Algebraic Laws for Concurrency and Separation

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Ongoing joint work with
T Hoare, B Moeller, G Struth, R Petersen...

Some Sources

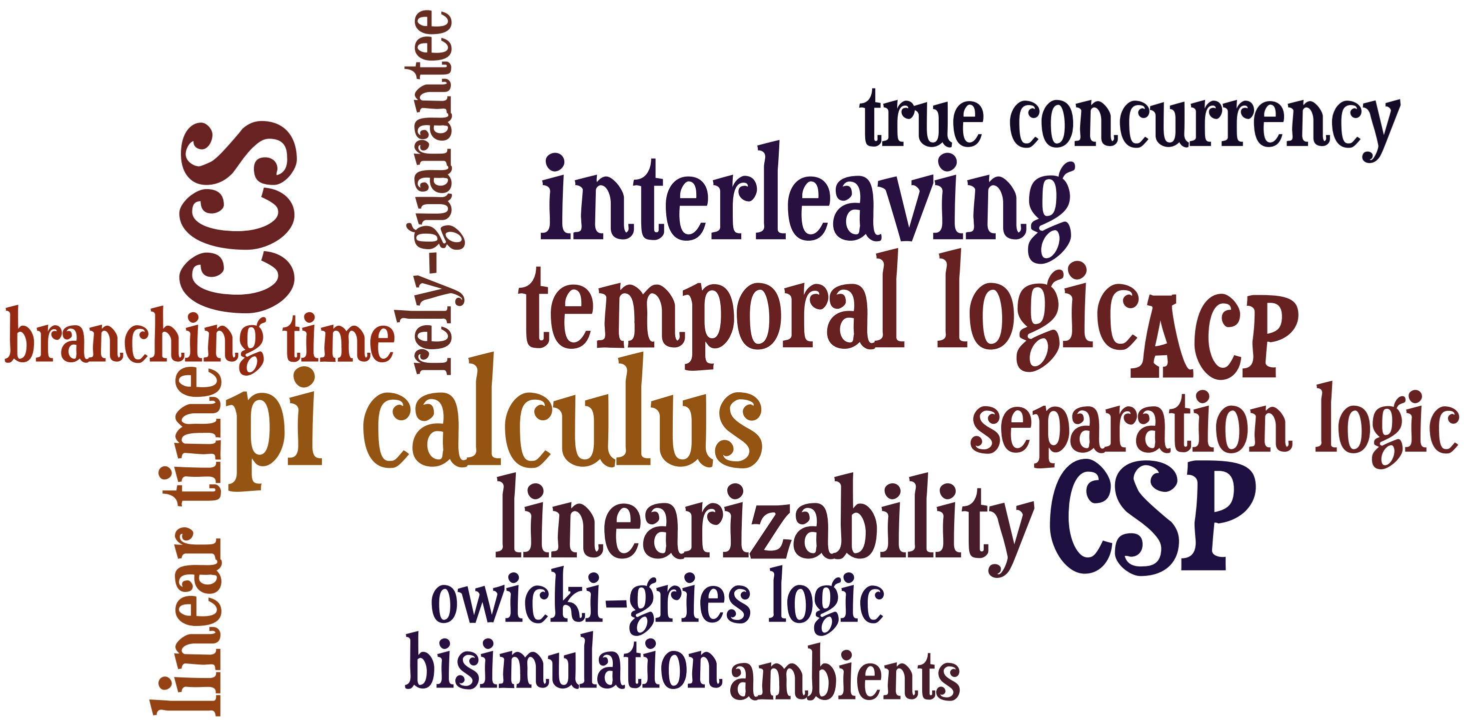
Resources, Concurrency and Local Reasoning

O'Hearn. TCS 2007

Concurrent Kleene Algebra and its Foundations

Hoare, Moeller, Struth, Wehrman. J Log Alg Prog, 2011

Diversity in theory of concurrency



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- based on a few simple axioms
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- Disclaimer: 'some' because 'all' is unrealistic as of yet: we are not in a position for a 'grand unified theory'... but will try for 'some' and see what we can do.
- This talk describes work in progress. Some parts are solid, others are in progress or are potential applications. I will say which as we go along.

