp add*: *Dlm fmeasurable.Un fmeasurable.finite_Union less.prems*(*2*) *subset_eq*)
qed *auto*
also have ... = *content* (*cbox a3 b3*) + *measure lebesgue* (⋃*D* − *cbox a3 b3*)
by (*simp add*: *Dlm fmeasurable.finite_Union less.prems*(*2*) *measure_Un2 subsetI*)
also have ... ≤ (*measure lebesgue D* + *measure lebesgue* (⋃*D*′)) * *3* ^ *DIM*('*a*)
proof −
have (⋃*D* − *cbox a3 b3*) ⊆ ⋃ *?E*
using *sub3* **by** *fastforce*
then have *measure lebesgue* (⋃*D* − *cbox a3 b3*) ≤ *measure lebesgue* (⋃ *?E*)
proof (*rule measure_mono_fmeasurable*)

show $\bigcup\ \mathcal{D} - \mathit{cbox}\ a3\ b3 \in \mathit{sets}\ \mathit{lebesgue}$
by (*simp add*: $\mathcal{D}\mathit{lm}$ *fmeasurableD less.prems*(2) *sets.Diff sets.finite_Union subsetI*)
show $\bigcup\ \{C \in \mathcal{D} - \{D\}.\ \mathit{disjnt}\ C\ D\} \in \mathit{lmeasurable}$
using $\mathcal{D}\mathit{lm}$ *less.prems*(2) **by** *auto*
qed
then have *measure lebesgue* $(\bigcup\mathcal{D} - \mathit{cbox}\ a3\ b3)\ /\ 3\ \hat{}\ \mathit{DIM}('a) \leq \mathit{measure}\ \mathit{lebesgue}\ (\bigcup\ \mathcal{D}')$
using $\mathcal{D}'m$ **by** (*simp add*: *field_split_simps*)
then show *?thesis*
by (*simp add*: *m3 field_simps*)
qed
also have $\ldots \leq \mathit{measure}\ \mathit{lebesgue}\ (\bigcup(\mathit{insert}\ D\ \mathcal{D}'))\ *\ 3\ \hat{}\ \mathit{DIM}('a)$
proof (*simp add*: $\mathcal{D}\mathit{lm}$ ‹$D \in \mathcal{D}$›)
show *measure lebesgue D* + *measure lebesgue* $(\bigcup\mathcal{D}') \leq \mathit{measure}\ \mathit{lebesgue}\ (D \cup \bigcup\ \mathcal{D}')$
proof (*subst measure_Un2*)
show $\bigcup\ \mathcal{D}' \in \mathit{lmeasurable}$
by (*meson* $\mathcal{D}\mathit{lm}$ ‹*insert D* $\mathcal{D}' \subseteq \mathcal{D}$› *fmeasurable.finite_Union less.prems*(2) *finite_subset subset_eq subset_insertI*)
show *measure lebesgue D* + *measure lebesgue* $(\bigcup\ \mathcal{D}') \leq \mathit{measure}\ \mathit{lebesgue}\ D$ + *measure lebesgue* $(\bigcup\ \mathcal{D}' - D)$
using ‹*insert D* $\mathcal{D}' \subseteq \mathcal{D}$› *infinite_super less.prems*(2) **by** *force*
qed (*simp add*: $\mathcal{D}\mathit{lm}$ ‹$D \in \mathcal{D}$›)
qed
finally show *measure lebesgue* $(\bigcup\mathcal{D})\ /\ 3\ \hat{}\ \mathit{DIM}('a) \leq \mathit{measure}\ \mathit{lebesgue}\ (\bigcup(\mathit{insert}\ D\ \mathcal{D}'))$
by (*simp add*: *field_split_simps*)
qed
qed
qed

9.16.2 A differentiability-like property of the indefinite integral.

proposition *integrable_ccontinuous_explicit*:
fixes $f :: 'a{::}\mathit{euclidean\_space} \Rightarrow 'b{::}\mathit{euclidean\_space}$
assumes $\bigwedge a\ b{::}'a.\ f\ \mathit{integrable\_on}\ \mathit{cbox}\ a\ b$
obtains N **where**
negligible N
$\bigwedge x\ e.\ [\![x \notin N;\ 0 < e]\!] \Longrightarrow$
$\exists\, d{>}0.\ \forall h.\ 0 < h \wedge h < d \longrightarrow$
$\mathit{norm}(\mathit{integral}\ (\mathit{cbox}\ x\ (x + h *_R \mathit{One}))\ f\ /_R\ h\ \hat{}\ \mathit{DIM}('a) - f\ x) < e$
proof −
define *BOX* **where** $\mathit{BOX} \equiv \lambda h.\ \lambda x{::}'a.\ \mathit{cbox}\ x\ (x + h *_R \mathit{One})$
define *BOX2* **where** $\mathit{BOX2} \equiv \lambda h.\ \lambda x{::}'a.\ \mathit{cbox}\ (x - h *_R \mathit{One})\ (x + h *_R \mathit{One})$
define i **where** $i \equiv \lambda h\ x.\ \mathit{integral}\ (\mathit{BOX}\ h\ x)\ f\ /_R\ h\ \hat{}\ \mathit{DIM}('a)$
define Ψ **where** $\Psi \equiv \lambda x\ r.\ \forall\, d{>}0.\ \exists h.\ 0 < h \wedge h < d \wedge r \leq \mathit{norm}(i\ h\ x - f\ x)$

```
  let ?N = {x. ∃e>0. Ψ x e}
  have ∃N. negligible N ∧ (∀x e. x ∉ N ∧ 0 < e ⟶ ¬ Ψ x e)
  proof (rule exI ; intro conjI allI impI)
    let ?M = ⋃n. {x. Ψ x (inverse(real n + 1))}
    have negligible ({x. Ψ x μ} ∩ cbox a b)
      if μ > 0 for a b μ
    proof (cases negligible(cbox a b))
      case True
      then show ?thesis
        by (simp add: negligible_Int)
    next
      case False
      then have box a b ≠ {}
        by (simp add: negligible_interval)
      then have ab: ⋀i. i ∈ Basis ⟹ a•i < b•i
        by (simp add: box_ne_empty)
      show ?thesis
        unfolding negligible_outer_le
      proof (intro allI impI)
        fix e::real
        let ?ee = (e * μ) / 2 / 6 ^ (DIM('a))
        assume e > 0
        then have gt0: ?ee > 0
          using ‹μ > 0› by auto
        have f': f integrable_on cbox (a - One) (b + One)
          using assms by blast
        obtain γ where gauge γ
          and γ: ⋀p. ⟦p tagged_partial_division_of (cbox (a - One) (b + One));
γ fine p⟧
                  ⟹ (∑(x, k)∈p. norm (content k *R f x - integral k f)) < ?ee
          using Henstock_lemma [OF f' gt0] that by auto
        let ?E = {x. x ∈ cbox a b ∧ Ψ x μ}
        have ∃h>0. BOX h x ⊆ γ x ∧
                  BOX h x ⊆ cbox (a - One) (b + One) ∧ μ ≤ norm (i h x - f x)
          if x ∈ cbox a b Ψ x μ for x
        proof -
          obtain d where d > 0 and d: ball x d ⊆ γ x
            using gaugeD [OF ‹gauge γ›, of x] openE by blast
          then obtain h where 0 < h h < 1 and hless: h < d / real DIM('a)
                        and mule: μ ≤ norm (i h x - f x)
            using ‹Ψ x μ› [unfolded Ψ_def, rule_format, of min 1 (d / DIM('a))]
            by auto
          show ?thesis
          proof (intro exI conjI)
            show 0 < h μ ≤ norm (i h x - f x) by fact+
            have BOX h x ⊆ ball x d
            proof (clarsimp simp: BOX_def mem_box dist_norm algebra_simps)
              fix y
              assume ∀i∈Basis. x • i ≤ y • i ∧ y • i ≤ h + x • i
```

```
            then have lt: |(x − y) · i| < d / real DIM('a) if i ∈ Basis for i
              using hless that by (force simp: inner_diff_left)
            have norm (x − y) ≤ (∑ i∈Basis. |(x − y) · i|)
              using norm_le_l1 by blast
            also have ... < d
                using sum_bounded_above_strict [of Basis λi. |(x − y) · i| d /
DIM('a), OF lt]
              by auto
            finally show norm (x − y) < d .
          qed
          with d show BOX h x ⊆ γ x
            by blast
          show BOX h x ⊆ cbox (a − One) (b + One)
            using that ‹h < 1›
          by (force simp: BOX_def mem_box algebra_simps intro: subset_box_imp)
        qed
      qed
      then obtain η where h0: ⋀x. x ∈ ?E ⟹ η x > 0
        and BOX_γ: ⋀x. x ∈ ?E ⟹ BOX (η x) x ⊆ γ x
        and ⋀x. x ∈ ?E ⟹ BOX (η x) x ⊆ cbox (a − One) (b + One) ∧ μ ≤
norm (i (η x) x − f x)
        by simp metis
      then have BOX_cbox: ⋀x. x ∈ ?E ⟹ BOX (η x) x ⊆ cbox (a − One) (b
+ One)
          and μ_le: ⋀x. x ∈ ?E ⟹ μ ≤ norm (i (η x) x − f x)
        by blast+
      define γ' where γ' ≡ λx. if x ∈ cbox a b ∧ Ψ x μ then ball x (η x) else γ x
      have gauge γ'
        using ‹gauge γ› by (auto simp: h0 gauge_def γ'_def)
      obtain 𝒟 where countable 𝒟
        and 𝒟: ⋃𝒟 ⊆ cbox a b
        ⋀K. K ∈ 𝒟 ⟹ interior K ≠ {} ∧ (∃ c d. K = cbox c d)
        and Dcovered: ⋀K. K ∈ 𝒟 ⟹ ∃x. x ∈ cbox a b ∧ Ψ x μ ∧ x ∈ K ∧ K
⊆ γ' x
        and subUD: ?E ⊆ ⋃𝒟
        by (rule covering_lemma [of ?E a b γ'] ) (simp_all add: Bex_def ‹box a b
≠ {}› ‹gauge γ'›)
      then have 𝒟 ⊆ sets lebesgue
        by fastforce
    show ∃ T. {x. Ψ x μ} ∩ cbox a b ⊆ T ∧ T ∈ lmeasurable ∧ measure lebesgue
T ≤ e
      proof (intro exI conjI)
        show {x. Ψ x μ} ∩ cbox a b ⊆ ⋃𝒟
          apply auto
          using subUD by auto
        have mUE: measure lebesgue (⋃ ℰ) ≤ measure lebesgue (cbox a b)
          if ℰ ⊆ 𝒟 finite ℰ for ℰ
        proof (rule measure_mono_fmeasurable)
          show ⋃ ℰ ⊆ cbox a b
```

```
          using 𝒟(1) that(1) by blast
        show ⋃ ℰ ∈ sets lebesgue
       by (metis 𝒟(2) fmeasurable.finite_Union fmeasurableD lmeasurable_cbox
subset_eq that)
      qed auto
      then show ⋃𝒟 ∈ lmeasurable
          by (metis 𝒟(2) ‹countable 𝒟› fmeasurable_Union_bound lmeasur-
able_cbox)
      then have leab: measure lebesgue (⋃𝒟) ≤ measure lebesgue (cbox a b)
     by (meson 𝒟(1) fmeasurableD lmeasurable_cbox measure_mono_fmeasurable)
      obtain ℱ where ℱ ⊆ 𝒟 finite ℱ
        and ℱ: measure lebesgue (⋃𝒟) ≤ 2 * measure lebesgue (⋃ℱ)
      proof (cases measure lebesgue (⋃𝒟) = 0)
        case True
        then show ?thesis
          by (force intro: that [where ℱ = {}])
      next
        case False
        obtain ℱ where ℱ ⊆ 𝒟 finite ℱ
          and ℱ: measure lebesgue (⋃𝒟)/2 < measure lebesgue (⋃ℱ)
           proof (rule measure_countable_Union_approachable [of 𝒟 measure
lebesgue (⋃𝒟) / 2 content (cbox a b)])
          show countable 𝒟
            by fact
          show 0 < measure lebesgue (⋃ 𝒟) / 2
            using False by (simp add: zero_less_measure_iff)
          show Dlm: D ∈ lmeasurable if D ∈ 𝒟 for D
            using 𝒟(2) that by blast
          show measure lebesgue (⋃ ℱ) ≤ content (cbox a b)
            if ℱ ⊆ 𝒟 finite ℱ for ℱ
          proof −
            have measure lebesgue (⋃ ℱ) ≤ measure lebesgue (⋃𝒟)
            proof (rule measure_mono_fmeasurable)
              show ⋃ ℱ ⊆ ⋃ 𝒟
                by (simp add: Sup_subset_mono ‹ℱ ⊆ 𝒟›)
              show ⋃ ℱ ∈ sets lebesgue
                by (meson Dlm fmeasurableD sets.finite_Union subset_eq that)
              show ⋃ 𝒟 ∈ lmeasurable
                by fact
            qed
            also have ... ≤ measure lebesgue (cbox a b)
            proof (rule measure_mono_fmeasurable)
              show ⋃ 𝒟 ∈ sets lebesgue
                by (simp add: ‹⋃ 𝒟 ∈ lmeasurable› fmeasurableD)
            qed (auto simp:𝒟(1))
            finally show ?thesis
              by simp
          qed
        qed auto
```

```
      then show ?thesis
        using that by auto
    qed
    obtain tag where tag_in_E: ⋀D. D ∈ 𝒟 ⟹ tag D ∈ ?E
      and tag_in_self: ⋀D. D ∈ 𝒟 ⟹ tag D ∈ D
      and tag_sub: ⋀D. D ∈ 𝒟 ⟹ D ⊆ γ' (tag D)
      using Dcovered by simp metis
    then have sub_ball_tag: ⋀D. D ∈ 𝒟 ⟹ D ⊆ ball (tag D) (η (tag D))
      by (simp add: γ'_def)
    define Φ where Φ ≡ λD. BOX (η(tag D)) (tag D)
    define Φ2 where Φ2 ≡ λD. BOX2 (η(tag D)) (tag D)
    obtain 𝒞 where 𝒞 ⊆ Φ2 ` ℱ pairwise disjnt 𝒞
      measure lebesgue (⋃𝒞) ≥ measure lebesgue (⋃(Φ2`ℱ)) / 3 ^ (DIM('a))
    proof (rule Austin_Lemma)
      show finite (Φ2`ℱ)
        using ‹finite ℱ› by blast
      have ∃k a b. Φ2 D = cbox a b ∧ (∀i∈Basis. b • i − a • i = k) if D ∈ ℱ
for D
        apply (rule_tac x=2 ∗ η(tag D) in exI)
        apply (rule_tac x=tag D − η(tag D) ∗R One in exI)
        apply (rule_tac x=tag D + η(tag D) ∗R One in exI)
        using that
        apply (auto simp: Φ2_def BOX2_def algebra_simps)
        done
      then show ⋀D. D ∈ Φ2 ` ℱ ⟹ ∃k a b. D = cbox a b ∧ (∀i∈Basis. b
• i − a • i = k)
        by blast
    qed auto
    then obtain 𝒢 where 𝒢 ⊆ ℱ and disj: pairwise disjnt (Φ2 ` 𝒢)
       and measure lebesgue (⋃(Φ2 ` 𝒢)) ≥ measure lebesgue (⋃(Φ2`ℱ)) / 3
^ (DIM('a))
      unfolding Φ2_def subset_image_iff
      by (meson empty_subsetI equals0D pairwise_imageI)
    moreover
    have measure lebesgue (⋃(Φ2 ` 𝒢)) ∗ 3 ^ DIM('a) ≤ e/2
    proof −
      have finite 𝒢
        using ‹finite ℱ› ‹𝒢 ⊆ ℱ› infinite_super by blast
      have BOX2_m: ⋀x. x ∈ tag ` 𝒢 ⟹ BOX2 (η x) x ∈ lmeasurable
        by (auto simp: BOX2_def)
      have BOX_m: ⋀x. x ∈ tag ` 𝒢 ⟹ BOX (η x) x ∈ lmeasurable
        by (auto simp: BOX_def)
      have BOX_sub: BOX (η x) x ⊆ BOX2 (η x) x for x
        by (auto simp: BOX_def BOX2_def subset_box algebra_simps)
       have DISJ2: BOX2 (η (tag X)) (tag X) ∩ BOX2 (η (tag Y)) (tag Y)
= {}
        if X ∈ 𝒢 Y ∈ 𝒢 tag X ≠ tag Y for X Y
      proof −
        obtain i where i: i ∈ Basis tag X • i ≠ tag Y • i
```

```
      using ‹tag X ≠ tag Y› by (auto simp: euclidean_eq_iff [of tag X])
    have XY: X ∈ 𝒟 Y ∈ 𝒟
      using ‹ℱ ⊆ 𝒟› ‹𝒢 ⊆ ℱ› that by auto
    then have 0 ≤ η (tag X) 0 ≤ η (tag Y)
      by (meson h0 le_cases not_le tag_in_E)+
    with XY i have BOX2 (η (tag X)) (tag X) ≠ BOX2 (η (tag Y)) (tag Y)
      unfolding eq_iff
      by (fastforce simp add: BOX2_def subset_box algebra_simps)
    then show ?thesis
      using disj that by (auto simp: pairwise_def disjnt_def Φ2_def)
  qed
 then have BOX2_disj: pairwise (λx y. negligible (BOX2 (η x) x ∩ BOX2 (η y) y)) (tag ` 𝒢)
    by (simp add: pairwise_imageI)
  then have BOX_disj: pairwise (λx y. negligible (BOX (η x) x ∩ BOX (η y) y)) (tag ` 𝒢)
  proof (rule pairwise_mono)
    show negligible (BOX (η x) x ∩ BOX (η y) y)
      if negligible (BOX2 (η x) x ∩ BOX2 (η y) y) for x y
     by (metis (no_types, opaque_lifting) that Int_mono negligible_subset BOX_sub)
  qed auto

  have eq: ⋀box. (λD. box (η (tag D)) (tag D)) ` 𝒢 = (λt. box (η t) t) ` tag ` 𝒢
    by (simp add: image_comp)
  have measure lebesgue (BOX2 (η t) t) * 3 ^ DIM('a)
      = measure lebesgue (BOX (η t) t) * (2*3) ^ DIM('a)
    if t ∈ tag ` 𝒢 for t
  proof −
    have content (cbox (t − η t *R One) (t + η t *R One))
       = content (cbox t (t + η t *R One)) * 2 ^ DIM('a)
    using that by (simp add: algebra_simps content_cbox_if box_eq_empty)
    then show ?thesis
      by (simp add: BOX2_def BOX_def flip: power_mult_distrib)
  qed
    then have measure lebesgue (⋃(Φ2 ` 𝒢)) * 3 ^ DIM('a) = measure lebesgue (⋃(Φ ` 𝒢)) * 6 ^ DIM('a)
    unfolding Φ_def Φ2_def eq
    by (simp add: measure_negligible_finite_Union_image
      ‹finite 𝒢› BOX2_m BOX_m BOX2_disj BOX_disj sum_distrib_right
        del: UN_simps)
  also have ... ≤ e/2
  proof −
   have μ * measure lebesgue (⋃D∈𝒢. Φ D) ≤ μ * (∑D∈Φ`𝒢. measure lebesgue D)
     using ‹μ > 0› ‹finite 𝒢› by (force simp: BOX_m Φ_def fmeasurableD intro: measure_Union_le)
```

```
          also have ... = (∑ D ∈ Φ ‘𝒢. measure lebesgue D * μ)
            by (metis mult.commute sum_distrib_right)
          also have ... ≤ (∑ (x, K) ∈ (λD. (tag D, Φ D)) ‘ 𝒢.  norm (content
K *R f x − integral K f))
          proof (rule sum_le_included; clarify?)
            fix D
            assume D ∈ 𝒢
            then have η (tag D) > 0
              using ‹ℱ ⊆ 𝒟› ‹𝒢 ⊆ ℱ› h0 tag_in_E by auto
            then have m_Φ: measure lebesgue (Φ D) > 0
              by (simp add: Φ_def BOX_def algebra_simps)
            have μ ≤ norm (i (η(tag D)) (tag D) − f(tag D))
              using μ_le ‹D ∈ 𝒢› ‹ℱ ⊆ 𝒟› ‹𝒢 ⊆ ℱ› tag_in_E by auto
          also have ... = norm ((content (Φ D) *R f(tag D) − integral (Φ D)
f) /R measure lebesgue (Φ D))
            using m_Φ
            unfolding i_def Φ_def BOX_def
          by (simp add: algebra_simps content_cbox_plus norm_minus_commute)
           finally have measure lebesgue (Φ D) * μ ≤ norm (content (Φ D) *R
f(tag D) − integral (Φ D) f)
            using m_Φ by simp (simp add: field_simps)
            then show ∃ y∈(λD. (tag D, Φ D)) ‘ 𝒢.
                  snd y = Φ D ∧ measure lebesgue (Φ D) * μ ≤ (case y of (x, k)
⇒ norm (content k *R f x − integral k f))
            using ‹D ∈ 𝒢› by auto
          qed (use ‹finite 𝒢› in auto)
          also have ... < ?ee
          proof (rule γ)
           show (λD. (tag D, Φ D)) ‘ 𝒢 tagged_partial_division_of cbox (a −
One) (b + One)
            unfolding tagged_partial_division_of_def
           proof (intro conjI allI impI ; clarify ?)
            show tag D ∈ Φ D
             if D ∈ 𝒢 for D
             using that ‹ℱ ⊆ 𝒟› ‹𝒢 ⊆ ℱ› h0 tag_in_E
                  by (auto simp: Φ_def BOX_def mem_box algebra_simps
eucl_less_le_not_le in_mono)
            show y ∈ cbox (a − One) (b + One) if D ∈ 𝒢 y ∈ Φ D for D y
              using that BOX_cbox Φ_def ‹ℱ ⊆ 𝒟› ‹𝒢 ⊆ ℱ› tag_in_E by
blast
            show tag D = tag E ∧ Φ D = Φ E
              if D ∈ 𝒢 E ∈ 𝒢 and ne: interior (Φ D) ∩ interior (Φ E) ≠ {}
for D E
            proof −
              have BOX2 (η (tag D)) (tag D) ∩ BOX2 (η (tag E)) (tag E) =
{} ∨ tag E = tag D
                using DISJ2 ‹D ∈ 𝒢› ‹E ∈ 𝒢› by force
             then have BOX (η (tag D)) (tag D) ∩ BOX (η (tag E)) (tag E)
= {} ∨ tag E = tag D
```

```
                using BOX_sub by blast
              then show tag D = tag E ∧ Φ D = Φ E
                by (metis Φ_def interior_Int interior_empty ne)
            qed
          qed (use ‹finite 𝒢› Φ_def BOX_def in auto)
          show γ fine (λD. (tag D, Φ D)) ‘ 𝒢
              unfolding fine_def Φ_def using BOX_γ ‹ℱ ⊆ 𝒟› ‹𝒢 ⊆ ℱ›
tag_in_E by blast
        qed
        finally show ?thesis
          using ‹μ > 0› by (auto simp: field_split_simps)
      qed
       finally show ?thesis .
      qed
      moreover
      have measure lebesgue (⋃ℱ) ≤ measure lebesgue (⋃(Φ2‘ℱ))
      proof (rule measure_mono_fmeasurable)
       have D ⊆ ball (tag D) (η(tag D)) if D ∈ ℱ for D
        using ‹ℱ ⊆ 𝒟› sub_ball_tag that by blast
       moreover have ball (tag D) (η(tag D)) ⊆ BOX2 (η (tag D)) (tag D) if
D ∈ ℱ for D
          proof (clarsimp simp: Φ2_def BOX2_def mem_box algebra_simps
dist_norm)
        fix x and i::'a
        assume norm (tag D − x) < η (tag D) and i ∈ Basis
        then have |tag D · i − x · i| ≤ η (tag D)
            by (metis eucl_less_le_not_le inner_commute inner_diff_right
norm_bound_Basis_le)
        then show tag D · i ≤ x · i + η (tag D) ∧ x · i ≤ η (tag D) + tag D
· i
          by (simp add: abs_diff_le_iff)
       qed
       ultimately show ⋃ℱ ⊆ ⋃(Φ2‘ℱ)
        by (force simp: Φ2_def)
       show ⋃ℱ ∈ sets lebesgue
        using ‹finite ℱ› ‹𝒟 ⊆ sets lebesgue› ‹ℱ ⊆ 𝒟› by blast
       show ⋃(Φ2‘ℱ) ∈ lmeasurable
        unfolding Φ2_def BOX2_def using ‹finite ℱ› by blast
      qed
      ultimately
      have measure lebesgue (⋃ℱ) ≤ e/2
       by (auto simp: field_split_simps)
      then show measure lebesgue (⋃𝒟) ≤ e
       using ℱ by linarith
     qed
    qed
   qed
   then have ⋀j. negligible {x. Ψ x (inverse(real j + 1))}
    using negligible_on_intervals
```

by (*metis* (*full_types*) *inverse_positive_iff_positive le_add_same_cancel1 linorder_not_le nat_le_real_less not_add_less1 of_nat_0*)
then have *negligible ?M*
by *auto*
moreover have *?N* ⊆ *?M*
proof (*clarsimp simp*: *dist_norm*)
fix *y e*
assume *0* < *e*
and *ye* [*rule_format*]: Ψ *y e*
then obtain *k* **where** *k*: *0* < *k inverse* (*real k* + *1*) < *e*
by (*metis One_nat_def add.commute less_add_same_cancel2 less_imp_inverse_less less_trans neq0_conv of_nat_1 of_nat_Suc reals_Archimedean zero_less_one*)
with *ye* **show** ∃ *n*. Ψ *y* (*inverse* (*real n* + *1*))
apply (*rule_tac x*=*k* **in** *exI*)
unfolding Ψ_*def*
by (*force intro*: *less_le_trans*)
qed
ultimately show *negligible ?N*
by (*blast intro*: *negligible_subset*)
show ¬ Ψ *x e* **if** *x* ∉ *?N* ∧ *0* < *e* **for** *x e*
using *that* **by** *blast*
qed
with *that* **show** *?thesis*
unfolding *i_def BOX_def* Ψ_*def* **by** (*fastforce simp add*: *not_le*)
qed

9.16.3 HOL Light measurability

definition *measurable_on* :: ('*a*::*euclidean_space* ⇒ '*b*::*real_normed_vector*) ⇒ '*a set* ⇒ *bool*
(**infixr** ‹*measurable'_on*› *46*)
where *f measurable_on S* ≡
∃ *N g*. *negligible N* ∧
(∀ *n*. *continuous_on UNIV* (*g n*)) ∧
(∀ *x*. *x* ∉ *N* ⟶ (λ*n*. *g n x*) ⟶⟶ (*if x* ∈ *S then f x else 0*))

lemma *measurable_on_UNIV*:
(λ*x*. *if x* ∈ *S then f x else 0*) *measurable_on UNIV* ⟷ *f measurable_on S*
by (*auto simp*: *measurable_on_def*)

lemma *measurable_on_spike_set*:
assumes *f*: *f measurable_on S* **and** *neg*: *negligible* ((*S* − *T*) ∪ (*T* − *S*))
shows *f measurable_on T*
proof −
obtain *N* **and** *F*
where *N*: *negligible N*
and *conF*: ⋀*n*. *continuous_on UNIV* (*F n*)
and *tendsF*: ⋀*x*. *x* ∉ *N* ⟹ (λ*n*. *F n x*) ⟶⟶ (*if x* ∈ *S then f x else 0*)
using *f* **by** (*auto simp*: *measurable_on_def*)

```
  show ?thesis
    unfolding measurable_on_def
  proof (intro exI conjI allI impI)
    show continuous_on UNIV (λx. F n x) for n
      by (intro conF continuous_intros)
    show negligible (N ∪ (S − T) ∪ (T − S))
      by (metis (full_types) N neg negligible_Un_eq)
    show (λn. F n x) ⟶ (if x ∈ T then f x else 0)
      if x ∉ (N ∪ (S − T) ∪ (T − S)) for x
      using that tendsF [of x] by auto
  qed
qed
```

Various common equivalent forms of function measurability.

```
lemma measurable_on_0 [simp]: (λx. 0) measurable_on S
  unfolding measurable_on_def
proof (intro exI conjI allI impI)
  show (λn. 0) ⟶ (if x ∈ S then 0::'b else 0) for x
    by force
qed auto

lemma measurable_on_scaleR_const:
  assumes f: f measurable_on S
  shows (λx. c *R f x) measurable_on S
proof −
  obtain NF and F
    where NF: negligible NF
      and conF: ⋀n. continuous_on UNIV (F n)
      and tendsF: ⋀x. x ∉ NF ⟹ (λn. F n x) ⟶ (if x ∈ S then f x else 0)
    using f by (auto simp: measurable_on_def)
  show ?thesis
    unfolding measurable_on_def
  proof (intro exI conjI allI impI)
    show continuous_on UNIV (λx. c *R F n x) for n
      by (intro conF continuous_intros)
    show (λn. c *R F n x) ⟶ (if x ∈ S then c *R f x else 0)
      if x ∉ NF for x
      using tendsto_scaleR [OF tendsto_const tendsF, of x] that by auto
  qed (auto simp: NF)
qed


lemma measurable_on_cmul:
  fixes c :: real
  assumes f measurable_on S
  shows (λx. c * f x) measurable_on S
  using measurable_on_scaleR_const [OF assms] by simp

lemma measurable_on_cdivide:
```

```
  fixes c :: real
  assumes f measurable_on S
  shows (λx. f x / c) measurable_on S
proof (cases c=0)
  case False
  then show ?thesis
    using measurable_on_cmul [of f S 1/c]
    by (simp add: assms)
qed auto


lemma measurable_on_minus:
   f measurable_on S ⟹ (λx. -(f x)) measurable_on S
  using measurable_on_scaleR_const [of f S -1] by auto


lemma continuous_imp_measurable_on:
   continuous_on UNIV f ⟹ f measurable_on UNIV
  unfolding measurable_on_def
  apply (rule_tac x={} in exI)
  apply (rule_tac x=λn. f in exI, auto)
  done

proposition integrable_subintervals_imp_measurable:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes ⋀a b. f integrable_on cbox a b
  shows f measurable_on UNIV
proof -
  define BOX where BOX ≡ λh. λx::'a. cbox x (x + h *R One)
  define i where i ≡ λh x. integral (BOX h x) f /R h ^ DIM('a)
  obtain N where negligible N
    and k: ⋀x e. ⟦x ∉ N; 0 < e⟧
          ⟹ ∃d>0. ∀h. 0 < h ∧ h < d ⟶
                norm (integral (cbox x (x + h *R One)) f /R h ^ DIM('a) - f x)
< e
    using integrable_ccontinuous_explicit assms by blast
  show ?thesis
    unfolding measurable_on_def
  proof (intro exI conjI allI impI)
    show continuous_on UNIV ((λn x. i (inverse(Suc n)) x) n) for n
    proof (clarsimp simp: continuous_on_iff)
      show ∃d>0. ∀x'. dist x' x < d ⟶ dist (i (inverse (1 + real n)) x') (i
(inverse (1 + real n)) x) < e
        if 0 < e
        for x e
      proof -
        let ?e = e / (1 + real n) ^ DIM('a)
        have ?e > 0
          using ‹e > 0› by auto
```

```
      moreover have x ∈ cbox (x − 2 *R One) (x + 2 *R One)
        by (simp add: mem_box inner_diff_left inner_left_distrib)
      moreover have x + One /R real (Suc n) ∈ cbox (x − 2 *R One) (x + 2
*R One)
        by (auto simp: mem_box inner_diff_left inner_left_distrib field_simps)
      ultimately obtain δ where δ > 0
        and δ: ⋀c' d'. ⟦c' ∈ cbox (x − 2 *R One) (x + 2 *R One);
                        d' ∈ cbox (x − 2 *R One) (x + 2 *R One);
                        norm(c' − x) ≤ δ; norm(d' − (x + One /R Suc n)) ≤ δ⟧
                        ⟹ norm(integral(cbox c' d') f − integral(cbox x (x + One
/R Suc n)) f) < ?e
        by (blast intro: indefinite_integral_continuous [of f _ _ x] assms)
      show ?thesis
      proof (intro exI impI conjI allI)
        show min δ 1 > 0
          using ‹δ > 0› by auto
        show dist (i (inverse (1 + real n)) y) (i (inverse (1 + real n)) x) < e
          if dist y x < min δ 1 for y
        proof −
          have no: norm (y − x) < 1
            using that by (auto simp: dist_norm)
          have le1: inverse (1 + real n) ≤ 1
            by (auto simp: field_split_simps)
          have norm (integral (cbox y (y + One /R real (Suc n))) f
              − integral (cbox x (x + One /R real (Suc n))) f)
              < e / (1 + real n) ^ DIM('a)
          proof (rule δ)
            show y ∈ cbox (x − 2 *R One) (x + 2 *R One)
           using no by (auto simp: mem_box algebra_simps dest: Basis_le_norm
[of _ y−x])
            show y + One /R real (Suc n) ∈ cbox (x − 2 *R One) (x + 2 *R One)
            proof (simp add: dist_norm mem_box algebra_simps, intro ballI conjI)
              fix i::'a
              assume i ∈ Basis
              then have 1: |y • i − x • i| < 1
              by (metis inner_commute inner_diff_right no norm_bound_Basis_lt)
               moreover have ... < (2 + inverse (1 + real n)) 1 ≤ 2 − inverse
(1 + real n)
                by (auto simp: field_simps)
              ultimately show x • i ≤ y • i + (2 + inverse (1 + real n))
                              y • i + inverse (1 + real n) ≤ x • i + 2
                by linarith+
            qed
            show norm (y − x) ≤ δ norm (y + One /R real (Suc n) − (x + One
/R real (Suc n))) ≤ δ
              using that by (auto simp: dist_norm)
          qed
          then show ?thesis
        using that by (simp add: dist_norm i_def BOX_def flip: scaleR_diff_right)
```

```
(simp add: field_simps)
        qed
       qed
      qed
    qed
    show negligible N
      by (simp add: ‹negligible N›)
    show (λn. i (inverse (Suc n)) x) ⟶ (if x ∈ UNIV then f x else 0)
      if x ∉ N for x
      unfolding lim_sequentially
    proof clarsimp
      show ∃no. ∀n≥no. dist (i (inverse (1 + real n)) x) (f x) < e
        if 0 < e for e
      proof −
        obtain d where d > 0
          and d: ⋀h. ⟦0 < h; h < d⟧ ⟹
              norm (integral (cbox x (x + h *R One)) f /R h ^ DIM('a) − f x) < e
          using k [of x e] ‹x ∉ N› ‹0 < e› by blast
        then obtain M where M: M ≠ 0 0 < inverse (real M) inverse (real M)
< d
          using real_arch_invD by auto
        show ?thesis
        proof (intro exI allI impI)
          show dist (i (inverse (1 + real n)) x) (f x) < e
            if M ≤ n for n
          proof −
            have ∗: 0 < inverse (1 + real n) inverse (1 + real n) ≤ inverse M
              using that ‹M ≠ 0› by auto
            show ?thesis
              using that M
              apply (simp add: i_def BOX_def dist_norm)
              apply (blast intro: le_less_trans ∗ d)
              done
          qed
        qed
      qed
    qed
  qed
qed
```

9.16.4 Composing continuous and measurable functions; a few variants

```
lemma measurable_on_compose_continuous:
  assumes f: f measurable_on UNIV and g: continuous_on UNIV g
  shows (g ∘ f) measurable_on UNIV
proof −
  obtain N and F
    where negligible N
```

```
    and conF: ⋀n. continuous_on UNIV (F n)
    and tendsF: ⋀x. x ∉ N ⟹ (λn. F n x) ⟶ f x
  using f by (auto simp: measurable_on_def)
 show ?thesis
  unfolding measurable_on_def
 proof (intro exI conjI allI impI)
  show negligible N
   by fact
  show continuous_on UNIV (g ∘ (F n)) for n
   using conF continuous_on_compose continuous_on_subset g by blast
  show (λn. (g ∘ F n) x) ⟶ (if x ∈ UNIV then (g ∘ f) x else 0)
   if x ∉ N for x :: 'a
  using that g tendsF by (auto simp: continuous_on_def intro: tendsto_compose)
 qed
qed

lemma measurable_on_compose_continuous_0:
  assumes f: f measurable_on S and g: continuous_on UNIV g and g 0 = 0
  shows (g ∘ f) measurable_on S
proof −
 have f': (λx. if x ∈ S then f x else 0) measurable_on UNIV
  using f measurable_on_UNIV by blast
 show ?thesis
  using measurable_on_compose_continuous [OF f' g]
  by (simp add: measurable_on_UNIV o_def if_distrib ‹g 0 = 0› cong: if_cong)
qed


lemma measurable_on_compose_continuous_box:
 assumes fm: f measurable_on UNIV and fab: ⋀x. f x ∈ box a b
   and contg: continuous_on (box a b) g
 shows (g ∘ f) measurable_on UNIV
proof −
  have ∃γ. (∀n. continuous_on UNIV (γ n)) ∧ (∀x. x ∉ N ⟶ (λn. γ n x)
⟶ g (f x))
   if negligible N
     and conth [rule_format]: ∀n. continuous_on UNIV (λx. h n x)
     and tends [rule_format]: ∀x. x ∉ N ⟶ (λn. h n x) ⟶ f x
   for N and h :: nat ⇒ 'a ⇒ 'b
 proof −
   define ϑ where ϑ ≡ λn x. (∑i∈Basis. (max (a•i + (b•i − a•i) / real (n+2))
                                        (min ((h n x)•i)
                                             (b•i − (b•i − a•i) / real (n+2)))) *R i)
   have aibi: ⋀i. i ∈ Basis ⟹ a • i < b • i
     using box_ne_empty(2) fab by auto
   then have ∗: ⋀i n. i ∈ Basis ⟹ a • i + real n ∗ (a • i) < b • i + real n ∗ (b
• i)
    by (meson add_mono_thms_linordered_field(3) less_eq_real_def mult_left_mono
of_nat_0_le_iff)
```

```
    show ?thesis
    proof (intro exI conjI allI impI)
      show continuous_on UNIV (g ∘ (ϑ n)) for n :: nat
        unfolding ϑ_def
       apply (intro continuous_on_compose2 [OF contg] continuous_intros conth)
         apply (auto simp: aibi * mem_box less_max_iff_disj min_less_iff_disj
field_split_simps)
        done
      show (λn. (g ∘ ϑ n) x) ⟶ g (f x)
        if x ∉ N for x
        unfolding o_def
      proof (rule isCont_tendsto_compose [where g=g])
        show isCont g (f x)
          using contg fab continuous_on_eq_continuous_at by blast
        have (λn. ϑ n x) ⟶ (∑ i∈Basis. max (a · i) (min (f x · i) (b · i)) *R
i)
          unfolding ϑ_def
        proof (intro tendsto_intros ‹x ∉ N› tends)
          fix i::'b
          assume i ∈ Basis
          have a: (λn. a · i + (b · i − a · i) / real n) ⟶ a·i + 0
            by (intro tendsto_add lim_const_over_n tendsto_const)
          show (λn. a · i + (b · i − a · i) / real (n + 2)) ⟶ a · i
            using LIMSEQ_ignore_initial_segment [where k=2, OF a] by simp
          have b: (λn. b·i − (b · i − a · i) / (real n)) ⟶ b·i − 0
            by (intro tendsto_diff lim_const_over_n tendsto_const)
          show (λn. b · i − (b · i − a · i) / real (n + 2)) ⟶ b · i
            using LIMSEQ_ignore_initial_segment [where k=2, OF b] by simp
        qed
       also have (∑ i∈Basis. max (a · i) (min (f x · i) (b · i)) *R i) = (∑ i∈Basis.
(f x · i) *R i)
          using fab by (auto simp add: mem_box intro: sum.cong)
        also have ... = f x
          using euclidean_representation by blast
        finally show (λn. ϑ n x) ⟶ f x .
      qed
    qed
  qed
  then show ?thesis
    using fm by (auto simp: measurable_on_def)
qed

lemma measurable_on_Pair:
  assumes f: f measurable_on S and g: g measurable_on S
  shows (λx. (f x, g x)) measurable_on S
proof −
  obtain NF and F
    where NF: negligible NF
      and conF: ⋀n. continuous_on UNIV (F n)
```

```
    and tendsF: ⋀x. x ∉ NF ⟹ (λn. F n x) ⟶ (if x ∈ S then f x else 0)
   using f by (auto simp: measurable_on_def)
  obtain NG and G
   where NG: negligible NG
    and conG: ⋀n. continuous_on UNIV (G n)
    and tendsG: ⋀x. x ∉ NG ⟹ (λn. G n x) ⟶ (if x ∈ S then g x else 0)
   using g by (auto simp: measurable_on_def)
  show ?thesis
   unfolding measurable_on_def
  proof (intro exI conjI allI impI)
   show negligible (NF ∪ NG)
    by (simp add: NF NG)
   show continuous_on UNIV (λx. (F n x, G n x)) for n
    using conF conG continuous_on_Pair by blast
   show (λn. (F n x, G n x)) ⟶ (if x ∈ S then (f x, g x) else 0)
    if x ∉ NF ∪ NG for x
    using tendsto_Pair [OF tendsF tendsG, of x x] that unfolding zero_prod_def
    by (simp add: split: if_split_asm)
  qed
qed

lemma measurable_on_combine:
  assumes f: f measurable_on S and g: g measurable_on S
   and h: continuous_on UNIV (λx. h (fst x) (snd x)) and h 0 0 = 0
  shows (λx. h (f x) (g x)) measurable_on S
proof −
  have ∗: (λx. h (f x) (g x)) = (λx. h (fst x) (snd x)) ∘ (λx. (f x, g x))
   by auto
  show ?thesis
   unfolding ∗ by (auto simp: measurable_on_compose_continuous_0 measur-
able_on_Pair assms)
qed

lemma measurable_on_add:
  assumes f: f measurable_on S and g: g measurable_on S
  shows (λx. f x + g x) measurable_on S
  by (intro continuous_intros measurable_on_combine [OF assms]) auto

lemma measurable_on_diff:
  assumes f: f measurable_on S and g: g measurable_on S
  shows (λx. f x − g x) measurable_on S
  by (intro continuous_intros measurable_on_combine [OF assms]) auto

lemma measurable_on_scaleR:
  assumes f: f measurable_on S and g: g measurable_on S
  shows (λx. f x ∗R g x) measurable_on S
  by (intro continuous_intros measurable_on_combine [OF assms]) auto

lemma measurable_on_sum:
```

assumes *finite I* $\bigwedge i.\ i \in I \Longrightarrow f\ i$ *measurable_on S*
shows $(\lambda x.\ sum\ \ (\lambda i.\ f\ i\ x)\ I)$ *measurable_on S*
using *assms* **by** (*induction I*) (*auto simp*: *measurable_on_add*)

lemma *measurable_on_spike*:
assumes *f*: *f measurable_on T* **and** *negligible S* **and** *gf*: $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
shows *g measurable_on T*
proof −
obtain *NF* **and** *F*
where *NF*: *negligible NF*
and *conF*: $\bigwedge n.$ *continuous_on UNIV* (*F n*)
and *tendsF*: $\bigwedge x.\ x \notin NF \Longrightarrow (\lambda n.\ F\ n\ x) \longrightarrow (if\ x \in T\ then\ f\ x\ else\ 0)$
using *f* **by** (*auto simp*: *measurable_on_def*)
show *?thesis*
unfolding *measurable_on_def*
proof (*intro exI conjI allI impI*)
show *negligible* (*NF* ∪ *S*)
by (*simp add*: *NF* ‹*negligible S*›)
show $\bigwedge x.\ x \notin NF \cup S \Longrightarrow (\lambda n.\ F\ n\ x) \longrightarrow (if\ x \in T\ then\ g\ x\ else\ 0)$
by (*metis* (*full_types*) *Diff_iff Un_iff gf tendsF*)
qed (*auto simp*: *conF*)
qed

proposition *indicator_measurable_on*:
assumes *S* ∈ *sets lebesgue*
shows *indicat_real S measurable_on UNIV*
proof −
{ fix *n*::*nat*
let *?ε* = (*1*::*real*) / (*2* ∗ *2*^*n*)
have *ε*: *?ε* > *0*
by *auto*
obtain *T* **where** *closed T T* ⊆ *S S*−*T* ∈ *lmeasurable* **and** *ST*: *emeasure lebesgue* (*S* − *T*) < *?ε*
by (*meson ε assms sets_lebesgue_inner_closed*)
obtain *U* **where** *open U S* ⊆ *U* (*U* − *S*) ∈ *lmeasurable* **and** *US*: *emeasure lebesgue* (*U* − *S*) < *?ε*
by (*meson ε assms sets_lebesgue_outer_open*)
have *eq*: −*T* ∩ *U* = (*S*−*T*) ∪ (*U* − *S*)
using ‹*T* ⊆ *S*› ‹*S* ⊆ *U*› **by** *auto*
have *emeasure lebesgue* ((*S*−*T*) ∪ (*U* − *S*)) ≤ *emeasure lebesgue* (*S* − *T*) + *emeasure lebesgue* (*U* − *S*)
using ‹*S* − *T* ∈ *lmeasurable*› ‹*U* − *S* ∈ *lmeasurable*› *emeasure_subadditive* **by** *blast*
also have ... < *?ε* + *?ε*
using *ST US add_mono_ennreal* **by** *metis*
finally have *le*: *emeasure lebesgue* (−*T* ∩ *U*) < *ennreal* (*1* / *2*^*n*)
by (*simp add*: *eq*)
have *1*: *continuous_on* (*T* ∪ −*U*) (*indicat_real S*)

```
    unfolding indicator_def of_bool_def
  proof (rule continuous_on_cases [OF ‹closed T›])
    show closed (− U)
      using ‹open U› by blast
    show continuous_on T (λx. 1::real) continuous_on (− U) (λx. 0::real)
      by (auto simp: continuous_on)
    show ∀x. x ∈ T ∧ x ∉ S ∨ x ∈ − U ∧ x ∈ S ⟶ (1::real) = 0
      using ‹T ⊆ S› ‹S ⊆ U› by auto
  qed
  have 2: closedin (top_of_set UNIV) (T ∪ −U)
    using ‹closed T› ‹open U› by auto
  obtain g where continuous_on UNIV g ⋀x. x ∈ T ∪ −U ⟹ g x = indicat_real S x ⋀x. norm(g x) ≤ 1
    by (rule Tietze [OF 1 2, of 1]) auto
  with le have ∃g E. continuous_on UNIV g ∧ (∀x ∈ −E. g x = indicat_real S x) ∧
                    (∀x. norm(g x) ≤ 1) ∧ E ∈ sets lebesgue ∧ emeasure lebesgue E < ennreal (1 / 2^n)
    apply (rule_tac x=g in exI)
    apply (rule_tac x=−T ∩ U in exI)
    using ‹S − T ∈ lmeasurable› ‹U − S ∈ lmeasurable› eq by auto
 }
 then obtain g E where cont: ⋀n. continuous_on UNIV (g n)
   and geq: ⋀n x. x ∈ − E n ⟹ g n x = indicat_real S x
   and ng1: ⋀n x. norm(g n x) ≤ 1
   and Eset: ⋀n. E n ∈ sets lebesgue
   and Em: ⋀n. emeasure lebesgue (E n) < ennreal (1 / 2^n)
   by metis
 have null: limsup E ∈ null_sets lebesgue
 proof (rule borel_cantelli_limsup1 [OF Eset])
   show emeasure lebesgue (E n) < ∞ for n
     by (metis Em infinity_ennreal_def order.asym top.not_eq_extremum)
   show summable (λn. measure lebesgue (E n))
   proof (rule summable_comparison_test' [OF summable_geometric, of 1/2 0])
     show norm (measure lebesgue (E n)) ≤ (1/2) ^ n for n
      using Em [of n] by (simp add: measure_def enn2real_leI power_one_over)
   qed auto
 qed
 have tends: (λn. g n x) ⟶ indicat_real S x if x ∉ limsup E for x
 proof −
   have ∀F n in sequentially. x ∈ − E n
     using that by (simp add: mem_limsup_iff not_frequently)
   then show ?thesis
     unfolding tendsto_iff dist_real_def
     by (simp add: eventually_mono geq)
 qed
 show ?thesis
   unfolding measurable_on_def
 proof (intro exI conjI allI impI)
```

```
    show negligible (limsup E)
      using negligible_iff_null_sets null by blast
    show continuous_on UNIV (g n) for n
      using cont by blast
  qed (use tends in auto)
qed

lemma measurable_on_restrict:
  assumes f: f measurable_on UNIV and S: S ∈ sets lebesgue
  shows (λx. if x ∈ S then f x else 0) measurable_on UNIV
proof −
  have indicat_real S measurable_on UNIV
    by (simp add: S indicator_measurable_on)
  then show ?thesis
    using measurable_on_scaleR [OF _ f, of indicat_real S]
    by (simp add: indicator_scaleR_eq_if)
qed

lemma measurable_on_const_UNIV: (λx. k) measurable_on UNIV
  by (simp add: continuous_imp_measurable_on)

lemma measurable_on_const [simp]: S ∈ sets lebesgue ⟹ (λx. k) measurable_on
S
 using measurable_on_UNIV measurable_on_const_UNIV measurable_on_restrict
by blast

lemma simple_function_indicator_representation_real:
  fixes f ::'a ⇒ real
  assumes f: simple_function M f and x: x ∈ space M and nn: ⋀x. f x ≥ 0
  shows f x = (∑ y ∈ f ‘ space M. y ∗ indicator (f −‘ {y} ∩ space M) x)
proof −
  have f': simple_function M (ennreal ∘ f)
    by (simp add: f)
  have ∗: f x =
     enn2real
      (∑ y∈ ennreal ‘ f ‘ space M.
         y ∗ indicator ((ennreal ∘ f) −‘ {y} ∩ space M) x)
    using arg_cong [OF simple_function_indicator_representation [OF f' x], of
enn2real, simplified nn o_def] nn
    unfolding o_def image_comp
    by (metis enn2real_ennreal)
  have enn2real (∑ y∈ennreal ‘ f ‘ space M. if ennreal (f x) = y ∧ x ∈ space M
then y else 0)
      = sum (enn2real ∘ (λy. if ennreal (f x) = y ∧ x ∈ space M then y else 0))
          (ennreal ‘ f ‘ space M)
    by (rule enn2real_sum) auto
  also have ... = sum (enn2real ∘ (λy. if ennreal (f x) = y ∧ x ∈ space M then
y else 0) ∘ ennreal)
                  (f ‘ space M)
```

by (*rule sum.reindex*) (*use nn* **in** ‹*auto simp*: *inj_on_def intro*: *sum.cong*›)
also have ... = ($\sum$ y∈f ‘ space M. *if* f x = y ∧ x ∈ space M *then* y *else* 0)
using *nn*
by (*auto simp*: *inj_on_def intro*: *sum.cong*)
finally show *?thesis*
by (*subst* ∗) (*simp add*: *enn2real_sum indicator_def of_bool_def if_distrib cong*: *if_cong*)
qed

lemma *simple_function_induct_real*
[*consumes 1*, *case_names cong set mult add*, *induct set*: *simple_function*]:
fixes u :: 'a ⇒ *real*
assumes *u*: *simple_function M u*
assumes *cong*: ⋀f g. *simple_function M f* ⟹ *simple_function M g* ⟹ (*AE x in M*. f x = g x) ⟹ *P f* ⟹ *P g*
assumes *set*: ⋀A. A ∈ *sets M* ⟹ *P* (*indicator A*)
assumes *mult*: ⋀u c. *P u* ⟹ *P* (λx. c ∗ u x)
assumes *add*: ⋀u v. *P u* ⟹ *P v* ⟹ *P* (λx. u x + v x)
and *nn*: ⋀x. u x ≥ 0
shows *P u*
proof (*rule cong*)
from *AE_space* **show** *AE x in M*. ($\sum$ y∈u ‘ space M. y ∗ indicator (u −‘ {y} ∩ space M) x) = u x
proof *eventually_elim*
fix x **assume** *x*: x ∈ space M
from *simple_function_indicator_representation_real*[*OF u x*] *nn*
show ($\sum$ y∈u ‘ space M. y ∗ indicator (u −‘ {y} ∩ space M) x) = u x
by *metis*
qed
next
from *u* **have** *finite* (u ‘ space M)
unfolding *simple_function_def* **by** *auto*
then show *P* (λx. $\sum$ y∈u ‘ space M. y ∗ indicator (u −‘ {y} ∩ space M) x)
proof *induct*
case *empty*
then show *?case*
using *set*[*of* {}] **by** (*simp add*: *indicator_def*[*abs_def*])
next
case (*insert a F*)
have *eq*: $\sum$ {y. u x = y ∧ (y = a ∨ y ∈ F) ∧ x ∈ space M}
= (*if* u x = a ∧ x ∈ space M *then* a *else* 0) + $\sum$ {y. u x = y ∧ y ∈ F ∧ x ∈ space M} **for** x
proof (*cases* x ∈ space M)
case *True*
have ∗: {y. u x = y ∧ (y = a ∨ y ∈ F)} = {y. u x = a ∧ y = a} ∪ {y. u x = y ∧ y ∈ F}
by *auto*
show *?thesis*
using *insert* **by** (*simp add*: ∗ *True*)

```
    qed auto
    have a: P (λx. a * indicator (u -` {a} ∩ space M) x)
    proof (intro mult set)
      show u -` {a} ∩ space M ∈ sets M
        using u by auto
    qed
    show ?case
      using nn insert a
      by (simp add: eq indicator_times_eq_if [where f = λx. a] add)
  qed
next
  show simple_function M (λx. (∑ y∈u ` space M. y * indicator (u -` {y} ∩
space M) x))
    apply (subst simple_function_cong)
    apply (rule simple_function_indicator_representation_real[symmetric])
    apply (auto intro: u nn)
    done
qed fact

proposition simple_function_measurable_on_UNIV:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes f: simple_function lebesgue f and nn: ⋀x. f x ≥ 0
  shows f measurable_on UNIV
  using f
proof (induction f)
  case (cong f g)
  then obtain N where negligible N {x. g x ≠ f x} ⊆ N
    by (auto simp: eventually_ae_filter_negligible eq_commute)
  then show ?case
    by (blast intro: measurable_on_spike cong)
next
  case (set S)
  then show ?case
    by (simp add: indicator_measurable_on)
next
  case (mult u c)
  then show ?case
    by (simp add: measurable_on_cmul)
  case (add u v)
  then show ?case
    by (simp add: measurable_on_add)
qed (auto simp: nn)

lemma simple_function_lebesgue_if:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes f: simple_function lebesgue f and S: S ∈ sets lebesgue
  shows simple_function lebesgue (λx. if x ∈ S then f x else 0)
proof -
  have ffin: finite (range f) and fsets: ∀ x. f -` {f x} ∈ sets lebesgue
```

using *f* **by** (*auto simp*: *simple_function_def*)
have *finite* (*f* ‘ *S*)
by (*meson finite_subset subset_image_iff ffin top_greatest*)
moreover have *finite* ((λ*x*. *0*::*real*) ‘ *T*) **for** *T* :: *'a set*
by (*auto simp*: *image_def*)
moreover have *if_sets*: (λ*x*. *if x* ∈ *S then f x else 0*) −‘ {*f a*} ∈ *sets lebesgue*
for *a*
proof −
have ∗: (λ*x*. *if x* ∈ *S then f x else 0*) −‘ {*f a*}
= (*if f a* = *0 then* −*S* ∪ *f* −‘ {*f a*} *else* (*f* −‘ {*f a*}) ∩ *S*)
by (*auto simp*: *split*: *if_split_asm*)
show *?thesis*
unfolding ∗ **by** (*metis Compl_in_sets_lebesgue S sets.Int sets.Un fsets*)
qed
moreover have (λ*x*. *if x* ∈ *S then f x else 0*) −‘ {*0*} ∈ *sets lebesgue*
proof (*cases 0* ∈ *range f*)
case *True*
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *if_sets rangeE*)
next
case *False*
then have (λ*x*. *if x* ∈ *S then f x else 0*) −‘ {*0*} = −*S*
by *auto*
then show *?thesis*
by (*simp add*: *Compl_in_sets_lebesgue S*)
qed
ultimately show *?thesis*
by (*auto simp*: *simple_function_def*)
qed

corollary *simple_function_measurable_on*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real*
assumes *f*: *simple_function lebesgue f* **and** *nn*: ⋀*x*. *f x* ≥ *0* **and** *S*: *S* ∈ *sets lebesgue*
shows *f measurable_on S*
by (*simp add*: *measurable_on_UNIV* [*symmetric*, *of f*] *S f simple_function_lebesgue_if nn simple_function_measurable_on_UNIV*)

lemma
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*ordered_euclidean_space*
assumes *f*: *f measurable_on S* **and** *g*: *g measurable_on S*
shows *measurable_on_sup*: (λ*x*. *sup* (*f x*) (*g x*)) *measurable_on S*
and *measurable_on_inf*: (λ*x*. *inf* (*f x*) (*g x*)) *measurable_on S*
proof −
obtain *NF* **and** *F*
where *NF*: *negligible NF*
and *conF*: ⋀*n*. *continuous_on UNIV* (*F n*)
and *tendsF*: ⋀*x*. *x* ∉ *NF* ⟹ (λ*n*. *F n x*) ⟶ (*if x* ∈ *S then f x else 0*)
using *f* **by** (*auto simp*: *measurable_on_def*)

```
  obtain NG and G
    where NG: negligible NG
      and conG: ⋀n. continuous_on UNIV (G n)
      and tendsG: ⋀x. x ∉ NG ⟹ (λn. G n x) ⟶ (if x ∈ S then g x else 0)
    using g by (auto simp: measurable_on_def)
  show (λx. sup (f x) (g x)) measurable_on S
    unfolding measurable_on_def
  proof (intro exI conjI allI impI)
    show continuous_on UNIV (λx. sup (F n x) (G n x)) for n
      unfolding sup_max eucl_sup  by (intro conF conG continuous_intros)
    show (λn. sup (F n x) (G n x)) ⟶ (if x ∈ S then sup (f x) (g x) else 0)
      if x ∉ NF ∪ NG for x
      using tendsto_sup [OF tendsF tendsG, of x x] that by auto
  qed (simp add: NF NG)
  show (λx. inf (f x) (g x)) measurable_on S
    unfolding measurable_on_def
  proof (intro exI conjI allI impI)
    show continuous_on UNIV (λx. inf (F n x) (G n x)) for n
      unfolding inf_min eucl_inf  by (intro conF conG continuous_intros)
    show (λn. inf (F n x) (G n x)) ⟶ (if x ∈ S then inf (f x) (g x) else 0)
      if x ∉ NF ∪ NG for x
      using tendsto_inf [OF tendsF tendsG, of x x] that by auto
  qed (simp add: NF NG)
qed

proposition measurable_on_componentwise_UNIV:
 f measurable_on UNIV ⟷ (∀ i∈Basis. (λx. (f x ⋅ i) *R i) measurable_on UNIV)
 (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  show ?rhs
  proof
    fix i::'b
    assume i ∈ Basis
    have cont: continuous_on UNIV (λx. (x ⋅ i) *R i)
      by (intro continuous_intros)
    show (λx. (f x ⋅ i) *R i) measurable_on UNIV
      using measurable_on_compose_continuous [OF L cont]
      by (simp add: o_def)
  qed
next
  assume ?rhs
  then have ∃N g. negligible N ∧
            (∀n. continuous_on UNIV (g n)) ∧
            (∀x. x ∉ N ⟶ (λn. g n x) ⟶ (f x ⋅ i) *R i)
    if i ∈ Basis for i
    by (simp add: measurable_on_def that)
  then obtain N g where N: ⋀i. i ∈ Basis ⟹ negligible (N i)
       and cont: ⋀i n. i ∈ Basis ⟹ continuous_on UNIV (g i n)
```

```
      and tends: ⋀i x. ⟦i ∈ Basis; x ∉ N i⟧ ⟹ (λn. g i n x) ⟶ (f x · i) *R i
    by metis
  show ?lhs
    unfolding measurable_on_def
  proof (intro exI conjI allI impI)
    show negligible (⋃ i ∈ Basis. N i)
      using N eucl.finite_Basis by blast
    show continuous_on UNIV (λx. (∑ i∈Basis. g i n x)) for n
      by (intro continuous_intros cont)
  next
    fix x
    assume x ∉ (⋃ i ∈ Basis. N i)
    then have ⋀i. i ∈ Basis ⟹ x ∉ N i
      by auto
    then have (λn. (∑ i∈Basis. g i n x)) ⟶ (∑ i∈Basis. (f x · i) *R i)
      by (intro tends tendsto_intros)
    then show (λn. (∑ i∈Basis. g i n x)) ⟶ (if x ∈ UNIV then f x else 0)
      by (simp add: euclidean_representation)
  qed
qed

corollary measurable_on_componentwise:
  f measurable_on S ⟷ (∀ i∈Basis. (λx. (f x · i) *R i) measurable_on S)
  apply (subst measurable_on_UNIV [symmetric])
  apply (subst measurable_on_componentwise_UNIV)
  apply (simp add: measurable_on_UNIV if_distrib [of λx. inner x _] if_distrib
[of λx. scaleR x _] cong: if_cong)
  done



lemma borel_measurable_implies_simple_function_sequence_real:
  fixes u :: 'a ⇒ real
  assumes u[measurable]: u ∈ borel_measurable M and nn: ⋀x. u x ≥ 0
  shows ∃f. incseq f ∧ (∀ i. simple_function M (f i)) ∧ (∀ x. bdd_above (range
(λi. f i x))) ∧
           (∀ i x. 0 ≤ f i x) ∧ u = (SUP i. f i)
proof −
  define f where [abs_def]:
    f i x = real_of_int (floor ((min i (u x)) * 2^i)) / 2^i for i x

  have [simp]: 0 ≤ f i x for i x
    by (auto simp: f_def intro!: divide_nonneg_nonneg mult_nonneg_nonneg nn)

  have *: 2^n * real_of_int x = real_of_int (2^n * x) for n x
    by simp

  have real_of_int ⌊real i * 2 ^ i⌋ = real_of_int ⌊i * 2 ^ i⌋ for i
    by (intro arg_cong[where f=real_of_int]) simp
```

```
  then have [simp]: real_of_int ⌊real i * 2 ^ i⌋ = i * 2 ^ i for i
    unfolding floor_of_nat by simp

  have bdd: bdd_above (range (λi. f i x)) for x
   by (rule bdd_aboveI [where M = u x]) (auto simp: f_def field_simps min_def)

  have incseq f
  proof (intro monoI le_funI)
    fix m n :: nat and x assume m ≤ n
    moreover
    { fix d :: nat
      have ⌊2^d::real⌋ * ⌊2^m * (min (of_nat m) (u x))⌋ ≤ ⌊2^d * (2^m * (min
(of_nat m) (u x)))⌋
        by (rule le_mult_floor) (auto simp: nn)
      also have ... ≤ ⌊2^d * (2^m *  (min (of_nat d + of_nat m) (u x)))⌋
        by (intro floor_mono mult_mono min.mono)
           (auto simp: nn min_less_iff_disj of_nat_less_top)
      finally have f m x ≤ f(m + d) x
        unfolding f_def
        by (auto simp: field_simps power_add * simp del: of_int_mult) }
    ultimately show f m x ≤ f n x
      by (auto simp: le_iff_add)
  qed
  then have inc_f: incseq (λi. f i x) for x
    by (auto simp: incseq_def le_fun_def)
  moreover
  have simple_function M (f i) for i
  proof (rule simple_function_borel_measurable)
    have ⌊(min (of_nat i) (u x)) * 2 ^ i⌋ ≤ ⌊int i * 2 ^ i⌋ for x
      by (auto split: split_min intro!: floor_mono)
    then have f i ‘ space M ⊆ (λn. real_of_int n / 2^i) ‘ {0 .. of_nat i * 2^i}
      unfolding floor_of_int by (auto simp: f_def nn intro!: imageI)
    then show finite (f i ‘ space M)
      by (rule finite_subset) auto
    show f i ∈ borel_measurable M
      unfolding f_def enn2real_def by measurable
  qed
  moreover
  { fix x
    have (SUP i. (f i x)) = u x
    proof −
      obtain n where u x ≤ of_nat n using real_arch_simple by auto
      then have min_eq_r: ∀F i in sequentially. min (real i) (u x) = u x
        by (auto simp: eventually_sequentially intro!: exI[of _ n] split: split_min)
      have (λi. real_of_int ⌊min (real i) (u x) * 2^i⌋ / 2^i) ⟶ u x
      proof (rule tendsto_sandwich)
        show (λn. u x − (1/2)^n) ⟶ u x
          by (auto intro!: tendsto_eq_intros LIMSEQ_power_zero)
        show ∀F n in sequentially. real_of_int ⌊min (real n) (u x) * 2 ^ n⌋ / 2 ^
```

```
n ≤ u x
        using min_eq_r by eventually_elim (auto simp: field_simps)
      have *: u x * (2 ^ n * 2 ^ n) ≤ 2^n + 2^n * real_of_int ⌊u x * 2 ^ n⌋ for
n
      using real_of_int_floor_ge_diff_one[of u x * 2^n, THEN mult_left_mono,
of 2^n]
        by (auto simp: field_simps)
      show ∀F n in sequentially. u x − (1/2)^n ≤ real_of_int ⌊min (real n) (u
x) * 2 ^ n⌋ / 2 ^ n
        using min_eq_r by eventually_elim (insert *, auto simp: field_simps)
    qed auto
    then have (λi. (f i x)) ⟶ u x
      by (simp add: f_def)
    from LIMSEQ_unique LIMSEQ_incseq_SUP [OF bdd inc_f] this
    show ?thesis
      by blast
  qed }
 ultimately show ?thesis
  by (intro exI [of _ λi x. f i x]) (auto simp: ‹incseq f› bdd image_comp)
qed


lemma homeomorphic_open_interval_UNIV:
 fixes a b:: real
 assumes a < b
 shows {a<..<b} homeomorphic (UNIV::real set)
proof −
 have {a<..<b} = ball ((b+a) / 2) ((b−a) / 2)
  using assms
  by (auto simp: dist_real_def abs_if field_split_simps split: if_split_asm)
 then show ?thesis
  by (simp add: homeomorphic_ball_UNIV assms)
qed

proposition homeomorphic_box_UNIV:
 fixes a b:: 'a::euclidean_space
 assumes box a b ≠ {}
 shows box a b homeomorphic (UNIV::'a set)
proof −
 have {a · i <..<b · i} homeomorphic (UNIV::real set) if i ∈ Basis for i
  using assms box_ne_empty that by (blast intro: homeomorphic_open_interval_UNIV)
 then have ∃f g. (∀x. a · i < x ∧ x < b · i ⟶ g (f x) = x) ∧
                 (∀y. a · i < g y ∧ g y < b · i ∧ f(g y) = y) ∧
                 continuous_on {a · i<..<b · i} f ∧
                 continuous_on (UNIV::real set) g
   if i ∈ Basis for i
   using that by (auto simp: homeomorphic_minimal mem_box Ball_def)
 then obtain f g where gf: ⋀i x. ⟦i ∈ Basis; a · i < x; x < b · i⟧ ⟹ g i (f i
x) = x
```

```
            and fg: ⋀i y. i ∈ Basis ⟹ a • i < g i y ∧ g i y < b • i ∧ f i (g i y)
= y
            and contf: ⋀i. i ∈ Basis ⟹ continuous_on {a • i<..<b • i} (f i)
            and contg: ⋀i. i ∈ Basis ⟹ continuous_on (UNIV::real set) (g i)
    by metis
  define F where F ≡ λx. ∑ i∈Basis. (f i (x • i)) *R i
  define G where G ≡ λx. ∑ i∈Basis. (g i (x • i)) *R i
  show ?thesis
    unfolding homeomorphic_minimal
  proof (intro exI conjI ballI)
    show G y ∈ box a b for y
      using fg by (simp add: G_def mem_box)
    show G (F x) = x if x ∈ box a b for x
     using that by (simp add: F_def G_def gf mem_box euclidean_representation)
    show F (G y) = y for y
      by (simp add: F_def G_def fg mem_box euclidean_representation)
    show continuous_on (box a b) F
      unfolding F_def
     proof (intro continuous_intros continuous_on_compose2 [OF contf continu-
ous_on_inner])
      show (λx. x • i) ‘ box a b ⊆ {a • i<..<b • i} if i ∈ Basis for i
        using that by (auto simp: mem_box)
    qed
    show continuous_on UNIV G
      unfolding G_def
       by (intro continuous_intros continuous_on_compose2 [OF contg continu-
ous_on_inner]) auto
  qed auto
qed



lemma diff_null_sets_lebesgue: ⟦N ∈ null_sets (lebesgue_on S); X−N ∈ sets
(lebesgue_on S); N ⊆ X⟧
    ⟹ X ∈ sets (lebesgue_on S)
  by (metis Int_Diff_Un inf.commute inf.orderE null_setsD2 sets.Un)

lemma borel_measurable_diff_null:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes N: N ∈ null_sets (lebesgue_on S) and S: S ∈ sets lebesgue
  shows f ∈ borel_measurable (lebesgue_on (S−N)) ⟷ f ∈ borel_measurable
(lebesgue_on S)
  unfolding in_borel_measurable space_lebesgue_on sets_restrict_UNIV
proof (intro ball_cong iffI)
  show f −‘ T ∩ S ∈ sets (lebesgue_on S)
    if f −‘ T ∩ (S−N) ∈ sets (lebesgue_on (S−N)) for T
  proof −
    have N ∩ S = N
      by (metis N S inf.orderE null_sets_restrict_space)
```

```
    moreover have N ∩ S ∈ sets lebesgue
      by (metis N S inf.orderE null_setsD2 null_sets_restrict_space)
    moreover have f −‘ T ∩ S ∩ (f −‘ T ∩ N) ∈ sets lebesgue
    by (metis N S completion.complete inf.absorb2 inf_le2 inf_mono null_sets_restrict_space)
    ultimately show ?thesis
    by (metis Diff_Int_distrib Int_Diff_Un S inf_le2 sets.Diff sets.Un sets_restrict_space_iff
space_lebesgue_on space_restrict_space that)
  qed
  show f −‘ T ∩ (S−N) ∈ sets (lebesgue_on (S−N))
    if f −‘ T ∩ S ∈ sets (lebesgue_on S) for T
  proof −
    have (S − N) ∩ f −‘ T = (S − N) ∩ (f −‘ T ∩ S)
      by blast
    then have (S − N) ∩ f −‘ T ∈ sets.restricted_space lebesgue (S − N)
      by (metis S image_iff sets.Int_space_eq2 sets_restrict_space_iff that)
    then show ?thesis
      by (simp add: inf.commute sets_restrict_space)
  qed
qed auto

lemma lebesgue_measurable_diff_null:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes N ∈ null_sets lebesgue
   shows f ∈ borel_measurable (lebesgue_on (−N)) ⟷ f ∈ borel_measurable
lebesgue
 by (simp add: Compl_eq_Diff_UNIV assms borel_measurable_diff_null lebesgue_on_UNIV_eq)



proposition measurable_on_imp_borel_measurable_lebesgue_UNIV:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f measurable_on UNIV
  shows f ∈ borel_measurable lebesgue
proof −
  obtain N and F
    where NF: negligible N
      and conF: ⋀n. continuous_on UNIV (F n)
      and tendsF: ⋀x. x ∉ N ⟹ (λn. F n x) ⟶ f x
    using assms by (auto simp: measurable_on_def)
  obtain N where N ∈ null_sets lebesgue f ∈ borel_measurable (lebesgue_on
(−N))
  proof
    show f ∈ borel_measurable (lebesgue_on (− N))
    proof (rule borel_measurable_LIMSEQ_metric)
      show F i ∈ borel_measurable (lebesgue_on (− N)) for i
     by (meson Compl_in_sets_lebesgue NF conF continuous_imp_measurable_on_sets_lebesgue
continuous_on_subset negligible_imp_sets subset_UNIV)
      show (λi. F i x) ⟶ f x if x ∈ space (lebesgue_on (− N)) for x
        using that
```

```
      by (simp add: tendsF)
    qed
    show N ∈ null_sets lebesgue
      using NF negligible_iff_null_sets by blast
  qed
  then show ?thesis
    using lebesgue_measurable_diff_null by blast
qed

corollary measurable_on_imp_borel_measurable_lebesgue:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f measurable_on S and S: S ∈ sets lebesgue
  shows f ∈ borel_measurable (lebesgue_on S)
proof −
  have (λx. if x ∈ S then f x else 0) measurable_on UNIV
    using assms(1) measurable_on_UNIV by blast
  then show ?thesis
   by (simp add: borel_measurable_if_D measurable_on_imp_borel_measurable_lebesgue_UNIV)
qed


proposition measurable_on_limit:
  fixes f :: nat ⇒ 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes f: ⋀n. f n measurable_on S and N: negligible N
    and lim: ⋀x. x ∈ S − N ⟹ (λn. f n x) ⟶ g x
  shows g measurable_on S
proof −
  have box (0::'b) One homeomorphic (UNIV::'b set)
    by (simp add: homeomorphic_box_UNIV)
  then obtain h h'::'b⇒'b where hh': ⋀x. x ∈ box 0 One ⟹ h (h' x) = x
                and h'im:  h' ‘ box 0 One = UNIV
                and conth: continuous_on UNIV h
                and conth': continuous_on (box 0 One) h'
                and h'h:   ⋀y. h' (h y) = y
                and rangeh: range h = box 0 One
    by (auto simp: homeomorphic_def homeomorphism_def)
  have norm y ≤ DIM('b) if y: y ∈ box 0 One for y::'b
  proof −
    have y01: 0 < y • i y • i < 1 if i ∈ Basis for i
      using that y by (auto simp: mem_box)
    have norm y ≤ (∑ i∈Basis. |y • i|)
      using norm_le_l1 by blast
    also have ... ≤ (∑ i::'b∈Basis. 1)
    proof (rule sum_mono)
      show |y • i| ≤ 1 if i ∈ Basis for i
        using y01 that by fastforce
    qed
    also have ... ≤ DIM('b)
      by auto
```

```
    finally show ?thesis .
  qed
  then have norm_le: norm(h y) ≤ DIM('b) for y
    by (metis UNIV_I image_eqI rangeh)
  have (h' ∘ (h ∘ (λx. if x ∈ S then g x else 0))) measurable_on UNIV
  proof (rule measurable_on_compose_continuous_box)
    let ?χ =  h ∘ (λx. if x ∈ S then g x else 0)
    let ?f = λn. h ∘ (λx. if x ∈ S then f n x else 0)
    show ?χ measurable_on UNIV
    proof (rule integrable_subintervals_imp_measurable)
      show ?χ integrable_on cbox a b for a b
      proof (rule integrable_spike_set)
        show ?χ integrable_on (cbox a b − N)
        proof (rule dominated_convergence_integrable)
          show const: (λx. DIM('b)) integrable_on cbox a b − N
            by (simp add: N has_integral_iff integrable_const integrable_negligible
integrable_setdiff negligible_diff)
          show norm ((h ∘ (λx. if x ∈ S then g x else 0)) x) ≤ DIM('b) if x ∈ cbox
a b − N for x
            using that norm_le  by (simp add: o_def)
          show (λk. ?f k x) ⟶ ?χ x if x ∈ cbox a b − N for x
            using that lim [of x] conth
            by (auto simp: continuous_on_def intro: tendsto_compose)
          show (?f n) absolutely_integrable_on cbox a b − N for n
        proof (rule measurable_bounded_by_integrable_imp_absolutely_integrable)
          show ?f n ∈ borel_measurable (lebesgue_on (cbox a b − N))
            proof (rule measurable_on_imp_borel_measurable_lebesgue [OF mea-
surable_on_spike_set])
              show ?f n measurable_on cbox a b
                unfolding measurable_on_UNIV [symmetric, of _ cbox a b]
              proof (rule measurable_on_restrict)
                have f': (λx. if x ∈ S then f n x else 0) measurable_on UNIV
                  by (simp add: f measurable_on_UNIV)
                show ?f n measurable_on UNIV
                  using measurable_on_compose_continuous [OF f' conth] by auto
              qed auto
              show negligible (sym_diff (cbox a b) (cbox a b − N))
                by (auto intro: negligible_subset [OF N])
              show cbox a b − N ∈ sets lebesgue
                by (simp add: N negligible_imp_sets sets.Diff)
            qed
            show cbox a b − N ∈ sets lebesgue
              by (simp add: N negligible_imp_sets sets.Diff)
            show norm (?f n x) ≤ DIM('b)
              if x ∈ cbox a b − N for x
              using that local.norm_le by simp
          qed (auto simp: const)
        qed
        show negligible {x ∈ cbox a b − N − cbox a b. ?χ x ≠ 0}
```

```
        by (auto simp: empty_imp_negligible)
      have {x ∈ cbox a b − (cbox a b − N). ?χ x ≠ 0} ⊆ N
        by auto
      then show negligible {x ∈ cbox a b − (cbox a b − N). ?χ x ≠ 0}
        using N negligible_subset by blast
    qed
  qed
  show ?χ x ∈ box 0 One for x
    using rangeh by auto
  show continuous_on (box 0 One) h'
    by (rule conth')
 qed
 then show ?thesis
  by (simp add: o_def h'h measurable_on_UNIV)
qed


lemma measurable_on_if_simple_function_limit:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 shows ⟦⋀n. g n measurable_on UNIV; ⋀n. finite (range (g n)); ⋀x. (λn. g n
x) ⟶ f x⟧
  ⟹ f measurable_on UNIV
 by (force intro: measurable_on_limit [where N={}])


lemma lebesgue_measurable_imp_measurable_on_nnreal_UNIV:
 fixes u :: 'a::euclidean_space ⇒ real
 assumes u: u ∈ borel_measurable lebesgue and nn: ⋀x. u x ≥ 0
 shows u measurable_on UNIV
proof −
 obtain f where incseq f and f: ∀ i. simple_function lebesgue (f i)
  and bdd: ⋀x. bdd_above (range (λi. f i x))
  and nnf: ⋀i x. 0 ≤ f i x and ∗: u = (SUP i. f i)
 using borel_measurable_implies_simple_function_sequence_real nn u by metis
 show ?thesis
  unfolding ∗
 proof (rule measurable_on_if_simple_function_limit [of concl: Sup (range f)])
  show (f i) measurable_on UNIV for i
   by (simp add: f nnf simple_function_measurable_on_UNIV)
  show finite (range (f i)) for i
   by (metis f simple_function_def space_borel space_completion space_lborel)
  show (λi. f i x) ⟶ Sup (range f) x for x
  proof −
   have incseq (λi. f i x)
    using ‹incseq f› apply (auto simp: incseq_def)
    by (simp add: le_funD)
   then show ?thesis
    by (metis SUP_apply bdd LIMSEQ_incseq_SUP)
  qed
```

qed
qed

lemma *lebesgue_measurable_imp_measurable_on_nnreal*:
fixes $u :: 'a::euclidean\_space \Rightarrow real$
assumes $u \in borel\_measurable\ lebesgue\ \bigwedge x.\ u\ x \geq 0 S \in sets\ lebesgue$
shows $u\ measurable\_on\ S$
unfolding *measurable_on_UNIV* [*symmetric*, *of u*]
using *assms*
by (*auto intro*: *lebesgue_measurable_imp_measurable_on_nnreal_UNIV*)

lemma *lebesgue_measurable_imp_measurable_on_real*:
fixes $u :: 'a::euclidean\_space \Rightarrow real$
assumes *u*: $u \in borel\_measurable\ lebesgue$ **and** *S*: $S \in sets\ lebesgue$
shows $u\ measurable\_on\ S$
proof −
let *?f* = $\lambda x.\ |u\ x| + u\ x$
let *?g* = $\lambda x.\ |u\ x| - u\ x$
have *?f measurable_on S ?g measurable_on S*
using *S u* **by** (*auto intro*: *lebesgue_measurable_imp_measurable_on_nnreal*)
then have $(\lambda x.\ (?f\ x - ?g\ x)\ /\ 2)\ measurable\_on\ S$
using *measurable_on_cdivide measurable_on_diff* **by** *blast*
then show *?thesis*
by *auto*
qed

proposition *lebesgue_measurable_imp_measurable_on*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *f*: $f \in borel\_measurable\ lebesgue$ **and** *S*: $S \in sets\ lebesgue$
shows $f\ measurable\_on\ S$
unfolding *measurable_on_componentwise* [*of f*]
proof
fix $i::'b$
assume $i \in Basis$
have $(\lambda x.\ (f\ x \cdot i)) \in borel\_measurable\ lebesgue$
using $\langle i \in Basis \rangle$ *borel_measurable_euclidean_space f* **by** *blast*
then have $(\lambda x.\ (f\ x \cdot i))\ measurable\_on\ S$
using *S lebesgue_measurable_imp_measurable_on_real* **by** *blast*
then show $(\lambda x.\ (f\ x \cdot i) *_R i)\ measurable\_on\ S$
by (*intro measurable_on_scaleR measurable_on_const S*)
qed

proposition *measurable_on_iff_borel_measurable*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $S \in sets\ lebesgue$
shows $f\ measurable\_on\ S \longleftrightarrow f \in borel\_measurable\ (lebesgue\_on\ S)$ (**is** *?lhs* = *?rhs*)
proof

show $f \in$ *borel_measurable* (*lebesgue_on S*)
if *f measurable_on S*
using *that* **by** (*simp add*: *assms measurable_on_imp_borel_measurable_lebesgue*)
next
assume $f \in$ *borel_measurable* (*lebesgue_on S*)
then have ($\lambda a.$ *if* $a \in S$ *then f a else 0*) *measurable_on UNIV*
by (*simp add*: *assms borel_measurable_if lebesgue_measurable_imp_measurable_on*)
then show *f measurable_on S*
using *measurable_on_UNIV* **by** *blast*
qed

9.16.5 Monotonic functions are Lebesgue integrable

lemma *integrable_mono_on_nonneg*:
fixes $f :: real \Rightarrow real$
assumes *mon*: *mono_on* $\{a..b\}$ *f* **and** *0*: $\bigwedge x.\ 0 \le f\ x$
shows *integrable* (*lebesgue_on* $\{a..b\}$) *f*
proof −
have *space lborel* = *space lebesgue sets borel* ⊆ *sets lebesgue*
by *force+*
then have *fborel*: $f \in$ *borel_measurable* (*lebesgue_on* $\{a..b\}$)
by (*metis mon borel_measurable_mono_on_fnc borel_measurable_subalgebra mono_restrict_space space_lborel space_restrict_space*)
then obtain *g* **where** *g*: *incseq g* **and** *simple*: $\bigwedge i.$ *simple_function* (*lebesgue_on* $\{a..b\}$) (*g i*)
and *bdd*: ($\forall x.$ *bdd_above* (*range* ($\lambda i.$ *g i x*))) **and** *nonneg*: $\forall\, i\ x.\ 0 \le g\ i\ x$
and *fsup*: $f = (SUP\ i.\ g\ i)$
by (*metis borel_measurable_implies_simple_function_sequence_real 0*)
have $f\ `\ \{a..b\} \subseteq \{f\ a..f\ b\}$
using *assms* **by** (*auto simp*: *mono_on_def*)
have *g_le_f*: $g\ i\ x \le f\ x$ **for** *i x*
proof −
have *bdd_above* ($(\lambda h.\ h\ x)$ ` *range g*)
using *bdd cSUP_lessD linorder_not_less* **by** *fastforce*
then show *?thesis*
by (*metis SUP_apply UNIV_I bdd cSUP_upper fsup*)
qed
then have *gfb*: $g\ i\ x \le f\ b$ **if** $x \in \{a..b\}$ **for** *i x*
by (*smt* (*verit*, *best*) *mon atLeastAtMost_iff mono_on_def that*)
have *g_le*: $g\ i\ x \le g\ j\ x$ **if** $i \le j$ **for** *i j x*
using *g* **by** (*simp add*: *incseq_def le_funD that*)
show *integrable* (*lebesgue_on* $\{a..b\}$) (*f*)
proof (*rule integrable_dominated_convergence*)
show $f \in$ *borel_measurable* (*lebesgue_on* $\{a..b\}$)
using *fborel* **by** *blast*
have $\bigwedge x.\ (\lambda i.\ g\ i\ x) \longrightarrow (SUP\ h \in range\ g.\ h\ \ x)$
proof (*rule order_tendstoI*)
show $\forall_F\ i$ *in sequentially*. $y < g\ i\ x$

```
      if y < (SUP h∈range g. h x) for x y
    proof −
      from that obtain h where h: h ∈ range g y < h x
        using g_le_f by (subst (asm)less_cSUP_iff) fastforce+
      then show ?thesis
        by (smt (verit, ccfv_SIG) eventually_sequentially g_le imageE)
    qed
    show ∀F i in sequentially. g i x < y
      if (SUP h∈range g. h x) < y for x y
      by (smt (verit, best) that Sup_apply g_le_f always_eventually fsup im-
age_cong)
  qed
  then show AE x in lebesgue_on {a..b}. (λi. g i x) ⟶ f x
    by (simp add: fsup)
  fix i
  show g i ∈ borel_measurable (lebesgue_on {a..b})
    using borel_measurable_simple_function simple by blast
  show AE x in lebesgue_on {a..b}. norm (g i x) ≤ f b
    by (simp add: gfb nonneg Measure_Space.AE_I' [of {}])
 qed auto
qed

lemma integrable_mono_on:
 fixes f :: real ⇒ real
 assumes mono_on {a..b} f
 shows integrable (lebesgue_on {a..b}) f
proof −
 define f' where f' ≡ λx. if x ∈ {a..b} then f x − f a else 0
 have mono_on {a..b} f'
   by (smt (verit, best) assms f'_def mono_on_def)
 moreover have 0: ⋀x. 0 ≤ f' x
   by (smt (verit, best) assms atLeastAtMost_iff f'_def mono_on_def)
 ultimately have integrable (lebesgue_on {a..b}) f'
   using integrable_mono_on_nonneg by presburger
 then have integrable (lebesgue_on {a..b}) (λx. f' x + f a)
   by force
 moreover have space lborel = space lebesgue sets borel ⊆ sets lebesgue
   by force+
 then have fborel: f ∈ borel_measurable (lebesgue_on {a..b})
  by (metis assms borel_measurable_mono_on_fnc borel_measurable_subalgebra
mono_restrict_space space_lborel space_restrict_space)
 ultimately show ?thesis
   by (rule integrable_cong_AE_imp) (auto simp add: f'_def)
qed

lemma integrable_on_mono_on:
 fixes f :: real ⇒ real
 assumes mono_on {a..b} f
 shows f integrable_on {a..b}
```

by (*simp add*: *assms integrable_mono_on integrable_on_lebesgue_on*)

9.16.6 Measurability on generalisations of the binary product

lemma *measurable_on_bilinear*:
fixes h :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*euclidean_space*
assumes *h*: *bilinear h* **and** *f*: *f measurable_on S* **and** *g*: *g measurable_on S*
shows (λ*x*. *h* (*f x*) (*g x*)) *measurable_on S*
proof (*rule measurable_on_combine* [**where** *h* = *h*])
show *continuous_on UNIV* (λ*x*. *h* (*fst x*) (*snd x*))
by (*simp add*: *bilinear_continuous_on_compose* [*OF continuous_on_fst continuous_on_snd h*])
show *h 0 0* = *0*
by (*simp add*: *bilinear_lzero h*)
qed (*auto intro*: *assms*)

lemma *borel_measurable_bilinear*:
fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*euclidean_space*
assumes *bilinear h f* ∈ *borel_measurable* (*lebesgue_on S*) *g* ∈ *borel_measurable* (*lebesgue_on S*)
and *S*: *S* ∈ *sets lebesgue*
shows (λ*x*. *h* (*f x*) (*g x*)) ∈ *borel_measurable* (*lebesgue_on S*)
using *assms measurable_on_bilinear* [*of h f S g*]
by (*simp flip*: *measurable_on_iff_borel_measurable*)

lemma *absolutely_integrable_bounded_measurable_product*:
fixes *h* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*euclidean_space*
assumes *bilinear h* **and** *f*: *f* ∈ *borel_measurable* (*lebesgue_on S*) *S* ∈ *sets lebesgue*
and *bou*: *bounded* (*f* ‘ *S*) **and** *g*: *g absolutely_integrable_on S*
shows (λ*x*. *h* (*f x*) (*g x*)) *absolutely_integrable_on S*
proof −
obtain *B* **where** *B* > *0* **and** *B*: ⋀*x y*. *norm* (*h x y*) ≤ *B* ∗ *norm x* ∗ *norm y*
using *bilinear_bounded_pos* ‹*bilinear h*› **by** *blast*
obtain *C* **where** *C* > *0* **and** *C*: ⋀*x*. *x* ∈ *S* ⟹ *norm* (*f x*) ≤ *C*
using *bounded_pos* **by** (*metis bou imageI*)
show *?thesis*
proof (*rule measurable_bounded_by_integrable_imp_absolutely_integrable* [*OF* _ ‹*S* ∈ *sets lebesgue*›])
show *norm* (*h* (*f x*) (*g x*)) ≤ *B* ∗ *C* ∗ *norm*(*g x*) **if** *x* ∈ *S* **for** *x*
by (*meson less_le mult_left_mono mult_right_mono norm_ge_zero order_trans that* ‹*B* > *0*› *B C*)
show (λ*x*. *h* (*f x*) (*g x*)) ∈ *borel_measurable* (*lebesgue_on S*)
using ‹*bilinear h*› *f g*
by (*blast intro*: *borel_measurable_bilinear dest*: *absolutely_integrable_measurable*)
show (λ*x*. *B* ∗ *C* ∗ *norm*(*g x*)) *integrable_on S*
using ‹*0* < *B*› ‹*0* < *C*› *absolutely_integrable_on_def g* **by** *auto*
qed

qed

lemma *absolutely_integrable_bounded_measurable_product_real*:
fixes $f :: real \Rightarrow real$
assumes $f \in borel\_measurable\ (lebesgue\_on\ S)$ $S \in sets\ lebesgue$
and *bounded* $(f\ `\ S)$ **and** g *absolutely_integrable_on* S
shows $(\lambda x.\ f\ x * g\ x)$ *absolutely_integrable_on* S
using *absolutely_integrable_bounded_measurable_product bilinear_times assms*
by *blast*

lemma *borel_measurable_AE*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $f \in borel\_measurable\ lebesgue$ **and** *ae*: $AE\ x\ in\ lebesgue.\ f\ x = g\ x$
shows $g \in borel\_measurable\ lebesgue$
proof −
obtain N **where** N: $N \in null\_sets\ lebesgue$ $\bigwedge x.\ x \notin N \Longrightarrow f\ x = g\ x$
using *ae* **unfolding** *completion.AE_iff_null_sets* **by** *auto*
have $f\ measurable\_on\ UNIV$
by (*simp add*: *assms lebesgue_measurable_imp_measurable_on*)
then have $g\ measurable\_on\ UNIV$
by (*metis Diff_iff N measurable_on_spike negligible_iff_null_sets*)
then show *?thesis*
using *measurable_on_imp_borel_measurable_lebesgue_UNIV* **by** *blast*
qed

lemma *has_bochner_integral_combine*:
fixes $f :: real \Rightarrow 'a::euclidean\_space$
assumes $a \le c$ $c \le b$
and *ac*: $has\_bochner\_integral\ (lebesgue\_on\ \{a..c\})\ f\ i$
and *cb*: $has\_bochner\_integral\ (lebesgue\_on\ \{c..b\})\ f\ j$
shows $has\_bochner\_integral\ (lebesgue\_on\ \{a..b\})\ f(i + j)$
proof −
have *i*: $has\_bochner\_integral\ lebesgue\ (\lambda x.\ indicator\ \{a..c\}\ x *_R f\ x)\ i$
and *j*: $has\_bochner\_integral\ lebesgue\ (\lambda x.\ indicator\ \{c..b\}\ x *_R f\ x)\ j$
using *assms* **by** (*auto simp*: *has_bochner_integral_restrict_space*)
have *AE*: $AE\ x\ in\ lebesgue.\ indicat\_real\ \{a..c\}\ x *_R f\ x + indicat\_real\ \{c..b\}\ x *_R f\ x = indicat\_real\ \{a..b\}\ x *_R f\ x$
proof (*rule AE_I'*)
have *eq*: $indicat\_real\ \{a..c\}\ x *_R f\ x + indicat\_real\ \{c..b\}\ x *_R f\ x = indicat\_real\ \{a..b\}\ x *_R f\ x$ **if** $x \ne c$ **for** x
using *assms that* **by** (*auto simp*: *indicator_def*)
then show $\{x \in space\ lebesgue.\ indicat\_real\ \{a..c\}\ x *_R f\ x + indicat\_real\ \{c..b\}\ x *_R f\ x \ne indicat\_real\ \{a..b\}\ x *_R f\ x\} \subseteq \{c\}$
by *auto*
qed *auto*
have $has\_bochner\_integral\ lebesgue\ (\lambda x.\ indicator\ \{a..b\}\ x *_R f\ x)\ (i + j)$
proof (*rule has_bochner_integralI_AE* [*OF has_bochner_integral_add* [*OF i j*] _ *AE*])

```
    have eq: indicat_real {a..c} x *R f x + indicat_real {c..b} x *R f x = indicat_real {a..b} x *R f x if x ≠ c for x
      using assms that by (auto simp: indicator_def)
    show (λx. indicat_real {a..b} x *R f x) ∈ borel_measurable lebesgue
    proof (rule borel_measurable_AE [OF borel_measurable_add AE])
      show (λx. indicator {a..c} x *R f x) ∈ borel_measurable lebesgue
           (λx. indicator {c..b} x *R f x) ∈ borel_measurable lebesgue
        using i j by auto
    qed
  qed
  then show ?thesis
    by (simp add: has_bochner_integral_restrict_space)
qed

lemma integrable_combine:
  fixes f :: real ⇒ 'a::euclidean_space
  assumes integrable (lebesgue_on {a..c}) f integrable (lebesgue_on {c..b}) f
    and a ≤ c c ≤ b
  shows integrable (lebesgue_on {a..b}) f
  using assms has_bochner_integral_combine has_bochner_integral_iff by blast

lemma integral_combine:
  fixes f :: real ⇒ 'a::euclidean_space
  assumes f: integrable (lebesgue_on {a..b}) f and a ≤ c c ≤ b
  shows integral^L (lebesgue_on {a..b}) f = integral^L (lebesgue_on {a..c}) f +
integral^L (lebesgue_on {c..b}) f
proof −
  have i: has_bochner_integral (lebesgue_on {a..c}) f(integral^L (lebesgue_on {a..c})
f)
    using integrable_subinterval ‹c ≤ b› f has_bochner_integral_iff by fastforce
  have j: has_bochner_integral (lebesgue_on {c..b}) f(integral^L (lebesgue_on {c..b})
f)
    using integrable_subinterval ‹a ≤ c› f has_bochner_integral_iff by fastforce
  show ?thesis
    by (meson ‹a ≤ c› ‹c ≤ b› has_bochner_integral_combine has_bochner_integral_iff
i j)
qed

lemma has_bochner_integral_null [intro]:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes N ∈ null_sets lebesgue
  shows has_bochner_integral (lebesgue_on N) f 0
  unfolding has_bochner_integral_iff — strange that the proof's so long
proof
  show integrable (lebesgue_on N) f
  proof (subst integrable_restrict_space)
    show N ∩ space lebesgue ∈ sets lebesgue
      using assms by force
    show integrable lebesgue (λx. indicat_real N x *R f x)
```

```
    proof (rule integrable_cong_AE_imp)
      show integrable lebesgue (λx. 0)
        by simp
      show ∗: AE x in lebesgue. 0 = indicat_real N x ∗R f x
        using assms
       by (simp add: indicator_def completion.null_sets_iff_AE eventually_mono)
      show (λx. indicat_real N x ∗R f x) ∈ borel_measurable lebesgue
        by (auto intro: borel_measurable_AE [OF _ ∗])
    qed
  qed
  show integral^L (lebesgue_on N) f = 0
  proof (rule integral_eq_zero_AE)
    show AE x in lebesgue_on N. f x = 0
    by (rule AE_I' [where N=N]) (auto simp: assms null_setsD2 null_sets_restrict_space)
  qed
qed

lemma has_bochner_integral_null_eq[simp]:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes N ∈ null_sets lebesgue
  shows has_bochner_integral (lebesgue_on N) f i ⟷ i = 0
  using assms has_bochner_integral_eq by blast

end
```

9.17 Embedding Measure Spaces with a Function

```
theory Embed_Measure
imports Binary_Product_Measure
begin
```

Given a measure space on some carrier set Ω and a function f, we can define a push-forward measure on the carrier set $f(\Omega)$ whose σ-algebra is the one generated by mapping f over the original sigma algebra.

This is useful e. g. when f is injective, i. e. it is some kind of "tagging" function. For instance, suppose we have some algebraaic datatype of values with various constructors, including a constructor *RealVal* for real numbers. Then *embed_measure* allows us to lift a measure on real numbers to the appropriate subset of that algebraic datatype.

```
definition embed_measure :: 'a measure ⇒ ('a ⇒ 'b) ⇒ 'b measure where
  embed_measure M f = measure_of (f ‘ space M) {f ‘ A |A. A ∈ sets M}
                        (λA. emeasure M (f −‘ A ∩ space M))

lemma space_embed_measure: space (embed_measure M f) = f ‘ space M
  unfolding embed_measure_def
  by (subst space_measure_of) (auto dest: sets.sets_into_space)

lemma sets_embed_measure':
```

assumes *inj*: *inj_on f* (*space M*)
shows *sets* (*embed_measure M f*) = {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
unfolding *embed_measure_def*
proof (*intro sigma_algebra.sets_measure_of_eq sigma_algebra_iff2*[*THEN iffD2*] *conjI allI ballI impI*)
fix *s* **assume** *s* ∈ {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
then obtain *s′* **where** *s′_props*: *s* = *f* ‘ *s′ s′* ∈ *sets M* **by** *auto*
hence *f* ‘ *space M* − *s* = *f* ‘ (*space M* − *s′*) **using** *inj*
by (*auto dest*: *inj_onD sets.sets_into_space*)
also have ... ∈ {*f* ‘ *A* |*A*. *A* ∈ *sets M*} **using** *s′_props* **by** *auto*
finally show *f* ‘ *space M* − *s* ∈ {*f* ‘ *A* |*A*. *A* ∈ *sets M*} **.**
next
fix *A* :: *nat* ⇒ _ **assume** *range A* ⊆ {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
then obtain *A′* **where** *A′*: ⋀*i*. *A i* = *f* ‘ *A′ i* ⋀*i*. *A′ i* ∈ *sets M*
by (*auto simp*: *subset_eq choice_iff*)
then have (⋃*x*. *f* ‘ *A′ x*) = *f* ‘ (⋃*x*. *A′ x*) **by** *blast*
with *A′* **show** (⋃*i*. *A i*) ∈ {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
by *simp blast*
qed (*auto dest*: *sets.sets_into_space*)

lemma *the_inv_into_vimage*:
inj_on f X ⟹ *A* ⊆ *X* ⟹ *the_inv_into X f* −‘ *A* ∩ (*f*‘*X*) = *f* ‘ *A*
by (*auto simp*: *the_inv_into_f_f*)

lemma *sets_embed_eq_vimage_algebra*:
assumes *inj_on f* (*space M*)
shows *sets* (*embed_measure M f*) = *sets* (*vimage_algebra* (*f*‘*space M*) (*the_inv_into* (*space M*) *f*) *M*)
by (*auto simp*: *sets_embed_measure′*[*OF assms*] *Pi_iff the_inv_into_f_f assms sets_vimage_algebra2 Setcompr_eq_image*
dest: *sets.sets_into_space*
intro!: *image_cong the_inv_into_vimage*[*symmetric*])

lemma *sets_embed_measure*:
assumes *inj*: *inj f*
shows *sets* (*embed_measure M f*) = {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
using *assms* **by** (*subst sets_embed_measure′*) (*auto intro*!: *inj_onI dest*: *injD*)

lemma *in_sets_embed_measure*: *A* ∈ *sets M* ⟹ *f* ‘ *A* ∈ *sets* (*embed_measure M f*)
unfolding *embed_measure_def*
by (*intro in_measure_of*) (*auto dest*: *sets.sets_into_space*)

lemma *measurable_embed_measure1*:
assumes *g*: (λ*x*. *g* (*f x*)) ∈ *measurable M N*
shows *g* ∈ *measurable* (*embed_measure M f*) *N*
unfolding *measurable_def*
proof *safe*
fix *A* **assume** *A* ∈ *sets N*

with *g* **have** (λ*x*. *g* (*f x*)) −‘ *A* ∩ *space M* ∈ *sets M*
by (*rule measurable_sets*)
then have *f* ‘ ((λ*x*. *g* (*f x*)) −‘ *A* ∩ *space M*) ∈ *sets* (*embed_measure M f*)
by (*rule in_sets_embed_measure*)
also have *f* ‘ ((λ*x*. *g* (*f x*)) −‘ *A* ∩ *space M*) = *g* −‘ *A* ∩ *space* (*embed_measure M f*)
by (*auto simp*: *space_embed_measure*)
finally show *g* −‘ *A* ∩ *space* (*embed_measure M f*) ∈ *sets* (*embed_measure M f*) **.**
qed (*insert measurable_space*[*OF assms*], *auto simp*: *space_embed_measure*)

lemma *measurable_embed_measure2′*:
assumes *inj_on f* (*space M*)
shows *f* ∈ *measurable M* (*embed_measure M f*)
proof−
{
fix *A* **assume** *A*: *A* ∈ *sets M*
also from *A* **have** *A* = *A* ∩ *space M* **by** *auto*
also have ... = *f* −‘ *f* ‘ *A* ∩ *space M* **using** *A assms*
by (*auto dest*: *inj_onD sets.sets_into_space*)
finally have *f* −‘ *f* ‘ *A* ∩ *space M* ∈ *sets M* **.**
}
thus *?thesis* **using** *assms* **unfolding** *embed_measure_def*
by (*intro measurable_measure_of*) (*auto dest*: *sets.sets_into_space*)
qed

lemma *measurable_embed_measure2*:
assumes [*simp*]: *inj f* **shows** *f* ∈ *measurable M* (*embed_measure M f*)
by (*auto simp*: *inj_vimage_image_eq embed_measure_def*
intro!: *measurable_measure_of dest*: *sets.sets_into_space*)

lemma *embed_measure_eq_distr′*:
assumes *inj_on f* (*space M*)
shows *embed_measure M f* = *distr M* (*embed_measure M f*) *f*
proof−
have *distr M* (*embed_measure M f*) *f* =
measure_of (*f* ‘ *space M*) {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
(λ*A*. *emeasure M* (*f* −‘ *A* ∩ *space M*)) **unfolding** *distr_def*
by (*simp add*: *space_embed_measure sets_embed_measure′*[*OF assms*])
also have ... = *embed_measure M f* **unfolding** *embed_measure_def* **..**
finally show *?thesis* **..**
qed

lemma *embed_measure_eq_distr*:
inj f ⟹ *embed_measure M f* = *distr M* (*embed_measure M f*) *f*
by (*rule embed_measure_eq_distr′*) (*auto intro*!: *inj_onI dest*: *injD*)

lemma *nn_integral_embed_measure′*:
inj_on f (*space M*) ⟹ *g* ∈ *borel_measurable* (*embed_measure M f*) ⟹

nn_integral (*embed_measure M f*) *g* = *nn_integral M* (λ*x*. *g* (*f x*))
apply (*subst embed_measure_eq_distr′*, *simp*)
apply (*subst nn_integral_distr*)
apply (*simp_all add*: *measurable_embed_measure2′*)
done

lemma *nn_integral_embed_measure*:
inj f ⟹ *g* ∈ *borel_measurable* (*embed_measure M f*) ⟹
nn_integral (*embed_measure M f*) *g* = *nn_integral M* (λ*x*. *g* (*f x*))
by(*erule nn_integral_embed_measure′*[*OF subset_inj_on*]) *simp*

lemma *emeasure_embed_measure′*:
assumes *inj_on f* (*space M*) *A* ∈ *sets* (*embed_measure M f*)
shows *emeasure* (*embed_measure M f*) *A* = *emeasure M* (*f* −‘ *A* ∩ *space M*)
by (*subst embed_measure_eq_distr′*[*OF assms*(*1*)])
(*simp add*: *emeasure_distr*[*OF measurable_embed_measure2′*[*OF assms*(*1*)] *assms*(*2*)])

lemma *emeasure_embed_measure*:
assumes *inj f A* ∈ *sets* (*embed_measure M f*)
shows *emeasure* (*embed_measure M f*) *A* = *emeasure M* (*f* −‘ *A* ∩ *space M*)
using *assms* **by** (*intro emeasure_embed_measure′*) (*auto intro*!: *inj_onI dest*: *injD*)

lemma *embed_measure_comp*:
assumes [*simp*]: *inj f inj g*
shows *embed_measure* (*embed_measure M f*) *g* = *embed_measure M* (*g* ∘ *f*)
proof−
have [*simp*]: *inj* (λ*x*. *g* (*f x*)) **by** (*subst o_def*[*symmetric*]) (*auto intro*: *inj_compose*)
note *measurable_embed_measure2*[*measurable*]
have *embed_measure* (*embed_measure M f*) *g* =
distr M (*embed_measure* (*embed_measure M f*) *g*) (*g* ∘ *f*)
by (*subst* (*1 2*) *embed_measure_eq_distr*)
(*simp_all add*: *distr_distr sets_embed_measure cong*: *distr_cong*)
also have ... = *embed_measure M* (*g* ∘ *f*)
by (*subst* (*3*) *embed_measure_eq_distr*, *simp add*: *o_def*, *rule distr_cong*)
(*auto simp*: *sets_embed_measure o_def image_image*[*symmetric*]
intro: *inj_compose cong*: *distr_cong*)
finally show *?thesis* **.**
qed

lemma *sigma_finite_embed_measure*:
assumes *sigma_finite_measure M* **and** *inj*: *inj f*
shows *sigma_finite_measure* (*embed_measure M f*)
proof −
from *assms*(*1*) **interpret** *sigma_finite_measure M* **.**
from *sigma_finite_countable* **obtain** *A* **where**
A_props: *countable A A* ⊆ *sets M* ⋃ *A* = *space M* ⋀*X*. *X*∈*A* ⟹ *emeasure M X* ≠ ∞ **by** *blast*

from *A_props* **have** *countable* ((‘) *f*‘*A*) **by** *auto*
moreover
from *inj* **and** *A_props* **have** (‘) *f*‘*A* ⊆ *sets* (*embed_measure M f*)
by (*auto simp*: *sets_embed_measure*)
moreover
from *A_props* **and** *inj* **have** ⋃((‘) *f*‘*A*) = *space* (*embed_measure M f*)
by (*auto simp*: *space_embed_measure intro*!: *imageI*)
moreover
from *A_props* **and** *inj* **have** ∀ *a*∈(‘) *f* ‘ *A*. *emeasure* (*embed_measure M f*) *a* ≠ ∞
by (*intro ballI*, *subst emeasure_embed_measure*)
(*auto simp*: *inj_vimage_image_eq intro*: *in_sets_embed_measure*)
ultimately show *?thesis* **by** − (*standard*, *blast*)
qed

lemma *embed_measure_count_space′*:
inj_on f A ⟹ *embed_measure* (*count_space A*) *f* = *count_space* (*f*‘*A*)
apply (*subst distr_bij_count_space*[*of f A f*‘*A*, *symmetric*])
apply (*simp add*: *inj_on_def bij_betw_def*)
apply (*subst embed_measure_eq_distr′*)
apply *simp*
apply(*auto 4 3 intro*!: *measure_eqI imageI simp add*: *sets_embed_measure′ subset_image_iff*)
apply (*subst* (*1 2*) *emeasure_distr*)
apply (*auto simp*: *space_embed_measure sets_embed_measure′*)
done

lemma *embed_measure_count_space*:
inj f ⟹ *embed_measure* (*count_space A*) *f* = *count_space* (*f*‘*A*)
by(*rule embed_measure_count_space′*)(*erule subset_inj_on*, *simp*)

lemma *sets_embed_measure_alt*:
inj f ⟹ *sets* (*embed_measure M f*) = ((‘) *f*) ‘ *sets M*
by (*auto simp*: *sets_embed_measure*)

lemma *emeasure_embed_measure_image′*:
assumes *inj_on f* (*space M*) *X* ∈ *sets M*
shows *emeasure* (*embed_measure M f*) (*f*‘*X*) = *emeasure M X*
proof−
from *assms* **have** *emeasure* (*embed_measure M f*) (*f*‘*X*) = *emeasure M* (*f* −‘ *f* ‘ *X* ∩ *space M*)
by (*subst emeasure_embed_measure′*) (*auto simp*: *sets_embed_measure′*)
also from *assms* **have** *f* −‘ *f* ‘ *X* ∩ *space M* = *X* **by** (*auto dest*: *inj_onD sets.sets_into_space*)
finally show *?thesis* **.**
qed

lemma *emeasure_embed_measure_image*:
inj f ⟹ *X* ∈ *sets M* ⟹ *emeasure* (*embed_measure M f*) (*f*‘*X*) = *emeasure*

```
M X
 by (simp_all add: emeasure_embed_measure in_sets_embed_measure inj_vimage_image_eq)

lemma embed_measure_eq_iff:
  assumes inj f
  shows embed_measure A f = embed_measure B f ⟷ A = B (is ?M = ?N
⟷ _)
proof
  from assms have I: inj ((`) f) by (auto intro: injI dest: injD)
  assume asm: ?M = ?N
  hence sets (embed_measure A f) = sets (embed_measure B f) by simp
 with assms have sets A = sets B by (simp only: I inj_image_eq_iff sets_embed_measure_alt)
  moreover {
    fix X assume X ∈ sets A
    from asm have emeasure ?M (f`X) = emeasure ?N (f`X) by simp
    with ‹X ∈ sets A› and ‹sets A = sets B› and assms
     have emeasure A X = emeasure B X by (simp add: emeasure_embed_measure_image)
  }
  ultimately show A = B by (rule measure_eqI)
qed simp

lemma the_inv_into_in_Pi: inj_on f A ⟹ the_inv_into A f ∈ f ` A → A
  by (auto simp: the_inv_into_f_f)

lemma map_prod_image: map_prod f g ` (A × B) = (f`A) × (g`B)
  using map_prod_surj_on[OF refl refl] .

lemma map_prod_vimage: map_prod f g -` (A × B) = (f-`A) × (g-`B)
  by auto

lemma embed_measure_prod:
 assumes f: inj f and g: inj g and [simp]: sigma_finite_measure M sigma_finite_measure
N
  shows embed_measure M f ⨂M embed_measure N g = embed_measure (M
⨂M N) (λ(x, y). (f x, g y))
    (is ?L = _)
  unfolding map_prod_def[symmetric]
proof (rule pair_measure_eqI)
  have fg[simp]: ⋀A. inj_on (map_prod f g) A ⋀A. inj_on f A ⋀A. inj_on g A
    using f g by (auto simp: inj_on_def)

  note complete_lattice_class.Sup_insert[simp del] ccSup_insert[simp del]
     ccSUP_insert[simp del]
  show sets: sets ?L = sets (embed_measure (M ⨂M N) (map_prod f g))
    unfolding map_prod_def[symmetric]
    apply (simp add: sets_pair_eq_sets_fst_snd sets_embed_eq_vimage_algebra
      cong: vimage_algebra_cong)
    apply (subst sets_vimage_Sup_eq[where Y=space (M ⨂M N)])
    apply (simp_all add: space_pair_measure[symmetric])
```

apply (*auto simp add*: *the_inv_into_f_f*
simp del: *map_prod_simp*
del: *prod_fun_imageE*) []
apply *auto* []
apply (*subst* (*1 2 3 4*) *vimage_algebra_vimage_algebra_eq*)
apply (*simp_all add*: *the_inv_into_in_Pi Pi_iff*[*of snd*] *Pi_iff*[*of fst*] *space_pair_measure*)
apply (*simp_all add*: *Pi_iff*[*of snd*] *Pi_iff*[*of fst*] *the_inv_into_in_Pi vimage_algebra_vimage_algebra_eq*
space_pair_measure[*symmetric*] *map_prod_image*[*symmetric*])
apply (*intro arg_cong*[**where** *f*=*sets*] *arg_cong*[**where** *f*=*Sup*] *arg_cong2*[**where** *f*=*insert*] *vimage_algebra_cong*)
apply (*auto simp*: *map_prod_image the_inv_into_f_f*
simp del: *map_prod_simp del*: *prod_fun_imageE*)
apply (*simp_all add*: *the_inv_into_f_f space_pair_measure*)
done

note *measurable_embed_measure2*[*measurable*]
fix *A B* **assume** *AB*: *A* ∈ *sets* (*embed_measure M f*) *B* ∈ *sets* (*embed_measure N g*)
moreover have *f* −‘ *A* × *g* −‘ *B* ∩ *space* (*M* ⨂$_M$ *N*) = (*f* −‘ *A* ∩ *space M*) × (*g* −‘ *B* ∩ *space N*)
by (*auto simp*: *space_pair_measure*)
ultimately show *emeasure* (*embed_measure M f*) *A* ∗ *emeasure* (*embed_measure N g*) *B* =
emeasure (*embed_measure* (*M* ⨂$_M$ *N*) (*map_prod f g*)) (*A* × *B*)
by (*simp add*: *map_prod_vimage sets*[*symmetric*] *emeasure_embed_measure*
sigma_finite_measure.emeasure_pair_measure_Times)
qed (*insert assms*, *simp_all add*: *sigma_finite_embed_measure*)

lemma *mono_embed_measure*:
space M = *space M′* ⟹ *sets M* ⊆ *sets M′* ⟹ *sets* (*embed_measure M f*) ⊆ *sets* (*embed_measure M′ f*)
unfolding *embed_measure_def*
apply (*subst* (*1 2*) *sets_measure_of*)
apply (*blast dest*: *sets.sets_into_space*)
apply (*blast dest*: *sets.sets_into_space*)
apply *simp*
apply (*intro sigma_sets_mono′*)
apply *safe*
apply (*simp add*: *subset_eq*)
apply *metis*
done

lemma *density_embed_measure*:
assumes *inj*: *inj f* **and** *Mg*[*measurable*]: *g* ∈ *borel_measurable* (*embed_measure M f*)
shows *density* (*embed_measure M f*) *g* = *embed_measure* (*density M* (*g* ∘ *f*)) *f*
(**is** *?M1* = *?M2*)
proof (*rule measure_eqI*)

fix X **assume** X: $X \in$ *sets ?M1*
from *inj* **have** *Mf*[*measurable*]: $f \in$ *measurable* M (*embed_measure* M f)
by (*rule measurable_embed_measure2*)
from *Mg* **and** X **have** *emeasure ?M1* $X = \int^+ x.\ g\ x * \mathit{indicator}\ X\ x\ \partial$*embed_measure* M f
by (*subst emeasure_density*) *simp_all*
also from X **have** ... $= \int^+ x.\ g\ (f\ x) * \mathit{indicator}\ X\ (f\ x)\ \partial M$
by (*subst embed_measure_eq_distr*[*OF inj*], *subst nn_integral_distr*) *auto*
also have ... $= \int^+ x.\ g\ (f\ x) * \mathit{indicator}\ (f\ -`\ X \cap \mathit{space}\ M)\ x\ \partial M$
by (*intro nn_integral_cong*) (*auto split*: *split_indicator*)
also from X **have** ... $=$ *emeasure* (*density* M $(g \circ f)$) $(f\ -`\ X \cap \mathit{space}\ M)$
by (*subst emeasure_density*) (*simp_all add*: *measurable_comp*[*OF Mf Mg*] *measurable_sets*[*OF Mf*])
also from X **and** *inj* **have** ... $=$ *emeasure ?M2 X*
by (*subst emeasure_embed_measure*) (*simp_all add*: *sets_embed_measure*)
finally show *emeasure ?M1 X* $=$ *emeasure ?M2 X* **.**
qed (*simp_all add*: *sets_embed_measure inj*)

lemma *density_embed_measure′*:
assumes *inj*: *inj f* **and** *inv*: $\bigwedge x.\ f'\ (f\ x) = x$ **and** *Mg*[*measurable*]: $g \in$ *borel_measurable* M
shows *density* (*embed_measure* M f) $(g \circ f') =$ *embed_measure* (*density* M g) f
proof−
have *density* (*embed_measure* M f) $(g \circ f') =$ *embed_measure* (*density* M $(g \circ f' \circ f)$) f
by (*rule density_embed_measure*[*OF inj*])
(*rule measurable_comp*, *rule measurable_embed_measure1*, *subst measurable_cong*,
rule inv, *rule measurable_ident_sets*, *simp*, *rule Mg*)
also have *density* M $(g \circ f' \circ f) =$ *density* M g
by (*intro density_cong*) (*subst measurable_cong*, *simp add*: *o_def inv*, *simp_all add*: *Mg inv*)
finally show *?thesis* **.**
qed

lemma *inj_on_image_subset_iff*:
assumes *inj_on f C A* $\subseteq C$ $B \subseteq C$
shows $f\ `\ A \subseteq f\ `\ B \longleftrightarrow A \subseteq B$
proof (*intro iffI subsetI*)
fix x **assume** A: $f\ `\ A \subseteq f\ `\ B$ **and** B: $x \in A$
from B **have** $f\ x \in f\ `\ A$ **by** *blast*
with A **have** $f\ x \in f\ `\ B$ **by** *blast*
then obtain y **where** $f\ x = f\ y$ **and** $y \in B$ **by** *blast*
with *assms* **and** B **have** $x = y$ **by** (*auto dest*: *inj_onD*)
with ‹$y \in B$› **show** $x \in B$ **by** *simp*
qed *auto*

lemma *AE_embed_measure′*:

assumes *inj*: *inj_on f* (*space M*)
shows (*AE x in embed_measure M f*. *P x*) ⟷ (*AE x in M*. *P* (*f x*))
proof
let *?M* = *embed_measure M f*
assume *AE x in ?M*. *P x*
then obtain *A* **where** *A_props*: *A* ∈ *sets ?M emeasure ?M A* = *0* {*x*∈*space ?M*. ¬*P x*} ⊆ *A*
by (*force elim*: *AE_E*)
then obtain *A′* **where** *A′_props*: *A* = *f* ‘ *A′ A′* ∈ *sets M* **by** (*auto simp*: *sets_embed_measure′ inj*)
moreover have *B*: {*x*∈*space ?M*. ¬*P x*} = *f* ‘ {*x*∈*space M*. ¬*P* (*f x*)}
by (*auto simp*: *inj space_embed_measure*)
from *A_props*(*3*) **have** {*x*∈*space M*. ¬*P* (*f x*)} ⊆ *A′*
by (*subst* (*asm*) *B*, *subst* (*asm*) *A′_props*, *subst* (*asm*) *inj_on_image_subset_iff*[*OF inj*])
(*insert A′_props*, *auto dest*: *sets.sets_into_space*)
moreover from *A_props A′_props* **have** *emeasure M A′* = *0*
by (*simp add*: *emeasure_embed_measure_image′ inj*)
ultimately show *AE x in M*. *P* (*f x*) **by** (*intro AE_I*)
next
let *?M* = *embed_measure M f*
assume *AE x in M*. *P* (*f x*)
then obtain *A* **where** *A_props*: *A* ∈ *sets M emeasure M A* = *0* {*x*∈*space M*. ¬*P* (*f x*)} ⊆ *A*
by (*force elim*: *AE_E*)
hence *f*‘*A* ∈ *sets ?M emeasure ?M* (*f*‘*A*) = *0* {*x*∈*space ?M*. ¬*P x*} ⊆ *f*‘*A*
by (*auto simp*: *space_embed_measure emeasure_embed_measure_image′ sets_embed_measure′ inj*)
thus *AE x in ?M*. *P x* **by** (*intro AE_I*)
qed

lemma *AE_embed_measure*:
assumes *inj*: *inj f*
shows (*AE x in embed_measure M f*. *P x*) ⟷ (*AE x in M*. *P* (*f x*))
using *assms* **by** (*intro AE_embed_measure′*) (*auto intro*!: *inj_onI dest*: *injD*)

lemma *nn_integral_monotone_convergence_SUP_countable*:
fixes *f* :: *′a* ⇒ *′b* ⇒ *ennreal*
assumes *nonempty*: *Y* ≠ {}
and *chain*: *Complete_Partial_Order.chain* (≤) (*f* ‘ *Y*)
and *countable*: *countable B*
shows (∫⁺ *x*. (*SUP i*∈*Y*. *f i x*) ∂*count_space B*) = (*SUP i*∈*Y*. (∫⁺ *x*. *f i x* ∂*count_space B*))
(**is** *?lhs* = *?rhs*)
proof −
let *?f* = (λ*i x*. *f i* (*from_nat_into B x*) ∗ *indicator* (*to_nat_on B* ‘ *B*) *x*)
have *?lhs* = ∫⁺ *x*. (*SUP i*∈*Y*. *f i* (*from_nat_into B* (*to_nat_on B x*))) ∂*count_space B*
by(*rule nn_integral_cong*)(*simp add*: *countable*)

also have ... = $\int^+ x.$ (*SUP* $i \in Y.$ *f i* (*from_nat_into B x*)) *∂count_space* (*to_nat_on B ‘ B*)
by(*simp add*: *embed_measure_count_space'*[*symmetric*] *inj_on_to_nat_on countable nn_integral_embed_measure' measurable_embed_measure1*)
also have ... = $\int^+ x.$ (*SUP* $i \in Y.$ *?f i x*) *∂count_space UNIV*
by(*simp add*: *nn_integral_count_space_indicator ennreal_indicator*[*symmetric*] *SUP_mult_right_ennreal nonempty*)
also have ... = (*SUP* $i \in Y.$ $\int^+ x.$ *?f i x ∂count_space UNIV*)
proof(*rule nn_integral_monotone_convergence_SUP_nat*)
show *Complete_Partial_Order.chain* (≤) (*?f ‘ Y*)
by(*rule chain_imageI*[*OF chain*, *unfolded image_image*])(*auto intro*!: *le_funI split*: *split_indicator dest*: *le_funD*)
qed *fact*
also have ... = (*SUP* $i \in Y.$ $\int^+ x.$ *f i* (*from_nat_into B x*) *∂count_space* (*to_nat_on B ‘ B*))
by(*simp add*: *nn_integral_count_space_indicator*)
also have ... = (*SUP* $i \in Y.$ $\int^+ x.$ *f i* (*from_nat_into B* (*to_nat_on B x*)) *∂count_space B*)
by(*simp add*: *embed_measure_count_space'*[*symmetric*] *inj_on_to_nat_on countable nn_integral_embed_measure' measurable_embed_measure1*)
also have ... = *?rhs*
by(*intro arg_cong2*[**where** $f = \lambda A\ f.$ *Sup* (*f ‘ A*)] *ext nn_integral_cong_AE*)(*simp_all add*: *AE_count_space countable*)
finally show *?thesis* **.**
qed

end

9.18 Brouwer's Fixed Point Theorem

theory *Brouwer_Fixpoint*
imports *Homeomorphism Derivative*
begin

9.18.1 Retractions

lemma *retract_of_contractible*:
assumes *contractible T S retract_of T*
shows *contractible S*
using *assms*
apply (*clarsimp simp add*: *retract_of_def contractible_def retraction_def homotopic_with image_subset_iff_funcset*)
apply (*rule_tac x*=*r a* **in** *exI*)
apply (*rule_tac x*=*r* ∘ *h* **in** *exI*)
apply (*intro conjI continuous_intros continuous_on_compose*)
apply (*erule continuous_on_subset* | *force*)+
done

lemma *retract_of_path_connected*:

⟦*path_connected T*; *S retract_of T*⟧ ⟹ *path_connected S*
by (*metis path_connected_continuous_image retract_of_def retraction*)

lemma *retract_of_simply_connected*:
assumes *T*: *simply_connected T* **and** *S retract_of T*
shows *simply_connected S*
proof −
obtain *r* **where** *r*: *retraction T S r*
using *assms* **by** (*metis retract_of_def*)
have *S* ⊆ *T*
by (*meson* ‹*retraction T S r*› *retraction*)
then have (λ*a*. *a*) ∈ *S* → *T*
by *blast*
then show *?thesis*
using *simply_connected_retraction_gen* [*OF T*]
by (*metis* (*no_types*) *r retraction retraction_refl*)
qed

lemma *retract_of_homotopically_trivial*:
assumes *ts*: *T retract_of S*
and *hom*: ⋀*f g*. ⟦*continuous_on U f*; *f* ∈ *U* → *S*;
continuous_on U g; *g* ∈ *U* → *S*⟧
⟹ *homotopic_with_canon* (λ*x*. *True*) *U S f g*
and *continuous_on U f f* ∈ *U* → *T*
and *continuous_on U g g* ∈ *U* → *T*
shows *homotopic_with_canon* (λ*x*. *True*) *U T f g*
proof −
obtain *r* **where** *r* ∈ *S* → *S continuous_on S r* ∀ *x*∈*S*. *r* (*r x*) = *r x T* = *r* ‘ *S*
using *ts* **by** (*auto simp*: *retract_of_def retraction*)
then obtain *k* **where** *Retracts S r T k*
unfolding *Retracts_def* **using** *continuous_on_id* **by** *blast*
then show *?thesis*
by (*rule Retracts.homotopically_trivial_retraction_gen*) (*use assms hom* **in** *force*)+
qed

lemma *retract_of_homotopically_trivial_null*:
assumes *ts*: *T retract_of S*
and *hom*: ⋀*f*. ⟦*continuous_on U f*; *f* ∈ *U* → *S*⟧
⟹ ∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *U S f* (λ*x*. *c*)
and *continuous_on U f f* ∈ *U* → *T*
obtains *c* **where** *homotopic_with_canon* (λ*x*. *True*) *U T f* (λ*x*. *c*)
proof −
obtain *r* **where** *r* ∈ *S* → *S continuous_on S r* ∀ *x*∈*S*. *r* (*r x*) = *r x T* = *r* ‘ *S*
using *ts* **by** (*auto simp*: *retract_of_def retraction*)
then obtain *k* **where** *Retracts S r T k*
unfolding *Retracts_def* **by** *fastforce*
then show *?thesis*
proof (*rule Retracts.homotopically_trivial_retraction_null_gen*)

show $\bigwedge f$. $[\![$*continuous_on U f*; $f \in U \to S]\!]$
$\Longrightarrow \exists c$. *homotopic_with_canon* $(\lambda a.\ True)\ U\ S\ f\ (\lambda x.\ c)$
using *hom* **by** *blast*
qed (*use assms that* **in** *auto*)
qed

lemma *retraction_openin_vimage_iff*:
openin (*top_of_set S*) (*S* $\cap$ *r* $-`$ *U*) $\longleftrightarrow$ *openin* (*top_of_set T*) *U*
if *retraction S T r* **and** *U* $\subseteq$ *T*
by (*simp add*: *retraction_openin_vimage_iff that*)

lemma *retract_of_locally_compact*:
fixes *S* :: *'a* :: {*heine_borel*,*real_normed_vector*} *set*
shows $[\![$ *locally compact S*; *T retract_of S*$]\!] \Longrightarrow$ *locally compact T*
by (*metis locally_compact_closedin closedin_retract*)

lemma *homotopic_into_retract*:
$[\![f \in S \to T$; $g \in S \to T$; *T retract_of U*; *homotopic_with_canon* $(\lambda x.\ True)$
S U f g$]\!]$
$\Longrightarrow$ *homotopic_with_canon* $(\lambda x.\ True)\ S\ T\ f\ g$
apply (*subst* (*asm*) *homotopic_with_def*)
apply (*simp add*: *homotopic_with retract_of_def retraction_def Pi_iff*, *clarify*)
apply (*rule_tac x*=*r* $\circ$ *h* **in** *exI*)
by (*smt* (*verit*, *ccfv_SIG*) *comp_def continuous_on_compose continuous_on_subset image_subset_iff*)

lemma *retract_of_locally_connected*:
assumes *locally connected T S retract_of T*
shows *locally connected S*
using *assms*
by (*metis retraction_openin_vimage_iff idempotent_imp_retraction locally_connected_quotient_imag retract_ofE*)

lemma *retract_of_locally_path_connected*:
assumes *locally path_connected T S retract_of T*
shows *locally path_connected S*
using *assms*
by (*metis retraction_openin_vimage_iff idempotent_imp_retraction locally_path_connected_quotient_ retract_ofE*)

A few simple lemmas about deformation retracts

lemma *deformation_retract_imp_homotopy_eqv*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *homotopic_with_canon* $(\lambda x.\ True)\ S\ S\ id\ r$ **and** *r*: *retraction S T r*
shows *S homotopy_eqv T*
proof −
have *homotopic_with_canon* $(\lambda x.\ True)\ S\ S\ (id \circ r)\ id$
by (*simp add*: *assms*(*1*) *homotopic_with_symD*)
moreover have *homotopic_with_canon* $(\lambda x.\ True)\ T\ T\ (r \circ id)\ id$

using *r* **unfolding** *retraction_def*
by (*metis eq_id_iff homotopic_with_id2 topspace_euclidean_subtopology*)
ultimately
show *?thesis*
unfolding *homotopy_equivalent_space_def*
by (*smt* (*verit*, *del_insts*) *continuous_map_id continuous_map_subtopology_eu id_def r retraction retraction_comp subset_refl*)
qed

lemma *deformation_retract*:
fixes *S* :: *'a*::*euclidean_space set*
shows (∃*r*. *homotopic_with_canon* (λ*x*. *True*) *S S id r* ∧ *retraction S T r*)
⟷
T retract_of S ∧ (∃*f*. *homotopic_with_canon* (λ*x*. *True*) *S S id f* ∧ *f* ∈ *S* → *T*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*auto simp*: *retract_of_def retraction_def*)
next
assume *R*: *?rhs*
have ⋀*r f*. ⟦*T* ⊆ *S*; *continuous_on S r*; *homotopic_with_canon* (λ*x*. *True*) *S S id f*;
f ∈ *S* → *T*; *r* ∈ *S* → *T*; ∀*x*∈*T*. *r x* = *x*⟧
⟹ *homotopic_with_canon* (λ*x*. *True*) *S S f r*
apply (*rule_tac f* = *r* ∘ *f* **and** *g*=*r* ∘ *id* **in** *homotopic_with_eq*)
apply (*rule_tac Y*=*S* **in** *homotopic_with_compose_continuous_left*)
apply (*auto simp*: *homotopic_with_sym Pi_iff*)
done
with *R homotopic_with_trans* **show** *?lhs*
unfolding *retract_of_def retraction_def* **by** *blast*
qed

lemma *deformation_retract_of_contractible_sing*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *contractible S a* ∈ *S*
obtains *r* **where** *homotopic_with_canon* (λ*x*. *True*) *S S id r retraction S* {*a*} *r*
proof −
have {*a*} *retract_of S*
by (*simp add*: ‹*a* ∈ *S*›)
moreover have *homotopic_with_canon* (λ*x*. *True*) *S S id* (λ*x*. *a*)
using *assms*
by (*auto simp*: *contractible_def homotopic_into_contractible image_subset_iff*)
moreover have (λ*x*. *a*) ∈ *S* → {*a*}
by (*simp add*: *image_subsetI*)
ultimately show *?thesis*
by (*metis that deformation_retract*)
qed

lemma *continuous_on_compact_surface_projection_aux*:
fixes S :: *'a::t2_space set*
assumes *compact S* $S \subseteq T$ *image q* $T \subseteq S$
and *contp*: *continuous_on* T p
and $\bigwedge x.\ x \in S \Longrightarrow q\ x = x$
and [*simp*]: $\bigwedge x.\ x \in T \Longrightarrow q(p\ x) = q\ x$
and $\bigwedge x.\ x \in T \Longrightarrow p(q\ x) = p\ x$
shows *continuous_on* T q
proof −
have $*$: *image* p T = *image* p S
using *assms* **by** *auto* (*metis imageI subset_iff*)
have *contp'*: *continuous_on* S p
by (*rule continuous_on_subset* [*OF contp* ‹$S \subseteq T$›])
have *continuous_on* (p ‘ T) q
by (*simp add*: $*$ *assms*(*1*) *assms*(*2*) *assms*(*5*) *continuous_on_inv contp' rev_subsetD*)
then have *continuous_on* T ($q \circ p$)
by (*rule continuous_on_compose* [*OF contp*])
then show *?thesis*
by (*rule continuous_on_eq* [*of* _ $q \circ p$]) (*simp add*: *o_def*)
qed

lemma *continuous_on_compact_surface_projection*:
fixes S :: *'a::real_normed_vector set*
assumes *compact S*
and S: $S \subseteq V - \{0\}$ **and** *cone V*
and *iff*: $\bigwedge x\ k.\ x \in V - \{0\} \Longrightarrow 0 < k \wedge (k *_R x) \in S \longleftrightarrow d\ x = k$
shows *continuous_on* ($V - \{0\}$) ($\lambda x.\ d\ x *_R x$)
proof (*rule continuous_on_compact_surface_projection_aux* [*OF* ‹*compact S*› *S*])
show ($\lambda x.\ d\ x *_R x$) ‘ ($V - \{0\}$) $\subseteq S$
using *iff* **by** *auto*
show *continuous_on* ($V - \{0\}$) ($\lambda x.$ *inverse*(*norm x*) $*_R$ x)
by (*intro continuous_intros*) *force*
show $\bigwedge x.\ x \in S \Longrightarrow d\ x *_R x = x$
by (*metis S zero_less_one local.iff scaleR_one subset_eq*)
show $d\ (x /_R norm\ x) *_R (x /_R norm\ x) = d\ x *_R x$ **if** $x \in V - \{0\}$ **for** x
using *iff* [*of inverse*(*norm x*) $*_R$ x *norm x* $*$ $d\ x$, *symmetric*] *iff that* ‹*cone V*›
by (*simp add*: *field_simps cone_def zero_less_mult_iff*)
show $d\ x *_R x /_R norm\ (d\ x *_R x) = x /_R norm\ x$ **if** $x \in V - \{0\}$ **for** x
proof −
have $0 < d\ x$
using *local.iff that* **by** *blast*
then show *?thesis*
by *simp*
qed
qed

9.18.2 Kuhn Simplices

lemma *bij_betw_singleton_eq*:
assumes *f*: *bij_betw f A B* **and** *g*: *bij_betw g A B* **and** *a*: $a \in A$
assumes *eq*: $(\bigwedge x.\ x \in A \Longrightarrow x \neq a \Longrightarrow f\ x = g\ x)$
shows $f\ a = g\ a$
proof −
have $f\ `\ (A - \{a\}) = g\ `\ (A - \{a\})$
by (*intro image_cong*) (*simp_all add*: *eq*)
then have $B - \{f\ a\} = B - \{g\ a\}$
using *f g a* **by** (*auto simp*: *bij_betw_def inj_on_image_set_diff set_eq_iff*)
moreover have $f\ a \in B\ g\ a \in B$
using *f g a* **by** (*auto simp*: *bij_betw_def*)
ultimately show *?thesis*
by *auto*
qed

lemmas *swap_apply1* = *swap_apply*(*1*)
lemmas *swap_apply2* = *swap_apply*(*2*)

lemma *pointwise_minimal_pointwise_maximal*:
fixes *s* :: (*nat* ⇒ *nat*) *set*
assumes *finite s*
and $s \neq \{\}$
and $\forall\, x{\in}s.\ \forall\, y{\in}s.\ x \leq y \vee y \leq x$
shows $\exists\, a{\in}s.\ \forall\, x{\in}s.\ a \leq x$
and $\exists\, a{\in}s.\ \forall\, x{\in}s.\ x \leq a$
using *assms*
proof (*induct s rule*: *finite_ne_induct*)
case (*insert b s*)
assume ∗: $\forall\, x{\in}$*insert b s*. $\forall\, y{\in}$*insert b s*. $x \leq y \vee y \leq x$
then obtain *u l* **where** $l \in s\ \forall\, b{\in}s.\ l \leq b\ u \in s\ \forall\, b{\in}s.\ b \leq u$
using *insert* **by** *auto*
with ∗ **show** $\exists\, a{\in}$*insert b s*. $\forall\, x{\in}$*insert b s*. $a \leq x\ \exists\, a{\in}$*insert b s*. $\forall\, x{\in}$*insert b s*. $x \leq a$
by (*metis insert_iff order.trans*)+
qed *auto*

lemma *kuhn_labelling_lemma*:
fixes *P Q* :: *'a*::*euclidean_space* ⇒ *bool*
assumes $\forall\, x.\ P\ x \longrightarrow P\ (f\ x)$
and $\forall\, x.\ P\ x \longrightarrow (\forall\, i{\in}Basis.\ Q\ i \longrightarrow 0 \leq x{\cdot}i \wedge x{\cdot}i \leq 1)$
shows $\exists\, l.\ (\forall\, x.\forall\, i{\in}Basis.\ l\ x\ i \leq (1{::}nat)) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (x{\cdot}i = 0) \longrightarrow (l\ x\ i = 0)) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (x{\cdot}i = 1) \longrightarrow (l\ x\ i = 1)) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (l\ x\ i = 0) \longrightarrow x{\cdot}i \leq f\ x{\cdot}i) \wedge$
$(\forall\, x.\forall\, i{\in}Basis.\ P\ x \wedge Q\ i \wedge (l\ x\ i = 1) \longrightarrow f\ x{\cdot}i \leq x{\cdot}i)$
proof −
{ fix *x i*
let *?R* = $\lambda y.\ (P\ x \wedge Q\ i \wedge x \cdot i = 0 \longrightarrow y = (0{::}nat)) \wedge$

$(P\ x \wedge Q\ i \wedge x \cdot i = 1 \longrightarrow y = 1) \wedge$
$(P\ x \wedge Q\ i \wedge y = 0 \longrightarrow x \cdot i \leq f\ x \cdot i) \wedge$
$(P\ x \wedge Q\ i \wedge y = 1 \longrightarrow f\ x \cdot i \leq x \cdot i)$
{ assume $P\ x\ Q\ i\ i \in Basis$ **with** *assms* **have** $0 \leq f\ x \cdot i \wedge f\ x \cdot i \leq 1$ **by** *auto* **}**
then have $i \in Basis \Longrightarrow ?R\ 0 \vee ?R\ 1$ **by** *auto* **}**
then show *?thesis*
unfolding *all_conj_distrib*[*symmetric*] *Ball_def*
by (*subst choice_iff*[*symmetric*])+ *blast*
qed

The key "counting" observation, somewhat abstracted

lemma *kuhn_counting_lemma*:
fixes *bnd compo compo' face S F*
defines $nF\ s == card\ \{f{\in}F.\ face\ f\ s \wedge compo'\ f\}$
assumes [*simp*, *intro*]: *finite F* — faces **and** [*simp*, *intro*]: *finite S* — simplices
and $\bigwedge f.\ f \in F \Longrightarrow bnd\ f \Longrightarrow card\ \{s{\in}S.\ face\ f\ s\} = 1$
and $\bigwedge f.\ f \in F \Longrightarrow \neg\ bnd\ f \Longrightarrow card\ \{s{\in}S.\ face\ f\ s\} = 2$
and $\bigwedge s.\ s \in S \Longrightarrow compo\ s \Longrightarrow nF\ s = 1$
and $\bigwedge s.\ s \in S \Longrightarrow \neg\ compo\ s \Longrightarrow nF\ s = 0 \vee nF\ s = 2$
and $odd\ (card\ \{f{\in}F.\ compo'\ f \wedge bnd\ f\})$
shows $odd\ (card\ \{s{\in}S.\ compo\ s\})$
proof −
have $(\sum s \mid s \in S \wedge \neg\ compo\ s.\ nF\ s) + (\sum s \mid s \in S \wedge compo\ s.\ nF\ s) = (\sum s{\in}S.\ nF\ s)$
by (*subst sum.union_disjoint*[*symmetric*]) (*auto intro*!: *sum.cong*)
also have $\ldots = (\sum s{\in}S.\ card\ \{f \in \{f{\in}F.\ compo'\ f \wedge bnd\ f\}.\ face\ f\ s\}) +$
$(\sum s{\in}S.\ card\ \{f \in \{f{\in}F.\ compo'\ f \wedge \neg\ bnd\ f\}.\ face\ f\ s\})$
unfolding *sum.distrib*[*symmetric*]
by (*subst card_Un_disjoint*[*symmetric*])
(*auto simp*: *nF_def intro*!: *sum.cong arg_cong*[**where** *f*=*card*])
also have $\ldots = 1 * card\ \{f{\in}F.\ compo'\ f \wedge bnd\ f\} + 2 * card\ \{f{\in}F.\ compo'\ f \wedge \neg\ bnd\ f\}$
using *assms*(*4*,*5*) **by** (*fastforce intro*!: *arg_cong2*[**where** *f*=(+)] *sum_multicount*)
finally have $odd\ ((\sum s \mid s \in S \wedge \neg\ compo\ s.\ nF\ s) + card\ \{s{\in}S.\ compo\ s\})$
using *assms*(*6*,*8*) **by** *simp*
moreover have $(\sum s \mid s \in S \wedge \neg\ compo\ s.\ nF\ s) =$
$(\sum s \mid s \in S \wedge \neg\ compo\ s \wedge nF\ s = 0.\ nF\ s) + (\sum s \mid s \in S \wedge \neg\ compo\ s \wedge nF\ s = 2.\ nF\ s)$
using *assms*(*7*) **by** (*subst sum.union_disjoint*[*symmetric*]) (*fastforce intro*!: *sum.cong*)+
ultimately show *?thesis*
by *auto*
qed

The odd/even result for faces of complete vertices, generalized

lemma *kuhn_complete_lemma*:
assumes [*simp*]: *finite simplices*

and *face*: $\bigwedge f\ s.\ face\ f\ s \longleftrightarrow (\exists\, a{\in}s.\ f = s - \{a\})$
and *card_s*[*simp*]: $\bigwedge s.\ s \in simplices \Longrightarrow card\ s = n + 2$
and *rl_bd*: $\bigwedge s.\ s \in simplices \Longrightarrow rl\ `\ s \subseteq \{..Suc\ n\}$
and *bnd*: $\bigwedge f\ s.\ s \in simplices \Longrightarrow face\ f\ s \Longrightarrow bnd\ f \Longrightarrow card\ \{s{\in}simplices.\ face\ f\ s\} = 1$
and *nbnd*: $\bigwedge f\ s.\ s \in simplices \Longrightarrow face\ f\ s \Longrightarrow \neg\ bnd\ f \Longrightarrow card\ \{s{\in}simplices.\ face\ f\ s\} = 2$
and *odd_card*: $odd\ (card\ \{f.\ (\exists\, s{\in}simplices.\ face\ f\ s) \wedge rl\ `\ f = \{..n\} \wedge bnd\ f\})$
shows $odd\ (card\ \{s{\in}simplices.\ (rl\ `\ s = \{..Suc\ n\})\})$
proof (*rule kuhn_counting_lemma*)
have *finite_s*[*simp*]: $\bigwedge s.\ s \in simplices \Longrightarrow finite\ s$
by (*metis add_is_0 zero_neq_numeral card.infinite assms*(*3*))

let *?F* = $\{f.\ \exists\, s{\in}simplices.\ face\ f\ s\}$
have *F_eq*: *?F* = $(\bigcup s{\in}simplices.\ \bigcup a{\in}s.\ \{s - \{a\}\})$
by (*auto simp*: *face*)
show *finite ?F*
using ‹*finite simplices*› **unfolding** *F_eq* **by** *auto*

show $card\ \{s \in simplices.\ face\ f\ s\} = 1$ **if** $f \in ?F\ bnd\ f$ **for** *f*
using *bnd that* **by** *auto*

show $card\ \{s \in simplices.\ face\ f\ s\} = 2$ **if** $f \in ?F\ \neg\ bnd\ f$ **for** *f*
using *nbnd that* **by** *auto*

show $odd\ (card\ \{f \in \{f.\ \exists\, s{\in}simplices.\ face\ f\ s\}.\ rl\ `\ f = \{..n\} \wedge bnd\ f\})$
using *odd_card* **by** *simp*

fix *s* **assume** *s*[*simp*]: $s \in simplices$
let *?S* = $\{f \in \{f.\ \exists\, s{\in}simplices.\ face\ f\ s\}.\ face\ f\ s \wedge rl\ `\ f = \{..n\}\}$
have *?S* = $(\lambda a.\ s - \{a\})\ `\ \{a{\in}s.\ rl\ `\ (s - \{a\}) = \{..n\}\}$
using *s* **by** (*fastforce simp*: *face*)
then have *card_S*: *card ?S* = $card\ \{a{\in}s.\ rl\ `\ (s - \{a\}) = \{..n\}\}$
by (*auto intro*!: *card_image inj_onI*)

{ assume *rl*: $rl\ `\ s = \{..Suc\ n\}$
then have *inj_rl*: *inj_on rl s*
by (*intro eq_card_imp_inj_on*) *auto*
moreover obtain *a* **where** $rl\ a = Suc\ n\ a \in s$
by (*metis atMost_iff image_iff le_Suc_eq rl*)
ultimately have *n*: $\{..n\} = rl\ `\ (s - \{a\})$
by (*auto simp*: *inj_on_image_set_diff rl*)
have $\{a{\in}s.\ rl\ `\ (s - \{a\}) = \{..n\}\} = \{a\}$
using *inj_rl* ‹$a \in s$› **by** (*auto simp*: *n inj_on_image_eq_iff*[*OF inj_rl*])
then show *card ?S* = *1*
unfolding *card_S* **by** *simp* **}**

{ assume *rl*: $rl\ `\ s \neq \{..Suc\ n\}$
show *card ?S* = *0* ∨ *card ?S* = *2*

```
    proof cases
      assume *: {..n} ⊆ rl ‘ s
      with rl rl_bd[OF s] have rl_s: rl ‘ s = {..n}
        by (auto simp: atMost_Suc subset_insert_iff split: if_split_asm)
      then have ¬ inj_on rl s
        by (intro pigeonhole) simp
      then obtain a b where ab: a ∈ s b ∈ s rl a = rl b a ≠ b
        by (auto simp: inj_on_def)
      then have eq: rl ‘ (s − {a}) = rl ‘ s
        by auto
      with ab have inj: inj_on rl (s − {a})
        by (intro eq_card_imp_inj_on) (auto simp: rl_s card_Diff_singleton_if)

      { fix x assume x ∈ s x ∉ {a, b}
        then have rl ‘ s − {rl x} = rl ‘ ((s − {a}) − {x})
          by (auto simp: eq inj_on_image_set_diff[OF inj])
        also have ... = rl ‘ (s − {x})
          using ab ‹x ∉ {a, b}› by auto
        also assume ... = rl ‘ s
        finally have False
          using ‹x∈s› by auto }
      moreover
      { fix x assume x ∈ {a, b} with ab have x ∈ s ∧ rl ‘ (s − {x}) = rl ‘ s
          by (simp add: set_eq_iff image_iff Bex_def) metis }
      ultimately have {a∈s. rl ‘ (s − {a}) = {..n}} = {a, b}
        unfolding rl_s[symmetric] by fastforce
      with ‹a ≠ b› show card ?S = 0 ∨ card ?S = 2
        unfolding card_S by simp
    next
      assume ¬ {..n} ⊆ rl ‘ s
      then have ⋀x. rl ‘ (s − {x}) ≠ {..n}
        by auto
      then show card ?S = 0 ∨ card ?S = 2
        unfolding card_S by simp
    qed }
qed fact

locale kuhn_simplex =
  fixes p n and base upd and S :: (nat ⇒ nat) set
  assumes base: base ∈ {..< n} → {..< p}
  assumes base_out: ⋀i. n ≤ i ⟹ base i = p
  assumes upd: bij_betw upd {..< n} {..< n}
  assumes s_pre: S = (λi j. if j ∈ upd‘{..< i} then Suc (base j) else base j) ‘ {..
n}
begin

definition enum i j = (if j ∈ upd‘{..< i} then Suc (base j) else base j)

lemma s_eq: S = enum ‘ {.. n}
```

unfolding *s_pre enum_def*[*abs_def*] **..**

lemma *upd_space*: $i < n \Longrightarrow upd\ i < n$
using *upd* **by** (*auto dest*!: *bij_betwE*)

lemma *s_space*: $S \subseteq \{..< n\} \to \{.. p\}$
proof −
{ fix i **assume** $i \leq n$ **then have** $enum\ i \in \{..< n\} \to \{.. p\}$
proof (*induct i*)
case *0* **then show** *?case*
using *base* **by** (*auto simp*: *Pi_iff less_imp_le enum_def*)
next
case (*Suc i*) **with** *base* **show** *?case*
by (*auto simp*: *Pi_iff Suc_le_eq less_imp_le enum_def intro*: *upd_space*)
qed }
then show *?thesis*
by (*auto simp*: *s_eq*)
qed

lemma *inj_upd*: *inj_on upd* $\{..< n\}$
using *upd* **by** (*simp add*: *bij_betw_def*)

lemma *inj_enum*: *inj_on enum* $\{.. n\}$
proof −
{ fix $x\ y :: nat$ **assume** $x \neq y\ x \leq n\ y \leq n$
with *upd* **have** *upd* ‘ $\{..< x\} \neq upd$ ‘ $\{..< y\}$
by (*subst inj_on_image_eq_iff*[**where** C=$\{..< n\}$]) (*auto simp*: *bij_betw_def*)
then have $enum\ x \neq enum\ y$
by (*auto simp*: *enum_def fun_eq_iff*) **}**
then show *?thesis*
by (*auto simp*: *inj_on_def*)
qed

lemma *enum_0*: $enum\ 0 = base$
by (*simp add*: *enum_def*[*abs_def*])

lemma *base_in_s*: $base \in S$
unfolding *s_eq* **by** (*subst enum_0*[*symmetric*]) *auto*

lemma *enum_in*: $i \leq n \Longrightarrow enum\ i \in S$
unfolding *s_eq* **by** *auto*

lemma *one_step*:
assumes *a*: $a \in S\ j < n$
assumes $*$: $\bigwedge a'.\ a' \in S \Longrightarrow a' \neq a \Longrightarrow a'\ j = p'$
shows $a\ j \neq p'$
proof
assume $a\ j = p'$
with $*$ a **have** $\bigwedge a'.\ a' \in S \Longrightarrow a'\ j = p'$

```
    by auto
  then have ⋀i. i ≤ n ⟹ enum i j = p'
    unfolding s_eq by auto
  from this[of 0] this[of n] have j ∉ upd ` {..< n}
    by (auto simp: enum_def fun_eq_iff split: if_split_asm)
  with upd ‹j < n› show False
    by (auto simp: bij_betw_def)
qed

lemma upd_inj: i < n ⟹ j < n ⟹ upd i = upd j ⟷ i = j
  using upd by (auto simp: bij_betw_def inj_on_eq_iff)

lemma upd_surj: upd ` {..< n} = {..< n}
  using upd by (auto simp: bij_betw_def)

lemma in_upd_image: A ⊆ {..< n} ⟹ i < n ⟹ upd i ∈ upd ` A ⟷ i ∈ A
  using inj_on_image_mem_iff[of upd {..< n}] upd
  by (auto simp: bij_betw_def)

lemma enum_inj: i ≤ n ⟹ j ≤ n ⟹ enum i = enum j ⟷ i = j
  using inj_enum by (auto simp: inj_on_eq_iff)

lemma in_enum_image: A ⊆ {.. n} ⟹ i ≤ n ⟹ enum i ∈ enum ` A ⟷ i ∈
A
  using inj_on_image_mem_iff[OF inj_enum] by auto

lemma enum_mono: i ≤ n ⟹ j ≤ n ⟹ enum i ≤ enum j ⟷ i ≤ j
  by (auto simp: enum_def le_fun_def in_upd_image Ball_def[symmetric])

lemma enum_strict_mono: i ≤ n ⟹ j ≤ n ⟹ enum i < enum j ⟷ i < j
  using enum_mono[of i j] enum_inj[of i j] by (auto simp: le_less)

lemma chain: a ∈ S ⟹ b ∈ S ⟹ a ≤ b ∨ b ≤ a
  by (auto simp: s_eq enum_mono)

lemma less: a ∈ S ⟹ b ∈ S ⟹ a i < b i ⟹ a < b
  using chain[of a b] by (auto simp: less_fun_def le_fun_def not_le[symmetric])

lemma enum_0_bot: a ∈ S ⟹ a = enum 0 ⟷ (∀ a'∈S. a ≤ a')
  unfolding s_eq by (auto simp: enum_mono Ball_def)

lemma enum_n_top: a ∈ S ⟹ a = enum n ⟷ (∀ a'∈S. a' ≤ a)
  unfolding s_eq by (auto simp: enum_mono Ball_def)

lemma enum_Suc: i < n ⟹ enum (Suc i) = (enum i)(upd i := Suc (enum i
(upd i)))
  by (auto simp: fun_eq_iff enum_def upd_inj)

lemma enum_eq_p: i ≤ n ⟹ n ≤ j ⟹ enum i j = p
```

by (*induct i*) (*auto simp*: *enum_Suc enum_0 base_out upd_space not_less*[*symmetric*])

lemma *out_eq_p*: $a \in S \Longrightarrow n \leq j \Longrightarrow a\ j = p$
unfolding *s_eq* **by** (*auto simp*: *enum_eq_p*)

lemma *s_le_p*: $a \in S \Longrightarrow a\ j \leq p$
using *out_eq_p*[*of a j*] *s_space* **by** (*cases* $j < n$) *auto*

lemma *le_Suc_base*: $a \in S \Longrightarrow a\ j \leq Suc\ (base\ j)$
unfolding *s_eq* **by** (*auto simp*: *enum_def*)

lemma *base_le*: $a \in S \Longrightarrow base\ j \leq a\ j$
unfolding *s_eq* **by** (*auto simp*: *enum_def*)

lemma *enum_le_p*: $i \leq n \Longrightarrow j < n \Longrightarrow enum\ i\ j \leq p$
using *enum_in*[*of i*] *s_space* **by** *auto*

lemma *enum_less*: $a \in S \Longrightarrow i < n \Longrightarrow enum\ i < a \longleftrightarrow enum\ (Suc\ i) \leq a$
unfolding *s_eq* **by** (*auto simp*: *enum_strict_mono enum_mono*)

lemma *ksimplex_0*:
$n = 0 \Longrightarrow S = \{(\lambda x.\ p)\}$
using *s_eq enum_def base_out* **by** *auto*

lemma *replace_0*:
assumes $j < n\ a \in S$ **and** p: $\forall x{\in}S - \{a\}.\ x\ j = 0$ **and** $x \in S$
shows $x \leq a$
proof *cases*
assume $x \neq a$
have $a\ j \neq 0$
using *assms* **by** (*intro one_step*[**where** $a{=}a$]) *auto*
with *less*[*OF* ‹$x{\in}S$› ‹$a{\in}S$›, *of j*] *p*[*rule_format*, *of x*] ‹$x \in S$› ‹$x \neq a$›
show *?thesis*
by *auto*
qed *simp*

lemma *replace_1*:
assumes $j < n\ a \in S$ **and** p: $\forall x{\in}S - \{a\}.\ x\ j = p$ **and** $x \in S$
shows $a \leq x$
proof *cases*
assume $x \neq a$
have $a\ j \neq p$
using *assms* **by** (*intro one_step*[**where** $a{=}a$]) *auto*
with *enum_le_p*[*of _ j*] ‹$j < n$› ‹$a{\in}S$›
have $a\ j < p$
by (*auto simp*: *less_le s_eq*)
with *less*[*OF* ‹$a{\in}S$› ‹$x{\in}S$›, *of j*] *p*[*rule_format*, *of x*] ‹$x \in S$› ‹$x \neq a$›
show *?thesis*
by *auto*

qed *simp*

end

locale *kuhn_simplex_pair* = *s*: *kuhn_simplex p n b_s u_s s* + *t*: *kuhn_simplex p n b_t u_t t*
  for *p n b_s u_s s b_t u_t t*
begin

lemma *enum_eq*:
  assumes *l*: $i \leq l\ l \leq j$ **and** $j + d \leq n$
  assumes *eq*: *s.enum* ‘ $\{i .. j\}$ = *t.enum* ‘ $\{i + d .. j + d\}$
  shows *s.enum l* = *t.enum* $(l + d)$
using *l* **proof** (*induct l rule*: *dec_induct*)
  case *base*
  then have *s*: *s.enum i* ∈ *t.enum* ‘ $\{i + d .. j + d\}$ **and** *t*: *t.enum* $(i + d)$ ∈ *s.enum* ‘ $\{i .. j\}$
    using *eq* **by** *auto*
  from *t* ‹$i \leq j$› ‹$j + d \leq n$› **have** *s.enum i* ≤ *t.enum* $(i + d)$
    by (*auto simp*: *s.enum_mono*)
  moreover from *s* ‹$i \leq j$› ‹$j + d \leq n$› **have** *t.enum* $(i + d) \leq$ *s.enum i*
    by (*auto simp*: *t.enum_mono*)
  ultimately show *?case*
    by *auto*
next
  case (*step l*)
  moreover from *step.prems* ‹$j + d \leq n$› **have**
      s.enum l < *s.enum* (*Suc l*)
      t.enum $(l + d)$ < *t.enum* (*Suc l* + *d*)
    by (*simp_all add*: *s.enum_strict_mono t.enum_strict_mono*)
  moreover have
      s.enum (*Suc l*) ∈ *t.enum* ‘ $\{i + d .. j + d\}$
      t.enum (*Suc l* + *d*) ∈ *s.enum* ‘ $\{i .. j\}$
    using *step* ‹$j + d \leq n$› *eq* **by** (*auto simp*: *s.enum_inj t.enum_inj*)
  ultimately have *s.enum* (*Suc l*) = *t.enum* (*Suc* $(l + d)$)
    using ‹$j + d \leq n$›
    by (*intro antisym s.enum_less*[*THEN iffD1*] *t.enum_less*[*THEN iffD1*])
      (*auto intro*!: *s.enum_in t.enum_in*)
  then show *?case* **by** *simp*
qed

lemma *ksimplex_eq_bot*:
  assumes *a*: $a \in s \bigwedge a'.\ a' \in s \Longrightarrow a \leq a'$
  assumes *b*: $b \in t \bigwedge b'.\ b' \in t \Longrightarrow b \leq b'$
  assumes *eq*: $s - \{a\} = t - \{b\}$
  shows $s = t$
proof *cases*
  assume *n* = *0* **with** *s.ksimplex_0 t.ksimplex_0* **show** *?thesis* **by** *simp*
next

```
  assume n ≠ 0
  have s.enum 0 = (s.enum (Suc 0)) (u_s 0 := s.enum (Suc 0) (u_s 0) − 1)
       t.enum 0 = (t.enum (Suc 0)) (u_t 0 := t.enum (Suc 0) (u_t 0) − 1)
    using ‹n ≠ 0› by (simp_all add: s.enum_Suc t.enum_Suc)
  moreover have e0: a = s.enum 0 b = t.enum 0
    using a b by (simp_all add: s.enum_0_bot t.enum_0_bot)
  moreover
  { fix j assume 0 < j j ≤ n
    moreover have s − {a} = s.enum ‘ {Suc 0 .. n} t − {b} = t.enum ‘ {Suc 0
.. n}
      unfolding s.s_eq t.s_eq e0 by (auto simp: s.enum_inj t.enum_inj)
    ultimately have s.enum j = t.enum j
      using enum_eq[of 1 j n 0] eq by auto }
  note enum_eq = this
  then have s.enum (Suc 0) = t.enum (Suc 0)
    using ‹n ≠ 0› by auto
  moreover
  { fix j assume Suc j < n
    with enum_eq[of Suc j] enum_eq[of Suc (Suc j)]
    have u_s (Suc j) = u_t (Suc j)
      using s.enum_Suc[of Suc j] t.enum_Suc[of Suc j]
      by (auto simp: fun_eq_iff split: if_split_asm) }
  then have ⋀j. 0 < j ⟹ j < n ⟹ u_s j = u_t j
    by (auto simp: gr0_conv_Suc)
  with ‹n ≠ 0› have u_t 0 = u_s 0
    by (intro bij_betw_singleton_eq[OF t.upd s.upd, of 0]) auto
  ultimately have a = b
    by simp
  with assms show s = t
    by auto
qed

lemma ksimplex_eq_top:
  assumes a: a ∈ s ⋀a'. a' ∈ s ⟹ a' ≤ a
  assumes b: b ∈ t ⋀b'. b' ∈ t ⟹ b' ≤ b
  assumes eq: s − {a} = t − {b}
  shows s = t
proof (cases n)
  assume n = 0 with s.ksimplex_0 t.ksimplex_0 show ?thesis by simp
next
  case (Suc n')
  have s.enum n = (s.enum n') (u_s n' := Suc (s.enum n' (u_s n')))
       t.enum n = (t.enum n') (u_t n' := Suc (t.enum n' (u_t n')))
    using Suc by (simp_all add: s.enum_Suc t.enum_Suc)
  moreover have en: a = s.enum n b = t.enum n
    using a b by (simp_all add: s.enum_n_top t.enum_n_top)
  moreover
  { fix j assume j < n
    moreover have s − {a} = s.enum ‘ {0 .. n'} t − {b} = t.enum ‘ {0 .. n'}
```

```
    unfolding s.s_eq t.s_eq en by (auto simp: s.enum_inj t.enum_inj Suc)
   ultimately have s.enum j = t.enum j
    using enum_eq[of 0 j n' 0] eq Suc by auto }
  note enum_eq = this
  then have s.enum n' = t.enum n'
   using Suc by auto
  moreover
  { fix j assume j < n'
   with enum_eq[of j] enum_eq[of Suc j]
   have u_s j = u_t j
    using s.enum_Suc[of j] t.enum_Suc[of j]
    by (auto simp: Suc fun_eq_iff split: if_split_asm) }
  then have ⋀j. j < n' ⟹ u_s j = u_t j
   by (auto simp: gr0_conv_Suc)
  then have u_t n' = u_s n'
   by (intro bij_betw_singleton_eq[OF t.upd s.upd, of n']) (auto simp: Suc)
  ultimately have a = b
   by simp
  with assms show s = t
   by auto
qed

end

inductive ksimplex for p n :: nat where
  ksimplex: kuhn_simplex p n base upd s ⟹ ksimplex p n s

lemma finite_ksimplexes: finite {s. ksimplex p n s}
proof (rule finite_subset)
  { fix a s assume ksimplex p n s a ∈ s
   then obtain b u where kuhn_simplex p n b u s by (auto elim: ksimplex.cases)
   then interpret kuhn_simplex p n b u s .
   from s_space ‹a ∈ s› out_eq_p[OF ‹a ∈ s›]
   have a ∈ (λf x. if n ≤ x then p else f x) ` ({..< n} →_E {.. p})
    by (auto simp: image_iff subset_eq Pi_iff split: if_split_asm
          intro!: bexI[of _ restrict a {..< n}]) }
  then show {s. ksimplex p n s} ⊆ Pow ((λf x. if n ≤ x then p else f x) ` ({..<
n} →_E {.. p}))
   by auto
qed (simp add: finite_PiE)

lemma ksimplex_card:
  assumes ksimplex p n s shows card s = Suc n
using assms proof cases
  case (ksimplex u b)
  then interpret kuhn_simplex p n u b s .
  show ?thesis
   by (simp add: card_image s_eq inj_enum)
qed
```

lemma *simplex_top_face*:
assumes $0 < p$ $\forall x{\in}s'.\ x\ n = p$
shows *ksimplex p n s'* $\longleftrightarrow$ $(\exists s\ a.$ *ksimplex p* (*Suc n*) $s \wedge a \in s \wedge s' = s - \{a\})$
using *assms*
proof *safe*
fix *s a* **assume** *ksimplex p* (*Suc n*) *s* **and** *a*: $a \in s$ **and** *na*: $\forall x{\in}s - \{a\}.\ x\ n = p$
then show *ksimplex p n* $(s - \{a\})$
proof *cases*
case (*ksimplex base upd*)
then interpret *kuhn_simplex p Suc n base upd s* **.**

have $a\ n < p$
using *one_step*[*of a n p*] *na* ‹$a{\in}s$› *s_space* **by** (*auto simp*: *less_le*)
then have $a = enum\ 0$
using ‹$a \in s$› *na* **by** (*subst enum_0_bot*) (*auto simp*: *le_less intro*!: *less*[*of a _ n*])
then have *s_eq*: $s - \{a\} = enum\ `\ Suc\ `\ \{..\ n\}$
using *s_eq* **by** (*simp add*: *atMost_Suc_eq_insert_0 insert_ident in_enum_image subset_eq*)
then have $enum\ 1 \in s - \{a\}$
by *auto*
then have $upd\ 0 = n$
using ‹$a\ n < p$› ‹$a = enum\ 0$› *na*[*rule_format*, *of enum 1*]
by (*auto simp*: *fun_eq_iff enum_Suc split*: *if_split_asm*)
then have *bij_betw upd* (*Suc* ‘ $\{..<n\}$) $\{..<n\}$
using *upd*
by (*subst notIn_Un_bij_betw3*[**where** *b=0*])
(*auto simp*: *lessThan_Suc*[*symmetric*] *lessThan_Suc_eq_insert_0*)
then have *bij_betw* $(upd{\circ}Suc)$ $\{..<n\}$ $\{..<n\}$
by (*rule bij_betw_trans*[*rotated*]) (*auto simp*: *bij_betw_def*)

have $a\ n = p - 1$
using *enum_Suc*[*of 0*] *na*[*rule_format*, *OF* ‹$enum\ 1 \in s - \{a\}$›] ‹$a = enum\ 0$› **by** (*auto simp*: ‹$upd\ 0 = n$›)

show *?thesis*
proof (*rule ksimplex.intros*, *standard*)
show *bij_betw* $(upd{\circ}Suc)$ $\{..<n\}$ $\{..<n\}$ **by** *fact*
show $base(n := p) \in \{..<n\} \to \{..<p\}$ $\bigwedge i.\ n{\le}i \Longrightarrow (base(n := p))\ i = p$
using *base base_out* **by** (*auto simp*: *Pi_iff*)

have $\bigwedge i.\ Suc\ `\ \{..<i\} = \{..<Suc\ i\} - \{0\}$
by (*auto simp*: *image_iff Ball_def*) *arith*
then have *upd_Suc*: $\bigwedge i.\ i \le n \Longrightarrow (upd{\circ}Suc)\ `\ \{..<i\} = upd\ `\ \{..<Suc\ i\} - \{n\}$
using ‹$upd\ 0 = n$› *upd_inj* **by** (*auto simp add*: *image_iff less_Suc_eq_0_disj*)
have *n_in_upd*: $\bigwedge i.\ n \in upd\ `\ \{..<Suc\ i\}$

```
      using ‹upd 0 = n› by auto

    define f' where f' i j =
      (if j ∈ (upd∘Suc)`{..< i} then Suc ((base(n := p)) j) else (base(n := p)) j)
for i j
    { fix x i
      assume i [arith]: i ≤ n
      with upd_Suc have (upd ∘ Suc) ` {..<i} = upd ` {..<Suc i} - {n} .
      with ‹a n < p› ‹a = enum 0› ‹upd 0 = n› ‹a n = p - 1›
      have enum (Suc i) x = f' i x
        by (auto simp add: f'_def enum_def) }
    then show s - {a} = f' ` {.. n}
      unfolding s_eq image_comp by (intro image_cong) auto
  qed
 qed
next
 assume ksimplex p n s' and ∗: ∀x∈s'. x n = p
 then show ∃s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ s' = s - {a}
 proof cases
  case (ksimplex base upd)
  then interpret kuhn_simplex p n base upd s' .
  define b where b = base (n := p - 1)
  define u where u i = (case i of 0 ⇒ n | Suc i ⇒ upd i) for i

  have ksimplex p (Suc n) (s' ∪ {b})
  proof (rule ksimplex.intros, standard)
   show b ∈ {..<Suc n} → {..<p}
    using base ‹0 < p› unfolding lessThan_Suc b_def by (auto simp: PiE_iff)
   show ⋀i. Suc n ≤ i ⟹ b i = p
    using base_out by (auto simp: b_def)

   have bij_betw u (Suc ` {..< n} ∪ {0}) ({..<n} ∪ {u 0})
    using upd
    by (intro notIn_Un_bij_betw) (auto simp: u_def bij_betw_def image_comp
comp_def inj_on_def)
   then show bij_betw u {..<Suc n} {..<Suc n}
    by (simp add: u_def lessThan_Suc[symmetric] lessThan_Suc_eq_insert_0)

   define f' where f' i j = (if j ∈ u`{..< i} then Suc (b j) else b j) for i j

   have u_eq: ⋀i. i ≤ n ⟹ u ` {..< Suc i} = upd ` {..< i} ∪ { n }
    by (auto simp: u_def image_iff upd_inj Ball_def split: nat.split) arith

   { fix x have x ≤ n ⟹ n ∉ upd ` {..<x}
     using upd_space by (simp add: image_iff neq_iff) }
   note n_not_upd = this

   have ∗: f' ` {.. Suc n} = f' ` (Suc ` {.. n} ∪ {0})
    unfolding atMost_Suc_eq_insert_0 by simp
```

```
    also have ... = (f' ∘ Suc) ` {.. n} ∪ {b}
      by (auto simp: f'_def)
    also have (f' ∘ Suc) ` {.. n} = s'
      using ‹0 < p› base_out[of n]
      unfolding s_eq enum_def[abs_def] f'_def[abs_def] upd_space
      by (intro image_cong) (simp_all add: u_eq b_def fun_eq_iff n_not_upd)
    finally show s' ∪ {b} = f' ` {.. Suc n} ..
  qed
  moreover have b ∉ s'
    using * ‹0 < p› by (auto simp: b_def)
  ultimately show ?thesis by auto
 qed
qed

lemma ksimplex_replace_0:
 assumes s: ksimplex p n s and a: a ∈ s
 assumes j: j < n and p: ∀x∈s - {a}. x j = 0
 shows card {s'. ksimplex p n s' ∧ (∃b∈s'. s' - {b} = s - {a})} = 1
 using s
proof cases
 case (ksimplex b_s u_s)

 { fix t b assume ksimplex p n t
   then obtain b_t u_t where kuhn_simplex p n b_t u_t t
     by (auto elim: ksimplex.cases)
   interpret kuhn_simplex_pair p n b_s u_s s b_t u_t t
     by intro_locales fact+

   assume b: b ∈ t t - {b} = s - {a}
   with a j p s.replace_0[of _ a] t.replace_0[of _ b] have s = t
     by (intro ksimplex_eq_top[of a b]) auto }
 then have {s'. ksimplex p n s' ∧ (∃b∈s'. s' - {b} = s - {a})} = {s}
   using s ‹a ∈ s› by auto
 then show ?thesis
   by simp
qed

lemma ksimplex_replace_1:
 assumes s: ksimplex p n s and a: a ∈ s
 assumes j: j < n and p: ∀x∈s - {a}. x j = p
 shows card {s'. ksimplex p n s' ∧ (∃b∈s'. s' - {b} = s - {a})} = 1
 using s
proof cases
 case (ksimplex b_s u_s)

 { fix t b assume ksimplex p n t
   then obtain b_t u_t where kuhn_simplex p n b_t u_t t
     by (auto elim: ksimplex.cases)
   interpret kuhn_simplex_pair p n b_s u_s s b_t u_t t
```

```
    by intro_locales fact+

  assume b: b ∈ t t − {b} = s − {a}
  with a j p s.replace_1[of _ a] t.replace_1[of _ b] have s = t
    by (intro ksimplex_eq_bot[of a b]) auto }
 then have {s'. ksimplex p n s' ∧ (∃ b∈s'. s' − {b} = s − {a})} = {s}
  using s ‹a ∈ s› by auto
 then show ?thesis
  by simp
qed

lemma ksimplex_replace_2:
 assumes s: ksimplex p n s and a ∈ s and n ≠ 0
  and lb: ∀ j<n. ∃ x∈s − {a}. x j ≠ 0
  and ub: ∀ j<n. ∃ x∈s − {a}. x j ≠ p
 shows card {s'. ksimplex p n s' ∧ (∃ b∈s'. s' − {b} = s − {a})} = 2
 using s
proof cases
 case (ksimplex base upd)
 then interpret kuhn_simplex p n base upd s .

 from ‹a ∈ s› obtain i where i ≤ n a = enum i
  unfolding s_eq by auto

 from ‹i ≤ n› have i = 0 ∨ i = n ∨ (0 < i ∧ i < n)
  by linarith
 then have ∃!s'. s' ≠ s ∧ ksimplex p n s' ∧ (∃ b∈s'. s − {a} = s'− {b})
 proof (elim disjE conjE)
  assume i = 0
  define rot where [abs_def]: rot i = (if i + 1 = n then 0 else i + 1) for i
  let ?upd = upd ∘ rot

  have rot: bij_betw rot {..< n} {..< n}
   by (auto simp: bij_betw_def inj_on_def image_iff Ball_def rot_def)
     arith+
  from rot upd have bij_betw ?upd {..<n} {..<n}
   by (rule bij_betw_trans)

  define f' where [abs_def]: f' i j =
   (if j ∈ ?upd‘{..< i} then Suc (enum (Suc 0) j) else enum (Suc 0) j) for i j

  interpret b: kuhn_simplex p n enum (Suc 0) upd ∘ rot f' ‘ {.. n}
  proof
   from ‹a = enum i› ub ‹n ≠ 0› ‹i = 0›
   obtain i' where i' ≤ n enum i' ≠ enum 0 enum i' (upd 0) ≠ p
    unfolding s_eq by (auto intro: upd_space simp: enum_inj)
   then have enum 1 ≤ enum i' enum i' (upd 0) < p
     using enum_le_p[of i' upd 0] by (auto simp: enum_inj enum_mono
upd_space)
```

```
    then have enum 1 (upd 0) < p
      by (auto simp: le_fun_def intro: le_less_trans)
    then show enum (Suc 0) ∈ {..<n} → {..<p}
       using base ‹n ≠ 0› by (auto simp: enum_0 enum_Suc PiE_iff extensional_def upd_space)

    { fix i assume n ≤ i then show enum (Suc 0) i = p
      using ‹n ≠ 0› by (auto simp: enum_eq_p) }
    show bij_betw ?upd {..<n} {..<n} by fact
  qed (simp add: f'_def)
  have ks_f': ksimplex p n (f' ‘ {.. n})
    by rule unfold_locales

  have b_enum: b.enum = f' unfolding f'_def b.enum_def[abs_def] ..
  with b.inj_enum have inj_f': inj_on f' {.. n} by simp

  have f'_eq_enum: f' j = enum (Suc j) if j < n for j
  proof −
    from that have rot ‘ {..< j} = {0 <..< Suc j}
      by (auto simp: rot_def image_Suc_lessThan cong: image_cong_simp)
    with that ‹n ≠ 0› show ?thesis
      by (simp only: f'_def enum_def fun_eq_iff image_comp [symmetric])
        (auto simp add: upd_inj)
  qed
  then have enum ‘ Suc ‘ {..< n} = f' ‘ {..< n}
    by (force simp: enum_inj)
  also have Suc ‘ {..< n} = {.. n} − {0}
    by (auto simp: image_iff Ball_def) arith
  also have {..< n} = {.. n} − {n}
    by auto
  finally have eq: s − {a} = f' ‘ {.. n} − {f' n}
    unfolding s_eq ‹a = enum i› ‹i = 0›
  by (simp add: inj_on_image_set_diff[OF inj_enum] inj_on_image_set_diff[OF inj_f'])

  have enum 0 < f' 0
    using ‹n ≠ 0› by (simp add: enum_strict_mono f'_eq_enum)
  also have ... < f' n
    using ‹n ≠ 0› b.enum_strict_mono[of 0 n] unfolding b_enum by simp
  finally have a ≠ f' n
    using ‹a = enum i› ‹i = 0› by auto

  { fix t c assume ksimplex p n t c ∈ t and eq_sma: s − {a} = t − {c}
    obtain b u where kuhn_simplex p n b u t
      using ‹ksimplex p n t› by (auto elim: ksimplex.cases)
    then interpret t: kuhn_simplex p n b u t .

    { fix x assume x ∈ s x ≠ a
      then have x (upd 0) = enum (Suc 0) (upd 0)
```

```
        by (auto simp: ‹a = enum i› ‹i = 0› s_eq enum_def enum_inj) }
    then have eq_upd0: ∀x∈t−{c}. x (upd 0) = enum (Suc 0) (upd 0)
      unfolding eq_sma[symmetric] by auto
    then have c (upd 0) ≠ enum (Suc 0) (upd 0)
      using ‹n ≠ 0› by (intro t.one_step[OF ‹c∈t› ]) (auto simp: upd_space)
    then have c (upd 0) < enum (Suc 0) (upd 0) ∨ c (upd 0) > enum (Suc 0)
(upd 0)
      by auto
    then have t = s ∨ t = f' ‘ {..n}
    proof (elim disjE conjE)
      assume ∗: c (upd 0) < enum (Suc 0) (upd 0)
      interpret st: kuhn_simplex_pair p n base upd s b u t ..
      { fix x assume x ∈ t with ∗ ‹c∈t› eq_upd0[rule_format, of x] have c ≤ x
          by (auto simp: le_less intro!: t.less[of _ _ upd 0]) }
      note top = this
      have s = t
        using ‹a = enum i› ‹i = 0› ‹c ∈ t›
        by (intro st.ksimplex_eq_bot[OF _ _ _ _ eq_sma])
           (auto simp: s_eq enum_mono t.s_eq t.enum_mono top)
      then show ?thesis by simp
    next
      assume ∗: c (upd 0) > enum (Suc 0) (upd 0)
      interpret st: kuhn_simplex_pair p n enum (Suc 0) upd ∘ rot f' ‘ {.. n} b
u t ..
      have eq: f' ‘ {..n} − {f' n} = t − {c}
        using eq_sma eq by simp
      { fix x assume x ∈ t with ∗ ‹c∈t› eq_upd0[rule_format, of x] have x ≤ c
          by (auto simp: le_less intro!: t.less[of _ _ upd 0]) }
      note top = this
      have f' ‘ {..n} = t
        using ‹a = enum i› ‹i = 0› ‹c ∈ t›
        by (intro st.ksimplex_eq_top[OF _ _ _ _ eq])
       (auto simp: b.s_eq b.enum_mono t.s_eq t.enum_mono b_enum[symmetric]
top)
      then show ?thesis by simp
    qed }
  with ks_f' eq ‹a ≠ f' n› ‹n ≠ 0› show ?thesis
    apply (intro ex1I[of _ f' ‘ {.. n}])
     apply auto []
    apply metis
    done
next
  assume i = n
  from ‹n ≠ 0› obtain n' where n': n = Suc n'
    by (cases n) auto

  define rot where rot i = (case i of 0 ⇒ n' | Suc i ⇒ i) for i
  let ?upd = upd ∘ rot
```

```
    have rot: bij_betw rot {..< n} {..< n}
      by (auto simp: bij_betw_def inj_on_def image_iff Bex_def rot_def n' split:
nat.splits)
         arith
    from rot upd have bij_betw ?upd {..<n} {..<n}
      by (rule bij_betw_trans)

    define b where b = base (upd n' := base (upd n') − 1)
    define f' where [abs_def]: f' i j = (if j ∈ ?upd‘{..< i} then Suc (b j) else b
j) for i j

    interpret b: kuhn_simplex p n b upd ∘ rot f' ‘ {.. n}
    proof
      { fix i assume n ≤ i then show b i = p
          using base_out[of i] upd_space[of n'] by (auto simp: b_def n') }
      show b ∈ {..<n} → {..<p}
        using base ‹n ≠ 0› upd_space[of n']
        by (auto simp: b_def PiE_def Pi_iff Ball_def upd_space extensional_def
n')

      show bij_betw ?upd {..<n} {..<n} by fact
    qed (simp add: f'_def)
    have f': b.enum = f' unfolding f'_def b.enum_def[abs_def] ..
    have ks_f': ksimplex p n (b.enum ‘ {.. n})
      unfolding f' by rule unfold_locales

    have 0 < n
      using ‹n ≠ 0› by auto

    { from ‹a = enum i› ‹n ≠ 0› ‹i = n› lb upd_space[of n']
      obtain i' where i' ≤ n enum i' ≠ enum n 0 < enum i' (upd n')
        unfolding s_eq by (auto simp: enum_inj n')
      moreover have enum i' (upd n') = base (upd n')
        unfolding enum_def using ‹i' ≤ n› ‹enum i' ≠ enum n› by (auto simp:
n' upd_inj enum_inj)
      ultimately have 0 < base (upd n')
        by auto }
    then have benum1: b.enum (Suc 0) = base
     unfolding b.enum_Suc[OF ‹0<n›] b.enum_0 by (auto simp: b_def rot_def)

    have [simp]: ⋀j. Suc j < n ⟹ rot ‘ {..< Suc j} = {n'} ∪ {..< j}
      by (auto simp: rot_def image_iff Ball_def split: nat.splits)
    have rot_simps: ⋀j. rot (Suc j) = j rot 0 = n'
      by (simp_all add: rot_def)

    { fix j assume j: Suc j ≤ n then have b.enum (Suc j) = enum j
      by (induct j) (auto simp: benum1 enum_0 b.enum_Suc enum_Suc rot_simps)
}
    note b_enum_eq_enum = this
```

```
    then have enum ` {..< n} = b.enum ` Suc ` {..< n}
      by (auto simp: image_comp intro!: image_cong)
    also have Suc ` {..< n} = {.. n} − {0}
      by (auto simp: image_iff Ball_def) arith
    also have {..< n} = {.. n} − {n}
      by auto
    finally have eq: s − {a} = b.enum ` {.. n} − {b.enum 0}
      unfolding s_eq ‹a = enum i› ‹i = n›
      using inj_on_image_set_diff[OF inj_enum Diff_subset, of {n}]
            inj_on_image_set_diff[OF b.inj_enum Diff_subset, of {0}]
      by (simp add: comp_def)

    have b.enum 0 ≤ b.enum n
      by (simp add: b.enum_mono)
    also have b.enum n < enum n
      using ‹n ≠ 0› by (simp add: enum_strict_mono b_enum_eq_enum n′)
    finally have a ≠ b.enum 0
      using ‹a = enum i› ‹i = n› by auto

    { fix t c assume ksimplex p n t c ∈ t and eq_sma: s − {a} = t − {c}
      obtain b′ u where kuhn_simplex p n b′ u t
        using ‹ksimplex p n t› by (auto elim: ksimplex.cases)
      then interpret t: kuhn_simplex p n b′ u t .

      { fix x assume x ∈ s x ≠ a
        then have x (upd n′) = enum n′ (upd n′)
          by (auto simp: ‹a = enum i› n′ ‹i = n› s_eq enum_def enum_inj
in_upd_image) }
      then have eq_upd0: ∀ x∈t−{c}. x (upd n′) = enum n′ (upd n′)
        unfolding eq_sma[symmetric] by auto
      then have c (upd n′) ≠ enum n′ (upd n′)
      using ‹n ≠ 0› by (intro t.one_step[OF ‹c∈t› ]) (auto simp: n′ upd_space[unfolded
n′])
      then have c (upd n′) < enum n′ (upd n′) ∨ c (upd n′) > enum n′ (upd n′)
        by auto
      then have t = s ∨ t = b.enum ` {..n}
      proof (elim disjE conjE)
        assume ∗: c (upd n′) > enum n′ (upd n′)
        interpret st: kuhn_simplex_pair p n base upd s b′ u t ..
        { fix x assume x ∈ t with ∗ ‹c∈t› eq_upd0[rule_format, of x] have x ≤ c
            by (auto simp: le_less intro!: t.less[of _ _ upd n′]) }
        note top = this
        have s = t
          using ‹a = enum i› ‹i = n› ‹c ∈ t›
          by (intro st.ksimplex_eq_top[OF _ _ _ _ eq_sma])
             (auto simp: s_eq enum_mono t.s_eq t.enum_mono top)
        then show ?thesis by simp
      next
        assume ∗: c (upd n′) < enum n′ (upd n′)
```

```
      interpret st: kuhn_simplex_pair p n b upd ∘ rot f' ‘ {.. n} b' u t ..
      have eq: f' ‘ {..n} − {b.enum 0} = t − {c}
        using eq_sma eq f' by simp
      { fix x assume x ∈ t with ∗ ‹c∈t› eq_upd0[rule_format, of x] have c ≤ x
          by (auto simp: le_less intro!: t.less[of _ _ upd n']) }
      note bot = this
      have f' ‘ {..n} = t
        using ‹a = enum i› ‹i = n› ‹c ∈ t›
        by (intro st.ksimplex_eq_bot[OF _ _ _ _ eq])
          (auto simp: b.s_eq b.enum_mono t.s_eq t.enum_mono bot)
      with f' show ?thesis by simp
    qed }
  with ks_f' eq ‹a ≠ b.enum 0› ‹n ≠ 0› show ?thesis
    apply (intro ex1I[of _ b.enum ‘ {.. n}])
    apply fastforce
    apply metis
    done
next
  assume i: 0 < i i < n
  define i' where i' = i − 1
  with i have Suc i' < n
    by simp
  with i have Suc_i': Suc i' = i
    by (simp add: i'_def)

  let ?upd = Fun.swap i' i upd
  from i upd have bij_betw ?upd {..< n} {..< n}
    by (subst bij_betw_swap_iff) (auto simp: i'_def)

  define f' where [abs_def]: f' i j = (if j ∈ ?upd‘{..< i} then Suc (base j) else
base j)
    for i j
  interpret b: kuhn_simplex p n base ?upd f' ‘ {.. n}
  proof
    show base ∈ {..<n} → {..<p} by (rule base)
    { fix i assume n ≤ i then show base i = p by (rule base_out) }
    show bij_betw ?upd {..<n} {..<n} by fact
  qed (simp add: f'_def)
  have f': b.enum = f' unfolding f'_def b.enum_def[abs_def] ..
  have ks_f': ksimplex p n (b.enum ‘ {.. n})
    unfolding f' by rule unfold_locales

  have {i} ⊆ {..n}
    using i by auto
  { fix j assume j ≤ n
    with i Suc_i' have enum j = b.enum j ⟷ j ≠ i
      unfolding fun_eq_iff enum_def b.enum_def image_comp [symmetric]
      apply (cases ‹i = j›)
      apply (metis imageI in_upd_image lessI lessThan_iff lessThan_subset_iff
```

```
order_less_le transpose_apply_first)
        by (metis lessThan_iff linorder_not_less not_less_eq_eq order_less_le
transpose_image_eq)
    }
   note enum_eq_benum = this
   then have enum ‘ ({.. n} − {i}) = b.enum ‘ ({.. n} − {i})
    by (intro image_cong) auto
   then have eq: s − {a} = b.enum ‘ {.. n} − {b.enum i}
    unfolding s_eq ‹a = enum i›
    using inj_on_image_set_diff[OF inj_enum Diff_subset ‹{i} ⊆ {..n}›]
        inj_on_image_set_diff[OF b.inj_enum Diff_subset ‹{i} ⊆ {..n}›]
    by (simp add: comp_def)

   have a ≠ b.enum i
    using ‹a = enum i› enum_eq_benum i by auto

   { fix t c assume ksimplex p n t c ∈ t and eq_sma: s − {a} = t − {c}
    obtain b′ u where kuhn_simplex p n b′ u t
     using ‹ksimplex p n t› by (auto elim: ksimplex.cases)
    then interpret t: kuhn_simplex p n b′ u t .
    have enum i′ ∈ s − {a} enum (i + 1) ∈ s − {a}
     using ‹a = enum i› i enum_in by (auto simp: enum_inj i′_def)
    then obtain l k where
     l: t.enum l = enum i′ l ≤ n t.enum l ≠ c and
     k: t.enum k = enum (i + 1) k ≤ n t.enum k ≠ c
     unfolding eq_sma by (auto simp: t.s_eq)
    with i have t.enum l < t.enum k
     by (simp add: enum_strict_mono i′_def)
    with ‹l ≤ n› ‹k ≤ n› have l < k
     by (simp add: t.enum_strict_mono)
    { assume Suc l = k
     have enum (Suc (Suc i′)) = t.enum (Suc l)
      using i by (simp add: k ‹Suc l = k› i′_def)
     then have False
      using ‹l < k› ‹k ≤ n› ‹Suc i′ < n›
        by (auto simp: t.enum_Suc enum_Suc l upd_inj fun_eq_iff split:
if_split_asm)
       (metis Suc_lessD n_not_Suc_n upd_inj) }
    with ‹l < k› have Suc l < k
     by arith
    have c_eq: c = t.enum (Suc l)
    proof (rule ccontr)
     assume c ≠ t.enum (Suc l)
     then have t.enum (Suc l) ∈ s − {a}
      using ‹l < k› ‹k ≤ n› by (simp add: t.s_eq eq_sma)
     then obtain j where t.enum (Suc l) = enum j j ≤ n enum j ≠ enum i
      unfolding s_eq ‹a = enum i› by auto
     with i have t.enum (Suc l) ≤ t.enum l ∨ t.enum k ≤ t.enum (Suc l)
      by (auto simp: i′_def enum_mono enum_inj l k)
```

```
      with ‹Suc l < k› ‹k ≤ n› show False
        by (simp add: t.enum_mono)
    qed

    { have t.enum (Suc (Suc l)) ∈ s − {a}
      unfolding eq_sma c_eq t.s_eq using ‹Suc l < k› ‹k ≤ n› by (auto simp:
t.enum_inj)
      then obtain j where eq: t.enum (Suc (Suc l)) = enum j and j ≤ n j ≠ i
        by (auto simp: s_eq ‹a = enum i›)
      moreover have enum i' < t.enum (Suc (Suc l))
        unfolding l(1)[symmetric] using ‹Suc l < k› ‹k ≤ n› by (auto simp:
t.enum_strict_mono)
      ultimately have i' < j
        using i by (simp add: enum_strict_mono i'_def)
      with ‹j ≠ i› ‹j ≤ n› have t.enum k ≤ t.enum (Suc (Suc l))
        unfolding i'_def by (simp add: enum_mono k eq)
      then have k ≤ Suc (Suc l)
        using ‹k ≤ n› ‹Suc l < k› by (simp add: t.enum_mono) }
    with ‹Suc l < k› have Suc (Suc l) = k by simp
    then have enum (Suc (Suc i')) = t.enum (Suc (Suc l))
      using i by (simp add: k i'_def)
     also have ... = (enum i') (u l := Suc (enum i' (u l)), u (Suc l) := Suc
(enum i' (u (Suc l))))
      using ‹Suc l < k› ‹k ≤ n› by (simp add: t.enum_Suc l t.upd_inj)
    finally have (u l = upd i' ∧ u (Suc l) = upd (Suc i')) ∨
      (u l = upd (Suc i') ∧ u (Suc l) = upd i')
     using ‹Suc i' < n› by (auto simp: enum_Suc fun_eq_iff split: if_split_asm)

    then have t = s ∨ t = b.enum ‘ {..n}
    proof (elim disjE conjE)
      assume u: u l = upd i'
      have c = t.enum (Suc l) unfolding c_eq ..
      also have t.enum (Suc l) = enum (Suc i')
       using u ‹l < k› ‹k ≤ n› ‹Suc i' < n› by (simp add: enum_Suc t.enum_Suc
l)
      also have ... = a
        using ‹a = enum i› i by (simp add: i'_def)
      finally show ?thesis
        using eq_sma ‹a ∈ s› ‹c ∈ t› by auto
    next
      assume u: u l = upd (Suc i')
      define B where B = b.enum ‘ {..n}
      have b.enum i' = enum i'
        using enum_eq_benum[of i'] i by (auto simp: i'_def gr0_conv_Suc)
      have c = t.enum (Suc l) unfolding c_eq ..
      also have t.enum (Suc l) = b.enum (Suc i')
        using u ‹l < k› ‹k ≤ n› ‹Suc i' < n›
          by (simp_all add: enum_Suc t.enum_Suc l b.enum_Suc ‹b.enum i' =
enum i'›)
```

```
        (simp add: Suc_i')
    also have ... = b.enum i
      using i by (simp add: i'_def)
    finally have c = b.enum i .
    then have t - {c} = B - {c} c ∈ B
      unfolding eq_sma[symmetric] eq B_def using i by auto
    with ‹c ∈ t› have t = B
      by auto
    then show ?thesis
      by (simp add: B_def)
  qed }
 with ks_f' eq ‹a ≠ b.enum i› ‹n ≠ 0› ‹i ≤ n› show ?thesis
  apply (intro ex1I[of _ b.enum ‘ {.. n}])
  apply auto []
  apply metis
  done
qed
then show ?thesis
 using s ‹a ∈ s› by (simp add: card_2_iff' Ex1_def) metis
qed
```

Hence another step towards concreteness.

```
lemma kuhn_simplex_lemma:
  assumes ∀ s. ksimplex p (Suc n) s ⟶ rl ‘ s ⊆ {.. Suc n}
    and odd (card {f. ∃ s a. ksimplex p (Suc n) s ∧ a ∈ s ∧ (f = s - {a}) ∧
      rl ‘ f = {..n} ∧ ((∃ j≤n. ∀ x∈f. x j = 0) ∨ (∃ j≤n. ∀ x∈f. x j = p))})
  shows odd (card {s. ksimplex p (Suc n) s ∧ rl ‘ s = {..Suc n}})
proof (rule kuhn_complete_lemma[OF finite_ksimplexes refl, unfolded mem_Collect_eq,
    where bnd=λf. (∃ j∈{..n}. ∀ x∈f. x j = 0) ∨ (∃ j∈{..n}. ∀ x∈f. x j = p)],
    safe del: notI)

  have *: ⋀x y. x = y ⟹ odd (card x) ⟹ odd (card y)
    by auto
  show odd (card {f. (∃ s∈{s. ksimplex p (Suc n) s}. ∃ a∈s. f = s - {a}) ∧
    rl ‘ f = {..n} ∧ ((∃ j∈{..n}. ∀ x∈f. x j = 0) ∨ (∃ j∈{..n}. ∀ x∈f. x j = p))})
    apply (rule *[OF _ assms(2)])
    apply (auto simp: atLeast0AtMost)
    done

next

  fix s assume s: ksimplex p (Suc n) s
  then show card s = n + 2
    by (simp add: ksimplex_card)

  fix a assume a: a ∈ s then show rl a ≤ Suc n
    using assms(1) s by (auto simp: subset_eq)

  let ?S = {t. ksimplex p (Suc n) t ∧ (∃ b∈t. s - {a} = t - {b})}
```

{ fix *j* **assume** *j*: $j \leq n \; \forall x{\in}s - \{a\}.\; x\; j = 0$
with *s a* **show** *card ?S = 1*
using *ksimplex_replace_0*[*of p n + 1 s a j*]
by (*subst eq_commute*) *simp* **}**

{ fix *j* **assume** *j*: $j \leq n \; \forall x{\in}s - \{a\}.\; x\; j = p$
with *s a* **show** *card ?S = 1*
using *ksimplex_replace_1*[*of p n + 1 s a j*]
by (*subst eq_commute*) *simp* **}**

{ assume *card ?S* $\neq$ *2* $\neg$ $(\exists j{\in}\{..n\}.\; \forall x{\in}s - \{a\}.\; x\; j = p)$
with *s a* **show** $\exists j{\in}\{..n\}.\; \forall x{\in}s - \{a\}.\; x\; j = 0$
using *ksimplex_replace_2*[*of p n + 1 s a*]
by (*subst* (*asm*) *eq_commute*) *auto* **}**
qed

Reduced labelling

definition *reduced* :: *nat* $\Rightarrow$ (*nat* $\Rightarrow$ *nat*) $\Rightarrow$ *nat* **where** *reduced n x* = (*LEAST k*. $k = n \vee x\; k \neq 0$)

lemma *reduced_labelling*:
shows *reduced n x* $\leq n$
and $\forall i{<}$*reduced n x*. *x i* = *0*
and *reduced n x* = $n \vee x$ (*reduced n x*) $\neq$ *0*
proof −
show *reduced n x* $\leq n$
unfolding *reduced_def* **by** (*rule LeastI2_wellorder*[**where** *a*=*n*]) *auto*
show $\forall i{<}$*reduced n x*. *x i* = *0*
unfolding *reduced_def* **by** (*rule LeastI2_wellorder*[**where** *a*=*n*]) *fastforce+*
show *reduced n x* = $n \vee x$ (*reduced n x*) $\neq$ *0*
unfolding *reduced_def* **by** (*rule LeastI2_wellorder*[**where** *a*=*n*]) *fastforce+*
qed

lemma *reduced_labelling_unique*:
$r \leq n \Longrightarrow \forall i{<}r.\; x\; i = 0 \Longrightarrow r = n \vee x\; r \neq 0 \Longrightarrow$ *reduced n x* = *r*
by (*metis linorder_less_linear linorder_not_le reduced_labelling*)

lemma *reduced_labelling_zero*: $j < n \Longrightarrow x\; j = 0 \Longrightarrow$ *reduced n x* $\neq$ *j*
using *reduced_labelling*[*of n x*] **by** *auto*

lemma *reduce_labelling_zero*[*simp*]: *reduced 0 x* = *0*
by (*rule reduced_labelling_unique*) *auto*

lemma *reduced_labelling_nonzero*: $j < n \Longrightarrow x\; j \neq 0 \Longrightarrow$ *reduced n x* $\leq j$
using *reduced_labelling*[*of n x*] **by** (*elim allE*[**where** *x*=*j*]) *auto*

lemma *reduced_labelling_Suc*: *reduced* (*Suc n*) *x* $\neq$ *Suc n* $\Longrightarrow$ *reduced* (*Suc n*) *x* = *reduced n x*

```
  using reduced_labelling[of Suc n x]
  by (intro reduced_labelling_unique[symmetric]) auto

lemma complete_face_top:
  assumes ∀x∈f. ∀j≤n. x j = 0 ⟶ lab x j = 0
    and ∀x∈f. ∀j≤n. x j = p ⟶ lab x j = 1
    and eq: (reduced (Suc n) ∘ lab) ‘ f = {..n}
  shows ((∃j≤n. ∀x∈f. x j = 0) ∨ (∃j≤n. ∀x∈f. x j = p)) ⟷ (∀x∈f. x n = p)
proof (safe del: disjCI)
  fix x j assume j: j ≤ n ∀x∈f. x j = 0
  { fix x assume x ∈ f with assms j have reduced (Suc n) (lab x) ≠ j
      by (intro reduced_labelling_zero) auto }
  moreover have j ∈ (reduced (Suc n) ∘ lab) ‘ f
    using j eq by auto
  ultimately show x n = p
    by force
next
  fix x j assume j: j ≤ n ∀x∈f. x j = p and x: x ∈ f
  have j = n
  proof (rule ccontr)
    assume ¬ ?thesis
    { fix x assume x ∈ f
      with assms j have reduced (Suc n) (lab x) ≤ j
        by (intro reduced_labelling_nonzero) auto
      then have reduced (Suc n) (lab x) ≠ n
        using ‹j ≠ n› ‹j ≤ n› by simp }
    moreover
    have n ∈ (reduced (Suc n) ∘ lab) ‘ f
      using eq by auto
    ultimately show False
      by force
  qed
  moreover have j ∈ (reduced (Suc n) ∘ lab) ‘ f
    using j eq by auto
  ultimately show x n = p
    using j x by auto
qed auto
```

Hence we get just about the nice induction.

```
lemma kuhn_induction:
  assumes 0 < p
    and lab_0: ∀x. ∀j≤n. (∀j. x j ≤ p) ∧ x j = 0 ⟶ lab x j = 0
    and lab_1: ∀x. ∀j≤n. (∀j. x j ≤ p) ∧ x j = p ⟶ lab x j = 1
    and odd: odd (card {s. ksimplex p n s ∧ (reduced n∘lab) ‘ s = {..n}})
  shows odd (card {s. ksimplex p (Suc n) s ∧ (reduced (Suc n)∘lab) ‘ s = {..Suc
n}})
proof −
  let ?rl = reduced (Suc n) ∘ lab and ?ext = λf v. ∃j≤n. ∀x∈f. x j = v
  let ?ext = λs. (∃j≤n. ∀x∈s. x j = 0) ∨ (∃j≤n. ∀x∈s. x j = p)
```

have $\forall s.\ ksimplex\ p\ (Suc\ n)\ s \longrightarrow ?rl\ `\ s \subseteq \{..Suc\ n\}$
by (*simp add*: *reduced_labelling subset_eq*)
moreover
have $\{s.\ ksimplex\ p\ n\ s \wedge (reduced\ n \circ lab)\ `\ s = \{..n\}\} =$
$\{f.\ \exists s\ a.\ ksimplex\ p\ (Suc\ n)\ s \wedge a \in s \wedge f = s - \{a\} \wedge ?rl\ `\ f = \{..n\} \wedge ?ext\ f\}$
proof (*intro set_eqI*, *safe del*: *disjCI equalityI disjE*)
fix s **assume** s: $ksimplex\ p\ n\ s$ **and** rl: $(reduced\ n \circ lab)\ `\ s = \{..n\}$
from s **obtain** $u\ b$ **where** $kuhn\_simplex\ p\ n\ u\ b\ s$ **by** (*auto elim*: *ksimplex.cases*)
then interpret $kuhn\_simplex\ p\ n\ u\ b\ s$ **.**
have all_eq_p: $\forall x{\in}s.\ x\ n = p$
by (*auto simp*: *out_eq_p*)
moreover
{ fix x **assume** $x \in s$
with $lab\_1[rule\_format, of\ n\ x]\ all\_eq\_p\ s\_le\_p[of\ x]$
have $?rl\ x \leq n$
by (*auto intro*!: *reduced_labelling_nonzero*)
then have $?rl\ x = reduced\ n\ (lab\ x)$
by (*auto intro*!: *reduced_labelling_Suc*) **}**
then have $?rl\ `\ s = \{..n\}$
using rl **by** (*simp cong*: *image_cong*)
moreover
obtain $t\ a$ **where** $ksimplex\ p\ (Suc\ n)\ t\ a \in t\ s = t - \{a\}$
using s **unfolding** $simplex\_top\_face[OF\ \langle 0 < p\rangle\ all\_eq\_p]$ **by** *auto*
ultimately
show $\exists t\ a.\ ksimplex\ p\ (Suc\ n)\ t \wedge a \in t \wedge s = t - \{a\} \wedge ?rl\ `\ s = \{..n\} \wedge ?ext\ s$
by *auto*
next
fix $x\ s\ a$ **assume** s: $ksimplex\ p\ (Suc\ n)\ s$ **and** rl: $?rl\ `\ (s - \{a\}) = \{..\ n\}$
and a: $a \in s$ **and** $?ext\ (s - \{a\})$
from s **obtain** $u\ b$ **where** $kuhn\_simplex\ p\ (Suc\ n)\ u\ b\ s$ **by** (*auto elim*: *ksimplex.cases*)
then interpret $kuhn\_simplex\ p\ Suc\ n\ u\ b\ s$ **.**
have all_eq_p: $\forall x{\in}s.\ x\ (Suc\ n) = p$
by (*auto simp*: *out_eq_p*)

{ fix x **assume** $x \in s - \{a\}$
then have $?rl\ x \in ?rl\ `\ (s - \{a\})$
by *auto*
then have $?rl\ x \leq n$
unfolding rl **by** *auto*
then have $?rl\ x = reduced\ n\ (lab\ x)$
by (*auto intro*!: *reduced_labelling_Suc*) **}**
then show rl': $(reduced\ n{\circ}lab)\ `\ (s - \{a\}) = \{..n\}$
unfolding $rl[symmetric]$ **by** (*intro image_cong*) *auto*

from $\langle ?ext\ (s - \{a\})\rangle$
have all_eq_p: $\forall x{\in}s - \{a\}.\ x\ n = p$

```
    proof (elim disjE exE conjE)
      fix j assume j ≤ n ∀x∈s − {a}. x j = 0
      with lab_0[rule_format, of j] all_eq_p s_le_p
      have ⋀x. x ∈ s − {a} ⟹ reduced (Suc n) (lab x) ≠ j
        by (intro reduced_labelling_zero) auto
      moreover have j ∈ ?rl ‘ (s − {a})
        using ‹j ≤ n› unfolding rl by auto
      ultimately show ?thesis
        by force
    next
      fix j assume j ≤ n and eq_p: ∀x∈s − {a}. x j = p
      show ?thesis
      proof cases
        assume j = n with eq_p show ?thesis by simp
      next
        assume j ≠ n
        { fix x assume x: x ∈ s − {a}
          have reduced n (lab x) ≤ j
          proof (rule reduced_labelling_nonzero)
            show lab x j ≠ 0
              using lab_1[rule_format, of j x] x s_le_p[of x] eq_p ‹j ≤ n› by auto
            show j < n
              using ‹j ≤ n› ‹j ≠ n› by simp
          qed
          then have reduced n (lab x) ≠ n
            using ‹j ≤ n› ‹j ≠ n› by simp }
        moreover have n ∈ (reduced n∘lab) ‘ (s − {a})
          unfolding rl′ by auto
        ultimately show ?thesis
          by force
      qed
    qed
    show ksimplex p n (s − {a})
      unfolding simplex_top_face[OF ‹0 < p› all_eq_p] using s a by auto
  qed
  ultimately show ?thesis
    using assms by (intro kuhn_simplex_lemma) auto
qed
```

And so we get the final combinatorial result.

```
lemma ksimplex_0: ksimplex p 0 s ⟷ s = {(λx. p)}
proof
  assume ksimplex p 0 s then show s = {(λx. p)}
    by (blast dest: kuhn_simplex.ksimplex_0 elim: ksimplex.cases)
next
  assume s: s = {(λx. p)}
  show ksimplex p 0 s
  proof (intro ksimplex, unfold_locales)
    show (λ_. p) ∈ {..<0::nat} → {..<p} by auto
```

```
    show bij_betw id {..<0} {..<0}
      by simp
  qed (auto simp: s)
qed

lemma kuhn_combinatorial:
  assumes 0 < p
    and ∀ x j. (∀ j. x j ≤ p) ∧ j < n ∧ x j = 0 ⟶ lab x j = 0
    and ∀ x j. (∀ j. x j ≤ p) ∧ j < n  ∧ x j = p ⟶ lab x j = 1
  shows odd (card {s. ksimplex p n s ∧ (reduced n∘lab) ‘ s = {..n}})
    (is odd (card (?M n)))
  using assms
proof (induct n)
  case 0 then show ?case
    by (simp add: ksimplex_0 cong: conj_cong)
next
  case (Suc n)
  then have odd (card (?M n))
    by force
  with Suc show ?case
    using kuhn_induction[of p n] by (auto simp: comp_def)
qed

lemma kuhn_lemma:
  fixes n p :: nat
  assumes 0 < p
    and ∀ x. (∀ i<n. x i ≤ p) ⟶ (∀ i<n. label x i = (0::nat) ∨ label x i = 1)
    and ∀ x. (∀ i<n. x i ≤ p) ⟶ (∀ i<n. x i = 0 ⟶ label x i = 0)
    and ∀ x. (∀ i<n. x i ≤ p) ⟶ (∀ i<n. x i = p ⟶ label x i = 1)
  obtains q where ∀ i<n. q i < p
    and ∀ i<n. ∃ r s. (∀ j<n. q j ≤ r j ∧ r j ≤ q j + 1) ∧ (∀ j<n. q j ≤ s j ∧ s j
≤ q j + 1) ∧ label r i ≠ label s i
proof −
  let ?rl = reduced n ∘ label
  let ?A = {s. ksimplex p n s ∧ ?rl ‘ s = {..n}}
  have odd (card ?A)
    using assms by (intro kuhn_combinatorial[of p n label]) auto
  then have ?A ≠ {}
    by (rule odd_card_imp_not_empty)
  then obtain s b u where kuhn_simplex p n b u s and rl: ?rl ‘ s = {..n}
    by (auto elim: ksimplex.cases)
  interpret kuhn_simplex p n b u s by fact

  show ?thesis
  proof (intro that[of b] allI impI)
    fix i
    assume i < n
    then show b i < p
      using base by auto
```

```
  next
    fix i
    assume i < n
    then have i ∈ {.. n} Suc i ∈ {.. n}
      by auto
    then obtain u v where u: u ∈ s Suc i = ?rl u and v: v ∈ s i = ?rl v
      unfolding rl[symmetric] by blast

    have label u i ≠ label v i
      using reduced_labelling [of n label u] reduced_labelling [of n label v]
        u(2)[symmetric] v(2)[symmetric] ‹i < n›
      by auto
    moreover
    have b j ≤ u j u j ≤ b j + 1 b j ≤ v j v j ≤ b j + 1 if j < n for j
      using that base_le[OF ‹u∈s›] le_Suc_base[OF ‹u∈s›] base_le[OF ‹v∈s›]
le_Suc_base[OF ‹v∈s›]
      by auto
    ultimately show ∃ r s. (∀ j<n. b j ≤ r j ∧ r j ≤ b j + 1) ∧
      (∀ j<n. b j ≤ s j ∧ s j ≤ b j + 1) ∧ label r i ≠ label s i
      by blast
  qed
qed
```

Main result for the unit cube

```
lemma kuhn_labelling_lemma':
  assumes (∀ x::nat⇒real. P x ⟶ P (f x))
    and ∀ x. P x ⟶ (∀ i::nat. Q i ⟶ 0 ≤ x i ∧ x i ≤ 1)
  shows ∃ l. (∀ x i. l x i ≤ (1::nat)) ∧
            (∀ x i. P x ∧ Q i ∧ x i = 0 ⟶ l x i = 0) ∧
            (∀ x i. P x ∧ Q i ∧ x i = 1 ⟶ l x i = 1) ∧
            (∀ x i. P x ∧ Q i ∧ l x i = 0 ⟶ x i ≤ f x i) ∧
            (∀ x i. P x ∧ Q i ∧ l x i = 1 ⟶ f x i ≤ x i)
  unfolding all_conj_distrib [symmetric]
  apply (subst choice_iff[symmetric])+
  by (metis assms choice_iff bot_nat_0.extremum nle_le zero_neq_one)
```

