Using Relation Algebraic Methods in the Coq Proof Assistant

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Relation algebra / point-free reasonning

- A wonderful tool to work with binary relations:
  - concise and expressive
  - allows calculational proofs
  - decidable fragments (KA, CKA, KAT, residuated lattices, ...)
- Can be exploited in other models
  - min-max algebras
  - languages, traces
  - matrices
- Very well suited to mechanised reasoning

Objectives

Exploit relation algebra tools and methodology

in the Coq proof assistant

Outline

Basic introduction to Coq

Two case-studies

- Commuting diagrams
- Compiler optimisations

Behind the scene

- How do `ka`/`kat` work?
- Types, untyping
- Algebraic hierarchy

Challenges for future work

What is Coq?

- A purely functional programming language
- A specification language
- An interactive proof assistant

What is it useful for?

- Build certified software

  Compcert (Leroy et al. '05-'10)

- Certify algorithms, prove mathematical theorems

  4 colours theorem (Gonthier '04)

  Feit-Thompson theorem (Gonthier et al. '08-)

What it's not?

- A Turing-complete programming language

Every program terminates

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- A Turing-complete programming language

  Every program terminates

- An automatic theorem prover (ATP)

  The user writes the proofs

- An oracle

  The user writes the programs

- Something always easy to work with

  The user writes the programs and the proofs

Curry-Howard-de Bruijn correspondence

- A proof of A is a lambda-term of type A
- Checking a proof amounts to type-checking a lambda-term

easy

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- Use tactics to produce lambda-terms (proofs)
- At "Qed", Coq type-checks the lambda-term

Small demo

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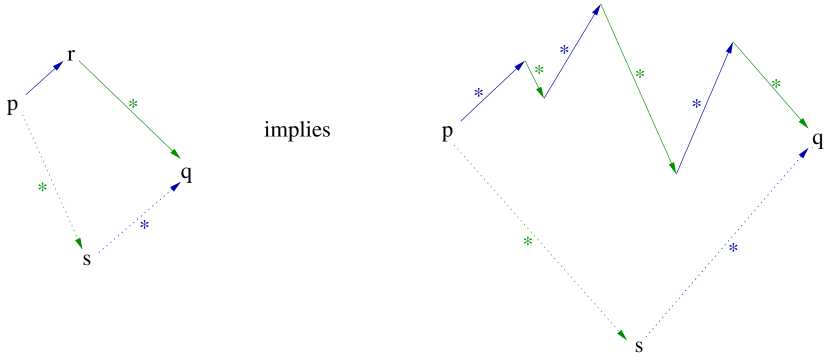
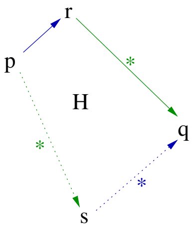
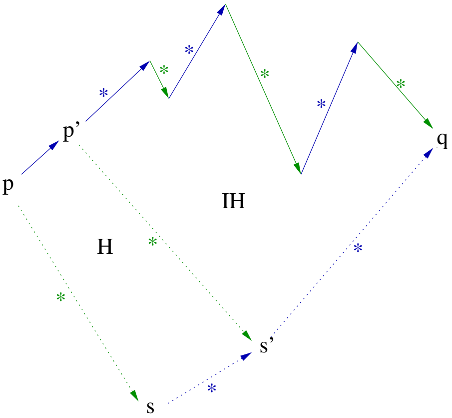
A Church-Rosser property



(Usually presented with $\textcolor{blue}{\rightarrow} \ = \ \textcolor{green}{\leftarrow}$)

Diagrammatic proof





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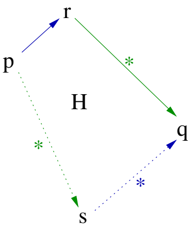
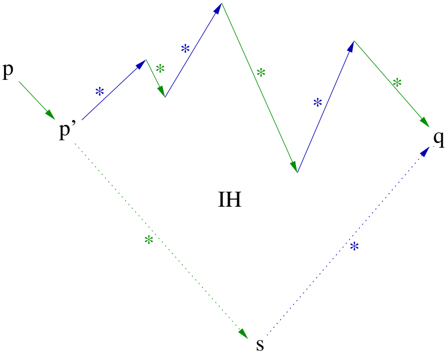
Diagrammatic proof