Minimalist theory

- A single poset $M, \sqsubseteq$ equipped with two structures:
  - ordered commutative monoid $(\|, \mathsf{nothing})$, and
  - an ordered monoid $(\mathbf{;}, \mathsf{skip})$
- ...

Example
Linearly-ordered model: The Interleaving Model

- We define $M, \sqsubseteq, \mathit{parallel}, \mathit{nothing}, \mathbf{;}, \mathsf{skip}$.
- $M = P(E^*)$, for a given set E of events. $\sqsubseteq = \subseteq$
- $\mathsf{nothing} = \mathsf{skip} = \{\epsilon\}$
- For $P, Q \subseteq E^*$, define

$$\begin{aligned} P \parallel Q &= \{t \mid \exists t_P \in P, t_Q \in Q \,.\, t \in \mathit{interleave}(t_P, t_Q)\} \\ P \mathbf{;} Q &= \{t \mid \exists t_P \in P, t_Q \in Q.\, t = t_P t_Q\} \end{aligned}$$

Example
Partially-ordered model: the Tracelet Model (aka Tony graphs)

- Start with a partially ordered set $E, \leq$. $M = P(P(E))$.
- For $X, Y \subseteq E$, define $X \preceq Y$ to mean that nothing in Y depends on anything in X. I.e., $\forall e_Y \in Y,\, e_X \in X.\, e_Y \not\leq e_X$.
- For $p, q \subseteq \mathcal{P}(E)$, define

$$
\begin{aligned}
p \parallel q &= \{X \uplus Y \mid X \in p,\, Y \in q,\, X \cap Y = \emptyset\} \\
p \,;\, q &= \{X \uplus Y \mid X \in p,\, Y \in q,\, X \cap Y = \emptyset,\, X \preceq Y\}
\end{aligned}
$$

I Wehrman, CAR Hoare, PW O'Hearn: Graphical models of separation logic. Inf. Process. Lett. 109(17): 1001-1004 (2009)

T Hoare, BMöller, G Struth, I Wehrman: Concurrent Kleene Algebra and its Foundations. J. Log. Algebr. Program. 80(6): 266-296 (2011)

Other models

- The pomset model (Pratt, Gisher). Sets of pomsets. $P; Q$ is (lifting of) strong sequential composition (everything in P precedes everything in Q), $\|$ is disjoint concurrency (no dependence).
- The fair interleaving model. Finite and infinite sequences, $\|$ is lifting of fair parallel composition.
- Failures/divergences model of CSP.
- ...

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The historic triple

- The historic triple $\{p\}\, c\, \{q\}$ is defined by

$$\{p\}\, c\, \{q\} \;\Leftrightarrow\; p; c \sqsubseteq q$$

  for p, c, q all elements of M.
- Consequence and sequencing rules of Hoare logic follow, interpreting entailment as $\sqsubseteq$

$$\frac{p' \sqsubseteq p \qquad p\,;c \sqsubseteq q \qquad q \sqsubseteq q'}{p'\,;c \sqsubseteq q'}$$

$$\frac{p\,;c_1 \sqsubseteq q \qquad q\,;c_2 \sqsubseteq r}{p\,;c_1\,;c_2 \sqsubseteq r}$$

The historic triple

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$$\{p\}\, c\, \{q\} \;\Leftrightarrow\; p; c \sqsubseteq q$$

  for p, c, q all elements of M.

- Suppose a pre or post represents 'traces up until now'. Then $\{p\}\, c\, \{q\}$ means

  > *q accounts for (overapproximates) the immediate past p followed by c.*

A potential use of the historic triple

- In work with Rinetzky and others we have been looking at highly-concurrent optimistic algorithms.
- In the case of a 'set' algorithm, the remarkable wait free traversal is the hardest operation to prove
- We do it by reasoning about the past, via a 'Hindsight lemma':

  any pointer link encountered in a list traversal was reachable from the head node sometime in the past, since the traversal started.

- No program logic as of PODC'10: We are working on formalization via historic triples.