9.18.3 Brouwer's fixed point theorem

We start proving Brouwer's fixed point theorem for the unit cube = *cbox 0 One*.

```
lemma brouwer_cube:
  fixes f :: 'a::euclidean_space ⇒ 'a
  assumes continuous_on (cbox 0 One) f
    and f ‘ cbox 0 One ⊆ cbox 0 One
  shows ∃ x∈cbox 0 One. f x = x
proof (rule ccontr)
  define n where n = DIM('a)
  have n: 1 ≤ n 0 < n n ≠ 0
```

```
    unfolding n_def by (auto simp: Suc_le_eq)
  assume ¬ ?thesis
  then have ∗: ¬ (∃ x∈cbox 0 One. f x − x = 0)
    by auto
  obtain d where
      d: d > 0 ⋀x. x ∈ cbox 0 One ⟹ d ≤ norm (f x − x)
    using brouwer_compactness_lemma[OF compact_cbox _ ∗] assms
    by (metis (no_types, lifting) continuous_on_cong continuous_on_diff contin-
uous_on_id)
  have ∗: ∀ x. x ∈ cbox 0 One ⟶ f x ∈ cbox 0 One
    ∀ x. x ∈ (cbox 0 One::'a set) ⟶ (∀ i∈Basis. True ⟶ 0 ≤ x · i ∧ x · i ≤ 1)
    using assms(2)[unfolded image_subset_iff Ball_def]
    unfolding cbox_def
    by auto
  obtain label :: 'a ⇒ 'a ⇒ nat where label [rule_format]:
    ∀ x. ∀ i∈Basis. label x i ≤ 1
    ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ x · i = 0 ⟶ label x i = 0
    ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ x · i = 1 ⟶ label x i = 1
    ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ label x i = 0 ⟶ x · i ≤ f x · i
    ∀ x. ∀ i∈Basis. x ∈ cbox 0 One ∧ label x i = 1 ⟶ f x · i ≤ x · i
    using kuhn_labelling_lemma[OF ∗] by auto
  note label = this [rule_format]
  have lem1: ∀ x∈cbox 0 One. ∀ y∈cbox 0 One. ∀ i∈Basis. label x i ≠ label y i ⟶
    |f x · i − x · i| ≤ norm (f y − f x) + norm (y − x)
  proof safe
    fix x y :: 'a
    assume x: x ∈ cbox 0 One and y: y ∈ cbox 0 One
    fix i
    assume i: label x i ≠ label y i i ∈ Basis
    have ∗: ⋀x y fx fy :: real. x ≤ fx ∧ fy ≤ y ∨ fx ≤ x ∧ y ≤ fy ⟹
      |fx − x| ≤ |fy − fx| + |y − x| by auto
    have |(f x − x) · i| ≤ |(f y − f x)·i| + |(y − x)·i|
    proof (cases label x i = 0)
      case True
      then have fxy: ¬ f y · i ≤ y · i ⟹ f x · i ≤ x · i
        by (metis True i label(1) label(5) le_antisym less_one not_le_imp_less y)
      show ?thesis
      unfolding inner_simps
      by (rule ∗) (auto simp: True i label x y fxy)
    next
      case False
      then show ?thesis
        using label [OF ‹i ∈ Basis›] i(1) x y
        by (smt (verit, ccfv_threshold) inner_diff_left less_one order_le_less)
    qed
    also have ... ≤ norm (f y − f x) + norm (y − x)
      by (simp add: add_mono i(2) norm_bound_Basis_le)
    finally show |f x · i − x · i| ≤ norm (f y − f x) + norm (y − x)
      unfolding inner_simps .
```

```
qed
have ∃ e>0. ∀ x∈cbox 0 One. ∀ y∈cbox 0 One. ∀ z∈cbox 0 One. ∀ i∈Basis.
  norm (x − z) < e ⟶ norm (y − z) < e ⟶ label x i ≠ label y i ⟶
    |(f(z) − z)•i| < d / (real n)
proof −
  have d': d / real n / 8 > 0
    using d(1) by (simp add: n_def)
  have ∗: uniformly_continuous_on (cbox 0 One) f
    by (rule compact_uniformly_continuous[OF assms(1) compact_cbox])
  obtain e where e:
      e > 0
      ⋀x x'. x ∈ cbox 0 One ⟹
        x' ∈ cbox 0 One ⟹
        norm (x' − x) < e ⟹
        norm (f x' − f x) < d / real n / 8
    using ∗[unfolded uniformly_continuous_on_def,rule_format,OF d']
    unfolding dist_norm
    by blast
  show ?thesis
  proof (intro exI conjI ballI impI)
    show 0 < min (e / 2) (d / real n / 8)
      using d' e by auto
    fix x y z i
    assume as:
      x ∈ cbox 0 One y ∈ cbox 0 One z ∈ cbox 0 One
      norm (x − z) < min (e / 2) (d / real n / 8)
      norm (y − z) < min (e / 2) (d / real n / 8)
      label x i ≠ label y i
    assume i: i ∈ Basis
    have ∗: ⋀z fz x fx n1 n2 n3 n4 d4 d :: real. |fx − x| ≤ n1 + n2 ⟹
      |fx − fz| ≤ n3 ⟹ |x − z| ≤ n4 ⟹
      n1 < d4 ⟹ n2 < 2 ∗ d4 ⟹ n3 < d4 ⟹ n4 < d4 ⟹
      (8 ∗ d4 = d) ⟹ |fz − z| < d
      by auto
    show |(f z − z) • i| < d / real n
      unfolding inner_simps
    proof (rule ∗)
      show |f x • i − x • i| ≤ norm (f y −f x) + norm (y − x)
        using as(1) as(2) as(6) i lem1 by blast
      show norm (f x − f z) < d / real n / 8
        using d' e as by auto
      show |f x • i − f z • i| ≤ norm (f x − f z) |x • i − z • i| ≤ norm (x − z)
        unfolding inner_diff_left[symmetric]
        by (rule Basis_le_norm[OF i])+
      have tria: norm (y − x) ≤ norm (y − z) + norm (x − z)
        using dist_triangle[of y x z, unfolded dist_norm]
        unfolding norm_minus_commute
        by auto
      also have ... < e / 2 + e / 2
```

```
      using as(4) as(5) by auto
    finally show norm (f y - f x) < d / real n / 8
      using as(1) as(2) e(2) by auto
    have norm (y - z) + norm (x - z) < d / real n / 8 + d / real n / 8
      using as(4) as(5) by auto
    with tria show norm (y - x) < 2 * (d / real n / 8)
      by auto
  qed (use as in auto)
 qed
qed
then
obtain e where e:
 e > 0
 ⋀x y z i. x ∈ cbox 0 One ⟹
  y ∈ cbox 0 One ⟹
  z ∈ cbox 0 One ⟹
  i ∈ Basis ⟹
  norm (x - z) < e ∧ norm (y - z) < e ∧ label x i ≠ label y i ⟹
  |(f z - z) • i| < d / real n
 by blast
obtain p :: nat where p: 1 + real n / e ≤ real p
 using real_arch_simple ..
have 1 + real n / e > 0
 using e(1) n by (simp add: add_pos_pos)
then have p > 0
 using p by auto

obtain b :: nat ⇒ 'a where b: bij_betw b {..< n} Basis
 by atomize_elim (auto simp: n_def intro!: finite_same_card_bij)
define b' where b' = inv_into {..< n} b
then have b': bij_betw b' Basis {..< n}
 using bij_betw_inv_into[OF b] by auto
then have b'_Basis: ⋀i. i ∈ Basis ⟹ b' i ∈ {..< n}
 unfolding bij_betw_def by (auto simp: set_eq_iff)
have bb'[simp]:⋀i. i ∈ Basis ⟹ b (b' i) = i
 unfolding b'_def
 using b
 by (auto simp: f_inv_into_f bij_betw_def)
have b'b[simp]:⋀i. i < n ⟹ b' (b i) = i
 unfolding b'_def
 using b
 by (auto simp: inv_into_f_eq bij_betw_def)
have *: ⋀x :: nat. x = 0 ∨ x = 1 ⟷ x ≤ 1
 by auto
have b'': ⋀j. j < n ⟹ b j ∈ Basis
 using b unfolding bij_betw_def by auto
have q1: 0 < p ∀x. (∀i<n. x i ≤ p) ⟶
 (∀i<n. (label (∑i∈Basis. (real (x (b' i)) / real p) *R i) ∘ b) i = 0 ∨
       (label (∑i∈Basis. (real (x (b' i)) / real p) *R i) ∘ b) i = 1)
```

```
    unfolding ∗
    using ‹p > 0› ‹n > 0›
    using label(1)[OF b'']
    by auto
  { fix x :: nat ⇒ nat and i assume ∀ i<n. x i ≤ p i < n x i = p ∨ x i = 0
    then have (∑ i∈Basis. (real (x (b' i)) / real p) ∗R i) ∈ (cbox 0 One::'a set)
      using b'_Basis
      by (auto simp: cbox_def bij_betw_def zero_le_divide_iff divide_le_eq_1) }
  note cube = this
  have q2: ∀ x. (∀ i<n. x i ≤ p) ⟶ (∀ i<n. x i = 0 ⟶
      (label (∑ i∈Basis. (real (x (b' i)) / real p) ∗R i) ∘ b) i = 0)
    unfolding o_def using cube ‹p > 0› by (intro allI impI label(2)) (auto simp:
b'')
  have q3: ∀ x. (∀ i<n. x i ≤ p) ⟶ (∀ i<n. x i = p ⟶
      (label (∑ i∈Basis. (real (x (b' i)) / real p) ∗R i) ∘ b) i = 1)
    using cube ‹p > 0› unfolding o_def by (intro allI impI label(3)) (auto simp:
b'')
  obtain q where q:
      ∀ i<n. q i < p
      ∀ i<n.
        ∃ r s. (∀ j<n. q j ≤ r j ∧ r j ≤ q j + 1) ∧
              (∀ j<n. q j ≤ s j ∧ s j ≤ q j + 1) ∧
              (label (∑ i∈Basis. (real (r (b' i)) / real p) ∗R i) ∘ b) i ≠
              (label (∑ i∈Basis. (real (s (b' i)) / real p) ∗R i) ∘ b) i
    by (rule kuhn_lemma[OF q1 q2 q3])
  define z :: 'a where z = (∑ i∈Basis. (real (q (b' i)) / real p) ∗R i)
  have ∃ i∈Basis. d / real n ≤ |(f z − z)•i|
  proof (rule ccontr)
    have ∀ i∈Basis. q (b' i) ∈ {0..p}
      using q(1) b'
      by (auto intro: less_imp_le simp: bij_betw_def)
    then have z ∈ cbox 0 One
      unfolding z_def cbox_def
      using b'_Basis
      by (auto simp: bij_betw_def zero_le_divide_iff divide_le_eq_1)
    then have d_fz_z: d ≤ norm (f z − z)
      by (rule d)
    assume ¬ ?thesis
    then have as: ∀ i∈Basis. |f z • i − z • i| < d / real n
      using ‹n > 0›
      by (auto simp: not_le inner_diff)
    have norm (f z − z) ≤ (∑ i∈Basis. |f z • i − z • i|)
      unfolding inner_diff_left[symmetric]
      by (rule norm_le_l1)
    also have ... < (∑ (i::'a) ∈ Basis. d / real n)
      by (meson as finite_Basis nonempty_Basis sum_strict_mono)
    also have ... = d
      using DIM_positive[where 'a='a] by (auto simp: n_def)
    finally show False
```

```
    using d_fz_z by auto
qed
then obtain i where i: i ∈ Basis d / real n ≤ |(f z − z) · i| ..
have ∗: b′ i < n
  using i and b′[unfolded bij_betw_def]
  by auto
obtain r s where rs:
  ⋀j. j < n ⟹ q j ≤ r j ∧ r j ≤ q j + 1
  ⋀j. j < n ⟹ q j ≤ s j ∧ s j ≤ q j + 1
  (label (∑ i∈Basis. (real (r (b′ i)) / real p) ∗R i) ∘ b) (b′ i) ≠
    (label (∑ i∈Basis. (real (s (b′ i)) / real p) ∗R i) ∘ b) (b′ i)
  using q(2)[rule_format,OF ∗] by blast
have b′_im: ⋀i. i ∈ Basis ⟹ b′ i < n
  using b′ unfolding bij_betw_def by auto
define r′ ::'a where r′ = (∑ i∈Basis. (real (r (b′ i)) / real p) ∗R i)
have ⋀i. i ∈ Basis ⟹ r (b′ i) ≤ p
  using b′_im q(1) rs(1) by fastforce
then have r′ ∈ cbox 0 One
  unfolding r′_def cbox_def
  using b′_Basis
  by (auto simp: bij_betw_def zero_le_divide_iff divide_le_eq_1)
define s′ :: 'a where s′ = (∑ i∈Basis. (real (s (b′ i)) / real p) ∗R i)
have ⋀i. i ∈ Basis ⟹ s (b′ i) ≤ p
  using b′_im q(1) rs(2) by fastforce
then have s′ ∈ cbox 0 One
  unfolding s′_def cbox_def
 using b′_Basis by (auto simp: bij_betw_def zero_le_divide_iff divide_le_eq_1)
have z ∈ cbox 0 One
  unfolding z_def cbox_def
  using b′_Basis q(1)[rule_format,OF b′_im] ‹p > 0›
  by (auto simp: bij_betw_def zero_le_divide_iff divide_le_eq_1 less_imp_le)
{
  have (∑ i∈Basis. |real (r (b′ i)) − real (q (b′ i))|) ≤ (∑ (i::'a)∈Basis. 1)
    by (rule sum_mono) (use rs(1)[OF b′_im] in force)
  also have ... < e ∗ real p
    using p ‹e > 0› ‹p > 0›
    by (auto simp: field_simps n_def)
  finally have (∑ i∈Basis. |real (r (b′ i)) − real (q (b′ i))|) < e ∗ real p .
}
moreover
{
  have (∑ i∈Basis. |real (s (b′ i)) − real (q (b′ i))|) ≤ (∑ (i::'a)∈Basis. 1)
    by (rule sum_mono) (use rs(2)[OF b′_im] in force)
  also have ... < e ∗ real p
    using p ‹e > 0› ‹p > 0›
    by (auto simp: field_simps n_def)
  finally have (∑ i∈Basis. |real (s (b′ i)) − real (q (b′ i))|) < e ∗ real p .
}
ultimately
```

```
  have norm (r' − z) < e and norm (s' − z) < e
    unfolding r'_def s'_def z_def
    using ‹p > 0›
    apply (rule_tac[!] le_less_trans[OF norm_le_l1])
    apply (auto simp: field_simps sum_divide_distrib[symmetric] inner_diff_left)
    done
  then have |(f z − z) • i| < d / real n
    using rs(3) i
    unfolding r'_def[symmetric] s'_def[symmetric] o_def bb'
    by (intro e(2)[OF ‹r'∈cbox 0 One› ‹s'∈cbox 0 One› ‹z∈cbox 0 One›]) auto
  then show False
    using i by auto
qed
```

Next step is to prove it for nonempty interiors.

```
lemma brouwer_weak:
  fixes f :: 'a::euclidean_space ⇒ 'a
  assumes compact S
    and convex S
    and interior S ≠ {}
    and continuous_on S f
    and f ∈ S → S
  obtains x where x ∈ S and f x = x
proof −
  let ?U = cbox 0 One :: 'a set
  have ∑ Basis /R 2 ∈ interior ?U
  proof (rule interiorI)
    let ?I = (⋂ i∈Basis. {x::'a. 0 < x • i} ∩ {x. x • i < 1})
    show open ?I
     by (intro open_INT finite_Basis ballI open_Int, auto intro: open_Collect_less
simp: continuous_on_inner)
    show ∑ Basis /R 2 ∈ ?I
      by simp
    show ?I ⊆ cbox 0 One
      unfolding cbox_def by force
  qed
  then have ∗: interior ?U ≠ {} by fast
  have ∗: ?U homeomorphic S
     using homeomorphic_convex_compact[OF convex_box(1) compact_cbox ∗
assms(2,1,3)] .
  have ∀ f. continuous_on ?U f ∧ f ∈ ?U → ?U ⟶ (∃ x∈?U. f x = x)
    using brouwer_cube by auto
  then show ?thesis
    unfolding homeomorphic_fixpoint_property[OF ∗]
    using assms
    by (auto intro: that)
qed
```

Then the particular case for closed balls.

lemma *brouwer_ball*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'a$
assumes $e > 0$
and *continuous_on* (*cball a e*) *f*
and $f \in cball\ a\ e \to cball\ a\ e$
obtains x **where** $x \in cball\ a\ e$ **and** $f\ x = x$
using *brouwer_weak*[*OF compact_cball convex_cball, of a e f*]
unfolding *interior_cball ball_eq_empty*
using *assms* **by** *auto*

And finally we prove Brouwer's fixed point theorem in its general version.

theorem *brouwer*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'a$
assumes *S*: *compact S convex S* $S \neq \{\}$
and *contf*: *continuous_on S f*
and *fim*: $f \in S \to S$
obtains x **where** $x \in S$ **and** $f\ x = x$
proof −
have $\exists e{>}0.\ S \subseteq cball\ 0\ e$
using *compact_imp_bounded*[*OF* ‹*compact S*›] **unfolding** *bounded_pos*
by *auto*
then obtain *e* **where** *e*: $e > 0$ $S \subseteq cball\ 0\ e$
by *blast*
have $\exists x{\in}\ cball\ 0\ e.\ (f \circ closest\_point\ S)\ x = x$
proof (*rule_tac brouwer_ball*[*OF e*(*1*)])
show *continuous_on* (*cball 0 e*) (*f* ∘ *closest_point S*)
by (*meson assms closest_point_in_set compact_eq_bounded_closed contf continuous_on_closest_point*
continuous_on_compose continuous_on_subset image_subsetI)
show $f \circ closest\_point\ S \in cball\ 0\ e \to cball\ 0\ e$
by (*smt* (*verit*) *Pi_iff assms*(*1*) *assms*(*3*) *closest_point_in_set comp_apply compact_eq_bounded_closed e*(*2*) *fim subset_eq*)
qed (*use assms* **in** *auto*)
then obtain x **where** x: $x \in cball\ 0\ e$ $(f \circ closest\_point\ S)\ x = x$ **..**
with *S* **have** $x \in S$
by (*metis PiE closest_point_in_set comp_apply compact_imp_closed fim*)
then have $*$: $closest\_point\ S\ x = x$
by (*rule closest_point_self*)
show *thesis*
proof
show $closest\_point\ S\ x \in S$
by (*simp add*: $*$ ‹$x \in S$›)
show $f\ (closest\_point\ S\ x) = closest\_point\ S\ x$
using $*$ *x* **by** *auto*
qed
qed

9.18.4 Applications

So we get the no-retraction theorem.

corollary *no_retraction_cball*:
fixes *a* :: *'a*::*euclidean_space*
assumes *e* > *0*
shows ¬ (*frontier* (*cball a e*) *retract_of* (*cball a e*))
proof
assume ∗: *frontier* (*cball a e*) *retract_of* (*cball a e*)
have ∗∗: ⋀*xa*. *a* − (*2* ∗$_R$ *a* − *xa*) = − (*a* − *xa*)
using *scaleR_left_distrib*[*of 1 1 a*] **by** *auto*
obtain *x* **where** *x*: *x* ∈ {*x*. *norm* (*a* − *x*) = *e*} *2* ∗$_R$ *a* − *x* = *x*
proof (*rule retract_fixpoint_property*[*OF* ∗, *of* λ*x*. *scaleR 2 a* − *x*])
show *continuous_on* (*frontier* (*cball a e*)) ((−) (*2* ∗$_R$ *a*))
by (*intro continuous_intros*)
show (−) (*2* ∗$_R$ *a*) ∈ *frontier* (*cball a e*) → *frontier* (*cball a e*)
by *clarsimp* (*metis* ∗∗ *dist_norm norm_minus_cancel*)
qed (*auto simp*: *dist_norm intro*: *brouwer_ball*[*OF assms*])
then have *scaleR 2 a* = *scaleR 1 x* + *scaleR 1 x*
by (*auto simp*: *algebra_simps*)
then have *a* = *x*
unfolding *scaleR_left_distrib*[*symmetric*] **by** *auto*
then show *False*
using *x assms* **by** *auto*
qed

corollary *contractible_sphere*:
fixes *a* :: *'a*::*euclidean_space*
shows *contractible*(*sphere a r*) ⟷ *r* ≤ *0*
proof (*cases 0* < *r*)
case *True*
then show *?thesis*
unfolding *contractible_def nullhomotopic_from_sphere_extension*
using *no_retraction_cball* [*OF True*, *of a*]
by (*auto simp*: *retract_of_def retraction_def*)
next
case *False*
then show *?thesis*
unfolding *contractible_def nullhomotopic_from_sphere_extension*
using *less_eq_real_def* **by** *auto*
qed

corollary *connected_sphere_eq*:
fixes *a* :: *'a* :: *euclidean_space*
shows *connected*(*sphere a r*) ⟷ *2* ≤ *DIM*(*'a*) ∨ *r* ≤ *0*
(**is** *?lhs* = *?rhs*)
proof (*cases r 0*::*real rule*: *linorder_cases*)
case *less*
then show *?thesis* **by** *auto*

next
case *equal*
then show *?thesis* **by** *auto*
next
case *greater*
show *?thesis*
proof
assume *L*: *?lhs*
have *False* **if** *1*: $DIM('a) = 1$
proof −
obtain *x y* **where** *xy*: *sphere a r* = {*x*,*y*} $x \neq y$
using *sphere_1D_doubleton* [*OF 1 greater*]
by (*metis dist_self greater insertI1 less_add_same_cancel1 mem_sphere mult_2 not_le zero_le_dist*)
then have *finite* (*sphere a r*)
by *auto*
with *L* ‹$r > 0$› *xy* **show** *False*
using *connected_finite_iff_sing* **by** *auto*
qed
with *greater* **show** *?rhs*
by (*metis DIM_ge_Suc0 One_nat_def Suc_1 le_antisym not_less_eq_eq*)
next
assume *?rhs*
then show *?lhs*
using *connected_sphere greater* **by** *auto*
qed
qed

corollary *path_connected_sphere_eq*:
fixes *a* :: $'a$ :: *euclidean_space*
shows *path_connected*(*sphere a r*) $\longleftrightarrow$ $2 \leq DIM('a) \vee r \leq 0$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *connected_sphere_eq path_connected_imp_connected* **by** *blast*
next
assume *R*: *?rhs*
then show *?lhs*
by (*auto simp*: *contractible_imp_path_connected contractible_sphere path_connected_sphere*)
qed

proposition *frontier_subset_retraction*:
fixes *S* :: $'a$::*euclidean_space set*
assumes *bounded S* **and** *fros*: *frontier S* $\subseteq$ *T*
and *contf*: *continuous_on* (*closure S*) *f*
and *fim*: $f \in S \to T$
and *fid*: $\bigwedge x.\ x \in T \Longrightarrow f\ x = x$
shows $S \subseteq T$

proof (*rule ccontr*)
assume ¬ $S \subseteq T$
then obtain a **where** $a \in S$ $a \notin T$ **by** *blast*
define g **where** $g \equiv \lambda z.$ *if* $z \in$ *closure* S *then* $f\ z$ *else* z
have *continuous_on* (*closure* $S \cup$ *closure*$(-S)$) g
unfolding *g_def* **using** *fros fid frontier_closures*
by (*intro continuous_on_cases*) (*auto simp*: *contf*)
moreover have *closure* $S \cup$ *closure*$(-S) =$ *UNIV*
using *closure_Un* **by** *fastforce*
ultimately have *contg*: *continuous_on UNIV g* **by** *metis*
obtain B **where** $0 < B$ **and** B: *closure* $S \subseteq$ *ball* a B
using ‹*bounded S*› *bounded_subset_ballD* **by** *blast*
have *notga*: $g\ x \neq a$ **for** x
unfolding *g_def* **using** *fros fim* ‹$a \notin T$›
by (*metis PiE Un_iff* ‹$a \in S$› *closure_Un_frontier fid subsetD*)
define h **where** $h \equiv (\lambda y.\ a + (B\ /\ norm(y - a)) *_R (y - a)) \circ g$
have ¬ (*frontier* (*cball* a B) *retract_of* (*cball* a B))
by (*metis no_retraction_cball* ‹$0 < B$›)
then have $\bigwedge k.$ ¬ *retraction* (*cball* a B) (*frontier* (*cball* a B)) k
by (*simp add*: *retract_of_def*)
moreover have *retraction* (*cball* a B) (*frontier* (*cball* a B)) h
unfolding *retraction_def*
proof (*intro conjI ballI*)
show *frontier* (*cball* a B) $\subseteq$ *cball* a B
by *force*
show *continuous_on* (*cball* a B) h
unfolding *h_def*
by (*intro continuous_intros*) (*use contg continuous_on_subset notga* **in** *auto*)
show $h \in$ *cball* a $B \to$ *frontier* (*cball* a B)
using ‹$0 < B$› **by** (*auto simp*: *h_def notga dist_norm*)
show $\bigwedge x.\ x \in$ *frontier* (*cball* a B) $\Longrightarrow h\ x = x$
using *notga* ‹$0 < B$›
apply (*simp add*: *g_def h_def field_simps*)
by (*metis B dist_commute dist_norm mem_ball order_less_irrefl subset_eq*)
qed
ultimately show *False* **by** *simp*
qed

Punctured affine hulls, etc

lemma *rel_frontier_deformation_retract_of_punctured_convex*:
fixes S :: *'a*::*euclidean_space set*
assumes *convex S convex T bounded S*
and *arelS*: $a \in$ *rel_interior S*
and *relS*: *rel_frontier S* $\subseteq T$
and *affS*: $T \subseteq$ *affine hull S*
obtains r **where** *homotopic_with_canon* ($\lambda x.$ *True*) ($T - \{a\}$) ($T - \{a\}$) *id r*
retraction ($T - \{a\}$) (*rel_frontier S*) r
proof −

```
  have ∃d. 0 < d ∧ (a + d *R l) ∈ rel_frontier S ∧
          (∀e. 0 ≤ e ∧ e < d ⟶ (a + e *R l) ∈ rel_interior S)
    if (a + l) ∈ affine hull S l ≠ 0 for l
    using ray_to_rel_frontier [OF ‹bounded S› arelS] that by metis
  then obtain dd
    where dd1: ⋀l. ⟦(a + l) ∈ affine hull S; l ≠ 0⟧ ⟹ 0 < dd l ∧ (a + dd l *R
l) ∈ rel_frontier S
      and dd2: ⋀l e. ⟦(a + l) ∈ affine hull S; e < dd l; 0 ≤ e; l ≠ 0⟧
                  ⟹ (a + e *R l) ∈ rel_interior S
    by metis+
  have aaffS: a ∈ affine hull S
    by (meson arelS subsetD hull_inc rel_interior_subset)
  have ((λz. z − a) ‘ (affine hull S − {a})) = ((λz. z − a) ‘ (affine hull S)) −
{0}
    by auto
  moreover have continuous_on (((λz. z − a) ‘ (affine hull S)) − {0}) (λx. dd
x *R x)
  proof (rule continuous_on_compact_surface_projection)
    show compact (rel_frontier ((λz. z − a) ‘ S))
    by (simp add: ‹bounded S› bounded_translation_minus compact_rel_frontier_bounded)
    have releq: rel_frontier ((λz. z − a) ‘ S) = (λz. z − a) ‘ rel_frontier S
      using rel_frontier_translation [of −a] add.commute by simp
    also have ... ⊆ (λz. z − a) ‘ (affine hull S) − {0}
      using rel_frontier_affine_hull arelS rel_frontier_def by fastforce
    finally show rel_frontier ((λz. z − a) ‘ S) ⊆ (λz. z − a) ‘ (affine hull S) −
{0} .
    show cone ((λz. z − a) ‘ (affine hull S))
      by (rule subspace_imp_cone)
       (use aaffS in ‹simp add: subspace_affine image_comp o_def affine_translation_aux
[of a]›)
    show (0 < k ∧ k *R x ∈ rel_frontier ((λz. z − a) ‘ S)) ⟷ (dd x = k)
        if x: x ∈ (λz. z − a) ‘ (affine hull S) − {0} for k x
    proof
      show dd x = k ⟹ 0 < k ∧ k *R x ∈ rel_frontier ((λz. z − a) ‘ S)
      using dd1 [of x] that image_iff by (fastforce simp add: releq)
    next
      assume k: 0 < k ∧ k *R x ∈ rel_frontier ((λz. z − a) ‘ S)
      have False if dd x < k
      proof −
        have k ≠ 0 a + k *R x ∈ closure S
          using k closure_translation [of −a]
          by (auto simp: rel_frontier_def cong: image_cong_simp)
        then have segsub: open_segment a (a + k *R x) ⊆ rel_interior S
          by (metis rel_interior_closure_convex_segment [OF ‹convex S› arelS])
        have x ≠ 0 and xaffS: a + x ∈ affine hull S
          using x by auto
        then have 0 < dd x and inS: a + dd x *R x ∈ rel_frontier S
          using dd1 by auto
        moreover have a + dd x *R x ∈ open_segment a (a + k *R x)
```

```
        unfolding in_segment
       proof (intro conjI exI)
        show a + dd x *R x = (1 - dd x / k) *R a + (dd x / k) *R (a + k *R x)
          using k by (simp add: that algebra_simps)
       qed (use ‹x ≠ 0› ‹0 < dd x› that in auto)
       ultimately show ?thesis
        using segsub by (auto simp: rel_frontier_def)
      qed
      moreover have False if k < dd x
       using x k that rel_frontier_def
       by (fastforce simp: algebra_simps releq dest!: dd2)
      ultimately show dd x = k
       by fastforce
     qed
    qed
    ultimately have *: continuous_on ((λz. z - a) ‘ (affine hull S - {a})) (λx. dd
x *R x)
     by auto
    have continuous_on (affine hull S - {a}) ((λx. a + dd x *R x) ∘ (λz. z - a))
     by (intro * continuous_intros continuous_on_compose)
    with affS have contdd: continuous_on (T - {a}) ((λx. a + dd x *R x) ∘ (λz.
z - a))
     by (blast intro: continuous_on_subset)
    show ?thesis
    proof
     show homotopic_with_canon (λx. True) (T - {a}) (T - {a}) id (λx. a + dd
(x-a) *R (x-a))
     proof (rule homotopic_with_linear)
      show continuous_on (T - {a}) id
       by (intro continuous_intros continuous_on_compose)
      show continuous_on (T - {a}) (λx. a + dd (x-a) *R (x-a))
       using contdd by (simp add: o_def)
      show closed_segment (id x) (a + dd (x-a) *R (x-a)) ⊆ T - {a}
          if x ∈ T - {a} for x
      proof (clarsimp simp: in_segment, intro conjI)
       fix u::real assume u: 0 ≤ u u ≤ 1
       have a + dd (x-a) *R (x-a) ∈ T
       by (metis DiffD1 DiffD2 add.commute add.right_neutral affS dd1 diff_add_cancel
relS singletonI subsetCE that)
       then show (1 - u) *R x + u *R (a + dd (x-a) *R (x-a)) ∈ T
        using convexD [OF ‹convex T›] that u by simp
       have iff: (1 - u) *R x + u *R (a + d *R (x-a)) = a ⟷
                 (1 - u + u * d) *R (x-a) = 0 for d
        by (auto simp: algebra_simps)
       have x ∈ T x ≠ a using that by auto
       then have axa: a + (x-a) ∈ affine hull S
        by (metis (no_types) add.commute affS diff_add_cancel rev_subsetD)
       then have ¬ dd (x-a) ≤ 0 ∧ a + dd (x-a) *R (x-a) ∈ rel_frontier S
        using ‹x ≠ a› dd1 by fastforce
```

```
      with ‹x ≠ a› show (1 − u) *R x + u *R (a + dd (x−a) *R (x−a)) ≠ a
        using less_eq_real_def mult_le_0_iff not_less u by (fastforce simp: iff)
      qed
    qed
    show retraction (T − {a}) (rel_frontier S) (λx. a + dd (x−a) *R (x−a))
    proof (simp add: retraction_def, intro conjI ballI)
      show rel_frontier S ⊆ T − {a}
        using arelS relS rel_frontier_def by fastforce
      show continuous_on (T − {a}) (λx. a + dd (x−a) *R (x−a))
        using contdd by (simp add: o_def)
      show (λx. a + dd (x−a) *R (x−a)) ∈ (T − {a}) → rel_frontier S
        unfolding Pi_iff using affS dd1 subset_eq by force
      show a + dd (x−a) *R (x−a) = x if x: x ∈ rel_frontier S for x
      proof −
        have x ≠ a
          using that arelS by (auto simp: rel_frontier_def)
        have False if dd (x−a) < 1
        proof −
          have x ∈ closure S
            using x by (auto simp: rel_frontier_def)
          then have segsub: open_segment a x ⊆ rel_interior S
            by (metis rel_interior_closure_convex_segment [OF ‹convex S› arelS])
          have  xaffS: x ∈ affine hull S
            using affS relS x by auto
         then have 0 < dd (x−a) and inS: a + dd (x−a) *R (x−a) ∈ rel_frontier
S
            using dd1 by (auto simp: ‹x ≠ a›)
          moreover have a + dd (x−a) *R (x−a) ∈ open_segment a x
            unfolding in_segment
          proof (intro exI conjI)
           show a + dd (x−a) *R (x−a) = (1 − dd (x−a)) *R a + (dd (x−a)) *R
x
              by (simp add: algebra_simps)
          qed (use ‹x ≠ a› ‹0 < dd (x−a)› that in auto)
          ultimately show ?thesis
            using segsub by (auto simp: rel_frontier_def)
        qed
        moreover have False if 1 < dd (x−a)
          using x that dd2 [of x − a 1] ‹x ≠ a› closure_affine_hull
          by (auto simp: rel_frontier_def)
        ultimately have dd (x−a) = 1 — similar to another proof above
          by fastforce
        with that show ?thesis
          by (simp add: rel_frontier_def)
      qed
    qed
  qed
qed
```

corollary *rel_frontier_retract_of_punctured_affine_hull*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *bounded S convex S a* $\in$ *rel_interior S*
shows *rel_frontier S retract_of* (*affine hull S* $-$ $\{a\}$)
by (*meson assms convex_affine_hull dual_order.refl rel_frontier_affine_hull*
rel_frontier_deformation_retract_of_punctured_convex retract_of_def)

corollary *rel_boundary_retract_of_punctured_affine_hull*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *compact S convex S a* $\in$ *rel_interior S*
shows (*S* $-$ *rel_interior S*) *retract_of* (*affine hull S* $-$ $\{a\}$)
by (*metis assms closure_closed compact_eq_bounded_closed rel_frontier_def*
rel_frontier_retract_of_punctured_affine_hull)

lemma *homotopy_eqv_rel_frontier_punctured_convex*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *convex S bounded S a* $\in$ *rel_interior S convex T rel_frontier S* $\subseteq$ *T T* $\subseteq$ *affine hull S*
shows (*rel_frontier S*) *homotopy_eqv* (*T* $-$ $\{a\}$)
by (*meson assms deformation_retract_imp_homotopy_eqv homotopy_equivalent_space_sym*
rel_frontier_deformation_retract_of_punctured_convex[*of S T*])

lemma *homotopy_eqv_rel_frontier_punctured_affine_hull*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *convex S bounded S a* $\in$ *rel_interior S*
shows (*rel_frontier S*) *homotopy_eqv* (*affine hull S* $-$ $\{a\}$)
by (*simp add*: *assms homotopy_eqv_rel_frontier_punctured_convex rel_frontier_affine_hull*)

lemma *path_connected_sphere_gen*:
assumes *convex S bounded S aff_dim S* $\neq$ *1*
shows *path_connected*(*rel_frontier S*)
proof $-$
have *convex* (*closure S*)
using *assms* **by** *auto*
then show *?thesis*
by (*metis Diff_empty aff_dim_affine_hull assms convex_affine_hull convex_imp_path_connected equals0I*
path_connected_punctured_convex rel_frontier_def rel_frontier_retract_of_punctured_affine_hull retract_of_path_connected)
qed

lemma *connected_sphere_gen*:
assumes *convex S bounded S aff_dim S* $\neq$ *1*
shows *connected*(*rel_frontier S*)
by (*simp add*: *assms path_connected_imp_connected path_connected_sphere_gen*)

Borsuk-style characterization of separation

lemma *continuous_on_Borsuk_map*:
  $a \notin S \Longrightarrow$ *continuous_on S* $(\lambda x.$ *inverse*$(norm\ (x-a)) *_R (x-a))$
by (*rule continuous_intros* | *force*)+

lemma *Borsuk_map_into_sphere*:
  $(\lambda x.$ *inverse*$(norm\ (x-a)) *_R (x-a)) \in S \to$ *sphere 0 1* $\longleftrightarrow (a \notin S)$
proof −
  have $\bigwedge x.\ [\![a \notin S;\ x \in S]\!] \Longrightarrow$ *inverse* $(norm\ (x-a)) * norm\ (x-a) = 1$
    by (*metis left_inverse norm_eq_zero right_minus_eq*)
  then show *?thesis*
    by *force*
qed

lemma *Borsuk_maps_homotopic_in_path_component*:
  assumes *path_component* $(-\ S)\ a\ b$
    shows *homotopic_with_canon* $(\lambda x.\ True)\ S$ *(sphere 0 1)*
                $(\lambda x.$ *inverse*$(norm(x-a)) *_R (x-a))$
                $(\lambda x.$ *inverse*$(norm(x-b)) *_R (x-b))$
proof −
  obtain *g* **where** *g*: *path g path_image g* $\subseteq -S$ *pathstart g* $= a$ *pathfinish g* $= b$
    using *assms* **by** (*auto simp*: *path_component_def*)
  define *h* **where** $h \equiv \lambda z.$ *(snd z* $-$ *(g* $\circ$ *fst) z)* $/_R$ *norm (snd z* $-$ *(g* $\circ$ *fst) z)*
  have *continuous_on* $(\{0..1\} \times S)$ *h*
    unfolding *h_def* **using** *g* **by** (*intro continuous_intros*) (*auto simp*: *path_defs*)
  moreover
  have *h* ‘ $(\{0..1\} \times S) \subseteq$ *sphere 0 1*
    unfolding *h_def* **using** *g* **by** (*auto simp*: *divide_simps path_defs*)
  ultimately show *?thesis*
    using *g* **by** (*auto simp*: *h_def path_defs homotopic_with_def*)
qed

lemma *non_extensible_Borsuk_map*:
  fixes *a* :: *'a* :: *euclidean_space*
  assumes *compact S* **and** *cin*: $C \in$ *components*$(-\ S)$ **and** *boc*: *bounded C* **and** *a* $\in C$
    shows $\neg\ (\exists g.$ *continuous_on* $(S \cup C)\ g \wedge$
                $g \in (S \cup C) \to$ *sphere 0 1* $\wedge$
                $(\forall x \in S.\ g\ x =$ *inverse*$(norm(x-a)) *_R (x-a)))$
proof −
  have *closed S* **using** *assms* **by** (*simp add*: *compact_imp_closed*)
  have $C \subseteq -S$
    using *assms* **by** (*simp add*: *in_components_subset*)
  with ‹$a \in C$› **have** $a \notin S$ **by** *blast*
  then have *ceq*: $C =$ *connected_component_set* $(-\ S)\ a$
    by (*metis* ‹$a \in C$› *cin components_iff connected_component_eq*)
  then have *bounded* $(S \cup$ *connected_component_set* $(-\ S)\ a)$
    using ‹*compact S*› *boc compact_imp_bounded* **by** *auto*
  with *bounded_subset_ballD* **obtain** *r* **where** $0 < r$ **and** *r*: $(S \cup$ *connected_component_set*

```
(− S) a) ⊆ ball a r
    by blast
  { fix g
    assume continuous_on (S ∪ C) g
           g ∈ (S ∪ C) → sphere 0 1
      and [simp]: ⋀x. x ∈ S ⟹ g x = (x−a) /R norm (x−a)
    then have norm_g1[simp]: ⋀x. x ∈ S ∪ C ⟹ norm (g x) = 1
      by force
    have cb_eq: cball a r = (S ∪ connected_component_set (− S) a) ∪
                    (cball a r − connected_component_set (− S) a)
      using ball_subset_cball [of a r] r by auto
    have cont1: continuous_on (S ∪ connected_component_set (− S) a)
                    (λx. a + r *R g x)
      using ‹continuous_on (S ∪ C) g› ceq
      by (intro continuous_intros) blast
    have cont2: continuous_on (cball a r − connected_component_set (− S) a)
           (λx. a + r *R ((x−a) /R norm (x−a)))
      by (rule continuous_intros | force simp: ‹a ∉ S›)+
    have 1: continuous_on (cball a r)
            (λx. if connected_component (− S) a x
                 then a + r *R g x
                 else a + r *R ((x−a) /R norm (x−a)))
      apply (subst cb_eq)
      apply (rule continuous_on_cases [OF _ _ cont1 cont2])
      using ‹closed S› ceq cin
     by (force simp: closed_Diff open_Compl closed_Un_complement_component
open_connected_component)+
    have 2: (λx. a + r *R g x) ‘ (cball a r ∩ connected_component_set (− S) a)
           ⊆ sphere a r
      using ‹0 < r› by (force simp: dist_norm ceq)
    have retraction (cball a r) (sphere a r)
           (λx. if x ∈ connected_component_set (− S) a
                then a + r *R g x
                else a + r *R ((x−a) /R norm (x−a)))
      using ‹0 < r› ‹a ∉ S› ‹a ∈ C› r
      by (auto simp: norm_minus_commute retraction_def Pi_iff ceq dist_norm
abs_if
         mult_less_0_iff divide_simps 1 2)
    then have False
      using no_retraction_cball
            [OF ‹0 < r›, of a, unfolded retract_of_def, simplified, rule_format,
             of λx. if x ∈ connected_component_set (− S) a
                    then a + r *R g x
                    else a + r *R inverse(norm(x−a)) *R (x−a)]
      by blast
  }
  then show ?thesis
    by blast
qed
```

Proving surjectivity via Brouwer fixpoint theorem

lemma *brouwer_surjective*:
  fixes f :: *'n::euclidean_space* ⇒ *'n*
  assumes T: *compact T convex T* $T \neq \{\}$
    and f: *continuous_on T f*
    and $\bigwedge x\ y.\ [\![x{\in}S;\ y{\in}T]\!] \Longrightarrow x + (y - f\ y) \in T$
    and $x \in S$
  shows $\exists\, y{\in}T.\ f\ y = x$
proof −
  have ∗: $\bigwedge x\ y.\ f\ y = x \longleftrightarrow x + (y - f\ y) = y$
    by (*auto simp add*: *algebra_simps*)
  show *?thesis*
    unfolding ∗
  proof (*rule brouwer*[*OF T*])
    show *continuous_on T* (λ*y*. *x* + (*y* − *f y*))
      by (*intro continuous_intros f*)
  qed (*use assms* **in** *auto*)
qed

lemma *brouwer_surjective_cball*:
  fixes f :: *'n::euclidean_space* ⇒ *'n*
  assumes *continuous_on* (*cball a e*) *f*
    and $e > 0$
    and $x \in S$
    and $\bigwedge x\ y.\ [\![x{\in}S;\ y{\in}\mathit{cball}\ a\ e]\!] \Longrightarrow x + (y - f\ y) \in \mathit{cball}\ a\ e$
  shows $\exists\, y{\in}\mathit{cball}\ a\ e.\ f\ y = x$
  by (*smt* (*verit*, *best*) *assms brouwer_surjective cball_eq_empty compact_cball convex_cball*)

Inverse function theorem

See Sussmann: "Multidifferential calculus", Theorem 2.1.1

lemma *sussmann_open_mapping*:
  fixes f :: *'a::real_normed_vector* ⇒ *'b::euclidean_space*
  assumes *open S*
    and *contf*: *continuous_on S f*
    and $x \in S$
    and *derf*: (*f has_derivative f'*) (*at x*)
    and *bounded_linear g' f'* ∘ *g'* = *id*
    and $T \subseteq S$
    and *x*: $x \in \mathit{interior}\ T$
  shows $f\ x \in \mathit{interior}\ (f\ `\ T)$
proof −
  interpret *f'*: *bounded_linear f'*
    using *assms* **unfolding** *has_derivative_def* **by** *auto*
  interpret *g'*: *bounded_linear g'*
    using *assms* **by** *auto*
  obtain *B* **where** *B*: $0 < B\ \forall x.\ \mathit{norm}\ (g'\ x) \leq \mathit{norm}\ x * B$

```
    using bounded_linear.pos_bounded[OF assms(5)] by blast
  hence ∗: 1 / (2 ∗ B) > 0 by auto
  obtain e0 where e0:
      0 < e0
      ∀ y. norm (y − x) < e0 ⟶ norm (f y − f x − f′ (y − x)) ≤ 1 / (2 ∗ B) ∗
norm (y − x)
    using derf unfolding has_derivative_at_alt
    using ∗ by blast
  obtain e1 where e1: 0 < e1 cball x e1 ⊆ T
    using mem_interior_cball x by blast
  have ∗: 0 < e0 / B 0 < e1 / B using e0 e1 B by auto
  obtain e where e: 0 < e e < e0 / B e < e1 / B
    using field_lbound_gt_zero[OF ∗] by blast
  have lem: ∃ y∈cball (f x) e. f (x + g′ (y − f x)) = z if z∈cball (f x) (e / 2) for
z
  proof (rule brouwer_surjective_cball)
    have z: z ∈ S if as: y ∈cball (f x) e z = x + (g′ y − g′ (f x)) for y z
    proof−
      have dist x z = norm (g′ (f x) − g′ y)
        unfolding as(2) and dist_norm by auto
      also have ... ≤ norm (f x − y) ∗ B
        by (metis B(2) g′.diff)
      also have ... ≤ e ∗ B
        by (metis B(1) dist_norm mem_cball mult_le_cancel_right_pos that(1))
      also have ... ≤ e1
        using B(1) e(3) pos_less_divide_eq by fastforce
      finally have z ∈ cball x e1
        by force
      then show z ∈ S
        using e1 assms(7) by auto
    qed
    show continuous_on (cball (f x) e) (λy. f (x + g′ (y − f x)))
      unfolding g′.diff
    proof (rule continuous_on_compose2 [OF _ _ order_refl, of _ _ f])
      show continuous_on ((λy. x + (g′ y − g′ (f x))) ‘ cball (f x) e) f
        by (rule continuous_on_subset[OF contf]) (use z in blast)
      show continuous_on (cball (f x) e) (λy. x + (g′ y − g′ (f x)))
       by (intro continuous_intros linear_continuous_on[OF ‹bounded_linear g′›])
    qed
  next
    fix y z
    assume y: y ∈ cball (f x) (e / 2) and z: z ∈ cball (f x) e
    have norm (g′ (z − f x)) ≤ norm (z − f x) ∗ B
      using B by auto
    also have ... ≤ e ∗ B
     by (metis B(1) z dist_norm mem_cball norm_minus_commute mult_le_cancel_right_pos)
    also have ... < e0
      using B(1) e(2) pos_less_divide_eq by blast
    finally have ∗: norm (x + g′ (z − f x) − x) < e0
```

```
    by auto
  have **: f x + f' (x + g' (z - f x) - x) = z
    using assms(6)[unfolded o_def id_def,THEN cong]
    by auto
  have norm (f x - (y + (z - f (x + g' (z - f x))))) ≤
      norm (f (x + g' (z - f x)) - z) + norm (f x - y)
    using norm_triangle_ineq[of f (x + g'(z - f x)) - z f x - y]
    by (auto simp add: algebra_simps)
  also have ... ≤ 1 / (B * 2) * norm (g' (z - f x)) + norm (f x - y)
    using e0(2)[rule_format, OF *]
    by (simp only: algebra_simps **) auto
  also have ... ≤ 1 / (B * 2) * norm (g' (z - f x)) + e/2
    using y by (auto simp: dist_norm)
  also have ... ≤ 1 / (B * 2) * B * norm (z - f x) + e/2
    using * B by (auto simp add: field_simps)
  also have ... ≤ 1 / 2 * norm (z - f x) + e/2
    by auto
  also have ... ≤ e/2 + e/2
    using B(1) ‹norm (z - f x) * B ≤ e * B› by auto
  finally show y + (z - f (x + g' (z - f x))) ∈ cball (f x) e
    by (auto simp: dist_norm)
 qed (use e that in auto)
 show ?thesis
  unfolding mem_interior
 proof (intro exI conjI subsetI)
  fix y
  assume y ∈ ball (f x) (e / 2)
  then have *: y ∈ cball (f x) (e / 2)
    by auto
  obtain z where z: z ∈ cball (f x) e f (x + g' (z - f x)) = y
    using lem * by blast
  then have norm (g' (z - f x)) ≤ norm (z - f x) * B
    using B
    by (auto simp add: field_simps)
  also have ... ≤ e * B
  by (metis B(1) dist_norm mem_cball norm_minus_commute mult_le_cancel_right_pos
z(1))
  also have ... ≤ e1
    using e B unfolding less_divide_eq by auto
  finally have x + g'(z - f x) ∈ T
  by (metis add_diff_cancel diff_diff_add dist_norm e1(2) mem_cball norm_minus_commute
subset_eq)
  then show y ∈ f ‘ T
    using z by auto
 qed (use e in auto)
qed
```

Hence the following eccentric variant of the inverse function theorem. This has no continuity assumptions, but we do need the inverse function. We

could put $f' \circ g = I$ but this happens to fit with the minimal linear algebra theory I've set up so far.

```
lemma has_derivative_inverse_strong:
  fixes f :: 'n::euclidean_space ⇒ 'n
  assumes S: open S x ∈ S
    and contf: continuous_on S f
    and gf: ⋀x. x ∈ S ⟹ g (f x) = x
    and derf: (f has_derivative f') (at x)
    and id: f' ∘ g' = id
  shows (g has_derivative g') (at (f x))
proof −
  have linf: bounded_linear f'
    using derf unfolding has_derivative_def by auto
  then have ling: bounded_linear g'
    unfolding linear_conv_bounded_linear[symmetric]
    using id right_inverse_linear by blast
  moreover have g' ∘ f' = id
    using id linear_inverse_left linear_linear linf ling by blast
  moreover have ∗: ⋀T. ⟦T ⊆ S; x ∈ interior T⟧ ⟹ f x ∈ interior (f ‘ T)
    using S derf contf id ling sussmann_open_mapping by blast
  have continuous (at (f x)) g
    unfolding continuous_at Lim_at
  proof (intro strip)
    fix e :: real
    assume e > 0
    then have f x ∈ interior (f ‘ (ball x e ∩ S))
      by (simp add: ∗ S interior_open)
    then obtain d where d: 0 < d ball (f x) d ⊆ f ‘ (ball x e ∩ S)
      unfolding mem_interior by blast
    show ∃ d>0. ∀ y. 0 < dist y (f x) ∧ dist y (f x) < d ⟶ dist (g y) (g (f x)) <
e
    proof (intro exI allI impI conjI)
      fix y
      assume 0 < dist y (f x) ∧ dist y (f x) < d
      then have g y ∈ g ‘ f ‘ (ball x e ∩ S)
        by (metis d(2) dist_commute mem_ball rev_image_eqI subset_iff)
      then show dist (g y) (g (f x)) < e
        using ‹x ∈ S› by (simp add: gf dist_commute image_iff)
    qed (use d in auto)
  qed
  moreover have f x ∈ interior (f ‘ S)
    using ∗ S interior_eq by blast
  moreover have f (g y) = y if y ∈ interior (f ‘ S) for y
    by (metis gf imageE interiorE subsetD that)
  ultimately show ?thesis using assms
    by (metis has_derivative_inverse_basic_x open_interior)
qed
```

A rewrite based on the other domain.

lemma *has_derivative_inverse_strong_x*:
fixes f :: *'a*::*euclidean_space* ⇒ *'a*
assumes *open S*
and $g\ y \in S$
and *continuous_on S f*
and $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
and (*f has_derivative f'*) (*at* (*g y*))
and $f' \circ g' = id$
and f: $f\ (g\ y) = y$
shows (*g has_derivative g'*) (*at y*)
using *has_derivative_inverse_strong*[*OF assms*(*1*−*6*)] **by** (*simp add*: *f*)

On a region.

theorem *has_derivative_inverse_on*:
fixes f :: *'n*::*euclidean_space* ⇒ *'n*
assumes *open S*
and $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f'*(*x*)) (*at x*)
and $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
and $f'\ x \circ g'\ x = id$
and $x \in S$
shows (*g has_derivative g'*(*x*)) (*at* (*f x*))
by (*meson assms continuous_on_eq_continuous_at has_derivative_continuous has_derivative_inverse_strong*)

end

9.19 Fashoda Meet Theorem

theory *Fashoda_Theorem*
imports *Brouwer_Fixpoint Path_Connected Cartesian_Euclidean_Space*
begin

9.19.1 Bijections between intervals

definition *interval_bij* :: *'a* × *'a* ⇒ *'a* × *'a* ⇒ *'a* ⇒ *'a*::*euclidean_space*
where *interval_bij* =
(λ(*a*, *b*) (*u*, *v*) *x*. ($\sum$ *i*∈*Basis*. (*u*•*i* + (*x*•*i* − *a*•*i*) / (*b*•*i* − *a*•*i*) ∗ (*v*•*i* − *u*•*i*)) ∗$_R$ *i*))

lemma *interval_bij_affine*:
interval_bij (*a*,*b*) (*u*,*v*) = (λ*x*. ($\sum$ *i*∈*Basis*. ((*v*•*i* − *u*•*i*) / (*b*•*i* − *a*•*i*) ∗ (*x*•*i*)) ∗$_R$ *i*) +
($\sum$ *i*∈*Basis*. (*u*•*i* − (*v*•*i* − *u*•*i*) / (*b*•*i* − *a*•*i*) ∗ (*a*•*i*)) ∗$_R$ *i*))
by (*simp add*: *interval_bij_def algebra_simps add_divide_distrib diff_divide_distrib flip*: *sum.distrib scaleR_add_left*)

lemma *continuous_interval_bij*:
fixes *a b* :: *'a*::*euclidean_space*

shows *continuous* (*at x*) (*interval_bij* (*a*, *b*) (*u*, *v*))
by (*auto simp add*: *divide_inverse interval_bij_def intro*!: *continuous_sum continuous_intros*)

lemma *continuous_on_interval_bij*: *continuous_on s* (*interval_bij* (*a*, *b*) (*u*, *v*))
by (*metis continuous_at_imp_continuous_on continuous_interval_bij*)

lemma *in_interval_interval_bij*:
fixes *a b u v x* :: *'a*::*euclidean_space*
assumes *x* ∈ *cbox a b*
and *cbox u v* ≠ {}
shows *interval_bij* (*a*, *b*) (*u*, *v*) *x* ∈ *cbox u v*
proof −
have ⋀*i*. *i* ∈ *Basis* ⟹ *u* · *i* ≤ *u* · *i* + (*x* · *i* − *a* · *i*) / (*b* · *i* − *a* · *i*) ∗ (*v* · *i* − *u* · *i*)
by (*smt* (*verit*) *assms box_ne_empty*(*1*) *divide_nonneg_nonneg mem_box*(*2*) *mult_nonneg_nonneg*)
moreover
have ⋀*i*. *i* ∈ *Basis* ⟹ *u* · *i* + (*x* · *i* − *a* · *i*) / (*b* · *i* − *a* · *i*) ∗ (*v* · *i* − *u* · *i*) ≤ *v* · *i*
apply (*simp add*: *divide_simps algebra_simps*)
by (*smt* (*verit*, *best*) *assms box_ne_empty*(*1*) *left_diff_distrib mem_box*(*2*) *mult.commute mult_left_mono*)
ultimately show *?thesis*
by (*force simp only*: *interval_bij_def split_conv mem_box inner_sum_left_Basis*)
qed

lemma *interval_bij_bij*:
∀(*i*::*'a*::*euclidean_space*)∈*Basis*. *a*•*i* < *b*•*i* ∧ *u*•*i* < *v*•*i* ⟹
interval_bij (*a*, *b*) (*u*, *v*) (*interval_bij* (*u*, *v*) (*a*, *b*) *x*) = *x*
by (*auto simp*: *interval_bij_def euclidean_eq_iff*[**where** *'a*=*'a*])

lemma *interval_bij_bij_cart*: **fixes** *x*::*real*^*'n* **assumes** ∀ *i*. *a*$*i* < *b*$*i* ∧ *u*$*i* < *v*$*i*
shows *interval_bij* (*a*,*b*) (*u*,*v*) (*interval_bij* (*u*,*v*) (*a*,*b*) *x*) = *x*
using *assms* **by** (*intro interval_bij_bij*) (*auto simp*: *Basis_vec_def inner_axis*)

9.19.2 Fashoda meet theorem

lemma *infnorm_2*:
fixes *x* :: *real*^*2*
shows *infnorm x* = *max* |*x*$*1*| |*x*$*2*|
unfolding *infnorm_cart UNIV_2* **by** (*rule cSup_eq*) *auto*

lemma *infnorm_eq_1_2*:
fixes *x* :: *real*^*2*
shows *infnorm x* = *1* ⟷
|*x*$*1*| ≤ *1* ∧ |*x*$*2*| ≤ *1* ∧ (*x*$*1* = −*1* ∨ *x*$*1* = *1* ∨ *x*$*2* = −*1* ∨ *x*$*2* = *1*)
unfolding *infnorm_2* **by** *auto*

lemma *infnorm_eq_1_imp*:
fixes $x :: real\hat{}2$
assumes *infnorm* $x = 1$
shows $|x\$1| \leq 1$ **and** $|x\$2| \leq 1$
using *assms* **unfolding** *infnorm_eq_1_2* **by** *auto*

proposition *fashoda_unit*:
fixes $f\ g :: real \Rightarrow real\hat{}2$
assumes $f\ `\ \{-1\ ..\ 1\} \subseteq cbox\ (-1)\ 1$
and $g\ `\ \{-1\ ..\ 1\} \subseteq cbox\ (-1)\ 1$
and *continuous_on* $\{-1\ ..\ 1\}\ f$
and *continuous_on* $\{-1\ ..\ 1\}\ g$
and $f\ (-\ 1)\$1 = -\ 1$
and $f\ 1\$1 = 1\ g\ (-\ 1)\ \$2 = -1$
and $g\ 1\ \$2 = 1$
shows $\exists s{\in}\{-1\ ..\ 1\}.\ \exists t{\in}\{-1\ ..\ 1\}.\ f\ s = g\ t$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
note *as* = *this*[*unfolded bex_simps*,*rule_format*]
define *sqprojection*
where [*abs_def*]: *sqprojection* $z = (inverse\ (infnorm\ z)) *_R z$ **for** $z :: real\hat{}2$
define *negatex* $:: real\hat{}2 \Rightarrow real\hat{}2$
where *negatex* $x = (vector\ [-(x\$1),\ x\$2])$ **for** x
have *inf_nega*: $\bigwedge z{::}real\hat{}2.\ infnorm\ (negatex\ z) = infnorm\ z$
unfolding *negatex_def infnorm_2 vector_2* **by** *auto*
have *inf_eq1*: $\bigwedge z.\ z \neq 0 \Longrightarrow infnorm\ (sqprojection\ z) = 1$
unfolding *sqprojection_def infnorm_mul*[*unfolded scalar_mult_eq_scaleR*]
by (*simp add*: *real_abs_infnorm infnorm_eq_0*)
let *?F* = $\lambda w{::}real\hat{}2.\ (f \circ (\lambda x.\ x\$1))\ w - (g \circ (\lambda x.\ x\$2))\ w$
have $*$: $\bigwedge i.\ (\lambda x{::}real\hat{}2.\ x\ \$\ i)\ `\ cbox\ (-\ 1)\ 1 = \{-1..1\}$
proof
show $(\lambda x{::}real\hat{}2.\ x\ \$\ i)\ `\ cbox\ (-\ 1)\ 1 \subseteq \{-1..1\}$ **for** i
by (*auto simp*: *mem_box_cart*)
show $\{-1..1\} \subseteq (\lambda x{::}real\hat{}2.\ x\ \$\ i)\ `\ cbox\ (-\ 1)\ 1$ **for** i
by (*clarsimp simp*: *image_iff mem_box_cart Bex_def*) (*metis* (*no_types*, *opaque_lifting*) *vec_component*)
qed
{
fix x
assume $x \in (\lambda w.\ (f \circ (\lambda x.\ x\ \$\ 1))\ w - (g \circ (\lambda x.\ x\ \$\ 2))\ w)\ `\ (cbox\ (-\ 1)\ (1{::}real\hat{}2))$
then obtain $w :: real\hat{}2$ **where** w:
$w \in cbox\ (-\ 1)\ 1$
$x = (f \circ (\lambda x.\ x\ \$\ 1))\ w - (g \circ (\lambda x.\ x\ \$\ 2))\ w$
unfolding *image_iff* **..**
then have $x \neq 0$
using *as*[*of w*$1 *w*$2] **by** (*auto simp*: *mem_box_cart atLeastAtMost_iff*)
} **note** *x0* = *this*

```
  let ?CB11 = cbox (− 1) (1::real^2)
  obtain x :: real^2 where x:
      x ∈ cbox (− 1) 1
      (negatex ∘ sqprojection ∘ (λw. (f ∘ (λx. x $ 1)) w − (g ∘ (λx. x $ 2)) w)) x
= x
  proof (rule brouwer_weak[of ?CB11 negatex ∘ sqprojection ∘ ?F])
    show compact ?CB11 convex ?CB11
      by (rule compact_cbox convex_box)+
    have box (− 1) (1::real^2) ≠ {}
      unfolding interval_eq_empty_cart by auto
    then show interior ?CB11 ≠ {}
      by simp
    have negatex (x + y) $ i = (negatex x + negatex y) $ i ∧ negatex (c *R x) $
i = (c *R negatex x) $ i
      for i x y c
      using exhaust_2 [of i] by (auto simp: negatex_def)
    then have bounded_linear negatex
      by (simp add: bounded_linearI' vec_eq_iff)
    then show continuous_on ?CB11 (negatex ∘ sqprojection ∘ ?F)
      unfolding sqprojection_def
      apply (intro continuous_intros continuous_on_component | use ∗ assms in
presburger)+
       apply (simp_all add: infnorm_eq_0 x0 linear_continuous_on)
      done
    have (negatex ∘ sqprojection ∘ ?F) ‘ ?CB11 ⊆ ?CB11
    proof clarsimp
      fix y :: real^2
      assume y: y ∈ ?CB11
      have ?F y ≠ 0
        by (rule x0) (use y in auto)
      then have ∗: infnorm (sqprojection (?F y)) = 1
        using inf_eq1 by blast
      show negatex (sqprojection (f (y $ 1) − g (y $ 2))) ∈ cbox (−1) 1
        unfolding mem_box_cart interval_cbox_cart infnorm_2
      by (smt (verit, del_insts) ∗ component_le_infnorm_cart inf_nega neg_one_index
o_apply one_index)
    qed
    then show negatex ∘ sqprojection ∘ ?F ∈ ?CB11 → ?CB11
      by blast
  qed
  have ?F x ≠ 0
    by (rule x0) (use x in auto)
  then have ∗: infnorm (sqprojection (?F x)) = 1
    using inf_eq1 by blast
  have nx: infnorm x = 1
    by (metis (no_types, lifting) ∗ inf_nega o_apply x(2))
  have iff: 0 < sqprojection x$i ⟷ 0 < x$i sqprojection x$i < 0 ⟷ x$i < 0
if x ≠ 0 for x i
  proof −
```

```
    have ∗: inverse (infnorm x) > 0
      by (simp add: infnorm_pos_lt that)
    then show (0 < sqprojection x $ i) = (0 < x $ i)
      by (simp add: sqprojection_def zero_less_mult_iff)
    show (sqprojection x $ i < 0) = (x $ i < 0)
      unfolding sqprojection_def
      by (metis ∗ pos_less_divideR_eq scaleR_zero_right vector_scaleR_component)
  qed
  have x1: x $ 1 ∈ {− 1..1::real} x $ 2 ∈ {− 1..1::real}
    using x(1) unfolding mem_box_cart by auto
  then have nz: f (x $ 1) − g (x $ 2) ≠ 0
    using as by auto
  consider x $ 1 = −1 | x $ 1 = 1 | x $ 2 = −1 | x $ 2 = 1
    using nx unfolding infnorm_eq_1_2 by auto
  then show False
  proof cases
    case 1
    then have ∗: f (x $ 1) $ 1 = − 1
      using assms(5) by auto
    have sqprojection (f (x$1) − g (x$2)) $ 1 > 0
      by (smt (verit) 1 negatex_def o_apply vector_2(1) x(2))
    moreover
    from x1 have g (x $ 2) ∈ cbox (−1) 1
      using assms(2) by blast
    ultimately show False
      unfolding iff[OF nz] vector_component_simps ∗ mem_box_cart
      using not_le by auto
  next
    case 2
    then have ∗: f (x $ 1) $ 1 = 1
      using assms(6) by auto
    have sqprojection (f (x$1) − g (x$2)) $ 1 < 0
      by (smt (verit) 2 negatex_def o_apply vector_2(1) x(2) zero_less_one)
    moreover have g (x $ 2) ∈ cbox (−1) 1
      using assms(2) x1 by blast
    ultimately show False
      unfolding iff[OF nz] vector_component_simps ∗ mem_box_cart
      using not_le by auto
  next
    case 3
    then have ∗: g (x $ 2) $ 2 = − 1
      using assms(7) by auto
    moreover have sqprojection (f (x$1) − g (x$2)) $ 2 < 0
      by (smt (verit, ccfv_SIG) 3 negatex_def o_apply vector_2(2) x(2))
    moreover from x1 have f (x $ 1) ∈ cbox (−1) 1
      using assms(1) by blast
    ultimately show False
      by (smt (verit, del_insts) iff(2) mem_box_cart(2) neg_one_index nz vec-
tor_minus_component)
```

```
  next
    case 4
    then have *: g (x $ 2) $ 2 = 1
      using assms(8) by auto
    have sqprojection (f (x$1) - g (x$2)) $ 2 > 0
      by (smt (verit, best) 4 negatex_def o_apply vector_2(2) x(2))
    moreover
    from x1 have f (x $ 1) ∈ cbox (-1) 1
      using assms(1) by blast
    ultimately show False
    by (smt (verit) * iff(1) mem_box_cart(2) nz one_index vector_minus_component)
  qed
qed

proposition fashoda_unit_path:
  fixes f g :: real ⇒ real^2
  assumes path f
    and path g
    and path_image f ⊆ cbox (-1) 1
    and path_image g ⊆ cbox (-1) 1
    and (pathstart f)$1 = -1
    and (pathfinish f)$1 = 1
    and (pathstart g)$2 = -1
    and (pathfinish g)$2 = 1
  obtains z where z ∈ path_image f and z ∈ path_image g
proof -
  note assms = assms[unfolded path_def pathstart_def pathfinish_def path_image_def]
  define iscale where [abs_def]: iscale z = inverse 2 *R (z + 1) for z :: real
  have isc: iscale ` {- 1..1} ⊆ {0..1}
    unfolding iscale_def by auto
  have ∃ s∈{- 1..1}. ∃ t∈{- 1..1}. (f ∘ iscale) s = (g ∘ iscale) t
  proof (rule fashoda_unit)
    show (f ∘ iscale) ` {- 1..1} ⊆ cbox (- 1) 1 (g ∘ iscale) ` {- 1..1} ⊆ cbox (-
1) 1
      using isc and assms(3-4) by (auto simp add: image_comp [symmetric])
    have *: continuous_on {- 1..1} iscale
      unfolding iscale_def by (rule continuous_intros)+
    show continuous_on {- 1..1} (f ∘ iscale)
     using * assms(1) continuous_on_compose continuous_on_subset isc by blast
    show continuous_on {- 1..1} (g ∘ iscale)
      by (meson * assms(2) continuous_on_compose continuous_on_subset isc)
    have *: (1 / 2) *R (1 + (1::real^1)) = 1
      unfolding vec_eq_iff by auto
    show (f ∘ iscale) (- 1) $ 1 = - 1
      and (f ∘ iscale) 1 $ 1 = 1
      and (g ∘ iscale) (- 1) $ 2 = -1
      and (g ∘ iscale) 1 $ 2 = 1
      unfolding o_def iscale_def using assms by (auto simp add: *)
  qed
```

```
  then obtain s t where st: s ∈ {− 1..1} t ∈ {− 1..1} (f ∘ iscale) s = (g ∘
iscale) t
    by auto
  show thesis
  proof
    show f (iscale s) ∈ path_image f
      by (metis image_eqI image_subset_iff isc path_image_def st(1))
    show f (iscale s) ∈ path_image g
      by (metis comp_def image_eqI image_subset_iff isc path_image_def st(2)
st(3))
  qed
qed

theorem fashoda:
  fixes b :: real^2
  assumes path f
    and path g
    and path_image f ⊆ cbox a b
    and path_image g ⊆ cbox a b
    and (pathstart f)$1 = a$1
    and (pathfinish f)$1 = b$1
    and (pathstart g)$2 = a$2
    and (pathfinish g)$2 = b$2
  obtains z where z ∈ path_image f and z ∈ path_image g
proof −
  fix P Q S
  presume P ∨ Q ∨ S P ⟹ thesis and Q ⟹ thesis and S ⟹ thesis
  then show thesis
    by auto
next
  have cbox a b ≠ {}
    using assms(3) using path_image_nonempty[of f] by auto
  then have a ≤ b
   unfolding interval_eq_empty_cart less_eq_vec_def by (auto simp add: not_less)
  then show a$1 = b$1 ∨ a$2 = b$2 ∨ (a$1 < b$1 ∧ a$2 < b$2)
    unfolding less_eq_vec_def forall_2 by auto
next
  assume as: a$1 = b$1
  have ∃ z∈path_image g. z$2 = (pathstart f)$2
  proof (rule connected_ivt_component_cart)
    show pathstart g $ 2 ≤ pathstart f $ 2
      by (metis assms(3) assms(7) mem_box_cart(2) pathstart_in_path_image
subset_iff)
    show pathstart f $ 2 ≤ pathfinish g $ 2
    by (metis assms(3) assms(8) in_mono mem_box_cart(2) pathstart_in_path_image)
    show connected (path_image g)
      using assms(2) by blast
  qed (auto simp: path_defs)
  then obtain z :: real^2 where z: z ∈ path_image g z $ 2 = pathstart f $ 2 ..
```

have $z \in cbox\ a\ b$
using *assms*(4) *z*(1) **by** *blast*
then have $z = f\ 0$
by (*smt* (*verit*) *as assms*(5) *exhaust_2 mem_box_cart*(2) *nle_le pathstart_def vec_eq_iff z*(2))
then show *thesis*
by (*metis path_defs*(2) *pathstart_in_path_image that z*(1))
next
assume *as*: $a\$2 = b\$2$
have $\exists z{\in}path\_image\ f.\ z\$1 = (pathstart\ g)\$1$
proof (*rule connected_ivt_component_cart*)
show $pathstart\ f\ \$\ 1 \le pathstart\ g\ \$\ 1$
using *assms*(4) *assms*(5) *mem_box_cart*(2) **by** *fastforce*
show $pathstart\ g\ \$\ 1 \le pathfinish\ f\ \$\ 1$
using *assms*(4) *assms*(6) *mem_box_cart*(2) *pathstart_in_path_image* **by** *fastforce*
show *connected* (*path_image f*)
by (*simp add*: *assms*(1) *connected_path_image*)
qed (*auto simp*: *path_defs*)
then obtain *z* **where** *z*: $z \in path\_image\ f\ z\ \$\ 1 = pathstart\ g\ \$\ 1$ **..**
have $z \in cbox\ a\ b$
using *assms*(3) *z*(1) **by** *auto*
then have $z = g\ 0$
by (*smt* (*verit*) *as assms*(7) *exhaust_2 mem_box_cart*(2) *pathstart_def vec_eq_iff z*(2))
then show *thesis*
by (*metis path_defs*(2) *pathstart_in_path_image that z*(1))
next
assume *as*: $a\ \$\ 1 < b\ \$\ 1 \wedge a\ \$\ 2 < b\ \$\ 2$
have *int_nem*: $cbox\ (-1)\ (1{::}real\hat{}2) \neq \{\}$
unfolding *interval_eq_empty_cart* **by** *auto*
obtain $z :: real\hat{}2$ **where** *z*:
$z \in (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ f)\ `\ \{0..1\}$
$z \in (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ g)\ `\ \{0..1\}$
proof (*rule fashoda_unit_path*)
show $path\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ f)$
by (*meson assms*(1) *continuous_on_interval_bij path_continuous_image*)
show $path\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ g)$
by (*meson assms*(2) *continuous_on_interval_bij path_continuous_image*)
show $path\_image\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ f) \subseteq cbox\ (-\ 1)\ 1$
using *assms*(3)
by (*simp add*: *path_image_def in_interval_interval_bij int_nem subset_eq*)
show $path\_image\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ g) \subseteq cbox\ (-\ 1)\ 1$
using *assms*(4)
by (*simp add*: *path_image_def in_interval_interval_bij int_nem subset_eq*)
show $pathstart\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ f)\ \$\ 1 = -\ 1$
$pathfinish\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ f)\ \$\ 1 = 1$
$pathstart\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ g)\ \$\ 2 = -\ 1$
$pathfinish\ (interval\_bij\ (a,\ b)\ (-\ 1,\ 1) \circ g)\ \$\ 2 = 1$

```
    using assms as
     by (simp_all add: cart_eq_inner_axis pathstart_def pathfinish_def interval_bij_def)
        (simp_all add: inner_axis)
  qed (auto simp: path_defs)
  then obtain zf zg where zf: zf ∈ {0..1} z = (interval_bij (a, b) (- 1, 1) ∘ f) zf
                      and zg: zg ∈ {0..1} z = (interval_bij (a, b) (- 1, 1) ∘ g) zg
    by blast
  have ∗: ∀ i. (- 1) $ i < (1::real^2) $ i ∧ a $ i < b $ i
    unfolding forall_2 using as by auto
  show thesis
  proof (rule_tac z=interval_bij (- 1,1) (a,b) z in that)
    show interval_bij (- 1, 1) (a, b) z ∈ path_image f
      using zf by (simp add: interval_bij_bij_cart[OF ∗] path_image_def)
    show interval_bij (- 1, 1) (a, b) z ∈ path_image g
      using zg by (simp add: interval_bij_bij_cart[OF ∗] path_image_def)
  qed
qed
```

9.19.3 Some slightly ad hoc lemmas I use below

```
lemma segment_vertical:
  fixes a :: real^2
  assumes a$1 = b$1
  shows x ∈ closed_segment a b ⟷
    x$1 = a$1 ∧ x$1 = b$1 ∧ (a$2 ≤ x$2 ∧ x$2 ≤ b$2 ∨ b$2 ≤ x$2 ∧ x$2 ≤ a$2)
  (is _ = ?R)
proof −
  let ?L = ∃ u. (x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1 ∧ x $ 2 = (1 − u) ∗ a $ 2 + u ∗ b $ 2) ∧ 0 ≤ u ∧ u ≤ 1
  {
    presume ?L ⟹ ?R and ?R ⟹ ?L
    then show ?thesis
      unfolding closed_segment_def mem_Collect_eq
    unfolding vec_eq_iff forall_2 scalar_mult_eq_scaleR[symmetric] vector_component_simps
      by blast
  }
  {
    assume ?L
    then obtain u where u:
        x $ 1 = (1 − u) ∗ a $ 1 + u ∗ b $ 1
        x $ 2 = (1 − u) ∗ a $ 2 + u ∗ b $ 2
        0 ≤ u u ≤ 1
      by blast
    { fix b a
      assume b + u ∗ a > a + u ∗ b
      then have (1 − u) ∗ b > (1 − u) ∗ a
```

```
      by (auto simp add:field_simps)
    then have b ≥ a
      using not_less_iff_gr_or_eq u(4) by fastforce
    then have u * a ≤ u * b
      by (simp add: mult_left_mono u(3))
  }
  moreover
  { fix a b
    assume u * b > u * a
    then have (1 - u) * a ≤ (1 - u) * b
      using less_eq_real_def u(3) u(4) by force
    then have a + u * b ≤ b + u * a
      by (auto simp add: field_simps)
  } ultimately show ?R
    by (force simp add: u assms field_simps not_le)
 }
 {
  assume ?R
  then show ?L
  proof (cases x$2 = b$2)
    case True
    with ‹?R› show ?L
      by (rule_tac x=(x$2 - a$2) / (b$2 - a$2) in exI) (auto simp add:
field_simps)
  next
    case False
    with ‹?R› show ?L
      by (rule_tac x=1 - (x$2 - b$2) / (a$2 - b$2) in exI) (auto simp add:
field_simps)
  qed
 }
qed
```

Essentially duplicate proof that could be done by swapping co-ordinates

```
lemma segment_horizontal:
 fixes a :: real^2
 assumes a$2 = b$2
 shows x ∈ closed_segment a b ⟷
   x$2 = a$2 ∧ x$2 = b$2 ∧ (a$1 ≤ x$1 ∧ x$1 ≤ b$1 ∨ b$1 ≤ x$1 ∧ x$1 ≤
a$1)
 (is _ = ?R)
proof -
 let ?L = ∃u. (x $ 1 = (1 - u) * a $ 1 + u * b $ 1 ∧ x $ 2 = (1 - u) * a $ 2
+ u * b $ 2) ∧ 0 ≤ u ∧ u ≤ 1
 {
  presume ?L ⟹ ?R and ?R ⟹ ?L
  then show ?thesis
    unfolding closed_segment_def mem_Collect_eq
  unfolding vec_eq_iff forall_2 scalar_mult_eq_scaleR[symmetric] vector_component_simps
```

```
    by blast
  }
  {
    assume ?L
    then obtain u where u:
      x $ 1 = (1 - u) * a $ 1 + u * b $ 1
      x $ 2 = (1 - u) * a $ 2 + u * b $ 2
      0 ≤ u u ≤ 1
      by blast
    { fix b a
      assume b + u * a > a + u * b
      then have (1 - u) * b > (1 - u) * a
        by (auto simp add: field_simps)
      then have b ≥ a
        by (smt (verit, best) mult_left_mono u(4))
      then have u * a ≤ u * b
        by (simp add: mult_left_mono u(3))
    }
    moreover
    { fix a b
      assume u * b > u * a
      then have (1 - u) * a ≤ (1 - u) * b
        using less_eq_real_def u(3) u(4) by force
      then have a + u * b ≤ b + u * a
        by (auto simp add: field_simps)
    }
    ultimately show ?R
      by (force simp add: u assms field_simps not_le intro: )
  }
  { assume ?R
    then show ?L
    proof (cases x$1 = b$1)
      case True
      with ‹?R› show ?L
        by (rule_tac x=(x$1 - a$1) / (b$1 - a$1) in exI) (auto simp add:
field_simps)
    next
      case False
      with ‹?R› show ?L
        by (rule_tac x=1 - (x$1 - b$1) / (a$1 - b$1) in exI) (auto simp add:
field_simps)
    qed
  }
qed
```

9.19.4 Useful Fashoda corollary pointed out to me by Tom Hales

corollary *fashoda_interlace*:

fixes *a* :: *real^2*
assumes *path f*
and *path g*
and *paf*: *path_image f* ⊆ *cbox a b*
and *pag*: *path_image g* ⊆ *cbox a b*
and (*pathstart f*)$*2* = *a*$*2*
and (*pathfinish f*)$*2* = *a*$*2*
and (*pathstart g*)$*2* = *a*$*2*
and (*pathfinish g*)$*2* = *a*$*2*
and (*pathstart f*)$*1* < (*pathstart g*)$*1*
and (*pathstart g*)$*1* < (*pathfinish f*)$*1*
and (*pathfinish f*)$*1* < (*pathfinish g*)$*1*
obtains *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*
proof −
have *cbox a b* ≠ {}
using *path_image_nonempty*[*of f*] **using** *assms*(*3*) **by** *auto*
note *ab*=*this*[*unfolded interval_eq_empty_cart not_ex forall_2 not_less*]
have *pathstart f* ∈ *cbox a b*
and *pathfinish f* ∈ *cbox a b*
and *pathstart g* ∈ *cbox a b*
and *pathfinish g* ∈ *cbox a b*
using *pathstart_in_path_image pathfinish_in_path_image*
using *assms*(*3*−*4*)
by *auto*
note *startfin* = *this*[*unfolded mem_box_cart forall_2*]
let *?P1* = *linepath* (*vector*[*a*$*1* − *2*, *a*$*2* − *2*]) (*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*]) +++
linepath(*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*])(*pathstart f*) +++ *f* +++
linepath(*pathfinish f*)(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*]) +++
linepath(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*])(*vector*[*b*$*1* + *2*,*a*$*2* − *2*])
let *?P2* = *linepath*(*vector*[(*pathstart g*)$*1*, (*pathstart g*)$*2* − *3*])(*pathstart g*) +++ *g* +++
linepath(*pathfinish g*)(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*]) +++
linepath(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*])(*vector*[*b*$*1* + *1*,*a*$*2* − *1*]) +++
linepath(*vector*[*b*$*1* + *1*,*a*$*2* − *1*])(*vector*[*b*$*1* + *1*,*b*$*2* + *3*])
let *?a* = *vector*[*a*$*1* − *2*, *a*$*2* − *3*]
let *?b* = *vector*[*b*$*1* + *2*, *b*$*2* + *3*]
have *P1P2*: *path_image ?P1* = *path_image* (*linepath* (*vector*[*a*$*1* − *2*, *a*$*2* − *2*]) (*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*])) ∪
path_image (*linepath*(*vector*[(*pathstart f*)$*1*,*a*$*2* − *2*])(*pathstart f*)) ∪ *path_image f* ∪
path_image (*linepath*(*pathfinish f*)(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*])) ∪
path_image (*linepath*(*vector*[(*pathfinish f*)$*1*,*a*$*2* − *2*])(*vector*[*b*$*1* + *2*,*a*$*2* − *2*]))
path_image ?P2 = *path_image*(*linepath*(*vector*[(*pathstart g*)$*1*, (*pathstart g*)$*2* − *3*])(*pathstart g*)) ∪ *path_image g* ∪
path_image(*linepath*(*pathfinish g*)(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*])) ∪
path_image(*linepath*(*vector*[(*pathfinish g*)$*1*,*a*$*2* − *1*])(*vector*[*b*$*1* + *1*,*a*$*2* − *1*])) ∪

```
      path_image(linepath(vector[b$1 + 1,a$2 − 1])(vector[b$1 + 1,b$2 + 3]))
using assms(1−2)
     by(auto simp add: path_image_join)
  have abab: cbox a b ⊆ cbox ?a ?b
    unfolding interval_cbox_cart[symmetric]
    by (auto simp add:less_eq_vec_def forall_2)
  obtain z where
    z ∈ path_image
          (linepath (vector [a $ 1 − 2, a $ 2 − 2]) (vector [pathstart f $ 1, a $ 2
− 2]) +++
           linepath (vector [pathstart f $ 1, a $ 2 − 2]) (pathstart f) +++
           f +++
           linepath (pathfinish f) (vector [pathfinish f $ 1, a $ 2 − 2]) +++
           linepath (vector [pathfinish f $ 1, a $ 2 − 2]) (vector [b $ 1 + 2, a $ 2
− 2]))
    z ∈ path_image
         (linepath (vector [pathstart g $ 1, pathstart g $ 2 − 3]) (pathstart g) +++
          g +++
          linepath (pathfinish g) (vector [pathfinish g $ 1, a $ 2 − 1]) +++
          linepath (vector [pathfinish g $ 1, a $ 2 − 1]) (vector [b $ 1 + 1, a $ 2
− 1]) +++
          linepath (vector [b $ 1 + 1, a $ 2 − 1]) (vector [b $ 1 + 1, b $ 2 + 3]))
    apply (rule fashoda[of ?P1 ?P2 ?a ?b])
   unfolding pathstart_join pathfinish_join pathstart_linepath pathfinish_linepath
vector_2
  proof −
    show path ?P1 and path ?P2
      using assms by auto
    show path_image ?P1 ⊆ cbox ?a ?b path_image ?P2 ⊆ cbox ?a ?b
      unfolding P1P2 path_image_linepath using startfin paf pag
     by (auto simp: mem_box_cart segment_horizontal segment_vertical forall_2)
    show a $ 1 − 2 = a $ 1 − 2
      and b $ 1 + 2 = b $ 1 + 2
      and pathstart g $ 2 − 3 = a $ 2 − 3
      and b $ 2 + 3 = b $ 2 + 3
      by (auto simp add: assms)
  qed
  note z=this[unfolded P1P2 path_image_linepath]
  show thesis
  proof (rule that[of z])
    have (z ∈ closed_segment (vector [a $ 1 − 2, a $ 2 − 2]) (vector [pathstart f
$ 1, a $ 2 − 2]) ∨
      z ∈ closed_segment (vector [pathstart f $ 1, a $ 2 − 2]) (pathstart f)) ∨
      z ∈ closed_segment (pathfinish f) (vector [pathfinish f $ 1, a $ 2 − 2]) ∨
      z ∈ closed_segment (vector [pathfinish f $ 1, a $ 2 − 2]) (vector [b $ 1 + 2,
a $ 2 − 2]) ⟹
    (((z ∈ closed_segment (vector [pathstart g $ 1, pathstart g $ 2 − 3]) (pathstart
g)) ∨
      z ∈ closed_segment (pathfinish g) (vector [pathfinish g $ 1, a $ 2 − 1])) ∨
```

```
    z ∈ closed_segment (vector [pathfinish g $ 1, a $ 2 − 1]) (vector [b $ 1 + 1,
a $ 2 − 1])) ∨
    z ∈ closed_segment (vector [b $ 1 + 1, a $ 2 − 1]) (vector [b $ 1 + 1, b $
2 + 3]) ⟹ False
  proof (simp only: segment_vertical segment_horizontal vector_2, goal_cases)
    case prems: 1
    have pathfinish f ∈ cbox a b
      using assms(3) pathfinish_in_path_image[of f] by auto
    then have 1 + b $ 1 ≤ pathfinish f $ 1 ⟹ False
      unfolding mem_box_cart forall_2 by auto
    then have z$1 ≠ pathfinish f$1
      using assms(10) assms(11) prems(2) by auto
    moreover have pathstart f ∈ cbox a b
      using assms(3) pathstart_in_path_image[of f]
      by auto
    then have 1 + b $ 1 ≤ pathstart f $ 1 ⟹ False
      unfolding mem_box_cart forall_2
      by auto
    then have z$1 ≠ pathstart f$1
      using prems(2) using assms ab
      by (auto simp add: field_simps)
    ultimately have ∗: z$2 = a$2 − 2
      using prems(1) by auto
    have z$1 ≠ pathfinish g$1
      using prems(2) assms ab
      by (auto simp add: field_simps ∗)
    moreover have pathstart g ∈ cbox a b
      using assms(4) pathstart_in_path_image[of g]
      by auto
    note this[unfolded mem_box_cart forall_2]
    then have z$1 ≠ pathstart g$1
      using prems(1) assms ab
      by (auto simp add: field_simps ∗)
    ultimately have a $ 2 − 1 ≤ z $ 2 ∧ z $ 2 ≤ b $ 2 + 3 ∨ b $ 2 + 3 ≤ z
$ 2 ∧ z $ 2 ≤ a $ 2 − 1
      using prems(2)  unfolding ∗ assms by (auto simp add: field_simps)
    then show False
      unfolding ∗ using ab by auto
  qed
  then have z ∈ path_image f ∨ z ∈ path_image g
    using z unfolding Un_iff by blast
  then have z′: z ∈ cbox a b
    using assms(3−4) by auto
  have a $ 2 = z $ 2 ⟹ (z $ 1 = pathstart f $ 1 ∨ z $ 1 = pathfinish f $ 1)
⟹
    z = pathstart f ∨ z = pathfinish f
    unfolding vec_eq_iff forall_2 assms
    by auto
  with z′ show z ∈ path_image f
```

using *z*(*1*)
unfolding *Un_iff mem_box_cart forall_2*
using *assms*(*5*) *assms*(*6*) *segment_horizontal segment_vertical* **by** *auto*
have *a* $ *2* = *z* $ *2* ⟹ (*z* $ *1* = *pathstart g* $ *1* ∨ *z* $ *1* = *pathfinish g* $ *1*)
⟹
z = *pathstart g* ∨ *z* = *pathfinish g*
unfolding *vec_eq_iff forall_2 assms*
by *auto*
with *z'* **show** *z* ∈ *path_image g*
using *z*(*2*)
unfolding *Un_iff mem_box_cart forall_2*
using *assms*(*7*) *assms*(*8*) *segment_horizontal segment_vertical* **by** *auto*
qed
qed

end

9.20 Vector Cross Products in 3 Dimensions

theory *Cross3*
imports *Determinants Cartesian_Euclidean_Space*
begin

context includes *no set_product_syntax*
begin — locally disable syntax for set product, to avoid warnings

definition *cross3* :: [*real^3*, *real^3*] ⇒ *real^3* (**infixr** ‹×› *80*)
where *a* × *b* ≡
vector [*a*$*2* ∗ *b*$*3* − *a*$*3* ∗ *b*$*2*,
a$*3* ∗ *b*$*1* − *a*$*1* ∗ *b*$*3*,
a$*1* ∗ *b*$*2* − *a*$*2* ∗ *b*$*1*]

end

open_bundle *cross3_syntax*
begin
notation *cross3* (**infixr** ‹×› *80*)
unbundle *no set_product_syntax*
end

9.20.1 Basic lemmas

lemmas *cross3_simps* = *cross3_def inner_vec_def sum_3 det_3 vec_eq_iff vector_def algebra_simps*

lemma *dot_cross_self*: *x* · (*x* × *y*) = *0* *x* · (*y* × *x*) = *0* (*x* × *y*) · *y* = *0* (*y* × *x*) · *y* = *0*
by (*simp_all add*: *orthogonal_def cross3_simps*)

lemma *orthogonal_cross*: *orthogonal* $(x \times y)$ x *orthogonal* $(x \times y)$ y
orthogonal y $(x \times y)$ *orthogonal* $(x \times y)$ x
by (*simp_all add*: *orthogonal_def dot_cross_self*)

lemma *cross_zero_left* [*simp*]: $0 \times x = 0$ **and** *cross_zero_right* [*simp*]: $x \times 0 = 0$ **for** x::*real^3*
by (*simp_all add*: *cross3_simps*)

lemma *cross_skew*: $(x \times y) = -(y \times x)$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_refl* [*simp*]: $x \times x = 0$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_add_left*: $(x + y) \times z = (x \times z) + (y \times z)$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_add_right*: $x \times (y + z) = (x \times y) + (x \times z)$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_mult_left*: $(c *_R x) \times y = c *_R (x \times y)$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_mult_right*: $x \times (c *_R y) = c *_R (x \times y)$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_minus_left* [*simp*]: $(-x) \times y = -(x \times y)$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *cross_minus_right* [*simp*]: $x \times -y = -(x \times y)$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *left_diff_distrib*: $(x - y) \times z = x \times z - y \times z$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

lemma *right_diff_distrib*: $x \times (y - z) = x \times y - x \times z$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

hide_fact (**open**) *left_diff_distrib right_diff_distrib*

proposition *Jacobi*: $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$ **for** x::*real^3*
by (*simp add*: *cross3_simps*)

proposition *Lagrange*: $x \times (y \times z) = (x \cdot z) *_R y - (x \cdot y) *_R z$
by (*simp add*: *cross3_simps*) (*metis* (*full_types*) *exhaust_3*)

proposition *cross_triple*: $(x \times y) \cdot z = (y \times z) \cdot x$
by (*simp add*: *cross3_def inner_vec_def sum_3 vec_eq_iff algebra_simps*)

lemma *cross_components*:
(*x* × *y*)$1 = *x*$2 * *y*$3 − *y*$2 * *x*$3 (*x* × *y*)$2 = *x*$3 * *y*$1 − *y*$3 * *x*$1 (*x* × *y*)$3 = *x*$1 * *y*$2 − *y*$1 * *x*$2
by (*simp_all add*: *cross3_def inner_vec_def sum_3 vec_eq_iff algebra_simps*)

lemma *cross_basis*: (*axis 1 1*) × (*axis 2 1*) = *axis 3 1* (*axis 2 1*) × (*axis 1 1*) = −(*axis 3 1*)
(*axis 2 1*) × (*axis 3 1*) = *axis 1 1* (*axis 3 1*) × (*axis 2 1*) = −(*axis 1 1*)
(*axis 3 1*) × (*axis 1 1*) = *axis 2 1* (*axis 1 1*) × (*axis 3 1*) = −(*axis 2 1*)
using *exhaust_3*
by (*force simp add*: *axis_def cross3_simps*)+

lemma *cross_basis_nonzero*:
u ≠ *0* ⟹ *u* × *axis 1 1* ≠ *0* ∨ *u* × *axis 2 1* ≠ *0* ∨ *u* × *axis 3 1* ≠ *0*
by (*clarsimp simp add*: *axis_def cross3_simps*) (*metis exhaust_3*)

lemma *cross_dot_cancel*:
fixes *x*::*real^3*
assumes *deq*: *x* · *y* = *x* · *z* **and** *veq*: *x* × *y* = *x* × *z* **and** *x*: *x* ≠ *0*
shows *y* = *z*
proof −
have *x* · *x* ≠ *0*
by (*simp add*: *x*)
then have *y* − *z* = *0*
using *veq*
by (*metis* (*no_types*, *lifting*) *Cross3.right_diff_distrib Lagrange deq eq_iff_diff_eq_0 inner_diff_right scale_eq_0_iff*)
then show *?thesis*
using *eq_iff_diff_eq_0* **by** *blast*
qed

lemma *norm_cross_dot*: (*norm* (*x* × *y*))^2 + (*x* · *y*)^2 = (*norm x* * *norm y*)^2
unfolding *power2_norm_eq_inner power_mult_distrib*
by (*simp add*: *cross3_simps power2_eq_square*)

lemma *dot_cross_det*: *x* · (*y* × *z*) = *det*(*vector*[*x*,*y*,*z*])
by (*simp add*: *cross3_simps*)

lemma *cross_cross_det*: (*w* × *x*) × (*y* × *z*) = *det*(*vector*[*w*,*x*,*z*]) *$_R$ *y* − *det*(*vector*[*w*,*x*,*y*]) *$_R$ *z*
using *exhaust_3* **by** (*force simp add*: *cross3_simps*)

proposition *dot_cross*: (*w* × *x*) · (*y* × *z*) = (*w* · *y*) * (*x* · *z*) − (*w* · *z*) * (*x* · *y*)
by (*force simp add*: *cross3_simps*)

proposition *norm_cross*: (*norm* (*x* × *y*))^2 = (*norm x*)^2 * (*norm y*)^2 − (*x* · *y*)^2
unfolding *power2_norm_eq_inner power_mult_distrib*

by (*simp add*: *cross3_simps power2_eq_square*)

lemma *cross_eq_0*: $x \times y = 0 \longleftrightarrow$ *collinear*$\{0,x,y\}$
proof −
have $x \times y = 0 \longleftrightarrow$ *norm* $(x \times y) = 0$
by *simp*
also have ... $\longleftrightarrow$ $(\textit{norm}\ x * \textit{norm}\ y)^2 = (x \cdot y)^2$
using *norm_cross* [*of x y*] **by** (*auto simp*: *power_mult_distrib*)
also have ... $\longleftrightarrow$ $|x \cdot y| = \textit{norm}\ x * \textit{norm}\ y$
using *power2_eq_iff*
by (*metis* (*mono_tags*, *opaque_lifting*) *abs_minus abs_norm_cancel abs_power2 norm_mult power_abs real_norm_def*)
also have ... $\longleftrightarrow$ *collinear* $\{0, x, y\}$
by (*rule norm_cauchy_schwarz_equal*)
finally show *?thesis* **.**
qed

lemma *cross_eq_self*: $x \times y = x \longleftrightarrow x = 0\ x \times y = y \longleftrightarrow y = 0$
apply (*metis cross_zero_left dot_cross_self*(*1*) *inner_eq_zero_iff*)
by (*metis cross_zero_right dot_cross_self*(*2*) *inner_eq_zero_iff*)

lemma *norm_and_cross_eq_0*:
$x \cdot y = 0 \wedge x \times y = 0 \longleftrightarrow x = 0 \vee y = 0$ (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*metis cross_dot_cancel cross_zero_right inner_zero_right*)
qed *auto*

lemma *bilinear_cross*: *bilinear*$(\times)$
apply (*auto simp add*: *bilinear_def linear_def*)
apply *unfold_locales*
apply (*simp add*: *cross_add_right*)
apply (*simp add*: *cross_mult_right*)
apply (*simp add*: *cross_add_left*)
apply (*simp add*: *cross_mult_left*)
done

9.20.2 Preservation by rotation, or other orthogonal transformation up to sign

lemma *cross_matrix_mult*: *transpose* $A *v ((A *v x) \times (A *v y)) = \textit{det}\ A *_R (x \times y)$
apply (*simp add*: *vec_eq_iff*)
apply (*simp add*: *vector_matrix_mult_def matrix_vector_mult_def forall_3 cross3_simps*)
done

lemma *cross_orthogonal_matrix*:

assumes *orthogonal_matrix A*
shows $(A *v x) \times (A *v y) = det\ A *_R (A *v (x \times y))$
proof −
have *mat 1 = transpose* $(A ** transpose\ A)$
by (*metis* (*no_types*) *assms orthogonal_matrix_def transpose_mat*)
then show *?thesis*
by (*metis* (*no_types*) *vector_matrix_mul_rid vector_transpose_matrix cross_matrix_mult matrix_vector_mul_assoc matrix_vector_mult_scaleR*)
qed

lemma *cross_rotation_matrix*: *rotation_matrix A* $\Longrightarrow (A *v x) \times (A *v y) = A *v (x \times y)$
by (*simp add*: *rotation_matrix_def cross_orthogonal_matrix*)

lemma *cross_rotoinversion_matrix*: *rotoinversion_matrix A* $\Longrightarrow (A *v x) \times (A *v y) = - A *v (x \times y)$
by (*simp add*: *rotoinversion_matrix_def cross_orthogonal_matrix scaleR_matrix_vector_assoc*)

lemma *cross_orthogonal_transformation*:
assumes *orthogonal_transformation f*
shows $(f x) \times (f y) = det(matrix\ f) *_R f(x \times y)$
proof −
have *orth*: *orthogonal_matrix* (*matrix f*)
using *assms orthogonal_transformation_matrix* **by** *blast*
have *matrix f* $*v z = f z$ **for** *z*
using *assms orthogonal_transformation_matrix* **by** *force*
with *cross_orthogonal_matrix* [*OF orth*] **show** *?thesis*
by *simp*
qed

lemma *cross_linear_image*:
$[\![linear\ f; \bigwedge x.\ norm(f\ x) = norm\ x; det(matrix\ f) = 1]\!] \Longrightarrow (f x) \times (f y) = f(x \times y)$
by (*simp add*: *cross_orthogonal_transformation orthogonal_transformation*)

9.20.3 Continuity

lemma *continuous_cross*: $[\![continuous\ F\ f; continuous\ F\ g]\!] \Longrightarrow continuous\ F\ (\lambda x.\ (f\ x) \times (g\ x))$
apply (*subst continuous_componentwise*)
apply (*clarsimp simp add*: *cross3_simps*)
apply (*intro continuous_intros*; *simp*)
done

lemma *continuous_on_cross*:
fixes *f* :: *'a*::*t2_space* ⇒ *real*^*3*
shows $[\![continuous\_on\ S\ f; continuous\_on\ S\ g]\!] \Longrightarrow continuous\_on\ S\ (\lambda x.\ (f\ x) \times (g\ x))$
by (*simp add*: *continuous_on_eq_continuous_within continuous_cross*)

unbundle *no cross3_syntax*

end

9.21 Bounded Continuous Functions

theory *Bounded_Continuous_Function*
imports
Topology_Euclidean_Space
Uniform_Limit
begin

9.21.1 Definition

definition *bcontfun* = {*f*. *continuous_on UNIV f* ∧ *bounded* (*range f*)}

typedef (**overloaded**) ('*a*, '*b*) *bcontfun* (‹(‹*notation*=‹*infix* ⇒$_C$›› _ ⇒$_C$ /_)› [*22*, *21*] *21*) =
bcontfun::('*a*::*topological_space* ⇒ '*b*::*metric_space*) *set*
morphisms *apply_bcontfun Bcontfun*
by (*auto intro*: *continuous_intros simp*: *bounded_def bcontfun_def*)

declare [[*coercion apply_bcontfun* :: ('*a*::*topological_space* ⇒$_C$ '*b*::*metric_space*) ⇒ '*a* ⇒ '*b*]]

setup_lifting *type_definition_bcontfun*

lemma *continuous_on_apply_bcontfun*[*intro*, *simp*]: *continuous_on T* (*apply_bcontfun x*)
and *bounded_apply_bcontfun*[*intro*, *simp*]: *bounded* (*range* (*apply_bcontfun x*))
using *apply_bcontfun*[*of x*]
by (*auto simp*: *bcontfun_def intro*: *continuous_on_subset*)

lemma *bcontfun_eqI*: (⋀*x*. *apply_bcontfun f x* = *apply_bcontfun g x*) ⟹ *f* = *g*
by *transfer auto*

lemma *bcontfunE*:
assumes *f* ∈ *bcontfun*
obtains *g* **where** *f* = *apply_bcontfun g*
by (*blast intro*: *apply_bcontfun_cases assms*)

lemma *const_bcontfun*: (λ*x*. *b*) ∈ *bcontfun*
by (*auto simp*: *bcontfun_def image_def*)

lift_definition *const_bcontfun*::'*b*::*metric_space* ⇒ ('*a*::*topological_space* ⇒$_C$ '*b*)
is λ*c* _. *c*
by (*rule const_bcontfun*)

instantiation *bcontfun* :: (*topological_space*, *metric_space*) *metric_space*
begin

lift_definition *dist_bcontfun* :: $'a \Rightarrow_C 'b \Rightarrow 'a \Rightarrow_C 'b \Rightarrow real$
is $\lambda f\ g.\ (SUP\ x.\ dist\ (f\ x)\ (g\ x))$ **.**

definition *uniformity_bcontfun* :: $('a \Rightarrow_C 'b \times 'a \Rightarrow_C 'b)\ filter$
where *uniformity_bcontfun* = $(INF\ e{\in}\{0 <..\}.\ principal\ \{(x, y).\ dist\ x\ y < e\})$

definition *open_bcontfun* :: $('a \Rightarrow_C 'b)\ set \Rightarrow bool$
where *open_bcontfun* $S = (\forall\, x{\in}S.\ \forall_F\ (x', y)\ in\ uniformity.\ x' = x \longrightarrow y \in S)$

lemma *bounded_dist_le_SUP_dist*:
bounded (*range f*) $\Longrightarrow$ *bounded* (*range g*) $\Longrightarrow$ *dist* (*f x*) (*g x*) $\leq$ (*SUP x. dist* (*f x*) (*g x*))
by (*auto intro*!: *cSUP_upper bounded_imp_bdd_above bounded_dist_comp*)

lemma *dist_bounded*:
fixes *f g* :: $'a \Rightarrow_C 'b$
shows *dist* (*f x*) (*g x*) $\leq$ *dist f g*
by *transfer* (*auto intro*!: *bounded_dist_le_SUP_dist simp*: *bcontfun_def*)

lemma *dist_bound*:
fixes *f g* :: $'a \Rightarrow_C 'b$
assumes $\bigwedge x.$ *dist* (*f x*) (*g x*) $\leq b$
shows *dist f g* $\leq b$
using *assms*
by *transfer* (*auto intro*!: *cSUP_least*)

lemma *dist_fun_lt_imp_dist_val_lt*:
fixes *f g* :: $'a \Rightarrow_C 'b$
assumes *dist f g* $< e$
shows *dist* (*f x*) (*g x*) $< e$
using *dist_bounded assms* **by** (*rule le_less_trans*)

instance
proof
fix *f g h* :: $'a \Rightarrow_C 'b$
show *dist f g* $= 0 \longleftrightarrow f = g$
proof
have $\bigwedge x.$ *dist* (*f x*) (*g x*) $\leq$ *dist f g*
by (*rule dist_bounded*)
also assume *dist f g* $= 0$
finally show $f = g$
by (*auto simp*: *apply_bcontfun_inject*[*symmetric*])
qed (*auto simp*: *dist_bcontfun_def intro*!: *cSup_eq*)
show *dist f g* $\leq$ *dist f h* + *dist g h*
proof (*rule dist_bound*)

```
    fix x
    have dist (f x) (g x) ≤ dist (f x) (h x) + dist (g x) (h x)
      by (rule dist_triangle2)
    also have dist (f x) (h x) ≤ dist f h
      by (rule dist_bounded)
    also have dist (g x) (h x) ≤ dist g h
      by (rule dist_bounded)
    finally show dist (f x) (g x) ≤ dist f h + dist g h
      by simp
  qed
qed (rule open_bcontfun_def uniformity_bcontfun_def)+

end

lift_definition PiC::'a::topological_space set ⇒ ('a ⇒ 'b set) ⇒ ('a ⇒C 'b::metric_space)
set
  is λI X. Pi I X ∩ bcontfun
  by auto

lemma mem_PiC_iff: x ∈ PiC I X ⟷ apply_bcontfun x ∈ Pi I X
  by transfer simp

lemmas mem_PiCD = mem_PiC_iff[THEN iffD1]
  and mem_PiCI = mem_PiC_iff[THEN iffD2]

lemma tendsto_bcontfun_uniform_limit:
  fixes f::'i ⇒ 'a::topological_space ⇒C 'b::metric_space
  assumes (f ⟶ l) F
  shows uniform_limit UNIV f l F
proof (rule uniform_limitI)
  fix e::real assume e > 0
  from tendstoD[OF assms this] have ∀F x in F. dist (f x) l < e .
  then show ∀F n in F. ∀x∈UNIV. dist ((f n) x) (l x) < e
    by eventually_elim (auto simp: dist_fun_lt_imp_dist_val_lt)
qed

lemma uniform_limit_tendsto_bcontfun:
  fixes f::'i ⇒ 'a::topological_space ⇒C 'b::metric_space
    and l::'a::topological_space ⇒C 'b::metric_space
  assumes uniform_limit UNIV f l F
  shows (f ⟶ l) F
proof (rule tendstoI)
  fix e::real assume e > 0
  then have e / 2 > 0 by simp
  from uniform_limitD[OF assms this]
  have ∀F i in F. ∀x. dist (f i x) (l x) < e / 2 by simp
  then have ∀F x in F. dist (f x) l ≤ e / 2
    by eventually_elim (blast intro: dist_bound less_imp_le)
  then show ∀F x in F. dist (f x) l < e
```

```
    by eventually_elim (use ‹0 < e› in auto)
qed

lemma uniform_limit_bcontfunE:
  fixes f::'i ⇒ 'a::topological_space ⇒_C 'b::metric_space
    and l::'a::topological_space ⇒ 'b::metric_space
  assumes uniform_limit UNIV f l F F ≠ bot
  obtains l'::'a::topological_space ⇒_C 'b::metric_space
  where l = l' (f ⟶ l') F
 by (metis (mono_tags, lifting) always_eventually apply_bcontfun apply_bcontfun_cases
assms
    bcontfun_def mem_Collect_eq uniform_limit_bounded uniform_limit_tendsto_bcontfun
      uniform_limit_theorem)

lemma closed_PiC:
  fixes I :: 'a::metric_space set
    and X :: 'a ⇒ 'b::complete_space set
  assumes ⋀i. i ∈ I ⟹ closed (X i)
  shows closed (PiC I X)
  unfolding closed_sequential_limits
proof safe
  fix f l
  assume seq: ∀ n. f n ∈ PiC I X and lim: f ⟶⟶ l
  show l ∈ PiC I X
  proof (safe intro!: mem_PiCI)
    fix x assume x ∈ I
    then have closed (X x)
      using assms by simp
    moreover have eventually (λi. f i x ∈ X x) sequentially
      using seq ‹x ∈ I›
      by (auto intro!: eventuallyI dest!: mem_PiCD simp: Pi_iff)
    moreover note sequentially_bot
    moreover have (λn. (f n) x) ⟶⟶ l x
      using tendsto_bcontfun_uniform_limit[OF lim]
      by (rule tendsto_uniform_limitI) simp
    ultimately show l x ∈ X x
      by (rule Lim_in_closed_set)
  qed
qed
```

9.21.2 Complete Space

```
instance bcontfun :: (metric_space, complete_space) complete_space
proof
  fix f :: nat ⇒ ('a, 'b) bcontfun
  assume Cauchy f — Cauchy equals uniform convergence
  then obtain g where uniform_limit UNIV f g sequentially
    using uniformly_convergent_eq_cauchy[of λ_. True f]
    unfolding Cauchy_def uniform_limit_sequentially_iff
```

by (*metis dist_fun_lt_imp_dist_val_lt*)

from *uniform_limit_bcontfunE*[*OF this sequentially_bot*]
obtain l' **where** g = *apply_bcontfun* l' ($f \longrightarrow l'$) **by** *metis*
then show *convergent f*
by (*intro convergentI*)
qed

9.21.3 Supremum norm for a normed vector space

instantiation *bcontfun* :: (*topological_space*, *real_normed_vector*) *real_vector*
begin

lemma *uminus_cont*: $f \in$ *bcontfun* $\Longrightarrow$ $(\lambda x.\ -f\ x) \in$ *bcontfun* **for** f::$'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros*)

lemma *plus_cont*: $f \in$ *bcontfun* $\Longrightarrow$ $g \in$ *bcontfun* $\Longrightarrow$ $(\lambda x.\ f\ x + g\ x) \in$ *bcontfun* **for** $f\ g$::$'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros bounded_plus_comp*)

lemma *minus_cont*: $f \in$ *bcontfun* $\Longrightarrow$ $g \in$ *bcontfun* $\Longrightarrow$ $(\lambda x.\ f\ x - g\ x) \in$ *bcontfun* **for** $f\ g$::$'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros bounded_minus_comp*)

lemma *scaleR_cont*: $f \in$ *bcontfun* $\Longrightarrow$ $(\lambda x.\ a *_R f\ x) \in$ *bcontfun* **for** $f :: 'a \Rightarrow 'b$
by (*auto simp*: *bcontfun_def intro*!: *continuous_intros bounded_scaleR_comp*)

lemma *bcontfun_normI*: *continuous_on UNIV f* $\Longrightarrow$ $(\bigwedge x.\ norm\ (f\ x) \leq b) \Longrightarrow f \in$ *bcontfun*
by (*auto simp*: *bcontfun_def intro*: *boundedI*)

lift_definition *uminus_bcontfun*::$('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda f\ x.\ -f\ x$
by (*rule uminus_cont*)

lift_definition *plus_bcontfun*::$('a \Rightarrow_C 'b) \Rightarrow ('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda f\ g\ x.\ f\ x + g\ x$
by (*rule plus_cont*)

lift_definition *minus_bcontfun*::$('a \Rightarrow_C 'b) \Rightarrow ('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda f\ g\ x.\ f\ x - g\ x$
by (*rule minus_cont*)

lift_definition *zero_bcontfun*::$'a \Rightarrow_C 'b$ **is** $\lambda\_.\ 0$
by (*rule const_bcontfun*)

lemma *const_bcontfun_0_eq_0*[*simp*]: *const_bcontfun 0* = *0*
by *transfer simp*

lift_definition *scaleR_bcontfun*::*real* $\Rightarrow$ $('a \Rightarrow_C 'b) \Rightarrow 'a \Rightarrow_C 'b$ **is** $\lambda r\ g\ x.\ r *_R$

g x
by (*rule scaleR_cont*)

lemmas [*simp*] =
const_bcontfun.rep_eq
uminus_bcontfun.rep_eq
plus_bcontfun.rep_eq
minus_bcontfun.rep_eq
zero_bcontfun.rep_eq
scaleR_bcontfun.rep_eq

instance
by *standard* (*auto intro*!: *bcontfun_eqI simp*: *algebra_simps*)

end

instantiation *bcontfun* :: (*topological_space*, *real_normed_vector*) *real_normed_vector*
begin

definition *norm_bcontfun* :: ('*a*, '*b*) *bcontfun* ⇒ *real*
where *norm_bcontfun f* = *dist f 0*

definition *sgn* (*f*::('*a*,'*b*) *bcontfun*) = *f* /$_R$ *norm f*

instance
proof
fix *a* :: *real*
fix *f g* :: ('*a*, '*b*) *bcontfun*
show *dist f g* = *norm* (*f* − *g*)
unfolding *norm_bcontfun_def*
by *transfer* (*simp add*: *dist_norm*)
show *norm* (*f* + *g*) ≤ *norm f* + *norm g*
unfolding *norm_bcontfun_def*
by *transfer*
(*auto intro*!: *cSUP_least norm_triangle_le add_mono bounded_norm_le_SUP_norm*
simp: *dist_norm bcontfun_def*)
show *norm* (*a* *$_R$ *f*) = |*a*| * *norm f*
unfolding *norm_bcontfun_def*
apply *transfer*
by (*rule trans*[*OF _ continuous_at_Sup_mono*[*symmetric*]])
(*auto intro*!: *monoI mult_left_mono continuous_intros bounded_imp_bdd_above*
simp: *bounded_norm_comp bcontfun_def image_comp*)
qed (*auto simp*: *norm_bcontfun_def sgn_bcontfun_def*)

end

lemma *norm_bounded*:
fixes *f* :: ('*a*::*topological_space*, '*b*::*real_normed_vector*) *bcontfun*

shows *norm* (*apply_bcontfun f x*) ≤ *norm f*
using *dist_bounded*[*of f x 0*]
by (*simp add*: *dist_norm*)

lemma *norm_bound*:
fixes *f* :: ('*a*::*topological_space*, '*b*::*real_normed_vector*) *bcontfun*
assumes ⋀*x*. *norm* (*apply_bcontfun f x*) ≤ *b*
shows *norm f* ≤ *b*
using *dist_bound*[*of f 0 b*] *assms*
by (*simp add*: *dist_norm*)

9.21.4 (bounded) continuous extenstion

lemma *continuous_on_cbox_bcontfunE*:
fixes *f*::'*a*::*euclidean_space* ⇒ '*b*::*metric_space*
assumes *continuous_on* (*cbox a b*) *f*
obtains *g*::'*a* ⇒$_C$ '*b* **where**
⋀*x*. *x* ∈ *cbox a b* ⟹ *g x* = *f x*
⋀*x*. *g x* = *f* (*clamp a b x*)
proof −
define *g* **where** *g* ≡ *ext_cont f a b*
have *g* ∈ *bcontfun*
using *assms*
by (*auto intro*!: *continuous_on_ext_cont simp*: *g_def bcontfun_def*)
(*auto simp*: *g_def ext_cont_def*
intro!: *clamp_bounded compact_imp_bounded*[*OF compact_continuous_image*]
assms)
then obtain *h* **where** *h*: *g* = *apply_bcontfun h* **by** (*rule bcontfunE*)
then have *h x* = *f x* **if** *x* ∈ *cbox a b* **for** *x*
by (*auto simp*: *h*[*symmetric*] *g_def that*)
moreover
have *h x* = *f* (*clamp a b x*) **for** *x*
by (*auto simp*: *h*[*symmetric*] *g_def ext_cont_def*)
ultimately show *?thesis* **..**
qed

lifting_update *bcontfun.lifting*
lifting_forget *bcontfun.lifting*

end

9.22 Infinite Products

theory *Infinite_Products*
imports *Topology_Euclidean_Space Complex_Transcendental*
begin

9.22.1 Preliminaries

lemma *sum_le_prod*:
fixes $f :: 'a \Rightarrow 'b :: linordered\_semidom$
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq 0$
shows $sum\ f\ A \leq (\prod x{\in}A.\ 1 + f\ x)$
using *assms*
proof (*induction A rule*: *infinite_finite_induct*)
case (*insert x A*)
from *insert.hyps* **have** $sum\ f\ A + f\ x * (\prod x{\in}A.\ 1) \leq (\prod x{\in}A.\ 1 + f\ x) + f\ x * (\prod x{\in}A.\ 1 + f\ x)$
by (*intro add_mono insert mult_left_mono prod_mono*) (*auto intro*: *insert.prems*)
with *insert.hyps* **show** *?case* **by** (*simp add*: *algebra_simps*)
qed *simp_all*

lemma *prod_le_exp_sum*:
fixes $f :: 'a \Rightarrow real$
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq 0$
shows $prod\ (\lambda x.\ 1 + f\ x)\ A \leq exp\ (sum\ f\ A)$
using *assms*
proof (*induction A rule*: *infinite_finite_induct*)
case (*insert x A*)
have $(1 + f\ x) * (\prod x{\in}A.\ 1 + f\ x) \leq exp\ (f\ x) * exp\ (sum\ f\ A)$
using *insert.prems* **by** (*intro mult_mono insert prod_nonneg exp_ge_add_one_self*) *auto*
with *insert.hyps* **show** *?case* **by** (*simp add*: *algebra_simps exp_add*)
qed *simp_all*

lemma *lim_ln_1_plus_x_over_x_at_0*: $(\lambda x{::}real.\ ln\ (1 + x)\ /\ x) \ -0\rightarrow\ 1$
proof (*rule lhopital*)
show $(\lambda x{::}real.\ ln\ (1 + x)) \ -0\rightarrow\ 0$
by (*rule tendsto_eq_intros refl | simp*)+
have *eventually* $(\lambda x{::}real.\ x \in \{-1/2<..<1/2\})\ (nhds\ 0)$
by (*rule eventually_nhds_in_open*) *auto*
hence $*$: *eventually* $(\lambda x{::}real.\ x \in \{-1/2<..<1/2\})\ (at\ 0)$
by (*rule filter_leD* [*rotated*]) (*simp_all add*: *at_within_def*)
show *eventually* $(\lambda x{::}real.\ ((\lambda x.\ ln\ (1 + x))\ has\_field\_derivative\ inverse\ (1 + x))\ (at\ x))\ (at\ 0)$
using $*$ **by** *eventually_elim* (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
show *eventually* $(\lambda x{::}real.\ ((\lambda x.\ x)\ has\_field\_derivative\ 1)\ (at\ x))\ (at\ 0)$
using $*$ **by** *eventually_elim* (*auto intro*!: *derivative_eq_intros simp*: *field_simps*)
show $\forall_F\ x\ in\ at\ 0.\ x \neq 0$ **by** (*auto simp*: *at_within_def eventually_inf_principal*)
show $(\lambda x{::}real.\ inverse\ (1 + x)\ /\ 1) \ -0\rightarrow\ 1$
by (*rule tendsto_eq_intros refl | simp*)+
qed *auto*

9.22.2 Definitions and basic properties

definition *raw_has_prod* :: $[nat \Rightarrow 'a{::}\{t2\_space,\ comm\_semiring\_1\},\ nat,\ 'a] \Rightarrow bool$

where *raw_has_prod f M p* ≡ (λ*n*. ∏ *i*≤*n*. *f* (*i*+*M*)) ⟶ *p* ∧ *p* ≠ *0*

The nonzero and zero cases, as in *Complex Analysis* by Joseph Bak and Donald J.Newman, page 241

definition
has_prod :: (*nat* ⇒ *'a*::{*t2_space*, *comm_semiring_1*}) ⇒ *'a* ⇒ *bool* (**infixr** ‹*has'_prod*› *80*)
where *f has_prod p* ≡ *raw_has_prod f 0 p* ∨ (∃ *i q*. *p* = *0* ∧ *f i* = *0* ∧ *raw_has_prod f* (*Suc i*) *q*)

definition *convergent_prod* :: (*nat* ⇒ *'a* :: {*t2_space*,*comm_semiring_1*}) ⇒ *bool*
where
convergent_prod f ≡ ∃ *M p*. *raw_has_prod f M p*

definition *prodinf* :: (*nat* ⇒ *'a*::{*t2_space*, *comm_semiring_1*}) ⇒ *'a*
(**binder** ‹∏› *10*)
where *prodinf f* = (*THE p*. *f has_prod p*)

lemmas *prod_defs* = *raw_has_prod_def has_prod_def convergent_prod_def prodinf_def*

lemma *has_prod_subst*[*trans*]: *f* = *g* ⟹ *g has_prod z* ⟹ *f has_prod z*
by *simp*

lemma *has_prod_cong*: (⋀*n*. *f n* = *g n*) ⟹ *f has_prod c* ⟷ *g has_prod c*
by *presburger*

lemma *raw_has_prod_nonzero* [*simp*]: ¬ *raw_has_prod f M 0*
by (*simp add*: *raw_has_prod_def*)

lemma *raw_has_prod_eq_0*:
fixes *f* :: *nat* ⇒ *'a*::{*semidom*,*t2_space*}
assumes *p*: *raw_has_prod f m p* **and** *i*: *f i* = *0 i* ≥ *m*
shows *p* = *0*
proof −
have *eq0*: (∏ *k*≤*n*. *f* (*k*+*m*)) = *0* **if** *i* − *m* ≤ *n* **for** *n*
proof −
have ∃ *k*≤*n*. *f* (*k* + *m*) = *0*
using *i that* **by** *auto*
then show *?thesis*
by *auto*
qed
have (λ*n*. ∏ *i*≤*n*. *f* (*i* + *m*)) ⟶ *0*
by (*rule LIMSEQ_offset* [**where** *k* = *i*−*m*]) (*simp add*: *eq0*)
with *p* **show** *?thesis*
unfolding *raw_has_prod_def*
using *LIMSEQ_unique* **by** *blast*
qed

lemma *raw_has_prod_Suc*:
raw_has_prod f (Suc M) a ⟷ raw_has_prod (λn. f (Suc n)) M a
unfolding *raw_has_prod_def* **by** *auto*

lemma *has_prod_0_iff*: *f has_prod 0 ⟷ (∃ i. f i = 0 ∧ (∃ p. raw_has_prod f (Suc i) p))*
by (*simp add*: *has_prod_def*)

lemma *has_prod_unique2*:
fixes *f* :: *nat* ⇒ *'a*::{*semidom*,*t2_space*}
assumes *f has_prod a f has_prod b* **shows** *a* = *b*
using *assms*
by (*auto simp*: *has_prod_def raw_has_prod_eq_0*) (*meson raw_has_prod_def sequentially_bot tendsto_unique*)

lemma *has_prod_unique*:
fixes *f* :: *nat* ⇒ *'a* :: {*semidom*,*t2_space*}
shows *f has_prod s* ⟹ *s* = *prodinf f*
by (*simp add*: *has_prod_unique2 prodinf_def the_equality*)

lemma *has_prod_eq_0_iff*:
fixes *f* :: *nat* ⇒ *'a* :: {*semidom*, *comm_semiring_1*, *t2_space*}
assumes *f has_prod P*
shows *P* = *0* ⟷ *0* ∈ *range f*
proof
assume *0* ∈ *range f*
then obtain *N* **where** *N*: *f N* = *0*
by *auto*
have *eventually* (*λn. n* > *N*) *at_top*
by (*rule eventually_gt_at_top*)
hence *eventually* (*λn.* (∏ *k*<*n. f k*) = *0*) *at_top*
by *eventually_elim* (*use N* **in** *auto*)
hence (*λn.* ∏ *k*<*n. f k*) ⟶ *0*
by (*simp add*: *tendsto_eventually*)
moreover have (*λn.* ∏ *k*<*n. f k*) ⟶ *P*
using *assms* **by** (*metis N calculation prod_defs*(*2*) *raw_has_prod_eq_0 zero_le*)
ultimately show *P* = *0*
using *tendsto_unique* **by** *force*
qed (*use assms* **in** ‹*auto simp*: *has_prod_def*›)

lemma *has_prod_0D*:
fixes *f* :: *nat* ⇒ *'a* :: {*semidom*, *comm_semiring_1*, *t2_space*}
shows *f has_prod 0* ⟹ *0* ∈ *range f*
using *has_prod_eq_0_iff*[*of f 0*] **by** *auto*

lemma *has_prod_zeroI*:
fixes *f* :: *nat* ⇒ *'a* :: {*semidom*, *comm_semiring_1*, *t2_space*}
assumes *f has_prod P f n* = *0*
shows *P* = *0*

```
  using assms by (auto simp: has_prod_eq_0_iff)

lemma raw_has_prod_in_Reals:
  assumes raw_has_prod (complex_of_real ∘ z) M p
  shows p ∈ ℝ
  using assms by (auto simp: raw_has_prod_def real_lim_sequentially)

lemma raw_has_prod_of_real_iff: raw_has_prod (complex_of_real ∘ z) M (of_real
p) ⟷ raw_has_prod z M p
  by (auto simp: raw_has_prod_def tendsto_of_real_iff simp flip: of_real_prod)

lemma convergent_prod_of_real_iff: convergent_prod (complex_of_real ∘ z) ⟷
convergent_prod z
  by (smt (verit, best) Reals_cases convergent_prod_def raw_has_prod_in_Reals
raw_has_prod_of_real_iff)

lemma convergent_prod_altdef:
  fixes f :: nat ⇒ 'a :: {t2_space,comm_semiring_1}
  shows convergent_prod f ⟷ (∃ M L. (∀ n≥M. f n ≠ 0) ∧ (λn. ∏ i≤n. f (i+M))
⟶ L ∧ L ≠ 0)
proof
  assume convergent_prod f
  then obtain M L where ∗: (λn. ∏ i≤n. f (i+M)) ⟶ L L ≠ 0
    by (auto simp: prod_defs)
  have f i ≠ 0 if i ≥ M for i
  proof
    assume f i = 0
    have ∗∗: eventually (λn. (∏ i≤n. f (i+M)) = 0) sequentially
      using eventually_ge_at_top[of i − M]
    proof eventually_elim
      case (elim n)
      with ‹f i = 0› and ‹i ≥ M› show ?case
        by (auto intro!: bexI[of _ i − M] prod_zero)
    qed
    have (λn. (∏ i≤n. f (i+M))) ⟶ 0
      unfolding filterlim_iff
      by (auto dest!: eventually_nhds_x_imp_x intro!: eventually_mono[OF ∗∗])
    from tendsto_unique[OF _ this ∗(1)] and ∗(2)
      show False by simp
  qed
  with ∗ show (∃ M L. (∀ n≥M. f n ≠ 0) ∧ (λn. ∏ i≤n. f (i+M)) ⟶ L ∧ L
≠ 0)
    by blast
qed (auto simp: prod_defs)

lemma raw_has_prod_norm:
  fixes a :: 'a ::real_normed_field
  assumes raw_has_prod f M a
  shows raw_has_prod (λn. norm (f n)) M (norm a)
```

using *assms* **by** (*auto simp*: *raw_has_prod_def prod_norm tendsto_norm*)

lemma *has_prod_norm*:
fixes *a* :: *'a* ::*real_normed_field*
assumes *f*: *f has_prod a*
shows (*λn. norm* (*f n*)) *has_prod* (*norm a*)
using *f* [*unfolded has_prod_def*]
proof (*elim disjE exE conjE*)
assume *f0*: *raw_has_prod f 0 a*
then show (*λn. norm* (*f n*)) *has_prod norm a*
using *has_prod_def raw_has_prod_norm* **by** *blast*
next
fix *i p*
assume *a* = *0* **and** *f i* = *0* **and** *p*: *raw_has_prod f* (*Suc i*) *p*
then have *Ex* (*raw_has_prod* (*λn. norm* (*f n*)) (*Suc i*))
using *raw_has_prod_norm* **by** *blast*
then show *?thesis*
by (*metis* ‹*a* = *0*› ‹*f i* = *0*› *has_prod_0_iff norm_zero*)
qed

9.22.3 Absolutely convergent products

definition *abs_convergent_prod* :: (*nat* ⇒ _) ⇒ *bool* **where**
abs_convergent_prod f ⟷ *convergent_prod* (*λi. 1* + *norm* (*f i* − *1*))

lemma *abs_convergent_prodI*:
assumes *convergent* (*λn.* ∏ *i*≤*n. 1* + *norm* (*f i* − *1*))
shows *abs_convergent_prod f*
proof −
from *assms* **obtain** *L* **where** *L*: (*λn.* ∏ *i*≤*n. 1* + *norm* (*f i* − *1*)) ⟶ *L*
by (*auto simp*: *convergent_def*)
have *L* ≥ *1*
proof (*rule tendsto_le*)
show *eventually* (*λn.* (∏ *i*≤*n. 1* + *norm* (*f i* − *1*)) ≥ *1*) *sequentially*
proof (*intro always_eventually allI*)
fix *n*
have (∏ *i*≤*n. 1* + *norm* (*f i* − *1*)) ≥ (∏ *i*≤*n. 1*)
by (*intro prod_mono*) *auto*
thus (∏ *i*≤*n. 1* + *norm* (*f i* − *1*)) ≥ *1* **by** *simp*
qed
qed (*use L* **in** *simp_all*)
hence *L* ≠ *0* **by** *auto*
with *L* **show** *?thesis* **unfolding** *abs_convergent_prod_def prod_defs*
by (*intro exI*[*of* _ *0*::*nat*] *exI*[*of* _ *L*]) *auto*
qed

lemma
fixes *f* :: *nat* ⇒ *'a* :: {*topological_semigroup_mult*,*t2_space*,*idom*}
assumes *convergent_prod f*

```
  shows   convergent_prod_imp_convergent:     convergent (λn. ∏ i≤n. f i)
    and   convergent_prod_to_zero_iff [simp]: (λn. ∏ i≤n. f i) ⟶ 0 ⟷
(∃ i. f i = 0)
proof −
  from assms obtain M L
    where M: ⋀n. n ≥ M ⟹ f n ≠ 0 and (λn. ∏ i≤n. f (i + M)) ⟶ L
and L ≠ 0
    by (auto simp: convergent_prod_altdef)
  note this(2)
  also have (λn. ∏ i≤n. f (i + M)) = (λn. ∏ i=M..M+n. f i)
    by (intro ext prod.reindex_bij_witness[of _ λn. n − M λn. n + M]) auto
  finally have (λn. (∏ i<M. f i) * (∏ i=M..M+n. f i)) ⟶ (∏ i<M. f i) * L
    by (intro tendsto_mult tendsto_const)
 also have (λn. (∏ i<M. f i) * (∏ i=M..M+n. f i)) = (λn. (∏ i∈{..<M}∪{M..M+n}.
f i))
    by (subst prod.union_disjoint) auto
  also have (λn. {..<M} ∪ {M..M+n}) = (λn. {..n+M}) by auto
  finally have lim: (λn. prod f {..n}) ⟶ prod f {..<M} * L
    by (rule LIMSEQ_offset)
  thus convergent (λn. ∏ i≤n. f i)
    by (auto simp: convergent_def)

  show (λn. ∏ i≤n. f i) ⟶ 0 ⟷ (∃ i. f i = 0)
  proof
    assume ∃ i. f i = 0
    then obtain i where f i = 0 by auto
    moreover with M have i < M by (cases i < M) auto
    ultimately have (∏ i<M. f i) = 0 by auto
    with lim show (λn. ∏ i≤n. f i) ⟶ 0 by simp
  next
    assume (λn. ∏ i≤n. f i) ⟶ 0
    from tendsto_unique[OF _ this lim] and ‹L ≠ 0›
    show ∃ i. f i = 0 by auto
  qed
qed

lemma convergent_prod_iff_nz_lim:
  fixes f :: nat ⇒ 'a :: {topological_semigroup_mult,t2_space,idom}
  assumes ⋀i. f i ≠ 0
  shows convergent_prod f ⟷ (∃ L. (λn. ∏ i≤n. f i) ⟶ L ∧ L ≠ 0)
    (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs then show ?rhs
   using assms convergentD convergent_prod_imp_convergent convergent_prod_to_zero_iff
by blast
next
  assume ?rhs then show ?lhs
    unfolding prod_defs
    by (rule_tac x=0 in exI) auto
```

qed

lemma *convergent_prod_iff_convergent*:
fixes $f :: nat \Rightarrow 'a :: \{topological\_semigroup\_mult,t2\_space,idom\}$
assumes $\bigwedge i.\ f\ i \neq 0$
shows *convergent_prod f* $\longleftrightarrow$ *convergent* $(\lambda n.\ \prod i{\leq}n.\ f\ i) \wedge lim\ (\lambda n.\ \prod i{\leq}n.\ f\ i) \neq 0$
by (*force simp*: *convergent_prod_iff_nz_lim assms convergent_def limI*)

lemma *bounded_imp_convergent_prod*:
fixes $a :: nat \Rightarrow real$
assumes *1*: $\bigwedge n.\ a\ n \geq 1$ **and** *bounded*: $\bigwedge n.\ (\prod i{\leq}n.\ a\ i) \leq B$
shows *convergent_prod a*
proof −
have *bdd_above* $(range(\lambda n.\ \prod i{\leq}n.\ a\ i))$
by (*meson bdd_aboveI2 bounded*)
moreover have *incseq* $(\lambda n.\ \prod i{\leq}n.\ a\ i)$
unfolding *mono_def* **by** (*metis 1 prod_mono2 atMost_subset_iff dual_order.trans finite_atMost zero_le_one*)
ultimately obtain *p* **where** *p*: $(\lambda n.\ \prod i{\leq}n.\ a\ i) \longrightarrow p$
using *LIMSEQ_incseq_SUP* **by** *blast*
then have $p \neq 0$
by (*metis 1 not_one_le_zero prod_ge_1 LIMSEQ_le_const*)
with *1 p* **show** *?thesis*
by (*metis convergent_prod_iff_nz_lim not_one_le_zero*)
qed

lemma *abs_convergent_prod_altdef*:
fixes $f :: nat \Rightarrow 'a :: \{one,real\_normed\_vector\}$
shows *abs_convergent_prod f* $\longleftrightarrow$ *convergent* $(\lambda n.\ \prod i{\leq}n.\ 1 + norm\ (f\ i - 1))$
proof
assume *abs_convergent_prod f*
thus *convergent* $(\lambda n.\ \prod i{\leq}n.\ 1 + norm\ (f\ i - 1))$
by (*auto simp*: *abs_convergent_prod_def intro*!: *convergent_prod_imp_convergent*)
qed (*auto intro*: *abs_convergent_prodI*)

lemma *Weierstrass_prod_ineq*:
fixes $f :: 'a \Rightarrow real$
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \in \{0..1\}$
shows $1 - sum\ f\ A \leq (\prod x{\in}A.\ 1 - f\ x)$
using *assms*
proof (*induction A rule*: *infinite_finite_induct*)
case (*insert x A*)
from *insert.hyps* **and** *insert.prems*
have $1 - sum\ f\ A + f\ x * (\prod x{\in}A.\ 1 - f\ x) \leq (\prod x{\in}A.\ 1 - f\ x) + f\ x * (\prod x{\in}A.\ 1)$
by (*intro insert.IH add_mono mult_left_mono prod_mono*) *auto*

```
  with insert.hyps show ?case by (simp add: algebra_simps)
qed simp_all

lemma norm_prod_minus1_le_prod_minus1:
  fixes f :: nat ⇒ 'a :: {real_normed_div_algebra,comm_ring_1}
  shows norm (prod (λn. 1 + f n) A − 1) ≤ prod (λn. 1 + norm (f n)) A − 1
proof (induction A rule: infinite_finite_induct)
  case (insert x A)
  from insert.hyps have
    norm ((∏n∈insert x A. 1 + f n) − 1) =
      norm ((∏n∈A. 1 + f n) − 1 + f x * (∏n∈A. 1 + f n))
    by (simp add: algebra_simps)
  also have ... ≤ norm ((∏n∈A. 1 + f n) − 1) + norm (f x * (∏n∈A. 1 + f
n))
    by (rule norm_triangle_ineq)
  also have norm (f x * (∏n∈A. 1 + f n)) = norm (f x) * (∏x∈A. norm (1 +
f x))
    by (simp add: prod_norm norm_mult)
  also have (∏x∈A. norm (1 + f x)) ≤ (∏x∈A. norm (1::'a) + norm (f x))
    by (intro prod_mono norm_triangle_ineq ballI conjI) auto
  also have norm (1::'a) = 1 by simp
  also note insert.IH
  also have (∏n∈A. 1 + norm (f n)) − 1 + norm (f x) * (∏x∈A. 1 + norm (f
x)) =
            (∏n∈insert x A. 1 + norm (f n)) − 1
    using insert.hyps by (simp add: algebra_simps)
  finally show ?case by − (simp_all add: mult_left_mono)
qed simp_all

lemma convergent_prod_imp_ev_nonzero:
  fixes f :: nat ⇒ 'a :: {t2_space,comm_semiring_1}
  assumes convergent_prod f
  shows   eventually (λn. f n ≠ 0) sequentially
 using assms by (auto simp: eventually_at_top_linorder convergent_prod_altdef)

lemma convergent_prod_imp_LIMSEQ:
  fixes f :: nat ⇒ 'a :: {real_normed_field}
  assumes convergent_prod f
  shows   f ⟶ 1
proof −
  from assms obtain M L where L: (λn. ∏i≤n. f (i+M)) ⟶ L ⋀n. n ≥
M ⟹ f n ≠ 0 L ≠ 0
    by (auto simp: convergent_prod_altdef)
 hence L': (λn. ∏i≤Suc n. f (i+M)) ⟶ L by (subst filterlim_sequentially_Suc)
  have (λn. (∏i≤Suc n. f (i+M)) / (∏i≤n. f (i+M))) ⟶ L / L
    using L L' by (intro tendsto_divide) simp_all
  also from L have L / L = 1 by simp
  also have (λn. (∏i≤Suc n. f (i+M)) / (∏i≤n. f (i+M))) = (λn. f (n + Suc
M))
```

```
    using assms L by (auto simp: fun_eq_iff atMost_Suc)
  finally show ?thesis by (rule LIMSEQ_offset)
qed

lemma abs_convergent_prod_imp_summable:
  fixes f :: nat ⇒ 'a :: real_normed_div_algebra
  assumes abs_convergent_prod f
  shows summable (λi. norm (f i − 1))
proof −
  from assms have convergent (λn. ∏ i≤n. 1 + norm (f i − 1))
   unfolding abs_convergent_prod_def by (rule convergent_prod_imp_convergent)
  then obtain L where L: (λn. ∏ i≤n. 1 + norm (f i − 1)) ⟶ L
    unfolding convergent_def by blast
  have convergent (λn. ∑ i≤n. norm (f i − 1))
  proof (rule Bseq_monoseq_convergent)
    have eventually (λn. (∏ i≤n. 1 + norm (f i − 1)) < L + 1) sequentially
      using L(1) by (rule order_tendstoD) simp_all
    hence ∀F x in sequentially. norm (∑ i≤x. norm (f i − 1)) ≤ L + 1
    proof eventually_elim
      case (elim n)
      have norm (∑ i≤n. norm (f i − 1)) = (∑ i≤n. norm (f i − 1))
       unfolding real_norm_def by (intro abs_of_nonneg sum_nonneg) simp_all
      also have ... ≤ (∏ i≤n. 1 + norm (f i − 1)) by (rule sum_le_prod) auto
      also have ... < L + 1 by (rule elim)
      finally show ?case by simp
    qed
    thus Bseq (λn. ∑ i≤n. norm (f i − 1)) by (rule BfunI)
  next
    show monoseq (λn. ∑ i≤n. norm (f i − 1))
      by (rule mono_SucI1) auto
  qed
  thus summable (λi. norm (f i − 1)) by (simp add: summable_iff_convergent')
qed

lemma summable_imp_abs_convergent_prod:
  fixes f :: nat ⇒ 'a :: real_normed_div_algebra
  assumes summable (λi. norm (f i − 1))
  shows   abs_convergent_prod f
proof (intro abs_convergent_prodI Bseq_monoseq_convergent)
  show monoseq (λn. ∏ i≤n. 1 + norm (f i − 1))
    by (intro mono_SucI1)
     (auto simp: atMost_Suc algebra_simps intro!: mult_nonneg_nonneg prod_nonneg)
next
  show Bseq (λn. ∏ i≤n. 1 + norm (f i − 1))
  proof (rule Bseq_eventually_mono)
    show eventually (λn. norm (∏ i≤n. 1 + norm (f i − 1)) ≤
          norm (exp (∑ i≤n. norm (f i − 1)))) sequentially
      by (intro always_eventually allI) (auto simp: abs_prod exp_sum intro!:
prod_mono)
```

```
  next
    from assms have (λn. ∑ i≤n. norm (f i − 1)) ⟶ (∑ i. norm (f i − 1))
      using sums_def_le by blast
    hence (λn. exp (∑ i≤n. norm (f i − 1))) ⟶ exp (∑ i. norm (f i − 1))
      by (rule tendsto_exp)
    hence convergent (λn. exp (∑ i≤n. norm (f i − 1)))
      by (rule convergentI)
    thus Bseq (λn. exp (∑ i≤n. norm (f i − 1)))
      by (rule convergent_imp_Bseq)
  qed
qed

theorem abs_convergent_prod_conv_summable:
  fixes f :: nat ⇒ 'a :: real_normed_div_algebra
  shows abs_convergent_prod f ⟷ summable (λi. norm (f i − 1))
 by (blast intro: abs_convergent_prod_imp_summable summable_imp_abs_convergent_prod)

lemma abs_convergent_prod_imp_LIMSEQ:
  fixes f :: nat ⇒ 'a :: {comm_ring_1,real_normed_div_algebra}
  assumes abs_convergent_prod f
  shows   f ⟶ 1
proof −
  from assms have summable (λn. norm (f n − 1))
    by (rule abs_convergent_prod_imp_summable)
  from summable_LIMSEQ_zero[OF this] have (λn. f n − 1) ⟶ 0
    by (simp add: tendsto_norm_zero_iff)
  from tendsto_add[OF this tendsto_const[of 1]] show ?thesis by simp
qed

lemma abs_convergent_prod_imp_ev_nonzero:
  fixes f :: nat ⇒ 'a :: {comm_ring_1,real_normed_div_algebra}
  assumes abs_convergent_prod f
  shows   eventually (λn. f n ≠ 0) sequentially
proof −
  from assms have f ⟶ 1
    by (rule abs_convergent_prod_imp_LIMSEQ)
  hence eventually (λn. dist (f n) 1 < 1) at_top
    by (auto simp: tendsto_iff)
  thus ?thesis by eventually_elim auto
qed
```

9.22.4 Ignoring initial segments

```
lemma convergent_prod_offset:
  assumes convergent_prod (λn. f (n + m))
  shows   convergent_prod f
proof −
  from assms obtain M L where (λn. ∏ k≤n. f (k + (M + m))) ⟶ L L ≠
0
```

```
    by (auto simp: prod_defs add.assoc)
  thus convergent_prod f
    unfolding prod_defs by blast
qed

lemma abs_convergent_prod_offset:
  assumes abs_convergent_prod (λn. f (n + m))
  shows   abs_convergent_prod f
 using assms unfolding abs_convergent_prod_def by (rule convergent_prod_offset)


lemma raw_has_prod_ignore_initial_segment:
  fixes f :: nat ⇒ 'a :: real_normed_field
  assumes raw_has_prod f M p N ≥ M
  obtains q where  raw_has_prod f N q
proof −
  have p: (λn. ∏k≤n. f (k + M)) ⟶ p and p ≠ 0
    using assms by (auto simp: raw_has_prod_def)
  then have nz: ⋀n. n ≥ M ⟹ f n ≠ 0
    using assms by (auto simp: raw_has_prod_eq_0)
  define C where C = (∏k<N−M. f (k + M))
  from nz have [simp]: C ≠ 0
    by (auto simp: C_def)

  from p have (λi. ∏k≤i + (N−M). f (k + M)) ⟶ p
    by (rule LIMSEQ_ignore_initial_segment)
  also have (λi. ∏k≤i + (N−M). f (k + M)) = (λn. C * (∏k≤n. f (k + N)))
  proof (rule ext, goal_cases)
    case (1 n)
    have {..n+(N−M)} = {..<(N−M)} ∪ {(N−M)..n+(N−M)} by auto
     also have (∏k∈.... f (k + M)) = C * (∏k=(N−M)..n+(N−M). f (k +
M))
      unfolding C_def by (rule prod.union_disjoint) auto
    also have (∏k=(N−M)..n+(N−M). f (k + M)) = (∏k≤n. f (k + (N−M)
+ M))
     by (intro ext prod.reindex_bij_witness[of_ λk. k + (N−M) λk. k − (N−M)])
auto
    finally show ?case
      using ‹N ≥ M› by (simp add: add_ac)
  qed
  finally have (λn. C * (∏k≤n. f (k + N)) / C) ⟶ p / C
    by (intro tendsto_divide tendsto_const) auto
  hence (λn. ∏k≤n. f (k + N)) ⟶ p / C by simp
  moreover from ‹p ≠ 0› have p / C ≠ 0 by simp
  ultimately show ?thesis
    using raw_has_prod_def that by blast
qed

corollary convergent_prod_ignore_initial_segment:
```

fixes $f :: nat \Rightarrow 'a :: real\_normed\_field$
assumes *convergent_prod f*
shows *convergent_prod* $(\lambda n.\ f\ (n + m))$
using *assms*
unfolding *convergent_prod_def*
apply *clarify*
apply (*erule_tac* $N{=}M{+}m$ **in** *raw_has_prod_ignore_initial_segment*)
apply (*auto simp add*: *raw_has_prod_def add_ac*)
done

corollary *convergent_prod_ignore_nonzero_segment*:
fixes $f :: nat \Rightarrow 'a :: real\_normed\_field$
assumes *f*: *convergent_prod f* **and** *nz*: $\bigwedge i.\ i \geq M \Longrightarrow f\ i \neq 0$
shows $\exists p.$ *raw_has_prod f M p*
using *convergent_prod_ignore_initial_segment* [*OF f*]
by (*metis convergent_LIMSEQ_iff convergent_prod_iff_convergent le_add_same_cancel2 nz prod_defs*(*1*) *zero_order*(*1*))

corollary *abs_convergent_prod_ignore_initial_segment*:
assumes *abs_convergent_prod f*
shows *abs_convergent_prod* $(\lambda n.\ f\ (n + m))$
using *assms* **unfolding** *abs_convergent_prod_def*
by (*rule convergent_prod_ignore_initial_segment*)

9.22.5 More elementary properties

theorem *abs_convergent_prod_imp_convergent_prod*:
fixes $f :: nat \Rightarrow 'a :: \{real\_normed\_div\_algebra, complete\_space, comm\_ring\_1\}$
assumes *abs_convergent_prod f*
shows *convergent_prod f*
proof −
from *assms* **have** *eventually* $(\lambda n.\ f\ n \neq 0)$ *sequentially*
by (*rule abs_convergent_prod_imp_ev_nonzero*)
then obtain *N* **where** *N*: $f\ n \neq 0$ **if** $n \geq N$ **for** *n*
by (*auto simp*: *eventually_at_top_linorder*)
let *?P* $= \lambda n.\ \prod i{\leq}n.\ f\ (i + N)$ **and** *?Q* $= \lambda n.\ \prod i{\leq}n.\ 1 + norm\ (f\ (i + N) - 1)$

have *Cauchy ?P*
proof (*rule CauchyI'*, *goal_cases*)
case (*1* ε)
from *assms* **have** *abs_convergent_prod* $(\lambda n.\ f\ (n + N))$
by (*rule abs_convergent_prod_ignore_initial_segment*)
hence *Cauchy ?Q*
unfolding *abs_convergent_prod_def*
by (*intro convergent_Cauchy convergent_prod_imp_convergent*)
from *CauchyD*[*OF this 1*] **obtain** *M* **where** *M*: $norm\ (?Q\ m - ?Q\ n) < \varepsilon$ **if** $m \geq M\ n \geq M$ **for** *m n*
by *blast*

```
    show ?case
    proof (rule exI[of _ M], safe, goal_cases)
      case (1 m n)
      have dist (?P m) (?P n) = norm (?P n − ?P m)
        by (simp add: dist_norm norm_minus_commute)
      also from 1 have {..n} = {..m} ∪ {m<..n} by auto
      hence norm (?P n − ?P m) = norm (?P m * (∏k∈{m<..n}. f (k + N)) −
?P m)
        by (subst prod.union_disjoint [symmetric]) (auto simp: algebra_simps)
      also have ... = norm (?P m * ((∏k∈{m<..n}. f (k + N)) − 1))
        by (simp add: algebra_simps)
      also have ... = (∏k≤m. norm (f (k + N))) * norm ((∏k∈{m<..n}. f (k
+ N)) − 1)
        by (simp add: norm_mult prod_norm)
      also have ... ≤ ?Q m * ((∏k∈{m<..n}. 1 + norm (f (k + N) − 1)) − 1)
       using norm_prod_minus1_le_prod_minus1[of λk. f (k + N) − 1 {m<..n}]
             norm_triangle_ineq[of 1 f k − 1 for k]
      by (intro mult_mono prod_mono ballI conjI norm_prod_minus1_le_prod_minus1
prod_nonneg) auto
      also have ... = ?Q m * (∏k∈{m<..n}. 1 + norm (f (k + N) − 1)) − ?Q
m
        by (simp add: algebra_simps)
      also have ?Q m * (∏k∈{m<..n}. 1 + norm (f (k + N) − 1)) =
                (∏k∈{..m}∪{m<..n}. 1 + norm (f (k + N) − 1))
        by (rule prod.union_disjoint [symmetric]) auto
      also from 1 have {..m}∪{m<..n} = {..n} by auto
      also have ?Q n − ?Q m ≤ norm (?Q n − ?Q m) by simp
      also from 1 have ... < ε by (intro M) auto
      finally show ?case .
    qed
  qed
  hence conv: convergent ?P by (rule Cauchy_convergent)
  then obtain L where L: ?P ⟶ L
    by (auto simp: convergent_def)

  have L ≠ 0
  proof
    assume [simp]: L = 0
   from tendsto_norm[OF L] have limit: (λn. ∏k≤n. norm (f (k + N))) ⟶
0
      by (simp add: prod_norm)

    from assms have (λn. f (n + N)) ⟶ 1
    by (intro abs_convergent_prod_imp_LIMSEQ abs_convergent_prod_ignore_initial_segment)
    hence eventually (λn. norm (f (n + N) − 1) < 1) sequentially
      by (auto simp: tendsto_iff dist_norm)
    then obtain M0 where M0: norm (f (n + N) − 1) < 1 if n ≥ M0 for n
      by (auto simp: eventually_at_top_linorder)
```

```
  {
    fix M assume M: M ≥ M0
    with M0 have M: norm (f (n + N) − 1) < 1 if n ≥ M for n using that
by simp

    have (λn. ∏k≤n. 1 − norm (f (k+M+N) − 1)) ⟶ 0
    proof (rule tendsto_sandwich)
      show eventually (λn. (∏k≤n. 1 − norm (f (k+M+N) − 1)) ≥ 0)
sequentially
      using M by (intro always_eventually prod_nonneg allI ballI) (auto intro:
less_imp_le)
      have norm (1::'a) − norm (f (i + M + N) − 1) ≤ norm (f (i + M +
N)) for i
      using norm_triangle_ineq3[of f (i + M + N) 1] by simp
      thus eventually (λn. (∏k≤n. 1 − norm (f (k+M+N) − 1)) ≤ (∏k≤n.
norm (f (k+M+N)))) at_top
      using M by (intro always_eventually allI prod_mono ballI conjI) (auto
intro: less_imp_le)

      define C where C = (∏k<M. norm (f (k + N)))
      from N have [simp]: C ≠ 0 by (auto simp: C_def)
      from L have (λn. norm (∏k≤n+M. f (k + N))) ⟶ 0
     by (intro LIMSEQ_ignore_initial_segment) (simp add: tendsto_norm_zero_iff)
      also have (λn. norm (∏k≤n+M. f (k + N))) = (λn. C * (∏k≤n. norm
(f (k + M + N))))
      proof (rule ext, goal_cases)
        case (1 n)
        have {..n+M} = {..<M} ∪ {M..n+M} by auto
        also have norm (∏k∈.... f (k + N)) = C * norm (∏k=M..n+M. f (k
+ N))
        unfolding C_def by (subst prod.union_disjoint) (auto simp: norm_mult
prod_norm)
        also have (∏k=M..n+M. f (k + N)) = (∏k≤n. f (k + N + M))
          by (intro prod.reindex_bij_witness[of _ λi. i + M λi. i − M]) auto
        finally show ?case by (simp add: add_ac prod_norm)
      qed
      finally have (λn. C * (∏k≤n. norm (f (k + M + N))) / C) ⟶ 0 /
C
        by (intro tendsto_divide tendsto_const) auto
      thus (λn. ∏k≤n. norm (f (k + M + N))) ⟶ 0 by simp
    qed simp_all

    have 1 − (∑i. norm (f (i + M + N) − 1)) ≤ 0
    proof (rule tendsto_le)
      show eventually (λn. 1 − (∑k≤n. norm (f (k+M+N) − 1)) ≤
                           (∏k≤n. 1 − norm (f (k+M+N) − 1))) at_top
        using M by (intro always_eventually allI Weierstrass_prod_ineq) (auto
intro: less_imp_le)
      show (λn. ∏k≤n. 1 − norm (f (k+M+N) − 1)) ⟶ 0 by fact
```

```
      show (λn. 1 − (∑k≤n. norm (f (k + M + N) − 1)))
            ⟶ 1 − (∑i. norm (f (i + M + N) − 1))
      by (intro tendsto_intros summable_LIMSEQ' summable_ignore_initial_segment

            abs_convergent_prod_imp_summable assms)
    qed simp_all
    hence (∑i. norm (f (i + M + N) − 1)) ≥ 1 by simp
    also have ... + (∑i<M. norm (f (i + N) − 1)) = (∑i. norm (f (i + N)
− 1))
    by (intro suminf_split_initial_segment [symmetric] summable_ignore_initial_segment
          abs_convergent_prod_imp_summable assms)
    finally have 1 + (∑i<M. norm (f (i + N) − 1)) ≤ (∑i. norm (f (i +
N) − 1)) by simp
  } note ∗ = this

  have 1 + (∑i. norm (f (i + N) − 1)) ≤ (∑i. norm (f (i + N) − 1))
  proof (rule tendsto_le)
    show (λM. 1 + (∑i<M. norm (f (i + N) − 1))) ⟶ 1 + (∑i. norm
(f (i + N) − 1))
    by (intro tendsto_intros summable_LIMSEQ summable_ignore_initial_segment

          abs_convergent_prod_imp_summable assms)
    show eventually (λM. 1 + (∑i<M. norm (f (i + N) − 1)) ≤ (∑i. norm
(f (i + N) − 1))) at_top
      using eventually_ge_at_top[of M0] by eventually_elim (use ∗ in auto)
  qed simp_all
  thus False by simp
 qed
 with L show ?thesis by (auto simp: prod_defs)
qed

lemma raw_has_prod_cases:
 fixes f :: nat ⇒ 'a :: {idom,topological_semigroup_mult,t2_space}
 assumes raw_has_prod f M p
 obtains i where i<M f i = 0 | p where raw_has_prod f 0 p
proof −
 have (λn. ∏i≤n. f (i + M)) ⟶ p p ≠ 0
  using assms unfolding raw_has_prod_def by blast+
 then have (λn. prod f {..<M} ∗ (∏i≤n. f (i + M))) ⟶ prod f {..<M} ∗
p
  by (metis tendsto_mult_left)
 moreover have prod f {..<M} ∗ (∏i≤n. f (i + M)) = prod f {..n+M} for n
 proof −
  have {..n+M} = {..<M} ∪ {M..n+M}
   by auto
  then have prod f {..n+M} = prod f {..<M} ∗ prod f {M..n+M}
   by simp (subst prod.union_disjoint; force)
  also have ... = prod f {..<M} ∗ (∏i≤n. f (i + M))
  by (metis (mono_tags, lifting) add.left_neutral atMost_atLeast0 prod.shift_bounds_cl_nat_ivl)
```

```
    finally show ?thesis by metis
  qed
  ultimately have (λn. prod f {..n}) ⟶ prod f {..<M} * p
    by (auto intro: LIMSEQ_offset [where k=M])
  then have raw_has_prod f 0 (prod f {..<M} * p) if ∀ i<M. f i ≠ 0
    using ‹p ≠ 0› assms that by (auto simp: raw_has_prod_def)
  then show thesis
    using that by blast
qed

corollary convergent_prod_offset_0:
  fixes f :: nat ⇒ 'a :: {idom,topological_semigroup_mult,t2_space}
  assumes convergent_prod f ⋀i. f i ≠ 0
  shows ∃ p. raw_has_prod f 0 p
  using assms convergent_prod_def raw_has_prod_cases by blast

lemma prodinf_eq_lim:
  fixes f :: nat ⇒ 'a :: {idom,topological_semigroup_mult,t2_space}
  assumes convergent_prod f ⋀i. f i ≠ 0
  shows prodinf f = lim (λn. ∏ i≤n. f i)
  using assms convergent_prod_offset_0 [OF assms]
 by (simp add: prod_defs lim_def) (metis (no_types) assms(1) convergent_prod_to_zero_iff)

lemma prodinf_eq_lim':
  fixes f :: nat ⇒ 'a :: {idom,topological_semigroup_mult,t2_space}
  assumes convergent_prod f ⋀i. f i ≠ 0
  shows prodinf f = lim (λn. ∏ i<n. f i)
 by (metis assms prodinf_eq_lim LIMSEQ_lessThan_iff_atMost convergent_prod_iff_nz_lim
limI)

lemma prodinf_eq_prod_lim:
  fixes a:: 'a :: {topological_semigroup_mult,t2_space,idom}
  assumes (λn. ∏ k≤n. f k) ⟶ a a ≠ 0
  shows(∏ k. f k) = a
 by (metis LIMSEQ_prod_0 LIMSEQ_unique assms convergent_prod_iff_nz_lim
limI prodinf_eq_lim)

lemma prodinf_eq_prod_lim':
  fixes a:: 'a :: {topological_semigroup_mult,t2_space,idom}
  assumes (λn. ∏ k<n. f k) ⟶ a a ≠ 0
  shows(∏ k. f k) = a
  using LIMSEQ_lessThan_iff_atMost assms prodinf_eq_prod_lim by blast

lemma has_prod_one[simp, intro]: (λn. 1) has_prod 1
  unfolding prod_defs by auto

lemma convergent_prod_one[simp, intro]: convergent_prod (λn. 1)
  unfolding prod_defs by auto
```

lemma *prodinf_cong*: $(\bigwedge n.\ f\ n = g\ n) \Longrightarrow prodinf\ f = prodinf\ g$
by *presburger*

lemma *convergent_prod_cong*:
fixes $f\ g :: nat \Rightarrow 'a::\{field,topological\_semigroup\_mult,t2\_space\}$
assumes *ev*: *eventually* $(\lambda x.\ f\ x = g\ x)$ *sequentially* **and** *f*: $\bigwedge i.\ f\ i \neq 0$ **and** *g*: $\bigwedge i.\ g\ i \neq 0$
shows $convergent\_prod\ f = convergent\_prod\ g$
proof −
from *assms* **obtain** *N* **where** *N*: $\forall\, n{\geq}N.\ f\ n = g\ n$
by (*auto simp*: *eventually_at_top_linorder*)
define *C* **where** $C = (\prod k{<}N.\ f\ k\ /\ g\ k)$
with *g* **have** $C \neq 0$
by (*simp add*: *f*)
have $*$: *eventually* $(\lambda n.\ prod\ f\ \{..n\} = C * prod\ g\ \{..n\})$ *sequentially*
using *eventually_ge_at_top*[*of N*]
proof *eventually_elim*
case (*elim n*)
then have $\{..n\} = \{..{<}N\} \cup \{N..n\}$
by *auto*
also have $prod\ f\ \ldots = prod\ f\ \{..{<}N\} * prod\ f\ \{N..n\}$
by (*intro prod.union_disjoint*) *auto*
also from *N* **have** $prod\ f\ \{N..n\} = prod\ g\ \{N..n\}$
by (*intro prod.cong*) *simp_all*
also have $prod\ f\ \{..{<}N\} * prod\ g\ \{N..n\} = C * (prod\ g\ \{..{<}N\} * prod\ g\ \{N..n\})$
unfolding *C_def* **by** (*simp add*: *g prod_dividef*)
also have $prod\ g\ \{..{<}N\} * prod\ g\ \{N..n\} = prod\ g\ (\{..{<}N\} \cup \{N..n\})$
by (*intro prod.union_disjoint* [*symmetric*]) *auto*
also from *elim* **have** $\{..{<}N\} \cup \{N..n\} = \{..n\}$
by *auto*
finally show $prod\ f\ \{..n\} = C * prod\ g\ \{..n\}$ **.**
qed
then have *cong*: $convergent\ (\lambda n.\ prod\ f\ \{..n\}) = convergent\ (\lambda n.\ C * prod\ g\ \{..n\})$
by (*rule convergent_cong*)
show *?thesis*
proof
assume *cf*: *convergent_prod f*
with *f* **have** $\neg\ (\lambda n.\ prod\ f\ \{..n\}) \longrightarrow 0$
by *simp*
then have $\neg\ (\lambda n.\ prod\ g\ \{..n\}) \longrightarrow 0$
using $*$ ‹$C \neq 0$› *filterlim_cong* **by** *fastforce*
then show *convergent_prod g*
by (*metis convergent_mult_const_iff* ‹$C \neq 0$› *cong cf convergent_LIMSEQ_iff convergent_prod_iff_convergent convergent_prod_imp_convergent g*)
next
assume *cg*: *convergent_prod g*
have $\exists\, a.\ C * a \neq 0 \wedge (\lambda n.\ prod\ g\ \{..n\}) \longrightarrow a$

```
    by (metis (no_types) ‹C ≠ 0› cg convergent_prod_iff_nz_lim divide_eq_0_iff
g nonzero_mult_div_cancel_right)
    then show convergent_prod f
      using ∗ tendsto_mult_left filterlim_cong
      by (fastforce simp add: convergent_prod_iff_nz_lim f)
  qed
qed

lemma has_prod_finite:
  fixes f :: nat ⇒ 'a::{semidom,t2_space}
  assumes [simp]: finite N
    and f: ⋀n. n ∉ N ⟹ f n = 1
  shows f has_prod (∏ n∈N. f n)
proof −
  have eq: prod f {..n + Suc (Max N)} = prod f N for n
  proof (rule prod.mono_neutral_right)
    show N ⊆ {..n + Suc (Max N)}
      by (auto simp: le_Suc_eq trans_le_add2)
    show ∀ i∈{..n + Suc (Max N)} − N. f i = 1
      using f by blast
  qed auto
  show ?thesis
  proof (cases ∀ n∈N. f n ≠ 0)
    case True
    then have prod f N ≠ 0
      by simp
    moreover have (λn. prod f {..n}) ⟶ prod f N
     by (rule LIMSEQ_offset[of _ Suc (Max N)]) (simp add: eq atLeast0LessThan
del: add_Suc_right)
    ultimately show ?thesis
      by (simp add: raw_has_prod_def has_prod_def)
  next
    case False
    then obtain k where k ∈ N f k = 0
      by auto
    let ?Z = {n ∈ N. f n = 0}
    have maxge: Max ?Z ≥ n if f n = 0 for n
      using Max_ge [of ?Z] ‹finite N› ‹f n = 0›
    by (metis (mono_tags) Collect_mem_eq f finite_Collect_conjI mem_Collect_eq
zero_neq_one)
    let ?q = prod f {Suc (Max ?Z)..Max N}
    have [simp]: ?q ≠ 0
      using maxge Suc_n_not_le_n le_trans by force
    have eq: (∏ i≤n + Max N. f (Suc (i + Max ?Z))) = ?q for n
    proof −
      have (∏ i≤n + Max N. f (Suc (i + Max ?Z))) = prod f {Suc (Max ?Z)..n
+ Max N + Suc (Max ?Z)}
       proof (rule prod.reindex_cong [where l = λi. i + Suc (Max ?Z), THEN
sym])
```

```
      show {Suc (Max ?Z)..n + Max N + Suc (Max ?Z)} = (λi. i + Suc (Max
?Z)) ` {..n + Max N}
        using le_Suc_ex by fastforce
    qed (auto simp: inj_on_def)
    also have ... = ?q
      by (rule prod.mono_neutral_right)
        (use Max.coboundedI [OF ‹finite N›] f in ‹force+›)
    finally show ?thesis .
  qed
  have q: raw_has_prod f (Suc (Max ?Z)) ?q
  proof (simp add: raw_has_prod_def)
    show (λn. ∏i≤n. f (Suc (i + Max ?Z))) ⟶ ?q
      by (rule LIMSEQ_offset[of _ (Max N)]) (simp add: eq)
  qed
  show ?thesis
    unfolding has_prod_def
  proof (intro disjI2 exI conjI)
    show prod f N = 0
      using ‹f k = 0› ‹k ∈ N› ‹finite N› prod_zero by blast
    show f (Max ?Z) = 0
      using Max_in [of ?Z] ‹finite N› ‹f k = 0› ‹k ∈ N› by auto
  qed (use q in auto)
 qed
qed

corollary has_prod_0:
 fixes f :: nat ⇒ 'a::{semidom,t2_space}
 assumes ⋀n. f n = 1
 shows f has_prod 1
 by (simp add: assms has_prod_cong)

lemma prodinf_zero[simp]: prodinf (λn. 1::'a::real_normed_field) = 1
 using has_prod_unique by force

lemma convergent_prod_finite:
 fixes f :: nat ⇒ 'a::{idom,t2_space}
 assumes finite N ⋀n. n ∉ N ⟹ f n = 1
 shows convergent_prod f
proof −
 have ∃ n p. raw_has_prod f n p
   using assms has_prod_def has_prod_finite by blast
 then show ?thesis
   by (simp add: convergent_prod_def)
qed

lemma has_prod_If_finite_set:
 fixes f :: nat ⇒ 'a::{idom,t2_space}
 shows finite A ⟹ (λr. if r ∈ A then f r else 1) has_prod (∏ r∈A. f r)
 using has_prod_finite[of A (λr. if r ∈ A then f r else 1)]
```

by *simp*

lemma *has_prod_If_finite*:
fixes $f :: nat \Rightarrow 'a::\{idom,t2\_space\}$
shows *finite* $\{r.\ P\ r\} \Longrightarrow (\lambda r.\ if\ P\ r\ then\ f\ r\ else\ 1)\ has\_prod\ (\prod r \mid P\ r.\ f\ r)$
using *has_prod_If_finite_set*[*of* $\{r.\ P\ r\}$] **by** *simp*

lemma *convergent_prod_If_finite_set*[*simp*, *intro*]:
fixes $f :: nat \Rightarrow 'a::\{idom,t2\_space\}$
shows *finite* $A \Longrightarrow convergent\_prod\ (\lambda r.\ if\ r \in A\ then\ f\ r\ else\ 1)$
by (*simp add*: *convergent_prod_finite*)

lemma *convergent_prod_If_finite*[*simp*, *intro*]:
fixes $f :: nat \Rightarrow 'a::\{idom,t2\_space\}$
shows *finite* $\{r.\ P\ r\} \Longrightarrow convergent\_prod\ (\lambda r.\ if\ P\ r\ then\ f\ r\ else\ 1)$
using *convergent_prod_def has_prod_If_finite has_prod_def* **by** *fastforce*

lemma *has_prod_single*:
fixes $f :: nat \Rightarrow 'a::\{idom,t2\_space\}$
shows $(\lambda r.\ if\ r = i\ then\ f\ r\ else\ 1)\ has\_prod\ f\ i$
using *has_prod_If_finite*[*of* $\lambda r.\ r = i$] **by** *simp*

The ge1 assumption can probably be weakened, at the expense of extra work

lemma *uniform_limit_prodinf*:
fixes f:: $nat \Rightarrow real \Rightarrow real$
assumes *uniformly_convergent_on* $X\ (\lambda n\ x.\ \prod k{<}n.\ f\ k\ x)$
and *ge1*: $\bigwedge x\ k\ .\ x \in X \Longrightarrow f\ k\ x \geq 1$
shows *uniform_limit* $X\ (\lambda n\ x.\ \prod k{<}n.\ f\ k\ x)\ (\lambda x.\ \prod k.\ f\ k\ x)$ *sequentially*
proof −
have *ul*: *uniform_limit* $X\ (\lambda n\ x.\ \prod k{<}n.\ f\ k\ x)\ (\lambda x.\ lim\ (\lambda n.\ \prod k{<}n.\ f\ k\ x))$ *sequentially*
using *assms uniformly_convergent_uniform_limit_iff* **by** *blast*
moreover have $(\prod k.\ f\ k\ x) = lim\ (\lambda n.\ \prod k{<}n.\ f\ k\ x)$ **if** $x \in X$ **for** x
proof (*intro prodinf_eq_lim'*)
have *tends*: $(\lambda n.\ \prod k{<}n.\ f\ k\ x) \longrightarrow lim\ (\lambda n.\ \prod k{<}n.\ f\ k\ x)$
using *tendsto_uniform_limitI* [*OF ul*] *that* **by** *metis*
moreover have $(\prod k{<}n.\ f\ k\ x) \geq 1$ **for** n
using *ge1* **by** (*simp add*: *prod_ge_1 that*)
ultimately have $lim\ (\lambda n.\ \prod k{<}n.\ f\ k\ x) \geq 1$
by (*meson LIMSEQ_le_const*)
then have *raw_has_prod* $(\lambda k.\ f\ k\ x)\ 0\ (lim\ (\lambda n.\ \prod k{<}n.\ f\ k\ x))$
using *LIMSEQ_lessThan_iff_atMost tends* **by** (*auto simp*: *raw_has_prod_def*)
then show *convergent_prod* $(\lambda k.\ f\ k\ x)$
unfolding *convergent_prod_def* **by** *blast*
show $\bigwedge k.\ f\ k\ x \neq 0$
by (*smt* (*verit*) *ge1 that*)
qed
ultimately show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *uniform_limit_cong'*)

qed

context
fixes $f :: nat \Rightarrow 'a :: real\_normed\_field$
begin

lemma *convergent_prod_imp_has_prod*:
assumes *convergent_prod f*
shows $\exists p.$ *f has_prod p*
proof −
obtain *M p* **where** *p*: *raw_has_prod f M p*
using *assms convergent_prod_def* **by** *blast*
then have $p \neq 0$
using *raw_has_prod_nonzero* **by** *blast*
with *p* **have** *fnz*: $f\ i \neq 0$ **if** $i \geq M$ **for** *i*
using *raw_has_prod_eq_0 that* **by** *blast*
define *C* **where** $C = (\prod n{<}M.\ f\ n)$
show *?thesis*
proof (*cases* $\forall n{\leq}M.\ f\ n \neq 0$)
case *True*
then have $C \neq 0$
by (*simp add*: *C_def*)
then show *?thesis*
by (*meson True assms convergent_prod_offset_0 fnz has_prod_def nat_le_linear*)
next
case *False*
let $?N = GREATEST\ n.\ f\ n = 0$
have *0*: $f\ ?N = 0$
using *fnz False*
by (*metis* (*mono_tags*, *lifting*) *GreatestI_ex_nat nat_le_linear*)
have $f\ i \neq 0$ **if** $i > ?N$ **for** *i*
by (*metis* (*mono_tags*, *lifting*) *Greatest_le_nat fnz leD linear that*)
then have $\exists p.$ *raw_has_prod f* (*Suc ?N*) *p*
using *assms* **by** (*auto simp*: *intro*!: *convergent_prod_ignore_nonzero_segment*)
then show *?thesis*
unfolding *has_prod_def* **using** *0* **by** *blast*
qed
qed

lemma *convergent_prod_has_prod* [*intro*]:
shows *convergent_prod f* $\Longrightarrow$ *f has_prod* (*prodinf f*)
unfolding *prodinf_def*
by (*metis convergent_prod_imp_has_prod has_prod_unique theI'*)

lemma *convergent_prod_LIMSEQ*:
shows *convergent_prod f* $\Longrightarrow$ $(\lambda n.\ \prod i{\leq}n.\ f\ i) \longrightarrow$ *prodinf f*
by (*metis convergent_LIMSEQ_iff convergent_prod_has_prod convergent_prod_imp_convergent*

convergent_prod_to_zero_iff raw_has_prod_eq_0 has_prod_def prodinf_eq_lim

zero_le)

theorem *has_prod_iff*: *f has_prod x* $\longleftrightarrow$ *convergent_prod f* $\wedge$ *prodinf f* $=$ *x*
proof
assume *f has_prod x*
then show *convergent_prod f* $\wedge$ *prodinf f* $=$ *x*
apply *safe*
using *convergent_prod_def has_prod_def* **apply** *blast*
using *has_prod_unique* **by** *blast*
qed *auto*

lemma *convergent_prod_has_prod_iff*: *convergent_prod f* $\longleftrightarrow$ *f has_prod prodinf f*
by (*auto simp*: *has_prod_iff convergent_prod_has_prod*)

lemma *prodinf_finite*:
assumes *N*: *finite N*
and *f*: $\bigwedge n.\ n \notin N \Longrightarrow f\ n = 1$
shows *prodinf f* $= (\prod n{\in}N.\ f\ n)$
using *has_prod_finite*[*OF assms*, *THEN has_prod_unique*] **by** *simp*

end

9.22.6 Infinite products on ordered topological monoids

context
fixes *f* :: *nat* $\Rightarrow$ *'a*::{*linordered_semidom*,*linorder_topology*}
begin

lemma *has_prod_nonzero*:
assumes *f has_prod a a* $\neq$ *0*
shows *f k* $\neq$ *0*
using *assms* **by** (*auto simp*: *has_prod_def raw_has_prod_def LIMSEQ_prod_0 LIMSEQ_unique*)

lemma *has_prod_le*:
assumes *f*: *f has_prod a* **and** *g*: *g has_prod b* **and** *le*: $\bigwedge n.\ 0 \leq f\ n \wedge f\ n \leq g\ n$
shows *a* $\leq$ *b*
proof (*cases a=0* $\vee$ *b=0*)
case *True*
then show *?thesis*
proof
assume [*simp*]: *a=0*
have *b* $\geq$ *0*
proof (*rule LIMSEQ_prod_nonneg*)
show ($\lambda n.$ *prod g* {*..n*}) $\longrightarrow$ *b*
using *g* **by** (*auto simp*: *has_prod_def raw_has_prod_def LIMSEQ_prod_0*)
qed (*use le order_trans* **in** *auto*)
then show *?thesis*

```
    by auto
  next
    assume [simp]: b=0
    then obtain i where g i = 0
      using g by (auto simp: prod_defs)
    then have f i = 0
      using antisym le by force
    then have a=0
      using f by (auto simp: prod_defs LIMSEQ_prod_0 LIMSEQ_unique)
    then show ?thesis
      by auto
  qed
next
  case False
  then show ?thesis
    using assms
    unfolding has_prod_def raw_has_prod_def
    by (force simp: LIMSEQ_prod_0 intro!: LIMSEQ_le prod_mono)
qed

lemma prodinf_le:
  assumes f: f has_prod a and g: g has_prod b and le: ⋀n. 0 ≤ f n ∧ f n ≤ g n
  shows prodinf f ≤ prodinf g
  using has_prod_le [OF assms] has_prod_unique f g  by blast

end


lemma prod_le_prodinf:
  fixes f :: nat ⇒ 'a::{linordered_idom,linorder_topology}
  assumes f has_prod a ⋀i. 0 ≤ f i ⋀i. i≥n ⟹ 1 ≤ f i
  shows prod f {..<n} ≤ prodinf f
  by(rule has_prod_le[OF has_prod_If_finite_set]) (use assms has_prod_unique
in auto)

lemma prodinf_nonneg:
  fixes f :: nat ⇒ 'a::{linordered_idom,linorder_topology}
  assumes f has_prod a ⋀i. 1 ≤ f i
  shows 1 ≤ prodinf f
  using prod_le_prodinf[of f a 0] assms
  by (metis order_trans prod_ge_1 zero_le_one)

lemma prodinf_le_const:
  fixes f :: nat ⇒ real
  assumes convergent_prod f ⋀n. n ≥ N ⟹ prod f {..<n} ≤ x
  shows prodinf f ≤ x
 by (metis lessThan_Suc_atMost assms convergent_prod_LIMSEQ LIMSEQ_le_const2
atMost_iff lessThan_iff less_le)
```

lemma *prodinf_eq_one_iff* [*simp*]:
fixes *f* :: *nat* ⇒ *real*
assumes *f*: *convergent_prod f* **and** *ge1*: ⋀*n*. *1* ≤ *f n*
shows *prodinf f* = *1* ⟷ (∀ *n*. *f n* = *1*)
proof
assume *prodinf f* = *1*
then have (λ*n*. ∏ *i*<*n*. *f i*) ⟶ *1*
using *convergent_prod_LIMSEQ*[*of f*] *assms* **by** (*simp add*: *LIMSEQ_lessThan_iff_atMost*)
then have ⋀*i*. (∏ *n*∈{*i*}. *f n*) ≤ *1*
proof (*rule LIMSEQ_le_const*)
have *1* ≤ *prod f n* **for** *n*
by (*simp add*: *ge1 prod_ge_1*)
have *prod f* {..<*n*} = *1* **for** *n*
by (*metis* ‹⋀*n*. *1* ≤ *prod f n*› ‹*prodinf f* = *1*› *antisym f convergent_prod_has_prod ge1 order_trans prod_le_prodinf zero_le_one*)
then have (∏ *n*∈{*i*}. *f n*) ≤ *prod f* {..<*n*} **if** *n* ≥ *Suc i* **for** *i n*
by (*metis mult.left_neutral order_refl prod.cong prod.neutral_const prod.lessThan_Suc*)
then show ∃ *N*. ∀ *n*≥*N*. (∏ *n*∈{*i*}. *f n*) ≤ *prod f* {..<*n*} **for** *i*
by *blast*
qed
with *ge1* **show** ∀ *n*. *f n* = *1*
by (*auto intro*!: *antisym*)
qed (*metis prodinf_zero fun_eq_iff*)

lemma *prodinf_pos_iff*:
fixes *f* :: *nat* ⇒ *real*
assumes *convergent_prod f* ⋀*n*. *1* ≤ *f n*
shows *1* < *prodinf f* ⟷ (∃ *i*. *1* < *f i*)
using *prod_le_prodinf*[*of f 1*] *prodinf_eq_one_iff*
by (*metis convergent_prod_has_prod assms less_le prodinf_nonneg*)

lemma *less_1_prodinf2*:
fixes *f* :: *nat* ⇒ *real*
assumes *convergent_prod f* ⋀*n*. *1* ≤ *f n* *1* < *f i*
shows *1* < *prodinf f*
proof −
have *1* < (∏ *n*<*Suc i*. *f n*)
using *assms* **by** (*intro less_1_prod2*[**where** *i*=*i*]) *auto*
also have ... ≤ *prodinf f*
by (*intro prod_le_prodinf*) (*use assms order_trans zero_le_one* **in** ‹*blast*+›)
finally show *?thesis* **.**
qed

lemma *less_1_prodinf*:
fixes *f* :: *nat* ⇒ *real*
shows ⟦*convergent_prod f*; ⋀*n*. *1* < *f n*⟧ ⟹ *1* < *prodinf f*
by (*intro less_1_prodinf2*[**where** *i*=*1*]) (*auto intro*: *less_imp_le*)

lemma *prodinf_nonzero*:

fixes $f :: nat \Rightarrow 'a :: \{idom,topological\_semigroup\_mult,t2\_space\}$
assumes *convergent_prod f* $\bigwedge i.\ f\ i \neq 0$
shows *prodinf f* $\neq 0$
by (*metis assms convergent_prod_offset_0 has_prod_unique raw_has_prod_def has_prod_def*)

lemma *less_0_prodinf*:
fixes $f :: nat \Rightarrow real$
assumes *f*: *convergent_prod f* **and** *0*: $\bigwedge i.\ f\ i > 0$
shows $0 <$ *prodinf f*
proof −
have *prodinf f* $\neq 0$
by (*metis assms less_irrefl prodinf_nonzero*)
moreover have $0 < (\prod n{<}i.\ f\ n)$ **for** *i*
by (*simp add*: *0 prod_pos*)
then have *prodinf f* ≥ 0
using *convergent_prod_LIMSEQ* [*OF f*] *LIMSEQ_prod_nonneg 0 less_le* **by** *blast*
ultimately show *?thesis*
by *auto*
qed

lemma *prod_less_prodinf2*:
fixes $f :: nat \Rightarrow real$
assumes *f*: *convergent_prod f* **and** *1*: $\bigwedge m.\ m{\geq}n \Longrightarrow 1 \leq f\ m$ **and** *0*: $\bigwedge m.\ 0 < f\ m$ **and** *i*: $n \leq i\ 1 < f\ i$
shows *prod f* $\{..{<}n\} <$ *prodinf f*
proof −
have *prod f* $\{..{<}n\} \leq$ *prod f* $\{..{<}i\}$
by (*rule prod_mono2*) (*use assms less_le* **in** *auto*)
then have *prod f* $\{..{<}n\} < f\ i *$ *prod f* $\{..{<}i\}$
using *mult_less_le_imp_less*[*of 1 f i prod f* $\{..{<}n\}$ *prod f* $\{..{<}i\}$] *assms*
by (*simp add*: *prod_pos*)
moreover have *prod f* $\{..{<}Suc\ i\} \leq$ *prodinf f*
using *prod_le_prodinf*[*of f _ Suc i*]
by (*meson 0 1 Suc_leD convergent_prod_has_prod f* ‹$n \leq i$› *le_trans less_eq_real_def*)
ultimately show *?thesis*
by (*metis le_less_trans mult.commute not_le prod.lessThan_Suc*)
qed

lemma *prod_less_prodinf*:
fixes $f :: nat \Rightarrow real$
assumes *f*: *convergent_prod f* **and** *1*: $\bigwedge m.\ m{\geq}n \Longrightarrow 1 < f\ m$ **and** *0*: $\bigwedge m.\ 0 < f\ m$
shows *prod f* $\{..{<}n\} <$ *prodinf f*
by (*meson 0 1 f le_less prod_less_prodinf2*)

lemma *raw_has_prodI_bounded*:
fixes $f :: nat \Rightarrow real$

assumes *pos*: $\bigwedge n.\ 1 \leq f\ n$
and *le*: $\bigwedge n.\ (\prod i{<}n.\ f\ i) \leq x$
shows $\exists p.$ *raw_has_prod f 0 p*
unfolding *raw_has_prod_def add_0_right*
proof (*rule exI LIMSEQ_incseq_SUP conjI*)+
show *bdd_above* (*range* ($\lambda n.$ *prod f* {..*n*}))
by (*metis bdd_aboveI2 le lessThan_Suc_atMost*)
then have (*SUP i. prod f* {..*i*}) > *0*
by (*metis UNIV_I cSUP_upper less_le_trans pos prod_pos zero_less_one*)
then show (*SUP i. prod f* {..*i*}) ≠ *0*
by *auto*
show *incseq* ($\lambda n.$ *prod f* {..*n*})
using *pos order_trans* [*OF zero_le_one*] **by** (*auto simp*: *mono_def intro*!: *prod_mono2*)
qed

lemma *convergent_prodI_nonneg_bounded*:
fixes *f* :: *nat* ⇒ *real*
assumes $\bigwedge n.\ 1 \leq f\ n$ $\bigwedge n.\ (\prod i{<}n.\ f\ i) \leq x$
shows *convergent_prod f*
using *convergent_prod_def raw_has_prodI_bounded* [*OF assms*] **by** *blast*

9.22.7 Infinite products on topological spaces

context
fixes *f g* :: *nat* ⇒ *'a*::{*t2_space*,*topological_semigroup_mult*,*idom*}
begin

lemma *raw_has_prod_mult*: ⟦*raw_has_prod f M a*; *raw_has_prod g M b*⟧ ⟹ *raw_has_prod* ($\lambda n.$ *f n* ∗ *g n*) *M* (*a* ∗ *b*)
by (*force simp add*: *prod.distrib tendsto_mult raw_has_prod_def*)

lemma *has_prod_mult_nz*: ⟦*f has_prod a*; *g has_prod b*; *a* ≠ *0*; *b* ≠ *0*⟧ ⟹ ($\lambda n.$ *f n* ∗ *g n*) *has_prod* (*a* ∗ *b*)
by (*simp add*: *raw_has_prod_mult has_prod_def*)

end

context
fixes *f g* :: *nat* ⇒ *'a*::*real_normed_field*
begin

lemma *has_prod_mult*:
assumes *f*: *f has_prod a* **and** *g*: *g has_prod b*
shows ($\lambda n.$ *f n* ∗ *g n*) *has_prod* (*a* ∗ *b*)
using *f* [*unfolded has_prod_def*]
proof (*elim disjE exE conjE*)
assume *f0*: *raw_has_prod f 0 a*

```
  show ?thesis
    using g [unfolded has_prod_def]
  proof (elim disjE exE conjE)
    assume g0: raw_has_prod g 0 b
    with f0 show ?thesis
    by (force simp add: has_prod_def prod.distrib tendsto_mult raw_has_prod_def)
  next
    fix j q
    assume b = 0 and g j = 0 and q: raw_has_prod g (Suc j) q
    obtain p where p: raw_has_prod f (Suc j) p
      using f0 raw_has_prod_ignore_initial_segment by blast
    then have Ex (raw_has_prod (λn. f n * g n) (Suc j))
      using q raw_has_prod_mult by blast
    then show ?thesis
      using ‹b = 0› ‹g j = 0› has_prod_0_iff by fastforce
  qed
next
  fix i p
  assume a = 0 and f i = 0 and p: raw_has_prod f (Suc i) p
  show ?thesis
    using g [unfolded has_prod_def]
  proof (elim disjE exE conjE)
    assume g0: raw_has_prod g 0 b
    obtain q where q: raw_has_prod g (Suc i) q
      using g0 raw_has_prod_ignore_initial_segment by blast
    then have Ex (raw_has_prod (λn. f n * g n) (Suc i))
      using raw_has_prod_mult p by blast
    then show ?thesis
      using ‹a = 0› ‹f i = 0› has_prod_0_iff by fastforce
  next
    fix j q
    assume b = 0 and g j = 0 and q: raw_has_prod g (Suc j) q
    obtain p' where p': raw_has_prod f (Suc (max i j)) p'
     by (metis raw_has_prod_ignore_initial_segment max_Suc_Suc max_def p)
    moreover
    obtain q' where q': raw_has_prod g (Suc (max i j)) q'
    by (metis raw_has_prod_ignore_initial_segment max.cobounded2 max_Suc_Suc
q)
    ultimately show ?thesis
       using ‹b = 0› by (simp add: has_prod_def) (metis ‹f i = 0› ‹g j = 0›
raw_has_prod_mult max_def)
  qed
qed

lemma convergent_prod_mult:
  assumes f: convergent_prod f and g: convergent_prod g
  shows convergent_prod (λn. f n * g n)
  unfolding convergent_prod_def
proof −
```

obtain $M\ p\ N\ q$ **where** p: *raw_has_prod f M p* **and** q: *raw_has_prod g N q*
using *convergent_prod_def f g* **by** *blast+*
then obtain $p'\ q'$ **where** p': *raw_has_prod f* $(max\ M\ N)\ p'$ **and** q': *raw_has_prod g* $(max\ M\ N)\ q'$
by (*meson raw_has_prod_ignore_initial_segment max.cobounded1 max.cobounded2*)
then show $\exists M\ p.$ *raw_has_prod* $(\lambda n.\ f\ n * g\ n)\ M\ p$
using *raw_has_prod_mult* **by** *blast*
qed

lemma *prodinf_mult*: *convergent_prod f* $\Longrightarrow$ *convergent_prod g* $\Longrightarrow$ *prodinf f* $*$ *prodinf g* $= (\prod n.\ f\ n * g\ n)$
by (*intro has_prod_unique has_prod_mult convergent_prod_has_prod*)

end

context
fixes $f :: 'i \Rightarrow nat \Rightarrow 'a$::*real_normed_field*
and $I :: 'i$ *set*
begin

lemma *has_prod_prod*: $(\bigwedge i.\ i \in I \Longrightarrow (f\ i)$ *has_prod* $(x\ i)) \Longrightarrow (\lambda n.\ \prod i{\in}I.\ f\ i\ n)$ *has_prod* $(\prod i{\in}I.\ x\ i)$
by (*induct I rule*: *infinite_finite_induct*) (*auto intro*!: *has_prod_mult*)

lemma *prodinf_prod*: $(\bigwedge i.\ i \in I \Longrightarrow$ *convergent_prod* $(f\ i)) \Longrightarrow (\prod n.\ \prod i{\in}I.\ f\ i\ n) = (\prod i{\in}I.\ \prod n.\ f\ i\ n)$
using *has_prod_unique*[*OF has_prod_prod*, *OF convergent_prod_has_prod*] **by** *simp*

lemma *convergent_prod_prod*: $(\bigwedge i.\ i \in I \Longrightarrow$ *convergent_prod* $(f\ i)) \Longrightarrow$ *convergent_prod* $(\lambda n.\ \prod i{\in}I.\ f\ i\ n)$
using *convergent_prod_has_prod_iff has_prod_prod prodinf_prod* **by** *force*

end

9.22.8 Infinite summability on real normed fields

context
fixes $f :: nat \Rightarrow 'a$::*real_normed_field*
begin

lemma *raw_has_prod_Suc_iff*: *raw_has_prod f M* $(a * f\ M) \longleftrightarrow$ *raw_has_prod* $(\lambda n.\ f\ (Suc\ n))\ M\ a \wedge f\ M \neq 0$
proof −
have *raw_has_prod f M* $(a * f\ M) \longleftrightarrow (\lambda i.\ \prod j{\leq}Suc\ i.\ f\ (j{+}M)) \longrightarrow a * f\ M \wedge a * f\ M \neq 0$
by (*subst filterlim_sequentially_Suc*) (*simp add*: *raw_has_prod_def*)
also have $\ldots \longleftrightarrow (\lambda i.\ (\prod j{\leq}i.\ f\ (Suc\ j + M)) * f\ M) \longrightarrow a * f\ M \wedge a * f\ M \neq 0$

```
  by (simp add: ac_simps atMost_Suc_eq_insert_0 image_Suc_atMost prod.atLeast1_atMost_eq
lessThan_Suc_atMost
                del: prod.cl_ivl_Suc)
  also have ... ⟷ raw_has_prod (λn. f (Suc n)) M a ∧ f M ≠ 0
  proof safe
    assume tends: (λi. (∏j≤i. f (Suc j + M)) * f M) ⟶ a * f M and 0: a
* f M ≠ 0
    with tendsto_divide[OF tends tendsto_const, of f M]
    show raw_has_prod (λn. f (Suc n)) M a
      by (simp add: raw_has_prod_def)
  qed (auto intro: tendsto_mult_right simp:  raw_has_prod_def)
  finally show ?thesis .
qed

lemma has_prod_Suc_iff:
  assumes f 0 ≠ 0 shows (λn. f (Suc n)) has_prod a ⟷ f has_prod (a * f 0)
proof (cases a = 0)
  case True
  then show ?thesis
  proof (simp add: has_prod_def, safe)
    fix i x
    assume f (Suc i) = 0 and raw_has_prod (λn. f (Suc n)) (Suc i) x
    then obtain y where raw_has_prod f (Suc (Suc i)) y
    by (metis (no_types) raw_has_prod_eq_0 Suc_n_not_le_n raw_has_prod_Suc_iff
raw_has_prod_ignore_initial_segment raw_has_prod_nonzero linear)
    then show ∃i. f i = 0 ∧ Ex (raw_has_prod f (Suc i))
      using ‹f (Suc i) = 0› by blast
  next
    fix i x
    assume f i = 0 and x: raw_has_prod f (Suc i) x
    then obtain j where j: i = Suc j
      by (metis assms not0_implies_Suc)
    moreover have ∃ y. raw_has_prod (λn. f (Suc n)) i y
      using x by (auto simp: raw_has_prod_def)
    then show ∃i. f (Suc i) = 0 ∧ Ex (raw_has_prod (λn. f (Suc n)) (Suc i))
      using ‹f i = 0› j by blast
  qed
next
  case False
  then show ?thesis
    by (auto simp: has_prod_def raw_has_prod_Suc_iff assms)
qed

lemma convergent_prod_Suc_iff [simp]:
  shows convergent_prod (λn. f (Suc n)) = convergent_prod f
proof
  assume convergent_prod f
  then obtain M L where M_nz:∀ n≥M. f n ≠ 0 and
       M_L:(λn. ∏i≤n. f (i + M)) ⟶ L and L ≠ 0
```

```
    unfolding convergent_prod_altdef by auto
  have (λn. ∏ i≤n. f (Suc (i + M))) ⟶ L / f M
  proof −
    have (λn. ∏ i∈{0..Suc n}. f (i + M)) ⟶ L
      using M_L
      apply (subst (asm) filterlim_sequentially_Suc[symmetric])
      using atLeast0AtMost by auto
    then have (λn. f M * (∏ i∈{0..n}. f (Suc (i + M)))) ⟶ L
      apply (subst (asm) prod.atLeast0_atMost_Suc_shift)
      by simp
    then have (λn. (∏ i∈{0..n}. f (Suc (i + M)))) ⟶ L/f M
      apply (drule_tac tendsto_divide)
      using M_nz[rule_format,of M,simplified] by auto
    then show ?thesis unfolding atLeast0AtMost .
  qed
  then show convergent_prod (λn. f (Suc n)) unfolding convergent_prod_altdef
    apply (rule_tac exI[where x=M])
    apply (rule_tac exI[where x=L/f M])
    using M_nz ‹L≠0› by auto
next
  assume convergent_prod (λn. f (Suc n))
  then obtain M where ∃ L. (∀ n≥M. f (Suc n) ≠ 0) ∧ (λn. ∏ i≤n. f (Suc (i
+ M))) ⟶ L ∧ L ≠ 0
    unfolding convergent_prod_altdef by auto
  then show convergent_prod f unfolding convergent_prod_altdef
    apply (rule_tac exI[where x=Suc M])
    using Suc_le_D by auto
qed

lemma raw_has_prod_inverse:
   assumes raw_has_prod f M a shows raw_has_prod (λn. inverse (f n)) M
(inverse a)
  using assms unfolding raw_has_prod_def by (auto dest: tendsto_inverse simp:
prod_inversef [symmetric])

lemma has_prod_inverse:
  assumes f has_prod a shows (λn. inverse (f n)) has_prod (inverse a)
using assms raw_has_prod_inverse unfolding has_prod_def by auto

lemma convergent_prod_inverse:
  assumes convergent_prod f
  shows convergent_prod (λn. inverse (f n))
  using assms unfolding convergent_prod_def by (blast intro: raw_has_prod_inverse
elim: )

end

context
  fixes f :: nat ⇒ 'a::real_normed_field
```

begin

lemma *raw_has_prod_Suc_iff'*: *raw_has_prod f M a* $\longleftrightarrow$ *raw_has_prod* ($\lambda n.$ *f* (*Suc n*)) *M* (*a* / *f M*) $\wedge$ *f M* $\neq$ *0*
by (*metis raw_has_prod_eq_0 add.commute add.left_neutral raw_has_prod_Suc_iff raw_has_prod_nonzero le_add1 nonzero_mult_div_cancel_right times_divide_eq_left*)

lemma *has_prod_divide*: *f has_prod a* $\Longrightarrow$ *g has_prod b* $\Longrightarrow$ ($\lambda n.$ *f n* / *g n*) *has_prod* (*a* / *b*)
unfolding *divide_inverse* **by** (*intro has_prod_inverse has_prod_mult*)

lemma *convergent_prod_divide*:
assumes *f*: *convergent_prod f* **and** *g*: *convergent_prod g*
shows *convergent_prod* ($\lambda n.$ *f n* / *g n*)
using *f g has_prod_divide has_prod_iff* **by** *blast*

lemma *prodinf_divide*: *convergent_prod f* $\Longrightarrow$ *convergent_prod g* $\Longrightarrow$ *prodinf f* / *prodinf g* = ($\prod n.$ *f n* / *g n*)
by (*intro has_prod_unique has_prod_divide convergent_prod_has_prod*)

lemma *prodinf_inverse*: *convergent_prod f* $\Longrightarrow$ ($\prod n.$ *inverse* (*f n*)) = *inverse* ($\prod n.$ *f n*)
by (*intro has_prod_unique* [*symmetric*] *has_prod_inverse convergent_prod_has_prod*)

lemma *has_prod_Suc_imp*:
assumes ($\lambda n.$ *f* (*Suc n*)) *has_prod a*
shows *f has_prod* (*a* $*$ *f 0*)
proof −
have *f has_prod* (*a* $*$ *f 0*) **when** *raw_has_prod* ($\lambda n.$ *f* (*Suc n*)) *0 a*
apply (*cases f 0=0*)
using *that* **unfolding** *has_prod_def raw_has_prod_Suc*
by (*auto simp add*: *raw_has_prod_Suc_iff*)
moreover have *f has_prod* (*a* $*$ *f 0*) **when**
($\exists i\ q.$ *a* = *0* $\wedge$ *f* (*Suc i*) = *0* $\wedge$ *raw_has_prod* ($\lambda n.$ *f* (*Suc n*)) (*Suc i*) *q*)
proof −
from *that*
obtain *i q* **where** *a* = *0 f* (*Suc i*) = *0 raw_has_prod* ($\lambda n.$ *f* (*Suc n*)) (*Suc i*) *q*
by *auto*
then show *?thesis* **unfolding** *has_prod_def*
by (*auto intro*!:*exI*[**where** *x*=*Suc i*] *simp*:*raw_has_prod_Suc*)
qed
ultimately show *f has_prod* (*a* $*$ *f 0*) **using** *assms* **unfolding** *has_prod_def*
by *auto*
qed

lemma *has_prod_iff_shift*:
assumes $\bigwedge i.$ *i* < *n* $\Longrightarrow$ *f i* $\neq$ *0*
shows ($\lambda i.$ *f* (*i* + *n*)) *has_prod a* $\longleftrightarrow$ *f has_prod* (*a* $*$ ($\prod i$<*n. f i*))
using *assms*

proof (*induct n arbitrary*: *a*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
then have (λ*i*. *f* (*Suc i* + *n*)) *has_prod a* ⟷ (λ*i*. *f* (*i* + *n*)) *has_prod* (*a* ∗ *f n*)
by (*subst has_prod_Suc_iff*) *auto*
with *Suc* **show** *?case*
by (*simp add*: *ac_simps*)
qed

corollary *has_prod_iff_shift'*:
assumes ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows (λ*i*. *f* (*i* + *n*)) *has_prod* (*a* / (∏ *i*<*n*. *f i*)) ⟷ *f has_prod a*
by (*simp add*: *assms has_prod_iff_shift*)

lemma *has_prod_one_iff_shift*:
assumes ⋀*i*. *i* < *n* ⟹ *f i* = *1*
shows (λ*i*. *f* (*i*+*n*)) *has_prod a* ⟷ (λ*i*. *f i*) *has_prod a*
by (*simp add*: *assms has_prod_iff_shift*)

lemma *convergent_prod_iff_shift* [*simp*]:
shows *convergent_prod* (λ*i*. *f* (*i* + *n*)) ⟷ *convergent_prod f*
apply *safe*
using *convergent_prod_offset* **apply** *blast*
using *convergent_prod_ignore_initial_segment convergent_prod_def* **by** *blast*

lemma *has_prod_split_initial_segment*:
assumes *f has_prod a* ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows (λ*i*. *f* (*i* + *n*)) *has_prod* (*a* / (∏ *i*<*n*. *f i*))
using *assms has_prod_iff_shift'* **by** *blast*

lemma *prodinf_divide_initial_segment*:
assumes *convergent_prod f* ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows (∏ *i*. *f* (*i* + *n*)) = (∏ *i*. *f i*) / (∏ *i*<*n*. *f i*)
by (*rule has_prod_unique*[*symmetric*]) (*auto simp*: *assms has_prod_iff_shift*)

lemma *prodinf_split_initial_segment*:
assumes *convergent_prod f* ⋀*i*. *i* < *n* ⟹ *f i* ≠ *0*
shows *prodinf f* = (∏ *i*. *f* (*i* + *n*)) ∗ (∏ *i*<*n*. *f i*)
by (*auto simp add*: *assms prodinf_divide_initial_segment*)

lemma *prodinf_split_head*:
assumes *convergent_prod f f 0* ≠ *0*
shows (∏ *n*. *f* (*Suc n*)) = *prodinf f* / *f 0*
using *prodinf_split_initial_segment*[*of 1*] *assms* **by** *simp*

lemma *has_prod_ignore_initial_segment'*:

```
  assumes convergent_prod f
  shows   f has_prod ((∏k<n. f k) * (∏k. f (k + n)))
proof (cases ∃k<n. f k = 0)
  case True
  hence [simp]: (∏k<n. f k) = 0
    by (meson finite_lessThan lessThan_iff prod_zero)
  thus ?thesis using True assms
    by (metis convergent_prod_has_prod_iff has_prod_zeroI mult_not_zero)
next
  case False
  hence (λi. f (i + n)) has_prod (prodinf f / prod f {..<n})
     using assms by (intro has_prod_split_initial_segment) (auto simp: convergent_prod_has_prod_iff)
  hence prodinf f = prod f {..<n} * (∏k. f (k + n))
    using False by (simp add: has_prod_iff divide_simps mult_ac)
  thus ?thesis
    using assms by (simp add: convergent_prod_has_prod_iff)
qed

end

context
  fixes f :: nat ⇒ 'a::real_normed_field
begin

lemma convergent_prod_inverse_iff [simp]: convergent_prod (λn. inverse (f n))
⟷ convergent_prod f
  by (auto dest: convergent_prod_inverse)

lemma convergent_prod_const_iff [simp]:
  fixes c :: 'a :: {real_normed_field}
  shows convergent_prod (λ_. c) ⟷ c = 1
proof
  assume convergent_prod (λ_. c)
  then show c = 1
    using convergent_prod_imp_LIMSEQ LIMSEQ_unique by blast
next
  assume c = 1
  then show convergent_prod (λ_. c)
    by auto
qed

lemma has_prod_power: f has_prod a ⟹ (λi. f i ^ n) has_prod (a ^ n)
  by (induction n) (auto simp: has_prod_mult)

lemma convergent_prod_power: convergent_prod f ⟹ convergent_prod (λi. f i
^ n)
  by (induction n) (auto simp: convergent_prod_mult)
```

lemma *prodinf_power*: *convergent_prod f* $\Longrightarrow$ *prodinf* $(\lambda i.\ f\ i\ \hat{}\ n) =$ *prodinf f* $\hat{}$ *n*
by (*metis has_prod_unique convergent_prod_imp_has_prod has_prod_power*)

end

9.22.9 Exponentials and logarithms

context
fixes *f* :: *nat* $\Rightarrow$ *'a*::{*real_normed_field*,*banach*}
begin

lemma *sums_imp_has_prod_exp*:
assumes *f sums s*
shows *raw_has_prod* $(\lambda i.\ exp\ (f\ i))\ 0\ (exp\ s)$
using *assms continuous_on_exp* [*of UNIV* λx::*'a*. *x*]
using *continuous_on_tendsto_compose* [*of UNIV exp* $(\lambda n.\ sum\ f\ \{..n\})\ s$]
by (*simp add*: *prod_defs sums_def_le exp_sum*)

lemma *convergent_prod_exp*:
assumes *summable f*
shows *convergent_prod* $(\lambda i.\ exp\ (f\ i))$
using *sums_imp_has_prod_exp assms* **unfolding** *summable_def convergent_prod_def*
by *blast*

lemma *prodinf_exp*:
assumes *summable f*
shows *prodinf* $(\lambda i.\ exp\ (f\ i)) = exp\ (suminf\ f)$
proof −
have *f sums suminf f*
using *assms* **by** *blast*
then have $(\lambda i.\ exp\ (f\ i))$ *has_prod exp* $(suminf\ f)$
by (*simp add*: *has_prod_def sums_imp_has_prod_exp*)
then show *?thesis*
by (*rule has_prod_unique* [*symmetric*])
qed

end

theorem *convergent_prod_iff_summable_real*:
fixes *a* :: *nat* $\Rightarrow$ *real*
assumes $\bigwedge n.\ a\ n > 0$
shows *convergent_prod* $(\lambda k.\ 1 + a\ k) \longleftrightarrow$ *summable a* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain *p* **where** *raw_has_prod* $(\lambda k.\ 1 + a\ k)\ 0\ p$
by (*metis assms add_less_same_cancel2 convergent_prod_offset_0 not_one_less_zero*)
then have *to_p*: $(\lambda n.\ \prod k{\le}n.\ 1 + a\ k) \longrightarrow p$
by (*auto simp*: *raw_has_prod_def*)

```
  moreover have le: (∑k≤n. a k) ≤ (∏k≤n. 1 + a k) for n
    by (rule sum_le_prod) (use assms less_le in force)
  have (∏k≤n. 1 + a k) ≤ p for n
  proof (rule incseq_le [OF _ to_p])
    show incseq (λn. ∏k≤n. 1 + a k)
      using assms by (auto simp: mono_def order.strict_implies_order intro!:
prod_mono2)
  qed
  with le have (∑k≤n. a k) ≤ p for n
    by (metis order_trans)
  with assms bounded_imp_summable show ?rhs
    by (metis not_less order.asym)
next
  assume R: ?rhs
  have (∏k≤n. 1 + a k) ≤ exp (suminf a) for n
  proof −
    have (∏k≤n. 1 + a k) ≤ exp (∑k≤n. a k) for n
      by (rule prod_le_exp_sum) (use assms less_le in force)
    moreover have exp (∑k≤n. a k) ≤ exp (suminf a) for n
      unfolding exp_le_cancel_iff
      by (meson sum_le_suminf R assms finite_atMost less_eq_real_def)
    ultimately show ?thesis
      by (meson order_trans)
  qed
  then obtain L where L: (λn. ∏k≤n. 1 + a k) ⟶ L
    by (metis assms bounded_imp_convergent_prod convergent_prod_iff_nz_lim
le_add_same_cancel1 le_add_same_cancel2 less_le not_le zero_le_one)
  moreover have L ≠ 0
  proof
    assume L = 0
    with L have (λn. ∏k≤n. 1 + a k) ⟶ 0
      by simp
    moreover have (∏k≤n. 1 + a k) > 1 for n
      by (simp add: assms less_1_prod)
    ultimately show False
      by (meson Lim_bounded2 not_one_le_zero less_imp_le)
  qed
  ultimately show ?lhs
    using assms convergent_prod_iff_nz_lim
    by (metis add_less_same_cancel1 less_le not_le zero_less_one)
qed

lemma exp_suminf_prodinf_real:
  fixes f :: nat ⇒ real
  assumes ge0:⋀n. f n ≥ 0 and ac: abs_convergent_prod (λn. exp (f n))
  shows prodinf (λi. exp (f i)) = exp (suminf f)
proof −
  have summable f
    using ac unfolding abs_convergent_prod_conv_summable
```

```
  proof (elim summable_comparison_test')
    fix n
    have |f n| = f n
      by (simp add: ge0)
    also have ... ≤ exp (f n) − 1
      by (metis diff_diff_add exp_ge_add_one_self ge_iff_diff_ge_0)
    finally show norm (f n) ≤ norm (exp (f n) − 1)
      by simp
  qed
  then show ?thesis
    by (simp add: prodinf_exp)
qed

lemma has_prod_imp_sums_ln_real:
  fixes f :: nat ⇒ real
  assumes raw_has_prod f 0 p and 0: ⋀x. f x > 0
  shows (λi. ln (f i)) sums (ln p)
proof −
  have p > 0
   using assms unfolding prod_defs by (metis LIMSEQ_prod_nonneg less_eq_real_def)
  moreover have ⋀x. f x ≠ 0
    by (smt (verit, best) 0)
  ultimately show ?thesis
    using assms continuous_on_ln [of {0<..} λx. x]
    using continuous_on_tendsto_compose [of {0<..} ln (λn. prod f {..n}) p]
    by (auto simp: prod_defs sums_def_le ln_prod order_tendstoD)
qed

lemma summable_ln_real:
  fixes f :: nat ⇒ real
  assumes f: convergent_prod f and 0: ⋀x. f x > 0
  shows summable (λi. ln (f i))
proof −
  obtain M p where raw_has_prod f M p
    using f convergent_prod_def by blast
  then consider i where i<M f i = 0 | p where raw_has_prod f 0 p
    using raw_has_prod_cases by blast
  then show ?thesis
  proof cases
    case 1
    with 0 show ?thesis
      by (metis less_irrefl)
  next
    case 2
    then show ?thesis
      using 0 has_prod_imp_sums_ln_real summable_def by blast
  qed
qed
```

lemma *suminf_ln_real*:
fixes $f :: nat \Rightarrow real$
assumes f: *convergent_prod* f **and** 0: $\bigwedge x.\ f\ x > 0$
shows *suminf* $(\lambda i.\ ln\ (f\ i)) = ln\ (prodinf\ f)$
proof −
have *f has_prod prodinf f*
by (*simp add*: *f has_prod_iff*)
then have *raw_has_prod f 0* (*prodinf f*)
by (*metis 0 has_prod_def less_irrefl*)
then have $(\lambda i.\ ln\ (f\ i))$ *sums ln* (*prodinf f*)
using *0 has_prod_imp_sums_ln_real* **by** *blast*
then show *?thesis*
by (*rule sums_unique* [*symmetric*])
qed

lemma *prodinf_exp_real*:
fixes $f :: nat \Rightarrow real$
assumes f: *convergent_prod* f **and** 0: $\bigwedge x.\ f\ x > 0$
shows *prodinf f* = *exp* (*suminf* $(\lambda i.\ ln\ (f\ i))$)
by (*simp add*: *0 f less_0_prodinf suminf_ln_real*)

theorem *Ln_prodinf_complex*:
fixes $z :: nat \Rightarrow complex$
assumes z: $\bigwedge j.\ z\ j \neq 0$ **and** ξ: $\xi \neq 0$
shows $((\lambda n.\ \prod j{\le}n.\ z\ j) \longrightarrow \xi) \longleftrightarrow (\exists k.\ (\lambda n.\ (\sum j{\le}n.\ Ln\ (z\ j))) \longrightarrow Ln\ \xi + of\_int\ k * (of\_real(2{*}pi) * \mathrm{i}))$ (**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
have *pnz*: $(\prod j{\le}n.\ z\ j) \neq 0$ **for** n
using z **by** *auto*
define Θ **where** $\Theta \equiv Arg\ \xi + 2{*}pi$
then have $\Theta > pi$
using *Arg_def mpi_less_Im_Ln* **by** *fastforce*
have ξ*_eq*: $\xi = cmod\ \xi * exp\ (\mathrm{i} * \Theta)$
using *Arg_def Arg_eq* ξ **unfolding** Θ*_def* **by** (*simp add*: *algebra_simps exp_add*)
define ϑ **where** $\vartheta \equiv \lambda n.\ THE\ t.\ is\_Arg\ (\prod j{\le}n.\ z\ j)\ t \wedge t \in \{\Theta{-}pi{<}..\Theta{+}pi\}$
have *uniq*: $\exists!s.\ is\_Arg\ (\prod j{\le}n.\ z\ j)\ s \wedge s \in \{\Theta{-}pi{<}..\Theta{+}pi\}$ **for** n
using *Argument_exists_unique* [*OF pnz*] **by** *metis*
have ϑ: *is_Arg* $(\prod j{\le}n.\ z\ j)\ (\vartheta\ n)$ **and** ϑ*_interval*: $\vartheta\ n \in \{\Theta{-}pi{<}..\Theta{+}pi\}$ **for** n
unfolding ϑ*_def*
using *theI'* [*OF uniq*] **by** *metis+*
have ϑ*_pos*: $\bigwedge j.\ \vartheta\ j > 0$
using ϑ*_interval* ‹$\Theta > pi$› **by** *simp* (*meson diff_gt_0_iff_gt less_trans*)
have $(\prod j{\le}n.\ z\ j) = cmod\ (\prod j{\le}n.\ z\ j) * exp\ (\mathrm{i} * \vartheta\ n)$ **for** n
using ϑ **by** (*auto simp*: *is_Arg_def*)
then have *eq*: $(\lambda n.\ \prod j{\le}n.\ z\ j) = (\lambda n.\ cmod\ (\prod j{\le}n.\ z\ j) * exp\ (\mathrm{i} * \vartheta\ n))$

```
  by simp
 then have (λn. (cmod (∏j≤n. z j)) * exp (i * (ϑ n))) ⟶ ξ
  using L by force
 then obtain k where k: (λj. ϑ j − of_int (k j) * (2 * pi)) ⟶ Θ
  using L by (subst (asm) ξ_eq) (auto simp add: eq z ξ polar_convergence)
 moreover have ∀F n in sequentially. k n = 0
 proof −
  have ∗: kj = 0 if dist (vj − real_of_int kj * 2) V < 1 vj ∈ {V − 1<..V +
1} for kj vj V
   using that by (auto simp: dist_norm)
  have ∀F j in sequentially. dist (ϑ j − of_int (k j) * (2 * pi)) Θ < pi
   using tendstoD [OF k] pi_gt_zero by blast
  then show ?thesis
  proof (rule eventually_mono)
   fix j
   assume d: dist (ϑ j − real_of_int (k j) * (2 * pi)) Θ < pi
   show k j = 0
    by (rule * [of ϑ j/pi _ Θ/pi])
      (use ϑ_interval [of j] d in ‹simp_all add: divide_simps dist_norm›)
  qed
 qed
 ultimately have ϑtoΘ: ϑ ⟶ Θ
  apply (simp only: tendsto_def)
  apply (erule all_forward imp_forward asm_rl)+
  apply (drule (1) eventually_conj)
  apply (auto elim: eventually_mono)
  done
 then have to0: (λn. |ϑ (Suc n) − ϑ n|) ⟶ 0
   by (metis (full_types) diff_self filterlim_sequentially_Suc tendsto_diff tend-
sto_rabs_zero)
 have ∃k. Im (∑j≤n. Ln (z j)) − of_int k * (2∗pi) = ϑ n for n
 proof (rule is_Arg_exp_diff_2pi)
  show is_Arg (exp (∑j≤n. Ln (z j))) (ϑ n)
   using pnz ϑ by (simp add: is_Arg_def exp_sum prod_norm)
 qed
 then have ∃k. (∑j≤n. Im (Ln (z j))) = ϑ n + of_int k * (2∗pi) for n
  by (simp add: algebra_simps)
 then obtain k where k: ⋀n. (∑j≤n. Im (Ln (z j))) = ϑ n + of_int (k n) *
(2∗pi)
  by metis
 obtain K where ∀F n in sequentially. k n = K
 proof −
  have k_le: (2∗pi) * |k (Suc n) − k n| ≤ |ϑ (Suc n) − ϑ n| + |Im (Ln (z (Suc
n)))| for n
  proof −
   have (∑j≤Suc n. Im (Ln (z j))) − (∑j≤n. Im (Ln (z j))) = Im (Ln (z
(Suc n)))
    by simp
   then show ?thesis
```

using k [*of* *Suc* n] k [*of* n] **by** (*auto simp*: *abs_if algebra_simps*)
qed
have $z \longrightarrow 1$
using $L\ \xi$ *convergent_prod_iff_nz_lim* z **by** (*blast intro*: *convergent_prod_imp_LIMSEQ*)
with z **have** $(\lambda n.\ Ln\ (z\ n)) \longrightarrow Ln\ 1$
using *isCont_tendsto_compose* [*OF continuous_at_Ln*] *nonpos_Reals_one_I*
by *blast*
then have $(\lambda n.\ Ln\ (z\ n)) \longrightarrow 0$
by *simp*
then have $(\lambda n.\ |Im\ (Ln\ (z\ (Suc\ n)))|) \longrightarrow 0$
by (*metis LIMSEQ_unique* ‹$z \longrightarrow 1$› *continuous_at_Ln filterlim_sequentially_Suc isCont_tendsto_compose nonpos_Reals_one_I tendsto_Im tendsto_rabs_zero_iff zero_complex.simps*(*2*))
then have $\forall_F\ n$ *in sequentially*. $|Im\ (Ln\ (z\ (Suc\ n)))| < 1$
by (*simp add*: *order_tendsto_iff*)
moreover have $\forall_F\ n$ *in sequentially*. $|\vartheta\ (Suc\ n) - \vartheta\ n| < 1$
using *to0* **by** (*simp add*: *order_tendsto_iff*)
ultimately have $\forall_F\ n$ *in sequentially*. $(2{*}pi) * |k\ (Suc\ n) - k\ n| < 1 + 1$
proof (*rule eventually_elim2*)
fix n
assume $|Im\ (Ln\ (z\ (Suc\ n)))| < 1$ **and** $|\vartheta\ (Suc\ n) - \vartheta\ n| < 1$
with *k_le* [*of* n] **show** $2 * pi * real\_of\_int\ |k\ (Suc\ n) - k\ n| < 1 + 1$
by *linarith*
qed
then have $\forall_F\ n$ *in sequentially*. $real\_of\_int|k\ (Suc\ n) - k\ n| < 1$
proof (*rule eventually_mono*)
fix $n :: nat$
assume $2 * pi * |k\ (Suc\ n) - k\ n| < 1 + 1$
then have $|k\ (Suc\ n) - k\ n| < 2\ /\ (2{*}pi)$
by (*simp add*: *field_simps*)
also have $... < 1$
using *pi_ge_two* **by** *auto*
finally show $real\_of\_int\ |k\ (Suc\ n) - k\ n| < 1$ **.**
qed
then obtain N **where** N: $\bigwedge n.\ n{\geq}N \Longrightarrow |k\ (Suc\ n) - k\ n| = 0$
using *eventually_sequentially less_irrefl of_int_abs* **by** *fastforce*
have $k\ (N{+}i) = k\ N$ **for** i
proof (*induction i*)
case (*Suc i*)
with N [*of* $N{+}i$] **show** *?case*
by *auto*
qed *simp*
then have $\bigwedge n.\ n{\geq}N \Longrightarrow k\ n = k\ N$
using *le_Suc_ex* **by** *auto*
then show *?thesis*
by (*force simp add*: *eventually_sequentially intro*: *that*)
qed
with *ϑtoΘ* **have** $(\lambda n.\ (\sum j{\leq}n.\ Im\ (Ln\ (z\ j)))) \longrightarrow \Theta + of\_int\ K * (2{*}pi)$
by (*simp add*: *k tendsto_add tendsto_mult tendsto_eventually*)

```
  moreover have (λn. (∑k≤n. Re (Ln (z k)))) ⟶ Re (Ln ξ)
    using assms continuous_imp_tendsto [OF isCont_ln tendsto_norm [OF L]]
    by (simp add: o_def flip: prod_norm ln_prod)
  ultimately show ?rhs
   by (rule_tac x=K+1 in exI) (auto simp: tendsto_complex_iff Θ_def Arg_def
assms algebra_simps)
next
  assume ?rhs
  then obtain r where r: (λn. (∑k≤n. Ln (z k))) ⟶ Ln ξ + of_int r *
(of_real(2*pi) * i) ..
  have (λn. exp (∑k≤n. Ln (z k))) ⟶ ξ
    using assms continuous_imp_tendsto [OF isCont_exp r] exp_integer_2pi [of
r]
    by (simp add: o_def exp_add algebra_simps)
  moreover have exp (∑k≤n. Ln (z k)) = (∏k≤n. z k) for n
    by (simp add: exp_sum add_eq_0_iff assms)
  ultimately show ?lhs
    by auto
qed
```