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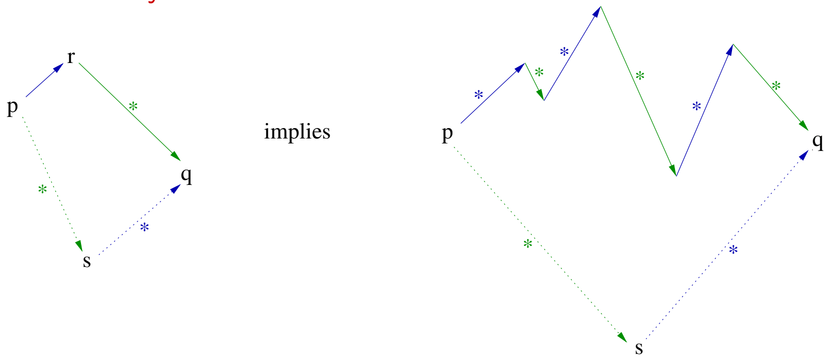
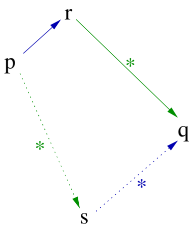
More formally

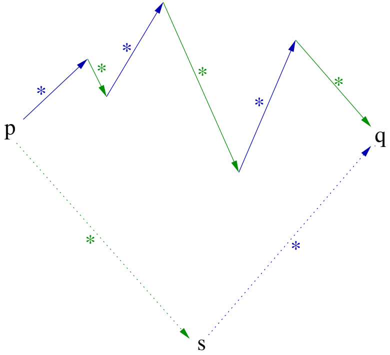


$$(\forall p, r, q, pRr, rS^\star q \Rightarrow \exists s, pS^\star s \wedge sR^\star q)$$
$$\Rightarrow \qquad (\forall p, q, p(R+S)^\star q \Rightarrow \exists s, pS^\star s \wedge sR^\star q)$$

More formally



implies



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$$\Rightarrow \qquad (\forall p, q, p(R+S)^\star q \Rightarrow \exists s, pS^\star s \wedge sR^\star q)$$

$$R; S^\star \subseteq S^\star; R^\star \Rightarrow (R+S)^\star \subseteq S^\star; R^\star$$

Point-free reasoning

- The point-free statement is much nicer than the expanded one
- The same holds for the corresponding proofs
- both with pen and pencil ...
  ... and with Coq

demo

Newman's Lemma

If R terminates, then

$$R^{\circ}; R \subseteq R^{\star}; R^{\circ\star} \quad \Rightarrow \quad R^{\circ\star}; R^{\star} \subseteq R^{\star}; R^{\circ\star}$$

$$\begin{tikzcd}
& \cdot \arrow[dl] \arrow[dr] & \\
{} \arrow[dr, dashed, "\star"'] & H & {} \arrow[dl, dashed, "\star"] \\
& \cdot &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \cdot \arrow[dl, "\star"'] \arrow[dr, "\star"] & \\
{} \arrow[dr, dashed, "\star"'] & & {} \arrow[dl, dashed, "\star"] \\
& \cdot &
\end{tikzcd}$$

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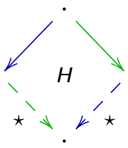

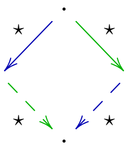

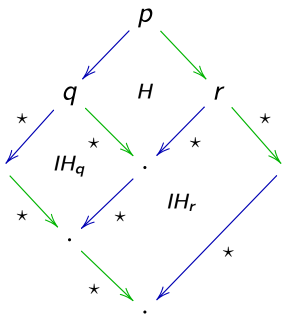

Abstract termination

How to express the termination hypothesis in an abstract setting?

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"A Calculational Approach to Mathematical Induction"
H. Doornbos, R. Backhouse, and J. van der Woude '97

(Alternative: ω-algebras)

Factors and monotype factors in division allegories

- Residuated semiring:

$$y \leq x \backslash z \qquad \Leftrightarrow \qquad x ; y \leq z \qquad \Leftrightarrow \qquad x \leq y / z$$

left and right factors

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- Left monotype factor:

$$B \leq x \mathbin{\backslash\!\!\!\cdot} A \qquad \Leftrightarrow \qquad x ; B \leq A ; x$$

With relations: $x \mathbin{\backslash\!\!\!\cdot} A$ is the set of points whose all predecessors by x belong to A

weakest precondition

Properties of monotype factors

- Cancellation: $x; (x \mathbin{\underline{\backslash}} A) \;\leq\; A; x$
- Duplication: $(x \mathbin{\underline{\backslash}} A); (x \mathbin{\underline{\backslash}} A) \;=\; x \mathbin{\underline{\backslash}} A$
- Reversal: $x \mathbin{\underline{\backslash}} A \;=\; A \mathbin{\underline{/}} x^\circ$

$A \mathbin{\underline{/}} x$ being defined symmetrically

Abstract well-foundedness [DBvdW'97]

Definition: Call t well-founded
if for all monotypes $A \leq 1$, $t \mathbin{\setminus\!\!\cdot} A \leq A$ entails $1 \leq A$.