PW O'Hearn, N Rinetzky, MT Vechev, E Yahav, G Yorsh: Verifying linearizability with hindsight. PODC 2010: 85-94

Slide courtesy of Noam Rinetzky

3. Hindsight

(no need for linearization: existence of past state)

```
bool contains(int k) {

    p,c=LOCATE(k);
    return (c.k==k)

}
```

```
LOCATE(k)
  p = H;
  c = H.n
  while (c.k < k) {
    p = c;
    c = p.n;
  }
```

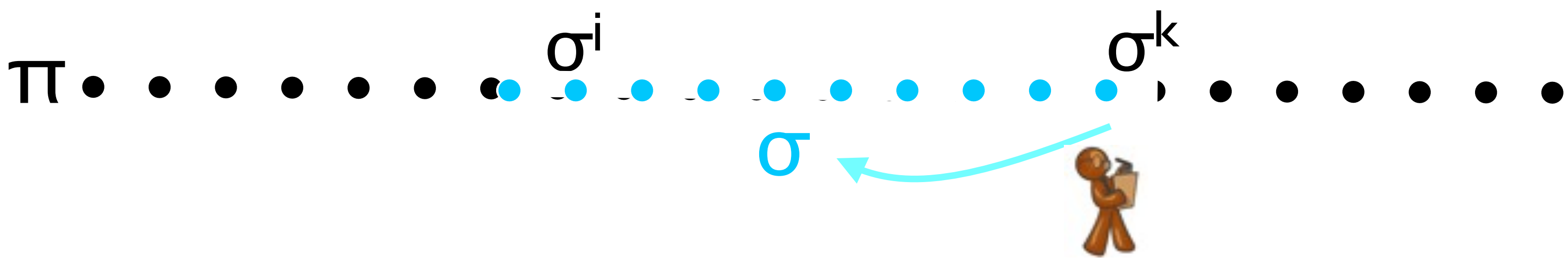
Tuesday, December 6, 2011

Hindsight Lemma

If $\pi \models \varphi$

$\sigma^i \models \varphi^i$

$\sigma^k \models \varphi^k$



then $\exists\, \sigma \in [\sigma^i \ldots\ \sigma^k]: \quad \sigma \models \psi$

Futuristic triples

- The futuristic triple $p \rightarrow_c q$ is defined by

$$p \rightarrow_c q \;\Leftrightarrow\; p \sqsupseteq c\,;q$$

Suppose a pre or post represents 'traces into the future'.
Then $p \rightarrow_c q$ means

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- Example (probable): Singularity OS has a concept of 'contract' in which preconditions and postconditions describe message passing protocols into the future.
- Formalized (Villard) with communicating automata + SL
- Likely connected as well to typestate and to session types.

M Fähndrich et. al.: Language support for fast and reliable message-based communication in singularity OS. EuroSys 2006: 177-190

J Villard: Heaps and Hops. Thèse de doctorat, ÉNS de Cachan, 2011

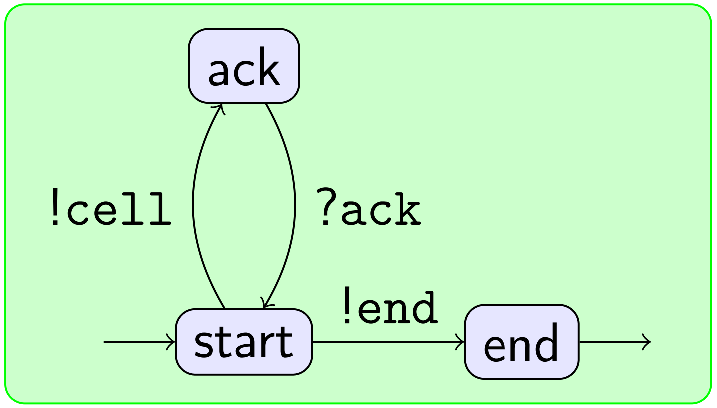
Slide courtesy of Jules Villard

Specs for List Passing

```
message cell [val ↦ −]
message ack [emp]
message endpoint [val ↦^ep (L{end}) ∧ val = src]

put(e,x) [e ↦^ep (L{start}) * list(x)] {
  local t;
  while(x != 0)
  [e ↦^ep (L{start}) * list(x)] {
    t = x->tl;
    send(cell,e,x);
    x = t;
    // e ↦^ep (L{ack}) * list(x)
    receive(ack,e);
  }
  send(endpoint,e,e);
} [emp]
```

L



Tuesday, December 6, 2011

So far...