Prop 17.2 of Bak and Newman, Complex Analysis, p.242

```
proposition convergent_prod_iff_summable_complex:
  fixes z :: nat ⇒ complex
  assumes ⋀k. z k ≠ 0
  shows convergent_prod (λk. z k) ⟷ summable (λk. Ln (z k)) (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain p where p: (λn. ∏k≤n. z k) ⟶ p and p ≠ 0
     using convergent_prod_LIMSEQ prodinf_nonzero add_eq_0_iff assms by
fastforce
  then show ?rhs
    using Ln_prodinf_complex assms
    by (auto simp: prodinf_nonzero summable_def sums_def_le)
next
  assume R: ?rhs
  have (∏k≤n. z k) = exp (∑k≤n. Ln (z k)) for n
    by (simp add: exp_sum add_eq_0_iff assms)
  then have (λn. ∏k≤n. z k) ⟶ exp (suminf (λk. Ln (z k)))
    using continuous_imp_tendsto [OF isCont_exp summable_LIMSEQ' [OF R]]
by (simp add: o_def)
  then show ?lhs
    by (subst convergent_prod_iff_convergent) (auto simp: convergent_def tend-
sto_Lim assms add_eq_0_iff)
qed
```

Prop 17.3 of Bak and Newman, Complex Analysis

```
proposition summable_imp_convergent_prod_complex:
  fixes z :: nat ⇒ complex
  assumes z: summable (λk. norm (z k)) and non0: ⋀k. z k ≠ −1
```

shows *convergent_prod* (λ*k*. *1* + *z k*)
proof −
obtain *N* **where** ⋀*k*. *k*≥*N* ⟹ *norm* (*z k*) < *1*/*2*
using *summable_LIMSEQ_zero* [*OF z*]
by (*metis diff_zero dist_norm half_gt_zero_iff less_numeral_extra*(*1*) *lim_sequentially tendsto_norm_zero_iff*)
then have *summable* (λ*k*. *Ln* (*1* + *z k*))
by (*metis norm_Ln_le summable_comparison_test summable_mult z*)
with *non0* **show** *?thesis*
by (*simp add*: *add_eq_0_iff convergent_prod_iff_summable_complex*)
qed

corollary *summable_imp_convergent_prod_real*:
fixes *z* :: *nat* ⇒ *real*
assumes *z*: *summable* (λ*k*. |*z k*|) **and** *non0*: ⋀*k*. *z k* ≠ −*1*
shows *convergent_prod* (λ*k*. *1* + *z k*)
proof −
have ⋀*k*. (*complex_of_real* ∘ *z*) *k* ≠ − *1*
by (*metis non0 o_apply of_real_1 of_real_eq_iff of_real_minus*)
with *z*
have *convergent_prod* (λ*k*. *1* + (*complex_of_real* ∘ *z*) *k*)
by (*auto intro*: *summable_imp_convergent_prod_complex*)
then show *?thesis*
using *convergent_prod_of_real_iff* [*of* λ*k*. *1* + *z k*] **by** (*simp add*: *o_def*)
qed

lemma *summable_Ln_complex*:
fixes *z* :: *nat* ⇒ *complex*
assumes *convergent_prod z* ⋀*k*. *z k* ≠ *0*
shows *summable* (λ*k*. *Ln* (*z k*))
using *convergent_prod_def assms convergent_prod_iff_summable_complex* **by** *blast*

9.22.10 Embeddings from the reals into some complete real normed field

lemma *tendsto_eq_of_real_lim*:
assumes (λ*n*. *of_real* (*f n*) :: *'a*::{*complete_space*,*real_normed_field*}) ⟶ *q*
shows *q* = *of_real* (*lim f*)
proof −
have *convergent* (λ*n*. *of_real* (*f n*) :: *'a*)
using *assms convergent_def* **by** *blast*
then have *convergent f*
unfolding *convergent_def*
by (*simp add*: *convergent_eq_Cauchy Cauchy_def*)
then show *?thesis*
by (*metis LIMSEQ_unique assms convergentD sequentially_bot tendsto_Lim tendsto_of_real*)
qed

lemma *tendsto_eq_of_real*:
assumes $(\lambda n.\ of\_real\ (f\ n) :: 'a::\{complete\_space,real\_normed\_field\}) \longrightarrow q$
obtains r **where** $q = of\_real\ r$
using *tendsto_eq_of_real_lim assms* **by** *blast*

lemma *has_prod_of_real_iff* [*simp*]:
$(\lambda n.\ of\_real\ (f\ n) :: 'a::\{complete\_space,real\_normed\_field\})$ *has_prod of_real* $c \longleftrightarrow f$ *has_prod* c
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
apply (*auto simp*: *prod_defs LIMSEQ_prod_0 tendsto_of_real_iff simp flip*: *of_real_prod*)
using *tendsto_eq_of_real*
by (*metis of_real_0 tendsto_of_real_iff*)
next
assume *?rhs*
with *tendsto_of_real_iff* **show** *?lhs*
by (*fastforce simp*: *prod_defs simp flip*: *of_real_prod*)
qed

end

9.23 Sums over Infinite Sets

theory *Infinite_Set_Sum*
imports *Set_Integral Infinite_Sum*
begin

Conflicting notation from *HOL−Analysis.Infinite_Sum*

no_notation *Infinite_Sum.abs_summable_on* (**infixr** ‹*abs'_summable'_on*› *46*)

lemma *sets_eq_countable*:
assumes *countable A space M* = *A* $\bigwedge x.\ x \in A \Longrightarrow \{x\} \in sets\ M$
shows *sets M* = *Pow A*
proof (*intro equalityI subsetI*)
fix *X* **assume** $X \in Pow\ A$
hence $(\bigcup x{\in}X.\ \{x\}) \in sets\ M$
by (*intro sets.countable_UN' countable_subset*[*OF _ assms*(*1*)]) (*auto intro*!: *assms*(*3*))
also have $(\bigcup x{\in}X.\ \{x\}) = X$ **by** *auto*
finally show $X \in sets\ M$ **.**
next
fix *X* **assume** $X \in sets\ M$
from *sets.sets_into_space*[*OF this*] **and** *assms*
show $X \in Pow\ A$ **by** *simp*

qed

lemma *measure_eqI_countable'*:
assumes *spaces*: *space* $M = A$ *space* $N = A$
assumes *sets*: $\bigwedge x.\ x \in A \Longrightarrow \{x\} \in$ *sets* M $\bigwedge x.\ x \in A \Longrightarrow \{x\} \in$ *sets* N
assumes A: *countable* A
assumes *eq*: $\bigwedge a.\ a \in A \Longrightarrow$ *emeasure* $M\ \{a\} =$ *emeasure* $N\ \{a\}$
shows $M = N$
proof (*rule measure_eqI_countable*)
show *sets* $M = Pow\ A$
by (*intro sets_eq_countable assms*)
show *sets* $N = Pow\ A$
by (*intro sets_eq_countable assms*)
qed *fact+*

lemma *count_space_PiM_finite*:
fixes $B :: 'a \Rightarrow 'b$ *set*
assumes *finite* A $\bigwedge i.$ *countable* $(B\ i)$
shows *PiM* A $(\lambda i.$ *count_space* $(B\ i)) =$ *count_space* $(PiE\ A\ B)$
proof (*rule measure_eqI_countable'*)
show *space* $(PiM\ A\ (\lambda i.$ *count_space* $(B\ i))) = PiE\ A\ B$
by (*simp add*: *space_PiM*)
show *space* $($*count_space* $(PiE\ A\ B)) = PiE\ A\ B$ **by** *simp*
next
fix f **assume** f: $f \in PiE\ A\ B$
hence $PiE\ A\ (\lambda x.\ \{f\ x\}) \in$ *sets* $(Pi_M\ A\ (\lambda i.$ *count_space* $(B\ i)))$
by (*intro sets_PiM_I_finite assms*) *auto*
also from f **have** $PiE\ A\ (\lambda x.\ \{f\ x\}) = \{f\}$
by (*intro PiE_singleton*) (*auto simp*: *PiE_def*)
finally show $\{f\} \in$ *sets* $(Pi_M\ A\ (\lambda i.$ *count_space* $(B\ i)))$ **.**
next
interpret *product_sigma_finite* $(\lambda i.$ *count_space* $(B\ i))$
by (*intro product_sigma_finite.intro sigma_finite_measure_count_space_countable assms*)
thm *sigma_finite_measure_count_space*
fix f **assume** f: $f \in PiE\ A\ B$
hence $\{f\} = PiE\ A\ (\lambda x.\ \{f\ x\})$
by (*intro PiE_singleton* [*symmetric*]) (*auto simp*: *PiE_def*)
also have *emeasure* $(Pi_M\ A\ (\lambda i.$ *count_space* $(B\ i))) \ldots =$
$(\prod i \in A.$ *emeasure* $($*count_space* $(B\ i))\ \{f\ i\})$
using f *assms* **by** (*subst emeasure_PiM*) *auto*
also have $\ldots = (\prod i \in A.\ 1)$
by (*intro prod.cong refl*, *subst emeasure_count_space_finite*) (*use* f **in** *auto*)
also have $\ldots =$ *emeasure* $($*count_space* $(PiE\ A\ B))\ \{f\}$
using f **by** (*subst emeasure_count_space_finite*) *auto*
finally show *emeasure* $(Pi_M\ A\ (\lambda i.$ *count_space* $(B\ i)))\ \{f\} =$
emeasure $($*count_space* $(Pi_E\ A\ B))\ \{f\}$ **.**
qed (*simp_all add*: *countable_PiE assms*)

definition *abs_summable_on* ::
(*'a* ⇒ *'b* :: {*banach*, *second_countable_topology*}) ⇒ *'a set* ⇒ *bool*
(**infix** ‹*abs'_summable'_on*› *50*)
where
f abs_summable_on A ⟷ *integrable* (*count_space A*) *f*

definition *infsetsum* ::
(*'a* ⇒ *'b* :: {*banach*, *second_countable_topology*}) ⇒ *'a set* ⇒ *'b*
where
infsetsum f A = *lebesgue_integral* (*count_space A*) *f*

syntax (*ASCII*)
_infsetsum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b*::{*banach*, *second_countable_topology*}
(‹(‹*indent=3 notation=‹binder INFSETSUM››INFSETSUM _:_./ _*)› [*0*, *51*, *10*] *10*)
syntax
_infsetsum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b*::{*banach*, *second_countable_topology*}
(‹(‹*indent=2 notation=‹binder* ∑ *a*›› ∑ *a*_∈_./ _*)› [*0*, *51*, *10*] *10*)
syntax_consts
_infsetsum ⇌ *infsetsum*
translations — Beware of argument permutation!
∑ *a i*∈*A*. *b* ⇌ *CONST infsetsum* (λ*i*. *b*) *A*

syntax (*ASCII*)
_uinfsetsum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b*::{*banach*, *second_countable_topology*}
(‹(‹*indent=3 notation=‹binder INFSETSUM››INFSETSUM _:_./ _*)› [*0*, *51*, *10*] *10*)
syntax
_uinfsetsum :: *pttrn* ⇒ *'b* ⇒ *'b*::{*banach*, *second_countable_topology*}
(‹(‹*indent=2 notation=‹binder* ∑ *a*›› ∑ *a*_./ _*)› [*0*, *10*] *10*)
syntax_consts
_uinfsetsum ⇌ *infsetsum*
translations — Beware of argument permutation!
∑ *a i*. *b* ⇌ *CONST infsetsum* (λ*i*. *b*) (*CONST UNIV*)

syntax (*ASCII*)
_qinfsetsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a*::{*banach*, *second_countable_topology*}
(‹(‹*indent=3 notation=‹binder INFSETSUM››INFSETSUM _ |/ _./ _*)› [*0*, *0*, *10*] *10*)
syntax
_qinfsetsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a*::{*banach*, *second_countable_topology*}
(‹(‹*indent=2 notation=‹binder* ∑ *a*›› ∑ *a*_ | (_)./ _*)› [*0*, *0*, *10*] *10*)
syntax_consts
_qinfsetsum ⇌ *infsetsum*
translations
∑ *a x*|*P*. *t* => *CONST infsetsum* (λ*x*. *t*) {*x*. *P*}

print_translation ‹
[(***const_syntax*** ‹*infsetsum*›, *K* (*Collect_binder_tr'* ***syntax_const*** ‹*_qinfsetsum*›))]
›

lemma *restrict_count_space_subset*:
$A \subseteq B \Longrightarrow$ *restrict_space* (*count_space B*) A = *count_space A*
by (*subst restrict_count_space*) (*simp_all add*: *Int_absorb2*)

lemma *abs_summable_on_restrict*:
fixes $f :: 'a \Rightarrow 'b :: \{banach, second\_countable\_topology\}$
assumes $A \subseteq B$
shows *f abs_summable_on A* $\longleftrightarrow$ $(\lambda x.$ *indicator A x* $*_R$ *f x*) *abs_summable_on B*
proof −
have *count_space A* = *restrict_space* (*count_space B*) *A*
by (*rule restrict_count_space_subset* [*symmetric*]) *fact+*
also have *integrable* ... *f* $\longleftrightarrow$ *set_integrable* (*count_space B*) *A f*
by (*simp add*: *integrable_restrict_space set_integrable_def*)
finally show *?thesis*
unfolding *abs_summable_on_def set_integrable_def* **.**
qed

lemma *abs_summable_on_altdef*: *f abs_summable_on A* $\longleftrightarrow$ *set_integrable* (*count_space UNIV*) *A f*
unfolding *abs_summable_on_def set_integrable_def*
by (*metis* (*no_types*) *inf_top.right_neutral integrable_restrict_space restrict_count_space sets_UNIV*)

lemma *abs_summable_on_altdef'*:
$A \subseteq B \Longrightarrow$ *f abs_summable_on A* $\longleftrightarrow$ *set_integrable* (*count_space B*) *A f*
unfolding *abs_summable_on_def set_integrable_def*
by (*metis* (*no_types*) *Pow_iff abs_summable_on_def inf.orderE integrable_restrict_space restrict_count_space_subset sets_count_space space_count_space*)

lemma *abs_summable_on_norm_iff* [*simp*]:
$(\lambda x.$ *norm* (*f x*)) *abs_summable_on A* $\longleftrightarrow$ *f abs_summable_on A*
by (*simp add*: *abs_summable_on_def integrable_norm_iff*)

lemma *abs_summable_on_normI*: *f abs_summable_on A* $\Longrightarrow$ $(\lambda x.$ *norm* (*f x*)) *abs_summable_on A*
by *simp*

lemma *abs_summable_complex_of_real* [*simp*]: $(\lambda n.$ *complex_of_real* (*f n*)) *abs_summable_on A* $\longleftrightarrow$ *f abs_summable_on A*
by (*simp add*: *abs_summable_on_def complex_of_real_integrable_eq*)

lemma *abs_summable_on_comparison_test*:
assumes *g abs_summable_on A*
assumes $\bigwedge x.\ x \in A \Longrightarrow$ *norm* (*f x*) $\leq$ *norm* (*g x*)

shows *f abs_summable_on A*
using *assms Bochner_Integration.integrable_bound*[*of count_space A g f*]
unfolding *abs_summable_on_def* **by** (*auto simp*: *AE_count_space*)

lemma *abs_summable_on_comparison_test'*:
assumes *g abs_summable_on A*
assumes $\bigwedge x.\ x \in A \Longrightarrow \mathit{norm}\ (f\ x) \leq g\ x$
shows *f abs_summable_on A*
proof (*rule abs_summable_on_comparison_test*[*OF assms*(*1*), *of f*])
fix *x* **assume** $x \in A$
with *assms*(*2*) **have** $\mathit{norm}\ (f\ x) \leq g\ x$ **.**
also have $\ldots \leq \mathit{norm}\ (g\ x)$ **by** *simp*
finally show $\mathit{norm}\ (f\ x) \leq \mathit{norm}\ (g\ x)$ **.**
qed

lemma *abs_summable_on_cong* [*cong*]:
$(\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x) \Longrightarrow A = B \Longrightarrow (f$ *abs_summable_on* $A) \longleftrightarrow (g$ *abs_summable_on* $B)$
unfolding *abs_summable_on_def* **by** (*intro integrable_cong*) *auto*

lemma *abs_summable_on_cong_neutral*:
assumes $\bigwedge x.\ x \in A - B \Longrightarrow f\ x = 0$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow g\ x = 0$
assumes $\bigwedge x.\ x \in A \cap B \Longrightarrow f\ x = g\ x$
shows *f abs_summable_on A* $\longleftrightarrow$ *g abs_summable_on B*
unfolding *abs_summable_on_altdef set_integrable_def* **using** *assms*
by (*intro Bochner_Integration.integrable_cong refl*)
(*auto simp*: *indicator_def split*: *if_splits*)

lemma *abs_summable_on_restrict'*:
fixes $f :: 'a \Rightarrow 'b :: \{\mathit{banach}, \mathit{second\_countable\_topology}\}$
assumes $A \subseteq B$
shows *f abs_summable_on A* $\longleftrightarrow$ $(\lambda x.\ \mathit{if}\ x \in A\ \mathit{then}\ f\ x\ \mathit{else}\ 0)$ *abs_summable_on B*
by (*subst abs_summable_on_restrict*[*OF assms*]) (*intro abs_summable_on_cong*, *auto*)

lemma *abs_summable_on_nat_iff*:
f abs_summable_on $(A :: \mathit{nat\ set}) \longleftrightarrow \mathit{summable}\ (\lambda n.\ \mathit{if}\ n \in A\ \mathit{then}\ \mathit{norm}\ (f\ n)\ \mathit{else}\ 0)$
proof −
have f *abs_summable_on* $A \longleftrightarrow \mathit{summable}\ (\lambda x.\ \mathit{norm}\ (\mathit{if}\ x \in A\ \mathit{then}\ f\ x\ \mathit{else}\ 0))$
by (*subst abs_summable_on_restrict'*[*of _ UNIV*])
(*simp_all add*: *abs_summable_on_def integrable_count_space_nat_iff*)
also have $(\lambda x.\ \mathit{norm}\ (\mathit{if}\ x \in A\ \mathit{then}\ f\ x\ \mathit{else}\ 0)) = (\lambda x.\ \mathit{if}\ x \in A\ \mathit{then}\ \mathit{norm}\ (f\ x)\ \mathit{else}\ 0)$
by *auto*
finally show *?thesis* **.**

qed

lemma *abs_summable_on_nat_iff'*:
f abs_summable_on (*UNIV* :: *nat set*) ⟷ *summable* (λ*n*. *norm* (*f n*))
by (*subst abs_summable_on_nat_iff*) *auto*

lemma *nat_abs_summable_on_comparison_test*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *second_countable_topology*}
assumes *g abs_summable_on I*
assumes ⋀*n*. ⟦*n*≥*N*; *n* ∈ *I*⟧ ⟹ *norm* (*f n*) ≤ *g n*
shows *f abs_summable_on I*
using *assms* **by** (*fastforce simp add*: *abs_summable_on_nat_iff intro*: *summable_comparison_test'*)

lemma *abs_summable_comparison_test_ev*:
assumes *g abs_summable_on I*
assumes *eventually* (λ*x*. *x* ∈ *I* ⟶ *norm* (*f x*) ≤ *g x*) *sequentially*
shows *f abs_summable_on I*
by (*metis* (*no_types*, *lifting*) *nat_abs_summable_on_comparison_test eventually_at_top_linorder assms*)

lemma *abs_summable_on_Cauchy*:
f abs_summable_on (*UNIV* :: *nat set*) ⟷ (∀ *e*>0. ∃ *N*. ∀ *m*≥*N*. ∀ *n*. (∑ *x* = *m*..<*n*. *norm* (*f x*)) < *e*)
by (*simp add*: *abs_summable_on_nat_iff' summable_Cauchy sum_nonneg*)

lemma *abs_summable_on_finite* [*simp*]: *finite A* ⟹ *f abs_summable_on A*
unfolding *abs_summable_on_def* **by** (*rule integrable_count_space*)

lemma *abs_summable_on_empty* [*simp*, *intro*]: *f abs_summable_on* {}
by *simp*

lemma *abs_summable_on_subset*:
assumes *f abs_summable_on B* **and** *A* ⊆ *B*
shows *f abs_summable_on A*
unfolding *abs_summable_on_altdef*
by (*rule set_integrable_subset*) (*insert assms*, *auto simp*: *abs_summable_on_altdef*)

lemma *abs_summable_on_union* [*intro*]:
assumes *f abs_summable_on A* **and** *f abs_summable_on B*
shows *f abs_summable_on* (*A* ∪ *B*)
using *assms* **unfolding** *abs_summable_on_altdef* **by** (*intro set_integrable_Un*) *auto*

lemma *abs_summable_on_insert_iff* [*simp*]:
f abs_summable_on insert x A ⟷ *f abs_summable_on A*
proof *safe*
assume *f abs_summable_on insert x A*
thus *f abs_summable_on A*
by (*rule abs_summable_on_subset*) *auto*

next
assume *f abs_summable_on A*
from *abs_summable_on_union*[*OF this*, *of* {*x*}]
show *f abs_summable_on insert x A* **by** *simp*
qed

lemma *abs_summable_sum*:
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x$ *abs_summable_on B*
shows $(\lambda y.\ \sum x{\in}A.\ f\ x\ y)$ *abs_summable_on B*
using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_sum*)

lemma *abs_summable_Re*: *f abs_summable_on A* $\Longrightarrow$ $(\lambda x.\ Re\ (f\,x))$ *abs_summable_on A*
by (*simp add*: *abs_summable_on_def*)

lemma *abs_summable_Im*: *f abs_summable_on A* $\Longrightarrow$ $(\lambda x.\ Im\ (f\,x))$ *abs_summable_on A*
by (*simp add*: *abs_summable_on_def*)

lemma *abs_summable_on_finite_diff*:
assumes *f abs_summable_on A A* $\subseteq$ *B finite* $(B - A)$
shows *f abs_summable_on B*
proof −
have *f abs_summable_on* $(A \cup (B - A))$
by (*intro abs_summable_on_union assms abs_summable_on_finite*)
also from *assms* **have** $A \cup (B - A) = B$ **by** *blast*
finally show *?thesis* **.**
qed

lemma *abs_summable_on_reindex_bij_betw*:
assumes *bij_betw g A B*
shows $(\lambda x.\ f\ (g\ x))$ *abs_summable_on A* $\longleftrightarrow$ *f abs_summable_on B*
proof −
have $*$: *count_space B* = *distr* (*count_space A*) (*count_space B*) *g*
by (*rule distr_bij_count_space* [*symmetric*]) *fact*
show *?thesis* **unfolding** *abs_summable_on_def*
by (*subst* $*$, *subst integrable_distr_eq*[*of* _ _ *count_space B*])
(*insert assms*, *auto simp*: *bij_betw_def*)
qed

lemma *abs_summable_on_reindex*:
assumes $(\lambda x.\ f\ (g\ x))$ *abs_summable_on A*
shows *f abs_summable_on* $(g\ `\ A)$
proof −
define g' **where** $g' = inv\_into\ A\ g$
from *assms* **have** $(\lambda x.\ f\ (g\ x))$ *abs_summable_on* $(g'\ `\ g\ `\ A)$
by (*rule abs_summable_on_subset*) (*auto simp*: g'*_def inv_into_into*)
also have *?this* $\longleftrightarrow$ $(\lambda x.\ f\ (g\ (g'\ x)))$ *abs_summable_on* $(g\ `\ A)$ **unfolding** g'*_def*

by (*intro abs_summable_on_reindex_bij_betw* [*symmetric*] *inj_on_imp_bij_betw inj_on_inv_into*) *auto*
also have ... $\longleftrightarrow$ *f abs_summable_on* (*g* ‘ *A*)
by (*intro abs_summable_on_cong refl*) (*auto simp*: *g'_def f_inv_into_f*)
finally show *?thesis* **.**
qed

lemma *abs_summable_on_reindex_iff*:
inj_on g A $\Longrightarrow$ (λx. *f* (*g x*)) *abs_summable_on A* $\longleftrightarrow$ *f abs_summable_on* (*g* ‘ *A*)
by (*intro abs_summable_on_reindex_bij_betw inj_on_imp_bij_betw*)

lemma *abs_summable_on_Sigma_project2*:
fixes *A* :: *'a set* **and** *B* :: *'a* $\Rightarrow$ *'b set*
assumes *f abs_summable_on* (*Sigma A B*) *x* $\in$ *A*
shows (λy. *f* (*x*, *y*)) *abs_summable_on* (*B x*)
proof −
from *assms*(*2*) **have** *f abs_summable_on* (*Sigma* {*x*} *B*)
by (*intro abs_summable_on_subset* [*OF assms*(*1*)]) *auto*
also have *?this* $\longleftrightarrow$ (λz. *f* (*x*, *snd z*)) *abs_summable_on* (*Sigma* {*x*} *B*)
by (*rule abs_summable_on_cong*) *auto*
finally have (λy. *f* (*x*, *y*)) *abs_summable_on* (*snd* ‘ *Sigma* {*x*} *B*)
by (*rule abs_summable_on_reindex*)
also have *snd* ‘ *Sigma* {*x*} *B* = *B x*
using *assms* **by** (*auto simp*: *image_iff*)
finally show *?thesis* **.**
qed

lemma *abs_summable_on_Times_swap*:
f abs_summable_on A × *B* $\longleftrightarrow$ (λ(*x*,*y*). *f* (*y*,*x*)) *abs_summable_on B* × *A*
proof −
have *bij*: *bij_betw* (λ(*x*,*y*). (*y*,*x*)) (*B* × *A*) (*A* × *B*)
by (*auto simp*: *bij_betw_def inj_on_def*)
show *?thesis*
by (*subst abs_summable_on_reindex_bij_betw*[*OF bij*, *of f*, *symmetric*])
(*simp_all add*: *case_prod_unfold*)
qed

lemma *abs_summable_on_0* [*simp*, *intro*]: (λ_. *0*) *abs_summable_on A*
by (*simp add*: *abs_summable_on_def*)

lemma *abs_summable_on_uminus* [*intro*]:
f abs_summable_on A $\Longrightarrow$ (λx. −*f x*) *abs_summable_on A*
unfolding *abs_summable_on_def* **by** (*rule Bochner_Integration.integrable_minus*)

lemma *abs_summable_on_add* [*intro*]:
assumes *f abs_summable_on A* **and** *g abs_summable_on A*
shows (λx. *f x* + *g x*) *abs_summable_on A*
using *assms* **unfolding** *abs_summable_on_def* **by** (*rule Bochner_Integration.integrable_add*)

lemma *abs_summable_on_diff* [*intro*]:
assumes *f abs_summable_on A* **and** *g abs_summable_on A*
shows $(\lambda x.\ f\ x - g\ x)$ *abs_summable_on A*
using *assms* **unfolding** *abs_summable_on_def* **by** (*rule Bochner_Integration.integrable_diff*)

lemma *abs_summable_on_scaleR_left* [*intro*]:
assumes $c \neq 0 \Longrightarrow$ *f abs_summable_on A*
shows $(\lambda x.\ f\ x *_R c)$ *abs_summable_on A*
using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_scaleR_left*)

lemma *abs_summable_on_scaleR_right* [*intro*]:
assumes $c \neq 0 \Longrightarrow$ *f abs_summable_on A*
shows $(\lambda x.\ c *_R f\ x)$ *abs_summable_on A*
using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_scaleR_righ*

lemma *abs_summable_on_cmult_right* [*intro*]:
fixes $f :: 'a \Rightarrow 'b :: \{$*banach*, *real_normed_algebra*, *second_countable_topology*$\}$
assumes $c \neq 0 \Longrightarrow$ *f abs_summable_on A*
shows $(\lambda x.\ c * f\ x)$ *abs_summable_on A*
using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_mult_right*)

lemma *abs_summable_on_cmult_left* [*intro*]:
fixes $f :: 'a \Rightarrow 'b :: \{$*banach*, *real_normed_algebra*, *second_countable_topology*$\}$
assumes $c \neq 0 \Longrightarrow$ *f abs_summable_on A*
shows $(\lambda x.\ f\ x * c)$ *abs_summable_on A*
using *assms* **unfolding** *abs_summable_on_def* **by** (*intro Bochner_Integration.integrable_mult_left*)

lemma *abs_summable_on_prod_PiE*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \{$*real_normed_field*,*banach*,*second_countable_topology*$\}$
assumes *finite*: *finite A* **and** *countable*: $\bigwedge x.\ x \in A \Longrightarrow$ *countable* $(B\ x)$
assumes *summable*: $\bigwedge x.\ x \in A \Longrightarrow f\ x$ *abs_summable_on B x*
shows $(\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x))$ *abs_summable_on PiE A B*
proof −
define B' **where** $B' = (\lambda x.$ *if* $x \in A$ *then B x else* $\{\})$
from *assms* **have** [*simp*]: *countable* $(B'\ x)$ **for** *x*
by (*auto simp*: *B'_def*)
then interpret *product_sigma_finite count_space* $\circ$ B'
unfolding *o_def* **by** (*intro product_sigma_finite.intro sigma_finite_measure_count_space_countable*
from *assms* **have** *integrable* (*PiM A* (*count_space* $\circ$ B')) $(\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x))$
by (*intro product_integrable_prod*) (*auto simp*: *abs_summable_on_def B'_def*)
also have *PiM A* (*count_space* $\circ$ B') = *count_space* (*PiE A B'*)
unfolding *o_def* **using** *finite* **by** (*intro count_space_PiM_finite*) *simp_all*
also have *PiE A B'* = *PiE A B* **by** (*intro PiE_cong*) (*simp_all add*: *B'_def*)
finally show *?thesis* **by** (*simp add*: *abs_summable_on_def*)
qed

lemma *not_summable_infsetsum_eq*:
 $\neg f$ *abs_summable_on* $A \Longrightarrow$ *infsetsum* $f\ A = 0$
 by (*simp add*: *abs_summable_on_def infsetsum_def not_integrable_integral_eq*)

lemma *infsetsum_altdef*:
 infsetsum $f\ A =$ *set_lebesgue_integral* (*count_space UNIV*) $A\ f$
 unfolding *set_lebesgue_integral_def*
 by (*subst integral_restrict_space* [*symmetric*])
 (*auto simp*: *restrict_count_space_subset infsetsum_def*)

lemma *infsetsum_altdef'*:
 $A \subseteq B \Longrightarrow$ *infsetsum* $f\ A =$ *set_lebesgue_integral* (*count_space* B) $A\ f$
 unfolding *set_lebesgue_integral_def*
 by (*subst integral_restrict_space* [*symmetric*])
 (*auto simp*: *restrict_count_space_subset infsetsum_def*)

lemma *nn_integral_conv_infsetsum*:
 assumes f *abs_summable_on* $A\ \bigwedge x.\ x \in A \Longrightarrow f\ x \geq 0$
 shows *nn_integral* (*count_space* A) $f =$ *ennreal* (*infsetsum* $f\ A$)
 using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
 by (*subst nn_integral_eq_integral*) *auto*

lemma *infsetsum_conv_nn_integral*:
 assumes *nn_integral* (*count_space* A) $f \neq \infty\ \bigwedge x.\ x \in A \Longrightarrow f\ x \geq 0$
 shows *infsetsum* $f\ A =$ *enn2real* (*nn_integral* (*count_space* A) f)
 unfolding *infsetsum_def* **using** *assms*
 by (*subst integral_eq_nn_integral*) *auto*

lemma *infsetsum_cong* [*cong*]:
 $(\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x) \Longrightarrow A = B \Longrightarrow$ *infsetsum* $f\ A =$ *infsetsum* $g\ B$
 unfolding *infsetsum_def* **by** (*intro Bochner_Integration.integral_cong*) *auto*

lemma *infsetsum_0* [*simp*]: *infsetsum* $(\lambda\_.\ 0)\ A = 0$
 by (*simp add*: *infsetsum_def*)

lemma *infsetsum_all_0*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x = 0) \Longrightarrow$ *infsetsum* $f\ A = 0$
 by *simp*

lemma *infsetsum_nonneg*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \geq (0::real)) \Longrightarrow$ *infsetsum* $f\ A \geq$
0
 unfolding *infsetsum_def* **by** (*rule Bochner_Integration.integral_nonneg*) *auto*

lemma *sum_infsetsum*:
 assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x$ *abs_summable_on* B
 shows $(\sum x{\in}A.\ \sum_a y{\in}B.\ f\ x\ y) = (\sum_a y{\in}B.\ \sum x{\in}A.\ f\ x\ y)$
 using *assms* **by** (*simp add*: *infsetsum_def abs_summable_on_def Bochner_Integration.integral_sum*)

lemma *Re_infsetsum*: f *abs_summable_on* $A \Longrightarrow$ *Re* (*infsetsum* $f\ A$) $= (\sum_a x{\in}A.$
Re $(f\ x))$

by (*simp add*: *infsetsum_def abs_summable_on_def*)

lemma *Im_infsetsum*: *f abs_summable_on A* $\Longrightarrow$ *Im* (*infsetsum f A*) = ($\sum_a x{\in}A$. *Im* (*f x*))
by (*simp add*: *infsetsum_def abs_summable_on_def*)

lemma *infsetsum_of_real*:
shows *infsetsum* (λ*x*. *of_real* (*f x*)
:: *'a* :: {*real_normed_algebra_1*,*banach*,*second_countable_topology*,*real_inner*}) *A* =
of_real (*infsetsum f A*)
unfolding *infsetsum_def*
by (*rule integral_bounded_linear'*[*OF bounded_linear_of_real bounded_linear_inner_left*[*of 1*]]) *auto*

lemma *infsetsum_finite* [*simp*]: *finite A* $\Longrightarrow$ *infsetsum f A* = ($\sum x{\in}A$. *f x*)
by (*simp add*: *infsetsum_def lebesgue_integral_count_space_finite*)

lemma *infsetsum_nat*:
assumes *f abs_summable_on A*
shows *infsetsum f A* = ($\sum$ *n*. *if n* ∈ *A then f n else 0*)
proof −
from *assms* **have** *infsetsum f A* = ($\sum$ *n*. *indicator A n* $*_R$ *f n*)
unfolding *infsetsum_altdef abs_summable_on_altdef set_lebesgue_integral_def set_integrable_def*
by (*subst integral_count_space_nat*) *auto*
also have (λ*n*. *indicator A n* $*_R$ *f n*) = (λ*n*. *if n* ∈ *A then f n else 0*)
by *auto*
finally show *?thesis* **.**
qed

lemma *infsetsum_nat'*:
assumes *f abs_summable_on UNIV*
shows *infsetsum f UNIV* = ($\sum$ *n*. *f n*)
using *assms* **by** (*subst infsetsum_nat*) *auto*

lemma *sums_infsetsum_nat*:
assumes *f abs_summable_on A*
shows (λ*n*. *if n* ∈ *A then f n else 0*) *sums infsetsum f A*
proof −
from *assms* **have** *summable* (λ*n*. *if n* ∈ *A then norm* (*f n*) *else 0*)
by (*simp add*: *abs_summable_on_nat_iff*)
also have (λ*n*. *if n* ∈ *A then norm* (*f n*) *else 0*) = (λ*n*. *norm* (*if n* ∈ *A then f n else 0*))
by *auto*
finally have *summable* (λ*n*. *if n* ∈ *A then f n else 0*)
by (*rule summable_norm_cancel*)
with *assms* **show** *?thesis*
by (*auto simp*: *sums_iff infsetsum_nat*)

qed

lemma *sums_infsetsum_nat'*:
assumes *f abs_summable_on UNIV*
shows *f sums infsetsum f UNIV*
using *sums_infsetsum_nat* [*OF assms*] **by** *simp*

lemma *infsetsum_Un_disjoint*:
assumes *f abs_summable_on A f abs_summable_on B A* ∩ *B* = {}
shows *infsetsum f* (*A* ∪ *B*) = *infsetsum f A* + *infsetsum f B*
using *assms* **unfolding** *infsetsum_altdef abs_summable_on_altdef*
by (*subst set_integral_Un*) *auto*

lemma *infsetsum_Diff*:
assumes *f abs_summable_on B A* ⊆ *B*
shows *infsetsum f* (*B* − *A*) = *infsetsum f B* − *infsetsum f A*
proof −
have *infsetsum f* ((*B* − *A*) ∪ *A*) = *infsetsum f* (*B* − *A*) + *infsetsum f A*
using *assms*(*2*) **by** (*intro infsetsum_Un_disjoint abs_summable_on_subset*[*OF assms*(*1*)]) *auto*
also from *assms*(*2*) **have** (*B* − *A*) ∪ *A* = *B*
by *auto*
ultimately show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *infsetsum_Un_Int*:
assumes *f abs_summable_on* (*A* ∪ *B*)
shows *infsetsum f* (*A* ∪ *B*) = *infsetsum f A* + *infsetsum f B* − *infsetsum f* (*A* ∩ *B*)
proof −
have *A* ∪ *B* = *A* ∪ (*B* − *A* ∩ *B*)
by *auto*
also have *infsetsum f* ... = *infsetsum f A* + *infsetsum f* (*B* − *A* ∩ *B*)
by (*intro infsetsum_Un_disjoint abs_summable_on_subset*[*OF assms*]) *auto*
also have *infsetsum f* (*B* − *A* ∩ *B*) = *infsetsum f B* − *infsetsum f* (*A* ∩ *B*)
by (*intro infsetsum_Diff abs_summable_on_subset*[*OF assms*]) *auto*
finally show *?thesis*
by (*simp add*: *algebra_simps*)
qed

lemma *infsetsum_reindex_bij_betw*:
assumes *bij_betw g A B*
shows *infsetsum* (λ*x*. *f* (*g x*)) *A* = *infsetsum f B*
proof −
have ∗: *count_space B* = *distr* (*count_space A*) (*count_space B*) *g*
by (*rule distr_bij_count_space* [*symmetric*]) *fact*
show *?thesis* **unfolding** *infsetsum_def*
by (*subst* ∗, *subst integral_distr*[*of* _ _ *count_space B*])

(*insert assms*, *auto simp*: *bij_betw_def*)
qed

theorem *infsetsum_reindex*:
assumes *inj_on g A*
shows *infsetsum f* (*g* ‘ *A*) = *infsetsum* (λ*x*. *f* (*g x*)) *A*
by (*intro infsetsum_reindex_bij_betw* [*symmetric*] *inj_on_imp_bij_betw assms*)

lemma *infsetsum_cong_neutral*:
assumes ⋀*x*. *x* ∈ *A* − *B* ⟹ *f x* = *0*
assumes ⋀*x*. *x* ∈ *B* − *A* ⟹ *g x* = *0*
assumes ⋀*x*. *x* ∈ *A* ∩ *B* ⟹ *f x* = *g x*
shows *infsetsum f A* = *infsetsum g B*
unfolding *infsetsum_altdef set_lebesgue_integral_def* **using** *assms*
by (*intro Bochner_Integration.integral_cong refl*)
(*auto simp*: *indicator_def split*: *if_splits*)

lemma *infsetsum_mono_neutral*:
fixes *f g* :: *'a* ⇒ *real*
assumes *f abs_summable_on A* **and** *g abs_summable_on B*
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≤ *g x*
assumes ⋀*x*. *x* ∈ *A* − *B* ⟹ *f x* ≤ *0*
assumes ⋀*x*. *x* ∈ *B* − *A* ⟹ *g x* ≥ *0*
shows *infsetsum f A* ≤ *infsetsum g B*
using *assms* **unfolding** *infsetsum_altdef set_lebesgue_integral_def abs_summable_on_altdef set_integrable_def*
by (*intro Bochner_Integration.integral_mono*) (*auto simp*: *indicator_def*)

lemma *infsetsum_mono_neutral_left*:
fixes *f g* :: *'a* ⇒ *real*
assumes *f abs_summable_on A* **and** *g abs_summable_on B*
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≤ *g x*
assumes *A* ⊆ *B*
assumes ⋀*x*. *x* ∈ *B* − *A* ⟹ *g x* ≥ *0*
shows *infsetsum f A* ≤ *infsetsum g B*
using ‹*A* ⊆ *B*› **by** (*intro infsetsum_mono_neutral assms*) *auto*

lemma *infsetsum_mono_neutral_right*:
fixes *f g* :: *'a* ⇒ *real*
assumes *f abs_summable_on A* **and** *g abs_summable_on B*
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≤ *g x*
assumes *B* ⊆ *A*
assumes ⋀*x*. *x* ∈ *A* − *B* ⟹ *f x* ≤ *0*
shows *infsetsum f A* ≤ *infsetsum g B*
using ‹*B* ⊆ *A*› **by** (*intro infsetsum_mono_neutral assms*) *auto*

lemma *infsetsum_mono*:
fixes *f g* :: *'a* ⇒ *real*
assumes *f abs_summable_on A* **and** *g abs_summable_on A*

assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$
shows *infsetsum f A* $\leq$ *infsetsum g A*
by (*intro infsetsum_mono_neutral assms*) *auto*

lemma *norm_infsetsum_bound*:
norm (*infsetsum f A*) $\leq$ *infsetsum* ($\lambda x.$ *norm* (*f x*)) *A*
unfolding *abs_summable_on_def infsetsum_def*
by (*rule Bochner_Integration.integral_norm_bound*)

theorem *infsetsum_Sigma*:
fixes *A* :: *'a set* **and** *B* :: *'a* $\Rightarrow$ *'b set*
assumes [*simp*]: *countable A* **and** $\bigwedge i.$ *countable* (*B i*)
assumes *summable*: *f abs_summable_on* (*Sigma A B*)
shows *infsetsum f* (*Sigma A B*) = *infsetsum* ($\lambda x.$ *infsetsum* ($\lambda y.$ *f* (*x*, *y*)) (*B x*)) *A*
proof −
define *B'* **where** *B'* = ($\bigcup i \in A.$ *B i*)
have [*simp*]: *countable B'*
unfolding *B'_def* **by** (*intro countable_UN assms*)
interpret *pair_sigma_finite count_space A count_space B'*
by (*intro pair_sigma_finite.intro sigma_finite_measure_count_space_countable*) *fact+*

have *integrable* (*count_space* (*A* $\times$ *B'*)) ($\lambda z.$ *indicator* (*Sigma A B*) *z* $*_R$ *f z*)
using *summable*
by (*metis* (*mono_tags*, *lifting*) *abs_summable_on_altdef abs_summable_on_def integrable_cong integrable_mult_indicator set_integrable_def sets_UNIV*)
also have *?this* $\longleftrightarrow$ *integrable* (*count_space A* $\bigotimes_M$ *count_space B'*) (λ(*x*, *y*). *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*))
by (*intro Bochner_Integration.integrable_cong*)
(*auto simp*: *pair_measure_countable indicator_def split*: *if_splits*)
finally have *integrable*: ... **.**

have *infsetsum* ($\lambda x.$ *infsetsum* ($\lambda y.$ *f* (*x*, *y*)) (*B x*)) *A* =
($\int x.$ *infsetsum* ($\lambda y.$ *f* (*x*, *y*)) (*B x*) *∂count_space A*)
unfolding *infsetsum_def* **by** *simp*
also have ... = ($\int x.$ $\int y.$ *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*) *∂count_space B' ∂count_space A*)
proof (*rule Bochner_Integration.integral_cong* [*OF refl*])
show $\bigwedge x.\ x \in$ *space* (*count_space A*) $\Longrightarrow$
($\sum_a y \in B\ x.$ *f* (*x*, *y*)) = *LINT y*|*count_space B'*. *indicat_real* (*B x*) *y* $*_R$ *f* (*x*, *y*)
using *infsetsum_altdef'*[*of _ B'*]
unfolding *set_lebesgue_integral_def B'_def*
by *auto*
qed
also have ... = ($\int$(*x*,*y*). *indicator* (*B x*) *y* $*_R$ *f* (*x*, *y*) *∂*(*count_space A* $\bigotimes_M$ *count_space B'*))
by (*subst integral_fst* [*OF integrable*]) *auto*

also have ... = ($\int$ z. *indicator* (*Sigma A B*) z $*_R$ f z ∂*count_space* ($A \times B'$))
by (*intro Bochner_Integration.integral_cong*)
(*auto simp*: *pair_measure_countable indicator_def split*: *if_splits*)
also have ... = *infsetsum f* (*Sigma A B*)
unfolding *set_lebesgue_integral_def* [*symmetric*]
by (*rule infsetsum_altdef'* [*symmetric*]) (*auto simp*: *B'_def*)
finally show *?thesis* **..**
qed

lemma *infsetsum_Sigma'*:
fixes *A* :: *'a set* **and** *B* :: *'a* ⇒ *'b set*
assumes [*simp*]: *countable A* **and** ⋀*i*. *countable* (*B i*)
assumes *summable*: (λ(*x*,*y*). *f x y*) *abs_summable_on* (*Sigma A B*)
shows *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f x y*) (*B x*)) *A* = *infsetsum* (λ(*x*,*y*). *f x y*) (*Sigma A B*)
using *assms* **by** (*subst infsetsum_Sigma*) *auto*

lemma *infsetsum_Times*:
fixes *A* :: *'a set* **and** *B* :: *'b set*
assumes [*simp*]: *countable A* **and** *countable B*
assumes *summable*: *f abs_summable_on* (*A* × *B*)
shows *infsetsum f* (*A* × *B*) = *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f* (*x*, *y*)) *B*) *A*
using *assms* **by** (*subst infsetsum_Sigma*) *auto*

lemma *infsetsum_Times'*:
fixes *A* :: *'a set* **and** *B* :: *'b set*
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c* :: {*banach*, *second_countable_topology*}
assumes [*simp*]: *countable A* **and** [*simp*]: *countable B*
assumes *summable*: (λ(*x*,*y*). *f x y*) *abs_summable_on* (*A* × *B*)
shows *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f x y*) *B*) *A* = *infsetsum* (λ(*x*,*y*). *f x y*) (*A* × *B*)
using *assms* **by** (*subst infsetsum_Times*) *auto*

lemma *infsetsum_swap*:
fixes *A* :: *'a set* **and** *B* :: *'b set*
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c* :: {*banach*, *second_countable_topology*}
assumes [*simp*]: *countable A* **and** [*simp*]: *countable B*
assumes *summable*: (λ(*x*,*y*). *f x y*) *abs_summable_on A* × *B*
shows *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f x y*) *B*) *A* = *infsetsum* (λ*y*. *infsetsum* (λ*x*. *f x y*) *A*) *B*
proof −
from *summable* **have** *summable'*: (λ(*x*,*y*). *f y x*) *abs_summable_on B* × *A*
by (*subst abs_summable_on_Times_swap*) *auto*
have *bij*: *bij_betw* (λ(*x*, *y*). (*y*, *x*)) (*B* × *A*) (*A* × *B*)
by (*auto simp*: *bij_betw_def inj_on_def*)
have *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f x y*) *B*) *A* = *infsetsum* (λ(*x*,*y*). *f x y*) (*A* × *B*)
using *summable* **by** (*subst infsetsum_Times*) *auto*
also have ... = *infsetsum* (λ(*x*,*y*). *f y x*) (*B* × *A*)

```
    by (subst infsetsum_reindex_bij_betw[OF bij, of λ(x,y). f x y, symmetric])
       (simp_all add: case_prod_unfold)
  also have ... = infsetsum (λy. infsetsum (λx. f x y) A) B
    using summable' by (subst infsetsum_Times) auto
  finally show ?thesis .
qed

theorem abs_summable_on_Sigma_iff:
  assumes [simp]: countable A and ⋀x. x ∈ A ⟹ countable (B x)
  shows   f abs_summable_on Sigma A B ⟷
            (∀ x∈A. (λy. f (x, y)) abs_summable_on B x) ∧
            ((λx. infsetsum (λy. norm (f (x, y))) (B x)) abs_summable_on A)
proof safe
  define B' where B' = (⋃ x∈A. B x)
  have [simp]: countable B'
    unfolding B'_def using assms by auto
  interpret pair_sigma_finite count_space A count_space B'
   by (intro pair_sigma_finite.intro sigma_finite_measure_count_space_countable)
fact+
  {
    assume ∗: f abs_summable_on Sigma A B
    thus (λy. f (x, y)) abs_summable_on B x if x ∈ A for x
      using that by (rule abs_summable_on_Sigma_project2)

    have set_integrable (count_space (A × B')) (Sigma A B) (λz. norm (f z))
      using abs_summable_on_normI[OF ∗]
      by (subst abs_summable_on_altdef' [symmetric]) (auto simp: B'_def)
    also have count_space (A × B') = count_space A ⨂M count_space B'
      by (simp add: pair_measure_countable)
    finally have integrable (count_space A)
                   (λx. lebesgue_integral (count_space B')
                     (λy. indicator (Sigma A B) (x, y) ∗R norm (f (x, y))))
      unfolding set_integrable_def by (rule integrable_fst')
    also have ?this ⟷ integrable (count_space A)
                   (λx. lebesgue_integral (count_space B')
                     (λy. indicator (B x) y ∗R norm (f (x, y))))
      by (intro integrable_cong refl) (simp_all add: indicator_def)
    also have ... ⟷ integrable (count_space A) (λx. infsetsum (λy. norm (f (x,
y))) (B x))
      unfolding set_lebesgue_integral_def [symmetric]
       by (intro integrable_cong refl infsetsum_altdef' [symmetric]) (auto simp:
B'_def)
   also have ... ⟷ (λx. infsetsum (λy. norm (f (x, y))) (B x)) abs_summable_on
A
      by (simp add: abs_summable_on_def)
    finally show ... .
  }
  {
    assume ∗: ∀ x∈A. (λy. f (x, y)) abs_summable_on B x
```

```
    assume (λx. ∑a y∈B x. norm (f (x, y))) abs_summable_on A
  also have ?this ⟷ (λx. ∫ y∈B x. norm (f (x, y)) ∂count_space B') abs_summable_on
A
    by (intro abs_summable_on_cong refl infsetsum_altdef') (auto simp: B'_def)
    also have ... ⟷ (λx. ∫ y. indicator (Sigma A B) (x, y) *R norm (f (x, y))
∂count_space B')
                        abs_summable_on A (is _ ⟷ ?h abs_summable_on _)
      unfolding set_lebesgue_integral_def
      by (intro abs_summable_on_cong) (auto simp: indicator_def)
    also have ... ⟷ integrable (count_space A) ?h
      by (simp add: abs_summable_on_def)
    finally have **: ... .

    have integrable (count_space A ⨂M count_space B') (λz. indicator (Sigma A
B) z *R f z)
    proof (rule Fubini_integrable, goal_cases)
      case 3
      {
        fix x assume x: x ∈ A
        with * have (λy. f (x, y)) abs_summable_on B x
          by blast
        also have ?this ⟷ integrable (count_space B')
                    (λy. indicator (B x) y *R f (x, y))
          unfolding set_integrable_def [symmetric]
         using x by (intro abs_summable_on_altdef') (auto simp: B'_def)
        also have (λy. indicator (B x) y *R f (x, y)) =
                  (λy. indicator (Sigma A B) (x, y) *R f (x, y))
          using x by (auto simp: indicator_def)
        finally have integrable (count_space B')
                      (λy. indicator (Sigma A B) (x, y) *R f (x, y)) .
      }
      thus ?case by (auto simp: AE_count_space)
    qed (insert **, auto simp: pair_measure_countable)
    moreover have count_space A ⨂M count_space B' = count_space (A × B')
      by (simp add: pair_measure_countable)
    moreover have set_integrable (count_space (A × B')) (Sigma A B) f ⟷
                f abs_summable_on Sigma A B
      by (rule abs_summable_on_altdef' [symmetric]) (auto simp: B'_def)
    ultimately show f abs_summable_on Sigma A B
      by (simp add: set_integrable_def)
  }
qed

lemma abs_summable_on_Sigma_project1:
  assumes (λ(x,y). f x y) abs_summable_on Sigma A B
  assumes [simp]: countable A and ⋀x. x ∈ A ⟹ countable (B x)
  shows   (λx. infsetsum (λy. norm (f x y)) (B x)) abs_summable_on A
  using assms by (subst (asm) abs_summable_on_Sigma_iff) auto
```

lemma *abs_summable_on_Sigma_project1'*:
assumes $(\lambda(x,y).\ f\ x\ y)$ *abs_summable_on Sigma A B*
assumes [*simp*]: *countable A* **and** $\bigwedge x.\ x \in A \Longrightarrow$ *countable* $(B\ x)$
shows $(\lambda x.$ *infsetsum* $(\lambda y.\ f\ x\ y)\ (B\ x))$ *abs_summable_on A*
by (*intro abs_summable_on_comparison_test'* [*OF abs_summable_on_Sigma_project1* [*OF assms*]]
norm_infsetsum_bound)

theorem *infsetsum_prod_PiE*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \{$*real_normed_field*,*banach*,*second_countable_topology*$\}$
assumes *finite*: *finite A* **and** *countable*: $\bigwedge x.\ x \in A \Longrightarrow$ *countable* $(B\ x)$
assumes *summable*: $\bigwedge x.\ x \in A \Longrightarrow f\ x$ *abs_summable_on B x*
shows *infsetsum* $(\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x))\ (PiE\ A\ B) = (\prod x{\in}A.$ *infsetsum* $(f\ x)\ (B\ x))$
proof −
define B' **where** $B' = (\lambda x.$ *if* $x \in A$ *then* $B\ x$ *else* $\{\})$
from *assms* **have** [*simp*]: *countable* $(B'\ x)$ **for** *x*
by (*auto simp*: B'*_def*)
then interpret *product_sigma_finite count_space* $\circ\ B'$
unfolding *o_def* **by** (*intro product_sigma_finite.intro sigma_finite_measure_count_space_countable*)
have *infsetsum* $(\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x))\ (PiE\ A\ B) =$
$(\int g.\ (\prod x{\in}A.\ f\ x\ (g\ x))\ \partial$*count_space* $(PiE\ A\ B))$
by (*simp add*: *infsetsum_def*)
also have $PiE\ A\ B = PiE\ A\ B'$
by (*intro PiE_cong*) (*simp_all add*: B'*_def*)
hence *count_space* $(PiE\ A\ B) =$ *count_space* $(PiE\ A\ B')$
by *simp*
also have $\ldots = PiM\ A$ (*count_space* $\circ\ B'$)
unfolding *o_def* **using** *finite* **by** (*intro count_space_PiM_finite* [*symmetric*]) *simp_all*
also have $(\int g.\ (\prod x{\in}A.\ f\ x\ (g\ x))\ \partial\ldots) = (\prod x{\in}A.$ *infsetsum* $(f\ x)\ (B'\ x))$
by (*subst product_integral_prod*)
(*insert summable finite*, *simp_all add*: *infsetsum_def B'_def abs_summable_on_def*)
also have $\ldots = (\prod x{\in}A.$ *infsetsum* $(f\ x)\ (B\ x))$
by (*intro prod.cong refl*) (*simp_all add*: B'*_def*)
finally show *?thesis* **.**
qed

lemma *infsetsum_uminus*: *infsetsum* $(\lambda x.\ -f\ x)\ A = -$*infsetsum f A*
unfolding *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_minus*)

lemma *infsetsum_add*:
assumes *f abs_summable_on A* **and** *g abs_summable_on A*
shows *infsetsum* $(\lambda x.\ f\ x + g\ x)\ A =$ *infsetsum f A* $+$ *infsetsum g A*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_add*)

lemma *infsetsum_diff*:

assumes *f abs_summable_on A* **and** *g abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* − *g x*) *A* = *infsetsum f A* − *infsetsum g A*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_diff*)

lemma *infsetsum_scaleR_left*:
assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* ∗*R* *c*) *A* = *infsetsum f A* ∗*R* *c*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_scaleR_left*)

lemma *infsetsum_scaleR_right*:
infsetsum (λ*x*. *c* ∗*R* *f x*) *A* = *c* ∗*R* *infsetsum f A*
unfolding *infsetsum_def abs_summable_on_def*
by (*subst Bochner_Integration.integral_scaleR_right*) *auto*

lemma *infsetsum_cmult_left*:
fixes *f* :: *'a* ⇒ *'b* :: {*banach*, *real_normed_algebra*, *second_countable_topology*}
assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* ∗ *c*) *A* = *infsetsum f A* ∗ *c*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_mult_left*)

lemma *infsetsum_cmult_right*:
fixes *f* :: *'a* ⇒ *'b* :: {*banach*, *real_normed_algebra*, *second_countable_topology*}
assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
shows *infsetsum* (λ*x*. *c* ∗ *f x*) *A* = *c* ∗ *infsetsum f A*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def*
by (*rule Bochner_Integration.integral_mult_right*)

lemma *infsetsum_cdiv*:
fixes *f* :: *'a* ⇒ *'b* :: {*banach*, *real_normed_field*, *second_countable_topology*}
assumes *c* ≠ *0* ⟹ *f abs_summable_on A*
shows *infsetsum* (λ*x*. *f x* / *c*) *A* = *infsetsum f A* / *c*
using *assms* **unfolding** *infsetsum_def abs_summable_on_def* **by** *auto*

lemma
fixes *f* :: *'a* ⇒ *'c* :: {*banach*, *real_normed_field*, *second_countable_topology*}
assumes [*simp*]: *countable A* **and** [*simp*]: *countable B*
assumes *f abs_summable_on A* **and** *g abs_summable_on B*
shows *abs_summable_on_product*: (λ(*x*,*y*). *f x* ∗ *g y*) *abs_summable_on A* ×
B
and *infsetsum_product*: *infsetsum* (λ(*x*,*y*). *f x* ∗ *g y*) (*A* × *B*) =
infsetsum f A ∗ *infsetsum g B*
proof −
from *assms* **show** (λ(*x*,*y*). *f x* ∗ *g y*) *abs_summable_on A* × *B*

```
    by (subst abs_summable_on_Sigma_iff)
     (auto intro!: abs_summable_on_cmult_right simp: norm_mult infsetsum_cmult_right)
  with assms show infsetsum (λ(x,y). f x * g y) (A × B) = infsetsum f A *
infsetsum g B
    by (subst infsetsum_Sigma)
      (auto simp: infsetsum_cmult_left infsetsum_cmult_right)
qed

lemma abs_summable_finite_sumsI:
  assumes ⋀F. finite F ⟹ F⊆S ⟹ sum (λx. norm (f x)) F ≤ B
  shows f abs_summable_on S
proof −
  have main: f abs_summable_on S ∧ infsetsum (λx. norm (f x)) S ≤ B if ‹B ≥
0› and ‹S ≠ {}›
  proof −
    define M normf where M = count_space S and normf x = ennreal (norm (f
x)) for x
    have sum normf F ≤ ennreal B
      if finite F and F ⊆ S and
        ⋀F. finite F ⟹ F ⊆ S ⟹ (∑ i∈F. ennreal (norm (f i))) ≤ ennreal B
and
        ennreal 0 ≤ ennreal B for F
      using that unfolding normf_def[symmetric] by simp
    hence normf_B: finite F ⟹ F⊆S ⟹ sum normf F ≤ ennreal B for F
      using assms[THEN ennreal_leI]
      by auto
    have integral^S M g ≤ B if simple_function M g and g ≤ normf for g
    proof −
      define gS where gS = g ‘ S
      have finite gS
       using that unfolding gS_def M_def simple_function_count_space by simp
      have gS ≠ {} unfolding gS_def using ‹S ≠ {}› by auto
      define part where part r = g −‘ {r} ∩ S for r
      have r_finite: r < ∞ if r : gS for r
        using ‹g ≤ normf› that unfolding gS_def le_fun_def normf_def apply
auto
        using ennreal_less_top neq_top_trans top.not_eq_extremum by blast
      define B' where B' r = (SUP F∈{F. finite F ∧ F⊆part r}. sum normf F)
for r
      have B'fin: B' r < ∞ for r
      proof −
        have B' r ≤ (SUP F∈{F. finite F ∧ F⊆part r}. sum normf F)
          unfolding B'_def
          by (metis (mono_tags, lifting) SUP_least SUP_upper)
        also have ... ≤ B
          using normf_B unfolding part_def
          by (metis (no_types, lifting) Int_subset_iff SUP_least mem_Collect_eq)
        also have ... < ∞
          by simp
```

```
    finally show ?thesis by simp
  qed
  have sumB': sum B' gS ≤ ennreal B + ε if ε>0 for ε
  proof −
    define N εN where N = card gS and εN = ε / N
    have N > 0
      unfolding N_def using ‹gS≠{}› ‹finite gS›
      by (simp add: card_gt_0_iff)
    from εN_def that have εN > 0
     by (simp add: ennreal_of_nat_eq_real_of_nat ennreal_zero_less_divide)
    have c1: ∃ y. B' r ≤ sum normf y + εN ∧ finite y ∧ y ⊆ part r
      if B' r = 0 for r
      using that by auto
    have c2: ∃ y. B' r ≤ sum normf y + εN ∧ finite y ∧ y ⊆ part r if B' r ≠
0 for r
    proof−
      have B' r − εN < B' r
        using B'fin ‹0 < εN› ennreal_between that by fastforce
      have B' r − εN < Sup (sum normf ‘ {F. finite F ∧ F ⊆ part r}) ⟹
          ∃ F. B' r − εN ≤ sum normf F ∧ finite F ∧ F ⊆ part r
       by (metis (no_types, lifting) leD le_cases less_SUP_iff mem_Collect_eq)
      hence B' r − εN < B' r ⟹ ∃ F. B' r − εN ≤ sum normf F ∧ finite F
∧ F ⊆ part r
        by (subst (asm) (2) B'_def)
       then obtain F where B' r − εN ≤ sum normf F and finite F and F
⊆ part r
        using ‹B' r − εN < B' r› by auto
      thus ∃ F. B' r ≤ sum normf F + εN ∧ finite F ∧ F ⊆ part r
        by (metis add.commute ennreal_minus_le_iff)
    qed
    have ∀ x. ∃ y. B' x ≤ sum normf y + εN ∧
      finite y ∧ y ⊆ part x
      using c1 c2
      by blast
    hence ∃ F. ∀ x. B' x ≤ sum normf (F x) + εN ∧ finite (F x) ∧ F x ⊆ part
x
      by metis
    then obtain F where F: sum normf (F r) + εN ≥ B' r and Ffin: finite
(F r) and Fpartr: F r ⊆ part r for r
      using atomize_elim by auto
    have w1: finite gS
      by (simp add: ‹finite gS›)
    have w2: ∀ i∈gS. finite (F i)
      by (simp add: Ffin)
    have False
      if ⋀r. F r ⊆ g −‘ {r} ∧ F r ⊆ S
        and i ∈ gS and j ∈ gS and i ≠ j and x ∈ F i and x ∈ F j
      for i j x
      by (metis subsetD that(1) that(4) that(5) that(6) vimage_singleton_eq)
```

```
      hence w3: ∀ i∈gS. ∀ j∈gS. i ≠ j ⟶ F i ∩ F j = {}
        using Fpartr[unfolded part_def] by auto
      have w4: sum normf (⋃ (F ‘ gS)) + ε = sum normf (⋃ (F ‘ gS)) + ε
        by simp
      have sum B' gS ≤ (∑ r∈gS. sum normf (F r) + εN)
        using F by (simp add: sum_mono)
      also have ... = (∑ r∈gS. sum normf (F r)) + (∑ r∈gS. εN)
        by (simp add: sum.distrib)
      also have ... = (∑ r∈gS. sum normf (F r)) + (card gS * εN)
        by auto
      also have ... = (∑ r∈gS. sum normf (F r)) + ε
        unfolding εN_def N_def[symmetric] using ‹N>0›
     by (simp add: ennreal_times_divide mult.commute mult_divide_eq_ennreal)
      also have ... = sum normf (⋃ (F ‘ gS)) + ε
        using w1 w2 w3 w4
        by (subst sum.UNION_disjoint[symmetric])
      also have ... ≤ B + ε
          using ‹finite gS› normf_B add_right_mono Ffin Fpartr unfolding
part_def
        by (simp add: ‹gS ≠ {}› cSUP_least)
      finally show ?thesis
        by auto
    qed
    hence sumB': sum B' gS ≤ B
      using ennreal_le_epsilon ennreal_less_zero_iff by blast
    have ∀ r. ∃ y. r ∈ gS ⟶ B' r = ennreal y
      using B'fin less_top_ennreal by auto
    hence ∃ B''. ∀ r. r ∈ gS ⟶ B' r = ennreal (B'' r)
      by (rule_tac choice)
    then obtain B'' where B'': B' r = ennreal (B'' r) if r ∈ gS for r
      by atomize_elim
    have cases[case_names zero finite infinite]: P if r=0 ⟹ P and finite (part
r) ⟹ P
      and infinite (part r) ⟹ r≠0 ⟹ P for P r
      using that by metis
    have emeasure_B': r * emeasure M (part r) ≤ B' r if r : gS for r
    proof (cases rule:cases[of r])
      case zero
      thus ?thesis by simp
    next
      case finite
      have s1: sum g F ≤ sum normf F
        if F ∈ {F. finite F ∧ F ⊆ part r}
        for F
        using ‹g ≤ normf›
        by (simp add: le_fun_def sum_mono)

      have r * of_nat (card (part r)) = r * (∑ x∈part r. 1) by simp
```

```
      also have ... = (∑ x∈part r. r)
        using mult.commute by auto
      also have ... = (∑ x∈part r. g x)
        unfolding part_def by auto
      also have ... ≤ (SUP F∈{F. finite F ∧ F⊆part r}. sum g F)
        using finite
        by (simp add: Sup_upper)
      also have ... ≤ B' r
        unfolding B'_def
        using s1 SUP_subset_mono by blast
      finally have r * of_nat (card (part r)) ≤ B' r by assumption
      thus ?thesis
        unfolding M_def
        using part_def finite by auto
    next
      case infinite
      from r_finite[OF ‹r : gS›] obtain r' where r': r = ennreal r'
        using ennreal_cases by auto
      with infinite have r' > 0
        using ennreal_less_zero_iff not_gr_zero by blast
      obtain N::nat where N:N > B / r' and real N > 0 apply atomize_elim
        using reals_Archimedean2
        by (metis less_trans linorder_neqE_linordered_idom)
      obtain F where finite F and card F = N and F ⊆ part r
        using infinite(1) infinite_arbitrarily_large by blast
      from ‹F ⊆ part r› have F ⊆ S unfolding part_def by simp
      have B < r * N
        unfolding r' ennreal_of_nat_eq_real_of_nat
        using N ‹0 < r'› ‹B ≥ 0› r'
            by (metis enn2real_ennreal enn2real_less_iff ennreal_less_top en-
nreal_mult' less_le mult_less_cancel_left_pos nonzero_mult_div_cancel_left times_divide_eq_right)
      also have r * N = (∑ x∈F. r)
        using ‹card F = N› by (simp add: mult.commute)
      also have (∑ x∈F. r) = (∑ x∈F. g x)
        using ‹F ⊆ part r›  part_def sum.cong subsetD by fastforce
      also have (∑ x∈F. g x) ≤ (∑ x∈F. ennreal (norm (f x)))
        by (metis (mono_tags, lifting) ‹g ≤ normf› ‹normf ≡ λx. ennreal (norm
(f x))› le_fun_def
            sum_mono)
      also have (∑ x∈F. ennreal (norm (f x))) ≤ B
        using ‹F ⊆ S› ‹finite F› ‹normf ≡ λx. ennreal (norm (f x))› normf_B
by blast
      finally have B < B by auto
      thus ?thesis by simp
    qed

    have integral^S M g = (∑ r ∈ gS. r * emeasure M (part r))
      unfolding simple_integral_def gS_def M_def part_def by simp
    also have ... ≤ (∑ r ∈ gS. B' r)
```

```
      by (simp add: emeasure_B' sum_mono)
    also have ... ≤ B
      using sumB' by blast
    finally show ?thesis by assumption
  qed
  hence int_leq_B: integral^N M normf ≤ B
  unfolding nn_integral_def by (metis (no_types, lifting) SUP_least mem_Collect_eq)
  hence integral^N M normf < ∞
    using le_less_trans by fastforce
  hence integrable M f
    unfolding M_def normf_def by (rule integrableI_bounded[rotated], simp)
  hence v1: f abs_summable_on S
    unfolding abs_summable_on_def M_def by simp

  have (λx. norm (f x)) abs_summable_on S
   using v1 Infinite_Set_Sum.abs_summable_on_norm_iff[where A = S and
f = f]
    by auto
  moreover have 0 ≤ norm (f x)
    if x ∈ S for x
    by simp
  moreover have (∫⁺ x. ennreal (norm (f x)) ∂count_space S) ≤ ennreal B
    using M_def ‹normf ≡ λx. ennreal (norm (f x))› int_leq_B by auto
  ultimately have ennreal (∑_a x∈S. norm (f x)) ≤ ennreal B
    by (simp add: nn_integral_conv_infsetsum)
  hence v2: (∑_a x∈S. norm (f x)) ≤ B
    by (subst ennreal_le_iff[symmetric], simp add: assms ‹B ≥ 0›)
  show ?thesis
    using v1 v2 by auto
 qed
 then show f abs_summable_on S
  by (metis abs_summable_on_finite assms empty_subsetI finite.emptyI sum_clauses(1))
qed


lemma infsetsum_nonneg_is_SUPREMUM_ennreal:
 fixes f :: 'a ⇒ real
 assumes summable: f abs_summable_on A
   and fnn: ⋀x. x∈A ⟹ f x ≥ 0
 shows ennreal (infsetsum f A) = (SUP F∈{F. finite F ∧ F ⊆ A}. (ennreal (sum
f F)))
proof −
 have sum_F_A: sum f F ≤ infsetsum f A
   if F ∈ {F. finite F ∧ F ⊆ A}
   for F
 proof−
   from that have finite F and F ⊆ A by auto
   from ‹finite F› have sum f F = infsetsum f F by auto
   also have ... ≤ infsetsum f A
```

```
    proof (rule infsetsum_mono_neutral_left)
      show f abs_summable_on F
        by (simp add: ‹finite F›)
      show f abs_summable_on A
        by (simp add: local.summable)
      show f x ≤ f x
        if x ∈ F
        for x :: 'a
        by simp
      show F ⊆ A
        by (simp add: ‹F ⊆ A›)
      show 0 ≤ f x
        if x ∈ A − F
        for x :: 'a
        using that fnn by auto
    qed
    finally show ?thesis by assumption
  qed
  hence geq: ennreal (infsetsum f A) ≥ (SUP F∈{G. finite G ∧ G ⊆ A}. (ennreal
(sum f F)))
    by (meson SUP_least ennreal_leI)

  define fe where fe x = ennreal (f x) for x

  have sum_f_int: infsetsum f A = ∫⁺ x. fe x ∂(count_space A)
    unfolding infsetsum_def fe_def
  proof (rule nn_integral_eq_integral [symmetric])
    show integrable (count_space A) f
      using abs_summable_on_def local.summable by blast
    show AE x in count_space A. 0 ≤ f x
      using fnn by auto
  qed
  also have ... = (SUP g ∈ {g. finite (g‘A) ∧ g ≤ fe}. integral^S (count_space A)
g)
    unfolding nn_integral_def simple_function_count_space by simp
  also have ... ≤ (SUP F∈{F. finite F ∧ F ⊆ A}. (ennreal (sum f F)))
  proof (rule Sup_least)
    fix x assume x ∈ integral^S (count_space A) ‘ {g. finite (g ‘ A) ∧ g ≤ fe}
    then obtain g where xg: x = integral^S (count_space A) g and fin_gA: finite
(g‘A)
      and g_fe: g ≤ fe by auto
    define F where F = {z:A. g z ≠ 0}
    hence F ⊆ A by simp

    have fin: finite {z:A. g z = t} if t ≠ 0 for t
    proof (rule ccontr)
      assume inf: infinite {z:A. g z = t}
      hence tgA: t ∈ g ‘ A
        by (metis (mono_tags, lifting) image_eqI not_finite_existsD)
```

```
    have x = (∑x∈g ‘ A. x * emeasure (count_space A) (g -‘ {x} ∩ A))
      unfolding xg simple_integral_def space_count_space by simp
    also have ... ≥ (∑x∈{t}. x * emeasure (count_space A) (g -‘ {x} ∩ A))
(is _ ≥ ...)
    proof (rule sum_mono2)
      show finite (g ‘ A)
        by (simp add: fin_gA)
      show {t} ⊆ g ‘ A
        by (simp add: tgA)
      show 0 ≤ b * emeasure (count_space A) (g -‘ {b} ∩ A)
        if b ∈ g ‘ A - {t}
        for b :: ennreal
        using that
        by simp
    qed
    also have ... = t * emeasure (count_space A) (g -‘ {t} ∩ A)
      by auto
    also have ... = t * ∞
    proof (subst emeasure_count_space_infinite)
      show g -‘ {t} ∩ A ⊆ A
        by simp
      have {a ∈ A. g a = t} = {a ∈ g -‘ {t}. a ∈ A}
        by auto
      thus infinite (g -‘ {t} ∩ A)
        by (metis (full_types) Int_def inf)
      show t * ∞ = t * ∞
        by simp
    qed
    also have ... = ∞ using ‹t ≠ 0›
      by (simp add: ennreal_mult_eq_top_iff)
    finally have x_inf: x = ∞
      using neq_top_trans by auto
    have x = integral^S (count_space A) g by (fact xg)
    also have ... = integral^N (count_space A) g
      by (simp add: fin_gA nn_integral_eq_simple_integral)
    also have ... ≤ integral^N (count_space A) fe
      using g_fe
      by (simp add: le_funD nn_integral_mono)
    also have ... < ∞
      by (metis sum_f_int ennreal_less_top infinity_ennreal_def)
    finally have x_fin: x < ∞ by simp
    from x_inf x_fin show False by simp
  qed
  have F: F = (⋃t∈g‘A-{0}. {z∈A. g z = t})
    unfolding F_def by auto
  hence finite F
    unfolding F using fin_gA fin by auto
  have x = integral^N (count_space A) g
    unfolding xg
```

```
    by (simp add: fin_gA nn_integral_eq_simple_integral)
  also have ... = set_nn_integral (count_space UNIV) A g
    by (simp add: nn_integral_restrict_space[symmetric] restrict_count_space)
  also have ... = set_nn_integral (count_space UNIV) F g
  proof -
    have ∀a. g a * (if a ∈ {a ∈ A. g a ≠ 0} then 1 else 0) = g a * (if a ∈ A
then 1 else 0)
      by auto
    hence (∫⁺ a. g a * (if a ∈ A then 1 else 0) ∂count_space UNIV)
        = (∫⁺ a. g a * (if a ∈ {a ∈ A. g a ≠ 0} then 1 else 0) ∂count_space
UNIV)
      by presburger
    thus ?thesis unfolding F_def indicator_def
      using mult.right_neutral mult_zero_right nn_integral_cong
      by (simp add: of_bool_def)
  qed
  also have ... = integral^N (count_space F) g
    by (simp add: nn_integral_restrict_space[symmetric] restrict_count_space)
  also have ... = sum g F
    using ‹finite F› by (rule nn_integral_count_space_finite)
  also have sum g F ≤ sum fe F
    using g_fe unfolding le_fun_def
    by (simp add: sum_mono)
  also have ... ≤ (SUP F ∈ {G. finite G ∧ G ⊆ A}. (sum fe F))
    using ‹finite F› ‹F⊆A›
    by (simp add: SUP_upper)
  also have ... = (SUP F ∈ {F. finite F ∧ F ⊆ A}. (ennreal (sum f F)))
  proof (rule SUP_cong [OF refl])
    have finite x ⟹ x ⊆ A ⟹ (∑ x∈x. ennreal (f x)) = ennreal (sum f x)
      for x
      by (metis fnn subsetCE sum_ennreal)
    thus sum fe x = ennreal (sum f x)
      if x ∈ {G. finite G ∧ G ⊆ A}
      for x :: 'a set
      using that unfolding fe_def by auto
  qed
  finally show x ≤ ... by simp
qed
finally have leq: ennreal (infsetsum f A) ≤ (SUP F∈{F. finite F ∧ F ⊆ A}.
(ennreal (sum f F)))
  by assumption
from leq geq show ?thesis by simp
qed

lemma infsetsum_nonneg_is_SUPREMUM_ereal:
fixes f :: 'a ⇒ real
assumes summable: f abs_summable_on A
  and fnn: ⋀x. x∈A ⟹ f x ≥ 0
shows ereal (infsetsum f A) = (SUP F∈{F. finite F ∧ F ⊆ A}. (ereal (sum f
```

F)))
proof −
have *ereal* (*infsetsum f A*) = *enn2ereal* (*ennreal* (*infsetsum f A*))
by (*simp add*: *fnn infsetsum_nonneg*)
also have ... = *enn2ereal* (*SUP F*∈{*F*. *finite F* ∧ *F* ⊆ *A*}. *ennreal* (*sum f F*))
apply (*subst infsetsum_nonneg_is_SUPREMUM_ennreal*)
using *fnn* **by** (*auto simp add*: *local.summable*)
also have ... = (*SUP F*∈{*F*. *finite F* ∧ *F* ⊆ *A*}. (*ereal* (*sum f F*)))
proof (*simp add*: *image_def Sup_ennreal.rep_eq*)
have *0* ≤ *Sup* {*y*. ∃ *x*. (∃ *xa*. *finite xa* ∧ *xa* ⊆ *A* ∧ *x* = *ennreal* (*sum f xa*)) ∧
y = *enn2ereal x*}
by (*metis* (*mono_tags*, *lifting*) *Sup_upper empty_subsetI ennreal_0 finite.emptyI*
mem_Collect_eq sum.empty zero_ennreal.rep_eq)
moreover have (∃ *x*. (∃ *y*. *finite y* ∧ *y* ⊆ *A* ∧ *x* = *ennreal* (*sum f y*)) ∧ *y* = *enn2ereal x*) =
(∃ *x*. *finite x* ∧ *x* ⊆ *A* ∧ *y* = *ereal* (*sum f x*)) **for** *y*
proof −
have (∃ *x*. (∃ *y*. *finite y* ∧ *y* ⊆ *A* ∧ *x* = *ennreal* (*sum f y*)) ∧ *y* = *enn2ereal x*) ⟷
(∃ *X x*. *finite X* ∧ *X* ⊆ *A* ∧ *x* = *ennreal* (*sum f X*) ∧ *y* = *enn2ereal x*)
by *blast*
also have ... ⟷ (∃ *X*. *finite X* ∧ *X* ⊆ *A* ∧ *y* = *ereal* (*sum f X*))
by (*rule arg_cong*[*of* _ _ *Ex*])
(*auto simp*: *fun_eq_iff intro*!: *enn2ereal_ennreal sum_nonneg enn2ereal_ennreal*[*symmetric*] *fnn*)
finally show *?thesis* **.**
qed
hence *Sup* {*y*. ∃ *x*. (∃ *y*. *finite y* ∧ *y* ⊆ *A* ∧ *x* = *ennreal* (*sum f y*)) ∧ *y* = *enn2ereal x*} =
Sup {*y*. ∃ *x*. *finite x* ∧ *x* ⊆ *A* ∧ *y* = *ereal* (*sum f x*)}
by *simp*
ultimately show *max 0* (*Sup* {*y*. ∃ *x*. (∃ *xa*. *finite xa* ∧ *xa* ⊆ *A* ∧ *x*
= *ennreal* (*sum f xa*)) ∧ *y* = *enn2ereal x*})
= *Sup* {*y*. ∃ *x*. *finite x* ∧ *x* ⊆ *A* ∧ *y* = *ereal* (*sum f x*)}
by *linarith*
qed
finally show *?thesis*
by *simp*
qed

The following theorem relates (*abs_summable_on*) with *Infinite_Sum.abs_summable_on*. Note that while this theorem expresses an equivalence, the notion on the lhs is more general nonetheless because it applies to a wider range of types. (The rhs requires second-countable Banach spaces while the lhs is well-defined on arbitrary real vector spaces.)

lemma *abs_summable_equivalent*: ‹*Infinite_Sum.abs_summable_on f A* ⟷ *f abs_summable_on A*›
proof (*rule iffI*)

```
  define n where ‹n x = norm (f x)› for x
  assume ‹n summable_on A›
  then have ‹sum n F ≤ infsum n A› if ‹finite F› and ‹F⊆A› for F
    using that by (auto simp flip: infsum_finite simp: n_def[abs_def] intro!:
infsum_mono_neutral)

  then show ‹f abs_summable_on A›
    by (auto intro!: abs_summable_finite_sumsI simp: n_def)
next
  define n where ‹n x = norm (f x)› for x
  assume ‹f abs_summable_on A›
  then have ‹n abs_summable_on A›
    by (simp add: ‹f abs_summable_on A› n_def)
  then have ‹sum n F ≤ infsetsum n A› if ‹finite F› and ‹F⊆A› for F
    using that by (auto simp flip: infsetsum_finite simp: n_def[abs_def] intro!:
infsetsum_mono_neutral)
  then show ‹n summable_on A›
    apply (rule_tac nonneg_bdd_above_summable_on)
    by (auto simp add: n_def bdd_above_def)
qed

lemma infsetsum_infsum:
  assumes f abs_summable_on A
  shows infsetsum f A = infsum f A
proof −
  have conv_sum_norm[simp]: (λx. norm (f x)) summable_on A
    using abs_summable_equivalent assms by blast
  have norm (infsetsum f A − infsum f A) ≤ ε if ε>0 for ε
  proof −
    define δ where δ = ε/2
    with that have [simp]: δ > 0 by simp
    obtain F1 where F1A: F1 ⊆ A and finF1: finite F1 and leq_eps: infsetsum
(λx. norm (f x)) (A−F1) ≤ δ
    proof −
      have sum_SUP: ereal (infsetsum (λx. norm (f x)) A) = (SUP F∈{F. finite
F ∧ F ⊆ A}. ereal (sum (λx. norm (f x)) F))
        (is _ = ?SUP)
        apply (rule infsetsum_nonneg_is_SUPREMUM_ereal)
        using assms by auto

      have (SUP F∈{F. finite F ∧ F ⊆ A}. ereal (∑x∈F. norm (f x))) − ereal δ
          < (SUP i∈{F. finite F ∧ F ⊆ A}. ereal (∑x∈i. norm (f x)))
        using ‹δ>0›
        by (metis diff_strict_left_mono diff_zero ereal_less_eq(3) ereal_minus(1)
not_le sum_SUP)
      then obtain F where F∈{F. finite F ∧ F ⊆ A} and ereal (sum (λx. norm
(f x)) F) > ?SUP − ereal (δ)
        by (meson less_SUP_iff)
```

```
  hence sum (λx. norm (f x)) F > infsetsum (λx. norm (f x)) A −  (δ)
    unfolding sum_SUP[symmetric] by auto
  hence δ > infsetsum (λx. norm (f x)) (A−F)
  proof (subst infsetsum_Diff)
    show (λx. norm (f x)) abs_summable_on A
      if (∑ a x∈A. norm (f x)) − δ < (∑ x∈F. norm (f x))
      using that
      by (simp add: assms)
    show F ⊆ A
      if (∑ a x∈A. norm (f x)) − δ < (∑ x∈F. norm (f x))
      using that ‹F ∈ {F. finite F ∧ F ⊆ A}› by blast
    show (∑ a x∈A. norm (f x)) − (∑ a x∈F. norm (f x)) < δ
      if (∑ a x∈A. norm (f x)) − δ < (∑ x∈F. norm (f x))
      using that ‹F ∈ {F. finite F ∧ F ⊆ A}› by auto
  qed
  thus ?thesis using that
    apply atomize_elim
    using ‹F ∈ {F. finite F ∧ F ⊆ A}› less_imp_le by blast
qed
obtain F2 where F2A: F2 ⊆ A and finF2: finite F2
  and dist: dist (sum (λx. norm (f x)) F2) (infsum (λx. norm (f x)) A) ≤ δ
  apply atomize_elim
  by (metis ‹0 < δ› conv_sum_norm infsum_finite_approximation)
have  leq_eps': infsum (λx. norm (f x)) (A−F2) ≤ δ
  apply (subst infsum_Diff)
  using finF2 F2A dist by (auto simp: dist_norm)
define F where F = F1 ∪ F2
have FA: F ⊆ A and finF: finite F
  unfolding F_def using F1A F2A finF1 finF2 by auto

have (∑ a x∈A − (F1 ∪ F2). norm (f x)) ≤ (∑ a x∈A − F1. norm (f x))
  apply (rule infsetsum_mono_neutral_left)
  using abs_summable_on_subset assms by fastforce+
hence leq_eps: infsetsum (λx. norm (f x)) (A−F) ≤ δ
  unfolding F_def
  using leq_eps by linarith
have infsum (λx. norm (f x)) (A − (F1 ∪ F2))
    ≤ infsum (λx. norm (f x)) (A − F2)
  apply (rule infsum_mono_neutral)
  using finF by (auto simp add: finF2 summable_on_cofin_subset F_def)
hence leq_eps': infsum (λx. norm (f x)) (A−F) ≤ δ
  unfolding F_def
  by (rule order.trans[OF _ leq_eps'])
have norm (infsetsum f A − infsetsum f F) = norm (infsetsum f (A−F))
  apply (subst infsetsum_Diff [symmetric])
  by (auto simp: FA assms)
also have ... ≤ infsetsum (λx. norm (f x)) (A−F)
  using norm_infsetsum_bound by blast
also have ... ≤ δ
```

```
    using leq_eps by simp
  finally have diff1: norm (infsetsum f A − infsetsum f F) ≤ δ
    by assumption
  have norm (infsum f A − infsum f F) = norm (infsum f (A−F))
    apply (subst infsum_Diff [symmetric])
    by (auto simp: assms abs_summable_summable finF FA)
  also have ... ≤ infsum (λx. norm (f x)) (A−F)
    by (simp add: finF summable_on_cofin_subset norm_infsum_bound)
  also have ... ≤ δ
    using leq_eps' by simp
  finally have diff2: norm (infsum f A − infsum f F) ≤ δ
    by assumption

  have x1: infsetsum f F = infsum f F
    using finF by simp
  have norm (infsetsum f A − infsum f A) ≤ norm (infsetsum f A − infsetsum
f F) + norm (infsum f A − infsum f F)
    apply (rule_tac norm_diff_triangle_le)
     apply auto
    by (simp_all add: x1 norm_minus_commute)
  also have ... ≤ ε
    using diff1 diff2 δ_def by linarith
  finally show ?thesis
    by assumption
 qed
 hence norm (infsetsum f A − infsum f A) = 0
  by (meson antisym_conv1 dense_ge norm_not_less_zero)
 thus ?thesis
  by auto
qed

end
```

9.24 Faces, Extreme Points, Polytopes, Polyhedra etc

Ported from HOL Light by L C Paulson

```
theory Polytope
imports Cartesian_Euclidean_Space Path_Connected
begin
```

9.24.1 Faces of a (usually convex) set

```
definition face_of :: ['a::real_vector set, 'a set] ⇒ bool (infixr ‹(face'_of)› 50)
 where
 T face_of S ⟷
      T ⊆ S ∧ convex T ∧
      (∀ a ∈ S. ∀ b ∈ S. ∀ x ∈ T. x ∈ open_segment a b ⟶ a ∈ T ∧ b ∈ T)
```

lemma *face_ofD*: ⟦*T face_of S*; *x* ∈ *open_segment a b*; *a* ∈ *S*; *b* ∈ *S*; *x* ∈ *T*⟧ ⟹ *a* ∈ *T* ∧ *b* ∈ *T*
unfolding *face_of_def* **by** *blast*

lemma *face_of_translation_eq* [*simp*]:
((+) *a* ‘ *T face_of* (+) *a* ‘ *S*) ⟷ *T face_of S*
proof −
have ∗: ⋀*a T S*. *T face_of S* ⟹ ((+) *a* ‘ *T face_of* (+) *a* ‘ *S*)
by (*simp add*: *face_of_def*)
show *?thesis*
by (*force simp*: *image_comp o_def dest*: ∗ [**where** *a* = −*a*] *intro*: ∗)
qed

lemma *face_of_linear_image*:
assumes *linear f inj f*
shows (*f* ‘ *c face_of f* ‘ *S*) ⟷ *c face_of S*
by (*simp add*: *face_of_def inj_image_subset_iff inj_image_mem_iff open_segment_linear_image assms*)

lemma *faces_of_linear_image*:
⟦*linear f*; *inj f*⟧ ⟹ {*T*. *T face_of* (*f* ‘ *S*)} = (*image f*) ‘ {*T*. *T face_of S*}
by (*smt* (*verit*) *Collect_cong face_of_def face_of_linear_image setcompr_eq_image subset_imageE*)

lemma *face_of_refl*: *convex S* ⟹ *S face_of S*
by (*auto simp*: *face_of_def*)

lemma *face_of_refl_eq*: *S face_of S* ⟷ *convex S*
by (*auto simp*: *face_of_def*)

lemma *empty_face_of* [*iff*]: {} *face_of S*
by (*simp add*: *face_of_def*)

lemma *face_of_empty* [*simp*]: *S face_of* {} ⟷ *S* = {}
by (*meson empty_face_of face_of_def subset_empty*)

lemma *face_of_trans* [*trans*]: ⟦*S face_of T*; *T face_of u*⟧ ⟹ *S face_of u*
unfolding *face_of_def* **by** (*safe*; *blast*)

lemma *face_of_face*: *T face_of S* ⟹ (*f face_of T* ⟷ *f face_of S* ∧ *f* ⊆ *T*)
unfolding *face_of_def* **by** (*safe*; *blast*)

lemma *face_of_subset*: ⟦*F face_of S*; *F* ⊆ *T*; *T* ⊆ *S*⟧ ⟹ *F face_of T*
unfolding *face_of_def* **by** (*safe*; *blast*)

lemma *face_of_slice*: ⟦*F face_of S*; *convex T*⟧ ⟹ (*F* ∩ *T*) *face_of* (*S* ∩ *T*)
unfolding *face_of_def* **by** (*blast intro*: *convex_Int*)

lemma *face_of_Int*: ⟦*t1 face_of S*; *t2 face_of S*⟧ ⟹ (*t1* ∩ *t2*) *face_of S*
unfolding *face_of_def* **by** (*blast intro*: *convex_Int*)

lemma *face_of_Inter*: ⟦*A* ≠ {}; ⋀*T*. *T* ∈ *A* ⟹ *T face_of S*⟧ ⟹ (⋂ *A*) *face_of S*
unfolding *face_of_def* **by** (*blast intro*: *convex_Inter*)

lemma *face_of_Int_Int*: ⟦*F face_of T*; *F′ face_of t′*⟧ ⟹ (*F* ∩ *F′*) *face_of* (*T* ∩ *t′*)
unfolding *face_of_def* **by** (*blast intro*: *convex_Int*)

lemma *face_of_imp_subset*: *T face_of S* ⟹ *T* ⊆ *S*
unfolding *face_of_def* **by** *blast*

proposition *face_of_imp_eq_affine_Int*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S*: *convex S* **and** *T*: *T face_of S*
shows *T* = (*affine hull T*) ∩ *S*
proof −
have *convex T* **using** *T* **by** (*simp add*: *face_of_def*)
have ∗: *False* **if** *x*: *x* ∈ *affine hull T* **and** *x* ∈ *S x* ∉ *T* **and** *y*: *y* ∈ *rel_interior T* **for** *x y*
proof −
obtain *e* **where** *e*>*0* **and** *e*: *cball y e* ∩ *affine hull T* ⊆ *T*
using *y* **by** (*auto simp*: *rel_interior_cball*)
have *y* ≠ *x y* ∈ *S y* ∈ *T*
using *face_of_imp_subset rel_interior_subset T that* **by** *blast+*
then have *zne*: ⋀*u*. ⟦*u* ∈ {*0*<..<*1*}; (*1* − *u*) ∗*R y* + *u* ∗*R x* ∈ *T*⟧ ⟹ *False*
using ‹*x* ∈ *S*› ‹*x* ∉ *T*› ‹*T face_of S*› **unfolding** *face_of_def*
by (*meson greaterThanLessThan_iff in_segment*(*2*))
define *u* **where** *u* ≡ *min* (*1*/*2*) (*e* / *norm* (*x* − *y*))
have *in01*: *u* ∈ {*0*<..<*1*}
using ‹*y* ≠ *x*› ‹*e* > *0*› **by** (*simp add*: *u_def*)
have *norm* (*u* ∗*R y* − *u* ∗*R x*) ≤ *e*
using ‹*e* > *0*›
by (*simp add*: *u_def norm_minus_commute min_mult_distrib_right flip*: *scaleR_diff_right*)
then have *dist y* ((*1* − *u*) ∗*R y* + *u* ∗*R x*) ≤ *e*
by (*simp add*: *dist_norm algebra_simps*)
then show *False*
using *zne* [*OF in01 e* [*THEN subsetD*]] **by** (*simp add*: ‹*y* ∈ *T*› *hull_inc mem_affine x*)
qed
show *?thesis*
proof (*rule subset_antisym*)
show *T* ⊆ *affine hull T* ∩ *S*
using *assms* **by** (*simp add*: *hull_subset face_of_imp_subset*)
show *affine hull T* ∩ *S* ⊆ *T*
using ∗ ‹*convex T*› *rel_interior_eq_empty* **by** *fastforce*

qed
qed

lemma *face_of_imp_closed*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *convex S closed S T face_of S* **shows** *closed T*
by (*metis affine_affine_hull affine_closed closed_Int face_of_imp_eq_affine_Int assms*)

lemma *face_of_Int_supporting_hyperplane_le_strong*:
assumes $convex(S \cap \{x.\ a \cdot x = b\})$ **and** *aleb*: $\bigwedge x.\ x \in S \Longrightarrow a \cdot x \le b$
shows $(S \cap \{x.\ a \cdot x = b\})$ *face_of S*
proof −
have ∗: $a \cdot u = a \cdot x$ **if** $x \in$ *open_segment u v u* $\in S$ $v \in S$ **and** *b*: $b = a \cdot x$
for *u v x*
proof (*rule antisym*)
show $a \cdot u \le a \cdot x$
using *aleb* ‹$u \in S$› ‹$b = a \cdot x$› **by** *blast*
next
obtain ξ **where** $b = a \cdot ((1 - \xi) *_R u + \xi *_R v)$ $0 < \xi$ $\xi < 1$
using ‹$b = a \cdot x$› ‹$x \in$ *open_segment u v*› *in_segment*
by (*auto simp*: *open_segment_image_interval split*: *if_split_asm*)
then have $b + \xi * (a \cdot u) \le a \cdot u + \xi * b$
using *aleb* [*OF* ‹$v \in S$›] **by** (*simp add*: *algebra_simps*)
then have $(1 - \xi) * b \le (1 - \xi) * (a \cdot u)$
by (*simp add*: *algebra_simps*)
then have $b \le a \cdot u$
using ‹$\xi < 1$› **by** *auto*
with *b* **show** $a \cdot x \le a \cdot u$ **by** *simp*
qed
show *?thesis*
using ∗ *open_segment_commute* **by** (*fastforce simp add*: *face_of_def assms*)
qed

lemma *face_of_Int_supporting_hyperplane_ge_strong*:
$[\![convex(S \cap \{x.\ a \cdot x = b\}); \bigwedge x.\ x \in S \Longrightarrow a \cdot x \ge b]\!]$
$\Longrightarrow (S \cap \{x.\ a \cdot x = b\})$ *face_of S*
using *face_of_Int_supporting_hyperplane_le_strong* [*of S* $-a$ $-b$] **by** *simp*

lemma *face_of_Int_supporting_hyperplane_le*:
$[\![convex\ S; \bigwedge x.\ x \in S \Longrightarrow a \cdot x \le b]\!] \Longrightarrow (S \cap \{x.\ a \cdot x = b\})$ *face_of S*
by (*simp add*: *convex_Int convex_hyperplane face_of_Int_supporting_hyperplane_le_strong*)

lemma *face_of_Int_supporting_hyperplane_ge*:
$[\![convex\ S; \bigwedge x.\ x \in S \Longrightarrow a \cdot x \ge b]\!] \Longrightarrow (S \cap \{x.\ a \cdot x = b\})$ *face_of S*
by (*simp add*: *convex_Int convex_hyperplane face_of_Int_supporting_hyperplane_ge_strong*)

lemma *face_of_imp_convex*: *T face_of S* $\Longrightarrow$ *convex T*
using *face_of_def* **by** *blast*

lemma *face_of_imp_compact*:
fixes $S :: 'a::euclidean\_space\ set$
shows $[\![convex\ S;\ compact\ S;\ T\ face\_of\ S]\!] \Longrightarrow compact\ T$
by (*meson bounded_subset compact_eq_bounded_closed face_of_imp_closed face_of_imp_subset*)

lemma *face_of_Int_subface*:
$[\![A \cap B\ face\_of\ A;\ A \cap B\ face\_of\ B;\ C\ face\_of\ A;\ D\ face\_of\ B]\!]$
$\Longrightarrow (C \cap D)\ face\_of\ C \wedge (C \cap D)\ face\_of\ D$
by (*meson face_of_Int_Int face_of_face inf_le1 inf_le2*)

lemma *subset_of_face_of*:
fixes $S :: 'a::real\_normed\_vector\ set$
assumes $T\ face\_of\ S\ u \subseteq S\ T \cap (rel\_interior\ u) \neq \{\}$
shows $u \subseteq T$
proof
fix c
assume $c \in u$
obtain b **where** $b \in T\ b \in rel\_interior\ u$ **using** *assms* **by** *auto*
then obtain e **where** $e>0\ b \in u$ **and** e: $cball\ b\ e \cap affine\ hull\ u \subseteq u$
by (*auto simp*: *rel_interior_cball*)
show $c \in T$
proof (*cases b=c*)
case *True* **with** ‹$b \in T$› **show** *?thesis* **by** *blast*
next
case *False*
define d **where** $d = b + (e\ /\ norm(b - c)) *_R (b - c)$
have $d \in cball\ b\ e \cap affine\ hull\ u$
using ‹$e > 0$› ‹$b \in u$› ‹$c \in u$›
by (*simp add*: *d_def dist_norm hull_inc mem_affine_3_minus False*)
with e **have** $d \in u$ **by** *blast*
have *nbc*: $norm\ (b - c) + e > 0$ **using** ‹$e > 0$›
by (*metis add.commute le_less_trans less_add_same_cancel2 norm_ge_zero*)
then have [*simp*]: $d \neq c$ **using** *False scaleR_cancel_left* [*of 1 + (e / norm (b − c)) b c*]
by (*simp add*: *algebra_simps d_def*) (*simp add*: *field_split_simps*)
have [*simp*]: $((e - e * e\ /\ (e + norm\ (b - c))))\ /\ norm\ (b - c)) = (e\ /\ (e + norm\ (b - c)))$
using *False nbc*
by (*simp add*: *divide_simps*) (*simp add*: *algebra_simps*)
have $b \in open\_segment\ d\ c$
apply (*simp add*: *open_segment_image_interval*)
apply (*simp add*: *d_def algebra_simps*)
apply (*rule_tac x=e / (e + norm (b − c))* **in** *image_eqI*)
using *False nbc* ‹$0 < e$› **by** (*auto simp*: *algebra_simps*)
then have $d \in T \wedge c \in T$
by (*meson* ‹$b \in T$› ‹$c \in u$› ‹$d \in u$› *assms face_ofD subset_iff*)
then show *?thesis* **..**

qed
qed

lemma *face_of_eq*:
fixes S :: *'a::real_normed_vector set*
assumes T *face_of* S U *face_of* S (*rel_interior* T) ∩ (*rel_interior* U) ≠ {}
shows $T = U$
using *assms*
unfolding *disjoint_iff_not_equal*
by (*metis IntI empty_iff face_of_imp_subset mem_rel_interior_ball subset_antisym subset_of_face_of*)

lemma *face_of_disjoint_rel_interior*:
fixes S :: *'a::real_normed_vector set*
assumes T *face_of* S $T \neq S$
shows $T \cap$ *rel_interior* $S = \{\}$
by (*meson assms subset_of_face_of face_of_imp_subset order_refl subset_antisym*)

lemma *face_of_disjoint_interior*:
fixes S :: *'a::real_normed_vector set*
assumes T *face_of* S $T \neq S$
shows $T \cap$ *interior* $S = \{\}$
using *assms face_of_disjoint_rel_interior interior_subset_rel_interior* **by** *fastforce*

lemma *face_of_subset_rel_boundary*:
fixes S :: *'a::real_normed_vector set*
assumes T *face_of* S $T \neq S$
shows $T \subseteq (S -$ *rel_interior* $S)$
by (*meson DiffI assms disjoint_iff_not_equal face_of_disjoint_rel_interior face_of_imp_subset subset_iff*)

lemma *face_of_subset_rel_frontier*:
fixes S :: *'a::real_normed_vector set*
assumes T *face_of* S $T \neq S$
shows $T \subseteq$ *rel_frontier* S
using *assms closure_subset face_of_disjoint_rel_interior face_of_imp_subset rel_frontier_def* **by** *fastforce*

lemma *face_of_aff_dim_lt*:
fixes S :: *'a::euclidean_space set*
assumes *convex* S T *face_of* S $T \neq S$
shows *aff_dim* $T <$ *aff_dim* S
proof −
have *aff_dim* $T \leq$ *aff_dim* S
by (*simp add*: *face_of_imp_subset aff_dim_subset assms*)
moreover have *aff_dim* $T \neq$ *aff_dim* S
by (*metis aff_dim_empty assms convex_rel_frontier_aff_dim face_of_imp_convex*

```
      face_of_subset_rel_frontier order_less_irrefl)
  ultimately show ?thesis
    by simp
qed

lemma subset_of_face_of_affine_hull:
    fixes S :: 'a::euclidean_space set
  assumes T: T face_of S and convex S U ⊆ S and dis: ¬ disjnt (affine hull T)
(rel_interior U)
  shows U ⊆ T
proof (rule subset_of_face_of [OF T ‹U ⊆ S›])
  show T ∩ rel_interior U ≠ {}
    using face_of_imp_eq_affine_Int [OF ‹convex S› T] rel_interior_subset dis
‹U ⊆ S› disjnt_def
    by fastforce
qed

lemma affine_hull_face_of_disjoint_rel_interior:
  fixes S :: 'a::euclidean_space set
  assumes convex S F face_of S F ≠ S
  shows affine hull F ∩ rel_interior S = {}
 by (meson antisym assms disjnt_def equalityD2 face_of_def subset_of_face_of_affine_hull)

lemma affine_diff_divide:
    assumes affine S k ≠ 0 k ≠ 1 and xy: x ∈ S y /R (1 − k) ∈ S
      shows (x − y) /R k ∈ S
proof −
  have inverse(k) ∗R (x − y) = (1 − inverse k) ∗R inverse(1 − k) ∗R y +
inverse(k) ∗R x
    using assms
      by (simp add: algebra_simps) (simp add: scaleR_left_distrib [symmetric]
field_split_simps)
  then show ?thesis
    using ‹affine S› xy by (auto simp: affine_alt)
qed

proposition face_of_conic:
  assumes conic S f face_of S
  shows conic f
  unfolding conic_def
proof (intro strip)
  fix x and c::real
  assume x ∈ f and 0 ≤ c
  have f: ⋀a b x. ⟦a ∈ S; b ∈ S; x ∈ f; x ∈ open_segment a b⟧ ⟹ a ∈ f ∧ b ∈ f
    using ‹f face_of S› face_ofD by blast
  show c ∗R x ∈ f
  proof (cases x=0 ∨ c=1)
    case True
    then show ?thesis
```

```
    using ‹x ∈ f› by auto
  next
    case False
    with ‹0 ≤ c› obtain d e where de: 0 ≤ d 0 ≤ e d < 1 1 < e d < e (d = c
∨ e = c)
      apply (simp add: neq_iff)
      by (metis gt_ex less_eq_real_def order_less_le_trans zero_less_one)
    then obtain [simp]: c *R x ∈ S e *R x ∈ S ‹x ∈ S›
      using ‹x ∈ f› assms conic_mul face_of_imp_subset by blast
    have x ∈ open_segment (d *R x) (e *R x) if c *R x ∉ f
      using de False that
      apply (simp add: in_segment)
      apply (rule_tac x=(1 − d) / (e − d) in exI)
      apply (simp add: field_simps)
      by (smt (verit, del_insts) add_divide_distrib divide_self scaleR_collapse)
    then show ?thesis
      using ‹conic S› f [of d *R x e *R x] de ‹x ∈ f›
      by (force simp: conic_def in_segment)
  qed
qed

proposition face_of_convex_hulls:
    assumes S: finite S T ⊆ S and disj: affine hull T ∩ convex hull (S − T) =
{}
    shows (convex hull T) face_of (convex hull S)
proof −
  have fin: finite T finite (S − T) using assms
    by (auto simp: finite_subset)
  have ∗: x ∈ convex hull T
        if x: x ∈ convex hull S and y: y ∈ convex hull S and w: w ∈ convex hull
T w ∈ open_segment x y
        for x y w
  proof −
    have waff: w ∈ affine hull T
      using convex_hull_subset_affine_hull w by blast
    obtain a b where a: ⋀i. i ∈ S ⟹ 0 ≤ a i and asum: sum a S = 1 and
aeqx: (∑ i∈S. a i *R i) = x
                and b: ⋀i. i ∈ S ⟹ 0 ≤ b i and bsum: sum b S = 1 and beqy:
(∑ i∈S. b i *R i) = y
      using x y by (auto simp: assms convex_hull_finite)
    obtain u where (1 − u) *R x + u *R y ∈ convex hull T x ≠ y and weq: w
= (1 − u) *R x + u *R y
              and u01: 0 < u u < 1
      using w by (auto simp: open_segment_image_interval split: if_split_asm)
    define c where c i = (1 − u) ∗ a i + u ∗ b i for i
    have cge0: ⋀i. i ∈ S ⟹ 0 ≤ c i
      using a b u01 by (simp add: c_def)
    have sumc1: sum c S = 1
      by (simp add: c_def sum.distrib sum_distrib_left [symmetric] asum bsum)
```

```
    have sumci_xy: (∑ i∈S. c i *R i) = (1 - u) *R x + u *R y
      apply (simp add: c_def sum.distrib scaleR_left_distrib)
     by (simp only: scaleR_scaleR [symmetric] Real_Vector_Spaces.scaleR_right.sum
[symmetric] aeqx beqy)
    show ?thesis
    proof (cases sum c (S - T) = 0)
      case True
      have ci0: ⋀i. i ∈ (S - T) ⟹ c i = 0
        using True cge0 fin(2) sum_nonneg_eq_0_iff by auto
      have a0: a i = 0 if i ∈ (S - T) for i
        using ci0 [OF that] u01 a [of i] b [of i] that
       by (simp add: c_def Groups.ordered_comm_monoid_add_class.add_nonneg_eq_0_iff)
      have  sum a T = 1
        using assms by (metis sum.mono_neutral_cong_right a0 asum)
      moreover have (∑ x∈T. a x *R x) = x
      using a0 assms by (auto simp: cge0 a aeqx [symmetric] sum.mono_neutral_right)
      ultimately show ?thesis
        using a assms(2) by (auto simp add: convex_hull_finite ‹finite T› )
    next
      case False
      define k where k = sum c (S - T)
      have k > 0 using False
      unfolding k_def by (metis DiffD1 antisym_conv cge0 sum_nonneg not_less)
      have weq_sumsum: w = sum (λx. c x *R x) T + sum (λx. c x *R x) (S -
T)
        by (metis (no_types) add.commute S(1) S(2) sum.subset_diff sumci_xy
weq)
      show ?thesis
      proof (cases k = 1)
        case True
        then have sum c T = 0
          by (simp add: S k_def sum_diff sumc1)
        then have sum c (S - T) = 1
          by (simp add: S sum_diff sumc1)
        moreover have ci0: ⋀i. i ∈ T ⟹ c i = 0
         by (meson ‹finite T› ‹sum c T = 0› ‹T ⊆ S› cge0 sum_nonneg_eq_0_iff
subsetCE)
        then have (∑ i∈S-T. c i *R i) = w
          by (simp add: weq_sumsum)
        ultimately have w ∈ convex hull (S - T)
          using cge0 by (auto simp add: convex_hull_finite fin)
        then show ?thesis
          using disj waff by blast
      next
        case False
        then have sumcf: sum c T = 1 - k
          by (simp add: S k_def sum_diff sumc1)
        have ⋀x. x ∈ T ⟹ 0 ≤ inverse (1 - k) * c x
       by (metis ‹T ⊆ S› cge0 inverse_nonnegative_iff_nonnegative mult_nonneg_nonneg
```

```
subsetD sum_nonneg sumcf)
      moreover have (∑ x∈T. inverse (1 − k) ∗ c x) = 1
      by (metis False eq_iff_diff_eq_0 mult.commute right_inverse sum_distrib_left
sumcf)
      ultimately have (∑ i∈T. c i ∗R i) /R (1 − k) ∈ convex hull T
        apply (simp add: convex_hull_finite fin)
       by (metis (mono_tags, lifting) scaleR_right.sum scaleR_scaleR sum.cong)
      with ‹0 < k› have inverse(k) ∗R (w − sum (λi. c i ∗R i) T) ∈ affine hull
T
          by (simp add: affine_diff_divide [OF affine_affine_hull] False waff
convex_hull_subset_affine_hull [THEN subsetD])
      moreover have inverse(k) ∗R (w − sum (λx. c x ∗R x) T) ∈ convex hull
(S − T)
        apply (simp add: weq_sumsum convex_hull_finite fin)
        apply (rule_tac x=λi. inverse k ∗ c i in exI)
        using ‹k > 0› cge0
        apply (auto simp: scaleR_right.sum simp flip: sum_distrib_left k_def)
        done
      ultimately show ?thesis
        using disj by blast
    qed
  qed
  have [simp]: convex hull T ⊆ convex hull S
    by (simp add: ‹T ⊆ S› hull_mono)
  show ?thesis
    using open_segment_commute by (auto simp: face_of_def intro: ∗)
qed

proposition face_of_convex_hull_insert:
  assumes finite S a ∉ affine hull S and T: T face_of convex hull S
  shows T face_of convex hull insert a S
proof −
  have convex hull S face_of convex hull insert a S
    by (simp add: assms face_of_convex_hulls insert_Diff_if subset_insertI)
  then show ?thesis
    using T face_of_trans by blast
qed

proposition face_of_affine_trivial:
    assumes affine S T face_of S
    shows T = {} ∨ T = S
proof (rule ccontr, clarsimp)
  assume T ≠ {} T ≠ S
  then obtain a where a ∈ T by auto
  then have a ∈ S
    using ‹T face_of S› face_of_imp_subset by blast
  have S ⊆ T
  proof
```

```
    fix b  assume b ∈ S
    show b ∈ T
    proof (cases a = b)
      case True with ‹a ∈ T› show ?thesis by auto
    next
      case False
      then have a ∈ open_segment (2 *R a − b) b
      by (metis diff_add_cancel inverse_eq_divide midpoint_def midpoint_in_open_segment

            scaleR_2 scaleR_half_double)
      moreover have 2 *R a − b ∈ S
        by (rule mem_affine [OF ‹affine S› ‹a ∈ S› ‹b ∈ S›, of 2 −1, simplified])
      moreover note ‹b ∈ S› ‹a ∈ T›
      ultimately show ?thesis
        by (rule face_ofD [OF ‹T face_of S›, THEN conjunct2])
    qed
  qed
  then show False
    using ‹T ≠ S› ‹T face_of S› face_of_imp_subset by blast
qed


lemma face_of_affine_eq:
   affine S ⟹ (T face_of S ⟷ T = {} ∨ T = S)
using affine_imp_convex face_of_affine_trivial face_of_refl by auto


proposition Inter_faces_finite_altbound:
    fixes T :: 'a::euclidean_space set set
    assumes cfaI: ⋀c. c ∈ T ⟹ c face_of S
    shows ∃F'. finite F' ∧ F' ⊆ T ∧ card F' ≤ DIM('a) + 2 ∧ ⋂F' = ⋂T
proof (cases ∀F'. finite F' ∧ F' ⊆ T ∧ card F' ≤ DIM('a) + 2 ⟶ (∃c. c ∈ T
∧ c ∩ (⋂F') ⊂ (⋂F')))
  case True
  then obtain c where c:
       ⋀F'. ⟦finite F'; F' ⊆ T; card F' ≤ DIM('a) + 2⟧ ⟹ c F' ∈ T ∧ c F' ∩
(⋂F') ⊂ (⋂F')
    by metis
  define d where d ≡ λn. ((λr. insert (c r) r)^^n) {c{}}
  note d_def [simp]
  have dSuc: ⋀n. d (Suc n) = insert (c (d n)) (d n)
    by simp
  have dn_notempty: d n ≠ {} for n
    by (induction n) auto
  have dn_le_Suc: d n ⊆ T ∧ finite(d n) ∧ card(d n) ≤ Suc n if n ≤ DIM('a) +
2 for n
  using that
  proof (induction n)
    case 0
```

```
    then show ?case by (simp add: c)
  next
    case (Suc n)
    then show ?case by (auto simp: c card_insert_if)
  qed
  have aff_dim_le: aff_dim(⋂(d n)) ≤ DIM('a) − int n if n ≤ DIM('a) + 2 for
n
  using that
  proof (induction n)
    case 0
    then show ?case
      by (simp add: aff_dim_le_DIM)
  next
    case (Suc n)
    have fs: ⋂(d (Suc n)) face_of S
     by (meson Suc.prems cfaI dn_le_Suc dn_notempty face_of_Inter subsetCE)
    have condn: convex (⋂(d n))
      using Suc.prems nat_le_linear not_less_eq_eq
      by (blast intro: face_of_imp_convex cfaI convex_Inter dest: dn_le_Suc)
    have fdn: ⋂(d (Suc n)) face_of ⋂(d n)
     by (metis (no_types, lifting) Inter_anti_mono Suc.prems dSuc cfaI dn_le_Suc
dn_notempty face_of_Inter face_of_imp_subset face_of_subset subset_iff sub-
set_insertI)
    have ne: ⋂(d (Suc n)) ≠ ⋂(d n)
     by (metis (no_types, lifting) Suc.prems Suc_leD c complete_lattice_class.Inf_insert
dSuc dn_le_Suc less_irrefl order.trans)
   have ∗: ⋀m::int. ⋀d. ⋀d'::int. d < d' ∧ d' ≤ m − n ⟹ d ≤ m − of_nat(n+1)
      by arith
    have aff_dim (⋂(d (Suc n))) < aff_dim (⋂(d n))
      by (rule face_of_aff_dim_lt [OF condn fdn ne])
    moreover have aff_dim (⋂(d n)) ≤ int (DIM('a)) − int n
      using Suc by auto
    ultimately
    have aff_dim (⋂(d (Suc n))) ≤ int (DIM('a)) − (n+1) by arith
    then show ?case by linarith
  qed
  have aff_dim (⋂(d (DIM('a) + 2))) ≤ −2
     using aff_dim_le [OF order_refl] by simp
  with aff_dim_geq [of ⋂(d (DIM('a) + 2))] show ?thesis
    using order.trans by fastforce
next
  case False
  then show ?thesis by fastforce
qed

lemma faces_of_translation:
   {F. F face_of (+) a ' S} = (image ((+) a)) ' {F. F face_of S}
proof −
  have ⋀F. F face_of (+) a ' S ⟹ ∃G. G face_of S ∧ F = (+) a ' G
```

```
    by (metis face_of_imp_subset face_of_translation_eq subset_imageE)
  then show ?thesis
    by (auto simp: image_iff)
qed

proposition face_of_Times:
  assumes F face_of S and F' face_of S'
    shows (F × F') face_of (S × S')
proof −
  have F × F' ⊆ S × S'
    using assms [unfolded face_of_def] by blast
  moreover
  have convex (F × F')
    using assms [unfolded face_of_def] by (blast intro: convex_Times)
  moreover
    have a ∈ F ∧ a' ∈ F' ∧ b ∈ F ∧ b' ∈ F'
      if a ∈ S b ∈ S a' ∈ S' b' ∈ S' x ∈ F × F' x ∈ open_segment (a,a') (b,b')
      for a b a' b' x
  proof (cases b=a ∨ b'=a')
    case True with that show ?thesis
      using assms
      by (force simp: in_segment dest: face_ofD)
  next
    case False with assms [unfolded face_of_def] that show ?thesis
      by (blast dest!: open_segment_PairD)
  qed
  ultimately show ?thesis
    unfolding face_of_def by blast
qed

corollary face_of_Times_decomp:
    fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
    shows C face_of (S × S') ⟷ (∃F F'. F face_of S ∧ F' face_of S' ∧ C =
F × F')
     (is ?lhs = ?rhs)
proof
  assume C: ?lhs
  show ?rhs
  proof (cases C = {})
    case True then show ?thesis by auto
  next
    case False
    have 1: fst ‘ C ⊆ S snd ‘ C ⊆ S'
      using C face_of_imp_subset by fastforce+
    have convex C
      using C by (metis face_of_imp_convex)
    have conv: convex (fst ‘ C) convex (snd ‘ C)
      by (simp_all add: ‹convex C› convex_linear_image linear_fst linear_snd)
    have fstab: a ∈ fst ‘ C ∧ b ∈ fst ‘ C
```

```
        if a ∈ S b ∈ S x ∈ open_segment a b (x,x') ∈ C for a b x x'
    proof −
      have ∗: (x,x') ∈ open_segment (a,x') (b,x')
        using that by (auto simp: in_segment)
      show ?thesis
        using face_ofD [OF C ∗] that face_of_imp_subset [OF C] by force
    qed
    have fst: fst ‘ C face_of S
      by (force simp: face_of_def 1 conv fstab)
    have sndab: a' ∈ snd ‘ C ∧ b' ∈ snd ‘ C
      if a' ∈ S' b' ∈ S' x' ∈ open_segment a' b' (x,x') ∈ C for a' b' x x'
    proof −
      have ∗: (x,x') ∈ open_segment (x,a') (x,b')
        using that by (auto simp: in_segment)
      show ?thesis
        using face_ofD [OF C ∗] that face_of_imp_subset [OF C] by force
    qed
    have snd: snd ‘ C face_of S'
      by (force simp: face_of_def 1 conv sndab)
    have cc: rel_interior C ⊆ rel_interior (fst ‘ C) × rel_interior (snd ‘ C)
      by (force simp: face_of_Times rel_interior_Times conv fst snd ‹convex C›
linear_fst linear_snd rel_interior_convex_linear_image [symmetric])
    have C = fst ‘ C × snd ‘ C
    proof (rule face_of_eq [OF C])
      show fst ‘ C × snd ‘ C face_of S × S'
        by (simp add: face_of_Times rel_interior_Times conv fst snd)
      show rel_interior C ∩ rel_interior (fst ‘ C × snd ‘ C) ≠ {}
        using False rel_interior_eq_empty ‹convex C› cc
        by (auto simp: face_of_Times rel_interior_Times conv fst)
    qed
    with fst snd show ?thesis by metis
  qed
qed (use face_of_Times in auto)

lemma face_of_Times_eq:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  shows (F × F') face_of (S × S') ⟷ F = {} ∨ F' = {} ∨ F face_of S ∧ F'
face_of S'
  by (auto simp: face_of_Times_decomp times_eq_iff)

lemma hyperplane_face_of_halfspace_le: {x. a • x = b} face_of {x. a • x ≤ b}
proof −
  have {x. a • x ≤ b} ∩ {x. a • x = b} = {x. a • x = b}
    by auto
  with face_of_Int_supporting_hyperplane_le [OF convex_halfspace_le [of a b],
of a b]
  show ?thesis by auto
qed
```

lemma *hyperplane_face_of_halfspace_ge*: $\{x.\ a \cdot x = b\}$ *face_of* $\{x.\ a \cdot x \geq b\}$
proof −
have $\{x.\ a \cdot x \geq b\} \cap \{x.\ a \cdot x = b\} = \{x.\ a \cdot x = b\}$
by *auto*
with *face_of_Int_supporting_hyperplane_ge* [*OF convex_halfspace_ge* [*of b a*], *of b a*]
show *?thesis* **by** *auto*
qed

lemma *face_of_halfspace_le*:
fixes $a :: 'n::euclidean\_space$
shows F *face_of* $\{x.\ a \cdot x \leq b\} \longleftrightarrow F = \{\} \vee F = \{x.\ a \cdot x = b\} \vee F = \{x.\ a \cdot x \leq b\}$
(**is** *?lhs* = *?rhs*)
proof (*cases* $a = 0$)
case *True* **then show** *?thesis*
using *face_of_affine_eq affine_UNIV* **by** *auto*
next
case *False*
then have *ine*: *interior* $\{x.\ a \cdot x \leq b\} \neq \{\}$
using *halfspace_eq_empty_lt interior_halfspace_le* **by** *blast*
show *?thesis*
proof
assume *L*: *?lhs*
have F *face_of* $\{x.\ a \cdot x = b\}$ **if** $F \neq \{x.\ a \cdot x \leq b\}$
proof −
have F *face_of rel_frontier* $\{x.\ a \cdot x \leq b\}$
proof (*rule face_of_subset* [*OF L*])
show $F \subseteq$ *rel_frontier* $\{x.\ a \cdot x \leq b\}$
by (*simp add*: *L face_of_subset_rel_frontier that*)
qed (*force simp*: *rel_frontier_def closed_halfspace_le*)
then show *?thesis*
using *False*
by (*simp add*: *frontier_halfspace_le rel_frontier_nonempty_interior* [*OF ine*])
qed
with *L* **show** *?rhs*
using *affine_hyperplane face_of_affine_eq* **by** *blast*
next
assume *?rhs*
then show *?lhs*
by (*metis convex_halfspace_le empty_face_of face_of_refl hyperplane_face_of_halfspace_le*)
qed
qed

lemma *face_of_halfspace_ge*:
fixes $a :: 'n::euclidean\_space$
shows F *face_of* $\{x.\ a \cdot x \geq b\} \longleftrightarrow F = \{\} \vee F = \{x.\ a \cdot x = b\} \vee F = \{x.\ a \cdot x \geq b\}$

using *face_of_halfspace_le* [*of F* $-a$ $-b$] **by** *simp*

9.24.2 Exposed faces

That is, faces that are intersection with supporting hyperplane

definition *exposed_face_of* :: ['*a*::*euclidean_space set*, '*a set*] ⇒ *bool*
(**infixr** ‹(*exposed'_face'_of*)› *50*)
where *T exposed_face_of S* ⟷
T face_of S ∧ (∃ *a b*. *S* ⊆ {*x*. *a* · *x* ≤ *b*} ∧ *T* = *S* ∩ {*x*. *a* · *x* = *b*})

lemma *empty_exposed_face_of* [*iff*]: {} *exposed_face_of S*
proof −
have *S* ⊆ {*x*. *0* · *x* ≤ *1*} ∧ {} = *S* ∩ {*x*. *0* · *x* = *1*}
by *force*
then show *?thesis*
using *exposed_face_of_def* **by** *blast*
qed

lemma *exposed_face_of_refl_eq* [*simp*]: *S exposed_face_of S* ⟷ *convex S*
proof
assume *S*: *convex S*
have *S* ⊆ {*x*. *0* · *x* ≤ *0*} ∧ *S* = *S* ∩ {*x*. *0* · *x* = *0*}
by *auto*
with *S* **show** *S exposed_face_of S*
using *exposed_face_of_def face_of_refl_eq* **by** *blast*
qed (*simp add*: *exposed_face_of_def face_of_refl_eq*)

lemma *exposed_face_of_refl*: *convex S* ⟹ *S exposed_face_of S*
by *simp*

lemma *exposed_face_of*:
T exposed_face_of S ⟷
T face_of S ∧ (*T* = {} ∨ *T* = *S* ∨
(∃ *a b*. *a* ≠ *0* ∧ *S* ⊆ {*x*. *a* · *x* ≤ *b*} ∧ *T* = *S* ∩ {*x*. *a* · *x* = *b*}))
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⟹ *?rhs*
by (*smt* (*verit*) *Collect_cong exposed_face_of_def hyperplane_eq_empty inf.absorb_iff1*
inf_bot_right inner_zero_left)
show *?rhs* ⟹ *?lhs*
using *exposed_face_of_def face_of_imp_convex* **by** *fastforce*
qed

lemma *exposed_face_of_Int_supporting_hyperplane_le*:
⟦*convex S*; ⋀*x*. *x* ∈ *S* ⟹ *a* · *x* ≤ *b*⟧ ⟹ (*S* ∩ {*x*. *a* · *x* = *b*}) *exposed_face_of*
S
by (*force simp*: *exposed_face_of_def face_of_Int_supporting_hyperplane_le*)

lemma *exposed_face_of_Int_supporting_hyperplane_ge*:

⟦*convex S*; ⋀*x*. *x* ∈ *S* ⟹ *a* · *x* ≥ *b*⟧ ⟹ (*S* ∩ {*x*. *a* · *x* = *b*}) *exposed_face_of S*
using *exposed_face_of_Int_supporting_hyperplane_le* [*of S* −*a* −*b*] **by** *simp*

proposition *exposed_face_of_Int*:
assumes *T exposed_face_of S*
and *U exposed_face_of S*
shows (*T* ∩ *U*) *exposed_face_of S*
proof −
obtain *a b* **where** *T*: *S* ∩ {*x*. *a* · *x* = *b*} *face_of S*
and *S*: *S* ⊆ {*x*. *a* · *x* ≤ *b*}
and *teq*: *T* = *S* ∩ {*x*. *a* · *x* = *b*}
using *assms* **by** (*auto simp*: *exposed_face_of_def*)
obtain *a′ b′* **where** *U*: *S* ∩ {*x*. *a′* · *x* = *b′*} *face_of S*
and *s′*: *S* ⊆ {*x*. *a′* · *x* ≤ *b′*}
and *ueq*: *U* = *S* ∩ {*x*. *a′* · *x* = *b′*}
using *assms* **by** (*auto simp*: *exposed_face_of_def*)
have *tu*: *T* ∩ *U face_of S*
using *T teq U ueq* **by** (*simp add*: *face_of_Int*)
have *ss*: *S* ⊆ {*x*. (*a* + *a′*) · *x* ≤ *b* + *b′*}
using *S s′* **by** (*force simp*: *inner_left_distrib*)
have *S* ⊆ {*x*. (*a* + *a′*) · *x* ≤ *b* + *b′*} ∧ *T* ∩ *U* = *S* ∩ {*x*. (*a* + *a′*) · *x* = *b* + *b′*}
using *S s′* **by** (*fastforce simp*: *ss inner_left_distrib teq ueq*)
then show *?thesis*
using *exposed_face_of_def tu* **by** *auto*
qed

proposition *exposed_face_of_Inter*:
fixes *P* :: *′a*::*euclidean_space set set*
assumes *P* ≠ {}
and ⋀*T*. *T* ∈ *P* ⟹ *T exposed_face_of S*
shows ⋂*P exposed_face_of S*
proof −
obtain *Q* **where** *finite Q* **and** *QsubP*: *Q* ⊆ *P card Q* ≤ *DIM*(*′a*) + *2* **and** *IntQ*: ⋂*Q* = ⋂*P*
using *Inter_faces_finite_altbound* [*of P S*] *assms* [*unfolded exposed_face_of*]
by *force*
show *?thesis*
proof (*cases Q* = {})
case *True* **then show** *?thesis*
by (*metis IntQ Inter_UNIV_conv*(*2*) *assms*(*1*) *assms*(*2*) *ex_in_conv*)
next
case *False*
have *Q* ⊆ {*T*. *T exposed_face_of S*}
using *QsubP assms* **by** *blast*
moreover have *Q* ⊆ {*T*. *T exposed_face_of S*} ⟹ ⋂*Q exposed_face_of S*
using ‹*finite Q*› *False*
by (*induction Q rule*: *finite_induct*; *use exposed_face_of_Int* **in** *fastforce*)
ultimately show *?thesis*

```
    by (simp add: IntQ)
  qed
qed

proposition exposed_face_of_sums:
  assumes convex S and convex T
      and F exposed_face_of {x + y | x y. x ∈ S ∧ y ∈ T}
          (is F exposed_face_of ?ST)
  obtains k l
    where k exposed_face_of S l exposed_face_of T
          F = {x + y | x y. x ∈ k ∧ y ∈ l}
proof (cases F = {})
  case True then show ?thesis
    using that by blast
next
  case False
  show ?thesis
  proof (cases F = ?ST)
    case True then show ?thesis
      using assms exposed_face_of_refl_eq that by blast
  next
    case False
    obtain p where p ∈ F using ‹F ≠ {}› by blast
    moreover
    obtain u z where T: ?ST ∩ {x. u • x = z} face_of ?ST
                and S: ?ST ⊆ {x. u • x ≤ z}
                and feq: F = ?ST ∩ {x. u • x = z}
      using assms by (auto simp: exposed_face_of_def)
    ultimately obtain a0 b0
          where p: p = a0 + b0 and a0 ∈ S b0 ∈ T and z: u • p = z
      by auto
    have lez: u • (x + y) ≤ z if x ∈ S y ∈ T for x y
      using S that by auto
    have sef: S ∩ {x. u • x = u • a0} exposed_face_of S
    proof (rule exposed_face_of_Int_supporting_hyperplane_le [OF ‹convex S›])
      show ⋀x. x ∈ S ⟹ u • x ≤ u • a0
        by (metis p z add_le_cancel_right inner_right_distrib lez [OF _ ‹b0 ∈
T›])
    qed
    have tef: T ∩ {x. u • x = u • b0} exposed_face_of T
    proof (rule exposed_face_of_Int_supporting_hyperplane_le [OF ‹convex T›])
      show ⋀x. x ∈ T ⟹ u • x ≤ u • b0
        by (metis p z add.commute add_le_cancel_right inner_right_distrib lez
[OF ‹a0 ∈ S›])
    qed
    have {x + y |x y. x ∈ S ∧ u • x = u • a0 ∧ y ∈ T ∧ u • y = u • b0} ⊆ F
      by (auto simp: feq) (metis inner_right_distrib p z)
    moreover have F ⊆ {x + y |x y. x ∈ S ∧ u • x = u • a0 ∧ y ∈ T ∧ u • y =
u • b0}
```

```
    proof −
      have ⋀x y. ⟦z = u · (x + y); x ∈ S; y ∈ T⟧
          ⟹ u · x = u · a0 ∧ u · y = u · b0
        by (smt (verit, best) z p ‹a0 ∈ S› ‹b0 ∈ T› inner_right_distrib lez)
      then show ?thesis
        using feq by blast
    qed
    ultimately have F = {x + y |x y. x ∈ S ∩ {x. u · x = u · a0} ∧ y ∈ T ∩
{x. u · x = u · b0}}
      by blast
    then show ?thesis
      by (rule that [OF sef tef])
  qed
qed

proposition exposed_face_of_parallel:
   T exposed_face_of S ⟷
        T face_of S ∧
        (∃ a b. S ⊆ {x. a · x ≤ b} ∧ T = S ∩ {x. a · x = b} ∧
              (T ≠ {} ⟶ T ≠ S ⟶ a ≠ 0) ∧
              (T ≠ S ⟶ (∀ w ∈ affine hull S. (w + a) ∈ affine hull S)))
  (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
  proof (clarsimp simp: exposed_face_of_def)
    fix a b
    assume faceS: S ∩ {x. a · x = b} face_of S and Ssub: S ⊆ {x. a · x ≤ b}
    show ∃ c d. S ⊆ {x. c · x ≤ d} ∧
              S ∩ {x. a · x = b} = S ∩ {x. c · x = d} ∧
              (S ∩ {x. a · x = b} ≠ {} ⟶ S ∩ {x. a · x = b} ≠ S ⟶ c ≠ 0) ∧
              (S ∩ {x. a · x = b} ≠ S ⟶ (∀ w ∈ affine hull S. w + c ∈ affine hull
S))
    proof (cases affine hull S ∩ {x. −a · x ≤ −b} = {} ∨ affine hull S ⊆ {x. − a
· x ≤ − b})
      case True
      then show ?thesis
      proof
        assume affine hull S ∩ {x. − a · x ≤ − b} = {}
        then show ?thesis
          apply (rule_tac x=0 in exI)
          apply (rule_tac x=1 in exI)
          using hull_subset by fastforce
      next
        assume affine hull S ⊆ {x. − a · x ≤ − b}
        then show ?thesis
          apply (rule_tac x=0 in exI)
          apply (rule_tac x=0 in exI)
          using Ssub hull_subset by fastforce
      qed
```

```
    next
    case False
    then obtain a' b' where a' ≠ 0
      and le: affine hull S ∩ {x. a' · x ≤ b'} = affine hull S ∩ {x. - a · x ≤ - b}
      and eq: affine hull S ∩ {x. a' · x = b'} = affine hull S ∩ {x. - a · x = - b}
      and mem: ⋀w. w ∈ affine hull S ⟹ w + a' ∈ affine hull S
      using affine_parallel_slice affine_affine_hull by metis
    show ?thesis
    proof (intro conjI impI allI ballI exI)
      have ∗: S ⊆ - (affine hull S ∩ {x. P x}) ∪ affine hull S ∩ {x. Q x} ⟹ S
⊆ {x. ¬ P x ∨ Q x}
        for P Q
        using hull_subset by fastforce
      have S ⊆ {x. ¬ (a' · x ≤ b') ∨ a' · x = b'}
        by (rule ∗) (use le eq Ssub in auto)
      then show S ⊆ {x. - a' · x ≤ - b'}
        by auto
      show S ∩ {x. a · x = b} = S ∩ {x. - a' · x = - b'}
        using eq hull_subset [of S affine] by force
      show ⟦S ∩ {x. a · x = b} ≠ {}; S ∩ {x. a · x = b} ≠ S⟧ ⟹ - a' ≠ 0
        using ‹a' ≠ 0› by auto
      show w + - a' ∈ affine hull S
        if S ∩ {x. a · x = b} ≠ S w ∈ affine hull S for w
      proof -
        have w + 1 *R (w - (w + a')) ∈ affine hull S
          using affine_affine_hull mem mem_affine_3_minus that(2) by blast
        then show ?thesis  by simp
      qed
    qed
  qed
qed
next
  assume ?rhs then show ?lhs
    unfolding exposed_face_of_def by blast
qed
```

9.24.3 Extreme points of a set: its singleton faces

```
definition extreme_point_of :: ['a::real_vector, 'a set] ⇒ bool
                               (infixr ‹(extreme'_point'_of)› 50)
  where x extreme_point_of S ⟷
        x ∈ S ∧ (∀ a ∈ S. ∀ b ∈ S. x ∉ open_segment a b)

lemma extreme_point_of_stillconvex:
   convex S ⟹ (x extreme_point_of S ⟷ x ∈ S ∧ convex(S - {x}))
  by (fastforce simp add: convex_contains_segment extreme_point_of_def open_segment_def)

lemma face_of_singleton:
  {x} face_of S ⟷ x extreme_point_of S
```

by (*fastforce simp add*: *extreme_point_of_def face_of_def*)

lemma *extreme_point_not_in_REL_INTERIOR*:
fixes S :: *'a*::*real_normed_vector set*
shows $[\![x$ *extreme_point_of* S; $S \neq \{x\}]\!] \Longrightarrow x \notin$ *rel_interior* S
by (*metis disjoint_iff face_of_disjoint_rel_interior face_of_singleton insertI1*)

lemma *extreme_point_not_in_interior*:
fixes S :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
assumes x *extreme_point_of* S **shows** $x \notin$ *interior* S
using *assms extreme_point_not_in_REL_INTERIOR interior_subset_rel_interior*
by *fastforce*

lemma *extreme_point_of_face*:
F *face_of* $S \Longrightarrow v$ *extreme_point_of* $F \longleftrightarrow v$ *extreme_point_of* $S \wedge v \in F$
by (*meson empty_subsetI face_of_face face_of_singleton insert_subset*)

lemma *extreme_point_of_convex_hull*:
x *extreme_point_of* (*convex hull* S) $\Longrightarrow x \in S$
using *hull_minimal* [*of* S (*convex hull* S) $-$ $\{x\}$ *convex*]
using *hull_subset* [*of* S *convex*]
by (*force simp add*: *extreme_point_of_stillconvex*)

proposition *extreme_points_of_convex_hull*:
$\{x.\ x$ *extreme_point_of* (*convex hull* $S)\} \subseteq S$
using *extreme_point_of_convex_hull* **by** *auto*

lemma *extreme_point_of_empty* [*simp*]: $\neg$ (x *extreme_point_of* $\{\}$)
by (*simp add*: *extreme_point_of_def*)

lemma *extreme_point_of_singleton* [*iff*]: x *extreme_point_of* $\{a\} \longleftrightarrow x = a$
using *extreme_point_of_stillconvex* **by** *auto*

lemma *extreme_point_of_translation_eq*:
($a + x$) *extreme_point_of* (*image* ($\lambda x.\ a + x$) S) $\longleftrightarrow x$ *extreme_point_of* S
by (*auto simp*: *extreme_point_of_def*)

lemma *extreme_points_of_translation*:
$\{x.\ x$ *extreme_point_of* (*image* ($\lambda x.\ a + x$) $S)\} =$
($\lambda x.\ a + x$) ' $\{x.\ x$ *extreme_point_of* $S\}$
using *extreme_point_of_translation_eq*
by *auto* (*metis* (*no_types*, *lifting*) *image_iff mem_Collect_eq minus_add_cancel*)

lemma *extreme_points_of_translation_subtract*:
$\{x.\ x$ *extreme_point_of* (*image* ($\lambda x.\ x - a$) $S)\} =$
($\lambda x.\ x - a$) ' $\{x.\ x$ *extreme_point_of* $S\}$
using *extreme_points_of_translation* [*of* $-$ a S]
by *simp*

lemma *extreme_point_of_Int*:
  ⟦*x extreme_point_of S*; *x extreme_point_of T*⟧ ⟹ *x extreme_point_of* (*S* ∩ *T*)
  by (*simp add*: *extreme_point_of_def*)

lemma *extreme_point_of_Int_supporting_hyperplane_le*:
  ⟦*S* ∩ {*x*. *a* ⋅ *x* = *b*} = {*c*}; ⋀*x*. *x* ∈ *S* ⟹ *a* ⋅ *x* ≤ *b*⟧ ⟹ *c extreme_point_of S*
 by (*metis convex_singleton face_of_Int_supporting_hyperplane_le_strong face_of_singleton*)

lemma *extreme_point_of_Int_supporting_hyperplane_ge*:
  ⟦*S* ∩ {*x*. *a* ⋅ *x* = *b*} = {*c*}; ⋀*x*. *x* ∈ *S* ⟹ *a* ⋅ *x* ≥ *b*⟧ ⟹ *c extreme_point_of S*
  using *extreme_point_of_Int_supporting_hyperplane_le* [*of S* −*a* −*b c*]
  by *simp*

lemma *exposed_point_of_Int_supporting_hyperplane_le*:
  ⟦*S* ∩ {*x*. *a* ⋅ *x* = *b*} = {*c*}; ⋀*x*. *x* ∈ *S* ⟹ *a* ⋅ *x* ≤ *b*⟧ ⟹ {*c*} *exposed_face_of S*
  unfolding *exposed_face_of_def*
 by (*force simp*: *face_of_singleton extreme_point_of_Int_supporting_hyperplane_le*)

lemma *exposed_point_of_Int_supporting_hyperplane_ge*:
  ⟦*S* ∩ {*x*. *a* ⋅ *x* = *b*} = {*c*}; ⋀*x*. *x* ∈ *S* ⟹ *a* ⋅ *x* ≥ *b*⟧ ⟹ {*c*} *exposed_face_of S*
  using *exposed_point_of_Int_supporting_hyperplane_le* [*of S* −*a* −*b c*]
  by *simp*

lemma *extreme_point_of_convex_hull_insert*:
  assumes *finite S a* ∉ *convex hull S*
  shows *a extreme_point_of* (*convex hull* (*insert a S*))
proof (*cases a* ∈ *S*)
  case *False*
  then show *?thesis*
   using *face_of_convex_hulls* [*of insert a S* {*a*}] *assms*
   by (*auto simp*: *face_of_singleton hull_same*)
qed (*use assms* **in** ‹*simp add*: *hull_inc*›)

lemma *extreme_point_of_conic*:
  assumes *conic S* **and** *x*: *x extreme_point_of S*
  shows *x* = *0*
proof −
  have {*x*} *face_of S*
    by (*simp add*: *face_of_singleton x*)
  then have *conic*{*x*}
    using *assms*(*1*) *face_of_conic* **by** *blast*
  then show *?thesis*
    by (*force simp*: *conic_def*)
qed

9.24.4 Facets

definition *facet_of* :: [*'a*::*euclidean_space set*, *'a set*] ⇒ *bool*
(**infixr** ‹*facet'_of*› *50*)
where *F facet_of S* ⟷ *F face_of S* ∧ *F* ≠ {} ∧ *aff_dim F* = *aff_dim S* − *1*

lemma *facet_of_empty* [*simp*]: ¬ *S facet_of* {}
by (*simp add*: *facet_of_def*)

lemma *facet_of_irrefl* [*simp*]: ¬ *S facet_of S*
by (*simp add*: *facet_of_def*)

lemma *facet_of_imp_face_of*: *F facet_of S* ⟹ *F face_of S*
by (*simp add*: *facet_of_def*)

lemma *facet_of_imp_subset*: *F facet_of S* ⟹ *F* ⊆ *S*
by (*simp add*: *face_of_imp_subset facet_of_def*)

lemma *hyperplane_facet_of_halfspace_le*:
a ≠ *0* ⟹ {*x*. *a* ∙ *x* = *b*} *facet_of* {*x*. *a* ∙ *x* ≤ *b*}
unfolding *facet_of_def hyperplane_eq_empty*
by (*auto simp*: *hyperplane_face_of_halfspace_ge hyperplane_face_of_halfspace_le*
Suc_leI of_nat_diff aff_dim_halfspace_le)

lemma *hyperplane_facet_of_halfspace_ge*:
a ≠ *0* ⟹ {*x*. *a* ∙ *x* = *b*} *facet_of* {*x*. *a* ∙ *x* ≥ *b*}
unfolding *facet_of_def hyperplane_eq_empty*
by (*auto simp*: *hyperplane_face_of_halfspace_le hyperplane_face_of_halfspace_ge*
Suc_leI of_nat_diff aff_dim_halfspace_ge)

lemma *facet_of_halfspace_le*:
F facet_of {*x*. *a* ∙ *x* ≤ *b*} ⟷ *a* ≠ *0* ∧ *F* = {*x*. *a* ∙ *x* = *b*}
(**is** *?lhs* = *?rhs*)
proof
assume *c*: *?lhs*
with *c facet_of_irrefl* **show** *?rhs*
by (*force simp*: *aff_dim_halfspace_le facet_of_def face_of_halfspace_le cong*:
conj_cong split: *if_split_asm*)
next
assume *?rhs* **then show** *?lhs*
by (*simp add*: *hyperplane_facet_of_halfspace_le*)
qed

lemma *facet_of_halfspace_ge*:
F facet_of {*x*. *a* ∙ *x* ≥ *b*} ⟷ *a* ≠ *0* ∧ *F* = {*x*. *a* ∙ *x* = *b*}
using *facet_of_halfspace_le* [*of F* −*a* −*b*] **by** *simp*

9.24.5 Edges: faces of affine dimension 1

definition *edge_of* :: [*'a*::*euclidean_space set*, *'a set*] ⇒ *bool* (**infixr** ‹(*edge'_of*)› *50*)
where *e edge_of S* ⟷ *e face_of S* ∧ *aff_dim e* = *1*

lemma *edge_of_imp_subset*:
S edge_of T ⟹ *S* ⊆ *T*
by (*simp add*: *edge_of_def face_of_imp_subset*)

9.24.6 Existence of extreme points

proposition *different_norm_3_collinear_points*:
fixes *a* :: *'a*::*euclidean_space*
assumes *x* ∈ *open_segment a b norm*(*a*) = *norm*(*b*) *norm*(*x*) = *norm*(*b*)
shows *False*
proof −
obtain *u* **where** *norm* ((*1* − *u*) *∗R a* + *u ∗R b*) = *norm b*
and *a* ≠ *b*
and *u01*: *0* < *u u* < *1*
using *assms* **by** (*auto simp*: *open_segment_image_interval if_splits*)
then have (*1* − *u*) *∗R a* ⋅ (*1* − *u*) *∗R a* + ((*1* − *u*) ∗ *2*) *∗R a* ⋅ *u ∗R b* =
(*1* − *u* ∗ *u*) *∗R* (*a* ⋅ *a*)
using *assms* **by** (*simp add*: *norm_eq algebra_simps inner_commute*)
then have (*1* − *u*) *∗R* ((*1* − *u*) *∗R a* ⋅ *a* + (*2* ∗ *u*) *∗R a* ⋅ *b*) =
(*1* − *u*) *∗R* ((*1* + *u*) *∗R* (*a* ⋅ *a*))
by (*simp add*: *algebra_simps*)
then have (*1* − *u*) *∗R* (*a* ⋅ *a*) + (*2* ∗ *u*) *∗R* (*a* ⋅ *b*) = (*1* + *u*) *∗R* (*a* ⋅ *a*)
using *u01* **by** *auto*
then have *a* ⋅ *b* = *a* ⋅ *a*
using *u01* **by** (*simp add*: *algebra_simps*)
then have *a* = *b*
using ‹*norm*(*a*) = *norm*(*b*)› *norm_eq vector_eq* **by** *fastforce*
then show *?thesis*
using ‹*a* ≠ *b*› **by** *force*
qed

proposition *extreme_point_exists_convex*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S convex S S* ≠ {}
obtains *x* **where** *x extreme_point_of S*
proof −
obtain *x* **where** *x* ∈ *S* **and** *xsup*: ⋀*y*. *y* ∈ *S* ⟹ *norm y* ≤ *norm x*
using *distance_attains_sup* [*of S 0*] *assms* **by** *auto*
have *False* **if** *a* ∈ *S b* ∈ *S* **and** *x*: *x* ∈ *open_segment a b* **for** *a b*
proof −
have *noax*: *norm a* ≤ *norm x* **and** *nobx*: *norm b* ≤ *norm x* **using** *xsup that*
by *auto*
have *a* ≠ *b*
using *empty_iff open_segment_idem x* **by** *auto*

show *False*
by (*metis dist_0_norm dist_decreases_open_segment noax nobx not_le x*)
qed
then show *?thesis*
by (*meson ‹x ∈ S› extreme_point_of_def that*)
qed

9.24.7 Krein-Milman, the weaker form

proposition *Krein_Milman*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S convex S*
shows *S* = *closure*(*convex hull* {*x*. *x extreme_point_of S*})
proof (*cases S* = {})
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
have *closed S*
by (*simp add*: ‹*compact S*› *compact_imp_closed*)
have *closure* (*convex hull* {*x*. *x extreme_point_of S*}) ⊆ *S*
by (*simp add*: ‹*closed S*› *assms closure_minimal extreme_point_of_def hull_minimal*)
moreover have *u* ∈ *closure* (*convex hull* {*x*. *x extreme_point_of S*})
if *u* ∈ *S* **for** *u*
proof (*rule ccontr*)
assume *unot*: *u* ∉ *closure*(*convex hull* {*x*. *x extreme_point_of S*})
then obtain *a b* **where** *a* · *u* < *b*
and *ab*: ⋀*x*. *x* ∈ *closure*(*convex hull* {*x*. *x extreme_point_of S*}) ⟹ *b* < *a* · *x*
using *separating_hyperplane_closed_point* [*of closure*(*convex hull* {*x*. *x extreme_point_of S*})]
by *blast*
have *continuous_on S* ((·) *a*)
by (*rule continuous_intros*)+
then obtain *m* **where** *m* ∈ *S* **and** *m*: ⋀*y*. *y* ∈ *S* ⟹ *a* · *m* ≤ *a* · *y*
using *continuous_attains_inf* [*of S* λ*x*. *a* · *x*] ‹*compact S*› ‹*u* ∈ *S*›
by *auto*
define *T* **where** *T* = *S* ∩ {*x*. *a* · *x* = *a* · *m*}
have *m* ∈ *T*
by (*simp add*: *T_def* ‹*m* ∈ *S*›)
moreover have *compact T*
by (*simp add*: *T_def compact_Int_closed* [*OF* ‹*compact S*› *closed_hyperplane*])
moreover have *convex T*
by (*simp add*: *T_def convex_Int* [*OF* ‹*convex S*› *convex_hyperplane*])
ultimately obtain *v* **where** *v*: *v extreme_point_of T*
using *extreme_point_exists_convex* [*of T*] **by** *auto*
then have {*v*} *face_of T*
by (*simp add*: *face_of_singleton*)
also have *T face_of S*
by (*simp add*: *T_def m face_of_Int_supporting_hyperplane_ge* [*OF* ‹*convex*

```
S›])
  finally have v extreme_point_of S
    by (simp add: face_of_singleton)
  then have b < a • v
    using closure_subset by (simp add: closure_hull hull_inc ab)
  then show False
     using ‹a • u < b› ‹{v} face_of T› face_of_imp_subset m T_def that by
fastforce
 qed
 ultimately show ?thesis
  by blast
qed
```

Now the sharper form.

```
lemma Krein_Milman_Minkowski_aux:
 fixes S :: 'a::euclidean_space set
 assumes n: dim S = n and S: compact S convex S 0 ∈ S
  shows 0 ∈ convex hull {x. x extreme_point_of S}
using n S
proof (induction n arbitrary: S rule: less_induct)
 case (less n S) show ?case
 proof (cases 0 ∈ rel_interior S)
  case True with Krein_Milman less.prems
  show ?thesis
   by (metis subsetD convex_convex_hull convex_rel_interior_closure rel_interior_subset)
 next
  case False
  have rel_interior S ≠ {}
   by (simp add: rel_interior_convex_nonempty_aux less)
  then obtain c where c: c ∈ rel_interior S by blast
  obtain a where a ≠ 0
          and le_ay: ⋀y. y ∈ S ⟹ a • 0 ≤ a • y
          and less_ay: ⋀y. y ∈ rel_interior S ⟹ a • 0 < a • y
   by (blast intro: supporting_hyperplane_rel_boundary intro!: less False)
  have face: S ∩ {x. a • x = 0} face_of S
   using face_of_Int_supporting_hyperplane_ge le_ay ‹convex S› by auto
  then have co: compact (S ∩ {x. a • x = 0}) convex (S ∩ {x. a • x = 0})
   using less.prems by (blast intro: face_of_imp_compact face_of_imp_convex)+
  have a • y = 0 if y ∈ span (S ∩ {x. a • x = 0}) for y
  proof −
   have y ∈ span {x. a • x = 0}
    by (metis inf.cobounded2 span_mono subsetCE that)
   then show ?thesis
    by (blast intro: span_induct [OF _ subspace_hyperplane])
  qed
  then have dim (S ∩ {x. a • x = 0}) < n
   by (metis (no_types) less_ay c subsetD dim_eq_span inf.strict_order_iff
       inf_le1 ‹dim S = n› not_le rel_interior_subset span_0 span_base)
  then have 0 ∈ convex hull {x. x extreme_point_of (S ∩ {x. a • x = 0})}
```

by (*rule less.IH*) (*auto simp*: *co less.prems*)
then show *?thesis*
by (*metis* (*mono_tags*, *lifting*) *Collect_mono_iff face extreme_point_of_face hull_mono subset_iff*)
qed
qed

theorem *Krein_Milman_Minkowski*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S convex S*
shows *S* = *convex hull* {*x*. *x extreme_point_of S*}
proof
show *S* ⊆ *convex hull* {*x*. *x extreme_point_of S*}
proof
fix *a* **assume** [*simp*]: *a* ∈ *S*
have *1*: *compact* ((+) (− *a*) ‘ *S*)
by (*simp add*: ‹*compact S*› *compact_translation_subtract cong*: *image_cong_simp*)
have *2*: *convex* ((+) (− *a*) ‘ *S*)
by (*simp add*: ‹*convex S*› *compact_translation_subtract*)
show *a_invex*: *a* ∈ *convex hull* {*x*. *x extreme_point_of S*}
using *Krein_Milman_Minkowski_aux* [*OF refl 1 2*]
convex_hull_translation [*of* −*a*]
by (*auto simp*: *extreme_points_of_translation_subtract translation_assoc cong*: *image_cong_simp*)
qed
next
show *convex hull* {*x*. *x extreme_point_of S*} ⊆ *S*
using ‹*convex S*› *extreme_point_of_stillconvex subset_hull* **by** *fastforce*
qed

9.24.8 Applying it to convex hulls of explicitly indicated finite sets

corollary *Krein_Milman_polytope*:
fixes *S* :: *'a*::*euclidean_space set*
shows
finite S
⟹ *convex hull S* =
convex hull {*x*. *x extreme_point_of* (*convex hull S*)}
by (*simp add*: *Krein_Milman_Minkowski finite_imp_compact_convex_hull*)

lemma *extreme_points_of_convex_hull_eq*:
fixes *S* :: *'a*::*euclidean_space set*
shows
⟦*compact S*; ⋀*T*. *T* ⊂ *S* ⟹ *convex hull T* ≠ *convex hull S*⟧
⟹ {*x*. *x extreme_point_of* (*convex hull S*)} = *S*
by (*metis* (*full_types*) *Krein_Milman_Minkowski compact_convex_hull convex_convex_hull extreme_points_of_convex_hull psubsetI*)

lemma *extreme_point_of_convex_hull_eq*:
fixes S :: *'a::euclidean_space set*
shows
$[\![compact\ S;\ \bigwedge T.\ T \subset S \Longrightarrow convex\ hull\ T \neq convex\ hull\ S]\!]$
$\Longrightarrow (x\ extreme\_point\_of\ (convex\ hull\ S) \longleftrightarrow x \in S)$
using *extreme_points_of_convex_hull_eq* **by** *auto*

lemma *extreme_point_of_convex_hull_convex_independent*:
fixes S :: *'a::euclidean_space set*
assumes *compact* S **and** S: $\bigwedge a.\ a \in S \Longrightarrow a \notin convex\ hull\ (S - \{a\})$
shows $(x\ extreme\_point\_of\ (convex\ hull\ S) \longleftrightarrow x \in S)$
proof −
have $convex\ hull\ T \neq convex\ hull\ S$ **if** $T \subset S$ **for** T
proof −
obtain a **where** $T \subseteq S\ a \in S\ a \notin T$ **using** ‹$T \subset S$› **by** *blast*
then show *?thesis*
by (*metis* (*full_types*) *Diff_eq_empty_iff Diff_insert0 S hull_mono hull_subset insert_Diff_single subsetCE*)
qed
then show *?thesis*
by (*rule extreme_point_of_convex_hull_eq* [*OF* ‹*compact S*›])
qed

lemma *extreme_point_of_convex_hull_affine_independent*:
fixes S :: *'a::euclidean_space set*
shows
$\neg$ *affine_dependent S*
$\Longrightarrow (x\ extreme\_point\_of\ (convex\ hull\ S) \longleftrightarrow x \in S)$
by (*metis aff_independent_finite affine_dependent_def affine_hull_convex_hull extreme_point_of_convex_hull_convex_independent finite_imp_compact hull_inc*)

Elementary proofs exist, not requiring Euclidean spaces and all this development

lemma *extreme_point_of_convex_hull_2*:
fixes x :: *'a::euclidean_space*
shows $x\ extreme\_point\_of\ (convex\ hull\ \{a,b\}) \longleftrightarrow x = a \vee x = b$
by (*simp add*: *extreme_point_of_convex_hull_affine_independent*)

lemma *extreme_point_of_segment*:
fixes x :: *'a::euclidean_space*
shows $x\ extreme\_point\_of\ closed\_segment\ a\ b \longleftrightarrow x = a \vee x = b$
by (*simp add*: *extreme_point_of_convex_hull_2 segment_convex_hull*)

lemma *face_of_convex_hull_subset*:
fixes S :: *'a::euclidean_space set*
assumes *compact* S **and** T: $T\ face\_of\ (convex\ hull\ S)$
obtains S' **where** $S' \subseteq S\ T = convex\ hull\ S'$

proof
show $\{x.\ x\ extreme\_point\_of\ T\} \subseteq S$
using T *extreme_point_of_convex_hull extreme_point_of_face* **by** *blast*
show $T = convex\ hull\ \{x.\ x\ extreme\_point\_of\ T\}$
by (*metis Krein_Milman_Minkowski assms compact_convex_hull convex_convex_hull*

face_of_imp_compact face_of_imp_convex)
qed

lemma *face_of_convex_hull_aux*:
assumes *eq*: $x *_R p = u *_R a + v *_R b + w *_R c$
and x: $u + v + w = x\ x \neq 0$ **and** S: $affine\ S\ a \in S\ b \in S\ c \in S$
shows $p \in S$
proof −
have $p = (u *_R a + v *_R b + w *_R c) /_R x$
by (*metis* ‹$x \neq 0$› *eq mult.commute right_inverse scaleR_one scaleR_scaleR*)
moreover have $affine\ hull\ \{a,b,c\} \subseteq S$
by (*simp add*: S *hull_minimal*)
moreover have $(u *_R a + v *_R b + w *_R c) /_R x \in affine\ hull\ \{a,b,c\}$
apply (*simp add*: *affine_hull_3*)
apply (*rule_tac x*=u/x **in** *exI*)
apply (*rule_tac x*=v/x **in** *exI*)
apply (*rule_tac x*=w/x **in** *exI*)
using x **apply** (*auto simp*: *field_split_simps*)
done
ultimately show *?thesis* **by** *force*
qed

proposition *face_of_convex_hull_insert_eq*:
fixes $a :: 'a :: euclidean\_space$
assumes *finite* S **and** a: $a \notin affine\ hull\ S$
shows ($F\ face\_of\ (convex\ hull\ (insert\ a\ S)) \longleftrightarrow$
$F\ face\_of\ (convex\ hull\ S) \vee$
$(\exists F'.\ F'\ face\_of\ (convex\ hull\ S) \wedge F = convex\ hull\ (insert\ a\ F'))$)
(**is** $F\ face\_of\ ?CAS \longleftrightarrow \_$)
proof *safe*
assume F: $F\ face\_of\ ?CAS$
and $*$: $\nexists F'.\ F'\ face\_of\ convex\ hull\ S \wedge F = convex\ hull\ insert\ a\ F'$
obtain T **where** T: $T \subseteq insert\ a\ S$ **and** $FeqT$: $F = convex\ hull\ T$
by (*metis F* ‹*finite S*› *compact_insert finite_imp_compact face_of_convex_hull_subset*)
show $F\ face\_of\ convex\ hull\ S$
proof (*cases* $a \in T$)
case *True*
have $F = convex\ hull\ insert\ a\ (convex\ hull\ T \cap convex\ hull\ S)$
proof
have $T \subseteq insert\ a\ (convex\ hull\ T \cap convex\ hull\ S)$
using T *hull_subset* **by** *fastforce*
then show $F \subseteq convex\ hull\ insert\ a\ (convex\ hull\ T \cap convex\ hull\ S)$

```
      by (simp add: FeqT hull_mono)
    show convex hull insert a (convex hull T ∩ convex hull S) ⊆ F
      by (simp add: FeqT True hull_inc hull_minimal)
  qed
  moreover have convex hull T ∩ convex hull S face_of convex hull S
  by (metis F FeqT convex_convex_hull face_of_slice hull_mono inf.absorb_iff2
subset_insertI)
  ultimately show ?thesis
    using ∗ by force
 next
  case False
  then show ?thesis
   by (metis FeqT F T face_of_subset hull_mono subset_insert subset_insertI)
 qed
next
 assume F face_of convex hull S
 show F face_of ?CAS
  by (simp add: ‹F face_of convex hull S› a face_of_convex_hull_insert ‹finite
S›)
next
 fix F
 assume F: F face_of convex hull S
 show convex hull insert a F face_of ?CAS
 proof (cases S = {})
  case True
  then show ?thesis
    using F face_of_affine_eq by auto
 next
  case False
  have anotc: a ∉ convex hull S
    by (metis (no_types) a affine_hull_convex_hull hull_inc)
  show ?thesis
  proof (cases F = {})
    case True show ?thesis
    using anotc by (simp add: ‹F = {}› ‹finite S› extreme_point_of_convex_hull_insert
face_of_singleton)
  next
    case False
    have convex hull insert a F ⊆ ?CAS
    by (simp add: F a ‹finite S› convex_hull_subset face_of_convex_hull_insert
face_of_imp_subset hull_inc)
    moreover
    have (∃ y v. (1 − ub) ∗R a + ub ∗R b = (1 − v) ∗R a + v ∗R y ∧
                 0 ≤ v ∧ v ≤ 1 ∧ y ∈ F) ∧
          (∃ x u. (1 − uc) ∗R a + uc ∗R c = (1 − u) ∗R a + u ∗R x ∧
                 0 ≤ u ∧ u ≤ 1 ∧ x ∈ F)
      if ∗: (1 − ux) ∗R a + ux ∗R x
           ∈ open_segment ((1 − ub) ∗R a + ub ∗R b) ((1 − uc) ∗R a + uc ∗R
c)
```

```
      and 0 ≤ ub ub ≤ 1 0 ≤ uc uc ≤ 1 0 ≤ ux ux ≤ 1
      and b: b ∈ convex hull S and c: c ∈ convex hull S and x ∈ F
    for b c ub uc ux x
  proof −
    have xah: x ∈ affine hull S
     using F convex_hull_subset_affine_hull face_of_imp_subset ‹x ∈ F› by
blast
    have ah: b ∈ affine hull S c ∈ affine hull S
     using b c convex_hull_subset_affine_hull by blast+
    obtain v where ne: (1 − ub) *R a + ub *R b ≠ (1 − uc) *R a + uc *R c
     and eq: (1 − ux) *R a + ux *R x =
            (1 − v) *R ((1 − ub) *R a + ub *R b) + v *R ((1 − uc) *R a +
uc *R c)
      and 0 < v v < 1
      using ∗ by (auto simp: in_segment)
    then have 0: ((1 − ux) − ((1 − v) ∗ (1 − ub) + v ∗ (1 − uc))) *R a +
                (ux *R x − (((1 − v) ∗ ub) *R b + (v ∗ uc) *R c)) = 0
     by (auto simp: algebra_simps)
    then have ((1 − ux) − ((1 − v) ∗ (1 − ub) + v ∗ (1 − uc))) *R a =
           ((1 − v) ∗ ub) *R b + (v ∗ uc) *R c + (−ux) *R x
     by (auto simp: algebra_simps)
    then have a ∈ affine hull S if 1 − ux − ((1 − v) ∗ (1 − ub) + v ∗ (1 −
uc)) ≠ 0
      by (rule face_of_convex_hull_aux) (use b c xah ah that in ‹auto simp:
algebra_simps›)
    then have 1 − ux − ((1 − v) ∗ (1 − ub) + v ∗ (1 − uc)) = 0
     using a by blast
    with 0 have equx: (1 − v) ∗ ub + v ∗ uc = ux
     and uxx: ux *R x = (((1 − v) ∗ ub) *R b + (v ∗ uc) *R c)
     by auto (auto simp: algebra_simps)
    show ?thesis
    proof (cases uc = 0)
     case True
     then show ?thesis
       using equx ‹0 ≤ ub› ‹ub ≤ 1› ‹v < 1› uxx ‹x ∈ F› by force
    next
     case False
     show ?thesis
     proof (cases ub = 0)
       case True
       then show ?thesis
         using equx ‹0 ≤ uc› ‹uc ≤ 1› ‹0 < v› uxx ‹x ∈ F› by force
     next
       case False
       then have 0 < ub 0 < uc
         using ‹uc ≠ 0› ‹0 ≤ ub› ‹0 ≤ uc› by auto
       then have (1 − v) ∗ ub > 0 v ∗ uc > 0
         by (simp_all add: ‹0 < uc› ‹0 < v› ‹v < 1›)
       then have ux ≠ 0
```

```
          using equx ‹0 < v› by auto
        have b ∈ F ∧ c ∈ F
        proof (cases b = c)
          case True
          then show ?thesis
          by (metis ‹ux ≠ 0› equx real_vector.scale_cancel_left scaleR_add_left
uxx ‹x ∈ F›)
        next
          case False
          have x = (((1 − v) ∗ ub) ∗R b + (v ∗ uc) ∗R c) /R ux
              by (metis ‹ux ≠ 0› uxx mult.commute right_inverse scaleR_one
scaleR_scaleR)
          also have ... = (1 − v ∗ uc / ux) ∗R b + (v ∗ uc / ux) ∗R c
            using ‹ux ≠ 0› equx apply (auto simp: field_split_simps)
        by (metis add.commute add_diff_eq add_divide_distrib diff_add_cancel
scaleR_add_left)
          finally have x = (1 − v ∗ uc / ux) ∗R b + (v ∗ uc / ux) ∗R c .
          then have x ∈ open_segment b c
            apply (simp add: in_segment ‹b ≠ c›)
            apply (rule_tac x=(v ∗ uc) / ux in exI)
            using ‹0 ≤ ux› ‹ux ≠ 0› ‹0 < uc› ‹0 < v› ‹0 < ub› ‹v < 1› equx
            apply (force simp: field_split_simps)
            done
          then show ?thesis
            by (rule face_ofD [OF F _ b c ‹x ∈ F›])
        qed
        with ‹0 ≤ ub› ‹ub ≤ 1› ‹0 ≤ uc› ‹uc ≤ 1› show ?thesis by blast
      qed
    qed
  qed
  moreover have convex hull F = F
    by (meson F convex_hull_eq face_of_imp_convex)
  ultimately show ?thesis
    unfolding face_of_def by (fastforce simp: convex_hull_insert_alt ‹S ≠
{}› ‹F ≠ {}›)
 qed
qed

lemma face_of_convex_hull_insert2:
  fixes a :: 'a :: euclidean_space
  assumes S: finite S and a: a ∉ affine hull S and F: F face_of convex hull S
  shows convex hull (insert a F) face_of convex hull (insert a S)
  by (metis F face_of_convex_hull_insert_eq [OF S a])

proposition face_of_convex_hull_affine_independent:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
    shows (T face_of (convex hull S) ⟷ (∃ c. c ⊆ S ∧ T = convex hull c))
```

```
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (meson ‹T face_of convex hull S› aff_independent_finite assms face_of_convex_hull_subset
finite_imp_compact)
next
  assume ?rhs
  then obtain C where C ⊆ S and T: T = convex hull C
    by blast
  have affine hull C ∩ affine hull (S − C) = {}
    by (intro disjoint_affine_hull [OF assms ‹C ⊆ S›], auto)
  then have affine hull C ∩ convex hull (S − C) = {}
    using convex_hull_subset_affine_hull by fastforce
  then show ?lhs
    by (metis face_of_convex_hulls ‹C ⊆ S› aff_independent_finite assms T)
qed

lemma facet_of_convex_hull_affine_independent:
  fixes S :: 'a::euclidean_space set
  assumes ¬ affine_dependent S
    shows T facet_of (convex hull S) ⟷
          T ≠ {} ∧ (∃u. u ∈ S ∧ T = convex hull (S − {u}))
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then have T face_of (convex hull S) T ≠ {}
       and afft: aff_dim T = aff_dim (convex hull S) − 1
    by (auto simp: facet_of_def)
  then obtain c where c ⊆ S and c: T = convex hull c
    by (auto simp: face_of_convex_hull_affine_independent [OF assms])
  then have affs: aff_dim S = aff_dim c + 1
    by (metis aff_dim_convex_hull afft eq_diff_eq)
  have ¬ affine_dependent c
    using ‹c ⊆ S› affine_dependent_subset assms by blast
  with affs have card (S − c) = 1
    by (smt (verit) ‹c ⊆ S› aff_dim_affine_independent aff_independent_finite
assms card_Diff_subset
        card_mono of_nat_diff of_nat_eq_1_iff)
  then obtain u where u: u ∈ S − c
   by (metis DiffI ‹c ⊆ S› aff_independent_finite assms cancel_comm_monoid_add_class.diff_cancel
              card_Diff_subset subsetI subset_antisym zero_neq_one)
  then have u: S = insert u c
    by (metis Diff_subset ‹c ⊆ S› ‹card (S − c) = 1› card_1_singletonE dou-
ble_diff insert_Diff insert_subset singletonD)
  have T = convex hull (c − {u})
   by (metis Diff_empty Diff_insert0 ‹T facet_of convex hull S› c facet_of_irrefl
insert_absorb u)
  with ‹T ≠ {}› show ?rhs
```

using *c u* **by** *auto*
next
assume *?rhs*
then obtain *u* **where** $T \neq \{\}$ $u \in S$ **and** *u*: $T = \textit{convex hull } (S - \{u\})$
by (*force simp*: *facet_of_def*)
then have $\neg\ S \subseteq \{u\}$
using ‹$T \neq \{\}$› *u* **by** *auto*
have $\textit{aff\_dim } (S - \{u\}) = \textit{aff\_dim } S - 1$
using *assms* ‹$u \in S$›
unfolding *affine_dependent_def*
by (*metis add_diff_cancel_right' aff_dim_insert insert_Diff* [*of u S*])
then have $\textit{aff\_dim } (\textit{convex hull } (S - \{u\})) = \textit{aff\_dim } (\textit{convex hull } S) - 1$
by (*simp add*: *aff_dim_convex_hull*)
then show *?lhs*
by (*metis Diff_subset* ‹$T \neq \{\}$› *assms face_of_convex_hull_affine_independent facet_of_def u*)
qed

lemma *facet_of_convex_hull_affine_independent_alt*:
fixes $S :: 'a::\textit{euclidean\_space set}$
assumes $\neg$ *affine_dependent S*
shows (T *facet_of* (*convex hull S*) $\longleftrightarrow$ $2 \leq \textit{card } S \wedge (\exists u.\ u \in S \wedge T = \textit{convex hull } (S - \{u\}))$)
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then obtain *x* **where**
$x \in S$ **and** *x*: $T = \textit{convex hull } (S - \{x\})$ **and** *finite S*
using *assms facet_of_convex_hull_affine_independent aff_independent_finite*
by *blast*
moreover have $\textit{Suc } (\textit{Suc } 0) \leq \textit{card } S$
using *L x* ‹$x \in S$› ‹*finite S*›
by (*metis Suc_leI assms card.remove convex_hull_eq_empty card_gt_0_iff facet_of_convex_hull_affine_independent finite_Diff not_less_eq_eq*)
ultimately show *?rhs*
by *auto*
next
assume *?rhs* **then show** *?lhs*
using *assms*
by (*auto simp*: *facet_of_convex_hull_affine_independent Set.subset_singleton_iff*)
qed

lemma *segment_face_of*:
assumes (*closed_segment a b*) *face_of S*
shows *a extreme_point_of S b extreme_point_of S*
proof −
have *as*: $\{a\}$ *face_of S*
by (*metis* (*no_types*) *assms convex_hull_singleton empty_iff extreme_point_of_convex_hull_insert face_of_face face_of_singleton finite.emptyI finite.insertI insert_absorb insert_iff*

segment_convex_hull)
moreover have {*b*} *face_of S*
proof −
have *b* ∈ *convex hull* {*a*} ∨ *b extreme_point_of convex hull* {*b*, *a*}
by (*meson extreme_point_of_convex_hull_insert finite.emptyI finite.insertI*)
moreover have *closed_segment a b* = *convex hull* {*b*, *a*}
using *closed_segment_commute segment_convex_hull* **by** *blast*
ultimately show *?thesis*
by (*metis as assms face_of_face convex_hull_singleton empty_iff face_of_singleton insertE*)
qed
ultimately show *a extreme_point_of S b extreme_point_of S*
using *face_of_singleton* **by** *blast*+
qed

proposition *Krein_Milman_frontier*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *convex S compact S*
shows *S* = *convex hull* (*frontier S*)
(**is** *?lhs* = *?rhs*)
proof
have *?lhs* ⊆ *convex hull* {*x*. *x extreme_point_of S*}
using *Krein_Milman_Minkowski assms* **by** *blast*
also have ... ⊆ *?rhs*
proof (*rule hull_mono*)
show {*x*. *x extreme_point_of S*} ⊆ *frontier S*
using *closure_subset*
by (*auto simp*: *frontier_def extreme_point_not_in_interior extreme_point_of_def*)
qed
finally show *?lhs* ⊆ *?rhs* **.**
next
have *?rhs* ⊆ *convex hull S*
by (*metis Diff_subset* ‹*compact S*› *closure_closed compact_eq_bounded_closed frontier_def hull_mono*)
also have ... ⊆ *?lhs*
by (*simp add*: ‹*convex S*› *hull_same*)
finally show *?rhs* ⊆ *?lhs* **.**
qed

9.24.9 Polytopes

definition *polytope* **where**
polytope S ≡ ∃ *v*. *finite v* ∧ *S* = *convex hull v*

lemma *polytope_translation_eq*: *polytope* ((+) *a* ‘ *S*) ⟷ *polytope S*
unfolding *polytope_def*
by (*metis* (*no_types*, *opaque_lifting*) *add.left_inverse convex_hull_translation finite_imageI image_add_0 translation_assoc*)

lemma *polytope_linear_image*: ⟦*linear f*; *polytope p*⟧ ⟹ *polytope*(*image f p*)
unfolding *polytope_def* **using** *convex_hull_linear_image* **by** *blast*

lemma *polytope_empty*: *polytope* {}
using *convex_hull_empty polytope_def* **by** *blast*

lemma *polytope_convex_hull*: *finite S* ⟹ *polytope*(*convex hull S*)
using *polytope_def* **by** *auto*

lemma *polytope_Times*: ⟦*polytope S*; *polytope T*⟧ ⟹ *polytope*(*S* × *T*)
unfolding *polytope_def*
by (*metis finite_cartesian_product convex_hull_Times*)

lemma *face_of_polytope_polytope*:
fixes *S* :: *'a*::*euclidean_space set*
shows ⟦*polytope S*; *F face_of S*⟧ ⟹ *polytope F*
unfolding *polytope_def*
by (*meson face_of_convex_hull_subset finite_imp_compact finite_subset*)

lemma *finite_polytope_faces*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *polytope S*
shows *finite* {*F*. *F face_of S*}
proof −
obtain *v* **where** *finite v S* = *convex hull v*
using *assms polytope_def* **by** *auto*
have *finite* ((*hull*) *convex* ‘ {*T*. *T* ⊆ *v*})
by (*simp add*: ‹*finite v*›)
moreover have {*F*. *F face_of S*} ⊆ ((*hull*) *convex* ‘ {*T*. *T* ⊆ *v*})
by (*metis* (*no_types*, *lifting*) ‹*finite v*› ‹*S* = *convex hull v*› *face_of_convex_hull_subset finite_imp_compact image_eqI mem_Collect_eq subsetI*)
ultimately show *?thesis*
by (*blast intro*: *finite_subset*)
qed

lemma *finite_polytope_facets*:
assumes *polytope S*
shows *finite* {*T*. *T facet_of S*}
by (*simp add*: *assms facet_of_def finite_polytope_faces*)

lemma *polytope_scaling*:
assumes *polytope S* **shows** *polytope* (*image* (λ*x*. *c* ∗*R* *x*) *S*)
by (*simp add*: *assms polytope_linear_image*)

lemma *polytope_imp_compact*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *polytope S* ⟹ *compact S*
by (*metis finite_imp_compact_convex_hull polytope_def*)

lemma *polytope_imp_convex*: *polytope S* $\Longrightarrow$ *convex S*
by (*metis convex_convex_hull polytope_def*)

lemma *polytope_imp_closed*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *polytope S* $\Longrightarrow$ *closed S*
by (*simp add*: *compact_imp_closed polytope_imp_compact*)

lemma *polytope_imp_bounded*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *polytope S* $\Longrightarrow$ *bounded S*
by (*simp add*: *compact_imp_bounded polytope_imp_compact*)

lemma *polytope_interval*: *polytope*(*cbox a b*)
unfolding *polytope_def* **by** (*meson closed_interval_as_convex_hull*)

lemma *polytope_sing*: *polytope* {*a*}
using *polytope_def* **by** *force*

lemma *face_of_polytope_insert*:
$[\![$*polytope S*; *a* $\notin$ *affine hull S*; *F face_of S*$]\!] \Longrightarrow$ *F face_of convex hull* (*insert a S*)
by (*metis* (*no_types*, *lifting*) *affine_hull_convex_hull face_of_convex_hull_insert hull_insert polytope_def*)

proposition *face_of_polytope_insert2*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *polytope S a* $\notin$ *affine hull S F face_of S*
shows *convex hull* (*insert a F*) *face_of convex hull* (*insert a S*)
proof −
obtain *V* **where** *finite V S* = *convex hull V*
using *assms* **by** (*auto simp*: *polytope_def*)
then have *convex hull* (*insert a F*) *face_of convex hull* (*insert a V*)
using *affine_hull_convex_hull assms face_of_convex_hull_insert2* **by** *blast*
then show *?thesis*
by (*metis* ‹*S* = *convex hull V*› *hull_insert*)
qed

9.24.10 Polyhedra

definition *polyhedron* **where**
polyhedron S ≡
$\exists F.$ *finite F* $\wedge$
$S = \bigcap F \wedge$
$(\forall h \in F. \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\})$

lemma *polyhedron_Int* [*intro*,*simp*]:
$[\![$*polyhedron S*; *polyhedron T*$]\!] \Longrightarrow$ *polyhedron* $(S \cap T)$

apply (*clarsimp simp add*: *polyhedron_def*)
subgoal for *F G*
by (*rule_tac x*=*F* ∪ *G* **in** *exI*, *auto*)
done

lemma *polyhedron_UNIV* [*iff*]: *polyhedron UNIV*
using *polyhedron_def* **by** *auto*

lemma *polyhedron_Inter* [*intro*,*simp*]:
⟦*finite F*; ⋀*S*. *S* ∈ *F* ⟹ *polyhedron S*⟧ ⟹ *polyhedron*(⋂ *F*)
by (*induction F rule*: *finite_induct*) *auto*

lemma *polyhedron_empty* [*iff*]: *polyhedron* ({} :: *'a* :: *euclidean_space set*)
proof −
define *i*::*'a* **where** (*i* ≡ *SOME i*. *i* ∈ *Basis*)
have ∃ *a*. *a* ≠ *0* ∧ (∃ *b*. {*x*. *i* ∙ *x* ≤ −*1*} = {*x*. *a* ∙ *x* ≤ *b*})
by (*rule_tac x*=*i* **in** *exI*) (*force simp*: *i_def SOME_Basis nonzero_Basis*)
moreover have ∃ *a b*. *a* ≠ *0* ∧ {*x*. −*i* ∙ *x* ≤ − *1*} = {*x*. *a* ∙ *x* ≤ *b*}
by (*metis Basis_zero SOME_Basis i_def neg_0_equal_iff_equal*)
ultimately show *?thesis*
unfolding *polyhedron_def*
by (*rule_tac x*={{*x*. *i* ∙ *x* ≤ −*1*}, {*x*. −*i* ∙ *x* ≤ −*1*}} **in** *exI*) *force*
qed

lemma *polyhedron_halfspace_le*:
fixes *a* :: *'a* :: *euclidean_space*
shows *polyhedron* {*x*. *a* ∙ *x* ≤ *b*}
proof (*cases a* = *0*)
case *True* **then show** *?thesis* **by** *auto*
next
case *False*
then show *?thesis*
unfolding *polyhedron_def*
by (*rule_tac x*={{*x*. *a* ∙ *x* ≤ *b*}} **in** *exI*) *auto*
qed

lemma *polyhedron_halfspace_ge*:
fixes *a* :: *'a* :: *euclidean_space*
shows *polyhedron* {*x*. *a* ∙ *x* ≥ *b*}
using *polyhedron_halfspace_le* [*of* −*a* −*b*] **by** *simp*

lemma *polyhedron_hyperplane*:
fixes *a* :: *'a* :: *euclidean_space*
shows *polyhedron* {*x*. *a* ∙ *x* = *b*}
proof −
have {*x*. *a* ∙ *x* = *b*} = {*x*. *a* ∙ *x* ≤ *b*} ∩ {*x*. *a* ∙ *x* ≥ *b*}
by *force*
then show *?thesis*

by (*simp add*: *polyhedron_halfspace_ge polyhedron_halfspace_le*)
qed

lemma *affine_imp_polyhedron*:
fixes $S :: 'a :: euclidean\_space\ set$
shows *affine* $S \Longrightarrow$ *polyhedron* S
by (*metis affine_hull_finite_intersection_hyperplanes hull_same polyhedron_Inter polyhedron_hyperplane*)

lemma *polyhedron_imp_closed*:
fixes $S :: 'a :: euclidean\_space\ set$
shows *polyhedron* $S \Longrightarrow$ *closed* S
by (*metis closed_Inter closed_halfspace_le polyhedron_def*)

lemma *polyhedron_imp_convex*:
fixes $S :: 'a :: euclidean\_space\ set$
shows *polyhedron* $S \Longrightarrow$ *convex* S
by (*metis convex_Inter convex_halfspace_le polyhedron_def*)

lemma *polyhedron_affine_hull*:
fixes $S :: 'a :: euclidean\_space\ set$
shows *polyhedron*(*affine hull* S)
by (*simp add*: *affine_imp_polyhedron*)

9.24.11 Canonical polyhedron representation making facial structure explicit

proposition *polyhedron_Int_affine*:
fixes $S :: 'a :: euclidean\_space\ set$
shows *polyhedron* $S \longleftrightarrow$
$(\exists F.\ finite\ F \wedge S = (affine\ hull\ S) \cap \bigcap F \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}))$
by (*metis hull_subset inf.absorb_iff2 polyhedron_Int polyhedron_affine_hull polyhedron_def*)

proposition *rel_interior_polyhedron_explicit*:
assumes *finite* F
and *seq*: $S = affine\ hull\ S \cap \bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset affine\ hull\ S \cap \bigcap F'$
shows *rel_interior* $S = \{x \in S.\ \forall h \in F.\ a\ h \cdot x < b\ h\}$
proof −
have *rels*: $\bigwedge x.\ x \in rel\_interior\ S \Longrightarrow x \in S$
by (*meson IntE mem_rel_interior*)
moreover have $a\ i \cdot x < b\ i$ **if** x: $x \in rel\_interior\ S$ **and** $i \in F$ **for** $x\ i$
proof −
have *fif*: $F - \{i\} \subset F$
using ‹$i \in F$› *Diff_insert_absorb Diff_subset set_insert psubsetI* **by** *blast*
then have $S \subset affine\ hull\ S \cap \bigcap (F - \{i\})$

```
    by (rule psub)
  then obtain z where ssub: S ⊆ ⋂(F − {i}) and zint: z ∈ ⋂(F − {i})
                and z ∉ S and zaff: z ∈ affine hull S
    by auto
  have z ≠ x
    using ‹z ∉ S› rels x by blast
  have z ∉ affine hull S ∩ ⋂F
    using ‹z ∉ S› seq by auto
  then have aiz: a i · z > b i
    using faceq zint zaff by fastforce
  obtain e where e > 0 x ∈ S and e: ball x e ∩ affine hull S ⊆ S
    using x by (auto simp: mem_rel_interior_ball)
  then have ins: ⋀y. ⟦norm (x − y) < e; y ∈ affine hull S⟧ ⟹ y ∈ S
    by (metis IntI subsetD dist_norm mem_ball)
  define ξ where ξ = min (1/2) (e / 2 / norm(z − x))
  have norm (ξ *R x − ξ *R z) = norm (ξ *R (x − z))
    by (simp add: ξ_def algebra_simps norm_mult)
  also have ... = ξ * norm (x − z)
    using ‹e > 0› by (simp add: ξ_def)
  also have ... < e
      using ‹z ≠ x› ‹e > 0› by (simp add: ξ_def min_def field_split_simps
norm_minus_commute)
  finally have lte: norm (ξ *R x − ξ *R z) < e .
  have ξ_aff: ξ *R z + (1 − ξ) *R x ∈ affine hull S
    by (simp add: ‹x ∈ S› hull_inc mem_affine zaff)
  have ξ *R z + (1 − ξ) *R x ∈ S
    using ins [OF _ ξ_aff] by (simp add: algebra_simps lte)
  then obtain l where l: 0 < l l < 1 and ls: (l *R z + (1 − l) *R x) ∈ S
    using ‹e > 0› ‹z ≠ x›
    by (rule_tac l = ξ in that) (auto simp: ξ_def)
  then have i: l *R z + (1 − l) *R x ∈ i
    using seq ‹i ∈ F› by auto
  have b i * l + (a i · x) * (1 − l) < a i · (l *R z + (1 − l) *R x)
    using l by (simp add: algebra_simps aiz)
  also have ... ≤ b i using i l
    using faceq mem_Collect_eq ‹i ∈ F› by blast
  finally have (a i · x) * (1 − l) < b i * (1 − l)
    by (simp add: algebra_simps)
  with l show ?thesis
    by simp
qed
moreover have x ∈ rel_interior S
      if x ∈ S and less: ⋀h. h ∈ F ⟹ a h · x < b h for x
proof −
  have 1: ⋀h. h ∈ F ⟹ x ∈ interior h
    by (metis interior_halfspace_le mem_Collect_eq less faceq)
  have 2: ⋀y. ⟦∀h∈F. y ∈ interior h; y ∈ affine hull S⟧ ⟹ y ∈ S
    by (metis IntI Inter_iff subsetD interior_subset seq)
  show ?thesis
```

```
    apply (simp add: rel_interior ‹x ∈ S›)
    apply (rule_tac x=⋂h∈F. interior h in exI)
    apply (auto simp: ‹finite F› open_INT 1 2)
    done
  qed
  ultimately show ?thesis by blast
qed


lemma polyhedron_Int_affine_parallel:
  fixes S :: 'a :: euclidean_space set
  shows polyhedron S ⟷
        (∃F. finite F ∧
             S = (affine hull S) ∩ (⋂F) ∧
             (∀h ∈ F. ∃a b. a ≠ 0 ∧ h = {x. a · x ≤ b} ∧
                            (∀x ∈ affine hull S. (x + a) ∈ affine hull S)))
    (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain F where finite F and seq: S = (affine hull S) ∩ ⋂F
                  and faces: ⋀h. h ∈ F ⟹ ∃a b. a ≠ 0 ∧ h = {x. a · x ≤ b}
    by (fastforce simp add: polyhedron_Int_affine)
  then obtain a b where ab: ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ h = {x. a h · x ≤ b h}
    by metis
  show ?rhs
  proof −
    have ∃a' b'. a' ≠ 0 ∧
                 affine hull S ∩ {x. a' · x ≤ b'} = affine hull S ∩ h ∧
                 (∀w ∈ affine hull S. (w + a') ∈ affine hull S)
      if h ∈ F ¬(affine hull S ⊆ h) for h
    proof −
      have a h ≠ 0 and h = {x. a h · x ≤ b h} h ∩ ⋂F = ⋂F
        using ‹h ∈ F› ab by auto
      then have (affine hull S) ∩ {x. a h · x ≤ b h} ≠ {}
        by (metis affine_hull_eq_empty inf.absorb_iff1 inf_assoc inf_bot_left seq
that(2))
      moreover have ¬ (affine hull S ⊆ {x. a h · x ≤ b h})
        using ‹h = {x. a h · x ≤ b h}› that(2) by blast
      ultimately show ?thesis
        using affine_parallel_slice [of affine hull S]
        by (metis ‹h = {x. a h · x ≤ b h}› affine_affine_hull)
    qed
    then obtain a b
        where ab: ⋀h. ⟦h ∈ F; ¬ (affine hull S ⊆ h)⟧
            ⟹ a h ≠ 0 ∧
                 affine hull S ∩ {x. a h · x ≤ b h} = affine hull S ∩ h ∧
                 (∀w ∈ affine hull S. (w + a h) ∈ affine hull S)
      by metis
    let ?F = (λh. {x. a h · x ≤ b h}) ‘ {h ∈ F. ¬ affine hull S ⊆ h}
```

```
    show ?thesis
    proof (intro exI conjI)
      show finite ?F
        using ‹finite F› by force
      show S = affine hull S ∩ ⋂ ?F
        by (subst seq) (auto simp: ab INT_extend_simps)
    qed (use ab in blast)
  qed
next
  assume ?rhs then show ?lhs
    by (metis polyhedron_Int_affine)
qed


proposition polyhedron_Int_affine_parallel_minimal:
  fixes S :: 'a :: euclidean_space set
  shows polyhedron S ⟷
        (∃F. finite F ∧
             S = (affine hull S) ∩ (⋂F) ∧
             (∀h ∈ F. ∃a b. a ≠ 0 ∧ h = {x. a · x ≤ b} ∧
                            (∀x ∈ affine hull S. (x + a) ∈ affine hull S)) ∧
             (∀F'. F' ⊂ F ⟶ S ⊂ (affine hull S) ∩ (⋂F')))
    (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain f0
          where f0: finite f0
                S = (affine hull S) ∩ (⋂f0)
                  (is ?P f0)
                ∀h ∈ f0. ∃a b. a ≠ 0 ∧ h = {x. a · x ≤ b} ∧
                               (∀x ∈ affine hull S. (x + a) ∈ affine hull S)
                  (is ?Q f0)
    by (force simp: polyhedron_Int_affine_parallel)
  define n where n = (LEAST n. ∃F. card F = n ∧ finite F ∧ ?P F ∧ ?Q F)
  have nf: ∃F. card F = n ∧ finite F ∧ ?P F ∧ ?Q F
    apply (simp add: n_def)
    apply (rule LeastI [where k = card f0])
    using f0 apply auto
    done
  then obtain F where F: card F = n finite F and seq: ?P F and aff: ?Q F
    by blast
  then have ¬ (finite g ∧ ?P g ∧ ?Q g) if card g < n for g
    using that by (auto simp: n_def dest!: not_less_Least)
  then have *: ¬ (?P g ∧ ?Q g) if g ⊂ F for g
    using that ‹finite F› psubset_card_mono ‹card F = n›
    by (metis finite_Int inf.strict_order_iff)
  have 1: ⋀F'. F' ⊂ F ⟹ S ⊆ affine hull S ∩ ⋂F'
    by (subst seq) blast
  have 2: S ≠ affine hull S ∩ ⋂F' if F' ⊂ F for F'
```

```
    using ∗ [OF that] by (metis IntE aff inf.strict_order_iff that)
  show ?rhs
    by (metis ‹finite F› seq aff psubsetI 1 2)
next
  assume ?rhs then show ?lhs
    by (auto simp: polyhedron_Int_affine_parallel)
qed


lemma polyhedron_Int_affine_minimal:
  fixes S :: 'a :: euclidean_space set
  shows polyhedron S ⟷
        (∃F. finite F ∧ S = (affine hull S) ∩ ⋂F ∧
             (∀h ∈ F. ∃a b. a ≠ 0 ∧ h = {x. a · x ≤ b}) ∧
             (∀F'. F' ⊂ F ⟶ S ⊂ (affine hull S) ∩ ⋂F'))
  by (metis polyhedron_Int_affine polyhedron_Int_affine_parallel_minimal)

proposition facet_of_polyhedron_explicit:
  assumes finite F
      and seq: S = affine hull S ∩ ⋂F
      and faceq: ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ h = {x. a h · x ≤ b h}
      and psub: ⋀F'. F' ⊂ F ⟹ S ⊂ affine hull S ∩ ⋂F'
    shows C facet_of S ⟷ (∃h. h ∈ F ∧ C = S ∩ {x. a h · x = b h})
proof (cases S = {})
  case True with psub show ?thesis by force
next
  case False
  have polyhedron S
    unfolding polyhedron_Int_affine by (metis ‹finite F› faceq seq)
  then have convex S
    by (rule polyhedron_imp_convex)
  with False rel_interior_eq_empty have rel_interior S ≠ {} by blast
  then obtain x where x ∈ rel_interior S by auto
  then obtain T where open T x ∈ T x ∈ S T ∩ affine hull S ⊆ S
    by (force simp: mem_rel_interior)
  then have xaff: x ∈ affine hull S and xint: x ∈ ⋂F
    using seq hull_inc by auto
  have rel_interior S = {x ∈ S. ∀h∈F. a h · x < b h}
    by (rule rel_interior_polyhedron_explicit [OF ‹finite F› seq faceq psub])
  with ‹x ∈ rel_interior S›
  have [simp]: ⋀h. h∈F ⟹ a h · x < b h by blast
  have ∗: (S ∩ {x. a h · x = b h}) facet_of S if h ∈ F for h
  proof −
    have S ⊂ affine hull S ∩ ⋂(F − {h})
      using psub that by (metis Diff_disjoint Diff_subset insert_disjoint(2) psub-
setI)
    then obtain z where zaff: z ∈ affine hull S and zint: z ∈ ⋂(F − {h}) and
z ∉ S
      by force
```

```
    then have z ≠ x z ∉ h using seq ‹x ∈ S› by auto
    have x ∈ h using that xint by auto
    then have able: a h · x ≤ b h
      using faceq that by blast
    also have ... < a h · z using ‹z ∉ h› faceq [OF that] xint by auto
    finally have xltz: a h · x < a h · z .
    define l where l = (b h − a h · x) / (a h · z − a h · x)
    define w where w = (1 − l) *R x + l *R z
    have 0 < l l < 1
      using able xltz ‹b h < a h · z› ‹h ∈ F›
      by (auto simp: l_def field_split_simps)
    have awlt: a i · w < b i if i ∈ F i ≠ h for i
    proof −
      have (1 − l) * (a i · x) < (1 − l) * b i
        by (simp add: ‹l < 1› ‹i ∈ F›)
      moreover have l * (a i · z) ≤ l * b i
      proof (rule mult_left_mono)
        show a i · z ≤ b i
          by (metis DiffI Inter_iff empty_iff faceq insertE mem_Collect_eq that
zint)
      qed (use ‹0 < l› in auto)
      ultimately show ?thesis by (simp add: w_def algebra_simps)
    qed
    have weq: a h · w = b h
      using xltz unfolding w_def l_def
      by (simp add: algebra_simps) (simp add: field_simps)
    let ?F = {x. a h · x = b h}
    have faceS: S ∩ ?F face_of S
    proof (rule face_of_Int_supporting_hyperplane_le)
      show ⋀x. x ∈ S ⟹ a h · x ≤ b h
        using faceq seq that by fastforce
    qed fact
    have w ∈ affine hull S
      by (simp add: w_def mem_affine xaff zaff)
    moreover have w ∈ ⋂F
      using ‹a h · w = b h› awlt faceq less_eq_real_def by blast
    ultimately have w ∈ S
      using seq by blast
    with weq have ne: S ∩ ?F ≠ {} by blast
    moreover have affine hull (S ∩ ?F) = (affine hull S) ∩ ?F
    proof
      show affine hull (S ∩ ?F) ⊆ affine hull S ∩ ?F
      proof −
        have affine hull (S ∩ ?F) ⊆ affine hull S
          by (simp add: hull_mono)
        then show ?thesis
          by (simp add: affine_hyperplane subset_hull)
      qed
    next
```

```
    show affine hull S ∩ ?F ⊆ affine hull (S ∩ ?F)
    proof
      fix y
      assume yaff: y ∈ affine hull S ∩ {y. a h · y = b h}
      obtain T where 0 < T
            and T: ⋀j. ⟦j ∈ F; j ≠ h⟧ ⟹ T * (a j · y − a j · w) ≤ b j − a j · w
      proof (cases F − {h} = {})
        case True then show ?thesis
          by (rule_tac T=1 in that) auto
      next
        case False
        then obtain h' where h': h' ∈ F − {h} by auto
        let ?body = (λj. if 0 < a j · y − a j · w
            then (b j − a j · w) / (a j · y − a j · w) else 1) ` (F − {h})
        define inff where inff = Inf ?body
        from ‹finite F› have finite ?body
          by blast
        moreover from h' have ?body ≠ {}
          by blast
        moreover have j > 0 if j ∈ ?body for j
        proof −
          from that obtain x where x ∈ F and x ≠ h and *: j =
            (if 0 < a x · y − a x · w
              then (b x − a x · w) / (a x · y − a x · w) else 1)
            by blast
          with awlt [of x] have a x · w < b x
            by simp
          with * show ?thesis
            by simp
        qed
        ultimately have 0 < inff
          by (simp_all add: finite_less_Inf_iff inff_def)
        moreover have inff * (a j · y − a j · w) ≤ b j − a j · w
                    if j ∈ F j ≠ h for j
        proof (cases a j · w < a j · y)
          case True
          then have inff ≤ (b j − a j · w) / (a j · y − a j · w)
            unfolding inff_def
              using ‹finite F› by (auto intro: cInf_le_finite simp add: that split:
if_split_asm)
          then show ?thesis
            using ‹0 < inff› awlt [OF that] mult_strict_left_mono
            by (fastforce simp add: field_split_simps split: if_split_asm)
        next
          case False
          with ‹0 < inff› have inff * (a j · y − a j · w) ≤ 0
            by (simp add: mult_le_0_iff)
          also have ... < b j − a j · w
            by (simp add: awlt that)
```

```
        finally show ?thesis by simp
      qed
      ultimately show ?thesis
        by (blast intro: that)
    qed
    define C where C = (1 - T) *R w + T *R y
    have (1 - T) *R w + T *R y ∈ j if j ∈ F for j
    proof (cases j = h)
      case True
      have (1 - T) *R w + T *R y ∈ {x. a h · x ≤ b h}
        using weq yaff by (auto simp: algebra_simps)
      with True faceq [OF that] show ?thesis by metis
    next
      case False
      with T that have (1 - T) *R w + T *R y ∈ {x. a j · x ≤ b j}
        by (simp add: algebra_simps)
      with faceq [OF that] show ?thesis by simp
    qed
    moreover have (1 - T) *R w + T *R y ∈ affine hull S
      using yaff ‹w ∈ affine hull S› affine_affine_hull affine_alt by blast
    ultimately have C ∈ S
      using seq by (force simp: C_def)
    moreover have a h · C = b h
      using yaff by (force simp: C_def algebra_simps weq)
    ultimately have caff: C ∈ affine hull (S ∩ {y. a h · y = b h})
      by (simp add: hull_inc)
    have waff: w ∈ affine hull (S ∩ {y. a h · y = b h})
      using ‹w ∈ S› weq by (blast intro: hull_inc)
    have yeq: y = (1 - inverse T) *R w + C /R T
      using ‹0 < T› by (simp add: C_def algebra_simps)
    show y ∈ affine hull (S ∩ {y. a h · y = b h})
      by (metis yeq affine_affine_hull [simplified affine_alt, rule_format, OF
waff caff])
  qed
 qed
 ultimately have aff_dim (affine hull (S ∩ ?F)) = aff_dim S - 1
  using ‹b h < a h · z› zaff by (force simp: aff_dim_affine_Int_hyperplane)
 then show ?thesis
  by (simp add: ne faceS facet_of_def)
qed
show ?thesis
proof
 show ∃h. h ∈ F ∧ C = S ∩ {x. a h · x = b h} ⟹ C facet_of S
  using ∗ by blast
next
 assume C facet_of S
 then have C face_of S convex C C ≠ {} and affc: aff_dim C = aff_dim S
- 1
  by (auto simp: facet_of_def face_of_imp_convex)
```

```
    then obtain x where x: x ∈ rel_interior C
      by (force simp: rel_interior_eq_empty)
    then have x ∈ C
      by (meson subsetD rel_interior_subset)
    then have x ∈ S
      using ‹C facet_of S› facet_of_imp_subset by blast
    have rels: rel_interior S = {x ∈ S. ∀h∈F. a h ⋅ x < b h}
      by (rule rel_interior_polyhedron_explicit [OF assms])
    have C ≠ S
      using ‹C facet_of S› facet_of_irrefl by blast
    then have x ∉ rel_interior S
    by (metis IntI empty_iff ‹x ∈ C› ‹C ≠ S› ‹C face_of S› face_of_disjoint_rel_interior)
    with rels ‹x ∈ S› obtain i where i ∈ F and i: a i ⋅ x ≥ b i
      by force
    have x ∈ {u. a i ⋅ u ≤ b i}
      by (metis IntD2 InterE ‹i ∈ F› ‹x ∈ S› faceq seq)
    then have a i ⋅ x ≤ b i by simp
    then have a i ⋅ x = b i using i by auto
    have C ⊆ S ∩ {x. a i ⋅ x = b i}
    proof (rule subset_of_face_of [of _ S])
      show S ∩ {x. a i ⋅ x = b i} face_of S
        by (simp add: ∗ ‹i ∈ F› facet_of_imp_face_of)
      show C ⊆ S
        by (simp add: ‹C face_of S› face_of_imp_subset)
      show S ∩ {x. a i ⋅ x = b i} ∩ rel_interior C ≠ {}
        using ‹a i ⋅ x = b i› ‹x ∈ S› x by blast
    qed
    then have cface: C face_of (S ∩ {x. a i ⋅ x = b i})
      by (meson ‹C face_of S› face_of_subset inf_le1)
    have con: convex (S ∩ {x. a i ⋅ x = b i})
      by (simp add: ‹convex S› convex_Int convex_hyperplane)
    show ∃h. h ∈ F ∧ C = S ∩ {x. a h ⋅ x = b h}
      apply (rule_tac x=i in exI)
     by (metis (no_types) ∗ ‹i ∈ F› affc facet_of_def less_irrefl face_of_aff_dim_lt
[OF con cface])
  qed
qed


lemma face_of_polyhedron_subset_explicit:
  fixes S :: 'a :: euclidean_space set
  assumes finite F
      and seq: S = affine hull S ∩ ⋂F
      and faceq: ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ h = {x. a h ⋅ x ≤ b h}
      and psub: ⋀F'. F' ⊂ F ⟹ S ⊂ affine hull S ∩ ⋂F'
      and C: C face_of S and C ≠ {} C ≠ S
    obtains h where h ∈ F C ⊆ S ∩ {x. a h ⋅ x = b h}
proof −
  have C ⊆ S using ‹C face_of S›
```

by (*simp add*: *face_of_imp_subset*)
have *polyhedron S*
by (*metis* ‹*finite F*› *faceq polyhedron_Int polyhedron_Inter polyhedron_affine_hull polyhedron_halfspace_le seq*)
then have *convex S*
by (*simp add*: *polyhedron_imp_convex*)
then have ∗: $(S \cap \{x.\ a\ h \cdot x = b\ h\})$ *face_of S* **if** $h \in F$ **for** *h*
using *faceq seq face_of_Int_supporting_hyperplane_le that* **by** *fastforce*
have *rel_interior C* ≠ {}
using *C* ‹*C* ≠ {}› *face_of_imp_convex rel_interior_eq_empty* **by** *blast*
then obtain *x* **where** *x* ∈ *rel_interior C* **by** *auto*
have *rels*: *rel_interior S* = $\{x \in S.\ \forall h{\in}F.\ a\ h \cdot x < b\ h\}$
by (*rule rel_interior_polyhedron_explicit* [*OF* ‹*finite F*› *seq faceq psub*])
then have *xnot*: *x* ∉ *rel_interior S*
by (*metis IntI* ‹*x* ∈ *rel_interior C*› *C* ‹*C* ≠ *S*› *contra_subsetD empty_iff face_of_disjoint_rel_interior rel_interior_subset*)
then have *x* ∈ *S*
using ‹*C* ⊆ *S*› ‹*x* ∈ *rel_interior C*› *rel_interior_subset* **by** *auto*
then have *xint*: *x* ∈ $\bigcap F$
using *seq* **by** *blast*
have *F* ≠ {} **using** *assms*
by (*metis affine_Int affine_Inter affine_affine_hull ex_in_conv face_of_affine_trivial*)
then obtain *i* **where** *i* ∈ *F* ¬ (*a i* · *x* < *b i*)
using ‹*x* ∈ *S*› *rels xnot* **by** *auto*
with *xint* **have** *a i* · *x* = *b i*
by (*metis eq_iff mem_Collect_eq not_le Inter_iff faceq*)
have *face*: $S \cap \{x.\ a\ i \cdot x = b\ i\}$ *face_of S*
by (*simp add*: ∗ ‹*i* ∈ *F*›)
show *?thesis*
proof
show $C \subseteq S \cap \{x.\ a\ i \cdot x = b\ i\}$
using *subset_of_face_of* [*OF face* ‹*C* ⊆ *S*›] ‹*a i* · *x* = *b i*› ‹*x* ∈ *rel_interior C*› ‹*x* ∈ *S*› **by** *blast*
qed *fact*
qed

Initial part of proof duplicates that above

proposition *face_of_polyhedron_explicit*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *finite F*
and *seq*: *S* = *affine hull S* ∩ $\bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset$ *affine hull S* ∩ $\bigcap F'$
and *C*: *C face_of S* **and** *C* ≠ {} *C* ≠ *S*
shows *C* = $\bigcap \{S \cap \{x.\ a\ h \cdot x = b\ h\} \mid h.\ h \in F \wedge C \subseteq S \cap \{x.\ a\ h \cdot x = b\ h\}\}$
proof −
let *?ab* = $\lambda h.\ \{x.\ a\ h \cdot x = b\ h\}$
have *C* ⊆ *S* **using** ‹*C face_of S*›

by (*simp add*: *face_of_imp_subset*)
have *polyhedron S*
by (*metis* ‹*finite F*› *faceq polyhedron_Int polyhedron_Inter polyhedron_affine_hull polyhedron_halfspace_le seq*)
then have *convex S*
by (*simp add*: *polyhedron_imp_convex*)
then have ∗: (*S* ∩ *?ab h*) *face_of S* **if** *h* ∈ *F* **for** *h*
using *faceq seq face_of_Int_supporting_hyperplane_le that* **by** *fastforce*
have *rel_interior C* ≠ {}
using *C* ‹*C* ≠ {}› *face_of_imp_convex rel_interior_eq_empty* **by** *blast*
then obtain *z* **where** *z*: *z* ∈ *rel_interior C* **by** *auto*
have *rels*: *rel_interior S* = {*z* ∈ *S*. ∀ *h*∈*F*. *a h* ⋅ *z* < *b h*}
by (*rule rel_interior_polyhedron_explicit* [*OF* ‹*finite F*› *seq faceq psub*])
then have *xnot*: *z* ∉ *rel_interior S*
by (*metis IntI* ‹*z* ∈ *rel_interior C*› *C* ‹*C* ≠ *S*› *contra_subsetD empty_iff face_of_disjoint_rel_interior rel_interior_subset*)
then have *z* ∈ *S*
using ‹*C* ⊆ *S*› ‹*z* ∈ *rel_interior C*› *rel_interior_subset* **by** *auto*
with *seq* **have** *xint*: *z* ∈ ⋂ *F* **by** *blast*
have *open* (⋂ *h*∈{*h* ∈ *F*. *a h* ⋅ *z* < *b h*}. {*w*. *a h* ⋅ *w* < *b h*})
by (*auto simp*: ‹*finite F*› *open_halfspace_lt open_INT*)
then obtain *e* **where** *0* < *e*
ball z e ⊆ (⋂ *h*∈{*h* ∈ *F*. *a h* ⋅ *z* < *b h*}. {*w*. *a h* ⋅ *w* < *b h*})
by (*auto intro*: *openE* [*of* _ *z*])
then have *e*: ⋀*h*. ⟦*h* ∈ *F*; *a h* ⋅ *z* < *b h*⟧ ⟹ *ball z e* ⊆ {*w*. *a h* ⋅ *w* < *b h*}
by *blast*
have *C* ⊆ (*S* ∩ *?ab h*) ⟷ *z* ∈ *S* ∩ *?ab h* **if** *h* ∈ *F* **for** *h*
proof
show *z* ∈ *S* ∩ *?ab h* ⟹ *C* ⊆ *S* ∩ *?ab h*
by (*metis* ∗ *Collect_cong IntI* ‹*C* ⊆ *S*› *empty_iff subset_of_face_of that z*)
next
show *C* ⊆ *S* ∩ *?ab h* ⟹ *z* ∈ *S* ∩ *?ab h*
using ‹*z* ∈ *rel_interior C*› *rel_interior_subset* **by** *force*
qed
then have ∗∗: {*S* ∩ *?ab h* | *h*. *h* ∈ *F* ∧ *C* ⊆ *S* ∧ *C* ⊆ *?ab h*} =
{*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *z* ∈ *S* ∩ *?ab h*}
by *blast*
have *bsub*: *ball z e* ∩ *affine hull* ⋂ {*S* ∩ *?ab h* |*h*. *h* ∈ *F* ∧ *a h* ⋅ *z* = *b h*}
⊆ *affine hull S* ∩ ⋂ *F* ∩ ⋂ {*?ab h* |*h*. *h* ∈ *F* ∧ *a h* ⋅ *z* = *b h*}
if *i* ∈ *F* **and** *i*: *a i* ⋅ *z* = *b i* **for** *i*
proof −
have *sub*: *ball z e* ∩ ⋂ {*?ab h* |*h*. *h* ∈ *F* ∧ *a h* ⋅ *z* = *b h*} ⊆ *j*
if *j* ∈ *F* **for** *j*
proof −
have *a j* ⋅ *z* ≤ *b j* **using** *faceq that xint* **by** *auto*
then consider *a j* ⋅ *z* < *b j* | *a j* ⋅ *z* = *b j* **by** *linarith*
then have ∃ *G*. *G* ∈ {*?ab h* |*h*. *h* ∈ *F* ∧ *a h* ⋅ *z* = *b h*} ∧ *ball z e* ∩ *G* ⊆ *j*
proof *cases*
assume *a j* ⋅ *z* < *b j*

```
      then have ball z e ∩ {x. a i · x = b i} ⊆ j
        using e [OF ‹j ∈ F›] faceq that
        by (fastforce simp: ball_def)
      then show ?thesis
        by (rule_tac x={x. a i · x = b i} in exI) (force simp: ‹i ∈ F› i)
    next
      assume eq: a j · z = b j
      with faceq that show ?thesis
        by (rule_tac x={x. a j · x = b j} in exI) (fastforce simp add: ‹j ∈ F›)
    qed
    then show ?thesis  by blast
  qed
  have 1: affine hull ⋂{S ∩ ?ab h |h. h ∈ F ∧ a h · z = b h} ⊆ affine hull S
    using that ‹z ∈ S› by (intro hull_mono) auto
  have 2: affine hull ⋂{S ∩ ?ab h |h. h ∈ F ∧ a h · z = b h}
        ⊆ ⋂{?ab h |h. h ∈ F ∧ a h · z = b h}
    by (rule hull_minimal) (auto intro: affine_hyperplane)
  have 3: ball z e ∩ ⋂{?ab h |h. h ∈ F ∧ a h · z = b h} ⊆ ⋂F
    by (iprover intro: sub Inter_greatest)
  have ∗: ⟦A ⊆ (B :: 'a set); A ⊆ C; E ∩ C ⊆ D⟧ ⟹ E ∩ A ⊆ (B ∩ D) ∩ C
        for A B C D E  by blast
  show ?thesis by (intro ∗ 1 2 3)
 qed
 have ∃h. h ∈ F ∧ C ⊆ ?ab h
   using assms
   by (metis face_of_polyhedron_subset_explicit [OF ‹finite F› seq faceq psub]
le_inf_iff)
 then have fac: ⋂{S ∩ ?ab h |h. h ∈ F ∧ C ⊆ S ∩ ?ab h} face_of S
   using ∗ by (force simp: ‹C ⊆ S› intro: face_of_Inter)
 have red: (⋀a. P a ⟹ T ⊆ S ∩ ⋂{F X |X. P X}) ⟹ T ⊆ ⋂{S ∩ F X |X::'a
set. P X} for P T F
   by blast
 have ball z e ∩ affine hull ⋂{S ∩ ?ab h |h. h ∈ F ∧ a h · z = b h}
      ⊆ ⋂{S ∩ ?ab h |h. h ∈ F ∧ a h · z = b h}
   by (rule red) (metis seq bsub)
 with ‹0 < e› have zinrel: z ∈ rel_interior
                  (⋂{S ∩ ?ab h |h. h ∈ F ∧ z ∈ S ∧ a h · z = b h})
   by (auto simp: mem_rel_interior_ball ‹z ∈ S›)
 show ?thesis
   using z zinrel
   by (intro face_of_eq [OF C fac]) (force simp: ∗∗)
qed
```

