On Completeness of ω-regular Algebras

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Introduction to Regular algebras

Definition A *regular algebra* is an algebra D with operations $0, +, ., ^+$ that satisfies the following axioms;

$(D, +, ., 0)$ is an idempotent semiring and

$$x + xx^+ = x^+, \qquad xy \leq y \Rightarrow x^+ y \leq y$$

hold, where

$$x \leq z \Leftrightarrow x + z = z.$$

A *unital regular algebra* is a regular algebra with a multiplicative identity 1.

Examples;

$\mathbf{Reg}_\Sigma^{\not 1}$ (the algebra of regular languages not containing ϵ) is a regular algebra,

$\mathbf{Reg}_\Sigma$ is a unital regular algebra.

The empty word property for Regular languages and Regular terms

We define

$$o(L) = \begin{cases} 1 & \epsilon \in L \\ 0 & \epsilon \notin L. \end{cases}$$

For regular expressions over alphabet Σ, we define

$$\begin{aligned} o(0) &= 0 \\ o(\sigma) &= 0 \text{ if } \sigma \in \Sigma \\ o(s+t) &= o(s) + o(t) \\ o(st) &= o(s)o(t) \\ o(s^+) &= o(s)^+. \end{aligned}$$

Hence $o(\llbracket t \rrbracket) = o(t)$ always holds.

Lemma. For every regular term s, there exists a term t with $o(t) = 0$ such that

$$s = o(s) + t$$

holds in unital regular algebra.

ω-Regular algebras

Definition Let R be a regular algebra. An R-module is a structure $(R, L, :)$ in which $(L, +)$ is a semilattice with least element 0_ω, and operation : has type $R \times L \to L$, and the following axioms hold:

$$
\begin{aligned}
x : (X + Y) &= x : X + x : Y, \\
(x + y) : Z &= x : Z + y : Z, \\
(xy) : Z &= x : (y : Z), \\
0 : Z &= 0_\omega, \\
x : 0_\omega &= 0_\omega
\end{aligned}
$$

An R-module is unital if R is unital and

$$1 : X = X$$

holds.

A unital R-module is a unital Wagner algebra if it has a further operation $^\omega : R \to L$ such that

$$o(s) = 0 \Rightarrow s^\omega = (ss^*)^\omega,$$
$$o(st) = 0 \Rightarrow (st)^\omega = s(ts)^\omega,$$

$$o(s+t) = 0 \wedge (s+t)^\omega = t(s+t)^\omega + S$$
$$\Rightarrow$$
$$(s+t)^\omega = t^\omega + t^*S.$$

A complete axiomatisation of valid equalities for ω-regular languages

Theorem[Wagner 1976] For an alphabet Σ,

$$[\![T]\!] = [\![T']\!]$$

holds for ω-regular terms T, T' over Σ iff

$$T = T'$$

is derivable from unital Wagner axioms.

Embedding a regular algebra in a unital regular algebra

Lemma

A regular algebra can be embedded in a unital regular algebra.

Sketch proof.

Let K be a regular algebra, then for $\mathbb{B} = \{0, 1\}$ (the 2-element Boolean algebra),

$$\mathbb{B} \times K$$

is a unital regular algebra, with the definitions

$$\begin{aligned}
(m,x) + (n,y) &= (m+n, x+y) \\
(m,x)(n,y) &= (mn, my + nx + xy) \\
(m,x)^{+} &= (m, x^{+})
\end{aligned}$$

and thus $\begin{cases} (1,0) & \text{is identity for multiplication} \\ (0,0) & \text{is zero element.} \end{cases}$

Corollary

Let s, t be terms with operations $0, +, ., ^+$ over a variable set; then

$$\begin{array}{ll} \text{regular algebras} & \models s = t \\ \Leftrightarrow & \\ \text{unital regular algebras} & \models s = t \end{array}$$

holds.

Non-unital Wagner algebras

Definition A (non-unital) Wagner algebra is an R-module (R, L) (for a non-unital regular algebra R) that has an additional function

$$^{\omega} : R \to L$$

satisfying the following;

$$\begin{aligned} x^{\omega} &= x^{+\omega}, && (1)\\ x^{\omega} &= xx^{\omega}, && (2)\\ (xy)^{\omega} &= x(yx)^{\omega}, && (3)\\ (xy+y)^{\omega} &= x(yx+y)^{\omega} + (yx+y)^{\omega}, && (4)\\ (xy+x)^{\omega} &= x(yx+x)^{\omega}, && (5) \end{aligned}$$

$$\begin{aligned} (x+y)^{\omega} &= y(x+y)^{\omega} + X \\ &\Rightarrow \\ (x+y)^{\omega} &= y^{\omega} + x^{+}X + X. && (6) \end{aligned}$$

Theorem Any non-unital Wagner algebra (K, L) can be 'embedded' in a Wagner algebra (K', L), where K' is unital regular algebra.

Sketch Proof.

Le $K' = \mathbb{B} \times K$, then define

$$\begin{aligned}
(0, x)(m, X) &= (m, xX), \\
(1, x)(m, X) &= (m, X) + (m, xX), \\
(0, x)^{\omega} &= (0, x^{\omega}).
\end{aligned}$$

Main Theorem

For 1-free ω-regular terms T, T',

non-unital Wagner algebras $\models T = T'$
$\Leftrightarrow$
unital Wagner algebras $\models T = T'$

holds.

Corollary

The non-unital Wagner algebras axiomatise the set of universal equalities $T = T'$ for which

$$[\![T]\!] = [\![T']\!]$$

holds.