- Trivial axioms (two ordered monoids), some particular models, and two unusual interpretations of pre/post specs.
- What we have is too little (just monotonicity, associativity...), and there are no axioms linking $\|$ and $;$.
- On our way to program logic, but we need more...

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- satisfying the exchange law

$$(p \| r)\mathbf{;}(q \| s) \sqsubseteq (p\mathbf{;}q) \| (r\mathbf{;}s)$$

- ...

Inequational exchange law emphasized by Hoare (2008-), mentioned before in concurrency by Gischer'88, Bloom-Ésik'95

Exchange law in Interleaving model

- exchange law: $(p \| r);(q \| s) \sqsubseteq (p;q) \| (r;s)$
- Writing a trace t for the singleton $\{t\}$, an instance is

$$(aa \| b);(cc \| d) \sqsubseteq (aa;cc) \| (b;d)$$

Then, for example,

$$aba \in interleave(aa, b) \text{ and } cdc \in interleave(cc, d)$$

Clearly $abacdc \in interleave(aacc, bd)$.

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  Then, for example,

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  Clearly $abacdc \in \mathit{interleave}(aacc, bd)$.
- The reverse inclusion does not hold:

$$aaccbd \in (aa;cc) \| (b;d)$$
$$\text{but}$$
$$aaccbd \notin (aa \| b);(cc \| d)$$

  so we cannot have the *equational* exchange law.

Exchange in Tracelet model

- Recall: $X \preceq Y$ means that nothing in Y depends on anything in X.
- For $p, q \subseteq \mathcal{P}(E)$, define

$$
\begin{aligned}
p \parallel q &= \{X \uplus Y \mid X \in p,\ Y \in q,\ X \cap Y = \emptyset\} \\
p \,;\, q &= \{X \uplus Y \mid X \in p,\ Y \in q,\ X \cap Y = \emptyset,\ X \preceq Y\}
\end{aligned}
$$

- Special case of exchange law ,

$$(X_1 \parallel Y_1);(X_2 \parallel Y_2) \sqsubseteq (X_1;X_2) \parallel (Y_1;Y_2)$$

boils down to

$$X_1 \uplus Y_1 \preceq X_2 \uplus Y_2 \ \Rightarrow\ \begin{array}{c} X_1 \preceq X_2 \\ \wedge \\ Y_1 \preceq Y_2 \end{array}$$

A negative example: fair $\|$ with subset order

- Consider finite and infinite traces with $\|$ being fair parallel composition.
- Without giving a definition of fairness, let us just assume that any trace of $a^\omega \| b$ *must* include b, and that $tt' = t$ if t is infinite.

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- Then

$$(a^\omega \| b);(c \| d) \not\sqsubseteq (a^\omega;c) \| (b;d)$$

because

$$ca^\omega \in (a^\omega;c) \| (b;d)$$

but it doesn't include a b, so

$$ca^\omega \notin (a^\omega \| b);(c \| d)$$

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- I attach no deep significance to this, but am just illustrating that our theory covers 'some' but not 'all' models of interest.

Exchange and logic: Locality on the cheap

- Historic triples $(\{p\}\, c\, \{q\} \;\Leftrightarrow\; p; c \sqsubseteq q)$

$$\dfrac{\dfrac{p_1 \,;\, c_1 \sqsubseteq q_1 \qquad p_2 \,;\, c_2 \sqsubseteq q_2}{(p_1; c_1) \parallel (p_2 \,;\, c_2) \sqsubseteq q_1 \parallel q_2}\ \parallel\ \textit{Monotone}}{(p_1 \parallel p_2) \,;\, (c_1 \parallel c_2) \sqsubseteq q_1 \parallel q_2}\ \textit{Exchange}$$

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- If we squint, this is the concurrency rule of concurrent separation logic

$$\frac{\{P_1\}\, C_1\, \{Q_1\} \quad \{P_2\}\, C_2\, \{Q_2\}}{\{P_1 * P_2\}\, C_1 \| C_2\, \{Q_1 * Q_2\}}$$

- And similar works for futuristic triples.