9.24.12 More general corollaries from the explicit representation

```
corollary facet_of_polyhedron:
 assumes polyhedron S and C facet_of S
 obtains a b where a ≠ 0 S ⊆ {x. a · x ≤ b} C = S ∩ {x. a · x = b}
```

proof −
obtain F **where** *finite* F **and** *seq*: $S = \mathit{affine\ hull}\ S \cap \bigcap F$
and *faces*: $\bigwedge h.\ h \in F \Longrightarrow \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}$
and *min*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset (\mathit{affine\ hull}\ S) \cap \bigcap F'$
using *assms* **by** (*simp add*: *polyhedron_Int_affine_minimal*) *meson*
then obtain $a\ b$ **where** *ab*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
by *metis*
obtain i **where** $i \in F$ **and** C: $C = S \cap \{x.\ a\ i \cdot x = b\ i\}$
using *facet_of_polyhedron_explicit* [*OF* ‹*finite* F› *seq ab min*] *assms*
by *force*
moreover have *ssub*: $S \subseteq \{x.\ a\ i \cdot x \leq b\ i\}$
using ‹$i \in F$› *ab* **by** (*subst seq*) *auto*
ultimately show *?thesis*
by (*rule_tac* $a = a\ i$ **and** $b = b\ i$ **in** *that*) (*simp_all add*: *ab*)
qed

corollary *face_of_polyhedron*:
assumes *polyhedron* S **and** C *face_of* S **and** $C \neq \{\}$ **and** $C \neq S$
shows $C = \bigcap \{F.\ F\ \mathit{facet\_of}\ S \wedge C \subseteq F\}$
proof −
obtain F **where** *finite* F **and** *seq*: $S = \mathit{affine\ hull}\ S \cap \bigcap F$
and *faces*: $\bigwedge h.\ h \in F \Longrightarrow \exists\, a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}$
and *min*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset (\mathit{affine\ hull}\ S) \cap \bigcap F'$
using *assms* **by** (*simp add*: *polyhedron_Int_affine_minimal*) *meson*
then obtain $a\ b$ **where** *ab*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
by *metis*
show *?thesis*
apply (*subst face_of_polyhedron_explicit* [*OF* ‹*finite* F› *seq ab min*])
apply (*auto simp*: *assms facet_of_polyhedron_explicit* [*OF* ‹*finite* F› *seq ab min*] *cong*: *Collect_cong*)
done
qed

lemma *face_of_polyhedron_subset_facet*:
assumes *polyhedron* S **and** C *face_of* S **and** $C \neq \{\}$ **and** $C \neq S$
obtains F **where** F *facet_of* S $C \subseteq F$
using *face_of_polyhedron assms*
by (*metis* (*no_types*, *lifting*) *Inf_greatest antisym_conv face_of_imp_subset mem_Collect_eq*)

lemma *exposed_face_of_polyhedron*:
assumes *polyhedron* S
shows F *exposed_face_of* $S \longleftrightarrow F$ *face_of* S
proof
show F *exposed_face_of* $S \Longrightarrow F$ *face_of* S
by (*simp add*: *exposed_face_of_def*)
next
assume F *face_of* S

```
  show F exposed_face_of S
  proof (cases F = {} ∨ F = S)
    case True then show ?thesis
      using ‹F face_of S› exposed_face_of by blast
  next
    case False
    then have {g. g facet_of S ∧ F ⊆ g} ≠ {}
    by (metis Collect_empty_eq_bot ‹F face_of S› assms empty_def face_of_polyhedron_subset_facet)
    moreover have ⋀T. ⟦T facet_of S; F ⊆ T⟧ ⟹ T exposed_face_of S
    by (metis assms exposed_face_of facet_of_imp_face_of facet_of_polyhedron)
    ultimately have ⋂{G. G facet_of S ∧ F ⊆ G} exposed_face_of S
      by (metis (no_types, lifting) mem_Collect_eq exposed_face_of_Inter)
    then show ?thesis
      using False ‹F face_of S› assms face_of_polyhedron by fastforce
  qed
qed

lemma face_of_polyhedron_polyhedron:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S c face_of S shows polyhedron c
by (metis assms face_of_imp_eq_affine_Int polyhedron_Int polyhedron_affine_hull
polyhedron_imp_convex)

lemma finite_polyhedron_faces:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
    shows finite {F. F face_of S}
proof −
  obtain F where finite F and seq: S = affine hull S ∩ ⋂F
            and faces: ⋀h. h ∈ F ⟹ ∃a b. a ≠ 0 ∧ h = {x. a · x ≤ b}
            and min:   ⋀F'. F' ⊂ F ⟹ S ⊂ (affine hull S) ∩ ⋂F'
    using assms by (simp add: polyhedron_Int_affine_minimal) meson
  then obtain a b where ab: ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ h = {x. a h · x ≤ b h}
    by metis
  have finite {⋂{S ∩ {x. a h · x = b h} |h. h ∈ F'}| F'. F' ∈ Pow F}
    by (simp add: ‹finite F›)
  moreover have {F. F face_of S} − {{}, S} ⊆ {⋂{S ∩ {x. a h · x = b h} |h.
h ∈ F'}| F'. F' ∈ Pow F}
    apply clarify
    apply (rename_tac c)
   apply (drule face_of_polyhedron_explicit [OF ‹finite F› seq ab min, simplified],
simp_all)
    apply (rule_tac x={h ∈ F. c ⊆ S ∩ {x. a h · x = b h}} in exI, auto)
    done
  ultimately show ?thesis
    by (meson finite.emptyI finite.insertI finite_Diff2 finite_subset)
qed

lemma finite_polyhedron_exposed_faces:
```

```
  polyhedron S ⟹ finite {F. F exposed_face_of S}
using exposed_face_of_polyhedron finite_polyhedron_faces by fastforce

lemma finite_polyhedron_extreme_points:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S shows finite {v. v extreme_point_of S}
proof −
  have finite {v. {v} face_of S}
   using assms by (intro finite_subset [OF _ finite_vimageI [OF finite_polyhedron_faces]],
auto)
  then show ?thesis
    by (simp add: face_of_singleton)
qed

lemma finite_polyhedron_facets:
  fixes S :: 'a :: euclidean_space set
  shows polyhedron S ⟹ finite {F. F facet_of S}
  unfolding facet_of_def
  by (blast intro: finite_subset [OF _ finite_polyhedron_faces])


proposition rel_interior_of_polyhedron:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
    shows rel_interior S = S − ⋃{F. F facet_of S}
proof −
  obtain F where finite F and seq: S = affine hull S ∩ ⋂F
            and faces: ⋀h. h ∈ F ⟹ ∃ a b. a ≠ 0 ∧ h = {x. a · x ≤ b}
            and min: ⋀F'. F' ⊂ F ⟹ S ⊂ (affine hull S) ∩ ⋂F'
    using assms by (simp add: polyhedron_Int_affine_minimal) meson
  then obtain a b where ab: ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ h = {x. a h · x ≤ b h}
    by metis
  have facet: (c facet_of S) ⟷ (∃h. h ∈ F ∧ c = S ∩ {x. a h · x = b h}) for c
    by (rule facet_of_polyhedron_explicit [OF ‹finite F› seq ab min])
  have rel: rel_interior S = {x ∈ S. ∀h∈F. a h · x < b h}
    by (rule rel_interior_polyhedron_explicit [OF ‹finite F› seq ab min])
  have a h · x < b h if x ∈ S h ∈ F and xnot: x ∉ ⋃{F. F facet_of S} for x h
  proof −
    have x ∈ ⋂F using seq that by force
    with ‹h ∈ F› ab have a h · x ≤ b h by auto
    then consider a h · x < b h | a h · x = b h by linarith
    then show ?thesis
    proof cases
      case 1 then show ?thesis .
    next
      case 2
      have Collect ((∈) x) ∉ Collect ((∈) (⋃{A. A facet_of S}))
        using xnot by fastforce
      then have F ∉ Collect ((∈) h)
```

```
      using 2 ‹x ∈ S› facet by blast
    with 2 that ‹x ∈ ⋂F› show ?thesis
      by blast
    qed
  qed
  moreover have ∃h∈F. a h · x ≥ b h if x ∈ ⋃{F. F facet_of S} for x
    using that by (force simp: facet)
  ultimately show ?thesis
    by (force simp: rel)
qed

lemma rel_boundary_of_polyhedron:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
    shows S - rel_interior S = ⋃ {F. F facet_of S}
using facet_of_imp_subset by (fastforce simp add: rel_interior_of_polyhedron
assms)

lemma rel_frontier_of_polyhedron:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
    shows rel_frontier S = ⋃ {F. F facet_of S}
by (simp add: assms rel_frontier_def polyhedron_imp_closed rel_boundary_of_polyhedron)

lemma rel_frontier_of_polyhedron_alt:
  fixes S :: 'a :: euclidean_space set
  assumes polyhedron S
  shows rel_frontier S = ⋃ {F. F face_of S ∧ F ≠ S}
proof
  show rel_frontier S ⊆ ⋃ {F. F face_of S ∧ F ≠ S}
    by (force simp: rel_frontier_of_polyhedron facet_of_def assms)
qed (use face_of_subset_rel_frontier in fastforce)
```

A characterization of polyhedra as having finitely many faces

```
proposition polyhedron_eq_finite_exposed_faces:
  fixes S :: 'a :: euclidean_space set
  shows polyhedron S ⟷ closed S ∧ convex S ∧ finite {F. F exposed_face_of S}
        (is ?lhs = ?rhs)
proof
  assume ?lhs
  then show ?rhs
   by (auto simp: polyhedron_imp_closed polyhedron_imp_convex finite_polyhedron_exposed_faces)
next
  assume ?rhs
  then have closed S convex S and fin: finite {F. F exposed_face_of S} by auto
  show ?lhs
  proof (cases S = {})
    case True then show ?thesis by auto
  next
```

```
    case False
    define F where F = {h. h exposed_face_of S ∧ h ≠ {} ∧ h ≠ S}
    have finite F by (simp add: fin F_def)
    have hface: h face_of S
      and ∃ a b. a ≠ 0 ∧ S ⊆ {x. a • x ≤ b} ∧ h = S ∩ {x. a • x = b}
      if h ∈ F for h
      using exposed_face_of F_def that by blast+
    then obtain a b where ab:
      ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ S ⊆ {x. a h • x ≤ b h} ∧ h = S ∩ {x. a h • x = b
h}
      by metis
    have ∗: False
      if paff: p ∈ affine hull S and p ∉ S
      and pint: p ∈ ⋂{{x. a h • x ≤ b h} |h. h ∈ F} for p
    proof −
      have rel_interior S ≠ {}
        by (simp add: ‹S ≠ {}› ‹convex S› rel_interior_eq_empty)
      then obtain c where c: c ∈ rel_interior S by auto
      with rel_interior_subset have c ∈ S  by blast
      have ccp: closed_segment c p ⊆ affine hull S
        by (meson affine_affine_hull affine_imp_convex c closed_segment_subset
hull_subset paff rel_interior_subset subsetCE)
       have oS: openin (top_of_set (closed_segment c p)) (closed_segment c p ∩
rel_interior S)
      by (force simp: openin_rel_interior openin_Int intro: openin_subtopology_Int_subset
[OF _ ccp])
       obtain x where xcl: x ∈ closed_segment c p and x ∈ S and xnot: x ∉
rel_interior S
        using connected_openin [of closed_segment c p]
        apply simp
        apply (drule_tac x=closed_segment c p ∩ rel_interior S in spec)
        apply (drule mp [OF _ oS])
        apply (drule_tac x=closed_segment c p ∩ (− S) in spec)
        using rel_interior_subset ‹closed S› c ‹p ∉ S› apply blast
        done
      then obtain μ where 0 ≤ μ μ ≤ 1 and xeq: x = (1 − μ) ∗R c + μ ∗R p
        by (auto simp: in_segment)
      show False
      proof (cases μ=0 ∨ μ=1)
        case True with xeq c xnot ‹x ∈ S› ‹p ∉ S›
        show False by auto
      next
        case False
        then have xos: x ∈ open_segment c p
          using ‹x ∈ S› c open_segment_def that(2) xcl xnot by auto
        have xclo: x ∈ closure S
          using ‹x ∈ S› closure_subset by blast
        obtain d where d ≠ 0
            and dle: ⋀y. y ∈ closure S ⟹ d • x ≤ d • y
```

```
            and dless: ⋀y. y ∈ rel_interior S ⟹ d • x < d • y
          by (metis supporting_hyperplane_relative_frontier [OF ‹convex S› xclo
xnot])
       have sex: S ∩ {y. d • y = d • x} exposed_face_of S
      by (simp add: ‹closed S› dle exposed_face_of_Int_supporting_hyperplane_ge
[OF ‹convex S›])
       have sne: S ∩ {y. d • y = d • x} ≠ {}
         using ‹x ∈ S› by blast
       have sns: S ∩ {y. d • y = d • x} ≠ S
           by (metis (mono_tags) Int_Collect c subsetD dless not_le order_refl
rel_interior_subset)
       obtain h where h ∈ F x ∈ h
         using F_def ‹x ∈ S› sex sns by blast
       have abface: {y. a h • y = b h} face_of {y. a h • y ≤ b h}
         using hyperplane_face_of_halfspace_le by blast
       then have c ∈ h
         using face_ofD [OF abface xos] ‹c ∈ S› ‹h ∈ F› ab pint ‹x ∈ h› by blast
       with c have h ∩ rel_interior S ≠ {} by blast
       then show False
         using ‹h ∈ F› F_def face_of_disjoint_rel_interior hface by auto
     qed
   qed
   let ?S' = affine hull S ∩ ⋂{{x. a h • x ≤ b h} |h. h ∈ F}
   have S ⊆ ?S'
     using ab by (auto simp: hull_subset)
   moreover have ?S' ⊆ S
     using ∗ by blast
   ultimately have S = ?S' ..
   moreover have polyhedron ?S'
     by (force intro: polyhedron_affine_hull polyhedron_halfspace_le simp: ‹finite
F›)
   ultimately show ?thesis
     by auto
 qed
qed

corollary polyhedron_eq_finite_faces:
 fixes S :: 'a :: euclidean_space set
 shows polyhedron S ⟷ closed S ∧ convex S ∧ finite {F. F face_of S}
       (is ?lhs = ?rhs)
proof
 assume ?lhs
 then show ?rhs
  by (simp add: finite_polyhedron_faces polyhedron_imp_closed polyhedron_imp_convex)
next
 assume ?rhs
 then show ?lhs
    by (force simp: polyhedron_eq_finite_exposed_faces exposed_face_of intro:
finite_subset)
```

qed

lemma *polyhedron_linear_image_eq*:
fixes $h :: 'a :: euclidean\_space \Rightarrow 'b :: euclidean\_space$
assumes *linear h bij h*
shows *polyhedron* $(h \text{ ` } S) \longleftrightarrow$ *polyhedron S*
proof −
have [*simp*]: *inj h* **using** *bij_is_inj assms* **by** *blast*
then have *injim*: *inj_on* $((\text{`})\ h)\ A$ **for** *A*
by (*simp add*: *inj_on_def inj_image_eq_iff*)
{ fix *P*
have $\bigwedge x.\ P\ x \Longrightarrow x \in (\text{`})\ h \text{ ` } \{f.\ P\ (h \text{ ` } f)\}$
using *bij_is_surj* [*OF* ‹*bij h*›]
by (*metis image_eqI mem_Collect_eq subset_imageE top_greatest*)
then have $\{f.\ P\ f\} = (image\ h) \text{ ` } \{f.\ P\ (h \text{ ` } f)\}$
by *force*
}
then have *finite* $\{F.\ F\ face\_of\ h \text{ ` } S\} =$*finite* $\{F.\ F\ face\_of\ S\}$
using ‹*linear h*›
by (*simp add*: *finite_image_iff injim flip*: *face_of_linear_image* [*of h _ S*])
then show *?thesis*
using ‹*linear h*›
by (*simp add*: *polyhedron_eq_finite_faces closed_injective_linear_image_eq*)
qed

lemma *polyhedron_negations*:
fixes $S :: 'a :: euclidean\_space\ set$
shows *polyhedron S* $\Longrightarrow$ *polyhedron*(*image uminus S*)
by (*subst polyhedron_linear_image_eq*) (*auto simp*: *bij_uminus intro*!: *linear_uminus*)

9.24.13 Relation between polytopes and polyhedra

proposition *polytope_eq_bounded_polyhedron*:
fixes $S :: 'a :: euclidean\_space\ set$
shows *polytope S* $\longleftrightarrow$ *polyhedron S* $\wedge$ *bounded S*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*simp add*: *finite_polytope_faces polyhedron_eq_finite_faces*
polytope_imp_closed polytope_imp_convex polytope_imp_bounded)
next
assume *R*: *?rhs*
then have *finite* $\{v.\ v\ extreme\_point\_of\ S\}$
by (*simp add*: *finite_polyhedron_extreme_points*)
moreover have $S = convex\ hull\ \{v.\ v\ extreme\_point\_of\ S\}$
using *R* **by** (*simp add*: *Krein_Milman_Minkowski compact_eq_bounded_closed*
polyhedron_imp_closed polyhedron_imp_convex)
ultimately show *?lhs*

unfolding *polytope_def* **by** *blast*
qed

lemma *polytope_Int*:
fixes $S :: 'a :: euclidean\_space\ set$
shows $[\![polytope\ S; polytope\ T]\!] \Longrightarrow polytope(S \cap T)$
by (*simp add*: *polytope_eq_bounded_polyhedron bounded_Int*)

lemma *polytope_Int_polyhedron*:
fixes $S :: 'a :: euclidean\_space\ set$
shows $[\![polytope\ S; polyhedron\ T]\!] \Longrightarrow polytope(S \cap T)$
by (*simp add*: *bounded_Int polytope_eq_bounded_polyhedron*)

lemma *polyhedron_Int_polytope*:
fixes $S :: 'a :: euclidean\_space\ set$
shows $[\![polyhedron\ S; polytope\ T]\!] \Longrightarrow polytope(S \cap T)$
by (*simp add*: *bounded_Int polytope_eq_bounded_polyhedron*)

lemma *polytope_imp_polyhedron*:
fixes $S :: 'a :: euclidean\_space\ set$
shows $polytope\ S \Longrightarrow polyhedron\ S$
by (*simp add*: *polytope_eq_bounded_polyhedron*)

lemma *polytope_facet_exists*:
fixes $p :: 'a :: euclidean\_space\ set$
assumes $polytope\ p\ 0 < aff\_dim\ p$
obtains F **where** $F\ facet\_of\ p$
proof (*cases* $p = \{\}$)
case *True* **with** *assms* **show** *?thesis* **by** *auto*
next
case *False*
then obtain v **where** $v\ extreme\_point\_of\ p$
using *extreme_point_exists_convex*
by (*blast intro*: ‹*polytope p*› *polytope_imp_compact polytope_imp_convex*)
then
show *?thesis*
by (*metis face_of_polyhedron_subset_facet polytope_imp_polyhedron aff_dim_sing all_not_in_conv assms face_of_singleton less_irrefl singletonI that*)
qed

lemma *polyhedron_interval* [*iff*]: $polyhedron(cbox\ a\ b)$
by (*metis polytope_imp_polyhedron polytope_interval*)

lemma *polyhedron_convex_hull*:
fixes $S :: 'a :: euclidean\_space\ set$
shows $finite\ S \Longrightarrow polyhedron(convex\ hull\ S)$
by (*simp add*: *polytope_convex_hull polytope_imp_polyhedron*)

9.24.14 Relative and absolute frontier of a polytope

lemma *rel_boundary_of_convex_hull*:
  fixes $S :: 'a::euclidean\_space\ set$
  assumes $\neg$ *affine_dependent S*
  shows (*convex hull S*) − *rel_interior*(*convex hull S*) = ($\bigcup a{\in}S$. *convex hull* (*S* − {*a*}))
proof −
 have *finite S* **by** (*metis assms aff_independent_finite*)
 then consider *card S* = *0* | *card S* = *1* | *2* ≤ *card S* **by** *arith*
 then show *?thesis*
 proof *cases*
  case *1* **then have** *S* = {} **by** (*simp add*: ‹*finite S*›)
  then show *?thesis* **by** *simp*
 next
  case *2* **show** *?thesis*
   by (*auto intro*: *card_1_singletonE* [*OF* ‹*card S* = *1*›])
 next
  case *3*
  with *assms* **show** *?thesis*
  by (*auto simp*: *polyhedron_convex_hull rel_boundary_of_polyhedron facet_of_convex_hull_affine_*
‹*finite S*›)
 qed
qed

proposition *frontier_of_convex_hull*:
  fixes $S :: 'a::euclidean\_space\ set$
  assumes *card S* = *Suc* (*DIM*('*a*))
  shows *frontier*(*convex hull S*) = $\bigcup$ {*convex hull* (*S* − {*a*}) | *a*. *a* ∈ *S*}
proof (*cases affine_dependent S*)
 case *True*
  have [*iff*]: *finite S*
   using *assms* **using** *card.infinite* **by** *force*
  then have *ccs*: *closed* (*convex hull S*)
   by (*simp add*: *compact_imp_closed finite_imp_compact_convex_hull*)
  { fix *x T*
   assume *int* (*card T*) ≤ *aff_dim S* + *1 finite T T* ⊆ *Sx* ∈ *convex hull T*
   then have *S* ≠ *T*
     using *True* ‹*finite S*› *aff_dim_le_card affine_independent_iff_card* **by**
fastforce
   then obtain *a* **where** *a* ∈ *S a* ∉ *T*
    using ‹*T* ⊆ *S*› **by** *blast*
   then have ∃ *y*∈*S*. *x* ∈ *convex hull* (*S* − {*y*})
    using *True affine_independent_iff_card* [*of S*]
    by (*metis* (*no_types*, *opaque_lifting*) *Diff_eq_empty_iff Diff_insert0* ‹*a* ∉
T› ‹*T* ⊆ *S*› ‹*x* ∈ *convex hull T*› *hull_mono insert_Diff_single subsetCE*)
  } **note** ∗ = *this*
  have *1*: *convex hull S* ⊆ ($\bigcup$ *a*∈*S*. *convex hull* (*S* − {*a*}))
   by (*subst caratheodory_aff_dim*) (*blast dest*: ∗)
  have *2*: $\bigcup$((λ*a*. *convex hull* (*S* − {*a*})) ‘ *S*) ⊆ *convex hull S*

by (*rule Union_least*) (*metis* (*no_types*, *lifting*) *Diff_subset hull_mono imageE*)
show *?thesis* **using** *True*
apply (*simp add*: *segment_convex_hull frontier_def*)
using *interior_convex_hull_eq_empty* [*OF assms*]
apply (*simp add*: *closure_closed* [*OF ccs*])
using *1 2* **by** *auto*
next
case *False*
then have *frontier* (*convex hull S*) = *closure* (*convex hull S*) − *interior* (*convex hull S*)
by (*simp add*: *rel_boundary_of_convex_hull frontier_def*)
also have ... = (*convex hull S*) − *rel_interior*(*convex hull S*)
by (*metis False aff_independent_finite assms closure_convex_hull finite_imp_compact_convex_hull hull_hull interior_convex_hull_eq_empty rel_interior_nonempty_interior*)
also have ... = ⋃{*convex hull* (*S* − {*a*}) |*a*. *a* ∈ *S*}
proof −
have *convex hull S* − *rel_interior* (*convex hull S*) = *rel_frontier* (*convex hull S*)
by (*simp add*: *False aff_independent_finite polyhedron_convex_hull rel_boundary_of_polyhedron rel_frontier_of_polyhedron*)
then show *?thesis*
by (*simp add*: *False rel_frontier_convex_hull_cases*)
qed
finally show *?thesis* **.**
qed

9.24.15 Special case of a triangle

proposition *frontier_of_triangle*:
fixes *a* :: *'a*::*euclidean_space*
assumes *DIM*(*'a*) = *2*
shows *frontier*(*convex hull* {*a*,*b*,*c*}) = *closed_segment a b* ∪ *closed_segment b c* ∪ *closed_segment c a*
(**is** *?lhs* = *?rhs*)
proof (*cases b* = *a* ∨ *c* = *a* ∨ *c* = *b*)
case *True* **then show** *?thesis*
by (*auto simp*: *assms segment_convex_hull frontier_def empty_interior_convex_hull insert_commute card_insert_le_m1 hull_inc insert_absorb*)
next
case *False* **then have** [*simp*]: *card* {*a*, *b*, *c*} = *Suc* (*DIM*(*'a*))
by (*simp add*: *card.insert_remove Set.insert_Diff_if assms*)
show *?thesis*
proof
show *?lhs* ⊆ *?rhs*
using *False*
by (*force simp*: *segment_convex_hull frontier_of_convex_hull insert_Diff_if insert_commute split*: *if_split_asm*)
show *?rhs* ⊆ *?lhs*

```
      using False
      apply (simp add: frontier_of_convex_hull segment_convex_hull)
      apply (intro conjI subsetI)
      apply (rule_tac X=convex hull {a,b} in UnionI; force simp: Set.insert_Diff_if)
       apply (rule_tac X=convex hull {b,c} in UnionI; force)
     apply (rule_tac X=convex hull {a,c} in UnionI; force simp: insert_commute
Set.insert_Diff_if)
      done
  qed
qed

corollary inside_of_triangle:
    fixes a :: 'a::euclidean_space
    assumes DIM('a) = 2
     shows inside (closed_segment a b ∪ closed_segment b c ∪ closed_segment c
a) = interior(convex hull {a,b,c})
by (metis assms frontier_of_triangle bounded_empty bounded_insert convex_convex_hull
inside_frontier_eq_interior bounded_convex_hull)

corollary interior_of_triangle:
    fixes a :: 'a::euclidean_space
    assumes DIM('a) = 2
    shows interior(convex hull {a,b,c}) =
            convex hull {a,b,c} − (closed_segment a b ∪ closed_segment b c ∪
closed_segment c a)
  using interior_subset
 by (force simp: frontier_of_triangle [OF assms, symmetric] frontier_def Diff_Diff_Int)
```

9.24.16 Subdividing a cell complex

```
lemma subdivide_interval:
  fixes x::real
  assumes a < |x − y| 0 < a
  obtains n where n ∈ ℤ x < n ∗ a ∧ n ∗ a < y ∨ y <  n ∗ a ∧ n ∗ a < x
proof −
  consider a + x < y | a + y < x
    using assms by linarith
  then show ?thesis
  proof cases
    case 1
    let ?n = of_int (floor (x/a)) + 1
    have x: x < ?n ∗ a
      by (meson ‹0 < a› divide_less_eq floor_eq_iff)
    have ?n ∗ a ≤ a + x
      using ‹a>0› by (simp add: distrib_right floor_divide_lower)
    also have ... < y
      by (rule 1)
    finally have ?n ∗ a < y .
    with x show ?thesis
```

```
    using Ints_1 Ints_add Ints_of_int that by blast
  next
    case 2
    let ?n = of_int (floor (y/a)) + 1
    have y: y < ?n * a
      by (meson ‹0 < a› divide_less_eq floor_eq_iff)
    have ?n * a ≤ a + y
      using ‹a>0› by (simp add: distrib_right floor_divide_lower)
    also have ... < x
      by (rule 2)
    finally have ?n * a < x .
    then show ?thesis
      using Ints_1 Ints_add Ints_of_int that y by blast
  qed
qed

lemma cell_subdivision_lemma:
  assumes finite ℱ
      and ⋀X. X ∈ ℱ ⟹ polytope X
      and ⋀X. X ∈ ℱ ⟹ aff_dim X ≤ d
      and ⋀X Y. ⟦X ∈ ℱ; Y ∈ ℱ⟧ ⟹ (X ∩ Y) face_of X
      and finite I
    shows ∃𝒢. ⋃𝒢 = ⋃ℱ ∧
                finite 𝒢 ∧
                (∀C ∈ 𝒢. ∃D. D ∈ ℱ ∧ C ⊆ D) ∧
                (∀C ∈ ℱ. ∀x ∈ C. ∃D. D ∈ 𝒢 ∧ x ∈ D ∧ D ⊆ C) ∧
                (∀X ∈ 𝒢. polytope X) ∧
                (∀X ∈ 𝒢. aff_dim X ≤ d) ∧
                (∀X ∈ 𝒢. ∀Y ∈ 𝒢. X ∩ Y face_of X) ∧
                (∀X ∈ 𝒢. ∀x ∈ X. ∀y ∈ X. ∀a b.
                      (a,b) ∈ I ⟶ a · x ≤ b ∧ a · y ≤ b ∨
                                  a · x ≥ b ∧ a · y ≥ b)
  using ‹finite I›
proof induction
  case empty
  then show ?case
    by (rule_tac x=ℱ in exI) (auto simp: assms)
next
  case (insert ab I)
  then obtain 𝒢 where eq: ⋃𝒢 = ⋃ℱ and finite 𝒢
                  and sub1: ⋀C. C ∈ 𝒢 ⟹ ∃D. D ∈ ℱ ∧ C ⊆ D
                   and sub2: ⋀C x. C ∈ ℱ ∧ x ∈ C ⟹ ∃D. D ∈ 𝒢 ∧ x ∈ D ∧ D
⊆ C
                  and poly: ⋀X. X ∈ 𝒢 ⟹ polytope X
                  and aff: ⋀X. X ∈ 𝒢 ⟹ aff_dim X ≤ d
                  and face: ⋀X Y. ⟦X ∈ 𝒢; Y ∈ 𝒢⟧ ⟹ X ∩ Y face_of X
                  and I: ⋀X x y a b. ⟦X ∈ 𝒢; x ∈ X; y ∈ X; (a,b) ∈ I⟧ ⟹
                                a · x ≤ b ∧ a · y ≤ b ∨ a · x ≥ b ∧ a · y ≥ b
    by (auto simp: that)
```

obtain a b **where** $ab = (a,b)$
by *fastforce*
let $?\mathcal{G} = (\lambda X.\ X \cap \{x.\ a \cdot x \leq b\})\ `\ \mathcal{G} \cup (\lambda X.\ X \cap \{x.\ a \cdot x \geq b\})\ `\ \mathcal{G}$
have *eqInt*: $(S \cap \mathit{Collect}\ P) \cap (T \cap \mathit{Collect}\ Q) = (S \cap T) \cap (\mathit{Collect}\ P \cap \mathit{Collect}\ Q)$ **for** $S\ T :: 'a\ \mathit{set}$ **and** $P\ Q$
by *blast*
show *?case*
proof (*intro conjI exI*)
show $\bigcup ?\mathcal{G} = \bigcup \mathcal{F}$
by (*force simp*: *eq* [*symmetric*])
show *finite* $?\mathcal{G}$
using ‹*finite* $\mathcal{G}$› **by** *force*
show $\forall X \in ?\mathcal{G}.\ \mathit{polytope}\ X$
by (*force simp*: *poly polytope_Int_polyhedron polyhedron_halfspace_le polyhedron_halfspace_ge*)
show $\forall X \in ?\mathcal{G}.\ \mathit{aff\_dim}\ X \leq d$
by (*auto*; *metis order_trans aff aff_dim_subset inf_le1*)
show $\forall X \in ?\mathcal{G}.\ \forall x \in X.\ \forall y \in X.\ \forall a\ b.$
$(a,b) \in \mathit{insert}\ ab\ I \longrightarrow a \cdot x \leq b \wedge a \cdot y \leq b \vee$
$a \cdot x \geq b \wedge a \cdot y \geq b$
using ‹$ab = (a,\ b)$› I **by** *fastforce*
show $\forall X \in ?\mathcal{G}.\ \forall Y \in ?\mathcal{G}.\ X \cap Y\ \mathit{face\_of}\ X$
by (*auto simp*: *eqInt halfspace_Int_eq face_of_Int_Int face face_of_halfspace_le face_of_halfspace_ge*)
show $\forall C \in ?\mathcal{G}.\ \exists D.\ D \in \mathcal{F} \wedge C \subseteq D$
using *sub1* **by** *force*
show $\forall C{\in}\mathcal{F}.\ \forall x{\in}C.\ \exists D.\ D \in ?\mathcal{G} \wedge x \in D \wedge D \subseteq C$
proof (*intro ballI*)
fix $C\ z$
assume $C \in \mathcal{F}\ z \in C$
with *sub2* **obtain** D **where** D: $D \in \mathcal{G}\ z \in D\ D \subseteq C$ **by** *blast*
have $D \in \mathcal{G} \wedge z \in D \cap \{x.\ a \cdot x \leq b\} \wedge D \cap \{x.\ a \cdot x \leq b\} \subseteq C \vee$
$D \in \mathcal{G} \wedge z \in D \cap \{x.\ a \cdot x \geq b\} \wedge D \cap \{x.\ a \cdot x \geq b\} \subseteq C$
using *linorder_class.linear* [*of* $a \cdot z\ b$] D **by** *blast*
then show $\exists D.\ D \in ?\mathcal{G} \wedge z \in D \wedge D \subseteq C$
by *blast*
qed
qed
qed

proposition *cell_complex_subdivision_exists*:
fixes $\mathcal{F} :: 'a{::}\mathit{euclidean\_space}\ \mathit{set}\ \mathit{set}$
assumes $0 < e$ *finite* $\mathcal{F}$
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow \mathit{polytope}\ X$
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow \mathit{aff\_dim}\ X \leq d$
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow X \cap Y\ \mathit{face\_of}\ X$
obtains $\mathcal{F}'$ **where** *finite* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$ $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow \mathit{diameter}\ X < e$
$\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow \mathit{polytope}\ X$ $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow \mathit{aff\_dim}\ X \leq d$

$\bigwedge X\ Y.\ [\![X \in \mathcal{F}';\ Y \in \mathcal{F}']\!] \Longrightarrow X \cap Y\ face\_of\ X$
$\bigwedge C.\ C \in \mathcal{F}' \Longrightarrow \exists D.\ D \in \mathcal{F} \wedge C \subseteq D$
$\bigwedge C\ x.\ C \in \mathcal{F} \wedge x \in C \Longrightarrow \exists D.\ D \in \mathcal{F}' \wedge x \in D \wedge D \subseteq C$

proof −
have *bounded*($\bigcup \mathcal{F}$)
by (*simp add*: ‹*finite* $\mathcal{F}$› *poly bounded_Union polytope_imp_bounded*)
then obtain B **where** $B > 0$ **and** B: $\bigwedge x.\ x \in \bigcup \mathcal{F} \Longrightarrow norm\ x < B$
by (*meson bounded_pos_less*)
define C **where** $C \equiv \{z \in \mathbb{Z}.\ |z * e\ /\ 2\ /\ real\ DIM('a)| \leq B\}$
define I **where** $I \equiv \bigcup i \in Basis.\ \bigcup j \in C.\ \{\ (i::'a,\ j * e\ /\ 2\ /\ DIM('a))\ \}$
have $C \subseteq \{x \in \mathbb{Z}.\ -B\ /\ (e\ /\ 2\ /\ real\ DIM('a)) \leq x \wedge x \leq B\ /\ (e\ /\ 2\ /\ real\ DIM('a))\}$
using ‹$0 < e$› **by** (*auto simp*: *field_split_simps C_def*)
then have *finite* C
using *finite_int_segment finite_subset* **by** *blast*
then have *finite* I
by (*simp add*: *I_def*)
obtain $\mathcal{F}'$ **where** *eq*: $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$ **and** *finite* $\mathcal{F}'$
and *poly*: $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow polytope\ X$
and *aff*: $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow aff\_dim\ X \leq d$
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F}';\ Y \in \mathcal{F}']\!] \Longrightarrow X \cap Y\ face\_of\ X$
and *I*: $\bigwedge X\ x\ y\ a\ b.\ [\![X \in \mathcal{F}';\ x \in X;\ y \in X;\ (a,b) \in I]\!] \Longrightarrow$
$a \cdot x \leq b \wedge a \cdot y \leq b \vee a \cdot x \geq b \wedge a \cdot y \geq b$
and *sub1*: $\bigwedge C.\ C \in \mathcal{F}' \Longrightarrow \exists D.\ D \in \mathcal{F} \wedge C \subseteq D$
and *sub2*: $\bigwedge C\ x.\ C \in \mathcal{F} \wedge x \in C \Longrightarrow \exists D.\ D \in \mathcal{F}' \wedge x \in D \wedge D \subseteq C$
apply (*rule exE* [*OF cell_subdivision_lemma*])
using *assms* ‹*finite* I› **by** *auto*
show *?thesis*
proof (*rule_tac* $\mathcal{F}'$=$\mathcal{F}'$ **in** *that*)
show *diameter* $X < e$ **if** $X \in \mathcal{F}'$ **for** X
proof −
have *diameter* $X \leq e/2$
proof (*rule diameter_le*)
show *norm* $(x - y) \leq e\ /\ 2$ **if** $x \in X$ $y \in X$ **for** $x\ y$
proof −
have *norm* $x < B$ *norm* $y < B$
using B ‹$X \in \mathcal{F}'$› *eq that* **by** *blast+*
have *norm* $(x - y) \leq (\sum b \in Basis.\ |(x-y) \cdot b|)$
by (*rule norm_le_l1*)
also have $\ldots \leq$ *of_nat* $(DIM('a)) * (e\ /\ 2\ /\ DIM('a))$
proof (*rule sum_bounded_above*)
fix $i::'a$
assume $i \in Basis$
then have I': $\bigwedge z\ b.\ [\![z \in C;\ b = z * e\ /\ (2 * real\ DIM('a))]\!] \Longrightarrow i \cdot x \leq b \wedge i \cdot y \leq b \vee i \cdot x \geq b \wedge i \cdot y \geq b$
using I[*of* $X\ x\ y$] ‹$X \in \mathcal{F}'$› *that* **unfolding** *I_def* **by** *auto*
show $|(x - y) \cdot i| \leq e\ /\ 2\ /\ real\ DIM('a)$
proof (*rule ccontr*)
assume $\neg\ |(x - y) \cdot i| \leq e\ /\ 2\ /\ real\ DIM('a)$

```
            then have xyi: |i · x − i · y| > e / 2 / real DIM('a)
              by (simp add: inner_commute inner_diff_right)
            obtain n where n ∈ ℤ and n: i · x < n * (e / 2 / real DIM('a)) ∧
n * (e / 2 / real DIM('a)) < i · y ∨ i · y < n * (e / 2 / real DIM('a)) ∧ n * (e
/ 2 / real DIM('a)) < i · x
              using subdivide_interval [OF xyi] DIM_positive ‹0 < e›
              by (auto simp: zero_less_divide_iff)
            have |i · x| < B
          by (metis ‹i ∈ Basis› ‹norm x < B› inner_commute norm_bound_Basis_lt)
            have |i · y| < B
          by (metis ‹i ∈ Basis› ‹norm y < B› inner_commute norm_bound_Basis_lt)
            have *: |n * e| ≤ B * (2 * real DIM('a))
                  if |ix| < B |iy| < B
                    and ix: ix * (2 * real DIM('a)) < n * e
                    and iy: n * e < iy * (2 * real DIM('a)) for ix iy
            proof (rule abs_leI)
              have iy * (2 * real DIM('a)) ≤ B * (2 * real DIM('a))
                by (rule mult_right_mono) (use ‹|iy| < B› in linarith)+
              then show n * e ≤ B * (2 * real DIM('a))
                using iy by linarith
            next
              have − ix * (2 * real DIM('a)) ≤ B * (2 * real DIM('a))
                by (rule mult_right_mono) (use ‹|ix| < B› in linarith)+
              then show − (n * e) ≤ B * (2 * real DIM('a))
                using ix by linarith
            qed
            have n ∈ C
             using ‹n ∈ ℤ› n  by (auto simp: C_def divide_simps intro: * ‹|i · x|
< B› ‹|i · y| < B›)
            show False
              using  I' [OF ‹n ∈ C› refl] n  by auto
          qed
        qed
        also have ... = e / 2
          by simp
        finally show ?thesis .
      qed
    qed (use ‹0 < e› in force)
    also have ... < e
      by (simp add: ‹0 < e›)
    finally show ?thesis .
  qed
 qed (auto simp: eq poly aff face sub1 sub2 ‹finite ℱ'›)
qed
```

9.24.17 Simplexes

The notion of n-simplex for integer $-1 \leq n$

definition *simplex* :: *int* ⇒ *'a::euclidean_space set* ⇒ *bool* (**infix** ‹*simplex*› *50*)

where *n simplex S* ≡ ∃ *C*. ¬ *affine_dependent C* ∧ *int*(*card C*) = *n* + *1* ∧ *S* = *convex hull C*

lemma *simplex*:
n simplex S ⟷ (∃ *C*. *finite C* ∧
¬ *affine_dependent C* ∧
int(*card C*) = *n* + *1* ∧
S = *convex hull C*)
by (*auto simp add*: *simplex_def intro*: *aff_independent_finite*)

lemma *simplex_convex_hull*:
¬ *affine_dependent C* ∧ *int*(*card C*) = *n* + *1* ⟹ *n simplex* (*convex hull C*)
by (*auto simp add*: *simplex_def*)

lemma *convex_simplex*: *n simplex S* ⟹ *convex S*
by (*metis convex_convex_hull simplex_def*)

lemma *compact_simplex*: *n simplex S* ⟹ *compact S*
unfolding *simplex*
using *finite_imp_compact_convex_hull* **by** *blast*

lemma *closed_simplex*: *n simplex S* ⟹ *closed S*
by (*simp add*: *compact_imp_closed compact_simplex*)

lemma *simplex_imp_polytope*:
n simplex S ⟹ *polytope S*
unfolding *simplex_def polytope_def*
using *aff_independent_finite* **by** *blast*

lemma *simplex_imp_polyhedron*:
n simplex S ⟹ *polyhedron S*
by (*simp add*: *polytope_imp_polyhedron simplex_imp_polytope*)

lemma *simplex_dim_ge*: *n simplex S* ⟹ −*1* ≤ *n*
by (*metis* (*no_types*, *opaque_lifting*) *aff_dim_geq affine_independent_iff_card diff_add_cancel diff_diff_eq2 simplex_def*)

lemma *simplex_empty* [*simp*]: *n simplex* {} ⟷ *n* = −*1*
proof
assume *n simplex* {}
then show *n* = −*1*
unfolding *simplex* **by** (*metis card.empty convex_hull_eq_empty diff_0 diff_eq_eq of_nat_0*)
next
assume *n* = −*1* **then show** *n simplex* {}
by (*fastforce simp*: *simplex*)
qed

lemma *simplex_minus_1* [*simp*]: −*1 simplex S* ⟷ *S* = {}

by (*metis simplex cancel_comm_monoid_add_class.diff_cancel card_0_eq diff_minus_eq_add of_nat_eq_0_iff simplex_empty*)

lemma *aff_dim_simplex*:
n simplex S $\Longrightarrow$ *aff_dim S* = *n*
by (*metis simplex add.commute add_diff_cancel_left' aff_dim_convex_hull affine_independent_iff_ca*

lemma *zero_simplex_sing*: *0 simplex* {*a*}
using *affine_independent_1 simplex_convex_hull* **by** *fastforce*

lemma *simplex_sing* [*simp*]: *n simplex* {*a*} $\longleftrightarrow$ *n* = *0*
using *aff_dim_simplex aff_dim_sing zero_simplex_sing* **by** *blast*

lemma *simplex_zero*: *0 simplex S* $\longleftrightarrow$ ($\exists$ *a*. *S* = {*a*})
by (*metis aff_dim_eq_0 aff_dim_simplex simplex_sing*)

lemma *one_simplex_segment*: *a* $\neq$ *b* $\Longrightarrow$ *1 simplex closed_segment a b*
unfolding *simplex_def*
by (*rule_tac x*={*a*,*b*} **in** *exI*) (*auto simp*: *segment_convex_hull*)

lemma *simplex_segment_cases*:
(*if a* = *b then 0 else 1*) *simplex closed_segment a b*
by (*auto simp*: *one_simplex_segment*)

lemma *simplex_segment*:
$\exists$ *n*. *n simplex closed_segment a b*
using *simplex_segment_cases* **by** *metis*

lemma *polytope_lowdim_imp_simplex*:
assumes *polytope P aff_dim P* $\leq$ *1*
obtains *n* **where** *n simplex P*
proof (*cases P* = {})
case *True*
then show *?thesis*
by (*simp add*: *that*)
next
case *False*
then show *?thesis*
by (*metis assms compact_convex_collinear_segment collinear_aff_dim polytope_imp_compact polytope_imp_convex simplex_segment_cases that*)
qed

lemma *simplex_insert_dimplus1*:
fixes *n*::*int*
assumes *n simplex S* **and** *a*: *a* $\notin$ *affine hull S*
shows (*n*+*1*) *simplex* (*convex hull* (*insert a S*))
proof −
obtain *C* **where** *C*: *finite C* ¬ *affine_dependent C int*(*card C*) = *n*+*1* **and** *S*:

S = *convex hull* C
using *assms* **unfolding** *simplex* **by** *force*
show *?thesis*
unfolding *simplex*
proof (*intro exI conjI*)
have *aff_dim* $S = n$
using *aff_dim_simplex assms*(*1*) **by** *blast*
moreover have $a \notin$ *affine hull* C
using S a *affine_hull_convex_hull* **by** *blast*
moreover have $a \notin C$
using S a *hull_inc* **by** *fastforce*
ultimately show ¬ *affine_dependent* (*insert a C*)
by (*simp add*: C S *aff_dim_convex_hull aff_dim_insert affine_independent_iff_card*)
next
have $a \notin C$
using S a *hull_inc* **by** *fastforce*
then show *int* (*card* (*insert a C*)) = $n + 1 + 1$
by (*simp add*: C)
next
show *convex hull insert a S* = *convex hull* (*insert a C*)
by (*simp add*: S *convex_hull_insert_segments*)
qed (*use* C **in** *auto*)
qed

9.24.18 Simplicial complexes and triangulations

definition *simplicial_complex* **where**
simplicial_complex $\mathcal{C} \equiv$
finite $\mathcal{C} \wedge$
$(\forall S \in \mathcal{C}.\ \exists n.\ n$ *simplex* $S) \wedge$
$(\forall F\ S.\ S \in \mathcal{C} \wedge F$ *face_of* $S \longrightarrow F \in \mathcal{C}) \wedge$
$(\forall S\ S'.\ S \in \mathcal{C} \wedge S' \in \mathcal{C} \longrightarrow (S \cap S')$ *face_of* $S)$

definition *triangulation* **where**
triangulation $\mathcal{T} \equiv$
finite $\mathcal{T} \wedge$
$(\forall T \in \mathcal{T}.\ \exists n.\ n$ *simplex* $T) \wedge$
$(\forall T\ T'.\ T \in \mathcal{T} \wedge T' \in \mathcal{T} \longrightarrow (T \cap T')$ *face_of* $T)$

9.24.19 Refining a cell complex to a simplicial complex

proposition *convex_hull_insert_Int_eq*:
fixes z :: $'a$ :: *euclidean_space*
assumes z: $z \in$ *rel_interior* S
and T: $T \subseteq$ *rel_frontier* S
and U: $U \subseteq$ *rel_frontier* S
and *convex* S *convex* T *convex* U
shows *convex hull* (*insert z T*) ∩ *convex hull* (*insert z U*) = *convex hull* (*insert z* ($T \cap U$))
(**is** *?lhs* = *?rhs*)

```
proof
  show ?lhs ⊆ ?rhs
  proof (cases T={} ∨ U={})
    case True then show ?thesis by auto
  next
    case False
    then have T ≠ {} U ≠ {} by auto
    have TU: convex (T ∩ U)
      by (simp add: ‹convex T› ‹convex U› convex_Int)
    have (⋃x∈T. closed_segment z x) ∩ (⋃x∈U. closed_segment z x)
        ⊆ (if T ∩ U = {} then {z} else ⋃((closed_segment z) ` (T ∩ U))) (is _
⊆ ?IF)
    proof clarify
      fix x t u
      assume xt: x ∈ closed_segment z t
        and xu: x ∈ closed_segment z u
        and t ∈ T u ∈ U
      then have ne: t ≠ z u ≠ z
        using T U z unfolding rel_frontier_def by blast+
      show x ∈ ?IF
      proof (cases x = z)
        case True then show ?thesis by auto
      next
        case False
        have t: t ∈ closure S
          using T ‹t ∈ T› rel_frontier_def by auto
        have u: u ∈ closure S
          using U ‹u ∈ U› rel_frontier_def by auto
        show ?thesis
        proof (cases t = u)
          case True
          then show ?thesis
            using ‹t ∈ T› ‹u ∈ U› xt by auto
        next
          case False
          have tnot: t ∉ closed_segment u z
          proof −
            have t ∈ closure S − rel_interior S
              using T ‹t ∈ T› rel_frontier_def by blast
            then have t ∉ open_segment z u
              by (meson DiffD2 rel_interior_closure_convex_segment [OF ‹convex
S› z u] subsetD)
            then show ?thesis
            by (simp add: ‹t ≠ u› ‹t ≠ z› open_segment_commute open_segment_def)
          qed
          moreover have u ∉ closed_segment z t
            using rel_interior_closure_convex_segment [OF ‹convex S› z t] ‹u ∈
U› ‹u ≠ z›
              U [unfolded rel_frontier_def] tnot
```

```
          by (auto simp: closed_segment_eq_open)
        ultimately
        have ¬(between (t,u) z | between (u,z) t | between (z,t) u) if x ≠ z
          using that xt xu
           by (meson between_antisym between_mem_segment between_trans_2
ends_in_segment(2))
        then have ¬ collinear {t, z, u} if x ≠ z
          by (auto simp: that collinear_between_cases between_commute)
        moreover have collinear {t, z, x}
        by (metis closed_segment_commute collinear_2 collinear_closed_segment
collinear_triples ends_in_segment(1) insert_absorb insert_absorb2 xt)
        moreover have collinear {z, x, u}
        by (metis closed_segment_commute collinear_2 collinear_closed_segment
collinear_triples ends_in_segment(1) insert_absorb insert_absorb2 xu)
        ultimately have False
          using collinear_3_trans [of t z x u] ‹x ≠ z› by blast
        then show ?thesis by metis
      qed
    qed
  qed
  then show ?thesis
    using False ‹convex T› ‹convex U› TU
    by (simp add: convex_hull_insert_segments hull_same split: if_split_asm)
 qed
 show ?rhs ⊆ ?lhs
  by (metis inf_greatest hull_mono inf.cobounded1 inf.cobounded2 insert_mono)
qed

lemma simplicial_subdivision_aux:
 assumes finite ℳ
    and ⋀C. C ∈ ℳ ⟹ polytope C
    and ⋀C. C ∈ ℳ ⟹ aff_dim C ≤ of_nat n
    and ⋀C F. ⟦C ∈ ℳ; F face_of C⟧ ⟹ F ∈ ℳ
    and ⋀C1 C2. ⟦C1 ∈ ℳ; C2 ∈ ℳ⟧ ⟹ C1 ∩ C2 face_of C1
  shows ∃𝒯. simplicial_complex 𝒯 ∧
            (∀K ∈ 𝒯. aff_dim K ≤ of_nat n) ∧
            ⋃𝒯 = ⋃ℳ ∧
            (∀C ∈ ℳ. ∃F. finite F ∧ F ⊆ 𝒯 ∧ C = ⋃F) ∧
            (∀K ∈ 𝒯. ∃C. C ∈ ℳ ∧ K ⊆ C)
 using assms
proof (induction n arbitrary: ℳ rule: less_induct)
 case (less n)
 then have polyℳ: ⋀C. C ∈ ℳ ⟹ polytope C
   and affℳ:   ⋀C. C ∈ ℳ ⟹ aff_dim C ≤ of_nat n
   and faceℳ:  ⋀C F. ⟦C ∈ ℳ; F face_of C⟧ ⟹ F ∈ ℳ
   and intfaceℳ: ⋀C1 C2. ⟦C1 ∈ ℳ; C2 ∈ ℳ⟧ ⟹ C1 ∩ C2 face_of C1
   by metis+
 show ?case
 proof (cases n ≤ 1)
```

case *True*
have $\bigwedge s.\ [\![n \leq 1;\ s \in \mathcal{M}]\!] \Longrightarrow \exists\, m.\ m\ simplex\ s$
using *poly*$\mathcal{M}$ *aff*$\mathcal{M}$ **by** (*force intro*: *polytope_lowdim_imp_simplex*)
then show *?thesis*
unfolding *simplicial_complex_def* **using** *True*
by (*rule_tac x*=$\mathcal{M}$ **in** *exI*) (*auto simp*: *less.prems*)
next
case *False*
define $\mathcal{S}$ **where** $\mathcal{S} \equiv \{C \in \mathcal{M}.\ aff\_dim\ C < n\}$
have *finite* $\mathcal{S}$ $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow$ *polytope* C $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow$ *aff_dim* $C \leq$ *int* $(n - 1)$
$\bigwedge C1\ C2.\ [\![C1 \in \mathcal{S};\ C2 \in \mathcal{S}]\!] \Longrightarrow C1 \cap C2\ face\_of\ C1$
using *less.prems* **by** (*auto simp*: $\mathcal{S}$*_def*)
moreover have §: $\bigwedge C\ F.\ [\![C \in \mathcal{S};\ F\ face\_of\ C]\!] \Longrightarrow F \in \mathcal{S}$
using *less.prems* **unfolding** $\mathcal{S}$*_def*
by (*metis* (*no_types*, *lifting*) *mem_Collect_eq aff_dim_subset face_of_imp_subset less_le not_le*)
ultimately obtain $\mathcal{U}$ **where** *simplicial_complex* $\mathcal{U}$
and *aff_dim*$\mathcal{U}$: $\bigwedge K.\ K \in \mathcal{U} \Longrightarrow aff\_dim\ K \leq int\ (n - 1)$
and $\bigcup \mathcal{U} = \bigcup \mathcal{S}$
and *fin*$\mathcal{U}$: $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow \exists\, F.\ finite\ F \wedge F \subseteq \mathcal{U} \wedge C = \bigcup F$
and *C*$\mathcal{U}$: $\bigwedge K.\ K \in \mathcal{U} \Longrightarrow \exists\, C.\ C \in \mathcal{S} \wedge K \subseteq C$
using *less.IH* [*of n*−*1* $\mathcal{S}$] *False* **by** *auto*
then have *finite* $\mathcal{U}$
and *simpl*$\mathcal{U}$: $\bigwedge S.\ S \in \mathcal{U} \Longrightarrow \exists\, n.\ n\ simplex\ S$
and *face*$\mathcal{U}$: $\bigwedge F\ S.\ [\![S \in \mathcal{U};\ F\ face\_of\ S]\!] \Longrightarrow F \in \mathcal{U}$
and *faceI*$\mathcal{U}$: $\bigwedge S\ S'.\ [\![S \in \mathcal{U};\ S' \in \mathcal{U}]\!] \Longrightarrow (S \cap S')\ face\_of\ S$
by (*auto simp*: *simplicial_complex_def*)
define $\mathcal{N}$ **where** $\mathcal{N} \equiv \{C \in \mathcal{M}.\ aff\_dim\ C = n\}$
have *finite* $\mathcal{N}$
by (*simp add*: $\mathcal{N}$*_def less.prems*(*1*))
have *poly*$\mathcal{N}$: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow$ *polytope* C
and *convex*$\mathcal{N}$: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow$ *convex* C
and *closed*$\mathcal{N}$: $\bigwedge C.\ C \in \mathcal{N} \Longrightarrow$ *closed* C
by (*auto simp*: $\mathcal{N}$*_def poly*$\mathcal{M}$ *polytope_imp_convex polytope_imp_closed*)
have *in_rel_interior*: (*SOME z*. $z \in$ *rel_interior* C) $\in$ *rel_interior* C **if** $C \in \mathcal{N}$ **for** C
using *that poly*$\mathcal{M}$ *polytope_imp_convex rel_interior_aff_dim some_in_eq* **by** (*fastforce simp*: $\mathcal{N}$*_def*)
have $*$: $\exists\, T.\ \neg$ *affine_dependent* $T \wedge$ *card* $T \leq n \wedge$ *aff_dim* $K < n \wedge K =$ *convex hull* T
if $K \in \mathcal{U}$ **for** K
proof −
obtain r **where** r: r *simplex* K
using ‹$K \in \mathcal{U}$› *simpl*$\mathcal{U}$ **by** *blast*
have r = *aff_dim* K
using ‹r *simplex* K› *aff_dim_simplex* **by** *blast*
with r
show *?thesis*

unfolding *simplex_def*
using *False* ‹$\bigwedge K.\ K \in \mathcal{U} \Longrightarrow$ *aff_dim* $K \leq$ *int* $(n - 1)$› *that* **by** *fastforce*
qed
have *ahK_C_disjoint*: *affine hull* $K \cap$ *rel_interior* $C = \{\}$
if $C \in \mathcal{N}$ $K \in \mathcal{U}$ $K \subseteq$ *rel_frontier* C **for** C K
proof −
have *convex* C *closed* C
by (*auto simp*: *convex*$\mathcal{N}$ *closed*$\mathcal{N}$ ‹$C \in \mathcal{N}$›)
obtain F **where** F: F *face_of* C **and** $F \neq C$ $K \subseteq F$
proof −
obtain L **where** $L \in \mathcal{S}$ $K \subseteq L$
using ‹$K \in \mathcal{U}$› $C\mathcal{U}$ **by** *blast*
have $K \leq$ *rel_frontier* C
by (*simp add*: ‹$K \subseteq$ *rel_frontier* C›)
also have $\ldots \leq C$
by (*simp add*: ‹*closed* C› *rel_frontier_def subset_iff*)
finally have $K \subseteq C$ **.**
have $L \cap C$ *face_of* C
using $\mathcal{N}$*_def* $\mathcal{S}$*_def* ‹$C \in \mathcal{N}$› ‹$L \in \mathcal{S}$› *intface*$\mathcal{M}$ **by** (*simp add*: *inf_commute*)
moreover have $L \cap C \neq C$
using ‹$C \in \mathcal{N}$› ‹$L \in \mathcal{S}$›
by (*metis* (*mono_tags*, *lifting*) $\mathcal{N}$*_def* $\mathcal{S}$*_def intface*$\mathcal{M}$ *mem_Collect_eq not_le order_refl* §)
moreover have $K \subseteq L \cap C$
using ‹$C \in \mathcal{N}$› ‹$L \in \mathcal{S}$› ‹$K \subseteq C$› ‹$K \subseteq L$› **by** (*auto simp*: $\mathcal{N}$*_def* $\mathcal{S}$*_def*)
ultimately show *?thesis* **using** *that* **by** *metis*
qed
have *affine hull* $F \cap$ *rel_interior* $C = \{\}$
by (*rule affine_hull_face_of_disjoint_rel_interior* [*OF* ‹*convex* C› F ‹$F \neq C$›])
with *hull_mono* [*OF* ‹$K \subseteq F$›]
show *affine hull* $K \cap$ *rel_interior* $C = \{\}$
by *fastforce*
qed
let *?*$\mathcal{T}$ $= (\bigcup C \in \mathcal{N}.\ \bigcup K \in \mathcal{U} \cap$ *Pow* (*rel_frontier* C).
$\{$*convex hull* (*insert* (*SOME* $z.\ z \in$ *rel_interior* C) K)$\})$
have $\exists \mathcal{T}.$ *simplicial_complex* $\mathcal{T} \wedge$
$(\forall K \in \mathcal{T}.$ *aff_dim* $K \leq$ *of_nat* $n) \wedge$
$(\forall C \in \mathcal{M}.\ \exists F.\ F \subseteq \mathcal{T} \wedge C = \bigcup F) \wedge$
$(\forall K \in \mathcal{T}.\ \exists C.\ C \in \mathcal{M} \wedge K \subseteq C)$
proof (*rule exI*, *intro conjI ballI*)
show *simplicial_complex* $(\mathcal{U} \cup$ *?*$\mathcal{T})$
unfolding *simplicial_complex_def*
proof (*intro conjI impI ballI allI*)
show *finite* $(\mathcal{U} \cup$ *?*$\mathcal{T})$
using ‹*finite* $\mathcal{U}$› ‹*finite* $\mathcal{N}$› **by** *simp*
show $\exists n.$ n *simplex* S **if** $S \in \mathcal{U} \cup$ *?*$\mathcal{T}$ **for** S
using *that ahK_C_disjoint in_rel_interior simpl*$\mathcal{U}$ *simplex_insert_dimplus1*

by *fastforce*
show $F \in \mathcal{U} \cup ?\mathcal{T}$ **if** S: $S \in \mathcal{U} \cup ?\mathcal{T} \wedge F$ *face_of* S **for** F S
proof −
have $F \in \mathcal{U}$ **if** $S \in \mathcal{U}$
using S *face*$\mathcal{U}$ *that* **by** *blast*
moreover have $F \in \mathcal{U} \cup ?\mathcal{T}$
if F *face_of* S $C \in \mathcal{N}$ $K \in \mathcal{U}$ **and** $K \subseteq$ *rel_frontier* C
and S: $S =$ *convex hull insert* (*SOME* z. $z \in$ *rel_interior* C) K **for** C K
proof −
let $?z =$ *SOME* z. $z \in$ *rel_interior* C
have $?z \in$ *rel_interior* C
by (*simp add*: *in_rel_interior* ‹$C \in \mathcal{N}$›)
moreover
obtain I **where** ¬ *affine_dependent* I *card* $I \leq n$ *aff_dim* $K <$ *int* n K = *convex hull* I
using $*$ [*OF* ‹$K \in \mathcal{U}$›] **by** *auto*
ultimately have $?z \notin$ *affine hull* I
using *ahK_C_disjoint affine_hull_convex_hull that* **by** *blast*
have *compact* I *finite* I
by (*auto simp*: ‹¬ *affine_dependent* I› *aff_independent_finite finite_imp_compact*)
moreover have F *face_of convex hull insert* $?z$ I
by (*metis* S ‹F *face_of* S› ‹$K =$ *convex hull* I› *convex_hull_eq_empty convex_hull_insert_segments hull_hull*)
ultimately obtain J **where** J: $J \subseteq$ *insert* $?z$ I $F =$ *convex hull* J
using *face_of_convex_hull_subset* [*of insert* $?z$ I F] **by** *auto*
show *?thesis*
proof (*cases* $?z \in J$)
case *True*
have $F \in (\bigcup K{\in}\mathcal{U} \cap$ *Pow* (*rel_frontier* C). {*convex hull insert* $?z$ K})
proof
have *convex hull* $(J - \{?z\})$ *face_of* K
by (*metis True* ‹$J \subseteq$ *insert* $?z$ I› ‹$K =$ *convex hull* I› ‹¬ *affine_dependent* I› *face_of_convex_hull_affine_independent subset_insert_iff*)
then have *convex hull* $(J - \{?z\}) \in \mathcal{U}$
by (*rule face*$\mathcal{U}$ [*OF* ‹$K \in \mathcal{U}$›])
moreover
have $\bigwedge x$. $x \in$ *convex hull* $(J - \{?z\}) \Longrightarrow x \in$ *rel_frontier* C
by (*metis True* ‹$J \subseteq$ *insert* $?z$ I› ‹$K =$ *convex hull* I› *subsetD hull_mono subset_insert_iff that*(4))
ultimately show *convex hull* $(J - \{?z\}) \in \mathcal{U} \cap$ *Pow* (*rel_frontier* C) **by** *auto*
let $?F =$ *convex hull insert* $?z$ (*convex hull* $(J - \{?z\})$)
have $F \subseteq ?F$
by (*simp add*: ‹$F =$ *convex hull* J› *hull_mono hull_subset subset_insert_iff*)
moreover have $?F \subseteq F$
by (*metis True* ‹$F =$ *convex hull* J› *hull_insert insert_Diff*

```
set_eq_subset)
              ultimately
              show F ∈ {?F} by auto
            qed
            with ‹C∈𝒩› show ?thesis by auto
          next
            case False
            then have F ∈ 𝒰
          using face_of_convex_hull_affine_independent [OF ‹¬ affine_dependent
I›]
              by (metis J ‹K = convex hull I› face𝒰 subset_insert ‹K ∈ 𝒰›)
            then show F ∈ 𝒰 ∪ ?𝒯
              by blast
          qed
        qed
        ultimately show ?thesis
          using that by auto
      qed
      have §: X ∩ Y face_of X ∧ X ∩ Y face_of Y
        if XY: X ∈ 𝒰 Y ∈ ?𝒯 for X Y
      proof −
        obtain C K
          where C ∈ 𝒩 K ∈ 𝒰 K ⊆ rel_frontier C
            and Y: Y = convex hull insert (SOME z. z ∈ rel_interior C) K
          using XY by blast
        have convex C
          by (simp add: ‹C ∈ 𝒩› convex𝒩)
        have K ⊆ C
          by (metis DiffE ‹C ∈ 𝒩› ‹K ⊆ rel_frontier C› closed𝒩 closure_closed
rel_frontier_def subset_iff)
        let ?z = (SOME z. z ∈ rel_interior C)
        have z: ?z ∈ rel_interior C
          using ‹C ∈ 𝒩› in_rel_interior by blast
        obtain D where D ∈ 𝒮 X ⊆ D
          using C𝒰 ‹X ∈ 𝒰› by blast
        have D ∩ rel_interior C = (C ∩ D) ∩ rel_interior C
          using rel_interior_subset by blast
        also have (C ∩ D) ∩ rel_interior C = {}
        proof (rule face_of_disjoint_rel_interior)
          show C ∩ D face_of C
            using 𝒩_def 𝒮_def ‹C ∈ 𝒩› ‹D ∈ 𝒮› intface𝒩 by blast
          show C ∩ D ≠ C
            by (metis (mono_tags, lifting) Int_lower2 𝒩_def 𝒮_def ‹C ∈ 𝒩› ‹D
∈ 𝒮› aff_dim_subset mem_Collect_eq not_le)
        qed
        finally have DC: D ∩ rel_interior C = {} .
        have eq: X ∩ convex hull (insert ?z K) = X ∩ convex hull K
        proof (rule Int_convex_hull_insert_rel_exterior [OF ‹convex C› ‹K ⊆
C› z])
```

```
          show disjnt X (rel_interior C)
            using DC by (meson ‹X ⊆ D› disjnt_def disjnt_subset1)
          qed
          obtain I where I: ¬ affine_dependent I
            and Keq: K = convex hull I and [simp]: convex hull K = K
            using ∗ ‹K ∈ 𝒰› by force
          then have ?z ∉ affine hull I
              using ahK_C_disjoint ‹C ∈ 𝒩› ‹K ∈ 𝒰› ‹K ⊆ rel_frontier C›
affine_hull_convex_hull z by blast
          have X ∩ K face_of K
            by (simp add: XY(1) ‹K ∈ 𝒰› faceI𝒰 inf_commute)
          also have ... face_of convex hull insert ?z K
              by (metis I Keq ‹?z ∉ affine hull I› aff_independent_finite con-
vex_convex_hull face_of_convex_hull_insert face_of_refl hull_insert)
          finally have X ∩ K face_of convex hull insert ?z K .
          then show ?thesis
            by (simp add: XY(1) Y ‹K ∈ 𝒰› eq faceI𝒰)
        qed

        show S ∩ S' face_of S
          if S ∈ 𝒰 ∪ ?𝒯 ∧ S' ∈ 𝒰 ∪ ?𝒯 for S S'
          using that
        proof (elim conjE UnE)
          fix X Y
          assume X ∈ 𝒰 and Y ∈ 𝒰
          then show X ∩ Y face_of X
            by (simp add: faceI𝒰)
        next
          fix X Y
          assume XY: X ∈ 𝒰 Y ∈ ?𝒯
          then show X ∩ Y face_of X Y ∩ X face_of Y
            using § [OF XY] by (auto simp: Int_commute)
        next
          fix X Y
          assume XY: X ∈ ?𝒯 Y ∈ ?𝒯
          show X ∩ Y face_of X
          proof −
            obtain C K D L
              where C ∈ 𝒩 K ∈ 𝒰 K ⊆ rel_frontier C
                and X: X = convex hull insert (SOME z. z ∈ rel_interior C) K
                and D ∈ 𝒩 L ∈ 𝒰 L ⊆ rel_frontier D
                and Y: Y = convex hull insert (SOME z. z ∈ rel_interior D) L
              using XY by blast
            let ?z = (SOME z. z ∈ rel_interior C)
            have z: ?z ∈ rel_interior C
              using ‹C ∈ 𝒩› in_rel_interior by blast
            have convex C
              by (simp add: ‹C ∈ 𝒩› convex𝒩)
            have convex K
```

using ∗ ‹$K \in \mathcal{U}$› **by** *blast*
have *convex L*
by (*meson* ‹$L \in \mathcal{U}$› *convex_simplex simplU*)
show *?thesis*
proof (*cases D=C*)
case *True*
then have $L \subseteq$ *rel_frontier C*
using ‹$L \subseteq$ *rel_frontier D*› **by** *auto*
have *convex hull insert* (*SOME z. z ∈ rel_interior C*) (*K ∩ L*) *face_of*
convex hull insert (*SOME z. z ∈ rel_interior C*) *K*
by (*metis IntI* ‹$C \in \mathcal{N}$› ‹$K \in \mathcal{U}$› ‹$K \subseteq$ *rel_frontier C*› ‹$L \in \mathcal{U}$› *ahK_C_disjoint empty_iff faceIU face_of_polytope_insert2 simplU simplex_imp_polytope z*)
then show *?thesis*
using *True X Y* ‹$K \subseteq$ *rel_frontier C*› ‹$L \subseteq$ *rel_frontier C*› ‹*convex C*› ‹*convex K*› ‹*convex L*› *convex_hull_insert_Int_eq z* **by** *force*
next
case *False*
have *convex D*
by (*simp add*: ‹$D \in \mathcal{N}$› *convexN*)
have $K \subseteq C$
by (*metis DiffE* ‹$C \in \mathcal{N}$› ‹$K \subseteq$ *rel_frontier C*› *closedN closure_closed rel_frontier_def subset_eq*)
have $L \subseteq D$
by (*metis DiffE* ‹$D \in \mathcal{N}$› ‹$L \subseteq$ *rel_frontier D*› *closedN closure_closed rel_frontier_def subset_eq*)
let *?w* = (*SOME w. w ∈ rel_interior D*)
have *w*: *?w ∈ rel_interior D*
using ‹$D \in \mathcal{N}$› *in_rel_interior* **by** *blast*
have $C \cap$ *rel_interior D* = $(D \cap C) \cap$ *rel_interior D*
using *rel_interior_subset* **by** *blast*
also have $(D \cap C) \cap$ *rel_interior D* = $\{\}$
proof (*rule face_of_disjoint_rel_interior*)
show $D \cap C$ *face_of D*
using *N_def* ‹$C \in \mathcal{N}$› ‹$D \in \mathcal{N}$› *intfaceM* **by** *blast*
have $D \in \mathcal{M} \wedge$ *aff_dim D* = *int n*
using *N_def* ‹$D \in \mathcal{N}$› **by** *blast*
moreover have $C \in \mathcal{M} \wedge$ *aff_dim C* = *int n*
using *N_def* ‹$C \in \mathcal{N}$› **by** *blast*
ultimately show $D \cap C \neq D$
by (*metis Int_commute False face_of_aff_dim_lt inf.idem inf_le1 intfaceM not_le polyM polytope_imp_convex*)
qed
finally have *CD*: $C \cap$ (*rel_interior D*) = $\{\}$ **.**
have *zKC*: (*convex hull insert ?z K*) $\subseteq C$
by (*metis* ‹$K \subseteq C$› ‹*convex C*› *in_mono insert_subsetI rel_interior_subset subset_hull z*)
have *disjnt* (*convex hull insert* (*SOME z. z ∈ rel_interior C*) *K*) (*rel_interior D*)

```
      using zKC CD by (force simp: disjnt_def)
    then have eq: convex hull (insert ?z K) ∩ convex hull (insert ?w L) =
                  convex hull (insert ?z K) ∩ convex hull L
      by (rule Int_convex_hull_insert_rel_exterior [OF ‹convex D› ‹L ⊆
D› w])
    have ch_id: convex hull K = K convex hull L = L
      using ∗ ‹K ∈ 𝒰› ‹L ∈ 𝒰› hull_same by auto
    have convex C
      by (simp add: ‹C ∈ 𝒩› convex𝒩)
    have convex hull (insert ?z K) ∩ L = L ∩ convex hull (insert ?z K)
      by blast
    also have ... = convex hull K ∩ L
    proof (subst Int_convex_hull_insert_rel_exterior [OF ‹convex C› ‹K
⊆ C› z])
      have (C ∩ D) ∩ rel_interior C = {}
      proof (rule face_of_disjoint_rel_interior)
        show C ∩ D face_of C
          using 𝒩_def ‹C ∈ 𝒩› ‹D ∈ 𝒩› intface𝒩 by blast
        have D ∈ 𝒩 aff_dim D = int n
          using 𝒩_def ‹D ∈ 𝒩› by fastforce+
        moreover have C ∈ 𝒩 aff_dim C = int n
          using 𝒩_def ‹C ∈ 𝒩› by fastforce+
        ultimately have aff_dim D + - 1 ∗ aff_dim C ≤ 0
          by fastforce
        then have ¬ C face_of D
          using False ‹convex D› face_of_aff_dim_lt by fastforce
        show C ∩ D ≠ C
            by (metis inf_commute ‹C ∈ 𝒩› ‹D ∈ 𝒩› ‹¬ C face_of D›
intface𝒩)
      qed
      then have D ∩ rel_interior C = {}
    by (metis inf.absorb_iff2 inf_assoc inf_sup_aci(1) rel_interior_subset)
      then show disjnt L (rel_interior C)
        by (meson ‹L ⊆ D› disjnt_def disjnt_subset1)
    next
      show L ∩ convex hull K = convex hull K ∩ L
        by force
    qed
    finally have chKL: convex hull (insert ?z K) ∩ L = convex hull K ∩
L .
    have convex hull insert ?z K ∩ convex hull L face_of K
      by (simp add: ‹K ∈ 𝒰› ‹L ∈ 𝒰› ch_id chKL face𝐼𝒰)
    also have ... face_of convex hull insert ?z K
    proof −
      obtain I where I: ¬ affine_dependent I K = convex hull I
        using ∗ [OF ‹K ∈ 𝒰›] by auto
      then have ⋀a. a ∉ rel_interior C ∨ a ∉ affine hull I
          using ahK_C_disjoint ‹C ∈ 𝒩› ‹K ∈ 𝒰› ‹K ⊆ rel_frontier C›
affine_hull_convex_hull by blast
```

```
            then show ?thesis
          by (metis I ‹convex K› aff_independent_finite face_of_convex_hull_insert_eq
face_of_refl hull_insert z)
           qed
           finally have 1: convex hull insert ?z K ∩ convex hull L face_of convex
hull insert ?z K .
           have convex hull insert ?z K ∩ convex hull L face_of L
            by (metis ‹K ∈ 𝒰› ‹L ∈ 𝒰› chKL ch_id faceI𝒰 inf_commute)
           also have ... face_of convex hull insert ?w L
           proof −
            obtain I where I: ¬ affine_dependent I L = convex hull I
              using ∗ [OF ‹L ∈ 𝒰›] by auto
            then have ⋀a. a ∉ rel_interior D ∨ a ∉ affine hull I
           using ‹D ∈ 𝒩› ‹L ∈ 𝒰› ‹L ⊆ rel_frontier D› affine_hull_convex_hull
ahK_C_disjoint by blast
            then show ?thesis
          by (metis I ‹convex L› aff_independent_finite face_of_convex_hull_insert
face_of_refl hull_insert w)
           qed
           finally have 2: convex hull insert ?z K ∩ convex hull L face_of convex
hull insert ?w L .
           show ?thesis
            by (simp add: X Y eq 1 2)
         qed
        qed
       qed
      qed
      show ∃ F ⊆ 𝒰 ∪ ?𝒯. C = ⋃ F if C ∈ ℳ for C
      proof (cases C ∈ 𝒮)
       case True
       then show ?thesis
        by (meson UnCI fin𝒰 subsetD subsetI)
      next
       case False
       then have C ∈ 𝒩
        by (simp add: 𝒩_def 𝒮_def aff ℳ less_le that)
       let ?z = SOME z. z ∈ rel_interior C
       have z: ?z ∈ rel_interior C
        using ‹C ∈ 𝒩› in_rel_interior by blast
       let ?F = ⋃ K ∈ 𝒰 ∩ Pow (rel_frontier C). {convex hull (insert ?z K)}
       have ?F ⊆ ?𝒯
        using ‹C ∈ 𝒩› by blast
       moreover have C ⊆ ⋃ ?F
       proof
        fix x
        assume x ∈ C
        have convex C
         using ‹C ∈ 𝒩› convex𝒩 by blast
        have bounded C
```

```
          using ‹C ∈ 𝒩› by (simp add: polyℳ polytope_imp_bounded that)
         have polytope C
          using ‹C ∈ 𝒩› poly𝒩 by auto
         have ¬ (?z = x ∧ C = {?z})
          using ‹C ∈ 𝒩› aff_dim_sing [of ?z] ‹¬ n ≤ 1› by (force simp: 𝒩_def)
        then obtain y where y: y ∈ rel_frontier C and xzy: x ∈ closed_segment
?z y
          and sub: open_segment ?z y ⊆ rel_interior C
          by (blast intro: segment_to_rel_frontier [OF ‹convex C› ‹bounded C› z
‹x ∈ C›])
         then obtain F where y ∈ F F face_of C F ≠ C
        by (auto simp: rel_frontier_of_polyhedron_alt [OF polytope_imp_polyhedron
[OF ‹polytope C›]])
         then obtain 𝒢 where finite 𝒢 𝒢 ⊆ 𝒰 F = ⋃𝒢
          by (metis (mono_tags, lifting) 𝒮_def ‹C ∈ ℳ› ‹convex C› affℳ faceℳ
face_of_aff_dim_lt fin𝒰 le_less_trans mem_Collect_eq not_less)
         then obtain K where y ∈ K K ∈ 𝒢
          using ‹y ∈ F› by blast
         moreover have x: x ∈ convex hull {?z,y}
          using segment_convex_hull xzy by auto
         moreover have convex hull {?z,y} ⊆ convex hull insert ?z K
             by (metis (full_types) ‹y ∈ K› hull_mono empty_subsetI insertCI
insert_subset)
         moreover have K ∈ 𝒰
          using ‹K ∈ 𝒢› ‹𝒢 ⊆ 𝒰› by blast
         moreover have K ⊆ rel_frontier C
        using ‹F = ⋃𝒢› ‹F ≠ C› ‹F face_of C› ‹K ∈ 𝒢› face_of_subset_rel_frontier
by fastforce
         ultimately show x ∈ ⋃ ?F
          by force
        qed
        moreover
        have convex hull insert (SOME z. z ∈ rel_interior C) K ⊆ C
         if K ∈ 𝒰 K ⊆ rel_frontier C for K
        proof (rule hull_minimal)
         show insert (SOME z. z ∈ rel_interior C) K ⊆ C
          using that ‹C ∈ 𝒩› in_rel_interior rel_interior_subset
          by (force simp: closure_eq rel_frontier_def closed𝒩)
         show convex C
          by (simp add: ‹C ∈ 𝒩› convex𝒩)
        qed
        then have ⋃ ?F ⊆ C
         by auto
        ultimately show ?thesis
         by blast
       qed
       have (∃ C. C ∈ ℳ ∧ L ⊆ C) ∧ aff_dim L ≤ int n  if L ∈ 𝒰 ∪ ?𝒯 for L
        using that
       proof
```

```
      assume L ∈ 𝒰
      then show ?thesis
        using C𝒰 𝒮_def * by fastforce
    next
      assume L ∈ ?𝒯
      then obtain C K where C ∈ 𝒩
        and L: L = convex hull insert (SOME z. z ∈ rel_interior C) K
        and K: K ∈ 𝒰 K ⊆ rel_frontier C
        by auto
      then have convex hull C = C
        by (meson convex𝒩 convex_hull_eq)
      then have convex C
        by (metis (no_types) convex_convex_hull)
      have rel_frontier C ⊆ C
       by (metis DiffE closed𝒩 ‹C ∈ 𝒩› closure_closed rel_frontier_def subsetI)
      have K ⊆ C
        using K ‹rel_frontier C ⊆ C› by blast
      have C ∈ ℳ
        using 𝒩_def ‹C ∈ 𝒩› by auto
      moreover have L ⊆ C
        using K L ‹C ∈ 𝒩›
          by (metis ‹K ⊆ C› ‹convex hull C = C› contra_subsetD hull_mono
in_rel_interior insert_subset rel_interior_subset)
      ultimately show ?thesis
         using ‹rel_frontier C ⊆ C› ‹L ⊆ C› aff ℳ aff_dim_subset ‹C ∈ ℳ›
dual_order.trans by blast
    qed
    then show ∃ C. C ∈ ℳ ∧ L ⊆ C aff_dim L ≤ int n if L ∈ 𝒰 ∪ ?𝒯 for L
      using that by auto
  qed
  then show ?thesis
    apply (rule ex_forward, safe)
     apply (meson Union_iff subsetCE, fastforce)
    by (meson infinite_super simplicial_complex_def)
 qed
qed


lemma simplicial_subdivision_of_cell_complex_lowdim:
 assumes finite ℳ
    and poly: ⋀C. C ∈ ℳ ⟹ polytope C
    and face: ⋀C1 C2. ⟦C1 ∈ ℳ; C2 ∈ ℳ⟧ ⟹ C1 ∩ C2 face_of C1
    and aff: ⋀C. C ∈ ℳ ⟹ aff_dim C ≤ d
 obtains 𝒯 where simplicial_complex 𝒯 ⋀K. K ∈ 𝒯 ⟹ aff_dim K ≤ d
                 ⋃𝒯 = ⋃ℳ
                 ⋀C. C ∈ ℳ ⟹ ∃ F. finite F ∧ F ⊆ 𝒯 ∧ C = ⋃F
                 ⋀K. K ∈ 𝒯 ⟹ ∃ C. C ∈ ℳ ∧ K ⊆ C
proof (cases d ≥ 0)
 case True
```

```
  then obtain n where n: d = of_nat n
    using zero_le_imp_eq_int by blast
  have ∃T. simplicial_complex T ∧
            (∀K∈T. aff_dim K ≤ int n) ∧
            ⋃T = ⋃(⋃C∈M. {F. F face_of C}) ∧
            (∀C∈⋃C∈M. {F. F face_of C}.
               ∃F. finite F ∧ F ⊆ T ∧ C = ⋃F) ∧
            (∀K∈T. ∃C. C ∈ (⋃C∈M. {F. F face_of C}) ∧ K ⊆ C)
  proof (rule simplicial_subdivision_aux)
    show finite (⋃C∈M. {F. F face_of C})
      using ‹finite M› poly polyhedron_eq_finite_faces polytope_imp_polyhedron
by fastforce
    show polytope F if F ∈ (⋃C∈M. {F. F face_of C}) for F
      using poly that face_of_polytope_polytope by blast
    show aff_dim F ≤ int n if F ∈ (⋃C∈M. {F. F face_of C}) for F
      using that
      by clarify (metis n aff_dim_subset aff face_of_imp_subset order_trans)
    show F ∈ (⋃C∈M. {F. F face_of C})
      if G ∈ (⋃C∈M. {F. F face_of C}) and F face_of G for F G
      using that face_of_trans by blast
  next
    fix F1 F2
   assume F1 ∈ (⋃C∈M. {F. F face_of C}) and F2 ∈ (⋃C∈M. {F. F face_of
C})
     then obtain C1 C2 where C1 ∈ M C2 ∈ M and F: F1 face_of C1 F2
face_of C2
      by auto
    show F1 ∩ F2 face_of F1
      using face_of_Int_subface [OF _ _ F]
      by (metis ‹C1 ∈ M› ‹C2 ∈ M› face inf_commute)
  qed
  moreover
  have ⋃(⋃C∈M. {F. F face_of C}) = ⋃M
    using face_of_imp_subset face by blast
  ultimately show ?thesis
    using face_of_imp_subset n
    by (fastforce intro!: that simp add: poly face_of_refl polytope_imp_convex)
next
  case False
  then have m1: ⋀C. C ∈ M ⟹ aff_dim C = -1
   by (metis aff aff_dim_empty_eq aff_dim_negative_iff dual_order.trans not_less)
  then have faceM: ⋀F S. ⟦S ∈ M; F face_of S⟧ ⟹ F ∈ M
    by (metis aff_dim_empty face_of_empty)
  show ?thesis
  proof
    have ⋀S. S ∈ M ⟹ ∃n. n simplex S
      by (metis (no_types) m1 aff_dim_empty simplex_minus_1)
    then show simplicial_complex M
      by (auto simp: simplicial_complex_def ‹finite M› face intro: faceM)
```

```
    show aff_dim K ≤ d if K ∈ ℳ for K
      by (simp add: that aff)
    show ∃F. finite F ∧ F ⊆ ℳ ∧ C = ⋃F if C ∈ ℳ for C
      using ‹C ∈ ℳ› equals0I by auto
    show ∃C. C ∈ ℳ ∧ K ⊆ C if K ∈ ℳ for K
      using ‹K ∈ ℳ› by blast
  qed auto
qed

proposition simplicial_subdivision_of_cell_complex:
  assumes finite ℳ
      and poly: ⋀C. C ∈ ℳ ⟹ polytope C
      and face: ⋀C1 C2. ⟦C1 ∈ ℳ; C2 ∈ ℳ⟧ ⟹ C1 ∩ C2 face_of C1
  obtains 𝒯 where simplicial_complex 𝒯
                  ⋃𝒯 = ⋃ℳ
                  ⋀C. C ∈ ℳ ⟹ ∃F. finite F ∧ F ⊆ 𝒯 ∧ C = ⋃F
                  ⋀K. K ∈ 𝒯 ⟹ ∃C. C ∈ ℳ ∧ K ⊆ C
  by (blast intro: simplicial_subdivision_of_cell_complex_lowdim [OF assms aff_dim_le_DIM])

corollary fine_simplicial_subdivision_of_cell_complex:
  assumes 0 < e finite ℳ
      and poly: ⋀C. C ∈ ℳ ⟹ polytope C
      and face: ⋀C1 C2. ⟦C1 ∈ ℳ; C2 ∈ ℳ⟧ ⟹ C1 ∩ C2 face_of C1
  obtains 𝒯 where simplicial_complex 𝒯
                  ⋀K. K ∈ 𝒯 ⟹ diameter K < e
                  ⋃𝒯 = ⋃ℳ
                  ⋀C. C ∈ ℳ ⟹ ∃F. finite F ∧ F ⊆ 𝒯 ∧ C = ⋃F
                  ⋀K. K ∈ 𝒯 ⟹ ∃C. C ∈ ℳ ∧ K ⊆ C
proof −
  obtain 𝒩 where 𝒩: finite 𝒩 ⋃𝒩 = ⋃ℳ
          and diapoly: ⋀X. X ∈ 𝒩 ⟹ diameter X < e ⋀X. X ∈ 𝒩 ⟹ polytope
X
            and      ⋀X Y. ⟦X ∈ 𝒩; Y ∈ 𝒩⟧ ⟹ X ∩ Y face_of X
            and 𝒩covers: ⋀C x. C ∈ ℳ ∧ x ∈ C ⟹ ∃D. D ∈ 𝒩 ∧ x ∈ D ∧ D
⊆ C
            and 𝒩covered: ⋀C. C ∈ 𝒩 ⟹ ∃D. D ∈ ℳ ∧ C ⊆ D
    by (blast intro: cell_complex_subdivision_exists [OF ‹0 < e› ‹finite ℳ› poly
aff_dim_le_DIM face])
  then obtain 𝒯 where 𝒯: simplicial_complex 𝒯 ⋃𝒯 = ⋃𝒩
                  and 𝒯covers: ⋀C. C ∈ 𝒩 ⟹ ∃F. finite F ∧ F ⊆ 𝒯 ∧ C = ⋃F
                  and 𝒯covered: ⋀K. K ∈ 𝒯 ⟹ ∃C. C ∈ 𝒩 ∧ K ⊆ C
    using simplicial_subdivision_of_cell_complex [OF ‹finite 𝒩›] by metis
  show ?thesis
  proof
    show simplicial_complex 𝒯
      by (rule 𝒯)
    show diameter K < e if K ∈ 𝒯 for K
    by (metis le_less_trans diapoly 𝒯covered diameter_subset polytope_imp_bounded
that)
```

show $\bigcup \mathcal{T} = \bigcup \mathcal{M}$
by (*simp add*: $\mathcal{N}(2)$ ‹$\bigcup \mathcal{T} = \bigcup \mathcal{N}$›)
show $\exists F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$ **if** $C \in \mathcal{M}$ **for** C
proof −
{ **fix** x
assume $x \in C$
then obtain D **where** $D \in \mathcal{T}$ $x \in D$ $D \subseteq C$
using $\mathcal{N}$ *covers* ‹$C \in \mathcal{M}$› $\mathcal{T}$ *covers* **by** *force*
then have $\exists X{\in}\mathcal{T} \cap \mathit{Pow}\ C.\ x \in X$
using ‹$D \in \mathcal{T}$› ‹$D \subseteq C$› ‹$x \in D$› **by** *blast*
}
moreover
have *finite* $(\mathcal{T} \cap \mathit{Pow}\ C)$
using ‹*simplicial_complex* $\mathcal{T}$› *simplicial_complex_def* **by** *auto*
ultimately show *?thesis*
by (*rule_tac* $x{=}(\mathcal{T} \cap \mathit{Pow}\ C)$ **in** *exI*) *auto*
qed
show $\exists C.\ C \in \mathcal{M} \wedge K \subseteq C$ **if** $K \in \mathcal{T}$ **for** K
by (*meson* $\mathcal{N}$ *covered* $\mathcal{T}$ *covered order_trans that*)
qed
qed

9.24.20 Some results on cell division with full-dimensional cells only

lemma *convex_Union_fulldim_cells*:
assumes *finite* $\mathcal{S}$ **and** *clo*: $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow$ *closed* C **and** *con*: $\bigwedge C.\ C \in \mathcal{S} \Longrightarrow$ *convex* C
and *eq*: $\bigcup \mathcal{S} = U$ **and** *convex* U
shows $\bigcup \{C \in \mathcal{S}.\ \mathit{aff\_dim}\ C = \mathit{aff\_dim}\ U\} = U$ (**is** *?lhs* $= U$)
proof −
have *closed* U
using ‹*finite* $\mathcal{S}$› *clo eq* **by** *blast*
have *?lhs* $\subseteq U$
using *eq* **by** *blast*
moreover have $U \subseteq$ *?lhs*
proof (*cases* $\forall C \in \mathcal{S}.\ \mathit{aff\_dim}\ C = \mathit{aff\_dim}\ U$)
case *True*
then show *?thesis*
using *eq* **by** *blast*
next
case *False*
have *closed ?lhs*
by (*simp add*: ‹*finite* $\mathcal{S}$› *clo closed_Union*)
moreover have $U \subseteq$ *closure ?lhs*
proof −
have $U \subseteq$ *closure*$(\bigcap \{U - C \mid C.\ C \in \mathcal{S} \wedge \mathit{aff\_dim}\ C < \mathit{aff\_dim}\ U\})$
proof (*rule Baire* [*OF* ‹*closed* U›])
show *countable* $\{U - C \mid C.\ C \in \mathcal{S} \wedge \mathit{aff\_dim}\ C < \mathit{aff\_dim}\ U\}$

```
        using ‹finite 𝒮› uncountable_infinite by fastforce
      have ⋀C. C ∈ 𝒮 ⟹ openin (top_of_set U) (U−C)
          by (metis Sup_upper clo closed_limpt closedin_limpt eq openin_diff
openin_subtopology_self)
      then show openin (top_of_set U) T ∧ U ⊆ closure T
        if T ∈ {U − C |C. C ∈ 𝒮 ∧ aff_dim C < aff_dim U} for T
        using that dense_complement_convex_closed ‹closed U› ‹convex U› by
auto
    qed
    also have ... ⊆ closure ?lhs
    proof −
      obtain C where C ∈ 𝒮 aff_dim C < aff_dim U
        by (metis False Sup_upper aff_dim_subset eq eq_iff not_le)
      then have ∃X. X ∈ 𝒮 ∧ aff_dim X = aff_dim U ∧ x ∈ X
        if ⋀V. (∃C. V = U − C ∧ C ∈ 𝒮 ∧ aff_dim C < aff_dim U) ⟹ x ∈
V for x
          by (metis Diff_iff Sup_upper UnionE aff_dim_subset eq order_less_le
that)
      then show ?thesis
        by (auto intro!: closure_mono)
    qed
    finally show ?thesis .
  qed
  ultimately show ?thesis
    using closure_subset_eq by blast
 qed
 ultimately show ?thesis by blast
qed

proposition fine_triangular_subdivision_of_cell_complex:
 assumes 0 < e finite ℳ
    and poly: ⋀C. C ∈ ℳ ⟹ polytope C
    and aff: ⋀C. C ∈ ℳ ⟹ aff_dim C = d
    and face: ⋀C1 C2. ⟦C1 ∈ ℳ; C2 ∈ ℳ⟧ ⟹ C1 ∩ C2 face_of C1
 obtains 𝒯 where triangulation 𝒯 ⋀k. k ∈ 𝒯 ⟹ diameter k < e
                 ⋀k. k ∈ 𝒯 ⟹ aff_dim k = d ⋃𝒯 = ⋃ℳ
                 ⋀C. C ∈ ℳ ⟹ ∃f. finite f ∧ f ⊆ 𝒯 ∧ C = ⋃f
                 ⋀k. k ∈ 𝒯 ⟹ ∃C. C ∈ ℳ ∧ k ⊆ C
proof −
 obtain 𝒯 where simplicial_complex 𝒯
           and diaT: ⋀K. K ∈ 𝒯 ⟹ diameter K < e
           and ⋃𝒯 = ⋃ℳ
           and inℳ: ⋀C. C ∈ ℳ ⟹ ∃F. finite F ∧ F ⊆ 𝒯 ∧ C = ⋃F
           and in𝒯: ⋀K. K ∈ 𝒯 ⟹ ∃C. C ∈ ℳ ∧ K ⊆ C
    by (blast intro: fine_simplicial_subdivision_of_cell_complex [OF ‹e > 0›
‹finite ℳ› poly face])
 let ?𝒯 = {K ∈ 𝒯. aff_dim K = d}
 show thesis
 proof
```

show *triangulation* $?\mathcal{T}$
using ‹*simplicial_complex* $\mathcal{T}$› **by** (*auto simp*: *triangulation_def simplicial_complex_def*)
show *diameter* $L < e$ **if** $L \in \{K \in \mathcal{T}.\ \mathit{aff\_dim}\ K = d\}$ **for** L
using *that* **by** (*auto simp*: *dia*$\mathcal{T}$)
show *aff_dim* $L = d$ **if** $L \in \{K \in \mathcal{T}.\ \mathit{aff\_dim}\ K = d\}$ **for** L
using *that* **by** *auto*
show $\exists F.$ *finite* $F \wedge F \subseteq \{K \in \mathcal{T}.\ \mathit{aff\_dim}\ K = d\} \wedge C = \bigcup F$ **if** $C \in \mathcal{M}$ **for** C
proof −
obtain F **where** *finite* F $F \subseteq \mathcal{T}$ $C = \bigcup F$
using *in*$\mathcal{M}$ [*OF* ‹$C \in \mathcal{M}$›] **by** *auto*
show *?thesis*
proof (*intro exI conjI*)
show *finite* $\{K \in F.\ \mathit{aff\_dim}\ K = d\}$
by (*simp add*: ‹*finite* F›)
show $\{K \in F.\ \mathit{aff\_dim}\ K = d\} \subseteq \{K \in \mathcal{T}.\ \mathit{aff\_dim}\ K = d\}$
using ‹$F \subseteq \mathcal{T}$› **by** *blast*
have $d = \mathit{aff\_dim}\ C$
by (*simp add*: *aff that*)
moreover have $\bigwedge K.\ K \in F \Longrightarrow$ *closed* $K \wedge$ *convex* K
using ‹*simplicial_complex* $\mathcal{T}$› ‹$F \subseteq \mathcal{T}$›
unfolding *simplicial_complex_def* **by** (*metis subsetCE* ‹$F \subseteq \mathcal{T}$› *closed_simplex convex_simplex*)
moreover have *convex* $(\bigcup F)$
using ‹$C = \bigcup F$› *poly polytope_imp_convex that* **by** *blast*
ultimately show $C = \bigcup \{K \in F.\ \mathit{aff\_dim}\ K = d\}$
by (*simp add*: *convex_Union_fulldim_cells* ‹$C = \bigcup F$› ‹*finite* F›)
qed
qed
then show $\bigcup \{K \in \mathcal{T}.\ \mathit{aff\_dim}\ K = d\} = \bigcup \mathcal{M}$
by *auto* (*meson in*$\mathcal{T}$ *subsetCE*)
show $\exists C.\ C \in \mathcal{M} \wedge L \subseteq C$
if $L \in \{K \in \mathcal{T}.\ \mathit{aff\_dim}\ K = d\}$ **for** L
using *that* **by** (*auto simp*: *in*$\mathcal{T}$)
qed
qed

9.25 Finitely generated cone is polyhedral, and hence closed

proposition *polyhedron_convex_cone_hull*:
fixes S :: *'a*::*euclidean_space set*
assumes *finite* S
shows *polyhedron*(*convex_cone hull* S)
proof (*cases* $S = \{\}$)
case *True*
then show *?thesis*

```
    by (simp add: affine_imp_polyhedron)
next
  case False
  then have polyhedron(convex hull (insert 0 S))
    by (simp add: assms polyhedron_convex_hull)
  then obtain F a b where finite F
        and F: convex hull (insert 0 S) = ⋂ F
        and ab: ⋀h. h ∈ F ⟹ a h ≠ 0 ∧ h = {x. a h · x ≤ b h}
    unfolding polyhedron_def by metis
  then have F ≠ {}
   by (metis bounded_convex_hull finite_imp_bounded Inf_empty assms finite_insert
not_bounded_UNIV)
  show ?thesis
    unfolding polyhedron_def
  proof (intro exI conjI)
    show convex_cone hull S = ⋂ {h ∈ F. b h = 0} (is ?lhs = ?rhs)
    proof
      show ?lhs ⊆ ?rhs
      proof (rule hull_minimal)
        show S ⊆ ⋂ {h ∈ F. b h = 0}
       by (smt (verit, best) F InterE InterI hull_subset insert_subset mem_Collect_eq
subset_eq)
        have ⋀S. ⟦S ∈ F; b S = 0⟧ ⟹ convex_cone S
          by (metis ab convex_cone_halfspace_le)
        then show convex_cone (⋂ {h ∈ F. b h = 0})
          by (force intro: convex_cone_Inter)
      qed
      have x ∈ convex_cone hull S
        if x: ⋀h. ⟦h ∈ F; b h = 0⟧ ⟹ x ∈ h for x
      proof −
        have ∃t. 0 < t ∧ (t *R x) ∈ h if h ∈ F for h
        proof (cases b h = 0)
          case True
          then show ?thesis
         by (metis x linordered_field_no_ub mult_1 scaleR_one that zero_less_mult_iff)
        next
          case False
          then have b h > 0
              by (smt (verit, del_insts) F InterE ab hull_subset inner_zero_right
insert_subset mem_Collect_eq that)
          then have 0 ∈ interior {x. a h · x ≤ b h}
            by (simp add: ab that)
          then have 0 ∈ interior h
            using ab that by auto
          then obtain ε where 0 < ε and ε: ball 0 ε ⊆ h
            using mem_interior by blast
          show ?thesis
          proof (cases x=0)
            case True
```

```
          then show ?thesis
            using ε ‹0 < ε› by auto
        next
          case False
          with ε ‹0 < ε› show ?thesis
            by (rule_tac x=ε / (2 * norm x) in exI) (auto simp: divide_simps)
        qed
      qed
      then obtain t where t: ⋀h. h ∈ F ⟹ 0 < t h ∧ (t h *R x) ∈ h
        by metis
      then have Inf (t ` F) *R x /R Inf (t ` F) = x
          by (smt (verit) ‹F ≠ {}› ‹finite F› divideR_right finite_imageI fi-
nite_less_Inf_iff image_iff image_is_empty)
      moreover have Inf (t ` F) *R x /R Inf (t ` F) ∈ convex_cone hull S
      proof (rule conicD [OF conic_convex_cone_hull])
        have Inf (t ` F) *R x ∈ ⋂ F
        proof clarify
          fix h
          assume  h ∈ F
          have eq: Inf (t ` F) *R x = (1 - Inf(t ` F) / t h) *R 0 + (Inf(t ` F) /
t h) *R t h *R x
            using ‹h ∈ F› t by force
          show Inf (t ` F) *R x ∈ h
            unfolding eq
          proof (rule convexD_alt)
            have h = {x. a h · x ≤ b h}
              by (simp add: ‹h ∈ F› ab)
            then show convex h
              by (metis convex_halfspace_le)
            show 0 ∈ h
              by (metis F InterE ‹h ∈ F› hull_subset insertCI subsetD)
            show t h *R x ∈ h
              by (simp add: ‹h ∈ F› t)
            show 0 ≤ Inf (t ` F) / t h
                by (metis ‹F ≠ {}› ‹h ∈ F› cINF_greatest divide_nonneg_pos
less_eq_real_def t)
            show Inf (t ` F) / t h ≤ 1
              by (simp add: ‹finite F› ‹h ∈ F› cInf_le_finite t)
          qed
        qed
        moreover have convex hull (insert 0 S) ⊆ convex_cone hull S
          by (simp add: convex_cone_hull_contains_0 convex_convex_cone_hull
hull_minimal hull_subset)
        ultimately show Inf (t ` F) *R x ∈ convex_cone hull S
          using F by blast
        show 0 ≤ inverse (Inf (t ` F))
       using t by (simp add: ‹F ≠ {}› ‹finite F› finite_less_Inf_iff less_eq_real_def)
      qed
      ultimately show ?thesis
```

```
        by auto
    qed
    then show ?rhs ⊆ ?lhs
      by auto
  qed
  show ∀h∈{h ∈ F. b h = 0}. ∃a b. a ≠ 0 ∧ h = {x. a · x ≤ b}
    using ab by blast
 qed (auto simp: ‹finite F›)
qed


lemma closed_convex_cone_hull:
 fixes S :: 'a::euclidean_space set
 shows finite S ⟹ closed(convex_cone hull S)
 by (simp add: polyhedron_convex_cone_hull polyhedron_imp_closed)

lemma polyhedron_convex_cone_hull_polytope:
 fixes S :: 'a::euclidean_space set
 shows polytope S ⟹ polyhedron(convex_cone hull S)
 by (metis convex_cone_hull_separate hull_hull polyhedron_convex_cone_hull
polytope_def)

lemma polyhedron_conic_hull_polytope:
 fixes S :: 'a::euclidean_space set
 shows polytope S ⟹ polyhedron(conic hull S)
 by (metis conic_hull_eq_empty convex_cone_hull_separate_nonempty hull_hull
polyhedron_convex_cone_hull_polytope polyhedron_empty polytope_def)

lemma closed_conic_hull_strong:
 fixes S :: 'a::euclidean_space set
 shows 0 ∈ rel_interior S ∨ polytope S ∨ compact S ∧ ~(0 ∈ S) ⟹ closed(conic
hull S)
 using closed_conic_hull polyhedron_conic_hull_polytope polyhedron_imp_closed
by blast

end
```

9.26 Absolute Retracts, Absolute Neighbourhood Retracts and Euclidean Neighbourhood Retracts

```
theory Retracts
imports
 Brouwer_Fixpoint
 Continuous_Extension
begin
```

Absolute retracts (AR), absolute neighbourhood retracts (ANR) and also

Euclidean neighbourhood retracts (ENR). We define AR and ANR by specializing the standard definitions for a set to embedding in spaces of higher dimension.

John Harrison writes: "This turns out to be sufficient (since any set in $\mathbb{R}^n$ can be embedded as a closed subset of a convex subset of $\mathbb{R}^{n+1}$) to derive the usual definitions, but we need to split them into two implications because of the lack of type quantifiers. Then ENR turns out to be equivalent to ANR plus local compactness."

definition *AR* :: *'a*::*topological_space set* ⇒ *bool* **where**
AR S ≡ ∀ *U*. ∀ *S'*::(*'a* ∗ *real*) *set*.
S homeomorphic S' ∧ *closedin* (*top_of_set U*) *S'* ⟶ *S' retract_of U*

definition *ANR* :: *'a*::*topological_space set* ⇒ *bool* **where**
ANR S ≡ ∀ *U*. ∀ *S'*::(*'a* ∗ *real*) *set*.
S homeomorphic S' ∧ *closedin* (*top_of_set U*) *S'*
⟶ (∃ *T*. *openin* (*top_of_set U*) *T* ∧ *S' retract_of T*)

definition *ENR* :: *'a*::*topological_space set* ⇒ *bool* **where**
ENR S ≡ ∃ *U*. *open U* ∧ *S retract_of U*

First, show that we do indeed get the "usual" properties of ARs and ANRs.

lemma *AR_imp_absolute_extensor*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *AR S* **and** *contf*: *continuous_on T f* **and** *f* ‘ *T* ⊆ *S*
and *cloUT*: *closedin* (*top_of_set U*) *T*
obtains *g* **where** *continuous_on U g g* ‘ *U* ⊆ *S* ⋀*x*. *x* ∈ *T* ⟹ *g x* = *f x*
proof −
have *aff_dim S* < *int* (*DIM*(*'b* × *real*))
using *aff_dim_le_DIM* [*of S*] **by** *simp*
then obtain *C* **and** *S'* :: (*'b* ∗ *real*) *set*
where *C*: *convex C C* ≠ {}
and *cloCS*: *closedin* (*top_of_set C*) *S'*
and *hom*: *S homeomorphic S'*
by (*metis that homeomorphic_closedin_convex*)
then have *S' retract_of C*
using ‹*AR S*› **by** (*simp add*: *AR_def*)
then obtain *r* **where** *S'* ⊆ *C* **and** *contr*: *continuous_on C r*
and *r* ‘ *C* ⊆ *S'* **and** *rid*: ⋀*x*. *x*∈*S'* ⟹ *r x* = *x*
by (*auto simp*: *retraction_def retract_of_def*)
obtain *g h* **where** *homeomorphism S S' g h*
using *hom* **by** (*force simp*: *homeomorphic_def*)
then have *continuous_on* (*f* ‘ *T*) *g*
by (*meson* ‹*f* ‘ *T* ⊆ *S*› *continuous_on_subset homeomorphism_def*)
then have *contgf*: *continuous_on T* (*g* ∘ *f*)
by (*metis continuous_on_compose contf*)
have *gfTC*: (*g* ∘ *f*) ‘ *T* ⊆ *C*
proof −
have *g* ‘ *S* = *S'*

```
    by (metis (no_types) ‹homeomorphism S S' g h› homeomorphism_def)
   with ‹S' ⊆ C› ‹f ` T ⊆ S› show ?thesis by force
 qed
 obtain f' where f': continuous_on U f'  f' ` U ⊆ C
                    ⋀x. x ∈ T ⟹ f' x = (g ∘ f) x
   by (metis Dugundji [OF C cloUT contgf gfTC])
 show ?thesis
 proof (rule_tac g = h ∘ r ∘ f' in that)
   show continuous_on U (h ∘ r ∘ f')
   proof (intro continuous_on_compose f')
     show continuous_on (f' ` U) r
       using continuous_on_subset contr f' by blast
     show continuous_on (r ` f' ` U) h
       using ‹homeomorphism S S' g h› ‹f' ` U ⊆ C›
       unfolding homeomorphism_def
       by (metis ‹r ` C ⊆ S'› continuous_on_subset image_mono)
   qed
   show (h ∘ r ∘ f') ` U ⊆ S
     using ‹homeomorphism S S' g h› ‹r ` C ⊆ S'› ‹f' ` U ⊆ C›
     by (fastforce simp: homeomorphism_def)
   show ⋀x. x ∈ T ⟹ (h ∘ r ∘ f') x = f x
     using ‹homeomorphism S S' g h› ‹f ` T ⊆ S› f'
     by (auto simp: rid homeomorphism_def)
 qed
qed

lemma AR_imp_absolute_retract:
 fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
 assumes AR S S homeomorphic S'
    and clo: closedin (top_of_set U) S'
   shows S' retract_of U
proof −
 obtain g h where hom: homeomorphism S S' g h
   using assms by (force simp: homeomorphic_def)
 obtain h: continuous_on S' h  h ` S' ⊆ S
   using hom homeomorphism_def by blast
 obtain h' where h': continuous_on U h' h' ` U ⊆ S
           and h'h: ⋀x. x ∈ S' ⟹ h' x = h x
   by (blast intro: AR_imp_absolute_extensor [OF ‹AR S› h clo])
 have [simp]: S' ⊆ U using clo closedin_limpt by blast
 show ?thesis
 proof (simp add: retraction_def retract_of_def, intro exI conjI)
   show continuous_on U (g ∘ h')
     by (meson continuous_on_compose continuous_on_subset h' hom homeomorphism_cont1)
   show (g ∘ h') ∈ U → S'
     using h'  by clarsimp (metis hom subsetD homeomorphism_def imageI)
   show ∀x∈S'. (g ∘ h') x = x
     by clarsimp (metis h'h hom homeomorphism_def)
```

```
  qed
qed

lemma AR_imp_absolute_retract_UNIV:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  assumes AR S S homeomorphic S' closed S'
    shows S' retract_of UNIV
  using AR_imp_absolute_retract assms by fastforce

lemma absolute_extensor_imp_AR:
  fixes S :: 'a::euclidean_space set
  assumes ⋀f :: 'a * real ⇒ 'a.
          ⋀U T. ⟦continuous_on T f;  f ` T ⊆ S;
                 closedin (top_of_set U) T⟧
                ⟹ ∃g. continuous_on U g ∧ g ` U ⊆ S ∧ (∀x ∈ T. g x = f x)
  shows AR S
proof (clarsimp simp: AR_def)
  fix U and T :: ('a * real) set
  assume S homeomorphic T and clo: closedin (top_of_set U) T
  then obtain g h where hom: homeomorphism S T g h
    by (force simp: homeomorphic_def)
  obtain h: continuous_on T h  h ` T ⊆ S
    using hom homeomorphism_def by blast
  obtain h' where h': continuous_on U h' h' ` U ⊆ S
             and h'h: ∀x∈T. h' x = h x
    using assms [OF h clo] by blast
  have [simp]: T ⊆ U
    using clo closedin_imp_subset by auto
  show T retract_of U
  proof (simp add: retraction_def retract_of_def, intro exI conjI)
    show continuous_on U (g ∘ h')
      by (meson continuous_on_compose continuous_on_subset h' hom homeo-
morphism_cont1)
    show (g ∘ h') ∈ U → T
      using h'  by clarsimp (metis hom subsetD homeomorphism_def imageI)
    show ∀x∈T. (g ∘ h') x = x
      by clarsimp (metis h'h hom homeomorphism_def)
  qed
qed

lemma AR_eq_absolute_extensor:
  fixes S :: 'a::euclidean_space set
  shows AR S ⟷
       (∀f :: 'a * real ⇒ 'a.
        ∀U T. continuous_on T f ⟶ f ` T ⊆ S ⟶
              closedin (top_of_set U) T ⟶
              (∃g. continuous_on U g ∧ g ` U ⊆ S ∧ (∀x ∈ T. g x = f x)))
 by (metis (mono_tags, opaque_lifting) AR_imp_absolute_extensor absolute_extensor_imp_AR)
```

lemma *AR_imp_retract*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $AR\ S \wedge closedin\ (top\_of\_set\ U)\ S$
shows $S\ retract\_of\ U$
using *AR_imp_absolute_retract assms homeomorphic_refl* **by** *blast*

lemma *AR_homeomorphic_AR*:
fixes $S :: 'a::euclidean\_space\ set$ **and** $T :: 'b::euclidean\_space\ set$
assumes $AR\ T\ S\ homeomorphic\ T$
shows $AR\ S$
unfolding *AR_def*
by (*metis assms AR_imp_absolute_retract homeomorphic_trans* [*of* _ *S*] *homeomorphic_sym*)

lemma *homeomorphic_AR_iff_AR*:
fixes $S :: 'a::euclidean\_space\ set$ **and** $T :: 'b::euclidean\_space\ set$
shows $S\ homeomorphic\ T \Longrightarrow AR\ S \longleftrightarrow AR\ T$
by (*metis AR_homeomorphic_AR homeomorphic_sym*)

lemma *ANR_imp_absolute_neighbourhood_extensor*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $ANR\ S$ **and** *contf*: $continuous\_on\ T\ f$ **and** $f \in T \to S$
and *cloUT*: $closedin\ (top\_of\_set\ U)\ T$
obtains $V\ g$ **where** $T \subseteq V\ openin\ (top\_of\_set\ U)\ V$
$continuous\_on\ V\ g$
$g \in V \to S \bigwedge x.\ x \in T \Longrightarrow g\ x = f\ x$
proof −
have $aff\_dim\ S < int\ (DIM('b \times real))$
using *aff_dim_le_DIM* [*of S*] **by** *simp*
then obtain C **and** $S' :: ('b * real)\ set$
where *C*: $convex\ C\ C \neq \{\}$
and *cloCS*: $closedin\ (top\_of\_set\ C)\ S'$
and *hom*: $S\ homeomorphic\ S'$
by (*metis that homeomorphic_closedin_convex*)
then obtain D **where** *opD*: $openin\ (top\_of\_set\ C)\ D$ **and** $S'\ retract\_of\ D$
using ‹$ANR\ S$› **by** (*auto simp*: *ANR_def*)
then obtain r **where** $S' \subseteq D$ **and** *contr*: $continuous\_on\ D\ r$
and $r\ `\ D \subseteq S'$ **and** *rid*: $\bigwedge x.\ x \in S' \Longrightarrow r\ x = x$
by (*auto simp*: *retraction_def retract_of_def*)
obtain $g\ h$ **where** *homgh*: $homeomorphism\ S\ S'\ g\ h$
using *hom* **by** (*force simp*: *homeomorphic_def*)
have $continuous\_on\ (f\ `\ T)\ g$
by (*metis PiE assms*(3) *continuous_on_subset homeomorphism_cont1 homgh image_subset_iff*)
then have *contgf*: $continuous\_on\ T\ (g \circ f)$
by (*intro continuous_on_compose contf*)
have *gfTC*: $(g \circ f)\ `\ T \subseteq C$
proof −

```
    have g ‘ S = S'
      by (metis (no_types) homeomorphism_def homgh)
    then show ?thesis
      by (metis PiE assms(3) cloCS closedin_def image_comp image_mono im-
age_subset_iff order.trans topspace_euclidean_subtopology)
  qed
  obtain f' where contf': continuous_on U f'
            and f' ‘ U ⊆ C
            and eq: ⋀x. x ∈ T ⟹ f' x = (g ∘ f) x
    by (metis Dugundji [OF C cloUT contgf gfTC])
  show ?thesis
  proof (rule_tac V = U ∩ f' −‘ D and g = h ∘ r ∘ f' in that)
    show T ⊆ U ∩ f' −‘ D
      using cloUT closedin_imp_subset ‹S' ⊆ D› ‹f ∈ T → S› eq homeomor-
phism_image1 homgh
      by fastforce
    show ope: openin (top_of_set U) (U ∩ f' −‘ D)
    by (meson ‹f' ‘ U ⊆ C› contf' continuous_openin_preimage image_subset_iff_funcset
opD)
    have conth: continuous_on (r ‘ f' ‘ (U ∩ f' −‘ D)) h
    proof (rule continuous_on_subset [of S'])
      show continuous_on S' h
        using homeomorphism_def homgh by blast
    qed (use ‹r ‘ D ⊆ S'› in blast)
    show continuous_on (U ∩ f' −‘ D) (h ∘ r ∘ f')
      by (blast intro: continuous_on_compose conth continuous_on_subset [OF
contr] continuous_on_subset [OF contf'])
    show (h ∘ r ∘ f') ∈ (U ∩ f' −‘ D) → S
      using ‹homeomorphism S S' g h› ‹f' ‘ U ⊆ C› ‹r ‘ D ⊆ S'›
      by (auto simp: homeomorphism_def)
    show ⋀x. x ∈ T ⟹ (h ∘ r ∘ f') x = f x
      using ‹homeomorphism S S' g h› ‹f ∈ T → S› eq
      by (metis PiE comp_apply homeomorphism_def image_iff rid)
  qed
qed


corollary ANR_imp_absolute_neighbourhood_retract:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  assumes ANR S S homeomorphic S'
      and clo: closedin (top_of_set U) S'
  obtains V where openin (top_of_set U) V S' retract_of V
proof −
  obtain g h where hom: homeomorphism S S' g h
    using assms by (force simp: homeomorphic_def)
  obtain h: continuous_on S' h  h ∈ S' → S
    using hom homeomorphism_def by blast
    from ANR_imp_absolute_neighbourhood_extensor [OF ‹ANR S› h clo]
  obtain V h' where S' ⊆ V and opUV: openin (top_of_set U) V
```

```
            and h': continuous_on V h' h' ` V ⊆ S
            and h'h:⋀x. x ∈ S' ⟹ h' x = h x
    by (blast intro: ANR_imp_absolute_neighbourhood_extensor [OF ‹ANR S› h clo])
  have S' retract_of V
  proof (simp add: retraction_def retract_of_def, intro exI conjI ‹S' ⊆ V›)
    show continuous_on V (g ∘ h')
      by (meson continuous_on_compose continuous_on_subset h'(1) h'(2) hom homeomorphism_cont1)
    show (g ∘ h') ∈ V → S'
      using h' by clarsimp (metis hom subsetD homeomorphism_def imageI)
    show ∀x∈S'. (g ∘ h') x = x
      by clarsimp (metis h'h hom homeomorphism_def)
  qed
  then show ?thesis
    by (rule that [OF opUV])
qed

corollary ANR_imp_absolute_neighbourhood_retract_UNIV:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  assumes ANR S and hom: S homeomorphic S' and clo: closed S'
  obtains V where open V S' retract_of V
  using ANR_imp_absolute_neighbourhood_retract [OF ‹ANR S› hom]
by (metis clo closed_closedin open_openin subtopology_UNIV)

corollary neighbourhood_extension_into_ANR:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and fim: f ∈ S → T and ANR T closed S
  obtains V g where S ⊆ V open V continuous_on V g
                    g ∈ V → T ⋀x. x ∈ S ⟹ g x = f x
  using ANR_imp_absolute_neighbourhood_extensor [OF ‹ANR T› contf fim]
  by (metis ‹closed S› closed_closedin open_openin subtopology_UNIV)

lemma absolute_neighbourhood_extensor_imp_ANR:
  fixes S :: 'a::euclidean_space set
  assumes ⋀f :: 'a * real ⇒ 'a.
          ⋀U T. ⟦continuous_on T f;  f ∈ T → S;
                 closedin (top_of_set U) T⟧
                ⟹ ∃V g. T ⊆ V ∧ openin (top_of_set U) V ∧
                     continuous_on V g ∧ g ∈ V → S ∧ (∀x ∈ T. g x = f x)
  shows ANR S
proof (clarsimp simp: ANR_def)
  fix U and T :: ('a * real) set
  assume S homeomorphic T and clo: closedin (top_of_set U) T
  then obtain g h where hom: homeomorphism S T g h
    by (force simp: homeomorphic_def)
  obtain h: continuous_on T h  h ∈ T → S
    using hom homeomorphism_def by blast
  obtain V h' where T ⊆ V and opV: openin (top_of_set U) V
```

```
            and h': continuous_on V h' h' ∈ V → S
           and h'h: ∀x∈T. h' x = h x
    using assms [OF h clo] by blast
  have [simp]: T ⊆ U
    using clo closedin_imp_subset by auto
  have T retract_of V
  proof (simp add: retraction_def retract_of_def, intro exI conjI ‹T ⊆ V›)
    show continuous_on V (g ∘ h')
      by (meson continuous_on_compose continuous_on_subset h' hom homeo-
morphism_def image_subset_iff_funcset)
    show (g ∘ h') ∈ V → T
      using h' hom homeomorphism_image1 by fastforce
    show ∀x∈T. (g ∘ h') x = x
      by clarsimp (metis h'h hom homeomorphism_def)
  qed
  then show ∃V. openin (top_of_set U) V ∧ T retract_of V
    using opV by blast
qed

lemma ANR_eq_absolute_neighbourhood_extensor:
  fixes S :: 'a::euclidean_space set
  shows ANR S ⟷
        (∀f :: 'a * real ⇒ 'a.
         ∀U T. continuous_on T f ⟶ f ∈ T → S ⟶
               closedin (top_of_set U) T ⟶
               (∃V g. T ⊆ V ∧ openin (top_of_set U) V ∧
                      continuous_on V g ∧ g ∈ V → S ∧ (∀x ∈ T. g x = f x))) (is
_ = ?rhs)
proof
  assume ANR S then show ?rhs
    by (metis ANR_imp_absolute_neighbourhood_extensor)
qed (simp add: absolute_neighbourhood_extensor_imp_ANR)

lemma ANR_imp_neighbourhood_retract:
  fixes S :: 'a::euclidean_space set
  assumes ANR S closedin (top_of_set U) S
  obtains V where openin (top_of_set U) V S retract_of V
using ANR_imp_absolute_neighbourhood_retract assms homeomorphic_refl by
blast

lemma ANR_imp_absolute_closed_neighbourhood_retract:
  fixes S :: 'a::euclidean_space set and S' :: 'b::euclidean_space set
  assumes ANR S S homeomorphic S' and US': closedin (top_of_set U) S'
  obtains V W
    where openin (top_of_set U) V
          closedin (top_of_set U) W
          S' ⊆ V V ⊆ W S' retract_of W
proof −
  obtain Z where openin (top_of_set U) Z and S'Z: S' retract_of Z
```

by (*blast intro*: *assms ANR_imp_absolute_neighbourhood_retract*)
then have *UUZ*: *closedin* (*top_of_set U*) (*U* − *Z*)
by *auto*
have *S′* ∩ (*U* − *Z*) = {}
using ‹*S′ retract_of Z*› *closedin_retract closedin_subtopology* **by** *fastforce*
then obtain *V W*
where *openin* (*top_of_set U*) *V*
and *openin* (*top_of_set U*) *W*
and *S′* ⊆ *V U* − *Z* ⊆ *W V* ∩ *W* = {}
using *separation_normal_local* [*OF US′ UUZ*] **by** *auto*
moreover have *S′ retract_of U* − *W*
proof (*rule retract_of_subset* [*OF S′Z*])
show *S′* ⊆ *U* − *W*
using *US′* ‹*S′* ⊆ *V*› ‹*V* ∩ *W* = {}› *closedin_subset* **by** *fastforce*
show *U* − *W* ⊆ *Z*
using *Diff_subset_conv* ‹*U* − *Z* ⊆ *W*› **by** *blast*
qed
ultimately show *?thesis*
by (*metis Diff_subset_conv Diff_triv Int_Diff_Un Int_absorb1 openin_closedin_eq that topspace_euclidean_subtopology*)
qed

lemma *ANR_imp_closed_neighbourhood_retract*:
fixes *S* :: *′a*::*euclidean_space set*
assumes *ANR S closedin* (*top_of_set U*) *S*
obtains *V W* **where** *openin* (*top_of_set U*) *V*
closedin (*top_of_set U*) *W*
S ⊆ *V V* ⊆ *W S retract_of W*
by (*meson ANR_imp_absolute_closed_neighbourhood_retract assms homeomorphic_refl*)

lemma *ANR_homeomorphic_ANR*:
fixes *S* :: *′a*::*euclidean_space set* **and** *T* :: *′b*::*euclidean_space set*
assumes *ANR T S homeomorphic T*
shows *ANR S*
unfolding *ANR_def*
by (*metis assms ANR_imp_absolute_neighbourhood_retract homeomorphic_trans* [*of _ S*] *homeomorphic_sym*)

lemma *homeomorphic_ANR_iff_ANR*:
fixes *S* :: *′a*::*euclidean_space set* **and** *T* :: *′b*::*euclidean_space set*
shows *S homeomorphic T* ⟹ *ANR S* ⟷ *ANR T*
by (*metis ANR_homeomorphic_ANR homeomorphic_sym*)

9.26.1 Analogous properties of ENRs

lemma *ENR_imp_absolute_neighbourhood_retract*:
fixes *S* :: *′a*::*euclidean_space set* **and** *S′* :: *′b*::*euclidean_space set*
assumes *ENR S* **and** *hom*: *S homeomorphic S′*

```
    and S' ⊆ U
  obtains V where openin (top_of_set U) V S' retract_of V
proof −
  obtain X where open X S retract_of X
    using ‹ENR S› by (auto simp: ENR_def)
  then obtain r where retraction X S r
    by (auto simp: retract_of_def)
  have locally compact S'
    using retract_of_locally_compact open_imp_locally_compact
        homeomorphic_local_compactness ‹S retract_of X› ‹open X› hom by blast
  then obtain W where UW: openin (top_of_set U) W
                  and WS': closedin (top_of_set W) S'
    apply (rule locally_compact_closedin_open)
     by (meson Int_lower2 assms(3) closedin_imp_subset closedin_subset_trans
le_inf_iff openin_open)
  obtain f g where hom: homeomorphism S S' f g
    using assms by (force simp: homeomorphic_def)
  have contg: continuous_on S' g
    using hom homeomorphism_def by blast
  moreover have g ‘ S' ⊆ S by (metis hom equalityE homeomorphism_def)
  ultimately obtain h where conth: continuous_on W h and hg: ⋀x. x ∈ S'
⟹ h x = g x
    using Tietze_unbounded [of S' g W] WS' by blast
  have W ⊆ U using UW openin_open by auto
  have S' ⊆ W using WS' closedin_closed by auto
  have him: ⋀x. x ∈ S' ⟹ h x ∈ X
   by (metis (no_types) ‹S retract_of X› hg hom homeomorphism_def image_insert
insert_absorb insert_iff retract_of_imp_subset subset_eq)
  have S' retract_of (W ∩ h −‘ X)
  proof (simp add: retraction_def retract_of_def, intro exI conjI)
    show S' ⊆ W S' ⊆ h −‘ X
      using him WS' closedin_imp_subset by blast+
    show continuous_on (W ∩ h −‘ X) (f ∘ r ∘ h)
    proof (intro continuous_on_compose)
      show continuous_on (W ∩ h −‘ X) h
        by (meson conth continuous_on_subset inf_le1)
      show continuous_on (h ‘ (W ∩ h −‘ X)) r
      proof −
        have h ‘ (W ∩ h −‘ X) ⊆ X
          by blast
        then show continuous_on (h ‘ (W ∩ h −‘ X)) r
          by (meson ‹retraction X S r› continuous_on_subset retraction)
      qed
      show continuous_on (r ‘ h ‘ (W ∩ h −‘ X)) f
      proof (rule continuous_on_subset [of S])
        show continuous_on S f
          using hom homeomorphism_def by blast
        show r ‘ h ‘ (W ∩ h −‘ X) ⊆ S
        by (metis ‹retraction X S r› image_mono image_subset_iff_subset_vimage
```

inf_le2 retraction)
    qed
  qed
  show $(f \circ r \circ h) \in (W \cap h -` X) \to S'$
    using ‹*retraction X S r*› *hom*
    by (*auto simp*: *retraction_def homeomorphism_def*)
  show $\forall x{\in}S'.\ (f \circ r \circ h)\ x = x$
    using ‹*retraction X S r*› *hom* **by** (*auto simp*: *retraction_def homeomorphism_def hg*)
 qed
 then show *?thesis*
  using *UW* ‹*open X*› *conth continuous_openin_preimage_eq openin_trans that*
by *blast*
qed

corollary *ENR_imp_absolute_neighbourhood_retract_UNIV*:
 fixes $S :: 'a{::}euclidean\_space\ set$ **and** $S' :: 'b{::}euclidean\_space\ set$
 assumes *ENR S S homeomorphic S'*
 obtains T' **where** *open T' S' retract_of T'*
by (*metis ENR_imp_absolute_neighbourhood_retract UNIV_I assms*(*1*) *assms*(*2*) *open_openin subsetI subtopology_UNIV*)

lemma *ENR_homeomorphic_ENR*:
 fixes $S :: 'a{::}euclidean\_space\ set$ **and** $T :: 'b{::}euclidean\_space\ set$
 assumes *ENR T S homeomorphic T*
  shows *ENR S*
unfolding *ENR_def*
by (*meson ENR_imp_absolute_neighbourhood_retract_UNIV assms homeomorphic_sym*)

lemma *homeomorphic_ENR_iff_ENR*:
 fixes $S :: 'a{::}euclidean\_space\ set$ **and** $T :: 'b{::}euclidean\_space\ set$
 assumes *S homeomorphic T*
  shows *ENR S* $\longleftrightarrow$ *ENR T*
by (*meson ENR_homeomorphic_ENR assms homeomorphic_sym*)

lemma *ENR_translation*:
 fixes $S :: 'a{::}euclidean\_space\ set$
 shows *ENR*(*image* $(\lambda x.\ a + x)$ *S*) $\longleftrightarrow$ *ENR S*
by (*meson homeomorphic_sym homeomorphic_translation homeomorphic_ENR_iff_ENR*)

lemma *ENR_linear_image_eq*:
 fixes $f :: 'a{::}euclidean\_space \Rightarrow 'b{::}euclidean\_space$
 assumes *linear f inj f*
 shows *ENR* (*image f S*) $\longleftrightarrow$ *ENR S*
 by (*meson assms homeomorphic_ENR_iff_ENR linear_homeomorphic_image*)

Some relations among the concepts. We also relate AR to being a retract of UNIV, which is often a more convenient proxy in the closed case.

lemma *AR_imp_ANR*: *AR S* $\Longrightarrow$ *ANR S*
using *ANR_def AR_def* **by** *fastforce*

lemma *ENR_imp_ANR*:
fixes *S* :: *'a*::*euclidean_space set*
shows *ENR S* $\Longrightarrow$ *ANR S*
by (*meson ANR_def ENR_imp_absolute_neighbourhood_retract closedin_imp_subset*)

lemma *ENR_ANR*:
fixes *S* :: *'a*::*euclidean_space set*
shows *ENR S* $\longleftrightarrow$ *ANR S* $\wedge$ *locally compact S*
proof
assume *ENR S*
then have *locally compact S*
using *ENR_def open_imp_locally_compact retract_of_locally_compact* **by**
auto
then show *ANR S* $\wedge$ *locally compact S*
using *ENR_imp_ANR* ‹*ENR S*› **by** *blast*
next
assume *ANR S* $\wedge$ *locally compact S*
then have *ANR S locally compact S* **by** *auto*
then obtain *T* :: (*'a* $*$ *real*) *set* **where** *closed T S homeomorphic T*
using *locally_compact_homeomorphic_closed*
by (*metis DIM_prod DIM_real Suc_eq_plus1 lessI*)
then show *ENR S*
using ‹*ANR S*›
by (*meson ANR_imp_absolute_neighbourhood_retract_UNIV ENR_def ENR_homeomorphic_ENR*)
qed

lemma *AR_ANR*:
fixes *S* :: *'a*::*euclidean_space set*
shows *AR S* $\longleftrightarrow$ *ANR S* $\wedge$ *contractible S* $\wedge$ *S* $\neq$ {}
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
have *aff_dim S* $<$ *int DIM*(*'a* $\times$ *real*)
using *aff_dim_le_DIM* [*of S*] **by** *auto*
then obtain *C* **and** *S'* :: (*'a* $*$ *real*) *set*
where *convex C C* $\neq$ {} *closedin* (*top_of_set C*) *S' S homeomorphic S'*
using *homeomorphic_closedin_convex* **by** *blast*
with ‹*AR S*› **have** *contractible S*
by (*meson AR_def convex_imp_contractible homeomorphic_contractible_eq*
retract_of_contractible)
with ‹*AR S*› **show** *?rhs*
using *AR_imp_ANR AR_imp_retract* **by** *fastforce*
next
assume *?rhs*
then obtain *a* **and** *h*:: *real* $\times$ *'a* $\Rightarrow$ *'a*

```
      where conth: continuous_on ({0..1} × S) h
        and hS: h ` ({0..1} × S) ⊆ S
        and [simp]: ⋀x. h(0, x) = x
        and [simp]: ⋀x. h(1, x) = a
        and ANR S S ≠ {}
    by (auto simp: contractible_def homotopic_with_def)
  then have a ∈ S
   by (metis all_not_in_conv atLeastAtMost_iff image_subset_iff mem_Sigma_iff
order_refl zero_le_one)
  have ∃g. continuous_on W g ∧ g ∈ W → S ∧ (∀x∈T. g x = f x)
        if      f: continuous_on T f f ∈ T → S
          and WT: closedin (top_of_set W) T
        for W T and f :: 'a × real ⇒ 'a
  proof −
    obtain U g
      where T ⊆ U and WU: openin (top_of_set W) U
        and contg: continuous_on U g
        and g ∈ U → S and gf: ⋀x. x ∈ T ⟹ g x = f x
         using iffD1 [OF ANR_eq_absolute_neighbourhood_extensor ‹ANR S›,
rule_format, OF f WT]
      by auto
    have WWU: closedin (top_of_set W) (W − U)
      using WU closedin_diff by fastforce
    moreover have (W − U) ∩ T = {}
      using ‹T ⊆ U› by auto
    ultimately obtain V V'
      where WV': openin (top_of_set W) V'
        and WV: openin (top_of_set W) V
        and W − U ⊆ V' T ⊆ V V' ∩ V = {}
      using separation_normal_local [of W W−U T] WT by blast
    then have WVT: T ∩ (W − V) = {}
      by auto
    have WWV: closedin (top_of_set W) (W − V)
      using WV closedin_diff by fastforce
    obtain j :: 'a × real ⇒ real
      where contj: continuous_on W j
        and j: ⋀x. x ∈ W ⟹ j x ∈ {0..1}
        and j0: ⋀x. x ∈ W − V ⟹ j x = 1
        and j1: ⋀x. x ∈ T ⟹ j x = 0
       by (rule Urysohn_local [OF WT WWV WVT, of 0 1::real]) (auto simp:
in_segment)
    have Weq: W = (W − V) ∪ (W − V')
      using ‹V' ∩ V = {}› by force
    show ?thesis
    proof (intro conjI exI)
      have ∗: continuous_on (W − V') (λx. h (j x, g x))
      proof (rule continuous_on_compose2 [OF conth continuous_on_Pair])
        show continuous_on (W − V') j
          by (rule continuous_on_subset [OF contj Diff_subset])
```

```
      show continuous_on (W − V′) g
        by (metis Diff_subset_conv ‹W − U ⊆ V′› contg continuous_on_subset
Un_commute)
      show (λx. (j x, g x)) ‘ (W − V′) ⊆ {0..1} × S
        using j ‹g ∈ U → S› ‹W − U ⊆ V′› by fastforce
    qed
    show continuous_on W (λx. if x ∈ W − V then a else h (j x, g x))
    proof (subst Weq, rule continuous_on_cases_local)
      show continuous_on (W − V′) (λx. h (j x, g x))
        using ∗ by blast
    qed (use WWV WV′ Weq j0 j1 in auto)
  next
    have h (j (x, y), g (x, y)) ∈ S if (x, y) ∈ W (x, y) ∈ V for x y
    proof −
      have j(x, y) ∈ {0..1}
        using j that by blast
      moreover have g(x, y) ∈ S
        using ‹V′ ∩ V = {}› ‹W − U ⊆ V′› ‹g ∈ U → S› that by fastforce
      ultimately show ?thesis
        using hS by blast
    qed
    with ‹a ∈ S› ‹g ∈ U → S›
    show (λx. if x ∈ W − V then a else h (j x, g x)) ∈ W → S
      by auto
  next
    show ∀ x∈T. (if x ∈ W − V then a else h (j x, g x)) = f x
      using ‹T ⊆ V› by (auto simp: j0 j1 gf)
  qed
 qed
 then show ?lhs
  by (simp add: AR_eq_absolute_extensor image_subset_iff_funcset)
qed


lemma ANR_retract_of_ANR:
 fixes S :: 'a::euclidean_space set
 assumes ANR T and ST: S retract_of T
 shows ANR S
proof (clarsimp simp add: ANR_eq_absolute_neighbourhood_extensor)
 fix f::'a × real ⇒ 'a and U W
 assume W: continuous_on W f f ∈ W → S closedin (top_of_set U) W
 then obtain r where S ⊆ T and r: continuous_on T r r ∈ T → S ∀ x∈S. r
x = x continuous_on W f f ∈ W → S
                                closedin (top_of_set U) W
   by (metis ST retract_of_def retraction_def)
 then have f ‘ W ⊆ T
   by blast
 with W obtain V g where V: W ⊆ V openin (top_of_set U) V continuous_on
V g g ∈ V → T ∀ x∈W. g x = f x
```

by (*smt* (*verit*) *ANR_imp_absolute_neighbourhood_extensor Pi_I assms*(*1*) *funcset_mem image_subset_iff_funcset*)
with *r* **have** *continuous_on V* (*r* ∘ *g*) ∧ (*r* ∘ *g*) ∈ *V* → *S* ∧ (∀ *x*∈*W*. (*r* ∘ *g*) *x* = *f x*)
by (*smt* (*verit*, *del_insts*) *Pi_iff comp_apply continuous_on_compose continuous_on_subset image_subset_iff_funcset*)
then show ∃ *V*. *W* ⊆ *V* ∧ *openin* (*top_of_set U*) *V* ∧ (∃ *g*. *continuous_on V g* ∧ *g* ∈ *V* → *S* ∧ (∀ *x*∈*W*. *g x* = *f x*))
by (*meson V*)
qed

lemma *AR_retract_of_AR*:
fixes *S* :: *'a*::*euclidean_space set*
shows ⟦*AR T*; *S retract_of T*⟧ ⟹ *AR S*
using *ANR_retract_of_ANR AR_ANR retract_of_contractible* **by** *fastforce*

lemma *ENR_retract_of_ENR*:
⟦*ENR T*; *S retract_of T*⟧ ⟹ *ENR S*
by (*meson ENR_def retract_of_trans*)

lemma *retract_of_UNIV*:
fixes *S* :: *'a*::*euclidean_space set*
shows *S retract_of UNIV* ⟷ *AR S* ∧ *closed S*
by (*metis AR_ANR AR_imp_retract ENR_def ENR_imp_ANR closed_UNIV closed_closedin contractible_UNIV empty_not_UNIV open_UNIV retract_of_closed retract_of_contractible retract_of_empty*(*1*) *subtopology_UNIV*)

lemma *compact_AR*:
fixes *S* :: *'a*::*euclidean_space set*
shows *compact S* ∧ *AR S* ⟷ *compact S* ∧ *S retract_of UNIV*
using *compact_imp_closed retract_of_UNIV* **by** *blast*

More properties of ARs, ANRs and ENRs

lemma *not_AR_empty* [*simp*]: ¬ *AR*({})
by (*auto simp*: *AR_def*)

lemma *ENR_empty* [*simp*]: *ENR* {}
by (*simp add*: *ENR_def*)

lemma *ANR_empty* [*simp*]: *ANR* ({} :: *'a*::*euclidean_space set*)
by (*simp add*: *ENR_imp_ANR*)

lemma *convex_imp_AR*:
fixes *S* :: *'a*::*euclidean_space set*
shows ⟦*convex S*; *S* ≠ {}⟧ ⟹ *AR S*
by (*metis* (*mono_tags*, *lifting*) *Dugundji absolute_extensor_imp_AR*)

lemma *convex_imp_ANR*:
fixes *S* :: *'a*::*euclidean_space set*

shows *convex S* $\Longrightarrow$ *ANR S*
using *ANR_empty AR_imp_ANR convex_imp_AR* **by** *blast*

lemma *ENR_convex_closed*:
fixes *S* :: *'a*::*euclidean_space set*
shows ⟦*closed S*; *convex S*⟧ $\Longrightarrow$ *ENR S*
using *ENR_def ENR_empty convex_imp_AR retract_of_UNIV* **by** *blast*

lemma *AR_UNIV* [*simp*]: *AR* (*UNIV* :: *'a*::*euclidean_space set*)
using *retract_of_UNIV* **by** *auto*

lemma *ANR_UNIV* [*simp*]: *ANR* (*UNIV* :: *'a*::*euclidean_space set*)
by (*simp add*: *AR_imp_ANR*)

lemma *ENR_UNIV* [*simp*]:*ENR UNIV*
using *ENR_def* **by** *blast*

lemma *AR_singleton*:
fixes *a* :: *'a*::*euclidean_space*
shows *AR* {*a*}
using *retract_of_UNIV* **by** *blast*

lemma *ANR_singleton*:
fixes *a* :: *'a*::*euclidean_space*
shows *ANR* {*a*}
by (*simp add*: *AR_imp_ANR AR_singleton*)

lemma *ENR_singleton*: *ENR* {*a*}
using *ENR_def* **by** *blast*

ARs closed under union

lemma *AR_closed_Un_local_aux*:
fixes *U* :: *'a*::*euclidean_space set*
assumes *closedin* (*top_of_set U*) *S*
closedin (*top_of_set U*) *T*
AR S AR T AR(*S* ∩ *T*)
shows (*S* ∪ *T*) *retract_of U*
proof −
have *S* ∩ *T* ≠ {}
using *assms AR_def* **by** *fastforce*
have *S* ⊆ *U T* ⊆ *U*
using *assms* **by** (*auto simp*: *closedin_imp_subset*)
define *S'* **where** *S'* ≡ {*x* ∈ *U*. *setdist* {*x*} *S* ≤ *setdist* {*x*} *T*}
define *T'* **where** *T'* ≡ {*x* ∈ *U*. *setdist* {*x*} *T* ≤ *setdist* {*x*} *S*}
define *W* **where** *W* ≡ {*x* ∈ *U*. *setdist* {*x*} *S* = *setdist* {*x*} *T*}
have *US'*: *closedin* (*top_of_set U*) *S'*
using *continuous_closedin_preimage* [*of U* λ*x*. *setdist* {*x*} *S* − *setdist* {*x*} *T* {..*0*}]
by (*simp add*: *S'_def vimage_def Collect_conj_eq continuous_on_diff contin-*

uous_on_setdist)
have *UT'*: *closedin* (*top_of_set U*) *T'*
using *continuous_closedin_preimage* [*of U* λ*x*. *setdist* {*x*} *T* − *setdist* {*x*} *S* {*..0*}]
by (*simp add*: *T'_def vimage_def Collect_conj_eq continuous_on_diff continuous_on_setdist*)
have *S* ⊆ *S'*
using *S'_def* ‹*S* ⊆ *U*› *setdist_sing_in_set* **by** *fastforce*
have *T* ⊆ *T'*
using *T'_def* ‹*T* ⊆ *U*› *setdist_sing_in_set* **by** *fastforce*
have *S* ∩ *T* ⊆ *W W* ⊆ *U*
using ‹*S* ⊆ *U*› **by** (*auto simp*: *W_def setdist_sing_in_set*)
have (*S* ∩ *T*) *retract_of W*
proof (*rule AR_imp_absolute_retract* [*OF* ‹*AR*(*S* ∩ *T*)›])
show *S* ∩ *T homeomorphic S* ∩ *T*
by (*simp add*: *homeomorphic_refl*)
show *closedin* (*top_of_set W*) (*S* ∩ *T*)
by (*meson* ‹*S* ∩ *T* ⊆ *W*› ‹*W* ⊆ *U*› *assms closedin_Int closedin_subset_trans*)
qed
then obtain *r0*
where *S* ∩ *T* ⊆ *W* **and** *contr0*: *continuous_on W r0*
and *r0* ‘ *W* ⊆ *S* ∩ *T*
and *r0* [*simp*]: ⋀*x*. *x* ∈ *S* ∩ *T* ⟹ *r0 x* = *x*
by (*auto simp*: *retract_of_def retraction_def*)
have *ST*: *x* ∈ *W* ⟹ *x* ∈ *S* ⟷ *x* ∈ *T* **for** *x*
using *setdist_eq_0_closedin* ‹*S* ∩ *T* ≠ {}› *assms*
by (*force simp*: *W_def setdist_sing_in_set*)
have *S'* ∩ *T'* = *W*
by (*auto simp*: *S'_def T'_def W_def*)
then have *cloUW*: *closedin* (*top_of_set U*) *W*
using *closedin_Int US' UT'* **by** *blast*
define *r* **where** *r* ≡ λ*x*. *if x* ∈ *W then r0 x else x*
have *contr*: *continuous_on* (*W* ∪ (*S* ∪ *T*)) *r*
unfolding *r_def*
proof (*rule continuous_on_cases_local* [*OF _ _ contr0 continuous_on_id*])
show *closedin* (*top_of_set* (*W* ∪ (*S* ∪ *T*))) *W*
using ‹*S* ⊆ *U*› ‹*T* ⊆ *U*› ‹*W* ⊆ *U*› ‹*closedin* (*top_of_set U*) *W*› *closedin_subset_trans* **by** *fastforce*
show *closedin* (*top_of_set* (*W* ∪ (*S* ∪ *T*))) (*S* ∪ *T*)
by (*meson* ‹*S* ⊆ *U*› ‹*T* ⊆ *U*› ‹*W* ⊆ *U*› *assms closedin_Un closedin_subset_trans sup.bounded_iff sup.cobounded2*)
show ⋀*x*. *x* ∈ *W* ∧ *x* ∉ *W* ∨ *x* ∈ *S* ∪ *T* ∧ *x* ∈ *W* ⟹ *r0 x* = *x*
by (*auto simp*: *ST*)
qed
have *rim*: *r* ‘ (*W* ∪ *S*) ⊆ *S r* ‘ (*W* ∪ *T*) ⊆ *T*
using ‹*r0* ‘ *W* ⊆ *S* ∩ *T*› *r_def* **by** *auto*
have *cloUWS*: *closedin* (*top_of_set U*) (*W* ∪ *S*)
by (*simp add*: *cloUW assms closedin_Un*)
obtain *g* **where** *contg*: *continuous_on U g*

```
    and g ‘ U ⊆ S and geqr: ⋀x. x ∈ W ∪ S ⟹ g x = r x
  proof (rule AR_imp_absolute_extensor [OF ‹AR S› _ _ cloUWS])
    show continuous_on (W ∪ S) r
      using continuous_on_subset contr sup_assoc by blast
  qed (use rim in auto)
  have cloUWT: closedin (top_of_set U) (W ∪ T)
    by (simp add: cloUW assms closedin_Un)
  obtain h where conth: continuous_on U h
            and h ‘ U ⊆ T and heqr: ⋀x. x ∈ W ∪ T ⟹ h x = r x
  proof (rule AR_imp_absolute_extensor [OF ‹AR T› _ _ cloUWT])
    show continuous_on (W ∪ T) r
      using continuous_on_subset contr sup_assoc by blast
  qed (use rim in auto)
  have U: U = S' ∪ T'
    by (force simp: S'_def T'_def)
  have cont: continuous_on U (λx. if x ∈ S' then g x else h x)
    unfolding U
    apply (rule continuous_on_cases_local)
    using US' UT' ‹S' ∩ T' = W› ‹U = S' ∪ T'›
          contg conth continuous_on_subset geqr heqr by auto
  have UST: (λx. if x ∈ S' then g x else h x) ‘ U ⊆ S ∪ T
    using ‹g ‘ U ⊆ S› ‹h ‘ U ⊆ T› by auto
  show ?thesis
    apply (simp add: retract_of_def retraction_def ‹S ⊆ U› ‹T ⊆ U›)
    apply (rule_tac x=λx. if x ∈ S' then g x else h x in exI)
    using ST UST ‹S ⊆ S'› ‹S' ∩ T' = W› ‹T ⊆ T'› cont geqr heqr r_def
    by (smt (verit, del_insts) IntI Pi_I Un_iff image_subset_iff r0 subsetD)
qed


lemma AR_closed_Un_local:
  fixes S :: 'a::euclidean_space set
  assumes STS: closedin (top_of_set (S ∪ T)) S
      and STT: closedin (top_of_set (S ∪ T)) T
      and AR S AR T AR(S ∩ T)
    shows AR(S ∪ T)
proof −
  have C retract_of U
       if hom: S ∪ T homeomorphic C and UC: closedin (top_of_set U) C
       for U and C :: ('a * real) set
  proof −
    obtain f g where hom: homeomorphism (S ∪ T) C f g
      using hom by (force simp: homeomorphic_def)
    have US: closedin (top_of_set U) (C ∩ g −‘ S)
    by (metis STS continuous_on_imp_closedin hom homeomorphism_def closedin_trans
[OF _ UC])
    have UT: closedin (top_of_set U) (C ∩ g −‘ T)
    by (metis STT continuous_on_closed hom homeomorphism_def closedin_trans
[OF _ UC])
```

```
    have homeomorphism (C ∩ g -‘ S) S g f
      using hom
      apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      apply (rule_tac x=f x in image_eqI, auto)
      done
    then have ARS: AR (C ∩ g -‘ S)
      using ‹AR S› homeomorphic_AR_iff_AR homeomorphic_def by blast
    have homeomorphism (C ∩ g -‘ T) T g f
      using hom
      apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      apply (rule_tac x=f x in image_eqI, auto)
      done
    then have ART: AR (C ∩ g -‘ T)
      using ‹AR T› homeomorphic_AR_iff_AR homeomorphic_def by blast
    have homeomorphism (C ∩ g -‘ S ∩ (C ∩ g -‘ T)) (S ∩ T) g f
      using hom
      apply (auto simp: homeomorphism_def elim!: continuous_on_subset)
      apply (rule_tac x=f x in image_eqI, auto)
      done
    then have ARI: AR ((C ∩ g -‘ S) ∩ (C ∩ g -‘ T))
     using ‹AR (S ∩ T)› homeomorphic_AR_iff_AR homeomorphic_def by blast
    have C = (C ∩ g -‘ S) ∪ (C ∩ g -‘ T)
      using hom  by (auto simp: homeomorphism_def)
    then show ?thesis
      by (metis AR_closed_Un_local_aux [OF US UT ARS ART ARI])
  qed
  then show ?thesis
    by (force simp: AR_def)
qed

corollary AR_closed_Un:
  fixes S :: 'a::euclidean_space set
  shows ⟦closed S; closed T; AR S; AR T; AR (S ∩ T)⟧ ⟹ AR (S ∪ T)
by (metis AR_closed_Un_local_aux closed_closedin retract_of_UNIV subtopology_UNIV)
```

ANRs closed under union

```
lemma ANR_closed_Un_local_aux:
  fixes U :: 'a::euclidean_space set
  assumes US: closedin (top_of_set U) S
      and UT: closedin (top_of_set U) T
      and ANR S ANR T ANR(S ∩ T)
  obtains V where openin (top_of_set U) V (S ∪ T) retract_of V
proof (cases S = {} ∨ T = {})
  case True with assms that show ?thesis
   by (metis ANR_imp_neighbourhood_retract Un_commute inf_bot_right sup_inf_absorb)
next
  case False
  then have [simp]: S ≠ {} T ≠ {} by auto
```

have $S \subseteq U$ $T \subseteq U$
using *assms* **by** (*auto simp*: *closedin_imp_subset*)
define S' **where** $S' \equiv \{x \in U.\ setdist\ \{x\}\ S \leq setdist\ \{x\}\ T\}$
define T' **where** $T' \equiv \{x \in U.\ setdist\ \{x\}\ T \leq setdist\ \{x\}\ S\}$
define W **where** $W \equiv \{x \in U.\ setdist\ \{x\}\ S = setdist\ \{x\}\ T\}$
have *cloUS'*: *closedin* (*top_of_set U*) S'
using *continuous_closedin_preimage* [*of U* $\lambda x.\ setdist\ \{x\}\ S - setdist\ \{x\}\ T$ *{..0}*]
by (*simp add*: S'*_def vimage_def Collect_conj_eq continuous_on_diff continuous_on_setdist*)
have *cloUT'*: *closedin* (*top_of_set U*) T'
using *continuous_closedin_preimage* [*of U* $\lambda x.\ setdist\ \{x\}\ T - setdist\ \{x\}\ S$ *{..0}*]
by (*simp add*: T'*_def vimage_def Collect_conj_eq continuous_on_diff continuous_on_setdist*)
have $S \subseteq S'$
using S'*_def* ‹$S \subseteq U$› *setdist_sing_in_set* **by** *fastforce*
have $T \subseteq T'$
using T'*_def* ‹$T \subseteq U$› *setdist_sing_in_set* **by** *fastforce*
have $S' \cup T' = U$
by (*auto simp*: S'*_def* T'*_def*)
have $W \subseteq S'$
by (*simp add*: *Collect_mono* S'*_def W_def*)
have $W \subseteq T'$
by (*simp add*: *Collect_mono* T'*_def W_def*)
have *ST_W*: $S \cap T \subseteq W$ **and** $W \subseteq U$
using ‹$S \subseteq U$› **by** (*force simp*: *W_def setdist_sing_in_set*)+
have $S' \cap T' = W$
by (*auto simp*: S'*_def* T'*_def W_def*)
then have *cloUW*: *closedin* (*top_of_set U*) W
using *closedin_Int cloUS' cloUT'* **by** *blast*
obtain W' *W0* **where** *openin* (*top_of_set W*) W'
and *cloWW0*: *closedin* (*top_of_set W*) *W0*
and $S \cap T \subseteq W'$ $W' \subseteq W0$
and *ret*: $(S \cap T)$ *retract_of W0*
by (*meson ANR_imp_closed_neighbourhood_retract ST_W US UT* ‹$W \subseteq U$› ‹$ANR(S \cap T)$› *closedin_Int closedin_subset_trans*)
then obtain *U0* **where** *openUU0*: *openin* (*top_of_set U*) *U0*
and *U0*: $S \cap T \subseteq U0$ $U0 \cap W \subseteq W0$
unfolding *openin_open* **using** ‹$W \subseteq U$› **by** *blast*
have $W0 \subseteq U$
using ‹$W \subseteq U$› *cloWW0 closedin_subset* **by** *fastforce*
obtain *r0*
where $S \cap T \subseteq W0$ **and** *contr0*: *continuous_on W0 r0* **and** $r0 \in W0 \to S \cap T$
and *r0* [*simp*]: $\bigwedge x.\ x \in S \cap T \Longrightarrow r0\ x = x$
using *ret* **by** (*force simp*: *retract_of_def retraction_def*)
have *ST*: $x \in W \Longrightarrow x \in S \longleftrightarrow x \in T$ **for** x
using *assms* **by** (*auto simp*: *W_def setdist_sing_in_set dest*!: *setdist_eq_0_closedin*)

define r **where** $r \equiv \lambda x.\ \mathit{if}\ x \in W0\ \mathit{then}\ r0\ x\ \mathit{else}\ x$
have $r\ `\ (W0 \cup S) \subseteq S\ r\ `\ (W0 \cup T) \subseteq T$
using $\langle r0 \in W0 \to S \cap T\rangle$ r_def **by** *auto*
have *contr*: $\mathit{continuous\_on}\ (W0 \cup (S \cup T))\ r$
unfolding r_def
proof (*rule continuous_on_cases_local* [*OF* _ _ *contr0 continuous_on_id*])
show $\mathit{closedin}\ (\mathit{top\_of\_set}\ (W0 \cup (S \cup T)))\ W0$
using *closedin_subset_trans* [*of U*]
by (*metis le_sup_iff order_refl cloWW0 cloUW closedin_trans* $\langle W0 \subseteq U\rangle$ $\langle S \subseteq U\rangle$ $\langle T \subseteq U\rangle$)
show $\mathit{closedin}\ (\mathit{top\_of\_set}\ (W0 \cup (S \cup T)))\ (S \cup T)$
by (*meson* $\langle S \subseteq U\rangle$ $\langle T \subseteq U\rangle$ $\langle W0 \subseteq U\rangle$ *assms closedin_Un closedin_subset_trans sup.bounded_iff sup.cobounded2*)
show $\bigwedge x.\ x \in W0 \wedge x \notin W0 \vee x \in S \cup T \wedge x \in W0 \Longrightarrow r0\ x = x$
using *ST cloWW0 closedin_subset* **by** *fastforce*
qed
have *cloS'WS*: $\mathit{closedin}\ (\mathit{top\_of\_set}\ S')\ (W0 \cup S)$
by (*meson closedin_subset_trans US cloUS'* $\langle S \subseteq S'\rangle$ $\langle W \subseteq S'\rangle$ *cloUW cloWW0*
closedin_Un closedin_imp_subset closedin_trans)
obtain $W1\ g$ **where** $W0 \cup S \subseteq W1$ **and** *contg*: $\mathit{continuous\_on}\ W1\ g$
and *opeSW1*: $\mathit{openin}\ (\mathit{top\_of\_set}\ S')\ W1$
and $g \in W1 \to S$ **and** *geqr*: $\bigwedge x.\ x \in W0 \cup S \Longrightarrow g\ x = r\ x$
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* $\langle \mathit{ANR}\ S\rangle$ _ _ *cloS'WS*])
show $\mathit{continuous\_on}\ (W0 \cup S)\ r$
using *continuous_on_subset contr sup_assoc* **by** *blast*
qed (*use* $\langle r\ `\ (W0 \cup S) \subseteq S\rangle$ **in** *auto*)
have *cloT'WT*: $\mathit{closedin}\ (\mathit{top\_of\_set}\ T')\ (W0 \cup T)$
by (*meson closedin_subset_trans UT cloUT'* $\langle T \subseteq T'\rangle$ $\langle W \subseteq T'\rangle$ *cloUW cloWW0*
closedin_Un closedin_imp_subset closedin_trans)
obtain $W2\ h$ **where** $W0 \cup T \subseteq W2$ **and** *conth*: $\mathit{continuous\_on}\ W2\ h$
and *opeSW2*: $\mathit{openin}\ (\mathit{top\_of\_set}\ T')\ W2$
and $h\ `\ W2 \subseteq T$ **and** *heqr*: $\bigwedge x.\ x \in W0 \cup T \Longrightarrow h\ x = r\ x$
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* $\langle \mathit{ANR}\ T\rangle$ _ _ *cloT'WT*])
show $\mathit{continuous\_on}\ (W0 \cup T)\ r$
using *continuous_on_subset contr sup_assoc* **by** *blast*
qed (*use* $\langle r\ `\ (W0 \cup T) \subseteq T\rangle$ **in** *auto*)
have $S' \cap T' = W$
by (*force simp*: $S'\_def\ T'\_def\ W\_def$)
obtain $O1\ O2$ **where** $O12$: $\mathit{open}\ O1\ W1 = S' \cap O1\ \mathit{open}\ O2\ W2 = T' \cap O2$
using *opeSW1 opeSW2* **by** (*force simp*: *openin_open*)
show *?thesis*
proof
have *eq*: $W1 - (W - U0) \cup (W2 - (W - U0))$
$= ((U - T') \cap O1 \cup (U - S') \cap O2 \cup U \cap O1 \cap O2) - (W - U0)$
(**is** *?WW1* $\cup$ *?WW2* = *?rhs*)

using ‹*U0* ∩ *W* ⊆ *W0*› ‹*W0* ∪ *S* ⊆ *W1*› ‹*W0* ∪ *T* ⊆ *W2*›
by (*auto simp*: ‹*S'* ∪ *T'* = *U*› [*symmetric*] ‹*S'* ∩ *T'* = *W*› [*symmetric*] ‹*W1* = *S'* ∩ *O1*› ‹*W2* = *T'* ∩ *O2*›)
show *openin* (*top_of_set U*) (*?WW1* ∪ *?WW2*)
by (*simp add*: *eq* ‹*open O1*› ‹*open O2*› *cloUS' cloUT' cloUW closedin_diff opeUU0 openin_Int_open openin_Un openin_diff*)
obtain *SU'* **where** *closed SU' S'* = *U* ∩ *SU'*
using *cloUS'* **by** (*auto simp add*: *closedin_closed*)
moreover have *?WW1* = (*?WW1* ∪ *?WW2*) ∩ *SU'*
using ‹*S'* = *U* ∩ *SU'*› ‹*W1* = *S'* ∩ *O1*› ‹*S'* ∪ *T'* = *U*› ‹*W2* = *T'* ∩ *O2*› ‹*S'* ∩ *T'* = *W*› ‹*W0* ∪ *S* ⊆ *W1*› *U0*
by *auto*
ultimately have *cloW1*: *closedin* (*top_of_set* (*W1* − (*W* − *U0*) ∪ (*W2* − (*W* − *U0*)))) (*W1* − (*W* − *U0*))
by (*metis closedin_closed_Int*)
obtain *TU'* **where** *closed TU' T'* = *U* ∩ *TU'*
using *cloUT'* **by** (*auto simp add*: *closedin_closed*)
moreover have *?WW2* = (*?WW1* ∪ *?WW2*) ∩ *TU'*
using ‹*T'* = *U* ∩ *TU'*› ‹*W1* = *S'* ∩ *O1*› ‹*S'* ∪ *T'* = *U*› ‹*W2* = *T'* ∩ *O2*› ‹*S'* ∩ *T'* = *W*› ‹*W0* ∪ *T* ⊆ *W2*› *U0*
by *auto*
ultimately have *cloW2*: *closedin* (*top_of_set* (*?WW1* ∪ *?WW2*)) *?WW2*
by (*metis closedin_closed_Int*)
let *?gh* = λ*x*. *if x* ∈ *S'* *then g x else h x*
have ∃*r*. *continuous_on* (*?WW1* ∪ *?WW2*) *r* ∧ *r* ‘ (*?WW1* ∪ *?WW2*) ⊆ *S* ∪ *T* ∧ (∀*x*∈*S* ∪ *T*. *r x* = *x*)
proof (*intro exI conjI*)
show ∀*x*∈*S* ∪ *T*. *?gh x* = *x*
using *ST* ‹*S'* ∩ *T'* = *W*› *geqr heqr O12*
by (*metis Int_iff Un_iff* ‹*W0* ∪ *S* ⊆ *W1*› ‹*W0* ∪ *T* ⊆ *W2*› *r0 r_def sup.order_iff*)
have ⋀*x*. *x* ∈ *?WW1* ∧ *x* ∉ *S'* ∨ *x* ∈ *?WW2* ∧ *x* ∈ *S'* ⟹ *g x* = *h x*
using *O12*
by (*metis* (*full_types*) *DiffD1 DiffD2 DiffI IntE IntI U0*(*2*) *UnCI* ‹*S'* ∩ *T'* = *W*› *geqr heqr in_mono*)
then show *continuous_on* (*?WW1* ∪ *?WW2*) *?gh*
using *continuous_on_cases_local* [*OF cloW1 cloW2 continuous_on_subset* [*OF contg*] *continuous_on_subset* [*OF conth*]]
by *simp*
show *?gh* ‘ (*?WW1* ∪ *?WW2*) ⊆ *S* ∪ *T*
using ‹*W1* = *S'* ∩ *O1*› ‹*W2* = *T'* ∩ *O2*› ‹*S'* ∩ *T'* = *W*› ‹*g* ∈ *W1* → *S*› ‹*h* ‘ *W2* ⊆ *T*› ‹*U0* ∩ *W* ⊆ *W0*› ‹*W0* ∪ *S* ⊆ *W1*›
by (*auto simp add*: *image_subset_iff*)
qed
then show *S* ∪ *T retract_of ?WW1* ∪ *?WW2*
using ‹*W0* ∪ *S* ⊆ *W1*› ‹*W0* ∪ *T* ⊆ *W2*› *ST opeUU0 U0*
by (*auto simp*: *retract_of_def retraction_def image_subset_iff_funcset*)
qed
qed

lemma *ANR_closed_Un_local*:
fixes S :: *'a*::*euclidean_space set*
assumes *STS*: *closedin* (*top_of_set* ($S \cup T$)) S
and *STT*: *closedin* (*top_of_set* ($S \cup T$)) T
and *ANR S ANR T ANR*($S \cap T$)
shows *ANR*($S \cup T$)
proof −
have $\exists T$. *openin* (*top_of_set U*) $T \wedge C$ *retract_of* T
if *hom*: $S \cup T$ *homeomorphic* C **and** *UC*: *closedin* (*top_of_set U*) C
for U **and** C :: (*'a* ∗ *real*) *set*
proof −
obtain f g **where** *hom*: *homeomorphism* ($S \cup T$) C f g
using *hom* **by** (*force simp*: *homeomorphic_def*)
have *US*: *closedin* (*top_of_set U*) ($C \cap g -` S$)
by (*metis STS UC closedin_trans continuous_on_imp_closedin hom homeomorphism_def*)
have *UT*: *closedin* (*top_of_set U*) ($C \cap g -` T$)
by (*metis STT UC closedin_trans continuous_on_imp_closedin hom homeomorphism_def*)
have *homeomorphism* ($C \cap g -` S$) S g f
using *hom*
apply (*auto simp*: *homeomorphism_def elim*!: *continuous_on_subset*)
by (*rule_tac x*=*f x* **in** *image_eqI*, *auto*)
then have *ANRS*: *ANR* ($C \cap g -` S$)
using ‹*ANR S*› *homeomorphic_ANR_iff_ANR homeomorphic_def* **by** *blast*
have *homeomorphism* ($C \cap g -` T$) T g f
using *hom* **apply** (*auto simp*: *homeomorphism_def elim*!: *continuous_on_subset*)
by (*rule_tac x*=*f x* **in** *image_eqI*, *auto*)
then have *ANRT*: *ANR* ($C \cap g -` T$)
using ‹*ANR T*› *homeomorphic_ANR_iff_ANR homeomorphic_def* **by** *blast*
have *homeomorphism* ($C \cap g -` S \cap (C \cap g -` T)$) ($S \cap T$) g f
using *hom*
apply (*auto simp*: *homeomorphism_def elim*!: *continuous_on_subset*)
by (*rule_tac x*=*f x* **in** *image_eqI*, *auto*)
then have *ANRI*: *ANR* (($C \cap g -` S$) ∩ ($C \cap g -` T$))
using ‹*ANR* ($S \cap T$)› *homeomorphic_ANR_iff_ANR homeomorphic_def*
by *blast*
have $C = (C \cap g -` S) \cup (C \cap g -` T)$
using *hom* **by** (*auto simp*: *homeomorphism_def*)
then show *?thesis*
by (*metis ANR_closed_Un_local_aux* [*OF US UT ANRS ANRT ANRI*])
qed
then show *?thesis*
by (*auto simp*: *ANR_def*)
qed

corollary *ANR_closed_Un*:

fixes $S :: 'a::euclidean\_space\ set$
shows $[\![closed\ S;\ closed\ T;\ ANR\ S;\ ANR\ T;\ ANR\ (S \cap T)]\!] \Longrightarrow ANR\ (S \cup T)$
by (*simp add*: *ANR_closed_Un_local closedin_def diff_eq open_Compl openin_open_Int*)

lemma *ANR_openin*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $ANR\ T$ **and** *opeTS*: $openin\ (top\_of\_set\ T)\ S$
shows $ANR\ S$
proof (*clarsimp simp only*: *ANR_eq_absolute_neighbourhood_extensor*)
fix $f :: 'a \times real \Rightarrow 'a$ **and** $U\ C$
assume *contf*: $continuous\_on\ C\ f$ **and** *fim*: $f \in C \to S$
and *cloUC*: $closedin\ (top\_of\_set\ U)\ C$
have $f \in C \to T$
using *fim opeTS openin_imp_subset* **by** *blast*
obtain $W\ g$ **where** $C \subseteq W$
and *UW*: $openin\ (top\_of\_set\ U)\ W$
and *contg*: $continuous\_on\ W\ g$
and *gim*: $g \in W \to T$
and *geq*: $\bigwedge x.\ x \in C \Longrightarrow g\ x = f\ x$
using *ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR T*› *contf* ‹$f \in C \to T$› *cloUC*] *fim* **by** *auto*
show $\exists V\ g.\ C \subseteq V \wedge openin\ (top\_of\_set\ U)\ V \wedge continuous\_on\ V\ g \wedge g \in V \to S \wedge (\forall x{\in}C.\ g\ x = f\ x)$
proof (*intro exI conjI*)
show $C \subseteq W \cap g -` S$
using ‹$C \subseteq W$› *fim geq* **by** *blast*
show $openin\ (top\_of\_set\ U)\ (W \cap g -` S)$
by (*metis* (*mono_tags*, *lifting*) *UW contg continuous_openin_preimage gim opeTS openin_trans*)
show $continuous\_on\ (W \cap g -` S)\ g$
by (*blast intro*: *continuous_on_subset* [*OF contg*])
show $g \in (W \cap g -` S) \to S$
using *gim* **by** *blast*
show $\forall x{\in}C.\ g\ x = f\ x$
using *geq* **by** *blast*
qed
qed

lemma *ENR_openin*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $ENR\ T\ openin\ (top\_of\_set\ T)\ S$
shows $ENR\ S$
by (*meson ANR_openin ENR_ANR assms locally_open_subset*)

lemma *ANR_neighborhood_retract*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $ANR\ U\ S\ retract\_of\ T\ openin\ (top\_of\_set\ U)\ T$
shows $ANR\ S$
using *ANR_openin ANR_retract_of_ANR assms* **by** *blast*

lemma *ENR_neighborhood_retract*:
  fixes *S* :: *'a*::*euclidean_space set*
  assumes *ENR U S retract_of T openin* (*top_of_set U*) *T*
  shows *ENR S*
 using *ENR_openin ENR_retract_of_ENR assms* **by** *blast*

lemma *ANR_rel_interior*:
 fixes *S* :: *'a*::*euclidean_space set*
 shows *ANR S* ⟹ *ANR*(*rel_interior S*)
  by (*blast intro*: *ANR_openin openin_set_rel_interior*)

lemma *ANR_delete*:
 fixes *S* :: *'a*::*euclidean_space set*
 shows *ANR S* ⟹ *ANR*(*S* − {*a*})
  by (*blast intro*: *ANR_openin openin_delete openin_subtopology_self*)

lemma *ENR_rel_interior*:
 fixes *S* :: *'a*::*euclidean_space set*
 shows *ENR S* ⟹ *ENR*(*rel_interior S*)
  by (*blast intro*: *ENR_openin openin_set_rel_interior*)

lemma *ENR_delete*:
 fixes *S* :: *'a*::*euclidean_space set*
 shows *ENR S* ⟹ *ENR*(*S* − {*a*})
  by (*blast intro*: *ENR_openin openin_delete openin_subtopology_self*)

lemma *open_imp_ENR*: *open S* ⟹ *ENR S*
  using *ENR_def* **by** *blast*

lemma *open_imp_ANR*:
  fixes *S* :: *'a*::*euclidean_space set*
  shows *open S* ⟹ *ANR S*
 by (*simp add*: *ENR_imp_ANR open_imp_ENR*)

lemma *ANR_ball* [*iff*]:
  fixes *a* :: *'a*::*euclidean_space*
  shows *ANR*(*ball a r*)
 by (*simp add*: *convex_imp_ANR*)

lemma *ENR_ball* [*iff*]: *ENR*(*ball a r*)
 by (*simp add*: *open_imp_ENR*)

lemma *AR_ball* [*simp*]:
  fixes *a* :: *'a*::*euclidean_space*
  shows *AR*(*ball a r*) ⟷ *0* < *r*
 by (*auto simp*: *AR_ANR convex_imp_contractible*)

lemma *ANR_cball* [*iff*]:

fixes $a :: 'a::euclidean\_space$
shows $ANR(cball\ a\ r)$
by (*simp add*: *convex_imp_ANR*)

lemma *ENR_cball*:
fixes $a :: 'a::euclidean\_space$
shows $ENR(cball\ a\ r)$
using *ENR_convex_closed* **by** *blast*

lemma *AR_cball* [*simp*]:
fixes $a :: 'a::euclidean\_space$
shows $AR(cball\ a\ r) \longleftrightarrow 0 \le r$
by (*auto simp*: *AR_ANR convex_imp_contractible*)

lemma *ANR_box* [*iff*]:
fixes $a :: 'a::euclidean\_space$
shows $ANR(cbox\ a\ b)\ ANR(box\ a\ b)$
by (*auto simp*: *convex_imp_ANR open_imp_ANR*)

lemma *ENR_box* [*iff*]:
fixes $a :: 'a::euclidean\_space$
shows $ENR(cbox\ a\ b)\ ENR(box\ a\ b)$
by (*simp_all add*: *ENR_convex_closed closed_cbox open_box open_imp_ENR*)

lemma *AR_box* [*simp*]:
$AR(cbox\ a\ b) \longleftrightarrow cbox\ a\ b \neq \{\}\ AR(box\ a\ b) \longleftrightarrow box\ a\ b \neq \{\}$
by (*auto simp*: *AR_ANR convex_imp_contractible*)

lemma *ANR_interior*:
fixes $S :: 'a::euclidean\_space\ set$
shows $ANR(interior\ S)$
by (*simp add*: *open_imp_ANR*)

lemma *ENR_interior*:
fixes $S :: 'a::euclidean\_space\ set$
shows $ENR(interior\ S)$
by (*simp add*: *open_imp_ENR*)

lemma *AR_imp_contractible*:
fixes $S :: 'a::euclidean\_space\ set$
shows $AR\ S \Longrightarrow contractible\ S$
by (*simp add*: *AR_ANR*)

lemma *ENR_imp_locally_compact*:
fixes $S :: 'a::euclidean\_space\ set$
shows $ENR\ S \Longrightarrow locally\ compact\ S$
by (*simp add*: *ENR_ANR*)

lemma *ANR_imp_locally_path_connected*:

```
  fixes S :: 'a::euclidean_space set
  assumes ANR S
    shows locally path_connected S
proof −
  obtain U and T :: ('a × real) set
     where convex U U ≠ {}
       and UT: closedin (top_of_set U) T and S homeomorphic T
  proof (rule homeomorphic_closedin_convex)
    show aff_dim S < int DIM('a × real)
      using aff_dim_le_DIM [of S] by auto
  qed auto
  then have locally path_connected T
    by (meson ANR_imp_absolute_neighbourhood_retract
      assms convex_imp_locally_path_connected locally_open_subset retract_of_locally_path_connected)
  then have S: locally path_connected S
      if openin (top_of_set U) V T retract_of V U ≠ {} for V
   using ‹S homeomorphic T› homeomorphic_locally homeomorphic_path_connectedness
by blast
  obtain Ta where (openin (top_of_set U) Ta ∧ T retract_of Ta)
    using ANR_def UT ‹S homeomorphic T› assms by atomize_elim (auto simp:
choice)
  then show ?thesis
    using S ‹U ≠ {}› by blast
qed

lemma ANR_imp_locally_connected:
  fixes S :: 'a::euclidean_space set
  assumes ANR S
    shows locally connected S
using locally_path_connected_imp_locally_connected ANR_imp_locally_path_connected
assms by auto

lemma AR_imp_locally_path_connected:
  fixes S :: 'a::euclidean_space set
  assumes AR S
    shows locally path_connected S
by (simp add: ANR_imp_locally_path_connected AR_imp_ANR assms)

lemma AR_imp_locally_connected:
  fixes S :: 'a::euclidean_space set
  assumes AR S
    shows locally connected S
using ANR_imp_locally_connected AR_ANR assms by blast

lemma ENR_imp_locally_path_connected:
  fixes S :: 'a::euclidean_space set
  assumes ENR S
    shows locally path_connected S
by (simp add: ANR_imp_locally_path_connected ENR_imp_ANR assms)
```

lemma *ENR_imp_locally_connected*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *ENR S*
shows *locally connected S*
using *ANR_imp_locally_connected ENR_ANR assms* **by** *blast*

lemma *ANR_Times*:
fixes $S :: 'a::euclidean\_space\ set$ **and** $T :: 'b::euclidean\_space\ set$
assumes *ANR S ANR T* **shows** $ANR(S \times T)$
proof (*clarsimp simp only*: *ANR_eq_absolute_neighbourhood_extensor*)
fix $f :: ('a \times 'b) \times real \Rightarrow 'a \times 'b$ **and** *U C*
assume *continuous_on C f* **and** *fim*: $f \in C \to S \times T$
and *cloUC*: *closedin* (*top_of_set U*) *C*
have *contf1*: *continuous_on C* (*fst* $\circ$ *f*)
by (*simp add*: ‹*continuous_on C f*› *continuous_on_fst*)
obtain *W1 g* **where** $C \subseteq W1$
and *UW1*: *openin* (*top_of_set U*) *W1*
and *contg*: *continuous_on W1 g*
and *gim*: $g\ `\ W1 \subseteq S$
and *geq*: $\bigwedge x.\ x \in C \Longrightarrow g\ x = (fst \circ f)\ x$
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR S*› *contf1* _ *cloUC*])
show $(fst \circ f) \in C \to S$
using *fim* **by** *force*
qed *auto*
have *contf2*: *continuous_on C* (*snd* $\circ$ *f*)
by (*simp add*: ‹*continuous_on C f*› *continuous_on_snd*)
obtain *W2 h* **where** $C \subseteq W2$
and *UW2*: *openin* (*top_of_set U*) *W2*
and *conth*: *continuous_on W2 h*
and *him*: $h \in W2 \to T$
and *heq*: $\bigwedge x.\ x \in C \Longrightarrow h\ x = (snd \circ f)\ x$
proof (*rule ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR T*› *contf2* _ *cloUC*])
show $(snd \circ f) \in C \to T$
using *fim* **by** *force*
qed *auto*
show $\exists V\ g.\ C \subseteq V \wedge$
openin (*top_of_set U*) *V* $\wedge$
continuous_on V g $\wedge$ $g \in V \to S \times T \wedge (\forall x \in C.\ g\ x = f\ x)$
proof (*intro exI conjI*)
show $C \subseteq W1 \cap W2$
by (*simp add*: ‹$C \subseteq W1$› ‹$C \subseteq W2$›)
show *openin* (*top_of_set U*) (*W1* $\cap$ *W2*)
by (*simp add*: *UW1 UW2 openin_Int*)
show *continuous_on* (*W1* $\cap$ *W2*) ($\lambda x.$ (*g x*, *h x*))
by (*metis* (*no_types*) *contg conth continuous_on_Pair continuous_on_subset inf_commute inf_le1*)

```
    show  (λx. (g x, h x)) ∈ (W1 ∩ W2) → S × T
      using gim him by blast
    show  (∀x∈C. (g x, h x) = f x)
      using geq heq by auto
  qed
qed

lemma AR_Times:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes AR S AR T shows AR(S × T)
  using assms by (simp add: AR_ANR ANR_Times contractible_Times)
```

9.26.2 More advanced properties of ANRs and ENRs

```
lemma ENR_rel_frontier_convex:
  fixes S :: 'a::euclidean_space set
  assumes bounded S convex S
    shows ENR(rel_frontier S)
proof (cases S = {})
  case True then show ?thesis
    by simp
next
  case False
  with assms have rel_interior S ≠ {}
    by (simp add: rel_interior_eq_empty)
  then obtain a where a: a ∈ rel_interior S
    by auto
  have ahS: affine hull S − {a} ⊆ {x. closest_point (affine hull S) x ≠ a}
    by (auto simp: closest_point_self)
  have rel_frontier S retract_of affine hull S − {a}
    by (simp add: assms a rel_frontier_retract_of_punctured_affine_hull)
  also have ... retract_of {x. closest_point (affine hull S) x ≠ a}
    unfolding retract_of_def retraction_def ahS
    apply (rule_tac x=closest_point (affine hull S) in exI)
   apply (auto simp: False closest_point_self affine_imp_convex closest_point_in_set
continuous_on_closest_point)
    done
  finally have rel_frontier S retract_of {x. closest_point (affine hull S) x ≠ a} .
  moreover have openin (top_of_set UNIV) (UNIV ∩ closest_point (affine hull
S) -` (- {a}))
   by (intro continuous_openin_preimage_gen) (auto simp: False affine_imp_convex
continuous_on_closest_point)
  ultimately show ?thesis
   by (meson ENR_convex_closed ENR_delete ENR_retract_of_ENR ‹rel_frontier
S retract_of affine hull S − {a}›
            closed_affine_hull convex_affine_hull)
qed

lemma ANR_rel_frontier_convex:
```

```
            fixes S :: 'a::euclidean_space set
  assumes bounded S convex S
    shows ANR(rel_frontier S)
by (simp add: ENR_imp_ANR ENR_rel_frontier_convex assms)

lemma ENR_closedin_Un_local:
  fixes S :: 'a::euclidean_space set
  shows ⟦ENR S; ENR T; ENR(S ∩ T);
         closedin (top_of_set (S ∪ T)) S; closedin (top_of_set (S ∪ T)) T⟧
        ⟹ ENR(S ∪ T)
by (simp add: ENR_ANR ANR_closed_Un_local locally_compact_closedin_Un)

lemma ENR_closed_Un:
  fixes S :: 'a::euclidean_space set
  shows ⟦closed S; closed T; ENR S; ENR T; ENR(S ∩ T)⟧ ⟹ ENR(S ∪ T)
by (auto simp: closed_subset ENR_closedin_Un_local)

lemma absolute_retract_Un:
  fixes S :: 'a::euclidean_space set
  shows ⟦S retract_of UNIV; T retract_of UNIV; (S ∩ T) retract_of UNIV⟧
        ⟹ (S ∪ T) retract_of UNIV
 by (meson AR_closed_Un_local_aux closed_subset retract_of_UNIV retract_of_imp_subset)

lemma retract_from_Un_Int:
  fixes S :: 'a::euclidean_space set
  assumes clS: closedin (top_of_set (S ∪ T)) S
      and clT: closedin (top_of_set (S ∪ T)) T
      and Un: (S ∪ T) retract_of U and Int: (S ∩ T) retract_of T
    shows S retract_of U
proof −
  obtain r where r: continuous_on T r r ‘ T ⊆ S ∩ T ∀x∈S ∩ T. r x = x
    using Int by (auto simp: retraction_def retract_of_def)
  have S retract_of S ∪ T
    unfolding retraction_def retract_of_def
  proof (intro exI conjI)
    show continuous_on (S ∪ T) (λx. if x ∈ S then x else r x)
      using r by (intro continuous_on_cases_local [OF clS clT]) auto
  qed (use r in auto)
  also have ... retract_of U
    by (rule Un)
  finally show ?thesis .
qed

lemma AR_from_Un_Int_local:
  fixes S :: 'a::euclidean_space set
  assumes clS: closedin (top_of_set (S ∪ T)) S
      and clT: closedin (top_of_set (S ∪ T)) T
      and Un: AR(S ∪ T) and Int: AR(S ∩ T)
    shows AR S
```

by (*meson AR_imp_retract AR_retract_of_AR Un assms closedin_closed_subset local.Int*
retract_from_Un_Int retract_of_refl sup_ge2)

lemma *AR_from_Un_Int_local'*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *closedin* (*top_of_set* $(S \cup T))\ S$
and *closedin* (*top_of_set* $(S \cup T))\ T$
and $AR(S \cup T)\ AR(S \cap T)$
shows $AR\ T$
using *AR_from_Un_Int_local* [*of* $T\ S$] *assms* **by** (*simp add*: *Un_commute Int_commute*)

lemma *AR_from_Un_Int*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *clo*: *closed* S *closed* T **and** *Un*: $AR(S \cup T)$ **and** *Int*: $AR(S \cap T)$
shows $AR\ S$
by (*metis AR_from_Un_Int_local* [*OF _ _ Un Int*] *Un_commute clo closed_closedin closedin_closed_subset inf_sup_absorb subtopology_UNIV top_greatest*)

lemma *ANR_from_Un_Int_local*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *clS*: *closedin* (*top_of_set* $(S \cup T))\ S$
and *clT*: *closedin* (*top_of_set* $(S \cup T))\ T$
and *Un*: $ANR(S \cup T)$ **and** *Int*: $ANR(S \cap T)$
shows $ANR\ S$
proof −
obtain V **where** *clo*: *closedin* (*top_of_set* $(S \cup T))\ (S \cap T)$
and *ope*: *openin* (*top_of_set* $(S \cup T))\ V$
and *ret*: $S \cap T$ *retract_of* V
using *ANR_imp_neighbourhood_retract* [*OF Int*] **by** (*metis clS clT closedin_Int*)
then obtain r **where** r: *continuous_on* $V\ r$ **and** *rim*: $r\ `\ V \subseteq S \cap T$ **and** *req*: $\forall x{\in}S \cap T.\ r\ x = x$
by (*auto simp*: *retraction_def retract_of_def*)
have *Vsub*: $V \subseteq S \cup T$
by (*meson ope openin_contains_cball*)
have *Vsup*: $S \cap T \subseteq V$
by (*simp add*: *retract_of_imp_subset ret*)
then have *eq*: $S \cup V = ((S \cup T) - T) \cup V$
by *auto*
have *eq'*: $S \cup V = S \cup (V \cap T)$
using *Vsub* **by** *blast*
have *continuous_on* $(S \cup V \cap T)\ (\lambda x.\ if\ x \in S\ then\ x\ else\ r\ x)$
proof (*rule continuous_on_cases_local*)
show *closedin* (*top_of_set* $(S \cup V \cap T))\ S$
using *clS closedin_subset_trans inf.boundedE* **by** *blast*
show *closedin* (*top_of_set* $(S \cup V \cap T))\ (V \cap T)$
using *clT Vsup* **by** (*auto simp*: *closedin_closed*)
show *continuous_on* $(V \cap T)\ r$

```
    by (meson Int_lower1 continuous_on_subset r)
  qed (use req continuous_on_id in auto)
  with rim have S retract_of S ∪ V
    unfolding retraction_def retract_of_def using eq' by fastforce
  then show ?thesis
    using ANR_neighborhood_retract [OF Un]
    using ‹S ∪ V = S ∪ T − T ∪ V› clT ope by fastforce
qed

lemma ANR_from_Un_Int:
  fixes S :: 'a::euclidean_space set
  assumes clo: closed S closed T and Un: ANR(S ∪ T) and Int: ANR(S ∩ T)
  shows ANR S
 by (metis ANR_from_Un_Int_local [OF _ _ Un Int] Un_commute clo closed_closedin
closedin_closed_subset inf_sup_absorb subtopology_UNIV top_greatest)

lemma ANR_finite_Union_convex_closed:
  fixes T :: 'a::euclidean_space set set
  assumes T: finite T and clo: ⋀C. C ∈ T ⟹ closed C and con: ⋀C. C ∈ T
⟹ convex C
  shows ANR(⋃T)
proof −
  have ANR(⋃T) if card T < n for n
  using assms that
  proof (induction n arbitrary: T)
    case 0 then show ?case by simp
  next
    case (Suc n)
    have ANR(⋃U) if finite U U ⊆ T for U
      using that
    proof (induction U)
      case empty
      then show ?case  by simp
    next
      case (insert C U)
      have ANR (C ∪ ⋃U)
      proof (rule ANR_closed_Un)
        show ANR (C ∩ ⋃U)
          unfolding Int_Union
        proof (rule Suc)
          show finite ((∩) C ‘ U)
            by (simp add: insert.hyps(1))
          show ⋀Ca. Ca ∈ (∩) C ‘ U ⟹ closed Ca
             by (metis (no_types, opaque_lifting) Suc.prems(2) closed_Int subsetD
imageE insert.prems insertI1 insertI2)
          show ⋀Ca. Ca ∈ (∩) C ‘ U ⟹ convex Ca
              by (metis (mono_tags, lifting) Suc.prems(3) convex_Int imageE in-
sert.prems insert_subset subsetCE)
          show card ((∩) C ‘ U) < n
```

```
        proof −
          have card 𝒯 ≤ n
            by (meson Suc.prems(4) not_less not_less_eq)
          then show ?thesis
              by (metis Suc.prems(1) card_image_le card_seteq insert.hyps in-
sert.prems insert_subset le_trans not_less)
        qed
      qed
      show closed (⋃𝒰)
        using Suc.prems(2) insert.hyps(1) insert.prems by blast
    qed (use Suc.prems convex_imp_ANR insert.prems insert.IH in auto)
    then show ?case
      by simp
  qed
  then show ?case
    using Suc.prems(1) by blast
 qed
 then show ?thesis
   by blast
qed


lemma finite_imp_ANR:
 fixes S :: 'a::euclidean_space set
 assumes finite S
 shows ANR S
proof −
 have ANR(⋃x ∈ S. {x})
   by (blast intro: ANR_finite_Union_convex_closed assms)
 then show ?thesis
   by simp
qed

lemma ANR_insert:
 fixes S :: 'a::euclidean_space set
 assumes ANR S closed S
 shows ANR(insert a S)
 by (metis ANR_closed_Un ANR_empty ANR_singleton Diff_disjoint Diff_insert_absorb
assms closed_singleton insert_absorb insert_is_Un)

lemma ANR_path_component_ANR:
 fixes S :: 'a::euclidean_space set
 shows ANR S ⟹ ANR(path_component_set S x)
 using ANR_imp_locally_path_connected ANR_openin openin_path_component_locally_path_connected
by blast

lemma ANR_connected_component_ANR:
 fixes S :: 'a::euclidean_space set
 shows ANR S ⟹ ANR(connected_component_set S x)
```

by (*metis ANR_openin openin_connected_component_locally_connected ANR_imp_locally_connected*

lemma *ANR_component_ANR*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *ANR S c* ∈ *components S*
shows *ANR c*
by (*metis ANR_connected_component_ANR assms componentsE*)

9.26.3 Original ANR material, now for ENRs

lemma *ENR_bounded*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *bounded S*
shows *ENR S* ⟷ (∃ *U*. *open U* ∧ *bounded U* ∧ *S retract_of U*)
(**is** *?lhs* = *?rhs*)
proof
obtain *r* **where** *0* < *r* **and** *r*: *S* ⊆ *ball 0 r*
using *bounded_subset_ballD assms* **by** *blast*
assume *?lhs*
then show *?rhs*
by (*meson ENR_def Elementary_Metric_Spaces.open_ball bounded_Int bounded_ball inf_le2 le_inf_iff*
open_Int r retract_of_imp_subset retract_of_subset)
next
assume *?rhs*
then show *?lhs*
using *ENR_def* **by** *blast*
qed

lemma *absolute_retract_imp_AR_gen*:
fixes *S* :: *'a*::*euclidean_space set* **and** *S'* :: *'b*::*euclidean_space set*
assumes *S retract_of T convex T T* ≠ {} *S homeomorphic S' closedin* (*top_of_set U*) *S'*
shows *S' retract_of U*
proof −
have *AR T*
by (*simp add*: *assms convex_imp_AR*)
then have *AR S*
using *AR_retract_of_AR assms* **by** *auto*
then show *?thesis*
using *assms AR_imp_absolute_retract* **by** *metis*
qed

lemma *absolute_retract_imp_AR*:
fixes *S* :: *'a*::*euclidean_space set* **and** *S'* :: *'b*::*euclidean_space set*
assumes *S retract_of UNIV S homeomorphic S' closed S'*
shows *S' retract_of UNIV*
using *AR_imp_absolute_retract_UNIV assms retract_of_UNIV* **by** *blast*

lemma *homeomorphic_compact_arness*:
fixes $S :: 'a::euclidean\_space\ set$ **and** $S' :: 'b::euclidean\_space\ set$
assumes S *homeomorphic* S'
shows *compact* $S \wedge S$ *retract_of UNIV* $\longleftrightarrow$ *compact* $S' \wedge S'$ *retract_of UNIV*
using *assms homeomorphic_compactness*
by (*metis compact_AR homeomorphic_AR_iff_AR*)

lemma *absolute_retract_from_Un_Int*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $(S \cup T)$ *retract_of UNIV* $(S \cap T)$ *retract_of UNIV closed* S *closed* T
shows S *retract_of UNIV*
using *AR_from_Un_Int assms retract_of_UNIV* **by** *auto*

lemma *ENR_from_Un_Int_gen*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *closedin* $(top\_of\_set\ (S \cup T))\ S$ *closedin* $(top\_of\_set\ (S \cup T))\ T$ $ENR(S \cup T)$ $ENR(S \cap T)$
shows *ENR* S
by (*meson ANR_from_Un_Int_local ANR_imp_neighbourhood_retract ENR_ANR ENR_neighborhood_retract assms*)

lemma *ENR_from_Un_Int*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *closed* S *closed* T $ENR(S \cup T)$ $ENR(S \cap T)$
shows *ENR* S
by (*meson ENR_from_Un_Int_gen assms closed_subset sup_ge1 sup_ge2*)

lemma *ENR_finite_Union_convex_closed*:
fixes $\mathcal{T} :: 'a::euclidean\_space\ set\ set$
assumes $\mathcal{T}$: *finite* $\mathcal{T}$ **and** *clo*: $\bigwedge C.\ C \in \mathcal{T} \Longrightarrow$ *closed* C **and** *con*: $\bigwedge C.\ C \in \mathcal{T} \Longrightarrow$ *convex* C
shows $ENR(\bigcup\ \mathcal{T})$
by (*simp add*: *ENR_ANR ANR_finite_Union_convex_closed* $\mathcal{T}$ *clo closed_Union closed_imp_locally_compact con*)

lemma *finite_imp_ENR*:
fixes $S :: 'a::euclidean\_space\ set$
shows *finite* $S \Longrightarrow$ *ENR* S
by (*simp add*: *ENR_ANR finite_imp_ANR finite_imp_closed closed_imp_locally_compact*)

lemma *ENR_insert*:
fixes $S :: 'a::euclidean\_space\ set$
assumes *closed* S *ENR* S
shows $ENR(insert\ a\ S)$
proof −
have *ENR* $(\{a\} \cup S)$
by (*metis ANR_insert ENR_ANR Un_commute Un_insert_right assms closed_imp_locally_compact*

closed_insert sup_bot_right)
then show *?thesis*
by *auto*
qed

lemma *ENR_path_component_ENR*:
fixes S :: *'a*::*euclidean_space set*
assumes *ENR S*
shows *ENR*(*path_component_set S x*)
by (*metis ANR_imp_locally_path_connected ENR_empty ENR_imp_ANR ENR_openin assms*
locally_path_connected_2 openin_subtopology_self path_component_eq_empty)

9.26.4 Finally, spheres are ANRs and ENRs

lemma *absolute_retract_homeomorphic_convex_compact*:
fixes S :: *'a*::*euclidean_space set* **and** U :: *'b*::*euclidean_space set*
assumes *S homeomorphic U S* $\neq \{\}$ $S \subseteq T$ *convex U compact U*
shows *S retract_of T*
by (*metis UNIV_I assms compact_AR convex_imp_AR homeomorphic_AR_iff_AR homeomorphic_compactness homeomorphic_empty*(*1*) *retract_of_subset subsetI*)

lemma *frontier_retract_of_punctured_universe*:
fixes S :: *'a*::*euclidean_space set*
assumes *convex S bounded S a* $\in$ *interior S*
shows (*frontier S*) *retract_of* $(- \{a\})$
using *rel_frontier_retract_of_punctured_affine_hull*
by (*metis Compl_eq_Diff_UNIV affine_hull_nonempty_interior assms empty_iff rel_frontier_frontier rel_interior_nonempty_interior*)

lemma *sphere_retract_of_punctured_universe_gen*:
fixes a :: *'a*::*euclidean_space*
assumes $b \in$ *ball a r*
shows *sphere a r retract_of* $(- \{b\})$
proof −
have *frontier* (*cball a r*) *retract_of* $(- \{b\})$
using *assms frontier_retract_of_punctured_universe interior_cball* **by** *blast*
then show *?thesis*
by *simp*
qed

lemma *sphere_retract_of_punctured_universe*:
fixes a :: *'a*::*euclidean_space*
assumes $0 < r$
shows *sphere a r retract_of* $(- \{a\})$
by (*simp add*: *assms sphere_retract_of_punctured_universe_gen*)

lemma *ENR_sphere*:
fixes a :: *'a*::*euclidean_space*

shows *ENR*(*sphere a r*)
proof (*cases* 0 < *r*)
case *True*
then have *sphere a r retract_of* −{*a*}
by (*simp add*: *sphere_retract_of_punctured_universe*)
with *open_delete* **show** *?thesis*
by (*auto simp*: *ENR_def*)
next
case *False*
then show *?thesis*
using *finite_imp_ENR*
by (*metis finite_insert infinite_imp_nonempty less_linear sphere_eq_empty sphere_trivial*)
qed

corollary *ANR_sphere*:
fixes *a* :: *'a*::*euclidean_space*
shows *ANR*(*sphere a r*)
by (*simp add*: *ENR_imp_ANR ENR_sphere*)

9.26.5 Spheres are connected, etc

lemma *locally_path_connected_sphere_gen*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *bounded S* **and** *convex S*
shows *locally path_connected* (*rel_frontier S*)
proof (*cases rel_interior S* = {})
case *True*
with *assms* **show** *?thesis*
by (*simp add*: *rel_interior_eq_empty*)
next
case *False*
then obtain *a* **where** *a*: *a* ∈ *rel_interior S*
by *blast*
show *?thesis*
proof (*rule retract_of_locally_path_connected*)
show *locally path_connected* (*affine hull S* − {*a*})
by (*meson convex_affine_hull convex_imp_locally_path_connected locally_open_subset openin_delete openin_subtopology_self*)
show *rel_frontier S retract_of affine hull S* − {*a*}
using *a assms rel_frontier_retract_of_punctured_affine_hull* **by** *blast*
qed
qed

lemma *locally_connected_sphere_gen*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *bounded S* **and** *convex S*
shows *locally connected* (*rel_frontier S*)
by (*simp add*: *ANR_imp_locally_connected ANR_rel_frontier_convex assms*)

lemma *locally_path_connected_sphere*:
fixes *a* :: *'a*::*euclidean_space*
shows *locally path_connected* (*sphere a r*)
using *ENR_imp_locally_path_connected ENR_sphere* **by** *blast*

lemma *locally_connected_sphere*:
fixes *a* :: *'a*::*euclidean_space*
shows *locally connected*(*sphere a r*)
using *ANR_imp_locally_connected ANR_sphere* **by** *blast*

9.26.6 Borsuk homotopy extension theorem

It's only this late so we can use the concept of retraction, saying that the domain sets or range set are ENRs.

theorem *Borsuk_homotopy_extension_homotopic*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *cloTS*: *closedin* (*top_of_set T*) *S*
and *anr*: (*ANR S* ∧ *ANR T*) ∨ *ANR U*
and *contf*: *continuous_on T f*
and *f* ∈ *T* → *U*
and *homotopic_with_canon* (λ*x*. *True*) *S U f g*
obtains *g'* **where** *homotopic_with_canon* (λ*x*. *True*) *T U f g'*
continuous_on T g' image g' T ⊆ *U*
⋀*x*. *x* ∈ *S* ⟹ *g' x* = *g x*
proof −
have *S* ⊆ *T* **using** *assms closedin_imp_subset* **by** *blast*
obtain *h* **where** *conth*: *continuous_on* ({*0..1*} × *S*) *h*
and *him*: *h* ∈ ({*0..1*} × *S*) → *U*
and [*simp*]: ⋀*x*. *h*(*0*, *x*) = *f x* ⋀*x*. *h*(*1*::*real*, *x*) = *g x*
using *assms* **by** (*fastforce simp*: *homotopic_with_def*)
define *h'* **where** *h'* ≡ λ*z*. *if snd z* ∈ *S then h z else* (*f* ∘ *snd*) *z*
define *B* **where** *B* ≡ {*0*::*real*} × *T* ∪ {*0..1*} × *S*
have *clo0T*: *closedin* (*top_of_set* ({*0..1*} × *T*)) ({*0*::*real*} × *T*)
by (*simp add*: *Abstract_Topology.closedin_Times*)
moreover have *cloT1S*: *closedin* (*top_of_set* ({*0..1*} × *T*)) ({*0..1*} × *S*)
by (*simp add*: *Abstract_Topology.closedin_Times assms*)
ultimately have *clo0TB*:*closedin* (*top_of_set* ({*0..1*} × *T*)) *B*
by (*auto simp*: *B_def*)
have *cloBS*: *closedin* (*top_of_set B*) ({*0..1*} × *S*)
by (*metis* (*no_types*) *Un_subset_iff B_def closedin_subset_trans* [*OF cloT1S*]
clo0TB closedin_imp_subset closedin_self)
moreover have *cloBT*: *closedin* (*top_of_set B*) ({*0*} × *T*)
using ‹*S* ⊆ *T*› *closedin_subset_trans* [*OF clo0T*]
by (*metis B_def Un_upper1 clo0TB closedin_closed inf_le1*)
moreover have *continuous_on* ({*0*} × *T*) (*f* ∘ *snd*)
proof (*rule continuous_intros*)+
show *continuous_on* (*snd* ' ({*0*} × *T*)) *f*
by (*simp add*: *contf*)

qed
ultimately have *continuous_on* $(\{0..1\} \times S \cup \{0\} \times T)$ $(\lambda x.$ *if snd* $x \in S$ *then* $h\ x$ *else* $(f \circ snd)\ x)$
by (*auto intro*!: *continuous_on_cases_local conth simp*: *B_def Un_commute* [*of* $\{0\} \times T$])
then have *conth'*: *continuous_on B h'*
by (*simp add*: *h'_def B_def Un_commute* [*of* $\{0\} \times T$])
have *image h' B* $\subseteq$ *U*
using ‹$f \in T \to U$› *him* **by** (*auto simp*: *h'_def B_def*)
obtain *V k* **where** $B \subseteq V$ **and** *opeTV*: *openin* (*top_of_set* $(\{0..1\} \times T)$) *V*
and *contk*: *continuous_on V k* **and** *kim*: $k \in V \to U$
and *keq*: $\bigwedge x.\ x \in B \Longrightarrow k\ x = h'\ x$
using *anr*
proof
assume *ST*: *ANR S* $\wedge$ *ANR T*
have *eq*: $(\{0\} \times T \cap \{0..1\} \times S) = \{0::real\} \times S$
using ‹$S \subseteq T$› **by** *auto*
have *ANR B*
unfolding *B_def*
proof (*rule ANR_closed_Un_local*)
show *closedin* (*top_of_set* $(\{0\} \times T \cup \{0..1\} \times S)$) $(\{0::real\} \times T)$
by (*metis cloBT B_def*)
show *closedin* (*top_of_set* $(\{0\} \times T \cup \{0..1\} \times S)$) $(\{0..1::real\} \times S)$
by (*metis Un_commute cloBS B_def*)
qed (*simp_all add*: *ANR_Times convex_imp_ANR ANR_singleton ST eq*)
note *Vk* = *that*
have $*$: *thesis* **if** *openin* (*top_of_set* $(\{0..1::real\} \times T)$) *V*
retraction V B r **for** *V r*
proof −
have *continuous_on V* $(h' \circ r)$
using *conth' continuous_on_compose retractionE that*(*2*) **by** *blast*
moreover have $(h' \circ r)$ ‘ *V* $\subseteq$ *U*
by (*metis* ‹*h'* ‘ *B* $\subseteq$ *U*› *image_comp retractionE that*(*2*))
ultimately show *?thesis*
using *Vk* [*of V h'* $\circ$ *r*] **by** (*metis comp_apply retraction image_subset_iff_funcset that*)
qed
show *thesis*
by (*meson* $*$ *ANR_imp_neighbourhood_retract* ‹*ANR B*› *clo0TB retract_of_def*)
next
assume *ANR U*
with *ANR_imp_absolute_neighbourhood_extensor* ‹*h'* ‘ *B* $\subseteq$ *U*› *clo0TB conth' image_subset_iff_funcset that*
show *?thesis*
by (*smt* (*verit*) *Pi_I funcset_mem*)
qed
define *S'* **where** $S' \equiv \{x.\ \exists u::real.\ u \in \{0..1\} \wedge (u, x::'a) \in \{0..1\} \times T - V\}$
have *closedin* (*top_of_set T*) *S'*
unfolding *S'_def* **using** *closedin_self opeTV*

```
    by (blast intro: closedin_compact_projection)
  have S'_def: S' = {x. ∃u::real.  (u, x::'a) ∈ {0..1} × T − V}
    by (auto simp: S'_def)
  have cloTS': closedin (top_of_set T) S'
    using S'_def ‹closedin (top_of_set T) S'› by blast
  have S ∩ S' = {}
    using S'_def B_def ‹B ⊆ V› by force
  obtain a :: 'a ⇒ real where conta: continuous_on T a
      and ⋀x. x ∈ T ⟹ a x ∈ closed_segment 1 0
      and a1: ⋀x. x ∈ S ⟹ a x = 1
      and a0: ⋀x. x ∈ S' ⟹ a x = 0
    by (rule Urysohn_local [OF cloTS cloTS' ‹S ∩ S' = {}›, of 1 0], blast)
  then have ain: ⋀x. x ∈ T ⟹ a x ∈ {0..1}
    using closed_segment_eq_real_ivl by auto
  have inV: (u * a t, t) ∈ V if t ∈ T 0 ≤ u u ≤ 1 for t u
  proof (rule ccontr)
    assume (u * a t, t) ∉ V
    with ain [OF ‹t ∈ T›] have a t = 0
      apply simp
        by (metis (no_types, lifting) a0 DiffI S'_def SigmaI atLeastAtMost_iff
mem_Collect_eq mult_le_one mult_nonneg_nonneg that)
    show False
      using B_def ‹(u * a t, t) ∉ V› ‹B ⊆ V› ‹a t = 0› that by auto
  qed
  show ?thesis
  proof
    show hom: homotopic_with_canon (λx. True) T U f (λx. k (a x, x))
    proof (simp add: homotopic_with, intro exI conjI)
      show continuous_on ({0..1} × T) (k ∘ (λz. (fst z *R (a ∘ snd) z, snd z)))
        apply (intro continuous_on_compose continuous_intros)
      apply (force intro: inV continuous_on_subset [OF contk] continuous_on_subset
[OF conta])+
        done
      show (k ∘ (λz. (fst z *R (a ∘ snd) z, snd z))) ` ({0..1} × T) ⊆ U
        using inV kim by auto
      show ∀x∈T. (k ∘ (λz. (fst z *R (a ∘ snd) z, snd z))) (0, x) = f x
        by (simp add: B_def h'_def keq)
      show ∀x∈T. (k ∘ (λz. (fst z *R (a ∘ snd) z, snd z))) (1, x) = k (a x, x)
        by auto
    qed
  show continuous_on T (λx. k (a x, x))
    using homotopic_with_imp_continuous_maps [OF hom] by auto
  show (λx. k (a x, x)) ` T ⊆ U
  proof clarify
    fix t
    assume t ∈ T
    show k (a t, t) ∈ U
    by (metis ‹t ∈ T› image_subset_iff inV kim not_one_le_zero linear mult_cancel_right1
image_subset_iff_funcset)
```

```
  qed
  show ⋀x. x ∈ S ⟹ k (a x, x) = g x
    by (simp add: B_def a1 h'_def keq)
  qed
qed


corollary nullhomotopic_into_ANR_extension:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes closed S
      and contf: continuous_on S f
      and ANR T
      and fim: f ` S ⊆ T
      and S ≠ {}
   shows (∃c. homotopic_with_canon (λx. True) S T f (λx. c)) ⟷
          (∃g. continuous_on UNIV g ∧ range g ⊆ T ∧ (∀x ∈ S. g x = f x))
       (is ?lhs = ?rhs)
proof
  assume ?lhs
  then obtain c where c: homotopic_with_canon (λx. True) S T (λx. c) f
    by (blast intro: homotopic_with_symD)
  have closedin (top_of_set UNIV) S
    using ‹closed S› closed_closedin by fastforce
  then obtain g where continuous_on UNIV g range g ⊆ T
                      ⋀x. x ∈ S ⟹ g x = f x
  proof (rule Borsuk_homotopy_extension_homotopic)
    show (λx. c) ∈ UNIV → T
      using ‹S ≠ {}› c homotopic_with_imp_subset1 by fastforce
  qed (use assms c in auto)
  then show ?rhs by blast
next
  assume ?rhs
  then obtain g where continuous_on UNIV g range g ⊆ T ⋀x. x∈S ⟹ g x =
f x
    by blast
  then obtain c where homotopic_with_canon (λh. True) UNIV T g (λx. c)
    using nullhomotopic_from_contractible [of UNIV g T] contractible_UNIV by
blast
  then have homotopic_with_canon (λx. True) S T g (λx. c)
    by (simp add: homotopic_from_subtopology)
  then show ?lhs
    by (force elim: homotopic_with_eq [of _ _ _ g λx. c] simp: ‹⋀x. x ∈ S ⟹
g x = f x›)
qed

corollary nullhomotopic_into_rel_frontier_extension:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes closed S
      and contf: continuous_on S f
```

```
      and convex T bounded T
      and fim: f ‘ S ⊆ rel_frontier T
      and S ≠ {}
    shows (∃ c. homotopic_with_canon (λx. True) S (rel_frontier T) f (λx. c))
⟷
          (∃ g. continuous_on UNIV g ∧ range g ⊆ rel_frontier T ∧ (∀ x ∈ S. g x
= f x))
by (simp add: nullhomotopic_into_ANR_extension assms ANR_rel_frontier_convex)

corollary nullhomotopic_into_sphere_extension:
  fixes f :: 'a::euclidean_space ⇒ 'b :: euclidean_space
  assumes closed S and contf: continuous_on S f
      and S ≠ {} and fim: f ‘ S ⊆ sphere a r
    shows ((∃ c. homotopic_with_canon (λx. True) S (sphere a r) f (λx. c)) ⟷
          (∃ g. continuous_on UNIV g ∧ range g ⊆ sphere a r ∧ (∀ x ∈ S. g x = f
x)))
          (is ?lhs = ?rhs)
proof (cases r = 0)
  case True with fim show ?thesis
   by (metis ANR_sphere ‹closed S› ‹S ≠ {}› contf nullhomotopic_into_ANR_extension)
next
  case False
  then have eq: sphere a r = rel_frontier (cball a r) by simp
  show ?thesis
     using fim nullhomotopic_into_rel_frontier_extension [OF ‹closed S› contf
convex_cball bounded_cball]
    by (simp add: ‹S ≠ {}› eq)
qed

proposition Borsuk_map_essential_bounded_component:
  fixes a :: 'a :: euclidean_space
  assumes compact S and a ∉ S
   shows bounded (connected_component_set (− S) a) ⟷
         ¬(∃ c. homotopic_with_canon (λx. True) S (sphere 0 1)
                                     (λx. inverse(norm(x − a)) *R (x − a)) (λx. c))
   (is ?lhs = ?rhs)
proof (cases S = {})
  case True then show ?thesis
    by (simp add: homotopic_on_emptyI)
next
  case False
  have closed S bounded S
    using ‹compact S› compact_eq_bounded_closed by auto
  have s01: (λx. (x − a) /R norm (x − a)) ‘ S ⊆ sphere 0 1
     using ‹a ∉ S›  by clarsimp (metis dist_eq_0_iff dist_norm mult.commute
right_inverse)
  have aincc: a ∈ connected_component_set (− S) a
    by (simp add: ‹a ∉ S›)
  obtain r where r>0 and r: S ⊆ ball 0 r
```

```
    using bounded_subset_ballD ‹bounded S› by blast
  have ¬ ?rhs ⟷ ¬ ?lhs
  proof
    assume notr: ¬ ?rhs
    have nog: ∄g. continuous_on (S ∪ connected_component_set (− S) a) g ∧
                  g ‘ (S ∪ connected_component_set (− S) a) ⊆ sphere 0 1 ∧
                  (∀x∈S. g x = (x − a) /R norm (x − a))
      if bounded (connected_component_set (− S) a)
      using non_extensible_Borsuk_map [OF ‹compact S› componentsI _ aincc]
‹a ∉ S› that by auto
    obtain g where range g ⊆ sphere 0 1 continuous_on UNIV g
                   ⋀x. x ∈ S ⟹ g x = (x − a) /R norm (x − a)
      using notr
      by (auto simp: nullhomotopic_into_sphere_extension
            [OF ‹closed S› continuous_on_Borsuk_map [OF ‹a ∉ S›] False s01])
    with ‹a ∉ S› show ¬ ?lhs
      by (metis UNIV_I continuous_on_subset image_subset_iff nog subsetI)
  next
    assume ¬ ?lhs
   then obtain b where b: b ∈ connected_component_set (− S) a and r ≤ norm
b
      using bounded_iff linear by blast
    then have bnot: b ∉ ball 0 r
      by simp
    have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. (x − a) /R norm
(x − a))
                                                        (λx. (x − b) /R norm (x − b))
    proof −
      have path_component (− S) a b
      by (metis (full_types) ‹closed S› b mem_Collect_eq open_Compl open_path_connected_component)
      then show ?thesis
        using Borsuk_maps_homotopic_in_path_component by blast
    qed
    moreover
    obtain c where homotopic_with_canon (λx. True) (ball 0 r) (sphere 0 1)
                                        (λx. inverse (norm (x − b)) *R (x − b)) (λx. c)
    proof (rule nullhomotopic_from_contractible)
      show contractible (ball (0::'a) r)
        by (metis convex_imp_contractible convex_ball)
      show continuous_on (ball 0 r) (λx. inverse(norm (x − b)) *R (x − b))
        by (rule continuous_on_Borsuk_map [OF bnot])
      show (λx. (x − b) /R norm (x − b)) ∈ ball 0 r → sphere 0 1
        using bnot Borsuk_map_into_sphere by blast
    qed blast
    ultimately have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. (x −
a) /R norm (x − a)) (λx. c)
      by (meson homotopic_with_subset_left homotopic_with_trans r)
    then show ¬ ?rhs
      by blast
```

```
  qed
  then show ?thesis by blast
qed

lemma homotopic_Borsuk_maps_in_bounded_component:
  fixes a :: 'a :: euclidean_space
  assumes compact S and a ∉ Sand b ∉ S
     and boc: bounded (connected_component_set (− S) a)
     and hom: homotopic_with_canon (λx. True) S (sphere 0 1)
                                   (λx. (x − a) /R norm (x − a))
                                   (λx. (x − b) /R norm (x − b))
   shows connected_component (− S) a b
proof (rule ccontr)
  assume notcc: ¬ connected_component (− S) a b
  let ?T = S ∪ connected_component_set (− S) a
  have ∄g. continuous_on (S ∪ connected_component_set (− S) a) g ∧
           g ∈ (S ∪ connected_component_set (− S) a) → sphere 0 1 ∧
           (∀x∈S. g x = (x − a) /R norm (x − a))
    using non_extensible_Borsuk_map [OF ‹compact S› _ boc] ‹a ∉ S›
    by (simp add: componentsI)
  moreover obtain g where continuous_on (S ∪ connected_component_set (−
S) a) g
                          g ' (S ∪ connected_component_set (− S) a) ⊆ sphere 0 1
                          ⋀x. x ∈ S ⟹ g x = (x − a) /R norm (x − a)
  proof (rule Borsuk_homotopy_extension_homotopic)
    show closedin (top_of_set ?T) S
      by (simp add: ‹compact S› closed_subset compact_imp_closed)
    show continuous_on ?T (λx. (x − b) /R norm (x − b))
      by (simp add: ‹b ∉ S› notcc continuous_on_Borsuk_map)
    show (λx. (x − b) /R norm (x − b)) ∈ ?T → sphere 0 1
      by (simp add: ‹b ∉ S› notcc Borsuk_map_into_sphere)
    show homotopic_with_canon (λx. True) S (sphere 0 1)
           (λx. (x − b) /R norm (x − b)) (λx. (x − a) /R norm (x − a))
      by (simp add: hom homotopic_with_symD)
    qed (auto simp: ANR_sphere intro: that)
  ultimately show False by blast
qed


lemma Borsuk_maps_homotopic_in_connected_component_eq:
  fixes a :: 'a :: euclidean_space
  assumes S: compact S a ∉ S b ∉ S and 2: 2 ≤ DIM('a)
    shows (homotopic_with_canon (λx. True) S (sphere 0 1)
                  (λx. (x − a) /R norm (x − a))
                  (λx. (x − b) /R norm (x − b)) ⟷
          connected_component (− S) a b)
        (is ?lhs = ?rhs)
proof
  assume L: ?lhs
```

```
  show ?rhs
  proof (cases bounded(connected_component_set (− S) a))
    case True
    show ?thesis
      by (rule homotopic_Borsuk_maps_in_bounded_component [OF S True L])
  next
    case not_bo_a: False
    show ?thesis
    proof (cases bounded(connected_component_set (− S) b))
      case True
      show ?thesis
        using homotopic_Borsuk_maps_in_bounded_component [OF S]
      by (simp add: L True assms connected_component_sym homotopic_Borsuk_maps_in_bounded_component
homotopic_with_sym)
    next
      case False
      then show ?thesis
        using cobounded_unique_unbounded_component [of −S a b] ‹compact S›
not_bo_a
      by (auto simp: compact_eq_bounded_closed assms connected_component_eq_eq)
    qed
  qed
next
  assume R: ?rhs
  then have path_component (− S) a b
   using assms(1) compact_eq_bounded_closed open_Compl open_path_connected_component_set
by fastforce
  then show ?lhs
    by (simp add: Borsuk_maps_homotopic_in_path_component)
qed
```