- This pointfree notion coincides with the usual notion of well-foundedness on binary relations.
- In particular, t is well-founded iff t° terminates.

Newman's Lemma using abstract well-founded induction

Setting $y = x\circ$, the proof reduces to showing that forall $A \leq 1$,

$$y^*; A; x^* \leq x^*; y^* \text{ (IH)} \quad \text{entails} \quad y^*; y; (y \backslash A); x; x^* \leq x^*; y^*$$

$$\begin{aligned}
& y^*; y; (y \backslash A); x; x^* & \\
= \; & y^*; y; (y \backslash A); (y \backslash A); x; x^* & \text{(duplication)} \\
= \; & y^*; y; (y \backslash A); (A / x); x; x^* & \text{(converse)} \\
\leq \; & y^*; A; y; (A / x); x; x^* & \text{(left cancellation)} \\
\leq \; & y^*; A; y; x; A; x^* & \text{(right cancellation)} \\
\leq \; & y^*; A; x^*; y^*; A; x^* & \text{(local confluence)} \\
\leq \; & x^*; y^*; y^*; A; x^* & \text{(IH)} \\
= \; & x^*; y^*; A; x^* & \text{(KA)} \\
\leq \; & x^*; x^*; y^* & \text{(IH)} \\
= \; & x^*; y^* & \text{(KA)}
\end{aligned}$$

demo

Compiler optimisations in KAT

"Certification of Compiler Optimizations using Kleene Algebra with Tests"

D. Kozen and M. C. Patron '00

demo

Summary

- fairly short point-free proofs
- thanks to several tactics:
  - `ka` for deciding Kleene algebra equations [Braibant, P. '09]
  - `kat/hkat` for deciding Kleene algebra with tests equations, under Hoare assumptions [P., to be released]
  - `mrewrite` for rewriting modulo associativity [Braibant, P. '11]
- proofs can even be searched this way, by exploiting the provided counter-examples

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- Algebraic hierarchy

Challenges for future work

Tactics by reflection

- Take a decidable property

  chose a decision procedure for it

- Coq is a programming language

  program the decision procedure in Coq

- Coq is a proof assistant

  prove the correctness of your decision procedure

- Coq knows how to compute

  let it go

An example

How do `ka/kat` work?

1. Implement an algorithm to check language equivalence of regular expressions with tests
2. Prove it correct

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   harder, need matrices

4. Formalise Kozen's reduction of KAT to KA
   ($KAT \vdash \widehat{e} = e$, $G(\widehat{e}) = L(\widehat{e})$)

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5. Pack everything into a tactic, using reflection

Types

- The completeness proof for KA requires matrices

  Rectangular matrices

- The corresponding algebra is "typed"

  Operations are partial

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Rectangular matrices

- The corresponding algebra is "typed"

Operations are partial

- Natural models are also "typed"

Heterogeneous relations

$$(x;y)^*;x \;=\; x;(y;x)^* \qquad \begin{cases} x : n \to m \\ y : m \to n \end{cases}$$

$\rightarrow$ Work in a categorical setting !

- really easy with Coq dependent types
- except for decision procedures

Untyping theorem

Theorem:

Let $e, f : n \to m$ be typed regular expressions.
Let $u(e), u(f)$ denote their untyped counterparts.
If $KA \vdash u(e) = u(f)$ then $KA \vdash e = f : n \to m$.

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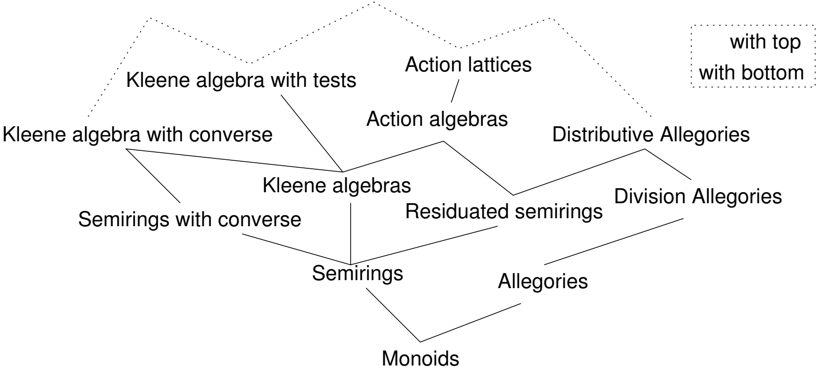
- Holds for various other fragments of relation algebra [P. '10]
  residuated semirings, allegories, cyclic linear logic
- Not all of them
  $\top \leq \top;\top$, but $\top_{A,B} \not\leq \top_{A,\emptyset};\top_{\emptyset,B}$