A CSL example: Parallel Mergesort

$$
\begin{array}{l}
\{\mathit{array}(a,i,j)\} \\
\texttt{procedure ms}(a,i,j) \\
\texttt{newvar } m := (i+j)/2;; \\
\texttt{if} \quad i<j \texttt{ then} \\
\qquad (\texttt{ms}(a,i,m) \ \| \ \texttt{ms}(a,m+1,j));; \\
\qquad \texttt{merge}(a,i,m+1,j);; \\
\{\mathit{sorted}(a,i,j)\}
\end{array}
$$

Main part of proof:

$$
\begin{array}{lcl}
\multicolumn{3}{c}{\{\mathit{array}(a,i,m) * \mathit{array}(a,m+1,j)\}} \\
\{\mathit{array}(a,i,m)\} & & \{\mathit{array}(a,m+1,j)\} \\
\texttt{ms}(a,i,m) & \| & \texttt{ms}(a,m+1,j) \\
\{\mathit{sorted}(a,i,m)\} & & \{\mathit{sorted}(a,m+1,j)\} \\
\multicolumn{3}{c}{\{\mathit{sorted}(a,i,m) * \mathit{sorted}(a,m+1,j)\}}
\end{array}
$$

Concurrency and Frame rules are linked

- Concurrency and Frame rules from SL

$$\frac{\{P_1\}\, C_1\, \{Q_1\} \quad \{P_2\}\, C_2\, \{Q_2\}}{\{P_1 * P_2\}\, C_1 \| C_2\, \{Q_1 * Q_2\}} \qquad \frac{\{P\}\, C\, \{Q\}}{\{P * F\}\, C\, \{Q * F\}}$$

- If $C = C \parallel \mathsf{skip}$ then we can derive Frame from Concurrency

$$\frac{\{P\}\, C\, \{Q\} \qquad \{F\}\, \mathsf{skip}\, \{F\}}{\{P * F\}\, C \parallel \mathsf{skip}\, \{Q * F\}}$$

- In the algebra, we will not *assume* that $C = C \parallel \mathsf{skip}$ for all C, but take this as the *definition* of locality

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CAR Hoare, A Hussain, B Möller, PW O'Hearn, RL Petersen, G Struth: On Locality and the Exchange Law for Concurrent Processes. CONCUR 2011