9.26.7 More extension theorems

```
lemma extension_from_clopen:
  assumes ope: openin (top_of_set S) T
      and clo: closedin (top_of_set S) T
      and contf: continuous_on T f and fim: f ‘ T ⊆ U and null: U = {} ⟹ S
= {}
 obtains g where continuous_on S g g ‘ S ⊆ U ⋀x. x ∈ T ⟹ g x = f x
proof (cases U = {})
  case True
  then show ?thesis
    by (simp add: null that)
next
  case False
  then obtain a where a ∈ U
    by auto
  let ?g = λx. if x ∈ T then f x else a
  have Seq: S = T ∪ (S − T)
```

```
    using clo closedin_imp_subset by fastforce
  show ?thesis
  proof
    have continuous_on (T ∪ (S − T)) ?g
      using Seq clo ope  by (intro continuous_on_cases_local) (auto simp: contf)
    with Seq show continuous_on S ?g
      by metis
    show ?g ‘ S ⊆ U
      using ‹a ∈ U› fim by auto
    show ⋀x. x ∈ T ⟹ ?g x = f x
      by auto
  qed
qed


lemma extension_from_component:
  fixes f :: 'a :: euclidean_space ⇒ 'b :: euclidean_space
  assumes S: locally connected S ∨ compact S and ANR U
    and C: C ∈ components S and contf: continuous_on C f and fim: f ∈ C →
U
 obtains g where continuous_on S g g ∈ S → U ⋀x. x ∈ C ⟹ g x = f x
proof −
  obtain T g where ope: openin (top_of_set S) T
              and clo: closedin (top_of_set S) T
              and C ⊆ T and contg: continuous_on T g and gim: g ∈ T → U
              and gf: ⋀x. x ∈ C ⟹ g x = f x
    using S
  proof
    assume locally connected S
    show ?thesis
      by (metis C ‹locally connected S› openin_components_locally_connected
closedin_component contf fim order_refl that)
  next
    assume compact S
    then obtain W g where C ⊆ W and opeW: openin (top_of_set S) W
                and contg: continuous_on W g
                and gim: g ∈ W → U and gf: ⋀x. x ∈ C ⟹ g x = f x
     using ANR_imp_absolute_neighbourhood_extensor [of U C f S] C ‹ANR U›
closedin_component contf fim by blast
    then obtain V where open V and V: W = S ∩ V
      by (auto simp: openin_open)
    moreover have locally compact S
     by (simp add: ‹compact S› closed_imp_locally_compact compact_imp_closed)
    ultimately obtain K where opeK: openin (top_of_set S) K and compact K
C ⊆ K K ⊆ V
      by (metis C Int_subset_iff ‹C ⊆ W› ‹compact S› compact_components
Sura_Bura_clopen_subset)
    show ?thesis
    proof
```

show *closedin* (*top_of_set S*) *K*
by (*meson* ‹*compact K*› ‹*compact S*› *closedin_compact_eq opeK openin_imp_subset*)
show *continuous_on K g*
by (*metis Int_subset_iff V* ‹*K* ⊆ *V*› *contg continuous_on_subset opeK openin_subtopology subset_eq*)
show *g* ∈ *K* → *U*
using *V* ‹*K* ⊆ *V*› *gim opeK openin_imp_subset* **by** *fastforce*
qed (*use opeK gf* ‹*C* ⊆ *K*› **in** *auto*)
qed
obtain *h* **where** *continuous_on S h h* ∈ *S* → *U* ⋀*x*. *x* ∈ *T* ⟹ *h x* = *g x*
using *extension_from_clopen*
by (*metis C bot.extremum_uniqueI clo contg gim fim image_is_empty in_components_nonempty ope image_subset_iff_funcset*)
then show *?thesis*
by (*metis* ‹*C* ⊆ *T*› *gf subset_eq that*)
qed

lemma *tube_lemma*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *compact S* **and** *S*: *S* ≠ {} (λ*x*. (*x*,*a*)) ‘ *S* ⊆ *U*
and *ope*: *openin* (*top_of_set* (*S* × *T*)) *U*
obtains *V* **where** *openin* (*top_of_set T*) *V a* ∈ *V S* × *V* ⊆ *U*
proof −
let *?W* = {*y*. ∃*x*. *x* ∈ *S* ∧ (*x*, *y*) ∈ (*S* × *T* − *U*)}
have *U* ⊆ *S* × *T closedin* (*top_of_set* (*S* × *T*)) (*S* × *T* − *U*)
using *ope* **by** (*auto simp*: *openin_closedin_eq*)
then have *closedin* (*top_of_set T*) *?W*
using ‹*compact S*› *closedin_compact_projection* **by** *blast*
moreover have *a* ∈ *T* − *?W*
using ‹*U* ⊆ *S* × *T*› *S* **by** *auto*
moreover have *S* × (*T* − *?W*) ⊆ *U*
by *auto*
ultimately show *?thesis*
by (*metis* (*no_types*, *lifting*) *Sigma_cong closedin_def that topspace_euclidean_subtopology*)
qed

lemma *tube_lemma_gen*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *compact S S* ≠ {} *T* ⊆ *T'* *S* × *T* ⊆ *U*
and *ope*: *openin* (*top_of_set* (*S* × *T'*)) *U*
obtains *V* **where** *openin* (*top_of_set T'*) *V T* ⊆ *V S* × *V* ⊆ *U*
proof −
have ⋀*x*. *x* ∈ *T* ⟹ ∃ *V*. *openin* (*top_of_set T'*) *V* ∧ *x* ∈ *V* ∧ *S* × *V* ⊆ *U*
using *assms* **by** (*auto intro*: *tube_lemma* [*OF* ‹*compact S*›])
then obtain *F* **where** *F*: ⋀*x*. *x* ∈ *T* ⟹ *openin* (*top_of_set T'*) (*F x*) ∧ *x* ∈ *F x* ∧ *S* × *F x* ⊆ *U*
by *metis*
show *?thesis*

```
  proof
    show openin (top_of_set T') (⋃(F ' T))
      using F by blast
    show T ⊆ ⋃(F ' T)
      using F by blast
    show S × ⋃(F ' T) ⊆ U
      using F by auto
  qed
qed

proposition homotopic_neighbourhood_extension:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and fim: f ' S ⊆ U
      and contg: continuous_on S g and gim: g ' S ⊆ U
      and clo: closedin (top_of_set S) T
      and ANR U and hom: homotopic_with_canon (λx. True) T U f g
    obtains V where T ⊆ V openin (top_of_set S) V
                    homotopic_with_canon (λx. True) V U f g
proof −
  have T ⊆ S
    using clo closedin_imp_subset by blast
  obtain h where conth: continuous_on ({0..1::real} × T) h
            and him: h ' ({0..1} × T) ⊆ U
            and h0: ⋀x. h(0, x) = f x and h1: ⋀x. h(1, x) = g x
    using hom by (auto simp: homotopic_with_def)
  define h' where h' ≡ λz. if fst z ∈ {0} then f(snd z)
                           else if fst z ∈ {1} then g(snd z)
                           else h z
  let ?S0 = {0::real} × S and ?S1 = {1::real} × S
  have continuous_on(?S0 ∪ (?S1 ∪ {0..1} × T)) h'
    unfolding h'_def
  proof (intro continuous_on_cases_local)
    show closedin (top_of_set (?S0 ∪ (?S1 ∪ {0..1} × T))) ?S0
         closedin (top_of_set (?S1 ∪ {0..1} × T)) ?S1
      using ‹T ⊆ S› by (force intro: closedin_Times closedin_subset_trans [of
{0..1} × S])+
    show closedin (top_of_set (?S0 ∪ (?S1 ∪ {0..1} × T))) (?S1 ∪ {0..1} × T)
         closedin (top_of_set (?S1 ∪ {0..1} × T)) ({0..1} × T)
      using ‹T ⊆ S› by (force intro: clo closedin_Times closedin_subset_trans [of
{0..1} × S])+
    show continuous_on (?S0) (λx. f (snd x))
      by (intro continuous_intros continuous_on_compose2 [OF contf]) auto
    show continuous_on (?S1) (λx. g (snd x))
      by (intro continuous_intros continuous_on_compose2 [OF contg]) auto
  qed (use h0 h1 conth in auto)
  then have continuous_on ({0,1} × S ∪ ({0..1} × T)) h'
    by (metis Sigma_Un_distrib1 Un_assoc insert_is_Un)
  moreover have h' ' ({0,1} × S ∪ {0..1} × T) ⊆ U
    using fim gim him ‹T ⊆ S› unfolding h'_def by force
```

moreover have *closedin* (*top_of_set* ({*0..1*::*real*} × *S*)) ({*0*,*1*} × *S* ∪ {*0..1*::*real*} × *T*)
by (*intro closedin_Times closedin_Un clo*) (*simp_all add*: *closed_subset*)
ultimately
obtain *W k* **where** *W*: ({*0*,*1*} × *S*) ∪ ({*0..1*} × *T*) ⊆ *W*
and *opeW*: *openin* (*top_of_set* ({*0..1*} × *S*)) *W*
and *contk*: *continuous_on W k*
and *kim*: *k* ∈ *W* → *U*
and *kh'*: ⋀*x*. *x* ∈ ({*0*,*1*} × *S*) ∪ ({*0..1*} × *T*) ⟹ *k x* = *h' x*
by (*metis ANR_imp_absolute_neighbourhood_extensor* [*OF* ‹*ANR U*›, *of* ({*0*,*1*} × *S*) ∪ ({*0..1*} × *T*) *h'* {*0..1*} × *S*] *image_subset_iff_funcset*)
obtain *T'* **where** *opeT'*: *openin* (*top_of_set S*) *T'*
and *T* ⊆ *T'* **and** *TW*: {*0..1*} × *T'* ⊆ *W*
using *tube_lemma_gen* [*of* {*0..1*::*real*} *T S W*] *W* ‹*T* ⊆ *S*› *opeW* **by** *auto*
moreover have *homotopic_with_canon* (λ*x*. *True*) *T' U f g*
proof (*simp add*: *homotopic_with*, *intro exI conjI*)
show *continuous_on* ({*0..1*} × *T'*) *k*
using *TW continuous_on_subset contk* **by** *auto*
show *k* ‘ ({*0..1*} × *T'*) ⊆ *U*
using *TW kim* **by** *fastforce*
have *T'* ⊆ *S*
by (*meson opeT' subsetD openin_imp_subset*)
then show ∀ *x*∈*T'*. *k* (*0*, *x*) = *f x* ∀ *x*∈*T'*. *k* (*1*, *x*) = *g x*
by (*auto simp*: *kh' h'_def*)
qed
ultimately show *?thesis*
by (*blast intro*: *that*)
qed

Homotopy on a union of closed-open sets.

proposition *homotopic_on_clopen_Union*:
fixes *F* :: *'a*::*euclidean_space set set*
assumes ⋀*S*. *S* ∈ *F* ⟹ *closedin* (*top_of_set* (⋃*F*)) *S*
and ⋀*S*. *S* ∈ *F* ⟹ *openin* (*top_of_set* (⋃*F*)) *S*
and ⋀*S*. *S* ∈ *F* ⟹ *homotopic_with_canon* (λ*x*. *True*) *S T f g*
shows *homotopic_with_canon* (λ*x*. *True*) (⋃*F*) *T f g*
proof −
obtain *V* **where** *V* ⊆ *F countable V* **and** *eqU*: ⋃*V* = ⋃*F*
using *Lindelof_openin assms* **by** *blast*
show *?thesis*
proof (*cases V* = {})
case *True*
then show *?thesis*
by (*metis Union_empty eqU homotopic_with_canon_on_empty*)
next
case *False*
then obtain *V* :: *nat* ⇒ *'a set* **where** *V*: *range V* = *V*
using *range_from_nat_into* ‹*countable V*› **by** *metis*
with ‹*V* ⊆ *F*› **have** *clo*: ⋀*n*. *closedin* (*top_of_set* (⋃*F*)) (*V n*)

```
            and ope: ⋀n. openin (top_of_set (⋃ℱ)) (V n)
            and hom: ⋀n. homotopic_with_canon (λx. True) (V n) T f g
    using assms by auto
  then obtain h where conth: ⋀n. continuous_on ({0..1::real} × V n) (h n)
            and him: ⋀n. h n ` ({0..1} × V n) ⊆ T
            and h0: ⋀n. ⋀x. x ∈ V n ⟹ h n (0, x) = f x
            and h1: ⋀n. ⋀x. x ∈ V n ⟹ h n (1, x) = g x
    by (simp add: homotopic_with) metis
  have wop: b ∈ V x ⟹ ∃k. b ∈ V k ∧ (∀j<k. b ∉ V j) for b x
    using nat_less_induct [where P = λi. b ∉ V i] by meson
  obtain ζ where cont: continuous_on ({0..1} × ⋃(V ` UNIV)) ζ
        and eq: ⋀x i. ⟦x ∈ {0..1} × ⋃(V ` UNIV) ∩
                        {0..1} × (V i − (⋃m<i. V m))⟧ ⟹ ζ x = h i x
  proof (rule pasting_lemma_exists)
    let ?X = top_of_set ({0..1::real} × ⋃(range V))
    show topspace ?X ⊆ (⋃i. {0..1::real} × (V i − (⋃m<i. V m)))
      by (force simp: Ball_def dest: wop)
    show openin (top_of_set ({0..1} × ⋃(V ` UNIV)))
           ({0..1::real} × (V i − (⋃m<i. V m))) for i
    proof (intro openin_Times openin_subtopology_self openin_diff)
      show openin (top_of_set (⋃(V ` UNIV))) (V i)
        using ope V eqU by auto
      show closedin (top_of_set (⋃(V ` UNIV))) (⋃m<i. V m)
        using V clo eqU by (force intro: closedin_Union)
    qed
    show continuous_map (subtopology ?X ({0..1} × (V i − ⋃ (V ` {..<i})))) 
euclidean (h i) for i
      by (auto simp add: subtopology_subtopology intro!: continuous_on_subset 
[OF conth])
    show ⋀i j x. x ∈ topspace ?X ∩ {0..1} × (V i − (⋃m<i. V m)) ∩ {0..1} 
× (V j − (⋃m<j. V m))
              ⟹ h i x = h j x
      by clarsimp (metis lessThan_iff linorder_neqE_nat)
  qed auto
  show ?thesis
  proof (simp add: homotopic_with eqU [symmetric], intro exI conjI ballI)
    show continuous_on ({0..1} × ⋃𝒱) ζ
      using V eqU by (blast intro!: continuous_on_subset [OF cont])
    show ζ ` ({0..1} × ⋃𝒱) ⊆ T
    proof clarsimp
      fix t :: real and y :: 'a and X :: 'a set
      assume y ∈ X X ∈ 𝒱 and t: 0 ≤ t t ≤ 1
      then obtain k where y ∈ V k and j: ∀j<k. y ∉ V j
        by (metis image_iff V wop)
      with him t show ζ(t, y) ∈ T
        by (subst eq) force+
    qed
    fix X y
    assume X ∈ 𝒱 y ∈ X
```

```
      then obtain k where y ∈ V k and j: ∀j<k. y ∉ V j
        by (metis image_iff V wop)
      then show ζ(0, y) = f y and ζ(1, y) = g y
        by (subst eq [where i=k]; force simp: h0 h1)+
    qed
  qed
qed

lemma homotopic_on_components_eq:
  fixes S :: 'a :: euclidean_space set and T :: 'b :: euclidean_space set
  assumes S: locally connected S ∨ compact S and ANR T
  shows homotopic_with_canon (λx. True) S T f g ⟷
        (continuous_on S f ∧ f ` S ⊆ T ∧ continuous_on S g ∧ g ` S ⊆ T) ∧
        (∀ C ∈ components S. homotopic_with_canon (λx. True) C T f g)
    (is ?lhs ⟷ ?C ∧ ?rhs)
proof −
  have continuous_on S f f ` S ⊆ T continuous_on S g g ` S ⊆ T if ?lhs
    using homotopic_with_imp_continuous homotopic_with_imp_subset1 homo-
topic_with_imp_subset2 that by blast+
  moreover have ?lhs ⟷ ?rhs
    if contf: continuous_on S f and fim: f ` S ⊆ T and contg: continuous_on S g
and gim: g ` S ⊆ T
  proof
    assume ?lhs
    with that show ?rhs
      by (simp add: homotopic_with_subset_left in_components_subset)
  next
    assume R: ?rhs
    have ∃ U. C ⊆ U ∧ closedin (top_of_set S) U ∧
              openin (top_of_set S) U ∧
               homotopic_with_canon (λx. True) U T f g if C: C ∈ components S
for C
    proof −
      have C ⊆ S
        by (simp add: in_components_subset that)
      show ?thesis
        using S
      proof
        assume locally connected S
        show ?thesis
        proof (intro exI conjI)
          show closedin (top_of_set S) C
            by (simp add: closedin_component that)
          show openin (top_of_set S) C
          by (simp add: ‹locally connected S› openin_components_locally_connected
that)
          show homotopic_with_canon (λx. True) C T f g
            by (simp add: R that)
        qed auto
```

```
      next
        assume compact S
        have hom: homotopic_with_canon (λx. True) C T f g
          using R that by blast
        obtain U where C ⊆ U and opeU: openin (top_of_set S) U
                   and hom: homotopic_with_canon (λx. True) U T f g
            using homotopic_neighbourhood_extension [OF contf fim contg gim _
‹ANR T› hom]
            ‹C ∈ components S› closedin_component by blast
        then obtain V where open V and V: U = S ∩ V
          by (auto simp: openin_open)
        moreover have locally compact S
       by (simp add: ‹compact S› closed_imp_locally_compact compact_imp_closed)
       ultimately obtain K where opeK: openin (top_of_set S) K and compact
K C ⊆ K K ⊆ V
          by (metis C Int_subset_iff Sura_Bura_clopen_subset ‹C ⊆ U› ‹compact
S› compact_components)
        show ?thesis
        proof (intro exI conjI)
          show closedin (top_of_set S) K
                 by (meson ‹compact K› ‹compact S› closedin_compact_eq opeK
openin_imp_subset)
          show homotopic_with_canon (λx. True) K T f g
         using V ‹K ⊆ V› hom homotopic_with_subset_left opeK openin_imp_subset
by fastforce
        qed (use opeK ‹C ⊆ K› in auto)
      qed
    qed
    then obtain φ where φ: ⋀C. C ∈ components S ⟹ C ⊆ φ C
                   and cloφ: ⋀C. C ∈ components S ⟹ closedin (top_of_set S) (φ
C)
                    and opeφ: ⋀C. C ∈ components S ⟹ openin (top_of_set S) (φ
C)
                     and homφ: ⋀C. C ∈ components S ⟹ homotopic_with_canon
(λx. True) (φ C) T f g
      by metis
    have Seq: S = ⋃ (φ ‘ components S)
    proof
      show S ⊆ ⋃ (φ ‘ components S)
        by (metis Sup_mono Union_components φ imageI)
      show ⋃ (φ ‘ components S) ⊆ S
        using opeφ openin_imp_subset by fastforce
    qed
    show ?lhs
      apply (subst Seq)
      using Seq cloφ opeφ homφ by (intro homotopic_on_clopen_Union) auto
  qed
  ultimately show ?thesis by blast
qed
```

lemma *cohomotopically_trivial_on_components*:
fixes $S :: 'a :: euclidean\_space\ set$ **and** $T :: 'b :: euclidean\_space\ set$
assumes S: *locally connected* $S \vee$ *compact* S **and** $ANR\ T$
shows
$(\forall f\ g.\ continuous\_on\ S\ f \longrightarrow f \in S \to T \longrightarrow continuous\_on\ S\ g \longrightarrow g \in S \to T \longrightarrow$
$homotopic\_with\_canon\ (\lambda x.\ True)\ S\ T\ f\ g)$
$\longleftrightarrow$
$(\forall\ C{\in}components\ S.$
$\forall f\ g.\ continuous\_on\ C\ f \longrightarrow f \in C \to T \longrightarrow continuous\_on\ C\ g \longrightarrow g \in C \to T \longrightarrow$
$homotopic\_with\_canon\ (\lambda x.\ True)\ C\ T\ f\ g)$
(**is** *?lhs* = *?rhs*)
proof
assume *L*[*rule_format*]: *?lhs*
show *?rhs*
proof *clarify*
fix $C\ f\ g$
assume *contf*: *continuous_on* $C\ f$ **and** *fim*: $f \in C \to T$
and *contg*: *continuous_on* $C\ g$ **and** *gim*: $g \in C \to T$ **and** C: $C \in components\ S$
obtain f' **where** *contf'*: *continuous_on* $S\ f'$ **and** *f'im*: $f' \in S \to T$ **and** *f'f*: $\bigwedge x.\ x \in C \Longrightarrow f'\ x = f\ x$
using *extension_from_component* [*OF S* ‹*ANR T*› *C contf fim*] **by** *metis*
obtain g' **where** *contg'*: *continuous_on* $S\ g'$ **and** *g'im*: $g' \in S \to T$ **and** *g'g*: $\bigwedge x.\ x \in C \Longrightarrow g'\ x = g\ x$
using *extension_from_component* [*OF S* ‹*ANR T*› *C contg gim*] **by** *metis*
have *homotopic_with_canon* $(\lambda x.\ True)\ C\ T\ f'\ g'$
using *L* [*OF contf' f'im contg' g'im*] *homotopic_with_subset_left C in_components_subset* **by** *fastforce*
then show *homotopic_with_canon* $(\lambda x.\ True)\ C\ T\ f\ g$
using *f'f g'g homotopic_with_eq* **by** *force*
qed
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
proof *clarify*
fix $f\ g$
assume *contf*: *continuous_on* $S\ f$ **and** *fim*: $f \in S \to T$
and *contg*: *continuous_on* $S\ g$ **and** *gim*: $g \in S \to T$
moreover have *homotopic_with_canon* $(\lambda x.\ True)\ C\ T\ f\ g$ **if** $C \in components\ S$ **for** C
using *R* [*OF that*] *contf contg continuous_on_subset fim gim in_components_subset*
by (*smt* (*verit*, *del_insts*) *Pi_anti_mono subsetD that*)
ultimately show *homotopic_with_canon* $(\lambda x.\ True)\ S\ T\ f\ g$
by (*subst homotopic_on_components_eq* [*OF S* ‹*ANR T*›]) *auto*
qed

qed

9.26.8 The complement of a set and path-connectedness

Complement in dimension N > 1 of set homeomorphic to any interval in any dimension is (path-)connected. This naively generalizes the argument in Ryuji Maehara's paper "The Jordan curve theorem via the Brouwer fixed point theorem", American Mathematical Monthly 1984.

lemma *unbounded_components_complement_absolute_retract*:
fixes S :: *'a*::*euclidean_space set*
assumes C: $C \in$ *components*$(- S)$ **and** S: *compact S AR S*
shows ¬ *bounded C*
proof −
obtain y **where** y: $C =$ *connected_component_set* $(- S)$ y **and** $y \notin S$
using C **by** (*auto simp*: *components_def*)
have *open*$(- S)$
using S **by** (*simp add*: *closed_open compact_eq_bounded_closed*)
have S *retract_of UNIV*
using S *compact_AR* **by** *blast*
then obtain r **where** *contr*: *continuous_on UNIV r* **and** *ontor*: *range* $r \subseteq S$
and r: $\bigwedge x.\ x \in S \Longrightarrow r\ x = x$
by (*auto simp*: *retract_of_def retraction_def*)
show *?thesis*
proof
assume *bounded C*
have *connected_component_set* $(- S)$ $y \subseteq S$
proof (*rule frontier_subset_retraction*)
show *bounded* (*connected_component_set* $(- S)$ y)
using ‹*bounded C*› y **by** *blast*
show *frontier* (*connected_component_set* $(- S)$ y) $\subseteq S$
using C ‹*compact S*› *compact_eq_bounded_closed frontier_of_components_closed_complement*
y **by** *blast*
show *continuous_on* (*closure* (*connected_component_set* $(- S)$ y)) r
by (*blast intro*: *continuous_on_subset* [*OF contr*])
qed (*use ontor r* **in** *auto*)
with ‹$y \notin S$› **show** *False* **by** *force*
qed
qed

lemma *connected_complement_absolute_retract*:
fixes S :: *'a*::*euclidean_space set*
assumes S: *compact S AR S* **and** *2*: $2 \leq$ *DIM*(*'a*)
shows *connected*$(- S)$
proof −
have S *retract_of UNIV*
using S *compact_AR* **by** *blast*
show *?thesis*
proof (*clarsimp simp*: *connected_iff_connected_component_eq*)

```
    have ¬ bounded (connected_component_set (− S) x) if x ∉ S for x
    by (meson Compl_iff assms componentsI that unbounded_components_complement_absolute_retract)
    then show connected_component_set (− S) x = connected_component_set
(− S) y
      if x ∉ S y ∉ S for x y
      using cobounded_unique_unbounded_component [OF _ 2]
      by (metis ‹compact S› compact_imp_bounded double_compl that)
  qed
qed

lemma path_connected_complement_absolute_retract:
  fixes S :: 'a::euclidean_space set
  assumes compact S AR S 2 ≤ DIM('a)
    shows path_connected(− S)
  using connected_complement_absolute_retract [OF assms]
 using ‹compact S› compact_eq_bounded_closed connected_open_path_connected
by blast

theorem connected_complement_homeomorphic_convex_compact:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes hom: S homeomorphic T and T: convex T compact T and 2: 2 ≤
DIM('a)
    shows connected(− S)
proof (cases S = {})
  case True
  then show ?thesis
    by (simp add: connected_UNIV)
next
  case False
  show ?thesis
  proof (rule connected_complement_absolute_retract)
    show compact S
      using ‹compact T› hom homeomorphic_compactness by auto
    show AR S
    by (meson AR_ANR False ‹convex T› convex_imp_ANR convex_imp_contractible
hom homeomorphic_ANR_iff_ANR homeomorphic_contractible_eq)
  qed (rule 2)
qed

corollary path_connected_complement_homeomorphic_convex_compact:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes hom: S homeomorphic T convex T compact T 2 ≤ DIM('a)
    shows path_connected(− S)
  using connected_complement_homeomorphic_convex_compact [OF assms]
 using ‹compact T› compact_eq_bounded_closed connected_open_path_connected
hom homeomorphic_compactness by blast

lemma path_connected_complement_homeomorphic_interval:
  fixes S :: 'a::euclidean_space set
```

assumes *S homeomorphic cbox a b* 2 ≤ *DIM*('*a*)
shows *path_connected*(−*S*)
using *assms compact_cbox convex_box*(1) *path_connected_complement_homeomorphic_convex_comp*
by *blast*

lemma *connected_complement_homeomorphic_interval*:
fixes *S* :: '*a*::*euclidean_space set*
assumes *S homeomorphic cbox a b* 2 ≤ *DIM*('*a*)
shows *connected*(−*S*)
using *assms path_connected_complement_homeomorphic_interval path_connected_imp_connected*
by *blast*

end

9.27 Extending Continous Maps, Invariance of Domain, etc

Ported from HOL Light (moretop.ml) by L C Paulson

theory *Further_Topology*
imports *Weierstrass_Theorems Polytope Complex_Transcendental Equivalence_Lebesgue_Henstock_I*
Retracts
begin

9.27.1 A map from a sphere to a higher dimensional sphere is nullhomotopic

lemma *spheremap_lemma1*:
fixes *f* :: '*a*::*euclidean_space* ⇒ '*a*::*euclidean_space*
assumes *subspace S subspace T* **and** *dimST*: *dim S* < *dim T*
and *S* ⊆ *T*
and *diff_f*: *f differentiable_on sphere* 0 1 ∩ *S*
shows *f* ' (*sphere* 0 1 ∩ *S*) ≠ *sphere* 0 1 ∩ *T*
proof
assume *fim*: *f* ' (*sphere* 0 1 ∩ *S*) = *sphere* 0 1 ∩ *T*
have *inS*: ⋀*x*. ⟦*x* ∈ *S*; *x* ≠ 0⟧ ⟹ (*x* /*R norm x*) ∈ *S*
using *subspace_mul* ‹*subspace S*› **by** *blast*
have *subS01*: (λ*x*. *x* /*R norm x*) ' (*S* − {0}) ⊆ *sphere* 0 1 ∩ *S*
using ‹*subspace S*› *subspace_mul* **by** *fastforce*
then have *diff_f'*: *f differentiable_on* (λ*x*. *x* /*R norm x*) ' (*S* − {0})
by (*rule differentiable_on_subset* [*OF diff_f*])
define *g* **where** *g* ≡ λ*x*. *norm x* ∗*R f*(*inverse*(*norm x*) ∗*R x*)
have *gdiff*: *g differentiable_on S* − {0}
unfolding *g_def*
by (*rule diff_f' derivative_intros differentiable_on_compose* [**where** *f*=*f*] |
force)+
have *geq*: *g* ' (*S* − {0}) = *T* − {0}
proof
have ⋀*u*. ⟦*u* ∈ *S*; *norm u* ∗*R f* (*u* /*R norm u*) ∉ *T*⟧ ⟹ *u* = 0

```
    by (metis (mono_tags, lifting) DiffI subS01 subspace_mul [OF ‹subspace T›]
fim image_subset_iff inf_le2 singletonD)
  then have g ∈ (S - {0}) → T
    using g_def by blast
  moreover have g ∈ (S - {0}) → UNIV - {0}
  proof (clarsimp simp: g_def)
    fix y
    assume y ∈ S and f0: f (y /R norm y) = 0
    then have y ≠ 0 ⟹ y /R norm y ∈ sphere 0 1 ∩ S
      by (auto simp: subspace_mul [OF ‹subspace S›])
    then show y = 0
    by (metis fim f0 Int_iff image_iff mem_sphere_0 norm_eq_zero zero_neq_one)
  qed
  ultimately show g ‘ (S - {0}) ⊆ T - {0}
    by auto
next
  have ∗: sphere 0 1 ∩ T ⊆ f ‘ (sphere 0 1 ∩ S)
    using fim by (simp add: image_subset_iff)
  have x ∈ (λx. norm x ∗R f (x /R norm x)) ‘ (S - {0})
       if x ∈ T x ≠ 0 for x
  proof -
    have x /R norm x ∈ T
      using ‹subspace T› subspace_mul that by blast
    then obtain u where u: f u ∈ T x /R norm x = f u norm u = 1 u ∈ S
      using ∗ [THEN subsetD, of x /R norm x] ‹x ≠ 0› by auto
    with that have [simp]: norm x ∗R f u = x
      by (metis divideR_right norm_eq_zero)
    moreover have norm x ∗R u ∈ S - {0}
      using ‹subspace S› subspace_scale that(2) u by auto
    with u show ?thesis
      by (simp add: image_eqI [where x=norm x ∗R u])
  qed
  then have T - {0} ⊆ (λx. norm x ∗R f (x /R norm x)) ‘ (S - {0})
    by force
  then show T - {0} ⊆ g ‘ (S - {0})
    by (simp add: g_def)
qed
define T' where T' ≡ {y. ∀x ∈ T. orthogonal x y}
have subspace T'
  by (simp add: subspace_orthogonal_to_vectors T'_def)
have dim_eq: dim T' + dim T = DIM('a)
  using dim_subspace_orthogonal_to_vectors [of T UNIV] ‹subspace T›
  by (simp add: T'_def)
have ∃v1 v2. v1 ∈ span T ∧ (∀w ∈ span T. orthogonal v2 w) ∧ x = v1 + v2
for x
  by (force intro: orthogonal_subspace_decomp_exists [of T x])
then obtain p1 p2 where p1span: p1 x ∈ span T
                    and ⋀w. w ∈ span T ⟹ orthogonal (p2 x) w
                    and eq: p1 x + p2 x = x for x
```

```
    by metis
  then have p1: ⋀z. p1 z ∈ T and ortho: ⋀w. w ∈ T ⟹ orthogonal (p2 x) w
for x
    using span_eq_iff ‹subspace T› by blast+
  then have p2: ⋀z. p2 z ∈ T'
    by (simp add: T'_def orthogonal_commute)
  have p12_eq: ⋀x y. ⟦x ∈ T; y ∈ T'⟧ ⟹ p1(x + y) = x ∧ p2(x + y) = y
  proof (rule orthogonal_subspace_decomp_unique [OF eq p1span, where T=T'])
    show ⋀x y. ⟦x ∈ T; y ∈ T'⟧ ⟹ p2 (x + y) ∈ span T'
      using span_eq_iff p2 ‹subspace T'› by blast
    show ⋀a b. ⟦a ∈ T; b ∈ T'⟧ ⟹ orthogonal a b
      using T'_def by blast
  qed (auto simp: span_base)
  then have ⋀c x. p1 (c *R x) = c *R p1 x ∧ p2 (c *R x) = c *R p2 x
  proof −
    fix c :: real and x :: 'a
    have f1: c *R x = c *R p1 x + c *R p2 x
      by (metis eq pth_6)
    have f2: c *R p2 x ∈ T'
      by (simp add: ‹subspace T'› p2 subspace_scale)
    have c *R p1 x ∈ T
      by (metis (full_types) assms(2) p1span span_eq_iff subspace_scale)
    then show p1 (c *R x) = c *R p1 x ∧ p2 (c *R x) = c *R p2 x
      using f2 f1 p12_eq by presburger
  qed
  moreover have lin_add: ⋀x y. p1 (x + y) = p1 x + p1 y ∧ p2 (x + y) = p2
x + p2 y
  proof (rule orthogonal_subspace_decomp_unique [OF _ p1span, where T=T'])
    show ⋀x y. p1 (x + y) + p2 (x + y) = p1 x + p1 y + (p2 x + p2 y)
      by (simp add: add.assoc add.left_commute eq)
    show  ⋀a b. ⟦a ∈ T; b ∈ T'⟧ ⟹ orthogonal a b
      using T'_def by blast
  qed (auto simp: p1span p2 span_base span_add)
  ultimately have linear p1 linear p2
    by unfold_locales auto
  have g differentiable_on p1 ` {x + y |x y. x ∈ S − {0} ∧ y ∈ T'}
    using p12_eq ‹S ⊆ T›  by (force intro: differentiable_on_subset [OF gdiff])
  then have (λz. g (p1 z)) differentiable_on {x + y |x y. x ∈ S − {0} ∧ y ∈ T'}
     by (rule differentiable_on_compose [OF linear_imp_differentiable_on [OF
‹linear p1›]])
  then have diff: (λx. g (p1 x) + p2 x) differentiable_on {x + y |x y. x ∈ S −
{0} ∧ y ∈ T'}
    by (intro derivative_intros linear_imp_differentiable_on [OF ‹linear p2›])
  have dim {x + y |x y. x ∈ S − {0} ∧ y ∈ T'} ≤ dim {x + y |x y. x ∈ S  ∧ y
∈ T'}
    by (blast intro: dim_subset)
  also have ... = dim S + dim T' − dim (S ∩ T')
    using dim_sums_Int [OF ‹subspace S› ‹subspace T'›]
    by (simp add: algebra_simps)
```

```
  also have ... < DIM('a)
    using dimST dim_eq by auto
  finally have neg: negligible {x + y |x y. x ∈ S - {0} ∧ y ∈ T'}
    by (rule negligible_lowdim)
  have negligible ((λx. g (p1 x) + p2 x) ` {x + y |x y. x ∈ S - {0} ∧ y ∈ T'})
    by (rule negligible_differentiable_image_negligible [OF order_refl neg diff])
  then have negligible {x + y |x y. x ∈ g ` (S - {0}) ∧ y ∈ T'}
  proof (rule negligible_subset)
    have ⟦t' ∈ T'; s ∈ S; s ≠ 0⟧
         ⟹ g s + t' ∈ (λx. g (p1 x) + p2 x) `
                      {x + t' |x t'. x ∈ S ∧ x ≠ 0 ∧ t' ∈ T'} for t' s
      using ‹S ⊆ T› p12_eq  by (rule_tac x=s + t' in image_eqI) auto
    then show {x + y |x y. x ∈ g ` (S - {0}) ∧ y ∈ T'}
         ⊆ (λx. g (p1 x) + p2 x) ` {x + y |x y. x ∈ S - {0} ∧ y ∈ T'}
      by auto
  qed
  moreover have - T' ⊆ {x + y |x y. x ∈ g ` (S - {0}) ∧ y ∈ T'}
  proof clarsimp
    fix z assume z ∉ T'
    show ∃x y. z = x + y ∧ x ∈ g ` (S - {0}) ∧ y ∈ T'
      by (metis Diff_iff ‹z ∉ T'› add.left_neutral eq geq p1 p2 singletonD)
  qed
  ultimately have negligible (-T')
    using negligible_subset by blast
  moreover have negligible T'
    using negligible_lowdim
     by (metis add.commute assms(3) diff_add_inverse2 diff_self_eq_0 dim_eq
le_add1 le_antisym linordered_semidom_class.add_diff_inverse not_less0)
  ultimately have  negligible (-T' ∪ T')
    by (metis negligible_Un_eq)
  then show False
    using negligible_Un_eq non_negligible_UNIV by simp
qed


lemma spheremap_lemma2:
  fixes f :: 'a::euclidean_space ⇒ 'a::euclidean_space
  assumes ST: subspace S subspace T dim S < dim T
      and S ⊆ T
      and contf: continuous_on (sphere 0 1 ∩ S) f
      and fim: f ∈ (sphere 0 1 ∩ S) → sphere 0 1 ∩ T
    shows ∃c. homotopic_with_canon (λx. True) (sphere 0 1 ∩ S) (sphere 0 1 ∩
T) f (λx. c)
proof -
  have [simp]: ⋀x. ⟦norm x = 1; x ∈ S⟧ ⟹ norm (f x) = 1
    using fim by auto
  have compact (sphere 0 1 ∩ S)
    by (simp add: ‹subspace S› closed_subspace compact_Int_closed)
  then obtain g where pfg: polynomial_function g and gim: g ∈ (sphere 0 1 ∩
```

```
S) → T
            and g12: ⋀x. x ∈ sphere 0 1 ∩ S ⟹ norm(f x − g x) < 1/2
    using Stone_Weierstrass_polynomial_function_subspace [OF _ contf _ ‹subspace T›, of 1/2] fim
    by (auto simp: image_subset_iff_funcset)
  have gnz: g x ≠ 0 if x ∈ sphere 0 1 ∩ S for x
    using g12 that by fastforce
  have diffg: g differentiable_on sphere 0 1 ∩ S
    by (metis pfg differentiable_on_polynomial_function)
  define h where h ≡ λx. inverse(norm(g x)) *_R g x
  have h: x ∈ sphere 0 1 ∩ S ⟹ h x ∈ sphere 0 1 ∩ T for x
    unfolding h_def using ‹subspace T› gim gnz subspace_mul by fastforce
  have diffh: h differentiable_on sphere 0 1 ∩ S
    unfolding h_def using gnz
   by (fastforce intro: derivative_intros diffg differentiable_on_compose [OF diffg])
  have homfg: homotopic_with_canon (λz. True) (sphere 0 1 ∩ S) (T − {0}) f g
  proof (rule homotopic_with_linear [OF contf])
    show continuous_on (sphere 0 1 ∩ S) g
      using pfg by (simp add: differentiable_imp_continuous_on diffg)
  next
    have non0fg: 0 ∉ closed_segment (f x) (g x) if norm x = 1 x ∈ S for x
    proof −
      have f x ∈ sphere 0 1
        using fim that by (simp add: image_subset_iff)
      moreover have norm(f x − g x) < 1/2
        using g12 that by auto
      ultimately show ?thesis
        by (auto simp: norm_minus_commute dest: segment_bound)
    qed
    show closed_segment (f x) (g x) ⊆ T − {0} if x ∈ sphere 0 1 ∩ S for x
    proof −
      have convex T
        by (simp add: ‹subspace T› subspace_imp_convex)
      then have convex hull {f x, g x} ⊆ T
        by (metis IntD2 PiE closed_segment_subset fim gim segment_convex_hull that)
      then show ?thesis
        using that non0fg segment_convex_hull by fastforce
    qed
  qed
  obtain d where d: d ∈ (sphere 0 1 ∩ T) − h ‘ (sphere 0 1 ∩ S)
    using h spheremap_lemma1 [OF ST ‹S ⊆ T› diffh] by force
  then have non0hd: 0 ∉ closed_segment (h x) (− d) if norm x = 1 x ∈ S for x
    using midpoint_between [of 0 h x −d] that h [of x]
    by (auto simp: between_mem_segment midpoint_def)
  have conth: continuous_on (sphere 0 1 ∩ S) h
    using differentiable_imp_continuous_on diffh by blast
  have hom_hd: homotopic_with_canon (λz. True) (sphere 0 1 ∩ S) (T − {0})
h (λx. −d)
```

```
  proof (rule homotopic_with_linear [OF conth continuous_on_const])
    fix x
    assume x: x ∈ sphere 0 1 ∩ S
    have convex hull {h x, − d} ⊆ T
    proof (rule hull_minimal)
      show {h x, − d} ⊆ T
        using h d x by (force simp: subspace_neg [OF ‹subspace T›])
    qed (simp add: subspace_imp_convex [OF ‹subspace T›])
    with x segment_convex_hull show closed_segment (h x) (− d) ⊆ T − {0}
      by (auto simp add: subset_Diff_insert non0hd)
  qed
  have conT0: continuous_on (T − {0}) (λy. inverse(norm y) *R y)
    by (intro continuous_intros) auto
  have sub0T: (λy. y /R norm y) ∈ (T − {0}) → sphere 0 1 ∩ T
    by (fastforce simp: assms(2) subspace_mul)
  obtain c where homhc: homotopic_with_canon (λz. True) (sphere 0 1 ∩ S)
(sphere 0 1 ∩ T) h (λx. c)
  proof
    show homotopic_with_canon (λz. True) (sphere 0 1 ∩ S) (sphere 0 1 ∩ T) h
(λx. − d)
      using d
      by (force simp: h_def
          intro: homotopic_with_eq homotopic_with_compose_continuous_left [OF
hom_hd conT0 sub0T])
  qed
  have homotopic_with_canon (λx. True) (sphere 0 1 ∩ S) (sphere 0 1 ∩ T) f h
    by (force simp: h_def
            intro: homotopic_with_eq homotopic_with_compose_continuous_left
[OF homfg conT0 sub0T])
  then show ?thesis
    by (metis homotopic_with_trans [OF _ homhc])
qed


lemma spheremap_lemma3:
  assumes bounded S convex S subspace U and affSU: aff_dim S ≤ dim U
  obtains T where subspace T T ⊆ U S ≠ {} ⟹ aff_dim T = aff_dim S
                (rel_frontier S) homeomorphic (sphere 0 1 ∩ T)
proof (cases S = {})
  case True
  with ‹subspace U› subspace_0 show ?thesis
    by (rule_tac T = {0} in that) auto
next
  case False
  then obtain a where a ∈ S
    by auto
  then have affS: aff_dim S = int (dim ((λx. −a+x) ‘ S))
    by (metis hull_inc aff_dim_eq_dim)
  with affSU have dim ((λx. −a+x) ‘ S) ≤ dim U
```

```
    by linarith
  with choose_subspace_of_subspace
  obtain T where subspace T T ⊆ span U and dimT: dim T = dim ((λx. −a+x)
‘ S) .
  show ?thesis
  proof (rule that [OF ‹subspace T›])
    show T ⊆ U
      using span_eq_iff ‹subspace U› ‹T ⊆ span U› by blast
    show aff_dim T = aff_dim S
      using dimT ‹subspace T› affS aff_dim_subspace by fastforce
    show rel_frontier S homeomorphic sphere 0 1 ∩ T
    proof −
      have aff_dim (ball 0 1 ∩ T) = aff_dim (T)
      by (metis IntI interior_ball ‹subspace T› aff_dim_convex_Int_nonempty_interior
centre_in_ball empty_iff inf_commute subspace_0 subspace_imp_convex zero_less_one)
      then have affS_eq: aff_dim S = aff_dim (ball 0 1 ∩ T)
        using ‹aff_dim T = aff_dim S› by simp
      have rel_frontier S homeomorphic rel_frontier(ball 0 1 ∩ T)
        using homeomorphic_rel_frontiers_convex_bounded_sets [OF ‹convex S›
‹bounded S›]
        by (simp add: ‹subspace T› affS_eq assms bounded_Int convex_Int
         homeomorphic_rel_frontiers_convex_bounded_sets subspace_imp_convex)
      also have ... = frontier (ball 0 1) ∩ T
      proof (rule convex_affine_rel_frontier_Int [OF convex_ball])
        show affine T
          by (simp add: ‹subspace T› subspace_imp_affine)
        show interior (ball 0 1) ∩ T ≠ {}
          using ‹subspace T› subspace_0 by force
      qed
      also have ... = sphere 0 1 ∩ T
        by auto
      finally show ?thesis .
    qed
  qed
qed


proposition inessential_spheremap_lowdim_gen:
  fixes f :: 'M::euclidean_space ⇒ 'a::euclidean_space
  assumes convex S bounded S convex T bounded T
      and affST: aff_dim S < aff_dim T
      and contf: continuous_on (rel_frontier S) f
      and fim: f ∈ (rel_frontier S) → rel_frontier T
  obtains c where homotopic_with_canon (λz. True) (rel_frontier S) (rel_frontier
T) f (λx. c)
proof (cases S = {})
  case True
  then show ?thesis
    using homotopic_with_canon_on_empty that by auto
```

next
case *False*
then show *?thesis*
proof (*cases* $T = \{\}$)
case *True*
then show *?thesis*
by (*smt* (*verit*, *best*) *False affST aff_dim_negative_iff*)
next
case *False*
obtain T':: *'a set*
where *subspace* T' **and** *affT'*: *aff_dim* $T' =$ *aff_dim* T
and *homT*: *rel_frontier T homeomorphic sphere 0 1* $\cap$ T'
using ‹$T \neq \{\}$› *spheremap_lemma3* [*OF* ‹*bounded T*› ‹*convex T*› *subspace_UNIV*, **where** *'b='a*]
by (*force simp add*: *aff_dim_le_DIM*)
with *homeomorphic_imp_homotopy_eqv*
have *relT*: *sphere 0 1* $\cap$ T' *homotopy_eqv rel_frontier T*
using *homotopy_equivalent_space_sym* **by** *blast*
have *aff_dim S* $\leq$ *int* (*dim T'*)
using *affT'* ‹*subspace T'*› *affST aff_dim_subspace* **by** *force*
with *spheremap_lemma3* [*OF* ‹*bounded S*› ‹*convex S*› ‹*subspace T'*›] ‹$S \neq \{\}$›
obtain S':: *'a set* **where** *subspace* S' $S' \subseteq T'$
and *affS'*: *aff_dim* $S' =$ *aff_dim S*
and *homT*: *rel_frontier S homeomorphic sphere 0 1* $\cap$ S'
by *metis*
with *homeomorphic_imp_homotopy_eqv*
have *relS*: *sphere 0 1* $\cap$ S' *homotopy_eqv rel_frontier S*
using *homotopy_equivalent_space_sym* **by** *blast*
have *dimST'*: *dim* $S' <$ *dim* T'
by (*metis* ‹$S' \subseteq T'$› ‹*subspace* S'› ‹*subspace* T'› *affS' affST affT' less_irrefl not_le subspace_dim_equal*)
have $\exists$ *c. homotopic_with_canon* (λz. *True*) (*rel_frontier S*) (*rel_frontier T*) *f* (λx. *c*)
using *spheremap_lemma2 homotopy_eqv_cohomotopic_triviality_null*[*OF relS*]
using *homotopy_eqv_homotopic_triviality_null_imp* [*OF relT contf fim*]
by (*metis* ‹$S' \subseteq T'$› ‹*subspace* S'› ‹*subspace* T'› *dimST' image_subset_iff_funcset*)
with *that* **show** *?thesis* **by** *blast*
qed
qed

lemma *inessential_spheremap_lowdim*:
fixes *f* :: *'M*::*euclidean_space* $\Rightarrow$ *'a*::*euclidean_space*
assumes
DIM(*'M*) $<$ *DIM*(*'a*) **and** *f*: *continuous_on* (*sphere a r*) *f f* $\in$ (*sphere a r*) $\to$ (*sphere b s*)
obtains *c* **where** *homotopic_with_canon* (λz. *True*) (*sphere a r*) (*sphere b s*) *f* (λx. *c*)
proof (*cases s* $\leq$ *0*)

case *True* **then show** *?thesis*
by (*meson nullhomotopic_into_contractible f contractible_sphere that*)
next
case *False*
show *?thesis*
proof (*cases* $r \leq 0$)
case *True* **then show** *?thesis*
by (*meson f nullhomotopic_from_contractible contractible_sphere that*)
next
case *False*
with ‹$\neg\ s \leq 0$› **have** $r > 0\ s > 0$ **by** *auto*
show *thesis*
using *inessential_spheremap_lowdim_gen* [*of cball a r cball b s f*]
using ‹$0 < r$› ‹$0 < s$› *assms*(*1*) *that* **by** (*auto simp add*: *f aff_dim_cball*)
qed
qed

9.27.2 Some technical lemmas about extending maps from cell complexes

lemma *extending_maps_Union_aux*:
assumes *fin*: *finite* $\mathcal{F}$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *closed* S
and $\bigwedge S\ T.\ [\![S \in \mathcal{F};\ T \in \mathcal{F};\ S \neq T]\!] \Longrightarrow S \cap T \subseteq K$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \exists g.$ *continuous_on* $S\ g \wedge g \in S \to T \wedge (\forall x \in S \cap K.\ g\ x = h\ x)$
shows $\exists g.$ *continuous_on* $(\bigcup \mathcal{F})\ g \wedge g \in (\bigcup \mathcal{F}) \to T \wedge (\forall x \in \bigcup \mathcal{F} \cap K.\ g\ x = h\ x)$
using *assms*
proof (*induction* $\mathcal{F}$)
case *empty* **show** *?case* **by** *simp*
next
case (*insert S* $\mathcal{F}$)
then obtain *f* **where** *contf*: *continuous_on S f* **and** *fim*: $f \in S \to T$ **and** *feq*: $\forall x \in S \cap K.\ f\ x = h\ x$
by (*metis funcset_image insert_iff*)
obtain *g* **where** *contg*: *continuous_on* $(\bigcup \mathcal{F})\ g$ **and** *gim*: $g \in (\bigcup \mathcal{F}) \to T$ **and** *geq*: $\forall x \in \bigcup \mathcal{F} \cap K.\ g\ x = h\ x$
using *insert* **by** *auto*
have *fg*: $f\ x = g\ x$ **if** $x \in T\ T \in \mathcal{F}\ x \in S$ **for** $x\ T$
proof −
have $T \cap S \subseteq K \vee S = T$
using *that* **by** (*metis* (*no_types*) *insert.prems*(*2*) *insertCI*)
then show *?thesis*
using *UnionI feq geq* ‹$S \notin \mathcal{F}$› *subsetD that* **by** *fastforce*
qed
moreover have *continuous_on* $(S \cup \bigcup \mathcal{F})$ $(\lambda x.$ *if* $x \in S$ *then* $f\ x$ *else* $g\ x)$
by (*auto simp*: *insert closed_Union contf contg intro*: *fg continuous_on_cases*)
moreover have $S \cup \bigcup \mathcal{F} = \bigcup$ (*insert S* $\mathcal{F}$)

by *auto*
ultimately show *?case*
by (*smt* (*verit*) *Int_iff Pi_iff UnE feq fim geq gim*)
qed

lemma *extending_maps_Union*:
assumes *fin*: *finite* $\mathcal{F}$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \exists g.$ *continuous_on* $S\ g \wedge g \in S \to T \wedge (\forall x \in S \cap K.\ g\ x = h\ x)$
and $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *closed* S
and K: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F};\ \neg\ X \subseteq Y;\ \neg\ Y \subseteq X]\!] \Longrightarrow X \cap Y \subseteq K$
shows $\exists g.$ *continuous_on* $(\bigcup \mathcal{F})\ g \wedge g \in (\bigcup \mathcal{F}) \to T \wedge (\forall x \in \bigcup \mathcal{F} \cap K.\ g\ x = h\ x)$
proof −
have $\bigwedge S\ T.\ [\![S \in \mathcal{F}; \forall\ U{\in}\mathcal{F}.\ \neg\ S \subset U;\ T \in \mathcal{F}; \forall\ U{\in}\mathcal{F}.\ \neg\ T \subset U;\ S \neq T]\!] \Longrightarrow S \cap T \subseteq K$
by (*metis K psubsetI*)
then show *?thesis*
apply (*simp flip*: *Union_maximal_sets* [*OF fin*])
apply (*rule extending_maps_Union_aux*, *simp_all add*: *Union_maximal_sets* [*OF fin*] *assms*)
done
qed

lemma *extend_map_lemma*:
assumes *finite* $\mathcal{F}$ $\mathcal{G} \subseteq \mathcal{F}$ *convex* T *bounded* T
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope* X
and *aff*: $\bigwedge X.\ X \in \mathcal{F} - \mathcal{G} \Longrightarrow$ *aff_dim* $X <$ *aff_dim* T
and *face*: $\bigwedge S\ T.\ [\![S \in \mathcal{F};\ T \in \mathcal{F}]\!] \Longrightarrow (S \cap T)$ *face_of* S
and *contf*: *continuous_on* $(\bigcup \mathcal{G})\ f$ **and** *fim*: $f \in (\bigcup \mathcal{G}) \to$ *rel_frontier* T
obtains g **where** *continuous_on* $(\bigcup \mathcal{F})\ g\ g \in (\bigcup \mathcal{F}) \to$ *rel_frontier* $T\ \bigwedge x.\ x \in \bigcup \mathcal{G} \Longrightarrow g\ x = f\ x$
proof (*cases* $\mathcal{F} - \mathcal{G} = \{\}$)
case *True*
show *?thesis*
using *True assms*(*2*) *contf fim that* **by** *force*
next
case *False*
then have $0 \le$ *aff_dim* T
by (*metis aff aff_dim_empty aff_dim_geq aff_dim_negative_iff all_not_in_conv not_less*)
then obtain i::*nat* **where** i: *int* $i =$ *aff_dim* T
by (*metis nonneg_eq_int*)
have *Union_empty_eq*: $\bigcup \{D.\ D = \{\} \wedge P\ D\} = \{\}$ **for** P :: *'a set* ⇒ *bool*
by *auto*
have *face'*: $\bigwedge S\ T.\ [\![S \in \mathcal{F};\ T \in \mathcal{F}]\!] \Longrightarrow (S \cap T)$ *face_of* $S \wedge (S \cap T)$ *face_of* T
by (*metis face inf_commute*)
have *extendf*: $\exists g.$ *continuous_on* $(\bigcup (\mathcal{G} \cup \{D.\ \exists\ C \in \mathcal{F}.\ D$ *face_of* $C \wedge$ *aff_dim* $D < i\}))\ g \wedge$

```
                    g ‘ (⋃ (𝒢 ∪ {D. ∃ C ∈ ℱ. D face_of C ∧ aff_dim D < i})) ⊆
rel_frontier T ∧
                    (∀ x ∈ ⋃𝒢. g x = f x)
      if i ≤ aff_dim T for i::nat
  using that
  proof (induction i)
    case 0
    show ?case
      using 0 contf fim by (auto simp add: Union_empty_eq)
  next
    case (Suc p)
    with ‹bounded T› have rel_frontier T ≠ {}
      by (auto simp: rel_frontier_eq_empty affine_bounded_eq_lowdim [of T])
    then obtain t where t: t ∈ rel_frontier T by auto
    have ple: int p ≤ aff_dim T using Suc.prems by force
    obtain h where conth: continuous_on (⋃(𝒢 ∪ {D. ∃ C ∈ ℱ. D face_of C ∧
aff_dim D < p})) h
              and him: h ‘ (⋃ (𝒢 ∪ {D. ∃ C ∈ ℱ. D face_of C ∧ aff_dim D < p}))
                       ⊆ rel_frontier T
               and heq: ⋀x. x ∈ ⋃𝒢 ⟹ h x = f x
      using Suc.IH [OF ple] by auto
    let ?Faces = {D. ∃ C ∈ ℱ. D face_of C ∧ aff_dim D ≤ p}
    have extendh: ∃ g. continuous_on D g ∧
                       g ∈ D → rel_frontier T ∧
                       (∀ x ∈ D ∩ ⋃(𝒢 ∪ {D. ∃ C ∈ ℱ. D face_of C ∧ aff_dim D <
p}). g x = h x)
      if D: D ∈ 𝒢 ∪ ?Faces for D
    proof (cases D ⊆ ⋃(𝒢 ∪ {D. ∃ C ∈ ℱ. D face_of C ∧ aff_dim D < p}))
      case True
      have continuous_on D h
        using True conth continuous_on_subset by blast
      moreover have h ∈ D → rel_frontier T
        using True him by blast
      ultimately show ?thesis
        by blast
    next
      case False
      note notDsub = False
      show ?thesis
      proof (cases ∃ a. D = {a})
        case True
        then obtain a where D = {a} by auto
        with notDsub t show ?thesis
          by (rule_tac x=λx. t in exI) simp
      next
        case False
        have D ≠ {} using notDsub by auto
        have Dnotin: D ∉ 𝒢 ∪ {D. ∃ C ∈ ℱ. D face_of C ∧ aff_dim D < p}
          using notDsub by auto
```

then have $D \notin \mathcal{G}$ **by** *simp*
have $D \in$ *?Faces* $- \{D.\ \exists\, C \in \mathcal{F}.\ D$ *face_of* $C \wedge$ *aff_dim* $D < p\}$
using *Dnotin that* **by** *auto*
then obtain C **where** $C \in \mathcal{F}$ D *face_of* C **and** *affD*: *aff_dim* $D =$ *int* p
by *auto*
then have *bounded* D
using *face_of_polytope_polytope poly polytope_imp_bounded* **by** *blast*
then have [*simp*]: $\neg$ *affine* D
using *affine_bounded_eq_trivial False* ‹$D \neq \{\}$› ‹*bounded* D› **by** *blast*
have $\{F.\ F$ *facet_of* $D\} \subseteq \{E.\ E$ *face_of* $C \wedge$ *aff_dim* $E <$ *int* $p\}$
by *clarify* (*metis* ‹D *face_of* C› *affD eq_iff face_of_trans facet_of_def zle_diff1_eq*)
moreover have *polyhedron* D
using ‹$C \in \mathcal{F}$› ‹D *face_of* C› *face_of_polytope_polytope poly polytope_imp_polyhedron* **by** *auto*
ultimately have *relf_sub*: *rel_frontier* $D \subseteq \bigcup \{E.\ E$ *face_of* $C \wedge$ *aff_dim* $E < p\}$
by (*simp add*: *rel_frontier_of_polyhedron Union_mono*)
then have *him_relf*: $h \in$ *rel_frontier* $D \to$ *rel_frontier* T
using ‹$C \in \mathcal{F}$› *him* **by** *blast*
have *convex* D
by (*simp add*: ‹*polyhedron* D› *polyhedron_imp_convex*)
have *affD_lessT*: *aff_dim* $D <$ *aff_dim* T
using *Suc.prems affD* **by** *linarith*
have *contDh*: *continuous_on* (*rel_frontier* D) h
using ‹$C \in \mathcal{F}$› *relf_sub* **by** (*blast intro*: *continuous_on_subset* [*OF conth*])
then have $*$: ($\exists\, c.$ *homotopic_with_canon* ($\lambda x.$ *True*) (*rel_frontier* D) (*rel_frontier* T) h ($\lambda x.$ c)) =
($\exists\, g.$ *continuous_on UNIV* $g \wedge$ *range* $g \subseteq$ *rel_frontier* $T \wedge$
($\forall\, x \in$*rel_frontier* $D.$ $g\ x = h\ x$))
by (*simp add*: *assms image_subset_iff_funcset rel_frontier_eq_empty him_relf nullhomotopic_into_rel_frontier_extension* [*OF closed_rel_frontier*])
have $\exists\, c.$ *homotopic_with_canon* ($\lambda x.$ *True*) (*rel_frontier* D) (*rel_frontier* T) h ($\lambda x.$ c)
by (*metis inessential_spheremap_lowdim_gen*
[*OF* ‹*convex* D› ‹*bounded* D› ‹*convex* T› ‹*bounded* T› *affD_lessT contDh him_relf*])
then obtain g **where** *contg*: *continuous_on UNIV* g
and *gim*: *range* $g \subseteq$ *rel_frontier* T
and *gh*: $\bigwedge x.\ x \in$ *rel_frontier* $D \Longrightarrow g\ x = h\ x$
by (*metis* $*$)
have $D \cap E \subseteq$ *rel_frontier* D
if $E \in \mathcal{G} \cup \{D.$ *Bex* $\mathcal{F}$ ((*face_of*) $D) \wedge$ *aff_dim* $D <$ *int* $p\}$ **for** E
proof (*rule face_of_subset_rel_frontier*)
show $D \cap E$ *face_of* D
using *that*
proof *safe*
assume $E \in \mathcal{G}$
then show $D \cap E$ *face_of* D

```
          by (meson ‹C ∈ ℱ› ‹D face_of C› assms(2) face' face_of_Int_subface
face_of_refl_eq poly polytope_imp_convex subsetD)
        next
          fix x
          assume aff_dim E < int p x ∈ ℱ E face_of x
          then show D ∩ E face_of D
            by (meson ‹C ∈ ℱ› ‹D face_of C› face' face_of_Int_subface that)
        qed
        show D ∩ E ≠ D
          using that notDsub by auto
      qed
      moreover have continuous_on D g
        using contg continuous_on_subset by blast
      ultimately show ?thesis
        by (rule_tac x=g in exI) (use gh gim in fastforce)
    qed
  qed
  have intle: i < 1 + int j ⟷ i ≤ int j for i j
    by auto
  have finite 𝒢
    using ‹finite ℱ› ‹𝒢 ⊆ ℱ› rev_finite_subset by blast
  moreover have finite (?Faces)
  proof −
    have §: finite (⋃ {{D. D face_of C} |C. C ∈ ℱ})
      by (auto simp: ‹finite ℱ› finite_polytope_faces poly)
    show ?thesis
      by (auto intro: finite_subset [OF _ §])
  qed
  ultimately have fin: finite (𝒢 ∪ ?Faces)
    by simp
  have clo: closed S if S ∈ 𝒢 ∪ ?Faces for S
    using that ‹𝒢 ⊆ ℱ› face_of_polytope_polytope poly polytope_imp_closed by
blast
  have K: X ∩ Y ⊆ ⋃(𝒢 ∪ {D. ∃C∈ℱ. D face_of C ∧ aff_dim D < int p})
          if X ∈ 𝒢 ∪ ?Faces Y ∈ 𝒢 ∪ ?Faces ¬ Y ⊆ X for X Y
  proof −
    have ff: X ∩ Y face_of X ∧ X ∩ Y face_of Y
      if XY: X face_of D Y face_of E and DE: D ∈ ℱ E ∈ ℱ for D E
      by (rule face_of_Int_subface [OF _ _ XY]) (auto simp: face' DE)
    show ?thesis
      using that
      apply clarsimp
    by (smt (verit) IntI face_of_aff_dim_lt face_of_imp_convex face_of_trans
ff inf_commute)
  qed
  obtain g where continuous_on (⋃(𝒢 ∪ ?Faces)) g
                 g ` ⋃(𝒢 ∪ ?Faces) ⊆ rel_frontier T
                 (∀x ∈ ⋃(𝒢 ∪ ?Faces) ∩
                       ⋃(𝒢 ∪ {D. ∃C∈ℱ. D face_of C ∧ aff_dim D < p}). g x =
```

```
h x)
      by (rule exE [OF extending_maps_Union [OF fin extendh clo K]], blast+)
    then show ?case
      by (simp add: intle local.heq [symmetric], blast)
  qed
  have eq: ⋃(𝒢 ∪ {D. ∃ C ∈ ℱ. D face_of C ∧ aff_dim D < i}) = ⋃ℱ
  proof
    show ⋃(𝒢 ∪ {D. ∃ C∈ℱ. D face_of C ∧ aff_dim D < int i}) ⊆ ⋃ℱ
      using ‹𝒢 ⊆ ℱ› face_of_imp_subset by fastforce
    show ⋃ℱ ⊆ ⋃(𝒢 ∪ {D. ∃ C∈ℱ. D face_of C ∧ aff_dim D < i})
       using face by (intro Union_mono) (fastforce simp: aff i)
  qed
  have int i ≤ aff_dim T by (simp add: i)
  then show ?thesis
    using extendf [of i] that unfolding eq by fastforce
qed

lemma extend_map_lemma_cofinite0:
  assumes finite ℱ
      and pairwise (λS T. S ∩ T ⊆ K) ℱ
      and ⋀S. S ∈ ℱ ⟹ ∃ a g. a ∉ U ∧ continuous_on (S − {a}) g ∧ g ∈ (S −
{a}) → T ∧ (∀ x ∈ S ∩ K. g x = h x)
      and ⋀S. S ∈ ℱ ⟹ closed S
    shows ∃ C g. finite C ∧ disjnt C U ∧ card C ≤ card ℱ ∧
                 continuous_on (⋃ℱ − C) g ∧ g ∈ (⋃ℱ − C) → T
                  ∧ (∀ x ∈ (⋃ℱ − C) ∩ K. g x = h x)
  using assms
proof induction
  case empty then show ?case
    by force
next
  case (insert X ℱ)
  then have closed X and clo: ⋀X. X ∈ ℱ ⟹ closed X
       and ℱ: ⋀S. S ∈ ℱ ⟹ ∃ a g. a ∉ U ∧ continuous_on (S − {a}) g ∧ g ∈
(S − {a}) → T ∧ (∀ x ∈ S ∩ K. g x = h x)
       and pwX: ⋀Y. Y ∈ ℱ ∧ Y ≠ X ⟶ X ∩ Y ⊆ K ∧ Y ∩ X ⊆ K
       and pwF: pairwise (λ S T. S ∩ T ⊆ K) ℱ
    by (simp_all add: pairwise_insert)
  obtain C g where C: finite C disjnt C U card C ≤ card ℱ
             and contg: continuous_on (⋃ℱ − C) g
             and gim: g ∈ (⋃ℱ − C) → T
             and gh:  ⋀x. x ∈ (⋃ℱ − C) ∩ K ⟹ g x = h x
    using insert.IH [OF pwF ℱ clo] by auto
  obtain a f where a ∉ U
             and contf: continuous_on (X − {a}) f
             and fim: f ∈ (X − {a}) → T
             and fh: (∀ x ∈ X ∩ K. f x = h x)
    using insert.prems by (meson insertI1)
  show ?case
```

proof (*intro exI conjI*)
show *finite* (*insert a C*)
by (*simp add*: *C*)
show *disjnt* (*insert a C*) *U*
using *C* ‹*a* ∉ *U*› **by** *simp*
show *card* (*insert a C*) ≤ *card* (*insert X* ℱ)
by (*simp add*: *C card_insert_if insert.hyps le_SucI*)
have *closed* (⋃ℱ)
using *clo insert.hyps* **by** *blast*
have *continuous_on* (*X* − *insert a C*) *f*
using *contf* **by** (*force simp*: *elim*: *continuous_on_subset*)
moreover have *continuous_on* (⋃ ℱ − *insert a C*) *g*
using *contg* **by** (*force simp*: *elim*: *continuous_on_subset*)
ultimately
have *continuous_on* (*X* − *insert a C* ∪ (⋃ℱ − *insert a C*)) (λ*x*. *if x* ∈ *X then f x else g x*)
apply (*intro continuous_on_cases_local*; *simp add*: *closedin_closed*)
using ‹*closed X*› **apply** *blast*
using ‹*closed* (⋃ℱ)› **apply** *blast*
using *fh gh insert.hyps pwX* **by** *fastforce*
then show *continuous_on* (⋃(*insert X* ℱ) − *insert a C*) (λ*a*. *if a* ∈ *X then f a else g a*)
by (*blast intro*: *continuous_on_subset*)
show ∀ *x*∈(⋃(*insert X* ℱ) − *insert a C*) ∩ *K*. (*if x* ∈ *X then f x else g x*) = *h x*
using *gh* **by** (*auto simp*: *fh*)
show (λ*a*. *if a* ∈ *X then f a else g a*) ∈ (⋃(*insert X* ℱ) − *insert a C*) → *T*
using *fim gim Pi_iff* **by** *fastforce*
qed
qed

lemma *extend_map_lemma_cofinite1*:
assumes *finite* ℱ
and ℱ: ⋀*X*. *X* ∈ ℱ ⟹ ∃ *a g*. *a* ∉ *U* ∧ *continuous_on* (*X* − {*a*}) *g* ∧ *g* ∈ (*X* − {*a*}) → *T* ∧ (∀ *x* ∈ *X* ∩ *K*. *g x* = *h x*)
and *clo*: ⋀*X*. *X* ∈ ℱ ⟹ *closed X*
and *K*: ⋀*X Y*. ⟦*X* ∈ ℱ; *Y* ∈ ℱ; ¬ *X* ⊆ *Y*; ¬ *Y* ⊆ *X*⟧ ⟹ *X* ∩ *Y* ⊆ *K*
obtains *C g* **where** *finite C disjnt C U card C* ≤ *card* ℱ *continuous_on* (⋃ℱ − *C*) *g*
g ∈ (⋃ℱ − *C*) → *T*
⋀*x*. *x* ∈ (⋃ℱ − *C*) ∩ *K* ⟹ *g x* = *h x*
proof −
let *?*ℱ = {*X* ∈ ℱ. ∀ *Y*∈ℱ. ¬ *X* ⊂ *Y*}
have [*simp*]: ⋃ *?*ℱ = ⋃ℱ
by (*simp add*: *Union_maximal_sets assms*)
have *fin*: *finite ?*ℱ
by (*force intro*: *finite_subset* [*OF* _ ‹*finite* ℱ›])
have *pw*: *pairwise* (λ *S T*. *S* ∩ *T* ⊆ *K*) *?*ℱ

```
    by (simp add: pairwise_def) (metis K psubsetI)
  have card {X ∈ ℱ. ∀ Y∈ℱ. ¬ X ⊂ Y} ≤ card ℱ
    by (simp add: ‹finite ℱ› card_mono)
  moreover
  obtain C g where finite C ∧ disjnt C U ∧ card C ≤ card ?ℱ ∧
                   continuous_on (⋃ ?ℱ − C) g ∧ g ∈ (⋃ ?ℱ − C) → T
                   ∧ (∀ x ∈ (⋃ ?ℱ − C) ∩ K. g x = h x)
    using extend_map_lemma_cofinite0 [OF fin pw, of U T h] by (fastforce intro!: clo ℱ)
  ultimately show ?thesis
    by (rule_tac C=C and g=g in that) auto
qed


lemma extend_map_lemma_cofinite:
  assumes finite ℱ 𝒢 ⊆ ℱ and T: convex T bounded T
      and poly: ⋀X. X ∈ ℱ ⟹ polytope X
      and contf: continuous_on (⋃𝒢) f and fim: f ∈ (⋃𝒢) → rel_frontier T
      and face: ⋀X Y. ⟦X ∈ ℱ; Y ∈ ℱ⟧ ⟹ (X ∩ Y) face_of X
      and aff: ⋀X. X ∈ ℱ − 𝒢 ⟹ aff_dim X ≤ aff_dim T
  obtains C g where
    finite C disjnt C (⋃𝒢) card C ≤ card ℱ continuous_on (⋃ℱ − C) g
    g ∈ (⋃ ℱ − C) → rel_frontier T ⋀x. x ∈ ⋃𝒢 ⟹ g x = f x
proof −
  define ℋ where ℋ ≡ 𝒢 ∪ {D. ∃ C ∈ ℱ − 𝒢. D face_of C ∧ aff_dim D <
aff_dim T}
  have finite 𝒢
    using assms finite_subset by blast
  have ∗: finite (⋃{{D. D face_of C} |C. C ∈ ℱ})
    using finite_polytope_faces poly ‹finite ℱ› by force
  then have finite ℋ
    by (auto simp: ℋ_def ‹finite 𝒢› intro: finite_subset [OF _ ∗])
  have face′: ⋀S T. ⟦S ∈ ℱ; T ∈ ℱ⟧ ⟹ (S ∩ T) face_of S ∧ (S ∩ T) face_of T
    by (metis face inf_commute)
  have ∗: ⋀X Y. ⟦X ∈ ℋ; Y ∈ ℋ⟧ ⟹ X ∩ Y face_of X
    by (simp add: ℋ_def) (smt (verit) ‹𝒢 ⊆ ℱ› DiffE face′ face_of_Int_subface
in_mono inf.idem)
  obtain h where conth: continuous_on (⋃ℋ) h and him: h ∈ (⋃ℋ) → rel_frontier
T
            and hf: ⋀x. x ∈ ⋃𝒢 ⟹ h x = f x
    proof (rule extend_map_lemma [OF ‹finite ℋ› [unfolded ℋ_def] Un_upper1
T])
    show ⋀X. ⟦X ∈ 𝒢 ∪ {D. ∃ C∈ℱ − 𝒢. D face_of C ∧ aff_dim D < aff_dim
T}⟧ ⟹ polytope X
      using ‹𝒢 ⊆ ℱ› face_of_polytope_polytope poly by fastforce
  qed (use ∗ ℋ_def contf fim in auto)
  have bounded (⋃𝒢)
    using ‹finite 𝒢› ‹𝒢 ⊆ ℱ› poly polytope_imp_bounded by blast
  then have ⋃𝒢 ≠ UNIV
```

```
  by auto
 then obtain a where a: a ∉ ⋃𝒢
  by blast
 have ℱ: ∃a g. a ∉ ⋃𝒢 ∧ continuous_on (D − {a}) g ∧
              g ∈ (D − {a}) → rel_frontier T ∧ (∀x ∈ D ∩ ⋃ℋ. g x = h x)
    if D ∈ ℱ for D
 proof (cases D ⊆ ⋃ℋ)
  case True
  then have h ∈ (D − {a}) → rel_frontier T continuous_on (D − {a}) h
   using him by (blast intro!: ‹a ∉ ⋃𝒢› continuous_on_subset [OF conth])+
  then show ?thesis
   using a by blast
 next
  case False
  note D_not_subset = False
  show ?thesis
  proof (cases D ∈ 𝒢)
   case True
   with D_not_subset show ?thesis
    by (auto simp: ℋ_def)
  next
   case False
   then have affD: aff_dim D ≤ aff_dim T
    by (simp add: ‹D ∈ ℱ› aff)
   show ?thesis
   proof (cases rel_interior D = {})
    case True
    with ‹D ∈ ℱ› poly a show ?thesis
     by (force simp: rel_interior_eq_empty polytope_imp_convex)
   next
    case False
    then obtain b where brelD: b ∈ rel_interior D
     by blast
    have polyhedron D
     by (simp add: poly polytope_imp_polyhedron that)
    have rel_frontier D retract_of affine hull D − {b}
     by (simp add: rel_frontier_retract_of_punctured_affine_hull poly poly-
tope_imp_bounded polytope_imp_convex that brelD)
    then obtain r where relfD: rel_frontier D ⊆ affine hull D − {b}
                    and contr: continuous_on (affine hull D − {b}) r
                    and rim: r ∈ (affine hull D − {b}) → rel_frontier D
                    and rid: ⋀x. x ∈ rel_frontier D ⟹ r x = x
     by (auto simp: retract_of_def retraction_def)
    show ?thesis
    proof (intro exI conjI ballI)
     show b ∉ ⋃𝒢
     proof clarify
      fix E
      assume b ∈ E E ∈ 𝒢
```

```
        then have E ∩ D face_of E ∧ E ∩ D face_of D
          using ‹𝒢 ⊆ ℱ› face' that by auto
      with face_of_subset_rel_frontier ‹E ∈ 𝒢› ‹b ∈ E› brelD rel_interior_subset
[of D]
            D_not_subset rel_frontier_def ℋ_def
        show False
          by blast
      qed
      have r ‘ (D − {b}) ⊆ r ‘ (affine hull D − {b})
        by (simp add: Diff_mono hull_subset image_mono)
      also have ... ⊆ rel_frontier D
        using rim by auto
      also have ... ⊆ ⋃{E. E face_of D ∧ aff_dim E < aff_dim T}
        using affD
            by (force simp: rel_frontier_of_polyhedron [OF ‹polyhedron D›]
facet_of_def)
      also have ... ⊆ ⋃(ℋ)
        using D_not_subset ℋ_def that by fastforce
      finally have rsub: r ‘ (D − {b}) ⊆ ⋃(ℋ) .
      show continuous_on (D − {b}) (h ∘ r)
      proof (rule continuous_on_compose)
        show continuous_on (D − {b}) r
          by (meson Diff_mono continuous_on_subset contr hull_subset or-
der_refl)
        show continuous_on (r ‘ (D − {b})) h
         by (simp add: Diff_mono hull_subset continuous_on_subset [OF conth
rsub])
      qed
      show (h ∘ r) ∈ (D − {b}) → rel_frontier T
        using brelD him rsub by fastforce
      show (h ∘ r) x = h x if x: x ∈ D ∩ ⋃ℋ for x
      proof −
        consider A where x ∈ D A ∈ 𝒢 x ∈ A
          | A B where x ∈ D A face_of B B ∈ ℱ B ∉ 𝒢 aff_dim A < aff_dim
T x ∈ A
          using x by (auto simp: ℋ_def)
        then have xrel: x ∈ rel_frontier D
        proof cases
          case 1 show ?thesis
          proof (rule face_of_subset_rel_frontier [THEN subsetD])
            show D ∩ A face_of D
              using ‹A ∈ 𝒢› ‹𝒢 ⊆ ℱ› face ‹D ∈ ℱ› by blast
            show D ∩ A ≠ D
              using ‹A ∈ 𝒢› D_not_subset ℋ_def by blast
          qed (auto simp: 1)
        next
          case 2 show ?thesis
          proof (rule face_of_subset_rel_frontier [THEN subsetD])
            have D face_of D
```

```
            by (simp add: ‹polyhedron D› polyhedron_imp_convex face_of_refl)
           then show D ∩ A face_of D
            by (meson 2(2) 2(3) ‹D ∈ ℱ› face' face_of_Int_Int face_of_face)
           show D ∩ A ≠ D
             using 2 D_not_subset ℋ_def by blast
         qed (auto simp: 2)
       qed
       show ?thesis
         by (simp add: rid xrel)
     qed
    qed
   qed
  qed
  have clo: ⋀S. S ∈ ℱ ⟹ closed S
   by (simp add: poly polytope_imp_closed)
  obtain C g where finite C disjnt C (⋃𝒢) card C ≤ card ℱ continuous_on (⋃ℱ
− C) g
                   g ∈ (⋃ℱ − C) → rel_frontier T
              and gh: ⋀x. x ∈ (⋃ℱ − C) ∩ ⋃ℋ ⟹ g x = h x
  proof (rule extend_map_lemma_cofinite1 [OF ‹finite ℱ› ℱ clo])
   show X ∩ Y ⊆ ⋃ℋ if XY: X ∈ ℱ Y ∈ ℱ and ¬ X ⊆ Y ¬ Y ⊆ X for X Y
   proof (cases X ∈ 𝒢)
     case True
     then show ?thesis
       by (auto simp: ℋ_def)
   next
     case False
     have X ∩ Y ≠ X
       using ‹¬ X ⊆ Y› by blast
     with XY
     show ?thesis
       by (clarsimp simp: ℋ_def)
          (metis Diff_iff Int_iff aff antisym_conv face face_of_aff_dim_lt
face_of_refl
                 not_le poly polytope_imp_convex)
   qed
  qed (blast)+
  with ‹𝒢 ⊆ ℱ› show ?thesis
    by (rule_tac C=C and g=g in that) (auto simp: disjnt_def hf [symmetric]
ℋ_def intro!: gh)
qed
```

The next two proofs are similar

```
theorem extend_map_cell_complex_to_sphere:
  assumes finite ℱ and S: S ⊆ ⋃ℱ closed S and T: convex T bounded T
      and poly: ⋀X. X ∈ ℱ ⟹ polytope X
      and aff: ⋀X. X ∈ ℱ ⟹ aff_dim X < aff_dim T
      and face: ⋀X Y. ⟦X ∈ ℱ; Y ∈ ℱ⟧ ⟹ (X ∩ Y) face_of X
```

```
    and contf: continuous_on S f and fim: f ∈ S → rel_frontier T
  obtains g where continuous_on (⋃ℱ) g
    g ∈ (⋃ℱ) → rel_frontier T ⋀x. x ∈ S ⟹ g x = f x
proof −
  obtain V g where S ⊆ V open V continuous_on V g and gim: g ∈ V →
rel_frontier T and gf: ⋀x. x ∈ S ⟹ g x = f x
   using neighbourhood_extension_into_ANR [OF contf fim _ ‹closed S›] ANR_rel_frontier_convex
T by blast
  have compact S
   by (meson assms compact_Union poly polytope_imp_compact seq_compact_closed_subset
seq_compact_eq_compact)
  then obtain d where d > 0 and d: ⋀x y. ⟦x ∈ S; y ∈ − V⟧ ⟹ d ≤ dist x y
    using separate_compact_closed [of S − V] ‹open V› ‹S ⊆ V› by force
  obtain 𝒢 where finite 𝒢 ⋃𝒢 = ⋃ℱ
            and diaG: ⋀X. X ∈ 𝒢 ⟹ diameter X < d
            and polyG: ⋀X. X ∈ 𝒢 ⟹ polytope X
            and affG: ⋀X. X ∈ 𝒢 ⟹ aff_dim X ≤ aff_dim T − 1
            and faceG: ⋀X Y. ⟦X ∈ 𝒢; Y ∈ 𝒢⟧ ⟹ X ∩ Y face_of X
  proof (rule cell_complex_subdivision_exists [OF ‹d>0› ‹finite ℱ› poly _ face])
    show ⋀X. X ∈ ℱ ⟹ aff_dim X ≤ aff_dim T − 1
      by (simp add: aff)
  qed auto
  obtain h where conth: continuous_on (⋃𝒢) h and him: h ‘ ⋃𝒢 ⊆ rel_frontier
T and hg: ⋀x. x ∈ ⋃(𝒢 ∩ Pow V) ⟹ h x = g x
  proof (rule extend_map_lemma [of 𝒢 𝒢 ∩ Pow V T g])
    show continuous_on (⋃(𝒢 ∩ Pow V)) g
      by (metis Union_Int_subset Union_Pow_eq ‹continuous_on V g› continu-
ous_on_subset le_inf_iff)
  qed (use ‹finite 𝒢› T polyG affG faceG gim image_subset_iff_funcset in auto)
  show ?thesis
  proof
    show continuous_on (⋃ℱ) h
      using ‹⋃𝒢 = ⋃ℱ› conth by auto
    show h ∈ ⋃ℱ → rel_frontier T
      using ‹⋃𝒢 = ⋃ℱ› him by auto
    show h x = f x if x ∈ S for x
    proof −
      have x ∈ ⋃𝒢
        using ‹⋃𝒢 = ⋃ℱ› ‹S ⊆ ⋃ℱ› that by auto
      then obtain X where x ∈ X X ∈ 𝒢 by blast
      then have diameter X < d bounded X
        by (auto simp: diaG ‹X ∈ 𝒢› polyG polytope_imp_bounded)
       then have X ⊆ V using d [OF ‹x ∈ S›] diameter_bounded_bound [OF
‹bounded X› ‹x ∈ X›]
        by fastforce
      have h x = g x
        using ‹X ∈ 𝒢› ‹X ⊆ V› ‹x ∈ X› hg by auto
      also have ... = f x
        by (simp add: gf that)
```

```
    finally show h x = f x .
  qed
 qed
qed


theorem extend_map_cell_complex_to_sphere_cofinite:
 assumes finite ℱ and S: S ⊆ ⋃ℱ closed S and T: convex T bounded T
    and poly: ⋀X. X ∈ ℱ ⟹ polytope X
    and aff: ⋀X. X ∈ ℱ ⟹ aff_dim X ≤ aff_dim T
    and face: ⋀X Y. ⟦X ∈ ℱ; Y ∈ ℱ⟧ ⟹ (X ∩ Y) face_of X
    and contf: continuous_on S f and fim: f ∈ S → rel_frontier T
 obtains C g where finite C disjnt C S continuous_on (⋃ℱ − C) g
   g ∈ (⋃ℱ − C) → rel_frontier T ⋀x. x ∈ S ⟹ g x = f x
proof −
 obtain V g where S ⊆ V open V continuous_on V g and gim: g ∈ V →
rel_frontier T and gf: ⋀x. x ∈ S ⟹ g x = f x
  using neighbourhood_extension_into_ANR [OF contf fim _ ‹closed S›] ANR_rel_frontier_convex
T by blast
 have compact S
  by (meson assms compact_Union poly polytope_imp_compact seq_compact_closed_subset
seq_compact_eq_compact)
 then obtain d where d > 0 and d: ⋀x y. ⟦x ∈ S; y ∈ − V⟧ ⟹ d ≤ dist x y
   using separate_compact_closed [of S −V] ‹open V› ‹S ⊆ V› by force
 obtain 𝒢 where finite 𝒢 ⋃𝒢 = ⋃ℱ
           and diaG: ⋀X. X ∈ 𝒢 ⟹ diameter X < d
           and polyG: ⋀X. X ∈ 𝒢 ⟹ polytope X
           and affG: ⋀X. X ∈ 𝒢 ⟹ aff_dim X ≤ aff_dim T
           and faceG: ⋀X Y. ⟦X ∈ 𝒢; Y ∈ 𝒢⟧ ⟹ X ∩ Y face_of X
   by (rule cell_complex_subdivision_exists [OF ‹d>0› ‹finite ℱ› poly aff face])
auto
 obtain C h where finite C and dis: disjnt C (⋃(𝒢 ∩ Pow V))
             and card: card C ≤ card 𝒢 and conth: continuous_on (⋃𝒢 − C) h
             and him: h ∈ (⋃𝒢 − C) → rel_frontier T
             and hg: ⋀x. x ∈ ⋃(𝒢 ∩ Pow V) ⟹ h x = g x
 proof (rule extend_map_lemma_cofinite [of 𝒢 𝒢 ∩ Pow V T g])
   show continuous_on (⋃(𝒢 ∩ Pow V)) g
     by (metis Union_Int_subset Union_Pow_eq ‹continuous_on V g› continu-
ous_on_subset le_inf_iff)
   show g ∈ ⋃(𝒢 ∩ Pow V) → rel_frontier T
     using gim by force
 qed (auto intro: ‹finite 𝒢› T polyG affG dest: faceG)
 have S ⊆ ⋃(𝒢 ∩ Pow V)
 proof
   fix x
   assume x ∈ S
   then have x ∈ ⋃𝒢
     using ‹⋃𝒢 = ⋃ℱ› ‹S ⊆ ⋃ℱ› by auto
   then obtain X where x ∈ X X ∈ 𝒢 by blast
```

then have *diameter X < d bounded X*
by (*auto simp*: *diaG ‹X ∈ 𝒢› polyG polytope_imp_bounded*)
then have *X ⊆ V* **using** *d* [*OF ‹x ∈ S›*] *diameter_bounded_bound* [*OF ‹bounded X› ‹x ∈ X›*]
by *fastforce*
then show *x ∈ ⋃(𝒢 ∩ Pow V)*
using *‹X ∈ 𝒢› ‹x ∈ X›* **by** *blast*
qed
then show *?thesis*
by (*metis PowI Union_Pow_eq ‹⋃ 𝒢 = ⋃ ℱ› ‹finite C› conth dis disjnt_Union2 gf hg him subsetD that*)
qed

9.27.3 Special cases and corollaries involving spheres

proposition *extend_map_affine_to_sphere_cofinite_simple*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *compact S convex U bounded U*
and *aff*: *aff_dim T ≤ aff_dim U*
and *S ⊆ T* **and** *contf*: *continuous_on S f*
and *fim*: *f ∈ S → rel_frontier U*
obtains *K g* **where** *finite K K ⊆ T disjnt K S continuous_on (T − K) g*
g ∈ (T − K) → rel_frontier U
⋀x. x ∈ S ⟹ g x = f x
proof −
have *∃K g. finite K ∧ disjnt K S ∧ continuous_on (T − K) g ∧*
g ∈ (T − K) → rel_frontier U ∧ (∀x ∈ S. g x = f x)
if *affine T S ⊆ T* **and** *aff*: *aff_dim T ≤ aff_dim U* **for** *T*
proof (*cases S = {}*)
case *True*
show *?thesis*
proof (*cases rel_frontier U = {}*)
case *True*
with *‹bounded U›* **have** *aff_dim U ≤ 0*
using *affine_bounded_eq_lowdim rel_frontier_eq_empty* **by** *auto*
with *aff* **have** *aff_dim T ≤ 0* **by** *auto*
then obtain *a* **where** *T ⊆ {a}*
using *‹affine T› affine_bounded_eq_lowdim affine_bounded_eq_trivial* **by** *auto*
then show *?thesis*
using *‹S = {}› fim*
by (*metis Diff_cancel contf disjnt_empty2 finite.emptyI finite_insert finite_subset*)
next
case *False*
then obtain *a* **where** *a ∈ rel_frontier U*
by *auto*
then show *?thesis*
using *continuous_on_const* [*of _ a*] *‹S = {}›* **by** *force*

```
  qed
 next
  case False
  have bounded S
   by (simp add: ‹compact S› compact_imp_bounded)
  then obtain b where b: S ⊆ cbox (−b) b
   using bounded_subset_cbox_symmetric by blast
  define bbox where bbox ≡ cbox (−(b+One)) (b+One)
  have cbox (−b) b ⊆ bbox
   by (auto simp: bbox_def algebra_simps intro!: subset_box_imp)
  with b ‹S ⊆ T› have S ⊆ bbox ∩ T
   by auto
  then have Ssub: S ⊆ ⋃{bbox ∩ T}
   by auto
  then have aff_dim (bbox ∩ T) ≤ aff_dim U
   by (metis aff aff_dim_subset inf_commute inf_le1 order_trans)
  obtain K g where K: finite K disjnt K S
            and contg: continuous_on (⋃{bbox ∩ T} − K) g
            and gim: g ∈ (⋃{bbox ∩ T} − K) → rel_frontier U
            and gf: ⋀x. x ∈ S ⟹ g x = f x
  proof (rule extend_map_cell_complex_to_sphere_cofinite
          [OF _ Ssub _ ‹convex U› ‹bounded U› _ _ _ contf fim])
   show closed S
    using ‹compact S› compact_eq_bounded_closed by auto
   show poly: ⋀X. X ∈ {bbox ∩ T} ⟹ polytope X
   by (simp add: polytope_Int_polyhedron bbox_def polytope_interval affine_imp_polyhedron
‹affine T›)
   show ⋀X Y. ⟦X ∈ {bbox ∩ T}; Y ∈ {bbox ∩ T}⟧ ⟹ X ∩ Y face_of X
    by (simp add:poly face_of_refl polytope_imp_convex)
   show ⋀X. X ∈ {bbox ∩ T} ⟹ aff_dim X ≤ aff_dim U
    by (simp add: ‹aff_dim (bbox ∩ T) ≤ aff_dim U›)
  qed auto
  define fro where fro ≡ λd. frontier(cbox (−(b + d *R One)) (b + d *R One))
  obtain d where d12: 1/2 ≤ d d ≤ 1 and dd: disjnt K (fro d)
  proof (rule disjoint_family_elem_disjnt [OF _ ‹finite K›])
   show infinite {1/2..1::real}
    by (simp add: infinite_Icc)
   have dis1: disjnt (fro x) (fro y) if x<y for x y
    by (auto simp: algebra_simps that subset_box_imp disjnt_Diff1 frontier_def
fro_def)
   then show disjoint_family_on fro {1/2..1}
    by (auto simp: disjoint_family_on_def disjnt_def neq_iff)
  qed auto
  define c where c ≡ b + d *R One
  have cbsub: cbox (−b) b ⊆ box (−c) c cbox (−b) b ⊆ cbox (−c) c  cbox (−c)
c ⊆ bbox
   using d12 by (auto simp: algebra_simps subset_box_imp c_def bbox_def)
  have clo_cbT: closed (cbox (− c) c ∩ T)
   by (simp add: affine_closed closed_Int closed_cbox ‹affine T›)
```

```
    have cpT_ne: cbox (− c) c ∩ T ≠ {}
      using ‹S ≠ {}› b cbsub(2) ‹S ⊆ T› by fastforce
    have closest_point (cbox (− c) c ∩ T) x ∉ K if x ∈ T x ∉ K for x
    proof (cases x ∈ cbox (−c) c)
      case True with that show ?thesis
        by (simp add: closest_point_self)
    next
      case False
      have int_ne: interior (cbox (−c) c) ∩ T ≠ {}
        using ‹S ≠ {}› ‹S ⊆ T› b ‹cbox (− b) b ⊆ box (− c) c› by force
      have convex T
        by (meson ‹affine T› affine_imp_convex)
      then have x ∈ affine hull (cbox (− c) c ∩ T)
          by (metis Int_commute Int_iff ‹S ≠ {}› ‹S ⊆ T› cbsub(1) ‹x ∈ T›
affine_hull_convex_Int_nonempty_interior all_not_in_conv b hull_inc inf.orderE
interior_cbox)
      then have x ∈ affine hull (cbox (− c) c ∩ T) − rel_interior (cbox (− c) c
∩ T)
        by (meson DiffI False Int_iff rel_interior_subset subsetCE)
      then have closest_point (cbox (− c) c ∩ T) x ∈ rel_frontier (cbox (− c) c
∩ T)
        by (rule closest_point_in_rel_frontier [OF clo_cbT cpT_ne])
      moreover have (rel_frontier (cbox (− c) c ∩ T)) ⊆ fro d
        by (subst convex_affine_rel_frontier_Int [OF _ ‹affine T› int_ne]) (auto
simp: fro_def c_def)
      ultimately show ?thesis
        using dd by (force simp: disjnt_def)
    qed
    then have cpt_subset: closest_point (cbox (− c) c ∩ T) ‘ (T − K) ⊆ ⋃{bbox
∩ T} − K
      using closest_point_in_set [OF clo_cbT cpT_ne] cbsub(3) by force
    show ?thesis
    proof (intro conjI ballI exI)
      have continuous_on (T − K) (closest_point (cbox (− c) c ∩ T))
      proof (rule continuous_on_closest_point)
        show convex (cbox (− c) c ∩ T)
          by (simp add: affine_imp_convex convex_Int ‹affine T›)
        show closed (cbox (− c) c ∩ T)
          using clo_cbT by blast
        show cbox (− c) c ∩ T ≠ {}
          using ‹S ≠ {}› cbsub(2) b that by auto
      qed
      then show continuous_on (T − K) (g ∘ closest_point (cbox (− c) c ∩ T))
      by (metis continuous_on_compose continuous_on_subset [OF contg cpt_subset])
      have (g ∘ closest_point (cbox (− c) c ∩ T)) ‘ (T − K) ⊆ g ‘ (⋃{bbox ∩ T}
− K)
        by (metis image_comp image_mono cpt_subset)
      also have ... ⊆ rel_frontier U
        using gim by blast
```

finally show (*g* ∘ *closest_point* (*cbox* (− *c*) *c* ∩ *T*)) ∈ (*T* − *K*) → *rel_frontier U*
by *blast*
show (*g* ∘ *closest_point* (*cbox* (− *c*) *c* ∩ *T*)) *x* = *f x* **if** *x* ∈ *S* **for** *x*
proof −
have (*g* ∘ *closest_point* (*cbox* (− *c*) *c* ∩ *T*)) *x* = *g x*
unfolding *o_def*
by (*metis IntI* ‹*S* ⊆ *T*› *b cbsub*(2) *closest_point_self subset_eq that*)
also have ... = *f x*
by (*simp add*: *that gf*)
finally show *?thesis* **.**
qed
qed (*auto simp*: *K*)
qed
then obtain *K g* **where** *finite K disjnt K S*
and *contg*: *continuous_on* (*affine hull T* − *K*) *g*
and *gim*: *g* ∈ (*affine hull T* − *K*) → *rel_frontier U*
and *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
by (*metis aff affine_affine_hull aff_dim_affine_hull order_trans* [*OF* ‹*S* ⊆ *T*› *hull_subset* [*of T affine*]])
then obtain *K g* **where** *finite K disjnt K S*
and *contg*: *continuous_on* (*T* − *K*) *g*
and *gim*: *g* ∈ (*T* − *K*) → *rel_frontier U*
and *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
by (*rule_tac K*=*K* **and** *g*=*g* **in** *that*) (*auto simp*: *hull_inc elim*: *continuous_on_subset*)
then show *?thesis*
by (*rule_tac K*=*K* ∩ *T* **and** *g*=*g* **in** *that*) (*auto simp*: *disjnt_iff Diff_Int contg*)
qed

9.27.4 Extending maps to spheres

lemma *extend_map_affine_to_sphere1*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*topological_space*
assumes *finite K affine U* **and** *contf*: *continuous_on* (*U* − *K*) *f*
and *fim*: *f* ∈ (*U* − *K*) → *T*
and *comps*: ⋀*C*. ⟦*C* ∈ *components*(*U* − *S*); *C* ∩ *K* ≠ {}⟧ ⟹ *C* ∩ *L* ≠ {}
and *clo*: *closedin* (*top_of_set U*) *S* **and** *K*: *disjnt K S K* ⊆ *U*
obtains *g* **where** *continuous_on* (*U* − *L*) *g g* ∈ (*U* − *L*) → *T* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*
proof (*cases K* = {})
case *True*
then show *?thesis*
by (*metis DiffD1 Diff_empty Diff_subset PiE Pi_I contf continuous_on_subset fim that*)
next
case *False*
have *S* ⊆ *U*

```
    using clo closedin_limpt by blast
  then have (U - S) ∩ K ≠ {}
    by (metis Diff_triv False Int_Diff K disjnt_def inf.absorb_iff2 inf_commute)
  then have ⋃(components (U - S)) ∩ K ≠ {}
    using Union_components by simp
  then obtain C0 where C0: C0 ∈ components (U - S) C0 ∩ K ≠ {}
    by blast
  have convex U
    by (simp add: affine_imp_convex ‹affine U›)
  then have locally connected U
    by (rule convex_imp_locally_connected)
  have ∃a g. a ∈ C ∧ a ∈ L ∧ continuous_on (S ∪ (C - {a})) g ∧
            g ` (S ∪ (C - {a})) ⊆ T ∧ (∀x ∈ S. g x = f x)
      if C: C ∈ components (U - S) and CK: C ∩ K ≠ {} for C
  proof -
    have C ⊆ U-S C ∩ L ≠ {}
      by (simp_all add: in_components_subset comps that)
    then obtain a where a: a ∈ C a ∈ L by auto
    have opeUC: openin (top_of_set U) C
    by (metis C ‹locally connected U› clo closedin_def locally_connected_open_component
topspace_euclidean_subtopology)
    then obtain d where C ⊆ U 0 < d and d: cball a d ∩ U ⊆ C
      using openin_contains_cball by (metis ‹a ∈ C›)
    then have ball a d ∩ U ⊆ C
      by auto
    obtain h k where homhk: homeomorphism (S ∪ C) (S ∪ C) h k
                and subC: {x. (¬ (h x = x ∧ k x = x))} ⊆ C
                and bou: bounded {x. (¬ (h x = x ∧ k x = x))}
                and hin: ⋀x. x ∈ C ∩ K ⟹ h x ∈ ball a d ∩ U
    proof (rule homeomorphism_grouping_points_exists_gen [of C ball a d ∩ U
C ∩ K S ∪ C])
      show openin (top_of_set C) (ball a d ∩ U)
       by (metis open_ball ‹C ⊆ U› ‹ball a d ∩ U ⊆ C› inf.absorb_iff2 inf.orderE
inf_assoc open_openin openin_subtopology)
      show openin (top_of_set (affine hull C)) C
      by (metis ‹a ∈ C› ‹openin (top_of_set U) C› affine_hull_eq affine_hull_openin
all_not_in_conv ‹affine U›)
      show ball a d ∩ U ≠ {}
        using ‹0 < d› ‹C ⊆ U› ‹a ∈ C› by force
      show finite (C ∩ K)
        by (simp add: ‹finite K›)
      show S ∪ C ⊆ affine hull C
        by (metis ‹S ⊆ U› ‹a ∈ C› affine_hull_eq affine_hull_openin assms(2)
empty_iff hull_subset le_sup_iff opeUC)
      show connected C
        by (metis C in_components_connected)
    qed auto
    have a_BU: a ∈ ball a d ∩ U
      using ‹0 < d› ‹C ⊆ U› ‹a ∈ C› by auto
```

have *rel_frontier* (*cball a d* ∩ *U*) *retract_of* (*affine hull* (*cball a d* ∩ *U*) − {*a*})
proof (*rule rel_frontier_retract_of_punctured_affine_hull*)
show *bounded* (*cball a d* ∩ *U*) *convex* (*cball a d* ∩ *U*)
by (*auto simp*: ‹*convex U*› *convex_Int*)
show *a* ∈ *rel_interior* (*cball a d* ∩ *U*)
by (*metis* ‹*affine U*› *convex_cball empty_iff interior_cball a_BU rel_interior_convex_Int_affine*)
qed
moreover have *rel_frontier* (*cball a d* ∩ *U*) = *frontier* (*cball a d*) ∩ *U*
by (*metis a_BU* ‹*affine U*› *convex_affine_rel_frontier_Int convex_cball equals0D interior_cball*)
moreover have *affine hull* (*cball a d* ∩ *U*) = *U*
by (*metis* ‹*convex U*› *a_BU affine_hull_convex_Int_nonempty_interior affine_hull_eq* ‹*affine U*› *equals0D inf.commute interior_cball*)
ultimately have *frontier* (*cball a d*) ∩ *U retract_of* (*U* − {*a*})
by *metis*
then obtain *r* **where** *contr*: *continuous_on* (*U* − {*a*}) *r*
and *rim*: *r* ∈ (*U* − {*a*}) → *sphere a d* *r* ∈ (*U* − {*a*}) → *U*
and *req*: ⋀*x*. *x* ∈ *sphere a d* ∩ *U* ⟹ *r x* = *x*
using ‹*affine U*› **by** (*force simp*: *retract_of_def retraction_def hull_same*)
define *j* **where** *j* ≡ λ*x*. *if x* ∈ *ball a d then r x else x*
have *kj*: ⋀*x*. *x* ∈ *S* ⟹ *k* (*j x*) = *x*
using ‹*C* ⊆ *U* − *S*› ‹*S* ⊆ *U*› ‹*ball a d* ∩ *U* ⊆ *C*› *j_def subC* **by** *auto*
have *Uaeq*: *U* − {*a*} = (*cball a d* − {*a*}) ∩ *U* ∪ (*U* − *ball a d*)
using ‹*0* < *d*› **by** *auto*
have *jim*: *j* ‘ (*S* ∪ (*C* − {*a*})) ⊆ (*S* ∪ *C*) − *ball a d*
proof *clarify*
fix *y* **assume** *y* ∈ *S* ∪ (*C* − {*a*})
then have *y* ∈ *U* − {*a*}
using ‹*C* ⊆ *U* − *S*› ‹*S* ⊆ *U*› ‹*a* ∈ *C*› **by** *auto*
then have *r y* ∈ *sphere a d*
using *rim* **by** *auto*
then show *j y* ∈ *S* ∪ *C* − *ball a d*
unfolding *j_def*
using ‹*y* ∈ *S* ∪ (*C* − {*a*})› ‹*y* ∈ *U* − {*a*}› *d rim*(*2*) **by** *auto*
qed
have *contj*: *continuous_on* (*U* − {*a*}) *j*
unfolding *j_def Uaeq*
proof (*intro continuous_on_cases_local continuous_on_id*, *simp_all add*: *req closedin_closed Uaeq* [*symmetric*])
show ∃ *T*. *closed T* ∧ (*cball a d* − {*a*}) ∩ *U* = (*U* − {*a*}) ∩ *T*
using *affine_closed* ‹*affine U*› **by** (*rule_tac x*=(*cball a d*) ∩ *U* **in** *exI*) *blast*
show ∃ *T*. *closed T* ∧ *U* − *ball a d* = (*U* − {*a*}) ∩ *T*
using ‹*0* < *d*› ‹*affine U*›
by (*rule_tac x*=*U* − *ball a d* **in** *exI*) (*force simp*: *affine_closed*)
show *continuous_on* ((*cball a d* − {*a*}) ∩ *U*) *r*
by (*force intro*: *continuous_on_subset* [*OF contr*])
qed
have *fT*: *x* ∈ *U* − *K* ⟹ *f x* ∈ *T* **for** *x*

```
    using fim by blast
  show ?thesis
  proof (intro conjI exI)
    show continuous_on (S ∪ (C − {a})) (f ∘ k ∘ j)
    proof (intro continuous_on_compose)
      have S ∪ (C − {a}) ⊆ U − {a}
        using ‹C ⊆ U − S› ‹S ⊆ U› ‹a ∈ C› by force
      then show continuous_on (S ∪ (C − {a})) j
        by (rule continuous_on_subset [OF contj])
      have j ‘ (S ∪ (C − {a})) ⊆ S ∪ C
        using jim ‹C ⊆ U − S› ‹S ⊆ U› ‹ball a d ∩ U ⊆ C› j_def by blast
      then show continuous_on (j ‘ (S ∪ (C − {a}))) k
       by (rule continuous_on_subset [OF homeomorphism_cont2 [OF homhk]])
      show continuous_on (k ‘ j ‘ (S ∪ (C − {a}))) f
      proof (clarify intro!: continuous_on_subset [OF contf])
        fix y  assume y ∈ S ∪ (C − {a})
        have ky: k y ∈ S ∪ C
          using homeomorphism_image2 [OF homhk] ‹y ∈ S ∪ (C − {a})› by
blast
        have jy: j y ∈ S ∪ C − ball a d
          using Un_iff ‹y ∈ S ∪ (C − {a})› jim by auto
        have k (j y) ∈ U
          using ‹C ⊆ U› ‹S ⊆ U›  homeomorphism_image2 [OF homhk] jy by
blast
        moreover have k (j y) ∉ K
          using K unfolding disjnt_iff
        by (metis DiffE Int_iff Un_iff hin homeomorphism_def homhk image_eqI
jy)
        ultimately show k (j y) ∈ U − K
          by blast
      qed
    qed
    have ST: ⋀x. x ∈ S ⟹ (f ∘ k ∘ j) x ∈ T
    proof (simp add: kj)
      show ⋀x. x ∈ S ⟹ f x ∈ T
        using K ‹S ⊆ U› fT unfolding disjnt_iff by auto
    qed
    moreover have (f ∘ k ∘ j) x ∈ T if x ∈ C x ≠ a x ∉ S for x
    proof −
      have rx: r x ∈ sphere a d
        using ‹C ⊆ U› rim that by fastforce
      have jj: j x ∈ S ∪ C − ball a d
        using jim that by blast
      have k (j x) = j x ⟶ k (j x) ∈ C ∨ j x ∈ C
      by (metis Diff_iff Int_iff Un_iff ‹S ⊆ U› subsetD d j_def jj rx sphere_cball
that(1))
      then have kj: k (j x) ∈ C
        using homeomorphism_apply2 [OF homhk, of j x]   ‹C ⊆ U› ‹S ⊆ U› a
rx
```

```
      by (metis (mono_tags, lifting) Diff_iff subsetD jj mem_Collect_eq subC)
    then show ?thesis
      by (metis DiffE DiffI IntD1 IntI ‹C ⊆ U› comp_apply fT hin homeomorphism_apply2 homhk jj kj subset_eq)
  qed
  ultimately show (f ∘ k ∘ j) ‘ (S ∪ (C − {a})) ⊆ T
    by force
  show ∀x∈S. (f ∘ k ∘ j) x = f x using kj by simp
 qed (auto simp: a)
qed
then obtain a h where
 ah: ⋀C. ⟦C ∈ components (U − S); C ∩ K ≠ {}⟧
       ⟹ a C ∈ C ∧ a C ∈ L ∧ continuous_on (S ∪ (C − {a C})) (h C) ∧
          h C ‘ (S ∪ (C − {a C})) ⊆ T ∧ (∀x ∈ S. h C x = f x)
 using that by metis
define F where F ≡ {C ∈ components (U − S). C ∩ K ≠ {}}
define G where G ≡ {C ∈ components (U − S). C ∩ K = {}}
define UF where UF ≡ (⋃C∈F. C − {a C})
have C0 ∈ F
 by (auto simp: F_def C0)
have finite F
proof (subst finite_image_iff [of λC. C ∩ K F, symmetric])
 show inj_on (λC. C ∩ K) F
  unfolding F_def inj_on_def
  using components_nonoverlap by blast
 show finite ((λC. C ∩ K) ‘ F)
  unfolding F_def
  by (rule finite_subset [of _ Pow K]) (auto simp: ‹finite K›)
qed
obtain g where contg: continuous_on (S ∪ UF) g
         and gh: ⋀x i. ⟦i ∈ F; x ∈ (S ∪ UF) ∩ (S ∪ (i − {a i}))⟧
                       ⟹ g x = h i x
proof (rule pasting_lemma_exists_closed [OF ‹finite F›])
 let ?X = top_of_set (S ∪ UF)
 show topspace ?X ⊆ (⋃C∈F. S ∪ (C − {a C}))
  using ‹C0 ∈ F› by (force simp: UF_def)
 show closedin (top_of_set (S ∪ UF)) (S ∪ (C − {a C}))
     if C ∈ F for C
 proof (rule closedin_closed_subset [of U S ∪ C])
  have C ∈ components (U − S)
   using F_def that by blast
  then show closedin (top_of_set U) (S ∪ C)
   by (rule closedin_Un_complement_component [OF ‹locally connected U› clo])
 next
  have x = a C' if C' ∈ F  x ∈ C' x ∉ U for x C'
  proof −
   have ∀A. x ∈ ⋃A ∨ C' ∉ A
    using ‹x ∈ C'› by blast
```

```
      with that show x = a C'
        by (metis (lifting) DiffD1 F_def Union_components mem_Collect_eq)
    qed
    then show S ∪ UF ⊆ U
      using ‹S ⊆ U› by (force simp: UF_def)
  next
    show S ∪ (C - {a C}) = (S ∪ C) ∩ (S ∪ UF)
      using F_def UF_def components_nonoverlap that by auto
  qed
  show continuous_map (subtopology ?X (S ∪ (C' - {a C'}))) euclidean (h C')
if C' ∈ F for C'
  proof -
    have C': C' ∈ components (U - S) C' ∩ K ≠ {}
      using F_def that by blast+
    show ?thesis
      using ah [OF C'] by (auto simp: F_def subtopology_subtopology intro:
continuous_on_subset)
  qed
  show ⋀i j x. ⟦i ∈ F; j ∈ F;
                x ∈ topspace ?X ∩ (S ∪ (i - {a i})) ∩ (S ∪ (j - {a j}))⟧
               ⟹ h i x = h j x
    using components_eq by (fastforce simp: components_eq F_def ah)
 qed auto
 have SU': S ∪ ⋃G ∪ (S ∪ UF) ⊆ U
  using ‹S ⊆ U› in_components_subset by (auto simp: F_def G_def UF_def)
 have clo1: closedin (top_of_set (S ∪ ⋃G ∪ (S ∪ UF))) (S ∪ ⋃G)
 proof (rule closedin_closed_subset [OF _ SU'])
  have *: ⋀C. C ∈ F ⟹ openin (top_of_set U) C
    unfolding F_def
      by (metis (no_types, lifting) ‹locally connected U› clo closedin_def lo-
cally_connected_open_component mem_Collect_eq topspace_euclidean_subtopology)
  show closedin (top_of_set U) (U - UF)
    unfolding UF_def by (force intro: openin_delete *)
  have (⋃x∈F. x - {a x}) ∩ S = {} ⋃ G ⊆ U
    using in_components_subset by (auto simp: F_def G_def)
  moreover have ⋃ G ∩ UF = {}
    using components_nonoverlap by (fastforce simp: F_def G_def UF_def)
  ultimately show S ∪ ⋃G = (U - UF) ∩ (S ∪ ⋃G ∪ (S ∪ UF))
    using UF_def ‹S ⊆ U› by auto
 qed
 have clo2: closedin (top_of_set (S ∪ ⋃G ∪ (S ∪ UF))) (S ∪ UF)
 proof (rule closedin_closed_subset [OF _ SU'])
  show closedin (top_of_set U) (⋃C∈F. S ∪ C)
  proof (rule closedin_Union)
    show ⋀T. T ∈ (∪) S ‘ F ⟹ closedin (top_of_set U) T
     using F_def ‹locally connected U› clo closedin_Un_complement_component
by blast
  qed (simp add: ‹finite F›)
  show S ∪ UF = (⋃C∈F. S ∪ C) ∩ (S ∪ ⋃G ∪ (S ∪ UF))
```

```
    proof
      show ⋃ ((∪) S ‘ F) ∩ (S ∪ ⋃ G ∪ (S ∪ UF)) ⊆ S ∪ UF
        using components_eq [of _ U−S]
        by (auto simp add: F_def G_def UF_def disjoint_iff_not_equal)
    qed (use UF_def ‹C0 ∈ F› in blast)
  qed
  have SUG: S ∪ ⋃ G ⊆ U − K
     using ‹S ⊆ U› K in_components_subset[of _ U−S] by (force simp: G_def
disjnt_iff)
  then have contf': continuous_on (S ∪ ⋃ G) f
    by (rule continuous_on_subset [OF contf])
  have contg': continuous_on (S ∪ UF) g
    by (simp add: contg)
  have  ⋀x. ⟦S ⊆ U; x ∈ S⟧ ⟹ f x = g x
    by (subst gh) (auto simp: ah C0 intro: ‹C0 ∈ F›)
  then have f_eq_g: ⋀x. x ∈ S ∪ UF ∧ x ∈ S ∪ ⋃ G ⟹ f x = g x
     using ‹S ⊆ U› components_eq [of _ U−S] by (fastforce simp add: F_def
G_def UF_def)
  have cont: continuous_on (S ∪ ⋃ G ∪ (S ∪ UF)) (λx. if x ∈ S ∪ ⋃ G then f x
else g x)
    by (blast intro: continuous_on_cases_local [OF clo1 clo2 contf' contg' f_eq_g,
of λx. x ∈ S ∪ ⋃ G])
  show ?thesis
  proof
    have UF: ⋃ F − L ⊆ UF
      unfolding F_def UF_def using ah by blast
    have U − S − L = ⋃(components (U − S)) − L
      by simp
    also have ... = ⋃ F ∪ ⋃ G − L
      unfolding F_def G_def by blast
    also have ... ⊆ UF ∪ ⋃ G
      using UF by blast
    finally have U − L ⊆ S ∪ ⋃ G ∪ (S ∪ UF)
      by blast
    then show continuous_on (U − L) (λx. if x ∈ S ∪ ⋃ G then f x else g x)
      by (rule continuous_on_subset [OF cont])
    have ((U − L) ∩ {x. x ∉ S ∧ (∀xa∈G. x ∉ xa)}) ⊆  ((U − L) ∩ (−S ∩ UF))
      using ‹U − L ⊆ S ∪ ⋃ G ∪ (S ∪ UF)› by auto
    moreover have g ‘ ((U − L) ∩ (−S ∩ UF)) ⊆ T
    proof −
      have g x ∈ T if x ∈ U x ∉ L x ∉ S C ∈ F x ∈ C x ≠ a C for x C
      proof (subst gh)
        show x ∈ (S ∪ UF) ∩ (S ∪ (C − {a C}))
          using that by (auto simp: UF_def)
        show h C x ∈ T
          using ah that by (fastforce simp add: F_def)
      qed (rule that)
      then show ?thesis
        by (force simp: UF_def)
```

```
    qed
    ultimately have g ‘ ((U − L) ∩ {x. x ∉ S ∧ (∀ xa∈G. x ∉ xa)}) ⊆ T
      using image_mono order_trans by blast
    moreover have f ‘ ((U − L) ∩ (S ∪ ⋃ G)) ⊆ T
      using fim SUG by blast
    ultimately show (λx. if x ∈ S ∪ ⋃ G then f x else g x) ∈ (U − L) → T
      by force
    show ⋀x. x ∈ S ⟹ (if x ∈ S ∪ ⋃ G then f x else g x) = f x
      by (simp add: F_def G_def)
  qed
qed


lemma extend_map_affine_to_sphere2:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes compact S convex U bounded U affine T S ⊆ T
      and affTU: aff_dim T ≤ aff_dim U
      and contf: continuous_on S f
      and fim: f ∈ S → rel_frontier U
      and ovlap: ⋀C. C ∈ components(T − S) ⟹ C ∩ L ≠ {}
    obtains K g where finite K K ⊆ L K ⊆ T disjnt K S
                      continuous_on (T − K) g g ∈ (T − K) → rel_frontier U
                      ⋀x. x ∈ S ⟹ g x = f x
proof −
  obtain K g where K: finite K K ⊆ T disjnt K S
             and contg: continuous_on (T − K) g
             and gim: g ∈ (T − K) → rel_frontier U
             and gf: ⋀x. x ∈ S ⟹ g x = f x
    using assms extend_map_affine_to_sphere_cofinite_simple by metis
  have ∃ y C. C ∈ components (T − S) ∧ x ∈ C ∧ y ∈ C ∧ y ∈ L if x ∈ K for x
  proof −
    have x ∈ T−S
      using ‹K ⊆ T› ‹disjnt K S› disjnt_def that by fastforce
    then obtain C where C ∈ components(T − S) x ∈ C
      by (metis UnionE Union_components)
    with ovlap [of C] show ?thesis
      by blast
  qed
  then obtain ξ where ξ: ⋀x. x ∈ K ⟹ ∃ C. C ∈ components (T − S) ∧ x ∈
C ∧ ξ x ∈ C ∧ ξ x ∈ L
    by metis
  obtain h where conth: continuous_on (T − ξ ‘ K) h
            and him: h ∈ (T − ξ ‘ K) → rel_frontier U
            and hg: ⋀x. x ∈ S ⟹ h x = g x
  proof (rule extend_map_affine_to_sphere1 [OF ‹finite K› ‹affine T› contg gim,
of S ξ ‘ K])
    show cloTS: closedin (top_of_set T) S
      by (simp add: ‹compact S› ‹S ⊆ T› closed_subset compact_imp_closed)
    show ⋀C. ⟦C ∈ components (T − S); C ∩ K ≠ {}⟧ ⟹ C ∩ ξ ‘ K ≠ {}
```

```
    using ξ components_eq by blast
  qed (use K in auto)
  show ?thesis
  proof
    show ∗: ξ ‘ K ⊆ L
      using ξ by blast
    show finite (ξ ‘ K)
      by (simp add: K)
    show ξ ‘ K ⊆ T
      by clarify (meson ξ Diff_iff contra_subsetD in_components_subset)
    show continuous_on (T − ξ ‘ K) h
      by (rule conth)
    show disjnt (ξ ‘ K) S
      using K ξ in_components_subset by (fastforce simp: disjnt_def)
  qed (simp_all add: him hg gf)
qed


proposition extend_map_affine_to_sphere_cofinite_gen:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes SUT: compact S convex U bounded U affine T S ⊆ T
      and aff: aff_dim T ≤ aff_dim U
      and contf: continuous_on S f
      and fim: f ∈ S → rel_frontier U
      and dis: ⋀C. ⟦C ∈ components(T − S); bounded C⟧ ⟹ C ∩ L ≠ {}
 obtains K g where finite K K ⊆ L K ⊆ T disjnt K S continuous_on (T − K)
g
                  g ∈ (T − K) → rel_frontier U
                  ⋀x. x ∈ S ⟹ g x = f x
proof (cases S = {})
  case True
  show ?thesis
  proof (cases rel_frontier U = {})
    case True
    with aff have aff_dim T ≤ 0
      using affine_bounded_eq_lowdim ‹bounded U› order_trans
      by (auto simp add: rel_frontier_eq_empty)
    with aff_dim_geq [of T] consider aff_dim T = −1 | aff_dim T = 0
      by linarith
    then show ?thesis
    proof cases
      assume aff_dim T = −1
      then have T = {}
        by (simp add: aff_dim_empty)
      then show ?thesis
        by (rule_tac K={} in that) auto
    next
      assume aff_dim T = 0
      then obtain a where T = {a}
```

```
      using aff_dim_eq_0 by blast
    then have a ∈ L
      using dis [of {a}] ‹S = {}› by (auto simp: in_components_self)
    with ‹S = {}› ‹T = {a}› show ?thesis
      by (rule_tac K={a} and g=f in that) auto
  qed
 next
  case False
  then obtain y where y ∈ rel_frontier U
    by auto
  with ‹S = {}› show ?thesis
    by (rule_tac K={} and g=λx. y in that)  (auto)
 qed
next
 case False
 have bounded S
   by (simp add: assms compact_imp_bounded)
 then obtain b where b: S ⊆ cbox (−b) b
   using bounded_subset_cbox_symmetric by blast
 define LU where LU ≡ L ∪ (⋃ {C ∈ components (T − S). ¬bounded C} −
cbox (−(b+One)) (b+One))
 obtain K g where finite K K ⊆ LU K ⊆ T disjnt K S
           and contg: continuous_on (T − K) g
           and gim: g ∈ (T − K) → rel_frontier U
           and gf:  ⋀x. x ∈ S ⟹ g x = f x
 proof (rule extend_map_affine_to_sphere2 [OF SUT aff contf fim])
   show C ∩ LU ≠ {} if C ∈ components (T − S) for C
   proof (cases bounded C)
     case True
     with dis that show ?thesis
       unfolding LU_def by fastforce
   next
     case False
     then have ¬ bounded (⋃{C ∈ components (T − S). ¬ bounded C})
       by (metis (no_types, lifting) Sup_upper bounded_subset mem_Collect_eq
that)
     then show ?thesis
     by (simp add: LU_def disjoint_iff) (meson False bounded_cbox bounded_subset
subset_iff that)
   qed
 qed blast
 have ∗: False if x ∈ cbox (− b − m ∗R One) (b + m ∗R One)
                  x ∉ box (− b − n ∗R One) (b + n ∗R One)
                  0 ≤ m m < n n ≤ 1 for m n x
   using that by (auto simp: mem_box algebra_simps)
 have disjoint_family_on (λd. frontier (cbox (− b − d ∗R One) (b + d ∗R One)))
{1 / 2..1}
   by (auto simp: disjoint_family_on_def neq_iff frontier_def dest: ∗)
 then obtain d where d12: 1/2 ≤ d d ≤ 1
```

```
                and ddis: disjnt K (frontier (cbox (−(b + d *R One)) (b + d *R
One)))
    using disjoint_family_elem_disjnt [of {1/2..1::real} K λd. frontier (cbox
(−(b + d *R One)) (b + d *R One))]
   by (auto simp: ‹finite K›)
  define c where c ≡ b + d *R One
  have cbsub: cbox (−b) b ⊆ box (−c) c
            cbox (−b) b ⊆ cbox (−c) c
            cbox (−c) c ⊆ cbox (−(b+One)) (b+One)
   using d12 by (simp_all add: subset_box c_def inner_diff_left inner_left_distrib)
  have clo_cT: closed (cbox (− c) c ∩ T)
   using affine_closed ‹affine T› by blast
  have cT_ne: cbox (− c) c ∩ T ≠ {}
   using ‹S ≠ {}› ‹S ⊆ T› b cbsub by fastforce
  have S_sub_cc: S ⊆ cbox (− c) c
   using ‹cbox (− b) b ⊆ cbox (− c) c› b by auto
  show ?thesis
  proof
   show finite (K ∩ cbox (−(b+One)) (b+One))
    using ‹finite K› by blast
   show K ∩ cbox (− (b + One)) (b + One) ⊆ L
    using ‹K ⊆ LU› by (auto simp: LU_def)
   show K ∩ cbox (− (b + One)) (b + One) ⊆ T
    using ‹K ⊆ T› by auto
   show disjnt (K ∩ cbox (− (b + One)) (b + One)) S
     using ‹disjnt K S›  by (simp add: disjnt_def disjoint_eq_subset_Compl
inf.coboundedI1)
   have cloTK: closest_point (cbox (− c) c ∩ T) x ∈ T − K
            if x ∈ T and Knot: x ∈ K ⟶ x ∉ cbox (− b − One) (b + One) for
x
   proof (cases x ∈ cbox (− c) c)
    case True
    with ‹x ∈ T› show ?thesis
     using cbsub(3) Knot by (force simp: closest_point_self)
   next
    case False
    have clo_in_rf: closest_point (cbox (− c) c ∩ T) x ∈ rel_frontier (cbox (−
c) c ∩ T)
    proof (intro closest_point_in_rel_frontier [OF clo_cT cT_ne] DiffI notI)
     have T ∩ interior (cbox (− c) c) ≠ {}
      using ‹S ≠ {}› ‹S ⊆ T› b cbsub(1) by fastforce
     then show x ∈ affine hull (cbox (− c) c ∩ T)
      by (simp add: Int_commute affine_hull_affine_Int_nonempty_interior
‹affine T› hull_inc that(1))
    next
     show False if x ∈ rel_interior (cbox (− c) c ∩ T)
     proof −
      have interior (cbox (− c) c) ∩ T ≠ {}
       using ‹S ≠ {}› ‹S ⊆ T› b cbsub(1) by fastforce
```

```
        then have affine hull (T ∩ cbox (− c) c) = T
          using affine_hull_convex_Int_nonempty_interior [of T cbox (− c) c]
          by (simp add: affine_imp_convex ‹affine T› inf_commute)
        then show ?thesis
          by (meson subsetD le_inf_iff rel_interior_subset that False)
      qed
    qed
    have closest_point (cbox (− c) c ∩ T) x ∉ K
    proof
      assume inK: closest_point (cbox (− c) c ∩ T) x ∈ K
       have ⋀x. x ∈ K ⟹ x ∉ frontier (cbox (− (b + d *R One)) (b + d *R
One))
        by (metis ddis disjnt_iff)
      then show False
        by (metis DiffI Int_iff ‹affine T› cT_ne c_def clo_cT clo_in_rf clos-
est_point_in_set
                  convex_affine_rel_frontier_Int convex_box(1) empty_iff fron-
tier_cbox inK interior_cbox)
    qed
    then show ?thesis
      using cT_ne clo_cT closest_point_in_set by blast
  qed
  have convex (cbox (− c) c ∩ T)
    by (simp add: affine_imp_convex assms(4) convex_Int)
   then show continuous_on (T − K ∩ cbox (− (b + One)) (b + One)) (g ∘
closest_point (cbox (−c) c ∩ T))
    using cloTK clo_cT cT_ne
      by (intro continuous_on_compose continuous_on_closest_point continu-
ous_on_subset [OF contg]; force)
  have g (closest_point (cbox (− c) c ∩ T) x) ∈ rel_frontier U
       if x ∈ T x ∈ K ⟶ x ∉ cbox (− b − One) (b + One) for x
    using cloTK gim that by auto
  then show (g ∘ closest_point (cbox (− c) c ∩ T)) ∈ (T − K ∩ cbox (− (b +
One)) (b + One))
            → rel_frontier U
    by force
  show ⋀x. x ∈ S ⟹ (g ∘ closest_point (cbox (− c) c ∩ T)) x = f x
      by simp (metis (mono_tags, lifting) IntI ‹S ⊆ T› cT_ne clo_cT clos-
est_point_refl gf subsetD S_sub_cc)
 qed
qed


corollary extend_map_affine_to_sphere_cofinite:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes SUT: compact S affine T S ⊆ T
    and aff: aff_dim T ≤ DIM('b) and 0 ≤ r
    and contf: continuous_on S f
    and fim: f ∈ S → sphere a r
```

```
    and dis: ⋀C. ⟦C ∈ components(T − S); bounded C⟧ ⟹ C ∩ L ≠ {}
  obtains K g where finite K K ⊆ L K ⊆ T disjnt K S continuous_on (T − K)
g
                g ∈ (T − K) → sphere a r ⋀x. x ∈ S ⟹ g x = f x
proof (cases r = 0)
  case True
  with fim show ?thesis
    by (rule_tac K={} and g = λx. a in that) (auto)
next
  case False
  show thesis
  proof (rule  extend_map_affine_to_sphere_cofinite_gen
      [OF ‹compact S› convex_cball bounded_cball ‹affine T› ‹S ⊆ T› _ contf])
    have 0 < r
      using assms False by auto
    then show aff_dim T ≤ aff_dim (cball a r)
      by (simp add: aff aff_dim_cball)
    show f ∈ S → rel_frontier (cball a r)
      by (simp add: False fim)
  qed (use dis False that in auto)
qed

corollary extend_map_UNIV_to_sphere_cofinite:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes DIM('a) ≤ DIM('b) and 0 ≤ r
      and compact S
      and continuous_on S f
      and f ∈ S → sphere a r
      and ⋀C. ⟦C ∈ components(− S); bounded C⟧ ⟹ C ∩ L ≠ {}
  obtains K g where finite K K ⊆ L disjnt K S continuous_on (− K) g
                   g ∈ (− K) → sphere a r ⋀x. x ∈ S ⟹ g x = f x
  using assms extend_map_affine_to_sphere_cofinite [OF ‹compact S› affine_UNIV
subset_UNIV]
  by (metis Compl_eq_Diff_UNIV aff_dim_UNIV of_nat_le_iff)

corollary extend_map_UNIV_to_sphere_no_bounded_component:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes aff: DIM('a) ≤ DIM('b) and 0 ≤ r
      and SUT: compact S
      and contf: continuous_on S f
      and fim: f ∈ S → sphere a r
      and dis: ⋀C. C ∈ components(− S) ⟹ ¬ bounded C
  obtains g where continuous_on UNIV g g ∈ UNIV → sphere a r ⋀x. x ∈ S
⟹ g x = f x
  using extend_map_UNIV_to_sphere_cofinite [OF aff ‹0 ≤ r› ‹compact S› contf
fim, of {}]
  by (metis Compl_empty_eq dis subset_empty)
```

theorem *Borsuk_separation_theorem_gen*:
fixes S :: *'a::euclidean_space set*
assumes *compact S*
shows $(\forall\, c \in \mathit{components}(-\ S).\ \neg \mathit{bounded}\ c) \longleftrightarrow$
$(\forall f.\ \mathit{continuous\_on}\ S\ f \wedge f \in S \to \mathit{sphere}\ (0::'a)\ 1$
$\longrightarrow (\exists c.\ \mathit{homotopic\_with\_canon}\ (\lambda x.\ \mathit{True})\ S\ (\mathit{sphere}\ 0\ 1)\ f\ (\lambda x.\ c)))$
(**is** *?lhs = ?rhs*)
proof
assume *L* [*rule_format*]: *?lhs*
show *?rhs*
proof *clarify*
fix *f* :: *'a* ⇒ *'a*
assume *contf*: *continuous_on S f* **and** *fim*: $f \in S \to \mathit{sphere}\ 0\ 1$
obtain *g* **where** *contg*: *continuous_on UNIV g* **and** *gim*: $g \in \mathit{UNIV} \to \mathit{sphere}\ 0\ 1$
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
by (*rule extend_map_UNIV_to_sphere_no_bounded_component* [*OF _ _* ‹*compact S*› *contf fim L*]) *auto*
then obtain *c* **where** *c*: *homotopic_with_canon* (λ*h*. *True*) *UNIV* (*sphere 0 1*) *g* (λ*x*. *c*)
by (*metis contractible_UNIV nullhomotopic_from_contractible*)
then show ∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*) *f* (λ*x*. *c*)
by (*metis assms compact_imp_closed contf contg contractible_empty fim gf gim nullhomotopic_from_contractible nullhomotopic_into_sphere_extension image_subset_iff_funcset*)
qed
next
assume *R* [*rule_format*]: *?rhs*
show *?lhs*
unfolding *components_def*
proof *clarify*
fix *a*
assume *a* ∉ *S* **and** *a*: *bounded* (*connected_component_set* (− *S*) *a*)
have ∀ *x*∈*S*. *norm* (*x* − *a*) ≠ *0*
using ‹*a* ∉ *S*› **by** *auto*
then have *cont*: *continuous_on S* (λ*x*. *inverse*(*norm*(*x* − *a*)) ∗$_R$ (*x* − *a*))
by (*intro continuous_intros*)
have *im*: (λ*x*. *inverse*(*norm*(*x* − *a*)) ∗$_R$ (*x* − *a*)) ‘ *S* ⊆ *sphere 0 1*
by *clarsimp* (*metis* ‹*a* ∉ *S*› *eq_iff_diff_eq_0 left_inverse norm_eq_zero*)
show *False*
using *R cont im Borsuk_map_essential_bounded_component* [*OF* ‹*compact S*› ‹*a* ∉ *S*›] *a* **by** *blast*
qed
qed

corollary *Borsuk_separation_theorem*:
fixes *S* :: *'a::euclidean_space set*

```
  assumes compact S and 2: 2 ≤ DIM('a)
    shows connected(− S) ⟷
          (∀f. continuous_on S f ∧ f ∈ S → sphere (0::'a) 1
                ⟶ (∃c. homotopic_with_canon (λx. True) S (sphere 0 1) f (λx.
c)))
       (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  show ?rhs
  proof (cases S = {})
    case True
    then show ?thesis
      using homotopic_with_canon_on_empty by blast
  next
    case False
    then have (∀c∈components (− S). ¬ bounded c)
        by (metis L assms(1) bounded_empty cobounded_imp_unbounded com-
pact_imp_bounded in_components_maximal order_refl)
    then show ?thesis
      by (simp add: Borsuk_separation_theorem_gen [OF ‹compact S›])
  qed
next
  assume R: ?rhs
  then have ∀c∈components (− S). ¬ bounded c ⟹ connected (− S)
   by (metis 2 assms(1) cobounded_has_bounded_component compact_imp_bounded
double_complement)
  with R show ?lhs
    by (simp add: Borsuk_separation_theorem_gen [OF ‹compact S›])
qed


lemma homotopy_eqv_separation:
  fixes S :: 'a::euclidean_space set and T :: 'a set
  assumes S homotopy_eqv T and compact S and compact T
  shows connected(− S) ⟷ connected(− T)
proof −
  consider DIM('a) = 1 | 2 ≤ DIM('a)
   by (metis DIM_ge_Suc0 One_nat_def Suc_1 dual_order.antisym not_less_eq_eq)
  then show ?thesis
  proof cases
    case 1
    then show ?thesis
     using bounded_connected_Compl_1 compact_imp_bounded homotopy_eqv_empty1
homotopy_eqv_empty2 assms by metis
  next
    case 2
    with assms show ?thesis
     by (simp add: Borsuk_separation_theorem homotopy_eqv_cohomotopic_triviality_null)
  qed
```

qed

proposition *Jordan_Brouwer_separation*:
fixes $S :: 'a::euclidean\_space\ set$ **and** $a::'a$
assumes *hom*: S *homeomorphic sphere* $a\ r$ **and** $0 < r$
shows $\neg$ *connected*$(-S)$
proof −
have $-$ *sphere* $a\ r \cap$ *ball* $a\ r \neq \{\}$
using ‹$0 < r$› **by** (*simp add*: *Int_absorb1 subset_eq*)
moreover
have *eq*: $-$ *sphere* $a\ r -$ *ball* $a\ r = -$ *cball* $a\ r$
by *auto*
have $-$ *cball* $a\ r \neq \{\}$
proof −
have *frontier* (*cball* $a\ r$) $\neq \{\}$
using ‹$0 < r$› **by** *auto*
then show *?thesis*
by (*metis frontier_complement frontier_empty*)
qed
with *eq* **have** $-$ *sphere* $a\ r -$ *ball* $a\ r \neq \{\}$
by *auto*
moreover
have *connected* $(-S)$ = *connected* $(-$ *sphere* $a\ r)$
by (*meson hom compact_sphere homeomorphic_compactness homeomorphic_imp_homotopy_eqv homotopy_eqv_separation*)
ultimately show *?thesis*
using *connected_Int_frontier* [*of* $-$ *sphere* $a\ r$ *ball* $a\ r$] **by** (*auto simp*: ‹$0 < r$›)
qed

proposition *Jordan_Brouwer_frontier*:
fixes $S :: 'a::euclidean\_space\ set$ **and** $a::'a$
assumes S: S *homeomorphic sphere* $a\ r$ **and** T: $T \in$ *components*$(-S)$ **and** *2*: $2 \leq DIM('a)$
shows *frontier* $T = S$
proof (*cases* r *rule*: *linorder_cases*)
assume $r < 0$
with $S\ T$ **show** *?thesis* **by** *auto*
next
assume $r = 0$
with $S\ T$ *card_eq_SucD* **obtain** b **where** $S = \{b\}$
by (*auto simp*: *homeomorphic_finite* [*of* $\{a\}\ S$])
have *components* $(-\{b\}) = \{-\{b\}\}$
using T ‹$S = \{b\}$› **by** (*auto simp*: *components_eq_sing_iff connected_punctured_universe 2*)
with T **show** *?thesis*
by (*metis* ‹$S = \{b\}$› *cball_trivial frontier_cball frontier_complement singletonD sphere_trivial*)

```
next
  assume r > 0
  have compact S
    using homeomorphic_compactness compact_sphere S by blast
  show ?thesis
  proof (rule frontier_minimal_separating_closed)
    show closed S
      using ‹compact S› compact_eq_bounded_closed by blast
    show ¬ connected (− S)
      using Jordan_Brouwer_separation S ‹0 < r› by blast
    obtain f g where hom: homeomorphism S (sphere a r) f g
      using S by (auto simp: homeomorphic_def)
    show connected (− T) if closed T T ⊂ S for T
    proof −
      have f ∈ T → sphere a r
        using ‹T ⊂ S› hom homeomorphism_image1 by blast
      moreover have f ‘ T ≠ sphere a r
        using ‹T ⊂ S› hom
        by (metis homeomorphism_image2 homeomorphism_of_subsets order_refl
psubsetE)
      ultimately have f ‘ T ⊂ sphere a r by blast
      then have connected (− f ‘ T)
        by (rule psubset_sphere_Compl_connected [OF _ ‹0 < r› 2])
      moreover have compact T
          using ‹compact S› bounded_subset compact_eq_bounded_closed that by
blast
      moreover then have compact (f ‘ T)
        by (meson compact_continuous_image continuous_on_subset hom homeo-
morphism_def psubsetE ‹T ⊂ S›)
      moreover have T homotopy_eqv f ‘ T
          by (meson hom homeomorphic_def homeomorphic_imp_homotopy_eqv
homeomorphism_of_subsets order.refl psubsetE that(2))
      ultimately show ?thesis
        using homotopy_eqv_separation [of T f‘T] by blast
    qed
  qed (rule T)
qed

proposition Jordan_Brouwer_nonseparation:
  fixes S :: 'a::euclidean_space set and a::'a
  assumes S: S homeomorphic sphere a r and T ⊂ S and 2: 2 ≤ DIM('a)
    shows connected(− T)
proof −
  have ∗: connected(C ∪ (S − T)) if C ∈ components(− S) for C
  proof (rule connected_intermediate_closure)
    show connected C
      using in_components_connected that by auto
    have S = frontier C
      using 2 Jordan_Brouwer_frontier S that by blast
```

```
    with closure_subset show C ∪ (S − T) ⊆ closure C
      by (auto simp: frontier_def)
  qed auto
  have components(− S) ≠ {}
   by (metis S bounded_empty cobounded_imp_unbounded compact_eq_bounded_closed
compact_sphere
            components_eq_empty homeomorphic_compactness)
  then have − T = (⋃ C ∈ components(− S). C ∪ (S − T))
    using Union_components [of −S] ‹T ⊂ S› by auto
  moreover have connected ...
    using ‹T ⊂ S› by (intro connected_Union) (auto simp: ∗)
  ultimately show ?thesis
    by simp
qed
```

9.27.5 Invariance of domain and corollaries

```
lemma invariance_of_domain_ball:
  fixes f :: 'a ⇒ 'a::euclidean_space
  assumes contf: continuous_on (cball a r) f and 0 < r
     and inj: inj_on f (cball a r)
   shows open(f ‘ ball a r)
proof (cases DIM('a) = 1)
  case True
  obtain h::'a⇒real and k
    where linear h linear k h ‘ UNIV = UNIV k ‘ UNIV = UNIV
      ⋀x. norm(h x) = norm x ⋀x. norm(k x) = norm x
      and kh: ⋀x. k(h x) = x and ⋀x. h(k x) = x
  proof (rule isomorphisms_UNIV_UNIV)
    show DIM('a) = DIM(real)
      using True by force
  qed (metis UNIV_I UNIV_eq_I imageI)
  have cont: continuous_on S h  continuous_on T k for S T
     by (simp_all add: ‹linear h› ‹linear k› linear_continuous_on linear_linear)
    have continuous_on (h ‘ cball a r) (h ∘ f ∘ k)
      by (intro continuous_on_compose cont continuous_on_subset [OF contf])
(auto simp: kh)
    moreover have is_interval (h ‘ cball a r)
      by (simp add: is_interval_connected_1 ‹linear h› linear_continuous_on
linear_linear connected_continuous_image)
    moreover have inj_on (h ∘ f ∘ k) (h ‘ cball a r)
      using inj by (simp add: inj_on_def) (metis ‹⋀x. k (h x) = x›)
    ultimately have ∗: ⋀T. ⟦open T; T ⊆ h ‘ cball a r⟧ ⟹ open ((h ∘ f ∘ k) ‘
T)
      using injective_eq_1d_open_map_UNIV by blast
    have open ((h ∘ f ∘ k) ‘ (h ‘ ball a r))
    by (rule ∗) (auto simp: ‹linear h› ‹range h = UNIV› open_surjective_linear_image)
    then have open ((h ∘ f) ‘ ball a r)
      by (simp add: image_comp ‹⋀x. k (h x) = x› cong: image_cong)
```

```
    then show ?thesis
      unfolding image_comp [symmetric]
      by (metis open_bijective_linear_image_eq ‹linear h› kh ‹range h = UNIV›
bijI inj_on_def)
next
  case False
  then have 2: DIM('a) ≥ 2
    by (metis DIM_ge_Suc0 One_nat_def Suc_1 antisym not_less_eq_eq)
  have fimsub: f ‘ ball a r ⊆ − f ‘ sphere a r
   using inj by clarsimp (metis inj_onD less_eq_real_def mem_cball order_less_irrefl)
  have hom: f ‘ sphere a r homeomorphic sphere a r
   by (meson compact_sphere contf continuous_on_subset homeomorphic_compact
homeomorphic_sym inj inj_on_subset sphere_cball)
  then have nconn: ¬ connected (− f ‘ sphere a r)
    by (rule Jordan_Brouwer_separation) (auto simp: ‹0 < r›)
  have bounded (f ‘ sphere a r)
   by (meson compact_imp_bounded compact_continuous_image_eq compact_sphere
contf inj sphere_cball)
  then obtain C where C: C ∈ components (− f ‘ sphere a r) and bounded C
    using cobounded_has_bounded_component [OF _ nconn] 2 by auto
  moreover have f ‘ (ball a r) = C
  proof
    have C ≠ {}
      by (rule in_components_nonempty [OF C])
    show C ⊆ f ‘ ball a r
    proof (rule ccontr)
      assume nonsub: ¬ C ⊆ f ‘ ball a r
      have − f ‘ cball a r ⊆ C
      proof (rule components_maximal [OF C])
        have f ‘ cball a r homeomorphic cball a r
          using compact_cball contf homeomorphic_compact homeomorphic_sym
inj by blast
        then show connected (− f ‘ cball a r)
          by (auto intro: connected_complement_homeomorphic_convex_compact
2)
        show − f ‘ cball a r ⊆ − f ‘ sphere a r
          by auto
        then show C ∩ − f ‘ cball a r ≠ {}
          using ‹C ≠ {}› in_components_subset [OF C] nonsub
          using image_iff by fastforce
      qed
      then have bounded (− f ‘ cball a r)
        using bounded_subset ‹bounded C› by auto
      then have ¬ bounded (f ‘ cball a r)
        using cobounded_imp_unbounded by blast
      then show False
      using compact_continuous_image [OF contf] compact_cball compact_imp_bounded
by blast
    qed
```

with ‹$C \neq \{\}$› **have** $C \cap f$ ‘ *ball a r* $\neq \{\}$
by (*simp add*: *inf.absorb_iff1*)
then show f ‘ *ball a r* $\subseteq C$
by (*metis components_maximal* [*OF C _ fimsub*] *connected_continuous_image ball_subset_cball connected_ball contf continuous_on_subset*)
qed
moreover have *open* $(-f$ ‘ *sphere a r*)
using *hom compact_eq_bounded_closed compact_sphere homeomorphic_compactness* **by** *blast*
ultimately show *?thesis*
using *open_components* **by** *blast*
qed

Proved by L. E. J. Brouwer (1912)

theorem *invariance_of_domain*:
fixes $f :: 'a \Rightarrow 'a$::*euclidean_space*
assumes *continuous_on S f open S inj_on f S*
shows *open*(f ‘ S)
unfolding *open_subopen* [*of f*‘*S*]
proof *clarify*
fix a
assume $a \in S$
obtain δ **where** $\delta > 0$ **and** δ: *cball a* $\delta \subseteq S$
using ‹*open S*› ‹$a \in S$› *open_contains_cball_eq* **by** *blast*
show $\exists T.$ *open* $T \wedge f\ a \in T \wedge T \subseteq f$ ‘ S
proof (*intro exI conjI*)
show *open* (f ‘ (*ball a* δ))
by (*meson* δ ‹$0 < \delta$› *assms continuous_on_subset inj_on_subset invariance_of_domain_ball*)
show $f\ a \in f$ ‘ *ball a* δ
by (*simp add*: ‹$0 < \delta$›)
show f ‘ *ball a* $\delta \subseteq f$ ‘ S
using δ *ball_subset_cball* **by** *blast*
qed
qed

lemma *inv_of_domain_ss0*:
fixes $f :: 'a \Rightarrow 'a$::*euclidean_space*
assumes *contf*: *continuous_on U f* **and** *injf*: *inj_on f U* **and** *fim*: $f \in U \to S$
and *subspace S* **and** *dimS*: *dim S* = *DIM*('*b*::*euclidean_space*)
and *ope*: *openin* (*top_of_set S*) *U*
shows *openin* (*top_of_set S*) (f ‘ U)
proof −
have $U \subseteq S$
using *ope openin_imp_subset* **by** *blast*
have (*UNIV*::'*b set*) *homeomorphic S*
by (*simp add*: ‹*subspace S*› *dimS homeomorphic_subspaces*)
then obtain *h k* **where** *homhk*: *homeomorphism* (*UNIV*::'*b set*) *S h k*
using *homeomorphic_def* **by** *blast*

```
  have homkh: homeomorphism S (k ‘ S) k h
    using homhk homeomorphism_image2 homeomorphism_sym by fastforce
  have open ((k ∘ f ∘ h) ‘ k ‘ U)
  proof (rule invariance_of_domain)
    show continuous_on (k ‘ U) (k ∘ f ∘ h)
    proof (intro continuous_intros)
      show continuous_on (k ‘ U) h
       by (meson continuous_on_subset [OF homeomorphism_cont1 [OF homhk]]
top_greatest)
      have h ‘ k ‘ U ⊆ U
        by (metis ‹U ⊆ S› dual_order.eq_iff homeomorphism_image2 homeomor-
phism_of_subsets homkh)
      then show continuous_on (h ‘ k ‘ U) f
        by (rule continuous_on_subset [OF contf])
      have f ‘ h ‘ k ‘ U ⊆ S
        using ‹h ‘ k ‘ U ⊆ U› fim by blast
      then show continuous_on (f ‘ h ‘ k ‘ U) k
        by (rule continuous_on_subset [OF homeomorphism_cont2 [OF homhk]])
    qed
    have ope_iff: ⋀T. open T ⟷ openin (top_of_set (k ‘ S)) T
      using homhk homeomorphism_image2 open_openin by fastforce
    show open (k ‘ U)
      by (simp add: ope_iff homeomorphism_imp_open_map [OF homkh ope])
    show inj_on (k ∘ f ∘ h) (k ‘ U)
      unfolding inj_on_def
      by (smt (verit, ccfv_threshold) PiE ‹U ⊆ S› assms(3) comp_apply homeo-
morphism_def homhk imageE inj_on_def injf subset_eq)
  qed
  moreover
  have eq: f ‘ U = h ‘ (k ∘ f ∘ h ∘ k) ‘ U
    unfolding image_comp [symmetric] using ‹U ⊆ S› fim
    by (metis homeomorphism_image2 homeomorphism_of_subsets homhk homkh
image_subset_iff_funcset top_greatest)
  ultimately show ?thesis
    by (metis (no_types, opaque_lifting) homeomorphism_imp_open_map homhk
image_comp open_openin subtopology_UNIV)
qed

lemma inv_of_domain_ss1:
  fixes f :: 'a ⇒ 'a::euclidean_space
  assumes contf: continuous_on U f and injf: inj_on f U and fim: f ∈ U → S
      and subspace S
      and ope: openin (top_of_set S) U
    shows openin (top_of_set S) (f ‘ U)
proof −
  define S' where S' ≡ {y. ∀x ∈ S. orthogonal x y}
  have subspace S'
    by (simp add: S'_def subspace_orthogonal_to_vectors)
  define g where g ≡ λz::'a*'a. ((f ∘ fst)z, snd z)
```

```
  have openin (top_of_set (S × S')) (g ‘ (U × S'))
  proof (rule inv_of_domain_ss0)
    show continuous_on (U × S') g
      unfolding g_def
      by (auto intro!: continuous_intros continuous_on_compose2 [OF contf con-
tinuous_on_fst])
    show g ∈ (U × S') → S × S'
      using fim  by (auto simp: g_def)
    show inj_on g (U × S')
      using injf by (auto simp: g_def inj_on_def)
    show subspace (S × S')
      by (simp add: ‹subspace S'› ‹subspace S› subspace_Times)
    show openin (top_of_set (S × S')) (U × S')
      by (simp add: openin_Times [OF ope])
    have dim (S × S') = dim S + dim S'
      by (simp add: ‹subspace S'› ‹subspace S› dim_Times)
    also have ... = DIM('a)
    using dim_subspace_orthogonal_to_vectors [OF ‹subspace S› subspace_UNIV]
      by (simp add: add.commute S'_def)
    finally show dim (S × S') = DIM('a) .
  qed
  moreover have g ‘ (U × S') = f ‘ U × S'
    by (auto simp: g_def image_iff)
  moreover have 0 ∈ S'
    using ‹subspace S'› subspace_affine by blast
  ultimately show ?thesis
    by (auto simp: openin_Times_eq)
qed


corollary invariance_of_domain_subspaces:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes ope: openin (top_of_set U) S
      and subspace U subspace V and VU: dim V ≤ dim U
      and contf: continuous_on S f and fim: f ∈ S → V
      and injf: inj_on f S
    shows openin (top_of_set V) (f ‘ S)
proof −
  obtain V' where subspace V' V' ⊆ U dim V' = dim V
    using choose_subspace_of_subspace [OF VU]
    by (metis span_eq_iff ‹subspace U›)
  then have V homeomorphic V'
    by (simp add: ‹subspace V› homeomorphic_subspaces)
  then obtain h k where homhk: homeomorphism V V' h k
    using homeomorphic_def by blast
  have eq: f ‘ S = k ‘ (h ∘ f) ‘ S
  proof −
    have k ‘ h ‘ f ‘ S = f ‘ S
      by (meson equalityD2 fim funcset_image homeomorphism_image2 homeo-
```

morphism_of_subsets homhk)
then show *?thesis*
by (*simp add*: *image_comp*)
qed
show *?thesis*
unfolding *eq*
proof (*rule homeomorphism_imp_open_map*)
show *homkh*: *homeomorphism V' V k h*
by (*simp add*: *homeomorphism_symD homhk*)
have *hfV'*: $(h \circ f)$ ` $S \subseteq V'$
using *fim homeomorphism_image1 homhk* **by** *fastforce*
moreover have *openin* (*top_of_set U*) ($(h \circ f)$ ` S)
proof (*rule inv_of_domain_ss1*)
show *continuous_on S* $(h \circ f)$
by (*meson contf continuous_on_compose continuous_on_subset fim funcset_image homeomorphism_cont2 homkh*)
show *inj_on* $(h \circ f)$ *S*
by (*smt* (*verit*, *ccfv_SIG*) *Pi_iff comp_apply fim homeomorphism_apply2 homkh inj_on_def injf*)
show $h \circ f \in S \to U$
using ‹$V' \subseteq U$› *hfV'* **by** *blast*
qed (*auto simp*: *assms*)
ultimately show *openin* (*top_of_set V'*) ($(h \circ f)$ ` S)
using *openin_subset_trans* ‹$V' \subseteq U$› **by** *force*
qed
qed

corollary *invariance_of_dimension_subspaces*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \to V$
and *injf*: *inj_on f S* **and** $S \neq \{\}$
shows *dim U* $\leq$ *dim V*
proof −
have *False* **if** *dim V* < *dim U*
proof −
obtain *T* **where** *subspace T T* $\subseteq$ *U dim T* = *dim V*
using *choose_subspace_of_subspace* [*of dim V U*]
by (*metis* ‹*dim V* < *dim U*› *assms*(*2*) *order.strict_implies_order span_eq_iff*)
then have *V homeomorphic T*
by (*simp add*: ‹*subspace V*› *homeomorphic_subspaces*)
then obtain *h k* **where** *homhk*: *homeomorphism V T h k*
using *homeomorphic_def* **by** *blast*
have *continuous_on S* $(h \circ f)$
by (*meson contf continuous_on_compose continuous_on_subset fim homeomorphism_def homhk image_subset_iff_funcset*)
moreover have $(h \circ f)$ ` $S \subseteq U$
using ‹$T \subseteq U$› *fim homeomorphism_image1 homhk* **by** *fastforce*

```
    moreover have inj_on (h ∘ f) S
      unfolding inj_on_def
       by (metis Pi_iff comp_apply fim homeomorphism_def homhk inj_on_def
injf)
    ultimately have ope_hf: openin (top_of_set U) ((h ∘ f) ‘ S)
      using invariance_of_domain_subspaces [OF ope ‹subspace U› ‹subspace U›]
by blast
    have (h ∘ f) ‘ S ⊆ T
      using fim homeomorphism_image1 homhk by fastforce
    then have dim ((h ∘ f) ‘ S) ≤ dim T
      by (rule dim_subset)
    also have dim ((h ∘ f) ‘ S) = dim U
      using ‹S ≠ {}› ‹subspace U›
      by (blast intro: dim_openin ope_hf)
    finally show False
      using ‹dim V < dim U› ‹dim T = dim V› by simp
  qed
  then show ?thesis
    using not_less by blast
qed

corollary invariance_of_domain_affine_sets:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes ope: openin (top_of_set U) S
      and aff: affine U affine V aff_dim V ≤ aff_dim U
      and contf: continuous_on S f and fim: f ∈ S → V
      and injf: inj_on f S
    shows openin (top_of_set V) (f ‘ S)
proof (cases S = {})
  case True
  then show ?thesis by auto
next
  case False
  obtain a b where a ∈ S a ∈ U b ∈ V
    using False fim ope openin_contains_cball by fastforce
  have openin (top_of_set ((+) (− b) ‘ V)) (((+) (− b) ∘ f ∘ (+) a) ‘ (+) (− a)
‘ S)
  proof (rule invariance_of_domain_subspaces)
    show openin (top_of_set ((+) (− a) ‘ U)) ((+) (− a) ‘ S)
      by (metis ope homeomorphism_imp_open_map homeomorphism_translation
translation_galois)
    show subspace ((+) (− a) ‘ U)
       by (simp add: ‹a ∈ U› affine_diffs_subspace_subtract ‹affine U› cong:
image_cong_simp)
    show subspace ((+) (− b) ‘ V)
       by (simp add: ‹b ∈ V› affine_diffs_subspace_subtract ‹affine V› cong:
image_cong_simp)
    show dim ((+) (− b) ‘ V) ≤ dim ((+) (− a) ‘ U)
     by (metis ‹a ∈ U› ‹b ∈ V› aff_dim_eq_dim affine_hull_eq aff of_nat_le_iff)
```

show *continuous_on* ((+) (− *a*) ‘ *S*) ((+) (− *b*) ∘ *f* ∘ (+) *a*)
by (*metis contf continuous_on_compose homeomorphism_cont2 homeomorphism_translation translation_galois*)
show (+) (− *b*) ∘ *f* ∘ (+) *a* ∈ (+) (− *a*) ‘ *S* → (+) (− *b*) ‘ *V*
using *fim* **by** *auto*
show *inj_on* ((+) (− *b*) ∘ *f* ∘ (+) *a*) ((+) (− *a*) ‘ *S*)
by (*auto simp*: *inj_on_def*) (*meson inj_onD injf*)
qed
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *homeomorphism_imp_open_map homeomorphism_translation image_comp translation_galois*)
qed

corollary *invariance_of_dimension_affine_sets*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V*
and *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *V*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *aff_dim U* ≤ *aff_dim V*
proof −
obtain *a b* **where** *a* ∈ *S a* ∈ *U b* ∈ *V*
using ‹*S* ≠ {}› *fim ope openin_contains_cball* **by** *fastforce*
have *dim* ((+) (− *a*) ‘ *U*) ≤ *dim* ((+) (− *b*) ‘ *V*)
proof (*rule invariance_of_dimension_subspaces*)
show *openin* (*top_of_set* ((+) (− *a*) ‘ *U*)) ((+) (− *a*) ‘ *S*)
by (*metis ope homeomorphism_imp_open_map homeomorphism_translation translation_galois*)
show *subspace* ((+) (− *a*) ‘ *U*)
by (*simp add*: ‹*a* ∈ *U*› *affine_diffs_subspace_subtract* ‹*affine U*› *cong*: *image_cong_simp*)
show *subspace* ((+) (− *b*) ‘ *V*)
by (*simp add*: ‹*b* ∈ *V*› *affine_diffs_subspace_subtract* ‹*affine V*› *cong*: *image_cong_simp*)
show *continuous_on* ((+) (− *a*) ‘ *S*) ((+) (− *b*) ∘ *f* ∘ (+) *a*)
by (*metis contf continuous_on_compose homeomorphism_cont2 homeomorphism_translation translation_galois*)
show (+) (− *b*) ∘ *f* ∘ (+) *a* ∈ (+) (− *a*) ‘ *S* → (+) (− *b*) ‘ *V*
using *fim* **by** *auto*
show *inj_on* ((+) (− *b*) ∘ *f* ∘ (+) *a*) ((+) (− *a*) ‘ *S*)
by (*auto simp*: *inj_on_def*) (*meson inj_onD injf*)
qed (*use* ‹*S* ≠ {}› **in** *auto*)
then show *?thesis*
by (*metis* ‹*a* ∈ *U*› ‹*b* ∈ *V*› *aff_dim_eq_dim affine_hull_eq aff of_nat_le_iff*)
qed

corollary *invariance_of_dimension*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *contf*: *continuous_on S f* **and** *open S*

```
    and injf: inj_on f S and S ≠ {}
  shows DIM('a) ≤ DIM('b)
 using invariance_of_dimension_subspaces [of UNIV S UNIV f] assms
 by auto


corollary continuous_injective_image_subspace_dim_le:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes subspace S subspace T
    and contf: continuous_on S f and fim: f ∈ S → T
    and injf: inj_on f S
  shows dim S ≤ dim T
 using invariance_of_dimension_subspaces [of S S _ f] assms by (auto simp:
subspace_affine)

lemma invariance_of_dimension_convex_domain:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes convex S
    and contf: continuous_on S f and fim: f ∈ S → affine hull T
    and injf: inj_on f S
  shows aff_dim S ≤ aff_dim T
proof (cases S = {})
 case True
 then show ?thesis by (simp add: aff_dim_geq)
next
 case False
 have aff_dim (affine hull S) ≤ aff_dim (affine hull T)
 proof (rule invariance_of_dimension_affine_sets)
  show openin (top_of_set (affine hull S)) (rel_interior S)
   by (simp add: openin_rel_interior)
  show continuous_on (rel_interior S) f
   using contf continuous_on_subset rel_interior_subset by blast
  show f ∈ rel_interior S → affine hull T
   using fim rel_interior_subset by blast
  show inj_on f (rel_interior S)
   using inj_on_subset injf rel_interior_subset by blast
  show rel_interior S ≠ {}
   by (simp add: False ‹convex S› rel_interior_eq_empty)
 qed auto
 then show ?thesis
  by simp
qed


lemma homeomorphic_convex_sets_le:
 assumes convex S S homeomorphic T
 shows aff_dim S ≤ aff_dim T
proof −
 obtain h k where homhk: homeomorphism S T h k
```

```
    using homeomorphic_def assms  by blast
  show ?thesis
  proof (rule invariance_of_dimension_convex_domain [OF ‹convex S›])
    show continuous_on S h
      using homeomorphism_def homhk by blast
    show h ∈ S → affine hull T
      using homeomorphism_image1 homhk hull_subset by fastforce
    show inj_on h S
      by (meson homeomorphism_apply1 homhk inj_on_inverseI)
  qed
qed

lemma homeomorphic_convex_sets:
  assumes convex S convex T S homeomorphic T
  shows aff_dim S = aff_dim T
 by (meson assms dual_order.antisym homeomorphic_convex_sets_le homeomor-
phic_sym)

lemma homeomorphic_convex_compact_sets_eq:
  assumes convex S compact S convex T compact T
  shows S homeomorphic T ⟷ aff_dim S = aff_dim T
 by (meson assms homeomorphic_convex_compact_sets homeomorphic_convex_sets)

lemma invariance_of_domain_gen:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f inj_on f S DIM('b) ≤ DIM('a)
    shows open(f ‘ S)
  using invariance_of_domain_subspaces [of UNIV S UNIV f] assms by auto

lemma injective_into_1d_imp_open_map_UNIV:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes open T continuous_on S f inj_on f S T ⊆ S
  shows open (f ‘ T)
proof −
  have DIM(real) ≤ DIM('a)
    by simp
  then show ?thesis
   using invariance_of_domain_gen assms continuous_on_subset subset_inj_on
by metis
qed

lemma continuous_on_inverse_open:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) and gf: ⋀x. x ∈ S ⟹
g(f x) = x
    shows continuous_on (f ‘ S) g
proof (clarsimp simp add: continuous_openin_preimage_eq)
  fix T :: 'a set
  assume open T
```

```
  have eq: f ‘ S ∩ g −‘ T = f ‘ (S ∩ T)
    by (auto simp: gf)
  have open (f ‘ S)
    by (rule invariance_of_domain_gen) (use assms inj_on_inverseI in auto)
  moreover have open (f ‘ (S ∩ T))
    using assms
      by (metis ‹open T› continuous_on_subset inj_onI inj_on_subset invariance_of_domain_gen openin_open openin_open_eq)
  ultimately show openin (top_of_set (f ‘ S)) (f ‘ S ∩ g −‘ T)
    unfolding eq by (auto intro: open_openin_trans)
qed

lemma invariance_of_domain_homeomorphism:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) inj_on f S
  obtains g where homeomorphism S (f ‘ S) f g
proof
  show homeomorphism S (f ‘ S) f (inv_into S f)
    by (simp add: assms continuous_on_inverse_open homeomorphism_def)
qed

corollary invariance_of_domain_homeomorphic:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) inj_on f S
  shows S homeomorphic (f ‘ S)
  using invariance_of_domain_homeomorphism [OF assms]
  by (meson homeomorphic_def)

lemma continuous_image_subset_interior:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes continuous_on S f inj_on f S DIM('b) ≤ DIM('a)
  shows f ∈ (interior S) → interior(f ‘ S)
proof −
  have open (f ‘ interior S)
    using assms
   by (intro invariance_of_domain_gen) (auto simp: subset_inj_on interior_subset continuous_on_subset)
  then show ?thesis
    by (simp add: image_mono interiorI interior_subset)
qed

lemma homeomorphic_interiors_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T and dimeq: DIM('a) = DIM('b)
  shows (interior S) homeomorphic (interior T)
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
```

```
  assume S: ∀x∈S. f x ∈ T ∧ g (f x) = x and T: ∀y∈T. g y ∈ S ∧ f (g y) = y
    and contf: continuous_on S f and contg: continuous_on T g
  then have fST: f ‘ S = T and gTS: g ‘ T = S and inj_on f S inj_on g T
    by (auto simp: inj_on_def intro: rev_image_eqI) metis+
  have fim: f ∈ interior S → interior T
    using continuous_image_subset_interior [OF contf ‹inj_on f S›] dimeq fST
by simp
  have gim: g ∈ interior T → interior S
    using continuous_image_subset_interior [OF contg ‹inj_on g T›] dimeq gTS
by simp
  show homeomorphism (interior S) (interior T) f g
    unfolding homeomorphism_def
  proof (intro conjI ballI)
    show ⋀x. x ∈ interior S ⟹ g (f x) = x
      by (meson ‹∀x∈S. f x ∈ T ∧ g (f x) = x› subsetD interior_subset)
    have interior T ⊆ f ‘ interior S
    proof
      fix x assume x ∈ interior T
      then have g x ∈ interior S
        using gim by blast
      then show x ∈ f ‘ interior S
        by (metis T ‹x ∈ interior T› image_iff interior_subset subsetCE)
    qed
    then show f ‘ interior S = interior T
      using fim by blast
    show continuous_on (interior S) f
      by (metis interior_subset continuous_on_subset contf)
    show ⋀y. y ∈ interior T ⟹ f (g y) = y
      by (meson T subsetD interior_subset)
    have interior S ⊆ g ‘ interior T
    proof
      fix x assume x ∈ interior S
      then have f x ∈ interior T
        using fim by blast
      then show x ∈ g ‘ interior T
        by (metis S ‹x ∈ interior S› image_iff interior_subset subsetCE)
    qed
    then show g ‘ interior T = interior S
      using gim by blast
    show continuous_on (interior T) g
      by (metis interior_subset continuous_on_subset contg)
  qed
qed

lemma homeomorphic_open_imp_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T open S S ≠ {} open T T ≠ {}
  shows DIM('a) = DIM('b)
    using assms
```

```
    apply (simp add: homeomorphic_minimal)
    apply (rule order_antisym; metis inj_onI invariance_of_dimension)
    done

proposition homeomorphic_interiors:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T interior S = {} ⟷ interior T = {}
    shows (interior S) homeomorphic (interior T)
proof (cases interior T = {})
  case False
  then have DIM('a) = DIM('b)
    using assms
    apply (simp add: homeomorphic_minimal)
   apply (rule order_antisym; metis continuous_on_subset inj_onI inj_on_subset
interior_subset invariance_of_dimension open_interior)
    done
  then show ?thesis
   by (rule homeomorphic_interiors_same_dimension [OF ‹S homeomorphic T›])
qed (use assms in auto)

lemma homeomorphic_frontiers_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T closed S closed T and dimeq: DIM('a) = DIM('b)
  shows (frontier S) homeomorphic (frontier T)
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
  assume S: ∀ x∈S. f x ∈ T ∧ g (f x) = x and T: ∀ y∈T. g y ∈ S ∧ f (g y) = y
     and contf: continuous_on S f and contg: continuous_on T g
  then have fST: f ‘ S = T and gTS: g ‘ T = S and inj_on f S inj_on g T
    by (auto simp: inj_on_def intro: rev_image_eqI) metis+
  have g ∈ interior T → interior S
    using continuous_image_subset_interior [OF contg ‹inj_on g T›] dimeq gTS
by simp
  then have fim: f ‘ frontier S ⊆ frontier T
    unfolding frontier_def using Pi_mem S assms by fastforce
  have f ∈ interior S → interior T
     using continuous_image_subset_interior [OF contf ‹inj_on f S›] dimeq fST
by simp
  then have gim: g ‘ frontier T ⊆ frontier S
    unfolding frontier_def using Pi_mem T assms by fastforce
  show homeomorphism (frontier S) (frontier T) f g
    unfolding homeomorphism_def
  proof (intro conjI ballI)
    show gf: ⋀x. x ∈ frontier S ⟹ g (f x) = x
      by (simp add: S assms(2) frontier_def)
    show fg: ⋀y. y ∈ frontier T ⟹ f (g y) = y
      by (simp add: T assms(3) frontier_def)
```

```
    have frontier T ⊆ f ‘ frontier S
    proof
      fix x assume x ∈ frontier T
      then have g x ∈ frontier S
        using gim by blast
      then show x ∈ f ‘ frontier S
        by (metis fg ‹x ∈ frontier T› imageI)
    qed
    then show f ‘ frontier S = frontier T
      using fim by blast
    show continuous_on (frontier S) f
      by (metis Diff_subset assms(2) closure_eq contf continuous_on_subset fron-
tier_def)
    have frontier S ⊆ g ‘ frontier T
    proof
      fix x assume x ∈ frontier S
      then have f x ∈ frontier T
        using fim by blast
      then show x ∈ g ‘ frontier T
        by (metis gf ‹x ∈ frontier S› imageI)
    qed
    then show g ‘ frontier T = frontier S
      using gim by blast
    show continuous_on (frontier T) g
      by (metis Diff_subset assms(3) closure_closed contg continuous_on_subset
frontier_def)
  qed
qed

lemma homeomorphic_frontiers:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T closed S closed T
          interior S = {} ⟷ interior T = {}
    shows (frontier S) homeomorphic (frontier T)
proof (cases interior T = {})
  case True
  then show ?thesis
    by (metis Diff_empty assms closure_eq frontier_def)
next
  case False
  then have DIM('a) = DIM('b)
   using assms homeomorphic_interiors homeomorphic_open_imp_same_dimension
by blast
  then show ?thesis
    using assms homeomorphic_frontiers_same_dimension by blast
qed

lemma continuous_image_subset_rel_interior:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
```

```
  assumes contf: continuous_on S f and injf: inj_on f S and fim: f ∈ S → T
      and TS: aff_dim T ≤ aff_dim S
    shows f ∈ (rel_interior S) → rel_interior(f ‘ S)
unfolding image_subset_iff_funcset [symmetric]
proof (rule rel_interior_maximal)
  show f ‘ rel_interior S ⊆ f ‘ S
    by(simp add: image_mono rel_interior_subset)
  show openin (top_of_set (affine hull f ‘ S)) (f ‘ rel_interior S)
  proof (rule invariance_of_domain_affine_sets)
    show openin (top_of_set (affine hull S)) (rel_interior S)
      by (simp add: openin_rel_interior)
    show aff_dim (affine hull f ‘ S) ≤ aff_dim (affine hull S)
    by (metis TS aff_dim_affine_hull aff_dim_subset fim image_subset_iff_funcset
order_trans)
    show f ∈ rel_interior S → affine hull f ‘ S
      using ‹f ‘ rel_interior S ⊆ f ‘ S› hull_subset by fastforce
    show continuous_on (rel_interior S) f
      using contf continuous_on_subset rel_interior_subset by blast
    show inj_on f (rel_interior S)
      using inj_on_subset injf rel_interior_subset by blast
  qed auto
qed

lemma homeomorphic_rel_interiors_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T and aff: aff_dim S = aff_dim T
  shows (rel_interior S) homeomorphic (rel_interior T)
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
  assume S: ∀ x∈S. f x ∈ T ∧ g (f x) = x and T: ∀ y∈T. g y ∈ S ∧ f (g y) = y
     and contf: continuous_on S f and contg: continuous_on T g
  then have fST: f ‘ S = T and gTS: g ‘ T = S and inj_on f S inj_on g T
    by (auto simp: inj_on_def intro: rev_image_eqI) metis+
  have fim: f ∈ rel_interior S → rel_interior T
   by (smt (verit, best) PiE Pi_I S ‹inj_on f S› aff contf continuous_image_subset_rel_interior
fST)
  have gim: g ∈ rel_interior T → rel_interior S
     by (metis T ‹inj_on g T› aff contg continuous_image_subset_rel_interior
dual_order.refl funcsetI gTS)
  show homeomorphism (rel_interior S) (rel_interior T) f g
    unfolding homeomorphism_def
  proof (intro conjI ballI)
    show gf: ⋀x. x ∈ rel_interior S ⟹ g (f x) = x
      using S rel_interior_subset by blast
    show fg: ⋀y. y ∈ rel_interior T ⟹ f (g y) = y
      using T mem_rel_interior_ball by blast
    have rel_interior T ⊆ f ‘ rel_interior S
```

```
    proof
      fix x assume x ∈ rel_interior T
      then have g x ∈ rel_interior S
        using gim by blast
      then show x ∈ f ‘ rel_interior S
        by (metis fg ‹x ∈ rel_interior T› imageI)
    qed
    moreover have f ‘ rel_interior S ⊆ rel_interior T
      using fim by blast
    ultimately show f ‘ rel_interior S = rel_interior T
      by blast
    show continuous_on (rel_interior S) f
      using contf continuous_on_subset rel_interior_subset by blast
    have rel_interior S ⊆ g ‘ rel_interior T
    proof
      fix x assume x ∈ rel_interior S
      then have f x ∈ rel_interior T
        using fim by blast
      then show x ∈ g ‘ rel_interior T
        by (metis gf ‹x ∈ rel_interior S› imageI)
    qed
    then show g ‘ rel_interior T = rel_interior S
      using gim by blast
    show continuous_on (rel_interior T) g
      using contg continuous_on_subset rel_interior_subset by blast
  qed
qed


lemma homeomorphic_aff_dim_le:
  fixes S :: 'a::euclidean_space set
  assumes S homeomorphic T rel_interior S ≠ {}
    shows aff_dim (affine hull S) ≤ aff_dim (affine hull T)
proof −
  obtain f g
    where S: ∀ x∈S. f x ∈ T ∧ g (f x) = x and T: ∀ y∈T. g y ∈ S ∧ f (g y) = y
      and contf: continuous_on S f and contg: continuous_on T g
    using assms [unfolded homeomorphic_minimal] by auto
  show ?thesis
  proof (rule invariance_of_dimension_affine_sets)
    show continuous_on (rel_interior S) f
      using contf continuous_on_subset rel_interior_subset by blast
    show f ∈ rel_interior S → affine hull T
      by (simp add: S hull_inc mem_rel_interior_ball)
    show inj_on f (rel_interior S)
      by (metis S inj_on_inverseI inj_on_subset rel_interior_subset)
  qed (simp_all add: openin_rel_interior assms)
qed
```

lemma *homeomorphic_rel_interiors*:
fixes S :: *'a*::*euclidean_space set* **and** T :: *'b*::*euclidean_space set*
assumes S *homeomorphic* T *rel_interior* $S = \{\} \longleftrightarrow$ *rel_interior* $T = \{\}$
shows (*rel_interior* S) *homeomorphic* (*rel_interior* T)
proof (*cases rel_interior* $T = \{\}$)
case *True*
with *assms* **show** *?thesis* **by** *auto*
next
case *False*
have *aff_dim* (*affine hull* S) $\leq$ *aff_dim* (*affine hull* T)
using *False assms homeomorphic_aff_dim_le* **by** *blast*
moreover have *aff_dim* (*affine hull* T) $\leq$ *aff_dim* (*affine hull* S)
using *False assms*(*1*) *homeomorphic_aff_dim_le homeomorphic_sym* **by** *auto*
ultimately have *aff_dim* S = *aff_dim* T **by** *force*
then show *?thesis*
by (*rule homeomorphic_rel_interiors_same_dimension* [*OF* ‹S *homeomorphic* T›])
qed

lemma *homeomorphic_rel_boundaries_same_dimension*:
fixes S :: *'a*::*euclidean_space set* **and** T :: *'b*::*euclidean_space set*
assumes S *homeomorphic* T **and** *aff*: *aff_dim* S = *aff_dim* T
shows (S − *rel_interior* S) *homeomorphic* (T − *rel_interior* T)
using *assms* [*unfolded homeomorphic_minimal*]
unfolding *homeomorphic_def*
proof (*clarify elim*!: *ex_forward*)
fix f g
assume S: $\forall x{\in}S.\ f\ x \in T \wedge g\ (f\ x) = x$ **and** T: $\forall y{\in}T.\ g\ y \in S \wedge f\ (g\ y) = y$
and *contf*: *continuous_on* S f **and** *contg*: *continuous_on* T g
then have *fST*: f ‘ $S = T$ **and** *gTS*: g ‘ $T = S$ **and** *inj_on* f S *inj_on* g T
by (*auto simp*: *inj_on_def intro*: *rev_image_eqI*) *metis*+
have *fim*: $f \in$ *rel_interior* $S \to$ *rel_interior* T
by (*metis* ‹*inj_on* f S› *aff contf continuous_image_subset_rel_interior dual_order.refl fST image_subset_iff_funcset*)
have *gim*: $g \in$ *rel_interior* $T \to$ *rel_interior* S
by (*metis* ‹*inj_on* g T› *aff contg continuous_image_subset_rel_interior dual_order.refl gTS image_subset_iff_funcset*)
show *homeomorphism* (S − *rel_interior* S) (T − *rel_interior* T) f g
unfolding *homeomorphism_def*
proof (*intro conjI ballI*)
show *gf*: $\bigwedge x.\ x \in S$ − *rel_interior* $S \Longrightarrow g\ (f\ x) = x$
using S *rel_interior_subset* **by** *blast*
show *fg*: $\bigwedge y.\ y \in T$ − *rel_interior* $T \Longrightarrow f\ (g\ y) = y$
using T *mem_rel_interior_ball* **by** *blast*
show f ‘ (S − *rel_interior* S) = T − *rel_interior* T
using S *fST fim gim image_subset_iff_funcset* **by** *fastforce*
show *continuous_on* (S − *rel_interior* S) f
using *contf continuous_on_subset rel_interior_subset* **by** *blast*

```
    show g ‘ (T − rel_interior T) = S − rel_interior S
      using T gTS gim fim image_subset_iff_funcset by fastforce
    show continuous_on (T − rel_interior T) g
      using contg continuous_on_subset rel_interior_subset by blast
  qed
qed

lemma homeomorphic_rel_boundaries:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T rel_interior S = {} ⟷ rel_interior T = {}
    shows (S − rel_interior S) homeomorphic (T − rel_interior T)
proof (cases rel_interior T = {})
  case True
  with assms show ?thesis by auto
next
  case False
  obtain f g
    where S: ∀ x∈S. f x ∈ T ∧ g (f x) = x and T: ∀ y∈T. g y ∈ S ∧ f (g y) = y
      and contf: continuous_on S f and contg: continuous_on T g
    using assms by (auto simp: homeomorphic_minimal)
  have aff_dim (affine hull S) ≤ aff_dim (affine hull T)
    using False assms homeomorphic_aff_dim_le by blast
  moreover have aff_dim (affine hull T) ≤ aff_dim (affine hull S)
    by (meson False assms(1) homeomorphic_aff_dim_le homeomorphic_sym)
  ultimately have aff_dim S = aff_dim T by force
  then show ?thesis
    by (rule homeomorphic_rel_boundaries_same_dimension [OF ‹S homeomor-
phic T›])
qed

proposition uniformly_continuous_homeomorphism_UNIV_trivial:
  fixes f :: 'a::euclidean_space ⇒ 'a
   assumes contf: uniformly_continuous_on S f and hom: homeomorphism S
UNIV f g
  shows S = UNIV
proof (cases S = {})
  case True
  then show ?thesis
    by (metis UNIV_I hom empty_iff homeomorphism_def image_eqI)
next
  case False
  have inj g
    by (metis UNIV_I hom homeomorphism_apply2 injI)
  then have open (g ‘ UNIV)
    by (blast intro: invariance_of_domain hom homeomorphism_cont2)
  then have open S
    using hom homeomorphism_image2 by blast
  moreover have complete S
    unfolding complete_def
```

proof *clarify*
fix σ
assume σ: $\forall n.\ \sigma\ n \in S$ **and** *Cauchy* σ
have *Cauchy* $(f\ o\ \sigma)$
using *uniformly_continuous_imp_Cauchy_continuous* ‹*Cauchy* σ› σ *contf*
unfolding *Cauchy_continuous_on_def* **by** *blast*
then obtain l **where** $(f \circ \sigma) \longrightarrow l$
by (*auto simp*: *convergent_eq_Cauchy* [*symmetric*])
show $\exists l{\in}S.\ \sigma \longrightarrow l$
proof
show $g\ l \in S$
using *hom homeomorphism_image2* **by** *blast*
have $(g \circ (f \circ \sigma)) \longrightarrow g\ l$
by (*meson UNIV_I* ‹$(f \circ \sigma) \longrightarrow l$› *continuous_on_sequentially hom homeomorphism_cont2*)
then show $\sigma \longrightarrow g\ l$
proof −
have $\forall n.\ \sigma\ n = (g \circ (f \circ \sigma))\ n$
by (*metis* (*no_types*) σ *comp_eq_dest_lhs hom homeomorphism_apply1*)
then show *?thesis*
by (*metis* (*no_types*) *LIMSEQ_iff* ‹$(g \circ (f \circ \sigma)) \longrightarrow g\ l$›)
qed
qed
qed
then have *closed S*
by (*simp add*: *complete_eq_closed*)
ultimately show *?thesis*
using *clopen* [*of S*] *False* **by** *simp*
qed

9.27.6 Formulation of loop homotopy in terms of maps out of type complex

lemma *homotopic_circlemaps_imp_homotopic_loops*:
assumes *homotopic_with_canon* $(\lambda h.\ True)$ (*sphere 0 1*) *S f g*
shows *homotopic_loops S* $(f \circ exp \circ (\lambda t.\ 2 * of\_real\ pi * of\_real\ t * \mathrm{i}))$
$(g \circ exp \circ (\lambda t.\ 2 * of\_real\ pi * of\_real\ t * \mathrm{i}))$
proof −
have *homotopic_with_canon* $(\lambda f.\ True)$ $\{z.\ cmod\ z = 1\}$ *S f g*
using *assms* **by** (*auto simp*: *sphere_def*)
moreover have *continuous_on* $\{0..1\}$ $(exp \circ (\lambda t.\ 2 * of\_real\ pi * of\_real\ t * \mathrm{i}))$
by (*intro continuous_intros*)
moreover have $(exp \circ (\lambda t.\ 2 * of\_real\ pi * of\_real\ t * \mathrm{i}))\ `\ \{0..1\} \subseteq \{z.\ cmod\ z = 1\}$
by (*auto simp*: *norm_mult*)
ultimately
show *?thesis*
apply (*simp add*: *homotopic_loops_def comp_assoc*)

```
    apply (rule homotopic_with_compose_continuous_right)
     apply (auto simp: pathstart_def pathfinish_def)
    done
qed

lemma homotopic_loops_imp_homotopic_circlemaps:
  assumes homotopic_loops S p q
    shows homotopic_with_canon (λh. True) (sphere 0 1) S
                        (p ∘ (λz. (Arg2pi z / (2 * pi))))
                        (q ∘ (λz. (Arg2pi z / (2 * pi))))
proof −
  obtain h where conth: continuous_on ({0..1::real} × {0..1}) h
            and him: h ∈ ({0..1} × {0..1}) → S
            and h0: (∀ x. h (0, x) = p x)
            and h1: (∀ x. h (1, x) = q x)
            and h01: (∀ t∈{0..1}. h (t, 1) = h (t, 0))
    using assms
     by (auto simp: homotopic_loops_def sphere_def homotopic_with_def path-
start_def pathfinish_def)
  define j where j ≡ λz. if 0 ≤ Im (snd z)
                        then h (fst z, Arg2pi (snd z) / (2 * pi))
                        else h (fst z, 1 − Arg2pi (cnj (snd z)) / (2 * pi))
  have Arg2pi_eq: 1 − Arg2pi (cnj y) / (2 * pi) = Arg2pi y / (2 * pi) ∨ Arg2pi
y = 0 ∧ Arg2pi (cnj y) = 0 if cmod y = 1 for y
   using that Arg2pi_eq_0_pi Arg2pi_eq_pi by (force simp: Arg2pi_cnj field_split_simps)
  show ?thesis
  proof (simp add: homotopic_with; intro conjI ballI exI)
    show continuous_on ({0..1} × sphere 0 1) (λw. h (fst w, Arg2pi (snd w) / (2
* pi)))
     proof (rule continuous_on_eq)
      show j: j x = h (fst x, Arg2pi (snd x) / (2 * pi)) if x ∈ {0..1} × sphere 0 1
for x
        using Arg2pi_eq that h01 by (force simp: j_def)
       have eq:  S = S ∩ (UNIV × {z. 0 ≤ Im z}) ∪ S ∩ (UNIV × {z. Im z ≤
0}) for S :: (real*complex)set
        by auto
       have §: Arg2pi z ≤ 2 * pi for z
        by (simp add: Arg2pi order_le_less)
       have c1: continuous_on ({0..1} × sphere 0 1 ∩ UNIV × {z. 0 ≤ Im z})
(λx. h (fst x, Arg2pi (snd x) / (2 * pi)))
        apply (intro continuous_intros continuous_on_compose2 [OF conth] con-
tinuous_on_compose2 [OF continuous_on_upperhalf_Arg2pi])
        by (auto simp: Arg2pi §)
       have c2: continuous_on ({0..1} × sphere 0 1 ∩ UNIV × {z. Im z ≤ 0})
(λx. h (fst x, 1 − Arg2pi (cnj (snd x)) / (2 * pi)))
        apply (intro continuous_intros continuous_on_compose2 [OF conth] con-
tinuous_on_compose2 [OF continuous_on_upperhalf_Arg2pi])
        by (auto simp: Arg2pi §)
      show continuous_on ({0..1} × sphere 0 1) j
```

```
      apply (simp add: j_def)
      apply (subst eq)
      apply (rule continuous_on_cases_local)
      using Arg2pi_eq h01
        by (force simp add: eq [symmetric] closedin_closed_Int closed_Times
closed_halfspace_Im_le closed_halfspace_Im_ge c1 c2)+
   qed
   have (λw. h (fst w, Arg2pi (snd w) / (2 * pi))) ‘ ({0..1} × sphere 0 1) ⊆ h ‘
({0..1} × {0..1})
     by (auto simp: Arg2pi_ge_0 Arg2pi_lt_2pi less_imp_le)
   also have ... ⊆ S
     using him by blast
   finally show (λw. h (fst w, Arg2pi (snd w) / (2 * pi))) ‘ ({0..1} × sphere 0
1) ⊆ S .
 qed (auto simp: h0 h1)
qed

lemma simply_connected_homotopic_loops:
 simply_connected S ⟷
     (∀ p q. homotopic_loops S p p ∧ homotopic_loops S q q ⟶ homotopic_loops
S p q)
unfolding simply_connected_def using homotopic_loops_refl by metis


lemma simply_connected_eq_homotopic_circlemaps1:
 fixes f :: complex ⇒ 'a::topological_space and g :: complex ⇒ 'a
 assumes S: simply_connected S
     and contf: continuous_on (sphere 0 1) f and fim: f ∈ (sphere 0 1) → S
     and contg: continuous_on (sphere 0 1) g and gim: g ∈ (sphere 0 1) → S
   shows homotopic_with_canon (λh. True) (sphere 0 1) S f g
proof −
 let ?h = (λt. complex_of_real (2 * pi * t) * i)
 have homotopic_loops S (f ∘ exp ∘ ?h) (f ∘ exp ∘ ?h) ∧ homotopic_loops S (g
∘ exp ∘ ?h) (g ∘ exp ∘ ?h)
    by (simp add: homotopic_circlemaps_imp_homotopic_loops contf fim contg
gim image_subset_iff_funcset)
 then have homotopic_loops S (f ∘ exp ∘ ?h) (g ∘ exp ∘ ?h)
   using S simply_connected_homotopic_loops by blast
 then show ?thesis
  apply (rule homotopic_with_eq [OF homotopic_loops_imp_homotopic_circlemaps])
    apply (auto simp: o_def complex_norm_eq_1_exp mult.commute)
   done
qed

lemma simply_connected_eq_homotopic_circlemaps2a:
 fixes h :: complex ⇒ 'a::topological_space
 assumes conth: continuous_on (sphere 0 1) h and him: h ∈ sphere 0 1 → S
     and hom: ⋀f g::complex ⇒ 'a.
              ⟦continuous_on (sphere 0 1) f; f ∈ (sphere 0 1) → S;
```

continuous_on (sphere 0 1) g; g ∈ (sphere 0 1) → S⟧
⟹ homotopic_with_canon (λh. True) (sphere 0 1) S f g
shows ∃ *a. homotopic_with_canon (λh. True) (sphere 0 1) S h (λx. a)*
by (*metis conth continuous_on_const him hom image_subset_iff image_subset_iff_funcset*)

lemma *simply_connected_eq_homotopic_circlemaps2b*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes ⋀*f g*::*complex* ⇒ *'a.*
⟦continuous_on (sphere 0 1) f; f ∈ (sphere 0 1) → S;
continuous_on (sphere 0 1) g; g ∈ (sphere 0 1) → S⟧
⟹ homotopic_with_canon (λh. True) (sphere 0 1) S f g
shows *path_connected S*
proof (*clarsimp simp add*: *path_connected_eq_homotopic_points*)
fix *a b*
assume *a ∈ S b ∈ S*
then show *homotopic_loops S* (*linepath a a*) (*linepath b b*)
using *homotopic_circlemaps_imp_homotopic_loops* [*OF assms* [*of λx. a λx. b*]]
by (*auto simp*: *o_def linepath_def*)
qed

lemma *simply_connected_eq_homotopic_circlemaps3*:
fixes *h* :: *complex* ⇒ *'a*::*real_normed_vector*
assumes *path_connected S*
and *hom*: ⋀*f*::*complex* ⇒ *'a.*
⟦continuous_on (sphere 0 1) f; f ‘(sphere 0 1) ⊆ S⟧
⟹ ∃ a. homotopic_with_canon (λh. True) (sphere 0 1) S f (λx. a)
shows *simply_connected S*
proof (*clarsimp simp add*: *simply_connected_eq_contractible_loop_some assms*)
fix *p*
assume *p*: *path p path_image p ⊆ S pathfinish p = pathstart p*
then have *homotopic_loops S p p*
by (*simp add*: *homotopic_loops_refl*)
then obtain *a* **where** *homp*: *homotopic_with_canon* (*λh. True*) (*sphere 0 1*) *S* (*p ∘ (λz. Arg2pi z / (2 * pi))*) (*λx. a*)
by (*metis homotopic_with_imp_subset2 homotopic_loops_imp_homotopic_circlemaps homotopic_with_imp_continuous hom*)
show ∃ *a. a ∈ S ∧ homotopic_loops S p* (*linepath a a*)
proof (*intro exI conjI*)
show *a ∈ S*
using *homotopic_with_imp_subset2* [*OF homp*]
by (*metis dist_0_norm image_subset_iff mem_sphere norm_one*)
have *teq*: ⋀*t*. *⟦0 ≤ t; t ≤ 1⟧*
⟹ t = Arg2pi (*exp* (*2 * of_real pi * of_real t * i*)) / (*2 * pi*) ∨ *t=1 ∧ Arg2pi* (*exp* (*2 * of_real pi * of_real t * i*)) = *0*
using *Arg2pi_of_real* [*of 1*] **by** (*force simp*: *Arg2pi_exp*)
have *homotopic_loops S p* (*p ∘ (λz. Arg2pi z / (2 * pi*)) ∘ *exp* ∘ (*λt. 2 * complex_of_real pi * complex_of_real t * i*))
using *p teq* **by** (*fastforce simp*: *pathfinish_def pathstart_def intro*: *homo-*

topic_loops_eq [*OF p*])
then show *homotopic_loops S p* (*linepath a a*)
by (*simp add*: *linepath_refl homotopic_loops_trans* [*OF _ homotopic_circlemaps_imp_homotopic_loops* [*OF homp*, *simplified K_record_comp*]])
qed
qed

proposition *simply_connected_eq_homotopic_circlemaps*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *simply_connected S* ⟷
(∀ *f g*::*complex* ⇒ *'a*.
continuous_on (*sphere 0 1*) *f* ∧ *f* ∈ (*sphere 0 1*) → *S* ∧
continuous_on (*sphere 0 1*) *g* ∧ *g* ∈ (*sphere 0 1*) → *S*
⟶ *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f g*)
by (*metis image_subset_iff_funcset simply_connected_eq_homotopic_circlemaps1 simply_connected_eq_homotopic_circlemaps2a*
simply_connected_eq_homotopic_circlemaps2b simply_connected_eq_homotopic_circlemaps3)

proposition *simply_connected_eq_contractible_circlemap*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *simply_connected S* ⟷
path_connected S ∧
(∀ *f*::*complex* ⇒ *'a*.
continuous_on (*sphere 0 1*) *f* ∧ *f* ‘(*sphere 0 1*) ⊆ *S*
⟶ (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) *S f* (λ*x*. *a*)))
by (*metis image_subset_iff_funcset simply_connected_eq_homotopic_circlemaps simply_connected_eq_homotopic_circlemaps2a*
simply_connected_eq_homotopic_circlemaps3 simply_connected_imp_path_connected)

corollary *homotopy_eqv_simple_connectedness*:
fixes *S* :: *'a*::*real_normed_vector set* **and** *T* :: *'b*::*real_normed_vector set*
shows *S homotopy_eqv T* ⟹ *simply_connected S* ⟷ *simply_connected T*
by (*simp add*: *simply_connected_eq_homotopic_circlemaps homotopy_eqv_homotopic_triviality image_subset_iff_funcset*)

9.27.7 Homeomorphism of simple closed curves to circles

proposition *homeomorphic_simple_path_image_circle*:
fixes *a* :: *complex* **and** *γ* :: *real* ⇒ *'a*::*t2_space*
assumes *simple_path γ* **and** *loop*: *pathfinish γ* = *pathstart γ* **and** *0* < *r*
shows (*path_image γ*) *homeomorphic sphere a r*
proof −
have *homotopic_loops* (*path_image γ*) *γ γ*
by (*simp add*: *assms homotopic_loops_refl simple_path_imp_path*)
then have *hom*: *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) (*path_image γ*)
(*γ* ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*))) (*γ* ∘ (λ*z*. *Arg2pi z* / (*2*∗*pi*)))
by (*rule homotopic_loops_imp_homotopic_circlemaps*)

```
  have ∃g. homeomorphism (sphere 0 1) (path_image γ) (γ ∘ (λz. Arg2pi z /
(2*pi))) g
  proof (rule homeomorphism_compact)
    show continuous_on (sphere 0 1) (γ ∘ (λz. Arg2pi z / (2*pi)))
      using hom homotopic_with_imp_continuous by blast
    show inj_on (γ ∘ (λz. Arg2pi z / (2*pi))) (sphere 0 1)
    proof
      fix x y
      assume xy: x ∈ sphere 0 1 y ∈ sphere 0 1
         and eq: (γ ∘ (λz. Arg2pi z / (2*pi))) x = (γ ∘ (λz. Arg2pi z / (2*pi))) y
      then have (Arg2pi x / (2*pi)) = (Arg2pi y / (2*pi))
      proof −
        have (Arg2pi x / (2*pi)) ∈ {0..1} (Arg2pi y / (2*pi)) ∈ {0..1}
          using Arg2pi_ge_0 Arg2pi_lt_2pi dual_order.strict_iff_order by fast-
force+
        with eq show ?thesis
          using ‹simple_path γ› Arg2pi_lt_2pi
          unfolding simple_path_def loop_free_def o_def
          by (metis eq_divide_eq_1 not_less_iff_gr_or_eq)
      qed
      with xy show x = y
        by (metis is_Arg_def Arg2pi Arg2pi_0 dist_0_norm divide_cancel_right
dual_order.strict_iff_order mem_sphere)
    qed
    have ⋀z. cmod z = 1 ⟹ ∃x∈{0..1}. γ (Arg2pi z / (2*pi)) = γ x
      by (metis Arg2pi_ge_0 Arg2pi_lt_2pi atLeastAtMost_iff divide_less_eq_1
less_eq_real_def zero_less_mult_iff pi_gt_zero zero_le_divide_iff zero_less_numeral)
    moreover have ∃z∈sphere 0 1. γ x = γ (Arg2pi z / (2*pi)) if 0 ≤ x x ≤ 1
for x
    proof (cases x=1)
      case True
      with Arg2pi_of_real [of 1] loop show ?thesis
        by (rule_tac x=1 in bexI) (auto simp: pathfinish_def pathstart_def ‹0 ≤
x›)
    next
      case False
      then have ∗: (Arg2pi (exp (i∗(2∗ of_real pi∗ of_real x))) / (2*pi)) = x
        using that by (auto simp: Arg2pi_exp field_split_simps)
      show ?thesis
        by (rule_tac x=exp(i ∗ of_real(2*pi*x)) in bexI) (auto simp: ∗)
    qed
    ultimately show (γ ∘ (λz. Arg2pi z / (2*pi))) ‘ sphere 0 1 = path_image γ
      by (auto simp: path_image_def image_iff)
    qed auto
    then have path_image γ homeomorphic sphere (0::complex) 1
      using homeomorphic_def homeomorphic_sym by blast
  also have ... homeomorphic sphere a r
    by (simp add: assms homeomorphic_spheres)
  finally show ?thesis .
```

qed

lemma *homeomorphic_simple_path_images*:
fixes $\gamma 1$ :: *real* $\Rightarrow$ *'a*::*t2_space* **and** $\gamma 2$ :: *real* $\Rightarrow$ *'b*::*t2_space*
assumes *simple_path* $\gamma 1$ **and** *loop*: *pathfinish* $\gamma 1$ = *pathstart* $\gamma 1$
assumes *simple_path* $\gamma 2$ **and** *loop*: *pathfinish* $\gamma 2$ = *pathstart* $\gamma 2$
shows (*path_image* $\gamma 1$) *homeomorphic* (*path_image* $\gamma 2$)
by (*meson assms homeomorphic_simple_path_image_circle homeomorphic_sym homeomorphic_trans loop pi_gt_zero*)

9.27.8 Dimension-based conditions for various homeomorphisms

lemma *homeomorphic_subspaces_eq*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *subspace S subspace T*
shows *S homeomorphic T* $\longleftrightarrow$ *dim S* = *dim T*
proof
assume *S homeomorphic T*
then obtain *f g* **where** *hom*: *homeomorphism S T f g*
using *homeomorphic_def* **by** *blast*
show *dim S* = *dim T*
by (*metis* ‹*S homeomorphic T*› *aff_dim_subspace assms homeomorphic_convex_sets of_nat_eq_iff subspace_imp_convex*)
next
assume *dim S* = *dim T*
then show *S homeomorphic T*
by (*simp add*: *assms homeomorphic_subspaces*)
qed

lemma *homeomorphic_affine_sets_eq*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *affine S affine T*
shows *S homeomorphic T* $\longleftrightarrow$ *aff_dim S* = *aff_dim T*
proof (*cases S* = {} $\vee$ *T* = {})
case *True*
then show *?thesis*
using *assms homeomorphic_affine_sets* **by** *force*
next
case *False*
then obtain *a b* **where** *a* $\in$ *S b* $\in$ *T*
by *blast*
then have *subspace* ((+) (− *a*) ‘ *S*) *subspace* ((+) (− *b*) ‘ *T*)
using *affine_diffs_subspace assms* **by** *blast*+
then show *?thesis*
by (*metis affine_imp_convex assms homeomorphic_affine_sets homeomorphic_convex_sets*)
qed

lemma *homeomorphic_hyperplanes_eq*:
fixes $a :: 'a::euclidean\_space$ **and** $c :: 'b::euclidean\_space$
assumes $a \neq 0$ $c \neq 0$
shows $(\{x.\ a \cdot x = b\}\ homeomorphic\ \{x.\ c \cdot x = d\} \longleftrightarrow DIM('a) = DIM('b))$
proof −
have $DIM('a) - Suc\ 0 = DIM('b) - Suc\ 0 \Longrightarrow DIM('a) = DIM('b)$
by (*metis DIM_positive Suc_pred*)
then show *?thesis*
by (*auto simp*: *homeomorphic_affine_sets_eq affine_hyperplane assms*)
qed

lemma *homeomorphic_UNIV_UNIV*:
shows $(UNIV::'a\ set)\ homeomorphic\ (UNIV::'b\ set) \longleftrightarrow$
$DIM('a::euclidean\_space) = DIM('b::euclidean\_space)$
by (*simp add*: *homeomorphic_subspaces_eq*)

lemma *simply_connected_sphere_gen*:
assumes *convex S bounded S* **and** *3*: $3 \leq aff\_dim\ S$
shows *simply_connected*(*rel_frontier S*)
proof −
have *pa*: *path_connected* (*rel_frontier S*)
using *assms* **by** (*simp add*: *path_connected_sphere_gen*)
show *?thesis*
proof (*clarsimp simp add*: *simply_connected_eq_contractible_circlemap pa*)
fix *f*
assume *f*: *continuous_on* (*sphere* (*0*::*complex*) *1*) *f f* ‘ *sphere 0 1* ⊆ *rel_frontier S*
have *eq*: *sphere* (*0*::*complex*) *1* = *rel_frontier*(*cball 0 1*)
by *simp*
have *convex* (*cball* (*0*::*complex*) *1*)
by (*rule convex_cball*)
then obtain *c* **where** *homotopic_with_canon* (λ*z*. *True*) (*sphere* (*0*::*complex*) *1*) (*rel_frontier S*) *f* (λ*x*. *c*)
apply (*rule inessential_spheremap_lowdim_gen* [*OF* _ *bounded_cball* ‹*convex S*› ‹*bounded S*›, **where** *f*=*f*])
using *f 3* **by** (*auto simp*: *aff_dim_cball*)
then show ∃ *a*. *homotopic_with_canon* (λ*h*. *True*) (*sphere 0 1*) (*rel_frontier S*) *f* (λ*x*. *a*)
by *blast*
qed
qed

9.27.9 more invariance of domain

proposition *invariance_of_domain_sphere_affine_set_gen*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *contf*: *continuous_on S f* **and** *injf*: *inj_on f S* **and** *fim*: $f \in S \to T$
and *U*: *bounded U convex U*
and *affine T* **and** *affTU*: $aff\_dim\ T < aff\_dim\ U$

```
      and ope: openin (top_of_set (rel_frontier U)) S
  shows openin (top_of_set T) (f ` S)
proof (cases rel_frontier U = {})
  case True
  then show ?thesis
    using ope openin_subset by force
next
  case False
  obtain b c where b: b ∈ rel_frontier U and c: c ∈ rel_frontier U and b ≠ c
    using ‹bounded U› rel_frontier_not_sing [of U] subset_singletonD False  by
fastforce
  obtain V :: 'a set where affine V and affV: aff_dim V = aff_dim U − 1
  proof (rule choose_affine_subset [OF affine_UNIV])
    show − 1 ≤ aff_dim U − 1
      by (metis aff_dim_empty aff_dim_geq aff_dim_negative_iff affTU diff_0
diff_right_mono not_le)
    show aff_dim U − 1 ≤ aff_dim (UNIV::'a set)
      by (metis aff_dim_UNIV aff_dim_le_DIM le_cases not_le zle_diff1_eq)
  qed auto
  have SU: S ⊆ rel_frontier U
    using ope openin_imp_subset by auto
  have homb: rel_frontier U − {b} homeomorphic V
   and homc: rel_frontier U − {c} homeomorphic V
    using homeomorphic_punctured_sphere_affine_gen [of U _ V]
    by (simp_all add: ‹affine V› affV U b c)
  then obtain g h j k
          where gh: homeomorphism (rel_frontier U − {b}) V g h
            and jk: homeomorphism (rel_frontier U − {c}) V j k
    by (auto simp: homeomorphic_def)
  with SU have hgsub: (h ` g ` (S − {b})) ⊆ S and kjsub: (k ` j ` (S − {c})) ⊆ S
    by (simp_all add: homeomorphism_def subset_eq)
  have [simp]: aff_dim T ≤ aff_dim V
    by (simp add: affTU affV)
  have openin (top_of_set T) ((f ∘ h) ` g ` (S − {b}))
  proof (rule invariance_of_domain_affine_sets [OF _ ‹affine V›])
    have openin (top_of_set (rel_frontier U − {b})) (S − {b})
     by (meson Diff_mono Diff_subset SU ope openin_delete openin_subset_trans
order_refl)
    then show openin (top_of_set V) (g ` (S − {b}))
      by (rule homeomorphism_imp_open_map [OF gh])
    show continuous_on (g ` (S − {b})) (f ∘ h)
    proof (rule continuous_on_compose)
      show continuous_on (g ` (S − {b})) h
       by (meson Diff_mono SU homeomorphism_def homeomorphism_of_subsets
gh set_eq_subset)
    qed (use contf continuous_on_subset hgsub in blast)
    show inj_on (f ∘ h) (g ` (S − {b}))
     by (smt (verit, del_insts) SU homeomorphism_def inj_on_def injf gh Diff_iff
comp_apply imageE subset_iff)
```

```
    show f ∘ h ∈ g ‘ (S − {b}) → T
      using fim hgsub by fastforce
  qed (auto simp: assms)
  moreover
  have openin (top_of_set T) ((f ∘ k) ‘ j ‘ (S − {c}))
  proof (rule invariance_of_domain_affine_sets [OF _ ‹affine V›])
    show openin (top_of_set V) (j ‘ (S − {c}))
     by (meson Diff_mono Diff_subset SU ope openin_delete openin_subset_trans
order_refl homeomorphism_imp_open_map [OF jk])
    show continuous_on (j ‘ (S − {c})) (f ∘ k)
    proof (rule continuous_on_compose)
      show continuous_on (j ‘ (S − {c})) k
       by (meson Diff_mono SU homeomorphism_def homeomorphism_of_subsets
jk set_eq_subset)
    qed (use contf continuous_on_subset kjsub in blast)
    show inj_on (f ∘ k) (j ‘ (S − {c}))
     by (smt (verit, del_insts) SU homeomorphism_def inj_on_def injf jk Diff_iff
comp_apply imageE subset_iff)
    show f ∘ k ∈ j ‘ (S − {c}) → T
      using fim kjsub by fastforce
  qed (auto simp: assms)
  ultimately have openin (top_of_set T) ((f ∘ h) ‘ g ‘ (S − {b}) ∪ ((f ∘ k) ‘ j
‘ (S − {c})))
    by (rule openin_Un)
  moreover have (f ∘ h) ‘ g ‘ (S − {b}) = f ‘ (S − {b})
  proof −
    have h ‘ g ‘ (S − {b}) = (S − {b})
      by (meson Diff_mono Diff_subset SU gh homeomorphism_def homeomor-
phism_of_subsets subset_singleton_iff)
    then show ?thesis
      by (metis image_comp)
  qed
  moreover have (f ∘ k) ‘ j ‘ (S − {c}) = f ‘ (S − {c})
  proof −
    have k ‘ j ‘ (S − {c}) = (S − {c})
      using homeomorphism_apply1 [OF jk] SU
      by (meson Diff_mono homeomorphism_def homeomorphism_of_subsets jk
subset_refl)
    then show ?thesis
      by (metis image_comp)
  qed
  moreover have f ‘ (S − {b}) ∪ f ‘ (S − {c}) = f ‘ (S)
    using ‹b ≠ c› by blast
  ultimately show ?thesis
    by simp
qed


lemma invariance_of_domain_sphere_affine_set:
```

```
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and injf: inj_on f S and fim: f ∈ S → T
      and r ≠ 0 affine T and affTU: aff_dim T < DIM('a)
      and ope: openin (top_of_set (sphere a r)) S
   shows openin (top_of_set T) (f ‘ S)
proof (cases sphere a r = {})
  case True
  then show ?thesis
    using ope openin_subset by force
next
  case False
  show ?thesis
   proof (rule invariance_of_domain_sphere_affine_set_gen [OF contf injf fim
bounded_cball convex_cball ‹affine T›])
    show aff_dim T < aff_dim (cball a r)
     by (metis False affTU aff_dim_cball assms(4) linorder_cases sphere_empty)
    show openin (top_of_set (rel_frontier (cball a r))) S
     by (simp add: ‹r ≠ 0› ope)
  qed
qed

lemma no_embedding_sphere_lowdim:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on (sphere a r) f and injf: inj_on f (sphere a r)
and r > 0
   shows DIM('a) ≤ DIM('b)
proof −
  have False if DIM('a) > DIM('b)
  proof −
    have compact (f ‘ sphere a r)
      using compact_continuous_image
      by (simp add: compact_continuous_image contf)
    then have ¬ open (f ‘ sphere a r)
      using compact_open
     by (metis assms(3) image_is_empty not_less_iff_gr_or_eq sphere_eq_empty)
    then have r=0
       by (metis Pi_I UNIV_I aff_dim_UNIV affine_UNIV contf injf invari-
ance_of_domain_sphere_affine_set
                 of_nat_less_iff open_openin openin_subtopology_self subtopol-
ogy_UNIV that)
    with ‹r>0› show False by auto
  qed
  then show ?thesis
    using not_less by blast
qed

lemma simply_connected_sphere:
  fixes a :: 'a::euclidean_space
  assumes 3 ≤ DIM('a)
```

```
    shows simply_connected(sphere a r)
proof (cases rule: linorder_cases [of r 0])
  case less
  then show ?thesis by simp
next
  case equal
  then show ?thesis  by (auto simp: convex_imp_simply_connected)
next
  case greater
  then show ?thesis
    using simply_connected_sphere_gen [of cball a r] assms
    by (simp add: aff_dim_cball)
qed

lemma simply_connected_sphere_eq:
  fixes a :: 'a::euclidean_space
  shows simply_connected(sphere a r) ⟷ 3 ≤ DIM('a) ∨ r ≤ 0  (is ?lhs =
?rhs)
proof (cases r ≤ 0)
  case True
  have simply_connected (sphere a r)
    using True less_eq_real_def by (auto intro: convex_imp_simply_connected)
  with True show ?thesis by auto
next
  case False
  show ?thesis
  proof
    assume L: ?lhs
    have False if DIM('a) = 1 ∨ DIM('a) = 2
      using that
    proof
      assume DIM('a) = 1
      with L show False
        using connected_sphere_eq simply_connected_imp_connected
        by (metis False Suc_1 not_less_eq_eq order_refl)
    next
      assume DIM('a) = 2
      then have sphere a r homeomorphic sphere (0::complex) 1
      by (metis DIM_complex False homeomorphic_spheres_gen not_less zero_less_one)
      then have simply_connected(sphere (0::complex) 1)
        using L homeomorphic_simply_connected_eq by blast
      then obtain a::complex where homotopic_with_canon (λh. True) (sphere 0
1) (sphere 0 1) id (λx. a)
      by (metis continuous_on_id' id_apply image_id subset_refl simply_connected_eq_contractible_cir
      then show False
        using contractible_sphere contractible_def not_one_le_zero by blast
    qed
    with False show ?rhs
    by (metis DIM_ge_Suc0 One_nat_def Suc_1 not_less_eq_eq numeral_3_eq_3
```

order_antisym_conv)
next
assume *?rhs*
with *False* **show** *?lhs* **by** (*simp add*: *simply_connected_sphere*)
qed
qed

lemma *simply_connected_punctured_universe_eq*:
fixes *a* :: *'a*::*euclidean_space*
shows *simply_connected*(− {*a*}) ⟷ *3* ≤ *DIM*(*'a*)
proof −
have [*simp*]: *a* ∈ *rel_interior* (*cball a 1*)
by (*simp add*: *rel_interior_nonempty_interior*)
have [*simp*]: *affine hull cball a 1* − {*a*} = −{*a*}
by (*metis Compl_eq_Diff_UNIV aff_dim_cball aff_dim_lt_full not_less_iff_gr_or_eq zero_less_one*)
have *sphere a 1 homotopy_eqv* − {*a*}
using *homotopy_eqv_rel_frontier_punctured_affine_hull* [*of cball a 1 a*] **by** *auto*
then have *simply_connected*(− {*a*}) ⟷ *simply_connected*(*sphere a 1*)
using *homotopy_eqv_simple_connectedness* **by** *blast*
also have ... ⟷ *3* ≤ *DIM*(*'a*)
by (*simp add*: *simply_connected_sphere_eq*)
finally show *?thesis* **.**
qed

lemma *not_simply_connected_circle*:
fixes *a* :: *complex*
shows *0* < *r* ⟹ ¬ *simply_connected*(*sphere a r*)
by (*simp add*: *simply_connected_sphere_eq*)

proposition *simply_connected_punctured_convex*:
fixes *a* :: *'a*::*euclidean_space*
assumes *convex S* **and** *3*: *3* ≤ *aff_dim S*
shows *simply_connected*(*S* − {*a*})
proof (*cases a* ∈ *rel_interior S*)
case *True*
then obtain *e* **where** *a* ∈ *S 0* < *e* **and** *e*: *cball a e* ∩ *affine hull S* ⊆ *S*
by (*auto simp*: *rel_interior_cball*)
have *con*: *convex* (*cball a e* ∩ *affine hull S*)
by (*simp add*: *convex_Int*)
have *bo*: *bounded* (*cball a e* ∩ *affine hull S*)
by (*simp add*: *bounded_Int*)
have *affine hull S* ∩ *interior* (*cball a e*) ≠ {}
using ‹*0* < *e*› ‹*a* ∈ *S*› *hull_subset* **by** *fastforce*
then have *3* ≤ *aff_dim* (*affine hull S* ∩ *cball a e*)
by (*simp add*: *3 aff_dim_convex_Int_nonempty_interior* [*OF convex_affine_hull*])

```
  also have ... = aff_dim (cball a e ∩ affine hull S)
    by (simp add: Int_commute)
  finally have 3 ≤ aff_dim (cball a e ∩ affine hull S) .
  moreover have rel_frontier (cball a e ∩ affine hull S) homotopy_eqv S − {a}
  proof (rule homotopy_eqv_rel_frontier_punctured_convex)
    show a ∈ rel_interior (cball a e ∩ affine hull S)
       by (meson IntI Int_mono ‹a ∈ S› ‹0 < e› e ‹cball a e ∩ affine hull S
⊆ S› ball_subset_cball centre_in_cball dual_order.strict_implies_order hull_inc
hull_mono mem_rel_interior_ball)
    have closed (cball a e ∩ affine hull S)
      by blast
    then show rel_frontier (cball a e ∩ affine hull S) ⊆ S
      by (metis Diff_subset closure_closed dual_order.trans e rel_frontier_def)
    show S ⊆ affine hull (cball a e ∩ affine hull S)
    by (metis (no_types, lifting) IntI ‹a ∈ S› ‹0 < e› affine_hull_convex_Int_nonempty_interior
centre_in_ball convex_affine_hull empty_iff hull_subset inf_commute interior_cball
subsetCE subsetI)
    qed (auto simp: assms con bo)
  ultimately show ?thesis
   using homotopy_eqv_simple_connectedness simply_connected_sphere_gen [OF
con bo]
    by blast
next
  case False
  then have rel_interior S ⊆ S − {a}
    by (simp add: False rel_interior_subset subset_Diff_insert)
  moreover have S − {a} ⊆ closure S
    by (meson Diff_subset closure_subset subset_trans)
  ultimately show ?thesis
   by (metis contractible_imp_simply_connected contractible_convex_tweak_boundary_points
[OF ‹convex S›])
qed

corollary simply_connected_punctured_universe:
  fixes a :: 'a::euclidean_space
  assumes 3 ≤ DIM('a)
  shows simply_connected(− {a})
proof −
  have [simp]: affine hull cball a 1 = UNIV
    by (simp add: aff_dim_cball affine_hull_UNIV)
  have a ∈ rel_interior (cball a 1)
    by (simp add: rel_interior_interior)
  then
  have simply_connected (rel_frontier (cball a 1)) = simply_connected (affine hull
cball a 1 − {a})
   using homotopy_eqv_rel_frontier_punctured_affine_hull homotopy_eqv_simple_connectedness
by blast
  then show ?thesis
   using simply_connected_sphere [of a 1, OF assms] by (auto simp: Compl_eq_Diff_UNIV)
```

qed

9.27.10 The power, squaring and exponential functions as covering maps

proposition *covering_space_power_punctured_plane*:
assumes *0 < n*
shows *covering_space (− {0}) (λz::complex. z^n) (− {0})*
proof −
consider *n = 1 | 2 ≤ n* **using** *assms* **by** *linarith*
then obtain *e* **where** *0 < e*
and *e*: ⋀*w z*. *cmod*(*w − z*) *< e ∗ cmod z ⟹* (*w^n = z^n ⟷ w = z*)
proof *cases*
assume *n = 1* **then show** *?thesis*
by (*rule_tac e=1* **in** *that*) *auto*
next
assume *2 ≤ n*
have *eq_if_pow_eq*:
w = z **if** *lt*: *cmod* (*w − z*) *< 2 ∗ sin* (*pi / real n*) *∗ cmod z*
and *eq*: *w^n = z^n* **for** *w z*
proof (*cases z = 0*)
case *True* **with** *eq assms* **show** *?thesis* **by** (*auto simp*: *power_0_left*)
next
case *False*
then have *z ≠ 0* **by** *auto*
have (*w/z*)*^n = 1*
by (*metis False divide_self_if eq power_divide power_one*)
then obtain *j*::*nat* **where** *j*: *w / z = exp* (*2 ∗ of_real pi ∗* i *∗ j / n*) **and** *j < n*
using *Suc_leI assms* ‹*2 ≤ n*› *complex_roots_unity* [*THEN eqset_imp_iff*, *of n w/z*]
by *force*
have *cmod* (*w/z − 1*) *< 2 ∗ sin* (*pi / real n*)
using *lt assms* ‹*z ≠ 0*› **by** (*simp add*: *field_split_simps norm_divide*)
then have *cmod* (*exp* (i *∗ of_real* (*2 ∗ pi ∗ j / n*)) *− 1*) *< 2 ∗ sin* (*pi / real n*)
by (*simp add*: *j field_simps*)
then have *2 ∗* |*sin*((*2 ∗ pi ∗ j / n*) */ 2*)| *< 2 ∗ sin* (*pi / real n*)
by (*simp only*: *dist_exp_i_1*)
then have *sin_less*: *sin*((*pi ∗ j / n*)) *< sin* (*pi / real n*)
by (*simp add*: *field_simps*)
then have *w / z = 1*
proof (*cases j = 0*)
case *True* **then show** *?thesis* **by** (*auto simp*: *j*)
next
case *False*
then have *sin* (*pi / real n*) *≤ sin*((*pi ∗ j / n*))
proof (*cases j / n ≤ 1/2*)

```
        case True
        show ?thesis
          using ‹j ≠ 0› ‹j < n› True
       by (intro sin_monotone_2pi_le) (auto simp: field_simps intro: order_trans
[of _ 0])
      next
        case False
        then have seq: sin(pi * j / n) = sin(pi * (n - j) / n)
              using ‹j < n› by (simp add: algebra_simps diff_divide_distrib
of_nat_diff)

        show ?thesis
          unfolding seq
        proof (intro sin_monotone_2pi_le)
          show - (pi / 2) ≤ pi / real n
            by (smt (verit) divide_nonneg_nonneg of_nat_0_le_iff pi_ge_zero)
        qed (use ‹j < n› False in ‹auto simp: divide_simps›)
      qed
      with sin_less show ?thesis by force
    qed
    then show ?thesis by simp
  qed
  show ?thesis
  proof
    show 0 < 2 * sin (pi / real n)
      by (force simp: ‹2 ≤ n› sin_pi_divide_n_gt_0)
  qed (meson eq_if_pow_eq)
 qed
 have zn1: continuous_on (- {0}) (λz::complex. z^n)
  by (rule continuous_intros)+
 have zn2: (λz::complex. z^n) ‘ (- {0}) = - {0}
   using assms by (auto simp: image_def elim: exists_complex_root_nonzero
[where n = n])
 have zn3: ∃ T. z^n ∈ T ∧ open T ∧ 0 ∉ T ∧
            (∃ v. ⋃ v = -{0} ∩ (λz. z ^ n) -‘ T ∧
                (∀ u∈v. open u ∧ 0 ∉ u) ∧
                pairwise disjnt v ∧
                (∀ u∈v. Ex (homeomorphism u T (λz. z^n))))
        if z ≠ 0 for z::complex
 proof -
  define d where d ≡ min (1/2) (e/4) * norm z
  have 0 < d
   by (simp add: d_def ‹0 < e› ‹z ≠ 0›)
  have iff_x_eq_y: x^n = y^n ⟷ x = y
      if eq: w^n = z^n and x: x ∈ ball w d and y: y ∈ ball w d for w x y
  proof -
   have [simp]: norm z = norm w using that
    by (simp add: assms power_eq_imp_eq_norm)
   show ?thesis
```

```
      proof (cases w = 0)
        case True with ‹z ≠ 0› assms eq
        show ?thesis by (auto simp: power_0_left)
      next
        case False
        have cmod (x − y) < 2∗d
          using x y
            by (simp add: dist_norm [symmetric]) (metis dist_commute mult_2
dist_triangle_less_add)
        also have ... ≤ 2 ∗ e / 4 ∗ norm w
          using ‹e > 0› by (simp add: d_def min_mult_distrib_right)
        also have ... = e ∗ (cmod w / 2)
          by simp
        also have ... ≤ e ∗ cmod y
        proof (rule mult_left_mono)
          have cmod (w − y) < cmod w / 2 ⟹ cmod w / 2 ≤ cmod y
            by (metis (no_types) dist_0_norm dist_norm norm_triangle_half_l
not_le order_less_irrefl)
          then show cmod w / 2 ≤ cmod y
            using y by (simp add: dist_norm d_def min_mult_distrib_right)
        qed (use ‹e > 0› in auto)
        finally have cmod (x − y) < e ∗ cmod y .
        then show ?thesis by (rule e)
      qed
    qed
    then have inj: inj_on (λw. w^n) (ball z d)
      by (simp add: inj_on_def)
    have cont: continuous_on (ball z d) (λw. w ^ n)
      by (intro continuous_intros)
    have noncon: ¬ (λw::complex. w^n) constant_on UNIV
     by (metis UNIV_I assms constant_on_def power_one zero_neq_one zero_power)
    have im_eq: (λw. w^n) ‘ ball z′ d = (λw. w^n) ‘ ball z d
              if z′: z′^n = z^n for z′
    proof −
      have nz′: norm z′ = norm z using that assms power_eq_imp_eq_norm by
blast
      have (w ∈ (λw. w^n) ‘ ball z′ d) = (w ∈ (λw. w^n) ‘ ball z d) for w
      proof (cases w=0)
        case True with assms show ?thesis
          by (simp add: image_def ball_def nz′)
      next
        case False
        have z′ ≠ 0 using ‹z ≠ 0› nz′ by force
        have 1: (z∗x / z′)^n = x^n if x ≠ 0 for x
          using z′ that by (simp add: field_simps ‹z ≠ 0›)
        have 2: cmod (z − z ∗ x / z′) = cmod (z′ − x) if x ≠ 0 for x
        proof −
          have cmod (z − z ∗ x / z′) = cmod z ∗ cmod (1 − x / z′)
         by (metis (no_types) ab_semigroup_mult_class.mult_ac(1) divide_complex_def
```

```
mult.right_neutral norm_mult right_diff_distrib')
        also have ... = cmod z' * cmod (1 - x / z')
          by (simp add: nz')
        also have ... = cmod (z' - x)
          by (simp add: ‹z' ≠ 0› diff_divide_eq_iff norm_divide)
        finally show ?thesis .
      qed
      have 3: (z'*x / z)^n = x^n if x ≠ 0 for x
        using z' that by (simp add: field_simps ‹z ≠ 0›)
      have 4: cmod (z' - z' * x / z) = cmod (z - x) if x ≠ 0 for x
      proof -
        have cmod (z * (1 - x * inverse z)) = cmod (z - x)
          by (metis ‹z ≠ 0› diff_divide_distrib divide_complex_def divide_self_if
nonzero_eq_divide_eq semiring_normalization_rules(7))
        then show ?thesis
          by (metis (no_types) mult.assoc divide_complex_def mult.right_neutral
norm_mult nz' right_diff_distrib')
      qed
      show ?thesis
        by (simp add: set_eq_iff image_def ball_def) (metis 1 2 3 4 diff_zero
dist_norm nz')
    qed
    then show ?thesis by blast
  qed

  have ex_ball: ∃B. (∃z'. B = ball z' d ∧ z'^n = z^n) ∧ x ∈ B
            if x ≠ 0 and eq: x^n = w^n and dzw: dist z w < d for x w
  proof -
    have w ≠ 0 by (metis assms power_eq_0_iff that(1) that(2))
    have [simp]: cmod x = cmod w
      using assms power_eq_imp_eq_norm eq by blast
    have [simp]: cmod (x * z / w - x) = cmod (z - w)
    proof -
      have cmod (x * z / w - x) = cmod x * cmod (z / w - 1)
        by (metis (no_types) mult.right_neutral norm_mult right_diff_distrib'
times_divide_eq_right)
      also have ... = cmod w * cmod (z / w - 1)
        by simp
      also have ... = cmod (z - w)
        by (simp add: ‹w ≠ 0› divide_diff_eq_iff nonzero_norm_divide)
      finally show ?thesis .
    qed
    show ?thesis
    proof (intro exI conjI)
      show (z / w * x) ^ n = z ^ n
        by (metis ‹w ≠ 0› eq nonzero_eq_divide_eq power_mult_distrib)
      show x ∈ ball (z / w * x) d
        using ‹d > 0› that
        by (simp add: ball_eq_ball_iff ‹z ≠ 0› ‹w ≠ 0› field_simps) (simp add:
```

```
dist_norm)
    qed auto
  qed

  show ?thesis
  proof (rule exI, intro conjI)
    show z ^ n ∈ (λw. w ^ n) ‘ ball z d
      using ‹d > 0› by simp
    show open ((λw. w ^ n) ‘ ball z d)
      by (rule invariance_of_domain [OF cont open_ball inj])
    show 0 ∉ (λw. w ^ n) ‘ ball z d
      using ‹z ≠ 0› assms by (force simp: d_def)
    show ∃v. ⋃v = - {0} ∩ (λz. z ^ n) -‘ (λw. w ^ n) ‘ ball z d ∧
             (∀u∈v. open u ∧ 0 ∉ u) ∧
             disjoint v ∧
             (∀u∈v. Ex (homeomorphism u ((λw. w ^ n) ‘ ball z d) (λz. z ^ n)))
    proof (rule exI, intro ballI conjI)
      show ⋃{ball z' d |z'. z'^n = z^n} = - {0} ∩ (λz. z ^ n) -‘ (λw. w ^ n) ‘
ball z d (is ?l = ?r)
      proof
        have ⋀z'. cmod z' < d ⟹ z' ^ n ≠ z ^ n
          by (auto simp add: assms d_def power_eq_imp_eq_norm that)
        then show ?l ⊆ ?r
          by auto (metis im_eq image_eqI mem_ball)
        show ?r ⊆ ?l
          by auto (meson ex_ball)
      qed
      show ⋀u. u ∈ {ball z' d |z'. z' ^ n = z ^ n} ⟹ 0 ∉ u
        by (force simp add: assms d_def power_eq_imp_eq_norm that)

      show disjoint {ball z' d |z'. z' ^ n = z ^ n}
      proof (clarsimp simp add: pairwise_def disjnt_iff)
        fix ξ ζ x
        assume ξ^n = z^n ζ^n = z^n ball ξ d ≠ ball ζ d
          and dist ξ x < d dist ζ x < d
        then have dist ξ ζ < d+d
          using dist_triangle_less_add by blast
        then have cmod (ξ - ζ) < 2*d
          by (simp add: dist_norm)
        also have ... ≤ e * cmod z
          using mult_right_mono ‹0 < e› that by (auto simp: d_def)
        finally have cmod (ξ - ζ) < e * cmod z .
        with e have ξ = ζ
          by (metis ‹ξ^n = z^n› ‹ζ^n = z^n› assms power_eq_imp_eq_norm)
        then show False
          using ‹ball ξ d ≠ ball ζ d› by blast
      qed
      show Ex (homeomorphism u ((λw. w ^ n) ‘ ball z d) (λz. z ^ n))
        if u ∈ {ball z' d |z'. z' ^ n = z ^ n} for u
```

```
      proof (rule invariance_of_domain_homeomorphism [of u λz. z^n])
        show open u
          using that by auto
        show continuous_on u (λz. z ^ n)
          by (intro continuous_intros)
        show inj_on (λz. z ^ n) u
          using that by (auto simp: iff_x_eq_y inj_on_def)
          show ⋀g. homeomorphism u ((λz. z ^ n) ` u) (λz. z ^ n) g ⟹ Ex
(homeomorphism u ((λw. w ^ n) ` ball z d) (λz. z ^ n))
          using im_eq that by clarify metis
      qed auto
    qed auto
  qed
  show ?thesis
    using assms
    apply (simp add: covering_space_def zn1 zn2)
    apply (subst zn2 [symmetric])
    apply (simp add: openin_open_eq open_Compl zn3)
    done
qed

corollary covering_space_square_punctured_plane:
  covering_space (− {0}) (λz::complex. z^2) (− {0})
  by (simp add: covering_space_power_punctured_plane)


proposition covering_space_exp_punctured_plane:
  covering_space UNIV (λz::complex. exp z) (− {0})
proof (simp add: covering_space_def, intro conjI ballI)
  show continuous_on UNIV (λz::complex. exp z)
    by (rule continuous_on_exp [OF continuous_on_id])
  show range exp = − {0::complex}
    by auto (metis exp_Ln range_eqI)
  show ∃ T. z ∈ T ∧ openin (top_of_set (− {0})) T ∧
            (∃ v. ⋃ v = exp −` T ∧ (∀ u∈v. open u) ∧ disjoint v ∧
                (∀ u∈v. ∃ q. homeomorphism u T exp q))
       if z ∈ − {0::complex} for z
  proof −
    have z ≠ 0
      using that by auto
    have ball (Ln z) 1 ⊆ ball (Ln z) pi
      using pi_ge_two by (simp add: ball_subset_ball_iff)
    then have inj_exp: inj_on exp (ball (Ln z) 1)
      using inj_on_exp_pi inj_on_subset by blast
    define twopi where twopi ≡ λn. of_real (2 * of_int n * pi) * i
    define V where V ≡ range (λn. (λx. x + twopi n) ` (ball(Ln z) 1))
    have exp_eq': exp w = exp z ⟷ (∃ n::int. w = z + twopi n) for z w
      by (simp add: exp_eq twopi_def)
```

```
    show ?thesis
    proof (intro exI conjI)
      show z ∈ exp ` (ball(Ln z) 1)
        by (metis ‹z ≠ 0› centre_in_ball exp_Ln rev_image_eqI zero_less_one)
      have open (− {0::complex})
        by blast
      with inj_exp show openin (top_of_set (− {0})) (exp ` ball (Ln z) 1)
       by (auto simp: openin_open_eq invariance_of_domain continuous_on_exp
[OF continuous_on_id])
      show UV: ⋃ 𝒱 = exp −` exp ` ball (Ln z) 1
      by (force simp: 𝒱_def twopi_def Complex_Transcendental.exp_eq image_iff)
      show ∀ V∈𝒱. open V
       by (auto simp: 𝒱_def inj_on_def continuous_intros invariance_of_domain)
      have 2 ≤ cmod (twopi m −twopi n) if m ≠ n for m n
      proof −
        have 1 ≤ abs (m − n)
          using that by linarith
        then have 1 ≤ cmod (of_int m − of_int n) * 1
          by (metis mult.right_neutral norm_of_int of_int_1_le_iff of_int_abs
of_int_diff)
        also have ... ≤ cmod (of_int m − of_int n) * of_real pi
          using pi_ge_two
          by (intro mult_left_mono) auto
        also have ... ≤ cmod ((of_int m − of_int n) * of_real pi * i)
          by (simp add: norm_mult)
       also have ... ≤ cmod (of_int m * of_real pi * i − of_int n * of_real pi * i)
          by (simp add: algebra_simps)
        finally have 1 ≤ cmod (of_int m * of_real pi * i − of_int n * of_real pi
* i) .
         then have 2 * 1 ≤ cmod (2 * (of_int m * of_real pi * i − of_int n *
of_real pi * i))
        by (metis mult_le_cancel_left_pos norm_mult_numeral1 zero_less_numeral)
         then show ?thesis
          by (simp add: algebra_simps twopi_def)
      qed
      then show disjoint 𝒱
        unfolding 𝒱_def pairwise_def disjnt_iff
          by (smt (verit, best) add.commute add_diff_cancel_left' add_diff_eq
dist_commute dist_complex_def dist_triangle imageE mem_ball)
      show ∀ u∈𝒱. ∃ q. homeomorphism u (exp ` ball (Ln z) 1) exp q
      proof
        fix u
        assume u ∈ 𝒱
        then obtain n where n: u = (λx. x + twopi n) ` (ball(Ln z) 1)
          by (auto simp: 𝒱_def)
        have compact (cball (Ln z) 1)
          by simp
        moreover have continuous_on (cball (Ln z) 1) exp
          by (rule continuous_on_exp [OF continuous_on_id])
```

```
      moreover have inj_on exp (cball (Ln z) 1)
        using pi_ge_two inj_on_subset [OF inj_on_exp_pi [of Ln z]]
        by (simp add: subset_iff)
      ultimately obtain γ where hom: homeomorphism (cball (Ln z) 1) (exp ‘
cball (Ln z) 1) exp γ
        using homeomorphism_compact by blast
      have eq1: exp ‘ u = exp ‘ ball (Ln z) 1
        by (smt (verit) n exp_eq′ image_cong image_image)
      have γexp: γ (exp x) + twopi n = x if x ∈ u for x
      proof −
        have exp x = exp (x − twopi n)
          using exp_eq′ by auto
        then have γ (exp x) = γ (exp (x − twopi n))
          by simp
        also have ... = x − twopi n
        using ‹x ∈ u› by (auto simp: n intro: homeomorphism_apply1 [OF hom])
        finally show ?thesis
          by simp
      qed
      have exp2n: exp (γ (exp x) + twopi n) = exp x if dist (Ln z) x < 1 for x
        by (metis γexp exp_eq′ imageI mem_ball n that)
      have continuous_on (exp ‘ ball (Ln z) 1) γ
           by (meson ball_subset_cball continuous_on_subset hom homeomor-
phism_cont2 image_mono)
      then have cont: continuous_on (exp ‘ ball (Ln z) 1) (λx. γ x + twopi n)
        by (intro continuous_intros)
      have homeomorphism u (exp ‘ ball (Ln z) 1) exp ((λx. x + twopi n) ∘ γ)
        unfolding homeomorphism_def
        apply (intro conjI ballI eq1 continuous_on_exp [OF continuous_on_id])
          apply (auto simp: γexp exp2n cont n)
         apply (force simp: image_iff homeomorphism_apply1 [OF hom])+
        done
      then show ∃ q. homeomorphism u (exp ‘ ball (Ln z) 1) exp q by metis
    qed
  qed
 qed
qed
```

9.27.11 Hence the Borsukian results about mappings into circles

```
lemma inessential_eq_continuous_logarithm:
 fixes f :: 'a::real_normed_vector ⇒ complex
 shows (∃ a. homotopic_with_canon (λh. True) S (−{0}) f (λt. a)) ⟷
       (∃ g. continuous_on S g ∧ (∀ x ∈ S. f x = exp(g x)))
 (is ?lhs ⟷ ?rhs)
proof
 assume ?lhs thus ?rhs
  by (metis covering_space_lift_inessential_function covering_space_exp_punctured_plane)
```

next
assume *?rhs*
then obtain *g* **where** *contg*: *continuous_on S g* **and** *f*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$
by *metis*
obtain *a* **where** *homotopic_with_canon* (λ*h*. *True*) *S* (− {*of_real 0*}) (*exp* ∘ *g*) (λ*x*. *a*)
proof (*rule nullhomotopic_through_contractible* [*OF contg _ _ _ contractible_UNIV*])
show *continuous_on* (*UNIV*::*complex set*) *exp*
by (*intro continuous_intros*)
show *exp* ∈ *UNIV* → −{*of_real 0*}
by *auto*
qed *force*
then have *homotopic_with_canon* (λ*h*. *True*) *S* (− {*0*}) *f* (λ*t*. *a*)
using *f homotopic_with_eq* **by** *fastforce*
then show *?lhs* **..**
qed

corollary *inessential_imp_continuous_logarithm_circle*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
assumes *homotopic_with_canon* (λ*h*. *True*) *S* (*sphere 0 1*) *f* (λ*t*. *a*)
obtains *g* **where** *continuous_on S g* **and** $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$
proof −
have *homotopic_with_canon* (λ*h*. *True*) *S* (− {*0*}) *f* (λ*t*. *a*)
using *assms homotopic_with_subset_right* **by** *fastforce*
then show *?thesis*
by (*metis inessential_eq_continuous_logarithm that*)
qed

lemma *inessential_eq_continuous_logarithm_circle*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
shows (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) *S* (*sphere 0 1*) *f* (λ*t*. *a*)) ⟷
(∃ *g*. *continuous_on S g* ∧ (∀ *x* ∈ *S*. *f x* = *exp*(i ∗ *of_real*(*g x*))))
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *L*: *?lhs*
then obtain *g* **where** *contg*: *continuous_on S g* **and** *g*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$
using *inessential_imp_continuous_logarithm_circle* **by** *blast*
have *f* ∈ *S* → *sphere 0 1*
by (*metis L image_subset_iff_funcset homotopic_with_imp_subset1*)
then have $\bigwedge x.\ x \in S \Longrightarrow Re\ (g\ x) = 0$
using *g* **by** (*simp add*: *Pi_iff*)
then show *?rhs*
by (*rule_tac x*=*Im* ∘ *g* **in** *exI*) (*auto simp*: *Euler g intro*: *contg continuous_intros*)
next
assume *?rhs*

then obtain *g* **where** *contg*: *continuous_on S g* **and** *g*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(\mathrm{i}*\ of\_real(g\ x))$
by *metis*
obtain *a* **where** *homotopic_with_canon* ($\lambda h.\ True$) *S* (*sphere 0 1*) ((*exp* $\circ$ ($\lambda z.\ \mathrm{i}*z$)) $\circ$ (*of_real* $\circ$ *g*)) ($\lambda x.\ a$)
proof (*rule nullhomotopic_through_contractible*)
show *continuous_on S* (*complex_of_real* $\circ$ *g*)
by (*intro conjI contg continuous_intros*)
show (*complex_of_real* $\circ$ *g*) $\in S \to \mathbb{R}$
by *auto*
show *continuous_on* $\mathbb{R}$ (*exp* $\circ$ (*)i)
by (*intro continuous_intros*)
show (*exp* $\circ$ (*)i) $\in \mathbb{R} \to$ *sphere 0 1*
by (*auto simp*: *complex_is_Real_iff*)
qed (*auto simp*: *convex_Reals convex_imp_contractible*)
moreover have $\bigwedge x.\ x \in S \Longrightarrow$ (*exp* $\circ$ (*)i $\circ$ (*complex_of_real* $\circ$ *g*)) $x = f\ x$
by (*simp add*: *g*)
ultimately have *homotopic_with_canon* ($\lambda h.\ True$) *S* (*sphere 0 1*) *f* ($\lambda t.\ a$)
using *homotopic_with_eq* **by** *force*
then show *?lhs* **..**
qed

proposition *homotopic_with_sphere_times*:
fixes *f* :: *'a*::*real_normed_vector* $\Rightarrow$ *complex*
assumes *hom*: *homotopic_with_canon* ($\lambda x.\ True$) *S* (*sphere 0 1*) *f g* **and** *conth*: *continuous_on S h*
and *hin*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in$ *sphere 0 1*
shows *homotopic_with_canon* ($\lambda x.\ True$) *S* (*sphere 0 1*) ($\lambda x.\ f\ x * h\ x$) ($\lambda x.\ g\ x * h\ x$)
proof −
obtain *k* **where** *contk*: *continuous_on* ({*0..1*::*real*} $\times$ *S*) *k*
and *kim*: $k \in (\{0..1\} \times S) \to$ *sphere 0 1*
and *k0*: $\bigwedge x.\ k(0, x) = f\ x$
and *k1*: $\bigwedge x.\ k(1, x) = g\ x$
using *hom* **by** (*auto simp*: *homotopic_with_def*)
show *?thesis*
apply (*simp add*: *homotopic_with*)
apply (*rule_tac x*=$\lambda z.\ k\ z * (h \circ snd)z$ **in** *exI*)
using *kim hin* **by** (*fastforce simp*: *conth norm_mult k0 k1 intro*!: *contk continuous_intros*)+
qed

proposition *homotopic_circlemaps_divide*:
fixes *f* :: *'a*::*real_normed_vector* $\Rightarrow$ *complex*
shows *homotopic_with_canon* ($\lambda x.\ True$) *S* (*sphere 0 1*) *f g* $\longleftrightarrow$
continuous_on S f $\wedge$ *f* $\in S \to$ *sphere 0 1* $\wedge$
continuous_on S g $\wedge$ *g* $\in S \to$ *sphere 0 1* $\wedge$
($\exists c.$ *homotopic_with_canon* ($\lambda x.\ True$) *S* (*sphere 0 1*) ($\lambda x.\ f\ x\ /\ g\ x$) ($\lambda x.\ c$))

```
proof −
  have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx. 1)
       if homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx. c)
for c
  proof −
    have S = {} ∨ path_component (sphere 0 1) 1 c
      using homotopic_with_imp_subset2 [OF that] path_connected_sphere [of
0::complex 1]
      by (auto simp: path_connected_component)
    with subtopology_empty_iff_trivial
    have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. 1) (λx. c)
      by (force simp add: homotopic_constant_maps)
    then show ?thesis
      using homotopic_with_symD homotopic_with_trans that by blast
  qed
  then have ∗: (∃ c. homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x /
g x) (λx. c)) ⟷
              homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx.
1)
    by auto
  have homotopic_with_canon (λx. True) S (sphere 0 1) f g ⟷
          continuous_on S f ∧ f ∈ S → sphere 0 1 ∧
          continuous_on S g ∧ g ∈ S → sphere 0 1 ∧
          homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx. 1)
       (is ?lhs ⟷ ?rhs)
  proof
    assume L: ?lhs
    have geq1 [simp]: ⋀x. x ∈ S ⟹ cmod (g x) = 1
      using homotopic_with_imp_subset2 [OF L]
      by (simp add: image_subset_iff)
    have cont: continuous_on S (inverse ∘ g)
    proof (rule continuous_intros)
      show continuous_on S g
        using homotopic_with_imp_continuous [OF L] by blast
      show continuous_on (g ‘ S) inverse
       by (rule continuous_on_subset [of sphere 0 1, OF continuous_on_inverse])
auto
    qed
    have [simp]: ⋀x. x ∈ S ⟹ g x ≠ 0
      using geq1 by fastforce
    have homotopic_with_canon (λx. True) S (sphere 0 1) (λx. f x / g x) (λx. 1)
      using homotopic_with_eq [OF homotopic_with_sphere_times [OF L cont]]
      by (auto simp: divide_inverse norm_inverse)
    with L show ?rhs
    by (simp add: homotopic_with_imp_continuous homotopic_with_imp_funspace1)
  next
    assume ?rhs then show ?lhs
      by (elim conjE homotopic_with_eq [OF homotopic_with_sphere_times];
force)
```

qed
then show *?thesis*
by (*simp add*: ∗)
qed

9.27.12 Upper and lower hemicontinuous functions

And relation in the case of preimage map to open and closed maps, and fact that upper and lower hemicontinuity together imply continuity in the sense of the Hausdorff metric (at points where the function gives a bounded and nonempty set).