The cloud of relation algebra fragments



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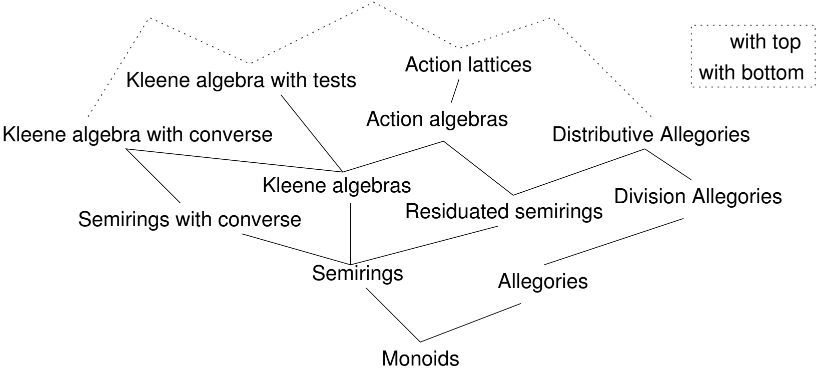
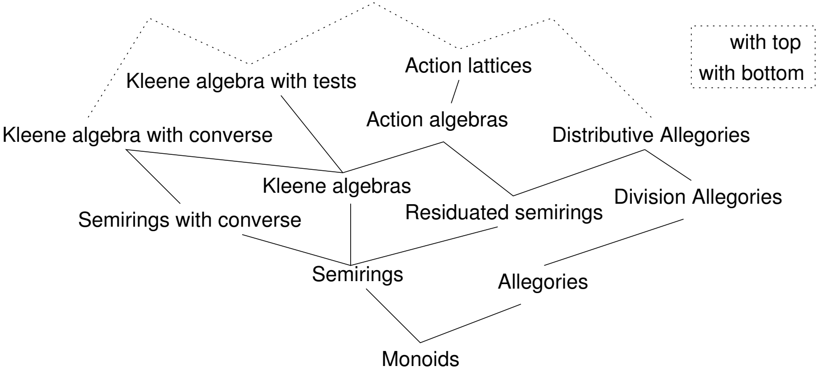


We need a modular presentation of the algebraic hierarchy:

- to capture the largest possible range of models
- to benefit from tools from lower structures when working in higher ones

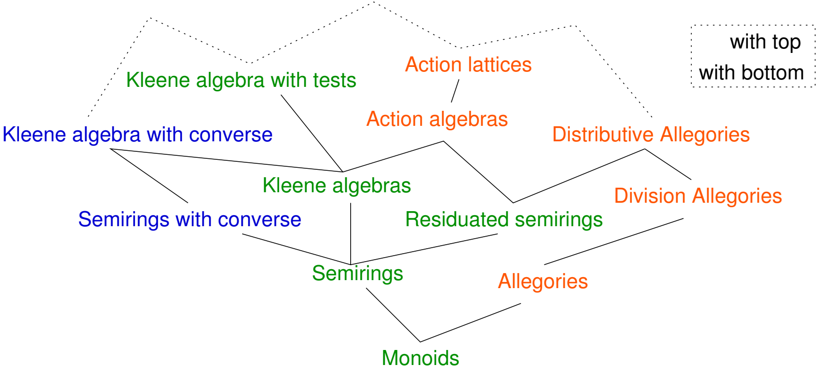
Modular algebraic hierarchy



- we failed using modules
- typeclasses do not scale
- current solution: exploit Coq's dependent types to make the relationships first-class

Outline

Algorithmics of relation algebra



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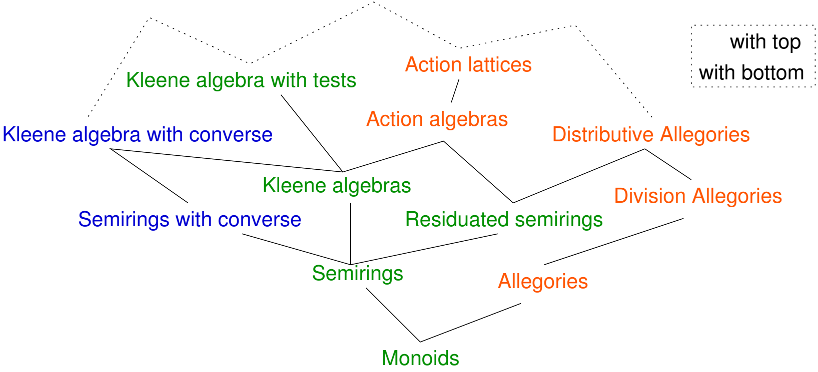

- Decidability:
  - tractable algorithm for Kleene algebra with converse?
  - decidability of action algebra? of allegories? ...
- Other properties:
  - elimination of hypotheses
  - matching / word problem
  - untyping theorem for action algebra?