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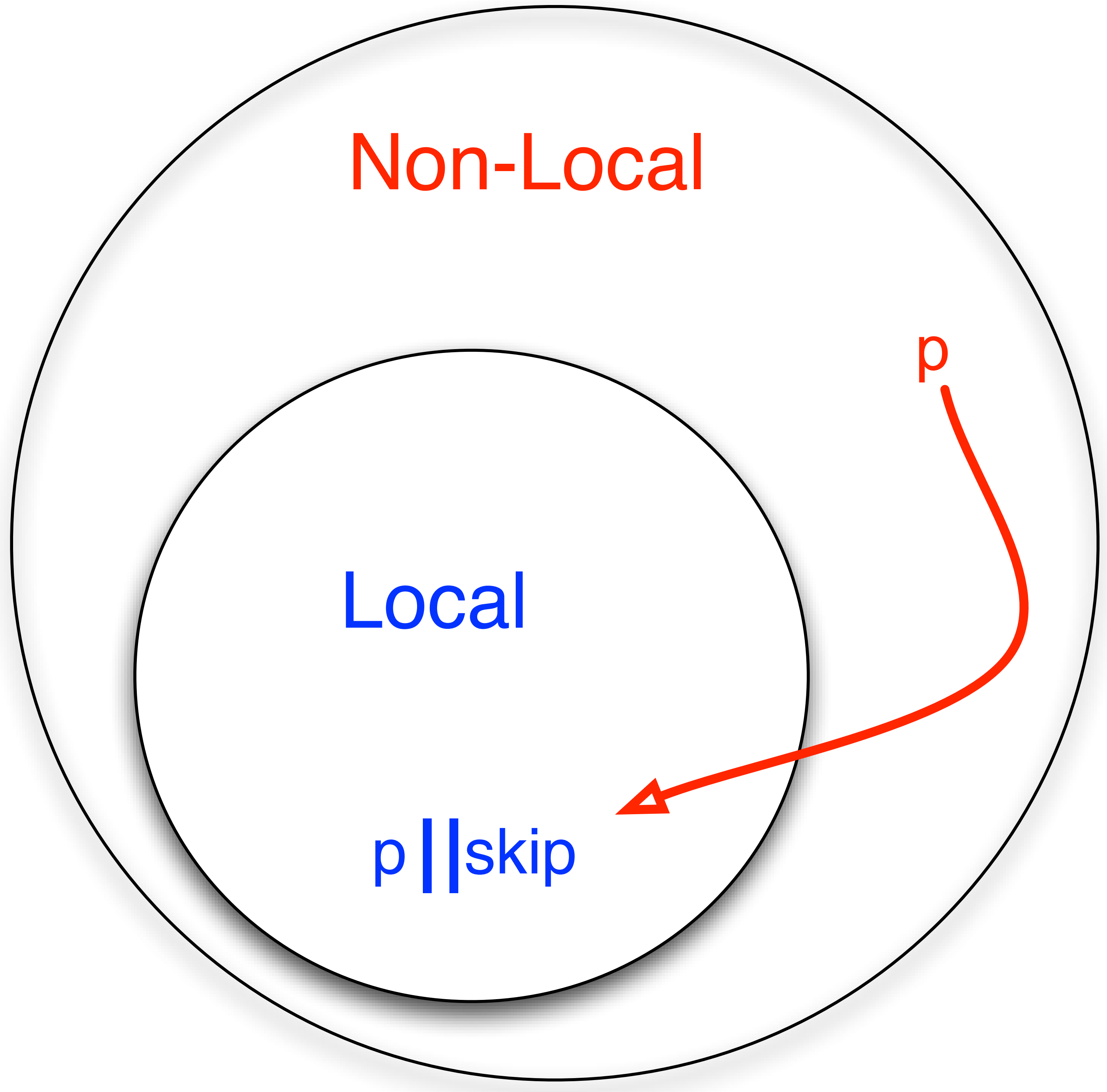
- Facts:
  - $\|$ and $\mathbf{;}$ preserve locality.
  - Let M_{loc} be the local elements. Galois connection with left adjoint $M_{loc} \hookrightarrow M$ and right adjoint $\lambda p.p \parallel \mathsf{skip}$
  - The SL concurrency rule holds in any locality bimonoid. The frame rule holds of historic triples of the form $\{p\}\, c\, \{q\}$ iff $c = c \parallel \mathsf{skip}$ (and similarly for futuristic triples)

CAR Hoare, A Hussain, B Möller, PW O'Hearn, RL Petersen, G Struth: On Locality and the Exchange Law for Concurrent Processes. CONCUR 2011


Non-Local
p
Local
p||skip

Perspective

- From our minimalist axioms, we automatically get lots of proof rules (Hoare and concurrent separation logic)
- For a range of models
- Wait a minute: do they mean what we expect? Is this cheating?

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- Turning the tables: Start from CSL, see if we can get locality bimonoid. If we succeed we confirm, no cheat in the logic, and we get lots of more example models.

Basic CSL

$$[\texttt{Skip}] \quad \frac{}{\{X\}\,\texttt{skip}\,\{X\}} \qquad [\texttt{Frame}] \quad \frac{\{X\}\,c\,\{Y\}}{\{X * F\}\,c\,\{Y * F\}}$$

$$[\texttt{Seq}] \quad \frac{\{X\}\,c_1\,\{Y\} \quad \{Y\}\,c_2\,\{Z\}}{\{X\}\,c_1; c_2\,\{Z\}} \qquad [\texttt{Par}] \quad \frac{\{X_1\}\,c_1\,\{Y_1\} \quad \{X_2\}\,c_2\,\{Y_2\}}{\{X_1 * X_2\}\,c_1 \parallel c_2\,\{Y_1 * Y_2\}}$$

$$[\texttt{Consequence}] \quad \frac{X' \vdash X \quad \{X\}\,c\,\{Y\} \quad Y \vdash Y'}{\{X'\}\,c\,\{Y'\}}$$

An *instance* of Basic CSL presumes a preordered commutative monoid $(Props, \vdash, *, \texttt{emp})$ and a set of axioms $\{X\}\,c_p\,\{Y\}$ for a set of primitive command c_p and $X, Y \in Prop$.

BCSL minus `Frame` can be interpreted in any locality bimonoid. `Frame` holds when primitive commands are local.