Many similar proofs below.

lemma *upper_hemicontinuous*:
assumes ⋀*x*. *x* ∈ *S* ⟹ *f x* ⊆ *T*
shows ((∀ *U*. *openin* (*top_of_set T*) *U*
⟶ *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}) ⟷
(∀ *U*. *closedin* (*top_of_set T*) *U*
⟶ *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ *U* ≠ {}}))
(**is** *?lhs* = *?rhs*)
proof (*intro iffI allI impI*)
fix *U*
assume ∗ [*rule_format*]: *?lhs* **and** *closedin* (*top_of_set T*) *U*
then have *openin* (*top_of_set T*) (*T* − *U*)
by (*simp add*: *openin_diff*)
then have *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *T* − *U*}
using ∗ [*of T*−*U*] **by** *blast*
moreover have *S* − {*x* ∈ *S*. *f x* ⊆ *T* − *U*} = {*x* ∈ *S*. *f x* ∩ *U* ≠ {}}
using *assms* **by** *blast*
ultimately show *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ *U* ≠ {}}
by (*simp add*: *openin_closedin_eq*)
next
fix *U*
assume ∗ [*rule_format*]: *?rhs* **and** *openin* (*top_of_set T*) *U*
then have *closedin* (*top_of_set T*) (*T* − *U*)
by (*simp add*: *closedin_diff*)
then have *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ (*T* − *U*) ≠ {}}
using ∗ [*of T*−*U*] **by** *blast*
moreover have {*x* ∈ *S*. *f x* ∩ (*T* − *U*) ≠ {}} = *S* − {*x* ∈ *S*. *f x* ⊆ *U*}
using *assms* **by** *auto*
ultimately show *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}
by (*simp add*: *openin_closedin_eq*)
qed

lemma *lower_hemicontinuous*:
assumes ⋀*x*. *x* ∈ *S* ⟹ *f x* ⊆ *T*
shows ((∀ *U*. *closedin* (*top_of_set T*) *U*
⟶ *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}) ⟷
(∀ *U*. *openin* (*top_of_set T*) *U*

```
            ⟶ openin (top_of_set S) {x ∈ S. f x ∩ U ≠ {}}))
        (is ?lhs = ?rhs)
proof (intro iffI allI impI)
  fix U
  assume * [rule_format]: ?lhs and openin (top_of_set T) U
  then have closedin (top_of_set T) (T − U)
    by (simp add: closedin_diff)
  then have closedin (top_of_set S) {x ∈ S. f x ⊆ T−U}
    using * [of T−U] by blast
  moreover have {x ∈ S. f x ⊆ T−U} = S − {x ∈ S. f x ∩ U ≠ {}}
    using assms by auto
  ultimately show openin (top_of_set S) {x ∈ S. f x ∩ U ≠ {}}
    by (simp add: openin_closedin_eq)
next
  fix U
  assume * [rule_format]: ?rhs and closedin (top_of_set T) U
  then have openin (top_of_set T) (T − U)
    by (simp add: openin_diff)
  then have openin (top_of_set S) {x ∈ S. f x ∩ (T − U) ≠ {}}
    using * [of T−U] by blast
  moreover have S − {x ∈ S. f x ∩ (T − U) ≠ {}} = {x ∈ S. f x ⊆ U}
    using assms by blast
  ultimately show closedin (top_of_set S) {x ∈ S. f x ⊆ U}
    by (simp add: openin_closedin_eq)
qed

lemma open_map_iff_lower_hemicontinuous_preimage:
  assumes f ∈ S → T
    shows ((∀ U. openin (top_of_set S) U
            ⟶ openin (top_of_set T) (f ‘ U)) ⟷
          (∀ U. closedin (top_of_set S) U
            ⟶ closedin (top_of_set T) {y ∈ T. {x. x ∈ S ∧ f x = y} ⊆ U}))
        (is ?lhs = ?rhs)
proof (intro iffI allI impI)
  fix U
  assume * [rule_format]: ?lhs and closedin (top_of_set S) U
  then have openin (top_of_set S) (S − U)
    by (simp add: openin_diff)
  then have openin (top_of_set T) (f ‘ (S − U))
    using * [of S−U] by blast
  moreover have T − (f ‘ (S − U)) = {y ∈ T. {x ∈ S. f x = y} ⊆ U}
    using assms by blast
  ultimately show closedin (top_of_set T) {y ∈ T. {x ∈ S. f x = y} ⊆ U}
    by (simp add: openin_closedin_eq)
next
  fix U
  assume * [rule_format]: ?rhs and opeSU: openin (top_of_set S) U
  then have closedin (top_of_set S) (S − U)
    by (simp add: closedin_diff)
```

```
  then have closedin (top_of_set T) {y ∈ T. {x ∈ S. f x = y} ⊆ S − U}
    using ∗ [of S−U] by blast
  moreover have {y ∈ T. {x ∈ S. f x = y} ⊆ S − U} = T − (f ‘ U)
    using assms openin_imp_subset [OF opeSU] by auto
  ultimately show openin (top_of_set T) (f ‘ U)
   using assms openin_imp_subset [OF opeSU] by (force simp: openin_closedin_eq)
qed

lemma closed_map_iff_upper_hemicontinuous_preimage:
  assumes f ∈ S → T
    shows ((∀ U. closedin (top_of_set S) U
                ⟶ closedin (top_of_set T) (f ‘ U)) ⟷
           (∀ U. openin (top_of_set S) U
                ⟶ openin (top_of_set T) {y ∈ T. {x. x ∈ S ∧ f x = y} ⊆ U}))
          (is ?lhs = ?rhs)
proof (intro iffI allI impI)
  fix U
  assume ∗ [rule_format]: ?lhs and opeSU: openin (top_of_set S) U
  then have closedin (top_of_set S) (S − U)
    by (simp add: closedin_diff)
  then have closedin (top_of_set T) (f ‘ (S − U))
    using ∗ [of S−U] by blast
  moreover have f ‘ (S − U) = T − {y ∈ T. {x. x ∈ S ∧ f x = y} ⊆ U}
    using assms openin_imp_subset [OF opeSU] by auto
  ultimately show openin (top_of_set T) {y ∈ T. {x. x ∈ S ∧ f x = y} ⊆ U}
   using assms openin_imp_subset [OF opeSU] by (force simp: openin_closedin_eq)
next
  fix U
  assume ∗ [rule_format]: ?rhs and cloSU: closedin (top_of_set S) U
  then have openin (top_of_set S) (S − U)
    by (simp add: openin_diff)
  then have openin (top_of_set T) {y ∈ T. {x ∈ S. f x = y} ⊆ S − U}
    using ∗ [of S−U] by blast
  moreover have (f ‘ U) = T − {y ∈ T. {x ∈ S. f x = y} ⊆ S − U}
    using assms closedin_imp_subset [OF cloSU] by auto
  ultimately show closedin (top_of_set T) (f ‘ U)
    by (simp add: openin_closedin_eq)
qed

proposition upper_lower_hemicontinuous_explicit:
  fixes T :: ('b::{real_normed_vector,heine_borel}) set
  assumes fST: ⋀x. x ∈ S ⟹ f x ⊆ T
      and ope: ⋀U. openin (top_of_set T) U
                   ⟹ openin (top_of_set S) {x ∈ S. f x ⊆ U}
      and clo: ⋀U. closedin (top_of_set T) U
                   ⟹ closedin (top_of_set S) {x ∈ S. f x ⊆ U}
      and x ∈ S 0 < e and bofx: bounded(f x) and fx_ne: f x ≠ {}
  obtains d where 0 < d
           ⋀x'. ⟦x' ∈ S; dist x x' < d⟧
```

$$\Longrightarrow (\forall\, y \in f\, x.\ \exists\, y'.\ y' \in f\, x' \wedge \mathit{dist}\ y\ y' < e) \wedge (\forall\, y' \in f\, x'.\ \exists\, y.\ y \in f\, x \wedge \mathit{dist}\ y'\ y < e)$$

proof −
have *openin* (*top_of_set T*) (*T* ∩ (⋃ *a*∈*f x*. ⋃ *b*∈*ball 0 e*. {*a* + *b*}))
by (*auto simp*: *open_sums openin_open_Int*)
with *ope* **have** *openin* (*top_of_set S*)
{*u* ∈ *S*. *f u* ⊆ *T* ∩ (⋃ *a*∈*f x*. ⋃ *b*∈*ball 0 e*. {*a* + *b*})} **by** *blast*
with ‹*0* < *e*› ‹*x* ∈ *S*› **obtain** *d1* **where** *d1* > *0* **and**
d1: ⋀*x'*. ⟦*x'* ∈ *S*; *dist x' x* < *d1*⟧ ⟹ *f x'* ⊆ *T* ∧ *f x'* ⊆ (⋃ *a* ∈ *f x*. ⋃ *b* ∈ *ball 0 e*. {*a* + *b*})
by (*force simp*: *openin_euclidean_subtopology_iff dest*: *fST*)
have *oo*: ⋀*U*. *openin* (*top_of_set T*) *U* ⟹
openin (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ *U* ≠ {}}
using *lower_hemicontinuous fST clo* **by** *blast*
have *compact* (*closure*(*f x*))
by (*simp add*: *bofx*)
moreover have *closure*(*f x*) ⊆ (⋃ *a* ∈ *f x*. *ball a* (*e*/*2*))
using ‹*0* < *e*› **by** (*force simp*: *closure_approachable simp del*: *divide_const_simps*)
ultimately obtain *C* **where** *C* ⊆ *f x finite C closure*(*f x*) ⊆ (⋃ *a* ∈ *C*. *ball a* (*e*/*2*))
by (*meson compactE finite_subset_image Elementary_Metric_Spaces.open_ball compactE_image*)
then have *fx_cover*: *f x* ⊆ (⋃ *a* ∈ *C*. *ball a* (*e*/*2*))
by (*meson closure_subset order_trans*)
with *fx_ne* **have** *C* ≠ {}
by *blast*
have *xin*: *x* ∈ (⋂ *a* ∈ *C*. {*x* ∈ *S*. *f x* ∩ *T* ∩ *ball a* (*e*/*2*) ≠ {}})
using ‹*x* ∈ *S*› ‹*0* < *e*› *fST* ‹*C* ⊆ *f x*› **by** *force*
have *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ∩ (*T* ∩ *ball a* (*e*/*2*)) ≠ {}} **for** *a*
by (*simp add*: *openin_open_Int oo*)
then have *openin* (*top_of_set S*) (⋂ *a* ∈ *C*. {*x* ∈ *S*. *f x* ∩ *T* ∩ *ball a* (*e*/*2*) ≠ {}})
by (*simp add*: *Int_assoc openin_INT2* [*OF* ‹*finite C*› ‹*C* ≠ {}›])
with *xin* **obtain** *d2* **where** *d2*>*0*
and *d2*: ⋀*u v*. ⟦*u* ∈ *S*; *dist u x* < *d2*; *v* ∈ *C*⟧ ⟹ *f u* ∩ *T* ∩ *ball v* (*e*/*2*) ≠ {}
unfolding *openin_euclidean_subtopology_iff* **using** *xin* **by** *fastforce*
show *?thesis*
proof (*intro that conjI ballI*)
show *0* < *min d1 d2*
using ‹*0* < *d1*› ‹*0* < *d2*› **by** *linarith*
next
fix *x' y*
assume *x'* ∈ *S dist x x'* < *min d1 d2 y* ∈ *f x*
then have *dd2*: *dist x' x* < *d2*
by (*auto simp*: *dist_commute*)
obtain *a* **where** *a* ∈ *C y* ∈ *ball a* (*e*/*2*)
using *fx_cover* ‹*y* ∈ *f x*› **by** *auto*
then show ∃ *y'*. *y'* ∈ *f x'* ∧ *dist y y'* < *e*

using *d2* [*OF* ‹$x' \in S$› *dd2*] *dist_triangle_half_r* **by** *fastforce*
next
fix x' y'
assume $x' \in S$ *dist* x $x' <$ *min d1 d2* $y' \in f\ x'$
then have *dist* x' $x <$ *d1*
by (*auto simp*: *dist_commute*)
then have $y' \in (\bigcup a{\in}f\ x.\ \bigcup b{\in}$*ball 0 e*. $\{a + b\})$
using *d1* [*OF* ‹$x' \in S$›] ‹$y' \in f\ x'$› **by** *force*
then show $\exists\, y.\ y \in f\ x \wedge$ *dist* y' $y <$ *e*
by *clarsimp* (*metis add_diff_cancel_left' dist_norm*)
qed
qed

9.27.13 Complex logs exist on various "well-behaved" sets

lemma *continuous_logarithm_on_contractible*:
fixes *f* :: 'a::*real_normed_vector* ⇒ *complex*
assumes *continuous_on S f contractible S* $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
obtains *g* **where** *continuous_on S g* $\bigwedge x.\ x \in S \Longrightarrow f\ x =$ *exp*(*g x*)
proof −
obtain *c* **where** *hom*: *homotopic_with_canon* (λ*h*. *True*) *S* (−{*0*}) *f* (λ*x*. *c*)
using *nullhomotopic_from_contractible assms*
by (*metis imageE subset_Compl_singleton image_subset_iff_funcset*)
then show *?thesis*
by (*metis inessential_eq_continuous_logarithm that*)
qed

lemma *continuous_logarithm_on_simply_connected*:
fixes *f* :: 'a::*real_normed_vector* ⇒ *complex*
assumes *contf*: *continuous_on S f* **and** *S*: *simply_connected S locally path_connected S*
and *f*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
obtains *g* **where** *continuous_on S g* $\bigwedge x.\ x \in S \Longrightarrow f\ x =$ *exp*(*g x*)
using *covering_space_lift* [*OF covering_space_exp_punctured_plane S contf*]
by (*metis* (*full_types*) *f imageE subset_Compl_singleton image_subset_iff_funcset*)

lemma *continuous_logarithm_on_cball*:
fixes *f* :: 'a::*real_normed_vector* ⇒ *complex*
assumes *continuous_on* (*cball a r*) *f* **and** $\bigwedge z.\ z \in$ *cball a r* $\Longrightarrow f\ z \neq 0$
obtains *h* **where** *continuous_on* (*cball a r*) *h* $\bigwedge z.\ z \in$ *cball a r* $\Longrightarrow f\ z =$ *exp*(*h z*)
using *assms continuous_logarithm_on_contractible convex_imp_contractible* **by** *blast*

lemma *continuous_logarithm_on_ball*:
fixes *f* :: 'a::*real_normed_vector* ⇒ *complex*
assumes *continuous_on* (*ball a r*) *f* **and** $\bigwedge z.\ z \in$ *ball a r* $\Longrightarrow f\ z \neq 0$
obtains *h* **where** *continuous_on* (*ball a r*) *h* $\bigwedge z.\ z \in$ *ball a r* $\Longrightarrow f\ z =$ *exp*(*h z*)

using *assms continuous_logarithm_on_contractible convex_imp_contractible* **by** *blast*

lemma *continuous_sqrt_on_contractible*:
fixes f :: *'a::real_normed_vector* ⇒ *complex*
assumes *continuous_on S f contractible S*
and $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
obtains g **where** *continuous_on S g* $\bigwedge x.\ x \in S \Longrightarrow f\ x = (g\ x)\ \hat{}\ 2$
proof −
obtain g **where** *contg*: *continuous_on S g* **and** *feq*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$
using *continuous_logarithm_on_contractible* [*OF assms*] **by** *blast*
show *?thesis*
proof
show *continuous_on S* (λ*z. exp* (*g z* / *2*))
by (*rule continuous_on_compose2* [*of UNIV exp*]; *intro continuous_intros contg subset_UNIV*) *auto*
show $\bigwedge x.\ x \in S \Longrightarrow f\ x = (exp\ (g\ x\ /\ 2))^2$
by (*metis exp_double feq nonzero_mult_div_cancel_left times_divide_eq_right zero_neq_numeral*)
qed
qed

lemma *continuous_sqrt_on_simply_connected*:
fixes f :: *'a::real_normed_vector* ⇒ *complex*
assumes *contf*: *continuous_on S f* **and** *S*: *simply_connected S locally path_connected S*
and *f*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
obtains g **where** *continuous_on S g* $\bigwedge x.\ x \in S \Longrightarrow f\ x = (g\ x)\ \hat{}\ 2$
proof −
obtain g **where** *contg*: *continuous_on S g* **and** *feq*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$
using *continuous_logarithm_on_simply_connected* [*OF assms*] **by** *blast*
show *?thesis*
proof
show *continuous_on S* (λ*z. exp* (*g z* / *2*))
by (*rule continuous_on_compose2* [*of UNIV exp*]; *intro continuous_intros contg subset_UNIV*) *auto*
show $\bigwedge x.\ x \in S \Longrightarrow f\ x = (exp\ (g\ x\ /\ 2))^2$
by (*metis exp_double feq nonzero_mult_div_cancel_left times_divide_eq_right zero_neq_numeral*)
qed
qed

9.27.14 Another simple case where sphere maps are nullhomotopic

lemma *inessential_spheremap_2_aux*:
fixes f :: *'a::euclidean_space* ⇒ *complex*

assumes *2*: *2 < DIM('a)* **and** *contf*: *continuous_on (sphere a r) f*
and *fim*: *f ∈ (sphere a r) → (sphere 0 1)*
obtains *c* **where** *homotopic_with_canon (λz. True) (sphere a r) (sphere 0 1) f (λx. c)*
proof −
obtain *g* **where** *contg*: *continuous_on (sphere a r) g*
and *feq*: *⋀x. x ∈ sphere a r ⟹ f x = exp(g x)*
proof (*rule continuous_logarithm_on_simply_connected* [*OF contf*])
show *simply_connected (sphere a r)*
using *2* **by** (*simp add*: *simply_connected_sphere_eq*)
show *locally path_connected (sphere a r)*
by (*simp add*: *locally_path_connected_sphere*)
show *⋀z. z ∈ sphere a r ⟹ f z ≠ 0*
using *fim* **by** *force*
qed *auto*
have *∃ g. continuous_on (sphere a r) g ∧ (∀ x∈sphere a r. f x = exp (i ∗ complex_of_real (g x)))*
proof (*intro exI conjI*)
show *continuous_on (sphere a r) (Im ∘ g)*
by (*intro contg continuous_intros continuous_on_compose*)
show *∀ x∈sphere a r. f x = exp (i ∗ complex_of_real ((Im ∘ g) x))*
using *exp_eq_polar feq fim norm_exp_eq_Re*
by (*auto simp flip*: *image_subset_iff_funcset*)
qed
with *inessential_eq_continuous_logarithm_circle that* **show** *?thesis*
by *metis*
qed

lemma *inessential_spheremap_2*:
fixes *f :: 'a::euclidean_space ⇒ 'b::euclidean_space*
assumes *a2*: *2 < DIM('a)* **and** *b2*: *DIM('b) = 2*
and *contf*: *continuous_on (sphere a r) f* **and** *fim*: *f ∈ (sphere a r) → (sphere b s)*
obtains *c* **where** *homotopic_with_canon (λz. True) (sphere a r) (sphere b s) f (λx. c)*
proof (*cases s ≤ 0*)
case *True*
then show *?thesis*
using *contf contractible_sphere fim nullhomotopic_into_contractible that* **by** *blast*
next
case *False*
then have *sphere b s homeomorphic sphere (0::complex) 1*
using *assms* **by** (*simp add*: *homeomorphic_spheres_gen*)
then obtain *h k* **where** *hk*: *homeomorphism (sphere b s) (sphere (0::complex) 1) h k*
by (*auto simp*: *homeomorphic_def*)
then have *conth*: *continuous_on (sphere b s) h*
and *contk*: *continuous_on (sphere 0 1) k*

and *him*: $h \in$ *sphere b s* $\to$ *sphere 0 1*
and *kim*: $k \in$ *sphere 0 1* $\to$ *sphere b s*
by (*force simp*: *homeomorphism_def*)+
obtain *c* **where** *homotopic_with_canon* ($\lambda z.$ *True*) (*sphere a r*) (*sphere 0 1*) ($h \circ f$) ($\lambda x.$ *c*)
proof (*rule inessential_spheremap_2_aux* [*OF a2*])
show *continuous_on* (*sphere a r*) ($h \circ f$)
by (*meson contf conth continuous_on_compose continuous_on_subset fim image_subset_iff_funcset*)
show ($h \circ f$) $\in$ *sphere a r* $\to$ *sphere 0 1*
using *fim him* **by** *force*
qed *auto*
then have *homotopic_with_canon* ($\lambda f.$ *True*) (*sphere a r*) (*sphere b s*) ($k \circ (h \circ f)$) ($k \circ (\lambda x.$ *c*))
by (*rule homotopic_with_compose_continuous_left* [*OF _ contk kim*])
moreover have $\bigwedge x.$ *r* = *dist a x* $\Longrightarrow$ *f x* = *k* (*h* (*f x*))
by (*metis fim hk homeomorphism_def image_subset_iff mem_sphere image_subset_iff_funcset*)
ultimately have *homotopic_with_canon* ($\lambda z.$ *True*) (*sphere a r*) (*sphere b s*) *f* ($\lambda x.$ *k c*)
by (*auto intro*: *homotopic_with_eq*)
then show *?thesis*
by (*metis that*)
qed

9.27.15 Holomorphic logarithms and square roots

lemma *g_imp_holomorphic_log*:
assumes *holf*: *f holomorphic_on S*
and *contg*: *continuous_on S g* **and** *feq*: $\bigwedge x.$ *x* $\in$ *S* $\Longrightarrow$ *f x* = *exp* (*g x*)
and *fnz*: $\bigwedge z.$ *z* $\in$ *S* $\Longrightarrow$ *f z* $\neq$ *0*
obtains *g* **where** *g holomorphic_on S* $\bigwedge z.$ *z* $\in$ *S* $\Longrightarrow$ *f z* = *exp*(*g z*)
proof −
have *contf*: *continuous_on S f*
by (*simp add*: *holf holomorphic_on_imp_continuous_on*)
have *g field_differentiable at z within S* **if** *f field_differentiable at z within S z* $\in$ *S* **for** *z*
proof −
obtain *f′* **where** *f′*: (($\lambda y.$ (*f y* − *f z*) / (*y* − *z*)) $\longrightarrow$ *f′*) (*at z within S*)
using ‹*f field_differentiable at z within S*› **by** (*auto simp*: *field_differentiable_def has_field_derivative_iff*)
then have *ee*: (($\lambda x.$ (*exp*(*g x*) − *exp*(*g z*)) / (*x* − *z*)) $\longrightarrow$ *f′*) (*at z within S*)
by (*simp add*: *feq* ‹*z* $\in$ *S*› *Lim_transform_within* [*OF _ zero_less_one*])
have ((($\lambda y.$ *if y* = *g z then exp* (*g z*) *else* (*exp y* − *exp* (*g z*)) / (*y* − *g z*)) $\circ$ *g*) $\longrightarrow$ *exp* (*g z*)) (*at z within S*)
proof (*rule tendsto_compose_at*)
show (*g* $\longrightarrow$ *g z*) (*at z within S*)
using *contg continuous_on* ‹*z* $\in$ *S*› **by** *blast*

```
    show (λy. if y = g z then exp (g z) else (exp y − exp (g z)) / (y − g z)) −g
z→ exp (g z)
    by (simp add: LIM_offset_zero_iff DERIV_D cong: if_cong Lim_cong_within)
    qed auto
  then have dd: ((λx. if g x = g z then exp(g z) else (exp(g x) − exp(g z)) / (g
x − g z)) ⟶ exp(g z)) (at z within S)
    by (simp add: o_def)
  have continuous (at z within S) g
    using contg continuous_on_eq_continuous_within ‹z ∈ S› by blast
  then have (∀F x in at z within S. dist (g x) (g z) < 2∗pi)
    by (simp add: continuous_within tendsto_iff)
  then have ∀F x in at z within S. exp (g x) = exp (g z) ⟶ g x ≠ g z ⟶ x
= z
    by (rule eventually_mono) (auto simp: exp_eq dist_norm norm_mult)
  then have ((λy. (g y − g z) / (y − z)) ⟶ f' / exp (g z)) (at z within S)
    by (auto intro!: Lim_transform_eventually [OF tendsto_divide [OF ee dd]])
  then show ?thesis
    by (auto simp: field_differentiable_def has_field_derivative_iff)
 qed
 then have g holomorphic_on S
  using holf holomorphic_on_def by auto
 then show ?thesis
  using feq that by auto
qed

lemma contractible_imp_holomorphic_log:
 assumes holf: f holomorphic_on S
    and S: contractible S
    and fnz: ⋀z. z ∈ S ⟹ f z ≠ 0
 obtains g where g holomorphic_on S ⋀z. z ∈ S ⟹ f z = exp(g z)
proof −
 have contf: continuous_on S f
  by (simp add: holf holomorphic_on_imp_continuous_on)
 obtain g where contg: continuous_on S g and feq: ⋀x. x ∈ S ⟹ f x = exp (g
x)
  by (metis continuous_logarithm_on_contractible [OF contf S fnz])
 then show thesis
  using fnz g_imp_holomorphic_log holf that by blast
qed

lemma simply_connected_imp_holomorphic_log:
 assumes holf: f holomorphic_on S
    and S: simply_connected S locally path_connected S
    and fnz: ⋀z. z ∈ S ⟹ f z ≠ 0
 obtains g where g holomorphic_on S ⋀z. z ∈ S ⟹ f z = exp(g z)
proof −
 have contf: continuous_on S f
  by (simp add: holf holomorphic_on_imp_continuous_on)
 obtain g where contg: continuous_on S g and feq: ⋀x. x ∈ S ⟹ f x = exp (g
```

x)
  by (*metis continuous_logarithm_on_simply_connected* [*OF contf S fnz*])
 then show *thesis*
  using *fnz g_imp_holomorphic_log holf that* **by** *blast*
qed

lemma *contractible_imp_holomorphic_sqrt*:
 assumes *holf*: *f holomorphic_on S*
    and *S*: *contractible S*
    and *fnz*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
 obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = g\ z\ \widehat{}\ 2$
proof −
 obtain *g* **where** *holg*: *g holomorphic_on S* **and** *feq*: $\bigwedge z.\ z \in S \Longrightarrow f\ z = exp(g\ z)$
  using *contractible_imp_holomorphic_log* [*OF assms*] **by** *blast*
 show *?thesis*
 proof
  show $exp \circ (\lambda z.\ z\ /\ 2) \circ g$ *holomorphic_on S*
   by (*intro holomorphic_on_compose holg holomorphic_intros*) *auto*
  show $\bigwedge z.\ z \in S \Longrightarrow f\ z = ((exp \circ (\lambda z.\ z\ /\ 2) \circ g)\ z)^2$
   by (*simp add*: *feq flip*: *exp_double*)
 qed
qed

lemma *simply_connected_imp_holomorphic_sqrt*:
 assumes *holf*: *f holomorphic_on S*
    and *S*: *simply_connected S locally path_connected S*
    and *fnz*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 0$
 obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = g\ z\ \widehat{}\ 2$
proof −
 obtain *g* **where** *holg*: *g holomorphic_on S* **and** *feq*: $\bigwedge z.\ z \in S \Longrightarrow f\ z = exp(g\ z)$
  using *simply_connected_imp_holomorphic_log* [*OF assms*] **by** *blast*
 show *?thesis*
 proof
  show $exp \circ (\lambda z.\ z\ /\ 2) \circ g$ *holomorphic_on S*
   by (*intro holomorphic_on_compose holg holomorphic_intros*) *auto*
  show $\bigwedge z.\ z \in S \Longrightarrow f\ z = ((exp \circ (\lambda z.\ z\ /\ 2) \circ g)\ z)^2$
   by (*simp add*: *feq flip*: *exp_double*)
 qed
qed

Related theorems about holomorphic inverse cosines.

lemma *contractible_imp_holomorphic_arccos*:
 assumes *holf*: *f holomorphic_on S* **and** *S*: *contractible S*
    and *non1*: $\bigwedge z.\ z \in S \Longrightarrow f\ z \neq 1 \wedge f\ z \neq -1$
 obtains *g* **where** *g holomorphic_on S* $\bigwedge z.\ z \in S \Longrightarrow f\ z = cos(g\ z)$
proof −
 have *hol1f*: $(\lambda z.\ 1 - f\ z\ \widehat{}\ 2)$ *holomorphic_on S*

```
    by (intro holomorphic_intros holf)
  obtain g where holg: g holomorphic_on S and eq: ⋀z. z ∈ S ⟹ 1 - (f z)^2 =
(g z)^2
    using contractible_imp_holomorphic_sqrt [OF hol1f S]
    by (metis eq_iff_diff_eq_0 non1 power2_eq_1_iff)
  have holfg: (λz. f z + i*g z) holomorphic_on S
    by (intro holf holg holomorphic_intros)
  have ⋀z. z ∈ S ⟹ f z + i*g z ≠ 0
   by (metis Arccos_body_lemma eq add.commute add.inverse_unique complex_i_mult_minus
power2_csqrt power2_eq_iff)
  then obtain h where holh: h holomorphic_on S and fgeq: ⋀z. z ∈ S ⟹ f z
+ i*g z = exp (h z)
    using contractible_imp_holomorphic_log [OF holfg S] by metis
  show ?thesis
  proof
    show (λz. -i*h z) holomorphic_on S
      by (intro holh holomorphic_intros)
    show f z = cos (- i*h z) if z ∈ S for z
    proof -
      have (f z + i*g z)*(f z - i*g z) = 1
        using that eq by (auto simp: algebra_simps power2_eq_square)
      then have f z - i*g z = inverse (f z + i*g z)
        using inverse_unique by force
      also have ... = exp (- h z)
        by (simp add: exp_minus fgeq that)
      finally have f z = exp (- h z) + i*g z
        by (simp add: diff_eq_eq)
      with that show ?thesis
        by (simp add: cos_exp_eq flip: fgeq)
    qed
  qed
qed


lemma contractible_imp_holomorphic_arccos_bounded:
  assumes holf: f holomorphic_on S and S: contractible S and a ∈ S
      and non1: ⋀z. z ∈ S ⟹ f z ≠ 1 ∧ f z ≠ -1
  obtains g where g holomorphic_on S norm(g a) ≤ pi + norm(f a) ⋀z. z ∈ S
⟹ f z = cos(g z)
proof -
  obtain g where holg: g holomorphic_on S and feq: ⋀z. z ∈ S ⟹ f z = cos (g
z)
    using contractible_imp_holomorphic_arccos [OF holf S non1] by blast
  obtain b where cos b = f a norm b ≤ pi + norm (f a)
    using cos_Arccos norm_Arccos_bounded by blast
  then have cos b = cos (g a)
    by (simp add: ‹a ∈ S› feq)
  then consider n where n ∈ ℤ b = g a + of_real(2*n*pi) | n where n ∈ ℤ b
= -g a + of_real(2*n*pi)
```

```
    by (auto simp: complex_cos_eq)
  then show ?thesis
  proof cases
    case 1
    show ?thesis
    proof
      show (λz. g z + of_real(2*n*pi)) holomorphic_on S
        by (intro holomorphic_intros holg)
      show cmod (g a + of_real(2*n*pi)) ≤ pi + cmod (f a)
        using 1 ‹cmod b ≤ pi + cmod (f a)› by blast
      show ⋀z. z ∈ S ⟹ f z = cos (g z + complex_of_real (2*n*pi))
        by (metis ‹n ∈ ℤ› complex_cos_eq feq)
    qed
  next
    case 2
    show ?thesis
    proof
      show (λz. -g z + of_real(2*n*pi)) holomorphic_on S
        by (intro holomorphic_intros holg)
      show cmod (-g a + of_real(2*n*pi)) ≤ pi + cmod (f a)
        using 2 ‹cmod b ≤ pi + cmod (f a)› by blast
      show ⋀z. z ∈ S ⟹ f z = cos (-g z + complex_of_real (2*n*pi))
        by (metis ‹n ∈ ℤ› complex_cos_eq feq)
    qed
  qed
qed
```

9.27.16 The "Borsukian" property of sets

This doesn't have a standard name. Kuratowski uses "contractible with respect to $[S^1]$" while Whyburn uses "property b". It's closely related to unicoherence.

```
definition Borsukian where
  Borsukian S ≡
    ∀f. continuous_on S f ∧ f ∈ S → (- {0::complex})
      ⟶ (∃a. homotopic_with_canon (λh. True) S (- {0}) f (λx. a))

lemma Borsukian_retraction_gen:
  assumes Borsukian S continuous_on S h h ' S = T
      continuous_on T k  k ∈ T → S  ⋀y. y ∈ T ⟹ h(k y) = y
    shows Borsukian T
proof -
  interpret R: Retracts S h T k
    using assms by (simp add: image_subset_iff_funcset Retracts.intro)
  show ?thesis
    using assms
    apply (clarsimp simp add: Borsukian_def)
    apply (rule R.cohomotopically_trivial_retraction_null_gen [OF TrueI TrueI
refl, of -{0}], auto)
```

done
qed

lemma *retract_of_Borsukian*: ⟦*Borsukian T*; *S retract_of T*⟧ ⟹ *Borsukian S*
by (*smt* (*verit*) *Borsukian_retraction_gen retract_of_def retraction retraction_def retraction_subset image_subset_iff_funcset*)

lemma *homeomorphic_Borsukian*: ⟦*Borsukian S*; *S homeomorphic T*⟧ ⟹ *Borsukian T*
using *Borsukian_retraction_gen order_refl*
by (*fastforce simp add*: *homeomorphism_def homeomorphic_def*)

lemma *homeomorphic_Borsukian_eq*:
S homeomorphic T ⟹ *Borsukian S* ⟷ *Borsukian T*
by (*meson homeomorphic_Borsukian homeomorphic_sym*)

lemma *Borsukian_translation*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *Borsukian* (*image* (λ*x*. *a* + *x*) *S*) ⟷ *Borsukian S*
using *homeomorphic_Borsukian_eq homeomorphic_translation* **by** *blast*

lemma *Borsukian_injective_linear_image*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *linear f inj f*
shows *Borsukian*(*f* ‘ *S*) ⟷ *Borsukian S*
using *assms homeomorphic_Borsukian_eq linear_homeomorphic_image* **by** *blast*

lemma *homotopy_eqv_Borsukianness*:
fixes *S* :: *'a*::*real_normed_vector set*
and *T* :: *'b*::*real_normed_vector set*
assumes *S homotopy_eqv T*
shows (*Borsukian S* ⟷ *Borsukian T*)
by (*meson Borsukian_def assms homotopy_eqv_cohomotopic_triviality_null image_subset_iff_funcset*)

lemma *Borsukian_alt*:
fixes *S* :: *'a*::*real_normed_vector set*
shows
Borsukian S ⟷
(∀ *f g*. *continuous_on S f* ∧ *f* ∈ *S* → −{*0*} ∧
continuous_on S g ∧ *g* ∈ *S* → −{*0*}
⟶ *homotopic_with_canon* (λ*h*. *True*) *S* (− {*0*::*complex*}) *f g*)
unfolding *Borsukian_def homotopic_triviality*
by (*force simp add*: *path_connected_punctured_universe*)

lemma *Borsukian_continuous_logarithm*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *Borsukian S* ⟷
(∀ *f*. *continuous_on S f* ∧ *f* ∈ *S* → (− {*0*::*complex*})

```
                ⟶ (∃g. continuous_on S g ∧ (∀x ∈ S. f x = exp(g x))))
  by (simp add: Borsukian_def inessential_eq_continuous_logarithm)

lemma Borsukian_continuous_logarithm_circle:
  fixes S :: 'a::real_normed_vector set
  shows Borsukian S ⟷
          (∀f. continuous_on S f ∧ f ∈ S → sphere (0::complex) 1
              ⟶ (∃g. continuous_on S g ∧ (∀x ∈ S. f x = exp(g x))))
   (is ?lhs = ?rhs)
proof
  assume ?lhs then show ?rhs
    by (force simp: Borsukian_continuous_logarithm)
next
  assume RHS [rule_format]: ?rhs
  show ?lhs
  proof (clarsimp simp: Borsukian_continuous_logarithm Pi_iff)
    fix f :: 'a ⇒ complex
    assume contf: continuous_on S f and 0: ∀i∈S. f i ≠ 0
    then have continuous_on S (λx. f x / complex_of_real (cmod (f x)))
      by (intro continuous_intros) auto
    moreover have (λx. f x / complex_of_real (cmod (f x))) ∈ S → sphere 0 1
      using 0 by (auto simp: norm_divide)
    ultimately obtain g where contg: continuous_on S g
                  and fg: ∀x ∈ S. f x / complex_of_real (cmod (f x)) = exp(g x)
      using RHS [of λx. f x / of_real(norm(f x))] by auto
    show ∃g. continuous_on S g ∧ (∀x∈S. f x = exp (g x))
    proof (intro exI ballI conjI)
      show continuous_on S (λx. (Ln ∘ of_real ∘ norm ∘ f)x + g x)
        by (intro continuous_intros contf contg conjI) (use 0 in auto)
      show f x = exp ((Ln ∘ complex_of_real ∘ cmod ∘ f) x + g x) if x ∈ S for x
        using 0 that
        apply (simp add: exp_add)
      by (metis div_by_0 exp_Ln exp_not_eq_zero fg mult.commute nonzero_eq_divide_eq)
    qed
  qed
qed


lemma Borsukian_continuous_logarithm_circle_real:
  fixes S :: 'a::real_normed_vector set
  shows Borsukian S ⟷
        (∀f. continuous_on S f ∧ f ∈ S → sphere (0::complex) 1
              ⟶ (∃g. continuous_on S (complex_of_real ∘ g) ∧ (∀x ∈ S. f x =
exp(i * of_real(g x)))))
   (is ?lhs = ?rhs)
proof
  assume LHS: ?lhs
  show ?rhs
  proof (clarify)
```

fix $f :: 'a \Rightarrow complex$
assume *continuous_on S f* **and** *f01*: $f \in S \to sphere\ 0\ 1$
then obtain g **where** *contg*: *continuous_on S g* **and** $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$
using *LHS* **by** (*auto simp*: *Borsukian_continuous_logarithm_circle*)
then have $\forall x{\in}S.\ f\ x = exp\ (\mathrm{i} * complex\_of\_real\ ((Im \circ g)\ x))$
using *f01 exp_eq_polar norm_exp_eq_Re* **by** (*fastforce simp*: *Pi_iff*)
then show $\exists g.\ continuous\_on\ S\ (complex\_of\_real \circ g) \wedge (\forall x{\in}S.\ f\ x = exp\ (\mathrm{i} * complex\_of\_real\ (g\ x)))$
by (*rule_tac x*=$Im \circ g$ **in** *exI*) (*force intro*: *continuous_intros contg*)
qed
next
assume *RHS* [*rule_format*]: *?rhs*
show *?lhs*
proof (*clarsimp simp*: *Borsukian_continuous_logarithm_circle*)
fix $f :: 'a \Rightarrow complex$
assume *continuous_on S f* **and** *f01*: $f \in S \to sphere\ 0\ 1$
then obtain g **where** *contg*: *continuous_on S* $(complex\_of\_real \circ g)$ **and** $\bigwedge x.\ x \in S \Longrightarrow f\ x = \ exp(\mathrm{i} * of\_real(g\ x))$
by (*metis RHS*)
then show $\exists g.\ continuous\_on\ S\ g \wedge (\forall x{\in}S.\ f\ x = exp\ (g\ x))$
by (*rule_tac x*=$\lambda x.\ \mathrm{i}*\ of\_real(g\ x)$ **in** *exI*) (*auto simp*: *continuous_intros contg*)
qed
qed

lemma *Borsukian_circle*:
fixes $S :: 'a{::}real\_normed\_vector\ set$
shows *Borsukian S* $\longleftrightarrow$
$(\forall f.\ continuous\_on\ S\ f \wedge f \in S \to sphere\ (0{::}complex)\ 1 \longrightarrow (\exists a.\ homotopic\_with\_canon\ (\lambda h.\ True)\ S\ (sphere\ (0{::}complex)\ 1)\ f\ (\lambda x.\ a)))$
by (*simp add*: *inessential_eq_continuous_logarithm_circle Borsukian_continuous_logarithm_circle_re*

lemma *contractible_imp_Borsukian*: *contractible S* $\Longrightarrow$ *Borsukian S*
by (*meson Borsukian_def nullhomotopic_from_contractible image_subset_iff_funcset*)

lemma *simply_connected_imp_Borsukian*:
fixes $S :: 'a{::}real\_normed\_vector\ set$
shows $[\![simply\_connected\ S; locally\ path\_connected\ S]\!] \Longrightarrow Borsukian\ S$
by (*smt* (*verit*, *del_insts*) *continuous_logarithm_on_simply_connected Borsukian_continuous_logarith PiE mem_sphere_0 norm_eq_zero zero_neq_one*)

lemma *starlike_imp_Borsukian*:
fixes $S :: 'a{::}real\_normed\_vector\ set$
shows *starlike S* $\Longrightarrow$ *Borsukian S*
by (*simp add*: *contractible_imp_Borsukian starlike_imp_contractible*)

lemma *Borsukian_empty*: *Borsukian* $\{\}$

by (*auto simp*: *contractible_imp_Borsukian*)

lemma *Borsukian_UNIV*: *Borsukian* (*UNIV* :: *'a*::*real_normed_vector set*)
by (*auto simp*: *contractible_imp_Borsukian*)

lemma *convex_imp_Borsukian*:
fixes *S* :: *'a*::*real_normed_vector set*
shows *convex S* ⟹ *Borsukian S*
by (*meson Borsukian_def convex_imp_contractible nullhomotopic_from_contractible*)

proposition *Borsukian_sphere*:
fixes *a* :: *'a*::*euclidean_space*
shows *3* ≤ *DIM*(*'a*) ⟹ *Borsukian* (*sphere a r*)
using *ENR_sphere*
by (*blast intro*: *simply_connected_imp_Borsukian ENR_imp_locally_path_connected simply_connected_sphere*)

lemma *Borsukian_Un_lemma*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *BS*: *Borsukian S* **and** *BT*: *Borsukian T* **and** *ST*: *connected*(*S* ∩ *T*)
and ∗: ⋀*f g*::*'a* ⇒ *complex*.
⟦*continuous_on S f*; *continuous_on T g*; ⋀*x*. *x* ∈ *S* ∧ *x* ∈ *T* ⟹ *f x* = *g x*⟧
⟹ *continuous_on* (*S* ∪ *T*) (λ*x*. *if x* ∈ *S then f x else g x*)
shows *Borsukian*(*S* ∪ *T*)
proof (*clarsimp simp add*: *Borsukian_continuous_logarithm Pi_iff*)
fix *f* :: *'a* ⇒ *complex*
assume *contf*: *continuous_on* (*S* ∪ *T*) *f* **and** *0*: ∀ *i*∈*S* ∪ *T*. *f i* ≠ *0*
then have *contfS*: *continuous_on S f* **and** *contfT*: *continuous_on T f*
using *continuous_on_subset* **by** *auto*
have ⟦*continuous_on S f*; *f* ∈ *S* → −{*0*}⟧ ⟹ ∃ *g*. *continuous_on S g* ∧ (∀ *x* ∈ *S*. *f x* = *exp*(*g x*))
using *BS* **by** (*auto simp*: *Borsukian_continuous_logarithm*)
then obtain *g* **where** *contg*: *continuous_on S g* **and** *fg*: ⋀*x*. *x* ∈ *S* ⟹ *f x* = *exp*(*g x*)
using *0 contfS* **by** *force*
have ⟦*continuous_on T f*; *f* ∈ *T* → −{*0*}⟧ ⟹ ∃ *g*. *continuous_on T g* ∧ (∀ *x* ∈ *T*. *f x* = *exp*(*g x*))
using *BT* **by** (*auto simp*: *Borsukian_continuous_logarithm*)
then obtain *h* **where** *conth*: *continuous_on T h* **and** *fh*: ⋀*x*. *x* ∈ *T* ⟹ *f x* = *exp*(*h x*)
using *0 contfT* **by** *force*
show ∃ *g*. *continuous_on* (*S* ∪ *T*) *g* ∧ (∀ *x*∈*S* ∪ *T*. *f x* = *exp* (*g x*))
proof (*cases S* ∩ *T* = {})
case *True*
show *?thesis*
proof (*intro exI conjI*)
show *continuous_on* (*S* ∪ *T*) (λ*x*. *if x* ∈ *S then g x else h x*)
using *True* ∗ [*OF contg conth*]

```
      by (meson disjoint_iff)
    show ∀x∈S ∪ T. f x = exp (if x ∈ S then g x else h x)
      using fg fh by auto
  qed
 next
  case False
  have (λx. g x − h x) constant_on S ∩ T
  proof (rule continuous_discrete_range_constant [OF ST])
    show continuous_on (S ∩ T) (λx. g x − h x)
      by (metis contg conth continuous_on_diff continuous_on_subset inf_le1
inf_le2)
    show ∃e>0. ∀y. y ∈ S ∩ T ∧ g y − h y ≠ g x − h x ⟶ e ≤ cmod (g y −
h y − (g x − h x))
        if x ∈ S ∩ T for x
    proof −
      have g y − g x = h y − h x
          if y ∈ S y ∈ T cmod (g y − g x − (h y − h x)) < 2 ∗ pi for y
      proof (rule exp_complex_eqI)
        have |Im (g y − g x) − Im (h y − h x)| ≤ cmod (g y − g x − (h y − h
x))
          by (metis abs_Im_le_cmod minus_complex.simps(2))
        then show |Im (g y − g x) − Im (h y − h x)| < 2 ∗ pi
          using that by linarith
        have exp (g x) = exp (h x) exp (g y) = exp (h y)
          using fg fh that ‹x ∈ S ∩ T› by fastforce+
        then show exp (g y − g x) = exp (h y − h x)
          by (simp add: exp_diff)
      qed
      then show ?thesis
        by (rule_tac x=2∗pi in exI) (fastforce simp add: algebra_simps)
    qed
  qed
  then obtain a where a: ⋀x. x ∈ S ∩ T ⟹ g x − h x = a
    by (auto simp: constant_on_def)
  with False have exp a = 1
   by (metis IntI disjoint_iff_not_equal divide_self_if exp_diff exp_not_eq_zero
fg fh)
  with a show ?thesis
    apply (rule_tac x=λx. if x ∈ S then g x else a + h x in exI)
    apply (intro ∗ contg conth continuous_intros conjI)
     apply (auto simp: algebra_simps fg fh exp_add)
    done
 qed
qed

proposition Borsukian_open_Un:
 fixes S :: 'a::real_normed_vector set
 assumes opeS: openin (top_of_set (S ∪ T)) S
    and opeT: openin (top_of_set (S ∪ T)) T
```

and *BS*: *Borsukian S* **and** *BT*: *Borsukian T* **and** *ST*: *connected*$(S \cap T)$
shows *Borsukian*$(S \cup T)$
by (*force intro*: *Borsukian_Un_lemma* [*OF BS BT ST*] *continuous_on_cases_local_open* [*OF opeS opeT*])

lemma *Borsukian_closed_Un*:
fixes S :: '*a*::*real_normed_vector set*
assumes *cloS*: *closedin* (*top_of_set* $(S \cup T)$) S
and *cloT*: *closedin* (*top_of_set* $(S \cup T)$) T
and *BS*: *Borsukian S* **and** *BT*: *Borsukian T* **and** *ST*: *connected*$(S \cap T)$
shows *Borsukian*$(S \cup T)$
by (*force intro*: *Borsukian_Un_lemma* [*OF BS BT ST*] *continuous_on_cases_local* [*OF cloS cloT*])

lemma *Borsukian_separation_compact*:
fixes S :: *complex set*
assumes *compact S*
shows *Borsukian S* $\longleftrightarrow$ *connected*$(-S)$
by (*simp add*: *Borsuk_separation_theorem Borsukian_circle assms*)

lemma *Borsukian_monotone_image_compact*:
fixes f :: '*a*::*euclidean_space* $\Rightarrow$ '*b*::*euclidean_space*
assumes *Borsukian S* **and** *contf*: *continuous_on S f* **and** *fim*: f ‘ $S = T$
and *compact S* **and** *conn*: $\bigwedge y.\ y \in T \Longrightarrow$ *connected* $\{x.\ x \in S \wedge f\,x = y\}$
shows *Borsukian T*
proof (*clarsimp simp*: *Borsukian_continuous_logarithm Pi_iff*)
fix g :: '*b* $\Rightarrow$ *complex*
assume *contg*: *continuous_on T g* **and** *0*: $\forall\, i{\in}T.\ g\ i \neq 0$
have *continuous_on S* $(g \circ f)$
using *contf contg continuous_on_compose fim* **by** *blast*
moreover have $(g \circ f)$ ‘ $S \subseteq -\{0\}$
using *fim 0* **by** *auto*
ultimately obtain h **where** *conth*: *continuous_on S h* **and** *gfh*: $\bigwedge x.\ x \in S \Longrightarrow (g \circ f)\ x = \exp(h\ x)$
using ‹*Borsukian S*› **by** (*auto simp*: *Borsukian_continuous_logarithm*)
have $\bigwedge y.\ \exists\, x.\ y \in T \longrightarrow x \in S \wedge f\,x = y$
using *fim* **by** *auto*
then obtain f' **where** f': $\bigwedge y.\ y \in T \longrightarrow f'\ y \in S \wedge f\ (f'\ y) = y$
by *metis*
have $*$: $(\lambda x.\ h\ x - h(f'\ y))$ *constant_on* $\{x.\ x \in S \wedge f\,x = y\}$ **if** $y \in T$ **for** y
proof (*rule continuous_discrete_range_constant* [*OF conn* [*OF that*], *of* $\lambda x.\ h\ x - h\ (f'\ y)$], *simp_all add*: *algebra_simps*)
show *continuous_on* $\{x \in S.\ f\,x = y\}$ $(\lambda x.\ h\ x - h\ (f'\ y))$
by (*intro continuous_intros continuous_on_subset* [*OF conth*]) *auto*
show $\exists\, e{>}0.\ \forall u.\ u \in S \wedge f\,u = y \wedge h\ u \neq h\ x \longrightarrow e \leq \mathit{cmod}\ (h\ u - h\ x)$
if x: $x \in S \wedge f\,x = y$ **for** x
proof −
have $h\ u = h\ x$ **if** $u \in S\ f\,u = y$ *cmod* $(h\ u - h\ x) < 2 * \mathit{pi}$ **for** u
proof (*rule exp_complex_eqI*)

```
      have |Im (h u) − Im (h x)| ≤ cmod (h u − h x)
        by (metis abs_Im_le_cmod minus_complex.simps(2))
      then show |Im (h u) − Im (h x)| < 2 * pi
        using that by linarith
      show exp (h u) = exp (h x)
        by (simp add: gfh [symmetric] x that)
    qed
    then show ?thesis
      by (rule_tac x=2*pi in exI) (fastforce simp add: algebra_simps)
  qed
 qed
 show ∃h. continuous_on T h ∧ (∀x∈T. g x = exp (h x))
 proof (intro exI conjI)
  show continuous_on T (h ∘ f')
  proof (rule continuous_from_closed_graph [of h ‘ S])
   show compact (h ‘ S)
    by (simp add: ‹compact S› compact_continuous_image conth)
   show (h ∘ f') ∈ T → h ‘ S
    by (auto simp: f')
   have h x = h (f' (f x)) if x ∈ S for x
    using * [of f x] fim that unfolding constant_on_def by clarsimp (metis f'
imageI right_minus_eq)
   moreover have ⋀x. x ∈ T ⟹ ∃u. u ∈ S ∧ x = f u ∧ h (f' x) = h u
    using f' by fastforce
   ultimately
   have eq: ((λx. (x, (h ∘ f') x)) ‘ T) =
          {p. ∃x. x ∈ S ∧ (x, p) ∈ (S × UNIV) ∩ ((λz. snd z − ((f ∘ fst) z,
(h ∘ fst) z)) −‘ {0})}
    using fim by (auto simp: image_iff)
   moreover have closed ...
   apply (intro closed_compact_projection [OF ‹compact S›] continuous_closed_preimage
                continuous_intros continuous_on_subset [OF contf] continu-
ous_on_subset [OF conth])
    by (auto simp: ‹compact S› closed_Times compact_imp_closed)
   ultimately show closed ((λx. (x, (h ∘ f') x)) ‘ T)
    by simp
  qed
 qed (use f' gfh in fastforce)
qed


lemma Borsukian_open_map_image_compact:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes Borsukian S and contf: continuous_on S f and fim: f ‘ S = T and
compact S
   and ope: ⋀U. openin (top_of_set S) U
            ⟹ openin (top_of_set T) (f ‘ U)
  shows Borsukian T
proof (clarsimp simp add: Borsukian_continuous_logarithm_circle_real)
```

fix $g :: 'b \Rightarrow complex$
assume *contg*: *continuous_on* T g **and** *gim*: $g \in T \to sphere\ 0\ 1$
have *continuous_on* S $(g \circ f)$
using *contf contg continuous_on_compose fim* **by** *blast*
moreover have $(g \circ f) \in S \to sphere\ 0\ 1$
using *fim gim* **by** *auto*
ultimately obtain h **where** *cont_cxh*: *continuous_on* S $(complex\_of\_real \circ h)$
and *gfh*: $\bigwedge x.\ x \in S \Longrightarrow (g \circ f)\ x = exp(\mathrm{i} * of\_real(h\ x))$
using ‹*Borsukian S*› *Borsukian_continuous_logarithm_circle_real* **by** *metis*
then have *conth*: *continuous_on* S h
by *simp*
have $\exists x.\ x \in S \wedge f\ x = y \wedge (\forall x' \in S.\ f\ x' = y \longrightarrow h\ x \le h\ x')$ **if** $y \in T$ **for** y
proof −
have *1*: *compact* $(h\ `\ \{x \in S.\ f\ x = y\})$
proof (*rule compact_continuous_image*)
show *continuous_on* $\{x \in S.\ f\ x = y\}$ h
by (*rule continuous_on_subset* [*OF conth*]) *auto*
have *compact* $(S \cap f\ -`\ \{y\})$
using *that proper_map_from_compact* [*OF contf _* ‹*compact S*›] *fim*
by *force*
then show *compact* $\{x \in S.\ f\ x = y\}$
by (*auto simp*: *vimage_def Int_def*)
qed
have *2*: $h\ `\ \{x \in S.\ f\ x = y\} \neq \{\}$
using *fim that* **by** *auto*
have $\exists s \in h\ `\ \{x \in S.\ f\ x = y\}.\ \forall t \in h\ `\ \{x \in S.\ f\ x = y\}.\ s \le t$
using *compact_attains_inf* [*OF 1 2*] **by** *blast*
then show *?thesis* **by** *auto*
qed
then obtain k **where** *kTS*: $\bigwedge y.\ y \in T \Longrightarrow k\ y \in S$
and *fk*: $\bigwedge y.\ y \in T \Longrightarrow f\ (k\ y) = y$
and *hle*: $\bigwedge x'\ y.\ [\![y \in T;\ x' \in S;\ f\ x' = y]\!] \Longrightarrow h\ (k\ y) \le h\ x'$
by *metis*
have *continuous_on* T $(h \circ k)$
proof (*clarsimp simp add*: *continuous_on_iff*)
fix y **and** e::*real*
assume $y \in T\ 0 < e$
moreover have *uniformly_continuous_on* S $(complex\_of\_real \circ h)$
using ‹*compact S*› *cont_cxh compact_uniformly_continuous* **by** *blast*
ultimately obtain d **where** $0 < d$
and d: $\bigwedge x\ x'.\ [\![x \in S;\ x' \in S;\ dist\ x'\ x < d]\!] \Longrightarrow dist\ (h\ x')\ (h\ x) < e$
by (*force simp*: *uniformly_continuous_on_def*)
obtain δ **where** $0 < \delta$ **and** δ:
$\bigwedge x'.\ [\![x' \in T;\ dist\ y\ x' < \delta]\!]$
$\Longrightarrow (\forall v \in \{z \in S.\ f\ z = y\}.\ \exists v'.\ v' \in \{z \in S.\ f\ z = x'\} \wedge dist\ v\ v' < d) \wedge$
$(\forall v' \in \{z \in S.\ f\ z = x'\}.\ \exists v.\ v \in \{z \in S.\ f\ z = y\} \wedge dist\ v'\ v < d)$
proof (*rule upper_lower_hemicontinuous_explicit* [*of T* $\lambda y.\ \{z \in S.\ f\ z = y\}$ *S*])

```
    show ⋀U. openin (top_of_set S) U
           ⟹ openin (top_of_set T) {x ∈ T. {z ∈ S. f z = x} ⊆ U}
     using closed_map_iff_upper_hemicontinuous_preimage [of f S T] fim contf
‹compact S›
      using Abstract_Topology_2.continuous_imp_closed_map by blast
    show ⋀U. closedin (top_of_set S) U ⟹
           closedin (top_of_set T) {x ∈ T. {z ∈ S. f z = x} ⊆ U}
      using ope open_map_iff_lower_hemicontinuous_preimage[of f S T] fim
[THEN equalityD1]
      by blast
    show bounded {z ∈ S. f z = y}
      by (metis (no_types, lifting) compact_imp_bounded [OF ‹compact S›]
bounded_subset mem_Collect_eq subsetI)
  qed (use ‹y ∈ T› ‹0 < d› fk kTS in ‹force+›)
  have dist (h (k y')) (h (k y)) < e if y' ∈ T dist y y' < δ for y'
  proof −
    have k1: k y ∈ S f (k y) = y and k2: k y' ∈ S f (k y') = y'
      by (auto simp: ‹y ∈ T› ‹y' ∈ T› kTS fk)
    have 1: ⋀v. ⟦v ∈ S; f v = y⟧ ⟹ ∃v'. v' ∈ {z ∈ S. f z = y'} ∧ dist v v' < d
     and 2: ⋀v'. ⟦v' ∈ S; f v' = y'⟧ ⟹ ∃v. v ∈ {z ∈ S. f z = y} ∧ dist v' v < d
      using δ [OF that] by auto
    then obtain w' w where w' ∈ S f w' = y' dist (k y) w' < d
      and w ∈ S f w = y dist (k y') w < d
      using 1 [OF k1] 2 [OF k2] by auto
    then show ?thesis
      using d [of w k y'] d [of w' k y] k1 k2 ‹y' ∈ T› ‹y ∈ T› hle
      by (fastforce simp: dist_norm abs_diff_less_iff algebra_simps)
  qed
  then show ∃d>0. ∀x'∈T. dist x' y < d ⟶ dist (h (k x')) (h (k y)) < e
    using ‹0 < δ› by (auto simp: dist_commute)
 qed
 then show ∃h. continuous_on T h ∧ (∀x∈T. g x = exp (i ∗ complex_of_real
(h x)))
  using fk gfh kTS by force
qed
```

If two points are separated by a closed set, there's a minimal one.

```
proposition closed_irreducible_separator:
 fixes a :: 'a::real_normed_vector
 assumes closed S and ab: ¬ connected_component (− S) a b
 obtains T where T ⊆ S closed T T ≠ {} ¬ connected_component (− T) a b
                ⋀U. U ⊂ T ⟹ connected_component (− U) a b
proof (cases a ∈ S ∨ b ∈ S)
 case True
 then show ?thesis
 proof
   assume ∗: a ∈ S
   show ?thesis
   proof
```

```
      show {a} ⊆ S
        using ∗ by blast
      show ¬ connected_component (− {a}) a b
        using connected_component_in by auto
      show ⋀U. U ⊂ {a} ⟹ connected_component (− U) a b
      by (metis connected_component_UNIV UNIV_I compl_bot_eq connected_component_eq_eq
less_le_not_le subset_singletonD)
    qed auto
  next
    assume ∗: b ∈ S
    show ?thesis
    proof
      show {b} ⊆ S
        using ∗ by blast
      show ¬ connected_component (− {b}) a b
        using connected_component_in by auto
      show ⋀U. U ⊂ {b} ⟹ connected_component (− U) a b
      by (metis connected_component_UNIV UNIV_I compl_bot_eq connected_component_eq_eq
less_le_not_le subset_singletonD)
    qed auto
  qed
next
  case False
  define A where A ≡ connected_component_set (− S) a
  define B where B ≡ connected_component_set (− (closure A)) b
  have a ∈ A
    using False A_def by auto
  have b ∈ B
    unfolding A_def B_def closure_Un_frontier
   using ab False ‹closed S› frontier_complement frontier_of_connected_component_subset
frontier_subset_closed by force
  have frontier B ⊆ frontier (connected_component_set (− closure A) b)
    using B_def by blast
  also have frsub: ... ⊆ frontier A
  proof −
    have ⋀A. closure (− closure (− A)) ⊆ closure A
      by (metis (no_types) closure_mono closure_subset compl_le_compl_iff dou-
ble_compl)
    then show ?thesis
      by (metis (no_types) closure_closure double_compl frontier_closures fron-
tier_of_connected_component_subset le_inf_iff subset_trans)
  qed
  finally have frBA: frontier B ⊆ frontier A .
  show ?thesis
  proof
    show frontier B ⊆ S
    proof −
      have frontier S ⊆ S
        by (simp add: ‹closed S› frontier_subset_closed)
```

```
    then show ?thesis
     using frsub frontier_complement frontier_of_connected_component_subset
      unfolding A_def B_def by blast
   qed
   show closed (frontier B)
    by simp
   show ¬ connected_component (− frontier B) a b
    unfolding connected_component_def
   proof clarify
    fix T
    assume connected T and TB: T ⊆ − frontier B and a ∈ T and b ∈ T
    have a ∉ B
       by (metis A_def B_def ComplD ‹a ∈ A› assms(1) closed_open con-
nected_component_subset in_closure_connected_component subsetD)
    have T ∩ B ≠ {}
     using ‹b ∈ B› ‹b ∈ T› by blast
    moreover have T − B ≠ {}
     using ‹a ∉ B› ‹a ∈ T› by blast
    ultimately show False
     using connected_Int_frontier [of T B] TB ‹connected T› by blast
   qed
   moreover have connected_component (− frontier B) a b if frontier B = {}
    using connected_component_eq_UNIV that by auto
   ultimately show frontier B ≠ {}
    by blast
   show connected_component (− U) a b if U ⊂ frontier B for U
   proof −
    obtain p where Usub: U ⊆ frontier B and p: p ∈ frontier B p ∉ U
     using ‹U ⊂ frontier B› by blast
    show ?thesis
     unfolding connected_component_def
    proof (intro exI conjI)
     have connected ((insert p A) ∪ (insert p B))
     proof (rule connected_Un)
      show connected (insert p A)
       by (metis A_def IntD1 frBA ‹p ∈ frontier B› closure_insert closure_subset
connected_connected_component connected_intermediate_closure frontier_closures
insert_absorb subsetCE subset_insertI)
      show connected (insert p B)
       by (metis B_def IntD1 ‹p ∈ frontier B› closure_insert closure_subset con-
nected_connected_component connected_intermediate_closure frontier_closures in-
sert_absorb subset_insertI)
     qed blast
     then show connected (insert p (B ∪ A))
      by (simp add: sup.commute)
     have A ⊆ − U
        using A_def Usub ‹frontier B ⊆ S› connected_component_subset by
fastforce
     moreover have B ⊆ − U
```

```
          using B_def Usub connected_component_subset frBA frontier_closures
by fastforce
        ultimately show insert p (B ∪ A) ⊆ - U
          using p by auto
      qed (auto simp: ‹a ∈ A› ‹b ∈ B›)
    qed
  qed
qed

lemma frontier_minimal_separating_closed_pointwise:
  fixes S :: 'a::real_normed_vector set
  assumes S: closed S a ∉ S and nconn: ¬ connected_component (- S) a b
      and conn: ⋀T. ⟦closed T; T ⊂ S⟧ ⟹ connected_component (- T) a b
    shows frontier(connected_component_set (- S) a) = S (is ?F = S)
proof -
  have ?F ⊆ S
    by (simp add: S componentsI frontier_of_components_closed_complement)
  moreover have False if ?F ⊂ S
  proof -
    have connected_component (- ?F) a b
      by (simp add: conn that)
    then obtain T where connected T T ⊆ -?F a ∈ T b ∈ T
      by (auto simp: connected_component_def)
    moreover have T ∩ ?F ≠ {}
    proof (rule connected_Int_frontier [OF ‹connected T›])
      show T ∩ connected_component_set (- S) a ≠ {}
        using ‹a ∉ S› ‹a ∈ T› by fastforce
      show T - connected_component_set (- S) a ≠ {}
        using ‹b ∈ T› nconn by blast
    qed
    ultimately show ?thesis
      by blast
  qed
  ultimately show ?thesis
    by blast
qed
```

9.27.17 Unicoherence (closed)

```
definition unicoherent where
  unicoherent U ≡
  ∀S T. connected S ∧ connected T ∧ S ∪ T = U ∧
        closedin (top_of_set U) S ∧ closedin (top_of_set U) T
        ⟶ connected (S ∩ T)

lemma unicoherentI [intro?]:
  assumes ⋀S T. ⟦connected S; connected T; U = S ∪ T; closedin (top_of_set
U) S; closedin (top_of_set U) T⟧
          ⟹ connected (S ∩ T)
```

shows *unicoherent U*
using *assms* **unfolding** *unicoherent_def* **by** *blast*

lemma *unicoherentD*:
assumes *unicoherent U connected S connected T U* = *S* ∪ *T closedin* (*top_of_set U*) *S closedin* (*top_of_set U*) *T*
shows *connected* (*S* ∩ *T*)
using *assms* **unfolding** *unicoherent_def* **by** *blast*

proposition *homeomorphic_unicoherent*:
assumes *ST*: *S homeomorphic T* **and** *S*: *unicoherent S*
shows *unicoherent T*
proof −
obtain *f g* **where** *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g* (*f x*) = *x* **and** *fim*: *T* = *f* ‘ *S* **and** *gfim*: *g* ‘ *f* ‘ *S* = *S*
and *contf*: *continuous_on S f* **and** *contg*: *continuous_on* (*f* ‘ *S*) *g*
using *ST* **by** (*auto simp*: *homeomorphic_def homeomorphism_def*)
show *?thesis*
proof
fix *U V*
assume *connected U connected V* **and** *T*: *T* = *U* ∪ *V*
and *cloU*: *closedin* (*top_of_set T*) *U*
and *cloV*: *closedin* (*top_of_set T*) *V*
have *f* ∈ (*g* ‘ *U* ∩ *g* ‘ *V*) → *U f* ∈ (*g* ‘ *U* ∩ *g* ‘ *V*) → *V*
using *gf fim T* **by** *auto* (*metis UnCI image_iff*)+
moreover have *U* ∩ *V* ⊆ *f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*)
using *gf fim* **by** (*force simp*: *image_iff T*)
ultimately have *U* ∩ *V* = *f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*) **by** *blast*
moreover have *connected* (*f* ‘ (*g* ‘ *U* ∩ *g* ‘ *V*))
proof (*rule connected_continuous_image*)
show *continuous_on* (*g* ‘ *U* ∩ *g* ‘ *V*) *f*
using *T fim gfim* **by** (*metis Un_upper1 contf continuous_on_subset image_mono inf_le1*)
show *connected* (*g* ‘ *U* ∩ *g* ‘ *V*)
proof (*intro conjI unicoherentD* [*OF S*])
show *connected* (*g* ‘ *U*) *connected* (*g* ‘ *V*)
using ‹*connected U*› *cloU* ‹*connected V*› *cloV*
by (*metis Topological_Spaces.connected_continuous_image closedin_imp_subset contg continuous_on_subset fim*)+
show *S* = *g* ‘ *U* ∪ *g* ‘ *V*
using *T fim gfim* **by** *auto*
have *hom*: *homeomorphism T S g f*
by (*simp add*: *contf contg fim gf gfim homeomorphism_def*)
have *closedin* (*top_of_set T*) *U closedin* (*top_of_set T*) *V*
by (*simp_all add*: *cloU cloV*)
then show *closedin* (*top_of_set S*) (*g* ‘ *U*)
closedin (*top_of_set S*) (*g* ‘ *V*)
by (*blast intro*: *homeomorphism_imp_closed_map* [*OF hom*])+
qed

```
    qed
    ultimately show connected (U ∩ V) by metis
  qed
qed


lemma homeomorphic_unicoherent_eq:
   S homeomorphic T ⟹ (unicoherent S ⟷ unicoherent T)
  by (meson homeomorphic_sym homeomorphic_unicoherent)

lemma unicoherent_translation:
  fixes S :: 'a::real_normed_vector set
  shows
   unicoherent (image (λx. a + x) S) ⟷ unicoherent S
  using homeomorphic_translation homeomorphic_unicoherent_eq by blast

lemma unicoherent_injective_linear_image:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f inj f
  shows (unicoherent(f ‘ S) ⟷ unicoherent S)
   using assms homeomorphic_unicoherent_eq linear_homeomorphic_image by
blast


lemma Borsukian_imp_unicoherent:
  fixes U :: 'a::euclidean_space set
  assumes Borsukian U  shows unicoherent U
  unfolding unicoherent_def
proof clarify
  fix S T
  assume connected S connected T U = S ∪ T
     and cloS: closedin (top_of_set (S ∪ T)) S
     and cloT: closedin (top_of_set (S ∪ T)) T
  show connected (S ∩ T)
    unfolding connected_closedin_eq
  proof clarify
    fix V W
    assume closedin (top_of_set (S ∩ T)) V
       and closedin (top_of_set (S ∩ T)) W
       and VW: V ∪ W = S ∩ T V ∩ W = {} and V ≠ {} W ≠ {}
    then have cloV: closedin (top_of_set U) V and cloW: closedin (top_of_set
U) W
      using ‹U = S ∪ T› cloS cloT closedin_trans by blast+
    obtain q where contq: continuous_on U q
        and q01: ⋀x. x ∈ U ⟹ q x ∈ {0..1::real}
        and qV: ⋀x. x ∈ V ⟹ q x = 0 and qW: ⋀x. x ∈ W ⟹ q x = 1
      by (rule Urysohn_local [OF cloV cloW ‹V ∩ W = {}›, of 0 1])
         (fastforce simp: closed_segment_eq_real_ivl)
    let ?h = λx. if x ∈ S then exp(pi * ii * q x) else 1 / exp(pi * ii * q x)
```

have *eqST*: *exp*(*pi* ∗ i ∗ *q x*) = *1* / *exp*(*pi* ∗ i ∗ *q x*) **if** *x* ∈ *S* ∩ *T* **for** *x*
proof −
have *x* ∈ *V* ∪ *W*
using *that* ‹*V* ∪ *W* = *S* ∩ *T*› **by** *blast*
with *qV qW* **show** *?thesis* **by** *force*
qed
obtain *g* **where** *contg*: *continuous_on U g*
and *circle*: *g* ∈ *U* → *sphere 0 1*
and *S*: ⋀*x*. *x* ∈ *S* ⟹ *g x* = *exp*(*pi* ∗ i ∗ *q x*)
and *T*: ⋀*x*. *x* ∈ *T* ⟹ *g x* = *1* / *exp*(*pi* ∗ i ∗ *q x*)
proof
show *continuous_on U ?h*
unfolding ‹*U* = *S* ∪ *T*›
proof (*rule continuous_on_cases_local* [*OF cloS cloT*])
show *continuous_on S* (λ*x*. *exp* (*pi* ∗ i ∗ *q x*))
proof (*intro continuous_intros*)
show *continuous_on S q*
using ‹*U* = *S* ∪ *T*› *continuous_on_subset contq* **by** *blast*
qed
show *continuous_on T* (λ*x*. *1* / *exp* (*pi* ∗ i ∗ *q x*))
proof (*intro continuous_intros*)
show *continuous_on T q*
using ‹*U* = *S* ∪ *T*› *continuous_on_subset contq* **by** *auto*
qed *auto*
qed (*use eqST* **in** *auto*)
qed (*use eqST* **in** ‹*auto simp*: *norm_divide*›)
then obtain *h* **where** *conth*: *continuous_on U h* **and** *heq*: ⋀*x*. *x* ∈ *U* ⟹ *g x* = *exp* (*h x*)
by (*metis Borsukian_continuous_logarithm_circle assms*)
obtain *v w* **where** *v* ∈ *V w* ∈ *W*
using ‹*V* ≠ {}› ‹*W* ≠ {}› **by** *blast*
then have *vw*: *v* ∈ *S* ∩ *T w* ∈ *S* ∩ *T*
using *VW* **by** *auto*
have *iff*: *2* ∗ *pi* ≤ *cmod* (*2* ∗ *of_int m* ∗ *of_real pi* ∗ i − *2* ∗ *of_int n* ∗ *of_real pi* ∗ i)
⟷ *1* ≤ *abs* (*m* − *n*) **for** *m n*
proof −
have *2* ∗ *pi* ≤ *cmod* (*2* ∗ *of_int m* ∗ *of_real pi* ∗ i − *2* ∗ *of_int n* ∗ *of_real pi* ∗ i)
⟷ *2* ∗ *pi* ≤ *cmod* ((*2* ∗ *pi* ∗ i) ∗ (*of_int m* − *of_int n*))
by (*simp add*: *algebra_simps*)
also have ... ⟷ *2* ∗ *pi* ≤ *2* ∗ *pi* ∗ *cmod* (*of_int m* − *of_int n*)
by (*simp add*: *norm_mult*)
also have ... ⟷ *1* ≤ *abs* (*m* − *n*)
by *simp* (*metis norm_of_int of_int_1_le_iff of_int_abs of_int_diff*)
finally show *?thesis* **.**
qed
have ∗: ∃ *n*::*int*. *h x* − (*pi* ∗ i ∗ *q x*) = (*of_int*(*2*∗*n*) ∗ *pi*) ∗ i **if** *x* ∈ *S* **for** *x*
using *that S* ‹*U* = *S* ∪ *T*› *heq exp_eq* [*symmetric*] **by** (*simp add*: *alge-*

```
bra_simps)
    moreover have (λx. h x − (pi ∗ i ∗ q x)) constant_on S
    proof (rule continuous_discrete_range_constant [OF ‹connected S›])
      have continuous_on S h continuous_on S q
        using ‹U = S ∪ T› continuous_on_subset conth contq by blast+
      then show continuous_on S (λx. h x − (pi ∗ i ∗ q x))
        by (intro continuous_intros)
      have 2∗pi ≤ cmod (h y − (pi ∗ i ∗ q y) − (h x − (pi ∗ i ∗ q x)))
        if x ∈ S y ∈ S and ne: h y − (pi ∗ i ∗ q y) ≠ h x − (pi ∗ i ∗ q x) for x y
        using ∗ [OF ‹x ∈ S›] ∗ [OF ‹y ∈ S›] ne by (auto simp: iff)
      then show ⋀x. x ∈ S ⟹
        ∃e>0. ∀y. y ∈ S ∧ h y − (pi ∗ i ∗ q y) ≠ h x − (pi ∗ i ∗ q x) ⟶
                e ≤ cmod (h y − (pi ∗ i ∗ q y) − (h x − (pi ∗ i ∗ q x)))
        by (rule_tac x=2∗pi in exI) auto
    qed
    ultimately
    obtain m where m: ⋀x. x ∈ S ⟹ h x − (pi ∗ i ∗ q x) = (of_int(2∗m) ∗ pi)
∗ i
      using vw by (force simp: constant_on_def)
    have ∗: ∃n::int. h x = − (pi ∗ i ∗ q x) + (of_int(2∗n) ∗ pi) ∗ i if x ∈ T for x
      unfolding exp_eq [symmetric]
        using that T ‹U = S ∪ T› by (simp add: exp_minus field_simps  heq
[symmetric])
    moreover have (λx. h x + (pi ∗ i ∗ q x)) constant_on T
    proof (rule continuous_discrete_range_constant [OF ‹connected T›])
      have continuous_on T h continuous_on T q
        using ‹U = S ∪ T› continuous_on_subset conth contq by blast+
      then show continuous_on T (λx. h x + (pi ∗ i ∗ q x))
        by (intro continuous_intros)
      have 2∗pi ≤ cmod (h y + (pi ∗ i ∗ q y) − (h x + (pi ∗ i ∗ q x)))
        if x ∈ T y ∈ T and ne: h y + (pi ∗ i ∗ q y) ≠ h x + (pi ∗ i ∗ q x) for x y
        using ∗ [OF ‹x ∈ T›] ∗ [OF ‹y ∈ T›] ne by (auto simp: iff)
      then show ⋀x. x ∈ T ⟹
        ∃e>0. ∀y. y ∈ T ∧ h y + (pi ∗ i ∗ q y) ≠ h x + (pi ∗ i ∗ q x) ⟶
                e ≤ cmod (h y + (pi ∗ i ∗ q y) − (h x + (pi ∗ i ∗ q x)))
        by (rule_tac x=2∗pi in exI) auto
    qed
    ultimately
    obtain n where n: ⋀x. x ∈ T ⟹ h x + (pi ∗ i ∗ q x) = (of_int(2∗n) ∗ pi)
∗ i
      using vw by (force simp: constant_on_def)
    show False
      using m [of v] m [of w] n [of v] n [of w] vw
      by (auto simp: algebra_simps ‹v ∈ V› ‹w ∈ W› qV qW)
  qed
qed


corollary contractible_imp_unicoherent:
```

fixes $U :: 'a::euclidean\_space\ set$
assumes *contractible U* **shows** *unicoherent U*
by (*simp add*: *Borsukian_imp_unicoherent assms contractible_imp_Borsukian*)

corollary *convex_imp_unicoherent*:
fixes $U :: 'a::euclidean\_space\ set$
assumes *convex U* **shows** *unicoherent U*
by (*simp add*: *Borsukian_imp_unicoherent assms convex_imp_Borsukian*)

If the type class constraint can be relaxed, I don't know how!

corollary *unicoherent_UNIV*: *unicoherent* (*UNIV* :: $'a$ :: *euclidean_space set*)
by (*simp add*: *convex_imp_unicoherent*)

lemma *unicoherent_monotone_image_compact*:
fixes $T :: 'b :: t2\_space\ set$
assumes *S*: *unicoherent S compact S* **and** *contf*: *continuous_on S f* **and** *fim*: *f ‘ S = T*
and *conn*: $\bigwedge y.\ y \in T \Longrightarrow$ *connected* $(S \cap f -` \{y\})$
shows *unicoherent T*
proof
fix *U V*
assume *UV*: *connected U connected V T = U ∪ V*
and *cloU*: *closedin* (*top_of_set T*) *U*
and *cloV*: *closedin* (*top_of_set T*) *V*
moreover have *compact T*
using ‹*compact S*› *compact_continuous_image contf fim* **by** *blast*
ultimately have *closed U closed V*
by (*auto simp*: *closedin_closed_eq compact_imp_closed*)
let *?SUV* = $(S \cap f -` U) \cap (S \cap f -` V)$
have *UV_eq*: *f ‘ ?SUV = U ∩ V*
using ‹*T = U ∪ V*› *fim* **by** *force+*
have *connected* (*f ‘ ?SUV*)
proof (*rule connected_continuous_image*)
show *continuous_on ?SUV f*
by (*meson contf continuous_on_subset inf_le1*)
show *connected ?SUV*
proof (*rule unicoherentD* [*OF* ‹*unicoherent S*›, *of S ∩ f -‘ U S ∩ f -‘ V*])
have $\bigwedge C.$ *closedin* (*top_of_set S*) *C* $\Longrightarrow$ *closedin* (*top_of_set T*) (*f ‘ C*)
by (*metis* ‹*compact S*› *closed_subset closedin_compact closedin_imp_subset compact_continuous_image compact_imp_closed contf continuous_on_subset fim image_mono*)
then show *connected* $(S \cap f -` U)$ *connected* $(S \cap f -` V)$
using *UV* **by** (*auto simp*: *conn intro*: *connected_closed_monotone_preimage* [*OF contf fim*])
show $S = (S \cap f -` U) \cup (S \cap f -` V)$
using *UV fim* **by** *blast*
show *closedin* (*top_of_set S*) $(S \cap f -` U)$
closedin (*top_of_set S*) $(S \cap f -` V)$

```
      by (auto simp: continuous_on_imp_closedin cloU cloV contf fim)
    qed
  qed
  with UV_eq show connected (U ∩ V)
    by simp
qed
```

9.27.18 Several common variants of unicoherence

```
lemma connected_frontier_simple:
  fixes S :: 'a :: euclidean_space set
  assumes connected S connected(− S) shows connected(frontier S)
  unfolding frontier_closures
  by (rule unicoherentD [OF unicoherent_UNIV]; simp add: assms connected_imp_connected_closure
flip: closure_Un)

lemma connected_frontier_component_complement:
  fixes S :: 'a :: euclidean_space set
  assumes connected S C ∈ components(− S) shows connected(frontier C)
  by (meson assms component_complement_connected connected_frontier_simple
in_components_connected)

lemma connected_frontier_disjoint:
  fixes S :: 'a :: euclidean_space set
  assumes connected S connected T disjnt S T and ST: frontier S ⊆ frontier T
  shows connected(frontier S)
proof (cases S = UNIV)
  case True then show ?thesis
    by simp
next
  case False
  then have −S ≠ {}
    by blast
  then obtain C where C: C ∈ components(− S) and T ⊆ C
    by (metis ComplI disjnt_iff subsetI exists_component_superset ‹disjnt S T›
‹connected T›)
  moreover have frontier S = frontier C
  proof −
    have frontier C ⊆ frontier S
      using C frontier_complement frontier_of_components_subset by blast
    moreover have x ∈ frontier C if x ∈ frontier S for x
    proof −
      have x ∈ closure C
        using that unfolding frontier_def
        by (metis (no_types) Diff_eq ST ‹T ⊆ C› closure_mono contra_subsetD
frontier_def le_inf_iff that)
      moreover have x ∉ interior C
        using that unfolding frontier_def
       by (metis C Compl_eq_Diff_UNIV Diff_iff subsetD in_components_subset
```

interior_diff interior_mono)
    ultimately show *?thesis*
      by (*auto simp*: *frontier_def*)
   qed
   ultimately show *?thesis*
    by *blast*
 qed
 ultimately show *?thesis*
   using ‹*connected S*› *connected_frontier_component_complement* **by** *auto*
qed

9.27.19 Some separation results

lemma *separation_by_component_closed_pointwise*:
 fixes *S* :: *'a* :: *euclidean_space set*
 assumes *closed S* ¬ *connected_component* (− *S*) *a b*
 obtains *C* **where** *C* ∈ *components S* ¬ *connected_component*(− *C*) *a b*
proof (*cases a* ∈ *S* ∨ *b* ∈ *S*)
 case *True*
 then show *?thesis*
   using *connected_component_in componentsI that* **by** *fastforce*
next
 case *False*
 obtain *T* **where** *T* ⊆ *S closed T T* ≠ {}
          and *nab*: ¬ *connected_component* (− *T*) *a b*
          and *conn*: ⋀*U*. *U* ⊂ *T* ⟹ *connected_component* (− *U*) *a b*
   using *closed_irreducible_separator* [*OF assms*] **by** *metis*
 moreover have *connected T*
 proof −
  have *ab*: *frontier*(*connected_component_set* (− *T*) *a*) = *T frontier*(*connected_component_set*
(− *T*) *b*) = *T*
     using *frontier_minimal_separating_closed_pointwise*
      by (*metis False* ‹*T* ⊆ *S*› ‹*closed T*› *connected_component_sym conn con-*
nected_component_eq_empty connected_component_intermediate_subset empty_subsetI
nab)+
   have *connected* (*frontier* (*connected_component_set* (− *T*) *a*))
   proof (*rule connected_frontier_disjoint*)
    show *disjnt* (*connected_component_set* (− *T*) *a*) (*connected_component_set*
(− *T*) *b*)
      unfolding *disjnt_iff*
       by (*metis connected_component_eq connected_component_eq_empty con-*
nected_component_idemp mem_Collect_eq nab)
   show *frontier* (*connected_component_set* (− *T*) *a*) ⊆ *frontier* (*connected_component_set*
(− *T*) *b*)
      by (*simp add*: *ab*)
   qed *auto*
   with *ab* ‹*closed T*› **show** *?thesis*
    by *simp*
 qed

```
  ultimately obtain C where C ∈ components S T ⊆ C
    using exists_component_superset [of T S] by blast
  then show ?thesis
    by (meson Compl_anti_mono connected_component_of_subset nab that)
qed


lemma separation_by_component_closed:
  fixes S :: 'a :: euclidean_space set
  assumes closed S ¬ connected(− S)
  obtains C where C ∈ components S ¬ connected(− C)
proof −
  obtain x y where closed S x ∉ S y ∉ S and ¬ connected_component (− S) x y
    using assms by (auto simp: connected_iff_connected_component)
  then obtain C where C ∈ components S ¬ connected_component(− C) x y
    using separation_by_component_closed_pointwise by metis
  then show thesis
   by (metis Compl_iff ‹x ∉ S› ‹y ∉ S› connected_component_eq_self in_components_subset
mem_Collect_eq subsetD that)
qed

lemma separation_by_Un_closed_pointwise:
  fixes S :: 'a :: euclidean_space set
  assumes ST: closed S closed T S ∩ T = {}
     and conS: connected_component (− S) a b and conT: connected_component
(− T) a b
    shows connected_component (− (S ∪ T)) a b
proof (rule ccontr)
  have a ∉ S b ∉ S a ∉ T b ∉ T
    using conS conT connected_component_in by auto
  assume ¬ connected_component (− (S ∪ T)) a b
 then obtain C where C ∈ components (S ∪ T) and C: ¬ connected_component(−
C) a b
    using separation_by_component_closed_pointwise assms by blast
  then have C ⊆ S ∨ C ⊆ T
  proof −
    have connected C C ⊆ S ∪ T
      using ‹C ∈ components (S ∪ T)› in_components_subset by (blast elim:
componentsE)+
    moreover then have C ∩ T = {} ∨ C ∩ S = {}
      by (metis Int_empty_right ST inf.commute connected_closed)
    ultimately show ?thesis
      by blast
  qed
  then show False
    by (meson Compl_anti_mono C conS conT connected_component_of_subset)
qed

lemma separation_by_Un_closed:
```

fixes $S :: 'a :: euclidean\_space\ set$
assumes ST: $closed\ S\ closed\ T\ S \cap T = \{\}$ **and** $conS$: $connected(-\ S)$ **and** $conT$: $connected(-\ T)$
shows $connected(-\ (S \cup T))$
using *assms separation_by_Un_closed_pointwise*
by (*fastforce simp add*: *connected_iff_connected_component*)

lemma *open_unicoherent_UNIV*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *open S open T connected S connected T* $S \cup T = UNIV$
shows $connected(S \cap T)$
proof −
have $connected(-\ (-S \cup -T))$
by (*metis closed_Compl compl_sup compl_top_eq double_compl separation_by_Un_closed assms*)
then show *?thesis*
by *simp*
qed

lemma *separation_by_component_open_aux*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes ST: $closed\ S\ closed\ T\ S \cap T = \{\}$
and $S \neq \{\}\ T \neq \{\}$
obtains C **where** $C \in components(-(S \cup T))\ C \neq \{\}$ *frontier* $C \cap S \neq \{\}$ *frontier* $C \cap T \neq \{\}$
proof (*rule ccontr*)
let *?S* $= S \cup \bigcup \{C \in components(-\ (S \cup T)).\ frontier\ C \subseteq S\}$
let *?T* $= T \cup \bigcup \{C \in components(-\ (S \cup T)).\ frontier\ C \subseteq T\}$
assume $\neg$ *thesis*
with *that* **have** $*$: *frontier* $C \cap S = \{\} \vee$ *frontier* $C \cap T = \{\}$
if C: $C \in components\ (-\ (S \cup T))\ C \neq \{\}$ **for** C
using C **by** *blast*
have $\exists A\ B{::}'a\ set.\ closed\ A \wedge closed\ B \wedge UNIV \subseteq A \cup B \wedge A \cap B = \{\} \wedge A \neq \{\} \wedge B \neq \{\}$
proof (*intro exI conjI*)
have *frontier* $(\bigcup \{C \in components\ (-\ S \cap -\ T).\ frontier\ C \subseteq S\}) \subseteq S$
using *subset_trans* [*OF frontier_Union_subset_closure*]
by (*metis* (*no_types*, *lifting*) *SUP_least* ‹*closed S*› *closure_minimal mem_Collect_eq*)
then have *frontier ?S* $\subseteq S$
by (*simp add*: *frontier_subset_eq assms subset_trans* [*OF frontier_Un_subset*])
then show *closed ?S*
using *frontier_subset_eq* **by** *fastforce*
have *frontier* $(\bigcup \{C \in components\ (-\ S \cap -\ T).\ frontier\ C \subseteq T\}) \subseteq T$
using *subset_trans* [*OF frontier_Union_subset_closure*]
by (*metis* (*no_types*, *lifting*) *SUP_least* ‹*closed T*› *closure_minimal mem_Collect_eq*)
then have *frontier ?T* $\subseteq T$
by (*simp add*: *frontier_subset_eq assms subset_trans* [*OF frontier_Un_subset*])
then show *closed ?T*
using *frontier_subset_eq* **by** *fastforce*

```
    have UNIV ⊆ (S ∪ T) ∪ ⋃(components(− (S ∪ T)))
      using Union_components by blast
    also have ... ⊆ ?S ∪ ?T
    proof −
      have C ∈ components (−(S ∪ T)) ∧ frontier C ⊆ S ∨
            C ∈ components (−(S ∪ T)) ∧ frontier C ⊆ T
        if C ∈ components (− (S ∪ T)) C ≠ {} for C
        using ∗ [OF that] that
        by clarify (metis (no_types, lifting) UnE ‹closed S› ‹closed T› closed_Un
disjoint_iff_not_equal frontier_of_components_closed_complement subsetCE)
      then show ?thesis
        by blast
    qed
    finally show UNIV ⊆ ?S ∪ ?T .
    have ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ S} ∪
         ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ T} ⊆ − (S ∪ T)
      using in_components_subset by fastforce
    moreover have ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ S} ∩
                  ⋃{C ∈ components (− (S ∪ T)). frontier C ⊆ T} = {}
    proof −
      have C ∩ C′ = {} if C ∈ components (− (S ∪ T)) frontier C ⊆ S
                          C′ ∈ components (− (S ∪ T)) frontier C′ ⊆ T for C C′
      proof −
        have NUN: − S ∩ − T ≠ UNIV
          using ‹T ≠ {}› by blast
        have C ≠ C′
        proof
          assume C = C′
          with that have frontier C′ ⊆ S ∩ T
            by simp
          also have ... = {}
            using ‹S ∩ T = {}› by blast
          finally have C′ = {} ∨ C′ = UNIV
            using frontier_eq_empty by auto
          then show False
        using ‹C = C′› NUN that by (force simp: dest: in_components_nonempty
in_components_subset)
        qed
        with that show ?thesis
          by (simp add: components_nonoverlap [of _ −(S ∪ T)])
      qed
      then show ?thesis
        by blast
    qed
    ultimately show ?S ∩ ?T = {}
      using ST by blast
    show ?S ≠ {} ?T ≠ {}
      using ‹S ≠ {}› ‹T ≠ {}› by blast+
  qed
```

then show *False*
by (*metis Compl_disjoint connected_UNIV compl_bot_eq compl_unique connected_closedD inf_sup_absorb sup_compl_top_left1 top.extremum_uniqueI*)
qed

proposition *separation_by_component_open*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *open S* **and** *non*: $\neg\ connected(-\ S)$
obtains C **where** $C \in components\ S\ \neg\ connected(-\ C)$
proof −
obtain $T\ U$
where *closed T closed U* **and** *TU*: $T \cup U = -\ S\ T \cap U = \{\}\ T \neq \{\}\ U \neq \{\}$
using *assms* **by** (*auto simp*: *connected_closed_set closed_def*)
then obtain C **where** C: $C \in components(-(T \cup U))\ C \neq \{\}$
and *frontier* $C \cap T \neq \{\}$ *frontier* $C \cap U \neq \{\}$
using *separation_by_component_open_aux* [*OF* ‹*closed T*› ‹*closed U*› ‹$T \cap U = \{\}$›] **by** *force*
show *thesis*
proof
show $C \in components\ S$
using *C*(*1*) *TU*(*1*) **by** *auto*
show $\neg\ connected\ (-\ C)$
proof
assume *connected* $(-\ C)$
then have *connected* (*frontier C*)
using *connected_frontier_simple* [*of C*] ‹$C \in components\ S$› *in_components_connected* **by** *blast*
then show *False*
unfolding *connected_closed*
by (*metis C*(*1*) *TU*(*2*) ‹*closed T*› ‹*closed U*› ‹*frontier* $C \cap T \neq \{\}$› ‹*frontier* $C \cap U \neq \{\}$› *closed_Un frontier_of_components_closed_complement inf_bot_right inf_commute*)
qed
qed
qed

lemma *separation_by_Un_open*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *open S open T* $S \cap T = \{\}$ **and** *cS*: *connected*$(-S)$ **and** *cT*: *connected*$(-T)$
shows *connected*$(-\ (S \cup T))$
using *assms unicoherent_UNIV* **unfolding** *unicoherent_def* **by** *force*

lemma *nonseparation_by_component_eq*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *open S* $\vee$ *closed S*

shows $((\forall\, C \in \mathit{components}\ S.\ \mathit{connected}(-C)) \longleftrightarrow \mathit{connected}(-\ S))$
by (*metis assms component_complement_connected double_complement separation_by_component_closed separation_by_component_open*)

Another interesting equivalent of an inessential mapping into C-0

proposition *inessential_eq_extensible*:
fixes $f :: 'a{::}\mathit{euclidean\_space} \Rightarrow \mathit{complex}$
assumes *closed S*
shows $(\exists\, a.\ \mathit{homotopic\_with\_canon}\ (\lambda h.\ \mathit{True})\ S\ (-\{0\})\ f\ (\lambda t.\ a)) \longleftrightarrow$
$(\exists\, g.\ \mathit{continuous\_on}\ \mathit{UNIV}\ g \wedge (\forall\, x \in S.\ g\ x = f\ x) \wedge (\forall\, x.\ g\ x \neq 0))$
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then obtain *a* **where** *a*: $\mathit{homotopic\_with\_canon}\ (\lambda h.\ \mathit{True})\ S\ (-\{0\})\ f\ (\lambda t.\ a)$
..
show *?rhs*
proof (*cases* $S = \{\}$)
case *True*
with *a* **show** *?thesis* **by** *force*
next
case *False*
have *anr*: $\mathit{ANR}\ (-\{0{::}\mathit{complex}\})$
by (*simp add*: *ANR_delete open_Compl open_imp_ANR*)
obtain *g* **where** *contg*: *continuous_on UNIV g* **and** *gim*: $g \in \mathit{UNIV} \to -\{0\}$
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof (*rule Borsuk_homotopy_extension_homotopic* [*OF _ _ continuous_on_const _ homotopic_with_symD* [*OF a*]])
show *closedin* (*top_of_set UNIV*) *S*
using *assms* **by** *auto*
show $(\lambda t.\ a) \in \mathit{UNIV} \to -\ \{0\}$
using *a homotopic_with_imp_subset2 False* **by** *blast*
qed (*use anr that* **in** ‹*force*+›)
then show *?thesis*
by *force*
qed
next
assume *?rhs*
then obtain *g* **where** *contg*: *continuous_on UNIV g*
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$ **and** *non0*: $\bigwedge x.\ g\ x \neq 0$
by *metis*
obtain *h k*::$'a \Rightarrow 'a$ **where** *hk*: *homeomorphism* (*ball 0 1*) *UNIV h k*
using *homeomorphic_ball01_UNIV homeomorphic_def* **by** *blast*
then have *continuous_on* (*ball 0 1*) $(g \circ h)$
by (*meson contg continuous_on_compose continuous_on_subset homeomorphism_cont1 top_greatest*)
then obtain *j* **where** *contj*: *continuous_on* (*ball 0 1*) *j*
and *j*: $\bigwedge z.\ z \in \mathit{ball}\ 0\ 1 \Longrightarrow \mathit{exp}(j\ z) = (g \circ h)\ z$
by (*metis* (*mono_tags*, *opaque_lifting*) *continuous_logarithm_on_ball comp_apply non0*)

```
  have [simp]: ⋀x. x ∈ S ⟹ h (k x) = x
    using hk homeomorphism_apply2 by blast
  have ∃ζ. continuous_on S ζ∧ (∀ x∈S. f x = exp (ζ x))
  proof (intro exI conjI ballI)
    show continuous_on S (j ∘ k)
    proof (rule continuous_on_compose)
      show continuous_on S k
        by (meson continuous_on_subset hk homeomorphism_cont2 top_greatest)
      show continuous_on (k ‘ S) j
         by (auto intro: continuous_on_subset [OF contj] simp flip: homeomorphism_image2 [OF hk])
    qed
    show f x = exp ((j ∘ k) x) if x ∈ S for x
      by (metis UNIV_I comp_apply gf hk homeomorphism_def image_eqI j that)
  qed
  then show ?lhs
    by (simp add: inessential_eq_continuous_logarithm)
qed

lemma inessential_on_clopen_Union:
  fixes ℱ :: 'a::euclidean_space set set
  assumes T: path_connected T
      and ⋀S. S ∈ ℱ ⟹ closedin (top_of_set (⋃ℱ)) S
      and ⋀S. S ∈ ℱ ⟹ openin (top_of_set (⋃ℱ)) S
     and hom: ⋀S. S ∈ ℱ ⟹ ∃ a. homotopic_with_canon (λx. True) S T f (λx. a)
  obtains a where homotopic_with_canon (λx. True) (⋃ℱ) T f (λx. a)
proof (cases ⋃ℱ = {})
  case True
  with that show ?thesis
    using homotopic_with_canon_on_empty by fastforce
next
  case False
  then obtain C where C ∈ ℱ C ≠ {}
    by blast
  then obtain a where clo: closedin (top_of_set (⋃ℱ)) C
    and ope: openin (top_of_set (⋃ℱ)) C
    and homotopic_with_canon (λx. True) C T f (λx. a)
    using assms by blast
  with ‹C ≠ {}› have f ∈ C → T a ∈ T
    using homotopic_with_imp_subset1 homotopic_with_imp_subset2 by blast+
  have homotopic_with_canon (λx. True) (⋃ℱ) T f (λx. a)
  proof (rule homotopic_on_clopen_Union)
    show ⋀S. S ∈ ℱ ⟹ closedin (top_of_set (⋃ℱ)) S
         ⋀S. S ∈ ℱ ⟹ openin (top_of_set (⋃ℱ)) S
      by (simp_all add: assms)
    show homotopic_with_canon (λx. True) S T f (λx. a) if S ∈ ℱ for S
    proof (cases S = {})
      case False
```

```
      then obtain b where b ∈ S
        by blast
      obtain c where c: homotopic_with_canon (λx. True) S T f (λx. c)
        using ‹S ∈ ℱ› hom by blast
      then have c ∈ T
        using ‹b ∈ S› homotopic_with_imp_subset2 by blast
      then have homotopic_with_canon (λx. True) S T (λx. a) (λx. c)
      using T ‹a ∈ T› by (simp add: homotopic_constant_maps path_connected_component)
      then show ?thesis
        using c homotopic_with_symD homotopic_with_trans by blast
    qed (simp add: homotopic_on_empty)
  qed
  then show ?thesis ..
qed

proposition Janiszewski_dual:
  fixes S :: complex set
  assumes compact S compact T connected S connected T connected(− (S ∪ T))
 shows connected(S ∩ T)
  by (meson Borsukian_imp_unicoherent Borsukian_separation_compact assms
closed_subset compact_Un
      compact_imp_closed sup_ge1 sup_ge2 unicoherentD)

end
```

9.28 The Jordan Curve Theorem and Applications

```
theory Jordan_Curve
  imports Arcwise_Connected Further_Topology
begin
```

9.28.1 Janiszewski's theorem

```
lemma Janiszewski_weak:
  fixes a b::complex
  assumes compact S compact T and conST: connected(S ∩ T)
      and ccS: connected_component (− S) a b and ccT: connected_component
(− T) a b
    shows connected_component (− (S ∪ T)) a b
proof −
  have [simp]: a ∉ S a ∉ T b ∉ S b ∉ T
    by (meson ComplD ccS ccT connected_component_in)+
  have clo: closedin (top_of_set (S ∪ T)) S closedin (top_of_set (S ∪ T)) T
    by (simp_all add: assms closed_subset compact_imp_closed)
  obtain g where contg: continuous_on S g
            and g: ⋀x. x ∈ S ⟹ exp (i* of_real (g x)) = (x − a) /R cmod (x −
a) / ((x − b) /R cmod (x − b))
    using ccS ‹compact S›
   apply (simp add: Borsuk_maps_homotopic_in_connected_component_eq [symmetric])
```

```
    apply (subst (asm) homotopic_circlemaps_divide)
    apply (auto simp: inessential_eq_continuous_logarithm_circle)
    done
  obtain h where conth: continuous_on T h
           and h: ⋀x. x ∈ T ⟹ exp (i* of_real (h x)) = (x − a) /R cmod (x −
a) / ((x − b) /R cmod (x − b))
    using ccT ‹compact T›
   apply (simp add: Borsuk_maps_homotopic_in_connected_component_eq [symmetric])
    apply (subst (asm) homotopic_circlemaps_divide)
    apply (auto simp: inessential_eq_continuous_logarithm_circle)
    done
  have continuous_on (S ∪ T) (λx. (x − a) /R cmod (x − a)) continuous_on (S
∪ T) (λx. (x − b) /R cmod (x − b))
    by (intro continuous_intros; force)+
  moreover have (λx. (x − a) /R cmod (x − a)) ‘ (S ∪ T) ⊆ sphere 0 1 (λx. (x
− b) /R cmod (x − b)) ‘ (S ∪ T) ⊆ sphere 0 1
    by (auto simp: divide_simps)
  moreover have ∃g. continuous_on (S ∪ T) g ∧
                 (∀x∈S ∪ T. (x − a) /R cmod (x − a) / ((x − b) /R cmod (x −
b)) = exp (i*complex_of_real (g x)))
  proof (cases S ∩ T = {})
    case True
    then have continuous_on (S ∪ T) (λx. if x ∈ S then g x else h x)
      using continuous_on_cases_local [OF clo contg conth]
      by (meson disjoint_iff)
    then show ?thesis
      by (rule_tac x=(λx. if x ∈ S then g x else h x) in exI) (auto simp: g h)
  next
    case False
    have diffpi: ∃n. g x = h x + 2* of_int n*pi if x ∈ S ∩ T for x
    proof −
      have exp (i* of_real (g x)) = exp (i* of_real (h x))
        using that by (simp add: g h)
      then obtain n where complex_of_real (g x) = complex_of_real (h x) + 2*
of_int n*complex_of_real pi
        apply (simp add: exp_eq)
        by (metis complex_i_not_zero distrib_left mult.commute mult_cancel_left)
      then show ?thesis
        using of_real_eq_iff by (fastforce intro!: exI [where x=n])
    qed
    have contgh: continuous_on (S ∩ T) (λx. g x − h x)
        by (intro continuous_intros continuous_on_subset [OF contg] continu-
ous_on_subset [OF conth]) auto
    moreover have disc:
          ∃e>0. ∀y. y ∈ S ∩ T ∧ g y − h y ≠ g x − h x ⟶ e ≤ norm ((g y − h
y) − (g x − h x))
        if x ∈ S ∩ T for x
    proof −
      obtain nx where nx: g x = h x + 2* of_int nx*pi
```

```
      using ‹x ∈ S ∩ T› diffpi by blast
    have 2*pi ≤ norm (g y − h y − (g x − h x)) if y: y ∈ S ∩ T and neq: g y
− h y ≠ g x − h x for y
    proof −
      obtain ny where ny: g y = h y + 2* of_int ny*pi
        using ‹y ∈ S ∩ T› diffpi by blast
      { assume nx ≠ ny
        then have 1 ≤ |real_of_int ny − real_of_int nx|
          by linarith
        then have (2*pi)*1 ≤ (2*pi)*|real_of_int ny − real_of_int nx|
          by simp
        also have ... = |2*real_of_int ny*pi − 2*real_of_int nx*pi|
          by (simp add: algebra_simps abs_if)
         finally have 2*pi ≤ |2*real_of_int ny*pi − 2*real_of_int nx*pi| by
simp
      }
      with neq show ?thesis
        by (simp add: nx ny)
    qed
    then show ?thesis
      by (rule_tac x=2*pi in exI) auto
  qed
  ultimately have (λx. g x − h x) constant_on S ∩ T
    using continuous_discrete_range_constant [OF conST contgh] by blast
  then obtain z where z: ⋀x. x ∈ S ∩ T ⟹ g x − h x = z
    by (auto simp: constant_on_def)
  obtain w where exp(i * of_real(h w)) = exp (i * of_real(z + h w))
    using disc z False
    by auto (metis diff_add_cancel g h of_real_add)
  then have [simp]: exp (i* of_real z) = 1
    by (metis cis_conv_exp cis_mult exp_not_eq_zero mult_cancel_right1)
  show ?thesis
  proof (intro exI conjI)
    show continuous_on (S ∪ T) (λx. if x ∈ S then g x else z + h x)
        by (intro continuous_intros continuous_on_cases_local [OF clo contg]
conth) (use z in force)
  qed (auto simp: g h algebra_simps exp_add)
 qed
 ultimately have homotopic_with_canon (λx. True) (S ∪ T) (sphere 0 1)
                  (λx. (x − a) /R cmod (x − a)) (λx. (x − b) /R cmod (x − b))
  by (subst homotopic_circlemaps_divide) (auto simp: inessential_eq_continuous_logarithm_circle)
 moreover have compact (S ∪ T)
   using assms by blast
 ultimately show ?thesis
   using assms Borsuk_maps_homotopic_in_connected_component_eq by fast-
force
qed
```

theorem *Janiszewski*:
fixes *a b* :: *complex*
assumes *compact S closed T* **and** *conST*: *connected* (*S* ∩ *T*)
and *ccS*: *connected_component* (− *S*) *a b* **and** *ccT*: *connected_component* (− *T*) *a b*
shows *connected_component* (− (*S* ∪ *T*)) *a b*
proof −
have *path_component*(− *T*) *a b*
by (*simp add*: ‹*closed T*› *ccT open_Compl open_path_connected_component*)
then obtain *g* **where** *g*: *path g path_image g* ⊆ − *T pathstart g* = *a pathfinish g* = *b*
by (*auto simp*: *path_component_def*)
then obtain *C* **where** *C*: *compact C connected C a* ∈ *C b* ∈ *C C* ∩ *T* = {}
by *fastforce*
obtain *r* **where** *0* < *r* **and** *r*: *C* ∪ *S* ⊆ *ball 0 r*
by (*metis* ‹*compact C*› ‹*compact S*› *bounded_Un compact_imp_bounded bounded_subset_ballD*)
have *connected_component* (− (*S* ∪ (*T* ∩ *cball 0 r* ∪ *sphere 0 r*))) *a b*
proof (*rule Janiszewski_weak* [*OF* ‹*compact S*›])
show *comT'*: *compact* ((*T* ∩ *cball 0 r*) ∪ *sphere 0 r*)
by (*simp add*: ‹*closed T*› *closed_Int_compact compact_Un*)
have *S* ∩ (*T* ∩ *cball 0 r* ∪ *sphere 0 r*) = *S* ∩ *T*
using *r* **by** *auto*
with *conST* **show** *connected* (*S* ∩ (*T* ∩ *cball 0 r* ∪ *sphere 0 r*))
by *simp*
show *connected_component* (− (*T* ∩ *cball 0 r* ∪ *sphere 0 r*)) *a b*
using *conST C r*
apply (*simp add*: *connected_component_def*)
apply (*rule_tac x*=*C* **in** *exI*)
by *auto*
qed (*simp add*: *ccS*)
then obtain *U* **where** *U*: *connected U U* ⊆ − *S U* ⊆ − *T* ∪ − *cball 0 r U* ⊆ − *sphere 0 r a* ∈ *U b* ∈ *U*
by (*auto simp*: *connected_component_def*)
show *?thesis*
unfolding *connected_component_def*
proof (*intro exI conjI*)
show *U* ⊆ − (*S* ∪ *T*)
using *U r* ‹*0* < *r*› ‹*a* ∈ *C*› *connected_Int_frontier* [*of U cball 0 r*]
apply *simp*
by (*metis ball_subset_cball compl_inf disjoint_eq_subset_Compl disjoint_iff_not_equal inf.orderE inf_sup_aci*(*3*) *subsetCE*)
qed (*auto simp*: *U*)
qed

lemma *Janiszewski_connected*:
fixes *S* :: *complex set*
assumes *ST*: *compact S closed T connected*(*S* ∩ *T*)
and *notST*: *connected* (− *S*) *connected* (− *T*)
shows *connected*(− (*S* ∪ *T*))

using *Janiszewski* [*OF ST*] **by** (*metis IntD1 IntD2 notST compl_sup connected_iff_connected_component*)

9.28.2 The Jordan Curve theorem

lemma *exists_double_arc*:
fixes *g* :: *real* ⇒ *'a*::*real_normed_vector*
assumes *simple_path g pathfinish g* = *pathstart g a* ∈ *path_image g b* ∈ *path_image g a* ≠ *b*
obtains *u d* **where** *arc u arc d pathstart u* = *a pathfinish u* = *b*
pathstart d = *b pathfinish d* = *a*
(*path_image u*) ∩ (*path_image d*) = {*a*,*b*}
(*path_image u*) ∪ (*path_image d*) = *path_image g*
proof −
obtain *u* **where** *u*: *0* ≤ *u u* ≤ *1 g u* = *a*
using *assms* **by** (*auto simp*: *path_image_def*)
define *h* **where** *h* ≡ *shiftpath u g*
have *simple_path h*
using ‹*simple_path g*› *simple_path_shiftpath* ‹*0* ≤ *u*› ‹*u* ≤ *1*› *assms*(*2*) *h_def*
by *blast*
have *pathstart h* = *g u*
by (*simp add*: ‹*u* ≤ *1*› *h_def pathstart_shiftpath*)
have *pathfinish h* = *g u*
by (*simp add*: ‹*0* ≤ *u*› *assms h_def pathfinish_shiftpath*)
have *pihg*: *path_image h* = *path_image g*
by (*simp add*: ‹*0* ≤ *u*› ‹*u* ≤ *1*› *assms h_def path_image_shiftpath*)
then obtain *v* **where** *v*: *0* ≤ *v v* ≤ *1 h v* = *b*
using *assms* **by** (*metis* (*mono_tags*, *lifting*) *atLeastAtMost_iff imageE path_image_def*)
show *?thesis*
proof
show *arc* (*subpath 0 v h*)
by (*metis* (*no_types*) ‹*pathstart h* = *g u*› ‹*simple_path h*› *arc_simple_path_subpath* ‹*a* ≠ *b*› *atLeastAtMost_iff zero_le_one order_refl pathstart_def u*(*3*) *v*)
show *arc* (*subpath v 1 h*)
by (*metis* (*no_types*) ‹*pathfinish h* = *g u*› ‹*simple_path h*› *arc_simple_path_subpath* ‹*a* ≠ *b*› *atLeastAtMost_iff zero_le_one order_refl pathfinish_def u*(*3*) *v*)
show *pathstart* (*subpath 0 v h*) = *a*
by (*metis* ‹*pathstart h* = *g u*› *pathstart_def pathstart_subpath u*(*3*))
show *pathfinish* (*subpath 0 v h*) = *b* *pathstart* (*subpath v 1 h*) = *b*
by (*simp_all add*: *v*(*3*))
show *pathfinish* (*subpath v 1 h*) = *a*
by (*metis* ‹*pathfinish h* = *g u*› *pathfinish_def pathfinish_subpath u*(*3*))
show *path_image* (*subpath 0 v h*) ∩ *path_image* (*subpath v 1 h*) = {*a*, *b*}
proof
have *loop_free h*
using ‹*simple_path h*› *simple_path_def* **by** *blast*
then show *path_image* (*subpath 0 v h*) ∩ *path_image* (*subpath v 1 h*) ⊆ {*a*, *b*}
using *v* ‹*pathfinish* (*subpath v 1 h*) = *a*›
apply (*clarsimp simp add*: *loop_free_def path_image_subpath Ball_def*)

```
      by (smt (verit))
    show {a, b} ⊆ path_image (subpath 0 v h) ∩ path_image (subpath v 1 h)
      using v ‹pathstart (subpath 0 v h) = a› ‹pathfinish (subpath v 1 h) = a›
      by (auto simp: path_image_subpath image_iff Bex_def)
  qed
  show path_image (subpath 0 v h) ∪ path_image (subpath v 1 h) = path_image
g
    using v path_image_subpath pihg path_image_def
    by (metis (full_types) image_Un ivl_disj_un_two_touch(4))
 qed
qed


theorem Jordan_curve:
 fixes c :: real ⇒ complex
 assumes simple_path c and loop: pathfinish c = pathstart c
 obtains inner outer where
            inner ≠ {} open inner connected inner
            outer ≠ {} open outer connected outer
            bounded inner ¬ bounded outer inner ∩ outer = {}
            inner ∪ outer = − path_image c
            frontier inner = path_image c
            frontier outer = path_image c
proof −
 have path c
   by (simp add: assms simple_path_imp_path)
 have hom: (path_image c) homeomorphic (sphere(0::complex) 1)
   by (simp add: assms homeomorphic_simple_path_image_circle)
 with Jordan_Brouwer_separation have ¬ connected (− (path_image c))
   by fastforce
 then obtain inner where inner: inner ∈ components (− path_image c) and
bounded inner
   using cobounded_has_bounded_component [of − (path_image c)]
  using ‹¬ connected (− path_image c)› ‹simple_path c› bounded_simple_path_image
by force
 obtain outer where outer: outer ∈ components (− path_image c) and ¬ bounded
outer
   using cobounded_unbounded_components [of − (path_image c)]
   using ‹path c› bounded_path_image by auto
 show ?thesis
 proof
   show inner ≠ {}
     using inner in_components_nonempty by auto
   show open inner
    by (meson ‹simple_path c› compact_imp_closed compact_simple_path_image
inner open_Compl open_components)
   show connected inner
     using in_components_connected inner by blast
   show outer ≠ {}
```

using *outer in_components_nonempty* **by** *auto*
show *open outer*
by (*meson ‹simple_path c› compact_imp_closed compact_simple_path_image outer open_Compl open_components*)
show *connected outer*
using *in_components_connected outer* **by** *blast*
show *inner_outer*: *inner ∩ outer = {}*
by (*meson ‹¬ bounded outer› ‹bounded inner› ‹connected outer› bounded_subset components_maximal in_components_subset inner outer*)
show *fro_inner*: *frontier inner = path_image c*
by (*simp add*: *Jordan_Brouwer_frontier* [*OF hom inner*])
show *fro_outer*: *frontier outer = path_image c*
by (*simp add*: *Jordan_Brouwer_frontier* [*OF hom outer*])
have *False* **if** *m*: *middle ∈ components (− path_image c)* **and** *middle ≠ inner middle ≠ outer* **for** *middle*
proof −
have *frontier middle = path_image c*
by (*simp add*: *Jordan_Brouwer_frontier* [*OF hom*] *that*)
obtain *middle*: *open middle connected middle middle ≠ {}*
by (*metis fro_inner frontier_closed in_components_maximal m open_Compl open_components*)
obtain *a0 b0* **where** *a0 ∈ path_image c b0 ∈ path_image c a0 ≠ b0*
using *simple_path_image_uncountable* [*OF ‹simple_path c›*]
by (*metis Diff_cancel countable_Diff_eq countable_empty insert_iff subsetI subset_singleton_iff*)
obtain *a b g* **where** *ab*: *a ∈ path_image c b ∈ path_image c a ≠ b*
and *g*: *arc g pathstart g = a pathfinish g = b*
and *pag_sub*: *path_image g − {a,b} ⊆ middle*
proof (*rule dense_accessible_frontier_point_pairs* [*OF ‹open middle› ‹connected middle›, of path_image c ∩ ball a0 (dist a0 b0) path_image c ∩ ball b0 (dist a0 b0)*])
show *openin (top_of_set (frontier middle)) (path_image c ∩ ball a0 (dist a0 b0))*
openin (top_of_set (frontier middle)) (path_image c ∩ ball b0 (dist a0 b0))
by (*simp_all add*: *‹frontier middle = path_image c› openin_open_Int*)
show *path_image c ∩ ball a0 (dist a0 b0) ≠ path_image c ∩ ball b0 (dist a0 b0)*
using *‹a0 ≠ b0› ‹b0 ∈ path_image c›* **by** *auto*
show *path_image c ∩ ball a0 (dist a0 b0) ≠ {}*
using *‹a0 ∈ path_image c› ‹a0 ≠ b0›* **by** *auto*
show *path_image c ∩ ball b0 (dist a0 b0) ≠ {}*
using *‹b0 ∈ path_image c› ‹a0 ≠ b0›* **by** *auto*
qed (*use arc_distinct_ends arc_imp_simple_path simple_path_endless that* **in** *fastforce*)
obtain *u d* **where** *arc u arc d*
and *pathstart u = a pathfinish u = b pathstart d = b pathfinish d = a*
and *ud_ab*: *(path_image u) ∩ (path_image d) = {a,b}*

```
                and ud_Un: (path_image u) ∪ (path_image d) = path_image c
      using exists_double_arc [OF assms ab] by blast
    obtain x y where x ∈ inner y ∈ outer
      using ‹inner ≠ {}› ‹outer ≠ {}› by auto
    have inner ∩ middle = {} middle ∩ outer = {}
      using components_nonoverlap inner outer m that by blast+
   have connected_component (− (path_image u ∪ path_image g ∪ (path_image
d ∪ path_image g))) x y
    proof (rule Janiszewski)
      show compact (path_image u ∪ path_image g)
        by (simp add: ‹arc g› ‹arc u› compact_Un compact_arc_image)
      show closed (path_image d ∪ path_image g)
        by (simp add: ‹arc d› ‹arc g› closed_Un closed_arc_image)
       show connected ((path_image u ∪ path_image g) ∩ (path_image d ∪
path_image g))
        using ud_ab
        by (metis Un_insert_left g connected_arc_image insert_absorb pathfin-
ish_in_path_image pathstart_in_path_image sup_bot_left sup_commute sup_inf_distrib1)
      show connected_component (− (path_image u ∪ path_image g)) x y
        unfolding connected_component_def
      proof (intro exI conjI)
        have connected ((inner ∪ (path_image c − path_image u)) ∪ (outer ∪
(path_image c − path_image u)))
        proof (rule connected_Un)
          show connected (inner ∪ (path_image c − path_image u))
            using connected_intermediate_closure [OF ‹connected inner›]
            by (metis Diff_subset closure_Un_frontier dual_order.refl fro_inner
sup.mono sup_ge1)
          show connected (outer ∪ (path_image c − path_image u))
            using connected_intermediate_closure [OF ‹connected outer›]
          by (simp add: Diff_eq closure_Un_frontier fro_outer sup_inf_distrib1)
          have (inner ∩ outer) ∪ (path_image c − path_image u) ≠ {}
          using ‹arc d› ‹pathfinish d = a› ‹pathstart d = b› arc_imp_simple_path
nonempty_simple_path_endless ud_Un ud_ab by fastforce
            then show (inner ∪ (path_image c − path_image u)) ∩ (outer ∪
(path_image c − path_image u)) ≠ {}
            by auto
        qed
        then show connected (inner ∪ outer ∪ (path_image c − path_image u))
          by (metis sup.right_idem sup_assoc sup_commute)
        have inner ⊆ − path_image u outer ⊆ − path_image u
          using in_components_subset inner outer ud_Un by auto
        moreover have inner ⊆ − path_image g outer ⊆ − path_image g
          using ‹inner ∩ middle = {}› ‹inner ⊆ − path_image u›
         using ‹middle ∩ outer = {}› ‹outer ⊆ − path_image u› pag_sub ud_ab
by fastforce+
        moreover have path_image c − path_image u ⊆ − path_image g
          using in_components_subset m pag_sub ud_ab by fastforce
        ultimately show inner ∪ outer ∪ (path_image c − path_image u) ⊆ −
```

(*path_image u* ∪ *path_image g*)
by *force*
show *x* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image u*)
by (*auto simp*: ‹*x* ∈ *inner*›)
show *y* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image u*)
by (*auto simp*: ‹*y* ∈ *outer*›)
qed
show *connected_component* (− (*path_image d* ∪ *path_image g*)) *x y*
unfolding *connected_component_def*
proof (*intro exI conjI*)
have *connected* ((*inner* ∪ (*path_image c* − *path_image d*)) ∪ (*outer* ∪ (*path_image c* − *path_image d*)))
proof (*rule connected_Un*)
show *connected* (*inner* ∪ (*path_image c* − *path_image d*))
using *connected_intermediate_closure* [*OF* ‹*connected inner*›] *fro_inner*
by (*simp add*: *closure_Un_frontier sup.coboundedI2*)
show *connected* (*outer* ∪ (*path_image c* − *path_image d*))
using *connected_intermediate_closure* [*OF* ‹*connected outer*›]
by (*simp add*: *closure_Un_frontier fro_outer sup.coboundedI2*)
have (*inner* ∩ *outer*) ∪ (*path_image c* − *path_image d*) ≠ {}
using ‹*arc u*› ‹*pathfinish u* = *b*› ‹*pathstart u* = *a*› *arc_imp_simple_path nonempty_simple_path_endless ud_Un ud_ab* **by** *fastforce*
then show (*inner* ∪ (*path_image c* − *path_image d*)) ∩ (*outer* ∪ (*path_image c* − *path_image d*)) ≠ {}
by *auto*
qed
then show *connected* (*inner* ∪ *outer* ∪ (*path_image c* − *path_image d*))
by (*metis sup.right_idem sup_assoc sup_commute*)
have *inner* ⊆ − *path_image d outer* ⊆ − *path_image d*
using *in_components_subset inner outer ud_Un* **by** *auto*
moreover have *inner* ⊆ − *path_image g outer* ⊆ − *path_image g*
using ‹*inner* ∩ *middle* = {}› ‹*inner* ⊆ − *path_image d*›
using ‹*middle* ∩ *outer* = {}› ‹*outer* ⊆ − *path_image d*› *pag_sub ud_ab* **by** *fastforce*+
moreover have *path_image c* − *path_image d* ⊆ − *path_image g*
using *in_components_subset m pag_sub ud_ab* **by** *fastforce*
ultimately show *inner* ∪ *outer* ∪ (*path_image c* − *path_image d*) ⊆ − (*path_image d* ∪ *path_image g*)
by *force*
show *x* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image d*)
by (*auto simp*: ‹*x* ∈ *inner*›)
show *y* ∈ *inner* ∪ *outer* ∪ (*path_image c* − *path_image d*)
by (*auto simp*: ‹*y* ∈ *outer*›)
qed
qed
then have *connected_component* (− (*path_image u* ∪ *path_image d* ∪ *path_image g*)) *x y*
by (*simp add*: *Un_ac*)
moreover have ¬(*connected_component* (− (*path_image c*)) *x y*)

by (*metis* (*no_types*, *lifting*) ‹¬ *bounded outer*› ‹*bounded inner*› ‹*x* ∈ *inner*› ‹*y* ∈ *outer*› *componentsE connected_component_eq inner mem_Collect_eq outer*)
ultimately show *False*
by (*auto simp*: *ud_Un* [*symmetric*] *connected_component_def*)
qed
then have *components* (− *path_image c*) = {*inner*,*outer*}
using *inner outer* **by** *blast*
then have *Union* (*components* (− *path_image c*)) = *inner* ∪ *outer*
by *simp*
then show *inner* ∪ *outer* = − *path_image c*
by *auto*
qed (*auto simp*: ‹*bounded inner*› ‹¬ *bounded outer*›)
qed

corollary *Jordan_disconnected*:
fixes *c* :: *real* ⇒ *complex*
assumes *simple_path c pathfinish c* = *pathstart c*
shows ¬ *connected*(− *path_image c*)
using *Jordan_curve* [*OF assms*]
by (*metis Jordan_Brouwer_separation assms homeomorphic_simple_path_image_circle zero_less_one*)

corollary *Jordan_inside_outside*:
fixes *c* :: *real* ⇒ *complex*
assumes *simple_path c pathfinish c* = *pathstart c*
shows *inside*(*path_image c*) ≠ {} ∧
open(*inside*(*path_image c*)) ∧
connected(*inside*(*path_image c*)) ∧
outside(*path_image c*) ≠ {} ∧
open(*outside*(*path_image c*)) ∧
connected(*outside*(*path_image c*)) ∧
bounded(*inside*(*path_image c*)) ∧
¬ *bounded*(*outside*(*path_image c*)) ∧
inside(*path_image c*) ∩ *outside*(*path_image c*) = {} ∧
inside(*path_image c*) ∪ *outside*(*path_image c*) =
− *path_image c* ∧
frontier(*inside*(*path_image c*)) = *path_image c* ∧
frontier(*outside*(*path_image c*)) = *path_image c*
proof −
obtain *inner outer*
where ∗: *inner* ≠ {} *open inner connected inner*
outer ≠ {} *open outer connected outer*
bounded inner ¬ *bounded outer inner* ∩ *outer* = {}
inner ∪ *outer* = − *path_image c*
frontier inner = *path_image c*
frontier outer = *path_image c*
using *Jordan_curve* [*OF assms*] **by** *blast*

then have *inner*: *inside*(*path_image c*) = *inner*
by (*metis dual_order.antisym inside_subset interior_eq interior_inside_frontier*)
have *outer*: *outside*(*path_image c*) = *outer*
using ‹*inner* ∪ *outer* = − *path_image c*› ‹*inside* (*path_image c*) = *inner*›
outside_inside ‹*inner* ∩ *outer* = {}› **by** *auto*
show *?thesis*
using ∗ **by** (*auto simp*: *inner outer*)
qed

Triple-curve or "theta-curve" theorem

Proof that there is no fourth component taken from Kuratowski's Topology vol 2, para 61, II.

theorem *split_inside_simple_closed_curve*:
fixes *c* :: *real* ⇒ *complex*
assumes *simple_path c1* **and** *c1*: *pathstart c1* = *a pathfinish c1* = *b*
and *simple_path c2* **and** *c2*: *pathstart c2* = *a pathfinish c2* = *b*
and *simple_path c* **and** *c*: *pathstart c* = *a pathfinish c* = *b*
and *a* ≠ *b*
and *c1c2*: *path_image c1* ∩ *path_image c2* = {*a*,*b*}
and *c1c*: *path_image c1* ∩ *path_image c* = {*a*,*b*}
and *c2c*: *path_image c2* ∩ *path_image c* = {*a*,*b*}
and *ne_12*: *path_image c* ∩ *inside*(*path_image c1* ∪ *path_image c2*) ≠ {}
obtains *inside*(*path_image c1* ∪ *path_image c*) ∩ *inside*(*path_image c2* ∪ *path_image c*) = {}
inside(*path_image c1* ∪ *path_image c*) ∪ *inside*(*path_image c2* ∪ *path_image c*) ∪
(*path_image c* − {*a*,*b*}) = *inside*(*path_image c1* ∪ *path_image c2*)
proof −
let *?*Θ = *path_image c* **let** *?*Θ*1* = *path_image c1* **let** *?*Θ*2* = *path_image c2*
have *sp*: *simple_path* (*c1* +++ *reversepath c2*) *simple_path* (*c1* +++ *reversepath c*) *simple_path* (*c2* +++ *reversepath c*)
using *assms* **by** (*auto simp*: *simple_path_join_loop_eq arc_simple_path simple_path_reversepath*)
then have *op_in12*: *open* (*inside* (*?*Θ*1* ∪ *?*Θ*2*))
and *op_out12*: *open* (*outside* (*?*Θ*1* ∪ *?*Θ*2*))
and *op_in1c*: *open* (*inside* (*?*Θ*1* ∪ *?*Θ))
and *op_in2c*: *open* (*inside* (*?*Θ*2* ∪ *?*Θ))
and *op_out1c*: *open* (*outside* (*?*Θ*1* ∪ *?*Θ))
and *op_out2c*: *open* (*outside* (*?*Θ*2* ∪ *?*Θ))
and *co_in1c*: *connected* (*inside* (*?*Θ*1* ∪ *?*Θ))
and *co_in2c*: *connected* (*inside* (*?*Θ*2* ∪ *?*Θ))
and *co_out12c*: *connected* (*outside* (*?*Θ*1* ∪ *?*Θ*2*))
and *co_out1c*: *connected* (*outside* (*?*Θ*1* ∪ *?*Θ))
and *co_out2c*: *connected* (*outside* (*?*Θ*2* ∪ *?*Θ))
and *pa_c*: *?*Θ − {*pathstart c*, *pathfinish c*} ⊆ − *?*Θ*1*
*?*Θ − {*pathstart c*, *pathfinish c*} ⊆ − *?*Θ*2*
and *pa_c1*: *?*Θ*1* − {*pathstart c1*, *pathfinish c1*} ⊆ − *?*Θ*2*
*?*Θ*1* − {*pathstart c1*, *pathfinish c1*} ⊆ − *?*Θ

```
    and pa_c2: ?Θ2 − {pathstart c2, pathfinish c2} ⊆ − ?Θ1
            ?Θ2 − {pathstart c2, pathfinish c2} ⊆ − ?Θ
    and co_c: connected(?Θ − {pathstart c,pathfinish c})
    and co_c1: connected(?Θ1 − {pathstart c1,pathfinish c1})
    and co_c2: connected(?Θ2 − {pathstart c2,pathfinish c2})
    and fr_in: frontier(inside(?Θ1 ∪ ?Θ2)) = ?Θ1 ∪ ?Θ2
          frontier(inside(?Θ2 ∪ ?Θ)) = ?Θ2 ∪ ?Θ
          frontier(inside(?Θ1 ∪ ?Θ)) = ?Θ1 ∪ ?Θ
    and fr_out: frontier(outside(?Θ1 ∪ ?Θ2)) = ?Θ1 ∪ ?Θ2
          frontier(outside(?Θ2 ∪ ?Θ)) = ?Θ2 ∪ ?Θ
          frontier(outside(?Θ1 ∪ ?Θ)) = ?Θ1 ∪ ?Θ
   using Jordan_inside_outside [of c1 +++ reversepath c2]
   using Jordan_inside_outside [of c1 +++ reversepath c]
   using Jordan_inside_outside [of c2 +++ reversepath c] assms
       apply (simp_all add: path_image_join closed_Un closed_simple_path_image
open_inside open_outside)
     apply (blast | metis connected_simple_path_endless)+
   done
 have inout_12: inside (?Θ1 ∪ ?Θ2) ∩ (?Θ − {pathstart c, pathfinish c}) ≠ {}
  by (metis (no_types, lifting) c c1c ne_12 Diff_Int_distrib Diff_empty Int_empty_right
Int_left_commute inf_sup_absorb inf_sup_aci(1) inside_no_overlap)
 have pi_disjoint: ?Θ ∩ outside(?Θ1 ∪ ?Θ2) = {}
 proof (rule ccontr)
   assume ?Θ ∩ outside (?Θ1 ∪ ?Θ2) ≠ {}
   then show False
     using connectedD [OF co_c, of inside(?Θ1 ∪ ?Θ2) outside(?Θ1 ∪ ?Θ2)]
     using c c1c2 pa_c op_in12 op_out12 inout_12
     apply clarsimp
       by (smt (verit, ccfv_threshold) Diff_Int_distrib Diff_cancel Diff_empty
Int_assoc inf_sup_absorb inf_sup_aci(1) outside_no_overlap)
 qed
  have out_sub12: outside(?Θ1 ∪ ?Θ2) ⊆ outside(?Θ1 ∪ ?Θ) outside(?Θ1 ∪
?Θ2) ⊆ outside(?Θ2 ∪ ?Θ)
   by (metis Un_commute pi_disjoint outside_Un_outside_Un)+
 have pa1_disj_in2: ?Θ1 ∩ inside (?Θ2 ∪ ?Θ) = {}
 proof (rule ccontr)
   assume ne: ?Θ1 ∩ inside (?Θ2 ∪ ?Θ) ≠ {}
   have 1: inside (?Θ ∪ ?Θ2) ∩ ?Θ = {}
   by (metis (no_types) Diff_Int_distrib Diff_cancel inf_sup_absorb inf_sup_aci(3)
inside_no_overlap)
   have 2: outside (?Θ ∪ ?Θ2) ∩ ?Θ = {}
      by (metis (no_types) Int_empty_right Int_left_commute inf_sup_absorb
outside_no_overlap)
   have path_image c1 ∩ outside (path_image c2 ∪ path_image c) = {}
     using connectedD [OF co_c1, of inside(?Θ2 ∪ ?Θ) outside(?Θ2 ∪ ?Θ)]
       pa_c1 op_in2c op_out2c ne c1 c2c 1 2 by (auto simp: inf_sup_aci)
   then have outside (?Θ2 ∪ ?Θ) ⊆ outside (?Θ1 ∪ ?Θ2)
     by (metis outside_Un_outside_Un sup_commute)
   with out_sub12
```

```
    have outside(?Θ1 ∪ ?Θ2) = outside(?Θ2 ∪ ?Θ) by blast
    then have frontier(outside(?Θ1 ∪ ?Θ2)) = frontier(outside(?Θ2 ∪ ?Θ))
      by simp
    then show False
      using inout_12 pi_disjoint c c1c c2c fr_out by auto
  qed
  have pa2_disj_in1: ?Θ2 ∩ inside(?Θ1 ∪ ?Θ) = {}
  proof (rule ccontr)
    assume ne: ?Θ2 ∩ inside (?Θ1 ∪ ?Θ) ≠ {}
    have 1: inside (?Θ ∪ ?Θ1) ∩ ?Θ = {}
    by (metis (no_types) Diff_Int_distrib Diff_cancel inf_sup_absorb inf_sup_aci(3)
inside_no_overlap)
    have 2: outside (?Θ ∪ ?Θ1) ∩ ?Θ = {}
      by (metis (no_types) Int_empty_right Int_left_commute inf_sup_absorb
outside_no_overlap)
    have outside (?Θ1 ∪ ?Θ) ⊆ outside (?Θ1 ∪ ?Θ2)
      apply (rule outside_Un_outside_Un)
      using connectedD [OF co_c2, of inside(?Θ1 ∪ ?Θ) outside(?Θ1 ∪ ?Θ)]
        pa_c2 op_in1c op_out1c ne c2 c1c 1 2 by (auto simp: inf_sup_aci)
    with out_sub12
    have outside(?Θ1 ∪ ?Θ2) = outside(?Θ1 ∪ ?Θ)
      by blast
    then have frontier(outside(?Θ1 ∪ ?Θ2)) = frontier(outside(?Θ1 ∪ ?Θ))
      by simp
    then show False
      using inout_12 pi_disjoint c c1c c2c fr_out by auto
  qed
  have in_sub_in1: inside(?Θ1 ∪ ?Θ) ⊆ inside(?Θ1 ∪ ?Θ2)
    using pa2_disj_in1 out_sub12 by (auto simp: inside_outside)
  have in_sub_in2: inside(?Θ2 ∪ ?Θ) ⊆ inside(?Θ1 ∪ ?Θ2)
    using pa1_disj_in2 out_sub12 by (auto simp: inside_outside)
  have in_sub_out12: inside(?Θ1 ∪ ?Θ) ⊆ outside(?Θ2 ∪ ?Θ)
  proof
    fix x
    assume x: x ∈ inside (?Θ1 ∪ ?Θ)
    then have xnot: x ∉ ?Θ
      by (simp add: inside_def)
    obtain z where zim: z ∈ ?Θ1 and zout: z ∈ outside(?Θ2 ∪ ?Θ)
      unfolding outside_inside
      using nonempty_simple_path_endless [OF ‹simple_path c1›] c1 c1c c1c2
pa1_disj_in2 by auto
    obtain e where e > 0 and e: ball z e ⊆ outside(?Θ2 ∪ ?Θ)
      using zout op_out2c open_contains_ball_eq by blast
    have z ∈ frontier (inside (?Θ1 ∪ ?Θ))
      using zim by (auto simp: fr_in)
    then obtain w where w1: w ∈ inside (?Θ1 ∪ ?Θ) and dwz: dist w z < e
      using zim ‹e > 0› by (auto simp: frontier_def closure_approachable)
    then have w2: w ∈ outside (?Θ2 ∪ ?Θ)
      by (metis e dist_commute mem_ball subsetCE)
```

```
    then have connected_component (− ?Θ2 ∩ − ?Θ) z w
      unfolding connected_component_def
      by (metis co_out2c compl_sup inside_Un_outside sup_ge2 zout)
    moreover have connected_component (− ?Θ2 ∩ − ?Θ) w x
      unfolding connected_component_def
      using pa2_disj_in1 co_in1c x w1 union_with_outside by fastforce
    ultimately have eq: connected_component_set (− ?Θ2 ∩ − ?Θ) x =
                        connected_component_set (− ?Θ2 ∩ − ?Θ) z
      by (metis (mono_tags, lifting) connected_component_eq mem_Collect_eq)
    show x ∈ outside (?Θ2 ∪ ?Θ)
      using zout x pa2_disj_in1 by (auto simp: outside_def eq xnot)
  qed
  have in_sub_out21: inside(?Θ2 ∪ ?Θ) ⊆ outside(?Θ1 ∪ ?Θ)
  proof
    fix x
    assume x: x ∈ inside (?Θ2 ∪ ?Θ)
    then have xnot: x ∉ ?Θ
      by (simp add: inside_def)
    obtain z where zim: z ∈ ?Θ2 and zout: z ∈ outside(?Θ1 ∪ ?Θ)
      unfolding outside_inside
       using nonempty_simple_path_endless [OF ‹simple_path c2›] c1c2 c2 c2c
pa2_disj_in1 by auto
    obtain e where e > 0 and e: ball z e ⊆ outside(?Θ1 ∪ ?Θ)
      using zout op_out1c open_contains_ball_eq by blast
    have z ∈ frontier (inside (?Θ2 ∪ ?Θ))
      using zim by (auto simp: fr_in)
    then obtain w where w2: w ∈ inside (?Θ2 ∪ ?Θ) and dwz: dist w z < e
      using zim ‹e > 0› by (auto simp: frontier_def closure_approachable)
    then have w1: w ∈ outside (?Θ1 ∪ ?Θ)
      by (metis e dist_commute mem_ball subsetCE)
    then have connected_component (− ?Θ1 ∩ − ?Θ) z w
      unfolding connected_component_def
      by (metis Compl_Un co_out1c inside_Un_outside sup_ge2 zout)
    moreover have connected_component (− ?Θ1 ∩ − ?Θ) w x
      unfolding connected_component_def
      using pa1_disj_in2 co_in2c x w2 union_with_outside by fastforce
    ultimately have eq: connected_component_set (− ?Θ1 ∩ − ?Θ) x =
                        connected_component_set (− ?Θ1 ∩ − ?Θ) z
      by (metis (no_types, lifting) connected_component_eq mem_Collect_eq)
    show x ∈ outside (?Θ1 ∪ ?Θ)
      using zout x pa1_disj_in2 by (auto simp: outside_def eq xnot)
  qed
  show ?thesis
  proof
    show inside (?Θ1 ∪ ?Θ) ∩ inside (?Θ2 ∪ ?Θ) = {}
   by (metis Int_Un_distrib in_sub_out12 bot_eq_sup_iff disjoint_eq_subset_Compl
outside_inside)
    have ∗: outside (?Θ1 ∪ ?Θ) ∩ outside (?Θ2 ∪ ?Θ) ⊆ outside (?Θ1 ∪ ?Θ2)
    proof (rule components_maximal)
```

show *out_in*: *outside* (*?Θ1* ∪ *?Θ2*) ∈ *components* (− (*?Θ1* ∪ *?Θ2*))
unfolding *outside_in_components co_out12c*
using *co_out12c fr_out*(*1*) **by** *force*
have *conn_U*: *connected* (− (*closure* (*inside* (*?Θ1* ∪ *?Θ*)) ∪ *closure* (*inside* (*?Θ2* ∪ *?Θ*))))
proof (*rule Janiszewski_connected*, *simp_all*)
show *bounded* (*inside* (*?Θ1* ∪ *?Θ*))
by (*simp add*: ‹*simple_path c1*› ‹*simple_path c*› *bounded_inside bounded_simple_path_image*)
have *if1*: − (*inside* (*?Θ1* ∪ *?Θ*) ∪ *frontier* (*inside* (*?Θ1* ∪ *?Θ*))) = − *?Θ1* ∩ − *?Θ* ∩ − *inside* (*?Θ1* ∪ *?Θ*)
by (*metis* (*no_types*, *lifting*) *Int_commute Jordan_inside_outside c c1 compl_sup path_image_join path_image_reversepath pathfinish_join pathfinish_reversepath pathstart_join pathstart_reversepath sp*(*2*) *closure_Un_frontier fr_out*(*3*))
then show *connected* (− *closure* (*inside* (*?Θ1* ∪ *?Θ*)))
by (*metis Compl_Un outside_inside co_out1c closure_Un_frontier*)
have *if2*: − (*inside* (*?Θ2* ∪ *?Θ*) ∪ *frontier* (*inside* (*?Θ2* ∪ *?Θ*))) = − *?Θ2* ∩ − *?Θ* ∩ − *inside* (*?Θ2* ∪ *?Θ*)
by (*metis* (*no_types*, *lifting*) *Int_commute Jordan_inside_outside c c2 compl_sup path_image_join path_image_reversepath pathfinish_join pathfinish_reversepath pathstart_join pathstart_reversepath sp*(*3*) *closure_Un_frontier fr_out*(*2*))
then show *connected* (− *closure* (*inside* (*?Θ2* ∪ *?Θ*)))
by (*metis Compl_Un outside_inside co_out2c closure_Un_frontier*)
have *connected*(*?Θ*)
by (*metis* ‹*simple_path c*› *connected_simple_path_image*)
moreover
have *closure* (*inside* (*?Θ1* ∪ *?Θ*)) ∩ *closure* (*inside* (*?Θ2* ∪ *?Θ*)) = *?Θ*
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
proof *clarify*
fix *x*
assume *x*: *x* ∈ *closure* (*inside* (*?Θ1* ∪ *?Θ*)) *x* ∈ *closure* (*inside* (*?Θ2* ∪ *?Θ*))
then have *x* ∉ *inside* (*?Θ1* ∪ *?Θ*)
by (*meson closure_iff_nhds_not_empty in_sub_out12 inside_Int_outside op_in1c*)
with *fr_in x* **show** *x* ∈ *?Θ*
by (*metis c1c c1c2 closure_Un_frontier pa1_disj_in2 Int_iff Un_iff insert_disjoint*(*2*) *insert_subset subsetI subset_antisym*)
qed
show *?rhs* ⊆ *?lhs*
using *if1 if2 closure_Un_frontier* **by** *fastforce*
qed
ultimately
show *connected* (*closure* (*inside* (*?Θ1* ∪ *?Θ*)) ∩ *closure* (*inside* (*?Θ2* ∪ *?Θ*)))
by *auto*

qed
show *connected* (*outside* (*?Θ1* ∪ *?Θ*) ∩ *outside* (*?Θ2* ∪ *?Θ*))
using *fr_in conn_U* **by** (*simp add*: *closure_Un_frontier outside_inside Un_commute*)
show *outside* (*?Θ1* ∪ *?Θ*) ∩ *outside* (*?Θ2* ∪ *?Θ*) ⊆ − (*?Θ1* ∪ *?Θ2*)
by *clarify* (*metis Diff_Compl Diff_iff Un_iff inf_sup_absorb outside_inside*)
show *outside* (*?Θ1* ∪ *?Θ2*) ∩
(*outside* (*?Θ1* ∪ *?Θ*) ∩ *outside* (*?Θ2* ∪ *?Θ*)) ≠ {}
by (*metis Int_assoc out_in inf.orderE out_sub12*(*1*) *out_sub12*(*2*) *outside_in_components*)
qed
show *inside* (*?Θ1* ∪ *?Θ*) ∪ *inside* (*?Θ2* ∪ *?Θ*) ∪ (*?Θ* − {*a*, *b*}) = *inside* (*?Θ1* ∪ *?Θ2*)
(**is** *?lhs* = *?rhs*)
proof
have *path_image c* − {*a*, *b*} ⊆ *inside* (*path_image c1* ∪ *path_image c2*)
using *c1c c2c inside_outside pi_disjoint* **by** *fastforce*
then show *?lhs* ⊆ *?rhs*
by (*simp add*: *in_sub_in1 in_sub_in2*)
have *inside* (*?Θ1* ∪ *?Θ2*) ⊆ *inside* (*?Θ1* ∪ *?Θ*) ∪ *inside* (*?Θ2* ∪ *?Θ*) ∪ (*?Θ*)
using *Compl_anti_mono* [*OF* ∗] **by** (*force simp*: *inside_outside*)
moreover have *inside* (*?Θ1* ∪ *?Θ2*) ⊆ −{*a*,*b*}
using *c1 union_with_outside* **by** *fastforce*
ultimately show *?rhs* ⊆ *?lhs* **by** *auto*
qed
qed
qed

end

9.29 Polynomial Functions: Extremal Behaviour and Root Counts

theory *Poly_Roots*
imports *Complex_Main*
begin

9.29.1 Basics about polynomial functions: extremal behaviour and root counts

lemma *sub_polyfun*:
fixes *x* :: *'a*::{*comm_ring*,*monoid_mult*}
shows (∑ *i*≤*n*. *a i* ∗ *x*^*i*) − (∑ *i*≤*n*. *a i* ∗ *y*^*i*) =
(*x* − *y*) ∗ (∑ *j*<*n*. ∑ *k*= *Suc j*..*n*. *a k* ∗ *y*^(*k* − *Suc j*) ∗ *x*^*j*)
proof −
have (∑ *i*≤*n*. *a i* ∗ *x*^*i*) − (∑ *i*≤*n*. *a i* ∗ *y*^*i*) =
(∑ *i*≤*n*. *a i* ∗ (*x*^*i* − *y*^*i*))
by (*simp add*: *algebra_simps sum_subtractf* [*symmetric*])

also have ... = $(\sum i{\le}n.\ a\ i * (x - y) * (\sum j{<}i.\ y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *power_diff_sumr2 ac_simps*)
also have ... = $(x - y) * (\sum i{\le}n.\ (\sum j{<}i.\ a\ i * y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *sum_distrib_left ac_simps*)
also have ... = $(x - y) * (\sum j{<}n.\ (\sum i{=}Suc\ j..n.\ a\ i * y\hat{}(i - Suc\ j) * x\hat{}j))$
by (*simp add*: *sum.nested_swap'*)
finally show *?thesis* **.**
qed

lemma *sub_polyfun_alt*:
fixes $x :: 'a{::}\{comm\_ring,monoid\_mult\}$
shows $(\sum i{\le}n.\ a\ i * x\hat{}i) - (\sum i{\le}n.\ a\ i * y\hat{}i) =$
$(x - y) * (\sum j{<}n.\ \sum k{<}n{-}j.\ a\ (j{+}k{+}1) * y\hat{}k * x\hat{}j)$
proof −
{ fix *j*
have $(\sum k = Suc\ j..n.\ a\ k * y\hat{}(k - Suc\ j) * x\hat{}j) =$
$(\sum k < n - j.\ a\ (Suc\ (j + k)) * y\hat{}k * x\hat{}j)$
by (*rule sum.reindex_bij_witness*[**where** $i{=}\lambda i.\ i + Suc\ j$ **and** $j{=}\lambda i.\ i - Suc\ j$]) *auto* **}**
then show *?thesis*
by (*simp add*: *sub_polyfun*)
qed

lemma *polyfun_linear_factor*:
fixes $a :: 'a{::}\{comm\_ring,monoid\_mult\}$
shows $\exists b.\ \forall z.\ (\sum i{\le}n.\ c\ i * z\hat{}i) =$
$(z{-}a) * (\sum i{<}n.\ b\ i * z\hat{}i) + (\sum i{\le}n.\ c\ i * a\hat{}i)$
proof −
{ fix *z*
have $(\sum i{\le}n.\ c\ i * z\hat{}i) - (\sum i{\le}n.\ c\ i * a\hat{}i) =$
$(z - a) * (\sum j{<}n.\ (\sum k = Suc\ j..n.\ c\ k * a\hat{}(k - Suc\ j)) * z\hat{}j)$
by (*simp add*: *sub_polyfun sum_distrib_right*)
then have $(\sum i{\le}n.\ c\ i * z\hat{}i) =$
$(z - a) * (\sum j{<}n.\ (\sum k = Suc\ j..n.\ c\ k * a\hat{}(k - Suc\ j)) * z\hat{}j)$
$+ (\sum i{\le}n.\ c\ i * a\hat{}i)$
by (*simp add*: *algebra_simps*) **}**
then show *?thesis*
by (*intro exI allI*)
qed

lemma *polyfun_linear_factor_root*:
fixes $a :: 'a{::}\{comm\_ring,monoid\_mult\}$
assumes $(\sum i{\le}n.\ c\ i * a\hat{}i) = 0$
shows $\exists b.\ \forall z.\ (\sum i{\le}n.\ c\ i * z\hat{}i) = (z{-}a) * (\sum i{<}n.\ b\ i * z\hat{}i)$
using *polyfun_linear_factor* [*of c n a*] *assms*
by *simp*

lemma *adhoc_norm_triangle*: $a + norm(y) \le b \Longrightarrow norm(x) \le a \Longrightarrow norm(x + y) \le b$

```
  by (metis norm_triangle_mono order.trans order_refl)

proposition polyfun_extremal_lemma:
  fixes c :: nat ⇒ 'a::real_normed_div_algebra
  assumes e > 0
    shows ∃M. ∀z. M ≤ norm z ⟶ norm(∑i≤n. c i * z^i) ≤ e * norm(z) ^
Suc n
proof (induction n)
  case 0
  show ?case
   by (rule exI [where x=norm (c 0) / e]) (auto simp: mult.commute pos_divide_le_eq
assms)
next
  case (Suc n)
  then obtain M where M: ∀z. M ≤ norm z ⟶ norm (∑i≤n. c i * z^i) ≤ e
* norm z ^ Suc n ..
  show ?case
  proof (rule exI [where x=max 1 (max M ((e + norm(c(Suc n))) / e))], clarify)
    fix z::'a
    assume max 1 (max M ((e + norm (c (Suc n))) / e)) ≤ norm z
    then have norm1: 0 < norm z M ≤ norm z (e + norm (c (Suc n))) / e ≤
norm z
      by auto
    then have norm2: (e + norm (c (Suc n))) ≤ e * norm z  (norm z * norm z
^ n) > 0
      apply (metis assms less_divide_eq mult.commute not_le)
      using norm1 apply (metis mult_pos_pos zero_less_power)
      done
    have e * (norm z * norm z ^ n) + norm (c (Suc n) * (z * z ^ n)) =
          (e + norm (c (Suc n))) * (norm z * norm z ^ n)
      by (simp add: norm_mult norm_power algebra_simps)
    also have ... ≤ (e * norm z) * (norm z * norm z ^ n)
      using norm2
      using assms mult_mono by fastforce
    also have ... = e * (norm z * (norm z * norm z ^ n))
      by (simp add: algebra_simps)
    finally have e * (norm z * norm z ^ n) + norm (c (Suc n) * (z * z ^ n))
                 ≤ e * (norm z * (norm z * norm z ^ n)) .
    then show norm (∑i≤Suc n. c i * z^i) ≤ e * norm z ^ Suc (Suc n) using
M norm1
      by (drule_tac x=z in spec) (auto simp: intro!: adhoc_norm_triangle)
    qed
qed

lemma norm_lemma_xy: assumes |b| + 1 ≤ norm(y) - a norm(x) ≤ a shows
b ≤ norm(x + y)
proof -
  have b ≤ norm y - norm x
    using assms by linarith
```

```
  then show ?thesis
    by (metis (no_types) add.commute norm_diff_ineq order_trans)
qed

proposition polyfun_extremal:
  fixes c :: nat ⇒ 'a::real_normed_div_algebra
  assumes ∃k. k ≠ 0 ∧ k ≤ n ∧ c k ≠ 0
    shows eventually (λz. norm(∑ i≤n. c i * z^i) ≥ B) at_infinity
using assms
proof (induction n)
  case 0 then show ?case
    by simp
next
  case (Suc n)
  show ?case
  proof (cases c (Suc n) = 0)
    case True
    with Suc show ?thesis
      by auto (metis diff_is_0_eq diffs0_imp_equal less_Suc_eq_le not_less_eq)
  next
    case False
    with polyfun_extremal_lemma [of norm(c (Suc n)) / 2 c n]
    obtain M where M: ⋀z. M ≤ norm z ⟹
            norm (∑ i≤n. c i * z^i) ≤ norm (c (Suc n)) / 2 * norm z ^ Suc n
      by auto
    show ?thesis
    unfolding eventually_at_infinity
    proof (rule exI [where x=max M (max 1 ((|B| + 1) / (norm (c (Suc n)) /
2)))], clarsimp)
      fix z::'a
      assume les: M ≤ norm z  1 ≤ norm z  (|B| * 2 + 2) / norm (c (Suc n)) ≤
norm z
      then have |B| * 2 + 2 ≤ norm z * norm (c (Suc n))
        by (metis False pos_divide_le_eq zero_less_norm_iff)
      then have |B| * 2 + 2 ≤ norm z ^ (Suc n) * norm (c (Suc n))
          by (metis ‹1 ≤ norm z› order.trans mult_right_mono norm_ge_zero
self_le_power zero_less_Suc)
      then show B ≤ norm ((∑ i≤n. c i * z^i) + c (Suc n) * (z * z ^ n)) using
M les
        apply auto
         apply (rule norm_lemma_xy [where a = norm (c (Suc n)) * norm z ^
(Suc n) / 2])
        apply (simp_all add: norm_mult norm_power)
        done
    qed
  qed
qed

proposition polyfun_rootbound:
```

```
 fixes c :: nat ⇒ 'a::{comm_ring,real_normed_div_algebra}
 assumes ∃k. k ≤ n ∧ c k ≠ 0
   shows finite {z. (∑i≤n. c i * z^i) = 0} ∧ card {z. (∑i≤n. c i * z^i) = 0}
≤ n
using assms
proof (induction n arbitrary: c)
 case (Suc n) show ?case
 proof (cases {z. (∑i≤Suc n. c i * z^i) = 0} = {})
   case False
   then obtain a where a: (∑i≤Suc n. c i * a^i) = 0
     by auto
   from polyfun_linear_factor_root [OF this]
   obtain b where ⋀z. (∑i≤Suc n. c i * z^i) = (z − a) * (∑i< Suc n. b i *
z^i)
     by auto
   then have b: ⋀z. (∑i≤Suc n. c i * z^i) = (z − a) * (∑i≤n. b i * z^i)
     by (metis lessThan_Suc_atMost)
   then have ins_ab: {z. (∑i≤Suc n. c i * z^i) = 0} = insert a {z. (∑i≤n. b i
* z^i) = 0}
     by auto
   have c0: c 0 = − (a * b 0) using  b [of 0]
     by simp
   then have extr_prem: ¬ (∃k≤n. b k ≠ 0) ⟹ ∃k. k ≠ 0 ∧ k ≤ Suc n ∧ c k
≠ 0
     by (metis Suc.prems le0 minus_zero mult_zero_right)
   have ∃k≤n. b k ≠ 0
     apply (rule ccontr)
     using polyfun_extremal [OF extr_prem, of 1]
     apply (auto simp: eventually_at_infinity b simp del: sum.atMost_Suc)
     apply (drule_tac x=of_real ba in spec, simp)
     done
   then show ?thesis using Suc.IH [of b] ins_ab
     by (auto simp: card_insert_if)
   qed simp
qed simp

corollary
  fixes c :: nat ⇒ 'a::{comm_ring,real_normed_div_algebra}
  assumes ∃k. k ≤ n ∧ c k ≠ 0
    shows polyfun_rootbound_finite: finite {z. (∑i≤n. c i * z^i) = 0}
      and polyfun_rootbound_card:   card {z. (∑i≤n. c i * z^i) = 0} ≤ n
using polyfun_rootbound [OF assms] by auto

proposition polyfun_finite_roots:
  fixes c :: nat ⇒ 'a::{comm_ring,real_normed_div_algebra}
    shows  finite {z. (∑i≤n. c i * z^i) = 0} ⟷ (∃k. k ≤ n ∧ c k ≠ 0)
proof (cases  ∃k≤n. c k ≠ 0)
  case True then show ?thesis
    by (blast intro: polyfun_rootbound_finite)
```

next
case *False* **then show** *?thesis*
by (*auto simp*: *infinite_UNIV_char_0*)
qed

lemma *polyfun_eq_0*:
fixes *c* :: *nat* ⇒ *'a*::{*comm_ring*,*real_normed_div_algebra*}
shows (∀ *z*. (∑ *i*≤*n*. *c i* ∗ *z*^*i*) = *0*) ⟷ (∀ *k*. *k* ≤ *n* ⟶ *c k* = *0*)
proof (*cases* (∀ *z*. (∑ *i*≤*n*. *c i* ∗ *z*^*i*) = *0*))
case *True*
then have ¬ *finite* {*z*. (∑ *i*≤*n*. *c i* ∗ *z*^*i*) = *0*}
by (*simp add*: *infinite_UNIV_char_0*)
with *True* **show** *?thesis*
by (*metis* (*poly_guards_query*) *polyfun_rootbound_finite*)
next
case *False*
then show *?thesis*
by *auto*
qed

theorem *polyfun_eq_const*:
fixes *c* :: *nat* ⇒ *'a*::{*comm_ring*,*real_normed_div_algebra*}
shows (∀ *z*. (∑ *i*≤*n*. *c i* ∗ *z*^*i*) = *k*) ⟷ *c 0* = *k* ∧ (∀ *k*. *k* ≠ *0* ∧ *k* ≤ *n* ⟶ *c k* = *0*)
proof −
{fix *z*
have (∑ *i*≤*n*. *c i* ∗ *z*^*i*) = (∑ *i*≤*n*. (*if i* = *0 then c 0* − *k else c i*) ∗ *z*^*i*) + *k*
by (*induct n*) *auto*
} then
have (∀ *z*. (∑ *i*≤*n*. *c i* ∗ *z*^*i*) = *k*) ⟷ (∀ *z*. (∑ *i*≤*n*. (*if i* = *0 then c 0* − *k else c i*) ∗ *z*^*i*) = *0*)
by *auto*
also have ... ⟷ *c 0* = *k* ∧ (∀ *k*. *k* ≠ *0* ∧ *k* ≤ *n* ⟶ *c k* = *0*)
by (*auto simp*: *polyfun_eq_0*)
finally show *?thesis* **.**
qed

end

9.30 Generalised Binomial Theorem

The proof of the Generalised Binomial Theorem and related results. We prove the generalised binomial theorem for complex numbers, following the proof at: https://proofwiki.org/wiki/Binomial_Theorem/General_Binomial_Theorem

theory *Generalised_Binomial_Theorem*
imports
Complex_Main

```
  Complex_Transcendental
  Summation_Tests
begin

lemma gbinomial_ratio_limit:
  fixes a :: 'a :: real_normed_field
  assumes a ∉ ℕ
  shows (λn. (a gchoose n) / (a gchoose Suc n)) ⟶ −1
proof (rule Lim_transform_eventually)
  let ?f = λn. inverse (a / of_nat (Suc n) − of_nat n / of_nat (Suc n))
  from eventually_gt_at_top[of 0::nat]
    show eventually (λn. ?f n = (a gchoose n) /(a gchoose Suc n)) sequentially
  proof eventually_elim
    fix n :: nat assume n: n > 0
    then obtain q where q: n = Suc q by (cases n) blast
    let ?P = ∏ i=0..<n. a − of_nat i
    from n have (a gchoose n) / (a gchoose Suc n) = (of_nat (Suc n) :: 'a) *
                  (?P / (∏ i=0..n. a − of_nat i))
      by (simp add: gbinomial_prod_rev atLeastLessThanSuc_atLeastAtMost)
    also from q have (∏ i=0..n. a − of_nat i) = ?P * (a − of_nat n)
     by (simp add: prod.atLeast0_atMost_Suc atLeastLessThanSuc_atLeastAtMost)
   also have ?P / ... = (?P / ?P) / (a − of_nat n) by (rule divide_divide_eq_left[symmetric])
    also from assms have ?P / ?P = 1 by auto
    also have of_nat (Suc n) * (1 / (a − of_nat n)) =
                  inverse (inverse (of_nat (Suc n)) * (a − of_nat n)) by (simp add:
field_simps)
    also have inverse (of_nat (Suc n)) * (a − of_nat n) = a / of_nat (Suc n) −
of_nat n / of_nat (Suc n)
      by (simp add: field_simps del: of_nat_Suc)
    finally show ?f n = (a gchoose n) / (a gchoose Suc n) by simp
  qed

  have (λn. norm a / (of_nat (Suc n))) ⟶ 0
    unfolding divide_inverse
    by (intro tendsto_mult_right_zero LIMSEQ_inverse_real_of_nat)
  hence (λn. a / of_nat (Suc n)) ⟶ 0
     by (subst tendsto_norm_zero_iff[symmetric]) (simp add: norm_divide del:
of_nat_Suc)
  hence ?f ⟶ inverse (0 − 1)
    by (intro tendsto_inverse tendsto_diff LIMSEQ_n_over_Suc_n) simp_all
  thus ?f ⟶ −1 by simp
qed

lemma conv_radius_gchoose:
  fixes a :: 'a :: {real_normed_field,banach}
  shows conv_radius (λn. a gchoose n) = (if a ∈ ℕ then ∞ else 1)
proof (cases a ∈ ℕ)
  assume a: a ∈ ℕ
  have eventually (λn. (a gchoose n) = 0) sequentially
```

```
    using eventually_gt_at_top[of nat ⌊norm a⌋]
   by eventually_elim (insert a, auto elim!: Nats_cases simp: binomial_gbinomial[symmetric])
  from conv_radius_cong'[OF this] a show ?thesis by simp
next
  assume a: a ∉ ℕ
  from tendsto_norm[OF gbinomial_ratio_limit[OF this]]
    have conv_radius (λn. a gchoose n) = 1
   by (intro conv_radius_ratio_limit_nonzero[of _ 1]) (simp_all add: norm_divide)
  with a show ?thesis by simp
qed

theorem gen_binomial_complex:
  fixes z :: complex
  assumes norm z < 1
  shows   (λn. (a gchoose n) * z^n) sums (1 + z) powr a
proof -
  define K where K = 1 - (1 - norm z) / 2
  from assms have K: K > 0 K < 1 norm z < K
     unfolding K_def by (auto simp: field_simps intro!: add_pos_nonneg)
  let ?f = λn. a gchoose n and ?f' = diffs (λn. a gchoose n)
  have summable_strong: summable (λn. ?f n * z ^ n) if norm z < 1 for z using
that
    by (intro summable_in_conv_radius) (simp_all add: conv_radius_gchoose)
  with K have summable: summable (λn. ?f n * z ^ n) if norm z < K for z using
that by auto
  hence summable': summable (λn. ?f' n * z ^ n) if norm z < K for z using that
    by (intro termdiff_converges[of _ K]) simp_all

  define f f' where [abs_def]: f z = (∑ n. ?f n * z ^ n) f' z = (∑ n. ?f' n * z ^
n) for z
  {
    fix z :: complex assume z: norm z < K
    from summable_mult2[OF summable'[OF z], of z]
      have summable1: summable (λn. ?f' n * z ^ Suc n) by (simp add: mult_ac)
    hence summable2: summable (λn. of_nat n * ?f n * z^n)
      unfolding diffs_def by (subst (asm) summable_Suc_iff)

    have (1 + z) * f' z = (∑ n. ?f' n * z^n) + (∑ n. ?f' n * z^Suc n)
      unfolding f_f'_def using summable' z by (simp add: algebra_simps sum-
inf_mult)
    also have (∑ n. ?f' n * z^n) = (∑ n. of_nat (Suc n) * ?f (Suc n) * z^n)
      by (intro suminf_cong) (simp add: diffs_def)
    also have (∑ n. ?f' n * z^Suc n) = (∑ n. of_nat n * ?f n * z ^ n)
      using summable1 suminf_split_initial_segment[OF summable1] unfolding
diffs_def
      by (subst suminf_split_head, subst (asm) summable_Suc_iff) simp_all
    also have (∑ n. of_nat (Suc n) * ?f (Suc n) * z^n) + (∑ n. of_nat n * ?f n
* z^n) =
                (∑ n. a * ?f n * z^n)
```

```
    by (subst gbinomial_mult_1, subst suminf_add)
       (insert summable'[OF z] summable2,
        simp_all add: summable_powser_split_head algebra_simps diffs_def)
   also have ... = a * f z unfolding f_f'_def
    by (subst suminf_mult[symmetric]) (simp_all add: summable[OF z] mult_ac)
   finally have a * f z = (1 + z) * f' z by simp
 } note deriv = this

 have [derivative_intros]: (f has_field_derivative f' z) (at z) if norm z < of_real
K for z
   unfolding f_f'_def using K that
   by (intro termdiffs_strong[of ?f K z] summable_strong) simp_all
 have f 0 = (∑ n. if n = 0 then 1 else 0) unfolding f_f'_def by (intro sum-
inf_cong) simp
 also have ... = 1 using sums_single[of 0 λ_. 1::complex] unfolding sums_iff
by simp
 finally have [simp]: f 0 = 1 .

 have ∃ c. ∀ z∈ball 0 K. f z * (1 + z) powr (−a) = c
 proof (rule has_field_derivative_zero_constant)
   fix z :: complex assume z': z ∈ ball 0 K
   hence z: norm z < K by simp
   with K have nz: 1 + z ≠ 0 by (auto dest!: minus_unique)
   from z K have norm z < 1 by simp
  hence (1 + z) ∉ ℝ≤0 by (cases z) (auto simp: Complex_eq complex_nonpos_Reals_iff)
   hence ((λz. f z * (1 + z) powr (−a)) has_field_derivative
           f' z * (1 + z) powr (−a) − a * f z * (1 + z) powr (−a−1)) (at z)
using z
     by (auto intro!: derivative_eq_intros)
   also from z have a * f z = (1 + z) * f' z by (rule deriv)
    finally show ((λz. f z * (1 + z) powr (−a)) has_field_derivative 0) (at z
within ball 0 K)
     using nz by (simp add: field_simps powr_diff at_within_open[OF z'])
 qed simp_all
 then obtain c where c: ⋀z. z ∈ ball 0 K ⟹ f z * (1 + z) powr (−a) = c by
blast
 from c[of 0] and K have c = 1 by simp
 with c[of z] have f z = (1 + z) powr a using K
   by (simp add: powr_minus field_simps dist_complex_def)
 with summable K show ?thesis unfolding f_f'_def by (simp add: sums_iff)
qed

lemma gen_binomial_complex':
 fixes x y :: real and a :: complex
 assumes |x| < |y|
 shows   (λn. (a gchoose n) * of_real x^n * of_real y powr (a − of_nat n)) sums
           of_real (x + y) powr a (is ?P x y)
proof −
 {
```

```
    fix x y :: real assume xy: |x| < |y| y ≥ 0
    hence y > 0 by simp
    note xy = xy this
    from xy have (λn. (a gchoose n) * of_real (x / y) ^ n) sums (1 + of_real (x
/ y)) powr a
        by (intro gen_binomial_complex) (simp add: norm_divide)
    hence (λn. (a gchoose n) * of_real (x / y) ^ n * y powr a) sums
              ((1 + of_real (x / y)) powr a * y powr a)
      by (rule sums_mult2)
     also have (1 + complex_of_real (x / y)) = complex_of_real (1 + x/y) by
simp
    also from xy have ... powr a * of_real y powr a = (... * y) powr a
      by (subst powr_times_real[symmetric]) (simp_all add: field_simps)
   also from xy have complex_of_real (1 + x / y) * complex_of_real y = of_real
(x + y)
      by (simp add: field_simps)
    finally have ?P x y using xy by (simp add: field_simps powr_diff powr_nat)
  } note A = this

  show ?thesis
  proof (cases y < 0)
    assume y: y < 0
    with assms have xy: x + y < 0 by simp
    with assms have |-x| < |-y| -y ≥ 0 by simp_all
    note A[OF this]
   also have complex_of_real (-x + -y) = - complex_of_real (x + y) by simp
   also from xy assms have ... powr a = (-1) powr -a * of_real (x + y) powr a
    by (subst powr_neg_real_complex) (simp add: abs_real_def split: if_split_asm)
    also {
      fix n :: nat
      from y have (a gchoose n) * of_real (-x) ^ n * of_real (-y) powr (a -
of_nat n) =
                    (a gchoose n) * (-of_real x / -of_real y) ^ n * (- of_real y)
powr a
        by (subst power_divide) (simp add: powr_diff powr_nat)
      also from y have (- of_real y) powr a = (-1) powr -a * of_real y powr a
        by (subst powr_neg_real_complex) simp
      also have -complex_of_real x / -complex_of_real y = complex_of_real x
/ complex_of_real y
        by simp
     also have ... ^ n = of_real x ^ n / of_real y ^ n by (simp add: power_divide)
      also have (a gchoose n) * ... * ((-1) powr -a * of_real y powr a) =
                  (-1) powr -a * ((a gchoose n) * of_real x ^ n * of_real y powr
(a - n))
        by (simp add: algebra_simps powr_diff powr_nat)
      finally have (a gchoose n) * of_real (- x) ^ n * of_real (- y) powr (a -
of_nat n) =
                    (-1) powr -a * ((a gchoose n) * of_real x ^ n * of_real y powr
(a - of_nat n)) .
```

```
    }
    note sums_cong[OF this]
    finally show ?thesis by (simp add: sums_mult_iff)
  qed (insert A[of x y] assms, simp_all add: not_less)
qed

lemma gen_binomial_complex'':
  fixes x y :: real and a :: complex
  assumes |y| < |x|
  shows   (λn. (a gchoose n) * of_real x powr (a - of_nat n) * of_real y ^ n) sums
              of_real (x + y) powr a
  using gen_binomial_complex'[OF assms] by (simp add: mult_ac add.commute)

lemma gen_binomial_real:
  fixes z :: real
  assumes |z| < 1
  shows   (λn. (a gchoose n) * z^n) sums (1 + z) powr a
proof -
  from assms have norm (of_real z :: complex) < 1 by simp
  from gen_binomial_complex[OF this]
    have (λn. (of_real a gchoose n :: complex) * of_real z ^ n) sums
            (of_real (1 + z)) powr (of_real a) by simp
  also have (of_real (1 + z) :: complex) powr (of_real a) = of_real ((1 + z) powr a)
    using assms by (subst powr_of_real) simp_all
  also have (of_real a gchoose n :: complex) = of_real (a gchoose n) for n
    by (simp add: gbinomial_prod_rev)
  hence (λn. (of_real a gchoose n :: complex) * of_real z ^ n) =
          (λn. of_real ((a gchoose n) * z ^ n)) by (intro ext) simp
  finally show ?thesis by (simp only: sums_of_real_iff)
qed

lemma gen_binomial_real':
  fixes x y a :: real
  assumes |x| < y
  shows   (λn. (a gchoose n) * x^n * y powr (a - of_nat n)) sums (x + y) powr a
proof -
  from assms have y > 0 by simp
  note xy = this assms
  from assms have |x / y| < 1 by simp
  hence (λn. (a gchoose n) * (x / y) ^ n) sums (1 + x / y) powr a
    by (rule gen_binomial_real)
  hence (λn. (a gchoose n) * (x / y) ^ n * y powr a) sums ((1 + x / y) powr a * y powr a)
    by (rule sums_mult2)
  with xy show ?thesis
    by (simp add: field_simps powr_divide powr_diff powr_realpow)
```

qed

lemma *one_plus_neg_powr_powser*:
fixes *z s* :: *complex*
assumes *norm* (*z* :: *complex*) < *1*
shows (λ*n*. (−*1*)^*n* ∗ ((*s* + *n* − *1*) *gchoose n*) ∗ *z*^*n*) *sums* (*1* + *z*) *powr* (−*s*)
using *gen_binomial_complex*[*OF assms*, *of* −*s*] **by** (*simp add*: *gbinomial_minus*)

lemma *gen_binomial_real′′*:
fixes *x y a* :: *real*
assumes |*y*| < *x*
shows (λ*n*. (*a gchoose n*) ∗ *x powr* (*a* − *of_nat n*) ∗ *y*^*n*) *sums* (*x* + *y*) *powr* *a*
using *gen_binomial_real′*[*OF assms*] **by** (*simp add*: *mult_ac add.commute*)

lemma *sqrt_series′*:
|*z*| < *a* ⟹ (λ*n*. ((*1*/*2*) *gchoose n*) ∗ *a powr* (*1*/*2* − *real_of_nat n*) ∗ *z* ^ *n*) *sums*
sqrt (*a* + *z* :: *real*)
using *gen_binomial_real′′*[*of z a 1*/*2*] **by** (*simp add*: *powr_half_sqrt*)

lemma *sqrt_series*:
|*z*| < *1* ⟹ (λ*n*. ((*1*/*2*) *gchoose n*) ∗ *z* ^ *n*) *sums sqrt* (*1* + *z*)
using *gen_binomial_real*[*of z 1*/*2*] **by** (*simp add*: *powr_half_sqrt*)

end

9.31 Vitali Covering Theorem and an Application to Negligibility

theory *Vitali_Covering_Theorem*
imports
HOL−Combinatorics.Permutations
Equivalence_Lebesgue_Henstock_Integration
begin

lemma *stretch_Galois*:
fixes *x* :: *real*^′*n*
shows (⋀*k*. *m k* ≠ *0*) ⟹ ((*y* = (χ *k*. *m k* ∗ *x*$*k*)) ⟷ (χ *k*. *y*$*k* / *m k*) = *x*)
by *auto*

lemma *lambda_swap_Galois*:
(*x* = (χ *i*. *y* $ *Transposition.transpose m n i*) ⟷ (χ *i*. *x* $ *Transposition.transpose m n i*) = *y*)
by (*auto*; *simp add*: *pointfree_idE vec_eq_iff*)

lemma *lambda_add_Galois*:
fixes *x* :: *real*^′*n*

shows $m \neq n \Longrightarrow (x = (\chi\ i.\ \mathit{if}\ i = m\ \mathit{then}\ y\$m + y\$n\ \mathit{else}\ y\$i) \longleftrightarrow (\chi\ i.\ \mathit{if}\ i = m\ \mathit{then}\ x\$m - x\$n\ \mathit{else}\ x\$i) = y)$
by (*safe*; *simp add*: *vec_eq_iff*)

lemma *Vitali_covering_lemma_cballs_balls*:
fixes $a :: 'a \Rightarrow 'b{::}\mathit{euclidean\_space}$
assumes $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i \wedge r\ i \leq B$
obtains C **where** $\mathit{countable}\ C\ C \subseteq K$
$\mathit{pairwise}\ (\lambda i\ j.\ \mathit{disjnt}\ (\mathit{cball}\ (a\ i)\ (r\ i))\ (\mathit{cball}\ (a\ j)\ (r\ j)))\ C$
$\bigwedge i.\ i \in K \Longrightarrow \exists j.\ j \in C \wedge$
$\neg\ \mathit{disjnt}\ (\mathit{cball}\ (a\ i)\ (r\ i))\ (\mathit{cball}\ (a\ j)\ (r\ j)) \wedge$
$\mathit{cball}\ (a\ i)\ (r\ i) \subseteq \mathit{ball}\ (a\ j)\ (5 * r\ j)$
proof (*cases* $K = \{\}$)
case *True*
with *that* **show** *?thesis*
by *auto*
next
case *False*
then have $B > 0$
using *assms less_le_trans* **by** *auto*
have *rgt0*[*simp*]: $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i$
using *assms* **by** *auto*
let *?djnt* = $\mathit{pairwise}\ (\lambda i\ j.\ \mathit{disjnt}\ (\mathit{cball}\ (a\ i)\ (r\ i))\ (\mathit{cball}\ (a\ j)\ (r\ j)))$
have $\exists C.\ \forall n.\ (C\ n \subseteq K \wedge$
$(\forall i \in C\ n.\ B/2\ \hat{}\ n \leq r\ i) \wedge \mathit{?djnt}\ (C\ n) \wedge$
$(\forall i \in K.\ B/2\ \hat{}\ n < r\ i$
$\longrightarrow (\exists j.\ j \in C\ n \wedge$
$\neg\ \mathit{disjnt}\ (\mathit{cball}\ (a\ i)\ (r\ i))\ (\mathit{cball}\ (a\ j)\ (r\ j)) \wedge$
$\mathit{cball}\ (a\ i)\ (r\ i) \subseteq \mathit{ball}\ (a\ j)\ (5 * r\ j)))) \wedge (C\ n \subseteq C(\mathit{Suc}\ n))$
proof (*rule dependent_nat_choice*, *safe*)
fix $C\ n$
define D **where** $D \equiv \{i \in K.\ B/2\ \hat{}\ \mathit{Suc}\ n < r\ i \wedge (\forall j \in C.\ \mathit{disjnt}\ (\mathit{cball}(a\ i)(r\ i))\ (\mathit{cball}\ (a\ j)\ (r\ j)))\}$
let *?cover_ar* = $\lambda i\ j.\ \neg\ \mathit{disjnt}\ (\mathit{cball}\ (a\ i)\ (r\ i))\ (\mathit{cball}\ (a\ j)\ (r\ j)) \wedge$
$\mathit{cball}\ (a\ i)\ (r\ i) \subseteq \mathit{ball}\ (a\ j)\ (5 * r\ j)$
assume $C \subseteq K$
and *Ble*: $\forall i \in C.\ B/2\ \hat{}\ n \leq r\ i$
and *djntC*: *?djnt C*
and *cov_n*: $\forall i \in K.\ B/2\ \hat{}\ n < r\ i \longrightarrow (\exists j.\ j \in C \wedge \mathit{?cover\_ar}\ i\ j)$
have $*$: $\forall C \in \mathit{chains}\ \{C.\ C \subseteq D \wedge \mathit{?djnt}\ C\}.\ \bigcup C \in \{C.\ C \subseteq D \wedge \mathit{?djnt}\ C\}$
proof (*clarsimp simp*: *chains_def*)
fix C
assume C: $C \subseteq \{C.\ C \subseteq D \wedge \mathit{?djnt}\ C\}$ **and** $\mathit{chain}_{\subseteq}\ C$
show $\bigcup C \subseteq D \wedge \mathit{?djnt}\ (\bigcup C)$
unfolding *pairwise_def*
proof (*intro ballI conjI impI*)
show $\bigcup C \subseteq D$
using C **by** *blast*

```
    next
      fix x y
      assume x ∈ ⋃ C and y ∈ ⋃ C and x ≠ y
      then obtain X Y where XY: x ∈ X X ∈ C y ∈ Y Y ∈ C
        by blast
      then consider X ⊆ Y | Y ⊆ X
        by (meson ‹chain⊆ C› chain_subset_def)
      then show disjnt (cball (a x) (r x)) (cball (a y) (r y))
      proof cases
        case 1
        with C XY ‹x ≠ y› show ?thesis
          unfolding pairwise_def by blast
      next
        case 2
        with C XY ‹x ≠ y› show ?thesis
          unfolding pairwise_def by blast
      qed
    qed
    obtain E where E ⊆ D and djntE: ?djnt E and maximalE: ⋀X. ⟦X ⊆ D;
?djnt X; E ⊆ X⟧ ⟹ X = E
      using Zorn_Lemma [OF *] by safe blast
    show ∃L. (L ⊆ K ∧
            (∀i∈L. B/2 ^ Suc n ≤ r i) ∧ ?djnt L ∧
           (∀i∈K. B/2 ^ Suc n < r i ⟶ (∃j. j ∈ L ∧ ?cover_ar i j))) ∧ C ⊆ L
    proof (intro exI conjI ballI)
      show C ∪ E ⊆ K
        using D_def ‹C ⊆ K› ‹E ⊆ D› by blast
      show B/2 ^ Suc n ≤ r i if i: i ∈ C ∪ E for i
        using i
      proof
        assume i ∈ C
        have B/2 ^ Suc n ≤ B/2 ^ n
          using ‹B > 0› by (simp add: field_split_simps)
        also have ... ≤ r i
          using Ble ‹i ∈ C› by blast
        finally show ?thesis .
      qed (use D_def ‹E ⊆ D› in auto)
      show ?djnt (C ∪ E)
        using D_def ‹C ⊆ K› ‹E ⊆ D› djntC djntE
        unfolding pairwise_def disjnt_def by blast
    next
      fix i
      assume i ∈ K
      show B/2 ^ Suc n < r i ⟶ (∃j. j ∈ C ∪ E ∧ ?cover_ar i j)
      proof (cases r i ≤ B/2^n)
        case False
        then show ?thesis
          using cov_n ‹i ∈ K› by auto
```

```
    next
      case True
      have cball (a i) (r i) ⊆ ball (a j) (5 * r j)
        if less: B/2 ^ Suc n < r i and j: j ∈ C ∪ E
          and nondis: ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) for j
      proof −
        obtain x where x: dist (a i) x ≤ r i dist (a j) x ≤ r j
          using nondis by (force simp: disjnt_def)
        have dist (a i) (a j) ≤ dist (a i) x + dist x (a j)
          by (simp add: dist_triangle)
        also have ... ≤ r i + r j
          by (metis add_mono_thms_linordered_semiring(1) dist_commute x)
        finally have aij: dist (a i) (a j) + r i < 5 * r j if r i < 2 * r j
          using that by auto
        show ?thesis
          using j
        proof
          assume j ∈ C
          have B/2^n < 2 * r j
            using Ble True ‹j ∈ C› less by auto
          with aij True show cball (a i) (r i) ⊆ ball (a j) (5 * r j)
            by (simp add: cball_subset_ball_iff)
        next
          assume j ∈ E
          then have B/2 ^ n < 2 * r j
            using D_def ‹E ⊆ D› by auto
          with True have r i < 2 * r j
            by auto
          with aij show cball (a i) (r i) ⊆ ball (a j) (5 * r j)
            by (simp add: cball_subset_ball_iff)
        qed
      qed
    moreover have ∃j. j ∈ C ∪ E ∧ ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j))
      if B/2 ^ Suc n < r i
    proof (rule classical)
      assume NON: ¬ ?thesis
      show ?thesis
      proof (cases i ∈ D)
        case True
        have insert i E = E
        proof (rule maximalE)
          show insert i E ⊆ D
            by (simp add: True ‹E ⊆ D›)
         show pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) (insert i
E)
            using False NON by (auto simp: pairwise_insert djntE disjnt_sym)
        qed auto
        then show ?thesis
          using ‹i ∈ K› assms by fastforce
```

```
      next
        case False
        with that show ?thesis
          by (auto simp: D_def disjnt_def ‹i ∈ K›)
      qed
    qed
    ultimately
    show B/2 ^ Suc n < r i ⟶
        (∃j. j ∈ C ∪ E ∧
            ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) ∧
            cball (a i) (r i) ⊆ ball (a j) (5 * r j))
      by blast
    qed
  qed auto
 qed (use assms in force)
 then obtain F where FK: ⋀n. F n ⊆ K
           and Fle: ⋀n i. i ∈ F n ⟹ B/2 ^ n ≤ r i
           and Fdjnt: ⋀n. ?djnt (F n)
           and FF: ⋀n i. ⟦i ∈ K; B/2 ^ n < r i⟧
                 ⟹ ∃j. j ∈ F n ∧ ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) ∧
                       cball (a i) (r i) ⊆ ball (a j) (5 * r j)
           and inc: ⋀n. F n ⊆ F(Suc n)
  by (force simp: all_conj_distrib)
 show thesis
 proof
   have *: countable I
     if I ⊆ K and pw: pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j)))
I for I
   proof −
     show ?thesis
     proof (rule countable_image_inj_on [of λi. cball(a i)(r i)])
       show countable ((λi. cball (a i) (r i)) ‘ I)
       proof (rule countable_disjoint_nonempty_interior_subsets)
         show disjoint ((λi. cball (a i) (r i)) ‘ I)
           by (auto simp: dest: pairwiseD [OF pw] intro: pairwise_imageI)
         show ⋀S. ⟦S ∈ (λi. cball (a i) (r i)) ‘ I; interior S = {}⟧ ⟹ S = {}
           using ‹I ⊆ K›
           by (auto simp: not_less [symmetric])
       qed
     next
       have ⋀x y. ⟦x ∈ I; y ∈ I; a x = a y; r x = r y⟧ ⟹ x = y
         using pw ‹I ⊆ K› assms
         apply (clarsimp simp: pairwise_def disjnt_def)
      by (metis assms centre_in_cball subsetD empty_iff inf.idem less_eq_real_def)
       then show inj_on (λi. cball (a i) (r i)) I
          using ‹I ⊆ K› by (fastforce simp: inj_on_def cball_eq_cball_iff dest:
assms)
     qed
   qed
```

```
    show (Union(range F)) ⊆ K
      using FK by blast
     moreover show pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j)))
(Union(range F))
    proof (rule pairwise_chain_Union)
      show chain⊆ (range F)
       unfolding chain_subset_def by clarify (meson inc lift_Suc_mono_le linear
subsetCE)
    qed (use Fdjnt in blast)
    ultimately show countable (Union(range F))
      by (blast intro: ∗)
  next
    fix i assume i ∈ K
    then obtain n where (1/2) ^ n < r i / B
      using ‹B > 0› assms real_arch_pow_inv by fastforce
    then have B2: B/2 ^ n < r i
      using ‹B > 0› by (simp add: field_split_simps)
    have 0 < r i r i ≤ B
      by (auto simp: ‹i ∈ K› assms)
    show ∃j. j ∈ (Union(range F)) ∧
          ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j)) ∧
          cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
      using FF [OF ‹i ∈ K› B2] by auto
  qed
qed
```

9.31.1 Vitali covering theorem

```
lemma Vitali_covering_lemma_cballs:
  fixes a :: 'a ⇒ 'b::euclidean_space
  assumes S: S ⊆ (⋃i∈K. cball (a i) (r i))
      and r: ⋀i. i ∈ K ⟹ 0 < r i ∧ r i ≤ B
  obtains C where countable C C ⊆ K
     pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
     S ⊆ (⋃i∈C. cball (a i) (5 ∗ r i))
proof −
  obtain C where C: countable C C ⊆ K
                    pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
          and cov: ⋀i. i ∈ K ⟹ ∃j. j ∈ C ∧ ¬ disjnt (cball (a i) (r i)) (cball (a
j) (r j)) ∧
                                cball (a i) (r i) ⊆ ball (a j) (5 ∗ r j)
    by (rule Vitali_covering_lemma_cballs_balls [OF r, where a=a]) (blast intro:
that)+
  show ?thesis
  proof
    have (⋃i∈K. cball (a i) (r i)) ⊆ (⋃i∈C. cball (a i) (5 ∗ r i))
      using cov subset_iff by fastforce
    with S show S ⊆ (⋃i∈C. cball (a i) (5 ∗ r i))
      by blast
```

qed (*use C* **in** *auto*)
qed

lemma *Vitali_covering_lemma_balls*:
fixes *a* :: *'a* ⇒ *'b*::*euclidean_space*
assumes *S*: $S \subseteq (\bigcup i{\in}K.\ ball\ (a\ i)\ (r\ i))$
and *r*: $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i \wedge r\ i \le B$
obtains *C* **where** *countable C* $C \subseteq K$
pairwise $(\lambda i\ j.\ disjnt\ (ball\ (a\ i)\ (r\ i))\ (ball\ (a\ j)\ (r\ j)))\ C$
$S \subseteq (\bigcup i{\in}C.\ ball\ (a\ i)\ (5 * r\ i))$
proof −
obtain *C* **where** *C*: *countable C* $C \subseteq K$
and *pw*: *pairwise* $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))\ C$
and *cov*: $\bigwedge i.\ i \in K \Longrightarrow \exists j.\ j \in C \wedge \neg\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)) \wedge$
$cball\ (a\ i)\ (r\ i) \subseteq ball\ (a\ j)\ (5 * r\ j)$
by (*rule Vitali_covering_lemma_cballs_balls* [*OF r*, **where** *a*=*a*]) (*blast intro*: *that*)+
show *?thesis*
proof
have $(\bigcup i{\in}K.\ ball\ (a\ i)\ (r\ i)) \subseteq (\bigcup i{\in}C.\ ball\ (a\ i)\ (5 * r\ i))$
using *cov subset_iff*
by *clarsimp* (*meson less_imp_le mem_ball mem_cball subset_eq*)
with *S* **show** $S \subseteq (\bigcup i{\in}C.\ ball\ (a\ i)\ (5 * r\ i))$
by *blast*
show *pairwise* $(\lambda i\ j.\ disjnt\ (ball\ (a\ i)\ (r\ i))\ (ball\ (a\ j)\ (r\ j)))\ C$
using *pw*
by (*clarsimp simp*: *pairwise_def*) (*meson ball_subset_cball disjnt_subset1 disjnt_subset2*)
qed (*use C* **in** *auto*)
qed

theorem *Vitali_covering_theorem_cballs*:
fixes *a* :: *'a* ⇒ *'n*::*euclidean_space*
assumes *r*: $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i$
and *S*: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!] \Longrightarrow \exists i.\ i \in K \wedge x \in cball\ (a\ i)\ (r\ i) \wedge r\ i < d$
obtains *C* **where** *countable C* $C \subseteq K$
pairwise $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))\ C$
negligible$(S - (\bigcup i \in C.\ cball\ (a\ i)\ (r\ i)))$
proof −
let *?μ* = *measure lebesgue*
have ∗: $\exists C.\ countable\ C \wedge C \subseteq K \wedge$
pairwise $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))\ C \wedge$
negligible$(S - (\bigcup i \in C.\ cball\ (a\ i)\ (r\ i)))$
if *r01*: $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i \wedge r\ i \le 1$
and *Sd*: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!] \Longrightarrow \exists i.\ i \in K \wedge x \in cball\ (a\ i)\ (r\ i) \wedge r\ i < d$

```
    for K r and a :: 'a ⇒ 'n
  proof −
    obtain C where C: countable C C ⊆ K
      and pwC: pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
      and cov: ⋀i. i ∈ K ⟹ ∃j. j ∈ C ∧ ¬ disjnt (cball (a i) (r i)) (cball (a j)
(r j)) ∧
                                cball (a i) (r i) ⊆ ball (a j) (5 * r j)
      by (rule Vitali_covering_lemma_cballs_balls [of K r 1 a]) (auto simp: r01)
    have ar_injective: ⋀x y. ⟦x ∈ C; y ∈ C; a x = a y; r x = r y⟧ ⟹ x = y
      using ‹C ⊆ K› pwC cov
      by (force simp: pairwise_def disjnt_def)
    show ?thesis
    proof (intro exI conjI)
      show negligible (S − (⋃i∈C. cball (a i) (r i)))
      proof (clarsimp simp: negligible_on_intervals [of S−T for T])
        fix l u
        show negligible ((S − (⋃i∈C. cball (a i) (r i))) ∩ cbox l u)
          unfolding negligible_outer_le
        proof (intro allI impI)
          fix e::real
          assume e > 0
          define D where D ≡ {i ∈ C. ¬ disjnt (ball(a i) (5 * r i)) (cbox l u)}
          then have D ⊆ C
            by auto
          have countable D
            unfolding D_def using ‹countable C› by simp
          have UD: (⋃i∈D. cball (a i) (r i)) ∈ lmeasurable
          proof (rule fmeasurableI2)
            show cbox (l − 6 *R One) (u + 6 *R One) ∈ lmeasurable
              by blast
            have y ∈ cbox (l − 6 *R One) (u + 6 *R One)
              if i ∈ C and x: x ∈ cbox l u and ai: dist (a i) y ≤ r i dist (a i) x < 5
* r i
              for i x y
            proof −
              have d6: dist y x < 6 * r i
                using dist_triangle3 [of y x a i] that by linarith
              show ?thesis
              proof (clarsimp simp: mem_box algebra_simps)
                fix j::'n
                assume j: j ∈ Basis
                then have xyj: |x • j − y • j| ≤ dist y x
                by (metis Basis_le_norm dist_commute dist_norm inner_diff_left)
                have l • j ≤ x • j
                  using ‹j ∈ Basis› mem_box ‹x ∈ cbox l u› by blast
                also have ... ≤ y • j + 6 * r i
                  using d6 xyj by (auto simp: algebra_simps)
                also have ... ≤ y • j + 6
                  using r01 [of i] ‹C ⊆ K› ‹i ∈ C› by auto
```

```
        finally have l: l · j ≤ y · j + 6 .
        have y · j ≤ x · j + 6 * r i
          using d6 xyj by (auto simp: algebra_simps)
        also have ... ≤ u · j + 6 * r i
          using j  x by (auto simp: mem_box)
        also have ... ≤ u · j + 6
          using r01 [of i] ‹C ⊆ K› ‹i ∈ C› by auto
        finally have u: y · j ≤ u · j + 6 .
        show l · j ≤ y · j + 6 ∧ y · j ≤ u · j + 6
          using l u by blast
      qed
    qed
   then show (⋃i∈D. cball (a i) (r i)) ⊆ cbox (l − 6 *R One) (u + 6 *R
One)
      by (force simp: D_def disjnt_def)
    show (⋃i∈D. cball (a i) (r i)) ∈ sets lebesgue
      using ‹countable D› by auto
  qed
  obtain D1 where D1 ⊆ D finite D1
    and measD1: ?μ (⋃i∈D. cball (a i) (r i)) − e / 5 ^ DIM('n) < ?μ
(⋃i∈D1. cball (a i) (r i))
  proof (rule measure_countable_Union_approachable [where e = e / 5 ^
(DIM('n))])
    show countable ((λi. cball (a i) (r i)) ‘ D)
      using ‹countable D› by auto
    show ⋀d. d ∈ (λi. cball (a i) (r i)) ‘ D ⟹ d ∈ lmeasurable
      by auto
    show ⋀D'. ⟦D' ⊆ (λi. cball (a i) (r i)) ‘ D; finite D'⟧ ⟹ ?μ (⋃ D') ≤
?μ (⋃i∈D. cball (a i) (r i))
      by (fastforce simp add: intro!: measure_mono_fmeasurable UD)
  qed (use ‹e > 0› in ‹auto dest: finite_subset_image›)
  show ∃ T. (S − (⋃i∈C. cball (a i) (r i))) ∩
          cbox l u ⊆ T ∧ T ∈ lmeasurable ∧ ?μ T ≤ e
  proof (intro exI conjI)
    show (S − (⋃i∈C. cball (a i) (r i))) ∩ cbox l u ⊆ (⋃i∈D − D1. ball
(a i) (5 * r i))
    proof clarify
      fix x
      assume x: x ∈ cbox l u x ∈ S x ∉ (⋃i∈C. cball (a i) (r i))
      have closed (⋃i∈D1. cball (a i) (r i))
        using ‹finite D1› by blast
      moreover have x ∉ (⋃j∈D1. cball (a j) (r j))
        using x ‹D1 ⊆ D› unfolding D_def by blast
      ultimately obtain q where q > 0 and q: ball x q ⊆ − (⋃i∈D1. cball
(a i) (r i))
        by (metis (no_types, lifting) ComplI open_contains_ball closed_def)
      obtain i where i ∈ K and xi: x ∈ cball (a i) (r i) and ri: r i < q/2
        using Sd [OF ‹x ∈ S›] ‹q > 0› half_gt_zero by blast
      then obtain j where j ∈ C
```

```
                    and nondisj: ¬ disjnt (cball (a i) (r i)) (cball (a j) (r j))
                    and sub5j:  cball (a i) (r i) ⊆ ball (a j) (5 * r j)
            using cov [OF ‹i ∈ K›] by metis
          show x ∈ (⋃ i∈D − D1. ball (a i) (5 * r i))
          proof
            show j ∈ D − D1
            proof
              show j ∈ D
                  using ‹j ∈ C› sub5j ‹x ∈ cbox l u› xi by (auto simp: D_def
disjnt_def)
              obtain y where yi: dist (a i) y ≤ r i and yj: dist (a j) y ≤ r j
                using disjnt_def nondisj by fastforce
              have dist x y ≤ r i + r i
                by (metis add_mono dist_commute dist_triangle_le mem_cball
xi yi)
              also have ... < q
                using ri by linarith
              finally have y ∈ ball x q
                by simp
              with yj q show j ∉ D1
                by (auto simp: disjoint_UN_iff)
            qed
            show x ∈ ball (a j) (5 * r j)
              using xi sub5j by blast
          qed
        qed
        have 3: ?μ (⋃ i∈D2. ball (a i) (5 * r i)) ≤ e
          if D2: D2 ⊆ D − D1 and finite D2 for D2
        proof −
          have rgt0: 0 < r i if i ∈ D2 for i
            using ‹C ⊆ K› D_def ‹i ∈ D2› D2 r01
            by (simp add: subset_iff)
          then have inj: inj_on (λi. ball (a i) (5 * r i)) D2
                using ‹C ⊆ K› D2 by (fastforce simp: inj_on_def D_def
ball_eq_ball_iff intro: ar_injective)
           have ?μ (⋃ i∈D2. ball (a i) (5 * r i)) ≤ sum (?μ) ((λi. ball (a i) (5
* r i)) ‘ D2)
            using that by (force intro: measure_Union_le)
          also have ... = (∑ i∈D2. ?μ (ball (a i) (5 * r i)))
            by (simp add: comm_monoid_add_class.sum.reindex [OF inj])
          also have ... = (∑ i∈D2. 5 ^ DIM('n) * ?μ (ball (a i) (r i)))
          proof (rule sum.cong [OF refl])
            fix i assume i ∈ D2
            thus ?μ (ball (a i) (5 * r i)) = 5 ^ DIM('n) * ?μ (ball (a i) (r i))
              using content_ball_conv_unit_ball[of 5 * r i a i]
                    content_ball_conv_unit_ball[of r i a i] rgt0[of i] by auto
          qed
          also have ... = (∑ i∈D2. ?μ (ball (a i) (r i))) * 5 ^ DIM('n)
            by (simp add: sum_distrib_left mult.commute)
```

```
          finally have ?μ (⋃i∈D2. ball (a i) (5 * r i)) ≤ (∑i∈D2. ?μ (ball (a
i) (r i))) * 5 ^ DIM('n) .
          moreover have (∑i∈D2. ?μ (ball (a i) (r i))) ≤ e / 5 ^ DIM('n)
          proof −
            have D12_dis: ((⋃x∈D1. cball (a x) (r x)) ∩ (⋃x∈D2. cball (a x)
(r x))) ≤ {}
            proof clarify
              fix w d1 d2
              assume d1 ∈ D1 w d1 d2 ∈ cball (a d1) (r d1) d2 ∈ D2 w d1 d2
∈ cball (a d2) (r d2)
              then show w d1 d2 ∈ {}
                  by (metis DiffE disjnt_iff subsetCE D2 ‹D1 ⊆ D› ‹D ⊆ C›
pairwiseD [OF pwC, of d1 d2])
            qed
            have inj: inj_on (λi. cball (a i) (r i)) D2
           using rgt0 D2 ‹D ⊆ C› by (force simp: inj_on_def cball_eq_cball_iff
intro!: ar_injective)
            have ds: disjoint ((λi. cball (a i) (r i)) ‘ D2)
             using D2 ‹D ⊆ C› by (auto intro: pairwiseI pairwiseD [OF pwC])
            have (∑i∈D2. ?μ (ball (a i) (r i))) = (∑i∈D2. ?μ (cball (a i) (r
i)))
              by (simp add: content_cball_conv_ball)
            also have ... = sum ?μ ((λi. cball (a i) (r i)) ‘ D2)
              by (simp add: comm_monoid_add_class.sum.reindex [OF inj])
            also have ... = ?μ (⋃i∈D2. cball (a i) (r i))
               by (auto intro: measure_Union' [symmetric] ds simp add: ‹finite
D2›)
            finally have ?μ (⋃i∈D1. cball (a i) (r i)) + (∑i∈D2. ?μ (ball (a
i) (r i))) =
                                 ?μ (⋃i∈D1. cball (a i) (r i)) + ?μ (⋃i∈D2. cball (a i)
(r i))
              by simp
            also have ... = ?μ (⋃i ∈ D1 ∪ D2. cball (a i) (r i))
             using D12_dis by (simp add: measure_Un3 ‹finite D1› ‹finite D2›
fmeasurable.finite_UN)
            also have ... ≤ ?μ (⋃i∈D. cball (a i) (r i))
           using D2 ‹D1 ⊆ D› by (fastforce intro!: measure_mono_fmeasurable
[OF _ _ UD] ‹finite D1› ‹finite D2›)
            finally have ?μ (⋃i∈D1. cball (a i) (r i)) + (∑i∈D2. ?μ (ball (a
i) (r i))) ≤ ?μ (⋃i∈D. cball (a i) (r i)) .
            with measD1 show ?thesis
              by simp
          qed
          ultimately show ?thesis
            by (simp add: field_split_simps)
        qed
        have co: countable (D − D1)
          by (simp add: ‹countable D›)
        show (⋃i∈D − D1. ball (a i) (5 * r i)) ∈ lmeasurable
```

```
          using ‹e > 0› by (auto simp: fmeasurable_UN_bound [OF co _ 3])
        show ?μ (⋃i∈D − D1. ball (a i) (5 * r i)) ≤ e
          using ‹e > 0› by (auto simp: measure_UN_bound [OF co _ 3])
      qed
    qed
  qed
 qed (use C pwC in auto)
qed
define K' where K' ≡ {i ∈ K. r i ≤ 1}
have 1: ⋀i. i ∈ K' ⟹ 0 < r i ∧ r i ≤ 1
 using K'_def r by auto
have 2: ∃ i. i ∈ K' ∧ x ∈ cball (a i) (r i) ∧ r i < d
 if x ∈ S ∧ 0 < d for x d
 using that by (auto simp: K'_def dest!: S [where d = min d 1])
have K' ⊆ K
 using K'_def by auto
then show thesis
 using * [OF 1 2] that by fastforce
qed


theorem Vitali_covering_theorem_balls:
 fixes a :: 'a ⇒ 'b::euclidean_space
 assumes S: ⋀x d. ⟦x ∈ S; 0 < d⟧ ⟹ ∃ i. i ∈ K ∧ x ∈ ball (a i) (r i) ∧ r i < d
 obtains C where countable C C ⊆ K
   pairwise (λi j. disjnt (ball (a i) (r i)) (ball (a j) (r j))) C
   negligible(S − (⋃ i ∈ C. ball (a i) (r i)))
proof −
 have 1: ∃ i. i ∈ {i ∈ K. 0 < r i} ∧ x ∈ cball (a i) (r i) ∧ r i < d
       if xd: x ∈ S d > 0 for x d
  by (metis (mono_tags, lifting) assms ball_eq_empty less_eq_real_def mem_Collect_eq
mem_ball mem_cball not_le xd(1) xd(2))
 obtain C where C: countable C C ⊆ K
           and pw: pairwise (λi j. disjnt (cball (a i) (r i)) (cball (a j) (r j))) C
           and neg: negligible(S − (⋃ i ∈ C. cball (a i) (r i)))
  by (rule Vitali_covering_theorem_cballs [of {i ∈ K. 0 < r i} r S a, OF _ 1])
auto
 show thesis
 proof
  show pairwise (λi j. disjnt (ball (a i) (r i)) (ball (a j) (r j))) C
   apply (rule pairwise_mono [OF pw])
   apply (auto simp: disjnt_def)
   by (meson disjoint_iff_not_equal less_imp_le mem_cball)
  have negligible (⋃i∈C. sphere (a i) (r i))
   by (auto intro: negligible_sphere ‹countable C›)
  then have negligible (S − (⋃ i ∈ C. cball(a i)(r i)) ∪ (⋃ i ∈ C. sphere (a i)
(r i)))
   by (rule negligible_Un [OF neg])
  then show negligible (S − (⋃i∈C. ball (a i) (r i)))
```

```
    by (rule negligible_subset) force
  qed (use C in auto)
qed


lemma negligible_eq_zero_density_alt:
    negligible S ⟷
     (∀x∈S. ∀e>0.
      ∃d U. 0 < d ∧ d ≤ e ∧ S ∩ ball x d ⊆ U ∧
            U ∈ lmeasurable ∧ measure lebesgue U < e * measure lebesgue (ball x
d))
    (is _ = (∀x∈S. ∀e>0. ?Q x e))
proof (intro iffI ballI allI impI)
  fix x and e :: real
  assume negligible S and x ∈ S and e > 0
  then
  show ∃d U. 0 < d ∧ d ≤ e ∧ S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧
            measure lebesgue U < e * measure lebesgue (ball x d)
    apply (rule_tac x=e in exI)
    apply (rule_tac x=S ∩ ball x e in exI)
   apply (auto simp: negligible_imp_measurable negligible_Int negligible_imp_measure0
zero_less_measure_iff
               intro: mult_pos_pos content_ball_pos)
    done
next
  assume R [rule_format]: ∀x∈S. ∀e>0. ?Q x e
  let ?μ = measure lebesgue
  have ∃U. openin (top_of_set S) U ∧ z ∈ U ∧ negligible U
    if z ∈ S for z
  proof (intro exI conjI)
    show openin (top_of_set S) (S ∩ ball z 1)
      by (simp add: openin_open_Int)
    show z ∈ S ∩ ball z 1
      using ‹z ∈ S› by auto
    show negligible (S ∩ ball z 1)
    proof (clarsimp simp: negligible_outer_le)
      fix e :: real
      assume e > 0
      let ?K = {(x,d). x ∈ S ∧ 0 < d ∧ ball x d ⊆ ball z 1 ∧
                    (∃U. S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧
                       ?μ U < e / ?μ (ball z 1) * ?μ (ball x d))}
      obtain C where countable C and Csub: C ⊆ ?K
        and pwC: pairwise (λi j. disjnt (ball (fst i) (snd i)) (ball (fst j) (snd j))) C
        and negC: negligible((S ∩ ball z 1) - (⋃i∈C. ball (fst i) (snd i)))
      proof (rule Vitali_covering_theorem_balls [of S ∩ ball z 1 ?K fst snd])
        fix x and d :: real
        assume x: x ∈ S ∩ ball z 1 and d > 0
        obtain k where k > 0 and k: ball x k ⊆ ball z 1
          by (meson Int_iff open_ball openE x)
```

```
      let ?ε = min (e / ?μ (ball z 1) / 2) (min (d / 2) k)
     obtain r U where r: r > 0 r ≤ ?ε and U: S ∩ ball x r ⊆ U U ∈ lmeasurable
       and mU: ?μ U < ?ε * ?μ (ball x r)
      using R [of x ?ε] ‹d > 0› ‹e > 0› ‹k > 0› x by (auto simp: content_ball_pos)
      show ∃ i. i ∈ ?K ∧ x ∈ ball (fst i) (snd i) ∧ snd i < d
      proof (rule exI [of _ (x,r)], simp, intro conjI exI)
       have ball x r ⊆ ball x k
        using r by (simp add: ball_subset_ball_iff)
       also have ... ⊆ ball z 1
        using ball_subset_ball_iff k by auto
       finally show ball x r ⊆ ball z 1 .
       have ?ε * ?μ (ball x r) ≤ e * content (ball x r) / content (ball z 1)
         using r ‹e > 0› by (simp add: ord_class.min_def field_split_simps
content_ball_pos)
       with mU show ?μ U < e * content (ball x r) / content (ball z 1)
        by auto
      qed (use r U x in auto)
     qed
     have ∃ U. case p of (x,d) ⇒ S ∩ ball x d ⊆ U ∧
                  U ∈ lmeasurable ∧ ?μ U < e / ?μ (ball z 1) * ?μ (ball x d)
      if p ∈ C for p
      using that Csub unfolding case_prod_unfold by blast
     then obtain U where U:
           ⋀p. p ∈ C ⟹
                case p of (x,d) ⇒ S ∩ ball x d ⊆ U p ∧
               U p ∈ lmeasurable ∧ ?μ (U p) < e / ?μ (ball z 1) * ?μ (ball x d)
      by (rule that [OF someI_ex])
     let ?T = ((S ∩ ball z 1) - (⋃(x,d)∈C. ball x d)) ∪ ⋃(U ‘ C)
     show ∃ T. S ∩ ball z 1 ⊆ T ∧ T ∈ lmeasurable ∧ ?μ T ≤ e
     proof (intro exI conjI)
      show S ∩ ball z 1 ⊆ ?T
       using U by fastforce
      { have Um: U i ∈ lmeasurable if i ∈ C for i
         using that U by blast
       have lee: ?μ (⋃i∈I. U i) ≤ e if I ⊆ C finite I for I
       proof −
        have ?μ (⋃(x,d)∈I. ball x d) ≤ ?μ (ball z 1)
         apply (rule measure_mono_fmeasurable)
            using ‹I ⊆ C› ‹finite I› Csub by (force simp: prod.case_eq_if
sets.finite_UN)+
        then have le1: (?μ (⋃(x,d)∈I. ball x d) / ?μ (ball z 1)) ≤ 1
         by (simp add: content_ball_pos)
        have ?μ (⋃i∈I. U i) ≤ (∑i∈I. ?μ (U i))
         using that U by (blast intro: measure_UNION_le)
        also have ... ≤ (∑(x,r)∈I. e / ?μ (ball z 1) * ?μ (ball x r))
         by (rule sum_mono) (use ‹I ⊆ C› U in force)
        also have ... = (e / ?μ (ball z 1)) * (∑(x,r)∈I. ?μ (ball x r))
         by (simp add: case_prod_app prod.case_distrib sum_distrib_left)
        also have ... = e * (?μ (⋃(x,r)∈I. ball x r) / ?μ (ball z 1))
```

```
        apply (subst measure_UNION')
       using that pwC by (auto simp: case_prod_unfold elim: pairwise_mono)
       also have ... ≤ e
        by (metis mult.commute mult.left_neutral mult_le_cancel_right_pos
‹e > 0› le1)
       finally show ?thesis .
     qed
     have ⋃(U ‘ C) ∈ lmeasurable ?μ (⋃(U ‘ C)) ≤ e
      using ‹e > 0› Um lee
          by(auto intro!: fmeasurable_UN_bound [OF ‹countable C›] mea-
sure_UN_bound [OF ‹countable C›])
    }
    moreover have ?μ ?T = ?μ (⋃(U ‘ C))
    proof (rule measure_negligible_symdiff [OF ‹⋃(U ‘ C) ∈ lmeasurable›])
     show negligible((⋃(U ‘ C) − ?T) ∪ (?T − ⋃(U ‘ C)))
      by (force intro!: negligible_subset [OF negC])
    qed
    ultimately show ?T ∈ lmeasurable ?μ ?T ≤ e
          by (simp_all add: fmeasurable.Un negC negligible_imp_measurable
split_def)
   qed
  qed
 qed
 with locally_negligible_alt show negligible S
  by metis
qed

proposition negligible_eq_zero_density:
  negligible S ⟷
   (∀ x∈S. ∀ r>0. ∀ e>0. ∃ d. 0 < d ∧ d ≤ r ∧
                (∃ U. S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧ measure lebesgue U
< e * measure lebesgue (ball x d)))
proof −
 let ?Q = λx d e. ∃ U. S ∩ ball x d ⊆ U ∧ U ∈ lmeasurable ∧ measure lebesgue
U < e * content (ball x d)
 have (∀ e>0. ∃ d>0. d ≤ e ∧ ?Q x d e) = (∀ r>0. ∀ e>0. ∃ d>0. d ≤ r ∧ ?Q
x d e)
   if x ∈ S for x
 proof (intro iffI allI impI)
   fix r :: real and e :: real
   assume L [rule_format]: ∀ e>0. ∃ d>0. d ≤ e ∧ ?Q x d e and r > 0 e > 0
   show ∃ d>0. d ≤ r ∧ ?Q x d e
     using L [of min r e] apply (rule ex_forward)
    using ‹r > 0› ‹e > 0› by (auto intro: less_le_trans elim!: ex_forward simp:
content_ball_pos)
 qed auto
 then show ?thesis
   by (force simp: negligible_eq_zero_density_alt)
qed
```

end

9.32 Change of Variables Theorems

theory *Change_Of_Vars*
imports *Vitali_Covering_Theorem Determinants*

begin

9.32.1 Measurable Shear and Stretch

proposition
fixes *a* :: *real^'n*
assumes $m \neq n$ **and** *ab_ne*: *cbox a b* $\neq \{\}$ **and** *an*: $0 \leq a\$n$
shows *measurable_shear_interval*: $(\lambda x.\ \chi\ i.\ \textit{if } i = m \textit{ then } x\$m + x\$n \textit{ else } x\$i)$ ‘ *(cbox a b)* ∈ *lmeasurable*
(**is** *?f ‘ _ ∈ _*)
and *measure_shear_interval*: *measure lebesgue* $((\lambda x.\ \chi\ i.\ \textit{if } i = m \textit{ then } x\$m + x\$n \textit{ else } x\$i)$ ‘ *cbox a b)*
= *measure lebesgue* (*cbox a b*) (**is** *?Q*)
proof −
have *lin*: *linear ?f*
by (*rule linearI*) (*auto simp*: *plus_vec_def scaleR_vec_def algebra_simps*)
show *fab*: *?f ‘ cbox a b* ∈ *lmeasurable*
by (*simp add*: *lin measurable_linear_image_interval*)
let *?c* = $\chi\ i.\ \textit{if } i = m \textit{ then } b\$m + b\$n \textit{ else } b\$i$
let *?mn* = *axis m 1* − *axis n* (*1*::*real*)
have *eq1*: *measure lebesgue* (*cbox a ?c*)
= *measure lebesgue* (*?f ‘ cbox a b*)
+ *measure lebesgue* (*cbox a ?c* ∩ $\{x.\ ?mn \cdot x \leq a\$m\}$)
+ *measure lebesgue* (*cbox a ?c* ∩ $\{x.\ ?mn \cdot x \geq b\$m\}$)
proof (*rule measure_Un3_negligible*)
show *cbox a ?c* ∩ $\{x.\ ?mn \cdot x \leq a\$m\}$ ∈ *lmeasurable cbox a ?c* ∩ $\{x.\ ?mn \cdot x \geq b\$m\}$ ∈ *lmeasurable*
by (*auto simp*: *convex_Int convex_halfspace_le convex_halfspace_ge bounded_Int measurable_convex*)
have *negligible* $\{x.\ ?mn \cdot x = a\$m\}$
by (*metis ‹m ≠ n› axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane*)
moreover have *?f ‘ cbox a b* ∩ (*cbox a ?c* ∩ $\{x.\ ?mn \cdot x \leq a\ \$\ m\}) \subseteq \{x.\ ?mn \cdot x = a\$m\}$
using *‹m ≠ n› antisym_conv* **by** (*fastforce simp*: *algebra_simps mem_box_cart inner_axis'*)
ultimately show *negligible* ((*?f ‘ cbox a b*) ∩ (*cbox a ?c* ∩ $\{x.\ ?mn \cdot x \leq a\ \$\ m\}$))
by (*rule negligible_subset*)
have *negligible* $\{x.\ ?mn \cdot x = b\$m\}$
by (*metis ‹m ≠ n› axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane*)
moreover have (*?f ‘ cbox a b*) ∩ (*cbox a ?c* ∩ $\{x.\ ?mn \cdot x \geq b\$m\}) \subseteq \{x.$

?mn · x = b$m}
using ‹*m ≠ n*› *antisym_conv* **by** (*fastforce simp*: *algebra_simps mem_box_cart inner_axis'*)
ultimately show *negligible* (*?f ‘ cbox a b ∩* (*cbox a ?c ∩* {*x*. *?mn · x ≥ b$m*}))
by (*rule negligible_subset*)
have *negligible* {*x*. *?mn · x = b$m*}
by (*metis* ‹*m ≠ n*› *axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane*)
moreover have (*cbox a ?c ∩* {*x*. *?mn · x ≤ a $ m*} *∩* (*cbox a ?c ∩* {*x*. *?mn · x ≥ b$m*})) *⊆* {*x*. *?mn · x = b$m*}
using ‹*m ≠ n*› *ab_ne*
apply (*clarsimp simp*: *algebra_simps mem_box_cart inner_axis'*)
by (*smt* (*verit*, *ccfv_SIG*) *interval_ne_empty_cart*(*1*))
ultimately show *negligible* (*cbox a ?c ∩* {*x*. *?mn · x ≤ a $ m*} *∩* (*cbox a ?c ∩* {*x*. *?mn · x ≥ b$m*}))
by (*rule negligible_subset*)
show *?f ‘ cbox a b ∪ cbox a ?c ∩* {*x*. *?mn · x ≤ a $ m*} *∪ cbox a ?c ∩* {*x*. *?mn · x ≥ b$m*} = *cbox a ?c* (**is** *?lhs = _*)
proof
show *?lhs ⊆ cbox a ?c*
by (*auto simp*: *mem_box_cart add_mono*) (*meson add_increasing2 an order_trans*)
show *cbox a ?c ⊆ ?lhs*
apply (*clarsimp simp*: *algebra_simps image_iff inner_axis' lambda_add_Galois* [*OF* ‹*m ≠ n*›])
by (*smt* (*verit*, *del_insts*) *mem_box_cart*(*2*) *vec_lambda_beta*)
qed
qed (*fact fab*)
let *?d* = *χ i*. *if i = m then a $ m − b $ m else 0*
have *eq2*: *measure lebesgue* (*cbox a ?c ∩* {*x*. *?mn · x ≤ a $ m*}) + *measure lebesgue* (*cbox a ?c ∩* {*x*. *?mn · x ≥ b$m*})
= *measure lebesgue* (*cbox a* (*χ i*. *if i = m then a $ m + b $ n else b $ i*))
proof (*rule measure_translate_add*[*of cbox a ?c ∩* {*x*. *?mn · x ≤ a$m*} *cbox a ?c ∩* {*x*. *?mn · x ≥ b$m*}
(*χ i*. *if i = m then a$m − b$m else 0*) *cbox a* (*χ i*. *if i = m then a$m + b$n else b$i*)])
show (*cbox a ?c ∩* {*x*. *?mn · x ≤ a$m*}) *∈ lmeasurable*
cbox a ?c ∩ {*x*. *?mn · x ≥ b$m*} *∈ lmeasurable*
by (*auto simp*: *convex_Int convex_halfspace_le convex_halfspace_ge bounded_Int measurable_convex*)
have ⋀*x*. ⟦*x $ n + a $ m ≤ x $ m*⟧
⟹ *x ∈* (+) (*χ i*. *if i = m then a $ m − b $ m else 0*) *‘* {*x*. *x $ n + b $ m ≤ x $ m*}
using ‹*m ≠ n*›
by (*rule_tac x*=*x − χ i*. *if i = m then a$m − b$m else 0*) **in** *image_eqI*)
(*simp_all add*: *mem_box_cart*)
then have *imeq*: (+) *?d ‘* {*x*. *b $ m ≤ ?mn · x*} = {*x*. *a $ m ≤ ?mn · x*}
using ‹*m ≠ n*› **by** (*auto simp*: *mem_box_cart inner_axis' algebra_simps*)
have ⋀*x*. ⟦*0 ≤ a $ n*; *x $ n + a $ m ≤ x $ m*;
∀ *i*. *i ≠ m ⟶ a $ i ≤ x $ i ∧ x $ i ≤ b $ i*⟧

```
        ⟹ a $ m ≤ x $ m
      using ‹m ≠ n›  by force
    then have (+) ?d ‘ (cbox a ?c ∩ {x. b $ m ≤ ?mn · x})
          = cbox a (χ i. if i = m then a $ m + b $ n else b $ i) ∩ {x. a $ m ≤
?mn · x}
      using an ab_ne
    apply (simp add: cbox_translation [symmetric] translation_Int interval_ne_empty_cart
imeq)
        apply (auto simp: mem_box_cart inner_axis' algebra_simps if_distrib
all_if_distrib)
      by (metis (full_types) add_mono mult_2_right)
    then show cbox a ?c ∩ {x. ?mn · x ≤ a $ m} ∪
         (+) ?d ‘ (cbox a ?c ∩ {x. b $ m ≤ ?mn · x}) =
         cbox a (χ i. if i = m then a $ m + b $ n else b $ i)  (is ?lhs = ?rhs)
      using an ‹m ≠ n›
        apply (auto simp: mem_box_cart inner_axis' algebra_simps if_distrib
all_if_distrib, force)
       apply (drule_tac x=n in spec)+
     by (meson ab_ne add_mono_thms_linordered_semiring(3) dual_order.trans
interval_ne_empty_cart(1))
    have negligible{x. ?mn · x = a$m}
     by (metis ‹m ≠ n› axis_index_axis eq_iff_diff_eq_0 negligible_hyperplane)
    moreover have (cbox a ?c ∩ {x. ?mn · x ≤ a $ m} ∩
                                  (+) ?d ‘ (cbox a ?c ∩ {x. b $ m ≤ ?mn · x})) ⊆ {x.
?mn · x = a$m}
    using ‹m ≠ n› antisym_conv by (fastforce simp: algebra_simps mem_box_cart
inner_axis')
    ultimately show negligible (cbox a ?c ∩ {x. ?mn · x ≤ a $ m} ∩
                                (+) ?d ‘ (cbox a ?c ∩ {x. b $ m ≤ ?mn · x}))
      by (rule negligible_subset)
  qed
  have ac_ne: cbox a ?c ≠ {}
   by (smt (verit, del_insts) ab_ne an interval_ne_empty_cart(1) vec_lambda_beta)
  have ax_ne: cbox a (χ i. if i = m then a $ m + b $ n else b $ i) ≠ {}
    using ab_ne an
    by (smt (verit, ccfv_threshold) interval_ne_empty_cart(1) vec_lambda_beta)
  have eq3: measure lebesgue (cbox a ?c) = measure lebesgue (cbox a (χ i. if i =
m then a$m + b$n else b$i)) + measure lebesgue (cbox a b)
   by (simp add: content_cbox_if_cart ab_ne ac_ne ax_ne algebra_simps prod.delta_remove
          if_distrib [of λu. u − z for z] prod.remove)
  show ?Q
    using eq1 eq2 eq3 by (simp add: algebra_simps)
qed


proposition
  fixes S :: (real^'n) set
  assumes S ∈ lmeasurable
  shows measurable_stretch: ((λx. χ k. m k * x$k) ‘ S) ∈ lmeasurable (is  ?f ‘ S
```

```
∈ _)
    and measure_stretch: measure lebesgue ((λx. χ k. m k * x$k) ' S) = |prod m
UNIV| * measure lebesgue S
    (is ?MEQ)
proof −
  have (?f ' S) ∈ lmeasurable ∧ ?MEQ
  proof (cases ∀ k. m k ≠ 0)
    case True
    have m0: 0 < |prod m UNIV|
      using True by simp
    have (indicat_real (?f ' S) has_integral |prod m UNIV| * measure lebesgue S)
UNIV
    proof (clarsimp simp add: has_integral_alt [where i=UNIV])
      fix e :: real
      assume e > 0
      have (indicat_real S has_integral (measure lebesgue S)) UNIV
        using assms lmeasurable_iff_has_integral by blast
      then obtain B where B>0
        and B: ⋀a b. ball 0 B ⊆ cbox a b ⟹
                     ∃ z. (indicat_real S has_integral z) (cbox a b) ∧
                          |z − measure lebesgue S| < e / |prod m UNIV|
          by (simp add: has_integral_alt [where i=UNIV]) (metis (full_types)
divide_pos_pos m0  m0 ‹e > 0›)
      show ∃ B>0. ∀ a b. ball 0 B ⊆ cbox a b ⟶
                (∃ z. (indicat_real (?f ' S) has_integral z) (cbox a b) ∧
                     |z − |prod m UNIV| * measure lebesgue S| < e)
      proof (intro exI conjI allI)
        let ?C = Max (range (λk. |m k|)) * B
        show ?C > 0
          using True ‹B > 0› by (simp add: Max_gr_iff)
        show ball 0 ?C ⊆ cbox u v ⟶
                (∃ z. (indicat_real (?f ' S) has_integral z) (cbox u v) ∧
                     |z − |prod m UNIV| * measure lebesgue S| < e) for u v
        proof
          assume uv: ball 0 ?C ⊆ cbox u v
          with ‹?C > 0› have cbox_ne: cbox u v ≠ {}
            using centre_in_ball by blast
          let ?α = λk. u$k / m k
          let ?β = λk. v$k / m k
          have invm0: ⋀k. inverse (m k) ≠ 0
            using True by auto
          have ball 0 B ⊆ (λx. χ k. x $ k / m k) ' ball 0 ?C
          proof clarsimp
            fix x :: real^'n
            assume x: norm x < B
            have [simp]: |Max (range (λk. |m k|))| = Max (range (λk. |m k|))
              by (meson Max_ge abs_ge_zero abs_of_nonneg finite finite_imageI
order_trans rangeI)
            have norm (χ k. m k * x $ k) ≤ norm (Max (range (λk. |m k|)) *R x)
```

by (*rule norm_le_componentwise_cart*) (*auto simp*: *abs_mult intro*: *mult_right_mono*)
also have ... < *?C*
using *x* ‹*0* < (*MAX k*. |*m k*|) * *B*› ‹*0* < *B*› *zero_less_mult_pos2* **by** *fastforce*
finally have *norm* (*χ k*. *m k* * *x* $ *k*) < *?C* **.**
then show *x* ∈ (*λx*. *χ k*. *x* $ *k* / *m k*) ‘ *ball 0 ?C*
using *stretch_Galois* [*of inverse* ∘ *m*] *True* **by** (*auto simp*: *image_iff field_simps*)
qed
then have *Bsub*: *ball 0 B* ⊆ *cbox* (*χ k*. *min* (*?α k*) (*?β k*)) (*χ k*. *max* (*?α k*) (*?β k*))
using *cbox_ne uv image_stretch_interval_cart* [*of inverse* ∘ *m u v*, *symmetric*]
by (*force simp*: *field_simps*)
obtain *z* **where** *zint*: (*indicat_real S has_integral z*) (*cbox* (*χ k*. *min* (*?α k*) (*?β k*)) (*χ k*. *max* (*?α k*) (*?β k*)))
and *zless*: |*z* − *measure lebesgue S*| < *e* / |*prod m UNIV*|
using *B* [*OF Bsub*] **by** *blast*
have *ind*: *indicat_real* (*?f* ‘ *S*) = (*λx*. *indicator S* (*χ k*. *x*$*k* / *m k*))
using *True stretch_Galois* [*of m*] **by** (*force simp*: *indicator_def*)
show ∃ *z*. (*indicat_real* (*?f* ‘ *S*) *has_integral z*) (*cbox u v*) ∧
|*z* − |*prod m UNIV*| * *measure lebesgue S*| < *e*
proof (*simp add*: *ind*, *intro conjI exI*)
have ((*λx*. *indicat_real S* (*χ k*. *x* $ *k*/ *m k*)) *has_integral z* *$_R$ |*prod m UNIV*|)
((*λx*. *χ k*. *x* $ *k* * *m k*) ‘ *cbox* (*χ k*. *min* (*?α k*) (*?β k*)) (*χ k*. *max* (*?α k*) (*?β k*)))
using *True has_integral_stretch_cart* [*OF zint*, *of inverse* ∘ *m*]
by (*simp add*: *field_simps prod_dividef*)
moreover have ((*λx*. *χ k*. *x* $ *k* * *m k*) ‘ *cbox* (*χ k*. *min* (*?α k*) (*?β k*)) (*χ k*. *max* (*?α k*) (*?β k*))) = *cbox u v*
using *True image_stretch_interval_cart* [*of inverse* ∘ *m u v*, *symmetric*]
image_stretch_interval_cart [*of λk*. *1 u v*, *symmetric*] ‹*cbox u v* ≠ {}›
by (*simp add*: *field_simps image_comp o_def*)
ultimately show ((*λx*. *indicat_real S* (*χ k*. *x* $ *k*/ *m k*)) *has_integral z* *$_R$ |*prod m UNIV*|) (*cbox u v*)
by *simp*
have |*z* *$_R$ |*prod m UNIV*| − |*prod m UNIV*| * *measure lebesgue S*|
= |*prod m UNIV*| * |*z* − *measure lebesgue S*|
by (*metis* (*no_types*, *opaque_lifting*) *abs_abs abs_scaleR mult.commute real_scaleR_def right_diff_distrib'*)
also have ... < *e*
using *zless True* **by** (*simp add*: *field_simps*)
finally show |*z* *$_R$ |*prod m UNIV*| − |*prod m UNIV*| * *measure lebesgue S*| < *e* **.**
qed
qed

```
    qed
   qed
   then show ?thesis
   by (auto simp: has_integral_integrable integral_unique lmeasure_integral_UNIV
measurable_integrable)
 next
   case False
   then obtain k where m k = 0 and prm: prod m UNIV = 0
    by auto
   have nfS: negligible (?f ` S)
    by (rule negligible_subset [OF negligible_standard_hyperplane_cart]) (use ‹m
k = 0› in auto)
   then show ?thesis
    by (simp add: negligible_iff_measure prm)
 qed
 then show (?f ` S) ∈ lmeasurable ?MEQ
   by metis+
qed


proposition
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes linear f S ∈ lmeasurable
  shows measurable_linear_image: (f ` S) ∈ lmeasurable
     and measure_linear_image: measure lebesgue (f ` S) = |det (matrix f)| *
measure lebesgue S (is ?Q f S)
proof −
  have ∀ S ∈ lmeasurable. (f ` S) ∈ lmeasurable ∧ ?Q f S
  proof (rule induct_linear_elementary [OF ‹linear f›]; intro ballI)
    fix f g and S :: (real,'n) vec set
    assume linear f and linear g
      and f [rule_format]: ∀ S ∈ lmeasurable. f ` S ∈ lmeasurable ∧ ?Q f S
      and g [rule_format]: ∀ S ∈ lmeasurable. g ` S ∈ lmeasurable ∧ ?Q g S
      and S: S ∈ lmeasurable
    then have gS: g ` S ∈ lmeasurable
      by blast
    show (f ∘ g) ` S ∈ lmeasurable ∧ ?Q (f ∘ g) S
      using f [OF gS] g [OF S] matrix_compose [OF ‹linear g› ‹linear f›]
      by (simp add: o_def image_comp abs_mult det_mul)
  next
    fix f :: real^'n::_ ⇒ real^'n::_ and i and S :: (real^'n::_) set
    assume linear f and 0: ⋀x. f x $ i = 0 and S ∈ lmeasurable
    then have ¬ inj f
      by (metis (full_types) linear_injective_imp_surjective one_neq_zero surjE
vec_component)
    have detf: det (matrix f) = 0
      using ‹¬ inj f› det_nz_iff_inj[OF ‹linear f›] by blast
    show f ` S ∈ lmeasurable ∧ ?Q f S
    proof
```

```
    show f ‘ S ∈ lmeasurable
      using lmeasurable_iff_indicator_has_integral ‹linear f› ‹¬ inj f› negligible_UNIV negligible_linear_singular_image by blast
    have measure lebesgue (f ‘ S) = 0
    by (meson ‹¬ inj f› ‹linear f› negligible_imp_measure0 negligible_linear_singular_image)
    also have ... = |det (matrix f)| * measure lebesgue S
      by (simp add: detf)
    finally show ?Q f S .
  qed
 next
  fix c and S :: (real^'n::_) set
  assume S ∈ lmeasurable
  show (λa. χ i. c i * a $ i) ‘ S ∈ lmeasurable ∧ ?Q (λa. χ i. c i * a $ i) S
  proof
    show (λa. χ i. c i * a $ i) ‘ S ∈ lmeasurable
      by (simp add: ‹S ∈ lmeasurable› measurable_stretch)
    show ?Q (λa. χ i. c i * a $ i) S
      by (simp add: measure_stretch [OF ‹S ∈ lmeasurable›, of c] axis_def matrix_def det_diagonal)
  qed
 next
  fix m :: 'n and n :: 'n and S :: (real, 'n) vec set
  assume m ≠ n and S ∈ lmeasurable
  let ?h = λv::(real, 'n) vec. χ i. v $ Transposition.transpose m n i
  have lin: linear ?h
    by (rule linearI) (simp_all add: plus_vec_def scaleR_vec_def)
  have meq: measure lebesgue ((λv::(real, 'n) vec. χ i. v $ Transposition.transpose m n i) ‘ cbox a b)
          = measure lebesgue (cbox a b) for a b
  proof (cases cbox a b = {})
    case True then show ?thesis
      by simp
  next
    case False
    then have him: ?h ‘ (cbox a b) ≠ {}
      by blast
    have eq: ?h ‘ (cbox a b) = cbox (?h a) (?h b)
      by (auto simp: image_iff lambda_swap_Galois mem_box_cart) (metis transpose_involutory)+
    show ?thesis
      using him prod.permute [OF permutes_swap_id, where S=UNIV and g=λi. (b - a)$i, symmetric]
      by (simp add: eq content_cbox_cart False)
  qed
  have (χ i j. if Transposition.transpose m n i = j then 1 else 0) = (χ i j. if j = Transposition.transpose m n i then 1 else (0::real))
    by (auto intro!: Cart_lambda_cong)
  then have matrix ?h = transpose(χ i j. mat 1 $ i $ Transposition.transpose m n j)
```

```
    by (auto simp: matrix_eq transpose_def axis_def mat_def matrix_def)
  then have 1: |det (matrix ?h)| = 1
    by (simp add: det_permute_columns permutes_swap_id sign_swap_id abs_mult)
  show ?h ` S ∈ lmeasurable ∧ ?Q ?h S
    using measure_linear_sufficient [OF lin ‹S ∈ lmeasurable›] meq 1 by force
 next
  fix m n :: 'n and S :: (real, 'n) vec set
  assume m ≠ n and S ∈ lmeasurable
  let ?h = λv::(real, 'n) vec. χ i. if i = m then v $ m + v $ n else v $ i
  have lin: linear ?h
    by (rule linearI) (auto simp: algebra_simps plus_vec_def scaleR_vec_def
vec_eq_iff)
  consider m < n | n < m
    using ‹m ≠ n› less_linear by blast
  then have 1: det(matrix ?h) = 1
  proof cases
    assume m < n
    have *: matrix ?h $ i $ j = (0::real) if j < i for i j :: 'n
    proof −
      have axis j 1 = (χ n. if n = j then 1 else (0::real))
        using axis_def by blast
      then have (χ p q. if p = m then axis q 1 $ m + axis q 1 $ n else axis q 1
$ p) $ i $ j = (0::real)
        using ‹j < i› axis_def ‹m < n› by auto
      with ‹m < n› show ?thesis
        by (auto simp: matrix_def axis_def cong: if_cong)
    qed
    show ?thesis
     using ‹m ≠ n› by (subst det_upperdiagonal [OF *]) (auto simp: matrix_def
axis_def cong: if_cong)
  next
    assume n < m
    have *: matrix ?h $ i $ j = (0::real) if j > i for i j :: 'n
    proof −
      have axis j 1 = (χ n. if n = j then 1 else (0::real))
        using axis_def by blast
      then have (χ p q. if p = m then axis q 1 $ m + axis q 1 $ n else axis q 1
$ p) $ i $ j = (0::real)
        using ‹j > i› axis_def ‹m > n› by auto
      with ‹m > n› show ?thesis
        by (auto simp: matrix_def axis_def cong: if_cong)
    qed
    show ?thesis
      using ‹m ≠ n›
     by (subst det_lowerdiagonal [OF *]) (auto simp: matrix_def axis_def cong:
if_cong)
  qed
  have meq: measure lebesgue (?h ` (cbox a b)) = measure lebesgue (cbox a b) for
a b
```

```
    proof (cases cbox a b = {})
      case True then show ?thesis by simp
    next
      case False
      then have ne: (+) (χ i. if i = n then − a $ n else 0) ‘ cbox a b ≠ {}
        by auto
      let ?v = χ i. if i = n then − a $ n else 0
      have ?h ‘ cbox a b
          = (+) (χ i. if i = m ∨ i = n then a $ n else 0) ‘ ?h ‘ (+) ?v ‘ (cbox a b)
        using ‹m ≠ n› unfolding image_comp o_def by (force simp: vec_eq_iff)
      then have measure lebesgue (?h ‘ (cbox a b))
            = measure lebesgue ((λv. χ i. if i = m then v $ m + v $ n else v $ i) ‘
                                (+) ?v ‘ cbox a b)
        by (rule ssubst) (rule measure_translation)
     also have ... = measure lebesgue ((λv. χ i. if i = m then v $ m + v $ n else
v $ i) ‘ cbox (?v +a) (?v + b))
        by (metis (no_types, lifting) cbox_translation)
      also have ... = measure lebesgue ((+) ?v ‘ cbox a b)
        apply (subst measure_shear_interval)
        using ‹m ≠ n› ne apply auto
        apply (simp add: cbox_translation)
        by (metis cbox_borel cbox_translation measure_completion sets_lborel)
      also have ... = measure lebesgue (cbox a b)
        by (rule measure_translation)
        finally show ?thesis .
      qed
    show ?h ‘ S ∈ lmeasurable ∧ ?Q ?h S
      using measure_linear_sufficient [OF lin ‹S ∈ lmeasurable›] meq 1 by force
  qed
  with assms show (f ‘ S) ∈ lmeasurable ?Q f S
    by metis+
qed


lemma
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes f: orthogonal_transformation f and S: S ∈ lmeasurable
  shows measurable_orthogonal_image: f ‘ S ∈ lmeasurable
    and measure_orthogonal_image: measure lebesgue (f ‘ S) = measure lebesgue
S
proof −
  have linear f
    by (simp add: f orthogonal_transformation_linear)
  then show f ‘ S ∈ lmeasurable
    by (metis S measurable_linear_image)
  show measure lebesgue (f ‘ S) = measure lebesgue S
    by (simp add: measure_linear_image ‹linear f› S f)
qed
```

proposition *measure_semicontinuous_with_hausdist_explicit*:
assumes *bounded S* **and** *neg*: *negligible*(*frontier S*) **and** *e* > *0*
obtains *d* **where** *d* > *0*
⋀*T*. ⟦*T* ∈ *lmeasurable*; ⋀*y*. *y* ∈ *T* ⟹ ∃ *x*. *x* ∈ *S* ∧ *dist x y* < *d*⟧
⟹ *measure lebesgue T* < *measure lebesgue S* + *e*
proof (*cases S* = {})
case *True*
with *that* ‹*e* > *0*› **show** *?thesis* **by** *force*
next
case *False*
then have *frS*: *frontier S* ≠ {}
using ‹*bounded S*› *frontier_eq_empty not_bounded_UNIV* **by** *blast*
have *S* ∈ *lmeasurable*
by (*simp add*: ‹*bounded S*› *measurable_Jordan neg*)
have *null*: (*frontier S*) ∈ *null_sets lebesgue*
by (*metis neg negligible_iff_null_sets*)
have *frontier S* ∈ *lmeasurable* **and** *mS0*: *measure lebesgue* (*frontier S*) = *0*
using *neg negligible_imp_measurable negligible_iff_measure* **by** *blast+*
with ‹*e* > *0*› *sets_lebesgue_outer_open*
obtain *U* **where** *open U*
and *U*: *frontier S* ⊆ *U U* − *frontier S* ∈ *lmeasurable emeasure lebesgue* (*U* − *frontier S*) < *e*
by (*metis fmeasurableD*)
with *null* **have** *U* ∈ *lmeasurable*
by (*metis borel_open measurable_Diff_null_set sets_completionI_sets sets_lborel*)
have *measure lebesgue* (*U* − *frontier S*) = *measure lebesgue U*
using *mS0* **by** (*simp add*: ‹*U* ∈ *lmeasurable*› *fmeasurableD measure_Diff_null_set null*)
with *U* **have** *mU*: *measure lebesgue U* < *e*
by (*simp add*: *emeasure_eq_measure2 ennreal_less_iff*)
show *?thesis*
proof
have *U* ≠ *UNIV*
using ‹*U* ∈ *lmeasurable*› **by** *auto*
then have − *U* ≠ {}
by *blast*
with ‹*open U*› ‹*frontier S* ⊆ *U*› **show** *setdist* (*frontier S*) (− *U*) > *0*
by (*auto simp*: ‹*bounded S*› *open_closed compact_frontier_bounded setdist_gt_0_compact_closed frS*)
fix *T*
assume *T* ∈ *lmeasurable*
and *T*: ⋀*t*. *t* ∈ *T* ⟹ ∃ *y*. *y* ∈ *S* ∧ *dist y t* < *setdist* (*frontier S*) (− *U*)
then have *measure lebesgue T* − *measure lebesgue S* ≤ *measure lebesgue* (*T* − *S*)
by (*simp add*: ‹*S* ∈ *lmeasurable*› *measure_diff_le_measure_setdiff*)
also have ... ≤ *measure lebesgue U*
proof −
have *T* − *S* ⊆ *U*
proof *clarify*

```
      fix x
      assume x ∈ T and x ∉ S
      then obtain y where y ∈ S and y: dist y x < setdist (frontier S) (- U)
        using T by blast
      have closed_segment x y ∩ frontier S ≠ {}
        using connected_Int_frontier ‹x ∉ S› ‹y ∈ S› by blast
      then obtain z where z: z ∈ closed_segment x y z ∈ frontier S
        by auto
      with y have dist z x < setdist(frontier S) (- U)
        by (auto simp: dist_commute dest!: dist_in_closed_segment)
      with z have False if x ∈ -U
        using setdist_le_dist [OF ‹z ∈ frontier S› that] by auto
      then show x ∈ U
        by blast
    qed
    then show ?thesis
      by (simp add: ‹S ∈ lmeasurable› ‹T ∈ lmeasurable› ‹U ∈ lmeasurable›
fmeasurableD measure_mono_fmeasurable sets.Diff)
  qed
  finally have measure lebesgue T - measure lebesgue S ≤ measure lebesgue U .
  with mU show measure lebesgue T < measure lebesgue S + e
    by linarith
 qed
qed

proposition
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes S: S ∈ lmeasurable
  and deriv: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
  and int: (λx. |det (matrix (f' x))|) integrable_on S
  and bounded: ⋀x. x ∈ S ⟹ |det (matrix (f' x))| ≤ B
  shows measurable_bounded_differentiable_image:
      f ‘ S ∈ lmeasurable
    and measure_bounded_differentiable_image:
      measure lebesgue (f ‘ S) ≤ B * measure lebesgue S (is ?M)
proof -
  have f ‘ S ∈ lmeasurable ∧ measure lebesgue (f ‘ S) ≤ B * measure lebesgue S
  proof (cases B < 0)
    case True
    then have S = {}
    by (meson abs_ge_zero bounded empty_iff equalityI less_le_trans linorder_not_less
subsetI)
    then show ?thesis
      by auto
  next
    case False
    then have B ≥ 0
      by arith
    let ?μ = measure lebesgue
```

```
    have f_diff: f differentiable_on S
      using deriv by (auto simp: differentiable_on_def differentiable_def)
    have eps: f ‘ S ∈ lmeasurable ?μ (f ‘ S) ≤ (B+e) * ?μ S (is ?ME)
           if e > 0 for e
    proof −
      have eps_d: f ‘ S ∈ lmeasurable  ?μ (f ‘ S) ≤ (B+e) * (?μ S + d) (is ?MD)
             if d > 0 for d
      proof −
        obtain T where T: open T S ⊆ T and TS: (T−S) ∈ lmeasurable and
emeasure lebesgue (T−S) < ennreal d
          using S ‹d > 0› sets_lebesgue_outer_open by blast
        then have ?μ (T−S) < d
          by (metis emeasure_eq_measure2 ennreal_leI not_less)
        with S T TS have T ∈ lmeasurable and Tless: ?μ T < ?μ S + d
          by (auto simp: measurable_measure_Diff dest!: fmeasurable_Diff_D)
        have ∃ r. 0 < r ∧ r < d ∧ ball x r ⊆ T ∧ f ‘ (S ∩ ball x r) ∈ lmeasurable ∧
                  ?μ (f ‘ (S ∩ ball x r)) ≤ (B + e) * ?μ (ball x r)
          if x ∈ S d > 0 for x d
        proof −
          have lin: linear (f′ x)
            and lim0: ((λy. (f y − (f x + f′ x (y − x))) /R norm(y − x)) ⟶ 0)
(at x within S)
            using deriv ‹x ∈ S› by (auto simp: has_derivative_within bounded_linear.linear
field_simps)
          have bo: bounded (f′ x ‘ ball 0 1)
            by (simp add: bounded_linear_image linear_linear lin)
          have neg: negligible (frontier (f′ x ‘ ball 0 1))
            using deriv has_derivative_linear ‹x ∈ S›
            by (auto intro!: negligible_convex_frontier [OF convex_linear_image])
          let ?unit_vol = content (ball (0 :: real ^ 'n :: {finite, wellorder}) 1)
          have 0: 0 < e * ?unit_vol
            using ‹e > 0› by (simp add: content_ball_pos)
          obtain k where k > 0 and k:
                ⋀U. ⟦U ∈ lmeasurable; ⋀y. y ∈ U ⟹ ∃ z. z ∈ f′ x ‘ ball 0 1 ∧ dist
z y < k⟧
                    ⟹ ?μ U < ?μ (f′ x ‘ ball 0 1) + e * ?unit_vol
            using measure_semicontinuous_with_hausdist_explicit [OF bo neg 0]
by blast
          obtain l where l > 0 and l: ball x l ⊆ T
            using ‹x ∈ S› ‹open T› ‹S ⊆ T› openE by blast
          obtain ζ where 0 < ζ
            and ζ: ⋀y. ⟦y ∈ S; y ≠ x; dist y x < ζ⟧
                    ⟹ norm (f y − (f x + f′ x (y − x))) / norm (y − x) < k
              using lim0 ‹k > 0› by (simp add: Lim_within) (auto simp add:
field_simps)
          define r where r ≡ min (min l (ζ/2)) (min 1 (d/2))
          show ?thesis
          proof (intro exI conjI)
            show r > 0 r < d
```

```
      using ‹l > 0› ‹ζ > 0› ‹d > 0› by (auto simp: r_def)
    have r ≤ l
      by (auto simp: r_def)
    with l show ball x r ⊆ T
      by auto
    have ex_lessK: ∃x'∈ball 0 1. dist (f' x x') ((f y - f x) /R r) < k
      if y ∈ S and dist x y < r for y
    proof (cases y = x)
      case True
      with lin linear_0 ‹k > 0› that show ?thesis
        by (rule_tac x=0 in bexI) (auto simp: linear_0)
    next
      case False
      then show ?thesis
      proof (rule_tac x=(y - x) /R r in bexI)
        have f' x ((y - x) /R r) = f' x (y - x) /R r
          by (simp add: lin linear_scale)
        then have dist (f' x ((y - x) /R r)) ((f y - f x) /R r) = norm (f'
x (y - x) /R r - (f y - f x) /R r)
          by (simp add: dist_norm)
        also have ... = norm (f' x (y - x) - (f y - f x)) / r
          using ‹r > 0› by (simp add: divide_simps scale_right_diff_distrib
[symmetric])
        also have ... ≤ norm (f y - (f x + f' x (y - x))) / norm (y - x)
          using that ‹r > 0› False by (simp add: field_split_simps dist_norm
norm_minus_commute mult_right_mono)
        also have ... < k
          using that ‹0 < ζ› by (simp add: dist_commute r_def ζ [OF ‹y
∈ S› False])
        finally show dist (f' x ((y - x) /R r)) ((f y - f x) /R r) < k .
        show (y - x) /R r ∈ ball 0 1
          using that ‹r > 0› by (simp add: dist_norm divide_simps
norm_minus_commute)
      qed
    qed
    let ?rfs = (λx. x /R r) ‘ (+) (- f x) ‘ f ‘ (S ∩ ball x r)
    have rfs_mble: ?rfs ∈ lmeasurable
    proof (rule bounded_set_imp_lmeasurable)
      have f differentiable_on S ∩ ball x r
        using f_diff by (auto simp: fmeasurableD differentiable_on_subset)
      with S show ?rfs ∈ sets lebesgue
        by (auto simp: sets.Int intro!: lebesgue_sets_translation differen-
tiable_image_in_sets_lebesgue)
      let ?B = (λ(x, y). x + y) ‘ (f' x ‘ ball 0 1 × ball 0 k)
      have bounded ?B
        by (simp add: bounded_plus [OF bo])
      moreover have ?rfs ⊆ ?B
        apply (auto simp: dist_norm image_iff dest!: ex_lessK)
        by (metis (no_types, opaque_lifting) add.commute diff_add_cancel
```

```
dist_0_norm dist_commute dist_norm mem_ball)
            ultimately show bounded (?rfs)
              by (rule bounded_subset)
          qed
          then have (λx. r *R x) ‘ ?rfs ∈ lmeasurable
            by (simp add: measurable_linear_image)
          with ‹r > 0› have (+) (− f x) ‘ f ‘ (S ∩ ball x r) ∈ lmeasurable
            by (simp add: image_comp o_def)
          then have (+) (f x) ‘ (+) (− f x) ‘ f ‘ (S ∩ ball x r) ∈ lmeasurable
            using  measurable_translation by blast
          then show fsb: f ‘ (S ∩ ball x r) ∈ lmeasurable
            by (simp add: image_comp o_def)
          have ?μ (f ‘ (S ∩ ball x r)) = ?μ (?rfs) * r ^ CARD('n)
            using ‹r > 0› fsb
              by (simp add: measure_linear_image measure_translation_subtract
measurable_translation_subtract field_simps cong: image_cong_simp)
        also have ... ≤ (|det (matrix (f' x))| * ?unit_vol + e * ?unit_vol) * r
^ CARD('n)
          proof −
            have ?μ (?rfs) < ?μ (f' x ‘ ball 0 1) + e * ?unit_vol
              using rfs_mble by (force intro: k dest!: ex_lessK)
            then have ?μ (?rfs) < |det (matrix (f' x))| * ?unit_vol + e * ?unit_vol
              by (simp add: lin measure_linear_image [of f' x])
            with ‹r > 0› show ?thesis
              by auto
          qed
          also have ... ≤ (B + e) * ?μ (ball x r)
            using bounded [OF ‹x ∈ S›] ‹r > 0›
                by (simp add: algebra_simps content_ball_conv_unit_ball[of r]
content_ball_pos)
          finally show ?μ (f ‘ (S ∩ ball x r)) ≤ (B + e) * ?μ (ball x r) .
        qed
      qed
      then obtain r where
        r0d: ⋀x d. ⟦x ∈ S; d > 0⟧ ⟹ 0 < r x d ∧ r x d < d
        and rT: ⋀x d. ⟦x ∈ S; d > 0⟧ ⟹ ball x (r x d) ⊆ T
        and r: ⋀x d. ⟦x ∈ S; d > 0⟧ ⟹
              (f ‘ (S ∩ ball x (r x d))) ∈ lmeasurable ∧
              ?μ (f ‘ (S ∩ ball x (r x d))) ≤ (B + e) * ?μ (ball x (r x d))
        by metis
      obtain C where countable C and Csub: C ⊆ {(x,r x t) |x t. x ∈ S ∧ 0 <
t}
        and pwC: pairwise (λi j. disjnt (ball (fst i) (snd i)) (ball (fst j) (snd j)))
C
        and negC: negligible(S − (⋃ i ∈ C. ball (fst i) (snd i)))
        apply (rule Vitali_covering_theorem_balls [of S {(x,r x t) |x t. x ∈ S ∧
0 < t} fst snd])
         apply auto
        by (metis dist_eq_0_iff r0d)
```

```
      let ?UB = (⋃(x,s) ∈ C. ball x s)
      have eq: f ‘ (S ∩ ?UB) = (⋃(x,s) ∈ C. f ‘ (S ∩ ball x s))
        by auto
      have mle: ?μ (⋃(x,s) ∈ K. f ‘ (S ∩ ball x s)) ≤ (B + e) * (?μ S + d)  (is ?l ≤ ?r)
        if K ⊆ C and finite K for K
      proof −
        have gt0: b > 0 if (a, b) ∈ K for a b
          using Csub that ‹K ⊆ C› r0d by auto
        have inj: inj_on (λ(x, y). ball x y) K
          by (force simp: inj_on_def ball_eq_ball_iff dest: gt0)
        have disjnt: disjoint ((λ(x, y). ball x y) ‘ K)
          using pwC that pairwise_image pairwise_mono by fastforce
        have ?l ≤ (∑ i∈K. ?μ (case i of (x, s) ⇒ f ‘ (S ∩ ball x s)))
        proof (rule measure_UNION_le [OF ‹finite K›], clarify)
          fix x r
          assume (x,r) ∈ K
          then have x ∈ S
            using Csub ‹K ⊆ C› by auto
          show f ‘ (S ∩ ball x r) ∈ sets lebesgue
            by (meson Int_lower1 S differentiable_on_subset f_diff fmeasurableD
lmeasurable_ball order_refl sets.Int differentiable_image_in_sets_lebesgue)
        qed
        also have ... ≤ (∑ (x,s) ∈ K. (B + e) * ?μ (ball x s))
          using Csub r ‹K ⊆ C›  by (intro sum_mono) auto
        also have ... = (B + e) * (∑ (x,s) ∈ K. ?μ (ball x s))
          by (simp add: prod.case_distrib sum_distrib_left)
        also have ... = (B + e) * sum ?μ ((λ(x, y). ball x y) ‘ K)
         using ‹B ≥ 0› ‹e > 0› by (simp add: inj sum.reindex prod.case_distrib)
        also have ... = (B + e) * ?μ (⋃(x,s) ∈ K. ball x s)
          using ‹B ≥ 0› ‹e > 0› that
          by (subst measure_Union′) (auto simp: disjnt measure_Union′)
        also have ... ≤ (B + e) * ?μ T
          using ‹B ≥ 0› ‹e > 0› that apply simp
         using measure_mono_fmeasurable [OF _ _ ‹T ∈ lmeasurable›] Csub rT
              by (smt (verit) SUP_least measure_nonneg measure_notin_sets
mem_Collect_eq old.prod.case subset_iff)
        also have ... ≤ (B + e) * (?μ S + d)
          using ‹B ≥ 0› ‹e > 0› Tless by simp
        finally show ?thesis .
      qed
      have fSUB_mble: (f ‘ (S ∩ ?UB)) ∈ lmeasurable
        unfolding eq using Csub r False ‹e > 0› that
       by (auto simp: intro!: fmeasurable_UN_bound [OF ‹countable C› _ mle])
      have fSUB_meas: ?μ (f ‘ (S ∩ ?UB)) ≤ (B + e) * (?μ S + d)  (is ?MUB)
        unfolding eq using Csub r False ‹e > 0› that
        by (auto simp: intro!: measure_UN_bound [OF ‹countable C› _ mle])
      have neg: negligible ((f ‘ (S ∩ ?UB) − f ‘ S) ∪ (f ‘ S − f ‘ (S ∩ ?UB)))
      proof (rule negligible_subset [OF negligible_differentiable_image_negligible
```

```
[OF order_refl negC, where f=f]])
        show f differentiable_on S - (⋃i∈C. ball (fst i) (snd i))
          by (meson DiffE differentiable_on_subset subsetI f_diff)
       qed force
       show f ‘ S ∈ lmeasurable
        by (rule lmeasurable_negligible_symdiff [OF fSUB_mble neg])
       show ?MD
         using fSUB_meas measure_negligible_symdiff [OF fSUB_mble neg] by
simp
     qed
     show f ‘ S ∈ lmeasurable
      using eps_d [of 1] by simp
     show ?ME
     proof (rule field_le_epsilon)
      fix δ :: real
      assume 0 < δ
      then show ?μ (f ‘ S) ≤ (B + e) * ?μ S + δ
       using eps_d [of δ / (B+e)] ‹e > 0› ‹B ≥ 0› by (auto simp: divide_simps
mult_ac)
     qed
   qed
   show ?thesis
   proof (cases ?μ S = 0)
     case True
     with eps have ?μ (f ‘ S) = 0
      by (metis mult_zero_right not_le zero_less_measure_iff)
     then show ?thesis
      using eps [of 1] by (simp add: True)
   next
     case False
     have ?μ (f ‘ S) ≤ B * ?μ S
     proof (rule field_le_epsilon)
      fix e :: real
      assume e > 0
      then show ?μ (f ‘ S) ≤ B * ?μ S + e
      using eps [of e / ?μ S] False by (auto simp: algebra_simps zero_less_measure_iff)
     qed
     with eps [of 1] show ?thesis by auto
   qed
  qed
  then show f ‘ S ∈ lmeasurable ?M by blast+
qed

lemma m_diff_image_weak:
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes S: S ∈ lmeasurable
    and deriv: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
    and int: (λx. |det (matrix (f' x))|) integrable_on S
   shows f ‘ S ∈ lmeasurable ∧ measure lebesgue (f ‘ S) ≤ integral S (λx. |det
```

```
(matrix (f' x))|)
proof −
  let ?μ = measure lebesgue
  have aint_S: (λx. |det (matrix (f' x))|) absolutely_integrable_on S
    using int unfolding absolutely_integrable_on_def by auto
  define m where m ≡ integral S (λx. |det (matrix (f' x))|)
  have ∗: f ‘ S ∈ lmeasurable ?μ (f ‘ S) ≤ m + e ∗ ?μ S
    if e > 0 for e
  proof −
    define T where T ≡ λn. {x ∈ S. n ∗ e ≤ |det (matrix (f' x))| ∧
                                |det (matrix (f' x))| < (Suc n) ∗ e}
    have meas_t: T n ∈ lmeasurable for n
    proof −
      have ∗: (λx. |det (matrix (f' x))|) ∈ borel_measurable (lebesgue_on S)
        using aint_S by (simp add: S borel_measurable_restrict_space_iff fmea-
surableD set_integrable_def)
      have [intro]: x ∈ sets (lebesgue_on S) ⟹ x ∈ sets lebesgue for x
        using S sets_restrict_space_subset by blast
      have {x ∈ S. real n ∗ e ≤ |det (matrix (f' x))|} ∈ sets lebesgue
      using ∗ by (auto simp: borel_measurable_iff_halfspace_ge space_restrict_space)
      then have 1: {x ∈ S. real n ∗ e ≤ |det (matrix (f' x))|} ∈ lmeasurable
        using S by (simp add: fmeasurableI2)
      have {x ∈ S. |det (matrix (f' x))| < (1 + real n) ∗ e} ∈ sets lebesgue
      using ∗ by (auto simp: borel_measurable_iff_halfspace_less space_restrict_space)
      then have 2: {x ∈ S. |det (matrix (f' x))| < (1 + real n) ∗ e} ∈ lmeasurable
        using S by (simp add: fmeasurableI2)
      show ?thesis
          using fmeasurable.Int [OF 1 2] by (simp add: T_def Int_def cong:
conj_cong)
    qed
    have aint_T: ⋀k. (λx. |det (matrix (f' x))|) absolutely_integrable_on T k
      using set_integrable_subset [OF aint_S] meas_t T_def by blast
    have Seq: S = (⋃ n. T n)
      apply (auto simp: T_def)
      apply (rule_tac x = nat ⌊|det (matrix (f' x))| / e⌋ in exI)
       by (smt (verit, del_insts) divide_nonneg_nonneg floor_eq_iff of_nat_nat
pos_divide_less_eq that zero_le_floor)
    have meas_ft: f ‘ T n ∈ lmeasurable for n
    proof (rule measurable_bounded_differentiable_image)
      show T n ∈ lmeasurable
        by (simp add: meas_t)
    next
      fix x :: (real,'n) vec
      assume x ∈ T n
      show (f has_derivative f' x) (at x within T n)
          by (metis (no_types, lifting) ‹x ∈ T n› deriv has_derivative_subset
mem_Collect_eq subsetI T_def)
      show |det (matrix (f' x))| ≤ (Suc n) ∗ e
        using ‹x ∈ T n› T_def by auto
```

```
    next
      show (λx. |det (matrix (f' x))|) integrable_on T n
        using aint_T absolutely_integrable_on_def by blast
    qed
    have disT: disjoint (range T)
      unfolding disjoint_def
    proof clarsimp
      show T m ∩ T n = {} if T m ≠ T n for m n
        using that
      proof (induction m n rule: linorder_less_wlog)
        case (less m n)
        with ‹e > 0› show ?case
          unfolding T_def
          proof (clarsimp simp add: Collect_conj_eq [symmetric])
            fix x
             assume e > 0  m < n  n * e ≤ |det (matrix (f' x))|  |det (matrix (f'
x))| < (1 + real m) * e
            then have n < 1 + real m
                 by (metis (no_types, opaque_lifting) less_le_trans mult.commute
not_le mult_le_cancel_left_pos)
            then show False
              using less.hyps by linarith
          qed
      qed auto
    qed
    have injT: inj_on T ({n. T n ≠ {}})
      unfolding inj_on_def
    proof clarsimp
      show m = n if T m = T n T n ≠ {} for m n
        using that
      proof (induction m n rule: linorder_less_wlog)
        case (less m n)
        have False if T n ⊆ T m x ∈ T n for x
          using ‹e > 0› ‹m < n› that
        apply (auto simp: T_def mult.commute intro: less_le_trans dest!: subsetD)
          by (smt (verit, best) mult_less_cancel_left_disj nat_less_real_le)
        then show ?case
          using less.prems by blast
      qed auto
    qed
    have sum_eq_Tim: (∑ k≤n. f (T k)) = sum f (T ‘ {..n}) if f {} = 0 for f ::
_ ⇒ real and n
    proof (subst sum.reindex_nontrivial)
      fix i j  assume i ∈ {..n} j ∈ {..n} i ≠ j T i = T j
      with that  injT [unfolded inj_on_def] show f (T i) = 0
        by simp metis
    qed (use atMost_atLeast0 in auto)
    let ?B = m + e * ?μ S
    have (∑ k≤n. ?μ (f ‘ T k)) ≤ ?B for n
```

```
  proof −
    have (∑ k≤n. ?μ (f ‘ T k)) ≤ (∑ k≤n. ((k+1) ∗ e) ∗ ?μ(T k))
    proof (rule sum_mono [OF measure_bounded_differentiable_image])
      show (f has_derivative f′ x) (at x within T k) if x ∈ T k for k x
       using that unfolding T_def by (blast intro: deriv has_derivative_subset)
      show (λx. |det (matrix (f′ x))|) integrable_on T k for k
        using absolutely_integrable_on_def aint_T by blast
      show |det (matrix (f′ x))| ≤ real (k + 1) ∗ e if x ∈ T k for k x
        using T_def that by auto
    qed (use meas_t in auto)
    also have ... ≤ (∑ k≤n. (k ∗ e) ∗ ?μ(T k)) + (∑ k≤n. e ∗ ?μ(T k))
      by (simp add: algebra_simps sum.distrib)
    also have ... ≤ ?B
    proof (rule add_mono)
      have (∑ k≤n. real k ∗ e ∗ ?μ (T k)) = (∑ k≤n. integral (T k) (λx. k ∗ e))
        by (simp add: lmeasure_integral [OF meas_t]
              flip: integral_mult_right integral_mult_left)
      also have ... ≤ (∑ k≤n. integral (T k) (λx.  (abs (det (matrix (f′ x))))))
      proof (rule sum_mono)
        fix k
        assume k ∈ {..n}
       show integral (T k) (λx. k ∗ e) ≤ integral (T k) (λx. |det (matrix (f′ x))|)
        proof (rule integral_le [OF integrable_on_const [OF meas_t]])
          show (λx. |det (matrix (f′ x))|) integrable_on T k
            using absolutely_integrable_on_def aint_T by blast
        next
          fix x assume x ∈ T k
          show k ∗ e ≤ |det (matrix (f′ x))|
            using ‹x ∈ T k› T_def by blast
        qed
      qed
      also have ... = sum (λT. integral T (λx. |det (matrix (f′ x))|)) (T ‘ {..n})
        by (auto intro: sum_eq_Tim)
      also have ... = integral (⋃ k≤n. T k) (λx. |det (matrix (f′ x))|)
      proof (rule integral_unique [OF has_integral_Union, symmetric])
        fix S  assume S ∈ T ‘ {..n}
         then show ((λx. |det (matrix (f′ x))|) has_integral integral S (λx. |det
(matrix (f′ x))|)) S
        using absolutely_integrable_on_def aint_T by blast
      next
        show pairwise (λS S′. negligible (S ∩ S′)) (T ‘ {..n})
           using disT unfolding disjnt_iff by (auto simp: pairwise_def intro!:
empty_imp_negligible)
      qed auto
      also have ... ≤ m
        unfolding m_def
      proof (rule integral_subset_le)
        have (λx. |det (matrix (f′ x))|) absolutely_integrable_on (⋃ k≤n. T k)
        proof (rule set_integrable_subset [OF aint_S])
```

```
        show ⋃ (T ‘ {..n}) ∈ sets lebesgue
          by (intro measurable meas_t fmeasurableD)
      qed (force simp: Seq)
      then show (λx. |det (matrix (f' x))|) integrable_on (⋃k≤n. T k)
        using absolutely_integrable_on_def by blast
    qed (use Seq int in auto)
    finally show (∑ k≤n. real k * e * ?μ (T k)) ≤ m .
  next
    have (∑ k≤n. ?μ (T k)) = sum ?μ (T ‘ {..n})
      by (auto intro: sum_eq_Tim)
    also have ... = ?μ (⋃k≤n. T k)
      using S disT by (auto simp: pairwise_def meas_t intro: measure_Union'
[symmetric])
    also have ... ≤ ?μ S
    using S by (auto simp: Seq intro: meas_t fmeasurableD measure_mono_fmeasurable)
    finally have (∑ k≤n. ?μ (T k)) ≤ ?μ S .
    then show (∑ k≤n. e * ?μ (T k)) ≤ e * ?μ S
    by (metis less_eq_real_def ordered_comm_semiring_class.comm_mult_left_mono
sum_distrib_left that)
  qed
  finally show (∑ k≤n. ?μ (f ‘ T k)) ≤ ?B .
qed
moreover have measure lebesgue (⋃k≤n. f ‘ T k) ≤ (∑ k≤n. ?μ (f ‘ T k))
for n
  by (simp add: fmeasurableD meas_ft measure_UNION_le)
ultimately have B_ge_m: ?μ (⋃k≤n. (f ‘ T k)) ≤ ?B for n
  by (meson order_trans)
have (⋃ n. f ‘ T n) ∈ lmeasurable
  by (rule fmeasurable_countable_Union [OF meas_ft B_ge_m])
moreover have ?μ (⋃ n. f ‘ T n) ≤ m + e * ?μ S
  by (rule measure_countable_Union_le [OF meas_ft B_ge_m])
ultimately show f ‘ S ∈ lmeasurable ?μ (f ‘ S) ≤ m + e * ?μ S
  by (auto simp: Seq image_Union)
qed
show ?thesis
proof
  show f ‘ S ∈ lmeasurable
    using * linordered_field_no_ub by blast
  let ?x = m − ?μ (f ‘ S)
  have False if ?μ (f ‘ S) > integral S (λx. |det (matrix (f' x))|)
  proof −
    have ml: m < ?μ (f ‘ S)
      using m_def that by blast
    then have ?μ S ≠ 0
      using *(2) bgauge_existence_lemma by fastforce
    with ml have 0: 0 < − (m − ?μ (f ‘ S))/2 / ?μ S
      using that zero_less_measure_iff by force
    then show ?thesis
    using * [OF 0] that by (auto simp: field_split_simps m_def split: if_split_asm)
```

```
    qed
    then show ?μ (f ‘ S) ≤ integral S (λx. |det (matrix (f' x))|)
      by fastforce
  qed
qed


theorem
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes S: S ∈ sets lebesgue
    and deriv: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
    and int: (λx. |det (matrix (f' x))|) integrable_on S
  shows measurable_differentiable_image: f ‘ S ∈ lmeasurable
    and measure_differentiable_image:
      measure lebesgue (f ‘ S) ≤ integral S (λx. |det (matrix (f' x))|) (is ?M)
proof −
  let ?I = λn::nat. cbox (vec (−n)) (vec n) ∩ S
  let ?μ = measure lebesgue
  have x ∈ cbox (vec (− real (nat ⌈norm x⌉))) (vec (real (nat ⌈norm x⌉))) for x
:: real^'n::_
    apply (simp add: mem_box_cart)
    by (smt (verit, best) component_le_norm_cart le_of_int_ceiling)
  then have Seq: S = (⋃ n. ?I n)
    by blast
  have fIn: f ‘ ?I n ∈ lmeasurable
      and mfIn: ?μ (f ‘ ?I n) ≤ integral S (λx. |det (matrix (f' x))|) (is ?MN) for
n
  proof −
    have In: ?I n ∈ lmeasurable
      by (simp add: S bounded_Int bounded_set_imp_lmeasurable sets.Int)
    moreover have ⋀x. x ∈ ?I n ⟹ (f has_derivative f' x) (at x within ?I n)
      by (meson Int_iff deriv has_derivative_subset subsetI)
    moreover have int_In: (λx. |det (matrix (f' x))|) integrable_on ?I n
    by (metis (mono_tags) Int_commute int integrable_altD(1) integrable_restrict_Int)
    ultimately have f ‘ ?I n ∈ lmeasurable ?μ (f ‘ ?I n) ≤ integral (?I n) (λx.
|det (matrix (f' x))|)
      using m_diff_image_weak by metis+
    moreover have integral (?I n) (λx. |det (matrix (f' x))|) ≤ integral S (λx.
|det (matrix (f' x))|)
      by (simp add: int_In int integral_subset_le)
    ultimately show f ‘ ?I n ∈ lmeasurable ?MN
      by auto
  qed
  have ?I k ⊆ ?I n if k ≤ n for k n
    by (rule Int_mono) (use that in ‹auto simp: subset_interval_imp_cart›)
  then have (⋃k≤n. f ‘ ?I k) = f ‘ ?I n for n
    by (fastforce simp add:)
  with mfIn have ?μ (⋃k≤n. f ‘ ?I k) ≤ integral S (λx. |det (matrix (f' x))|) for
n
```

```
    by simp
  then have (⋃ n. f ‘ ?I n) ∈ lmeasurable ?μ (⋃ n. f ‘ ?I n) ≤ integral S (λx. |det
(matrix (f' x))|)
    by (rule fmeasurable_countable_Union [OF fIn] measure_countable_Union_le
[OF fIn])+
  then show f ‘ S ∈ lmeasurable ?M
    by (metis Seq image_UN)+
qed


lemma borel_measurable_simple_function_limit_increasing:
  fixes f :: 'a::euclidean_space ⇒ real
  shows (f ∈ borel_measurable lebesgue ∧ (∀ x. 0 ≤ f x)) ⟷
        (∃ g. (∀ n x. 0 ≤ g n x ∧ g n x ≤ f x) ∧ (∀ n x. g n x ≤ (g(Suc n) x)) ∧
            (∀ n. g n ∈ borel_measurable lebesgue) ∧ (∀ n. finite(range (g n))) ∧
            (∀ x. (λn. g n x) ⟶ f x))
        (is ?lhs = ?rhs)
proof
  assume f: ?lhs
  have leb_f: {x. a ≤ f x ∧ f x < b} ∈ sets lebesgue for a b
  proof −
    have {x. a ≤ f x ∧ f x < b} = {x. f x < b} − {x. f x < a}
      by auto
    also have ... ∈ sets lebesgue
      using borel_measurable_vimage_halfspace_component_lt [of f UNIV] f by
auto
    finally show ?thesis .
  qed
  have g n x ≤ f x
       if inc_g: ⋀n x. 0 ≤ g n x ∧ g n x ≤ g (Suc n) x
          and meas_g: ⋀n. g n ∈ borel_measurable lebesgue
          and fin: ⋀n. finite(range (g n)) and lim: ⋀x. (λn. g n x) ⟶ f x for
g n x
  proof −
    have ∃ r>0. ∀ N. ∃ n≥N. dist (g n x) (f x) ≥ r if g n x > f x
    proof −
      have g: g n x ≤ g (N + n) x for N
        by (rule transitive_stepwise_le) (use inc_g in auto)
      have ∃ m≥N. g n x − f x ≤ dist (g m x) (f x) for N
      proof
        show N ≤ N + n ∧ g n x − f x ≤ dist (g (N + n) x) (f x)
          using g [of N] by (auto simp: dist_norm)
      qed
      with that show ?thesis
        using diff_gt_0_iff_gt by blast
    qed
    with lim show ?thesis
      unfolding lim_sequentially
      by (meson less_le_not_le not_le_imp_less)
```

qed
moreover
let *?Ω* = *λn k. indicator* {*y. k/2^n ≤ f y ∧ f y < (k+1)/2^n*}
let *?g* = *λn x.* (∑ *k*::*real* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2*n). k/2^n * ?Ω n k x*)
have ∃ *g.* (∀ *n x. 0 ≤ g n x ∧ g n x ≤* (*g*(*Suc n*) *x*)) ∧
(∀ *n. g n* ∈ *borel_measurable lebesgue*) ∧ (∀ *n. finite*(*range* (*g n*))) ∧(∀ *x.* (*λn. g n x*) ⟶ *f x*)
proof (*intro exI allI conjI*)
show *0 ≤ ?g n x* **for** *n x*
proof (*clarify intro*!: *ordered_comm_monoid_add_class.sum_nonneg*)
fix *k*::*real*
assume *k* ∈ ℤ **and** *k*: |*k*| ≤ *2 ^ (2*n)*
show *0 ≤ k/2^n * ?Ω n k x*
using *f ‹k ∈ ℤ›* **apply** (*clarsimp simp*: *indicator_def field_split_simps Ints_def*)
by (*smt* (*verit*) *int_less_real_le mult_nonneg_nonneg of_int_0 zero_le_power*)
qed
show *?g n x ≤ ?g* (*Suc n*) *x* **for** *n x*
proof −
have *?g n x* =
(∑ *k* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2*n).*
*k/2^n * (indicator* {*y. k/2^n ≤ f y ∧ f y < (k+1/2)/2^n*} *x* +
indicator {*y. (k+1/2)/2^n ≤ f y ∧ f y < (k+1)/2^n*} *x*))
by (*rule sum.cong* [*OF refl*]) (*simp add*: *indicator_def field_split_simps*)
also have … = (∑ *k* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2*n). k/2^n * indicator* {*y. k/2^n ≤ f y ∧ f y < (k+1/2)/2^n*} *x*) +
(∑ *k* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2*n). k/2^n * indicator* {*y. (k+1/2)/2^n ≤ f y ∧ f y < (k+1)/2^n*} *x*)
by (*simp add*: *comm_monoid_add_class.sum.distrib algebra_simps*)
also have … = (∑ *k* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2*n). (2 * k)/2 ^ Suc n * indicator* {*y. (2 * k)/2 ^ Suc n ≤ f y ∧ f y < (2 * k+1)/2 ^ Suc n*} *x*) +
(∑ *k* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2*n). (2 * k)/2 ^ Suc n * indicator* {*y. (2 * k+1)/2 ^ Suc n ≤ f y ∧ f y < ((2 * k+1) + 1)/2 ^ Suc n*} *x*)
by (*force simp*: *field_simps indicator_def intro*: *sum.cong*)
also have … ≤ (∑ *k* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2 * Suc n). k/2 ^ Suc n * (indicator* {*y. k/2 ^ Suc n ≤ f y ∧ f y < (k+1)/2 ^ Suc n*} *x*))
(**is** *?a + _ ≤ ?b*)
proof −
have *: ⟦*sum f I ≤ sum h I*; *a + sum h I ≤ b*⟧ ⟹ *a + sum f I ≤ b* **for** *I a b f* **and** *h* :: *real*⇒*real*
by *linarith*
let *?h* = *λk. (2*k+1)/2 ^ Suc n * (indicator* {*y. (2 * k+1)/2 ^ Suc n ≤ f y ∧ f y < ((2*k+1) + 1)/2 ^ Suc n*} *x*)
show *?thesis*
proof (*rule* *)
show (∑ *k* | *k* ∈ ℤ ∧ |*k*| ≤ *2 ^ (2*n).*
*2 * k/2 ^ Suc n * indicator* {*y. (2 * k+1)/2 ^ Suc n ≤ f y ∧ f y < (2 * k+1 + 1)/2 ^ Suc n*} *x*)

```
            ≤ sum ?h {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}
          by (rule sum_mono) (simp add: indicator_def field_split_simps)
        next
          have α: ?a = (∑ k ∈ (*)2 ` {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}.
                  k/2 ^ Suc n * indicator {y. k/2 ^ Suc n ≤ f y ∧ f y < (k+1)/2
^ Suc n} x)
          by (auto simp: inj_on_def field_simps comm_monoid_add_class.sum.reindex)
          have β: sum ?h {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}
                = (∑ k ∈ (λx. 2*x + 1) ` {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}.
                  k/2 ^ Suc n * indicator {y. k/2 ^ Suc n ≤ f y ∧ f y < (k+1)/2
^ Suc n} x)
          by (auto simp: inj_on_def field_simps comm_monoid_add_class.sum.reindex)
          have 0: (*) 2 ` {k ∈ ℤ. P k} ∩ (λx. 2 * x + 1) ` {k ∈ ℤ. P k} = {} for
P :: real ⇒ bool
          proof −
            have 2 * i ≠ 2 * j + 1 for i j :: int by arith
            thus ?thesis
              unfolding Ints_def by auto (use of_int_eq_iff in fastforce)
          qed
          have ?a + sum ?h {k ∈ ℤ. |k| ≤ 2 ^ (2*n)}
              = (∑ k ∈ (*)2 ` {k ∈ ℤ. |k| ≤ 2 ^ (2*n)} ∪ (λx. 2*x + 1) ` {k ∈ ℤ.
|k| ≤ 2 ^ (2*n)}.
                  k/2 ^ Suc n * indicator {y. k/2 ^ Suc n ≤ f y ∧ f y < (k+1)/2 ^
Suc n} x)
            unfolding α β
            using finite_abs_int_segment [of 2 ^ (2*n)]
            by (subst sum_Un) (auto simp: 0)
          also have ... ≤ ?b
          proof (rule sum_mono2)
            show finite {k::real. k ∈ ℤ ∧ |k| ≤ 2 ^ (2 * Suc n)}
              by (rule finite_abs_int_segment)
            show (*) 2 ` {k::real. k ∈ ℤ ∧ |k| ≤ 2^(2*n)} ∪ (λx. 2*x + 1) ` {k ∈
ℤ. |k| ≤ 2^(2*n)} ⊆ {k ∈ ℤ. |k| ≤ 2 ^ (2 * Suc n)}
              apply (clarsimp simp: image_subset_iff)
              using one_le_power [of 2::real 2*n] by linarith
            have *: ⟦x ∈ (S ∪ T) − U; ⋀x. x ∈ S ⟹ x ∈ U; ⋀x. x ∈ T ⟹ x ∈
U⟧ ⟹ P x for S T U P
              by blast
            have 0 ≤ b if b ∈ ℤ f x * (2 * 2^n) < b + 1 for b
          by (smt (verit, ccfv_SIG) Ints_cases f int_le_real_less mult_nonneg_nonneg
of_int_add one_le_power that)
            then show 0 ≤ b/2 ^ Suc n * indicator {y. b/2 ^ Suc n ≤ f y ∧ f y <
(b + 1)/2 ^ Suc n} x
              if b ∈ {k ∈ ℤ. |k| ≤ 2 ^ (2 * Suc n)} −
                   ((*) 2 ` {k ∈ ℤ. |k| ≤ 2 ^ (2*n)} ∪ (λx. 2*x + 1) ` {k ∈ ℤ.
|k| ≤ 2 ^ (2*n)}) for b
              using that by (simp add: indicator_def divide_simps)
          qed
          finally show ?a + sum ?h {k ∈ ℤ. |k| ≤ 2 ^ (2*n)} ≤ ?b .
```

```
      qed
    qed
    finally show ?thesis .
  qed
  show ?g n ∈ borel_measurable lebesgue for n
  apply (intro borel_measurable_indicator borel_measurable_times borel_measurable_sum)
    using leb_f sets_restrict_UNIV by auto
  show finite (range (?g n)) for n
  proof −
    have (∑ k | k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n). k/2^n ∗ ?Ω n k x)
          ∈ (λk. k/2^n) ‘ {k ∈ ℤ. |k| ≤ 2 ^ (2∗n)} for x
   proof (cases ∃ k. k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n) ∧ k/2^n ≤ f x ∧ f x < (k+1)/2^n)
      case True
      then show ?thesis
        by (blast intro: indicator_sum_eq)
    next
      case False
      then have (∑ k | k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n). k/2^n ∗ ?Ω n k x) = 0
        by auto
      then show ?thesis by force
    qed
    then have range (?g n) ⊆ ((λk. (k/2^n)) ‘ {k. k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n)})
      by auto
    moreover have finite ((λk::real. (k/2^n)) ‘ {k ∈ ℤ. |k| ≤ 2 ^ (2∗n)})
      by (intro finite_imageI finite_abs_int_segment)
    ultimately show ?thesis
      by (rule finite_subset)
  qed
  show (λn. ?g n x) ⟶ f x for x
  proof (clarsimp simp add: lim_sequentially)
    fix e::real
    assume e > 0
    obtain N1 where N1: 2 ^ N1 > abs(f x)
      using real_arch_pow by fastforce
    obtain N2 where N2: (1/2) ^ N2 < e
      using real_arch_pow_inv ‹e > 0› by fastforce
   have dist (∑ k | k ∈ ℤ ∧ |k| ≤ 2 ^ (2∗n). k/2^n ∗ ?Ω n k x) (f x) < e if N1
+ N2 ≤ n for n
    proof −
      let ?m = real_of_int ⌊2^n ∗ f x⌋
      have |?m| ≤ 2^n ∗ 2^N1
        using N1 apply (simp add: f)
     by (meson floor_mono le_floor_iff less_le_not_le mult_le_cancel_left_pos
zero_less_numeral zero_less_power)
      also have ... ≤ 2 ^ (2∗n)
        by (metis that add_leD1 add_le_cancel_left mult.commute mult_2_right
one_less_numeral_iff
                  power_add power_increasing_iff semiring_norm(76))
      finally have m_le: |?m| ≤ 2 ^ (2∗n) .
```

```
      have ?m/2^n ≤ f x f x < (?m + 1)/2^n
        by (auto simp: mult.commute pos_divide_le_eq mult_imp_less_div_pos)
      then have eq: dist (∑ k | k ∈ ℤ ∧ |k| ≤ 2 ^ (2*n). k/2^n * ?Ω n k x) (f x)
                  = dist (?m/2^n) (f x)
        by (subst indicator_sum_eq [of ?m]) (auto simp: m_le)
      have |2^n| * |?m/2^n - f x| = |2^n * (?m/2^n - f x)|
        by (simp add: abs_mult)
      also have ... < 2 ^ N2 * e
        using N2 by (simp add: divide_simps mult.commute) linarith
      also have ... ≤ |2^n| * e
        using that ‹e > 0› by auto
      finally show ?thesis
        using eq by (simp add: dist_real_def)
    qed
    then show ∃ no. ∀ n≥no. dist (∑ k | k ∈ ℤ ∧ |k| ≤ 2 ^ (2*n). k * ?Ω n k
x/2^n) (f x) < e
      by force
  qed
 qed
 ultimately show ?rhs
  by metis
next
 assume RHS: ?rhs
 with borel_measurable_simple_function_limit [of f UNIV, unfolded lebesgue_on_UNIV_eq]
 show ?lhs
  by (blast intro: order_trans)
qed
```

9.32.2 Borel measurable Jacobian determinant

```
lemma lemma_partial_derivatives0:
 fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
 assumes linear f and lim0: ((λx. f x /R norm x) ⟶ 0) (at 0 within S)
  and lb: ⋀v. v ≠ 0 ⟹ (∃ k>0. ∀ e>0. ∃ x. x ∈ S - {0} ∧ norm x < e ∧ k *
norm x ≤ |v · x|)
 shows f x = 0
proof -
 interpret linear f by fact
 have dim {x. f x = 0} ≤ DIM('a)
  by (rule dim_subset_UNIV)
 moreover have False if less: dim {x. f x = 0} < DIM('a)
 proof -
  obtain d where d ≠ 0 and d: ⋀y. f y = 0 ⟹ d · y = 0
    using orthogonal_to_subspace_exists [OF less] orthogonal_def
    by (metis (mono_tags, lifting) mem_Collect_eq span_base)
  then obtain k where k > 0
    and k: ⋀e. e > 0 ⟹ ∃ y. y ∈ S - {0} ∧ norm y < e ∧ k * norm y ≤ |d ·
y|
    using lb by blast
```

```
  have ∃h. ∀n. ((h n ∈ S ∧ h n ≠ 0 ∧ k * norm (h n) ≤ |d • h n|) ∧ norm (h
n) < 1 / real (Suc n)) ∧
            norm (h (Suc n)) < norm (h n)
  proof (rule dependent_nat_choice)
   show ∃y. (y ∈ S ∧ y ≠ 0 ∧ k * norm y ≤ |d • y|) ∧ norm y < 1 / real (Suc
0)
     by simp (metis DiffE insertCI k not_less not_one_le_zero)
  qed (use k [of min (norm x) (1/(Suc n + 1)) for x n] in auto)
  then obtain α where α: ⋀n. α n ∈ S − {0} and kd: ⋀n. k * norm(α n) ≤
|d • α n|
      and norm_lt: ⋀n. norm(α n) < 1/(Suc n)
    by force
  let ?β = λn. α n /R norm (α n)
  have com: ⋀g. (∀n. g n ∈ sphere (0::'a) 1)
          ⟹ ∃l ∈ sphere 0 1. ∃ϱ::nat⇒nat. strict_mono ϱ ∧ (g ∘ ϱ) ⟶ l
    using compact_sphere compact_def by metis
  moreover have ∀n. ?β n ∈ sphere 0 1
    using α by auto
  ultimately obtain l::'a and ϱ::nat⇒nat
    where l: l ∈ sphere 0 1 and strict_mono ϱ and to_l: (?β ∘ ϱ) ⟶ l
    by meson
  moreover have continuous (at l) (λx. (|d • x| − k))
    by (intro continuous_intros)
  ultimately have lim_dl: ((λx. (|d • x| − k)) ∘ (?β ∘ ϱ)) ⟶ (|d • l| − k)
    by (meson continuous_imp_tendsto)
  have ∀F i in sequentially. 0 ≤ ((λx. |d • x| − k) ∘ ((λn. α n /R norm (α n))
∘ ϱ)) i
    using α kd by (auto simp: field_split_simps)
  then have k ≤ |d • l|
    using tendsto_lowerbound [OF lim_dl, of 0] by auto
  moreover have d • l = 0
  proof (rule d)
    show f l = 0
    proof (rule LIMSEQ_unique [of f ∘ ?β ∘ ϱ])
      have isCont f l
        using ‹linear f› linear_continuous_at linear_conv_bounded_linear by
blast
      then show (f ∘ (λn. α n /R norm (α n)) ∘ ϱ) ⟶ f l
        unfolding comp_assoc
        using to_l continuous_imp_tendsto by blast
      have α ⟶ 0
        using norm_lt LIMSEQ_norm_0 by metis
      with ‹strict_mono ϱ› have (α ∘ ϱ) ⟶ 0
        by (metis LIMSEQ_subseq_LIMSEQ)
      with lim0 α have ((λx. f x /R norm x) ∘ (α ∘ ϱ)) ⟶ 0
        by (force simp: tendsto_at_iff_sequentially)
      then show (f ∘ (λn. α n /R norm (α n)) ∘ ϱ) ⟶ 0
        by (simp add: o_def scale)
    qed
```

```
    qed
    ultimately show False
      using ‹k > 0› by auto
  qed
  ultimately have dim: dim {x. f x = 0} = DIM('a)
    by force
  then show ?thesis
   by (metis (mono_tags, lifting) dim_eq_full UNIV_I eq_0_on_span mem_Collect_eq
span_raw_def)
qed

lemma lemma_partial_derivatives:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes linear f and lim: ((λx. f (x − a) /R norm (x − a)) ⟶ 0) (at a
within S)
    and lb: ⋀v. v ≠ 0 ⟹ (∃k>0. ∀e>0. ∃x ∈ S − {a}. norm(a − x) < e ∧ k
* norm(a − x) ≤ |v • (x − a)|)
  shows f x = 0
proof −
  have ((λx. f x /R norm x) ⟶ 0) (at 0 within (λx. x−a) ‘ S)
    using lim by (simp add: Lim_within dist_norm)
  then show ?thesis
  proof (rule lemma_partial_derivatives0 [OF ‹linear f›])
    fix v :: 'a
    assume v: v ≠ 0
    show ∃k>0. ∀e>0. ∃x. x ∈ (λx. x − a) ‘ S − {0} ∧ norm x < e ∧ k * norm
x ≤ |v • x|
      using lb [OF v] by (force simp: norm_minus_commute)
  qed
qed


proposition borel_measurable_partial_derivatives:
  fixes f :: real^'m::{finite,wellorder} ⇒ real^'n
  assumes S: S ∈ sets lebesgue
    and f: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
  shows (λx. (matrix(f' x)$m$n)) ∈ borel_measurable (lebesgue_on S)
proof −
  have contf: continuous_on S f
    using continuous_on_eq_continuous_within f has_derivative_continuous by
blast
  have {x ∈ S. (matrix (f' x)$m$n) ≤ b} ∈ sets lebesgue for b
  proof (rule sets_negligible_symdiff)
    let ?T = {x ∈ S. ∀e>0. ∃d>0. ∃A. A$m$n < b ∧ (∀i j. A$i$j ∈ ℚ) ∧
                        (∀y ∈ S. norm(y − x) < d ⟶ norm(f y − f x − A *v (y −
x)) ≤ e * norm(y − x))}
    let ?U = S ∩
             (⋂e ∈ {e ∈ ℚ. e > 0}.
               ⋃A ∈ {A. A$m$n < b ∧ (∀i j. A$i$j ∈ ℚ)}.
```

$\bigcup d \in \{d \in \mathbb{Q}.\ 0 < d\}.$
$S \cap (\bigcap y \in S.\ \{x \in S.\ norm(y - x) < d \longrightarrow norm(f\ y - f\ x - A *v (y - x)) \leq e * norm(y - x)\}))$
have *?T = ?U*
proof (*intro set_eqI iffI*)
fix *x*
assume *xT*: $x \in$ *?T*
then show $x \in$ *?U*
proof (*clarsimp simp add*:)
fix *q* :: *real*
assume $q \in \mathbb{Q}$ $q > 0$
then obtain *d A* **where** $d > 0$ **and** *A*: $A \$ m \$ n < b \bigwedge i\ j.\ A \$ i \$ j \in \mathbb{Q}$
$\bigwedge y.\ [\![y \in S;\ norm\ (y - x) < d]\!] \Longrightarrow norm\ (f\ y - f\ x - A *v (y - x)) \leq q * norm\ (y - x)$
using *xT* **by** *auto*
then obtain δ **where** $d > \delta$ $\delta > 0$ $\delta \in \mathbb{Q}$
using *Rats_dense_in_real* **by** *blast*
with *A* **show** $\exists A.\ A \$ m \$ n < b \wedge (\forall i\ j.\ A \$ i \$ j \in \mathbb{Q}) \wedge$
$(\exists s.\ s \in \mathbb{Q} \wedge 0 < s \wedge (\forall y \in S.\ norm\ (y - x) < s \longrightarrow norm\ (f\ y - f\ x - A *v (y - x)) \leq q * norm\ (y - x)))$
by *force*
qed
next
fix *x*
assume *xU*: $x \in$ *?U*
then show $x \in$ *?T*
proof *clarsimp*
fix *e* :: *real*
assume $e > 0$
then obtain ε **where** ε: $e > \varepsilon$ $\varepsilon > 0$ $\varepsilon \in \mathbb{Q}$
using *Rats_dense_in_real* **by** *blast*
with *xU* **obtain** *A r* **where** $x \in S$ **and** *Ar*: $A \$ m \$ n < b \forall i\ j.\ A \$ i \$ j \in \mathbb{Q}$ $r \in \mathbb{Q}$ $r > 0$
and $\forall y \in S.\ norm\ (y - x) < r \longrightarrow norm\ (f\ y - f\ x - A *v (y - x)) \leq \varepsilon * norm\ (y - x)$
by (*auto simp*: *split*: *if_split_asm*)
then have $\forall y \in S.\ norm\ (y - x) < r \longrightarrow norm\ (f\ y - f\ x - A *v (y - x)) \leq e * norm\ (y - x)$
by (*meson* ‹$e > \varepsilon$› *less_eq_real_def mult_right_mono norm_ge_zero order_trans*)
then show $\exists d{>}0.\ \exists A.\ A \$ m \$ n < b \wedge (\forall i\ j.\ A \$ i \$ j \in \mathbb{Q}) \wedge (\forall y \in S.\ norm\ (y - x) < d \longrightarrow norm\ (f\ y - f\ x - A *v (y - x)) \leq e * norm\ (y - x))$
using ‹$x \in S$› *Ar* **by** *blast*
qed
qed
moreover have *?U* $\in$ *sets lebesgue*
proof −
have *coQ*: *countable* $\{e \in \mathbb{Q}.\ 0 < e\}$
using *countable_Collect countable_rat* **by** *blast*

```
    have ne: {e ∈ ℚ. (0::real) < e} ≠ {}
      using zero_less_one Rats_1 by blast
    have coA: countable {A. A $ m $ n < b ∧ (∀ i j. A $ i $ j ∈ ℚ)}
    proof (rule countable_subset)
      show countable {A. ∀ i j. A $ i $ j ∈ ℚ}
        using countable_vector [OF countable_vector, of λi j. ℚ] by (simp add:
countable_rat)
    qed blast
    have *: ⟦U ≠ {} ⟹ closedin (top_of_set S) (S ∩ ⋂ U)⟧
          ⟹ closedin (top_of_set S) (S ∩ ⋂ U) for U
      by fastforce
    have eq: {x::(real,'m)vec. P x ∧ (Q x ⟶ R x)} = {x. P x ∧ ¬ Q x} ∪ {x.
P x ∧ R x} for P Q R
      by auto
    have sets: S ∩ (⋂y∈S. {x ∈ S. norm (y − x) < d ⟶ norm (f y − f x − A
*v (y − x)) ≤ e * norm (y − x)})
            ∈ sets lebesgue for e A d
    proof −
      have clo: closedin (top_of_set S)
              {x ∈ S. norm (y − x) < d ⟶ norm (f y − f x − A *v (y − x))
≤ e * norm (y − x)}
        for y
      proof −
        have cont1: continuous_on S (λx. norm (y − x))
        and  cont2: continuous_on S (λx. e * norm (y − x) − norm (f y − f x
− (A *v y − A *v x)))
          by (force intro: contf continuous_intros)+
        have clo1: closedin (top_of_set S) {x ∈ S. d ≤ norm (y − x)}
        using continuous_closedin_preimage [OF cont1, of {d..}] by (simp add:
vimage_def Int_def)
        have clo2: closedin (top_of_set S)
                {x ∈ S. norm (f y − f x − (A *v y − A *v x)) ≤ e * norm (y
− x)}
        using continuous_closedin_preimage [OF cont2, of {0..}] by (simp add:
vimage_def Int_def)
        show ?thesis
          by (auto simp: eq not_less matrix_vector_mult_diff_distrib intro: clo1
clo2)
      qed
      show ?thesis
        by (rule lebesgue_closedin [of S]) (force intro: * S clo)+
    qed
    show ?thesis
     by (intro sets sets.Int S sets.countable_UN'' sets.countable_INT'' coQ coA)
auto
  qed
  ultimately show ?T ∈ sets lebesgue
    by simp
  let ?M = (?T − {x ∈ S. matrix (f' x) $ m $ n ≤ b} ∪ ({x ∈ S. matrix (f' x)
```

```
$ m $ n ≤ b} - ?T))
    let ?Θ = λx v. ∀ξ>0. ∃e>0. ∀y ∈ S-{x}. norm (x - y) < e ⟶ |v • (y -
x)| < ξ * norm (x - y)
    have nN: negligible {x ∈ S. ∃v≠0. ?Θ x v}
      unfolding negligible_eq_zero_density
    proof clarsimp
      fix x v and r e :: real
      assume x ∈ S v ≠ 0 r > 0 e > 0
      and Theta [rule_format]: ?Θ x v
      moreover have (norm v * e / 2) / CARD('m) ^ CARD('m) > 0
        by (simp add: ‹v ≠ 0› ‹e > 0›)
      ultimately obtain d where d > 0
        and dless: ⋀y. ⟦y ∈ S - {x}; norm (x - y) < d⟧ ⟹
                    |v • (y - x)| < ((norm v * e / 2) / CARD('m) ^ CARD('m))
* norm (x - y)
        by metis
      let ?W = ball x (min d r) ∩ {y. |v • (y - x)| < (norm v * e/2 * min d r) /
CARD('m) ^ CARD('m)}
      have open {x. |v • (x - a)| < b} for a b
        by (intro open_Collect_less continuous_intros)
      show ∃d>0. d ≤ r ∧
          (∃U. {x' ∈ S. ∃v≠0. ?Θ x' v} ∩ ball x d ⊆ U ∧
              U ∈ lmeasurable ∧ measure lebesgue U < e * content (ball x d))
      proof (intro exI conjI)
        show 0 < min d r min d r ≤ r
          using ‹r > 0› ‹d > 0› by auto
        show {x' ∈ S. ∃v. v ≠ 0 ∧ (∀ξ>0. ∃e>0. ∀z∈S - {x'}. norm (x' - z)
< e ⟶ |v • (z - x')| < ξ * norm (x' - z))} ∩ ball x (min d r) ⊆ ?W
          proof (clarsimp simp: dist_norm norm_minus_commute)
            fix y w
            assume y ∈ S w ≠ 0
              and less [rule_format]:
                  ∀ξ>0. ∃e>0. ∀z∈S - {y}. norm (y - z) < e ⟶ |w • (z - y)|
< ξ * norm (y - z)
              and d: norm (y - x) < d and r: norm (y - x) < r
             show |v • (y - x)| < norm v * e * min d r / (2 * real CARD('m) ^
CARD('m))
            proof (cases y = x)
              case True
              with ‹r > 0› ‹d > 0› ‹e > 0› ‹v ≠ 0› show ?thesis
                by simp
            next
              case False
              have |v • (y - x)| < norm v * e / 2 / real (CARD('m) ^ CARD('m))
* norm (x - y)
                  by (metis Diff_iff False ‹y ∈ S› d dless empty_iff insert_iff
norm_minus_commute)
              also have ... ≤ norm v * e * min d r / (2 * real CARD('m) ^
CARD('m))
```

```
          using d r ‹e > 0› by (simp add: field_simps norm_minus_commute
mult_left_mono)
         finally show ?thesis .
       qed
     qed
     show ?W ∈ lmeasurable
       by (simp add: fmeasurable_Int_fmeasurable borel_open)
     obtain k::'m where True
       by metis
     obtain T where T: orthogonal_transformation T and v: v = T(norm v
*R axis k (1::real))
       using rotation_rightward_line by metis
     define b where b ≡ norm v
     have b > 0
       using ‹v ≠ 0› by (auto simp: b_def)
     obtain eqb: inv T v = b *R axis k (1::real) and inj T bij T and invT:
orthogonal_transformation (inv T)
         by (metis UNIV_I b_def  T v bij_betw_inv_into_left orthogo-
nal_transformation_inj orthogonal_transformation_bij orthogonal_transformation_inv)
     let ?v = χ i. min d r / CARD('m)
     let ?v' = χ i. if i = k then (e/2 * min d r) / CARD('m) ^ CARD('m)
else min d r
     let ?x' = inv T x
     let ?W' = (ball ?x' (min d r) ∩ {y. |(y - ?x')$k| < e * min d r / (2 *
CARD('m) ^ CARD('m))})
     have abs: x - e ≤ y ∧ y ≤ x + e ⟷ abs(y - x) ≤ e for x y e::real
       by auto
     have ?W = T ` ?W'
     proof -
       have 1: T ` (ball (inv T x) (min d r)) = ball x (min d r)
         by (simp add: T image_orthogonal_transformation_ball orthogo-
nal_transformation_surj surj_f_inv_f)
       have 2: {y. |v • (y - x)| < b * e * min d r / (2 * real CARD('m) ^
CARD('m))} =
               T ` {y. |y $ k - ?x' $ k| < e * min d r / (2 * real CARD('m) ^
CARD('m))}
       proof -
         have *: |T (b *R axis k 1) • (y - x)| = b * |inv T y $ k - ?x' $ k| for
y
         proof -
           have |T (b *R axis k 1) • (y - x)| = |(b *R axis k 1) • inv T (y - x)|
             by (metis (no_types, opaque_lifting) b_def eqb invT orthogo-
nal_transformation_def v)
           also have ... = b * |(axis k 1) • inv T (y - x)|
             using ‹b > 0› by (simp add: abs_mult)
           also have ... = b * |inv T y $ k - ?x' $ k|
             using orthogonal_transformation_linear [OF invT]
             by (simp add: inner_axis' linear_diff)
           finally show ?thesis
```

```
            by simp
          qed
          show ?thesis
            using v b_def [symmetric]
            using ‹b > 0› by (simp add: * bij_image_Collect_eq [OF ‹bij T›]
mult_less_cancel_left_pos times_divide_eq_right [symmetric] del: times_divide_eq_right)
        qed
        show ?thesis
          using ‹b > 0› by (simp add: image_Int ‹inj T› 1 2 b_def [symmetric])
      qed
      moreover have ?W' ∈ lmeasurable
        by (auto intro: fmeasurable_Int_fmeasurable)
      ultimately have measure lebesgue ?W = measure lebesgue ?W'
        by (metis measure_orthogonal_image T)
      also have ... ≤ measure lebesgue (cbox (?x' − ?v') (?x' + ?v'))
      proof (rule measure_mono_fmeasurable)
        show ?W' ⊆ cbox (?x' − ?v') (?x' + ?v')
      apply (clarsimp simp add: mem_box_cart abs dist_norm norm_minus_commute
simp del: min_less_iff_conj min.bounded_iff)
          by (metis component_le_norm_cart less_eq_real_def le_less_trans
vector_minus_component)
      qed auto
      also have ... ≤ e/2 * measure lebesgue (cbox (?x' − ?v) (?x' + ?v))
      proof −
        have cbox (?x' − ?v) (?x' + ?v) ≠ {}
          using ‹r > 0› ‹d > 0› by (auto simp: interval_eq_empty_cart
divide_less_0_iff)
        with ‹r > 0› ‹d > 0› ‹e > 0› show ?thesis
          apply (simp add: content_cbox_if_cart mem_box_cart)
          apply (auto simp: prod_nonneg)
      apply (simp add: abs if_distrib prod.delta_remove field_simps power_diff
split: if_split_asm)
        done
      qed
      also have ... ≤ e/2 * measure lebesgue (cball ?x' (min d r))
      proof (rule mult_left_mono [OF measure_mono_fmeasurable])
        have *: norm (?x' − y) ≤ min d r
          if y: ⋀i. |?x' $ i − y $ i| ≤ min d r / real CARD('m) for y
        proof −
          have norm (?x' − y) ≤ (∑ i∈UNIV. |(?x' − y) $ i|)
            by (rule norm_le_l1_cart)
          also have ... ≤ real CARD('m) * (min d r / real CARD('m))
            by (rule sum_bounded_above) (use y in auto)
          finally show ?thesis
            by simp
        qed
        show cbox (?x' − ?v) (?x' + ?v) ⊆ cball ?x' (min d r)
            apply (clarsimp simp only: mem_box_cart dist_norm mem_cball
intro!: *)
```

```
          by (simp add: abs_diff_le_iff abs_minus_commute)
        qed (use ‹e > 0› in auto)
        also have ... < e * content (cball ?x' (min d r))
          using ‹r > 0› ‹d > 0› ‹e > 0› by (auto intro: content_cball_pos)
        also have ... = e * content (ball x (min d r))
         using ‹r > 0› ‹d > 0› content_ball_conv_unit_ball[of min d r inv T x]
               content_ball_conv_unit_ball[of min d r x]
          by (simp add: content_cball_conv_ball)
        finally show measure lebesgue ?W < e * content (ball x (min d r)) .
      qed
    qed
    have *: (⋀x. (x ∉ S) ⟹ (x ∈ T ⟷ x ∈ U)) ⟹ (T - U) ∪ (U - T) ⊆ S
for S T U :: (real,'m) vec set
      by blast
    have MN: ?M ⊆ {x ∈ S. ∃ v≠0. ?Θ x v}
    proof (rule *)
      fix x
      assume x: x ∉ {x ∈ S. ∃ v≠0. ?Θ x v}
      show (x ∈ ?T) ⟷ (x ∈ {x ∈ S. matrix (f' x) $ m $ n ≤ b})
      proof (cases x ∈ S)
        case True
        then have x: ¬ ?Θ x v if v ≠ 0 for v
          using x that by force
        show ?thesis
        proof (rule iffI; clarsimp)
          assume b: ∀ e>0. ∃ d>0. ∃ A. A $ m $ n < b ∧ (∀ i j. A $ i $ j ∈ ℚ) ∧
                                       (∀ y∈S. norm (y - x) < d ⟶ norm (f y - f x - A
*v (y - x)) ≤ e * norm (y - x))
                    (is ∀ e>0. ∃ d>0. ∃ A. ?Φ e d A)
          then have ∀ k. ∃ d>0. ∃ A. ?Φ (1 / Suc k) d A
         by (metis (no_types, opaque_lifting) less_Suc_eq_0_disj of_nat_0_less_iff
zero_less_divide_1_iff)
          then obtain δ A where δ: ⋀k. δ k > 0
                           and Ab: ⋀k. A k $ m $ n < b
                           and A: ⋀k y. ⟦y ∈ S; norm (y - x) < δ k⟧ ⟹
                                        norm (f y - f x - A k *v (y - x)) ≤ 1/(Suc k)
* norm (y - x)
            by metis
          have ∀ i j. ∃ a. (λn. A n $ i $ j) ⟶⟶ a
          proof (intro allI)
            fix i j
            have vax: (A n *v axis j 1) $ i = A n $ i $ j for n
              by (metis cart_eq_inner_axis matrix_vector_mul_component)
            let ?CA = {x. Cauchy (λn. (A n) *v x)}
            have subspace ?CA
              unfolding subspace_def convergent_eq_Cauchy [symmetric]
                by (force simp: algebra_simps intro: tendsto_intros)
            then have CA_eq: ?CA = span ?CA
              by (metis span_eq_iff)
```

```
          also have ... = UNIV
          proof −
            have dim ?CA ≤ CARD('m)
              using dim_subset_UNIV[of ?CA] by auto
            moreover have False if less: dim ?CA < CARD('m)
            proof −
             obtain d where d ≠ 0 and d: ⋀y. y ∈ span ?CA ⟹ orthogonal d y
               using less by (force intro: orthogonal_to_subspace_exists [of ?CA])
              with x [OF ‹d ≠ 0›] obtain ξ where ξ > 0
              and ξ: ⋀e. e > 0 ⟹ ∃ y ∈ S − {x}. norm (x − y) < e ∧ ξ * norm
(x − y) ≤ |d • (y − x)|
               by (fastforce simp: not_le Bex_def)
              obtain γ z where γSx: ⋀i. γ i ∈ S − {x}
                           and γle:  ⋀i. ξ * norm(γ i − x) ≤ |d • (γ i − x)|
                           and γx:   γ ⟶ x
                           and z:    (λn. (γ n − x) /R norm (γ n − x)) ⟶ z
              proof −
                have ∃γ. (∀ i. (γ i ∈ S − {x} ∧
                                ξ * norm(γ i − x) ≤ |d • (γ i − x)| ∧ norm(γ i − x)
< 1/Suc i) ∧
                                norm(γ(Suc i) − x) < norm(γ i − x))
                proof (rule dependent_nat_choice)
                  show ∃ y. y ∈ S − {x} ∧ ξ * norm (y − x) ≤ |d • (y − x)| ∧
norm (y − x) < 1 / Suc 0
                   using ξ [of 1] by (auto simp: dist_norm norm_minus_commute)
                next
                  fix y i
                 assume y ∈ S − {x} ∧ ξ * norm (y − x) ≤ |d • (y − x)| ∧ norm
(y − x) < 1/Suc i
                  then have min (norm(y − x)) (1/((Suc i) + 1)) > 0
                    by auto
                   then obtain y' where y' ∈ S − {x} and y': norm (x − y') <
min (norm (y − x)) (1/((Suc i) + 1))
                                        ξ * norm (x − y') ≤ |d • (y' − x)|
                    using ξ by metis
                  with ξ show ∃ y'. (y' ∈ S − {x} ∧ ξ * norm (y' − x) ≤ |d • (y'
− x)| ∧
                                norm (y' − x) < 1/(Suc (Suc i))) ∧ norm (y' − x) <
norm (y − x)
                    by (auto simp: dist_norm norm_minus_commute)
                qed
                then obtain γ where
                     γSx: ⋀i. γ i ∈ S − {x}
                     and γle: ⋀i. ξ * norm(γ i − x) ≤ |d • (γ i − x)|
                     and γconv: ⋀i. norm(γ i − x) < 1/(Suc i)
                  by blast
                let ?f = λi. (γ i − x) /R norm (γ i − x)
                have ?f i ∈ sphere 0 1 for i
                  using γSx by auto
```

```
            then obtain l ϱ where l ∈ sphere 0 1 strict_mono ϱ and l: (?f ∘
ϱ) ⟶ l
            using compact_sphere [of 0::(real,'m) vec 1] unfolding compact_def
by meson
            show thesis
            proof
              show (γ ∘ ϱ) i ∈ S − {x} ξ * norm ((γ ∘ ϱ) i − x) ≤ |d • ((γ ∘
ϱ) i − x)| for i
                using γSx γle by auto
              have γ ⟶ x
              proof (clarsimp simp add: LIMSEQ_def dist_norm)
                fix r :: real
                assume r > 0
                with real_arch_invD obtain no where no ≠ 0 real no > 1/r
                      by (metis divide_less_0_1_iff not_less_iff_gr_or_eq
of_nat_0_eq_iff reals_Archimedean2)
                with γconv show ∃ no. ∀ n≥no. norm (γ n − x) < r
               by (metis ‹r > 0› add.commute divide_inverse inverse_inverse_eq
inverse_less_imp_less less_trans mult.left_neutral nat_le_real_less of_nat_Suc)
              qed
              with ‹strict_mono ϱ› show (γ ∘ ϱ) ⟶ x
                by (metis LIMSEQ_subseq_LIMSEQ)
              show (λn. ((γ ∘ ϱ) n − x) /R norm ((γ ∘ ϱ) n − x)) ⟶ l
                using l by (auto simp: o_def)
            qed
          qed
          have isCont (λx. (|d • x| − ξ)) z
            by (intro continuous_intros)
          from isCont_tendsto_compose [OF this z]
          have lim: (λy. |d • ((γ y − x) /R norm (γ y − x))| − ξ) ⟶ |d •
z| − ξ
            by auto
           moreover have ∀F i in sequentially. 0 ≤ |d • ((γ i − x) /R norm
(γ i − x))| − ξ
          proof (rule eventuallyI)
            fix n
            show 0 ≤ |d • ((γ n − x) /R norm (γ n − x))| − ξ
              using γle [of n] γSx by (auto simp: abs_mult divide_simps)
          qed
          ultimately have ξ ≤ |d • z|
            using tendsto_lowerbound [where a=0] by fastforce
          have Cauchy (λn. (A n) *v z)
          proof (clarsimp simp add: Cauchy_def)
            fix ε :: real
            assume 0 < ε
            then obtain N::nat where N > 0 and N: ε/2 > 1/N
        by (metis half_gt_zero inverse_eq_divide neq0_conv real_arch_inverse)
            show ∃ M. ∀ m≥M. ∀ n≥M. dist (A m *v z) (A n *v z) < ε
            proof (intro exI allI impI)
```

```
                fix i j
                assume ij: N ≤ i N ≤ j
                let ?V = λi k. A i *v ((γ k − x) /R norm (γ k − x))
                have ∀F k in sequentially. dist (γ k) x < min (δ i) (δ j)
                 using γx [unfolded tendsto_iff] by (meson min_less_iff_conj δ)
               then have even: ∀F k in sequentially. norm (?V i k − ?V j k) −
2 / N ≤ 0
                proof (rule eventually_mono, clarsimp)
                  fix p
                  assume p: dist (γ p) x < δ i dist (γ p) x < δ j
                  let ?C = λk. f (γ p) − f x − A k *v (γ p − x)
                  have norm ((A i − A j) *v (γ p − x)) = norm (?C j − ?C i)
                    by (simp add: algebra_simps)
                  also have ... ≤ norm (?C j) + norm (?C i)
                    using norm_triangle_ineq4 by blast
                     also have ... ≤ 1/(Suc j) * norm (γ p − x) + 1/(Suc i) *
norm (γ p − x)
                    by (metis A Diff_iff γSx dist_norm p add_mono)
                   also have ... ≤ 1/N * norm (γ p − x) + 1/N * norm (γ p −
x)
                       using ij ‹N > 0› by (intro add_mono mult_right_mono)
(auto simp: field_simps)
                  also have ... = 2 / N * norm (γ p − x)
                    by simp
                  finally have no_le: norm ((A i − A j) *v (γ p − x)) ≤ 2 / N
* norm (γ p − x) .
                  have norm (?V i p − ?V j p) =
                        norm ((A i − A j) *v ((γ p − x) /R norm (γ p − x)))
                    by (simp add: algebra_simps)
                  also have ... = norm ((A i − A j) *v (γ p − x)) / norm (γ p
− x)
                    by (simp add: divide_inverse matrix_vector_mult_scaleR)
                  also have ... ≤ 2 / N
                    using no_le by (auto simp: field_split_simps)
                  finally show norm (?V i p − ?V j p) ≤ 2 / N .
                qed
                have isCont (λw. (norm(A i *v w − A j *v w) − 2 / N)) z
                  by (intro continuous_intros)
                from isCont_tendsto_compose [OF this z]
                have lim: (λw. norm (A i *v ((γ w − x) /R norm (γ w − x)) −
                                A j *v ((γ w − x) /R norm (γ w − x))) − 2 / N)
                          ⟶ norm (A i *v z − A j *v z) − 2 / N
                  by auto
                have dist (A i *v z) (A j *v z) ≤ 2 / N
                       using tendsto_upperbound [OF lim even] by (auto simp:
dist_norm)
                with N show dist (A i *v z) (A j *v z) < ε
                  by linarith
               qed
```

```
            qed
            then have d · z = 0
              using CA_eq d orthogonal_def by auto
            then show False
              using ‹0 < ξ› ‹ξ ≤ |d · z|› by auto
          qed
          ultimately show ?thesis
            using dim_eq_full by fastforce
        qed
        finally have ?CA = UNIV .
        then have Cauchy (λn. (A n) *v axis j 1)
          by auto
        then obtain L where (λn. A n *v axis j 1) ⟶ L
          by (auto simp: Cauchy_convergent_iff convergent_def)
        then have (λx. (A x *v axis j 1) $ i) ⟶ L $ i
          by (rule tendsto_vec_nth)
        then show ∃ a. (λn. A n $ i $ j) ⟶ a
          by (force simp: vax)
      qed
      then obtain B where B: ⋀i j. (λn. A n $ i $ j) ⟶ B $ i $ j
        by (auto simp: lambda_skolem)
      have lin_df: linear (f' x)
           and lim_df: ((λy. (1 / norm (y − x)) *R (f y − (f x + f' x (y −
x)))) ⟶ 0) (at x within S)
      using ‹x ∈ S› assms by (auto simp: has_derivative_within linear_linear)
      moreover
      interpret linear f' x by fact
      have (matrix (f' x) − B) *v w = 0 for w
      proof (rule lemma_partial_derivatives [of (*v) (matrix (f' x) − B)])
        show linear ((*v) (matrix (f' x) − B))
          by (rule matrix_vector_mul_linear)
        have ((λy. ((f x + f' x (y − x)) − f y) /R norm (y − x)) ⟶ 0) (at
x within S)
          using tendsto_minus [OF lim_df] by (simp add: field_split_simps)
          then show ((λy. (matrix (f' x) − B) *v (y − x) /R norm (y − x))
⟶ 0) (at x within S)
        proof (rule Lim_transform)
          have ((λy. ((f y + B *v x − (f x + B *v y)) /R norm (y − x))) ⟶
0) (at x within S)
          proof (clarsimp simp add: Lim_within dist_norm)
            fix e :: real
            assume e > 0
            then obtain q::nat where q ≠ 0 and qe2: 1/q < e/2
              by (metis divide_pos_pos inverse_eq_divide real_arch_inverse
zero_less_numeral)
            let ?g = λp. sum (λi. sum (λj. abs((A p − B)$i$j)) UNIV) UNIV
            have (λk. onorm (λy. (A k − B) *v y)) ⟶ 0
            proof (rule Lim_null_comparison)
              show ∀F k in sequentially. norm (onorm (λy. (A k − B) *v y)) ≤
```

```
?g k
                proof (rule eventually_sequentiallyI)
                  fix k :: nat
                  assume 0 ≤ k
                  have 0 ≤ onorm ((∗v) (A k − B))
                    using matrix_vector_mul_bounded_linear
                    by (rule onorm_pos_le)
                     then show norm (onorm ((∗v) (A k − B))) ≤ (∑ i∈UNIV.
∑ j∈UNIV. |(A k − B) $ i $ j|)
                        by (simp add: onorm_le_matrix_component_sum del:
vector_minus_component)
                qed
              next
                show ?g ⟶ 0
                    using B Lim_null tendsto_rabs_zero_iff by (fastforce intro!:
tendsto_null_sum)
              qed
              with ‹e > 0› obtain p where ⋀n. n ≥ p ⟹ |onorm ((∗v) (A n −
B))| < e/2
                  unfolding lim_sequentially by (metis diff_zero dist_real_def
divide_pos_pos zero_less_numeral)
              then have pqe2: |onorm ((∗v) (A (p + q) − B))| < e/2
                using le_add1 by blast
              show ∃ d>0. ∀ y∈S. y ≠ x ∧ norm (y − x) < d ⟶
                    inverse (norm (y − x)) ∗ norm (f y + B ∗v x − (f x + B ∗v
y)) < e
              proof (intro exI, safe)
                show 0 < δ(p + q)
                  by (simp add: δ)
              next
                fix y
                assume y: y ∈ S norm (y − x) < δ(p + q) and y ≠ x
                have ∗: ⟦norm(b − c) < e − d; norm(y − x − b) ≤ d⟧ ⟹ norm(y
− x − c) < e
                  for b c d e x and y:: real^'n
                  using norm_triangle_ineq2 [of y − x − c y − x − b] by simp
                have norm (f y − f x − B ∗v (y − x)) < e ∗ norm (y − x)
                proof (rule ∗)
                  show norm (f y − f x − A (p + q) ∗v (y − x)) ≤ norm (y − x)
/ (Suc (p + q))
                    using A [OF y] by simp
                  have norm (A (p + q) ∗v (y − x) − B ∗v (y − x)) ≤ onorm(λx.
(A(p + q) − B) ∗v x) ∗ norm(y − x)
                      by (metis linear_linear matrix_vector_mul_linear ma-
trix_vector_mult_diff_rdistrib onorm)
                  also have ... < (e/2) ∗ norm (y − x)
                    using ‹y ≠ x› pqe2 by auto
                  also have ... ≤ (e − 1 / (Suc (p + q))) ∗ norm (y − x)
                  proof (rule mult_right_mono)
```

```
                have 1 / Suc (p + q) ≤ 1 / q
                  using ‹q ≠ 0› by (auto simp: field_split_simps)
                also have ... < e/2
                  using qe2 by auto
                finally show e / 2 ≤ e − 1 / real (Suc (p + q))
                  by linarith
              qed auto
             finally show norm (A (p + q) *v (y − x) − B *v (y − x)) < e *
norm (y − x) − norm (y − x) / real (Suc (p + q))
               by (simp add: algebra_simps)
            qed
           then show inverse (norm (y − x)) * norm (f y + B *v x − (f x +
B *v y)) < e
             using ‹y ≠ x› by (simp add: field_split_simps algebra_simps)
          qed
        qed
        then show ((λy. (matrix (f' x) − B) *v (y − x) /R
                    norm (y − x) − (f x + f' x (y − x) − f y) /R norm (y −
x)) ⟶ 0)
                  (at x within S)
          by (simp add: algebra_simps diff lin_df scalar_mult_eq_scaleR)
      qed
    qed (use x in ‹simp; auto simp: not_less›)
    ultimately have f' x = (*v) B
      by (force simp: algebra_simps scalar_mult_eq_scaleR)
    show matrix (f' x) $ m $ n ≤ b
    proof (rule tendsto_upperbound [of λi. (A i $ m $ n) _ sequentially])
      show (λi. A i $ m $ n) ⟶ matrix (f' x) $ m $ n
        by (simp add: B ‹f' x = (*v) B›)
      show ∀F i in sequentially. A i $ m $ n ≤ b
        by (simp add: Ab less_eq_real_def)
    qed auto
  next
    fix e :: real
    assume x ∈ S and b: matrix (f' x) $ m $ n ≤ b and e > 0
    then obtain d where d>0
      and d: ⋀y. y∈S ⟹ 0 < dist y x ∧ dist y x < d ⟶ norm (f y − f x
− f' x (y − x)) / (norm (y − x))
          < e/2
      using f [OF ‹x ∈ S›]
      by (simp add: Deriv.has_derivative_at_within Lim_within)
        (auto simp add: field_simps dest: spec [of _ e/2])
    let ?A = matrix(f' x) − (χ i j. if i = m ∧ j = n then e / 4 else 0)
    obtain B where BRats: ⋀i j. B$i$j ∈ ℚ and Bo_e6: onorm((*v) (?A −
B)) < e/6
      using matrix_rational_approximation ‹e > 0›
      by (metis zero_less_divide_iff zero_less_numeral)
    show ∃d>0. ∃A. A $ m $ n < b ∧ (∀i j. A $ i $ j ∈ ℚ) ∧
        (∀y∈S. norm (y − x) < d ⟶ norm (f y − f x − A *v (y − x)) ≤ e
```

```
* norm (y - x))
        proof (intro exI conjI ballI allI impI)
          show d>0
            by (rule ‹d>0›)
          show B $ m $ n < b
          proof -
            have |matrix ((*v) (?A - B)) $ m $ n| ≤ onorm ((*v) (?A - B))
             using component_le_onorm [OF matrix_vector_mul_linear, of _ m
n] by metis
            then show ?thesis
              using b Bo_e6 by simp
          qed
          show B $ i $ j ∈ ℚ for i j
            using BRats by auto
          show norm (f y - f x - B *v (y - x)) ≤ e * norm (y - x)
            if y ∈ S and y: norm (y - x) < d for y
          proof (cases y = x)
            case True then show ?thesis
              by simp
          next
            case False
              have *: norm(d' - d) ≤ e/2 ⟹ norm(y - (x + d')) < e/2 ⟹
norm(y - x - d) ≤ e for d d' e and x y::real^'n
              using norm_triangle_le [of d' - d y - (x + d')] by simp
            show ?thesis
            proof (rule *)
              have split246: ⟦norm y ≤ e / 6; norm(x - y) ≤ e / 4⟧ ⟹ norm x
≤ e/2 if e > 0 for e and x y :: real^'n
                using norm_triangle_le [of y x-y e/2] ‹e > 0› by simp
              have linear (f' x)
                using True f has_derivative_linear by blast
             then have norm (f' x (y - x) - B *v (y - x)) = norm ((matrix (f'
x) - B) *v (y - x))
                by (simp add: matrix_vector_mult_diff_rdistrib)
              also have ... ≤ (e * norm (y - x)) / 2
              proof (rule split246)
                have norm ((?A - B) *v (y - x)) / norm (y - x) ≤ onorm(λx.
(?A - B) *v x)
                  by (rule le_onorm) auto
                also have  ... < e/6
                  by (rule Bo_e6)
              finally have norm ((?A - B) *v (y - x)) / norm (y - x) < e / 6 .
              then show norm ((?A - B) *v (y - x)) ≤ e * norm (y - x) / 6
                by (simp add: field_split_simps False)
              have norm ((matrix (f' x) - B) *v (y - x) - ((?A - B) *v (y -
x))) = norm ((χ i j. if i = m ∧ j = n then e / 4 else 0) *v (y - x))
                by (simp add: algebra_simps)
              also have ... = norm((e/4) *R (y - x)$n *R axis m (1::real))
              proof -
```

```
                have (∑j∈UNIV. (if i = m ∧ j = n then e / 4 else 0) * (y $ j
− x $ j)) * 4 = e * (y $ n − x $ n) * axis m 1 $ i for i
                proof (cases i=m)
                  case True then show ?thesis
                    by (auto simp: if_distrib [of λz. z * _] cong: if_cong)
                next
                  case False then show ?thesis
                    by (simp add: axis_def)
                qed
                then have (χ i j. if i = m ∧ j = n then e / 4 else 0) *v (y − x)
= (e/4) *R (y − x)$n *R axis m (1::real)
                  by (auto simp: vec_eq_iff matrix_vector_mult_def)
                then show ?thesis
                  by metis
              qed
              also have ... ≤ e * norm (y − x) / 4
              proof −
                have |y $ n − x $ n| ≤ norm (y − x)
                  by (metis component_le_norm_cart vector_minus_component)
                with ‹e > 0› show ?thesis
                  by (simp add: norm_mult abs_mult)
              qed
              finally show norm ((matrix (f' x) − B) *v (y − x) − ((?A − B)
*v (y − x))) ≤ e * norm (y − x) / 4 .
              show 0 < e * norm (y − x)
                by (simp add: False ‹e > 0›)
            qed
            finally show norm (f' x (y − x) − B *v (y − x)) ≤ (e * norm (y −
x)) / 2 .
            show norm (f y − (f x + f' x (y − x))) < (e * norm (y − x)) / 2
              using False d [OF ‹y ∈ S›] y by (simp add: dist_norm field_simps)
          qed
        qed
      qed
    qed
  qed auto
 qed
 show negligible ?M
  using negligible_subset [OF nN MN] .
qed
then show ?thesis
 by (simp add: borel_measurable_vimage_halfspace_component_le sets_restrict_space_iff
assms)
qed


theorem borel_measurable_det_Jacobian:
 fixes f :: real^'n::{finite,wellorder} ⇒ real^'n::_
  assumes S: S ∈ sets lebesgue and f: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at
```

x within S)
shows (λ*x*. *det*(*matrix*(*f'* *x*))) ∈ *borel_measurable* (*lebesgue_on S*)
unfolding *det_def*
by (*intro measurable*) (*auto intro*: *f borel_measurable_partial_derivatives* [*OF S*])

The localisation wrt S uses the same argument for many similar results.

theorem *borel_measurable_lebesgue_on_preimage_borel*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S* ∈ *sets lebesgue*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*) ⟷
(∀ *T*. *T* ∈ *sets borel* ⟶ {*x* ∈ *S*. *f x* ∈ *T*} ∈ *sets lebesgue*)
proof −
have {*x*. (*if x* ∈ *S then f x else* 0) ∈ *T*} ∈ *sets lebesgue* ⟷ {*x* ∈ *S*. *f x* ∈ *T*} ∈ *sets lebesgue*
if *T* ∈ *sets borel* **for** *T*
proof (*cases* 0 ∈ *T*)
case *True*
then have {*x* ∈ *S*. *f x* ∈ *T*} = {*x*. (*if x* ∈ *S then f x else* 0) ∈ *T*} ∩ *S*
{*x*. (*if x* ∈ *S then f x else* 0) ∈ *T*} = {*x* ∈ *S*. *f x* ∈ *T*} ∪ −*S*
by *auto*
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *Compl_in_sets_lebesgue assms sets.Int sets.Un*)
next
case *False*
then have {*x*. (*if x* ∈ *S then f x else* 0) ∈ *T*} = {*x* ∈ *S*. *f x* ∈ *T*}
by *auto*
then show *?thesis*
by *auto*
qed
then show *?thesis*
unfolding *borel_measurable_lebesgue_preimage_borel borel_measurable_if* [*OF assms*, *symmetric*]
by *blast*
qed

lemma *sets_lebesgue_almost_borel*:
assumes *S* ∈ *sets lebesgue*
obtains *B N* **where** *B* ∈ *sets borel negligible N B* ∪ *N* = *S*
by (*metis assms negligible_iff_null_sets negligible_subset null_sets_completionI sets_completionE sets_lborel*)

lemma *double_lebesgue_sets*:
assumes *S*: *S* ∈ *sets lebesgue* **and** *T*: *T* ∈ *sets lebesgue* **and** *fim*: *f* ‘ *S* ⊆ *T*
shows (∀ *U*. *U* ∈ *sets lebesgue* ∧ *U* ⊆ *T* ⟶ {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets lebesgue*) ⟷
f ∈ *borel_measurable* (*lebesgue_on S*) ∧
(∀ *U*. *negligible U* ∧ *U* ⊆ *T* ⟶ {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets lebesgue*)
(**is** *?lhs* ⟷ _ ∧ *?rhs*)

unfolding *borel_measurable_lebesgue_on_preimage_borel* [*OF S*]
proof (*intro iffI allI conjI impI*, *safe*)
fix *V* :: *'b set*
assume ∗: ∀ *U*. *U* ∈ *sets lebesgue* ∧ *U* ⊆ *T* ⟶ {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets lebesgue*
and *V* ∈ *sets borel*
then have *V*: *V* ∈ *sets lebesgue*
by *simp*
have {*x* ∈ *S*. *f x* ∈ *V*} = {*x* ∈ *S*. *f x* ∈ *T* ∩ *V*}
using *fim* **by** *blast*
also have {*x* ∈ *S*. *f x* ∈ *T* ∩ *V*} ∈ *sets lebesgue*
using *T V* ∗ *le_inf_iff* **by** *blast*
finally show {*x* ∈ *S*. *f x* ∈ *V*} ∈ *sets lebesgue* **.**
next
fix *U* :: *'b set*
assume ∀ *U*. *U* ∈ *sets lebesgue* ∧ *U* ⊆ *T* ⟶ {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets lebesgue*
negligible U U ⊆ *T*
then show {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets lebesgue*
using *negligible_imp_sets* **by** *blast*
next
fix *U* :: *'b set*
assume *1* [*rule_format*]: (∀ *T*. *T* ∈ *sets borel* ⟶ {*x* ∈ *S*. *f x* ∈ *T*} ∈ *sets lebesgue*)
and *2* [*rule_format*]: ∀ *U*. *negligible U* ∧ *U* ⊆ *T* ⟶ {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets lebesgue*
and *U* ∈ *sets lebesgue U* ⊆ *T*
then obtain *C N* **where** *C*: *C* ∈ *sets borel* ∧ *negligible N* ∧ *C* ∪ *N* = *U*
using *sets_lebesgue_almost_borel* **by** *metis*
then have {*x* ∈ *S*. *f x* ∈ *C*} ∈ *sets lebesgue*
by (*blast intro*: *1*)
moreover have {*x* ∈ *S*. *f x* ∈ *N*} ∈ *sets lebesgue*
using *C* ‹*U* ⊆ *T*› **by** (*blast intro*: *2*)
moreover have {*x* ∈ *S*. *f x* ∈ *C* ∪ *N*} = {*x* ∈ *S*. *f x* ∈ *C*} ∪ {*x* ∈ *S*. *f x* ∈ *N*}
by *auto*
ultimately show {*x* ∈ *S*. *f x* ∈ *U*} ∈ *sets lebesgue*
using *C* **by** *auto*
qed

9.32.3 Simplest case of Sard's theorem (we don't need continuity of derivative)

lemma *Sard_lemma00*:
fixes *P* :: *'b*::*euclidean_space set*
assumes *a* ≥ *0* **and** *a*: *a* ∗*R i* ≠ *0* **and** *i*: *i* ∈ *Basis*
and *P*: *P* ⊆ {*x*. *a* ∗*R i* · *x* = *0*}
and *0* ≤ *m 0* ≤ *e*
obtains *S* **where** *S* ∈ *lmeasurable*
and {*z*. *norm z* ≤ *m* ∧ (∃ *t* ∈ *P*. *norm*(*z* − *t*) ≤ *e*)} ⊆ *S*
and *measure lebesgue S* ≤ (*2* ∗ *e*) ∗ (*2* ∗ *m*) ^(*DIM*(*'b*) − *1*)
proof −

```
  have a > 0
    using assms by simp
  let ?v = (∑j∈Basis. (if j = i then e else m) *R j)
  show thesis
  proof
    have − e ≤ x • i x • i ≤ e
      if t ∈ P norm (x − t) ≤ e for x t
      using ‹a > 0› that Basis_le_norm [of i x−t] P i
      by (auto simp: inner_commute algebra_simps)
    moreover have − m ≤ x • j x • j ≤ m
      if norm x ≤ m t ∈ P norm (x − t) ≤ e j ∈ Basis and j ≠ i
      for x t j
      using that Basis_le_norm [of j x] by auto
    ultimately
    show {z. norm z ≤ m ∧ (∃t∈P. norm (z − t) ≤ e)} ⊆ cbox (−?v) ?v
      by (auto simp: mem_box)
    have *: ∀k∈Basis. − ?v • k ≤ ?v • k
      using ‹0 ≤ m› ‹0 ≤ e› by (auto simp: inner_Basis)
    have 2: 2 ^ DIM('b) = 2 * 2 ^ (DIM('b) − Suc 0)
      by (metis DIM_positive Suc_pred power_Suc)
    show measure lebesgue (cbox (−?v) ?v) ≤ 2 * e * (2 * m) ^ (DIM('b) − 1)
      using ‹i ∈ Basis›
    by (simp add: content_cbox [OF *] prod.distrib prod.If_cases Diff_eq [symmetric]
2)
  qed blast
qed
```

As above, but reorienting the vector (HOL Light's @textGEOM_BASIS_MULTIPLE_TAC)

```
lemma Sard_lemma0:
  fixes P :: (real^'n::{finite,wellorder}) set
  assumes a ≠ 0
    and P: P ⊆ {x. a • x = 0} and 0 ≤ m 0 ≤ e
  obtains S where S ∈ lmeasurable
    and {z. norm z ≤ m ∧ (∃ t ∈ P. norm(z − t) ≤ e)} ⊆ S
    and measure lebesgue S ≤ (2 * e) * (2 * m) ^ (CARD('n) − 1)
proof −
  obtain T and k::'n where T: orthogonal_transformation T and a: a = T
(norm a *R axis k (1::real))
    using rotation_rightward_line by metis
  have Tinv [simp]: T (inv T x) = x for x
    by (simp add: T orthogonal_transformation_surj surj_f_inv_f)
  obtain S where S: S ∈ lmeasurable
    and subS: {z. norm z ≤ m ∧ (∃ t ∈ T−‘P. norm(z − t) ≤ e)} ⊆ S
    and mS: measure lebesgue S ≤ (2 * e) * (2 * m) ^ (CARD('n) − 1)
  proof (rule Sard_lemma00 [of norm a axis k (1::real) T−‘P m e])
    have norm a *R axis k 1 • x = 0 if T x ∈ P for x
    by (smt (verit, del_insts) P T a mem_Collect_eq orthogonal_transformation_def
subset_eq that)
```

```
    then show T −‘ P ⊆ {x. norm a ∗R axis k 1 · x = 0}
      by auto
  qed (use assms T in auto)
  show thesis
  proof
    show T ‘ S ∈ lmeasurable
      using S measurable_orthogonal_image T by blast
    have {z. norm z ≤ m ∧ (∃ t∈P. norm (z − t) ≤ e)} ⊆ T ‘ {z. norm z ≤ m ∧
(∃ t∈T −‘ P. norm (z − t) ≤ e)}
    proof clarsimp
      fix x t
      assume §: norm x ≤ m t ∈ P norm (x − t) ≤ e
      then have norm (inv T x) ≤ m
          using orthogonal_transformation_inv [OF T] by (simp add: orthogo-
nal_transformation_norm)
      moreover have ∃ t∈T −‘ P. norm (inv T x − t) ≤ e
      by (smt (verit, del_insts) T Tinv § linear_diff orthogonal_transformation_def
orthogonal_transformation_norm vimage_eq)
      ultimately show x ∈ T ‘ {z. norm z ≤ m ∧ (∃ t∈T −‘ P. norm (z − t) ≤
e)}
        by force
    qed
    then show {z. norm z ≤ m ∧ (∃ t∈P. norm (z − t) ≤ e)} ⊆ T ‘ S
      using image_mono [OF subS] by (rule order_trans)
    show measure lebesgue (T ‘ S) ≤ 2 ∗ e ∗ (2 ∗ m) ^(CARD('n) − 1)
      using mS T by (simp add: S measure_orthogonal_image)
  qed
qed
```

As above, but translating the sets (HOL Light's @textGEN_GEOM_ORIGIN_TAC)

```
lemma Sard_lemma1:
  fixes P :: (real^'n::{finite,wellorder}) set
   assumes P: dim P < CARD('n) and 0 ≤ m 0 ≤ e
 obtains S where S ∈ lmeasurable
          and {z. norm(z − w) ≤ m ∧ (∃ t ∈ P. norm(z − w − t) ≤ e)} ⊆ S
          and measure lebesgue S ≤ (2 ∗ e) ∗ (2 ∗ m) ^(CARD('n) − 1)
proof −
  obtain a where a ≠ 0 P ⊆ {x. a · x = 0}
    using lowdim_subset_hyperplane [of P] P span_base by auto
  then obtain S where S: S ∈ lmeasurable
    and subS: {z. norm z ≤ m ∧ (∃ t ∈ P. norm(z − t) ≤ e)} ⊆ S
    and mS: measure lebesgue S ≤ (2 ∗ e) ∗ (2 ∗ m) ^(CARD('n) − 1)
    by (rule Sard_lemma0 [OF _ _ ‹0 ≤ m› ‹0 ≤ e›])
  show thesis
  proof
    show (+)w ‘ S ∈ lmeasurable
      by (metis measurable_translation S)
    show {z. norm (z − w) ≤ m ∧ (∃ t∈P. norm (z − w − t) ≤ e)} ⊆ (+)w ‘ S
```

```
    using subS by force
  show measure lebesgue ((+)w ` S) ≤ 2 * e * (2 * m) ^ (CARD('n) - 1)
    by (metis measure_translation mS)
 qed
qed

lemma Sard_lemma2:
 fixes f :: real^'m::{finite,wellorder} ⇒ real^'n::{finite,wellorder}
 assumes mlen: CARD('m) ≤ CARD('n) (is ?m ≤ ?n)
   and B > 0 bounded S
   and derS: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
   and rank: ⋀x. x ∈ S ⟹ rank(matrix(f' x)) < CARD('n)
   and B: ⋀x. x ∈ S ⟹ onorm(f' x) ≤ B
 shows negligible(f ` S)
proof -
 have lin_f': ⋀x. x ∈ S ⟹ linear(f' x)
   using derS has_derivative_linear by blast
 show ?thesis
 proof (clarsimp simp add: negligible_outer_le)
   fix e :: real
   assume e > 0
   obtain c where csub: S ⊆ cbox (- (vec c)) (vec c) and c > 0
   proof -
     obtain b where b: ⋀x. x ∈ S ⟹ norm x ≤ b
       using ‹bounded S› by (auto simp: bounded_iff)
     show thesis
     proof
       have - |b| - 1 ≤ x $ i ∧ x $ i ≤ |b| + 1 if x ∈ S for x i
         using component_le_norm_cart [of x i] b [OF that] by auto
       then show S ⊆ cbox (- vec (|b| + 1)) (vec (|b| + 1))
         by (auto simp: mem_box_cart)
     qed auto
   qed
   then have box_cc: box (- (vec c)) (vec c) ≠ {} and cbox_cc: cbox (- (vec
c)) (vec c) ≠ {}
     by (auto simp: interval_eq_empty_cart)
   obtain d where d > 0 d ≤ B
         and d: (d * 2) * (4 * B) ^ (?n - 1) ≤ e / (2*c) ^ ?m / ?m ^ ?m
     apply (rule that [of min B (e / (2*c) ^ ?m / ?m ^ ?m / (4 * B) ^ (?n -
1) / 2)])
     using ‹B > 0› ‹c > 0› ‹e > 0›
     by (simp_all add: divide_simps min_mult_distrib_right)
   have ∃r. 0 < r ∧ r ≤ 1/2 ∧
           (x ∈ S
            ⟶ (∀y. y ∈ S ∧ norm(y - x) < r
                  ⟶ norm(f y - f x - f' x (y - x)) ≤ d * norm(y - x))) for x
   proof (cases x ∈ S)
     case True
     then obtain r where r > 0
```

```
          and ⋀y. ⟦y ∈ S; norm (y − x) < r⟧
                ⟹ norm (f y − f x − f′ x (y − x)) ≤ d ∗ norm (y − x)
      using derS ‹d > 0› by (force simp: has_derivative_within_alt)
    then show ?thesis
      by (rule_tac x=min r (1/2) in exI) simp
  next
    case False
    then show ?thesis
      by (rule_tac x=1/2 in exI) simp
  qed
  then obtain r where r12: ⋀x. 0 < r x ∧ r x ≤ 1/2
        and r: ⋀x y. ⟦x ∈ S; y ∈ S; norm(y − x) < r x⟧
                  ⟹ norm(f y − f x − f′ x (y − x)) ≤ d ∗ norm(y − x)
    by metis
  then have ga: gauge (λx. ball x (r x))
    by (auto simp: gauge_def)
  obtain 𝒟 where 𝒟: countable 𝒟 and sub_cc: ⋃𝒟 ⊆ cbox (− vec c) (vec c)
    and cbox: ⋀K. K ∈ 𝒟 ⟹ interior K ≠ {} ∧ (∃u v. K = cbox u v)
    and djointish: pairwise (λA B. interior A ∩ interior B = {}) 𝒟
    and covered: ⋀K. K ∈ 𝒟 ⟹ ∃x ∈ S ∩ K. K ⊆ ball x (r x)
    and close: ⋀u v. cbox u v ∈ 𝒟 ⟹ ∃n. ∀i::'m. v $ i − u $ i = 2∗c / 2^n
    and covers: S ⊆ ⋃𝒟
    apply (rule covering_lemma [OF csub box_cc ga])
    apply (auto simp: Basis_vec_def cart_eq_inner_axis [symmetric])
    done
  let ?μ = measure lebesgue
  have ∃T. T ∈ lmeasurable ∧ f ‘ (K ∩ S) ⊆ T ∧ ?μ T ≤ e / (2∗c) ^ ?m ∗ ?μ
K
    if K ∈ 𝒟 for K
  proof −
    obtain u v where uv: K = cbox u v
      using cbox ‹K ∈ 𝒟› by blast
    then have uv_ne: cbox u v ≠ {}
      using cbox that by fastforce
    obtain x where x: x ∈ S ∩ cbox u v cbox u v ⊆ ball x (r x)
      using ‹K ∈ 𝒟› covered uv by blast
    then have dim (range (f′ x)) < ?n
      using rank_dim_range [of matrix (f′ x)] x rank[of x]
      by (auto simp: matrix_works scalar_mult_eq_scaleR lin_f′)
    then obtain T where T: T ∈ lmeasurable
        and subT: {z. norm(z − f x) ≤ (2 ∗ B) ∗ norm(v − u) ∧ (∃t ∈ range
(f′ x). norm(z − f x − t) ≤ d ∗ norm(v − u))} ⊆ T
        and measT: ?μ T ≤ (2 ∗ (d ∗ norm(v − u))) ∗ (2 ∗ ((2 ∗ B) ∗ norm(v
− u))) ^ (?n − 1)
                  (is _ ≤ ?DVU)
      using Sard_lemma1 [of range (f′ x) (2 ∗ B) ∗ norm(v − u) d ∗ norm(v −
u)]
      using ‹B > 0› ‹d > 0› by auto
    show ?thesis
```

```
    proof (intro exI conjI)
      have f ‘ (K ∩ S) ⊆ {z. norm(z − f x) ≤ (2 ∗ B) ∗ norm(v − u) ∧ (∃ t ∈
range (f′ x). norm(z − f x − t) ≤ d ∗ norm(v − u))}
        unfolding uv
      proof (clarsimp simp: mult.assoc, intro conjI)
        fix y
        assume y: y ∈ cbox u v and y ∈ S
        then have norm (y − x) < r x
          by (metis dist_norm mem_ball norm_minus_commute subsetCE x(2))
        then have le_dyx: norm (f y − f x − f′ x (y − x)) ≤ d ∗ norm (y − x)
          using r [of x y] x ‹y ∈ S› by blast
        have yx_le: norm (y − x) ≤ norm (v − u)
        proof (rule norm_le_componentwise_cart)
          show norm ((y − x) $ i) ≤ norm ((v − u) $ i) for i
            using x y by (force simp: mem_box_cart dest!: spec [where x=i])
        qed
        have ∗: ⟦norm(y − x − z) ≤ d; norm z ≤ B; d ≤ B⟧ ⟹ norm(y − x)
≤ 2 ∗ B
          for x y z :: real^'n::_ and d B
          using norm_triangle_ineq2 [of y − x z] by auto
        show norm (f y − f x) ≤ 2 ∗ (B ∗ norm (v − u))
        proof (rule ∗ [OF le_dyx])
          have norm (f′ x (y − x)) ≤ onorm (f′ x) ∗ norm (y − x)
            using onorm [of f′ x y−x] by (meson IntE lin_f′ linear_linear x(1))
          also have ... ≤ B ∗ norm (v − u)
              by (meson B IntE lin_f′ linear_linear mult_mono′ norm_ge_zero
onorm_pos_le x(1) yx_le)
          finally show norm (f′ x (y − x)) ≤ B ∗ norm (v − u) .
          show d ∗ norm (y − x) ≤ B ∗ norm (v − u)
            using ‹B > 0› by (auto intro: mult_mono [OF ‹d ≤ B› yx_le])
        qed
        show ∃ t. norm (f y − f x − f′ x t) ≤ d ∗ norm (v − u)
          by (smt (verit, best) ‹0 < d› le_dyx mult_le_cancel_left_pos yx_le)
      qed
      with subT show f ‘ (K ∩ S) ⊆ T by blast
      show ?μ T ≤ e / (2∗c) ^ ?m ∗ ?μ K
      proof (rule order_trans [OF measT])
        have ?DVU = (d ∗ 2 ∗ (4 ∗ B) ^ (?n − 1)) ∗ norm (v − u)^?n
          using ‹c > 0›
          apply (simp add: algebra_simps)
          by (metis Suc_pred power_Suc zero_less_card_finite)
        also have ... ≤ (e / (2∗c) ^ ?m / (?m ^ ?m)) ∗ norm(v − u) ^ ?n
          by (rule mult_right_mono [OF d]) auto
        also have ... ≤ e / (2∗c) ^ ?m ∗ ?μ K
        proof −
          have u ∈ ball (x) (r x) v ∈ ball x (r x)
          using box_ne_empty(1) contra_subsetD [OF x(2)] mem_box(2) uv_ne
by fastforce+
          moreover have r x ≤ 1/2
```

```
          using r12 by auto
        ultimately have norm (v − u) ≤ 1
          using norm_triangle_half_r [of x u 1 v]
              by (metis (no_types, opaque_lifting) dist_commute dist_norm
less_eq_real_def less_le_trans mem_ball)
        then have norm (v − u) ^ ?n ≤ norm (v − u) ^ ?m
          by (simp add: power_decreasing [OF mlen])
        also have ... ≤ ?μ K * real (?m ^ ?m)
        proof −
          obtain n where n: ⋀i. v$i − u$i = 2 * c / 2^n
            using close [of u v] ‹K ∈ 𝒟› uv by blast
          have norm (v − u) ^ ?m ≤ (∑ i∈UNIV. |(v − u) $ i|) ^ ?m
            by (intro norm_le_l1_cart power_mono) auto
           also have ... ≤ (∏ i∈UNIV. v $ i − u $ i) * real CARD('m) ^
CARD('m)
            by (simp add: n field_simps ‹c > 0› less_eq_real_def)
          also have ... = ?μ K * real (?m ^ ?m)
            by (simp add: uv uv_ne content_cbox_cart)
          finally show ?thesis .
        qed
        finally have *: 1 / real (?m ^ ?m) * norm (v − u) ^ ?n ≤ ?μ K
          by (simp add: field_split_simps)
        show ?thesis
          using mult_left_mono [OF *, of e / (2*c) ^ ?m] ‹c > 0› ‹e > 0› by
auto
      qed
      finally show ?DVU ≤ e / (2*c) ^ ?m * ?μ K .
    qed
  qed (use T in auto)
qed
then obtain g where meas_g: ⋀K. K ∈ 𝒟 ⟹ g K ∈ lmeasurable
                and sub_g: ⋀K. K ∈ 𝒟 ⟹ f ‘ (K ∩ S) ⊆ g K
                and le_g: ⋀K. K ∈ 𝒟 ⟹ ?μ (g K) ≤ e / (2*c)^?m * ?μ K
  by metis
have le_e: ?μ (⋃ i∈ℱ. g i) ≤ e
  if ℱ ⊆ 𝒟 finite ℱ for ℱ
proof −
  have ?μ (⋃ i∈ℱ. g i) ≤ (∑ i∈ℱ. ?μ (g i))
    using meas_g ‹ℱ ⊆ 𝒟› by (auto intro: measure_UNION_le [OF ‹finite
ℱ›])
  also have ... ≤ (∑ K∈ℱ. e / (2*c) ^ ?m * ?μ K)
   using ‹ℱ ⊆ 𝒟› sum_mono [OF le_g] by (meson le_g subsetCE sum_mono)
  also have ... = e / (2*c) ^ ?m * (∑ K∈ℱ. ?μ K)
    by (simp add: sum_distrib_left)
  also have ... ≤ e
  proof −
    have ℱ division_of ⋃ℱ
    proof (rule division_ofI)
      show K ⊆ ⋃ℱ  K ≠ {} ∃ a b. K = cbox a b if K ∈ ℱ for K
```

```
        using ‹K ∈ ℱ› covered cbox ‹ℱ ⊆ 𝒟› by (auto simp: Union_upper)
      show interior K ∩ interior L = {} if K ∈ ℱ and L ∈ ℱ and K ≠ L for
K L
       by (metis (mono_tags, lifting) ‹ℱ ⊆ 𝒟› pairwiseD djointish pairwise_subset
that)
     qed (use that in auto)
     then have sum ?μ ℱ ≤ ?μ (⋃ℱ)
       by (simp add: content_division)
     also have ... ≤ ?μ (cbox (− vec c) (vec c) :: (real, 'm) vec set)
     proof (rule measure_mono_fmeasurable)
       show ⋃ℱ ⊆ cbox (− vec c) (vec c)
         by (meson Sup_subset_mono sub_cc order_trans ‹ℱ ⊆ 𝒟›)
     qed (use ‹ℱ division_of ⋃ℱ› lmeasurable_division in auto)
     also have ... = content (cbox (− vec c) (vec c) :: (real, 'm) vec set)
       by simp
     also have ... ≤ (2 ^ ?m * c ^ ?m)
       using ‹c > 0› by (simp add: content_cbox_if_cart)
     finally have sum ?μ ℱ ≤ (2 ^ ?m * c ^ ?m) .
     then show ?thesis
       using ‹e > 0› ‹c > 0› by (auto simp: field_split_simps)
   qed
   finally show ?thesis .
  qed
  show ∃ T. f ‘ S ⊆ T ∧ T ∈ lmeasurable ∧ ?μ T ≤ e
  proof (intro exI conjI)
   show f ‘ S ⊆ ⋃ (g ‘ 𝒟)
     using covers sub_g by force
   show ⋃ (g ‘ 𝒟) ∈ lmeasurable
     by (rule fmeasurable_UN_bound [OF ‹countable 𝒟› meas_g le_e])
   show ?μ (⋃ (g ‘ 𝒟)) ≤ e
     by (rule measure_UN_bound [OF ‹countable 𝒟› meas_g le_e])
  qed
 qed
qed


theorem baby_Sard:
 fixes f :: real^'m::{finite,wellorder} ⇒ real^'n::{finite,wellorder}
 assumes mlen: CARD('m) ≤ CARD('n)
   and der: ⋀x. x ∈ S ⟹ (f has_derivative f' x) (at x within S)
   and rank: ⋀x. x ∈ S ⟹ rank(matrix(f' x)) < CARD('n)
 shows negligible(f ‘ S)
proof −
 let ?U = λn. {x ∈ S. norm(x) ≤ n ∧ onorm(f' x) ≤ real n}
 have ⋀x. x ∈ S ⟹ ∃ n. norm x ≤ real n ∧ onorm (f' x) ≤ real n
   by (meson linear order_trans real_arch_simple)
 then have eq: S = (⋃ n. ?U n)
   by auto
 have negligible (f ‘ ?U n) for n
```

```
  proof (rule Sard_lemma2 [OF mlen])
    show 0 < real n + 1
      by auto
    show bounded (?U n)
      using bounded_iff by blast
    show (f has_derivative f' x) (at x within ?U n) if x ∈ ?U n for x
      using der that by (force intro: has_derivative_subset)
  qed (use rank in auto)
  then show ?thesis
    by (subst eq) (simp add: image_Union negligible_Union_nat)
qed
```

9.32.4 A one-way version of change-of-variables not assuming injectivity.

```
lemma integral_on_image_ubound_weak:
  fixes f :: real^'n::{finite,wellorder} ⇒ real
  assumes S: S ∈ sets lebesgue
      and f: f ∈ borel_measurable (lebesgue_on (g ` S))
      and nonneg_fg: ⋀x. x ∈ S ⟹ 0 ≤ f(g x)
      and der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
      and det_int_fg: (λx. |det (matrix (g' x))| * f(g x)) integrable_on S
      and meas_gim: ⋀T. ⟦T ⊆ g ` S; T ∈ sets lebesgue⟧ ⟹ {x ∈ S. g x ∈ T}
∈ sets lebesgue
    shows f integrable_on (g ` S) ∧
          integral (g ` S) f ≤ integral S (λx. |det (matrix (g' x))| * f(g x))
        (is _ ∧ _ ≤ ?b)
proof −
  let ?D = λx. |det (matrix (g' x))|
  have cont_g: continuous_on S g
    using der_g has_derivative_continuous_on by blast
  have [simp]: space (lebesgue_on S) = S
    by (simp add: S)
  have gS_in_sets_leb: g ` S ∈ sets lebesgue
    apply (rule differentiable_image_in_sets_lebesgue)
    using der_g by (auto simp: S differentiable_def differentiable_on_def)
  obtain h where nonneg_h: ⋀n x. 0 ≤ h n x
    and h_le_f: ⋀n x. x ∈ S ⟹ h n (g x) ≤ f (g x)
    and h_inc: ⋀n x. h n x ≤ h (Suc n) x
    and h_meas: ⋀n. h n ∈ borel_measurable lebesgue
    and fin_R: ⋀n. finite(range (h n))
    and lim: ⋀x. x ∈ g ` S ⟹ (λn. h n x) ⟶ f x
  proof −
    let ?f = λx. if x ∈ g ` S then f x else 0
    have ?f ∈ borel_measurable lebesgue ∧ (∀x. 0 ≤ ?f x)
      by (auto simp: gS_in_sets_leb f nonneg_fg measurable_restrict_space_iff
[symmetric])
    then show ?thesis
    apply (clarsimp simp add: borel_measurable_simple_function_limit_increasing)
```

```
    apply (rename_tac h)
    by (rule_tac h=h in that) (auto split: if_split_asm)
  qed
  have h_lmeas: {t. h n (g t) = y} ∩ S ∈ sets lebesgue for y n
  proof −
    have space (lebesgue_on (UNIV::(real,'n) vec set)) = UNIV
      by simp
    then have ((h n) −‘{y} ∩ g ‘ S) ∈ sets (lebesgue_on (g ‘ S))
    by (metis Int_commute borel_measurable_vimage h_meas image_eqI inf_top.right_neutral
sets_restrict_space space_borel space_completion space_lborel)
    then have ({u. h n u = y} ∩ g ‘ S) ∈ sets lebesgue
      using gS_in_sets_leb
      by (simp add: integral_indicator fmeasurableI2 sets_restrict_space_iff vim-
age_def)
    then have {x ∈ S. g x ∈ ({u. h n u = y} ∩ g ‘ S)} ∈ sets lebesgue
      using meas_gim[of ({u. h n u = y} ∩ g ‘ S)] by force
    moreover have {t. h n (g t) = y} ∩ S = {x ∈ S. g x ∈ ({u. h n u = y} ∩ g ‘
S)}
      by blast
    ultimately show ?thesis
      by auto
  qed
  have hint: h n integrable_on g ‘ S ∧ integral (g ‘ S) (h n) ≤ integral S (λx. ?D
x ∗ h n (g x))
        (is ?INT ∧ ?lhs ≤ ?rhs) for n
  proof −
    let ?R = range (h n)
    have hn_eq: h n = (λx. ∑ y∈?R. y ∗ indicat_real {x. h n x = y} x)
      by (simp add: indicator_def if_distrib fin_R cong: if_cong)
    have yind: (λt. y ∗ indicator{x. h n x = y} t) integrable_on (g ‘ S) ∧
              (integral (g ‘ S) (λt. y ∗ indicator {x. h n x = y} t))
              ≤ integral S (λt. |det (matrix (g′ t))| ∗ y ∗ indicator {x. h n x = y}
(g t))
      if y: y ∈ ?R for y::real
    proof (cases y=0)
      case True
      then show ?thesis using gS_in_sets_leb integrable_0 by force
    next
      case False
      with that have y > 0
        using less_eq_real_def nonneg_h by fastforce
      have (λx. if x ∈ {t. h n (g t) = y} then ?D x else 0) integrable_on S
      proof (rule measurable_bounded_by_integrable_imp_integrable)
        have (λx. ?D x) ∈ borel_measurable (lebesgue_on ({t. h n (g t) = y} ∩ S))
        proof −
          have (λv. det (matrix (g′ v))) ∈ borel_measurable (lebesgue_on (S ∩ {v.
h n (g v) = y}))
            by (metis Int_lower1 S assms(4) borel_measurable_det_Jacobian
measurable_restrict_mono)
```

```
      then show ?thesis
        by (simp add: Int_commute)
    qed
      then have (λx. if x ∈ {t. h n (g t) = y} ∩ S then ?D x else 0) ∈
borel_measurable lebesgue
      by (rule borel_measurable_if_I [OF _ h_lmeas])
  then show (λx. if x ∈ {t. h n (g t) = y} then ?D x else 0) ∈ borel_measurable
(lebesgue_on S)
       by (simp add: if_if_eq_conj Int_commute borel_measurable_if [OF S,
symmetric])
    show (λx. ?D x *R f (g x) /R y) integrable_on S
      by (rule integrable_cmul) (use det_int_fg in auto)
    show norm (if x ∈ {t. h n (g t) = y} then ?D x else 0) ≤ ?D x *R f (g x)
/R y
      if x ∈ S for x
      using nonneg_h [of n x] ‹y > 0› nonneg_fg [of x] h_le_f [of x n] that
      by (auto simp: divide_simps mult_left_mono)
  qed (use S in auto)
  then have int_det: (λt. |det (matrix (g' t))|) integrable_on ({t. h n (g t) =
y} ∩ S)
    using integrable_restrict_Int by force
  have (g ` ({t. h n (g t) = y} ∩ S)) ∈ lmeasurable
  by (blast intro: has_derivative_subset [OF der_g] measurable_differentiable_image
[OF h_lmeas] int_det)
  moreover have g ` ({t. h n (g t) = y} ∩ S) = {x. h n x = y} ∩ g ` S
    by blast
  moreover have measure lebesgue (g ` ({t. h n (g t) = y} ∩ S))
                ≤ integral ({t. h n (g t) = y} ∩ S) (λt. |det (matrix (g' t))|)
  by (blast intro: has_derivative_subset [OF der_g] measure_differentiable_image
[OF h_lmeas _ int_det])
  ultimately show ?thesis
   using ‹y > 0› integral_restrict_Int [of S {t. h n (g t) = y} λt. |det (matrix
(g' t))| * y]
    apply (simp add: integrable_on_indicator integral_indicator)
    apply (simp add: indicator_def of_bool_def if_distrib cong: if_cong)
    done
qed
show ?thesis
proof
  show h n integrable_on g ` S
    apply (subst hn_eq)
    using yind by (force intro: integrable_sum [OF fin_R])
  have ?lhs = integral (g ` S) (λx. ∑ y∈range (h n). y * indicat_real {x. h n
x = y} x)
    by (metis hn_eq)
  also have ... = (∑ y∈range (h n). integral (g ` S) (λx. y * indicat_real {x.
h n x = y} x))
    by (rule integral_sum [OF fin_R]) (use yind in blast)
  also have ... ≤ (∑ y∈range (h n). integral S (λu. |det (matrix (g' u))| * y
```

```
* indicat_real {x. h n x = y} (g u)))
      using yind by (force intro: sum_mono)
    also have ... = integral S (λu. ∑ y∈range (h n). |det (matrix (g' u))| * y *
indicat_real {x. h n x = y} (g u))
    proof (rule integral_sum [OF fin_R, symmetric])
      fix y assume y: y ∈ ?R
      with nonneg_h have y ≥ 0
        by auto
      show (λu. |det (matrix (g' u))| * y * indicat_real {x. h n x = y} (g u))
integrable_on S
      proof (rule measurable_bounded_by_integrable_imp_integrable)
          have (λx. indicat_real {x. h n x = y} (g x)) ∈ borel_measurable
(lebesgue_on S)
            using h_lmeas S
        by (auto simp: indicator_vimage [symmetric] borel_measurable_indicator_iff
sets_restrict_space_iff)
        then show (λu. |det (matrix (g' u))| * y * indicat_real {x. h n x = y} (g
u)) ∈ borel_measurable (lebesgue_on S)
        by (intro borel_measurable_times borel_measurable_abs borel_measurable_const
borel_measurable_det_Jacobian [OF S der_g])
      next
        fix x
        assume x ∈ S
        then have y * indicat_real {x. h n x = y} (g x) ≤ f (g x)
            by (metis (full_types) h_le_f indicator_simps mem_Collect_eq
mult.right_neutral mult_zero_right nonneg_fg)
        with ‹y ≥ 0› show norm (?D x * y * indicat_real {x. h n x = y} (g x))
≤ ?D x * f(g x)
          by (simp add: abs_mult mult.assoc mult_left_mono)
      qed (use S det_int_fg in auto)
    qed
    also have ... = integral S (λT. |det (matrix (g' T))| *
                                (∑ y∈range (h n). y * indicat_real {x. h n x = y}
(g T)))
      by (simp add: sum_distrib_left mult.assoc)
    also have ... = ?rhs
      by (metis hn_eq)
    finally show integral (g ' S) (h n) ≤ ?rhs .
  qed
 qed
 have le: integral S (λT. |det (matrix (g' T))| * h n (g T)) ≤ ?b for n
 proof (rule integral_le)
   show (λT. |det (matrix (g' T))| * h n (g T)) integrable_on S
   proof (rule measurable_bounded_by_integrable_imp_integrable)
    have (λT. |det (matrix (g' T))| *R h n (g T)) ∈ borel_measurable (lebesgue_on
S)
    proof (intro borel_measurable_scaleR borel_measurable_abs borel_measurable_det_Jacobian
‹S ∈ sets lebesgue›)
      have eq: {x ∈ S. f x ≤ a} = (⋃ b ∈ (f ' S) ∩ atMost a. {x. f x = b} ∩ S)
```

for f **and** a::*real*
by *auto*
have *finite* $((\lambda x.\ h\ n\ (g\ x))\ `\ S \cap \{..a\})$ **for** a
by (*force intro*: *finite_subset* [*OF* _ *fin_R*])
with *h_lmeas* [*of n*] **show** $(\lambda x.\ h\ n\ (g\ x)) \in$ *borel_measurable* (*lebesgue_on S*)
apply (*simp add*: *borel_measurable_vimage_halfspace_component_le* ‹$S \in$ *sets lebesgue*› *sets_restrict_space_iff eq*)
by (*metis* (*mono_tags*) *SUP_inf sets.finite_UN*)
qed (*use der_g* **in** *blast*)
then show $(\lambda T.\ |det\ (matrix\ (g'\ T))| * h\ n\ (g\ T)) \in$ *borel_measurable* (*lebesgue_on S*)
by *simp*
show *norm* $(?D\ x * h\ n\ (g\ x)) \le ?D\ x *_R f\ (g\ x)$
if $x \in S$ **for** x
by (*simp add*: *h_le_f mult_left_mono nonneg_h that*)
qed (*use S det_int_fg* **in** *auto*)
show $?D\ x * h\ n\ (g\ x) \le ?D\ x * f\ (g\ x)$ **if** $x \in S$ **for** x
by (*simp add*: ‹$x \in S$› *h_le_f mult_left_mono*)
show $(\lambda x.\ ?D\ x * f\ (g\ x))$ *integrable_on S*
using *det_int_fg* **by** *blast*
qed
have *f integrable_on g ` S* $\wedge$ $(\lambda k.$ *integral* $(g\ `\ S)\ (h\ k)) \longrightarrow$ *integral* $(g\ `\ S)\ f$
proof (*rule monotone_convergence_increasing*)
have $|integral\ (g\ `\ S)\ (h\ n)| \le integral\ S\ (\lambda x.\ ?D\ x * f\ (g\ x))$ **for** n
proof −
have $|integral\ (g\ `\ S)\ (h\ n)| = integral\ (g\ `\ S)\ (h\ n)$
using *hint* **by** (*simp add*: *integral_nonneg nonneg_h*)
also have $\ldots \le integral\ S\ (\lambda x.\ ?D\ x * f\ (g\ x))$
using *hint le* **by** (*meson order_trans*)
finally show *?thesis* **.**
qed
then show *bounded* (*range* $(\lambda k.$ *integral* $(g\ `\ S)\ (h\ k)))$
by (*force simp*: *bounded_iff*)
qed (*use h_inc lim hint* **in** *auto*)
moreover have *integral* $(g\ `\ S)\ (h\ n) \le integral\ S\ (\lambda x.\ ?D\ x * f\ (g\ x))$ **for** n
using *hint* **by** (*blast intro*: *le order_trans*)
ultimately show *?thesis*
by (*auto intro*: *Lim_bounded*)
qed

lemma *integral_on_image_ubound_nonneg*:
fixes $f :: real^{\frown}{}'n::\{finite,wellorder\} \Rightarrow real$
assumes *nonneg_fg*: $\bigwedge x.\ x \in S \Longrightarrow 0 \le f(g\ x)$
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g$ *has_derivative* $g'\ x)$ *(at x within S)*
and *intS*: $(\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))$ *integrable_on S*
shows *f integrable_on* $(g\ `\ S) \wedge integral\ (g\ `\ S)\ f \le integral\ S\ (\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))$

```
        (is _ ∧ _ ≤ ?b)
proof −
  let ?D = λx. det (matrix (g′ x))
  define S′ where S′ ≡ {x ∈ S. ?D x ∗ f(g x) ≠ 0}
  then have der_gS′: ⋀x. x ∈ S′ ⟹ (g has_derivative g′ x) (at x within S′)
    by (metis (mono_tags, lifting) der_g has_derivative_subset mem_Collect_eq
subset_iff)
  have (λx. if x ∈ S then |?D x| ∗ f (g x) else 0) integrable_on UNIV
    by (simp add: integrable_restrict_UNIV intS)
 then have Df_borel: (λx. if x ∈ S then |?D x| ∗ f (g x) else 0) ∈ borel_measurable
lebesgue
    using integrable_imp_measurable lebesgue_on_UNIV_eq by force
  have S′: S′ ∈ sets lebesgue
  proof −
    from Df_borel borel_measurable_vimage_open [of _ UNIV]
    have {x. (if x ∈ S then |?D x| ∗ f (g x) else 0) ∈ T} ∈ sets lebesgue
      if open T for T
      using that unfolding lebesgue_on_UNIV_eq
      by (fastforce simp add: dest!: spec)
    then have {x. (if x ∈ S then |?D x| ∗ f (g x) else 0) ∈ −{0}} ∈ sets lebesgue
      using open_Compl by blast
    then show ?thesis
      by (simp add: S′_def conj_ac split: if_split_asm cong: conj_cong)
  qed
  then have gS′: g ‘ S′ ∈ sets lebesgue
  proof (rule differentiable_image_in_sets_lebesgue)
    show g differentiable_on S′
      using der_g unfolding S′_def differentiable_def differentiable_on_def
      by (blast intro: has_derivative_subset)
  qed auto
  have f: f ∈ borel_measurable (lebesgue_on (g ‘ S′))
  proof (clarsimp simp add: borel_measurable_vimage_open)
    fix T :: real set
    assume open T
    have {x ∈ g ‘ S′. f x ∈ T} = g ‘ {x ∈ S′. f(g x) ∈ T}
      by blast
    moreover have g ‘ {x ∈ S′. f(g x) ∈ T} ∈ sets lebesgue
    proof (rule differentiable_image_in_sets_lebesgue)
      let ?h = λx. |?D x| ∗ f (g x) /R |?D x|
      have (λx. if x ∈ S′ then |?D x| ∗ f (g x) else 0) = (λx. if x ∈ S then |?D x|
∗ f (g x) else 0)
        by (auto simp: S′_def)
      also have ... ∈ borel_measurable lebesgue
        by (rule Df_borel)
      finally have ∗: (λx. |?D x| ∗ f (g x)) ∈ borel_measurable (lebesgue_on S′)
        by (simp add: borel_measurable_if_D)
      have (λv. det (matrix (g′ v))) ∈ borel_measurable (lebesgue_on S′)
        using S′ borel_measurable_det_Jacobian der_gS′ by blast
      then have ?h ∈ borel_measurable (lebesgue_on S′)
```

using ∗ *borel_measurable_abs borel_measurable_inverse borel_measurable_scaleR*
by *blast*
moreover have *?h x* = *f*(*g x*) **if** *x* ∈ *S′* **for** *x*
using *that* **by** (*auto simp*: *S′_def*)
ultimately have (λ*x*. *f*(*g x*)) ∈ *borel_measurable* (*lebesgue_on S′*)
by (*metis* (*no_types*, *lifting*) *measurable_lebesgue_cong*)
then show {*x* ∈ *S′*. *f* (*g x*) ∈ *T*} ∈ *sets lebesgue*
by (*simp add*: ‹*S′* ∈ *sets lebesgue*› ‹*open T*› *borel_measurable_vimage_open*
sets_restrict_space_iff)
show *g differentiable_on* {*x* ∈ *S′*. *f* (*g x*) ∈ *T*}
using *der_g* **unfolding** *S′_def differentiable_def differentiable_on_def*
by (*blast intro*: *has_derivative_subset*)
qed *auto*
ultimately have {*x* ∈ *g* ‘ *S′*. *f x* ∈ *T*} ∈ *sets lebesgue*
by *metis*
then show {*x* ∈ *g* ‘ *S′*. *f x* ∈ *T*} ∈ *sets* (*lebesgue_on* (*g* ‘ *S′*))
by (*simp add*: ‹*g* ‘ *S′* ∈ *sets lebesgue*› *sets_restrict_space_iff*)
qed
have *intS′*: (λ*x*. |*?D x*| ∗ *f* (*g x*)) *integrable_on S′*
using *intS*
by (*rule integrable_spike_set*) (*auto simp*: *S′_def intro*: *empty_imp_negligible*)
have *lebS′*: {*x* ∈ *S′*. *g x* ∈ *T*} ∈ *sets lebesgue* **if** *T* ⊆ *g* ‘ *S′ T* ∈ *sets lebesgue*
for *T*
proof −
have *g* ∈ *borel_measurable* (*lebesgue_on S′*)
using *der_gS′ has_derivative_continuous_on S′*
by (*blast intro*: *continuous_imp_measurable_on_sets_lebesgue*)
moreover have {*x* ∈ *S′*. *g x* ∈ *U*} ∈ *sets lebesgue* **if** *negligible U U* ⊆ *g* ‘ *S′*
for *U*
proof (*intro negligible_imp_sets negligible_differentiable_vimage that*)
fix *x*
assume *x*: *x* ∈ *S′*
then have *linear* (*g′ x*)
using *der_gS′ has_derivative_linear* **by** *blast*
with *x* **show** *inj* (*g′ x*)
by (*auto simp*: *S′_def det_nz_iff_inj*)
qed (*use der_gS′* **in** *auto*)
ultimately show *?thesis*
using *double_lebesgue_sets* [*OF S′ gS′ order_refl*] *that* **by** *blast*
qed
have *int_gS′*: *f integrable_on g* ‘ *S′* ∧ *integral* (*g* ‘ *S′*) *f* ≤ *integral S′* (λ*x*. |*?D*
x| ∗ *f*(*g x*))
using *integral_on_image_ubound_weak* [*OF S′ f nonneg_fg der_gS′ intS′*
lebS′] *S′_def* **by** *blast*
have *negligible* (*g* ‘ {*x* ∈ *S*. *det*(*matrix*(*g′ x*)) = *0*})
proof (*rule baby_Sard*, *simp_all*)
fix *x*
assume *x*: *x* ∈ *S* ∧ *det* (*matrix* (*g′ x*)) = *0*
then show (*g has_derivative g′ x*) (*at x within* {*x* ∈ *S*. *det* (*matrix* (*g′ x*)) =

```
0})
      by (metis (no_types, lifting) der_g has_derivative_subset mem_Collect_eq
subsetI)
    then show rank (matrix (g' x)) < CARD('n)
      using det_nz_iff_inj matrix_vector_mul_linear x
      by (fastforce simp add: less_rank_noninjective)
  qed
  then have negg: negligible (g ` S - g ` {x ∈ S. ?D x ≠ 0})
    by (rule negligible_subset) (auto simp: S'_def)
  have null: g ` {x ∈ S. ?D x ≠ 0} - g ` S = {}
    by (auto simp: S'_def)
  let ?F = {x ∈ S. f (g x) ≠ 0}
  have eq: g ` S' = g ` ?F ∩ g ` {x ∈ S. ?D x ≠ 0}
    by (auto simp: S'_def image_iff)
  show ?thesis
  proof
    have ((λx. if x ∈ g ` ?F then f x else 0) integrable_on g ` {x ∈ S. ?D x ≠ 0})
      using int_gS' eq integrable_restrict_Int [where f=f]
      by simp
    then have f integrable_on g ` {x ∈ S. ?D x ≠ 0}
      by (auto simp: image_iff elim!: integrable_eq)
    then show f integrable_on g ` S
      using negg null
       by (auto intro: integrable_spike_set [OF _ empty_imp_negligible negligi-
ble_subset])
    have integral (g ` S) f = integral (g ` {x ∈ S. ?D x ≠ 0}) f
      using negg by (auto intro: negligible_subset integral_spike_set)
    also have ... = integral (g ` {x ∈ S. ?D x ≠ 0}) (λx. if x ∈ g ` ?F then f x
else 0)
      by (auto simp: image_iff intro!: integral_cong)
    also have ... = integral (g ` S') f
      using  eq integral_restrict_Int by simp
    also have ... ≤ integral S' (λx. |?D x| * f(g x))
      by (metis int_gS')
    also have ... ≤ ?b
       by (rule integral_subset_le [OF _ intS' intS]) (use nonneg_fg S'_def in
auto)
    finally show integral (g ` S) f ≤ ?b .
  qed
qed


lemma absolutely_integrable_on_image_real:
  fixes f :: real^'n::{finite,wellorder} ⇒ real and g :: real^'n::_ ⇒ real^'n::_
  assumes der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and intS: (λx. |det (matrix (g' x))| * f(g x)) absolutely_integrable_on S
  shows f absolutely_integrable_on (g ` S)
proof −
  let ?D = λx. |det (matrix (g' x))| * f (g x)
```

```
  let ?N = {x ∈ S. f (g x) < 0} and ?P = {x ∈ S. f (g x) > 0}
  have eq: {x. (if x ∈ S then ?D x else 0) > 0} = {x ∈ S. ?D x > 0}
          {x. (if x ∈ S then ?D x else 0) < 0} = {x ∈ S. ?D x < 0}
    by auto
  have ?D integrable_on S
    using intS absolutely_integrable_on_def by blast
  then have (λx. if x ∈ S then ?D x else 0) integrable_on UNIV
    by (simp add: integrable_restrict_UNIV)
 then have D_borel: (λx. if x ∈ S then ?D x else 0) ∈ borel_measurable (lebesgue_on
UNIV)
    using integrable_imp_measurable lebesgue_on_UNIV_eq by blast
  then have Dlt: {x ∈ S. ?D x < 0} ∈ sets lebesgue
    unfolding borel_measurable_vimage_halfspace_component_lt
    by (drule_tac x=0 in spec) (auto simp: eq)
  from D_borel have Dgt: {x ∈ S. ?D x > 0} ∈ sets lebesgue
    unfolding borel_measurable_vimage_halfspace_component_gt
    by (drule_tac x=0 in spec) (auto simp: eq)

  have dfgbm: ?D ∈ borel_measurable (lebesgue_on S)
    using intS absolutely_integrable_on_def integrable_imp_measurable by blast
  have der_gN: (g has_derivative g' x) (at x within ?N) if x ∈ ?N for x
     using der_g has_derivative_subset that by force
  have (λx. − f x) integrable_on g ‘ ?N ∧
        integral (g ‘ ?N) (λx. − f x) ≤ integral ?N (λx. |det (matrix (g' x))| ∗ − f
(g x))
  proof (rule integral_on_image_ubound_nonneg [OF _ der_gN])
    have 1: ?D integrable_on {x ∈ S. ?D x < 0}
      using Dlt
     by (auto intro: set_lebesgue_integral_eq_integral [OF set_integrable_subset]
intS)
    have uminus ∘ (λx. |det (matrix (g' x))| ∗ − f (g x)) integrable_on ?N
    by (simp add: o_def mult_less_0_iff empty_imp_negligible integrable_spike_set
[OF 1])
    then show (λx. |det (matrix (g' x))| ∗ − f (g x)) integrable_on ?N
      by (simp add: integrable_neg_iff o_def)
  qed auto
  then have f integrable_on g ‘ ?N
    by (simp add: integrable_neg_iff)
  moreover have g ‘ ?N = {y ∈ g ‘ S. f y < 0}
    by auto
  ultimately have f integrable_on {y ∈ g ‘ S. f y < 0}
    by simp
  then have N: f absolutely_integrable_on {y ∈ g ‘ S. f y < 0}
    by (rule absolutely_integrable_absolutely_integrable_ubound) auto

  have der_gP: (g has_derivative g' x) (at x within ?P) if x ∈ ?P for x
     using der_g has_derivative_subset that by force
    have f integrable_on g ‘ ?P ∧ integral (g ‘ ?P) f ≤ integral ?P ?D
    proof (rule integral_on_image_ubound_nonneg [OF _ der_gP])
```

```
      show ?D integrable_on ?P
      proof (rule integrable_spike_set)
        show ?D integrable_on {x ∈ S. 0 < ?D x}
          using Dgt
       by (auto intro: set_lebesgue_integral_eq_integral [OF set_integrable_subset]
intS)
      qed (auto simp: zero_less_mult_iff empty_imp_negligible)
    qed auto
  then have f integrable_on g ‘ ?P
    by metis
  moreover have g ‘ ?P = {y ∈ g ‘ S. f y > 0}
    by auto
  ultimately have f integrable_on {y ∈ g ‘ S. f y > 0}
    by simp
  then have P: f absolutely_integrable_on {y ∈ g ‘ S. f y > 0}
    by (rule absolutely_integrable_absolutely_integrable_lbound) auto
  have (λx. if x ∈ g ‘ S ∧ f x < 0 ∨ x ∈ g ‘ S ∧ 0 < f x then f x else 0) = (λx.
if x ∈ g ‘ S then f x else 0)
    by auto
  then show ?thesis
   using absolutely_integrable_Un [OF N P] absolutely_integrable_restrict_UNIV
[symmetric, where f=f]
    by simp
qed


proposition absolutely_integrable_on_image:
  fixes f :: real^'m::{finite,wellorder} ⇒ real^'n and g :: real^'m::_ ⇒ real^'m::_
  assumes der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and intS: (λx. |det (matrix (g' x))| *_R f(g x)) absolutely_integrable_on S
  shows f absolutely_integrable_on (g ‘ S)
 apply (rule absolutely_integrable_componentwise [OF absolutely_integrable_on_image_real
[OF der_g]])
  using absolutely_integrable_component [OF intS] by auto

proposition integral_on_image_ubound:
  fixes f :: real^'n::{finite,wellorder} ⇒ real and g :: real^'n::_ ⇒ real^'n::_
  assumes⋀x. x ∈ S ⟹ 0 ≤ f(g x)
    and ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and (λx. |det (matrix (g' x))| * f(g x)) integrable_on S
  shows integral (g ‘ S) f ≤ integral S (λx. |det (matrix (g' x))| * f(g x))
  using integral_on_image_ubound_nonneg [OF assms] by simp
```