Embedding

Theorem. From the proof theory of BCSL one can construct a locality bimonoid (model of minimalist theory) together with

- embeddings of propositions and programs into the bimonoid,
- sending $*$ to $\|$ and preserving and reflecting order,
- sending programs to elements of the bimonoid, such that

$$\{p\}\, c\, \{q\} \text{ is provable in BCSL} \iff$$
$$embed(p) \,;\, embed(c) \sqsubseteq embed(q)$$

Ideas in the proof

- Use ideal completion: map a proposition p to everything that entails it $p\Downarrow$. Down-closed subsets have rich structure: complete Heyting algebra, residuated monoid (cf. BI algebra).
- Intuitionistic BI semantics of $*$ on down-closed sets (call it $\circledast$)

$$\begin{array}{lcl} P \circledast Q & = & \{X \mid Y \in P \wedge Z \in Q \wedge X \vdash Y * Z\} \\ I & = & \{p \mid p \vdash emp\} \end{array}$$

- Monotone function space $[Preds \rightarrow Preds]$ is carrier of our algebra

$$\begin{array}{lcl} (F_1 \| F_2)Y & = & \bigcup\{F_1 Y_1 \circledast F_2 Y_2 \mid Y_1 \circledast Y_2 \subseteq Y\} \\ \text{nothing } Y & = & Y \cap I \\ (F_1 ; F_2)Y & = & F_1(F_2(Y)) \\ \text{skip } Y & = & Y \end{array}$$

- Inject predicate P into predicate transformers using greatest transformer F satisfying $\mathtt{emp} \subseteq F(P)$. This maps $\circledast$ to $\|$.

Ack to H Yang: suggestion of $F_1 \| F_2$.

Sum up

- Minimalist theory with a few axioms:

  A single poset $M, \sqsubseteq$ *equipped withan ordered commutative monoid* $(\|, \mathsf{nothing})$ *and an ordered monoid* $(\mathbf{;}, \mathsf{skip})$, *satisfying the exchange law, where* $\mathsf{skip} \parallel \mathsf{skip} = \mathsf{skip}$.

- Connection with program logic: generalized CSL.
- Lots of models: interleaving, independence, resource separation...
- Temporal readings of triples which we are exploring

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- **Speculation**: programs and assertions are part of the same space. I wonder if we can push this and make a more genuine logic encompassing both, also bringing out the temporal aspect?

Maximalist model (tentative.. speculation)

- The traces model $P(E^*)$ has lots more structure. Ditto for tracelet model.
- $G = (M, \sqsubseteq, *, \mathsf{nothing}, \mathbf{;}, \mathsf{skip})$ is an ordered **residuated** commutative monoid $(*, \mathsf{nothing})$ and a ordered **residuated** monoid $(\mathbf{;}, \mathsf{skip})$ on the same **complete boolean algebra** $(M, \sqsubseteq)$, satisfying exchange, where $\mathsf{skip} * \mathsf{skip} = \mathsf{skip}$.
- Residuation means that we have the adjoint cousins

$$p * q \sqsubseteq r \quad \Leftrightarrow \quad p \sqsubseteq q \mathrel{-\!\!*} r$$

$$p; q \sqsubseteq r \quad \Leftrightarrow \quad p \sqsubseteq q \triangleright r \quad \Leftrightarrow \quad q \sqsubseteq q \triangleleft r$$

- We have classical predicate logic (complete bool alg), alongside full-strength substructural logics (like in BI/SL).
- These connectives have a declarative reading given by a Kripke semantics (a la bunched/separation logic), where $\mathbf{;}, \triangleleft, \triangleright$ have a temporal flavour

- E.g., in the tracelet model
  (recall that X, Y etc are subsets of a given poset $E, \leq$)

  $Y \preceq X$ means that nothing in X depends on anything in Y.
  Then,

  $$X \models p \lhd q \;\text{ iff }\; \forall Y.\, Y \preceq X \text{ and } Y \models p \text{ implies } Y \uplus X \models q$$

- $$\begin{aligned}\mathsf{previous}(p) &= \neg(p \rhd \mathsf{false}) \\ &= \exists Y. \neg(Y \preceq X \; \mathit{and} \; Y \models p \; \mathit{implies} \; \mathsf{false}) \\ &= \exists Y. Y \preceq X \; \mathit{and} \; Y \models p\end{aligned}$$