9.32.5 Change-of-variables theorem

The classic change-of-variables theorem. We have two versions with quite general hypotheses, the first that the transforming function has a continuous inverse, the second that the base set is Lebesgue measurable.

```
lemma cov_invertible_nonneg_le:
```

```
  fixes f :: real^'n::{finite,wellorder} ⇒ real and g :: real^'n::_ ⇒ real^'n::_
  assumes der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and der_h: ⋀y. y ∈ T ⟹ (h has_derivative h' y) (at y within T)
    and f0: ⋀y. y ∈ T ⟹ 0 ≤ f y
    and hg: ⋀x. x ∈ S ⟹ g x ∈ T ∧ h(g x) = x
    and gh: ⋀y. y ∈ T ⟹ h y ∈ S ∧ g(h y) = y
    and id: ⋀y. y ∈ T ⟹ h' y ∘ g'(h y) = id
  shows f integrable_on T ∧ (integral T f) ≤ b ⟷
          (λx. |det (matrix (g' x))| * f(g x)) integrable_on S ∧
          integral S (λx. |det (matrix (g' x))| * f(g x)) ≤ b
      (is ?lhs = ?rhs)
proof −
  have Teq: T = g`S and Seq: S = h`T
    using hg gh image_iff by fastforce+
  have gS: g differentiable_on S
    by (meson der_g differentiable_def differentiable_on_def)
  let ?D = λx. |det (matrix (g' x))| * f (g x)
  show ?thesis
  proof
    assume ?lhs
    then have fT: f integrable_on T and intf: integral T f ≤ b
      by blast+
    show ?rhs
    proof
      let ?fgh = λx. |det (matrix (h' x))| * (|det (matrix (g' (h x)))| * f (g (h x)))
      have ddf: ?fgh x = f x
        if x ∈ T for x
      proof −
        have matrix (h' x) ** matrix (g' (h x)) = mat 1
          by (metis der_g der_h gh has_derivative_linear local.id matrix_compose
matrix_id_mat_1 that)
        then have |det (matrix (h' x))| * |det (matrix (g' (h x)))| = 1
          by (metis abs_1 abs_mult det_I det_mul)
        then show ?thesis
          by (simp add: gh that)
      qed
      have ?D integrable_on (h ` T)
    proof (intro set_lebesgue_integral_eq_integral absolutely_integrable_on_image_real)
        show (λx. ?fgh x) absolutely_integrable_on T
          by (smt (verit, del_insts) abs_absolutely_integrableI_1 ddf f0 fT inte-
grable_eq)
      qed (use der_h in auto)
      with Seq show (λx. ?D x) integrable_on S
        by simp
      have integral S (λx. ?D x) ≤ integral T (λx. ?fgh x)
        unfolding Seq
      proof (rule integral_on_image_ubound)
        show (λx. ?fgh x) integrable_on T
        using ddf fT integrable_eq by force
```

```
    qed (use f0 gh der_h in auto)
    also have ... = integral T f
      by (force simp: ddf intro: integral_cong)
    finally show integral S (λx. ?D x) ≤ b
      using intf by linarith
  qed
 next
  assume R: ?rhs
  then have f integrable_on g ‘ S
    using der_g f0 hg integral_on_image_ubound_nonneg by blast
  moreover have integral (g ‘ S) f ≤ integral S (λx. ?D x)
    by (rule integral_on_image_ubound [OF f0 der_g]) (use R Teq in auto)
  ultimately show ?lhs
    using R by (simp add: Teq)
 qed
qed


lemma cov_invertible_nonneg_eq:
 fixes f :: real^'n::{finite,wellorder} ⇒ real and g :: real^'n::_ ⇒ real^'n::_
 assumes ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and ⋀y. y ∈ T ⟹ (h has_derivative h' y) (at y within T)
    and ⋀y. y ∈ T ⟹ 0 ≤ f y
    and ⋀x. x ∈ S ⟹ g x ∈ T ∧ h(g x) = x
    and ⋀y. y ∈ T ⟹ h y ∈ S ∧ g(h y) = y
    and ⋀y. y ∈ T ⟹ h' y ∘ g'(h y) = id
 shows ((λx. |det (matrix (g' x))| * f(g x)) has_integral b) S ⟷ (f has_integral
b) T
 using cov_invertible_nonneg_le [OF assms]
 by (simp add: has_integral_iff) (meson eq_iff)


lemma cov_invertible_real:
 fixes f :: real^'n::{finite,wellorder} ⇒ real and g :: real^'n::_ ⇒ real^'n::_
 assumes der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and der_h: ⋀y. y ∈ T ⟹ (h has_derivative h' y) (at y within T)
    and hg: ⋀x. x ∈ S ⟹ g x ∈ T ∧ h(g x) = x
    and gh: ⋀y. y ∈ T ⟹ h y ∈ S ∧ g(h y) = y
    and id: ⋀y. y ∈ T ⟹ h' y ∘ g'(h y) = id
 shows (λx. |det (matrix (g' x))| * f(g x)) absolutely_integrable_on S ∧
        integral S (λx. |det (matrix (g' x))| * f(g x)) = b ⟷
       f absolutely_integrable_on T ∧ integral T f = b
       (is ?lhs = ?rhs)
proof −
 have Teq: T = g‘S and Seq: S = h‘T
   using hg gh image_iff by fastforce+
 let ?DP = λx. |det (matrix (g' x))| * f(g x) and ?DN = λx. |det (matrix (g'
x))| * −f(g x)
 have +: (?DP has_integral b) {x ∈ S. f (g x) > 0} ⟷ (f has_integral b) {y ∈
```

```
T. f y > 0} for b
  proof (rule cov_invertible_nonneg_eq)
    have ∗: (λx. f (g x)) −‘ Y ∩ {x ∈ S. f (g x) > 0}
          = ((λx. f (g x)) −‘ Y ∩ S) ∩ {x ∈ S. f (g x) > 0} for Y
      by auto
    show (g has_derivative g' x) (at x within {x ∈ S. f (g x) > 0}) if x ∈ {x ∈ S.
f (g x) > 0} for x
      using that der_g has_derivative_subset by fastforce
    show (h has_derivative h' y) (at y within {y ∈ T. f y > 0}) if y ∈ {y ∈ T. f
y > 0} for y
      using that der_h has_derivative_subset by fastforce
  qed (use gh hg id in auto)
  have −: (?DN has_integral b) {x ∈ S. f (g x) < 0} ⟷ ((λx. − f x) has_integral
b) {y ∈ T. f y < 0} for b
  proof (rule cov_invertible_nonneg_eq)
    have ∗: (λx. − f (g x)) −‘ y ∩ {x ∈ S. f (g x) < 0}
          = ((λx. f (g x)) −‘ uminus ‘ y ∩ S) ∩ {x ∈ S. f (g x) < 0} for y
      using image_iff by fastforce
    show (g has_derivative g' x) (at x within {x ∈ S. f (g x) < 0}) if x ∈ {x ∈ S.
f (g x) < 0} for x
      using that der_g has_derivative_subset by fastforce
    show (h has_derivative h' y) (at y within {y ∈ T. f y < 0}) if y ∈ {y ∈ T. f
y < 0} for y
      using that der_h has_derivative_subset by fastforce
  qed (use gh hg id in auto)
  show ?thesis
  proof
    assume LHS: ?lhs
    have eq: {x. (if x ∈ S then ?DP x else 0) > 0} = {x ∈ S. ?DP x > 0}
      {x. (if x ∈ S then ?DP x else 0) < 0} = {x ∈ S. ?DP x < 0}
      by auto
    have ?DP integrable_on S
      using LHS absolutely_integrable_on_def by blast
    then have (λx. if x ∈ S then ?DP x else 0) integrable_on UNIV
      by (simp add: integrable_restrict_UNIV)
     then have D_borel: (λx. if x ∈ S then ?DP x else 0) ∈ borel_measurable
(lebesgue_on UNIV)
      using integrable_imp_measurable lebesgue_on_UNIV_eq by blast
    then have SN: {x ∈ S. ?DP x < 0} ∈ sets lebesgue
      unfolding borel_measurable_vimage_halfspace_component_lt
      by (drule_tac x=0 in spec) (auto simp: eq)
    from D_borel have SP: {x ∈ S. ?DP x > 0} ∈ sets lebesgue
      unfolding borel_measurable_vimage_halfspace_component_gt
      by (drule_tac x=0 in spec) (auto simp: eq)
    have ?DP absolutely_integrable_on {x ∈ S. ?DP x > 0}
      using LHS by (fast intro!: set_integrable_subset [OF _, of _ S] SP)
    then have aP: ?DP absolutely_integrable_on {x ∈ S. f (g x) > 0}
       by (rule absolutely_integrable_spike_set) (auto simp: zero_less_mult_iff
empty_imp_negligible)
```

```
    have ?DP absolutely_integrable_on {x ∈ S. ?DP x < 0}
      using LHS by (fast intro!: set_integrable_subset [OF _, of _ S] SN)
    then have aN: ?DP absolutely_integrable_on {x ∈ S. f (g x) < 0}
    by (rule absolutely_integrable_spike_set) (auto simp: mult_less_0_iff empty_imp_negligible)
    have fN: f integrable_on {y ∈ T. f y < 0}
            integral {y ∈ T. f y < 0} f = integral {x ∈ S. f (g x) < 0} ?DP
      using - [of integral {x ∈ S. f(g x) < 0} ?DN] aN
    by (auto simp: set_lebesgue_integral_eq_integral has_integral_iff integrable_neg_iff)
    have faN: f absolutely_integrable_on {y ∈ T. f y < 0}
    proof (rule absolutely_integrable_integrable_bound)
      show (λx. - f x) integrable_on {y ∈ T. f y < 0}
        using fN by (auto simp: integrable_neg_iff)
    qed (use fN in auto)
    have fP: f integrable_on {y ∈ T. f y > 0}
            integral {y ∈ T. f y > 0} f = integral {x ∈ S. f (g x) > 0} ?DP
      using + [of integral {x ∈ S. f(g x) > 0} ?DP] aP
    by (auto simp: set_lebesgue_integral_eq_integral has_integral_iff integrable_neg_iff)
    have faP: f absolutely_integrable_on {y ∈ T. f y > 0}
      using fP(1) nonnegative_absolutely_integrable_1 by fastforce
    have fa: f absolutely_integrable_on ({y ∈ T. f y < 0} ∪ {y ∈ T. f y > 0})
      by (rule absolutely_integrable_Un [OF faN faP])
    show ?rhs
    proof
      have eq: ((if x ∈ T ∧ f x < 0 ∨ x ∈ T ∧ 0 < f x then 1 else 0) * f x)
            = (if x ∈ T then 1 else 0) * f x for x
        by auto
      show f absolutely_integrable_on T
        using fa by (simp add: indicator_def of_bool_def set_integrable_def eq)
       have [simp]: {y ∈ T. f y < 0} ∩ {y ∈ T. 0 < f y} = {} for T and f ::
(real^'n::_) ⇒ real
        by auto
      have integral T f = integral ({y ∈ T. f y < 0} ∪ {y ∈ T. f y > 0}) f
        by (intro empty_imp_negligible integral_spike_set) (auto simp: eq)
      also have ... = integral {y ∈ T. f y < 0} f + integral {y ∈ T. f y > 0} f
        using fN fP by simp
      also have ... = integral {x ∈ S. f (g x) < 0} ?DP + integral {x ∈ S. 0 <
f (g x)} ?DP
        by (simp add: fN fP)
      also have ... = integral ({x ∈ S. f (g x) < 0} ∪ {x ∈ S. 0 < f (g x)}) ?DP
        using aP aN by (simp add: set_lebesgue_integral_eq_integral)
      also have ... = integral S ?DP
        by (intro empty_imp_negligible integral_spike_set) auto
      also have ... = b
        using LHS by simp
      finally show integral T f = b .
    qed
  next
    assume RHS: ?rhs
    have eq: {x. (if x ∈ T then f x else 0) > 0} = {x ∈ T. f x > 0}
```

```
        {x. (if x ∈ T then f x else 0) < 0} = {x ∈ T. f x < 0}
    by auto
  have f integrable_on T
    using RHS absolutely_integrable_on_def by blast
  then have (λx. if x ∈ T then f x else 0) integrable_on UNIV
    by (simp add: integrable_restrict_UNIV)
    then have D_borel: (λx. if x ∈ T then f x else 0) ∈ borel_measurable
(lebesgue_on UNIV)
    using integrable_imp_measurable lebesgue_on_UNIV_eq by blast
  then have TN: {x ∈ T. f x < 0} ∈ sets lebesgue
    unfolding borel_measurable_vimage_halfspace_component_lt
    by (drule_tac x=0 in spec) (auto simp: eq)
  from D_borel have TP: {x ∈ T. f x > 0} ∈ sets lebesgue
    unfolding borel_measurable_vimage_halfspace_component_gt
    by (drule_tac x=0 in spec) (auto simp: eq)
  have aint: f absolutely_integrable_on {y. y ∈ T ∧ 0 < (f y)}
          f absolutely_integrable_on {y. y ∈ T ∧ (f y) < 0}
     and intT: integral T f = b
    using set_integrable_subset [of _ T] TP TN RHS by blast+
  show ?lhs
  proof
    have fN: f integrable_on {v ∈ T. f v < 0}
      using absolutely_integrable_on_def aint by blast
    then have DN: (?DN has_integral integral {y ∈ T. f y < 0} (λx. − f x)) {x
∈ S. f (g x) < 0}
      using − [of integral {y ∈ T. f y < 0} (λx. − f x)]
      by (simp add: has_integral_neg_iff integrable_integral)
    have aDN: ?DP absolutely_integrable_on {x ∈ S. f (g x) < 0}
      apply (rule absolutely_integrable_integrable_bound [where g = ?DN])
      using DN hg by (fastforce simp: abs_mult integrable_neg_iff)+
    have fP: f integrable_on {v ∈ T. f v > 0}
      using absolutely_integrable_on_def aint by blast
    then have DP: (?DP has_integral integral {y ∈ T. f y > 0} f) {x ∈ S. f (g
x) > 0}
      using + [of integral {y ∈ T. f y > 0} f]
      by (simp add: has_integral_neg_iff integrable_integral)
    have aDP: ?DP absolutely_integrable_on {x ∈ S. f (g x) > 0}
      apply (rule absolutely_integrable_integrable_bound [where g = ?DP])
      using DP hg by (fastforce simp: integrable_neg_iff)+
    have eq: (if x ∈ S then 1 else 0) * ?DP x = (if x ∈ S ∧ f (g x) < 0 ∨ x ∈ S
∧ f (g x) > 0 then 1 else 0) * ?DP x for x
      by force
    have ?DP absolutely_integrable_on ({x ∈ S. f (g x) < 0} ∪ {x ∈ S. f (g x)
> 0})
      by (rule absolutely_integrable_Un [OF aDN aDP])
    then show I: ?DP absolutely_integrable_on S
      by (simp add: indicator_def of_bool_def eq set_integrable_def)
     have [simp]: {y ∈ S. f y < 0} ∩ {y ∈ S. 0 < f y} = {} for S and f ::
(real^'n::_) ⇒ real
```

```
      by auto
    have integral S ?DP = integral ({x ∈ S. f (g x) < 0} ∪ {x ∈ S. f (g x) >
0}) ?DP
      by (intro empty_imp_negligible integral_spike_set) auto
    also have ... = integral {x ∈ S. f (g x) < 0} ?DP + integral {x ∈ S. 0 <
f (g x)} ?DP
      using aDN aDP by (simp add: set_lebesgue_integral_eq_integral)
    also have ... = − integral {y ∈ T. f y < 0} (λx. − f x) + integral {y ∈ T.
f y > 0} f
      using DN DP by (auto simp: has_integral_iff)
    also have ... = integral ({x ∈ T. f x < 0} ∪ {x ∈ T. 0 < f x}) f
      by (simp add: fN fP)
    also have ... = integral T f
      by (intro empty_imp_negligible integral_spike_set) auto
    also have ... = b
      using intT by simp
    finally show integral S ?DP = b .
  qed
 qed
qed


lemma cv_inv_version3:
 fixes f :: real^'m::{finite,wellorder} ⇒ real^'n and g :: real^'m::_ ⇒ real^'m::_
 assumes der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
   and der_h: ⋀y. y ∈ T ⟹ (h has_derivative h' y) (at y within T)
   and hg: ⋀x. x ∈ S ⟹ g x ∈ T ∧ h(g x) = x
   and gh: ⋀y. y ∈ T ⟹ h y ∈ S ∧ g(h y) = y
   and id: ⋀y. y ∈ T ⟹ h' y ∘ g'(h y) = id
 shows (λx. |det (matrix (g' x))| *R f(g x)) absolutely_integrable_on S ∧
          integral S (λx. |det (matrix (g' x))| *R f(g x)) = b
       ⟷ f absolutely_integrable_on T ∧ integral T f = b
proof −
 let ?D = λx. |det (matrix (g' x))| *R f(g x)
 have ((λx. |det (matrix (g' x))| * f(g x) $ i) absolutely_integrable_on S ∧ integral
S (λx. |det (matrix (g' x))| * (f(g x) $ i)) = b $ i) ⟷
       ((λx. f x $ i) absolutely_integrable_on T ∧ integral T (λx. f x $ i) = b $
i) for i
   by (rule cov_invertible_real [OF der_g der_h hg gh id])
 then have ?D absolutely_integrable_on S ∧ (?D has_integral b) S ⟷
      f absolutely_integrable_on T ∧ (f has_integral b) T
  unfolding absolutely_integrable_componentwise_iff [where f=f] has_integral_componentwise_iff
[of f]
        absolutely_integrable_componentwise_iff [where f=?D] has_integral_componentwise_iff
[of ?D]
  by (auto simp: all_conj_distrib Basis_vec_def cart_eq_inner_axis [symmetric]
       has_integral_iff set_lebesgue_integral_eq_integral)
 then show ?thesis
   using absolutely_integrable_on_def by blast
```

qed

lemma *cv_inv_version4*:
fixes $f :: real^{\prime}m::\{finite,wellorder\} \Rightarrow real^{\prime}n$ **and** $g :: real^{\prime}m::\_ \Rightarrow real^{\prime}m::\_$
assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S) \wedge invertible(matrix(g'\ x))$
and *hg*: $\bigwedge x.\ x \in S \Longrightarrow continuous\_on\ (g\ `\ S)\ h \wedge h(g\ x) = x$
shows $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))\ absolutely\_integrable\_on\ S \wedge$
$integral\ S\ (\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x)) = b$
$\longleftrightarrow f\ absolutely\_integrable\_on\ (g\ `\ S) \wedge integral\ (g\ `\ S)\ f = b$
proof −
have $\forall x.\ \exists h'.\ x \in S$
$\longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S) \wedge linear\ h' \wedge g'\ x \circ h' = id \wedge h' \circ g'\ x = id$
using *der_g matrix_invertible has_derivative_linear* **by** *blast*
then obtain h' **where** h':
$\bigwedge x.\ x \in S$
$\Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S) \wedge$
$linear\ (h'\ x) \wedge g'\ x \circ (h'\ x) = id \wedge (h'\ x) \circ g'\ x = id$
by *metis*
show *?thesis*
proof (*rule cv_inv_version3*)
show $\bigwedge y.\ y \in g\ `\ S \Longrightarrow (h\ has\_derivative\ h'\ (h\ y))\ (at\ y\ within\ g\ `\ S)$
using h' *hg*
by (*force simp*: *continuous_on_eq_continuous_within intro*!: *has_derivative_inverse_within*)
qed (*use* h' *hg* **in** *auto*)
qed

theorem *has_absolute_integral_change_of_variables_invertible*:
fixes $f :: real^{\prime}m::\{finite,wellorder\} \Rightarrow real^{\prime}n$ **and** $g :: real^{\prime}m::\_ \Rightarrow real^{\prime}m::\_$
assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *hg*: $\bigwedge x.\ x \in S \Longrightarrow h(g\ x) = x$
and *conth*: $continuous\_on\ (g\ `\ S)\ h$
shows $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))\ absolutely\_integrable\_on\ S \wedge integral\ S\ (\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x)) = b \longleftrightarrow$
$f\ absolutely\_integrable\_on\ (g\ `\ S) \wedge integral\ (g\ `\ S)\ f = b$
(**is** *?lhs = ?rhs*)
proof −
let *?S* $= \{x \in S.\ invertible\ (matrix\ (g'\ x))\}$ **and** *?D* $= \lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x)$
have $*$: *?D absolutely_integrable_on ?S* $\wedge$ *integral ?S ?D* $= b$
$\longleftrightarrow f\ absolutely\_integrable\_on\ (g\ `\ ?S) \wedge integral\ (g\ `\ ?S)\ f = b$
proof (*rule cv_inv_version4*)
show $(g\ has\_derivative\ g'\ x)\ (at\ x\ within\ ?S) \wedge invertible\ (matrix\ (g'\ x))$
if $x \in ?S$ **for** x
using *der_g that has_derivative_subset that* **by** *fastforce*
show $continuous\_on\ (g\ `\ ?S)\ h \wedge h\ (g\ x) = x$

if $x \in ?S$ **for** x
using *that continuous_on_subset* [*OF conth*] **by** (*simp add*: *hg image_mono*)
qed
have (*g has_derivative g' x*) (*at x within* $\{x \in S.\ rank\ (matrix\ (g'\ x)) < CARD('m)\}$) **if** $x \in S$ **for** x
by (*metis* (*no_types*, *lifting*) *der_g has_derivative_subset mem_Collect_eq subsetI that*)
then have *negligible* (g ‘ $\{x \in S.\ \neg$ *invertible* (*matrix* (g' x))$\}$)
by (*auto simp*: *invertible_det_nz det_eq_0_rank intro*: *baby_Sard*)
then have *neg*: *negligible* $\{x \in g$ ‘ $S.\ x \notin g$ ‘ $?S \wedge f\ x \neq 0\}$
by (*auto intro*: *negligible_subset*)
have [*simp*]: $\{x \in g$ ‘ $?S.\ x \notin g$ ‘ $S \wedge f\ x \neq 0\} = \{\}$
by *auto*
have *?D absolutely_integrable_on ?S* $\wedge$ *integral ?S ?D* $= b$
$\longleftrightarrow$ *?D absolutely_integrable_on S* $\wedge$ *integral S ?D* $= b$
apply (*intro conj_cong absolutely_integrable_spike_set_eq*)
apply(*auto simp*: *integral_spike_set invertible_det_nz empty_imp_negligible neg*)
done
moreover
have *f absolutely_integrable_on* (g ‘ $?S$) $\wedge$ *integral* (g ‘ $?S$) $f = b$
$\longleftrightarrow$ *f absolutely_integrable_on* (g ‘ S) $\wedge$ *integral* (g ‘ S) $f = b$
by (*auto intro*!: *conj_cong absolutely_integrable_spike_set_eq integral_spike_set neg*)
ultimately
show *?thesis*
using $*$ **by** *blast*
qed

theorem *has_absolute_integral_change_of_variables_compact*:
fixes f :: *real^'m*::{*finite*,*wellorder*} $\Rightarrow$ *real^'n* **and** g :: *real^'m*::_ $\Rightarrow$ *real^'m*::_
assumes *compact S*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
shows $((\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))$ *absolutely_integrable_on S* $\wedge$
integral S $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x)) = b$
$\longleftrightarrow$ *f absolutely_integrable_on* (g ‘ S) $\wedge$ *integral* (g ‘ S) $f = b$)
proof −
obtain h **where** *hg*: $\bigwedge x.\ x \in S \Longrightarrow h(g\ x) = x$
using *inj* **by** (*metis the_inv_into_f_f*)
have *conth*: *continuous_on* (g ‘ S) h
by (*metis* ‹*compact S*› *continuous_on_inv der_g has_derivative_continuous_on hg*)
show *?thesis*
by (*rule has_absolute_integral_change_of_variables_invertible* [*OF der_g hg conth*])
qed

lemma *has_absolute_integral_change_of_variables_compact_family*:
fixes $f :: real\hat{}'m::\{finite,wellorder\} \Rightarrow real\hat{}'n$ **and** $g :: real\hat{}'m::\_ \Rightarrow real\hat{}'m::\_$
assumes *compact*: $\bigwedge n::nat.\ compact\ (F\ n)$
and *der_g*: $\bigwedge x.\ x \in (\bigcup n.\ F\ n) \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ (\bigcup n.\ F\ n))$
and *inj*: $inj\_on\ g\ (\bigcup n.\ F\ n)$
shows $((\lambda x.\ |det\ (matrix\ (g'\ x))|\ *_R\ f(g\ x))\ absolutely\_integrable\_on\ (\bigcup n.\ F\ n) \wedge$
$integral\ (\bigcup n.\ F\ n)\ (\lambda x.\ |det\ (matrix\ (g'\ x))|\ *_R\ f(g\ x)) = b$
$\longleftrightarrow f\ absolutely\_integrable\_on\ (g\ `\ (\bigcup n.\ F\ n)) \wedge integral\ (g\ `\ (\bigcup n.\ F\ n))\ f = b)$
proof −
let $?D = \lambda x.\ |det\ (matrix\ (g'\ x))|\ *_R\ f\ (g\ x)$
let $?U = \lambda n.\ \bigcup m{\le}n.\ F\ m$
let $?lift = vec::real \Rightarrow real\hat{}1$
have *F_leb*: $F\ m \in sets\ lebesgue$ **for** m
by (*simp add*: *compact borel_compact*)
have *iff*: $(\lambda x.\ |det\ (matrix\ (g'\ x))|\ *_R\ f\ (g\ x))\ absolutely\_integrable\_on\ (?U\ n) \wedge$
$integral\ (?U\ n)\ (\lambda x.\ |det\ (matrix\ (g'\ x))|\ *_R\ f\ (g\ x)) = b$
$\longleftrightarrow f\ absolutely\_integrable\_on\ (g\ `\ (?U\ n)) \wedge integral\ (g\ `\ (?U\ n))\ f = b$
for $n\ b$ **and** $f :: real\hat{}'m::\_ \Rightarrow real\hat{}'k$
proof (*rule has_absolute_integral_change_of_variables_compact*)
show $compact\ (?U\ n)$
by (*simp add*: *compact compact_UN*)
show $(g\ has\_derivative\ g'\ x)\ (at\ x\ within\ (?U\ n))$
if $x \in ?U\ n$ **for** x
using *that* **by** (*blast intro*!: *has_derivative_subset* [*OF der_g*])
show $inj\_on\ g\ (?U\ n)$
using *inj* **by** (*auto simp*: *inj_on_def*)
qed
show *?thesis*
unfolding *image_UN*
proof *safe*
assume *DS*: $?D\ absolutely\_integrable\_on\ (\bigcup n.\ F\ n)$
and *b*: $b = integral\ (\bigcup n.\ F\ n)\ ?D$
have *DU*: $\bigwedge n.\ ?D\ absolutely\_integrable\_on\ (?U\ n)$
$(\lambda n.\ integral\ (?U\ n)\ ?D) \longrightarrow integral\ (\bigcup n.\ F\ n)\ ?D$
using *integral_countable_UN* [*OF DS F_leb*] **by** *auto*
with *iff* **have** *fag*: $f\ absolutely\_integrable\_on\ g\ `\ (?U\ n)$
and *fg_int*: $integral\ (\bigcup m{\le}n.\ g\ `\ F\ m)\ f = integral\ (?U\ n)\ ?D$ **for** n
by (*auto simp*: *image_UN*)
let $?h = \lambda x.\ if\ x \in (\bigcup m.\ g\ `\ F\ m)\ then\ norm(f\ x)\ else\ 0$
have $(\lambda x.\ if\ x \in (\bigcup m.\ g\ `\ F\ m)\ then\ f\ x\ else\ 0)\ absolutely\_integrable\_on\ UNIV$
proof (*rule dominated_convergence_absolutely_integrable*)
show $(\lambda x.\ if\ x \in (\bigcup m{\le}k.\ g\ `\ F\ m)\ then\ f\ x\ else\ 0)\ absolutely\_integrable\_on\ UNIV$ **for** k

```
      unfolding absolutely_integrable_restrict_UNIV
      using fag by (simp add: image_UN)
    let ?nf = λn x. if x ∈ (⋃ m≤n. g ‘ F m) then norm(f x) else 0
    show ?h integrable_on UNIV
    proof (rule monotone_convergence_increasing [THEN conjunct1])
      show ?nf k integrable_on UNIV for k
        using fag
          unfolding integrable_restrict_UNIV absolutely_integrable_on_def by
(simp add: image_UN)
      { fix n
        have (norm ∘ ?D) absolutely_integrable_on ?U n
          by (intro absolutely_integrable_norm DU)
        then have integral (g ‘ ?U n) (norm ∘ f) = integral (?U n) (norm ∘ ?D)
          using iff [of n vec ∘ norm ∘ f integral (?U n) (λx. |det (matrix (g′ x))|
*R (?lift ∘ norm ∘ f) (g x))]
          unfolding absolutely_integrable_on_1_iff integral_on_1_eq by (auto
simp: o_def)
      }
      moreover have bounded (range (λk. integral (?U k) (norm ∘ ?D)))
        unfolding bounded_iff
      proof (rule exI, clarify)
        fix k
        show norm (integral (?U k) (norm ∘ ?D)) ≤ integral (⋃ n. F n) (norm ∘
?D)
          unfolding integral_restrict_UNIV [of _ norm ∘ ?D, symmetric]
        proof (rule integral_norm_bound_integral)
          show (λx. if x ∈ ⋃ (F ‘ {..k}) then (norm ∘ ?D) x else 0) integrable_on
UNIV
            (λx. if x ∈ (⋃ n. F n) then (norm ∘ ?D) x else 0) integrable_on UNIV
            using DU(1) DS
          unfolding absolutely_integrable_on_def o_def integrable_restrict_UNIV
by auto
        qed auto
      qed
      ultimately show bounded (range (λk. integral UNIV (?nf k)))
        by (simp add: integral_restrict_UNIV image_UN [symmetric] o_def)
    next
      show (λk. if x ∈ (⋃ m≤k. g ‘ F m) then norm (f x) else 0)
          ⟶ (if x ∈ (⋃ m. g ‘ F m) then norm (f x) else 0) for x
        by (force intro: tendsto_eventually eventually_sequentiallyI)
    qed auto
  next
    show (λk. if x ∈ (⋃ m≤k. g ‘ F m) then f x else 0)
        ⟶ (if x ∈ (⋃ m. g ‘ F m) then f x else 0) for x
    proof clarsimp
      fix m y
      assume y ∈ F m
      show (λk. if ∃ x∈{..k}. g y ∈ g ‘ F x then f (g y) else 0) ⟶ f (g y)
      using ‹y ∈ F m› by (force intro: tendsto_eventually eventually_sequentiallyI
```

```
[of m])
      qed
    qed auto
    then show fai: f absolutely_integrable_on (⋃m. g ‘ F m)
      using absolutely_integrable_restrict_UNIV by blast
    show integral ((⋃x. g ‘ F x)) f = integral (⋃n. F n) ?D
    proof (rule LIMSEQ_unique)
      show (λn. integral (?U n) ?D) ⟶ integral (⋃x. g ‘ F x) f
        unfolding fg_int [symmetric]
      proof (rule integral_countable_UN [OF fai])
        show g ‘ F m ∈ sets lebesgue for m
        proof (rule differentiable_image_in_sets_lebesgue [OF F_leb])
          show g differentiable_on F m
            by (meson der_g differentiableI UnionI differentiable_on_def differen-
tiable_on_subset rangeI subsetI)
        qed auto
      qed
    qed (use DU in metis)
  next
    assume fs: f absolutely_integrable_on (⋃x. g ‘ F x)
      and b: b = integral ((⋃x. g ‘ F x)) f
    have gF_leb: g ‘ F m ∈ sets lebesgue for m
    proof (rule differentiable_image_in_sets_lebesgue [OF F_leb])
      show g differentiable_on F m
        using der_g unfolding differentiable_def differentiable_on_def
        by (meson Sup_upper UNIV_I UnionI has_derivative_subset image_eqI)
    qed auto
    have fgU: ⋀n. f absolutely_integrable_on (⋃m≤n. g ‘ F m)
     (λn. integral (⋃m≤n. g ‘ F m) f) ⟶ integral (⋃m. g ‘ F m) f
      using integral_countable_UN [OF fs gF_leb] by auto
    with iff have DUn: ?D absolutely_integrable_on ?U n
      and D_int: integral (?U n) ?D = integral (⋃m≤n. g ‘ F m) f for n
      by (auto simp: image_UN)
    let ?h = λx. if x ∈ (⋃n. F n) then norm(?D x) else 0
    have (λx. if x ∈ (⋃n. F n) then ?D x else 0) absolutely_integrable_on UNIV
    proof (rule dominated_convergence_absolutely_integrable)
      show (λx. if x ∈ ?U k then ?D x else 0) absolutely_integrable_on UNIV for
k
        unfolding absolutely_integrable_restrict_UNIV using DUn by simp
      let ?nD = λn x. if x ∈ ?U n then norm(?D x) else 0
      show ?h integrable_on UNIV
      proof (rule monotone_convergence_increasing [THEN conjunct1])
        show ?nD k integrable_on UNIV for k
          using DUn
            unfolding integrable_restrict_UNIV absolutely_integrable_on_def by
(simp add: image_UN)
        { fix n::nat
          have (norm ∘ f) absolutely_integrable_on (⋃m≤n. g ‘ F m)
            using absolutely_integrable_norm fgU by blast
```

```
        then have integral (?U n) (norm ∘ ?D) = integral (g ‘ ?U n) (norm ∘ f)
          using iff [of n ?lift ∘ norm ∘ f integral (g ‘ ?U n) (?lift ∘ norm ∘ f)]
         unfolding absolutely_integrable_on_1_iff integral_on_1_eq image_UN
by (auto simp: o_def)
       }
       moreover have bounded (range (λk. integral (g ‘ ?U k) (norm ∘ f)))
        unfolding bounded_iff
       proof (rule exI, clarify)
        fix k
         show norm (integral (g ‘ ?U k) (norm ∘ f)) ≤ integral (g ‘ (⋃ n. F n))
(norm ∘ f)
          unfolding integral_restrict_UNIV [of _ norm ∘ f, symmetric]
         proof (rule integral_norm_bound_integral)
          show (λx. if x ∈ g ‘ ?U k then (norm ∘ f) x else 0) integrable_on UNIV
                (λx. if x ∈ g ‘ (⋃ n. F n) then (norm ∘ f) x else 0) integrable_on
UNIV
            using fgU fs
          unfolding absolutely_integrable_on_def o_def integrable_restrict_UNIV
            by (auto simp: image_UN)
         qed auto
       qed
       ultimately show bounded (range (λk. integral UNIV (?nD k)))
        unfolding integral_restrict_UNIV image_UN [symmetric] o_def by simp
     next
       show (λk. if x ∈ ?U k then norm (?D x) else 0) ⟶ (if x ∈ (⋃ n. F n)
then norm (?D x) else 0) for x
         by (force intro: tendsto_eventually eventually_sequentiallyI)
     qed auto
   next
     show (λk. if x ∈ ?U k then ?D x else 0) ⟶ (if x ∈ (⋃ n. F n) then ?D x
else 0) for x
      proof clarsimp
       fix n
       assume x ∈ F n
       show (λm. if ∃j∈{..m}. x ∈ F j then ?D x else 0) ⟶ ?D x
      using ‹x ∈ F n› by (auto intro!: tendsto_eventually eventually_sequentiallyI
[of n])
     qed
   qed auto
   then show Dai: ?D absolutely_integrable_on (⋃ n. F n)
     unfolding absolutely_integrable_restrict_UNIV by simp
   show integral (⋃ n. F n) ?D = integral ((⋃ x. g ‘ F x)) f
   proof (rule LIMSEQ_unique)
     show (λn. integral (⋃ m≤n. g ‘ F m) f) ⟶ integral (⋃ n. F n) ?D
       unfolding D_int [symmetric] by (rule integral_countable_UN [OF Dai
F_leb])
   qed (use fgU in metis)
 qed
qed
```

theorem *has_absolute_integral_change_of_variables*:
fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** g :: *real^'m::_* ⇒ *real^'m::_*
assumes S: $S \in$ *sets lebesgue*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g$ *has_derivative* $g'\ x)$ *(at x within S)*
and *inj*: *inj_on g S*
shows $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))$ *absolutely_integrable_on S* ∧
integral S $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x)) = b$
⟷ *f absolutely_integrable_on* $(g$ ‘ $S) \wedge$ *integral* $(g$ ‘ $S)\ f = b$
proof −
obtain *C N* **where** *fsigma C* **and** *N*: $N \in$ *null_sets lebesgue* **and** *CNS*: $C \cup N = S$ **and** *disjnt C N*
using *lebesgue_set_almost_fsigma* [*OF S*] **.**
then obtain F :: *nat* ⇒ *(real^'m::_) set*
where F: *range F* ⊆ *Collect compact* **and** *Ceq*: $C =$ *Union*(*range F*)
using *fsigma_Union_compact* **by** *metis*
have *negligible N*
using *N* **by** (*simp add*: *negligible_iff_null_sets*)
let *?D* = $\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f\ (g\ x)$
have *?D absolutely_integrable_on C* ∧ *integral C ?D* = *b*
⟷ *f absolutely_integrable_on* $(g$ ‘ $C) \wedge$ *integral* $(g$ ‘ $C)\ f = b$
unfolding *Ceq*
proof (*rule has_absolute_integral_change_of_variables_compact_family*)
fix *n x*
assume $x \in \bigcup (F$ ‘ $UNIV)$
then show $(g$ *has_derivative* $g'\ x)$ *(at x within* $\bigcup (F$ ‘ $UNIV))$
using *Ceq* ‹$C \cup N = S$› *der_g has_derivative_subset* **by** *blast*
next
have $\bigcup (F$ ‘ $UNIV) \subseteq S$
using *Ceq* ‹$C \cup N = S$› **by** *blast*
then show *inj_on g* $(\bigcup (F$ ‘ $UNIV))$
using *inj* **by** (*meson inj_on_subset*)
qed (*use F* **in** *auto*)
moreover
have *?D absolutely_integrable_on C* ∧ *integral C ?D* = *b*
⟷ *?D absolutely_integrable_on S* ∧ *integral S ?D* = *b*
proof (*rule conj_cong*)
have *neg*: *negligible* $\{x \in C - S.\ ?D\ x \neq 0\}$ *negligible* $\{x \in S - C.\ ?D\ x \neq 0\}$
using *CNS* **by** (*blast intro*: *negligible_subset* [*OF* ‹*negligible N*›])+
then show (*?D absolutely_integrable_on C*) = (*?D absolutely_integrable_on S*)
by (*rule absolutely_integrable_spike_set_eq*)
show (*integral C ?D* = *b*) ⟷ (*integral S ?D* = *b*)
using *integral_spike_set* [*OF neg*] **by** *simp*
qed
moreover
have *f absolutely_integrable_on* $(g$ ‘ $C) \wedge$ *integral* $(g$ ‘ $C)\ f = b$

```
    ⟷ f absolutely_integrable_on (g ‘ S) ∧ integral (g ‘ S) f = b
  proof (rule conj_cong)
    have g differentiable_on N
    by (metis CNS der_g differentiable_def differentiable_on_def differentiable_on_subset
sup.cobounded2)
    with ‹negligible N›
    have neg_gN: negligible (g ‘ N)
      by (blast intro: negligible_differentiable_image_negligible)
    have neg: negligible {x ∈ g ‘ C − g ‘ S. f x ≠ 0}
              negligible {x ∈ g ‘ S − g ‘ C. f x ≠ 0}
      using CNS by (blast intro: negligible_subset [OF neg_gN])+
    then show (f absolutely_integrable_on g ‘ C) = (f absolutely_integrable_on g
‘ S)
      by (rule absolutely_integrable_spike_set_eq)
    show (integral (g ‘ C) f = b) ⟷ (integral (g ‘ S) f = b)
      using integral_spike_set [OF neg] by simp
  qed
  ultimately show ?thesis
    by simp
qed


corollary absolutely_integrable_change_of_variables:
  fixes f :: real^'m::{finite,wellorder} ⇒ real^'n and g :: real^'m::_ ⇒ real^'m::_
  assumes S ∈ sets lebesgue
    and ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and inj_on g S
  shows f absolutely_integrable_on (g ‘ S)
     ⟷ (λx. |det (matrix (g' x))| *_R f(g x)) absolutely_integrable_on S
  using assms has_absolute_integral_change_of_variables by blast

corollary integral_change_of_variables:
  fixes f :: real^'m::{finite,wellorder} ⇒ real^'n and g :: real^'m::_ ⇒ real^'m::_
  assumes S: S ∈ sets lebesgue
    and der_g: ⋀x. x ∈ S ⟹ (g has_derivative g' x) (at x within S)
    and inj: inj_on g S
    and disj: (f absolutely_integrable_on (g ‘ S) ∨
        (λx. |det (matrix (g' x))| *_R f(g x)) absolutely_integrable_on S)
  shows integral (g ‘ S) f = integral S (λx. |det (matrix (g' x))| *_R f(g x))
  using has_absolute_integral_change_of_variables [OF S der_g inj] disj
  by blast

lemma has_absolute_integral_change_of_variables_1:
  fixes f :: real ⇒ real^'n::{finite,wellorder} and g :: real ⇒ real
  assumes S: S ∈ sets lebesgue
    and der_g: ⋀x. x ∈ S ⟹ (g has_vector_derivative g' x) (at x within S)
    and inj: inj_on g S
  shows (λx. |g' x| *_R f(g x)) absolutely_integrable_on S ∧
        integral S (λx. |g' x| *_R f(g x)) = b
```

```
    ⟷ f absolutely_integrable_on (g ‘ S) ∧ integral (g ‘ S) f = b
proof −
  let ?lift = vec :: real ⇒ real^1
  let ?drop = (λx::real^1. x $ 1)
  have S': ?lift ‘ S ∈ sets lebesgue
   by (auto intro: differentiable_image_in_sets_lebesgue [OF S] differentiable_vec)
  have ((λx. vec (g (x $ 1))) has_derivative (∗R) (g' z)) (at (vec z) within ?lift ‘
S)
    if z ∈ S for z
    using der_g [OF that]
    by (simp add: has_vector_derivative_def has_derivative_vector_1)
  then have der': ⋀x. x ∈ ?lift ‘ S ⟹
       (?lift ∘ g ∘ ?drop has_derivative (∗R) (g' (?drop x))) (at x within ?lift ‘ S)
    by (auto simp: o_def)
  have inj': inj_on (vec ∘ g ∘ ?drop) (vec ‘ S)
    using inj by (simp add: inj_on_def)
  let ?fg = λx. |g' x| ∗R f(g x)
  have ((λx. ?fg x $ i) absolutely_integrable_on S ∧ ((λx. ?fg x $ i) has_integral
b $ i) S
    ⟷ (λx. f x $ i) absolutely_integrable_on g ‘ S ∧ ((λx. f x $ i) has_integral
b $ i) (g ‘ S)) for i
     using has_absolute_integral_change_of_variables [OF S' der' inj', of λx.
?lift(f (?drop x) $ i) ?lift (b$i)]
   unfolding integrable_on_1_iff integral_on_1_eq absolutely_integrable_on_1_iff
absolutely_integrable_drop absolutely_integrable_on_def
    by (auto simp: image_comp o_def integral_vec1_eq has_integral_iff)
  then have ?fg absolutely_integrable_on S ∧ (?fg has_integral b) S
        ⟷ f absolutely_integrable_on (g ‘ S) ∧ (f has_integral b) (g ‘ S)
    unfolding has_integral_componentwise_iff [where y=b]
          absolutely_integrable_componentwise_iff [where f=f]
          absolutely_integrable_componentwise_iff [where f = ?fg]
    by (force simp: Basis_vec_def cart_eq_inner_axis)
  then show ?thesis
    using absolutely_integrable_on_def by blast
qed


corollary absolutely_integrable_change_of_variables_1:
  fixes f :: real ⇒ real^'n::{finite,wellorder} and g :: real ⇒ real
  assumes S: S ∈ sets lebesgue
    and der_g: ⋀x. x ∈ S ⟹ (g has_vector_derivative g' x) (at x within S)
    and inj: inj_on g S
  shows (f absolutely_integrable_on g ‘ S ⟷
           (λx. |g' x| ∗R f(g x)) absolutely_integrable_on S)
  using has_absolute_integral_change_of_variables_1 [OF assms] by auto
```

when $n = 1$

```
lemma has_absolute_integral_change_of_variables_1':
  fixes f :: real ⇒ real and g :: real ⇒ real
```

assumes S: $S \in sets\ lebesgue$
and der_g: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_field\_derivative\ g'\ x)\ (at\ x\ within\ S)$
and inj: $inj\_on\ g\ S$
shows $(\lambda x.\ |g'\ x| *_R f(g\ x))\ absolutely\_integrable\_on\ S \wedge$
$integral\ S\ (\lambda x.\ |g'\ x| *_R f(g\ x)) = b$
$\longleftrightarrow f\ absolutely\_integrable\_on\ (g\ `\ S) \wedge integral\ (g\ `\ S)\ f = b$
proof −
have $(\lambda x.\ |g'\ x| *_R vec\ (f(g\ x)) :: real\ \hat{}\ 1)\ absolutely\_integrable\_on\ S \wedge$
$integral\ S\ (\lambda x.\ |g'\ x| *_R vec\ (f(g\ x))) = (vec\ b :: real\ \hat{}\ 1)$
$\longleftrightarrow (\lambda x.\ vec\ (f\ x) :: real\ \hat{}\ 1)\ absolutely\_integrable\_on\ (g\ `\ S) \wedge$
$integral\ (g\ `\ S)\ (\lambda x.\ vec\ (f\ x)) = (vec\ b :: real\ \hat{}\ 1)$
using *assms* **unfolding** *has_real_derivative_iff_has_vector_derivative*
by (*intro has_absolute_integral_change_of_variables_1 assms*) *auto*
thus *?thesis*
by (*simp add*: *absolutely_integrable_on_1_iff integral_on_1_eq*)
qed

9.32.6 Change of variables for integrals: special case of linear function

lemma *has_absolute_integral_change_of_variables_linear*:
fixes $f :: real\hat{}'m::\{finite,wellorder\} \Rightarrow real\hat{}'n$ **and** $g :: real\hat{}'m::\_ \Rightarrow real\hat{}'m::\_$
assumes *linear g*
shows $(\lambda x.\ |det\ (matrix\ g)| *_R f(g\ x))\ absolutely\_integrable\_on\ S \wedge$
$integral\ S\ (\lambda x.\ |det\ (matrix\ g)| *_R f(g\ x)) = b$
$\longleftrightarrow f\ absolutely\_integrable\_on\ (g\ `\ S) \wedge integral\ (g\ `\ S)\ f = b$
proof (*cases* $det(matrix\ g) = 0$)
case *True*
then have $negligible(g\ `\ S)$
using *assms det_nz_iff_inj negligible_linear_singular_image* **by** *blast*
with *True* **show** *?thesis*
by (*auto simp*: *absolutely_integrable_on_def integrable_negligible integral_negligible*)
next
case *False*
then obtain h **where** h: $\bigwedge x.\ x \in S \Longrightarrow h\ (g\ x) = x\ linear\ h$
using *assms det_nz_iff_inj linear_injective_isomorphism* **by** *metis*
show *?thesis*
proof (*rule has_absolute_integral_change_of_variables_invertible*)
show $(g\ has\_derivative\ g)\ (at\ x\ within\ S)$ **for** x
by (*simp add*: *assms linear_imp_has_derivative*)
show $continuous\_on\ (g\ `\ S)\ h$
using *continuous_on_eq_continuous_within has_derivative_continuous linear_imp_has_derivative h* **by** *blast*
qed (*use h* **in** *auto*)
qed

lemma *absolutely_integrable_change_of_variables_linear*:
fixes $f :: real\hat{}'m::\{finite,wellorder\} \Rightarrow real\hat{}'n$ **and** $g :: real\hat{}'m::\_ \Rightarrow real\hat{}'m::\_$
assumes *linear g*

shows $(\lambda x.\ |det\ (matrix\ g)|\ *_R\ f(g\ x))\ absolutely\_integrable\_on\ S$
$\longleftrightarrow f\ absolutely\_integrable\_on\ (g\ `\ S)$
using *assms has_absolute_integral_change_of_variables_linear* **by** *blast*

lemma *absolutely_integrable_on_linear_image*:
fixes $f :: real\hat{}'m::\{finite,wellorder\} \Rightarrow real\hat{}'n$ **and** $g :: real\hat{}'m::\_ \Rightarrow real\hat{}'m::\_$
assumes *linear g*
shows $f\ absolutely\_integrable\_on\ (g\ `\ S)$
$\longleftrightarrow (f \circ g)\ absolutely\_integrable\_on\ S \vee det(matrix\ g) = 0$
unfolding *assms absolutely_integrable_change_of_variables_linear* [*OF assms*, *symmetric*] *absolutely_integrable_on_scaleR_iff*
by (*auto simp*: *set_integrable_def*)

lemma *integral_change_of_variables_linear*:
fixes $f :: real\hat{}'m::\{finite,wellorder\} \Rightarrow real\hat{}'n$ **and** $g :: real\hat{}'m::\_ \Rightarrow real\hat{}'m::\_$
assumes *linear g*
and $f\ absolutely\_integrable\_on\ (g\ `\ S) \vee (f \circ g)\ absolutely\_integrable\_on\ S$
shows $integral\ (g\ `\ S)\ f = |det\ (matrix\ g)|\ *_R\ integral\ S\ (f \circ g)$
proof −
have $((\lambda x.\ |det\ (matrix\ g)|\ *_R\ f\ (g\ x))\ absolutely\_integrable\_on\ S) \vee (f\ absolutely\_integrable\_on\ g\ `\ S)$
using *absolutely_integrable_on_linear_image assms* **by** *blast*
moreover
have *?thesis* **if** $((\lambda x.\ |det\ (matrix\ g)|\ *_R\ f\ (g\ x))\ absolutely\_integrable\_on\ S)\ (f\ absolutely\_integrable\_on\ g\ `\ S)$
using *has_absolute_integral_change_of_variables_linear* [*OF* ‹*linear g*›] *that*
by (*auto simp*: *o_def*)
ultimately show *?thesis*
using *absolutely_integrable_change_of_variables_linear* [*OF* ‹*linear g*›]
by *blast*
qed

9.32.7 Change of variable for measure

lemma *has_measure_differentiable_image*:
fixes $f :: real\hat{}'n::\{finite,wellorder\} \Rightarrow real\hat{}'n::\_$
assumes $S \in sets\ lebesgue$
and $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
and $inj\_on\ f\ S$
shows $f\ `\ S \in lmeasurable \wedge measure\ lebesgue\ (f\ `\ S) = m$
$\longleftrightarrow ((\lambda x.\ |det\ (matrix\ (f'\ x))|)\ has\_integral\ m)\ S$
using *has_absolute_integral_change_of_variables* [*OF assms*, *of* $\lambda x.\ (1::real\hat{}1)$ *vec m*]
unfolding *absolutely_integrable_on_1_iff integral_on_1_eq integrable_on_1_iff absolutely_integrable_on_def*
by (*auto simp*: *has_integral_iff lmeasurable_iff_integrable_on lmeasure_integral*)

lemma *measurable_differentiable_image_eq*:
fixes $f :: real\hat{}'n::\{finite,wellorder\} \Rightarrow real\hat{}'n::\_$

assumes $S \in sets\ lebesgue$
and $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
and $inj\_on\ f\ S$
shows $f\ `\ S \in lmeasurable \longleftrightarrow (\lambda x.\ |det\ (matrix\ (f'\ x))|)\ integrable\_on\ S$
using *has_measure_differentiable_image* [*OF assms*]
by *blast*

lemma *measurable_differentiable_image_alt*:
fixes $f :: real\hat{}'n::\{finite,wellorder\} \Rightarrow real\hat{}'n::\_$
assumes $S \in sets\ lebesgue$
and $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
and $inj\_on\ f\ S$
shows $f\ `\ S \in lmeasurable \longleftrightarrow (\lambda x.\ |det\ (matrix\ (f'\ x))|)\ absolutely\_integrable\_on\ S$
using *measurable_differentiable_image_eq* [*OF assms*]
by (*simp only*: *absolutely_integrable_on_iff_nonneg*)

lemma *measure_differentiable_image_eq*:
fixes $f :: real\hat{}'n::\{finite,wellorder\} \Rightarrow real\hat{}'n::\_$
assumes *S*: $S \in sets\ lebesgue$
and *der_f*: $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
and *inj*: $inj\_on\ f\ S$
and *intS*: $(\lambda x.\ |det\ (matrix\ (f'\ x))|)\ integrable\_on\ S$
shows $measure\ lebesgue\ (f\ `\ S) = integral\ S\ (\lambda x.\ |det\ (matrix\ (f'\ x))|)$
using *measurable_differentiable_image_eq* [*OF S der_f inj*]
assms has_measure_differentiable_image **by** *blast*

end

9.33 Lipschitz Continuity

theory *Lipschitz*
imports
Derivative Abstract_Metric_Spaces
begin

definition *lipschitz_on*
where $lipschitz\_on\ C\ U\ f \longleftrightarrow (0 \le C \wedge (\forall x \in U.\ \forall y{\in}U.\ dist\ (f\ x)\ (f\ y) \le C * dist\ x\ y))$

open_bundle *lipschitz_syntax*
begin
notation
$lipschitz\_on\ (\langle(\langle open\_block\ notation{=}\langle postfix\ lipschitz\_on\rangle\rangle\_-lipschitz'\_on)\rangle\ [1000])$
end

lemma *lipschitz_onI*: $L{-}lipschitz\_on\ X\ f$
if $\bigwedge x\ y.\ x \in X \Longrightarrow y \in X \Longrightarrow dist\ (f\ x)\ (f\ y) \le L * dist\ x\ y\ 0 \le L$

using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_onD*:
dist (*f x*) (*f y*) ≤ *L* ∗ *dist x y*
if *L*−*lipschitz_on X f x* ∈ *X y* ∈ *X*
using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_nonneg*:
0 ≤ *L* **if** *L*−*lipschitz_on X f*
using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_normD*:
norm (*f x* − *f y*) ≤ *L* ∗ *norm* (*x* − *y*)
if *lipschitz_on L X f x* ∈ *X y* ∈ *X*
using *lipschitz_onD*[*OF that*]
by (*simp add*: *dist_norm*)

lemma *lipschitz_on_mono*: *L*−*lipschitz_on D f* **if** *M*−*lipschitz_on E f D* ⊆ *E M*
≤ *L*
using *that*
by (*force simp*: *lipschitz_on_def intro*: *order_trans*[*OF _ mult_right_mono*])

lemmas *lipschitz_on_subset* = *lipschitz_on_mono*[*OF _ _ order_refl*]
and *lipschitz_on_le* = *lipschitz_on_mono*[*OF _ order_refl*]

lemma *lipschitz_on_leI*:
L−*lipschitz_on X f*
if ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ *x* ≤ *y* ⟹ *dist* (*f x*) (*f y*) ≤ *L* ∗ *dist x y*
0 ≤ *L*
for *f*::*'a*::{*linorder_topology*, *ordered_real_vector*, *metric_space*} ⇒ *'b*::*metric_space*
proof (*rule lipschitz_onI*)
fix *x y* **assume** *xy*: *x* ∈ *X y* ∈ *X*
consider *y* ≤ *x* | *x* ≤ *y*
by (*rule le_cases*)
then show *dist* (*f x*) (*f y*) ≤ *L* ∗ *dist x y*
proof *cases*
case *1*
then have *dist* (*f y*) (*f x*) ≤ *L* ∗ *dist y x*
by (*auto intro*!: *that xy*)
then show *?thesis*
by (*simp add*: *dist_commute*)
qed (*auto intro*!: *that xy*)
qed *fact*

lemma *lipschitz_on_concat*:
fixes *a b c*::*real*
assumes *f*: *L*−*lipschitz_on* {*a* .. *b*} *f*
assumes *g*: *L*−*lipschitz_on* {*b* .. *c*} *g*
assumes *fg*: *f b* = *g b*

```
  shows lipschitz_on L {a .. c} (λx. if x ≤ b then f x else g x)
    (is lipschitz_on _ _ ?f)
proof (rule lipschitz_on_leI)
  fix x y
  assume x: x ∈ {a..c} and y: y ∈ {a..c} and xy: x ≤ y
  consider x ≤ b ∧ b < y | x ≥ b ∨ y ≤ b by arith
  then show dist (?f x) (?f y) ≤ L * dist x y
  proof cases
    case 1
    have dist (f x) (g y) ≤ dist (f x) (f b) + dist (g b) (g y)
      unfolding fg by (rule dist_triangle)
    also have dist (f x) (f b) ≤ L * dist x b
      using 1 x
      by (auto intro!: lipschitz_onD[OF f])
    also have dist (g b) (g y) ≤ L * dist b y
      using 1 x y
      by (auto intro!: lipschitz_onD[OF g] lipschitz_onD[OF f])
    finally have dist (f x) (g y) ≤ L * dist x b + L * dist b y
      by simp
    also have ... = L * (dist x b + dist b y)
      by (simp add: algebra_simps)
    also have dist x b + dist b y = dist x y
      using 1 x y
      by (auto simp: dist_real_def abs_real_def)
    finally show ?thesis
      using 1 by simp
  next
    case 2
    with lipschitz_onD[OF f, of x y] lipschitz_onD[OF g, of x y] x y xy
    show ?thesis
      by (auto simp: fg)
  qed
qed (rule lipschitz_on_nonneg[OF f])

lemma lipschitz_on_concat_max:
  fixes a b c::real
  assumes f: L−lipschitz_on {a .. b} f
  assumes g: M−lipschitz_on {b .. c} g
  assumes fg: f b = g b
  shows (max L M)−lipschitz_on {a .. c} (λx. if x ≤ b then f x else g x)
proof −
  have lipschitz_on (max L M) {a .. b} f lipschitz_on (max L M) {b .. c} g
    by (auto intro!: lipschitz_on_mono[OF f order_refl] lipschitz_on_mono[OF g
order_refl])
  from lipschitz_on_concat[OF this fg] show ?thesis .
qed
```

Equivalence between "abstract" and "type class" Lipschitz notions, for all types in the metric space class

lemma *Lipschitz_map_euclidean* [*simp*]:
Lipschitz_continuous_map euclidean_metric euclidean_metric f
$\longleftrightarrow$ ($\exists$ *B*. *lipschitz_on B UNIV f*) (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*force simp add*: *Lipschitz_continuous_map_pos lipschitz_on_def less_le_not_le*)
show *?rhs* $\Longrightarrow$ *?lhs*
by (*auto simp*: *Lipschitz_continuous_map_def lipschitz_on_def*)
qed

Continuity

proposition *lipschitz_on_uniformly_continuous*:
assumes *L*−*lipschitz_on X f*
shows *uniformly_continuous_on X f*
unfolding *uniformly_continuous_on_def*
proof *safe*
fix *e*::*real*
assume *0* < *e*
from *assms* **have** *l*: (*L*+*1*)−*lipschitz_on X f*
by (*rule lipschitz_on_mono*) *auto*
show $\exists$ *d*>*0*. $\forall$ *x*∈*X*. $\forall$ *x'*∈*X*. *dist x' x* < *d* $\longrightarrow$ *dist* (*f x'*) (*f x*) < *e*
using *lipschitz_onD*[*OF l*] *lipschitz_on_nonneg*[*OF assms*] ‹*0* < *e*›
by (*force intro*!: *exI*[**where** *x*=*e*/(*L* + *1*)] *simp*: *field_simps*)
qed

proposition *lipschitz_on_continuous_on*:
continuous_on X f **if** *L*−*lipschitz_on X f*
by (*rule uniformly_continuous_imp_continuous*[*OF lipschitz_on_uniformly_continuous*[*OF that*]])

lemma *lipschitz_on_continuous_within*:
continuous (*at x within X*) *f* **if** *L*−*lipschitz_on X f x* ∈ *X*
using *lipschitz_on_continuous_on*[*OF that*(*1*)] *that*(*2*)
by (*auto simp*: *continuous_on_eq_continuous_within*)

Differentiable functions

proposition *bounded_derivative_imp_lipschitz*:
assumes $\bigwedge$*x*. *x* ∈ *X* $\Longrightarrow$ (*f has_derivative f' x*) (*at x within X*)
assumes *convex*: *convex X*
assumes $\bigwedge$*x*. *x* ∈ *X* $\Longrightarrow$ *onorm* (*f' x*) ≤ *C* *0* ≤ *C*
shows *C*−*lipschitz_on X f*
proof (*rule lipschitz_onI*)
show $\bigwedge$*x y*. *x* ∈ *X* $\Longrightarrow$ *y* ∈ *X* $\Longrightarrow$ *dist* (*f x*) (*f y*) ≤ *C* ∗ *dist x y*
by (*auto intro*!: *assms differentiable_bound*[*unfolded dist_norm*[*symmetric*], *OF convex*])
qed *fact*

Structural introduction rules

named_theorems *lipschitz_intros structural introduction rules for Lipschitz controls*

lemma *lipschitz_on_compose* [*lipschitz_intros*]:
(*D* ∗ *C*)−*lipschitz_on U* (*g o f*)
if *f*: *C*−*lipschitz_on U f* **and** *g*: *D*−*lipschitz_on* (*f*‘*U*) *g*
proof (*rule lipschitz_onI*)
show *D*∗ *C* ≥ *0* **using** *lipschitz_on_nonneg*[*OF f*] *lipschitz_on_nonneg*[*OF g*] **by** *auto*
fix *x y* **assume** *H*: *x* ∈ *U y* ∈ *U*
have *dist* (*g* (*f x*)) (*g* (*f y*)) ≤ *D* ∗ *dist* (*f x*) (*f y*)
apply (*rule lipschitz_onD*[*OF g*]) **using** *H* **by** *auto*
also have ... ≤ *D* ∗ *C* ∗ *dist x y*
using *mult_left_mono*[*OF lipschitz_onD*(*1*)[*OF f H*] *lipschitz_on_nonneg*[*OF g*]] **by** *auto*
finally show *dist* ((*g* ∘ *f*) *x*) ((*g* ∘ *f*) *y*) ≤ *D* ∗ *C*∗ *dist x y*
unfolding *comp_def* **by** (*auto simp add*: *mult.commute*)
qed

lemma *lipschitz_on_compose2*:
(*D* ∗ *C*)−*lipschitz_on U* (λ*x*. *g* (*f x*))
if *C*−*lipschitz_on U f D*−*lipschitz_on* (*f*‘*U*) *g*
using *lipschitz_on_compose*[*OF that*] **by** (*simp add*: *o_def*)

lemma *lipschitz_on_cong*[*cong*]:
C−*lipschitz_on U g* ⟷ *D*−*lipschitz_on V f*
if *C* = *D U* = *V* ⋀*x*. *x* ∈ *V* ⟹ *g x* = *f x*
using *that* **by** (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_transform*:
C−*lipschitz_on U g*
if *C*−*lipschitz_on U f*
⋀*x*. *x* ∈ *U* ⟹ *g x* = *f x*
using *that*
by *simp*

lemma *lipschitz_on_empty_iff*[*simp*]: *C*−*lipschitz_on* {} *f* ⟷ *C* ≥ *0*
by (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_insert_iff*[*simp*]:
C−*lipschitz_on* (*insert y X*) *f* ⟷
C−*lipschitz_on X f* ∧ (∀ *x* ∈ *X*. *dist* (*f x*) (*f y*) ≤ *C* ∗ *dist x y*)
by (*auto simp*: *lipschitz_on_def dist_commute*)

lemma *lipschitz_on_singleton* [*lipschitz_intros*]: *C* ≥ *0* ⟹ *C*−*lipschitz_on* {*x*} *f*
and *lipschitz_on_empty* [*lipschitz_intros*]: *C* ≥ *0* ⟹ *C*−*lipschitz_on* {} *f*
by *simp_all*

lemma *lipschitz_on_id* [*lipschitz_intros*]: *1−lipschitz_on U* (*λx. x*)
by (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_constant* [*lipschitz_intros*]: *0−lipschitz_on U* (*λx. c*)
by (*auto simp*: *lipschitz_on_def*)

lemma *lipschitz_on_add* [*lipschitz_intros*]:
fixes *f*::*'a*::*metric_space* ⇒*'b*::*real_normed_vector*
assumes *C−lipschitz_on U f*
D−lipschitz_on U g
shows (*C*+*D*)*−lipschitz_on U* (*λx. f x* + *g x*)
proof (*rule lipschitz_onI*)
show *C* + *D* ≥ *0*
using *lipschitz_on_nonneg*[*OF assms*(*1*)] *lipschitz_on_nonneg*[*OF assms*(*2*)]
by *auto*
fix *x y* **assume** *H*: *x* ∈ *U y* ∈ *U*
have *dist* (*f x* + *g x*) (*f y* + *g y*) ≤ *dist* (*f x*) (*f y*) + *dist* (*g x*) (*g y*)
by (*simp add*: *dist_triangle_add*)
also have ... ≤ *C* ∗ *dist x y* + *D* ∗ *dist x y*
using *lipschitz_onD*(*1*)[*OF assms*(*1*) *H*] *lipschitz_onD*(*1*)[*OF assms*(*2*) *H*] **by** *auto*
finally show *dist* (*f x* + *g x*) (*f y* + *g y*) ≤ (*C*+*D*) ∗ *dist x y* **by** (*auto simp add*: *algebra_simps*)
qed

lemma *lipschitz_on_cmult* [*lipschitz_intros*]:
fixes *f*::*'a*::*metric_space* ⇒ *'b*::*real_normed_vector*
assumes *C−lipschitz_on U f*
shows (*abs*(*a*) ∗ *C*)*−lipschitz_on U* (*λx. a* ∗*R f x*)
proof (*rule lipschitz_onI*)
show *abs*(*a*) ∗ *C* ≥ *0* **using** *lipschitz_on_nonneg*[*OF assms*(*1*)] **by** *auto*
fix *x y* **assume** *H*: *x* ∈ *U y* ∈ *U*
have *dist* (*a* ∗*R f x*) (*a* ∗*R f y*) = *abs*(*a*) ∗ *dist* (*f x*) (*f y*)
by (*metis dist_norm norm_scaleR real_vector.scale_right_diff_distrib*)
also have ... ≤ *abs*(*a*) ∗ *C* ∗ *dist x y*
using *lipschitz_onD*(*1*)[*OF assms*(*1*) *H*] **by** (*simp add*: *Groups.mult_ac*(*1*) *mult_left_mono*)
finally show *dist* (*a* ∗*R f x*) (*a* ∗*R f y*) ≤ |*a*| ∗ *C* ∗ *dist x y* **by** *auto*
qed

lemma *lipschitz_on_cmult_real* [*lipschitz_intros*]:
fixes *f*::*'a*::*metric_space* ⇒ *real*
assumes *C−lipschitz_on U f*
shows (*abs*(*a*) ∗ *C*)*−lipschitz_on U* (*λx. a* ∗ *f x*)
using *lipschitz_on_cmult*[*OF assms*] **by** *auto*

lemma *lipschitz_on_cmult_nonneg* [*lipschitz_intros*]:
fixes *f*::*'a*::*metric_space* ⇒ *'b*::*real_normed_vector*

assumes $C-lipschitz\_on\ U\ f$
$a \geq 0$
shows $(a * C)-lipschitz\_on\ U\ (\lambda x.\ a *_R f\ x)$
using *lipschitz_on_cmult*[*OF assms*(*1*), *of a*] *assms*(*2*) **by** *auto*

lemma *lipschitz_on_cmult_real_nonneg* [*lipschitz_intros*]:
fixes $f::'a::metric\_space \Rightarrow real$
assumes $C-lipschitz\_on\ U\ f$
$a \geq 0$
shows $(a * C)-lipschitz\_on\ U\ (\lambda x.\ a * f\ x)$
using *lipschitz_on_cmult_nonneg*[*OF assms*] **by** *auto*

lemma *lipschitz_on_cmult_upper* [*lipschitz_intros*]:
fixes $f::'a::metric\_space \Rightarrow 'b::real\_normed\_vector$
assumes $C-lipschitz\_on\ U\ f$
$abs(a) \leq D$
shows $(D * C)-lipschitz\_on\ U\ (\lambda x.\ a *_R f\ x)$
apply (*rule lipschitz_on_mono*[*OF lipschitz_on_cmult*[*OF assms*(*1*), *of a*], *of* _ $D * C$])
using *assms*(*2*) *lipschitz_on_nonneg*[*OF assms*(*1*)] *mult_right_mono* **by** *auto*

lemma *lipschitz_on_cmult_real_upper* [*lipschitz_intros*]:
fixes $f::'a::metric\_space \Rightarrow real$
assumes $C-lipschitz\_on\ U\ f$
$abs(a) \leq D$
shows $(D * C)-lipschitz\_on\ U\ (\lambda x.\ a * f\ x)$
using *lipschitz_on_cmult_upper*[*OF assms*] **by** *auto*

lemma *lipschitz_on_minus*[*lipschitz_intros*]:
fixes $f::'a::metric\_space \Rightarrow 'b::real\_normed\_vector$
assumes $C-lipschitz\_on\ U\ f$
shows $C-lipschitz\_on\ U\ (\lambda x.\ -\ f\ x)$
by (*metis* (*mono_tags*, *lifting*) *assms dist_minus lipschitz_on_def*)

lemma *lipschitz_on_minus_iff*[*simp*]:
$L-lipschitz\_on\ X\ (\lambda x.\ -\ f\ x) \longleftrightarrow L-lipschitz\_on\ X\ f$
$L-lipschitz\_on\ X\ (-\ f) \longleftrightarrow L-lipschitz\_on\ X\ f$
for $f::'a::metric\_space \Rightarrow 'b::real\_normed\_vector$
using *lipschitz_on_minus*[*of L X f*] *lipschitz_on_minus*[*of L X* $-f$]
by *auto*

lemma *lipschitz_on_diff*[*lipschitz_intros*]:
fixes $f::'a::metric\_space \Rightarrow 'b::real\_normed\_vector$
assumes $C-lipschitz\_on\ U\ f\ D-lipschitz\_on\ U\ g$
shows $(C + D)-lipschitz\_on\ U\ (\lambda x.\ f\ x - g\ x)$
using *lipschitz_on_add*[*OF assms*(*1*) *lipschitz_on_minus*[*OF assms*(*2*)]] **by** *auto*

lemma *lipschitz_on_closure* [*lipschitz_intros*]:

assumes $C-lipschitz\_on\ U\ f\ continuous\_on\ (closure\ U)\ f$
shows $C-lipschitz\_on\ (closure\ U)\ f$
proof (*rule lipschitz_onI*)
show $C \geq 0$ **using** *lipschitz_on_nonneg*[*OF assms*(*1*)] **by** *simp*
fix $x\ y$ **assume** $x \in closure\ U\ y \in closure\ U$
obtain $u\ v{::}nat \Rightarrow 'a$ **where** $*$: $\bigwedge n.\ u\ n \in U\ u \longrightarrow x$
$\bigwedge n.\ v\ n \in U\ v \longrightarrow y$
using ‹$x \in closure\ U$› ‹$y \in closure\ U$› **unfolding** *closure_sequential* **by** *blast*
have *a*: $(\lambda n.\ f\ (u\ n)) \longrightarrow f\ x$
using $*(1)$ $*(2)$ ‹$x \in closure\ U$› ‹*continuous_on* $(closure\ U)\ f$›
unfolding *comp_def continuous_on_closure_sequentially*[*of U f*] **by** *auto*
have *b*: $(\lambda n.\ f\ (v\ n)) \longrightarrow f\ y$
using $*(3)$ $*(4)$ ‹$y \in closure\ U$› ‹*continuous_on* $(closure\ U)\ f$›
unfolding *comp_def continuous_on_closure_sequentially*[*of U f*] **by** *auto*
have *l*: $(\lambda n.\ C * dist\ (u\ n)\ (v\ n) - dist\ (f\ (u\ n))\ (f\ (v\ n))) \longrightarrow C * dist\ x\ y - dist\ (f\ x)\ (f\ y)$
by (*intro tendsto_intros* $*$ *a b*)
have $C * dist\ (u\ n)\ (v\ n) - dist\ (f\ (u\ n))\ (f\ (v\ n)) \geq 0$ **for** n
using *lipschitz_onD*(*1*)[*OF assms*(*1*) ‹$u\ n \in U$› ‹$v\ n \in U$›] **by** *simp*
then have $C * dist\ x\ y - dist\ (f\ x)\ (f\ y) \geq 0$ **using** *LIMSEQ_le_const*[*OF l*, *of 0*] **by** *auto*
then show $dist\ (f\ x)\ (f\ y) \leq C * dist\ x\ y$ **by** *auto*
qed

lemma *lipschitz_on_Pair*[*lipschitz_intros*]:
assumes *f*: $L-lipschitz\_on\ A\ f$
assumes *g*: $M-lipschitz\_on\ A\ g$
shows $(sqrt\ (L^2 + M^2))-lipschitz\_on\ A\ (\lambda a.\ (f\ a,\ g\ a))$
proof (*rule lipschitz_onI*, *goal_cases*)
case (*1 x y*)
have $dist\ (f\ x,\ g\ x)\ (f\ y,\ g\ y) = sqrt\ ((dist\ (f\ x)\ (f\ y))^2 + (dist\ (g\ x)\ (g\ y))^2)$
by (*auto simp add*: *dist_Pair_Pair real_le_lsqrt*)
also have $\ldots \leq sqrt\ ((L * dist\ x\ y)^2 + (M * dist\ x\ y)^2)$
by (*auto intro*!: *real_sqrt_le_mono add_mono power_mono 1 lipschitz_onD f g*)
also have $\ldots \leq sqrt\ (L^2 + M^2) * dist\ x\ y$
by (*auto simp*: *power_mult_distrib ring_distribs*[*symmetric*] *real_sqrt_mult*)
finally show *?case* **.**
qed *simp*

lemma *lipschitz_extend_closure*:
fixes $f{::}('a{::}metric\_space) \Rightarrow ('b{::}complete\_space)$
assumes $C-lipschitz\_on\ U\ f$
shows $\exists g.\ C-lipschitz\_on\ (closure\ U)\ g \wedge (\forall x{\in}U.\ g\ x = f\ x)$
proof −
obtain *g* **where** *g*: $\bigwedge x.\ x \in U \Longrightarrow g\ x = f\ x\ uniformly\_continuous\_on\ (closure\ U)\ g$
using *uniformly_continuous_on_extension_on_closure*[*OF lipschitz_on_uniformly_continuous*[*OF assms*]] **by** *metis*

```
  have C−lipschitz_on (closure U) g
    apply (rule lipschitz_on_closure, rule lipschitz_on_transform[OF assms])
    using g uniformly_continuous_imp_continuous[OF g(2)] by auto
  then show ?thesis using g(1) by auto
qed

lemma (in bounded_linear) lipschitz_boundE:
  obtains B where B−lipschitz_on A f
proof −
  from nonneg_bounded
  obtain B where B: B ≥ 0 ⋀x. norm (f x) ≤ B ∗ norm x
    by (auto simp: ac_simps)
  have B−lipschitz_on A f
    by (auto intro!: lipschitz_onI B simp: dist_norm diff[symmetric])
  thus ?thesis ..
qed
```

9.33.1 Local Lipschitz continuity

Given a function defined on a real interval, it is Lipschitz-continuous if and only if it is locally so, as proved in the following lemmas. It is useful especially for piecewise-defined functions: if each piece is Lipschitz, then so is the whole function. The same goes for functions defined on geodesic spaces, or more generally on geodesic subsets in a metric space (for instance convex subsets in a real vector space), and this follows readily from the real case, but we will not prove it explicitly.

We give several variations around this statement. This is essentially a connectedness argument.

```
lemma locally_lipschitz_imp_lipschitz_aux:
  fixes f::real ⇒ ('a::metric_space)
  assumes a ≤ b
          continuous_on {a..b} f
          ⋀x. x ∈ {a..<b} ⟹ ∃ y ∈ {x<..b}. dist (f y) (f x) ≤ M ∗ (y−x)
  shows dist (f b) (f a) ≤ M ∗ (b−a)
proof −
  define A where A = {x ∈ {a..b}. dist (f x) (f a) ≤ M ∗ (x−a)}
  have ∗: A = (λx. M ∗ (x−a) − dist (f x) (f a))−‘{0..} ∩ {a..b}
    unfolding A_def by auto
  have a ∈ A unfolding A_def using ‹a ≤ b› by auto
  then have A ≠ {} by auto
  moreover have bdd_above A unfolding A_def by auto
  moreover have closed A unfolding ∗ by (rule closed_vimage_Int, auto intro!:
continuous_intros assms)
  ultimately have Sup A ∈ A by (rule closed_contains_Sup)
  have Sup A = b
  proof (rule ccontr)
    assume Sup A ≠ b
    define x where x = Sup A
```

have *I*: *dist* (*f x*) (*f a*) ≤ *M* ∗ (*x*−*a*) **using** ‹*Sup A* ∈ *A*› *x_def A_def* **by** *auto*
have *x* ∈ {*a*..<*b*} **unfolding** *x_def* **using** ‹*Sup A* ∈ *A*› ‹*Sup A* ≠ *b*› *A_def* **by** *auto*
then obtain *y* **where** *J*: *y* ∈ {*x*<..*b*} *dist* (*f y*) (*f x*) ≤ *M* ∗ (*y*−*x*) **using** *assms*(*3*) **by** *blast*
have *dist* (*f y*) (*f a*) ≤ *dist* (*f y*) (*f x*) + *dist* (*f x*) (*f a*) **by** (*rule dist_triangle*)
also have ... ≤ *M* ∗ (*y*−*x*) + *M* ∗ (*x*−*a*) **using** *I J*(*2*) **by** *auto*
finally have *dist* (*f y*) (*f a*) ≤ *M* ∗ (*y*−*a*) **by** (*auto simp add*: *algebra_simps*)
then have *y* ∈ *A* **unfolding** *A_def* **using** ‹*y* ∈ {*x*<..*b*}› ‹*x* ∈ {*a*..<*b*}› **by** *auto*
then have *y* ≤ *Sup A* **by** (*rule cSup_upper*, *auto simp*: *A_def*)
then show *False* **using** ‹*y* ∈ {*x*<..*b*}› *x_def* **by** *auto*
qed
then show *?thesis* **using** ‹*Sup A* ∈ *A*› *A_def* **by** *auto*
qed

lemma *locally_lipschitz_imp_lipschitz*:
fixes *f*::*real* ⇒ ('*a*::*metric_space*)
assumes *continuous_on* {*a*..*b*} *f*
⋀*x y*. *x* ∈ {*a*..<*b*} ⟹ *y* > *x* ⟹ ∃ *z* ∈ {*x*<..*y*}. *dist* (*f z*) (*f x*) ≤ *M* ∗ (*z*−*x*)
M ≥ *0*
shows *lipschitz_on M* {*a*..*b*} *f*
proof (*rule lipschitz_onI*[*OF* _ ‹*M* ≥ *0*›])
have ∗: *dist* (*f t*) (*f s*) ≤ *M* ∗ (*t*−*s*) **if** *s* ≤ *t s* ∈ {*a*..*b*} *t* ∈ {*a*..*b*} **for** *s t*
proof (*rule locally_lipschitz_imp_lipschitz_aux*, *simp add*: ‹*s* ≤ *t*›)
show *continuous_on* {*s*..*t*} *f* **using** *continuous_on_subset*[*OF assms*(*1*)] *that* **by** *auto*
fix *x* **assume** *x* ∈ {*s*..<*t*}
then have *x* ∈ {*a*..<*b*} **using** *that* **by** *auto*
show ∃ *z*∈{*x*<..*t*}. *dist* (*f z*) (*f x*) ≤ *M* ∗ (*z* − *x*)
using *assms*(*2*)[*OF* ‹*x* ∈ {*a*..<*b*}›, *of t*] ‹*x* ∈ {*s*..<*t*}› **by** *auto*
qed
fix *x y* **assume** *x* ∈ {*a*..*b*} *y* ∈ {*a*..*b*}
consider *x* ≤ *y* | *y* ≤ *x* **by** *linarith*
then show *dist* (*f x*) (*f y*) ≤ *M* ∗ *dist x y*
apply (*cases*)
using ∗[*OF* _ ‹*x* ∈ {*a*..*b*}› ‹*y* ∈ {*a*..*b*}›] ∗[*OF* _ ‹*y* ∈ {*a*..*b*}› ‹*x* ∈ {*a*..*b*}›]
by (*auto simp add*: *dist_commute dist_real_def*)
qed

We deduce that if a function is Lipschitz on finitely many closed sets on the real line, then it is Lipschitz on any interval contained in their union. The difficulty in the proof is to show that any point z in this interval (except the maximum) has a point arbitrarily close to it on its right which is contained in a common initial closed set. Otherwise, we show that there is a small interval $(z,\ T)$ which does not intersect any of the initial closed sets, a contradiction.

proposition *lipschitz_on_closed_Union*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *lipschitz_on* M $(U\ i)$ f
$\bigwedge i.\ i \in I \Longrightarrow$ *closed* $(U\ i)$
finite I
$M \geq 0$
$\{u..(v::real)\} \subseteq (\bigcup i{\in}I.\ U\ i)$
shows *lipschitz_on* M $\{u..v\}$ f
proof (*rule locally_lipschitz_imp_lipschitz*[*OF* _ _ ‹$M \geq 0$›])
have $*$: *continuous_on* $(U\ i)$ f **if** $i \in I$ **for** i
by (*rule lipschitz_on_continuous_on*[*OF assms*(*1*)[*OF* ‹$i{\in}$ I›]])
have *continuous_on* $(\bigcup i{\in}I.\ U\ i)$ f
apply (*rule continuous_on_closed_Union*) **using** ‹*finite* I› $*$ *assms*(*2*) **by** *auto*
then show *continuous_on* $\{u..v\}$ f
using ‹$\{u..(v::real)\} \subseteq (\bigcup i{\in}I.\ U\ i)$› *continuous_on_subset* **by** *auto*

fix z Z **assume** z: $z \in \{u..<v\}$ $z < Z$
then have $u \leq v$ **by** *auto*
define T **where** $T = min\ Z\ v$
then have T: $T > z$ $T \leq v$ $T \geq u$ $T \leq Z$ **using** z **by** *auto*
define A **where** $A = (\bigcup i{\in}\ I \cap \{i.\ U\ i \cap \{z<..T\} \neq \{\}\}.\ U\ i \cap \{z..T\})$
have a: *closed* A
unfolding A_def **apply** (*rule closed_UN*) **using** ‹*finite* I› ‹$\bigwedge i.\ i \in I \Longrightarrow$ *closed* $(U\ i)$› **by** *auto*
have b: *bdd_below* A **unfolding** A_def **using** ‹*finite* I› **by** *auto*
have $\exists\, i \in I.\ T \in U\ i$ **using** ‹$\{u..v\} \subseteq (\bigcup i{\in}I.\ U\ i)$› T **by** *auto*
then have c: $T \in A$ **unfolding** A_def **using** T **by** (*auto*, *fastforce*)
have *Inf* $A \geq z$
apply (*rule cInf_greatest*, *auto*) **using** c **unfolding** A_def **by** *auto*
moreover have *Inf* $A \leq z$
proof (*rule ccontr*)
assume $\neg$(*Inf* $A \leq z$)
then obtain w **where** w: $w > z$ $w <$ *Inf* A **by** (*meson dense not_le_imp_less*)
have *Inf* $A \leq T$ **using** a b c **by** (*simp add*: *cInf_lower*)
then have $w \leq T$ **using** w **by** *auto*
then have $w \in \{u..v\}$ **using** w ‹$z \in \{u..<v\}$› T **by** *auto*
then obtain j **where** j: $j \in I$ $w \in U\ j$ **using** ‹$\{u..v\} \subseteq (\bigcup i{\in}I.\ U\ i)$› **by** *fastforce*
then have $w \in U\ j \cap \{z..T\}$ $U\ j \cap \{z<..T\} \neq \{\}$ **using** j T w ‹$w \leq T$› **by** *auto*
then have $w \in A$ **unfolding** A_def **using** ‹$j \in I$› **by** *auto*
then have *Inf* $A \leq w$ **using** a b **by** (*simp add*: *cInf_lower*)
then show *False* **using** w **by** *auto*
qed
ultimately have *Inf* $A = z$ **by** *simp*
moreover have *Inf* $A \in A$
apply (*rule closed_contains_Inf*) **using** a b c **by** *auto*
ultimately have $z \in A$ **by** *simp*
then obtain i **where** i: $i \in I$ $U\ i \cap \{z<..T\} \neq \{\}$ $z \in U\ i$ **unfolding** A_def **by** *auto*

then obtain t **where** $t \in U\ i \cap \{z<..T\}$ **by** *blast*
then have *dist* $(f\ t)\ (f\ z) \leq M * (t - z)$
using *lipschitz_onD*(1)[*OF assms*(1)[*of i*], *of t z*] *i dist_real_def* **by** *auto*
then show $\exists\, t{\in}\{z<..Z\}.$ *dist* $(f\ t)\ (f\ z) \leq M * (t - z)$ **using** ‹$T \leq Z$› ‹$t \in U\ i \cap \{z<..T\}$› **by** *auto*
qed

9.33.2 Local Lipschitz continuity (uniform for a family of functions)

definition *local_lipschitz*::
$'a$::*metric_space set* ⇒ $'b$::*metric_space set* ⇒ ($'a$ ⇒ $'b$ ⇒ $'c$::*metric_space*) ⇒ *bool*
where
local_lipschitz $T\ X\ f \equiv \forall\, x \in X.\ \forall\, t \in T.$
$\exists\, u{>}0.\ \exists\, L.\ \forall\, t \in$ *cball* $t\ u \cap T.$ $L-$*lipschitz_on* (*cball* $x\ u \cap X$) $(f\ t)$

lemma *local_lipschitzI*:
assumes $\bigwedge t\ x.\ t \in T \Longrightarrow x \in X \Longrightarrow \exists\, u{>}0.\ \exists\, L.\ \forall\, t \in$ *cball* $t\ u \cap T.$ $L-$*lipschitz_on* (*cball* $x\ u \cap X$) $(f\ t)$
shows *local_lipschitz* $T\ X\ f$
using *assms*
unfolding *local_lipschitz_def*
by *auto*

lemma *local_lipschitzE*:
assumes *local_lipschitz*: *local_lipschitz* $T\ X\ f$
assumes $t \in T\ x \in X$
obtains $u\ L$ **where** $u > 0\ \bigwedge s.\ s \in$ *cball* $t\ u \cap T \Longrightarrow L-$*lipschitz_on* (*cball* $x\ u \cap X$) $(f\ s)$
using *assms local_lipschitz_def*
by *metis*

lemma *local_lipschitz_continuous_on*:
assumes *local_lipschitz*: *local_lipschitz* $T\ X\ f$
assumes $t \in T$
shows *continuous_on* $X\ (f\ t)$
unfolding *continuous_on_def*
proof *safe*
fix x **assume** $x \in X$
from *local_lipschitzE*[*OF local_lipschitz* ‹$t \in T$› ‹$x \in X$›] **obtain** $u\ L$
where $0 < u$
and L: $\bigwedge s.\ s \in$ *cball* $t\ u \cap T \Longrightarrow L-$*lipschitz_on* (*cball* $x\ u \cap X$) $(f\ s)$
by *metis*
have $x \in$ *ball* $x\ u$ **using** ‹$0 < u$› **by** *simp*
from *lipschitz_on_continuous_on*[*OF L*]
have *tendsto*: $(f\ t \longrightarrow f\ t\ x)$ (*at* x *within cball* $x\ u \cap X$)
using ‹$0 < u$› ‹$x \in X$› ‹$t \in T$›
by (*auto simp*: *continuous_on_def*)

moreover have $\forall_F$ *xa in at x*. (*xa* ∈ *cball x u* ∩ *X*) = (*xa* ∈ *X*)
using *eventually_at_ball*[*OF* ‹0 < *u*›, *of x UNIV*]
by *eventually_elim auto*
ultimately show (*f t* ⟶ *f t x*) (*at x within X*)
by (*rule Lim_transform_within_set*)
qed

lemma
local_lipschitz_compose1:
assumes *ll*: *local_lipschitz* (*g* ‘ *T*) *X* (λ*t*. *f t*)
assumes *g*: *continuous_on T g*
shows *local_lipschitz T X* (λ*t*. *f* (*g t*))
proof (*rule local_lipschitzI*)
fix *t x*
assume *t* ∈ *T x* ∈ *X*
then have *g t* ∈ *g* ‘ *T* **by** *simp*
from *local_lipschitzE*[*OF assms*(*1*) *this* ‹*x* ∈ *X*›]
obtain *u L* **where** *0* < *u* **and** *l*: (⋀*s*. *s* ∈ *cball* (*g t*) *u* ∩ *g* ‘ *T* ⟹ *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f s*))
by *auto*
from *g*[*unfolded continuous_on_eq_continuous_within*, *rule_format*, *OF* ‹*t* ∈ *T*›,
unfolded continuous_within_eps_delta, *rule_format*, *OF* ‹*0* < *u*›]
obtain *d* **where** *d*: *d*>*0* ⋀*x*′. *x*′∈*T* ⟹ *dist x*′ *t* < *d* ⟹ *dist* (*g x*′) (*g t*) < *u*
by (*auto*)
show ∃ *u*>*0*. ∃ *L*. ∀ *t*∈*cball t u* ∩ *T*. *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f* (*g t*))
using *d* ‹*0* < *u*›
by (*fastforce intro*: *exI*[**where** *x*=(*min d u*)/*2*] *exI*[**where** *x*=*L*]
intro!: *less_imp_le*[*OF d*(*2*)] *lipschitz_on_subset*[*OF l*] *simp*: *dist_commute*)
qed

context
fixes *T*::′*a*::*metric_space set* **and** *X f*
assumes *local_lipschitz*: *local_lipschitz T X f*
begin

lemma *continuous_on_TimesI*:
assumes *y*: ⋀*x*. *x* ∈ *X* ⟹ *continuous_on T* (λ*t*. *f t x*)
shows *continuous_on* (*T* × *X*) (λ(*t*, *x*). *f t x*)
unfolding *continuous_on_iff*
proof (*safe*, *simp*)
fix *a b* **and** *e*::*real*
assume *H*: *a* ∈ *T b* ∈ *X* *0* < *e*
hence *0* < *e*/*2* **by** *simp*
from *y*[*unfolded continuous_on_iff*, *OF* ‹*b* ∈ *X*›, *rule_format*, *OF* ‹*a* ∈ *T*› ‹*0* < *e*/*2*›]
obtain *d* **where** *d*: *d* > *0* ⋀*t*. *t* ∈ *T* ⟹ *dist t a* < *d* ⟹ *dist* (*f t b*) (*f a b*) < *e*/*2*
by *auto*

```
from ‹a : T› ‹b ∈ X›
obtain u L where u: 0 < u
  and L: ⋀t. t ∈ cball a u ∩ T ⟹ L−lipschitz_on  (cball b u ∩ X) (f t)
  by (erule local_lipschitzE[OF local_lipschitz])

have a ∈ cball a u ∩ T by (auto simp: ‹0 < u› ‹a ∈ T› less_imp_le)
from lipschitz_on_nonneg[OF L[OF ‹a ∈ cball _ _ ∩ _›]] have 0 ≤ L .

let ?d = Min {d, u, (e/2/(L + 1))}
show ∃ d>0. ∀ x∈T. ∀ y∈X. dist (x, y) (a, b) < d ⟶ dist (f x y) (f a b) < e
proof (rule exI[where x = ?d], safe)
  show 0 < ?d
    using ‹0 ≤ L› ‹0 < u› ‹0 < e› ‹0 < d›
    by (auto intro!: divide_pos_pos )
  fix x y
  assume x ∈ T y ∈ X
  assume dist_less: dist (x, y) (a, b) < ?d
  have dist y b ≤ dist (x, y) (a, b)
    using dist_snd_le[of (x, y) (a, b)]
    by auto
  also
  note dist_less
  also
  {
    note calculation
    also have ?d ≤ u by simp
    finally have dist y b < u .
  }
  have ?d ≤ e/2/(L + 1) by simp
  also have (L + 1) * ... ≤ e / 2
    using ‹0 < e› ‹L ≥ 0›
    by (auto simp: field_split_simps)
  finally have le1: (L + 1) * dist y b < e / 2 using ‹L ≥ 0› by simp

  have dist x a ≤ dist (x, y) (a, b)
    using dist_fst_le[of (x, y) (a, b)]
    by auto
  also note dist_less
  finally have dist x a < ?d .
  also have ?d ≤ d by simp
  finally have dist x a < d .
  note ‹dist x a < ?d›
  also have ?d ≤ u by simp
  finally have dist x a < u .
  then have x ∈ cball a u ∩ T
    using ‹x ∈ T›
    by (auto simp: dist_commute)
  have dist (f x y) (f a b) ≤ dist (f x y) (f x b) + dist (f x b) (f a b)
```

```
    by (rule dist_triangle)
   also have (L + 1)−lipschitz_on (cball b u ∩ X) (f x)
    using L[OF ‹x ∈ cball a u ∩ T›]
    by (rule lipschitz_on_le) simp
   then have dist (f x y) (f x b) ≤ (L + 1) ∗ dist y b
    apply (rule lipschitz_onD)
    subgoal
     using ‹y ∈ X› ‹dist y b < u›
     by (simp add: dist_commute)
    subgoal
     using ‹0 < u› ‹b ∈ X›
     by simp
    done
   also have (L + 1) ∗ dist y b ≤ e / 2
    using le1 ‹0 ≤ L› by simp
   also have dist (f x b) (f a b) < e / 2
    by (rule d; fact)
   also have e / 2 + e / 2 = e by simp
   finally show dist (f x y) (f a b) < e by simp
  qed
qed

lemma local_lipschitz_compact_implies_lipschitz:
  assumes compact X compact T
  assumes cont: ⋀x. x ∈ X ⟹ continuous_on T (λt. f t x)
  obtains L where ⋀t. t ∈ T ⟹ L−lipschitz_on X (f t)
proof −
  {
   assume ∗: ⋀n::nat. ¬(∀ t∈T. n−lipschitz_on X (f t))
   {
    fix n::nat
    from ∗[of n] have ∃ x y t. t ∈ T ∧ x ∈ X ∧ y ∈ X ∧ dist (f t y) (f t x) > n
∗ dist y x
     by (force simp: lipschitz_on_def)
   } then obtain t and x y::nat ⇒ 'b where xy: ⋀n. x n ∈ X ⋀n. y n ∈ X
    and t: ⋀n. t n ∈ T
    and d: ⋀n. dist (f (t n) (y n)) (f (t n) (x n)) > n ∗ dist (y n) (x n)
    by metis
   from xy assms obtain lx rx where lx': lx ∈ X strict_mono (rx :: nat ⇒ nat)
(x o rx) ⟶ lx
    by (metis compact_def)
   with xy have ⋀n. (y o rx) n ∈ X by auto
   with assms obtain ly ry where ly': ly ∈ X strict_mono (ry :: nat ⇒ nat) ((y
o rx) o ry) ⟶ ly
    by (metis compact_def)
   with t have ⋀n. ((t o rx) o ry) n ∈ T by simp
   with assms obtain lt rt where lt': lt ∈ T strict_mono (rt :: nat ⇒ nat) (((t
o rx) o ry) o rt) ⟶ lt
    by (metis compact_def)
```

```
    from lx' ly'
    have lx: (x o (rx o ry o rt)) ⟶ lx (is ?x ⟶ _)
      and ly: (y o (rx o ry o rt)) ⟶ ly (is ?y ⟶ _)
      and lt: (t o (rx o ry o rt)) ⟶ lt (is ?t ⟶ _)
      subgoal by (simp add: LIMSEQ_subseq_LIMSEQ o_assoc lt'(2))
      subgoal by (simp add: LIMSEQ_subseq_LIMSEQ ly'(3) o_assoc lt'(2))
      subgoal by (simp add: o_assoc lt'(3))
      done
    hence (λn. dist (?y n) (?x n)) ⟶ dist ly lx
      by (metis tendsto_dist)
    moreover
    let ?S = (λ(t, x). f t x) ` (T × X)
    have eventually (λn::nat. n > 0) sequentially
      by (metis eventually_at_top_dense)
    hence eventually (λn. norm (dist (?y n) (?x n)) ≤ norm (|diameter ?S| / n)
* 1) sequentially
    proof eventually_elim
      case (elim n)
      have 0 < rx (ry (rt n)) using ‹0 < n›
        by (metis dual_order.strict_trans1 lt'(2) lx'(2) ly'(2) seq_suble)
      have compact: compact ?S
          by (auto intro!: compact_continuous_image continuous_on_subset[OF
continuous_on_TimesI]
        compact_Times ‹compact X› ‹compact T› cont)
      have norm (dist (?y n) (?x n)) = dist (?y n) (?x n) by simp
      also
      from this elim d[of rx (ry (rt n))]
      have ... < dist (f (?t n) (?y n)) (f (?t n) (?x n)) / rx (ry (rt (n)))
        using lx'(2) ly'(2) lt'(2) ‹0 < rx _›
        by (auto simp add: field_split_simps strict_mono_def)
      also have ... ≤ diameter ?S / n
      proof (rule frac_le)
        show diameter ?S ≥ 0
          using compact compact_imp_bounded diameter_ge_0 by blast
        show dist (f (?t n) (?y n)) (f (?t n) (?x n)) ≤ diameter ((λ(t,x). f t x) `
(T × X))
        by (metis (no_types) compact compact_imp_bounded diameter_bounded_bound
image_eqI mem_Sigma_iff o_apply split_conv t xy(1) xy(2))
        show real n ≤ real (rx (ry (rt n)))
        by (meson le_trans lt'(2) lx'(2) ly'(2) of_nat_mono strict_mono_imp_increasing)
      qed (use ‹n > 0› in auto)
      also have ... ≤ abs (diameter ?S) / n
        by (auto intro!: divide_right_mono)
      finally show ?case by simp
    qed
    with _ have (λn. dist (?y n) (?x n)) ⟶ 0
      by (rule tendsto_0_le)
     (metis tendsto_divide_0[OF tendsto_const] filterlim_at_top_imp_at_infinity
        filterlim_real_sequentially)
```

```
    ultimately have lx = ly
      using LIMSEQ_unique by fastforce
    with assms lx' have lx ∈ X by auto
    from ‹lt ∈ T› this obtain u L where L: u > 0 ⋀t. t ∈ cball lt u ∩ T ⟹
L−lipschitz_on (cball lx u ∩ X) (f t)
      by (erule local_lipschitzE[OF local_lipschitz])
    hence L ≥ 0 by (force intro!: lipschitz_on_nonneg ‹lt ∈ T›)

    from L lt ly lx ‹lx = ly›
    have
      eventually (λn. ?t n ∈ ball lt u) sequentially
      eventually (λn. ?y n ∈ ball lx u) sequentially
      eventually (λn. ?x n ∈ ball lx u) sequentially
      by (auto simp: dist_commute Lim)
    moreover have eventually (λn. n > L) sequentially
      by (metis filterlim_at_top_dense filterlim_real_sequentially)
    ultimately
    have eventually (λ_. False) sequentially
    proof eventually_elim
      case (elim n)
      hence dist (f (?t n) (?y n)) (f (?t n) (?x n)) ≤ L * dist (?y n) (?x n)
        using assms xy t
        unfolding dist_norm[symmetric]
        by (intro lipschitz_onD[OF L(2)]) (auto)
      also have ... ≤ n * dist (?y n) (?x n)
        using elim by (intro mult_right_mono) auto
      also have ... ≤ rx (ry (rt n)) * dist (?y n) (?x n)
        by (intro mult_right_mono[OF _ zero_le_dist])
           (meson lt'(2) lx'(2) ly'(2) of_nat_le_iff order_trans seq_suble)
      also have ... < dist (f (?t n) (?y n)) (f (?t n) (?x n))
        by (auto intro!: d)
      finally show ?case by simp
    qed
    hence False
      by simp
  } then obtain L where ⋀t. t ∈ T ⟹ L−lipschitz_on X (f t)
    by metis
  thus ?thesis ..
qed

lemma local_lipschitz_subset:
  assumes S ⊆ T Y ⊆ X
  shows local_lipschitz S Y f
proof (rule local_lipschitzI)
  fix t x assume t ∈ S x ∈ Y
  then have t ∈ T x ∈ X using assms by auto
  from local_lipschitzE[OF local_lipschitz, OF this]
  obtain u L where u: 0 < u and L: ⋀s. s ∈ cball t u ∩ T ⟹ L−lipschitz_on
(cball x u ∩ X) (f s)
```

```
    by blast
  show ∃u>0. ∃L. ∀t∈cball t u ∩ S. L−lipschitz_on (cball x u ∩ Y) (f t)
    using assms
    by (auto intro: exI[where x=u] exI[where x=L]
        intro!: u lipschitz_on_subset[OF _ Int_mono[OF order_refl ‹Y ⊆ X›]] L)
qed

end

lemma local_lipschitz_minus:
  fixes f::'a::metric_space ⇒ 'b::metric_space ⇒ 'c::real_normed_vector
  shows local_lipschitz T X (λt x. − f t x) = local_lipschitz T X f
  by (auto simp: local_lipschitz_def lipschitz_on_minus)

lemma local_lipschitz_PairI:
  assumes f: local_lipschitz A B (λa b. f a b)
  assumes g: local_lipschitz A B (λa b. g a b)
  shows local_lipschitz A B (λa b. (f a b, g a b))
proof (rule local_lipschitzI)
  fix t x assume t ∈ A x ∈ B
  from local_lipschitzE[OF f this] local_lipschitzE[OF g this]
  obtain u L v M where 0 < u (⋀s. s ∈ cball t u ∩ A ⟹ L−lipschitz_on (cball
x u ∩ B) (f s))
    0 < v (⋀s. s ∈ cball t v ∩ A ⟹ M−lipschitz_on (cball x v ∩ B) (g s))
    by metis
  then show ∃u>0. ∃L. ∀t∈cball t u ∩ A. L−lipschitz_on (cball x u ∩ B) (λb.
(f t b, g t b))
    by (intro exI[where x=min u v])
      (force intro: lipschitz_on_subset intro!: lipschitz_on_Pair)
qed

lemma local_lipschitz_constI: local_lipschitz S T (λt x. f t)
  by (auto simp: intro!: local_lipschitzI lipschitz_on_constant intro: exI[where
x=1])

lemma (in bounded_linear) local_lipschitzI:
  shows local_lipschitz A B (λ_. f)
proof (rule local_lipschitzI, goal_cases)
  case (1 t x)
  from lipschitz_boundE[of (cball x 1 ∩ B)] obtain C where C−lipschitz_on
(cball x 1 ∩ B) f by auto
  then show ?case
    by (auto intro: exI[where x=1])
qed

proposition c1_implies_local_lipschitz:
  fixes T::real set and X::'a::{banach,heine_borel} set
    and f::real ⇒ 'a ⇒ 'a
  assumes f': ⋀t x. t ∈ T ⟹ x ∈ X ⟹ (f t has_derivative blinfun_apply (f'
```

```
(t, x))) (at x)
  assumes cont_f': continuous_on (T × X) f'
  assumes open T
  assumes open X
  shows local_lipschitz T X f
proof (rule local_lipschitzI)
  fix t x
  assume t ∈ T x ∈ X
  from open_contains_cball[THEN iffD1, OF ‹open X›, rule_format, OF ‹x ∈
X›]
  obtain u where u: u > 0 cball x u ⊆ X by auto
  moreover
  from open_contains_cball[THEN iffD1, OF ‹open T›, rule_format, OF ‹t ∈
T›]
  obtain v where v: v > 0 cball t v ⊆ T by auto
  ultimately
  have compact (cball t v × cball x u) cball t v × cball x u ⊆ T × X
    by (auto intro!: compact_Times)
  then have compact (f' ` (cball t v × cball x u))
   by (auto intro!: compact_continuous_image continuous_on_subset[OF cont_f'])
  then obtain B where B: B > 0 ⋀s y. s ∈ cball t v ⟹ y ∈ cball x u ⟹ norm
(f' (s, y)) ≤ B
    by (auto dest!: compact_imp_bounded simp: bounded_pos)

  have lipschitz: B−lipschitz_on (cball x (min u v) ∩ X) (f s) if s: s ∈ cball t v
for s
  proof −
    note s
    also note ‹cball t v ⊆ T›
    finally
    have deriv: ⋀y. y ∈ cball x u ⟹ (f s has_derivative blinfun_apply (f' (s, y)))
(at y within cball x u)
      using ‹_ ⊆ X›
      by (auto intro!: has_derivative_at_withinI[OF f'])
    have norm (f s y − f s z) ≤ B * norm (y − z)
      if y ∈ cball x u z ∈ cball x u
      for y z
      using s that
      by (intro differentiable_bound[OF convex_cball deriv])
        (auto intro!: B  simp: norm_blinfun.rep_eq[symmetric])
    then show ?thesis
      using ‹0 < B›
      by (auto intro!: lipschitz_onI simp: dist_norm)
  qed
  show ∃u>0. ∃L. ∀t∈cball t u ∩ T. L−lipschitz_on (cball x u ∩ X) (f t)
    by (force intro: exI[where x=min u v] exI[where x=B] intro!: lipschitz simp:
u v)
qed
```

end
theory
Multivariate_Analysis
imports
Ordered_Euclidean_Space
Determinants
Cross3
Lipschitz
Starlike
begin

Entry point excluding integration and complex analysis.

end

9.34 Volume of a Simplex

theory *Simplex_Content*
imports *Change_Of_Vars*
begin

lemma *fact_neq_top_ennreal* [*simp*]: *fact n* ≠ (*top* :: *ennreal*)
by (*induction n*) (*auto simp*: *ennreal_mult_eq_top_iff*)

lemma *ennreal_fact*: *ennreal* (*fact n*) = *fact n*
by (*induction n*) (*auto simp*: *ennreal_mult algebra_simps ennreal_of_nat_eq_real_of_nat*)

context
fixes *S* :: *'a set* ⇒ *real* ⇒ (*'a* ⇒ *real*) *set*
defines *S* ≡ (λ*A t*. {*x*. (∀ *i*∈*A*. *0* ≤ *x i*) ∧ *sum x A* ≤ *t*})
begin

lemma *emeasure_std_simplex_aux_step*:
assumes *b* ∉ *A finite A*
shows *x*(*b* := *y*) ∈ *S* (*insert b A*) *t* ⟷ *y* ∈ {*0*..*t*} ∧ *x* ∈ *S A* (*t* − *y*)
using *assms sum_nonneg*[*of A x*] **unfolding** *S_def*
by (*force simp*: *sum_delta_notmem algebra_simps*)

lemma *emeasure_std_simplex_aux*:
fixes *t* :: *real*
assumes *finite* (*A* :: *'a set*) *t* ≥ *0*
shows *emeasure* (*Pi*$_M$ *A* (λ_. *lborel*))
(*S A t* ∩ *space* (*Pi*$_M$ *A* (λ_. *lborel*))) = *t* ^ *card A* / *fact* (*card A*)
using *assms*(*1*,*2*)
proof (*induction arbitrary*: *t rule*: *finite_induct*)
case (*empty t*)
thus *?case* **by** (*simp add*: *PiM_empty S_def*)
next
case (*insert b A t*)
define *n* **where** *n* = *Suc* (*card A*)

```
  have n_pos: n > 0 by (simp add: n_def)
  let ?M = λA. (Pi_M A (λ_. lborel))
  {
    fix A :: 'a set and t :: real assume finite A
    have S A t ∩ space (Pi_M A (λ_. lborel)) =
          Pi_E A (λ_. {0..}) ∩ (λx. sum x A) -` {..t} ∩ space (Pi_M A (λ_. lborel))
      by (auto simp: S_def space_PiM)
    also have ... ∈ sets (Pi_M A (λ_. lborel))
      using ‹finite A› by measurable
    finally have S A t ∩ space (Pi_M A (λ_. lborel)) ∈ sets (Pi_M A (λ_. lborel)) .
  } note meas [measurable] = this

  interpret product_sigma_finite λ_. lborel
    by standard
  have emeasure (?M (insert b A)) (S (insert b A) t ∩ space (?M (insert b A)))
=
        nn_integral (?M (insert b A))
          (λx. indicator (S (insert b A) t ∩ space (?M (insert b A))) x)
    using insert.hyps by (subst nn_integral_indicator) auto
  also have ... = (∫⁺ y. ∫⁺ x. indicator (S (insert b A) t ∩ space (?M (insert
b A)))
                    (x(b := y)) ∂?M A ∂lborel)
   using insert.prems insert.hyps by (intro product_nn_integral_insert_rev) auto
  also have ... = (∫⁺ y. ∫⁺ x. indicator {0..t} y * indicator (S A (t - y) ∩
space (?M A)) x
                    ∂?M A ∂lborel)
    using insert.hyps insert.prems emeasure_std_simplex_aux_step[of b A]
    by (intro nn_integral_cong)
       (auto simp: fun_eq_iff indicator_def space_PiM PiE_def extensional_def)
  also have ... = (∫⁺ y. indicator {0..t} y * (∫⁺ x. indicator (S A (t - y) ∩
space (?M A)) x
                    ∂?M A) ∂lborel) using ‹finite A›
    by (subst nn_integral_cmult) auto
  also have ... = (∫⁺ y. indicator {0..t} y * emeasure (?M A) (S A (t - y) ∩
space (?M A)) ∂lborel)
    using ‹finite A› by (subst nn_integral_indicator) auto
  also have ... = (∫⁺ y. indicator {0..t} y * (t - y) ^ card A / ennreal (fact
(card A)) ∂lborel)
     using insert.IH by (intro nn_integral_cong) (auto simp: indicator_def di-
vide_ennreal)
  also have ... = (∫⁺ y. indicator {0..t} y * (t - y) ^ card A ∂lborel) / ennreal
(fact (card A))
    using ‹finite A› by (subst nn_integral_divide) auto
  also have (∫⁺ y. indicator {0..t} y * (t - y) ^ card A ∂lborel) =
              (∫⁺y∈{0..t}. ennreal ((t - y) ^ (n - 1)) ∂lborel)
    by (intro nn_integral_cong) (auto simp: indicator_def n_def)
  also have ((λx. - ((t - x) ^ n / n)) has_real_derivative (t - x) ^ (n - 1))
(at x)
    if x ∈ {0..t} for x by (rule derivative_eq_intros refl | simp add: n_pos)+
```

hence $(\int^+ y{\in}\{0..t\}.\ ennreal\ ((t - y)\ \hat{}\ (n - 1))\ \partial lborel) =$
$ennreal\ (-((t - t)\ \hat{}\ n\ /\ n) - (-((t - 0)\ \hat{}\ n\ /\ n)))$
using *insert.prems insert.hyps* **by** (*intro nn_integral_FTC_Icc*) *auto*
also have $\ldots = ennreal\ (t\ \hat{}\ n\ /\ n)$ **using** *n_pos* **by** (*simp add*: *zero_power*)
also have $\ldots\ /\ ennreal\ (fact\ (card\ A)) = ennreal\ (t\ \hat{}\ n\ /\ n\ /\ fact\ (card\ A))$
using *n_pos* ‹$t \geq 0$› **by** (*subst divide_ennreal*) *auto*
also have $t\ \hat{}\ n\ /\ n\ /\ fact\ (card\ A) = t\ \hat{}\ n\ /\ fact\ n$
by (*simp add*: *n_def*)
also have $n = card\ (insert\ b\ A)$
using *insert.hyps* **by** (*subst card.insert_remove*) (*auto simp*: *n_def*)
finally show *?case* **.**
qed

end

lemma *emeasure_std_simplex*:
emeasure lborel (*convex hull* (*insert 0 Basis* :: *'a* :: *euclidean_space set*)) =
ennreal (*1 / fact DIM*('*a*))
proof −
have *emeasure lborel* $\{x::'a.\ (\forall i{\in}Basis.\ 0 \leq x \cdot i) \wedge sum\ ((\cdot)\ x)\ Basis \leq 1\} =$
emeasure (*distr* (*Pi*$_M$ *Basis* (λ*b*. *lborel*)) *borel* (λ*f*. $\sum b{\in}Basis.\ f\ b *_R b$))
$\{x::'a.\ (\forall i{\in}Basis.\ 0 \leq x \cdot i) \wedge sum\ ((\cdot)\ x)\ Basis \leq 1\}$
by (*subst lborel_eq*) *simp*
also have $\ldots =$ *emeasure* (*Pi*$_M$ *Basis* (λ*b*. *lborel*))
$(\{y::'a \Rightarrow real.\ (\forall i{\in}Basis.\ 0 \leq y\ i) \wedge sum\ y\ Basis \leq 1\} \cap$
space (*Pi*$_M$ *Basis* (λ*b*. *lborel*)))
by (*subst emeasure_distr*) *auto*
also have $\ldots =$ *ennreal* (*1 / fact DIM*('*a*))
by (*subst emeasure_std_simplex_aux*) *auto*
finally show *?thesis* **by** (*simp only*: *std_simplex*)
qed

theorem *content_std_simplex*:
measure lborel (*convex hull* (*insert 0 Basis* :: *'a* :: *euclidean_space set*)) =
1 / fact DIM('*a*)
by (*simp add*: *measure_def emeasure_std_simplex*)

proposition *measure_lebesgue_linear_transformation*:
fixes *A* :: (*real* ^ *'n* :: {*finite*, *wellorder*}) *set*
fixes *f* :: *_* ⇒ *real* ^ *'n* :: {*finite*, *wellorder*}
assumes *bounded A A* ∈ *sets lebesgue linear f*
shows *measure lebesgue* (*f* ‘ *A*) = |*det* (*matrix f*)| ∗ *measure lebesgue A*
proof −
from *assms* **have** [*intro*]: *A* ∈ *lmeasurable*
by (*intro bounded_set_imp_lmeasurable*) *auto*
hence [*intro*]: *f* ‘ *A* ∈ *lmeasurable*
by (*intro lmeasure_integral measurable_linear_image assms*)

```
  have measure lebesgue (f ‘ A) = integral (f ‘ A) (λ_. 1)
    by (intro lmeasure_integral measurable_linear_image assms) auto
  also have ... = integral (f ‘ A) (λ_. 1 :: real ^ 1) $ 0
   by (subst integral_component_eq_cart [symmetric]) (auto intro: integrable_on_const)
  also have ... = |det (matrix f)| * integral A (λx. 1 :: real ^ 1) $ 0
    using assms
    by (subst integral_change_of_variables_linear)
       (auto simp: o_def absolutely_integrable_on_def intro: integrable_on_const)
  also have integral A (λx. 1 :: real ^ 1) $ 0 = integral A (λx. 1)
   by (subst integral_component_eq_cart [symmetric]) (auto intro: integrable_on_const)
  also have ... = measure lebesgue A
    by (intro lmeasure_integral [symmetric]) auto
  finally show ?thesis .
qed

theorem content_simplex:
  fixes X :: (real ^ 'n :: {finite, wellorder}) set and f :: 'n :: _ ⇒ real ^ ('n :: _)
  assumes finite X card X = Suc CARD('n) and x0: x0 ∈ X and bij: bij_betw f
UNIV (X - {x0})
  defines M ≡ (χ i. χ j. f j $ i - x0 $ i)
  shows content (convex hull X) = |det M| / fact (CARD('n))
proof -
  define g where g = (λx. M *v x)
  have [simp]: M *v axis i 1 = f i - x0 for i :: 'n
    by (simp add: M_def matrix_vector_mult_basis column_def vec_eq_iff)
  define std where std = (convex hull insert 0 Basis :: (real ^ 'n :: _) set)
  have compact: compact std unfolding std_def
    by (intro finite_imp_compact_convex_hull) auto

  have measure lebesgue (convex hull X) = measure lebesgue (((+) (-x0)) ‘ (convex
hull X))
    by (rule measure_translation [symmetric])
  also have ((+) (-x0)) ‘ (convex hull X) = convex hull (((+) (-x0)) ‘ X)
    by (rule convex_hull_translation [symmetric])
  also have ((+) (-x0)) ‘ X = insert 0 ((λx. x - x0) ‘ (X - {x0}))
    using x0 by (auto simp: image_iff)
  finally have eq: measure lebesgue (convex hull X) = measure lebesgue (convex
hull ...) .

  from compact have measure lebesgue (g ‘ std) = |det M| * measure lebesgue std
    by (subst measure_lebesgue_linear_transformation)
       (auto intro: finite_imp_bounded_convex_hull dest: compact_imp_closed
simp: g_def std_def)
  also have measure lebesgue std = content std using compact
    by (intro measure_completion) (auto dest: compact_imp_closed)
  also have content std = 1 / fact CARD('n) unfolding std_def
    by (simp add: content_std_simplex)
  also have g ‘ std = convex hull (g ‘ insert 0 Basis) unfolding std_def
    by (rule convex_hull_linear_image) (auto simp: g_def)
```

```
  also have g ‘ insert 0 Basis = insert 0 (g ‘ Basis)
    by (auto simp: g_def)
  also have g ‘ Basis = (λx. x − x0) ‘ range f
    by (auto simp: g_def Basis_vec_def image_iff)
  also have range f = X − {x0} using bij
    using bij_betw_imp_surj_on by blast
  also note eq [symmetric]
  finally show ?thesis
     using finite_imp_compact_convex_hull[OF ‹finite X›] by (auto dest: com-
pact_imp_closed)
qed

theorem content_triangle:
  fixes A B C :: real ^ 2
  shows content (convex hull {A, B, C}) =
          |(C $ 1 − A $ 1) ∗ (B $ 2 − A $ 2) − (B $ 1 − A $ 1) ∗ (C $ 2 − A
$ 2)| / 2
proof −
  define M :: real ^ 2 ^ 2 where M ≡ (χ i. χ j. (if j = 1 then B else C) $ i −
A $ i)
  define g where g = (λx. M ∗v x)
  define std where std = (convex hull insert 0 Basis :: (real ^ 2) set)
  have [simp]: M ∗v axis i 1 = (if i = 1 then B − A else C − A) for i
    by (auto simp: M_def matrix_vector_mult_basis column_def vec_eq_iff)
  have compact: compact std unfolding std_def
    by (intro finite_imp_compact_convex_hull) auto

  have measure lebesgue (convex hull {A, B, C}) =
         measure lebesgue (((+) (−A)) ‘ (convex hull {A, B, C}))
    by (rule measure_translation [symmetric])
   also have ((+) (−A)) ‘ (convex hull {A, B, C}) = convex hull (((+) (−A)) ‘
{A, B, C})
    by (rule convex_hull_translation [symmetric])
  also have ((+) (−A)) ‘ {A, B, C} = {0, B − A, C − A}
    by (auto simp: image_iff)
  finally have eq: measure lebesgue (convex hull {A, B, C}) =
                   measure lebesgue (convex hull {0, B − A, C − A}) .

  from compact have measure lebesgue (g ‘ std) = |det M| ∗ measure lebesgue std
    by (subst measure_lebesgue_linear_transformation)
       (auto intro: finite_imp_bounded_convex_hull dest: compact_imp_closed
simp: g_def std_def)
  also have measure lebesgue std = content std using compact
    by (intro measure_completion) (auto dest: compact_imp_closed)
  also have content std = 1 / 2 unfolding std_def
    by (simp add: content_std_simplex)
  also have g ‘ std = convex hull (g ‘ insert 0 Basis) unfolding std_def
    by (rule convex_hull_linear_image) (auto simp: g_def)
  also have g ‘ insert 0 Basis = insert 0 (g ‘ Basis)
```

```
    by (auto simp: g_def)
  also have (2 :: 2) ≠ 1 by auto
  hence ¬(∀ y::2. y = 1) by blast
  hence g ‘ Basis = {B − A, C − A}
    by (auto simp: g_def Basis_vec_def image_iff)
  also note eq [symmetric]
  finally show ?thesis
    using finite_imp_compact_convex_hull[of {A, B, C}]
    by (auto dest!: compact_imp_closed simp: det_2 M_def)
qed

theorem heron:
  fixes A B C :: real ^ 2
  defines a ≡ dist B C and b ≡ dist A C and c ≡ dist A B
  defines s ≡ (a + b + c) / 2
  shows   content (convex hull {A, B, C}) = sqrt (s * (s − a) * (s − b) * (s −
c))
proof −
  have [simp]: (UNIV :: 2 set) = {1, 2}
    using exhaust_2 by auto
  have dist_eq: dist (A :: real ^ 2) B ^ 2 = (A $ 1 − B $ 1) ^ 2 + (A $ 2 − B
$ 2) ^ 2
    for A B by (simp add: dist_vec_def dist_real_def)
  have nonneg: s * (s − a) * (s − b) * (s − c) ≥ 0
    using dist_triangle[of A B C] dist_triangle[of A C B] dist_triangle[of B C A]
   by (intro mult_nonneg_nonneg) (auto simp: s_def a_def b_def c_def dist_commute)

  have 16 * content (convex hull {A, B, C}) ^ 2 =
        4 * ((C $ 1 − A $ 1) * (B $ 2 − A $ 2) − (B $ 1 − A $ 1) * (C $ 2 −
A $ 2)) ^ 2
    by (subst content_triangle) (simp add: power_divide)
  also have ... = (2 * (dist A B ^ 2 * dist A C ^ 2 + dist A B ^ 2 * dist B C
^ 2 +
      dist A C ^ 2 * dist B C ^ 2) − (dist A B ^ 2) ^ 2 − (dist A C ^ 2) ^ 2 −
(dist B C ^ 2) ^ 2)
    unfolding dist_eq unfolding power2_eq_square by algebra
  also have ... = (a + b + c) * ((a + b + c) − 2 * a) * ((a + b + c) − 2 * b) *
                  ((a + b + c) − 2 * c)
    unfolding power2_eq_square by (simp add: s_def a_def b_def c_def alge-
bra_simps)
  also have ... = 16 * s * (s − a) * (s − b) * (s − c)
    by (simp add: s_def field_split_simps)
  finally have content (convex hull {A, B, C}) ^ 2 = s * (s − a) * (s − b) * (s
− c)
    by simp
  also have ... = sqrt (s * (s − a) * (s − b) * (s − c)) ^ 2
    by (intro real_sqrt_pow2 [symmetric] nonneg)
  finally show ?thesis using nonneg
    by (subst (asm) power2_eq_iff_nonneg) auto
```

qed

end

9.35 Convergence of Formal Power Series

theory *FPS_Convergence*
imports
Generalised_Binomial_Theorem
HOL−Computational_Algebra.Formal_Power_Series
begin

In this theory, we will connect formal power series (which are algebraic objects) with analytic functions. This will become more important in complex analysis, and indeed some of the less trivial results will only be proven there.

9.35.1 Balls with extended real radius

The following is a variant of *ball* that also allows an infinite radius.

definition *eball* :: *'a* :: *metric_space* ⇒ *ereal* ⇒ *'a set* **where**
eball z r = {*z'*. *ereal* (*dist z z'*) < *r*}

lemma *in_eball_iff* [*simp*]: *z* ∈ *eball z0 r* ⟷ *ereal* (*dist z0 z*) < *r*
by (*simp add*: *eball_def*)

lemma *eball_ereal* [*simp*]: *eball z* (*ereal r*) = *ball z r*
by *auto*

lemma *eball_inf* [*simp*]: *eball z* ∞ = *UNIV*
by *auto*

lemma *eball_empty* [*simp*]: *r* ≤ *0* ⟹ *eball z r* = {}
proof *safe*
fix *x* **assume** *r* ≤ *0 x* ∈ *eball z r*
hence *dist z x* < *r* **by** *simp*
also have ... ≤ *ereal 0* **using** ‹*r* ≤ *0*› **by** (*simp add*: *zero_ereal_def*)
finally show *x* ∈ {} **by** *simp*
qed

lemma *eball_conv_UNION_balls*:
eball z r = (⋃ *r'*∈{*r'*. *ereal r'* < *r*}. *ball z r'*)
by (*cases r*) (*use dense gt_ex* **in** *force*)+

lemma *eball_mono*: *r* ≤ *r'* ⟹ *eball z r* ≤ *eball z r'*
by *auto*

lemma *ball_eball_mono*: *ereal r* ≤ *r'* ⟹ *ball z r* ≤ *eball z r'*
using *eball_mono*[*of ereal r r'*] **by** *simp*

lemma *open_eball* [*simp*, *intro*]: *open* (*eball z r*)
by (*cases r*) *auto*

9.35.2 Basic properties of convergent power series

definition *fps_conv_radius* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *ereal* **where**
fps_conv_radius f = *conv_radius* (*fps_nth f*)

definition *eval_fps* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *'a* ⇒ *'a* **where**
eval_fps f z = (∑ *n*. *fps_nth f n* ∗ *z* ^ *n*)

lemma *norm_summable_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
shows *norm z* < *fps_conv_radius f* ⟹ *summable* (λ*n*. *norm* (*fps_nth f n* ∗ *z* ^ *n*))
by (*rule abs_summable_in_conv_radius*) (*simp_all add*: *fps_conv_radius_def*)

lemma *summable_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
shows *norm z* < *fps_conv_radius f* ⟹ *summable* (λ*n*. *fps_nth f n* ∗ *z* ^ *n*)
by (*rule summable_in_conv_radius*) (*simp_all add*: *fps_conv_radius_def*)

theorem *sums_eval_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
assumes *norm z* < *fps_conv_radius f*
shows (λ*n*. *fps_nth f n* ∗ *z* ^ *n*) *sums eval_fps f z*
using *assms* **unfolding** *eval_fps_def fps_conv_radius_def*
by (*intro summable_sums summable_in_conv_radius*) *simp_all*

lemma *continuous_on_eval_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
shows *continuous_on* (*eball 0* (*fps_conv_radius f*)) (*eval_fps f*)
proof (*subst continuous_on_eq_continuous_at* [*OF open_eball*], *safe*)
fix *x* :: *'a* **assume** *x*: *x* ∈ *eball 0* (*fps_conv_radius f*)
define *r* **where** *r* = (*if fps_conv_radius f* = ∞ *then norm x* + *1 else*
(*norm x* + *real_of_ereal* (*fps_conv_radius f*)) / *2*)
have *r*: *norm x* < *r* ∧ *ereal r* < *fps_conv_radius f*
using *x* **by** (*cases fps_conv_radius f*)
(*auto simp*: *r_def eball_def split*: *if_splits*)

have *continuous_on* (*cball 0 r*) (λ*x*. ∑ *i*. *fps_nth f i* ∗ (*x* − *0*) ^ *i*)
by (*rule powser_continuous_suminf*) (*insert r*, *auto simp*: *fps_conv_radius_def*)
hence *continuous_on* (*cball 0 r*) (*eval_fps f*)
by (*simp add*: *eval_fps_def*)
thus *isCont* (*eval_fps f*) *x*
by (*rule continuous_on_interior*) (*use r* **in** *auto*)

qed

lemma *continuous_on_eval_fps'* [*continuous_intros*]:
assumes *continuous_on A g*
assumes *g ' A ⊆ eball 0 (fps_conv_radius f)*
shows *continuous_on A (λx. eval_fps f (g x))*
using *continuous_on_compose2*[*OF continuous_on_eval_fps assms*] **.**

lemma *has_field_derivative_powser*:
fixes *z* :: *'a* :: {*banach*, *real_normed_field*}
assumes *ereal (norm z) < conv_radius f*
shows *((λz. ∑ n. f n * z ^ n) has_field_derivative (∑ n. diffs f n * z ^ n)) (at z within A)*
proof −
define *K* **where** *K = (if conv_radius f = ∞ then norm z + 1*
else (norm z + real_of_ereal (conv_radius f)) / 2)
have *K*: *norm z < K ∧ ereal K < conv_radius f*
using *assms* **by** (*cases conv_radius f*) (*auto simp*: *K_def*)
have *0 ≤ norm z* **by** *simp*
also from *K* **have** *... < K* **by** *simp*
finally have *K_pos*: *K > 0* **by** *simp*

have *summable (λn. f n * of_real K ^ n)*
using *K* **and** *K_pos* **by** (*intro summable_in_conv_radius*) *auto*
moreover from *K* **and** *K_pos* **have** *norm z < norm (of_real K* :: *'a)* **by** *auto*
ultimately show *?thesis*
by (*rule has_field_derivative_at_within* [*OF termdiffs_strong*])
qed

lemma *has_field_derivative_eval_fps*:
fixes *z* :: *'a* :: {*banach*, *real_normed_field*}
assumes *norm z < fps_conv_radius f*
shows *(eval_fps f has_field_derivative eval_fps (fps_deriv f) z) (at z within A)*
proof −
have *(eval_fps f has_field_derivative eval_fps (Abs_fps (diffs (fps_nth f))) z) (at z within A)*
using *assms* **unfolding** *eval_fps_def fps_nth_Abs_fps fps_conv_radius_def*
by (*intro has_field_derivative_powser*) *auto*
also have *Abs_fps (diffs (fps_nth f)) = fps_deriv f*
by (*simp add*: *fps_eq_iff diffs_def*)
finally show *?thesis* **.**
qed

lemma *holomorphic_on_eval_fps* [*holomorphic_intros*]:
fixes *z* :: *'a* :: {*banach*, *real_normed_field*}
assumes *A ⊆ eball 0 (fps_conv_radius f)*
shows *eval_fps f holomorphic_on A*
proof (*rule holomorphic_on_subset* [*OF _ assms*])

```
  show eval_fps f holomorphic_on eball 0 (fps_conv_radius f)
  proof (subst holomorphic_on_open [OF open_eball], safe, goal_cases)
    case (1 x)
    thus ?case
    by (intro exI[of _ eval_fps (fps_deriv f) x]) (auto intro: has_field_derivative_eval_fps)
  qed
qed

lemma analytic_on_eval_fps:
  fixes z :: 'a :: {banach, real_normed_field}
  assumes A ⊆ eball 0 (fps_conv_radius f)
  shows   eval_fps f analytic_on A
proof (rule analytic_on_subset [OF _ assms])
  show eval_fps f analytic_on eball 0 (fps_conv_radius f)
    using holomorphic_on_eval_fps[of eball 0 (fps_conv_radius f)]
    by (subst analytic_on_open) auto
qed

lemma continuous_eval_fps [continuous_intros]:
  fixes z :: 'a::{real_normed_field,banach}
  assumes norm z < fps_conv_radius F
  shows   continuous (at z within A) (eval_fps F)
proof -
  from ereal_dense2[OF assms] obtain K :: real where K: norm z < K K <
fps_conv_radius F
    by auto
  have 0 ≤ norm z by simp
  also have norm z < K by fact
  finally have K > 0 .
  from K and ‹K > 0› have summable (λn. fps_nth F n * of_real K ^ n)
    by (intro summable_fps) auto
  from this have isCont (eval_fps F) z unfolding eval_fps_def
    by (rule isCont_powser) (use K in auto)
  thus continuous (at z within A)  (eval_fps F)
    by (simp add: continuous_at_imp_continuous_within)
qed
```

9.35.3 Lower bounds on radius of convergence

```
lemma fps_conv_radius_deriv:
  fixes f :: 'a :: {banach, real_normed_field} fps
  shows fps_conv_radius (fps_deriv f) ≥ fps_conv_radius f
  unfolding fps_conv_radius_def
proof (rule conv_radius_geI_ex)
  fix r :: real assume r: r > 0 ereal r < conv_radius (fps_nth f)
  define K where K = (if conv_radius (fps_nth f) = ∞ then r + 1
                      else (real_of_ereal (conv_radius (fps_nth f)) + r) / 2)
  have K: r < K ∧ ereal K < conv_radius (fps_nth f)
    using r by (cases conv_radius (fps_nth f)) (auto simp: K_def)
```

```
  have summable (λn. diffs (fps_nth f) n * of_real r ^ n)
  proof (rule termdiff_converges)
    fix x :: 'a assume norm x < K
    hence ereal (norm x) < ereal K by simp
    also have ... < conv_radius (fps_nth f) using K by simp
    finally show summable (λn. fps_nth f n * x ^ n)
      by (intro summable_in_conv_radius) auto
  qed (insert K r, auto)
  also have ... = (λn. fps_nth (fps_deriv f) n * of_real r ^ n)
    by (simp add: fps_deriv_def diffs_def)
  finally show ∃z::'a. norm z = r ∧ summable (λn. fps_nth (fps_deriv f) n * z
^ n)
    using r by (intro exI[of _ of_real r]) auto
qed

lemma eval_fps_at_0: eval_fps f 0 = fps_nth f 0
  by (simp add: eval_fps_def)

lemma fps_conv_radius_norm [simp]:
  fps_conv_radius (Abs_fps (λn. norm (fps_nth f n))) = fps_conv_radius f
  by (simp add: fps_conv_radius_def)

lemma fps_conv_radius_const [simp]: fps_conv_radius (fps_const c) = ∞
proof -
  have fps_conv_radius (fps_const c) = conv_radius (λ_. 0 :: 'a)
    unfolding fps_conv_radius_def
   by (intro conv_radius_cong eventually_mono[OF eventually_gt_at_top[of 0]])
auto
  thus ?thesis by simp
qed

lemma fps_conv_radius_0 [simp]: fps_conv_radius 0 = ∞
  by (simp only: fps_const_0_eq_0 [symmetric] fps_conv_radius_const)

lemma fps_conv_radius_1 [simp]: fps_conv_radius 1 = ∞
  by (simp only: fps_const_1_eq_1 [symmetric] fps_conv_radius_const)

lemma fps_conv_radius_numeral [simp]: fps_conv_radius (numeral n) = ∞
  by (simp add: numeral_fps_const)

lemma fps_conv_radius_fps_X_power [simp]: fps_conv_radius (fps_X ^ n) =
∞
proof -
  have fps_conv_radius (fps_X ^ n :: 'a fps) = conv_radius (λ_. 0 :: 'a)
    unfolding fps_conv_radius_def
   by (intro conv_radius_cong eventually_mono[OF eventually_gt_at_top[of n]])

      (auto simp: fps_X_power_iff)
  thus ?thesis by simp
```

qed

lemma *fps_conv_radius_fps_X* [*simp*]: *fps_conv_radius fps_X* = ∞
using *fps_conv_radius_fps_X_power*[*of 1*] **by** (*simp only*: *power_one_right*)

lemma *fps_conv_radius_shift* [*simp*]:
fps_conv_radius (*fps_shift n f*) = *fps_conv_radius f*
by (*simp add*: *fps_conv_radius_def fps_shift_def conv_radius_shift*)

lemma *fps_conv_radius_cmult_left*:
c ≠ *0* ⟹ *fps_conv_radius* (*fps_const c* ∗ *f*) = *fps_conv_radius f*
unfolding *fps_conv_radius_def* **by** (*simp add*: *conv_radius_cmult_left*)

lemma *fps_conv_radius_cmult_right*:
c ≠ *0* ⟹ *fps_conv_radius* (*f* ∗ *fps_const c*) = *fps_conv_radius f*
unfolding *fps_conv_radius_def* **by** (*simp add*: *conv_radius_cmult_right*)

lemma *fps_conv_radius_uminus* [*simp*]:
fps_conv_radius (−*f*) = *fps_conv_radius f*
using *fps_conv_radius_cmult_left*[*of −1 f*]
by (*simp flip*: *fps_const_neg*)

lemma *fps_conv_radius_add*: *fps_conv_radius* (*f* + *g*) ≥ *min* (*fps_conv_radius f*) (*fps_conv_radius g*)
unfolding *fps_conv_radius_def* **using** *conv_radius_add_ge*[*of fps_nth f fps_nth g*]
by *simp*

lemma *fps_conv_radius_diff*: *fps_conv_radius* (*f* − *g*) ≥ *min* (*fps_conv_radius f*) (*fps_conv_radius g*)
using *fps_conv_radius_add*[*of f* −*g*] **by** *simp*

lemma *fps_conv_radius_mult*: *fps_conv_radius* (*f* ∗ *g*) ≥ *min* (*fps_conv_radius f*) (*fps_conv_radius g*)
using *conv_radius_mult_ge*[*of fps_nth f fps_nth g*]
by (*simp add*: *fps_mult_nth fps_conv_radius_def atLeast0AtMost*)

lemma *fps_conv_radius_power*: *fps_conv_radius* (*f* ^ *n*) ≥ *fps_conv_radius f*
proof (*induction n*)
case (*Suc n*)
hence *fps_conv_radius f* ≤ *min* (*fps_conv_radius f*) (*fps_conv_radius* (*f* ^ *n*))
by *simp*
also have ... ≤ *fps_conv_radius* (*f* ∗ *f* ^ *n*)
by (*rule fps_conv_radius_mult*)
finally show *?case* **by** *simp*
qed *simp_all*

context
begin

lemma *natfun_inverse_bound*:
fixes $f :: 'a :: \{real\_normed\_field\}$ *fps*
assumes *fps_nth f 0 = 1* **and** $\delta > 0$
and *summable*: *summable* $(\lambda n.\ norm\ (fps\_nth\ f\ (Suc\ n)) * \delta\ \hat{}\ Suc\ n)$
and *le*: $(\sum n.\ norm\ (fps\_nth\ f\ (Suc\ n)) * \delta\ \hat{}\ Suc\ n) \leq 1$
shows *norm (natfun_inverse f n) ≤ inverse (δ ^ n)*
proof (*induction n rule*: *less_induct*)
case (*less m*)
show *?case*
proof (*cases m*)
case *0*
thus *?thesis* **using** *assms* **by** (*simp add*: *field_split_simps norm_inverse norm_divide*)
next
case [*simp*]: (*Suc n*)
have *norm (natfun_inverse f (Suc n)) =*
norm $(\sum i = Suc\ 0..Suc\ n.\ fps\_nth\ f\ i * natfun\_inverse\ f\ (Suc\ n - i))$
(**is** *_ = norm ?S*) **using** *assms*
by (*simp add*: *field_simps norm_mult norm_divide del*: *sum.cl_ivl_Suc*)
also have *norm ?S ≤* $(\sum i = Suc\ 0..Suc\ n.\ norm\ (fps\_nth\ f\ i * natfun\_inverse\ f\ (Suc\ n - i)))$
by (*rule norm_sum*)
also have $\ldots \leq (\sum i = Suc\ 0..Suc\ n.\ norm\ (fps\_nth\ f\ i)\ /\ \delta\ \hat{}\ (Suc\ n - i))$
proof (*intro sum_mono*, *goal_cases*)
case (*1 i*)
have *norm (fps_nth f i * natfun_inverse f (Suc n − i)) =*
*norm (fps_nth f i) * norm (natfun_inverse f (Suc n − i))*
by (*simp add*: *norm_mult*)
also have $\ldots \leq norm\ (fps\_nth\ f\ i) * inverse\ (\delta\ \hat{}\ (Suc\ n - i))$
using *1* **by** (*intro mult_left_mono less.IH*) *auto*
also have $\ldots = norm\ (fps\_nth\ f\ i)\ /\ \delta\ \hat{}\ (Suc\ n - i)$
by (*simp add*: *field_split_simps*)
finally show *?case* **.**
qed
also have $\ldots = (\sum i = Suc\ 0..Suc\ n.\ norm\ (fps\_nth\ f\ i) * \delta\ \hat{}\ i)\ /\ \delta\ \hat{}\ Suc\ n$
by (*subst sum_divide_distrib*, *rule sum.cong*)
(*insert ‹δ > 0›*, *auto simp*: *field_simps power_diff*)
also have $(\sum i = Suc\ 0..Suc\ n.\ norm\ (fps\_nth\ f\ i) * \delta\ \hat{}\ i) =$
$(\sum i{=}0..n.\ norm\ (fps\_nth\ f\ (Suc\ i)) * \delta\ \hat{}\ (Suc\ i))$
by (*subst sum.atLeast_Suc_atMost_Suc_shift*) *simp_all*
also have $\{0..n\} = \{..<Suc\ n\}$ **by** *auto*
also have $(\sum i< Suc\ n.\ norm\ (fps\_nth\ f\ (Suc\ i)) * \delta\ \hat{}\ (Suc\ i)) \leq$
$(\sum n.\ norm\ (fps\_nth\ f\ (Suc\ n)) * \delta\ \hat{}\ (Suc\ n))$
using ‹δ > 0› **by** (*intro sum_le_suminf ballI mult_nonneg_nonneg zero_le_power summable*) *auto*
also have $\ldots \leq 1$ **by** *fact*
finally show *?thesis* **using** ‹δ > 0›
by (*simp add*: *divide_right_mono field_split_simps*)

```
  qed
qed

private lemma fps_conv_radius_inverse_pos_aux:
  fixes f :: 'a :: {banach, real_normed_field} fps
  assumes fps_nth f 0 = 1 fps_conv_radius f > 0
  shows   fps_conv_radius (inverse f) > 0
proof −
  let ?R = fps_conv_radius f
  define h where h = Abs_fps (λn. norm (fps_nth f n))
  have [simp]: fps_conv_radius h = ?R by (simp add: h_def)
  have continuous_on (eball 0 (fps_conv_radius h)) (eval_fps h)
    by (intro continuous_on_eval_fps)
  hence ∗: open (eval_fps h −‘ A ∩ eball 0 ?R) if open A for A
    using that by (subst (asm) continuous_on_open_vimage) auto
  have open (eval_fps h −‘ {..<2} ∩ eball 0 ?R)
    by (rule ∗) auto
  moreover have 0 ∈ eval_fps h −‘ {..<2} ∩ eball 0 ?R
    using assms by (auto simp: eball_def zero_ereal_def eval_fps_at_0 h_def)
  ultimately obtain ε where ε: ε > 0 ball 0 ε ⊆ eval_fps h −‘ {..<2} ∩ eball 0
?R
    by (subst (asm) open_contains_ball_eq) blast+

  define δ where δ = real_of_ereal (min (ereal ε / 2) (?R / 2))
  have δ: 0 < δ ∧ δ < ε ∧ ereal δ < ?R
    using ‹ε > 0› and assms by (cases ?R) (auto simp: δ_def min_def)

  have summable: summable (λn. norm (fps_nth f n) ∗ δ ^ n)
   using δ by (intro summable_in_conv_radius) (simp_all add: fps_conv_radius_def)
  hence (λn. norm (fps_nth f n) ∗ δ ^ n) sums eval_fps h δ
    by (simp add: eval_fps_def summable_sums h_def)
  hence (λn. norm (fps_nth f (Suc n)) ∗ δ ^ Suc n) sums (eval_fps h δ − 1)
    by (subst sums_Suc_iff) (auto simp: assms)
  moreover {
    from δ have δ ∈ ball 0 ε by auto
    also have ... ⊆ eval_fps h −‘ {..<2} ∩ eball 0 ?R by fact
    finally have eval_fps h δ < 2 by simp
  }
  ultimately have le: (∑ n. norm (fps_nth f (Suc n)) ∗ δ ^ Suc n) ≤ 1
    by (simp add: sums_iff)
  from summable have summable: summable (λn. norm (fps_nth f (Suc n)) ∗ δ
^ Suc n)
    by (subst summable_Suc_iff)

  have 0 < δ using δ by blast
  also have δ = inverse (limsup (λn. ereal (inverse δ)))
    using δ by (subst Limsup_const) auto
  also have ... ≤ conv_radius (natfun_inverse f)
    unfolding conv_radius_def
```

```
  proof (intro ereal_inverse_antimono Limsup_mono
         eventually_mono[OF eventually_gt_at_top[of 0]])
    fix n :: nat assume n: n > 0
    have root n (norm (natfun_inverse f n)) ≤ root n (inverse (δ ^ n))
      using n assms δ le summable
      by (intro real_root_le_mono natfun_inverse_bound) auto
    also have ... = inverse δ
      using n δ by (simp add: power_inverse [symmetric] real_root_pos2)
    finally show ereal (inverse δ) ≥ ereal (root n (norm (natfun_inverse f n)))
      by (subst ereal_less_eq)
  next
    have 0 = limsup (λn. 0::ereal)
      by (rule Limsup_const [symmetric]) auto
    also have ... ≤ limsup (λn. ereal (root n (norm (natfun_inverse f n))))
      by (intro Limsup_mono) (auto simp: real_root_ge_zero)
    finally show 0 ≤ ... by simp
  qed
  also have ... = fps_conv_radius (inverse f)
    using assms by (simp add: fps_conv_radius_def fps_inverse_def)
  finally show ?thesis by (simp add: zero_ereal_def)
qed

lemma fps_conv_radius_inverse_pos:
  fixes f :: 'a :: {banach, real_normed_field} fps
  assumes fps_nth f 0 ≠ 0 and fps_conv_radius f > 0
  shows   fps_conv_radius (inverse f) > 0
proof -
  let ?c = fps_nth f 0
  have fps_conv_radius (inverse f) = fps_conv_radius (fps_const ?c * inverse f)
    using assms by (subst fps_conv_radius_cmult_left) auto
  also have fps_const ?c * inverse f = inverse (fps_const (inverse ?c) * f)
    using assms by (simp add: fps_inverse_mult fps_const_inverse)
  also have fps_conv_radius ... > 0 using assms
    by (intro fps_conv_radius_inverse_pos_aux)
       (auto simp: fps_conv_radius_cmult_left)
  finally show ?thesis .
qed

end

lemma fps_conv_radius_exp [simp]:
  fixes c :: 'a :: {banach, real_normed_field}
  shows fps_conv_radius (fps_exp c) = ∞
  unfolding fps_conv_radius_def
proof (rule conv_radius_inftyI'')
  fix z :: 'a
  have (λn. norm (c * z) ^ n /R fact n) sums exp (norm (c * z))
    by (rule exp_converges)
  also have (λn. norm (c * z) ^ n /R fact n) = (λn. norm (fps_nth (fps_exp c)
```

$n * z \mathbin{\hat{}} n))$
by (*rule ext*) (*simp add*: *norm_divide norm_mult norm_power field_split_simps*)
finally have *summable* ... **by** (*simp add*: *sums_iff*)
thus *summable* (λn. *fps_nth* (*fps_exp c*) $n * z \mathbin{\hat{}} n$)
by (*rule summable_norm_cancel*)
qed

9.35.4 Evaluating power series

theorem *eval_fps_deriv*:
assumes *norm z* < *fps_conv_radius f*
shows *eval_fps* (*fps_deriv f*) *z* = *deriv* (*eval_fps f*) *z*
by (*intro DERIV_imp_deriv* [*symmetric*] *has_field_derivative_eval_fps assms*)

theorem *fps_nth_conv_deriv*:
fixes *f* :: *complex fps*
assumes *fps_conv_radius f* > *0*
shows *fps_nth f n* = (*deriv* $\mathbin{\hat{}\hat{}}$ *n*) (*eval_fps f*) *0* / *fact n*
using *assms*
proof (*induction n arbitrary*: *f*)
case *0*
thus *?case* **by** (*simp add*: *eval_fps_def*)
next
case (*Suc n f*)
have (*deriv* $\mathbin{\hat{}\hat{}}$ *Suc n*) (*eval_fps f*) *0* = (*deriv* $\mathbin{\hat{}\hat{}}$ *n*) (*deriv* (*eval_fps f*)) *0*
unfolding *funpow_Suc_right o_def* **..**
also have *eventually* (λz::*complex*. *z* ∈ *eball 0* (*fps_conv_radius f*)) (*nhds 0*)
using *Suc.prems* **by** (*intro eventually_nhds_in_open*) (*auto simp*: *zero_ereal_def*)
hence *eventually* (λz. *deriv* (*eval_fps f*) *z* = *eval_fps* (*fps_deriv f*) *z*) (*nhds 0*)
by *eventually_elim* (*simp add*: *eval_fps_deriv*)
hence (*deriv* $\mathbin{\hat{}\hat{}}$ *n*) (*deriv* (*eval_fps f*)) *0* = (*deriv* $\mathbin{\hat{}\hat{}}$ *n*) (*eval_fps* (*fps_deriv f*)) *0*
by (*intro higher_deriv_cong_ev refl*)
also have ... / *fact n* = *fps_nth* (*fps_deriv f*) *n*
using *Suc.prems fps_conv_radius_deriv*[*of f*]
by (*intro Suc.IH* [*symmetric*]) *auto*
also have ... / *of_nat* (*Suc n*) = *fps_nth f* (*Suc n*)
by (*simp add*: *fps_deriv_def del*: *of_nat_Suc*)
finally show *?case* **by** (*simp add*: *field_split_simps*)
qed

theorem *eval_fps_eqD*:
fixes *f g* :: *complex fps*
assumes *fps_conv_radius f* > *0 fps_conv_radius g* > *0*
assumes *eventually* (λz. *eval_fps f z* = *eval_fps g z*) (*nhds 0*)
shows *f* = *g*
proof (*rule fps_ext*)
fix *n* :: *nat*
have *fps_nth f n* = (*deriv* $\mathbin{\hat{}\hat{}}$ *n*) (*eval_fps f*) *0* / *fact n*

```
    using assms by (intro fps_nth_conv_deriv)
  also have (deriv ^^ n) (eval_fps f) 0 = (deriv ^^ n) (eval_fps g) 0
    by (intro higher_deriv_cong_ev refl assms)
  also have ... / fact n = fps_nth g n
    using assms by (intro fps_nth_conv_deriv [symmetric])
  finally show fps_nth f n = fps_nth g n .
qed

lemma eval_fps_const [simp]:
  fixes c :: 'a :: {banach, real_normed_div_algebra}
  shows eval_fps (fps_const c) z = c
proof -
  have (λn::nat. if n ∈ {0} then c else 0) sums (∑ n∈{0::nat}. c)
    by (rule sums_If_finite_set) auto
  also have ?this ⟷ (λn::nat. fps_nth (fps_const c) n * z ^ n) sums (∑ n∈{0::nat}.
c)
    by (intro sums_cong) auto
  also have (∑ n∈{0::nat}. c) = c
    by simp
  finally show ?thesis
    by (simp add: eval_fps_def sums_iff)
qed

lemma eval_fps_0 [simp]:
  eval_fps (0 :: 'a :: {banach, real_normed_div_algebra} fps) z = 0
  by (simp only: fps_const_0_eq_0 [symmetric] eval_fps_const)

lemma eval_fps_1 [simp]:
  eval_fps (1 :: 'a :: {banach, real_normed_div_algebra} fps) z = 1
  by (simp only: fps_const_1_eq_1 [symmetric] eval_fps_const)

lemma eval_fps_numeral [simp]:
  eval_fps (numeral n :: 'a :: {banach, real_normed_div_algebra} fps) z = numeral
n
  by (simp only: numeral_fps_const eval_fps_const)

lemma eval_fps_X_power [simp]:
  eval_fps (fps_X ^ m :: 'a :: {banach, real_normed_div_algebra} fps) z = z ^ m
proof -
  have (λn::nat. if n ∈ {m} then z ^ n else 0 :: 'a) sums (∑ n∈{m::nat}. z ^ n)
    by (rule sums_If_finite_set) auto
  also have ?this ⟷ (λn::nat. fps_nth (fps_X ^ m) n * z ^ n) sums (∑ n∈{m::nat}.
z ^ n)
    by (intro sums_cong) (auto simp: fps_X_power_iff)
  also have (∑ n∈{m::nat}. z ^ n) = z ^ m
    by simp
  finally show ?thesis
    by (simp add: eval_fps_def sums_iff)
qed
```

lemma *eval_fps_X* [*simp*]:
eval_fps (*fps_X* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*) *z* = *z*
using *eval_fps_X_power*[*of 1 z*] **by** (*simp only*: *power_one_right*)

lemma *eval_fps_minus*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
assumes *norm z* < *fps_conv_radius f*
shows *eval_fps* (−*f*) *z* = −*eval_fps f z*
using *assms* **unfolding** *eval_fps_def*
by (*subst suminf_minus* [*symmetric*]) (*auto intro*!: *summable_fps*)

lemma *eval_fps_add*:
fixes *f g* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
assumes *norm z* < *fps_conv_radius f norm z* < *fps_conv_radius g*
shows *eval_fps* (*f* + *g*) *z* = *eval_fps f z* + *eval_fps g z*
using *assms* **unfolding** *eval_fps_def*
by (*subst suminf_add*) (*auto simp*: *ring_distribs intro*!: *summable_fps*)

lemma *eval_fps_diff*:
fixes *f g* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
assumes *norm z* < *fps_conv_radius f norm z* < *fps_conv_radius g*
shows *eval_fps* (*f* − *g*) *z* = *eval_fps f z* − *eval_fps g z*
using *assms* **unfolding** *eval_fps_def*
by (*subst suminf_diff*) (*auto simp*: *ring_distribs intro*!: *summable_fps*)

lemma *eval_fps_mult*:
fixes *f g* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*} *fps*
assumes *norm z* < *fps_conv_radius f norm z* < *fps_conv_radius g*
shows *eval_fps* (*f* ∗ *g*) *z* = *eval_fps f z* ∗ *eval_fps g z*
proof −
have *eval_fps f z* ∗ *eval_fps g z* =
(∑ *k*. ∑ *i*≤*k*. *fps_nth f i* ∗ *fps_nth g* (*k* − *i*) ∗ (*z* ^ *i* ∗ *z* ^ (*k* − *i*)))
unfolding *eval_fps_def*
proof (*subst Cauchy_product*)
show *summable* (λ*k*. *norm* (*fps_nth f k* ∗ *z* ^ *k*)) *summable* (λ*k*. *norm* (*fps_nth g k* ∗ *z* ^ *k*))
by (*rule norm_summable_fps assms*)+
qed (*simp_all add*: *algebra_simps*)
also have (λ*k*. ∑ *i*≤*k*. *fps_nth f i* ∗ *fps_nth g* (*k* − *i*) ∗ (*z* ^ *i* ∗ *z* ^ (*k* − *i*))) =
(λ*k*. ∑ *i*≤*k*. *fps_nth f i* ∗ *fps_nth g* (*k* − *i*) ∗ *z* ^ *k*)
by (*intro ext sum.cong refl*) (*simp add*: *power_add* [*symmetric*])
also have *suminf* ... = *eval_fps* (*f* ∗ *g*) *z*
by (*simp add*: *eval_fps_def fps_mult_nth atLeast0AtMost sum_distrib_right*)
finally show *?thesis* **..**
qed

lemma *eval_fps_shift*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*} *fps*

assumes $n \leq$ *subdegree f norm z* $<$ *fps_conv_radius f*
shows *eval_fps* (*fps_shift n f*) *z* = (*if z* = *0 then fps_nth f n else eval_fps f z* / *z* ^ *n*)
proof (*cases z* = *0*)
case *False*
have *eval_fps* (*fps_shift n f* ∗ *fps_X* ^ *n*) *z* = *eval_fps* (*fps_shift n f*) *z* ∗ *z* ^ *n*
using *assms* **by** (*subst eval_fps_mult*) *simp_all*
also from *assms* **have** *fps_shift n f* ∗ *fps_X* ^ *n* = *f*
by (*simp add*: *fps_shift_times_fps_X_power*)
finally show *?thesis* **using** *False* **by** (*simp add*: *field_simps*)
qed (*simp_all add*: *eval_fps_at_0*)

lemma *eval_fps_exp* [*simp*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_field*}
shows *eval_fps* (*fps_exp c*) *z* = *exp* (*c* ∗ *z*) **unfolding** *eval_fps_def exp_def*
by (*simp add*: *eval_fps_def exp_def scaleR_conv_of_real field_split_simps*)

The case of division is more complicated and will therefore not be handled here. Handling division becomes much more easy using complex analysis, and we will do so once that is available.

9.35.5 Power series expansions of analytic functions

This predicate contains the notion that the given formal power series converges in some disc of positive radius around the origin and is equal to the given complex function there.

This relationship is unique in the sense that no complex function can have more than one formal power series to which it expands, and if two holomorphic functions that are holomorphic on a connected open set around the origin and have the same power series expansion, they must be equal on that set.

More concrete statements about the radius of convergence can usually be made, but for many purposes, the statment that the series converges to the function in some neighbourhood of the origin is enough, and that can be shown almost fully automatically in most cases, as there are straightforward introduction rules to show this.

In particular, when one wants to relate the coefficients of the power series to the values of the derivatives of the function at the origin, or if one wants to approximate the coefficients of the series with the singularities of the function, this predicate is enough.

definition
has_fps_expansion :: (*'a* :: {*banach*,*real_normed_div_algebra*} ⇒ *'a*) ⇒ *'a fps* ⇒ *bool*
(**infixl** ‹*has'_fps'_expansion*› *60*)
where (*f has_fps_expansion F*) ⟷
fps_conv_radius F > *0* ∧ *eventually* (λ*z*. *eval_fps F z* = *f z*) (*nhds 0*)

named_theorems *fps_expansion_intros*

lemma *has_fps_expansion_schematicI*:
f has_fps_expansion A $\Longrightarrow$ *A = B* $\Longrightarrow$ *f has_fps_expansion B*
by *simp*

lemma *fps_nth_fps_expansion*:
fixes *f* :: *complex* $\Rightarrow$ *complex*
assumes *f has_fps_expansion F*
shows *fps_nth F n = (deriv* $\hat{}\hat{}$ *n) f 0 / fact n*
proof −
have *fps_nth F n = (deriv* $\hat{}\hat{}$ *n) (eval_fps F) 0 / fact n*
using *assms* **by** (*intro fps_nth_conv_deriv*) (*auto simp*: *has_fps_expansion_def*)
also have *(deriv* $\hat{}\hat{}$ *n) (eval_fps F) 0 = (deriv* $\hat{}\hat{}$ *n) f 0*
using *assms* **by** (*intro higher_deriv_cong_ev*) (*auto simp*: *has_fps_expansion_def*)
finally show *?thesis* **.**
qed

lemma *has_fps_expansion_imp_continuous*:
fixes *F* :: *'a*::{*real_normed_field*,*banach*} *fps*
assumes *f has_fps_expansion F*
shows *continuous (at 0 within A) f*
proof −
from *assms* **have** *isCont (eval_fps F) 0*
by (*intro continuous_eval_fps*) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
also have *?this* $\longleftrightarrow$ *isCont f 0* **using** *assms*
by (*intro isCont_cong*) (*auto simp*: *has_fps_expansion_def*)
finally have *isCont f 0* **.**
thus *continuous (at 0 within A) f*
by (*simp add*: *continuous_at_imp_continuous_within*)
qed

lemma *has_fps_expansion_const* [*simp*, *intro*, *fps_expansion_intros*]:
(λ*_. c) has_fps_expansion fps_const c*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_0* [*simp*, *intro*, *fps_expansion_intros*]:
(λ*_. 0) has_fps_expansion 0*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_1* [*simp*, *intro*, *fps_expansion_intros*]:
(λ*_. 1) has_fps_expansion 1*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_numeral* [*simp*, *intro*, *fps_expansion_intros*]:
(λ*_. numeral n) has_fps_expansion numeral n*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_fps_X_power* [*fps_expansion_intros*]:
(λ*x*. *x* ^ *n*) *has_fps_expansion* (*fps_X* ^ *n*)
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_fps_X* [*fps_expansion_intros*]:
(λ*x*. *x*) *has_fps_expansion fps_X*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_cmult_left* [*fps_expansion_intros*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*}
assumes *f has_fps_expansion F*
shows (λ*x*. *c* ∗ *f x*) *has_fps_expansion fps_const c* ∗ *F*
proof (*cases c* = *0*)
case *False*
from *assms* **have** *eventually* (λ*z*. *z* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
moreover from *assms* **have** *eventually* (λ*z*. *eval_fps F z* = *f z*) (*nhds 0*)
by (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*z*. *eval_fps* (*fps_const c* ∗ *F*) *z* = *c* ∗ *f z*) (*nhds 0*)
by *eventually_elim* (*simp_all add*: *eval_fps_mult*)
with *assms* **and** *False* **show** *?thesis*
by (*auto simp*: *has_fps_expansion_def fps_conv_radius_cmult_left*)
qed *auto*

lemma *has_fps_expansion_cmult_right* [*fps_expansion_intros*]:
fixes *c* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*}
assumes *f has_fps_expansion F*
shows (λ*x*. *f x* ∗ *c*) *has_fps_expansion F* ∗ *fps_const c*
proof −
have *F* ∗ *fps_const c* = *fps_const c* ∗ *F*
by (*intro fps_ext*) (*auto simp*: *mult.commute*)
with *has_fps_expansion_cmult_left* [*OF assms*] **show** *?thesis*
by (*simp add*: *mult.commute*)
qed

lemma *has_fps_expansion_minus* [*fps_expansion_intros*]:
assumes *f has_fps_expansion F*
shows (λ*x*. − *f x*) *has_fps_expansion* −*F*
proof −
from *assms* **have** *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
moreover from *assms* **have** *eventually* (λ*x*. *eval_fps F x* = *f x*) (*nhds 0*)
by (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*x*. *eval_fps* (−*F*) *x* = −*f x*) (*nhds 0*)
by *eventually_elim* (*auto simp*: *eval_fps_minus*)
thus *?thesis* **using** *assms* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_add* [*fps_expansion_intros*]:
assumes *f has_fps_expansion F g has_fps_expansion G*
shows (λ*x*. *f x* + *g x*) *has_fps_expansion F* + *G*
proof −
from *assms* **have** *0* < *min* (*fps_conv_radius F*) (*fps_conv_radius G*)
by (*auto simp*: *has_fps_expansion_def*)
also have ... ≤ *fps_conv_radius* (*F* + *G*)
by (*rule fps_conv_radius_add*)
finally have *radius*: ... > *0* **.**

from *assms* **have** *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
eventually (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius G*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*; *force simp*: *has_fps_expansion_def zero_ereal_def*)+
moreover have *eventually* (λ*x*. *eval_fps F x* = *f x*) (*nhds 0*)
and *eventually* (λ*x*. *eval_fps G x* = *g x*) (*nhds 0*)
using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*x*. *eval_fps* (*F* + *G*) *x* = *f x* + *g x*) (*nhds 0*)
by *eventually_elim* (*auto simp*: *eval_fps_add*)
with *radius* **show** *?thesis* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_diff* [*fps_expansion_intros*]:
assumes *f has_fps_expansion F g has_fps_expansion G*
shows (λ*x*. *f x* − *g x*) *has_fps_expansion F* − *G*
using *has_fps_expansion_add*[*of f F* λ*x*. − *g x* −*G*] *assms*
by (*simp add*: *has_fps_expansion_minus*)

lemma *has_fps_expansion_mult* [*fps_expansion_intros*]:
fixes *F G* :: *'a* :: {*banach*, *real_normed_div_algebra*, *comm_ring_1*} *fps*
assumes *f has_fps_expansion F g has_fps_expansion G*
shows (λ*x*. *f x* ∗ *g x*) *has_fps_expansion F* ∗ *G*
proof −
from *assms* **have** *0* < *min* (*fps_conv_radius F*) (*fps_conv_radius G*)
by (*auto simp*: *has_fps_expansion_def*)
also have ... ≤ *fps_conv_radius* (*F* ∗ *G*)
by (*rule fps_conv_radius_mult*)
finally have *radius*: ... > *0* **.**

from *assms* **have** *eventually* (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius F*)) (*nhds 0*)
eventually (λ*x*. *x* ∈ *eball 0* (*fps_conv_radius G*)) (*nhds 0*)
by (*intro eventually_nhds_in_open*; *force simp*: *has_fps_expansion_def zero_ereal_def*)+
moreover have *eventually* (λ*x*. *eval_fps F x* = *f x*) (*nhds 0*)
and *eventually* (λ*x*. *eval_fps G x* = *g x*) (*nhds 0*)
using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* (λ*x*. *eval_fps* (*F* ∗ *G*) *x* = *f x* ∗ *g x*) (*nhds 0*)
by *eventually_elim* (*auto simp*: *eval_fps_mult*)
with *radius* **show** *?thesis* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_inverse* [*fps_expansion_intros*]:
fixes $F :: 'a :: \{banach, real\_normed\_field\}$ *fps*
assumes *f has_fps_expansion F*
assumes *fps_nth F 0* $\neq$ *0*
shows $(\lambda x.\ inverse\ (f\ x))$ *has_fps_expansion inverse F*
proof −
have *radius*: *fps_conv_radius* (*inverse F*) > *0*
using *assms* **unfolding** *has_fps_expansion_def*
by (*intro fps_conv_radius_inverse_pos*) *auto*
let *?R* = *min* (*fps_conv_radius F*) (*fps_conv_radius* (*inverse F*))
from *assms radius*
have *eventually* $(\lambda x.\ x \in eball\ 0\ (fps\_conv\_radius\ F))$ (*nhds 0*)
eventually $(\lambda x.\ x \in eball\ 0\ (fps\_conv\_radius\ (inverse\ F)))$ (*nhds 0*)
by (*intro eventually_nhds_in_open*; *force simp*: *has_fps_expansion_def zero_ereal_def*)+
moreover have *eventually* $(\lambda z.\ eval\_fps\ F\ z = f\ z)$ (*nhds 0*)
using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* $(\lambda z.\ eval\_fps\ (inverse\ F)\ z = inverse\ (f\ z))$ (*nhds 0*)
proof *eventually_elim*
case (*elim z*)
hence *eval_fps* (*inverse F* ∗ *F*) *z* = *eval_fps* (*inverse F*) *z* ∗ *f z*
by (*subst eval_fps_mult*) *auto*
also have *eval_fps* (*inverse F* ∗ *F*) *z* = *1*
using *assms* **by** (*simp add*: *inverse_mult_eq_1*)
finally show *?case* **by** (*auto simp*: *field_split_simps*)
qed
with *radius* **show** *?thesis* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_exp* [*fps_expansion_intros*]:
fixes $c :: 'a :: \{banach, real\_normed\_field\}$
shows $(\lambda x.\ exp\ (c * x))$ *has_fps_expansion fps_exp c*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_exp1* [*fps_expansion_intros*]:
$(\lambda x{::}'a :: \{banach, real\_normed\_field\}.\ exp\ x)$ *has_fps_expansion fps_exp 1*
using *has_fps_expansion_exp*[*of 1*] **by** *simp*

lemma *has_fps_expansion_exp_neg1* [*fps_expansion_intros*]:
$(\lambda x{::}'a :: \{banach, real\_normed\_field\}.\ exp\ (-x))$ *has_fps_expansion fps_exp* (−*1*)
using *has_fps_expansion_exp*[*of* −*1*] **by** *simp*

lemma *has_fps_expansion_deriv* [*fps_expansion_intros*]:
assumes *f has_fps_expansion F*
shows *deriv f has_fps_expansion fps_deriv F*
proof −
have *eventually* $(\lambda z.\ z \in eball\ 0\ (fps\_conv\_radius\ F))$ (*nhds 0*)
using *assms* **by** (*intro eventually_nhds_in_open*)

```
                (auto simp: has_fps_expansion_def zero_ereal_def)
  moreover from assms have eventually (λz. eval_fps F z = f z) (nhds 0)
    by (auto simp: has_fps_expansion_def)
  then obtain s where open s 0 ∈ s and s: ⋀w. w ∈ s ⟹ eval_fps F w = f w
    by (auto simp: eventually_nhds)
  hence eventually (λw. w ∈ s) (nhds 0)
    by (intro eventually_nhds_in_open) auto
  ultimately have eventually (λz. eval_fps (fps_deriv F) z = deriv f z) (nhds 0)
  proof eventually_elim
    case (elim z)
    hence eval_fps (fps_deriv F) z = deriv (eval_fps F) z
      by (simp add: eval_fps_deriv)
    also have eventually (λw. w ∈ s) (nhds z)
      using elim and ‹open s› by (intro eventually_nhds_in_open) auto
    hence eventually (λw. eval_fps F w = f w) (nhds z)
      by eventually_elim (simp add: s)
    hence deriv (eval_fps F) z = deriv f z
      by (intro deriv_cong_ev refl)
    finally show ?case .
  qed
  with assms and fps_conv_radius_deriv[of F] show ?thesis
    by (auto simp: has_fps_expansion_def)
qed

lemma fps_conv_radius_binomial:
  fixes c :: 'a :: {real_normed_field,banach}
  shows fps_conv_radius (fps_binomial c) = (if c ∈ ℕ then ∞ else 1)
  unfolding fps_conv_radius_def by (simp add: conv_radius_gchoose)

lemma fps_conv_radius_ln:
  fixes c :: 'a :: {banach, real_normed_field, field_char_0}
  shows fps_conv_radius (fps_ln c) = (if c = 0 then ∞ else 1)
proof (cases c = 0)
  case False
  have conv_radius (λn. 1 / of_nat n :: 'a) = 1
  proof (rule conv_radius_ratio_limit_nonzero)
    show (λn. norm (1 / of_nat n :: 'a) / norm (1 / of_nat (Suc n) :: 'a)) ⟶
1
      using LIMSEQ_Suc_n_over_n by (simp add: norm_divide del: of_nat_Suc)
  qed auto
  also have conv_radius (λn. 1 / of_nat n :: 'a) =
              conv_radius (λn. if n = 0 then 0 else (- 1) ^ (n - 1) / of_nat n ::
'a)
    by (intro conv_radius_cong eventually_mono[OF eventually_gt_at_top[of 0]])
       (simp add: norm_mult norm_divide norm_power)
  finally show ?thesis using False unfolding fps_ln_def
    by (subst fps_conv_radius_cmult_left) (simp_all add: fps_conv_radius_def)
qed (auto simp: fps_ln_def)
```

lemma *fps_conv_radius_ln_nonzero* [*simp*]:
assumes $c \neq (0 :: 'a :: \{banach, real\_normed\_field, field\_char\_0\})$
shows *fps_conv_radius* (*fps_ln c*) = *1*
using *assms* **by** (*simp add*: *fps_conv_radius_ln*)

lemma *fps_conv_radius_sin* [*simp*]:
fixes $c :: 'a :: \{banach, real\_normed\_field, field\_char\_0\}$
shows *fps_conv_radius* (*fps_sin c*) = ∞
proof (*cases c = 0*)
case *False*
have ∞ = *conv_radius* ($\lambda n.$ *of_real* (*sin_coeff n*) :: *'a*)
proof (*rule sym*, *rule conv_radius_inftyI''*, *rule summable_norm_cancel*, *goal_cases*)
case (*1 z*)
show *?case* **using** *summable_norm_sin*[*of z*] **by** (*simp add*: *norm_mult*)
qed
also have ... / *norm c* = *conv_radius* ($\lambda n.$ *c* ^ *n* * *of_real* (*sin_coeff n*) :: *'a*)
using *False* **by** (*subst conv_radius_mult_power*) *auto*
also have ... = *fps_conv_radius* (*fps_sin c*) **unfolding** *fps_conv_radius_def*
by (*rule conv_radius_cong_weak*) (*auto simp add*: *fps_sin_def sin_coeff_def*)
finally show *?thesis* **by** *simp*
qed *simp_all*

lemma *fps_conv_radius_cos* [*simp*]:
fixes $c :: 'a :: \{banach, real\_normed\_field, field\_char\_0\}$
shows *fps_conv_radius* (*fps_cos c*) = ∞
proof (*cases c = 0*)
case *False*
have ∞ = *conv_radius* ($\lambda n.$ *of_real* (*cos_coeff n*) :: *'a*)
proof (*rule sym*, *rule conv_radius_inftyI''*, *rule summable_norm_cancel*, *goal_cases*)
case (*1 z*)
show *?case* **using** *summable_norm_cos*[*of z*] **by** (*simp add*: *norm_mult*)
qed
also have ... / *norm c* = *conv_radius* ($\lambda n.$ *c* ^ *n* * *of_real* (*cos_coeff n*) :: *'a*)
using *False* **by** (*subst conv_radius_mult_power*) *auto*
also have ... = *fps_conv_radius* (*fps_cos c*) **unfolding** *fps_conv_radius_def*
by (*rule conv_radius_cong_weak*) (*auto simp add*: *fps_cos_def cos_coeff_def*)
finally show *?thesis* **by** *simp*
qed *simp_all*

lemma *eval_fps_sin* [*simp*]:
fixes $z :: 'a :: \{banach, real\_normed\_field, field\_char\_0\}$
shows *eval_fps* (*fps_sin c*) *z* = *sin* (*c* * *z*)
proof −
have ($\lambda n.$ *sin_coeff n* *$_R$ (*c* * *z*) ^ *n*) *sums sin* (*c* * *z*) **by** (*rule sin_converges*)
also have ($\lambda n.$ *sin_coeff n* *$_R$ (*c* * *z*) ^ *n*) = ($\lambda n.$ *fps_nth* (*fps_sin c*) *n* * *z* ^ *n*)
by (*rule ext*) (*auto simp*: *sin_coeff_def fps_sin_def power_mult_distrib scaleR_conv_of_real*)
finally show *?thesis* **by** (*simp add*: *sums_iff eval_fps_def*)
qed

lemma *eval_fps_cos* [*simp*]:
fixes $z :: 'a :: \{banach, real\_normed\_field, field\_char\_0\}$
shows *eval_fps* (*fps_cos c*) *z* = *cos* (*c* ∗ *z*)
proof −
have (λ*n*. *cos_coeff n* ∗$_R$ (*c* ∗ *z*) ^ *n*) *sums cos* (*c* ∗ *z*) **by** (*rule cos_converges*)
also have (λ*n*. *cos_coeff n* ∗$_R$ (*c* ∗ *z*) ^ *n*) = (λ*n*. *fps_nth* (*fps_cos c*) *n* ∗ *z* ^ *n*)
by (*rule ext*) (*auto simp*: *cos_coeff_def fps_cos_def power_mult_distrib scaleR_conv_of_real*)
finally show *?thesis* **by** (*simp add*: *sums_iff eval_fps_def*)
qed

lemma *cos_eq_zero_imp_norm_ge*:
assumes *cos* (*z* :: *complex*) = *0*
shows *norm z* ≥ *pi* / *2*
proof −
from *assms* **obtain** *n* **where** *z* = *complex_of_real* ((*of_int n* + *1* / *2*) ∗ *pi*)
by (*auto simp*: *cos_eq_0 algebra_simps*)
also have *norm* ... = |*real_of_int n* + *1* / *2*| ∗ *pi*
by (*subst norm_of_real*) (*simp_all add*: *abs_mult*)
also have *real_of_int n* + *1* / *2* = *of_int* (*2* ∗ *n* + *1*) / *2* **by** *simp*
also have |...| = *of_int* |*2* ∗ *n* + *1*| / *2* **by** (*subst abs_divide*) *simp_all*
also have ... ∗ *pi* = *of_int* |*2* ∗ *n* + *1*| ∗ (*pi* / *2*) **by** *simp*
also have ... ≥ *of_int 1* ∗ (*pi* / *2*)
by (*intro mult_right_mono*, *subst of_int_le_iff*) (*auto simp*: *abs_if*)
finally show *?thesis* **by** *simp*
qed

lemma *eval_fps_binomial*:
fixes *c* :: *complex*
assumes *norm z* < *1*
shows *eval_fps* (*fps_binomial c*) *z* = (*1* + *z*) *powr c*
using *gen_binomial_complex*[*OF assms*] **by** (*simp add*: *sums_iff eval_fps_def*)

lemma *has_fps_expansion_binomial_complex* [*fps_expansion_intros*]:
fixes *c* :: *complex*
shows (λ*x*. (*1* + *x*) *powr c*) *has_fps_expansion fps_binomial c*
proof −
have ∗: *eventually* (λ*z*::*complex*. *z* ∈ *eball 0 1*) (*nhds 0*)
by (*intro eventually_nhds_in_open*) *auto*
thus *?thesis*
by (*auto simp*: *has_fps_expansion_def eval_fps_binomial fps_conv_radius_binomial*
intro!: *eventually_mono* [*OF* ∗])
qed

lemma *has_fps_expansion_sin* [*fps_expansion_intros*]:
fixes *c* :: $'a :: \{banach, real\_normed\_field, field\_char\_0\}$

shows $(\lambda x.\ sin\ (c * x))$ *has_fps_expansion fps_sin c*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_sin'* [*fps_expansion_intros*]:
$(\lambda x{::}'a :: \{banach, real\_normed\_field\}.\ sin\ x)$ *has_fps_expansion fps_sin 1*
using *has_fps_expansion_sin*[*of 1*] **by** *simp*

lemma *has_fps_expansion_cos* [*fps_expansion_intros*]:
fixes $c :: 'a :: \{banach, real\_normed\_field, field\_char\_0\}$
shows $(\lambda x.\ cos\ (c * x))$ *has_fps_expansion fps_cos c*
by (*auto simp*: *has_fps_expansion_def*)

lemma *has_fps_expansion_cos'* [*fps_expansion_intros*]:
$(\lambda x{::}'a :: \{banach, real\_normed\_field\}.\ cos\ x)$ *has_fps_expansion fps_cos 1*
using *has_fps_expansion_cos*[*of 1*] **by** *simp*

lemma *has_fps_expansion_shift* [*fps_expansion_intros*]:
fixes $F :: 'a :: \{banach, real\_normed\_field\}$ *fps*
assumes *f has_fps_expansion F* **and** $n \leq$ *subdegree F*
assumes $c =$ *fps_nth F n*
shows $(\lambda x.\ if\ x = 0\ then\ c\ else\ f\ x\ /\ x\ \hat{}\ n)$ *has_fps_expansion* (*fps_shift n F*)
proof −
have *eventually* $(\lambda x.\ x \in eball\ 0\ (fps\_conv\_radius\ F))$ (*nhds 0*)
using *assms* **by** (*intro eventually_nhds_in_open*) (*auto simp*: *has_fps_expansion_def zero_ereal_def*)
moreover have *eventually* $(\lambda x.\ eval\_fps\ F\ x = f\ x)$ (*nhds 0*)
using *assms* **by** (*auto simp*: *has_fps_expansion_def*)
ultimately have *eventually* $(\lambda x.\ eval\_fps\ (fps\_shift\ n\ F)\ x =$
$(if\ x = 0\ then\ c\ else\ f\ x\ /\ x\ \hat{}\ n))$ (*nhds 0*)
by *eventually_elim* (*auto simp*: *eval_fps_shift assms*)
with *assms* **show** *?thesis* **by** (*auto simp*: *has_fps_expansion_def*)
qed

lemma *has_fps_expansion_divide* [*fps_expansion_intros*]:
fixes $F\ G :: 'a :: \{banach, real\_normed\_field\}$ *fps*
assumes *f has_fps_expansion F* **and** *g has_fps_expansion G* **and**
$subdegree\ G \leq subdegree\ F\ G \neq 0$
$c = fps\_nth\ F\ (subdegree\ G)\ /\ fps\_nth\ G\ (subdegree\ G)$
shows $(\lambda x.\ if\ x = 0\ then\ c\ else\ f\ x\ /\ g\ x)$ *has_fps_expansion* $(F\ /\ G)$
proof −
define *n* **where** $n =$ *subdegree G*
define F' **and** G' **where** $F' =$ *fps_shift n F* **and** $G' =$ *fps_shift n G*
have $F = F' * fps\_X\ \hat{}\ n\ G = G' * fps\_X\ \hat{}\ n$ **unfolding** $F'\_def\ G'\_def\ n\_def$

by (*rule fps_shift_times_fps_X_power* [*symmetric*] *le_refl* | *fact*)+
moreover from *assms* **have** $fps\_nth\ G'\ 0 \neq 0$
by (*simp add*: $G'\_def\ n\_def$)
ultimately have *FG*: $F\ /\ G = F' * inverse\ G'$
by (*simp add*: *fps_divide_unit*)

```
  have (λx. (if x = 0 then fps_nth F n else f x / x ^ n) *
        inverse (if x = 0 then fps_nth G n else g x / x ^ n)) has_fps_expansion
F / G
    (is ?h has_fps_expansion _) unfolding FG F'_def G'_def n_def using ‹G
≠ 0›
    by (intro has_fps_expansion_mult has_fps_expansion_inverse
          has_fps_expansion_shift assms) auto
  also have ?h = (λx. if x = 0 then c else f x / g x)
    using assms(5) unfolding n_def
    by (intro ext) (auto split: if_splits simp: field_simps)
  finally show ?thesis .
qed

lemma has_fps_expansion_divide' [fps_expansion_intros]:
  fixes F G :: 'a :: {banach, real_normed_field} fps
  assumes f has_fps_expansion F and g has_fps_expansion G and fps_nth G 0
≠ 0
  shows   (λx. f x / g x) has_fps_expansion (F / G)
proof −
  have (λx. if x = 0 then fps_nth F 0 / fps_nth G 0 else f x / g x) has_fps_expansion
(F / G)
    (is ?h has_fps_expansion _) using assms(3) by (intro has_fps_expansion_divide
assms) auto
  also from assms have fps_nth F 0 = f 0 fps_nth G 0 = g 0
    by (auto simp: has_fps_expansion_def eval_fps_at_0 dest: eventually_nhds_x_imp_x)
  hence ?h = (λx. f x / g x) by auto
  finally show ?thesis .
qed

lemma has_fps_expansion_tan [fps_expansion_intros]:
  fixes c :: 'a :: {banach, real_normed_field, field_char_0}
  shows (λx. tan (c * x)) has_fps_expansion fps_tan c
proof −
  have (λx. sin (c * x) / cos (c * x)) has_fps_expansion fps_sin c / fps_cos c
    by (intro fps_expansion_intros) auto
  thus ?thesis by (simp add: tan_def fps_tan_def)
qed

lemma has_fps_expansion_tan' [fps_expansion_intros]:
  tan has_fps_expansion fps_tan (1 :: 'a :: {banach, real_normed_field, field_char_0})
  using has_fps_expansion_tan[of 1] by simp

lemma has_fps_expansion_imp_holomorphic:
  assumes f has_fps_expansion F
  obtains s where open s 0 ∈ s f holomorphic_on s ⋀z. z ∈ s ⟹ f z = eval_fps
F z
proof −
  from assms obtain s where s: open s 0 ∈ s ⋀z. z ∈ s ⟹ eval_fps F z = f z
```

```
    unfolding has_fps_expansion_def eventually_nhds by blast
  let ?s' = eball 0 (fps_conv_radius F) ∩ s
  have eval_fps F holomorphic_on ?s'
    by (intro holomorphic_intros) auto
  also have ?this ⟷ f holomorphic_on ?s'
    using s by (intro holomorphic_cong) auto
  finally show ?thesis using s assms
    by (intro that[of ?s']) (auto simp: has_fps_expansion_def zero_ereal_def)
qed

end
```

9.36 Smooth paths

```
theory Smooth_Paths
  imports Retracts
begin
```

9.36.1 Homeomorphisms of arc images

```
lemma path_connected_arc_complement:
  fixes γ :: real ⇒ 'a::euclidean_space
  assumes arc γ 2 ≤ DIM('a)
  shows path_connected(− path_image γ)
proof −
  have path_image γ homeomorphic {0..1::real}
    by (simp add: assms homeomorphic_arc_image_interval)
  then show ?thesis
   by (intro path_connected_complement_homeomorphic_convex_compact) (auto
simp: assms)
qed

lemma connected_arc_complement:
  fixes γ :: real ⇒ 'a::euclidean_space
  assumes arc γ 2 ≤ DIM('a)
  shows connected(− path_image γ)
 by (simp add: assms path_connected_arc_complement path_connected_imp_connected)

lemma inside_arc_empty:
  fixes γ :: real ⇒ 'a::euclidean_space
  assumes arc γ
    shows inside(path_image γ) = {}
proof (cases DIM('a) = 1)
  case True
  then show ?thesis
   using assms connected_arc_image connected_convex_1_gen inside_convex by
blast
next
  case False
```

then have *connected* (− *path_image* γ)
by (*metis DIM_ge_Suc0 One_nat_def Suc_1 antisym assms connected_arc_complement not_less_eq_eq*)
then
show *?thesis*
by (*simp add*: *assms bounded_arc_image inside_bounded_complement_connected_empty*)
qed

lemma *inside_simple_curve_imp_closed*:
fixes γ :: *real* ⇒ *'a*::*euclidean_space*
shows ⟦*simple_path* γ; $x \in$ *inside*(*path_image* γ)⟧ ⟹ *pathfinish* γ = *pathstart* γ
using *arc_simple_path inside_arc_empty* **by** *blast*

9.36.2 Piecewise differentiability of paths

lemma *continuous_on_joinpaths_D1*:
assumes *continuous_on* {*0..1*} (*g1* +++ *g2*)
shows *continuous_on* {*0..1*} *g1*
proof (*rule continuous_on_eq*)
have *continuous_on* {*0..1/2*} (*g1* +++ *g2*)
using *assms continuous_on_subset split_01* **by** *auto*
then show *continuous_on* {*0..1*} (*g1* +++ *g2* ∘ (∗) (*inverse 2*))
by (*intro continuous_intros*) *force*
qed (*auto simp*: *joinpaths_def*)

lemma *continuous_on_joinpaths_D2*:
⟦*continuous_on* {*0..1*} (*g1* +++ *g2*); *pathfinish g1* = *pathstart g2*⟧ ⟹ *continuous_on* {*0..1*} *g2*
using *path_def path_join* **by** *blast*

lemma *piecewise_differentiable_D1*:
assumes (*g1* +++ *g2*) *piecewise_differentiable_on* {*0..1*}
shows *g1 piecewise_differentiable_on* {*0..1*}
proof −
obtain *S* **where** *cont*: *continuous_on* {*0..1*} *g1* **and** *finite S*
and *S*: ⋀*x*. *x* ∈ {*0..1*} − *S* ⟹ *g1* +++ *g2 differentiable at x within* {*0..1*}
using *assms* **unfolding** *piecewise_differentiable_on_def*
by (*blast dest*!: *continuous_on_joinpaths_D1*)
show *?thesis*
unfolding *piecewise_differentiable_on_def*
proof (*intro exI conjI ballI cont*)
show *finite* (*insert 1* (((∗)*2*) ‘ *S*))
by (*simp add*: ‹*finite S*›)
show *g1 differentiable at x within* {*0..1*} **if** *x* ∈ {*0..1*} − *insert 1* ((∗) *2* ‘ *S*)
for *x*
proof (*rule_tac d*=*dist* (*x*/*2*) (*1*/*2*) **in** *differentiable_transform_within*)
have *g1* +++ *g2 differentiable at* (*x* / *2*) *within* {*0..1/2*}
by (*rule differentiable_subset* [*OF S* [*of x*/*2*]] | *use that* **in** *force*)+

```
      then show g1 +++ g2 ∘ (*) (inverse 2) differentiable at x within {0..1}
        using image_affinity_atLeastAtMost_div [of 2 0 0::real 1]
        by (auto intro: differentiable_chain_within)
    qed (use that in ‹auto simp: joinpaths_def›)
  qed
qed

lemma piecewise_differentiable_D2:
  assumes (g1 +++ g2) piecewise_differentiable_on {0..1} and eq: pathfinish g1
= pathstart g2
  shows g2 piecewise_differentiable_on {0..1}
proof −
  have [simp]: g1 1 = g2 0
    using eq by (simp add: pathfinish_def pathstart_def)
  obtain S where cont: continuous_on {0..1} g2 and finite S
    and S: ⋀x. x ∈ {0..1} − S ⟹ g1 +++ g2 differentiable at x within {0..1}
    using assms unfolding piecewise_differentiable_on_def
    by (blast dest!: continuous_on_joinpaths_D2)
  show ?thesis
    unfolding piecewise_differentiable_on_def
  proof (intro exI conjI ballI cont)
    show finite (insert 0 ((λx. 2∗x−1) ‘S))
      by (simp add: ‹finite S›)
     show g2 differentiable at x within {0..1} if x ∈ {0..1} − insert 0 ((λx.
2∗x−1) ‘S) for x
    proof (rule_tac d=dist ((x+1)/2) (1/2) in differentiable_transform_within)
      have x2: (x + 1) / 2 ∉ S
        using that
        apply (clarsimp simp: image_iff)
       by (metis add.commute add_diff_cancel_left' mult_2 field_sum_of_halves)
      have g1 +++ g2 ∘ (λx. (x+1) / 2) differentiable at x within {0..1}
     by (rule differentiable_chain_within differentiable_subset [OF S [of (x+1)/2]]
| use x2 that in force)+
      then show g1 +++ g2 ∘ (λx. (x+1) / 2) differentiable at x within {0..1}
        by (auto intro: differentiable_chain_within)
      show (g1 +++ g2 ∘ (λx. (x + 1) / 2)) x′ = g2 x′ if x′ ∈ {0..1} dist x′ x
< dist ((x + 1) / 2) (1/2) for x′
      proof −
        have [simp]: (2∗x′+2)/2 = x′+1
          by (simp add: field_split_simps)
        show ?thesis
          using that by (auto simp: joinpaths_def)
      qed
    qed (use that in ‹auto simp: joinpaths_def›)
  qed
qed

lemma piecewise_C1_differentiable_D1:
  fixes g1 :: real ⇒ 'a::real_normed_field
```

```
  assumes (g1 +++ g2) piecewise_C1_differentiable_on {0..1}
    shows g1 piecewise_C1_differentiable_on {0..1}
proof −
  obtain S where finite S
          and co12: continuous_on ({0..1} − S) (λx. vector_derivative (g1 +++
g2) (at x))
            and g12D: ∀ x∈{0..1} − S. g1 +++ g2 differentiable at x
   using assms by (auto simp: piecewise_C1_differentiable_on_def C1_differentiable_on_eq)
  have g1D: g1 differentiable at x if x ∈ {0..1} − insert 1 ((∗) 2 ‘ S) for x
  proof (rule differentiable_transform_within)
    show g1 +++ g2 ∘ (∗) (inverse 2) differentiable at x
      using that g12D
      unfolding joinpaths_def
      by (intro differentiable_chain_at derivative_intros | force)+
    show ⋀x′. ⟦dist x′ x < dist (x/2) (1/2)⟧
         ⟹ (g1 +++ g2 ∘ (∗) (inverse 2)) x′ = g1 x′
      using that by (auto simp: dist_real_def joinpaths_def)
  qed (use that in ‹auto simp: dist_real_def›)
  have [simp]: vector_derivative (g1 ∘ (∗) 2) (at (x/2)) = 2 ∗R vector_derivative
g1 (at x)
              if x ∈ {0..1} − insert 1 ((∗) 2 ‘ S) for x
    apply (subst vector_derivative_chain_at)
    using that
    apply (rule derivative_eq_intros g1D | simp)+
    done
   have continuous_on ({0..1/2} − insert (1/2) S) (λx. vector_derivative (g1
+++ g2) (at x))
    using co12 by (rule continuous_on_subset) force
   then have coDhalf: continuous_on ({0..1/2} − insert (1/2) S) (λx. vec-
tor_derivative (g1 ∘ (∗)2) (at x))
  proof (rule continuous_on_eq [OF _ vector_derivative_at])
     show (g1 +++ g2 has_vector_derivative vector_derivative (g1 ∘ (∗) 2) (at
x)) (at x)
      if x ∈ {0..1/2} − insert (1/2) S for x
    proof (rule has_vector_derivative_transform_within)
      show (g1 ∘ (∗) 2 has_vector_derivative vector_derivative (g1 ∘ (∗) 2) (at
x)) (at x)
        using that
        by (force intro: g1D differentiable_chain_at simp: vector_derivative_works
[symmetric])
      show ⋀x′. ⟦dist x′ x < dist x (1/2)⟧ ⟹ (g1 ∘ (∗) 2) x′ = (g1 +++ g2) x′
        using that by (auto simp: dist_norm joinpaths_def)
    qed (use that in ‹auto simp: dist_norm›)
  qed
  have continuous_on ({0..1} − insert 1 ((∗) 2 ‘ S))
                     ((λx. 1/2 ∗ vector_derivative (g1 ∘ (∗)2) (at x)) ∘ (∗)(1/2))
    using coDhalf
    apply (intro continuous_intros)
    by (simp add: scaleR_conv_of_real image_set_diff image_image)
```

```
  then have con_g1: continuous_on ({0..1} − insert 1 ((∗) 2 ‘ S)) (λx. vec-
tor_derivative g1 (at x))
    by (rule continuous_on_eq) (simp add: scaleR_conv_of_real)
  have continuous_on {0..1} g1
   using continuous_on_joinpaths_D1 assms piecewise_C1_differentiable_on_def
by blast
  with ‹finite S› show ?thesis
   apply (clarsimp simp add: piecewise_C1_differentiable_on_def C1_differentiable_on_eq)
    apply (rule_tac x=insert 1 (((∗)2)‘S) in exI)
    apply (simp add: g1D con_g1)
  done
qed

lemma piecewise_C1_differentiable_D2:
  fixes g2 :: real ⇒ 'a::real_normed_field
  assumes (g1 +++ g2) piecewise_C1_differentiable_on {0..1} pathfinish g1 =
pathstart g2
    shows g2 piecewise_C1_differentiable_on {0..1}
proof −
  obtain S where finite S
          and co12: continuous_on ({0..1} − S) (λx. vector_derivative (g1 +++
g2) (at x))
            and g12D: ∀x∈{0..1} − S. g1 +++ g2 differentiable at x
   using assms by (auto simp: piecewise_C1_differentiable_on_def C1_differentiable_on_eq)
  have g2D: g2 differentiable at x if x ∈ {0..1} − insert 0 ((λx. 2∗x−1) ‘ S) for
x
  proof (rule differentiable_transform_within)
    show g1 +++ g2 ∘ (λx. (x + 1) / 2) differentiable at x
      using g12D that
      unfolding joinpaths_def
      apply (drule_tac x= (x+1) / 2 in bspec, force simp: field_split_simps)
      apply (rule differentiable_chain_at derivative_intros | force)+
      done
    show ⋀x′. dist x′ x < dist ((x + 1) / 2) (1/2) ⟹ (g1 +++ g2 ∘ (λx. (x +
1) / 2)) x′ = g2 x′
      using that by (auto simp: dist_real_def joinpaths_def field_simps)
    qed (use that in ‹auto simp: dist_norm›)
  have [simp]: vector_derivative (g2 ∘ (λx. 2∗x−1)) (at ((x+1)/2)) = 2 ∗R
vector_derivative g2 (at x)
            if x ∈ {0..1} − insert 0 ((λx. 2∗x−1) ‘ S) for x
   using that by (auto simp: vector_derivative_chain_at field_split_simps g2D)
  have continuous_on ({1/2..1} − insert (1/2) S) (λx. vector_derivative (g1
+++ g2) (at x))
    using co12 by (rule continuous_on_subset) force
  then have coDhalf: continuous_on ({1/2..1} − insert (1/2) S) (λx. vec-
tor_derivative (g2 ∘ (λx. 2∗x−1)) (at x))
  proof (rule continuous_on_eq [OF _ vector_derivative_at])
    show (g1 +++ g2 has_vector_derivative vector_derivative (g2 ∘ (λx. 2 ∗ x
− 1)) (at x))
```

(*at x*)
if $x \in \{1\ /\ 2..1\} - insert\ (1\ /\ 2)\ S$ **for** *x*
proof (*rule_tac f=g2* $\circ$ ($\lambda x.\ 2*x-1$) **and** *d=dist (3/4) ((x+1)/2)* **in** *has_vector_derivative_transfo*
show (*g2* $\circ$ ($\lambda x.\ 2 * x - 1$) *has_vector_derivative vector_derivative* (*g2* $\circ$ ($\lambda x.\ 2 * x - 1$)) (*at x*))
(*at x*)
using *that* **by** (*force intro*: *g2D differentiable_chain_at simp*: *vector_derivative_works* [*symmetric*])
show $\bigwedge x'.\ [\![dist\ x'\ x < dist\ (3\ /\ 4)\ ((x + 1)\ /\ 2)]\!] \Longrightarrow$ (*g2* $\circ$ ($\lambda x.\ 2 * x - 1$)) $x' =$ (*g1 +++ g2*) x'
using *that* **by** (*auto simp*: *dist_norm joinpaths_def add_divide_distrib*)
qed (*use that* **in** ‹*auto simp*: *dist_norm*›)
qed
have [*simp*]: (($\lambda x.\ (x+1)\ /\ 2$) ‘ ($\{0..1\} - insert\ 0\ ((\lambda x.\ 2 * x - 1)$ ‘ *S*))) = ($\{1/2..1\} - insert\ (1/2)\ S$)
apply (*simp add*: *image_set_diff inj_on_def image_image*)
apply (*auto simp*: *image_affinity_atLeastAtMost_div add_divide_distrib*)
done
have *continuous_on* ($\{0..1\} - insert\ 0\ ((\lambda x.\ 2*x-1)$ ‘ *S*))
(($\lambda x.\ 1/2 *$ *vector_derivative* (*g2* $\circ$ ($\lambda x.\ 2*x-1$)) (*at x*)) $\circ$ ($\lambda x.\ (x+1)/2$))
by (*rule continuous_intros* | *simp add*: *coDhalf*)+
then have *con_g2*: *continuous_on* ($\{0..1\} - insert\ 0\ ((\lambda x.\ 2*x-1)$ ‘ *S*)) ($\lambda x.$ *vector_derivative g2* (*at x*))
by (*rule continuous_on_eq*) (*simp add*: *scaleR_conv_of_real*)
have *continuous_on* $\{0..1\}$ *g2*
using *continuous_on_joinpaths_D2 assms piecewise_C1_differentiable_on_def*
by *blast*
with ‹*finite S*› **show** *?thesis*
by (*meson C1_differentiable_on_eq con_g2 finite_imageI finite_insert g2D piecewise_C1_differentiable_on_def*)
qed

9.36.3 Valid paths, and their start and finish

definition *valid_path* :: (*real* $\Rightarrow$ *'a* :: *real_normed_vector*) $\Rightarrow$ *bool*
where *valid_path f* $\equiv$ *f piecewise_C1_differentiable_on* $\{0..1::real\}$

definition *closed_path* :: (*real* $\Rightarrow$ *'a* :: *real_normed_vector*) $\Rightarrow$ *bool*
where *closed_path g* $\equiv$ *g 0* = *g 1*

In particular, all results for paths apply

lemma *valid_path_imp_path*: *valid_path g* $\Longrightarrow$ *path g*
by (*simp add*: *path_def piecewise_C1_differentiable_on_def valid_path_def*)

lemma *connected_valid_path_image*: *valid_path g* $\Longrightarrow$ *connected*(*path_image g*)
by (*metis connected_path_image valid_path_imp_path*)

lemma *compact_valid_path_image*: *valid_path g* $\Longrightarrow$ *compact*(*path_image g*)

by (*metis compact_path_image valid_path_imp_path*)

lemma *bounded_valid_path_image*: *valid_path g* ⟹ *bounded*(*path_image g*)
by (*metis bounded_path_image valid_path_imp_path*)

lemma *closed_valid_path_image*: *valid_path g* ⟹ *closed*(*path_image g*)
by (*metis closed_path_image valid_path_imp_path*)

lemma *valid_path_translation_eq*: *valid_path* ((+)*d* ∘ *p*) ⟷ *valid_path p*
by (*simp add*: *valid_path_def piecewise_C1_differentiable_on_translation_eq*)

lemma *valid_path_compose*:
assumes *valid_path g*
and *der*: ⋀*x*. *x* ∈ *path_image g* ⟹ *f field_differentiable* (*at x*)
and *con*: *continuous_on* (*path_image g*) (*deriv f*)
shows *valid_path* (*f* ∘ *g*)
proof −
obtain *S* **where** *finite S* **and** *g_diff*: *g C1_differentiable_on* {*0..1*} − *S*
using ‹*valid_path g*› **unfolding** *valid_path_def piecewise_C1_differentiable_on_def*
by *auto*
have *f* ∘ *g differentiable at t* **when** *t*∈{*0..1*} − *S* **for** *t*
proof (*rule differentiable_chain_at*)
show *g differentiable at t* **using** ‹*valid_path g*›
by (*meson C1_differentiable_on_eq* ‹*g C1_differentiable_on* {*0..1*} − *S*›
that)
next
have *g t*∈*path_image g* **using** *that DiffD1 image_eqI path_image_def* **by**
metis
then show *f differentiable at* (*g t*)
using *der*[*THEN field_differentiable_imp_differentiable*] **by** *auto*
qed
moreover have *continuous_on* ({*0..1*} − *S*) (λ*x*. *vector_derivative* (*f* ∘ *g*) (*at*
x))
proof (*rule continuous_on_eq* [**where** *f* = λ*x*. *vector_derivative g* (*at x*) ∗
deriv f (*g x*)],
rule continuous_intros)
show *continuous_on* ({*0..1*} − *S*) (λ*x*. *vector_derivative g* (*at x*))
using *g_diff C1_differentiable_on_eq* **by** *auto*
next
have *continuous_on* {*0..1*} (λ*x*. *deriv f* (*g x*))
using *continuous_on_compose*[*OF _ con*[*unfolded path_image_def*],*unfolded*
comp_def]
‹*valid_path g*› *piecewise_C1_differentiable_on_def valid_path_def*
by *blast*
then show *continuous_on* ({*0..1*} − *S*) (λ*x*. *deriv f* (*g x*))
using *continuous_on_subset* **by** *blast*
next
show *vector_derivative g* (*at t*) ∗ *deriv f* (*g t*) = *vector_derivative* (*f* ∘ *g*)
(*at t*)

```
        when t ∈ {0..1} − S for t
    by (metis C1_differentiable_on_eq DiffD1 der g_diff imageI path_image_def
that
                vector_derivative_chain_at_general)
  qed
 ultimately have f ∘ g C1_differentiable_on {0..1} − S
  using C1_differentiable_on_eq by blast
 moreover have path (f ∘ g)
  using der
  by (simp add: path_continuous_image[OF valid_path_imp_path[OF ‹valid_path
g›]] continuous_at_imp_continuous_on field_differentiable_imp_continuous_at)
 ultimately show ?thesis unfolding valid_path_def piecewise_C1_differentiable_on_def
path_def
  using ‹finite S› by auto
qed

lemma valid_path_uminus_comp[simp]:
 fixes g::real ⇒ 'a ::real_normed_field
 shows valid_path (uminus ∘ g) ⟷ valid_path g
proof
 show valid_path g ⟹ valid_path (uminus ∘ g) for g::real ⇒ 'a
  by (auto intro!: valid_path_compose derivative_intros)
 then show valid_path g when valid_path (uminus ∘ g)
  by (metis fun.map_comp group_add_class.minus_comp_minus id_comp that)
qed

lemma valid_path_offset[simp]:
 shows valid_path (λt. g t − z) ⟷ valid_path g
proof
 show ∗: valid_path (g::real⇒'a) ⟹ valid_path (λt. g t − z) for g z
  unfolding valid_path_def
   by (fastforce intro:derivative_intros C1_differentiable_imp_piecewise piece-
wise_C1_differentiable_diff)
 show valid_path (λt. g t − z) ⟹ valid_path g
  using ∗[of λt. g t − z −z,simplified] .
qed

lemma valid_path_imp_reverse:
 assumes valid_path g
  shows valid_path(reversepath g)
proof −
 obtain S where finite S and S: g C1_differentiable_on ({0..1} − S)
  using assms by (auto simp: valid_path_def piecewise_C1_differentiable_on_def)
 then have finite ((−) 1 ‘ S)
  by auto
 moreover have (reversepath g C1_differentiable_on ({0..1} − (−) 1 ‘ S))
  unfolding reversepath_def
  apply (rule C1_differentiable_compose [of λx::real. 1−x _ g, unfolded o_def])
  using S
```

by (*force simp*: *finite_vimageI inj_on_def C1_differentiable_on_eq elim*!: *continuous_on_subset*)+
ultimately show *?thesis* **using** *assms*
by (*auto simp*: *valid_path_def piecewise_C1_differentiable_on_def path_def* [*symmetric*])
qed

lemma *valid_path_reversepath* [*simp*]: *valid_path*(*reversepath g*) ⟷ *valid_path g*
using *valid_path_imp_reverse* **by** *force*

lemma *valid_path_join*:
assumes *valid_path g1 valid_path g2 pathfinish g1 = pathstart g2*
shows *valid_path*(*g1 +++ g2*)
proof −
have *g1 1 = g2 0*
using *assms* **by** (*auto simp*: *pathfinish_def pathstart_def*)
moreover have (*g1* ∘ (λ*x*. *2*∗*x*)) *piecewise_C1_differentiable_on* {*0..1/2*}
apply (*rule piecewise_C1_differentiable_compose*)
using *assms*
apply (*auto simp*: *valid_path_def piecewise_C1_differentiable_on_def continuous_on_joinpaths*)
apply (*force intro*: *finite_vimageI* [**where** *h* = (∗)*2*] *inj_onI*)
done
moreover have (*g2* ∘ (λ*x*. *2*∗*x*−*1*)) *piecewise_C1_differentiable_on* {*1/2..1*}
apply (*rule piecewise_C1_differentiable_compose*)
using *assms* **unfolding** *valid_path_def piecewise_C1_differentiable_on_def*
by (*auto intro*!: *continuous_intros finite_vimageI* [**where** *h* = (λ*x*. *2*∗*x* − *1*)] *inj_onI*
simp: *image_affinity_atLeastAtMost_diff continuous_on_joinpaths*)
ultimately show *?thesis*
unfolding *valid_path_def continuous_on_joinpaths joinpaths_def*
by (*intro piecewise_C1_differentiable_cases*) (*auto simp*: *o_def*)
qed

lemma *valid_path_join_D1*:
fixes *g1* :: *real* ⇒ *'a*::*real_normed_field*
shows *valid_path* (*g1 +++ g2*) ⟹ *valid_path g1*
unfolding *valid_path_def*
by (*rule piecewise_C1_differentiable_D1*)

lemma *valid_path_join_D2*:
fixes *g2* :: *real* ⇒ *'a*::*real_normed_field*
shows ⟦*valid_path* (*g1 +++ g2*); *pathfinish g1 = pathstart g2*⟧ ⟹ *valid_path g2*
unfolding *valid_path_def*
by (*rule piecewise_C1_differentiable_D2*)

lemma *valid_path_join_eq* [*simp*]:

fixes *g2* :: *real* ⇒ *'a*::*real_normed_field*
shows *pathfinish g1* = *pathstart g2* ⟹ (*valid_path*(*g1* +++ *g2*) ⟷ *valid_path g1* ∧ *valid_path g2*)
using *valid_path_join_D1 valid_path_join_D2 valid_path_join* **by** *blast*

lemma *valid_path_shiftpath* [*intro*]:
assumes *valid_path g pathfinish g* = *pathstart g a* ∈ {*0..1*}
shows *valid_path*(*shiftpath a g*)
using *assms*
unfolding *valid_path_def shiftpath_alt_def*
apply (*intro piecewise_C1_differentiable_cases*)
apply (*simp_all add*: *add.commute*)
apply (*rule piecewise_C1_differentiable_affine* [*of g 1 a*, *simplified o_def scaleR_one*])
apply (*force simp*: *pathfinish_def pathstart_def elim*: *piecewise_C1_differentiable_on_subset*)
apply (*rule piecewise_C1_differentiable_affine* [*of g 1 a−1*, *simplified o_def scaleR_one algebra_simps*])
apply (*auto simp*: *pathfinish_def pathstart_def elim*: *piecewise_C1_differentiable_on_subset*)
done

lemma *vector_derivative_linepath_within*:
x ∈ {*0..1*} ⟹ *vector_derivative* (*linepath a b*) (*at x within* {*0..1*}) = *b* − *a*
by (*simp add*: *has_vector_derivative_linepath_within vector_derivative_at_within_ivl*)

lemma *vector_derivative_linepath_at* [*simp*]: *vector_derivative* (*linepath a b*) (*at x*) = *b* − *a*
by (*simp add*: *has_vector_derivative_linepath_within vector_derivative_at*)

lemma *valid_path_linepath* [*iff*]: *valid_path* (*linepath a b*)
using *C1_differentiable_on_eq piecewise_C1_differentiable_on_def valid_path_def*
by *fastforce*

lemma *valid_path_subpath*:
fixes *g* :: *real* ⇒ *'a* :: *real_normed_vector*
assumes *valid_path g u* ∈ {*0..1*} *v* ∈ {*0..1*}
shows *valid_path*(*subpath u v g*)
proof (*cases v*=*u*)
case *True*
then show *?thesis*
unfolding *valid_path_def subpath_def*
by (*force intro*: *C1_differentiable_on_const C1_differentiable_imp_piecewise*)
next
case *False*
let *?f* = λ*x*. ((*v*−*u*) ∗ *x* + *u*)
have (*g* ∘ *?f*) *piecewise_C1_differentiable_on* {*0..1*}
proof (*rule piecewise_C1_differentiable_compose*)
show *?f piecewise_C1_differentiable_on* {*0..1*}
by (*simp add*: *C1_differentiable_imp_piecewise*)
have *g piecewise_C1_differentiable_on* (*if u* ≤ *v then* {*u..v*} *else* {*v..u*})

```
    using assms piecewise_C1_differentiable_on_subset valid_path_def by force
  then show g piecewise_C1_differentiable_on ?f ` {0..1}
    by (simp add: image_affinity_atLeastAtMost split: if_split_asm)
  show ⋀x. finite ({0..1} ∩ ?f -` {x})
    using False
      by (simp add: Int_commute [of {0..1}] inj_on_def crossproduct_eq finite_vimage_IntI)
 qed
 then show ?thesis
  by (auto simp: o_def valid_path_def subpath_def)
qed

lemma valid_path_rectpath [simp, intro]: valid_path (rectpath a b)
 by (simp add: Let_def rectpath_def)

lemma linear_image_valid_path:
 fixes p :: real ⇒ 'a :: euclidean_space
 assumes valid_path p linear f
 shows   valid_path (f ∘ p)
 unfolding valid_path_def piecewise_C1_differentiable_on_def
proof (intro conjI)
 from assms have path p
  by (simp add: valid_path_imp_path)
 thus continuous_on {0..1} (f ∘ p)
   unfolding o_def path_def by (intro linear_continuous_on_compose[OF _ assms(2)])
 from assms(1) obtain S where S: finite S p C1_differentiable_on {0..1} - S
  by (auto simp: valid_path_def piecewise_C1_differentiable_on_def)
 from S(2) obtain p' :: real ⇒ 'a
  where p': ⋀x. x ∈ {0..1} - S ⟹ (p has_vector_derivative p' x) (at x)
          continuous_on ({0..1} - S) p'
  by (fastforce simp: C1_differentiable_on_def)

 have (f ∘ p has_vector_derivative f (p' x)) (at x) if x ∈ {0..1} - S for x
  by (rule vector_derivative_diff_chain_within [OF p'(1)[OF that]]
        linear_imp_has_derivative assms)+
 moreover have continuous_on ({0..1} - S) (λx. f (p' x))
  by (rule linear_continuous_on_compose [OF p'(2) assms(2)])
 ultimately have f ∘ p C1_differentiable_on {0..1} - S
  unfolding C1_differentiable_on_def by (intro exI[of _ λx. f (p' x)]) fast
 thus ∃S. finite S ∧ f ∘ p C1_differentiable_on {0..1} - S
  using ‹finite S› by blast
qed

lemma valid_path_times:
 fixes γ::real ⇒ 'a ::real_normed_field
 assumes c≠0
 shows valid_path ((*) c ∘ γ) = valid_path γ
proof
```

```
  assume valid_path ((*) c ∘ γ)
  then have valid_path ((*) (1/c) ∘ ((*) c ∘ γ))
    by (simp add: valid_path_compose)
  then show valid_path γ
    unfolding comp_def using ‹c≠0› by auto
next
  assume valid_path γ
  then show valid_path ((*) c ∘ γ)
    by (simp add: valid_path_compose)
qed

lemma path_compose_cnj_iff [simp]: path (cnj ∘ p) ⟷ path p
proof −
  have path (cnj ∘ p) if path p for p
    by (intro path_continuous_image continuous_intros that)
  from this[of p] and this[of cnj ∘ p] show ?thesis
    by (auto simp: o_def)
qed

lemma valid_path_cnj:
  fixes g::real ⇒ complex
    shows valid_path (cnj ∘ g) = valid_path g
proof
  show valid:valid_path (cnj ∘ g) if valid_path g for g
  proof −
    obtain S where finite S and g_diff: g C1_differentiable_on {0..1} − S
    using ‹valid_path g› unfolding valid_path_def piecewise_C1_differentiable_on_def
by auto

    have g_diff':g differentiable at t when t∈{0..1} − S for t
      by (meson C1_differentiable_on_eq ‹g C1_differentiable_on {0..1} − S›
that)
    then have (cnj ∘ g) differentiable at t when t∈{0..1} − S for t
    using bounded_linear_cnj bounded_linear_imp_differentiable differentiable_chain_at
that by blast
    moreover have continuous_on ({0..1} − S)
       (λx. vector_derivative (cnj ∘ g) (at x))
    proof −
      have continuous_on ({0..1} − S)
         (λx. vector_derivative (cnj ∘ g) (at x))
       = continuous_on ({0..1} − S)
         (λx. cnj (vector_derivative g (at x)))
       apply (rule continuous_on_cong[OF refl])
       unfolding comp_def using g_diff'
     using has_vector_derivative_cnj vector_derivative_at vector_derivative_works
by blast
     also have ...
       apply (intro continuous_intros)
       using C1_differentiable_on_eq g_diff by blast
```

```
    finally show ?thesis .
  qed
  ultimately have cnj ∘ g C1_differentiable_on {0..1} − S
    using C1_differentiable_on_eq by blast
  moreover have path (cnj ∘ g)
  apply (rule path_continuous_image[OF valid_path_imp_path[OF ‹valid_path
g›]])
    by (intro continuous_intros)
 ultimately show ?thesis unfolding valid_path_def piecewise_C1_differentiable_on_def
path_def
    using ‹finite S› by auto
 qed
 from this[of cnj o g]
 show valid_path (cnj ∘ g) ⟹ valid_path g
   unfolding comp_def by simp
qed

end
```

9.37 Metrics on product spaces

```
theory Function_Metric
 imports
   Function_Topology
   Elementary_Metric_Spaces
begin
```

In general, the product topology is not metrizable, unless the index set is countable. When the index set is countable, essentially any (convergent) combination of the metrics on the factors will do. We use below the simplest one, based on L^1, but L^2 would also work, for instance.

What is not completely trivial is that the distance thus defined induces the same topology as the product topology. This is what we have to prove to show that we have an instance of *metric_space*.

The proofs below would work verbatim for general countable products of metric spaces. However, since distances are only implemented in terms of type classes, we only develop the theory for countable products of the same space.

```
instantiation fun :: (countable, metric_space) metric_space
begin

definition dist_fun_def:
  dist x y = (∑ n. (1/2)^n * min (dist (x (from_nat n)) (y (from_nat n))) 1)

definition uniformity_fun_def:
  (uniformity::(('a ⇒ 'b) × ('a ⇒ 'b)) filter) = (INF e∈{0<..}. principal {(x, y).
dist (x::('a⇒'b)) y < e})
```

Except for the first one, the auxiliary lemmas below are only useful when proving the instance: once it is proved, they become trivial consequences of the general theory of metric spaces. It would thus be desirable to hide them once the instance is proved, but I do not know how to do this.

```
lemma dist_fun_le_dist_first_terms:
  dist x y ≤ 2 * Max {dist (x (from_nat n)) (y (from_nat n))|n. n ≤ N} +
(1/2)^N
proof −
  have (∑ n. (1 / 2) ^ (n+Suc N) * min (dist (x (from_nat (n+Suc N))) (y
(from_nat (n+Suc N)))) 1)
        = (∑ n. (1 / 2) ^ (Suc N) * ((1/2) ^ n * min (dist (x (from_nat (n+Suc
N))) (y (from_nat (n+Suc N)))) 1))
    by (rule suminf_cong, simp add: power_add)
  also have ... = (1/2)^(Suc N) * (∑ n. (1 / 2) ^ n * min (dist (x (from_nat
(n+Suc N))) (y (from_nat (n+Suc N)))) 1)
    apply (rule suminf_mult)
   by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
  also have ... ≤ (1/2)^(Suc N) * (∑ n. (1 / 2) ^ n)
    apply (simp, rule suminf_le, simp)
   by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
  also have ... = (1/2)^(Suc N) * 2
    using suminf_geometric[of 1/2] by auto
  also have ... = (1/2)^N
    by auto
  finally have ∗: (∑ n. (1 / 2) ^ (n+Suc N) * min (dist (x (from_nat (n+Suc
N))) (y (from_nat (n+Suc N)))) 1) ≤ (1/2)^N
    by simp

  define M where M = Max {dist (x (from_nat n)) (y (from_nat n))|n. n ≤ N}
  have dist (x (from_nat 0)) (y (from_nat 0)) ≤ M
    unfolding M_def by (rule Max_ge, auto)
  then have [simp]: M ≥ 0 by (meson dual_order.trans zero_le_dist)
  have dist (x (from_nat n)) (y (from_nat n)) ≤ M if n≤N for n
    unfolding M_def apply (rule Max_ge) using that by auto
  then have i: min (dist (x (from_nat n)) (y (from_nat n))) 1 ≤ M if n≤N for
n
    using that by force
  have (∑ n< Suc N. (1 / 2) ^ n * min (dist (x (from_nat n)) (y (from_nat
n))) 1) ≤
      (∑ n< Suc N. M * (1 / 2) ^ n)
    by (rule sum_mono, simp add: i)
  also have ... = M * (∑ n<Suc N. (1 / 2) ^ n)
    by (rule sum_distrib_left[symmetric])
  also have ... ≤ M * (∑ n. (1 / 2) ^ n)
   by (rule mult_left_mono, rule sum_le_suminf, auto simp add: summable_geometric_iff)
  also have ... = M * 2
    using suminf_geometric[of 1/2] by auto
  finally have ∗∗: (∑ n< Suc N. (1 / 2) ^ n * min (dist (x (from_nat n)) (y
(from_nat n))) 1) ≤ 2 * M
```

```
    by simp

  have dist x y = (∑ n. (1 / 2) ^ n * min (dist (x (from_nat n)) (y (from_nat
n))) 1)
    unfolding dist_fun_def by simp
  also have ... = (∑ n. (1 / 2) ^ (n+Suc N) * min (dist (x (from_nat (n+Suc
N))) (y (from_nat (n+Suc N)))) 1)
                  + (∑ n<Suc N. (1 / 2) ^ n * min (dist (x (from_nat n)) (y
(from_nat n))) 1)
    apply (rule suminf_split_initial_segment)
   by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
  also have ... ≤ 2 * M + (1/2)^N
    using * ** by auto
  finally show ?thesis unfolding M_def by simp
qed

lemma open_fun_contains_ball_aux:
  assumes open (U::(('a ⇒ 'b) set))
          x ∈ U
  shows ∃ e>0. {y. dist x y < e} ⊆ U
proof −
  have ∗: openin (product_topology (λi. euclidean) UNIV) U
    using open_fun_def assms by auto
  obtain X where H: Pi_E UNIV X ⊆ U
                    ⋀i. openin euclidean (X i)
                    finite {i. X i ≠ topspace euclidean}
                    x ∈ Pi_E UNIV X
    using product_topology_open_contains_basis[OF * ‹x ∈ U›] by auto
  define I where I = {i. X i ≠ topspace euclidean}
  have finite I unfolding I_def using H(3) by auto
  {
    fix i
    have x i ∈ X i using ‹x ∈ U› H by auto
    then have ∃ e. e>0 ∧ ball (x i) e ⊆ X i
      using ‹openin euclidean (X i)› open_openin open_contains_ball by blast
    then obtain e where e>0 ball (x i) e ⊆ X i by blast
    define f where f = min e (1/2)
    have f>0 f<1 unfolding f_def using ‹e>0› by auto
    moreover have ball (x i) f ⊆ X i unfolding f_def using ‹ball (x i) e ⊆ X
i› by auto
    ultimately have ∃f. f > 0 ∧ f < 1 ∧ ball (x i) f ⊆ X i by auto
  } note ∗ = this
  have ∃ e. ∀ i. e i > 0 ∧ e i < 1 ∧ ball (x i) (e i) ⊆ X i
    by (rule choice, auto simp add: ∗)
  then obtain e where ⋀i. e i > 0 ⋀i. e i < 1 ⋀i. ball (x i) (e i) ⊆ X i
    by blast
  define m where m = Min {(1/2)^(to_nat i) * e i|i. i ∈ I}
  have m > 0 if I≠{}
    unfolding m_def Min_gr_iff using ‹finite I› ‹I ≠ {}› ‹⋀i. e i > 0› by auto
```

```
  moreover have {y. dist x y < m} ⊆ U
  proof (auto)
    fix y assume dist x y < m
    have y i ∈ X i if i ∈ I for i
    proof −
      have ∗: summable (λn. (1/2)^n ∗ min (dist (x (from_nat n)) (y (from_nat
n))) 1)
          by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add:
summable_geometric_iff)
      define n where n = to_nat i
      have (1/2)^n ∗ min (dist (x (from_nat n)) (y (from_nat n))) 1 < m
        using ‹dist x y < m› unfolding dist_fun_def using sum_le_suminf[OF
∗, of {n}] by auto
      then have (1/2)^(to_nat i) ∗ min (dist (x i) (y i)) 1 < m
        using ‹n = to_nat i› by auto
      also have ... ≤ (1/2)^(to_nat i) ∗ e i
        unfolding m_def apply (rule Min_le) using ‹finite I› ‹i ∈ I› by auto
      ultimately have min (dist (x i) (y i)) 1 < e i
        by (auto simp add: field_split_simps)
      then have dist (x i) (y i) < e i
        using ‹e i < 1› by auto
      then show y i ∈ X i using ‹ball (x i) (e i) ⊆ X i› by auto
    qed
    then have y ∈ Pi_E UNIV X using H(1) unfolding I_def topspace_euclidean
by (auto simp add: PiE_iff)
    then show y ∈ U using ‹Pi_E UNIV X ⊆ U› by auto
  qed
  ultimately have ∗: ∃ m>0. {y. dist x y < m} ⊆ U if I ≠ {} using that by
auto

  have U = UNIV if I = {}
     using that H(1) unfolding I_def topspace_euclidean by (auto simp add:
PiE_iff)
  then have ∃ m>0. {y. dist x y < m} ⊆ U if I = {} using that zero_less_one
by blast
  then show ∃ m>0. {y. dist x y < m} ⊆ U using ∗ by blast
qed

lemma ball_fun_contains_open_aux:
  fixes x::('a ⇒ 'b) and e::real
  assumes e>0
  shows ∃ U. open U ∧ x ∈ U ∧ U ⊆ {y. dist x y < e}
proof −
  have ∃ N::nat. 2^N > 8/e
    by (simp add: real_arch_pow)
  then obtain N::nat where 2^N > 8/e by auto
  define f where f = e/4
  have [simp]: e>0 f > 0 unfolding f_def using assms by auto
  define X::('a ⇒ 'b set) where X = (λi. if (∃ n≤N. i = from_nat n) then {z.
```

```
dist (x i) z < f} else UNIV)
  have {i. X i ≠ UNIV} ⊆ from_nat‘{0..N}
    unfolding X_def by auto
  then have finite {i. X i ≠ topspace euclidean}
    unfolding topspace_euclidean using finite_surj by blast
  have ⋀i. open (X i)
    unfolding X_def using metric_space_class.open_ball open_UNIV by auto
  then have ⋀i. openin euclidean (X i)
    using open_openin by auto
  define U where U = Pi_E UNIV X
  have open U
   unfolding open_fun_def product_topology_def apply (rule topology_generated_by_Basis)
    unfolding U_def using ‹⋀i. openin euclidean (X i)› ‹finite {i. X i ≠ topspace
euclidean}›
    by auto
  moreover have x ∈ U
    unfolding U_def X_def by (auto simp add: PiE_iff)
  moreover have dist x y < e if y ∈ U for y
  proof −
    have ∗: dist (x (from_nat n)) (y (from_nat n)) ≤ f if n ≤ N for n
      using ‹y ∈ U› unfolding U_def apply (auto simp add: PiE_iff)
      unfolding X_def using that by (metis less_imp_le mem_Collect_eq)
    have ∗∗: Max {dist (x (from_nat n)) (y (from_nat n))|n. n ≤ N} ≤ f
      apply (rule Max.boundedI) using ∗ by auto

    have dist x y ≤ 2 ∗ Max {dist (x (from_nat n)) (y (from_nat n))|n. n ≤ N}
+ (1/2)^N
      by (rule dist_fun_le_dist_first_terms)
    also have ... ≤ 2 ∗ f + e / 8
    apply (rule add_mono) using ∗∗ ‹2^N > 8/e› by(auto simp add: field_split_simps)
    also have ... ≤ e/2 + e/8
      unfolding f_def by auto
    also have ... < e
      by auto
    finally show dist x y < e by simp
  qed
  ultimately show ?thesis by auto
qed

lemma fun_open_ball_aux:
  fixes U::('a ⇒ 'b) set
  shows open U ⟷ (∀ x∈U. ∃ e>0. ∀ y. dist x y < e ⟶ y ∈ U)
proof (auto)
  assume open U
  fix x assume x ∈ U
  then show ∃ e>0. ∀ y. dist x y < e ⟶ y ∈ U
    using open_fun_contains_ball_aux[OF ‹open U› ‹x ∈ U›] by auto
next
  assume H: ∀ x∈U. ∃ e>0. ∀ y. dist x y < e ⟶ y ∈ U
```

```
  define K where K = {V. open V ∧ V ⊆ U}
  {
    fix x assume x ∈ U
    then obtain e where e: e>0 {y. dist x y < e} ⊆ U using H by blast
    then obtain V where V: open V x ∈ V V ⊆ {y. dist x y < e}
      using ball_fun_contains_open_aux[OF ‹e>0›, of x] by auto
    have V ∈ K
      unfolding K_def using e(2) V(1) V(3) by auto
    then have x ∈ (⋃K) using ‹x ∈ V› by auto
  }
  then have (⋃K) = U
    unfolding K_def by auto
  moreover have open (⋃K)
    unfolding K_def by auto
  ultimately show open U by simp
qed

instance proof
  fix x y::'a ⇒ 'b show (dist x y = 0) = (x = y)
  proof
    assume x = y
    then show dist x y = 0 unfolding dist_fun_def using ‹x = y› by auto
  next
    assume dist x y = 0
    have ∗: summable (λn. (1/2)^n ∗ min (dist (x (from_nat n)) (y (from_nat
n))) 1)
    by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
    have (1/2)^n ∗ min (dist (x (from_nat n)) (y (from_nat n))) 1 = 0 for n
        using ‹dist x y = 0› unfolding dist_fun_def by (simp add: ∗ sum-
inf_eq_zero_iff)
    then have dist (x (from_nat n)) (y (from_nat n)) = 0 for n
      by (metis div_0 min_def nonzero_mult_div_cancel_left power_eq_0_iff
            zero_eq_1_divide_iff zero_neq_numeral)
    then have x (from_nat n) = y (from_nat n) for n
      by auto
    then have x i = y i for i
      by (metis from_nat_to_nat)
    then show x = y
      by auto
  qed
next
```

The proof of the triangular inequality is trivial, modulo the fact that we are dealing with infinite series, hence we should justify the convergence at each step. In this respect, the following simplification rule is extremely handy.

```
  have [simp]: summable (λn. (1/2)^n ∗ min (dist (u (from_nat n)) (v (from_nat
n))) 1) for u v::'a ⇒ 'b
   by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add: summable_geometric_iff)
  fix x y z::'a ⇒ 'b
```

```
  {
    fix n
    have *: dist (x (from_nat n)) (y (from_nat n)) ≤
              dist (x (from_nat n)) (z (from_nat n)) + dist (y (from_nat n)) (z
(from_nat n))
      by (simp add: dist_triangle2)
    have 0 ≤ dist (y (from_nat n)) (z (from_nat n))
      using zero_le_dist by blast
    moreover
    {
     assume min (dist (y (from_nat n)) (z (from_nat n))) 1 ≠ dist (y (from_nat
n)) (z (from_nat n))
      then have 1 ≤ min (dist (x (from_nat n)) (z (from_nat n))) 1 + min (dist
(y (from_nat n)) (z (from_nat n))) 1
      by (metis (no_types) diff_le_eq diff_self min_def zero_le_dist zero_le_one)
    }
    ultimately have min (dist (x (from_nat n)) (y (from_nat n))) 1 ≤
          min (dist (x (from_nat n)) (z (from_nat n))) 1 + min (dist (y (from_nat
n)) (z (from_nat n))) 1
      using * by linarith
  } note ineq = this
  have dist x y = (∑ n. (1/2)^n * min (dist (x (from_nat n)) (y (from_nat n)))
1)
    unfolding dist_fun_def by simp
  also have ... ≤ (∑ n. (1/2)^n * min (dist (x (from_nat n)) (z (from_nat n)))
1
                      + (1/2)^n * min (dist (y (from_nat n)) (z (from_nat n))) 1)
    apply (rule suminf_le)
      using ineq apply (metis (no_types, opaque_lifting) add.right_neutral dis-
trib_left
    le_divide_eq_numeral1(1) mult_2_right mult_left_mono zero_le_one zero_le_power)
    by (auto simp add: summable_add)
  also have ... = (∑ n. (1/2)^n * min (dist (x (from_nat n)) (z (from_nat n)))
1)
                + (∑ n. (1/2)^n * min (dist (y (from_nat n)) (z (from_nat n)))
1)
    by (rule suminf_add[symmetric], auto)
  also have ... = dist x z + dist y z
    unfolding dist_fun_def by simp
  finally show dist x y ≤ dist x z + dist y z
    by simp
next
```

Finally, we show that the topology generated by the distance and the product topology coincide. This is essentially contained in Lemma *fun_open_ball_aux*, except that the condition to prove is expressed with filters. To deal with this, we copy some mumbo jumbo from Lemma *eventually_uniformity_metric* in `~~/src/HOL/Real_Vector_Spaces.thy`

```
  fix U::('a ⇒ 'b) set
```

```
  have eventually P uniformity ⟷ (∃ e>0. ∀ x (y::('a ⇒ 'b)). dist x y < e ⟶
P (x, y)) for P
  unfolding uniformity_fun_def apply (subst eventually_INF_base)
    by (auto simp: eventually_principal subset_eq intro: bexI[of _ min _ _])
  then show open U = (∀ x∈U. ∀ F (x', y) in uniformity. x' = x ⟶ y ∈ U)
    unfolding fun_open_ball_aux by simp
qed (simp add: uniformity_fun_def)

end
```

Nice properties of spaces are preserved under countable products. In addition to first countability, second countability and metrizability, as we have seen above, completeness is also preserved, and therefore being Polish.

```
instance fun :: (countable, complete_space) complete_space
proof
  fix u::nat ⇒ ('a ⇒ 'b) assume Cauchy u
  have Cauchy (λn. u n i) for i
    unfolding Cauchy_def
  proof (intro strip)
    fix e::real assume e>0
    obtain k where i = from_nat k
      using from_nat_to_nat[of i] by metis
    have (1/2)^k * min e 1 > 0 using ‹e>0› by auto
    then have ∃ N. ∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (u m) (u n) < (1/2)^k *
min e 1
      using ‹Cauchy u› by (meson Cauchy_def)
    then obtain N::nat where C: ∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (u m) (u n)
< (1/2)^k * min e 1
      by blast
    have ∀ m n. N ≤ m ∧ N ≤ n ⟶ dist (u m i) (u n i) < e
    proof (auto)
      fix m n::nat assume m ≥ N n ≥ N
      have (1/2)^k * min (dist (u m i) (u n i)) 1
            = sum (λp. (1/2)^p * min (dist (u m (from_nat p)) (u n (from_nat
p))) 1) {k}
        using ‹i = from_nat k› by auto
     also have ... ≤ (∑ p. (1/2)^p * min (dist (u m (from_nat p)) (u n (from_nat
p))) 1)
        apply (rule sum_le_suminf)
          by (rule summable_comparison_test'[of λn. (1/2)^n], auto simp add:
summable_geometric_iff)
      also have ... = dist (u m) (u n)
        unfolding dist_fun_def by auto
      also have ... < (1/2)^k * min e 1
        using C ‹m ≥ N› ‹n ≥ N› by auto
      finally have min (dist (u m i) (u n i)) 1 < min e 1
        by (auto simp add: field_split_simps)
      then show dist (u m i) (u n i) < e by auto
    qed
```

```
    then show ∃M. ∀m≥M. ∀n≥M. dist (u m i) (u n i) < e
      by blast
  qed
  then have ∃x. (λn. u n i) ⟶ x for i
    using Cauchy_convergent convergent_def by auto
  then have ∃x. ∀i. (λn. u n i) ⟶ x i
    using choice by force
  then obtain x where *: ⋀i. (λn. u n i) ⟶ x i by blast
  have u ⟶ x
  proof (rule metric_LIMSEQ_I)
    fix e assume [simp]: e>(0::real)
    have i: ∃K. ∀n≥K. dist (u n i) (x i) < e/4 for i
      by (rule metric_LIMSEQ_D, auto simp add: *)
    have ∃K. ∀i. ∀n≥K i. dist (u n i) (x i) < e/4
      apply (rule choice) using i by auto
    then obtain K where K: ⋀i n. n ≥ K i ⟹ dist (u n i) (x i) < e/4
      by blast

    have ∃N::nat. 2^N > 4/e
      by (simp add: real_arch_pow)
    then obtain N::nat where 2^N > 4/e by auto
    define L where L = Max {K (from_nat n)|n. n ≤ N}
    have dist (u k) x < e if k ≥ L for k
    proof −
      have *: dist (u k (from_nat n)) (x (from_nat n)) ≤ e / 4 if n ≤ N for n
      proof −
        have K (from_nat n) ≤ L
          unfolding L_def apply (rule Max_ge) using ‹n ≤ N› by auto
        then have k ≥ K (from_nat n) using ‹k ≥ L› by auto
        then show ?thesis using K less_imp_le by auto
      qed
      have **: Max {dist (u k (from_nat n)) (x (from_nat n)) |n. n ≤ N} ≤ e/4
        apply (rule Max.boundedI) using * by auto
      have dist (u k) x ≤ 2 * Max {dist (u k (from_nat n)) (x (from_nat n)) |n.
n ≤ N} + (1/2)^N
        using dist_fun_le_dist_first_terms by auto
      also have ... ≤ 2 * e/4 + e/4
        apply (rule add_mono)
        using ** ‹2^N > 4/e› less_imp_le by (auto simp add: field_split_simps)
      also have ... < e by auto
      finally show ?thesis by simp
    qed
    then show ∃L. ∀k≥L. dist (u k) x < e by blast
  qed
  then show convergent u using convergent_def by blast
qed

instance fun :: (countable, polish_space) polish_space
  by standard
```

end
theory *Analysis*
imports

Convex
Determinants

FSigma
Sum_Topology
Abstract_Topological_Spaces
Abstract_Metric_Spaces
Urysohn
Connected
Abstract_Limits
Isolated
Sparse_In

Elementary_Normed_Spaces
Norm_Arith

Convex_Euclidean_Space
Operator_Norm

Line_Segment
Derivative
Cartesian_Euclidean_Space
Kronecker_Approximation_Theorem
Weierstrass_Theorems

Ball_Volume
Integral_Test
Improper_Integral
Equivalence_Measurable_On_Borel
Lebesgue_Integral_Substitution
Embed_Measure
Complete_Measure
Radon_Nikodym
Fashoda_Theorem
Cross3
Homeomorphism
Bounded_Continuous_Function
Abstract_Topology
Product_Topology
Lindelof_Spaces
Infinite_Products
Infinite_Sum
Infinite_Set_Sum
Polytope

Jordan_Curve
Poly_Roots
Generalised_Binomial_Theorem
Gamma_Function
Change_Of_Vars
Multivariate_Analysis
Simplex_Content
FPS_Convergence
Smooth_Paths
Abstract_Euclidean_Space
Function_Metric
begin

end

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Bibliography

[1]

[2] J. B. Conway. *A course in functional analysis*, volume 96. Springer Science & Business Media, 2013.

[3] J. Dugundji. An extension of Tietze's theorem. *Pacific J. Math.*, 1(3):353–367, 1951.

[4] R. Engelking. *General Topology.* Sigma series in pure mathematics. Heldermann, 1989.

[5] S. Lang. *Real and Functional Analysis.* Springer, 1993.

[6] M. Maggesi. A formalization of metric spaces in HOL light. *J. Autom. Reasoning*, 60(2):237–254, 2018.